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COURSE IN MATHEMATICS 3

for the IIT-JEE & Other Engineering Entrance Examinations

Trigonometry

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COURSE IN MATHEMATICS

(FOR IIT JEE AND OTHER ENGINEERING ENTRANCE EXAMINATIONS)

TRIGONOMETRY

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PEARSON

Chandigarh • Delhi • Chennai

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PREFACE

When a new book is written on a well known subject like *Trigonometry* for class XI/XII Academics/ AIEEE/IIT/State engineering entrance exams and NDA, several questions arise like—why, what, how and for whom? What is new in it? How is it different from other books? For whom is it meant? The answers to these questions are often not mutually exclusive. Neither are they entirely satisfactory except perhaps to the authors. We are certainly not under the illusion that there are no good books. There are many good books available in the market.

However, none of them caters specifically to the needs of students. Students find it difficult to solve most of the problems of any of the books in the absence of proper planning. This inspired us to write this book *Trigonometry*, to address the requirements of students of class XI/XII CBSE and State Board Academics. In this book, we have tried to give a connected and simple account of the subject. It gives a detailed, lecture wise description of basic concepts with many numerical problems and innovative tricks and tips. Theory and problems have been designed in such a way that the students can themselves pursue the subject. We have also tried to keep this book self contained. In each lecture all relevant concepts, prerequisites and definitions have been discussed in a lucid manner and also explained with suitable illustrated examples including tests.

Due care has been taken regarding the Board (CBSE/ State) examination need of students and nearly 100 per cent articles and problems set in various examinations including the IIT-JEE have been included.

The presentation of the subject matter is lecturewise, intelligent and systematic, the style is lucid and rational, and the approach is comprehensible with emphasis on improving speed and accuracy. The basic motive is to attract students towards the study of mathematics by making it simple, easy and interesting and on a day-to-day basis. The instructions and method for grasping the lectures are clearly outlined topic wise. The presentation of each lecture is planned for better experiential learning of mathematics which is as follows:

1. Basic Concepts: Lecture Wise
2. Solved Subjective Problems (XII Board (C.B.S.E./State): For Better Understanding and Concept Building of the Topic.
3. Unsolved Subjective Problems (XII Board (C.B.S.E./State): To Grasp the Lecture Solve These Problems.
4. Solved Objective Problems: Helping Hand.
5. Objective Problem: Important Questions with Solutions.
6. Unsolved Objective Problems (Identical Problems for Practice)
   For Improving Speed with Accuracy.

7. Worksheet: To Check Preparation Level
8. Assertion-Reason Problems : Topic Wise Important Questions and Solutions with Reasoning
9. Mental Preparation Test: 01
10. Mental Preparation Test: 02
11. Topic Wise Warm Up Test: 01: Objective Test
12. Topic Wise Warm Up Test: 02: Objective Test
13. Objective Question Bank Topic Wise: Solve These to Master.

This book will serve the need of the students of class XI/XII board, NDA, AIEEE and SLEEE (state level engineering entrance exam) and IIT-JEE. We suggest each student to attempt as many exercises as possible without looking up the solutions. However, one should not feel discouraged if one needs frequent help of the solutions as there are many questions that are either tough or lengthy. Students should not get frustrated if they fail to understand some of the solutions in the first attempt. Instead they should go back to the beginning of the solution and try to figure out what is being done At the end of every topic, some harder problems with 100 per cent solutions and Question Bank are also given for better understanding of the subject.

There is no end and limit to the improvement of the book. So, suggestions for improving the book are always welcome.

We thank our publisher, Pearson Education for their support and guidance in completing the project in record time.

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PART A

Trigonometric Functions and Identities

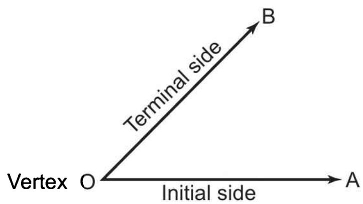
LECTURE 1

Measurement of Angles

BASIC CONCEPTS

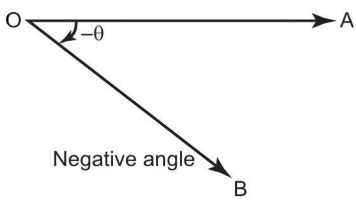
1. MEASUREMENT OF AN ANGLE

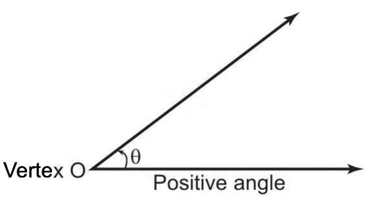
The measurement of an angle is the amount of rotation from the initial side to the terminal side.



2. SENSE OF AN ANGLE

The sense of angle is positive or negative according as the initial side rotates in anti-clockwise or clockwise direction to get the terminal side.





2.1 Types of Angles

An angle θ is defined to be

(i) an acute angel, if $0° \leq \theta < 90°$
(ii) an obtuse angle, if $90° < \theta < 180°$
(iii) a reflex angle, if $180° < \theta < 360°$
(iv) a right angle, if $\theta = 90°$
(v) a straight angle, if $\theta = 180°$

2.2 Coterminal Angles

Two angles having different measures but same initial and terminal sides are said to be coterminal angles. e.g., (i) Angles with measure –30° and 330° are coterminal. (ii) Angles with measure 45° and –315° are coterminal.

3. SYSTEMS OF MEASURING ANGLES

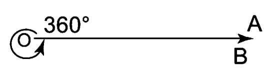
There are three systems of measuring angles:

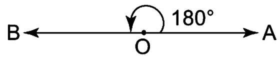
(i) Sexagesimal system
(ii) Centesimal system
(iii) Circular system

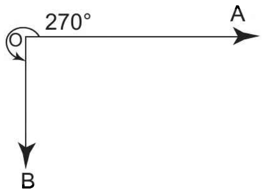
3.1 Sexagesimal System

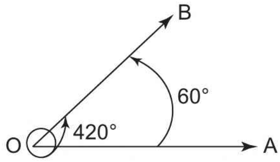
In sexagesimal system, we have 1 right angle = 90 degrees (90°), 1° = 60 minutes = (60′) 1′ = 60 seconds (= 60″)

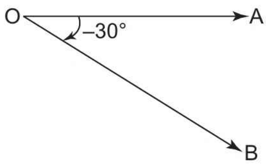
Some of the angles whose measures are 360°, 180°, 270°, 420° and –30° are shown in figures below:









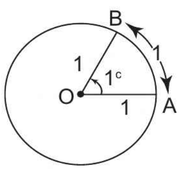


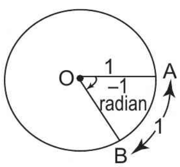
3.2 Centesimal System

In centesimal system, we have 1 right angle = 100 grades (= 100^g) 1^g = 100 minutes (= 100′); 1′ = 100 seconds (= 100″)

3.3 Circular System

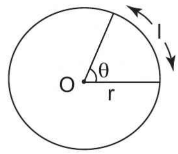
In circular system, the unit of measurement is radian. One radian, written as 1^c, is the measure of an angle subtended at the centre of a circle by an arc of length equal to the radius of the circle. Following figures show the angles whose measure are: 1 radian and –1 radian.





4. RELATION BETWEEN θ, *l* AND *r*

If in a circle of radius r, an arc of length l subtends an angle of θ radians, then $l = r\,\theta$ or $\theta = l/r$



NOTE

Here l, r are in the same units and θ is always in radians.

5. RELATION BETWEEN THREE SYSTEMS OF MEASUREMENT OF AN ANGLE

π Radian = π^c = 180° = 200^g = $2rt\ \angle s$ = 2 right angles

5.1 Conversion Formula

$$x^\circ = \left(x \times \frac{\pi}{180}\right) \text{ radians} = \left(x \times \frac{200}{180}\right) \text{ grades}$$

$$x^C = \left(x \times \frac{180}{\pi}\right) \text{ degrees} = \left(x \times \frac{200}{\pi}\right) \text{ grades}$$

6. RELATION BETWEEN DEGREE AND RADIAN

Since a circle subtends at centre, an angle whose radian measure is 2π and its degree measure is 360°, it follows that

2 π radian = 360° or π radian = 180°

or 1^c radian $= \dfrac{180°}{\pi}$ = 57° 17′ 44.8″, nearly

Also $1° = \dfrac{\pi}{180}$ radian ≈ 0.01746 radian

6.1 Value of $\pi = \dfrac{22}{7}$ or $\dfrac{355}{113}$ or 3.141592.........

π is an irrational number.

Some Useful Points

(i) The angle between two consecutive digits in a clock is $30°\left(=\dfrac{\pi}{6}\text{radians}\right)$.

(ii) The hour hand rotates through an angle of 30° in one hour i.e., $\left(\dfrac{1}{2}\right)^\circ$ in one minute.

(iii) The minute hand rotates through an angle of 6° in one minute.

(iv) Area of a circular sector which has a central angle of θ is $\frac{1}{2}r^2\theta$.

(v) Each inferior angle of a regular polygon of side n is = $(n-2)\frac{\pi}{n}$

NOTES

a, b, c are in A.P. then

(i) $2b = a + c$

(ii) $b - c = c - b$ = common difference

(iii) 1, 3, 5 are in A.P.

(iv) $60° - \theta, 60°, 60° + \theta$ are in A.P.

(v) $a - d, a, a + d$ are in A.P.

$\frac{\pi}{6} = (30)°$, $\frac{\pi}{4} = (45)°$, $\frac{\pi}{8} = \left(22\frac{1}{2}\right)°$,

$\frac{\pi}{12} = (15)°$, $\left(\frac{\pi}{15}\right) = 12°$, $\frac{3\pi}{8} = \left(67\frac{1}{2}\right)°$,

$\frac{\pi}{5} = (36)°$, $\left(\frac{\pi}{9}\right) = 20°$, $\frac{\pi}{10} = (18)°$

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Find the degree measure corresponding to the following radian measures:

(i) $\left(\frac{1}{4}\right)$ (ii) $-2°$

Solution

(i) $\left(\frac{1}{4}\right)° = \left(\frac{1}{4}\times\frac{180}{\pi}\right)° = \left(\frac{1}{4}\times\frac{180}{22}\times 7\right)°$

$= \left(\frac{315}{22}\right)° = \left(14\frac{7}{22}\right)°$

$= 14°\left(\frac{7}{22}\times 60\right)' = 14°\left(19\frac{1}{11}\right)'$

$= 14°19'\left(\frac{1}{11}\times 60\right)'' = 14°19'5''$

(ii) $(-2)^c = \left(\frac{180}{\pi}\times -2\right)° = \left(\frac{180}{22}\times 7\times(-2)\right)°$

$= \left(-114\frac{6}{11}\right)° = \left(-114°\left(\frac{6}{11}\times 60\right)'\right)$ γ

$= -\left[114°\left(32\frac{8}{11}\right)'\right]$

$= -\left[114°32'\left(\frac{8}{11}\times 60\right)''\right]$

$= -[114°32'44'']$

2. The circular measures of two angles of a triangle are $\frac{1}{2}$ and $\frac{1}{3}$, find the third angle in English system. (Take $\pi = \frac{22}{7}$)

Solution

We know that the sum of three angles of a triangle is 180°, i.e., π radians.

∴ The third angle = $\left(\pi - \frac{1}{2} - \frac{1}{3}\right)$

radian = $\left(\frac{22}{7} - \frac{1}{2} - \frac{1}{3}\right)^c$

$= \left(\frac{132-21-14}{42}\right)^c = \left(\frac{97}{42}\times\frac{180°}{\pi}\right)°$

($\because \pi^c = 180°$)

$= \left(\frac{97\times 30}{22}\right)° = \frac{1455}{11}$ degree

$= \left(132\frac{3}{11}\right)° = 132°\left(\frac{3\times 60}{11}\right)'$

$= 132°\left(16+\frac{4}{11}\right)' = 132°16'\left(\frac{4}{11}\times 60\right)''$

3. A wheel makes 360 revolutions in one minute. Through how many radius does it turn in one second?

Solution

Number of revolutions made in 60 seconds = 360

($\therefore$ 1 minute = 60 second)

Number of revolutions made in 1 seconds = $\frac{360}{60} = 6$.

When the wheel makes one revolution, it turns through 360° or 2π radians

$\therefore$ Number of radians turned by the wheel in one second = $6 \times 2\pi = 12\pi$.

4. Find the degree measure of the angle subtended at the centre of a circle of radius 100 cm by an arc of length 22 cm.

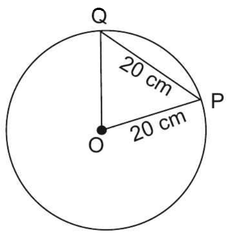
$\left(\text{Use } \pi = \frac{22}{7}\right)$

Solution

Here 'l' = 22 cm and 'r' = 100 cm

Let θ be the circular measure of the angle subtended, then

$$\theta = \frac{l}{r} = \frac{22\text{ cm}}{100\text{ cm}} = \frac{11}{50}$$



Hence, the angle subtended = $\frac{11}{50}$ radian

$$= \frac{11}{50} \times \left(\frac{180}{\pi}\right)^{\circ}$$

$$\left(\because \pi \text{ radian} = 180°, \therefore 1 \text{ radian} = \left(\frac{180}{\pi}\right)^{\circ}\right)$$

$$= \left(\frac{11}{50} \times \frac{180 \times 7}{22}\right)^{\circ} = \left(\frac{63}{5}\right)^{\circ} = \left(12 + \frac{3}{5}\right)^{\circ}$$

$$= 12° \left(\frac{3}{5} \times 60\right)' = 12°36'.$$

5. In a circle of diameter 40 cm, the length of a chord is 20 cm. Find the length of minor arc of the chord.

Solution

Radius of the circle = $\frac{1}{2}$(diameter)

$$= \frac{1}{2}\text{ (40 cm)} = 20\text{ cm}$$

Length of chord = 20 cm.

If O is the centre of the circle and chord is PQ, then

$\angle POQ = 60°$($\therefore$ $\Delta\, OPQ$ is an equilateral triangle)

$$= \left(\frac{60}{180}\pi\right) \text{ radian}$$

($\because$ 180° = π radian $\therefore 1° = \frac{\pi}{180}$ radian)

$$= \frac{\pi}{3} \text{ radian}$$

$\Rightarrow$ Circular measure of $\angle POQ = \frac{\pi}{3}$.

If l is the length of arc PQ, then $\frac{l}{20\text{ cm}} = \frac{\pi}{3}$

$$\left(\because \theta = \frac{l}{r}\right)$$

$$\Rightarrow l = \frac{\pi}{3} \times 20\text{ cm} = \frac{22 \times 20}{3 \times 7}\text{ cm} = 20.95\text{ cm}.$$

6. A horse is tied to a post by a rope. If the horse moves along a circular path always keeping the rope tight and describes 88 metres when it was traced out 72° at the centre. Find the length of the rope.

Solution

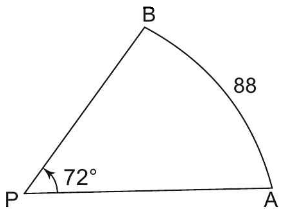
Let the post be at point P and let PA be the length of the rope in tight position. Suppose the horse moves along the arc AB so that $\angle APB = 72°$ and arc AB = 88 m. Let r be the length of the rope i.e., $PA = r$ metres.

Now, $\theta = 72° = \left(72 \times \frac{\pi}{180}\right)^c = \left(\frac{2\pi}{5}\right)^c$ and

S = 88 m

$$\therefore \quad \theta = \frac{\text{arc}}{\text{radius}}$$

$$\Rightarrow \quad \frac{2\pi}{5} = \frac{88}{r}$$



$$\Rightarrow \quad r = \frac{88 \times 5}{2\pi} = 70 \text{ metres.}$$

7. Find the angle between the minute hand of a clock and the hour hand when the time is 7 : 20 AM.

Solution

We know that the hour hand completes one rotation in 12 hours, while the minutes hand completes one rotation in 60 minutes.

$\therefore$ Angle traced by the hour hand in 12 hours = 360°.

$\Rightarrow$ Angle traced by the hour hand in 7 hours 20 minute i.e.,

$$\frac{22}{3} \text{ h}$$

$$= \left(\frac{360}{12} \times \frac{22}{3}\right)^\circ = 220°$$

Also, the angle traced by the minutes hand in 60 minutes = 360°.

$\Rightarrow$ The angle traced by the minutes hand in 20 minutes

$$= \left(\frac{360}{60} \times 20\right)^\circ = 120°$$

Hence, the required angle between two hands

$$= 220° - 120° = 100°$$

8. The large hand of a big clock is 70 cm long. How many cm does its extremity move in 6 minute time?

Solution

Angle traced by the large minute hand in 60 minutes = 360°

$\therefore$ Angle traced by it in 6 min.

$$= \left(\frac{360}{60} \times 6\right)^\circ = 36°$$

Thus $\theta = 36° = \left(\frac{\pi}{180} \times 36\right)^c = \left(\frac{\pi}{5}\right)^c$ and

$r = 70$ cm

Let s be the arc length moved by the tip of the minute hand.

Then, $\theta = \frac{s}{r} \Rightarrow s = r\theta = \left(70 \times \frac{\pi}{5}\right)$ cm

$$\Rightarrow \quad s = \left(\frac{70 \times 22}{7} \times \frac{1}{5}\right) \text{cm} = 44 \text{ cm}$$

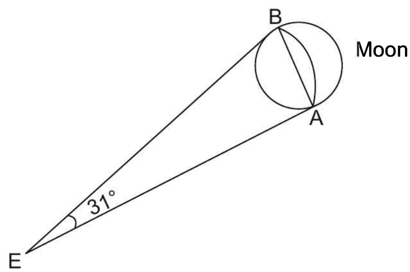
9. The moon's distance from the earth is 360000 km and its diameter subtends an angle of 31° at the eye of the observer. Find the diameter of the moon.

Solution

Let AB be the diameter of the moon and let E be the eye of the observer. Since the distance between the earth and the moon is quite large, so we take diameter AB as arc AB.

Let d be the diameter of the moon.

Then, $d =$ arc AB.



Now, $\theta = 31' = \left(\frac{31}{60}\right)^\circ = \left(\frac{31}{60} \times \frac{\pi}{180}\right)^c$

and $r = 360000$ km

$$\therefore \quad \theta = \frac{\text{arc}}{\text{radius}} \Rightarrow \frac{31}{60} \times \frac{\pi}{180} = \frac{d}{360000}$$

$$\Rightarrow \quad d = \left(\frac{31}{60} \times \frac{22}{7 \times 180} \times 360000\right) \text{km}$$

$$= 3247.62 \text{ km}$$

Hence, the diameter of the moon is 3247.62 km.

10. The angles of a triangle are in A.P. The number of grades in the least, is to the number of radians in the greatest as 40 : π. Find the angles in degrees.

Solution

Let the angles of the triangle be $(a - d)°$, $a°$ and $(a + d)°$.
Then, $a - d + a + a + d = 180°$
$3a = 180° \Rightarrow a = 60°$
So, the angles are $(60 - d)°$, $60°$ and $(60 + d)°$
Clearly, the least angle is $(60 - d)°$ and the greatest angle is $(60 + d)°$ Since, $90° = 100^g$

$$\Rightarrow 1° = \left(\frac{10}{9}\right)^g = (60 - d)° = \left[\frac{10}{9}(60-d)^g\right]$$

$$= \left[\frac{600-10d}{9}\right]^g$$

Also, $(60 + d)°$ $= \left[(60+d)\times\frac{\pi}{180}\right]^{\circ}$

It is given that

$$\frac{\text{Number of grades in the least angle}}{\text{Number of radians in the greatest angle}} = \frac{40}{\pi}$$

$$\Rightarrow \frac{\frac{600-10d}{9}}{(60+d)\frac{\pi}{180}} = \frac{40}{\pi}$$

$$\Rightarrow \frac{600-10d}{9}\times\frac{180}{(60+d)\pi} = \frac{40}{\pi}$$

$\Rightarrow$ $600 - 10d = 120 + 2d$
$\Rightarrow$ $12d = 480$
$\Rightarrow$ $d = 40$

Hence, the angles of the triangle are 20°, 60° and 100°.

11. In a circle of diameter 40 cm, the length of a chord is 20 cm. Find the length of minor corresponding to the chord.

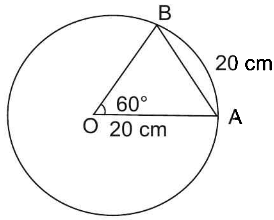
Solution

Let arc $AB = s$.

It is given that OA = 20 cm and Chord AB = 20 cm.
Therefore, ΔOAB is an equilateral triangle.

Hence, $\Delta OAB = 60° = \left(60\times\frac{\pi}{180}\right)^{\circ} = \left(\frac{\pi}{3}\right)^{\circ}$

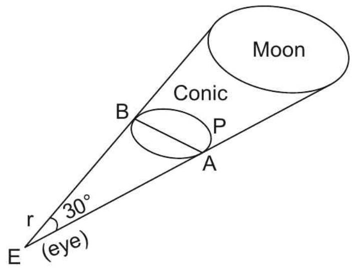
Now, $\theta = \frac{arc}{radius}$



$$\Rightarrow \frac{\pi}{3} = \frac{s}{20} \Rightarrow s = \frac{20\pi}{3}\text{ cm}$$

12. If the angular diameter of the moon be 30°, how far from the eye a coin of diameter 2.2 cm be kept to hide the moon?

Solution



Suppose the coin is kept at a distance r form the eye to hide the moon completely. Let E be the eye of the observer and let AB be the diameter of the coin.
Then, arc APB = diameter AB = 2.2 cm
Now, $\theta = 30°$

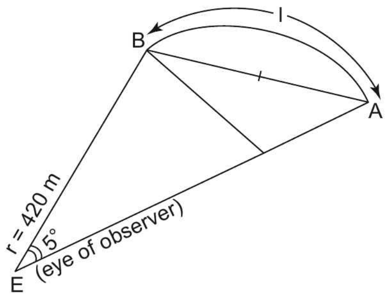
$$= \left(\frac{30}{60}\right)^{\circ} = \left(\frac{1}{2}\times\frac{\pi}{180}\right)^{c} = \left(\frac{\pi}{360}\right)^{\circ}$$

$$\therefore \quad \theta = \frac{\text{arc}}{\text{radius}}$$

$$\Rightarrow \frac{\pi}{360} = \frac{2.2}{r} \Rightarrow r = \frac{2.2\times 360}{\pi}$$

$$\Rightarrow \quad r = \frac{2.2\times 360\times 7}{22} = 252\text{ cm}$$

13. A person of normal eye sight can read a point at such a distance that the letters subtend an angle of 5° at the eye. Find the height of the letters (in cm) he can read at a distance of 420 m.



Here, $\theta = 5' = \left(\frac{5}{60}\right)^\circ = \left(\frac{1}{12}\right)^\circ = \frac{1}{12} \times \frac{\pi}{180}$ rad.

$r = 420$ m $= 42000$ cm.

$\theta = \frac{l}{r} \Rightarrow l = r\theta = 42000 \times \frac{1}{12} \times \frac{\pi}{180}$ cm

$\Rightarrow l = 42000 \times \frac{1}{12} \times \frac{22}{7 \times 180} = \frac{550}{9}$ cm

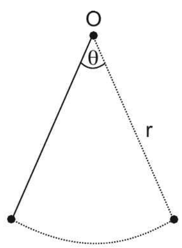
= 61.1 cm. Now θ being small; height of letter

$= AOB =$ chord $AB = \overparen{AB} = l = 61.1$ cm

14. Find the angle in degrees through which a pendulum swings if its length is 75 cm and tip describes an arc of 10 cm.

Solution

Here, $r = 75$ cm and $s = 10$ cm



$\therefore \theta = \left(\frac{s}{r}\right)^c \Rightarrow \theta = \left(\frac{10}{75}\right)^c = \left(\frac{2}{15}\right)^c$

$= \left(\frac{2}{15} \times \frac{180}{\pi}\right)^\circ = \left(\frac{2 \times 180 \times 7}{15 \times 22}\right)^\circ = \left(\frac{84}{11}\right)^\circ$

$= \left(7\frac{7}{11}\right)^\circ = 7^\circ\left(\frac{7}{11} \times 60\right)'$

$= 7^\circ\left(38\frac{2}{11}\right)' = 7^\circ 38'\left(\frac{2}{11} \times 60\right)''$

$= 7^\circ 38' 11''$

15. One angle of a triangle is $\frac{4x}{3}$ grades and another is $3x$ degrees, while the third is $\frac{2\pi x}{75}$ radians. Express them all in degrees.

Solution

First angle $= \left(\frac{4x}{3}\right)^g$

$= \left\{\left(\frac{4x}{3}\right)\left(\frac{90}{100}\right)\right\}^\circ$

$= \left(\frac{30x}{25}\right)^\circ = \left(\frac{6x}{5}\right)^\circ$

second angle $= (3x)^\circ$

Third angle $= \frac{2\pi x}{75}$ rad

$= \left(\frac{2\pi x}{75} \times \frac{180}{\pi}\right)^\circ = \left(\frac{24x}{5}\right)^\circ$

Sum of three angles = 180°

$\Rightarrow \quad \frac{6x}{5} + 3x + \frac{24x}{5} = 180$

$\Rightarrow \quad 6x + 15x + 24x = 900$

$\Rightarrow \quad 45x = 900$

$\Rightarrow \quad x = 20$

$\therefore$ First angle $= \left(\frac{6x}{5}\right)^\circ$

$= \left(\frac{6 \times 20}{5}\right)^\circ = 24^\circ$

Second angle $= (3x)^\circ = (3 \times 20)^\circ = 60^\circ$

Third angle $= \left(\frac{24x}{5}\right)^\circ$

$= \left(\frac{24 \times 20}{5}\right)^\circ = 96^\circ$

16. Find the ratio of the radii of two circles at the centres of which two equal arcs subtend angles of 30° and 63°.

Solution

Let r_1 and r_2 be the radii of the given circles and let their arcs of same length s subtend angles of 30° and 63° at their centres.

Now, $30° = \left(30\times\frac{\pi}{180}\right)^c = \left(\frac{\pi}{6}\right)^c$

and $63° = \left(63\times\frac{\pi}{180}\right) = \left(\frac{21\pi}{60}\right)^c$

$\therefore \quad \frac{\pi}{6} = \frac{s}{r_1}$ and $\frac{21\pi}{60} = \frac{s}{r_2}$

$\Rightarrow \quad \frac{\pi}{6}r_1 = s$ and $\frac{21\pi}{60}r_2 = s$

$\Rightarrow \quad \frac{\pi}{6}r_1 = \frac{21\pi}{60}r_2$

$\Rightarrow \quad 10r_1 = 21r_2$

$\Rightarrow \quad r_1 : r_2 = 21 : 10$

17. A circular wire of radius 7.5 cm is cut and bent so as to lie along the circumference of a hoop whose radius is 120 cm. Find in degrees the angle which is subtended at the centre of the hoop.

Solution

Radius of the circular wire = 7.5 cm

$\therefore$ Length of the circular wire = $2\pi \times 7.5$ = 15π cm

[$\because$ Using; circumference = $2\pi r$]

Radius of the hoop = 120 cm

Let θ be the angle subtended by the wire at the centre of the hoop. Then,

$$\theta = \frac{\text{arc}}{\text{radius}} \Rightarrow \theta = \left(\frac{15\pi}{20}\right)^c = \left(\frac{\pi}{8}\right)^c = \left(\frac{\pi}{8}\times\frac{180}{\pi}\right)^c$$

$= 22°30'$

18. Assuming that a person of normal sight can read print, at such a distance that the letters subtend an angle of 5° at his eye, find what is the height of the letters that he can read at a distance of 12 metres.

Solution

Let h be the required height in metres. Here h can be considered as the arc of a circle of radius 12 m, which subtends an angle of 5° at the centre.

Now, $\theta = 5' = \left(\frac{5}{60}\right)^{\circ} = \left(\frac{1}{12}\times\frac{\pi}{180}\right)^c$

and $r = 12$ metre

$$\theta = \frac{\text{arc}}{\text{radius}} \Rightarrow \frac{\pi}{12\times180} = \frac{h}{12}$$

$\Rightarrow \quad h = \left(\frac{\pi}{180}\right)$ metre

$= 1.7$ cm

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): SOLVE THESE PROBLEMS TO GRASP THE TOPIC

EXERCISE 1

Directions for questions 1 to 3:

Find the degree measure corresponding to the following radian measures:

1. $\left(\frac{2\pi}{15}\right)^C$

2. $(6)^C$

3. $\left(\frac{\pi}{8}\right)^C$

Directions for questions 4 to 7:

Find the radian measure corresponding to the following degree measures:

4. 15°

5. −22° 30′

6. 340°

7. 75°

8. Find the radius of a circle in which a central angle of 45° intercepts an arc of 187 cm.

9. If the arcs of same length in two circles subtend angles of 60° and 75° at the center. Find the ratio of their radii.

10. Find the length of an arc of a circle of radius 5 cm, subtending a central angle measuring 15°.

11. Find the degrees the angle subtended at the center of a circle of diameter 50 cm by an arc of length 11 cm.

12. The angles of a triangle are in A.P. The number of degrees in the least is to the number of radians in the greatest as 60 : π. Find the angles in degrees.

13. Find the degrees the angle through which a pendulum swings if its length is 50 cm and the tip describes an arc of length 10 cm.

14. Find in degrees and radians the angle between the hour hand and the minute hand of a clock at half past three.

15. In a right angled triangle, the difference between two acute angles is $\frac{\pi}{9}$ in circular measure. Express the angles in degrees.

16. Express in degrees and in radians the angles of a regular polygon of *n* sides.

17. Find in degrees and radians the angles of regular
    (i) Hexagon
    (ii) Heptagon
    (iii) Octagon

18. The angles of a quadrilateral are in A.P. and the greatest is double the least, express the least angle in radians.
    (If a, b, c are in A.P.; then $b - c = c - b$ or $2b = a + c$)
    (Four consecutive terms of an A.P. are $\alpha - 3d$, $\alpha - d$, $\alpha + d$ and $\alpha + 3d$)

19. Find the radius of the circle in which a central angle of 60° intercepts an arc of length 37.4 cm $\left(\text{use } \pi = \frac{22}{7}\right)$

20. The minute hand of a watch is 1.5 cm long. How far does its tip move in 40 minutes? (use $\pi = 3.14$)

21. If the arcs of the same lengths in two circles subtend angles 65° and 110° at the center, find the ratio of their radii.

EXERCISE 2

1. A wheel rotates marking 20 revolutions per second. If the radius of the wheel is 35 cm, what linear distance does a point of its rim transverse in three minutes? (Take $\pi = 22/7$)

2. In a right angle triangle, the difference between two acute angles is $\frac{\pi}{6}$ in circular measure. Express the angles in degrees.

3. Find the distance from the eye at which a coin of diameter 1 cm be placed so as just to hid the full moon, it being given that the diameter of the moon subtends an angle of 31° at the eye of the observer.

4. The number of sides of two regular polygons are in the ratio 5 : 4 and the difference between their interior angles is 6°. Find the number of sides in the two polygons.

5. If the angle between the hands of a clock be 54° and the time it reads be between 7 and 8, find the time indicated by the clock.

6. The difference of two angles is 1° while their sum is 1 in circular measure. Find the angles in degrees.

7. Find the ratio of the radii of two circles at the centres of which two equals arcs subtend angles of 30° and 70°.

8. Find the angle between the hands of a clock at 6:30 pm.

ANSWERS

EXERCISE 1

1. $24°$
2. $343°38'\ 11''$
3. $22°\ 30°$
4. $\left(\frac{\pi}{12}\right)^C$
5. $\left(-\frac{\pi}{8}\right)^C$
6. $\left(\frac{17\pi}{9}\right)^C$
7. $\left(\frac{5\pi}{12}\right)^C$
8. 238 cm
9. $r_1 : r_2 = 5 : 4$
10. $\frac{5\pi}{12}$ cm
11. $25°12'$
12. $30°, 60°, 90°$
13. $11°\ 27'\ 16''$
14. $\frac{5\pi}{12}$ radians
15. $55°, 35°$
16. (i) Each interior angle = $(n-2)\,\frac{\pi}{n}$
    (ii) Each exterior angle = $\pi - (n-2)\,\frac{\pi}{n} = \frac{2\pi}{n}$
    (iii) Sum of interior and exterior angles is 180° or π
17. (i) $120°, \left(\frac{2\pi}{3}\right)^C$
    (ii) $\left(\frac{900}{7}\right)^\circ, \left(\frac{5\pi}{7}\right)^C$
    (iii) $\left(\frac{3\pi}{4}\right)^C$, 135°
18. $\pi/3$ radians
19. 35.7 cm
20. 6.28 cm
21. $r_1 : r_2 = 22 : 13$

EXERCISE 2

1. 7.92 km
2. $\frac{\pi}{3}, \frac{\pi}{6}$ or $60°, 30°$
3. 110.9 cm
4. 3
5. 7:48
6. $\left(\frac{180+\pi}{2\pi}\right)^\circ, \left(\frac{180-\pi}{2\pi}\right)^\circ$
7. 21 : 10
8. 15°

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The Answer sheet is immediately below the worksheet.
2. The worksheet is of 15 minutes.
3. The worksheet consists of 15 questions. The maximum marks are 45.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. If the length of a chord of circle is equal to that of the radius of the circle, then the angle subtended, in radians, at the centre of the circle by chord is

(a) 1 (b) $\pi/2$
(c) $\pi/3$ (d) $\pi/4$

2. A cow is tied to a post by a rope. If the cow moves along a circular path always keeping the rope tight, and describes 44 metres, when it has traced out 72° at the centre, the length of the rope is

(a) 35 metres (b) 22 metres
(c) 56 metres (d) 45 metres

3. The perimeter of a certain sector of a circle is equal to half of the circle of which, it is a part. The circular measure of the angle of the sector is

(a) 2 (b) $\pi/2$
(c) $\pi - 2$ (d) $\pi + 2$

4. The radius of the circle whose arc of length 15 cm makes an angle of 3/4 radian at the centre is

[Karnataka CET-2002]

(a) 10 cm (b) 20 cm
(c) $11\frac{1}{4}$ cm (d) $22\frac{1}{2}$ cm

5. A circular wire of radius 7 cm is cut and bend again into an arc of a circle of radius 12 cm. The angle subtended by the arc at the centre is

[Kerala (Engg.) 2002]

(a) 50° (b) 210°
(c) 100° (d) 60°

6. The angle subtended at the centre of a circle of radius 3 metres by an arc of length 1 metre is equal to ***[MNR-1973]***

(a) 20°
(b) 60°
(c) 1/3 radian
(d) 3 radians

7. The angle subtended at the centre of a circle of diameter 50 cm by an arc of length 11 cm is

(a) 25°27' (b) 23°12'
(c) 25°12' (d) None of these

8. If the arcs of same length in two circles subtended angles of 60° and 75° at their centres. Then the ratio of their radii is

(a) 3 : 2 (b) 6 : 5
(c) 5 : 4 (d) None of these

9. Assuming the distance of the earth from the moon to be 38400 km and the angle subtended by the moon at the eye of a person on the earth to be 31' then the diameter of the moon is

(a) $3464\frac{8}{63}$ km

(b) $2656\frac{7}{65}$ km

(c) $4464\frac{8}{63}$ km

(d) None of these

10. The angle between the minute hand of a clock and the hour hand when the time is 7:20 AM is

(a) 108° (b) 100°
(c) 112° (d) None of these

11. The greatest angle of a cyclic quarilateral is 3 times the least. The circular measure of the least angle is

(a) 45° (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{3}$ (d) None of these

12. The angle between two hands of a clock at quarter past one is

(a) 60° (b) $\left(52\frac{1}{2}\right)^\circ$
(c) $\left(\frac{\pi}{3}\right)^c$ (d) None of these

13. If the sum of two angles is 1 radian and the difference between them is 1°, then the smaller angle is

(a) $\left(\frac{90}{\pi}-\frac{1}{2}\right)^\circ$

(b) $\left(\frac{90}{\pi}+\frac{1}{2}\right)^\circ$

(c) $\left(\frac{180}{\pi}-1\right)^\circ$

(d) $\left(\frac{180}{\pi}+1\right)^\circ$

14. The distance between 6.00 A.M. and 3.15 P.M. by the tip of the 12 cm long hour hand in a clock is

(a) $\frac{35}{2}\pi$ cm (b) 18π cm

(c) $\frac{37}{2}\pi$ cm (d) 19π cm

15. A circular wire of radius 3 cm is cut and bent so as to lie along the circumference at a loop whose radius is 48 cm. The angle in grades which is subtended at the centre of the loop is

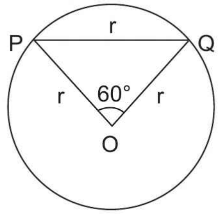
(a) 50 grades (b) 20 grades
(c) 25 grades (d) None of these

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	**6.** ⓐ ⓑ ⓒ ⓓ	**11.** ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	**7.** ⓐ ⓑ ⓒ ⓓ	**12.** ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	**8.** ⓐ ⓑ ⓒ ⓓ	**13.** ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	**9.** ⓐ ⓑ ⓒ ⓓ	**14.** ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ	**10.** ⓐ ⓑ ⓒ ⓓ	**15.** ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

1. (c)



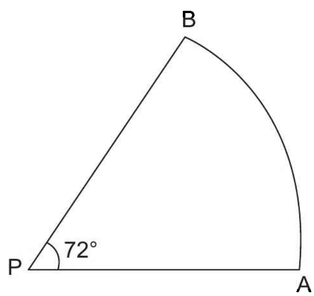
Here, length of chord, say PQ, is equal to the radius, therefore, ΔOPQ is equilateral

$\therefore \quad \angle POQ = 60°$

Also, $\quad 180° = \pi$

$\therefore \quad 60° = \frac{\pi}{180°} \times 60°$ radian.

2. (a) Let the post be at point P and let PA be the length of the rope in tight position. Suppose the cow moves along the arc AB so that $\angle APB = 72°$ and *arc* AB = 44 m. Let r be the length of the rope i.e., $PA = r$ metres.



Now, $\theta = 72° = \left(72 \times \frac{\pi}{180}\right)^c = \left(\frac{2\pi}{5}\right)^c$ and s = 44 m

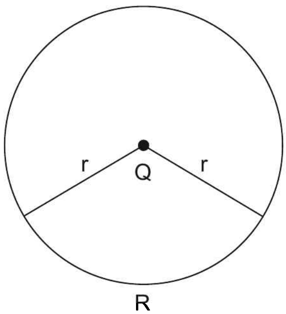
$$\therefore \quad \theta = \frac{arc}{radius} \Rightarrow \frac{2\pi}{5} = \frac{44}{r}$$

$$\Rightarrow \quad r = 44 \times \frac{5}{2\pi}$$

$$= \frac{44 \times 5 \times 7}{2 \times 22} = 35 \text{ metres.}$$

3. (c) Given perimeter of sector = $\frac{1}{2}$ (perimeter of circle)

$$\Rightarrow \quad r + r + l = \frac{1}{2}(2\pi r)$$



$$\Rightarrow \quad l = \pi r - 2r = (\pi - 2)r$$

$$\Rightarrow \quad \frac{l}{r} = \pi - 2 \qquad \therefore \theta = \frac{l}{r} = \pi - 2$$

Hence $\theta = \pi - 2$

4. (b) Using $\theta = \frac{l}{r}$,

we get $r = \frac{1}{\theta} = \frac{15 \text{ cm}}{\frac{3}{4}} = 20$ cm

5. (b) Given that diameter of circular wire = 14 cm

Therefore length of circular wire = 14 π cm

$\therefore$ Required angle

$$= \frac{arc}{radius} = \frac{14\pi}{12} = \frac{7\pi}{6} = \frac{7}{6}\pi . \frac{180°}{\pi} = 210°$$

6. (c) By using $\theta = \frac{1}{r} = \frac{1}{3}$ radian

7. (c) Here, $r = 25$ and $s = 11$ cm

$$\therefore \quad \theta = \left(\frac{s}{r}\right)^R$$

$$\Rightarrow \quad \theta = \left(\frac{11}{25}\right)^R = \left(\frac{11}{25} \times \frac{180}{\pi}\right)^\circ = \left(\frac{11}{25} \times \frac{180}{22} \times 7\right)^\circ$$

$$= \left(\frac{126}{5}\right)^\circ = \left(25\frac{1}{5}\right)^\circ = 25°\left(\frac{1}{5} \times 60\right)' = 25°12'$$

8. (c) Let r_1 and r_2 be the radii of the given circles and let their arcs of same length s subtend angles of 60° and 75° at their centres.

Now, $60° = \left(60 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)^R$ and

$$75° = \left(75 \times \frac{\pi}{180}\right)^R = \left(\frac{5\pi}{12}\right)^R$$

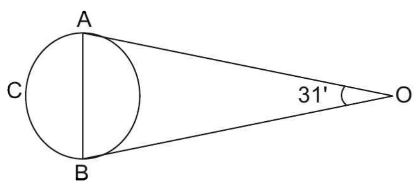
$$\therefore \quad \frac{\pi}{3} = \frac{s}{r_1} \text{ and } \frac{5\pi}{12} = \frac{s}{r_2} \Rightarrow \frac{\pi}{3}r_1 = s$$

$$\text{and } \frac{5\pi}{12}r_2 = s \Rightarrow \frac{\pi}{3}r_1 = \frac{5\pi}{12}r_2$$

$$\Rightarrow 4r_1 = 5r_2 \Rightarrow r_1 : r_2 = 5 : 4$$

Hence, $r_1 : r_2 = 5:4$

9. (a) Let AB be the diameter of the moon and O be the observer.



Given $\angle AOB = 31' = \frac{31}{60} \times \frac{\pi}{180}$ radians

Since angles subtended by the moon is very small, therefore its diameter will be approximately equal to a small arc of a circle whose centre is the eye of the observer and radius is the distance of the earth from the moon. Also the moon subtends an angle of 31' at the centre of this circle.

$$\therefore\ \theta = \frac{1}{r} \therefore \frac{31}{60} \times \frac{\pi}{180} = \frac{AB}{38400}$$

$$\Rightarrow AB = \frac{31}{60} \times \frac{22}{7 \times 180} \times 38400 = 3464\frac{8}{63}\text{km.}$$

10. (b) We know that the hour hand completes one rotation in 12 hours while the minute hand competes one rotation in 60 minuts.

$\therefore$ Angle traced by the hour hand in 12 hours = 360°

$\Rightarrow$ Angle traced by the hour hand in 7 hours 20 minutes

i.e., $\frac{22}{3}\text{h} = \left(\frac{360}{12} \times \frac{22}{3}\right)^{\circ} = 220°$

Also, the angle traced by the minute hand in 60 min = 360°

$\Rightarrow$ The angle traced by the minute hand in

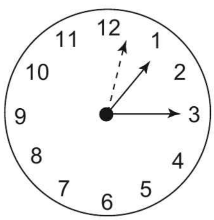
20 min $= \left(\frac{360}{12} \times 20\right)^{\circ} = 120°$

Hence, the required angle between two hands

= 220° – 120° = 100°

11. (b) The greatest angle is always opposite to the smallest angle. Hence, the greatest angle + the smallest angle = 180°.
So, least angle = 45° whose circular measure is $\pi/4$

12. (b)



If hours hand were at 1 and minutes hand at 3, the angle between the two hands would have been 60°. In 15 minutes, hours hand revolves through $\left(\frac{360 \times 15}{720}\right)^{\circ} = \left(7\frac{1}{2}\right)^{\circ}$

($\because$ In 12 hours, i.e., 720 min, hours hand revolves through 360°)

$\therefore$ Required angle between the hands of clock

$$= 60° - \left(7\frac{1}{2}\right)^{\circ} = \left(55\frac{1}{2}\right)^{\circ}$$

13. (a) Let the two angles be A and B, where $A > B$.

Then $A + B = 1$ radian $= \left(\frac{180}{\pi}\right)^{\circ}$ and $A - B = 1°$.

Subtracting, we get 2B $= \left(\frac{180}{\pi} - 1\right)^{\circ}$

$$\Rightarrow B = \left(\frac{90}{\pi} - \frac{1}{2}\right)^{\circ}$$

14. (c) Angle covered from 6 A.M. to 3.15 P.M.

$$\Rightarrow 277\frac{1}{2}^{\circ} = \frac{555}{2} \times \frac{\pi}{180} = \frac{111}{2} \times \frac{\pi}{36}$$

$$\therefore\ \theta = \frac{37\pi}{24} \text{ radian}$$

Length of hour hand = 12 cm

i.e., $r = 12$ cm

Since $\theta = \frac{l}{r}$

$$\therefore\ l = r\theta = \frac{12 \times 37\pi}{24} = \frac{37\pi}{2}$$

Hence required distance $= \frac{37\pi}{2}$cm.

15. (c) Circumference of a circle of radius = 3 cm

= 2π(3) = 6π cm.

$\therefore$ length of the wire (l) = 6π cm.

Radius (r) = 48 cm.

Since $\theta = \frac{l}{r} = \frac{6\pi}{48} = \frac{\pi}{8}$ radians

$= \frac{200}{8}$ grades = 25 grades.

LECTURE

2

Trigonometric Functions

BASIC CONCEPTS

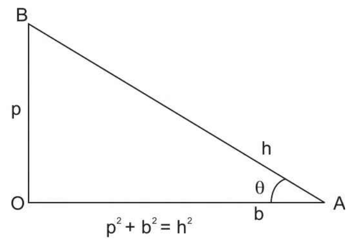
1. TABLE REPRESENTING ALL TRIGONAL RATIOS (TRs) IN TERMS OF ONE TRIGONAL RATIO

	sin θ	cos θ	tan θ	cot θ	sec θ	cosec θ
sin θ	sin θ	$\sqrt{1-\cos^2\theta}$	$\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$	$\frac{1}{\sqrt{1+\cot^2\theta}}$	$\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\frac{1}{\operatorname{cosec}\theta}$
cos θ	$\sqrt{1-\sin^2\theta}$	cosθ	$\frac{1}{\sqrt{1+\tan^2\theta}}$	$\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$	$\frac{1}{\sec\theta}$	$\frac{\sqrt{\operatorname{cosec}^2\theta-1}}{\operatorname{cosec}\theta}$
tan θ	$\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	tanθ	$\frac{1}{\cot\theta}$	$\sqrt{\sec^2\theta-1}$	$\frac{1}{\sqrt{\operatorname{cosec}^2\theta-1}}$
cot θ	$\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\frac{1}{\tan\theta}$	cotθ	$\frac{1}{\sqrt{\sec^2\theta-1}}$	$\sqrt{\operatorname{cosec}^2\theta-1}$
sec θ	$\frac{1}{\sqrt{1-\sin^2\theta}}$	$\frac{1}{\cos\theta}$	$\sqrt{1+\tan^2\theta}$	$\frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$	secθ	$\frac{\operatorname{cosec}\theta}{\sqrt{\operatorname{cosec}^2-1}}$

2. T-RATIOS (OR TRIGONOMETRIC FUNCTIONS)

$$\sin\theta=\frac{p}{h},\cos\theta=\frac{b}{h},\tan\theta=\frac{p}{b},$$

$$\operatorname{cosec}\theta=\frac{h}{p},\sec\theta=\frac{h}{b},\cot\theta=\frac{b}{p};$$



'p' perpendicular', '*b*' base and '*h*' stands for hypotenuse.

3. FOLLOWING ARE SOME OF THE FUNDAMENTAL TRIGONOMETRIC IDENTITIES

(i) $\sin\theta = \dfrac{1}{\text{cosec}\,\theta}$

or $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$

(ii) $\cos\theta = \dfrac{1}{\sec\theta}$

or $\sec\theta = \dfrac{1}{\cos\theta}$

(iii) $\cot\theta = \dfrac{1}{\tan\theta}$

or $\tan\theta = \dfrac{1}{\cot\theta}$

(iv) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

or $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

(v) $\sin^2\theta + \cos^2\theta = 1$ or $\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$

(vi) $\sec^2\theta - \tan^2\theta = 1$

or $1 + \tan^2\theta - \sin^2\theta$

or $\sec\theta - \tan\theta = \dfrac{1}{\sec\theta + \tan\theta}$

(vii) $\text{cosec}^2\theta - \cot^2\theta = 1$ or $1 + \cot^2\theta = \text{cosec}^2\theta$

or $\text{cosec}\,\theta - \cot\theta = \dfrac{1}{\text{cosec}\,\theta + \cot\theta}$

4. MAXIMUM AND MINIMUM VALUES OF TRIGONOMETRICAL FUNCTIONS

(i) $-1 \le \sin x \le 1$

(ii) $-1 \le \cos x \le 1$

(iii) $-\infty < \tan x < \infty$

(iv) $-\infty < \cot x < \infty$

(v) $|\sec x| \ge 1$ i.e., $\sec x \le -1$ or $\sec x \ge 1$

(vi) $|\text{cosec}\,x| \ge 1$ i.e., $\text{cosec}\,x \le -1$ or $\text{cosec}\,x \ge 1$

(vii) $\sin^2 x + \text{cosec}^2 x \ge 2,\ \forall\, x \in R$

(viii) $\cos^2 x + \sec^2 x \ge 2,\ \forall\, x \in R$

(ix) $\tan^2 x + \cot^2 x \ge 2,\ \forall\, x \in R$

(x) $|\sin x + \text{cosec}\,x| \ge 2$

(xi) $|\cos x + \sec x| \ge 2$

(xii) $|\tan x + \cot x| \ge 2$

(xiii) If $a \sin x + b \cos x = c$, then $(a \cos x - b \sin x)^2 = a^2 + b^2 - c^2$

5. SOME USEFUL RESULTS

(i) $\sin^2\theta + \cos^4\theta = \sin^4\theta + \cos^2\theta = 1 - \sin^2\theta\cos^2\theta$

(ii) $\sin^4\theta + \cos^4\theta = 1 - 2\sin^2\theta\cos^2\theta$

(iii) $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$

(iv) $\sin^4\theta - \cos^4\theta = 1 - 2\cos^2\theta$

(v) $\sin^8\theta - \cos^8\theta = (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta\cos^2\theta)$

(vi) $\sec^2\theta + \text{cosec}^2\theta = \sec^2\theta - \text{cosec}^2\theta = \tan^2\theta + \cot^2\theta + 2$

(vii) $\dfrac{1}{\sec\theta - \tan\theta} = \sec\theta + \tan\theta = \sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}}$

(viii) $\dfrac{1}{\sec\theta + \tan\theta} = \sec\theta - \tan\theta = \sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}$

(ix) $\dfrac{1}{\text{cosec}\theta - \cot\theta} = \text{cosec}\theta + \cot\theta = \sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}}$

(x) $\dfrac{1}{\text{cosec}\theta + \cot\theta} = \text{cosec}\theta - \cot\theta = \sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$

(xi) $\dfrac{1-\sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1+\sin\theta} = \sec\theta - \tan\theta$

(xii) $\dfrac{1-\cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1+\cos\theta} = \text{cosec}\,\theta - \cot\theta$

(xiii) $\dfrac{1-\sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1+\sin\theta} = \sec\theta - \tan\theta$

(xiv) $\dfrac{1-\cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1+\cos\theta} = \text{cosec}\,\theta - \cot\theta$

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Show that $\dfrac{(1+\tan\theta+\cot\theta)(\sin\theta-\cos\theta)}{\sec^3\theta-\text{cosec}^3\theta}$

Solution

$$\text{L.H.S.} = \frac{(1+\tan\theta+\cot\theta)(\sin\theta-\cos\theta)}{\sec^3\theta-\text{cosec}^3\theta}$$

$$= \frac{\left(1+\dfrac{\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{\sin\theta}\right)(\sin\theta-\cos\theta)}{\dfrac{1}{\cos^3\theta}-\dfrac{1}{\sin^3\theta}}$$

$$= \frac{(\sin\theta\cos\theta+\sin^2\theta+\cos^2\theta)(\sin\theta-\cos\theta)}{\sin\theta\cos\theta\left(\dfrac{\sin^3\theta-\cos^3\theta}{\sin^3\theta\cos^3\theta}\right)}$$

$$= \frac{(\sin^3\theta-\cos^3\theta)\sin^3\theta\cos^3\theta}{\sin\theta\cos\theta(\sin^3\theta-\cos^3\theta)}$$

$(\because\ (a^2+b^2+ab)(a-b)=a^3-b^3)$

$$= \sin^2\theta\cos^2\theta = \text{R.H.S.}$$

2. If $\cos^2\alpha - \sin^2\alpha = \tan^2\beta$, then show that $\tan^2\alpha = \cos^2\beta - \sin^2\beta$.

Solution

Given, $\cos^2\alpha - \sin^2\alpha = \tan^2\beta$

$$\Rightarrow \frac{\cos^2\alpha-\sin^2\alpha}{\cos^2\alpha+\sin^2\alpha} = \frac{\sin^2\beta}{\cos^2\beta}$$

$$\Rightarrow \frac{\cos^2\alpha+\sin^2\alpha}{\cos^2\alpha-\sin^2\alpha} = \frac{\cos^2\beta}{\sin^2\beta}$$

$(\because\ \cos^2\alpha+\sin^2\alpha=1)$

Applying componendo and dividendo, we get

$$\frac{\cos^2\alpha+\sin^2\alpha+\cos^2\alpha-\sin^2\alpha}{\cos^2\alpha+\sin^2\alpha-\cos^2\alpha+\sin^2\alpha} = \frac{\cos^2\beta+\sin^2\beta}{\cos^2\beta-\sin^2\beta}$$

$$\Rightarrow \frac{2\cos^2\alpha}{2\sin^2\alpha} = \frac{1}{\cos^2\beta-\sin^2\beta}$$

$$\Rightarrow \frac{\sin^2\alpha}{\cos^2\alpha} = \cos^2\beta-\sin^2\beta$$

$\Rightarrow \tan^2\alpha = \cos^2\beta - \sin^2\beta$, as desired.

3. If $\dfrac{\sin A}{\sin B} = m$ and $\dfrac{\cos A}{\cos B} = n$, find the value of $\tan B$; $n^2 < 1 < m^2$.

Solution

Given $\dfrac{\sin A}{\sin B} = m \Rightarrow \sin A = m\sin B$ ....(1)

and $\dfrac{\cos A}{\cos B} = n \Rightarrow \cos A = n\cos B$ ....(2)

By squaring (1) and (2) and adding, we get

$$1 = m^2\sin^2 B + n^2\cos^2 B$$

$\Rightarrow \dfrac{1}{\cos^2 B} = m^2\dfrac{\sin^2 B}{\cos^2 B} + n^2$ (Dividing by $\cos^2 B$)

$$\Rightarrow \sec^2 B = m^2\tan^2 B + n^2$$

$$\Rightarrow 1+\tan^2 B = m^2\tan^2 B + n^2$$

$$\Rightarrow 1-n^2 = (m^2-1)\tan^2 B \Rightarrow \tan^2 B = \frac{1-n^2}{m^2-1}$$

$$\Rightarrow \tan B = \pm\sqrt{\frac{1-n^2}{m^2-1}}$$

4. Eliminate θ between $\text{cosec}\theta - \sin\theta = a$, $\sec\theta - \cos\theta = b$.

Solution

Given $\text{cosec}\theta - \sin\theta = a$ ....(1)

and $\sec\theta - \cos\theta = b$ ....(2)

From (1), $\dfrac{1}{\sin\theta} - \sin\theta = a \Rightarrow \dfrac{\cos^2\theta}{\sin\theta} = a$ ....(3)

From (2), $\dfrac{1}{\cos\theta} - \cos\theta = b \Rightarrow \dfrac{\sin^2\theta}{\cos\theta} = b$ ....(4)

Squaring (3) and multiplying by (4), we get

$\cos^3\theta = a^2 b \Rightarrow \cos\theta = (a^2 b)^{1/3}$ ....(5)

Squaring (4) and multiplying by (3), we get

$\sin^3\theta = ab^2 \Rightarrow \sin\theta = (ab^2)^{1/3}$ ....(6)

5. Eliminate θ and ϕ between

$x = r\cos\theta\cos\phi,\ y = r\cos\theta\sin\phi,\ z = r\sin\theta.$

Solution

Given $x = r\cos\theta\cos\phi,\ y = r\cos\theta\sin\phi$ and $z = r\sin\theta.$

Squaring and adding, we get

$x^2 + y^2 + z^2$

$= r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta$

$= r^2\cos^2\theta\,(\cos^2\phi + \sin^2\phi) + r^2\sin^2\theta$

$= r^2(\sin^2\theta + \cos^2\theta) = r^2\ [\because \sin^2\theta + \cos^2\theta = 1]$

6. If $\dfrac{\sin\alpha}{\sin\beta} = m$ and $\dfrac{\cos\alpha}{\cos\beta} = n$, than prove that

$\tan\alpha = \pm\dfrac{m}{n}\sqrt{\dfrac{1-n^2}{m^2-1}}; n^2 < 1 < m^2.$

Solution

Given $\dfrac{\sin\alpha}{\sin\beta} = m$ ....(1)

and $\dfrac{\cos\alpha}{\cos\beta} = n$ ....(2)

From (1), $\sin\beta = \dfrac{\sin\alpha}{m}$ ....(3)

and from (2), $\cos\beta = \dfrac{\cos\alpha}{n}$ ....(4)

Squaring (3) and (4) and adding, we get

$1 = \dfrac{\sin^2\alpha}{m^2} + \dfrac{\cos^2\alpha}{n^2}$

$\Rightarrow m^2n^2 = n^2\sin^2\alpha + m^2\cos^2\alpha$

Divide throughtout by $\cos^2\alpha$

$\Rightarrow \dfrac{m^2n^2}{\cos^2\alpha} = \dfrac{n^2\sin^2\alpha}{\cos^2\alpha} + \dfrac{m^2\cos^2\alpha}{\cos^2\alpha}$

$\Rightarrow m^2n^2\sec^2\alpha = n^2\tan^2\alpha + m^2$

$\Rightarrow m^2n^2(1+\tan^2\alpha) = n^2\tan^2\alpha + m^2$

$\Rightarrow m^2n^2 + m^2n^2\tan^2\alpha = n^2\tan^2\alpha + m^2$

$\Rightarrow m^2n^2 - m^2 = n^2\tan^2\alpha - m^2n^2\tan^2\alpha$

$\Rightarrow \tan^2\alpha = \dfrac{m^2}{n^2}\times\dfrac{n^2-1}{1-m^2}$

$\Rightarrow \tan^2\alpha = \pm\dfrac{m}{n}\sqrt{\dfrac{1-n^2}{m^2-1}}.$ (Proved)

7. If $\dfrac{\sin^4 A}{a} + \dfrac{\cos^4 A}{b} = \dfrac{1}{a+b}$, then prove that

$\dfrac{\sin^8 A}{a^3} + \dfrac{\cos^8 A}{b^3} = \dfrac{1}{(a+b)^3}$

Solution

$(a + b)\,(b\sin^4 A + a\cos^4 A) - ab = 0$

or $ab\,[\sin^4 A + \cos^4 A - 1] + a^2\cos^4 A + b^2\sin^4 A = 0$

or $ab\,[1 - 2\sin^2 A\cos^2 A - 1] + a^2\cos^4 A + b^2\sin^4 A = 0$

or $(a\cos^2 A - b\sin^2 A)^2 = 0$

or $\dfrac{\cos^2 A}{b} = \dfrac{\sin^2 A}{a} = \dfrac{1}{a+b}$

$\therefore \dfrac{\sin^8 A}{a^3} + \dfrac{\cos^8 A}{b^3} = \dfrac{1}{a^3}.\dfrac{a^4}{(a+b)^4} + \dfrac{1}{b^3}.\dfrac{b^4}{(a+b)^4}$

$= \dfrac{(a+b)}{(a+b)^4} = \dfrac{1}{(a+b)^3}$

8. If $\sin x + \sin^2 x + \sin^3 x = 1$, then

$\cos^6 x - 4\cos^4 x + 8\cos^2 x =$ ....

Solution

The given relation can be written as

$\sin x\,(1 + \sin^2 x) = 1 - \sin^2 x = \cos^2 x$

$\Rightarrow \sin x\,(2 - \cos^2 x) = \cos^2 x$

$\Rightarrow \sin^2 x\,(2 - \cos^2 x)^2 = \cos^4 x$

(squaring both sides)

$\Rightarrow (1 - \cos^2 x)(4 - 4\cos^2 x + \cos^4 x) = \cos^4 x$

$\Rightarrow \cos^6 x - 4\cos^4 x + 8\cos^2 x = 4.$

9. If $8\sin\theta = 4 + \cos\theta$, then find the value of $\sin\theta$.

Solution

$8\sin\theta = 4 + \cos\theta$

$\Rightarrow 8\sin\theta - 4 = \cos\theta$

$\Rightarrow (8\sin\theta - 4)^2 = \cos^2\theta$

$\Rightarrow (8\sin\theta - 4)^2 = 1 - \sin^2\theta$

$\Rightarrow 64\sin^2\theta + 16 - 64\sin\theta = 1 - \sin^2\theta$

$\Rightarrow 65\sin^2\theta - 64\sin\theta + 15 = 0$

$\Rightarrow 65\sin^2\theta - 39\sin\theta - 25\sin\theta + 15 = 0$

$\Rightarrow \quad 13 \sin\theta (5 \sin \theta - 3) - 5 (5 \sin\theta - 3) = 0$

$\Rightarrow \quad (13 \sin\theta - 5) (5 \sin \theta - 3) = 0$

$\Rightarrow \quad \sin\theta = \frac{5}{13}$ or $\frac{3}{5}$.

$\Rightarrow \quad \sin\theta = \frac{3}{5}$ in first quadrant

and $\sin\theta = \frac{5}{13}$ in second quadrant

10. Prove that,

$$\frac{\tan\theta}{1-\cot\theta} + \frac{\cot\theta}{1-\tan\theta} = \sec\theta \operatorname{cosec}\theta + 1$$

Solution

$$\text{L.H.S.} = \frac{\tan\theta}{1-\cot\theta} + \frac{\cot\theta}{1-\tan\theta}$$

$$= \frac{\frac{\sin\theta}{\cos\theta}}{\frac{\sin\theta-\cos\theta}{\sin\theta}} + \frac{\frac{\cos\theta}{\sin\theta}}{\frac{\cos\theta-\sin\theta}{\cos\theta}}$$

$$= \frac{\sin^2\theta}{\cos\theta(\sin\theta-\cos\theta)} + \frac{\cos^2\theta}{\sin\theta(\cos\theta-\sin\theta)}$$

$$= \frac{\sin^3\theta-\cos^3\theta}{\cos\theta\sin\theta(\sin\theta-\cos\theta)}$$

$$[a^3 - b^3 = (a-b)(a^2+b^2+ab)]$$

$$= \frac{(\sin\theta-\cos\theta)(\sin^2\theta+\cos^2\theta+\sin\theta\cos\theta)}{\cos\theta\sin\theta(\sin\theta-\cos\theta)}$$

$$= \frac{1+\sin\theta\cos\theta}{\sin\theta\cos\theta} = \sec\theta \operatorname{cosec}\theta + 1$$

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): SOLVE THESE PROBLEMS TO GRASP THE TOPIC

EXERCISE 1

1. Prove that, $\sec^4\theta - \sec^2\theta = \tan^4\theta + \tan^2\theta$
2. Prove that, $\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = 2\operatorname{cosec}\theta$
3. Prove that, $\frac{\operatorname{cosec}\theta}{\operatorname{cosec}-1} + \frac{\operatorname{cosec}\theta}{\operatorname{cosec}+1} = 2\sec^2\theta$
4. Prove that, $(\operatorname{cosec}\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta) = 1$
5. If $\tan\theta = \frac{3}{4}$, then find all the trigonometric ratios.
6. If $\sec\theta + \tan\theta = 4$, find the values of $\sin\theta$, $\cos\theta$, $\sec\theta$ and $\tan\theta$.
7. Prove that, $\frac{1+\cos x}{1-\cos x} = (\operatorname{cosec}x + \cot x)^2$
8. If $\sin x + \sin^2 x = 1$, then prove that $\cos^2 x + \cos^4 x = 1$
9. If $\sec\theta + \tan\theta = p$, obtain the values of $\sec\theta$, $\tan\theta$ in terms of p.
10. Prove that, $\sin\theta\cot\theta + \sin\theta\operatorname{cosec}\theta = 1+\cos\theta$
11. Prove that,
$$\frac{\tan^2\theta}{\tan^2\theta-1} + \frac{\operatorname{cosec}^2\theta}{\sec^2\theta-\operatorname{cosec}^2\theta} = \frac{1}{\sin^2\theta-\cos^2\theta}$$
12. Prove that,
$(1-\tan x)^2 + (1-\cot x)^2 = (\sec x - \operatorname{cosec}x)^2$.

EXERCISE 2

1. Prove that $\tan^2\theta - \sin^2\theta = \tan^2\theta - \sin^2\theta$
2. Prove that
$$\frac{\sec\theta-\tan\theta}{\sec\theta+\tan\theta} = 1 - 2\sec\theta\tan\theta + 2\tan^2\theta$$
3. Prove that $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = (\operatorname{cosec}\theta + \cot\theta)$

4. Prove that $\dfrac{1-\cos\theta}{\sin\theta}=\dfrac{\sin\theta}{1+\cos\theta}$

5. Prove that $\cot^4\theta+\cot^2\theta=\operatorname{cosec}^4\theta-\operatorname{cosec}^2\theta$

6. Prove that (Sin A + cos A) (tan A + cot A) = sec A + cosec A

7. Prove that
$$\frac{\cos A\operatorname{cosec} A-\sin A\sec A}{\cos A+\sin A}=\operatorname{cosec} A-\sec A$$

8. Prove that $2\sin^2 A+\cos^4 A=1+\sin^4 A$

9. Prove that
$$\frac{1}{\operatorname{cosec}\theta-\cot\theta}-\frac{1}{\sin\theta}=\frac{1}{\sin\theta}-\frac{1}{\operatorname{cosec}\theta+\cot\theta}$$

10. Prove that, $\dfrac{1+\cos x}{1-\cos x}=(\operatorname{cosec} x+\cot x)^2$

11. If $\sin x+\sin^2 x=1$, prove that $\cos^2 x+\cos^4 x=1$

12. If $\sec\theta+\tan\theta=p$, obtain the values of $\sec\theta$, $\tan\theta$ in terms of p.

ANSWERS

EXERCISE 1

5. $\sin\theta=\dfrac{3}{5},\cos\theta=\dfrac{4}{5},$

$\cot\theta=\dfrac{4}{3},\ \operatorname{cosec}\theta=\dfrac{5}{3},$

$\sec\theta=\dfrac{5}{4}$

6. $\sin\theta=\dfrac{15}{17},\cos\theta=\dfrac{8}{17},$

$\tan\theta=\dfrac{15}{8},$

$\sec\theta=\dfrac{17}{8}$

9. $\sec\theta=\dfrac{p^2+1}{2p},\tan\theta=\dfrac{p^2-1}{2p}$

and $\sin\theta=\dfrac{p^2-1}{p^2+1}$

EXERCISE 2

12. $=\sec\theta=\dfrac{p^2+1}{2p},\tan\theta=\dfrac{p^2-1}{2p}$

and $\sin\theta$ $=\dfrac{p^2-1}{p^2+1}$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. If $y=\dfrac{2\sin a}{1+\cos\alpha+\sin\alpha}$ then $\dfrac{1-\cos\alpha+\sin\alpha}{1+\sin\alpha}$ is equal to *[UPSEAT, 99]*

(a) $1/y$ (b) y

(c) $1-y$ (d) $1+y$

Solution

(b) $\dfrac{1-\cos\alpha+\sin\alpha}{1+\sin\alpha}$

$$=\frac{1-\cos\alpha+\sin\alpha}{1+\sin\alpha}\cdot\frac{1+\cos\alpha+\sin\alpha}{1+\cos\alpha+\sin\alpha}$$

$$=\frac{(1+\sin\alpha)^2-\cos^2\alpha}{(1+\sin\alpha)(1+\cos\alpha+\sin\alpha)}$$

$$=\frac{(1+\sin^2\alpha+2\sin\alpha)-(1-\sin^2\alpha)}{(1+\sin\alpha)(1+\cos\alpha+\sin\alpha)}$$

$$=\frac{2\sin\alpha(1+\sin\alpha)}{(1+\sin\alpha)(1+\cos\alpha+\sin\alpha)}$$

$$=\frac{2\sin\alpha}{1+\cos\alpha+\sin\alpha}=y$$

2. If $a\cos^3\alpha + 3a\cos\alpha\sin^2\alpha = m$ and $a\sin^3\alpha + 3a\cos^2\alpha\sin\alpha = n$, then $(m+n)^{2/3} + (m-n)^{2/3}$ is equal to

(a) $2a^2$ (b) $2a^{1/3}$
(c) $2a^{2/3}$ (d) $2a^3$

Solution

(c) From the given relations, we get

$m + n = a\cos^3\alpha + 3a\cos\alpha\sin^2\alpha + 3a\cos^2\alpha\sin\alpha + a\sin^3\alpha$

$= a(\cos\alpha + \sin\alpha)^3$ similarly, $m - n = a(\cos\alpha - \sin\alpha)^3$

$\therefore\ (m+n)^{2/3} + (m-n)^{2/3} = a^{2/3}[(\cos\alpha + \sin\alpha)^2 + (\cos\alpha - \sin\alpha)^2]$

$= a^{2/3}[2(\cos^2\alpha + \sin^2\alpha)] = 2a^{2/3}$

3. $(m+2)\sin\theta + (2m-1)\cos\theta = 2m+1$, if

(a) $\tan\theta = 3/4$
(b) $\tan\theta = 4/3$
(c) $\tan\theta = 2m/(m^2 - 1)$
(d) $\tan\theta = 2m/(m^2 + 1)$

Solution

(b, c) The given relation can be written as

$(m+2)\tan\theta + (2m-1) = (2m+1)\sec\theta$

$\Rightarrow (m+2)^2\tan^2\theta + 2(m+2)(2m-1)\tan\theta + (2m-1)^2$

$= (2m+1)^2(1+\tan^2\theta)$

$\Rightarrow [(m+2)^2 - (2m+1)^2]\tan^2\theta + 2(m+2)(2m-1)\tan\theta$

$+ (2m-1)^2 - (2m+1)^2 = 0$

$\Rightarrow 3(1-m^2)\tan^2\theta + (4m^2 + 6m - 4)\tan\theta - 8m = 0$

$\Rightarrow [(m+2)^2 - (2m+1)^2]\tan^2\theta + 2(m+2)(2m-1)\tan\theta$

$+ (2m-1)^2 - (2m+1)^2 = 0$

$\Rightarrow 3(1-m^2)\tan^2\theta + (4m^2 + 6m - 4)\tan\theta - 8m = 0$

$\Rightarrow (3\tan\theta - 4)[(1-m^2)\tan\theta + 2m] = 0$

Which is true if $\tan\theta = 4/3$ or $\tan\theta = 2m/(m^2 - 1)$

4. If $x = \sec\varphi - \tan\varphi$ and $y = \operatorname{cosec}\varphi + \cot\varphi$, then

(a) $x = \dfrac{y+1}{y-1}$

(b) $x = \dfrac{y-1}{y+1}$

(c) $y = \dfrac{1+x}{1-x}$

(d) $xy + x - y + 1 = 0$

Solution

(b, c, d) We have $x = \dfrac{1-\sin\phi}{\cos\phi}, y = \dfrac{1+\cos\phi}{\sin\phi}$

By multiplying, we get

$$xy = \frac{(1-\sin\phi)(1+\cos\phi)}{\cos\phi\ \sin\phi}$$

$\Rightarrow\ xy + 1$

$$= \frac{1-\sin\phi+\cos\phi-\sin\phi\cos\phi+\sin\phi\cos\phi}{\cos\phi\ \sin\phi}$$

$$= \frac{1-\sin\phi+\cos\phi}{\cos\phi\ \sin\phi} \text{ and}$$

$$x - y = \frac{(1-\sin\phi)\sin\phi - \cos\phi(1+\cos\phi)}{\cos\phi\ \sin\phi}$$

$$= \frac{\sin\phi - \sin^2\phi - \cos\phi - \cos^2\phi}{\cos\phi\ \sin\phi}$$

$$= \frac{\sin\phi - \cos\phi - 1}{\cos\phi\sin\phi} = -(xy+1)$$

Thus, $xy = x - y + 1 = 0 \Rightarrow x = \dfrac{y-1}{y+1}$ and

$$y = \frac{1+x}{1-x}$$

5. If $1 + \sin^2 A = 3\sin A\cos A$, then possible values of tan A are ***[NDA-2005]***

(a) 1, 1/2 (b) 2, 1/4
(c) 3, 1/6 (d) 4, 1/8

Solution

(a) $1 + \sin^2 A = 3\sin A\cos A$

$\Rightarrow (\sin^2 A + \cos^2 A) + \sin^2 A = 3\sin A\cos A$

$\Rightarrow 2\sin^2 A - 3\sin A\cos A + \cos^2 A = 0$

$\Rightarrow 2\tan^2 A - 3\tan A + 1 = 0$

$\Rightarrow (\tan A - 1)(2\tan A - 1) = 0$

$\Rightarrow \tan A - 1 = 0, 2\tan A - 1 = 0$

$\Rightarrow \tan A - 1 = 0,\ 2\tan A - 1 = 0$

$\Rightarrow \tan A = 1,\ 1/2$

6. For what values of x is the equation $2\sin\theta = x + 1/x$ is valid? *[NDA-2006]*

(a) $x = \pm 1$
(b) all real values of x
(c) $-1 < x < 1$
(d) $x > 1$ and $x < -1$

Solution

(a) $x^2 - 2x\sin\theta + 1 = 0$

$$\Rightarrow x = \frac{2\sin\theta \pm \sqrt{4\sin^2\theta - 4}}{2}$$

$$= \sin\theta \pm \sqrt{\sin^2\theta - 1}$$

Which is valid when $\sin\theta = \pm 1$ Then $x = \pm 1$.

7. If $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$ and $z = r\cos\theta$, then $x^2 + y^2 + z^2$ is independent of which of the following? *[NDA-2007]*

(a) r only (b) r, φ
(c) θ, φ (d) r, θ

Solution

(c) $x^2 + y^2 + z^2$

$= r^2\sin^2\theta\cos^2 \quad + r^2\sin^2\theta\sin^2 \quad + r^2\cos^2\theta$

$= r^2\sin^2\theta(\cos^2 \quad + \sin^2 \quad) + r^2\cos^2\theta$

$= r^2(\sin^2\theta + \cos^2\theta) = r^2$.

8. What is the value of
$(\sec\theta - \cos\theta)(\mathrm{cosec}\theta - \sin\theta)(\cot\theta + \tan\theta)$? *[NDA-07]*

(a) 1 (b) 2
(c) $\sin\theta$ (d) $\cos\theta$

Solution

(a) $(\sec\theta - \cos\theta)(\mathrm{cosec}\theta - \sin\theta)(\cot\theta + \tan\theta)$

$$= \frac{(1-\cos^2\theta)}{\cos\theta} \times \frac{(1-\sin^2\theta)}{\sin\theta} \times \frac{(1+\tan^2\theta)}{\tan\theta}$$

$$= \frac{\sin^2\theta.\cos^2\theta.\sec^2\theta.\cos\theta}{\cos\theta.\sin\theta.\sin\theta} = 1.$$

9. If $\sin x + \sin^2 x = 1$, then $\cos^6 x + \cos^{12} x + 3\cos^{10} x + 3\cos^8 x$ is equal to *[Pb. CET-2002; MPPET-2006]*

(a) 1 (b) $\cos^3 x \sin^3 x$
(c) 0 (d) ∞

Solution

(a) $\because \sin x + \sin^2 x = 1$

$\Rightarrow \sin x = 1 - \sin^2 x$

$\Rightarrow \sin x = \cos^2 x$

$\therefore \cos^6 x + \cos^{12} x + 3\cos^{10} x + 3\cos^8 x$

$= \sin^3 x + \sin^6 x + 3\sin^5 x + 3\sin^4 x = (\sin x + \sin^2 x)^3 = 1.$

10. The value of the expression *[Bihar EE-1990]*

$$1 - \frac{\sin^2 y}{1+\cos y} + \frac{1+\cos y}{\sin y} - \frac{\sin y}{1-\cos y}$$ is equal to

(a) 0 (b) 1
(c) $\sin y$ (d) $\cos y$

Solution

(d) The given value

$$= 1 - (1 - \cos y) + \frac{1-\cos^2 y - \sin^2 y}{\sin y(1-\cos y)} = \cos y + 0 = \cos y$$

11. $\dfrac{2\sin\theta\tan\theta(1-\tan\theta) + 2\sin\theta\sec^2\theta}{(1+\tan\theta)^2}$ is equal to *[Roorkee-1975]*

(a) $\dfrac{\sin\theta}{1+\tan\theta}$ (b) $\dfrac{2\sin\theta}{1+\tan\theta}$

(c) $\dfrac{2\sin\theta}{(1+\tan\theta)^2}$ (d) None of these

Solution

(b) Given that expression

$$= \frac{2\sin\theta}{(1+\tan\theta)^2}\{\tan\theta(1-\tan\theta) + \sec^2\theta\}$$

$$= \frac{2\sin\theta}{(1+\tan\theta)^2}\{\tan\theta - \tan^2\theta + 1 + \tan^2\theta\}$$

$$= \frac{2\sin\theta}{1+\tan\theta}$$

12. If $\cot\theta + \tan\theta = m$ and $\sec\theta - \cos\theta = n$, then which of the following is correct?

(a) $m(mn^2)^{1/3} - n\,(nm^2)^{1/3} = 1$
(b) $m(m^2n)^{1/3} - n\,(mn^2)^{1/3} = 1$
(c) $n(mn^2)^{1/3} - m\,(nm^2)^{1/3} = 1$
(d) $n(m^2n)^{1/3} - m\,(mn^2)^{1/3} = 1$

Solution

As given,

$$\frac{1}{\tan\theta} + \tan\theta = m \Rightarrow 1 + \tan^2\theta = m\tan\theta$$

$\Rightarrow \sec^2\theta = m\tan\theta$ .... (i)

and $\sec\theta - \cos\theta = n$

$\Rightarrow \sec^2\theta - 1 = n\sec\theta$

$\Rightarrow \tan^2\theta = n\sec\theta$

$\Rightarrow \tan^4\theta = n^2\sec^2\theta = n^2 m\tan\theta$ {by (i)}

$\Rightarrow \tan^3\theta = n^2m,$ $[\because \tan\theta \neq 0)$

$\Rightarrow \tan\theta = (n^2m)^{1/3}$ .... (ii)

Also, $\sec^2\theta = m\tan\theta = m(n^2m)^{1/3}$

{by (i) and (ii)}

using the identity $\sec^2\theta - \tan^2\theta = 1$

$\Rightarrow m(n^2m)^{1/3} - (n^2m)^{2/3} = 1$

$\Rightarrow m(mn^2)^{1/3} - n\,(nm^2)^{1/3} = 1$

13. $\sin^2\theta = \dfrac{4xy}{(x+y)^2}$ is true, if and only if

[AIEEE-2002]

(a) $x - y \neq 0$ (b) $x = -y$
(c) $x = y$ (d) $x \neq 0, y \neq 0$

Solution

(c) Since, $\theta \leq 1$

$$\Rightarrow \frac{4xy}{(x+y)^2} \leq 1 \left(\because \sin^2\theta = \frac{4xy}{(x+y)^2} \text{ given}\right)$$

$\Rightarrow x^2 + y^2 + 2xy - 4xy \geq 0$

$\Rightarrow (x - y)^2 \geq 0$

Which is true for all real values of x and y provided $x + y \neq 0$, otherwise, $\dfrac{4xy}{(x+y)^2}$ will be meaningless.

14. If $y = \sin^2\theta + \operatorname{cosec}^2\theta$, $\theta \neq 0$, then

[AIEEE-2002]

(a) $y = 0$ (b) $y \leq 0$
(c) $y \geq -2$ (d) $y \geq 2$

Solution

(d) Given that, $y = \sin^2\theta + \operatorname{cosec}^2\theta$

$\therefore$ $y = (\sin\theta - \operatorname{cosec}\theta)^2 + 2$

$\Rightarrow$ $y \geq 2, \theta \neq 0$

15. If $\cos x = \tan y$, $\cos y = \tan z$, $\cos z = \tan x$; than prove that $\sin x = \sin y = \sin z = 2\sin 18°$.

Solution

Making use of given relations, we have

$\cos^2 x = \tan^2 y = \sec^2 y - 1 = \cot^2 z - 1.$

or $$1 + \cos^2 x = \frac{\cos^2 z}{1 - \cos^2 z} = \frac{\tan^2 x}{1 - \tan^2 x}$$

or $$1 + \cos^2 x = \frac{\sin^2 x}{\cos^2 x - \sin^2 x}$$

or changing to sin x, we get $(2 - \sin^2 x)(1 - 2\sin^2 x) = \sin^2 x$

or $2\sin^4 x - 6\sin^2 x + 2 = 0$ or $\sin^4 x - 3\sin^2 x + 1 = 0$

$$\therefore \sin^2 x = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2} = \frac{3 - \sqrt{5}}{2}$$

We have rejected the value $\dfrac{3+\sqrt{5}}{2}$ as it is > 1.

or $$\sin^2 x = \frac{6 - 2\sqrt{5}}{4} = \left(\frac{\sqrt{5}-1}{2}\right)^2$$

$$\therefore \sin x = \frac{\sqrt{5}-1}{2} = 2.\frac{(\sqrt{5}-1)}{4} = 2\sin 18°$$

By symmetry, we can say that $\sin x = \sin y = \sin z = 2\sin 18°$.

16. Show that, $\cos(\sin\theta) > \sin(\cos\theta), 0 \leq \theta \leq \dfrac{\pi}{2}$.

[IIT-JEE-1981]

Solution

$$\sin\theta + \cos\theta \leq \sqrt{2} < \frac{\pi}{2}$$

$$\Rightarrow \sin\theta < \frac{\pi}{2} - \cos\theta$$

$$\Rightarrow \cos(\sin\theta) > \cos\left(\frac{\pi}{2} - \cos\theta\right) = \sin(\cos\theta)$$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. If $\sin\theta = \dfrac{2t}{1+t^2}$, then $\cos\theta$ is equal to

(a) $\dfrac{2t}{1-t^2}$ (b) $\dfrac{2t}{1+t^2}$

(c) $\dfrac{1-t^2}{1+t^2}$ (d) $\dfrac{1+t^2}{1-t^2}$

2. $\dfrac{\sin\theta}{1-\cot\theta}+\dfrac{\cos\theta}{1-\tan\theta}$ is equal to *[Karnataka CET-1998]*

(a) 0 (b) 1

(c) $\cos\theta - \sin\theta$ (d) $\cos\theta + \sin\theta$

3. If for real values of x, $\cos\theta = x + \dfrac{1}{x}$, then *[MPPET-1996]*

(a) θ is an acute angle

(b) θ is a right angle

(c) θ is an obtuse angle

(d) No values of θ is possible

4. The equation $\sec^2\theta = \dfrac{4xy}{(x+y)^2}$ is only possible when *[MPPET-1986; IIT-1996]*

(a) $x = y$ (b) $x < y$

(c) $x > y$ (d) None of these

5. Which of the following relations is correct? *[WBJEE-91]*

(a) $\sin 1 < \sin 1°$

(b) $\sin 1 > \sin 1°$

(c) $\sin 1 = \sin 1°$

(d) $\dfrac{\pi}{180}\sin 1 = \sin 1°$

6. If $\sin\theta + \operatorname{cosec}\theta = 2$, then $\sin^2\theta + \operatorname{cosec}^2\theta$ is equal to *[MPPET-1992; MNR-1990; UPSEAT-2002]*

(a) 1 (b) 4

(c) 2 (d) None of these

7. If $\tan\theta = \dfrac{20}{21}$, $\cos\theta$ will be *[MPPET-1994]*

(a) $\pm\dfrac{20}{41}$ (b) $\pm\dfrac{1}{21}$

(c) $\pm\dfrac{21}{29}$ (d) $\pm\dfrac{20}{21}$

8. If $x = a\cos^3\theta$, $y = b\sin^3\theta$, then

(a) $(a/x)^{2/3} + (b/y)^{2/3} = 1$

(b) $(b/x)^{2/3} + (a/y)^{2/3} = 1$

(c) $(x/a)^{2/3} + (y/b)^{2/3} = 1$

(d) $(x/b)^{2/3} + (y/a)^{2/3} = 1$

9. If $x = \sec\theta + \tan\theta$, then $x + \dfrac{1}{x}$ is equal to *[MPPET-1986]*

(a) 1 (b) $2\sec\theta$

(c) 2 (d) $2\tan\theta$

10. If $(1+\sin A)(1+\sin B)(1+\sin C) = (1-\sin A)(1-\sin B)(1-\sin C)$, then each side is equal to

(a) $\pm\sin A\sin B\sin C$

(b) $\pm\cos A\cos B\cos C$

(c) $\pm\sin A\cos B\cos C$

(d) $\pm\cos A\sin B\sin C$

11. If $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 3$, then $\cos\theta_1 + \cos\theta_2 + \cos\theta_3$ is equal to *[EAMCET-1994]*

(a) 3 (b) 2

(c) 1 (d) 0

12. If $\tan\theta - \cot\theta = a$ and $\sin\theta + \cos\theta = b$, then $(b^2-1)^2(a^2+4)$ is equal to *[WB JEE-1979]*

(a) 2 (b) -4

(c) ± 4 (d) 4

13. If $\tan\theta = \dfrac{x\sin\phi}{1-x\cos\phi}$ and $\tan\phi = \dfrac{y\sin\theta}{1-y\cos\theta}$, then $\dfrac{x}{y}$ is equal to *[MPPET-1991]*

(a) $\dfrac{\sin\phi}{\sin\theta}$ (b) $\dfrac{\sin\theta}{\sin\phi}$

(c) $\dfrac{\sin\phi}{1-\cos\theta}$ (d) $\dfrac{\sin\theta}{1-\cos\phi}$

14. If sec θ + tan θ = p, then tan θ is equal to **[MPPET-1994]**

(a) $\dfrac{2p}{p^2-1}$ (b) $\dfrac{p^2-1}{2p}$

(c) $\dfrac{p^2+1}{2p}$ (d) $\dfrac{2p}{p^2+1}$

15. If $p=\dfrac{2\sin\theta}{1+\cos\theta+\sin\theta}$ and $q=\dfrac{\cos\theta}{1+\sin\theta}$, then **[MPPET-2001]**

(a) $pq=1$ (b) $\dfrac{q}{p}=1$

(c) $q-p=1$ (d) $q+p=1$

16. The minimum value of 9 tan² θ + 4 cot² θ is

(a) 13 (b) 9

(c) 6 (d) 12

17. The maximum value of 4 sin² x + 3 cos² x is **[Karnataka CET-2003]**

(a) 3 (b) 4

(c) 5 (d) 7

18. Which value of k, $(\cos x+\sin x)^2+k\sin x\cos x-1=0$ is identity? **[Kerala (Engg.)-2001]**

(a) –1 (b) –2

(c) 0 (d) 1

20. If $\sin x+\sin^2 x=1$, then $\cos^8 x+2\cos^6 x+\cos^4 x$ is equal to

(a) 0 (b) –1

(c) 2 (d) 1

21. The least value of $2\sin^2\theta+3\cos^2\theta$ is **[MPPET-2010]**

(a) 1 (b) 2

(c) 3 (d) 5

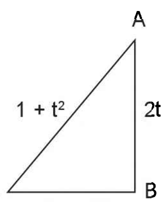
22. If $\sec A=x+\dfrac{1}{4x}$, then the value of sec A + tan A is **[MPPET-2010]**

(a) $3x$ (b) $\dfrac{x}{3}$

(c) $\dfrac{x}{2}$ (d) $2x$

HINTS AND EXPLANATIONS

1. (c) Given $\sin\theta=\dfrac{2t}{1+t^2}$



By Pythagoras theorem

$AC^2=AB^2+BC^2$

$(1+t^2)^2=(2t)^2+BC^2$

$(1+t^2)^2=4t^2+BC^2$

$\Rightarrow\ BC^2=(1+t^2)^2-4t^2=1+t^4+2t^2-4t^2=1+t^4-2t^2$

$\Rightarrow\ BC^2=(1-t^2)\ \Rightarrow BC=1-t^2$

$\therefore\ \cos\theta=\dfrac{BC}{AC}=\dfrac{1-t^2}{1+t^2}$

Second Method

$\sin\theta=\dfrac{2t}{1+t^2}=\dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$. If $t=\tan\dfrac{\theta}{2}$

$\therefore\ \cos\theta=\dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}=\dfrac{1-t^2}{1+t^2}$

2. (d) Given $\dfrac{\sin\theta}{1-\cot\theta}+\dfrac{\cos\theta}{1-\tan\theta}$

$=\dfrac{\sin\theta}{1-\dfrac{\cos\theta}{\sin\theta}}+\dfrac{\cos\theta}{1-\dfrac{\sin\theta}{\cos\theta}}$

$=\dfrac{\sin\theta}{\dfrac{\sin\theta-\cos\theta}{\sin\theta}}+\dfrac{\cos\theta}{\dfrac{\cos\theta-\sin\theta}{\cos\theta}}$

$\Rightarrow\ \dfrac{\sin^2\theta}{\sin\theta-\cos\theta}+\dfrac{\cos^2\theta}{\cos\theta-\sin\theta}$

$\Rightarrow \quad \dfrac{\sin^2\theta}{\sin\theta-\cos\theta}-\dfrac{\cos^2\theta}{\sin\theta-\cos\theta}$

$\Rightarrow \quad \dfrac{\sin^2\theta-\cos^2\theta}{\sin\theta-\cos\theta}$

$\Rightarrow \quad \dfrac{(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)}{(\sin\theta-\cos\theta)}$

$\Rightarrow \quad \sin\theta+\cos\theta$

3. (d) Given equation is $\cos\theta = x + 1/x$
or $x^2 - x\cos\theta + 1 = 0$
for real value of x By $B^2 > 4AC$

$\because \quad A = 1, B = -\cos\theta, C = 1$

$\cos^2\theta > 4(1)(1)$ or $|\cos\theta| \le 2$

Which is impossible because $|\cos\theta| < 1$

$\therefore$ No real value of θ is possible.

4. (a) $\sec^2\theta \ge 1\ \forall\ \theta \Rightarrow \dfrac{4xy}{(x+y)^2} \ge 1$

$\Rightarrow \quad 4xy \ge (x+y)^2$

$\Rightarrow \quad (x+y)^2 - 4xy \le 0$

$\Rightarrow \quad (x-y)^2 \le 0$. But $(x-y)^2 \ge 0$

$\Rightarrow \quad (x-y)^2 = 0 \Rightarrow x = y$

5. (a) 1 radian $= \dfrac{180}{\pi}$ degrees = 57° (approximately)

$\Rightarrow$ $\sin x$ is increasing function

$1° < 1^c$, $\sin(1°) < \sin(1^c)$

6. (c) $\sin\theta + \text{cosec}(\theta) = 2$

$\Rightarrow \quad \sin\theta + \dfrac{1}{\sin\theta} = 2$

$\Rightarrow \quad \sin^2\theta - 2\sin\theta + 1 = 0$

$\Rightarrow \quad (\sin\theta - 1)^2 = 0$

$\Rightarrow \quad \sin\theta = 1$

$\Rightarrow \quad \theta = \pi/2$

$\therefore \quad \sin^2(\theta) + \text{cosec}^2(\theta) = 1 + 1 = 2$

7. (c) Given $\tan\theta = \dfrac{20}{21}$ Squaring both sides

$\tan^2\theta = \dfrac{400}{441}$

By $\sec^2\theta = 1 + \tan^2\theta$

$\therefore \quad \sec^2\theta = 1 + \dfrac{400}{441}$

or $\cos^2\theta = \dfrac{441}{841}$ or $\cos\theta = \pm\dfrac{21}{29}$

8. (c) Given $x = a\cos^3\theta$, $y = b\sin^3\theta$

$\Rightarrow \quad \dfrac{x}{a} = \cos^3\theta, \dfrac{y}{b} = \sin^3\theta$

taking cube root on both sides.

$\Rightarrow \quad \left(\dfrac{x}{a}\right)^{1/3} = \cos\theta, \left(\dfrac{y}{b}\right)^{1/3} = \sin\theta$

Now square and add

$$\left(\left(\frac{x}{a}\right)^{\frac{1}{3}}\right)^2 + \left(\left(\frac{y}{b}\right)^{\frac{1}{3}}\right)^2 = \sin^2\theta + \cos^2\theta$$

$\Rightarrow \quad \left(\dfrac{x}{a}\right)^{2/3} + \left(\dfrac{y}{b}\right)^{2/3} = 1$

9. (b) $x + \dfrac{1}{x} = (\sec\theta + \tan\theta) + \dfrac{1}{\sec\theta + \tan\theta}$

$= (\sec\theta + \tan\theta) + \dfrac{(\sec\theta - \tan\theta)}{\sec^2\theta - \tan^2\theta}$

$\Rightarrow \quad x + \dfrac{1}{x} = 2\sec(\theta)$

10. (b) Multiplying both sides by $(1 - \sin A)(1 - \sin^2 B)(1 - \sin^2 C)$

We have $(1 - \sin^2 A)(1 - \sin^2 B)(1 - \sin^2 C)$

$= (1 - \sin A)^2 (1 - \sin B)^2 (1 - \sin C)^2$

$\Rightarrow \quad (1 - \sin A)(1 - \sin B)(1 - \sin C) = \pm\cos A\cos B\cos C$

Similarly, $(1 + \sin A)(1 + \sin B)(1 + \sin C)$

$= \pm\cos A\cos B\cos C$

11. (d) Given $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 3$

$\Rightarrow \quad \sin\theta_1 = 1 = \sin\theta_2 = \sin\theta_3$

$\Rightarrow \quad \theta_1 = \theta_2 = \theta_3 = \pi/2$

$\Rightarrow \quad \cos\theta_1 = \cos\theta_2 = \cos\theta_3 = 0$

$\Rightarrow \quad \cos\theta_1 + \cos\theta_2 + \cos\theta_3 = 0$

12. (d) Given, $\tan\theta - \cot\theta = a$ ……. (1)

$\sin\theta + \cos\theta = b$ ……….. (2)

(2) $\Rightarrow b^2 = 1 + 2\sin\theta\cos\theta$

$\Rightarrow \quad b^2 - 1 = \sin(2\theta)$ .... (3)

(1) $\Rightarrow a^2 = \tan^2\theta + \cot^2\theta - 2\tan\theta\cot\theta$

$= \tan^2\theta + \cot^2\theta - 2$

$= a^2 + 4 = (\tan\theta + \cot\theta)^2$

$$= a^2 + 4 = \left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)^2$$

$$= \left(\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}\right) = \left(\frac{2}{\sin 2\theta}\right)^2$$

$$= \frac{4}{(b^2 - 1)} \Rightarrow (a^2 + 4)(b^2 - 1)^2 = 4$$

13. (b) $\tan\theta = \dfrac{x\sin\phi}{1 - x\cos\phi}$

$\Rightarrow \quad (1 - x\cos\varphi)\tan\theta = x\sin\varphi$

$$\Rightarrow \quad x = \frac{\tan\theta}{\sin\phi + \cos\phi\tan\phi} \quad \text{.......... (1)}$$

Similarly, $\tan\phi = \dfrac{y\sin\theta}{1 - y\sin\theta}$

$$\Rightarrow y = \frac{\tan\phi}{\sin\theta + \cos\theta\tan\theta} \quad \text{.......... (2)}$$

$$\frac{(1)}{(2)} \Rightarrow \frac{x}{y} = \frac{\tan\theta}{\tan\phi}\cdot\left(\frac{\sin\theta + \cos\theta\tan\phi}{\sin\phi + \cos\phi\tan\phi}\right)$$

$$= \frac{\sin\theta\cot\phi + \cos\theta}{\sin\phi\cot\theta + \cos\phi}$$

$$= \frac{\sin\theta(\cot\phi + \cot\theta)}{\sin\phi(\cot\theta + \cot\phi)} = \frac{\sin\theta}{\sin\phi}$$

14. (b) $\sec\theta + \tan\theta = p$ .... (1)

$$\Rightarrow \quad \sec\theta - \tan\theta = \frac{1}{p} \quad \text{............ (2)}$$

$$(1) - (2) \Rightarrow p - \frac{1}{p} = 2\tan\theta \Rightarrow \tan\theta = \frac{p^2 - 1}{2p}$$

15. (d) $p + q = \dfrac{2\sin\theta}{1 + \sin\theta + \cos\theta} + \dfrac{\cos\theta}{1 + \sin\theta}$

$$= \frac{2\sin\theta + 2\sin^2\theta + \cos\theta + \sin\theta\cos\theta + \cos^2\theta}{(1 + \sin\theta + \cos\theta)(1 + \sin\theta)}$$

$$= \frac{(1 + \sin\theta + \cos\theta) + (\sin^2\theta + \sin\theta + \sin\theta\cos\theta)}{(1 + \sin\theta + \cos\theta)(1 + \sin\theta)}$$

$$= \frac{(1 + \sin\theta + \cos\theta)(1 + \sin\theta)}{(1 + \sin\theta + \cos\theta)(1 + \sin\theta)} = 1$$

16. (d) Since, A.M. $\geq$ G.M.

Therefore,

$$\frac{9\tan^2\theta + 4\cot^2\theta}{2} \geq \sqrt{9\tan^2\theta \times 4\cot^2\theta}$$

$$\Rightarrow \quad 9\tan^2\theta + 4\cot^2\theta \geq 2 \times \sqrt{36} = 12$$

17. (b) $f(x) = 4\sin^2 x + 3\cos^2 x = \sin^2 x + 3$ and $0 \leq |\sin x| \leq 1$

$\therefore$ maximum value of $\sin^2 x + 3$ is 4.

18. (b) Given

$(\cos x + \sin x)^2 + k\sin x\cos x - 1 = 0 \forall x \in R$

$\Rightarrow \quad \cos^2 x + \sin^2 x + 2\sin x\cos x + k\sin x\cos x - 1 = 0 \forall x$

$\Rightarrow \quad (k + 2)\sin x\cos x = 0 \forall x$

$\Rightarrow \quad k + 2 = 0 \Rightarrow k = -2$

19. (c) $x\sin^3\alpha + y\cos^3\alpha = \sin\alpha \,.\, \cos\alpha$ ...(1)

$x\sin\alpha - y\cos\alpha = 0$ .... (2)

Now, $a^2 = \tan^2\theta + \cot^2\theta - 2\tan\theta \,.\, \cot\theta = \tan^2\theta + \cot^2\theta - 2$

$\Rightarrow \quad a^2 + 4 = (\tan\theta + \cot\theta)^2$

$$\Rightarrow \quad a^2 + 4 = \left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)^2$$

$$= \left(\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}\right) = \left(\frac{2}{\sin 2\theta}\right)^2$$

$$= \frac{4}{(b^2 - 1)} \Rightarrow (a^2 + 4)(b^2 - 1)^2 = 4$$

20. (d) We have,

$\sin x + \sin^2 x = 1 \Rightarrow \sin x = 1 - \sin^2 x$

$\Rightarrow \quad \sin x = \cos^2 x$

$\Rightarrow \quad \cos^8 x + 2\cos^6 x + \cos^4 x$

$\Rightarrow \quad \sin^4 x + 2\sin^3 x + \sin^2 x = (\sin x + \sin^2 x)^2 = 1$

21. (b) $2\sin^2\theta + 3(1 - \sin^2\theta) = 3 - \sin^2\theta$

$\therefore$ least value $= 3 - 1 = 2$

22. (d) Let $\sec A + \tan A = t$

$\therefore \quad \sec A - \tan A = \dfrac{1}{t}$

By adding both $2\sec A = t + \dfrac{1}{t}$

$2x + \dfrac{1}{2x} = t + \dfrac{1}{t} \quad \therefore \; t = 2x$ or $\dfrac{1}{2x}$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. If 5 tan θ = 4, then $\frac{5\sin\theta - 3\cos\theta}{\sin\theta + 2\cos\theta} =$

(a) 5/9 (b) 14/5
(c) 9/5 (d) 5/14

2. $(\sec^2\theta - 1)(\text{cosec}^2\theta - 1) =$ ***[Karnataka CET-1998]***

(a) 0 (b) 1
(c) sec θ.cosec θ (d) $\sin^2\theta - \cos^2\theta$

3. The incorrect statement is ***[MNR-1993]***

(a) sin θ = 1/5 (b) cos θ = 1
(c) sin θ = 2 (d) tan θ = 20

4. Which of the following relations is possible?

(a) sin θ = 5/3
(b) tan θ =1002
(c) $\cos\theta = \frac{1+p^2}{1-p^2}(p \neq \pm 1)$
(d) sec θ =1/2

5. If sin θ + cos θ = *m* and sec θ + cosec θ = *n*, then *n* (*m* + 1)(*m* – 1) = ***[MPPET-1986]***

(a) *m* (b) *n*
(c) 2*m* (d) 2*n*

6. If cosec*A* + cot *A* = $\frac{11}{2}$, then tan *A* = ***[Roorkee-1995]***

(a) 21/22 (b) 15/16
(c) 44/117 (d) 117/43

7. If *a*cos θ + *b*sin θ = *m* and *a*sin θ – *b*cos θ = *n*, then $a^2 + b^2 =$

(a) $m + n$ (b) $m^2 - n^2$
(c) $m^2 + n^2$ (d) None of these

8. The value of $2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1$ is ***[MPPET-1997; UPSEAT-2002]***

(a) 2 (b) 0
(c) 4 (d) 6

9. (sec *A* + tan *A* – 1) (sec *A* – tan *A* + 1) – 2 tan *A* = ***[Roorkee-1972]***

(a) sec *A* (b) 2 sec *A*
(c) 0 (d) 1

10. If tan θ + sin θ = *m* and tan θ – sin θ = *n*, then ***[IIT-1970]***

(a) $m^2 - n^2 = 4mn$
(b) $m^2 + n^2 = 4mn$
(c) $m^2 - n^2 = m^2 + n^2$
(d) $m^2 - n^2 = 4\sqrt{mn}$

11. If $0 < x < \pi$ and $\cos x + \sin x = \frac{1}{2}$, then the value of tan *x* is ***[MPPET-2009]***

(a) $\frac{2-\sqrt{7}}{3}$ (b) $-\frac{4+\sqrt{7}}{3}$
(c) $-\frac{1+\sqrt{7}}{3}$ (d) $-\frac{2+\sqrt{7}}{3}$

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The worksheet is of 15 minutes.
3. The worksheet consists of 15 questions. The maximum marks are 45.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. Which of the following is equal to 1?
(a) $\cos^2\theta - \sin^2\theta$
(b) $\sec^2\theta - \text{cosec}^2\theta$
(c) $\cot^2\theta - \tan^2\theta$
(d) $\sec^2\theta - \tan^2\theta$

2. The value of $\dfrac{\cos A}{\sec A} + \dfrac{\sin A}{\text{cosec} A}$ is
(a) $\sec^2 A + \tan^2 A$
(b) $\sec^2 A - \tan^2 A$
(c) $\cot^2 A - \text{cosec}^2 A$
(d) $\text{cosec}^2 A + \cot^2 A$

3. $\dfrac{1+\cos\theta}{\sin^2\theta}$
(a) 0 (b) 1
(c) $\dfrac{1}{1-\cos\theta}$ (d) $\dfrac{1}{1+\cos\theta}$

4. $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}$ is equal to
(a) 0 (b) 1
(c) $\sec\theta \,.\, \tan\theta$ (d) $\sec\theta - \tan\theta$

5. The equation $(a+b)^2 = 4ab\sin^2\theta$ is possible only when
(a) $2a = b$ (b) $a = b$
(c) $a = 2b$ (d) None of these

6. $\sin^6\theta + \cos^6\theta + 3\sin^2\theta\cos^2\theta$ is equal to ***[MPPET-1995, 2002; DCE-2005]***
(a) 0 (b) –1
(c) 1 (d) None

7. If $f(x) = \cos^2 x + \sec^2 x$, then ***[MNR-1996]***
(a) $f(x) < 1$ (b) $f(x) = 1$
(c) $1 < f(x) < 2$ (d) $f(x) \geq 2$

8. If $\sin\theta + \cos\theta = a$, then the value of $|\sin\theta \cdot \cos\theta|$ is ***[Pb. CET-92]***
(a) $\sqrt{2-a^2}$ (b) $\sqrt{2+a^2}$
(c) $\sqrt{a^2-2}$ (d) None

9. Which of the following is possible?
(a) $\cos\theta = \dfrac{7}{5}$ (b) $\sin\theta = \dfrac{a^2+b^2}{a^2-b^2}$
(c) $5\sec\theta = 4$ (d) $\tan\theta = 45$

10. If $\cos x + \cos^2 x = 1$, then the value of $\sin^{12}x + 3\sin^{10}x + 3\sin^8 x + \sin^6 x - 1$ is equal to ***[VIT-2007]***
(a) 2 (b) 1
(c) – 1 (d) 0

11. If $\sin\theta + \cos\theta = 1$, then $\sin\theta\cos\theta$ is equal to ***[Karnataka CET-1998]***
(a) 0 (b) 1
(c) 2 (d) 1/2

12. If $(\sec\alpha + \tan\alpha)(\sec\beta + \tan\beta)(\sec\gamma + \tan\gamma) = \tan\alpha\tan\beta\tan\gamma$, then $(\sec\alpha - \tan\alpha)(\sec\beta - \tan\beta)(\sec\gamma - \tan\gamma)$ is equal to ***[Krukshetra CEE-1998]***
(a) $\cot\alpha\cot\beta\cot\gamma$
(b) $\tan\alpha\tan\beta\tan\gamma$
(c) $\cot\alpha + \cot\beta + \cot\gamma$
(d) $\tan\alpha + \tan\beta + \tan\gamma$

13. The value of $6(\sin^6\theta + \cos^6\theta) - 9(\sin^4\theta + \cos^4\theta) + 4$ is ***[MPPET-2001]***
(a) –3 (b) 0
(c) 1 (d) 3

14. If $y = \cos^2 x + \sec^2 x$, then ***[MP PET-2004]***
(a) $y \leq 2$ (b) $y \leq 1$
(c) $y \geq 2$ (d) $1 < y < 2$

15. If $\sin\theta + \text{cosec}\,\theta = 2$, the value of $\sin^{10}\theta + \text{cosec}^{10}\theta$ is ***[MP PET-2004]***
(a) 2 (b) 2^{10}
(c) 2^9 (d) 10

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	6. ⓐ ⓑ ⓒ ⓓ	11. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	7. ⓐ ⓑ ⓒ ⓓ	12. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	8. ⓐ ⓑ ⓒ ⓓ	13. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	9. ⓐ ⓑ ⓒ ⓓ	14. ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ	10. ⓐ ⓑ ⓒ ⓓ	15. ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

1. (d) from 2nd trigonometric identities

$1 + \tan^2\theta = \sec^2\theta$

$\Rightarrow \quad \sec^2\theta - \tan^2\theta = 1$

2. (b) $\dfrac{\cos A}{\sec A} + \dfrac{\sin A}{\operatorname{cosec} A} = \dfrac{\cos A}{\frac{1}{\cos A}} + \dfrac{\sin A}{\frac{1}{\sin A}}$

$= \cos^2 A + \sin^2 A = 1 = \sec^2 A - \tan^2 A$

3. (c) Given, $\dfrac{1+\cos\theta}{\sin^2\theta}$

$= \dfrac{1+\cos\theta}{1-\cos^2\theta}$ $\quad [\because \sin^2\theta = 1 - \cos^2\theta]$

$= \dfrac{1+\cos\theta}{(1-\cos\theta)(1+\cos\theta)} = \dfrac{1}{1-\cos\theta}$

4. (d) Given

$\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} \times \sqrt{\dfrac{1-\sin\theta}{1-\sin\theta}}$

$= \sqrt{\dfrac{(1-\sin\theta)^2}{1-\sin^2\theta}} = \sqrt{\dfrac{(1-\sin\theta)^2}{\cos^2\theta}}$

$= \sqrt{\left(\dfrac{1-\sin\theta}{\cos\theta}\right)^2}$

$= \dfrac{1}{\cos\theta} - \dfrac{\sin\theta}{\cos\theta} = \sec\theta - \tan\theta$

5. (b) Try yourself.

6. (c) $\sin^6\theta + \cos^6\theta + 3\sin^2\theta\cos^2\theta$

$= (\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta + 3\sin^2\theta$ $\cos^2\theta = 1$

Short Method

Put $\theta = 0°$, we get the values of expression equal to 1. Again put $\theta = 45°$, the value remains 1, it means that the expression is independent of θ and is equal to 1.

7. (d) $f(x) = (\sec x - \cos x)^2 + 2$

$\Rightarrow \quad f(x) \geq 2$ for all x.

8. (a) $|\sin\theta - \cos\theta|^2 = (\sin\theta - \text{cso}\,\theta)^2 = 1 - \sin 2\theta$ and $\sin\theta + \cos\theta = a$

$\Rightarrow \quad 1 + \sin 2\theta = a^2$

$\Rightarrow \quad \sin 2\theta = a^2 - 1$

$\Rightarrow \quad 1 - \sin 2\theta = 2 - a^2$

$\therefore \quad |\sin\theta - \cos\theta| = \sqrt{2-a^2}$

9. (d) For option (a): $\cos\theta$ cannot be greater than 1.

For option (b): $\sin\theta$ also cannot be greater than 1

For option (c): $\sec\theta = \dfrac{4}{5}$ is also not possible as $\sec\theta$ cannot be less then 1.

For option (d): $\tan\theta$ can assume any real value.

$\therefore$ option (d) is correct.

10. (d) $\therefore \cos x + \cos^2 x = 1$

$\Rightarrow \quad \cos x = 1 - \cos^2 x = \sin^2 x$

$\therefore \quad \sin^{12} x + 3\sin^{10} x + 3\sin^8 x + \sin^6 x - 1$

$= \cos^6 x + 3\cos^5 x + 3\cos^4 x + \cos^3 x - 1$

$= (\cos^2 x + \cos x)^3 - 1 = 1 - 1 = 0$

11. (a) Given, $\sin\theta + \cos\theta = 1$

By squaring both sides, we get

$(\sin\theta + \cos\theta)^2 = 1$

$\Rightarrow \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 1$

$\Rightarrow 1 + 2\sin\theta\cos\theta = 1$ $[\because \sin^2\theta + \cos^2\theta = 1]$

$\Rightarrow 2\sin\theta\cos\theta = 0 \Rightarrow \sin\theta\cos\theta = 0$

12. (a) Given

$(\sec\alpha + \tan\alpha)(\sec\beta + \tan\beta)(\sec\gamma + \tan\gamma)$

$= \tan\alpha\tan\beta\tan\gamma$ .... (1)

Let $x = (\sec\alpha - \tan\alpha)(\sec\beta - \tan\beta)(\sec\gamma - \tan\gamma)$ .... (2)

By multiplying both equations (1) and (2), we get

$(\sec^2\alpha - \tan^2\alpha)(\sec^2\beta - \tan^2\beta)(\sec^2\gamma - \tan^2\gamma)$

$= x.(\tan\alpha\tan\beta\tan\gamma)$

$$= x = \frac{1}{\tan\alpha\tan\beta\tan\gamma}$$

$\therefore x = \cot\alpha\cot\beta\cot\gamma$

13. (c) $6[(\sin^2\theta)^3 + (\cos^2\theta)^3] - 9\sin^4\theta - 9\cos^4\theta + 4$

$= 6[(\sin^2\theta + \cos^2\theta)(\sin^4\theta - \sin^2\theta\cos^2\theta) + (\cos^4\theta)] - 9\sin^4\theta + 4$

$= -3(\sin^4\theta + \cos^4\theta + 2\sin^2\theta\cos^2\theta) + 4$

$= -3(\sin^2\theta + \cos^2\theta)^2 + 4 = -3 + 4 = 1$

14. (c) $y = \cos^2 x + \sec^2 x$ may be written as

$y = (\cos^2 x + \sec^2 x - 2) + 2$

or $y = (\cos x - \sec x)^2 + 2$

As $(\cos x - \sec x)^2$ is 0 or +ve

$\therefore y = 2 +$ (positive or zero)

$\therefore y \geq 2.$

15. (a) We have,

$\sin\theta + \operatorname{cosec}\theta = 2$

$$\Rightarrow \sin\theta + \frac{1}{\sin\theta} = 2 \quad \left[\because \operatorname{cosec}\theta = \frac{1}{\sin\theta}\right]$$

$\Rightarrow \sin^2\theta + 1 = 2\sin\theta$

$\Rightarrow \sin^2\theta - 2\sin\theta + 1 = 0$

$\Rightarrow (\sin\theta - 1)^2 = 0 \Rightarrow \sin\theta = 1$

Required value of $\sin^{10}\theta + \operatorname{cosec}^{10}\theta$

$$= (1)^{10} + \frac{1}{(1)^{10}} = 2$$

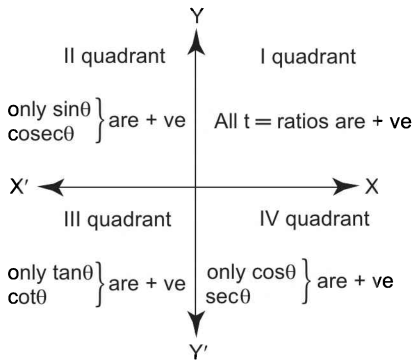
LECTURE 3

Quadrants

BASIC CONCEPTS

1. QUADRANTS

A system of rectangular coordinate axes divides a plane into four quadrants. An angle θ lies in one and only one of these quadrants. The values of the trigonometric ratios in the various quadrants are shown.



i.e., Sign of trigonometrical functions in different quadrants may be memorized as follows:

(i) I quadrant: All t-ratios are positive.
(ii) II quadrant: sin and cosec are positive and all others negative.
(iii) III quadrant: tan and cot are positive and all others are negative.
(iv) IV quadrant: cos and sec are positive and all others are negative.

2. CHANGE IN FUNCTION CONNECTED WITH RELATED ANGLES

$n\frac{\pi}{2} \pm \theta$ are called Related or allied angles. Related angles can be divided into two groups quadrant-wise namely, as follows

(i) When n is odd then related angles quadrant-wise are as mentioned below
Value of the trigonometric ratio of any angle for odd n results is the co-function when effect of any trigonometric function is studied e.g.,

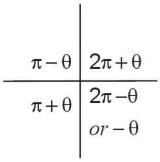
$$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta,\ \sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta$$

$$\cos\left(\frac{3\pi}{2}-\theta\right)=-\sin\theta,\ \cos\left(\frac{3\pi}{2}+\theta\right)=\sin\theta$$

$\frac{\pi}{2}+\theta$	$\frac{\pi}{2}-\theta$
$\frac{3\pi}{2}-\theta$	$\frac{3\pi}{2}+\theta$

(ii) When n is even then related angles quadrant-wise are as follows:
Values of the trigonometric ratio of any angle for even n results in the same function when effect of any trigonometric function is studied

e.g.,



$\sin(\pi - \theta) = \sin\theta$, $\sin(\pi + \theta) = -\sin\theta$
$\sin(2\pi - \theta) = -\sin\theta$, $\sin(2\pi + \theta) = \sin\theta$

Example

(a) To determine $\sin(540° - \theta)$, we note that $540° - \theta = 6 \times 90° - \theta$ is a second quadrant angle if $0° < \theta < 90°$.
In this quadrant, sine is positive and, since the given angle contains an even multiple of $\pi/2$, the sine function is retained.
Hence, $\sin(540° - \theta) = \sin\theta$

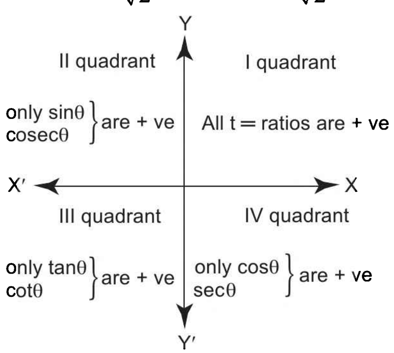
(b) To determine $\cos(630° - \theta)$, we note $630° - \theta = 7 \times 90° - \theta$ is a third quadrant angle if $0° < \theta < 90°$. In this quadrant, cosine is negative and, since the given angle contains an odd multiple of $\pi/2$, cosine is replaced by sine. Hence, $\cos(630° - \theta) = -\sin\theta$.

3. VALUE OF TR IN TERMS OF AN ANGLE IN THE FIRST QUADRANT

The value of trigonometric ratios of any angle can be expressed in terms of an angle in first quadrant, by the use of quadrant-wise angle and sign convention

$$\sin 45° = +\frac{1}{\sqrt{2}},\ \sin 135° = +\frac{1}{\sqrt{2}},$$

$$\sin 225° = -\frac{1}{\sqrt{2}},\ \sin 315° = -\frac{1}{\sqrt{2}}$$



NOTE

The value of the trigonometric ratios at each of the above quadrant-wise angles are numerically same and sign is as per sign convention.

4. IMPORTANT TR VALUE'S

(i) $\sin(-\theta) = -\sin\theta$ $\text{cosec}(-\theta) = -\text{cosec}\,\theta$

(ii) $\cos(-\theta) = \cos\theta$ $\sec(-\theta) = \sec\theta$

(iii) $\tan(-\theta) = -\tan\theta$ $\cot(-\theta) = -\cot\theta$

(iv) $\sin(90° - \theta) = \cot\theta$, $\cos(90° - \theta) = \sin\theta$
$\tan(90° - \theta) = \cot\theta$, $\cot(90° - \theta) = \tan\theta$
$\sec(90° - \theta) = \text{cosec}\,\theta$, $\text{cosec}(90° - \theta) = \sec\theta$

(v) $\sin(90° + \theta) = \cos\theta$, $\cos(90° + \theta) = -\sin\theta$
$\tan(90° + \theta) = -\cot\theta$, $\cot(90° + \theta) = -\tan\theta$
$\sec(90° + \theta) = -\text{cosec}\,\theta$, $\text{cosec}(90° + \theta) = \sec\theta$

(vi) $\sin(180° - \theta) = \sin\theta$, $\cos(180° - \theta) = -\cos\theta$
$\tan(180° - \theta) = -\tan\theta$, $\cot(180° - \theta) = -\cot\theta$
$\sec(180° - \theta) = -\sec\theta$, $\text{cosec}(180° - \theta) = \text{cosec}\,\theta$

(vii) $\sin(180° + \theta) = -\sin\theta$, $\cos(180° + \theta) = -\cos\theta$
$\tan(180° + \theta) = \tan\theta$, $\cot(180° + \theta) = \cot\theta$
$\sec(180° + \theta) = -\sec\theta$, $\text{cosec}(180° + \theta) = -\text{cosec}\,\theta$

(viii) $\sin(270° - \theta) = -\cos\theta$, $\cos(270° - \theta) = -\sin\theta$
$\tan(270° - \theta) = \cot\theta$, $\cot(270° - \theta) = -\tan\theta$
$\text{cosec}(270° - \theta) = -\sec\theta$, $\sec(270° - \theta) = -\text{cosec}\,\theta$

(xi) $\sin(270° + \theta) = -\cos\theta$, $\cos(270° + \theta) = \sin\theta$
$\tan(270° + \theta) = -\cot\theta$, $\cot(270° + \theta) = -\tan\theta$
$\text{cosec}(270° + \theta) = -\sec\theta$, $\sec(270° + \theta) = \text{cosec}\,\theta$

(x) $\sin(360° - \theta) = -\sin\theta$, $\cos(360° - \theta) = \cos\theta$
$\tan(360° - \theta) = -\tan\theta$, $\cot(360° - \theta) = -\cot\theta$
$\text{cosec}(360° - \theta) = -\text{cosec}\,\theta$, $\sec(360° - \theta) = \sec\theta$

5. TRIGONOMETRY RATIOS OF STANDARD ANGLES

5.1 Table

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°
sin	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0
cos	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0	–1/2	$-1/\sqrt{2}$	$-\sqrt{3}/2$	–1
tan	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	–1	$-1/\sqrt{3}$	0
cot	∞	$\sqrt{3}$	1	$1/\sqrt{3}$	0	$-1/\sqrt{3}$	–1	$-\sqrt{3}$	∞
sec	1	$2/\sqrt{3}$	$\sqrt{2}$	2	∞	–2	$-\sqrt{2}$	$-2/\sqrt{3}$	–1
cosec	∞	2	$\sqrt{2}$	$2/\sqrt{3}$	1	$2/\sqrt{3}$	$\sqrt{2}$	2	∞

5.2 Important relations

1. $\sin^2\alpha + \cos^2\alpha = 1$
2. $\sin^2\alpha + \sin^2(90° - \alpha) = 1$
3. $\cos^2\alpha + \cos^2(90° - \alpha) = 1$
4. $\sin^2\alpha + \sin^2\beta = 1$ if $\alpha + \beta = 90°$
5. $\cos^2\alpha + \cos^2\beta = 1$ if $\alpha + \beta = 90°$
6. $\sin^2\alpha + \sin^2\left(\frac{\pi}{2} - \alpha\right) + \sin^2\left(\frac{\pi}{2} + \alpha\right) + \sin^2(\pi - \alpha) = 2$
7. $\cos^2\alpha + \cos^2\left(\frac{\pi}{2} - \alpha\right) + \cos^2\left(\frac{\pi}{2} + \alpha\right) + \cos^2(\pi - \alpha) = 2$
8. $\tan\theta \cot\theta = 1$
9. $\tan A \tan B = 1$ if $A + B = 90°$ or $270°$
10. $\sin\alpha + \sin\beta = 0$ if $\alpha + \beta = 0°$ or $360°$ or 2π
11. $\cos\alpha + \cos\beta = 0$ if $\alpha + \beta = \pi = 180°$
12. $\sin\alpha + \sin\beta$ if $\alpha + \beta = \pi = 180°$
13. $\tan\alpha + \tan\beta = 0$ if $\alpha + \beta = \pi = 180°$ or 2π
14. $\sin\alpha = \cos\beta$ if $\alpha + \beta = 90°$
15. $\cos\alpha = \sin\beta$ if $\alpha + \beta = 90°$

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. If $\cos x = -\frac{3}{5}$, x lies in the third quadrant, find the values of other five trigonometric functions. *[NCERT]*

Solution

Since $\cos x = -\frac{3}{5}$, we have $\sec x = -\frac{5}{3}$

Now, $\sin^2 x + \cos^2 x = 1$

i.e., $\sin^2 x = 1 - \cos^2 x$

or $\sin^2 x = 1 - \frac{9}{25} = \frac{16}{25}$

Hence, $\sin x = \pm\frac{4}{5}$

Since x lies in third quadrant, $\sin x$ is negative.

Therefore, $\sin x = -\frac{4}{5}$

which also gives,

$\text{cosec } x = -\frac{5}{4}$

Further, we have

$\tan x = \frac{\sin x}{\cos x} = \frac{4}{3}$ and $\cos x = \frac{\cos x}{\sin x} = \frac{3}{4}$

2. If $\cot x = -\frac{5}{12}$, x lies in second quadrant, find the values of other five trigonometric functions. *[NCERT]*

Solution

Since, $\cot x = -\frac{5}{12}$, we have $\tan = -\frac{12}{5}$

Now, $\sec^2 x = 1 + \tan^2 x = 1 + \frac{144}{25} = \frac{169}{25}$

Hence, $\sec x = \pm\frac{13}{5}$

Since x lines in second quadrant, sec x will be negative. Therefore,

$$\sec x = -\frac{13}{5}$$

which also gives $\cos x = -\frac{5}{13}$

Further, we have

$\sin x = \tan x \cos x = \left(-\frac{12}{5}\right)\times\left(-\frac{5}{13}\right) = \frac{12}{13}$

and cosec $x = \frac{1}{\sin x} = \frac{13}{12}$

3. Find the value of $\sin\frac{31\pi}{3}$. *[NCERT]*

Solution

We know that values of sin x repeats after an interval of 2π. therefore,

$$\sin\frac{31\pi}{3} = \sin\left(10\pi + \frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

4. Find the value of cos(–1710°).

Solution

We know that values of cos x repeats after an interval of 2π or 360°. Therefore, cos(–1710°) = cos(–1710° + 5 × 360°) = cos(–1710° + 1800°) = cos90° = 0

5. cot15° + cot75° + cot135° – cosec 30° is equal to

Solution

The given expression is equal to

cot 15° + cot(90° – 15°) + cot(90° + 45°) – cosec 30°

= cot 15° + tan 15° – tan 45° – cosec 30°

$= \frac{2}{\sin 30°} - 1 - 2 = 4 - 3 = 1$

6. If B + C = 60°, prove that sin (120° – B) = sin(120° – C).

Solution

L.H.S. = sin(120° – B)
= sin [120° - (60° – C)] [∵ B + C = 60°]
= sin (60° + C)
= sin [180° – (60° + C)] [∵ sin θ = sin(180° – θ)]
= sin (120° – C) = R.H.S.

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): SOLVE THESE PROBLEMS TO GRASP THE TOPIC

EXERCISE 1

Directions for questions 1 to 6:
Find the values of the following trigonometric ratios.

1. sin 315°

2. cos (–480°)

3. sin (–1125°)

4. cot 570°

5. cos 270°

6. $\sin\left(\frac{-41\pi}{4}\right)$

Directions for questions 7 to 19:
Prove the following:

7. tan 225° cot 405° + tan 765° cot 675° = 0

8. sin(270° – θ) sin (90° – θ) – cos(270° – θ) cos(90° + θ) + 1 = 0

9. $\sin(-420°)(\cos 390°) + \cos(-660°)$
$\sin 330° = -1$

10. $\dfrac{\cos(90°+\theta)\sec(-\theta)\tan(180°-\theta)}{\sec(360°-\theta)\sin(180°+\theta)\cot(90°-\theta)} = -1$

11. $\text{Sin } 600° \tan(-690°) + \sec 840° \cot(-945°)$
$= \dfrac{3}{2}$

12. $\left[1+\cot\alpha - \sec\left(\alpha+\dfrac{\pi}{2}\right)\right]$
$\left[1+\cot\alpha + \sec\left(\alpha+\dfrac{\pi}{2}\right)\right] = 2\cot\alpha$

13. Simplify $\dfrac{\cos\theta}{\sin(90°+\theta)} + \dfrac{\sin(-\theta)}{\sin(180°+\theta)} - \dfrac{\tan(90°+\theta)}{\cot\theta}$.

14. Find all other trigonometrical ratios if $\sin\theta = -\dfrac{2\sqrt{6}}{5}$ and θ lies in quadrant III.

15. If $\sec\theta = \sqrt{2}$ and $3\pi/2 < \theta < 2\pi$ find the value of $\dfrac{1+\tan\theta+\text{cosec }\theta}{1+\cot\theta-\text{cosec }\theta}$.

16. Prove that cos 150° + cos 15° = sin 75° – sin 15°

17. Prove that
cos 34° + cos 43° + cos 137° + cos 214° + cos 300° $= \dfrac{1}{2}$

18. Prove that $\cos^2\dfrac{\pi}{8} + \cos^2\dfrac{3\pi}{8} + \cos^2\dfrac{5\pi}{8} + \cos^2\dfrac{7\pi}{8} = 2$

19. Find all angles between 0° and 360° satisfying $3\tan^2 x = 1$.

EXERCISE 2

1. Find the values of the following trigonometric ratios:

(a) cos 210° (b) cos 480°
(c) sin (–π/2) (d) cos –π

2. Prove that

(a) $\dfrac{\sin(-\theta)\tan(90°+\theta)\tan(180°+\theta)}{\sin(180°-\theta)\cos(90°-\theta)\cos(360°-\theta)} = \sec\theta\,\text{cosec }\theta$

(b) cos 510° cos 330° + sin 390° cos 120° = –1

(c) $\dfrac{\cos(\pi+\theta)\cos(-\theta)}{\sin(\pi-\theta)\cos\left(\dfrac{\pi}{2}+\theta\right)} = \cot^2\theta$

(d) $\cos\left(\dfrac{3\pi}{2}+\theta\right)\cos(2\pi+\theta)\left[\cot\left(\dfrac{3\pi}{2}-\theta\right)+\cot(2\pi+\theta)\right] = 1$

3. Simplify
$\dfrac{\tan(90°-\theta)\sec(180°-\theta)\sin(-\theta)}{\sin(180+\theta)\cot(360°-\theta)\text{cosec }(90°+\theta)}$.

4. Find sin θ and tan θ if cos θ = –12/13 and θ lies in the third quadrant.

5. If cos θ = –1/2
and π < θ 3π/2,
find the value of $\tan^2\theta - 3\,\text{cosec}^2\,\theta$.

6. If $\sin\theta = 1/\sqrt{2}$ and θ lies in the second quadrant, find all other trigonometric ratios.

7. Find all values of θ between 0° and 360° for which

(i) sin θ = 1/2,

(ii) $\text{cosec }\theta = -\dfrac{2}{\sqrt{3}}$.

8. If tan θ = 3/4 lies in 3rd quadrant, find all trigonometric ratios.

9. cosec (90° + *A*) + *x* cos *A* cot (90° + *A*) = sin (90° + *A*)

10. If $\cot x = \dfrac{-5}{12}$, *x* lies in second quadrant, find the values of other five trigonometric functions.

11. Prove that

(a) $\cot^2\dfrac{\pi}{6} + \text{cosec}\,\dfrac{5\pi}{6} + 3\tan^2\dfrac{\pi}{6} = 6$

(b) $2\sin^2\dfrac{3\pi}{4} + 2\cos^2\dfrac{\pi}{4} + 2\sec^2\dfrac{\pi}{3} = 10$

(c) $\cos^2\left(\dfrac{\pi}{4}+\theta\right) - \sin^2\left(\dfrac{\pi}{4}-\theta\right) = 0$

ANSWERS

EXERCISE 1

1. $\frac{-1}{\sqrt{2}}$
2. $\frac{-1}{2}$
3. $\frac{-1}{\sqrt{2}}$
4. $\sqrt{3}$
5. 0
6. $-\frac{1}{\sqrt{2}}$

13. 3

14. $\cos\theta = \frac{-1}{5}$, $\tan\theta = 2\sqrt{6}$,

$\text{cosec}\ \theta = \frac{-5}{2\sqrt{6}}$, $\sec\theta = -5$,

$\cot\theta = \frac{1}{2\sqrt{6}}$

15. –1

19. (i) 30° or 210° when $\tan x = \frac{1}{\sqrt{3}}$

(ii) 150°, 330° when $\tan x = \frac{-1}{\sqrt{3}}$

EXERCISE 2

1. (a) $\frac{-\sqrt{3}}{2}$

(b) $-\sqrt{3}$

(c) –1

(d) –1

3. 1

4. $\sin\theta = -5/13$, $\tan\theta = 5/12$

5. 8

6. $\cos\theta = -1/\sqrt{2}$

$\tan\theta = -1$

$\cot\theta = -1$

$\text{cosec}\ \theta = \sqrt{2}$

$\sec\theta = -\sqrt{2}$

7. (i) 30°, 150°

(ii) 240°, 300°

8. $\sin\theta = -\frac{3}{5}$

$\cos\theta = -\frac{4}{5}$

$\cot\theta = \frac{4}{3}$

$\sec\theta = -\frac{5}{4}$

$\text{cosec}\ \theta = -\frac{5}{3}$

9. $\tan A$

10. $\tan x = -\frac{5}{12}$,

$\sec x = -\frac{13}{5}$

$\cos x = -\frac{5}{13}$

$\sin x = \frac{12}{13}$

$\text{cosec}\ x = \frac{13}{12}$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. If $\frac{3\pi}{4} < \alpha < \pi$, then $\sqrt{2\cot\alpha + \frac{1}{\sin^2\alpha}}$ is equal to ***[Pb. CET-2000; AMU-2001]***

(a) $1 + \cot\alpha$ (b) $-1 - \cot\alpha$
(c) $1 - \cot\alpha$ (d) $-1 + \cot\alpha$

Solution

(b) $\sqrt{2\cot\alpha + \frac{1}{\sin^2\alpha}}$

$= \sqrt{2\cot\alpha + \text{cosec}^2\alpha}$

$= \sqrt{2\cot\alpha + 1 + \cot^2\alpha}$

$= \sqrt{(1+\cot\alpha)^2} = |1 + \cot\alpha|$

Since, $\cot\alpha < -1$ when $\frac{3\pi}{4} < \alpha < \pi$,

we have $|1 + \cot\alpha| = -1 - \cot\alpha$

2. If $\tan\theta = -\frac{4}{3}$, then $\sin\theta$ is ***[AIEEE-2002]***

(a) $-\frac{4}{5}$ but not $\frac{4}{5}$ (b) $-\frac{4}{5}$ or $\frac{4}{5}$

(c) $\frac{4}{5}$ but not $-\frac{4}{5}$ (d) None of these

Solution

Since, $\tan\theta = -\frac{4}{3}$

$\therefore \quad \sin\theta = \frac{BC}{AC} = \frac{4}{5}$

But, tan θ is negative which is possible only, if θ lie in second and fourth quadrants.

3. If α is a root of $25\cos^2\theta + 5\cos\theta - 12 = 0$, $\frac{\pi}{2} < \alpha < \pi$, then sin 2α is equal to

[AIEEE-2002]

(a) $\frac{24}{25}$ (b) $-\frac{24}{25}$

(c) $\frac{13}{18}$ (d) $-\frac{13}{18}$

Solution

Since, α is a root of $25\cos^2\theta + 5\cos\theta - 12 = 0$

$\therefore \quad 25\cos^2\alpha + 5\cos\alpha - 12 = 0$

$\Rightarrow \quad (5\cos\alpha - 3)(5\cos\alpha + 4) = 0$

$\Rightarrow \quad \cos\alpha = -\frac{4}{5}, \frac{3}{5}$

But $\frac{\pi}{2} < \alpha < \pi$ i.e., in second quadrant

$\therefore \quad \cos\alpha = -\frac{4}{5}$

$\Rightarrow \quad \sin\alpha = \frac{3}{5}$

Now, $\sin 2\alpha = 2\sin\alpha\cos\alpha$

$$= 2 \times \frac{3}{5} \times \left(-\frac{4}{5}\right) = -\frac{24}{25}$$

4. If $\sin(30° - \theta) = \cos(60° - \phi)$ then

[C.D.S.-1998]

(a) $\phi - \theta = 30°$ (b) $\phi - \theta = 0°$

(c) $\phi + \theta = 60°$ (d) $\phi - \theta = 60°$

Solution

$\sin(30° - \theta) = \cos(60° - \theta)$

$= \sin[90° - (60° - \phi) = \sin(30° - \phi)]$

$\therefore \quad 30° - \theta = 30° - \phi$

$\Rightarrow \quad \theta = \phi \Rightarrow \theta - \phi = 0$

5. The value of $\sin(-870°)$ is

[S.C.R.A. Exam.-1999]

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$

(c) 1 (d) $-\frac{1}{2}$

Solution

$\sin[-870°] = -\sin(870°)$

$= -\sin(9 \times 90° + 60°)$

$= -\cos 60° = -\frac{1}{2}$

6. If $\theta < 90°$, then $\sin\theta + \cos\theta$ is

[N.D.A. Sept-1998 Type]

(a) less then 1
(b) equal to 1
(c) greater than 1
(d) none of these

Solution

Since, $0 < \theta < 90°$

$\therefore \quad 0 < \sin\theta < 1$ and $0 < \cos\theta < 1$

$\therefore \quad \sin\theta + \cos\theta$ lies between 0 and 2

Also, $\sin\theta + \cos\theta = \sqrt{2}\sin(\theta + \pi/4) \le \sqrt{2}$

$\therefore \quad 0 \le \sin\theta + \cos\theta \le \sqrt{2}$

7. The value of $\tan(180° + \theta)\tan(90° - \theta)$ is

[N.D.A. Sept-1998]

(a) 1 (b) –1
(c) 0 (d) None

Solution

$\tan(180° + \theta)\tan(90° - \theta) = \tan\theta \cdot \cot\theta = 1$

8. If we convert $\sin(-566°)$ to some trigonometrical ratio of a positive angle lying between 0° and 45°, then we get

[N.D.A. Sept-1998]

(a) cos 26° (b) – cos 26°
(c) sin 26° (d) – sin 26°

Solution

sin (–566°) = –sin 566°
= –sin (6 × 90° + 26°)
= –(–sin 26°) = sin 26°

9. The value of $\sin^2 75° - \sin^2 15°$ is ***[Delhi C.E.E.-1995, Pb. C.E.T.-2001]***

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$
(c) 1 (d) 0

Solution

$\sin^2 75° - \sin^2 15°$

$= \sin(75° + 15°)\sin(75° - 15°)$

$= \sin 90° \sin 60° = 1.\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$

10. If $A > 0, B > 0$ and $A + B = \frac{\pi}{3}$, find the maximum value of $\tan A \tan B$. ***[IIT-JEE-1993]***

Solution

Let $\tan A \,.\, \tan B = x$

$\tan(A + B) = \sqrt{3}$

$\Rightarrow \quad \sqrt{3}(1 - x) = \tan A + \tan B \geq 2\sqrt{x}$

Using A.M. – G.M. inequality

$\therefore \quad x + \frac{2}{\sqrt{3}}\sqrt{x} - 1 \leq 0$

or $\quad (\sqrt{x} + \sqrt{3})\left(\sqrt{x} - \frac{1}{\sqrt{3}}\right) \leq 0$

$\therefore \quad \sqrt{x} - \frac{1}{\sqrt{3}} \leq 0 \quad$ or $\quad x \leq \frac{1}{3}$

Maximum value of $\tan a.\, \tan B = \frac{1}{3}$

11. sin 47° – sin 25° + sin 61° – sin 11° is equal to

(a) cos 7° (b) sin 7°
(c) 2 cos 7° (d) 2 sin 7°

Solution

(a) We have,

sin 47° – sin 25° + sin 61° – sin 11°

= 2 sin 11° cos 36° + 2 sin 25° cos 36°

= 2 cos 36° (sin 25° + sin 11°)

= 2 cos 36° × 2 sin 18° cos 7°

$= 4\left(\frac{\sqrt{5}+4}{4}\right)\left(\frac{\sqrt{5}-1}{4}\right)\cos 7° = \cos 7°$

12. cos 1° + cos 2° + cos 3° + …….. + cos 180° is equal to

(a) 1 (b) 0
(c) 2 (d) –1

Solution

(d) We have,

cos 1° + cos 2° + cos 3° + …….. + cos 180°

$= \sum_{\theta=1}^{90°}\{\cos\theta + \cos(180° - \theta)\} + \cos 180°$

$= \sum_{\theta=1}^{90°}(\cos\theta - \cos\theta) - 1 = -1$

OBJECTIVE PROBLEM: IMPORTANT QUESTIONS WITH SOLUTIONS

1. If cosec θ + 2 = 0 and 0 < θ < 360°, then θ is equal to ***[EAMCET-1994]***

(a) 210°, 300° (b) 240°, 300°
(c) 210°, 240° (d) 210°, 330°

2. tan 1° tan 2° tan 3° tan 4° …. tan 89° is equal to ***[MPPET-1998, 2001; AMU-1999; Pb. CET-1994; DCE-2003]***

(a) 1 (b) 0
(c) ∞ (d) ½

3. If $\sin\theta = \frac{24}{25}$ and θ lies in the second quadrant, then sec θ + tan θ is equal to

(a) –3 (b) –5
(c) –7 (d) –9

4. If $\tan\theta = \frac{-4}{3}$, then sin θ is equal to

[IIT-79; Pb-95; Orissa JEE-02]

(a) –4/5 but not 4/5 (b) –4/5 or 4/5
(c) 4/5 but not –4/5 (d) None of these

5. tan A + cot(180° + A) + cot(90° + A) + cot (360° – A) ***[MPPET-1992]***

(a) 0 (b) 2 tan A
(c) 2 cot A (d) 2 (tan A – cot A)

6. The value of cos A – sin A when $A = \frac{5\pi}{4}$, is

[MPPET-1990]

(a) $\sqrt{2}$ (b) $1/\sqrt{2}$
(c) 0 (d) 1

7. The value of cos (270° + θ) cos (90° – θ) – sin (270° – θ) cos θ is

[Karnataka CET-2005]

(a) 0 (b) –1
(c) 1/2 (d) 1

8. The value of $\cos^2\frac{\pi}{12}+\cos^2\frac{\pi}{4}+\cos^2\frac{5\pi}{12}$ is

[Karnataka CET-2002]

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$
(c) $\frac{3+\sqrt{3}}{2}$ (d) $\frac{2}{3+\sqrt{3}}$

9. The value of sin 600° cos 330° + cos 120° sin 150° is

(a) –1 (b) 1
(c) $1/\sqrt{2}$ (d) $\sqrt{3}/2$

10. If sec θ $= -2/\sqrt{3}$ and cosec θ = 2, then the quadrants in which θ lies is

(a) I (b) II
(c) III (d) IV

11. cosec θ – cos θ $= \frac{1}{2}$, $0 < \theta < \frac{\pi}{2}$, then cos θ is equal to ***[KCET-2000]***

(a) –3/5 (b) –5/3
(c) 5/3 (d) 3/5

12. If $0 \le \theta \le \pi$ and $81^{\sin^2\theta} + 81^{\cos^2\theta} = 30$ then θ is

(a) 30° (b) 60°
(c) 120° (d) 150°

13. The value cos 105° + sin 105° is

(a) $\frac{1}{2}$ (b) 1
(c) $\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$

14. The numerical value of $\tan\frac{\pi}{3}+2\tan\frac{2\pi}{3}+4\tan\frac{4\pi}{3}+8\tan\frac{8\pi}{3}$ is equal to

(a) $-5\sqrt{3}$ (b) $-\frac{5}{\sqrt{3}}$
(c) $5\sqrt{3}$ (d) $\frac{5}{\sqrt{3}}$

15. The value of $\sin^2\frac{\pi}{8}+\sin^2\frac{3\pi}{8}+\sin^2\frac{5\pi}{8}+\sin^2\frac{7\pi}{8}$ is

(a) 1 (b) 2
(c) $1\frac{1}{8}$ (d) $2\frac{1}{8}$

16. The value of π radian is equal to

[MPPET-2010]

(a) 60° (b) 180°
(c) 90° (d) 360°

HINTS AND EXPLANATIONS

1. (d) cosec= –2

$\Rightarrow \sin\theta - \frac{1}{2} = \sin(-30°)$

$= \sin(\pi + 30°)$ or $\sin(2\pi - 30°)$

$\Rightarrow \theta = 210°, 330°$

2. (a) tan 1° tan 2° tan 3° tan 4° --------- tan 89°

(tan 1° . tan 89°) (tan 2° . tan 88°) ---------

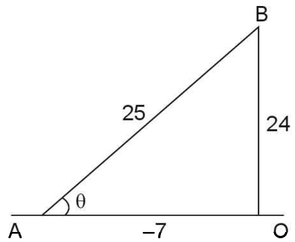
(tan 44° . tan 46°) tan 45°

= (tan 1° . cot 1°) (tan 2° . cot 2°) ---------tan 44° . cot 44°) .1 = (1) (1) ------- (1) = 1

($\because \tan 89° = \tan 90° - 1) = \cot 1°$
$\tan 88° = \tan (90° - 2) = \cot 2°$
and $\tan 46° = \tan (90° - 44) = \cot 44°$)

3. (c) Given $\sin\theta = \dfrac{24}{25}$ (in II quadrants)

as $\sin\theta = \dfrac{\text{Perp.}}{\text{Hyp.}} = \dfrac{24}{25}$



therefore, Base = –7 (As in II quadrant)

$\therefore \quad \sec\theta + \tan\theta = -\dfrac{25}{7} - \dfrac{24}{7} = -\dfrac{49}{7} = -7$

4. (b) Since $\text{cosec}^2\theta = 1 + \cot^2\theta$

$= 1 + \dfrac{9}{16} = \dfrac{25}{16} \qquad \left(\therefore \tan\theta = \dfrac{-4}{3}\right)$

$\sin^2 = \dfrac{1}{\text{cosec}^2\theta} = \dfrac{16}{25} \Rightarrow \sin\theta = \pm\dfrac{4}{5}$

Both the values are aceeptable.

Since $\tan\theta = \dfrac{-4}{3}$

i.e., θ lies in 2nd or 4th quadrants.

5. (a) $\tan A + \cot A - \tan A - \cos A = 0$

$\cot(180° + A) = \cot A$, $\cot(90° + A) = \cot(360° - A) = -\cot A$

[$\therefore$ According to quadrant rule]

6. (c) $\cos\dfrac{5\pi}{4} - \sin\dfrac{5\pi}{4} = \dfrac{-1}{\sqrt{2}} - \left(\dfrac{-1}{\sqrt{2}}\right) = 0$

7. (d) Given

$\cos(270° + \theta)\cos(90° - \theta) - \sin(270° - \theta)\cos\theta$

$= \sin\theta.\sin\theta + \cos\theta.\cos\theta$

$= \sin^2\theta + \cos^2\theta = 1$

[According to the quadrant rule, $\cos(270° + \theta) = \sin\theta$, $\cos(90° - \theta) = \sin\theta$, $\sin(270° - \theta) = \cos\theta$]

8. (a) $\cos^2\dfrac{\pi}{12} + \cos^2\dfrac{\pi}{4} + \cos^2\dfrac{5\pi}{12}$

$= \dfrac{1 + \cos\dfrac{\pi}{6}}{2} + \dfrac{1 + \cos\dfrac{5\pi}{6}}{2}$

$= \dfrac{3}{2} + \dfrac{1}{2}\left[\cos\dfrac{\pi}{6} + \cos\dfrac{5\pi}{6}\right]$

$= \dfrac{3}{2} + \dfrac{1}{2}\left[\cos\dfrac{\pi}{6} + \cos\left(\pi - \dfrac{\pi}{6}\right)\right]$

$= \dfrac{3}{2} + \dfrac{1}{2}\left[\cos\dfrac{\pi}{6} - \cos\dfrac{\pi}{6}\right]$

$= \dfrac{3}{2} + \dfrac{1}{2}(0) = \dfrac{3}{2}$

9. (a) $\sin 600° . \cos 330° + \cos 120° \sin 150° =$

$\sin(720° - 120°)\cos(180° + 150°) + \cos 120° \sin 150°$

$= (-\sin 120°)(-\cos 150°) + \cos 120° \sin 150°$

$= \sin 120° \cos 150° + \cos 120° \sin 150°$

$= \sin(120° + 150°) = \sin 270° = -1$

10. (b) Since sec θ i.e., cos θ is negative

$\therefore$ θ lies in IInd or IIIrd quadrant.

Since, sin θ is +ve $\therefore$ θ lies in Ist or IInd quadrant.

Hence, θ lies in IInd quadrants.

11. (d) $\text{cosec}\,\theta - \cot\theta = \dfrac{1}{2}$

Also, $\text{cosec}^2\theta - \cot^2\theta = 1$

$\Rightarrow \quad \text{cosec}\,\theta + \cot\theta = 2$

$\therefore \quad 2\,\text{cosec}\,\theta = 2 + \dfrac{1}{2} = \dfrac{5}{2}$

$\Rightarrow \quad \sin\theta = \dfrac{4}{5} \Rightarrow 2\cot\theta = 2 - \dfrac{1}{2} = \dfrac{3}{2}$

$\Rightarrow \quad \cot\theta = \dfrac{3}{4} \Rightarrow \cos\theta = \dfrac{3}{4}\sin\theta$

$= \dfrac{3}{4} \times \dfrac{4}{5} = \dfrac{3}{5}$

12. (a, b) $\cos^2\theta = 1 - \sin^2\theta$. Let $81^{\sec^2\theta} = t$

$\Rightarrow \quad t + \dfrac{81}{t} = 30 \quad \Rightarrow t^2 - 30t + 81 = 0$

$\Rightarrow \quad (t-27)(t-3)=0$

$\Rightarrow \quad 81^{\sin^2\theta} = 3^{4\sin^2\theta} = 3^3 \text{ or } 3^1$

$\Rightarrow \quad \sin^2\theta = \frac{3}{4}, \frac{1}{4} \quad \text{or} \quad \sin\theta = \pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2}$

$\Rightarrow \quad \theta = \frac{\pi}{6}, \frac{\pi}{3} \quad \text{or } 30°, 60°$

13. (d) cos 105° + sin 105°

= cos (90° + 15°) + sin (90° + 15°)

= cos 15° – sin 15°

$= \frac{\sqrt{3}+1}{2\sqrt{2}} - \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$

14. (a) $\tan\frac{\pi}{3} + 2\tan\frac{2\pi}{3} + \; + \tan\frac{4\pi}{3} + 8\tan\frac{8\pi}{3}$

$= \tan\frac{\pi}{3} + 2\tan\left(\pi - \frac{\pi}{3}\right) + 4\tan$

$\left(\pi + \frac{\pi}{3}\right) + 8\tan\left(3\pi - \frac{\pi}{3}\right)$

$= \tan\frac{\pi}{3} - 2\tan\frac{\pi}{3} + 4\tan\frac{\pi}{3} - 8\tan\frac{\pi}{3}$

$= -5\tan\frac{\pi}{3} = -5\sqrt{3}$

15. (b) $\sin\frac{7\pi}{8} = \sin\left(\pi - \frac{\pi}{8}\right) = \sin\frac{\pi}{8}$

$\sin\frac{5\pi}{8} = \sin\left(\pi - \frac{3\pi}{8}\right) = \sin\frac{3\pi}{8}$

$\therefore$ The given value $= 2\left[\sin^2\frac{\pi}{8} + \sin^2\frac{3\pi}{8}\right]$

$= 2\left[\sin^2\frac{\pi}{8} + \cos^2\frac{\pi}{8}\right]$

$\left[\because \sin\frac{3\pi}{8} = \sin\left(\frac{\pi}{2} - \frac{\pi}{8}\right) = \cos\frac{\pi}{8}\right]$

= 2 (1) = 2

16. (b) Obvious

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. If $\sin\theta = -1/\sqrt{2}$ and tan θ = 1, then θ lies in which quadrant

(a) First (b) Second
(c) Third (d) Fourth

2. cos 1° . cos 2° . cos 3° .... cos 179° is equal to ***[Karnataka CET-1999; DCE-2005]***

(a) 0 (b) 1
(c) 2 (d) 1/2

3. The value of $\frac{\cot 54°}{\tan 36°} + \frac{\tan 20°}{\cot 70°}$ is ***[Karnataka CET-1999]***

(a) 2 (b) 3
(c) 1 (d) 0

4. The value of sin 10° + sin 20° + sin 30° + ... + sin 360° is ***[Pb. CET-2003]***

(a) 1 (b) 0
(c) –1 (d) None

5. cos 1° + cos 2° + cos 3° + ... + cos 180° is equal to ***[Karnataka CET-2003]***

(a) 0 (b) 1
(c) –1 (d) 2

6. cos *A* + sin (270° + *A*) – sin (270° – *A*) + cos (180° + *A*) is equal to ***[MPPET-1990]***

(a) –1 (b) 0
(c) 1 (d) None

7. If $\pi < \alpha < \frac{3\pi}{2}$, then $\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} + \sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}$ is equal to ***[MNR-1997]***

(a) 2/sin α (b) –2/sin α
(c) 1/sin α (d) –1/sin α

8. cot (45° + θ) cot (45° – θ) is equal to

(a) –1 (b) 0
(c) 1 (d) ∞

9. $\tan\frac{\pi}{20}\tan\frac{3\pi}{20}\tan\frac{5\pi}{20}\tan\frac{7\pi}{20}\tan\frac{9\pi}{20}$ is equal to

(a) –1 (b) 1/2
(c) 1 (d) ∞

10. If $\tan\theta = \tan 240°$ and θ is in first quadrant, then θ is equal to *[MPPET-97]*
(a) 60° (b) 45°
(c) 30° (d) 15°

11. The value of $\tan(-945°)$ is
(a) -1 (b) -2
(c) -3 (d) -4

12. If $\sin(\alpha-\beta) = \frac{1}{2}$ and $\cos(\alpha+\beta) = \frac{1}{2}$, where α, β are positive acute angle, then
(a) $\alpha = 45°, \beta = 15°$ (b) $\alpha = 15°, \beta = 45°$
(c) $\alpha = 60°, \beta = 15°$ (d) None of these

13. Which of the following is the correct identity?
(a) $\cot\left(\frac{\pi}{2}+A\right) = \tan A$
(b) $\sec\left(\frac{7\pi}{2}-A\right) = -\text{cosec}\, A$
(c) $\sin(n\pi + A) = \sin A$
(d) $\sin(\pi - A) = -\cos A$

14. $\frac{\sin(180°-A)\cos(270°-A)}{\sin(180°+A)\cos(270°+A)}$ is equal to
(a) 1 (b) -1
(c) $\tan A$ (d) $\cot A$

15. $\tan\theta\sin\left(\frac{\pi}{2}+\theta\right)\cos\left(\frac{\pi}{2}-\theta\right)$ is equal to *[EAMCET-81]*
(a) 1 (b) 0
(c) $1/\sqrt{2}$ (d) None

16. $\sin^2\frac{\pi}{18}+\sin^2\frac{\pi}{9}+\sin^2\frac{7\pi}{18}+\sin^2\frac{4\pi}{9}$ is equal to
(a) 1 (b) 4
(c) 2 (d) 0

17. If θ lies in 3rd quadrant, then $\sin\theta + \cos\theta$ is
(a) Negative
(b) Positive
(c) Zero
(d) Zero or Positive

18. Angles between 0 and 2π whose sin is 1/2 are
(a) $\frac{\pi}{3}, \frac{4\pi}{3}$ (b) $\frac{3\pi}{4}, \frac{7\pi}{3}$
(c) $\frac{\pi}{6}, \frac{5\pi}{6}$ (d) $\frac{2\pi}{3}, \frac{7\pi}{6}$

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet
2. The worksheet is of 15 minutes.
3. The worksheet consists of 15 questions. The maximum marks are 45.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. cosec 390°
(a) 2 (b) –2
(c) ±2 (d) None of these

2. If $\sin\theta = -1/2$ and $\cos\theta = \sqrt{3}/2$ then θ lies in
(a) I (b) II
(c) III (d) IV

3. Angles between 0 and 2π whose tangent is –1 are
(a) $\frac{\pi}{4}, \frac{3\pi}{4}$ (b) $\frac{\pi}{6}, \frac{5\pi}{6}$
(c) $\frac{2\pi}{3}, \frac{7\pi}{6}$ (d) $\frac{3\pi}{4}, \frac{7\pi}{4}$

4. If 0 < θ < 90°. The sec θ is
(a) Greater than 1 (b) Less than 1
(c) Equal to 1 (d) None of these

5. If $\sin A = \cos A$, 0° < A < 90° then A is
(a) 15° (b) 30°
(c) 45° (d) 60°

6. If $\sin A = 12/13$ and A lies in the first quadrant then, sec A + tan A is equal to
(a) 1/5 (b) –1/5
(c) –5 (d) 5

7. If $\sin x = \frac{-24}{25}$, then the value of tan x is *[UPSEAT-2003]*
(a) 24/25 (b) –24/27
(c) 25/24 (d) None of these

8. If A lies in the second quadrant and 3tan A + 4 = 0, the value of 2 cot A – 5 cos A + sin A is equal to *[Pb CET-2000; NDA-07]*
(a) –53/10 (b) –7/10
(c) 7/10 (d) 23/10

9. cos 24° + cos 5° + cos 175° + cos 204° + cos 300° is equal to
(a) 1/2 (b) –1/2
(c) $\sqrt{3}/2$ (d) 1

10. If sinθ = 1/2 and θ is an obtuse angle, then cotθ is equal to
(a) $\frac{1}{\sqrt{3}}$ (b) $-\frac{1}{\sqrt{3}}$
(c) $\sqrt{3}$ (d) $-\sqrt{3}$

11. sec (270° – A) is (90° – A) – tan (270° – A) + tan (90° + A) is equal
(a) 0 (b) 1
(c) –1 (d) None of these

12. If θ lies in the second quadrant, then the value of $\sqrt{\left(\frac{1-\sin\theta}{1+\sin\theta}\right)} + \sqrt{\left(\frac{1+\sin\theta}{1-\sin\theta}\right)}$
(a) 2 sec θ (b) –2 sec θ
(c) 2cosec θ (d) None

13. If tan θ = $-1/\sqrt{10}$ and θ lies in the fourth quadrant, then cos θ is equal to
(a) $1/\sqrt{11}$ (b) $-1/\sqrt{11}$
(c) $\sqrt{10/11}$ (d) $-\sqrt{10/11}$

14. The value of $\tan^2 30° + \tan^2 45° + \tan^2 60°$ is
(a) 13/3 (b) –13/3
(c) 3/13 (d) –3/13

15. If $x \sin 45° \cos^2 60° = \frac{\tan^2 60° \operatorname{cosec} 30°}{\sec 45° \cot^2 30°}$, then x is equal to
(a) 2 (b) 4
(c) 8 (d) 16

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ
6. ⓐ ⓑ ⓒ ⓓ
7. ⓐ ⓑ ⓒ ⓓ
8. ⓐ ⓑ ⓒ ⓓ
9. ⓐ ⓑ ⓒ ⓓ
10. ⓐ ⓑ ⓒ ⓓ
11. ⓐ ⓑ ⓒ ⓓ
12. ⓐ ⓑ ⓒ ⓓ
13. ⓐ ⓑ ⓒ ⓓ
14. ⓐ ⓑ ⓒ ⓓ
15. ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

8. (d) $3\tan A + 4 = 0$; $\tan A = \dfrac{-4}{3}$

$\because$ A is in II quadrant

$\cot A = \dfrac{-3}{4}$, $\cos A = \dfrac{-3}{5}$,

$\sin A = \dfrac{4}{5}$

$\therefore$ $2\cot A - 5\cos A + \sin A$

$$= 2\left(-\frac{3}{4}\right) - 5\left(\frac{-3}{5}\right) + \frac{4}{5} = \frac{23}{10}$$

9. (a) $\cos 24° + \cos 5° + \cos(180° - 5°) + \cos(180° + 24°) + \cos(360° - 60°) = \cos 24° + \cos 5° - \cos 5° - \cos 24° + \cos 60° = \dfrac{1}{2}$

12. (b) $\dfrac{1-\sin\theta+1+\sin\theta}{\sqrt{1-\sin^2\theta}} = \dfrac{2}{\sqrt{\cos^2\theta}}$

$= \dfrac{2}{|\cos\theta|} = \dfrac{-2}{\cos\theta}$

($\because$ $\theta \in$ II Quadrant)

LECTURE 4

Trigonometric Functions of Compound Angles

BASIC CONCEPTS

1. TRIGONOMETRIC FUNCTIONS FOR SUM AND DIFFERENCE OF THE TWO ANGLES

(i) $\text{Sin}\,(A+B) = \sin A \cos B + \cos A \sin B$

(ii) $\text{Sin}\,(A-B) = \sin A \cos B \;\; \cos A \sin B$

(iii) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

(iv) $\cos(A-B) = \cos A \cos B + \sin A \sin B$

(v) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(vi) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(vii) $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$

(viii) $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

(ix) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(x) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

(xi) $2A = (A+B) + (A-B)$

(xii) $2B = (A+B) - (A-B)$

2. TRIGONOMETRIC RATIOS OF SOME USEFUL ANGLES

(i) $\sin 15° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$, $\cos 15° = \dfrac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75°$

(ii) $\tan 15° = \dfrac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} = \dfrac{1}{2+\sqrt{3}}$ $\cot 75°$

(iii) $\sin 75° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$,

$\cos 75° = \dfrac{\sqrt{3}-1}{2\sqrt{2}} = \sin 15°$

(iv) $\tan 75° = \dfrac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3}$

$= \dfrac{1}{2-\sqrt{3}} = \cot 15°$

3. MAXIMUM AND MINIMUM VALUES OF SOME FUNCTIONS

1. $a \sin x + b \cos x = \sqrt{a^2+b^2}\sin\left(x + \tan^{-1}\dfrac{b}{a}\right)$

or $\sqrt{a^2+b^2}\cos\left(x - \tan^{-1}\dfrac{a}{b}\right)$ So

(i) $\sqrt{a^2+b^2} \le a \sin x + b \cos x \le \sqrt{a^2+b^2}$ and

(ii) $c - \sqrt{a^2+b^2} \le a \sin x + b \cos x + c \le c + \sqrt{a^2+b^2}$

NOTE

Maximum value of a sin x + b cos x occurs at $x = \tan^{-1}\dfrac{a}{b}$ and minimum value is defined at $\pi + \tan^{-1}\left(\dfrac{a}{b}\right)$

2. $|a \sec x + b \tan x| \geq \sqrt{a^2 + b^2}$

4. SOME IMPORTANT FUNCTIONS

(i) $\tan\left(\dfrac{\pi}{4}+\theta\right)=\dfrac{1+\tan\theta}{1-\tan\theta}$

(ii) $\tan\left(\dfrac{\pi}{4}-\theta\right)=\dfrac{1-\tan\theta}{1+\tan\theta}$

(iii) $\dfrac{1}{\sec\theta-\tan\theta} = \sec\theta + \tan\theta = \sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}}$

(iv) $\dfrac{1}{\sec\theta+\tan\theta} = \sec\theta - \tan\theta = \sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}$

(v) $\dfrac{1}{\operatorname{cosec}\theta-\cot\theta} = \operatorname{cosec}\theta + \cot\theta = \sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}}$

(vi) $\dfrac{1}{\operatorname{cosec}\theta+\cot\theta} = \operatorname{cosec}\theta - \cot\theta = \sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Prove that $\tan\theta\tan\left(\theta+\dfrac{\pi}{3}\right) + \tan\left(\theta+\dfrac{\pi}{3}\right)\tan\left(\theta-\dfrac{\pi}{3}\right) + \tan\theta\tan\left(\theta-\dfrac{\pi}{3}\right) = -3.$

Solution

$$\text{L.H.S} = \tan\theta\left\{\tan\left(\theta+\frac{\pi}{3}\right)+\tan\left(\theta-\frac{\pi}{3}\right)\right\}$$

$$+\tan\left(\theta+\frac{\pi}{3}\right)\tan\left(\theta-\frac{\pi}{3}\right)$$

$$=\tan\theta\left\{\frac{\tan\theta+\sqrt{3}}{1-\sqrt{3}\tan\theta}+\frac{\tan\theta-\sqrt{3}}{1+\sqrt{3}\tan\theta}\right\}$$

$$+\left(\frac{\tan\theta+\sqrt{3}}{1-\sqrt{3}\tan\theta}\right)\left(\frac{\tan\theta-\sqrt{3}}{1+\sqrt{3}\tan\theta}\right)$$

$$=\tan\theta\left\{\frac{8\tan\theta}{1-3\tan\theta}\right\}+\frac{\tan^2\theta-3}{1-3\tan^2\theta}$$

$$=\frac{9\tan^2\theta-3}{1-3\tan^2\theta} = -3 = \text{R.H.S}$$

2. 5cos θ + 3cps $\left(\theta+\dfrac{\pi}{3}\right)$ + 3 lies between –4 and 10.

Solution

$$5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right) + 3 = 5\cos\theta + 3$$

$$\left(\cos\theta\cos\frac{\pi}{3}-\sin\theta\sin\frac{\pi}{3}\right) + 3$$

$$= 5\cos\theta + 3\left((\cos\theta)\frac{1}{2}-(\sin\theta)\frac{\sqrt{3}}{2}\right) + 3$$

$$= \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3$$

$$\left(\text{Put } \frac{13}{2} = r\cos\alpha, \frac{3\sqrt{3}}{2} = r\sin\alpha\right.$$

$$\left.\Rightarrow r^2 = \frac{169}{4}+\frac{27}{4} = 49 \Rightarrow r = 7\right)$$

$= r\cos\alpha\cos\theta - r\sin\alpha\sin\theta + 3 = 7\cos(\theta+\alpha) + 3$

Now, $-1 \leq \cos(\theta+\alpha) \leq 1$

$\Rightarrow \quad -7 \leq 7\cos(\theta+\alpha) \leq 7$

$\Rightarrow \quad -4 \leq 7\cos(\theta+\alpha) + 3 \leq 10$

Hence, the given expression lies between –4 and 10.

3. If α and β are the solutions of the equation and $\tan\theta + b\sec\theta = c$, then show that $\tan(\alpha+\beta) = 2ac/a^2 - c^2)$.

Solution

Here, $b\sec\theta = c - a\tan\theta$.

Squaring both sides

$b^2(1+\tan^2\theta) = c^2 - 2\,ca\tan\theta + a^2\tan^2\theta$

$(a^2 - b^2)\tan^2\theta - 2\,ca\tan\theta + (c^2 - b^2) = 0$

$\therefore\quad \tan\alpha + \tan\beta = \dfrac{2ca}{a^2-b^2},\ \tan\alpha\tan\beta = \dfrac{c^2-b^2}{a^2-b^2}$

$\therefore\quad \tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$$= \frac{2ca}{(a^2-b^2)-(c^2-b^2)} = \frac{2ac}{a^2-c^2}$$

4. If $\sin x + \sin y = 3(\cos y - \cos x)$, prove that $\sin 3x + \sin 3y = 0$.

Solution

F e have

$3\cos x + \sin x = 3\cos y - \sin y$ …………(1)

Put $3 = r\cos\alpha$, $1 = r\sin\alpha$,

$\therefore\quad r = \sqrt{10},\ \tan\alpha = \dfrac{1}{3}.$

$\therefore\quad r\cos(x-\alpha) = r\cos(y+\alpha)$

$\therefore\quad x - \alpha = \pm(y+\alpha)\ \therefore\ x = -y$ or $x = y + 2\alpha$

clearly $x = -y$ satisfies (1)

$\therefore\quad 3x = -3y$ or $\sin 3x = \sin(-3y) = -\sin 3y$

or $\quad \sin 3x + \sin 3y = 0$

5. If $\tan\alpha = \dfrac{Q\sin\beta}{P+Q\cos\beta}$, prove that

$\tan(\beta-\alpha) = \dfrac{P\sin\beta}{Q+P\cos\beta}$

Solution

$$\tan(\beta-\alpha) = \frac{\tan\beta-\tan\alpha}{1+\tan\beta\tan\alpha}$$

$$= \frac{\dfrac{\sin\beta}{\cos\beta} - \dfrac{Q\sin\beta}{P+Q\cos\beta}}{1+\dfrac{\sin\beta}{\cos\beta}\left(\dfrac{Q\sin\beta}{P+Q\cos\beta}\right)}$$

$$= \frac{\sin\beta(P+Q\cos\beta) - Q\sin\beta\cos\beta}{\cos\beta(P+Q\cos\beta)+Q\sin^2\beta}$$

$$= \frac{P\sin\beta}{Q+P\cos\beta}$$

6. Prove that, $\sin A \pm \cos A = \sqrt{2}\sin\left(\dfrac{\pi}{4}\pm A\right) = \sqrt{2}\cos\left(\dfrac{\pi}{4}\mp A\right)$.

Solution

LHS $= \sin A + \cos A$ (multiplying and dividing by $\sqrt{2}$)

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin A + \frac{1}{\sqrt{2}}\cos A\right)$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4}\sin A + \sin\frac{\pi}{4}\cos A\right)$$

$$= \sqrt{2}\sin\left(A+\frac{\pi}{4}\right)$$

or

$$= \sqrt{2}\left(\sin A\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\cos A\right)$$

$$= \sqrt{2}\cos\left(\frac{\pi}{4}-A\right)$$

Similarly, $\sin - \cos A = \sqrt{2}\sin\left(\dfrac{\pi}{4}-A\right)$ or $\sqrt{2}\cos\left(\dfrac{\pi}{4}+A\right)$

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): SOLVE THESE PROBLEMS TO GRASP THE TOPIC

EXERCISE 1

1. If $\cos A = \frac{4}{5}$, $\cos B = \frac{12}{13}, \frac{3\pi}{2} < A, B < 2\pi$, find the values of the following:
   (i) $\cos (A + B)$ (ii) $\sin (A - B)$

2. Find the values of the following:
   (i) $\sin 75°$ (ii) $\cos 75°$
   (iii) $\sin 15°$ (iv) $\cos 15°$

3. Evaluate the following:
   (i) $\sin\frac{7\pi}{12}\cos\frac{\pi}{4} - \cos\frac{7\pi}{12}\sin\frac{\pi}{4}$
   (ii) $\sin\frac{\pi}{4}\cos\frac{\pi}{12} + \cos\frac{\pi}{4}\sin\frac{\pi}{12}$

4. $\frac{\tan(45° + x)}{\tan(45° - x)} = \left(\frac{1 + \tan x}{1 - \tan x}\right)^2$

5. If $\tan \alpha = \frac{m}{m+1}$ and $\tan \beta = \frac{1}{2m+1}$ prove that $\alpha + \beta = \frac{\pi}{4}$

EXERCISE 2

1. Evaluate the following:
   (i) $\sin\frac{7\pi}{12}\cos\frac{\pi}{4} - \cos\frac{7\pi}{12}\sin\frac{\pi}{4}$

2. If $A + B = \frac{\pi}{4}$, prove that $(1 + \tan A)(1 + \tan B) = 2$

3. Prove that, $\tan 70° = \tan 20° + 2 \tan 50°$

4. Prove that,
   (i) $\tan 15° + \cot 15° = 4$
   (ii) $\tan 75° + \cot 75° = 4$

5. Prove that, $\cos \theta - \sin \theta = \sqrt{2} \cos\left(\theta + \frac{\pi}{4}\right)$.

6. Prove that,
   $$\frac{\sin(A-B)}{\sin A \sin B} + \frac{\sin(B-C)}{\sin B \sin C} + \frac{\sin(C-A)}{\sin C \sin A} = 0$$

7. Prove that,
   $\cos (n + 1)A \cos (n - 1)A + \sin (n + 1)A \sin (n - 1)A = \cos 2A$

8. If $\cot A \cot B = 3$, show that,
   $$\frac{\cos(A+B)}{\cos(A-B)} = \frac{1}{2}$$

9. Prove that,
   $\cos 2x \cos 2y + \cos^2 (x + y) - \cos^2 (x - y) = \cos (2x + 2y)$.

ANSWERS

EXERCISE 1

1. (i) 33/65
   (ii) −16/65

2. (i) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ (ii) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (iii) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (iv) $\frac{\sqrt{3}+1}{2\sqrt{2}}$

3. (i) $\sqrt{3}/2$
   (ii) $\sqrt{3}$

EXERCISE 2

1. $\sqrt{3}/2$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. If $\frac{\pi}{2} < \alpha < \pi$, $\pi < \beta < \frac{3\pi}{2}$, $\sin\alpha = \frac{15}{17}$ and tan $\beta = \frac{12}{15}$ then sin (β – α) is equal to

[Roorkee Screening-2000]

(a) $-\frac{17}{221}$ (b) $-\frac{21}{221}$

(c) $\frac{21}{221}$ (d) $\frac{171}{221}$

Solution

(d) sin α = 15/17 => cos α = –8/17

[∵ cos α < 0 then π/2 < α < π]

tan β = 12/5 => sin β = –12/13, cos β = –5/13

[∵ π < β < 3π/2 => sin β < 0, cos β < 0]

Now sin (β – α) = sin β cos α – sin α cos β
= (–12/13) (–8/17) – (15/17) (–15/13)
= 171/221

2. sec50° + tan50° is equal to *[DEC-2002]*

(a) tan 20° + tan 50°
(b) 2 tan 20° + tan 50°
(c) tan 20° + 2 tan 50°
(d) 2 tan 20° + 2 tan 50°

Solution

(c) tan 50 = tan(70° – 20°)

$$= \frac{\tan 70° - \tan 20°}{1 + \tan 70° \tan 20°}$$

=> tan 50° + tan 70° tan 20° tan 50°
= tan 70° – tan 20°

=> tan 50° + tan 50° = tan 70° – tan 20°

[∵ tan 70° = cot 20°]

=>2 tan 50° + tan 20° = tan 70°

=> 2 tan 50° + tan 20° = tan 50° + sec 50°

(∵ tan 50° + sec 50° = tan 70°)

3. What is the value of cosec (13π/12)?

[NDA-2007]

(a) $\sqrt{6}+\sqrt{2}$ (b) $-\sqrt{6}+\sqrt{2}$

(c) $\sqrt{6}-\sqrt{2}$ (d) $-\sqrt{6}-\sqrt{2}$

Solution

(d) cosec (π + π/12) = – cosec π/12
= –1/sin 15°

$$= \frac{-1}{\sin 45° \cos 30° - \cos 45° \sin 30°}$$

$$= \frac{-1}{\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}}$$

$$= \frac{-2\sqrt{2}(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$= \frac{-2\sqrt{2}(\sqrt{3}+1)}{2} = -\sqrt{6}-\sqrt{2}$$

4. If tan α, tan β are roots of the equation $x^2 + px + q = 0$ $(p \neq 0)$, then

(a) $\sin^2(\alpha+\beta) + p\sin(\alpha+\beta)\cos(\alpha+\beta) + q\cos^2(\alpha+\beta) = q$

(b) $\tan(\alpha+\beta) = \frac{p}{q-1}$

(c) $\cos(\alpha+\beta) = 1 - q$

(d) $\sin(\alpha+\beta) = -p$

Solution

(a, b) Since tan α, tan β are the roots of the equation $x^2 + px + q = 0$.

∴ tan α + tan β = –p, tan α + tan β = q

∴ $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \frac{p}{q-1}$,

which is given in (b)

Also, when $\tan(\alpha+\beta) = \frac{p}{q-1}$

L.H.S. of the expression given in (a)

$= \cos^2(\alpha+\beta)\,[\tan^2(\alpha+\beta) + p\tan(\alpha+\beta) + q]$

$$= \frac{1}{1+\tan^2(\alpha+\beta)}\left[\frac{p^2}{(q-1)^2} + \frac{p^2}{q-1} + q\right]$$

$$= \frac{(q-1)^2}{(q-1)^2+p^2}\left[\frac{p^2+p^2(q-1)+q(q-1)^2}{(q-1)^2}\right]$$

$$= \frac{q\{p^2+(q-1)^2\}}{p^2+(q-1)^2} = q = \text{R.H.S of (a)}$$

i.e., relation given in (a) is also satisfied.

5. What is the value of $\sin(A+B)\sin(A-B)+\sin(B+C)\sin(B-C)+\sin(C+A)\sin(C-A)$? *[NDA-2007]*

(a) 0
(b) $\sin A+\sin B+\sin C$
(c) $\cos A+\cos B+\cos C$
(d) 1

Solution

(a) $\sin(A+B)\sin(A-B)+\sin(B+C)\sin(B-C)+\sin(C+A)\sin(C-A)$

$=\sin^2 A-\sin^2 B+\sin^2 B-\sin^2 C+\sin^2 C-\sin^2 A=0$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. $\cos^2\left(\frac{\pi}{4}-\beta\right)-\sin^2\left(\alpha-\frac{\pi}{4}\right)$ is equal to

(a) $\sin(\alpha+\beta)\sin(\alpha-\beta)$
(b) $\cos(\alpha+\beta)\cos(\alpha-\beta)$
(c) $\sin(\alpha-\beta)\cos(\alpha+\beta)$
(d) $\sin(\alpha+\beta)\cos(\alpha-\beta)$

2. $\dfrac{\cos 12^\circ-\sin 12^\circ}{\cos 12^\circ+\sin 12^\circ}+\dfrac{\sin 147^\circ}{\cos 147^\circ}$ is equal to *[MPPET-1991]*

(a) 1 (b) –1
(c) 0 (d) None

3. If $\tan A=-\frac{1}{2}$ and $\tan B=-\frac{1}{3}$, then $A+B$ is equal to *[IIT-1967; MNR-1987; MPPET-1989]*

(a) $\pi/4$ (b) $3\pi/4$
(c) $5\pi/4$ (d) None of these

4. If $\cos(A-B)=3/5$ and $\tan A\tan B=2$, then *[MPPET-1997]*

(a) $\cos A\cos B=\frac{1}{5}$
(b) $\sin A\sin B=-\frac{2}{5}$
(c) $\cos A\cos B=-\frac{1}{5}$
(d) $\sin A\sin B=-\frac{1}{5}$

5. $\tan\frac{2\pi}{5}-\tan\frac{\pi}{15}-\sqrt{3}\tan\frac{2\pi}{5}\tan\frac{\pi}{15}$ is equal to

(a) $-\sqrt{3}$ (b) $1/\sqrt{3}$
(c) 1 (d) $\sqrt{3}$

6. $\tan 3A-\tan 2A-\tan A$ is equal to *[MNR-1982; Pb. CET-1991]*

(a) $\tan 3A\tan 2A\tan A$
(b) $-\tan 3A\tan 2A\tan A$
(c) $\tan A\tan 2A-\tan 2A\tan 3A-\tan 3A\tan A$
(d) None of these

7. $\sin 163^\circ\cos 347^\circ+\sin 73^\circ\sin 167^\circ$ is equal to *[MPPET-2000]*

(a) 0 (b) 1/2
(c) 1 (d) None of these

8. If $\cos P=1/7$ and $\cos Q=13/14$, when P and Q both are acute angle then the value of $P-Q$ is *[Karnataka CET-2002]*

(a) 30° (b) 60°
(c) 45° (d) 75°

9. The minimum value of $3\cos x+4\sin x+8$ is

(a) 8 (b) 9
(c) 7 (d) 3

10. Maximum value of $\sin\theta+\cos\theta$, when

(a) $\theta=30^\circ$ (b) $\theta=45^\circ$
(c) $\theta=60^\circ$ (d) $\theta=90^\circ$

11. The value of $\cos 15^\circ-\sin 15^\circ$ is equal to *[MNR-1975; MPPET-1994, 2002]*

(a) $1/\sqrt{2}$ (b) 1/2
(c) $-1/\sqrt{2}$ (d) 0

12. The maximum value of $24\sin\theta+7\cos\theta$ is *[MPPET-2010]*

(a) 1 (b) 24
(c) 25 (d) 7

13. If $A+B=225^\circ$, then $\dfrac{\cot A}{1+\cot A}\cdot\dfrac{\cot B}{1+\cot B}$ is equal to *[MNR-1974]*

(a) 1 (b) –1
(c) 0 (d) 1/2

HINTS AND EXPLANATIONS

1. (d) We know that $\cos^2 A - \sin^2 B = \cos(A+B)\cos(A-B)$

$\therefore \quad \cos^2\left(\frac{\pi}{4}-\beta\right)-\sin^2\left(\alpha-\frac{\pi}{4}\right)$

$= \cos\left(\frac{\pi}{4}-\beta+\alpha-\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}-\beta-\alpha+\frac{\pi}{4}\right)$

$= \cos(\alpha-\beta)\cos\left(\frac{\pi}{2}-(\alpha+\beta)\right)$

$= \cos(\alpha-\beta)\sin(\alpha+\beta)$

2. (c) $\frac{\cos 12°-\sin 12°}{\cos 12°+\sin 12°}=\frac{1-\tan 12°}{1+\tan 12°}$

$=\frac{\tan 45°-\tan 12°}{1+\tan 45°\tan 12°}$

$= \tan(45°-12°) = \tan 33°$

$\therefore \quad \frac{\cos 12°-\sin 12°}{\cos 12°+\sin 12°}+\tan 147$

$= \tan 33° + \tan(180° - 30°)$

$= \tan 33° - \tan 33° = 0$

3. (b) $\tan A=\frac{-1}{2}$, $\tan B=\frac{-1}{3}$, $\tan(A+B)$

$=\frac{\tan A+\tan B}{1-\tan A\tan B}$

$\therefore \quad \tan(A+B) = -1$

$\Rightarrow \quad A+B=\frac{3\pi}{4}$

4. (a) $\cos(A-B)=\frac{3}{5}$, $\tan A\tan B = 2$

Now, $\cos A\cos B+\sin A\sin B=\frac{3}{5}$

$\cos A\cos B(1+\tan A\tan B)=\frac{3}{5}$

$\Rightarrow \quad \cos A\cos B(1+2)=\frac{3}{5}$

$\Rightarrow \quad \cos A\cos B=\frac{1}{5}$

$\therefore \quad \sin A\sin B=\frac{3}{5}-\cos A\cos B$

$=\frac{3}{5}-\frac{1}{5}=\frac{2}{5}$

5. (d) $\frac{2\pi}{5}-\frac{\pi}{15}=\frac{\pi}{3} \Rightarrow \tan\left(\frac{2\pi}{5}-\frac{\pi}{15}\right)=\tan\frac{\pi}{3}$

$$\frac{\tan\frac{2\pi}{5}-\tan\frac{\pi}{15}}{1+\tan\frac{2\pi}{5}\tan\frac{\pi}{15}}=\sqrt{3}$$

$\therefore \quad \tan\frac{2\pi}{5}-\tan\frac{\pi}{15}=\sqrt{3}+\sqrt{3}\tan\frac{2\pi}{5}\tan\frac{\pi}{15}$

$\tan\frac{2\pi}{5}-\tan\frac{\pi}{15}-\sqrt{3}\tan\frac{2\pi}{5}\tan\frac{\pi}{15}=\sqrt{3}$

6. (a) $3A = 2A + A \Rightarrow \tan 3A=\frac{\tan 2A+\tan A}{1-\tan 2A\tan A}$

By cross multiplying and solving, we get

$\tan 3A - \tan 2A - \tan A = \tan 3A\tan 2A\tan A$

7. (b) $\sin 163°\cos 347° + \sin 73°\sin 167°$

$= \sin(90° + 73°)\cos(360° - 13°) + \sin 73° \sin(180° - 13°)$

$= \cos 73°\cos 13° + \sin 73°\sin 13°$

$= \cos(73° - 13°) = \cos 60° = \frac{1}{2}$

8. (b) $\cos(P-Q) = \cos P\cos Q + \sin P\sin Q$

$=\frac{1}{7}\cdot\frac{13}{14}+\frac{4\sqrt{3}}{7}\cdot\frac{3\sqrt{3}}{14}=\frac{1}{2}$

$\left(\because \cos P=\frac{1}{7}, \sin P=\frac{4\sqrt{3}}{7}, \cos Q=\frac{13}{14}, \sin Q=\frac{3\sqrt{3}}{14}\right)$

$\therefore \quad P-Q=60°$

9. (d) $(3\cos x+4\sin x+8)=-\sqrt{3^2+4^2}+8$

Minimum value $= -5 + 8 = 3$

10. (b) $\sin\theta + \cos\theta = \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta\right)$

$= \sqrt{2}\sin(\theta + 45°)$ is maximum when

$\theta + 45° = 90°$

$\Rightarrow \quad \theta = 45°$

11. (a) cos 15° – sin 15°

$= \frac{\sqrt{3}+1}{2\sqrt{2}} - \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$

12. (c) Maximum value $= \sqrt{24^2 + 7^2} = 25$

13. (d) $A + B = 225° \Rightarrow \cot(A + B) = \cot 225°$

$= \frac{\cot \cot B - 1}{\cot B + \cot A} = 1$

or $\cot A \cot B - 1 = \cot B + \cot A$ ..... (1)

Now, $\frac{\cot A \cot B}{(1 + \cot A)(1 + \cot B)}$

$= \frac{\cot A \cot B}{1 + \cot B + \cot A + \cot A \cot B}$

Using (1)

$= \frac{\cot A \cot B}{1 + \cot B \cot B + \cot A \cot B - 1} = \frac{1}{2}$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. $\frac{\cos 17° + \sin 17°}{\cos 17° - \sin 17°}$ is equal to ***[MPPET-1998]***

(a) tan 62° (b) tan 56°
(c) tan 54° (d) tan 73°

2. If $\alpha + \beta = \frac{\pi}{2}$ and $\beta + \gamma = \alpha$, then tan α is equal to

[IIT Screening 2001; DCE-2005; NDA-2007]

(a) 2(tan β + tan γ) (b) tan β + tan γ
(c) tan β + 2 tan γ (d) 2 tan β + tan γ

3. $\frac{\cos 10° + \sin 10°}{\cos 10° - \sin 10°}$ is equal to

[MPPET-2002; DCE-2002]

(a) tan 55° (b) cot 55°
(c) –tan 35° (d) –cot 35°

4. If $\sin A = \frac{4}{5}$ and $\cos B = -\frac{12}{13}$, where *A* and *B* lies in first and third quadrant respectively, then cos (*A* + *B*) is equal to

(a) 56/65 (b) –56/65
(c) 16/65 (d) –16/65

5. cos² α + cos ² (α + 120°) + cos² (α – 120°) is equal to ***[MPPET-93]***

(a) 3/2 (b) 1
(c) 1/2 (d) 0

6. The value of the exepression sin θ + cos θ lies between ***[IIT-JEE-88]***

(a) –2 and 2 both inclusive
(b) 0 and $\sqrt{2}$ both inclusive
(c) $-\sqrt{2}$ and $\sqrt{2}$ both inclusive
(d) 0 and 2 both inclusive

7. The value of the expression *a* cos θ + *b* sin θ lies between

(a) *a* – *b* and *a* + *b*
(b) *a* and *b*
(c) $-(a^2 + b^2)$ and $(a^2 + b^2)$
(d) $-\sqrt{a^2+b^2}$ and $\sqrt{a^2+b^2}$

8. cos 105° + sin 105° is equal to

[MNR-1975, 76]

(a) 1/2 (b) 1
(c) $\sqrt{2}$ (d) $1/\sqrt{2}$

9. sin (45° + θ) – cos (45° – θ) is equal to

(a) 2 cos θ (b) 2 sin θ
(c) 1 (d) 0

10. If $\sin\alpha = \frac{12}{13}\left(0 < \alpha < \frac{\pi}{2}\right)$ and $\cos\beta = -\frac{3}{5}\left(\pi < \beta < \frac{3}{2}\pi\right)$, then $\sin(\alpha + \beta)$ is equal to

(a) −56/65 (b) 16/65

(c) 56/65 (d) −16/65

11. $\frac{\cos 9° + \sin 9°}{\cos 9° - \sin 9°}$ is equal to

[EAMCET-1992; Kerala (Engg.)-2005]

(a) tan 54°

(b) tan 36°

(c) tan 18°

(d) None of these

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The worksheet is of 16 minutes.
3. The worksheet consists of 16 questions. The maximum marks are 48.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. If $\tan A = \frac{a}{b}$ and $\tan B = \frac{c}{d}$, then $\tan(A + B)$ is equal to

(a) $\frac{ad+bc}{bd-ac}$ (b) $\frac{ac+bd}{bc-ad}$

(c) $\frac{bd-ac}{ad+bc}$ (d) $\frac{ad-bc}{ad+bc}$

2. If $\sin A = \frac{1}{\sqrt{5}}$ and $\sin B = \frac{1}{\sqrt{10}}$, then $A + B$ is equal to

(a) 45° (b) 90°

(c) 60° (d) 30°

3. sin 75° is equal to *[MNR-1979]*

(a) $\frac{2-\sqrt{3}}{2}$ (b) $\frac{\sqrt{3}+1}{2\sqrt{2}}$

(c) $\frac{\sqrt{3}-1}{-2\sqrt{2}}$ (d) $\frac{\sqrt{3}-1}{2\sqrt{2}}$

4. $\frac{\sin(B+A)+\cos(B-A)}{\sin(B-A)+\cos(B+A)}$ is equal to

[Roorkee-70, IIT-66]

(a) $\frac{\cos B+\sin B}{\cos B-\sin B}$ (b) $\frac{\cos A+\sin A}{\cos A-\sin A}$

(c) 0 (d) None of these

5. The value of sin 28° cos 17° cos 28° sin 17° is

(a) $1/\sqrt{2}$ (b) 1

(c) $-1/\sqrt{2}$ (d) 0

6. The value of cos 53° cos 37° – sin 53° sin 37°

(a) 1 (b) $1/\sqrt{2}$

(c) 0 (d) None of these

7. $\frac{\cos 21° - \sin 21°}{\cos 21° + \sin 21°}$ is equal to

(a) tan 21° (b) tan 66°

(c) tan 24° (d) tan 69°

8. tan 15° + cot 15° is equal to

(a) $\sqrt{3}$ (b) $2\sqrt{3}$

(c) 4 (d) –4

9. If $\cos(A + B) = \alpha \cos A \cos B + \beta \sin A \sin B$, then (α, β) is equal to

[MPPET-1992]

(a) (–1, –1) (b) (–1, 1)

(c) (1, –1) (b) (1, 1)

10. If A – B = π/4, then $(1 + \tan A)(1 - \tan B)$ is equal to

(a) 1 (b) 2

(c) –1 (d) –2

11. tan 75° – cot 75° is equal to

[MNR-1982; Pb. CET-1990, 2000]

(a) $2\sqrt{3}$ (b) $2+\sqrt{3}$

(c) $2-\sqrt{3}$ (d) None of these

12. tan 20° + tan 40° + $\sqrt{3}$ tan 20° tan 40° is equal to

(a) $1/\sqrt{3}$ (b) $\sqrt{3}$

(c) $-1/\sqrt{3}$ (d) $-\sqrt{3}$

13. $\cos^2 48° - \sin^2 12$ is equal to

[MNR-1977]

(a) $\frac{\sqrt{5}-1}{4}$ (b) $\frac{\sqrt{5}+1}{8}$

(c) $\frac{\sqrt{3}-1}{4}$ (d) $\frac{\sqrt{3}+1}{2\sqrt{2}}$

14. Maximum value of 3 cos θ + 4 sin θ is

[MNR-1990]

(a) 3 (b) 4

(c) 5 (d) None of these

15. $\dfrac{\tan 70° - \tan 20°}{\tan 50°}$ is equal to *[Karnataka CET-2003]*

(a) 1 (b) 2
(c) 3 (d) 0

16. $\cos 2\theta \cos 2\phi + \sin^2(\theta - \phi) - \sin^2(\theta + \phi)$ is equal to

(a) $\sin 2(\theta + \phi)$ (b) $\cos 2(\theta + \phi)$
(c) $\sin 2(\theta - \phi)$ (d) $\cos 2(\theta - \phi)$

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	**7.** ⓐ ⓑ ⓒ ⓓ	**13.** ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	**8.** ⓐ ⓑ ⓒ ⓓ	**14.** ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	**9.** ⓐ ⓑ ⓒ ⓓ	**15.** ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	**10.** ⓐ ⓑ ⓒ ⓓ	**16.** ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ	**11.** ⓐ ⓑ ⓒ ⓓ	
6. ⓐ ⓑ ⓒ ⓓ	**12.** ⓐ ⓑ ⓒ ⓓ	

HINTS AND EXPLANATIONS

4. (b)

$$\frac{\sin B \cos A + \sin A \cos B + \cos A \cos B + \sin A \sin B}{\sin B \cos A - \sin A \cos B + \cos A \cos B - \sin A \sin B}$$

$$= \frac{(\sin B + \cos B)(\cos A + \sin A)}{(\sin B + \cos B)(\cos A - \sin A)}$$

$$= \frac{\cos A + \sin A}{\cos A - \sin A}$$

10. (b) $\tan(A - B) = \tan\left(\dfrac{\pi}{4}\right)$

$$\Rightarrow \frac{\tan A - \tan B}{1 + \tan A \tan B} = 1$$

$\tan A - \tan B = 1 + \tan A \tan B$

$\Rightarrow 1 + \tan A - \tan B - \tan A \tan B = 2$

$\Rightarrow (1 + \tan A)(1 - \tan B) = 2$

LECTURE 5

Sum and Difference of Two Angles

BASIC CONCEPTS

1. FORMULAE CONVERTING PRODUCT INTO SUM OR DIFFERENCE

(i) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

(ii) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(iii) $2\cos A\cos B = \cos(A-B) + \cos(A+B)$

(iv) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

2. FORMULAE CONVERTING SUM AND DIFFERENCE INTO PRODUCT

(i) $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(ii) $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

(iii) $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(iv) $\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

(v) $\tan A + \tan B = \frac{\sin(A+B)}{\cos A\cos B}$

(vi) $\tan A - \tan B = \frac{\sin(A-B)}{\cos A\cos B}$

(vii) $\frac{\tan A+\tan B}{\tan A-\tan B} = \frac{\sin(A+B)}{\sin(A-B)}$

(viii) $\cot A + \cot B = \frac{\sin(A+B)}{\sin A\sin B}$

(ix) $\cot A - \cot B = \frac{-\sin(A-B)}{\cos A\cos B}$

(x) $\frac{\cot A+\cot B}{\cot A-\cot B} = \frac{-\sin(A+B)}{\sin(A-B)}$

(xi) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots\ldots + \cos\{\alpha+(n-1)\beta\}$

$$\cos\left[\frac{\text{First angle}+\text{Last angle}}{2}\right]$$

$$= \frac{\sin n\left(\frac{\text{common difference}}{2}\right)}{\sin\left(\frac{\text{common difference}}{2}\right)}$$

(xii) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots + \sin\{\alpha+(n-1)\beta\}$

$$= \frac{\sin\left[\frac{\alpha+\alpha+(n-1)\beta}{2}\right]\sin n\left(\frac{\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

3. TRIGONOMETRIC FUNCTIONS FOR SUM OF THREE OF MORE ANGLES

$\tan(A+B+C)$

$$= \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$$

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Sum of the sine and cosine series when the angles are in A.P.

$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\, n$ terms

$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots\, n$ terms

$$= \frac{\sin n.\frac{diff}{2}}{\sin\frac{diff}{2}} \begin{matrix}\sin\\ \text{or}\\ \cos\end{matrix}\left[\frac{\text{1st angle} + \text{last angle}}{2}\right]$$

$$= \frac{\sin(n\beta/2)}{\sin(\beta/2)} \begin{matrix}\sin\\ \text{or}\\ \cos\end{matrix}\left[\frac{\alpha+\alpha+(n-1)\beta}{2}\right]$$

$$= \frac{\sin(n\beta/2)}{\sin\beta/2} \begin{matrix}\sin\\ \text{or}\\ \cos\end{matrix}\left[\alpha+(n-1)\frac{\beta}{2}\right] \quad \ldots\ldots\ldots\ (A)$$

(Remember this as standard result)

Solution

Let $S = \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots$

Multiply both sides by $2\sin(\beta/2)$ and write

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$\therefore\ 2\sin(\beta/2)\, S = [\cos(\alpha-\beta/2) - \cos(\alpha+\beta/2)]$

$+ [\cos(\alpha+\beta/2) - \cos(\alpha+3\beta/2)]$

$+ [\cos(\alpha+3\beta/2) - \cos(\alpha+5\beta/2)]$

....

$+ [\cos\{\alpha+(2n-3)\beta/2 - \cos\{\alpha+2n-1)\beta/2)\}]$

$\therefore\ 2\sin\frac{\beta}{2}.\ S = \cos(\alpha-\beta/2) - \cos[\alpha+(2n-1)\beta/2)]$

$= 2\sin(\alpha+(n-1)\beta/2\}\sin(n\beta/2).$

2. Prove that, $\cos 2\theta\cos\frac{\theta}{2} - \cos 3\theta\cos\frac{9\theta}{2}$

$= \sin 5\theta \sin\frac{5\theta}{2}$

Solution

$$\text{L.H.S.} = \cos 2\theta\cos\frac{\theta}{2} - \cos 3\theta\cos\frac{9\theta}{2}$$

$$= \frac{1}{2}\left[2\cos 2\theta\cos\frac{\theta}{2} - 2\cos 3\theta\cos\frac{9\theta}{2}\right]$$

$$= \frac{1}{2}\left[\cos\left(2\theta+\frac{\theta}{2}\right) + \cos\left(2\theta-\frac{\theta}{2}\right) - \left\{\cos\left(3\theta+\frac{9\theta}{2}\right)+\cos\left(3\theta-\frac{9\theta}{2}\right)\right\}\right]$$

$$= \frac{1}{2}\left[\cos\frac{5\theta}{2} + \cos\frac{3\theta}{2} - \cos\frac{15\theta}{2} - \cos\frac{3\theta}{2}\right]$$

$$= \frac{1}{2}\left[\cos\frac{5\theta}{2} - \cos\frac{15\theta}{2}\right]$$

$$= \frac{1}{2}\times 2\sin\left(\frac{\frac{5\theta}{2}+\frac{15\theta}{2}}{2}\right)\sin\left(\frac{\frac{15\theta}{2}-\frac{5\theta}{2}}{2}\right)$$

$$= \sin 5\theta \sin\frac{15\theta}{2}$$

L.H.S. = R.H.S.

3. Prove that, $\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma)$

$$= 4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\alpha+\gamma}{2}$$

Solution

L.H.S. $= \cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma)$

$= (\cos\alpha + \cos\beta) + [\cos\alpha + \cos(\alpha+\beta+\gamma)]$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + 2\cos\left(\frac{\alpha+\beta+\gamma+\gamma}{2}\right)\cos\left(\frac{\alpha+\beta+\gamma-\gamma}{2}\right)$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\left[\cos\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta+2\gamma}{2}\right)\right]$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\times 2\cos\left[\frac{\alpha-\beta+\alpha+\beta+2\gamma}{4}\right]\cos\left[\frac{\alpha+\beta+2\gamma-\alpha+\beta}{4}\right]$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\times 2\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)$$

$$= 4\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\alpha+\gamma}{2}\right)$$

4. Prove that

$$\frac{\cos 2A\cos 3A - \cos 2A\cos 7A + \cos A\cos 10A}{\sin 4A\sin 3A - \sin 2A\sin 5A + \sin 4A\sin 7A}$$

$= \cot 6A \cot 5A$

Solution

L.H.S. =

$$\frac{2\cos 3A\cos 2A - 2\cos 7A\cos 2A + 2\cos A\cos 10A}{2\sin 4A\sin 3A - 2\sin 5A\sin 2A + 2\sin 4A\sin 7A}$$

$\{\cos(3A + 2A) + \cos(3A - 2A)\} - \{\cos(7A + 2A) + \cos(7A - 2A)\}$

$$= \frac{\{\cos(3A+2A)+\cos(3A-2A)\}-\{\cos(7A+2A)+\cos(7A-2A)\}+\{\cos(10A+A)+\cos(10A-A)\}}{\{\cos(4A-3A)-\cos(4A+3A)\}-\{\cos(5A-2A)-\cos(5A+2A)\}+\{\cos(7A-4A)-\cos(7A+4A)\}}$$

$$= \frac{\cos 5A + \cos A - \cos 9A - \cos 5A + \cos 11A + \cos 9A}{\cos A - \cos 7A - \cos 3A + \cos 7A + \cos 3A - \cos 11A}$$

$$= \frac{\cos A + \cos 11A}{\cos A - \cos 11A}$$

$$= \frac{2\cos\left(\frac{11A+A}{2}\right)\cos\left(\frac{11A-A}{2}\right)}{2\sin\left(\frac{A+11A}{2}\right)\sin\left(\frac{11A-A}{2}\right)}$$

$$= \frac{\cos 6A\cos 5A}{\sin 6A\sin 5A} = \cot 6A\cos 5A$$

5. Prove that

$$\left(\frac{\cos A+\cos B}{\sin A-\sin B}\right)^n + \left(\frac{\sin A+\sin B}{\cos A-\cos B}\right)^n$$

$$= \begin{cases} 2\cot^n\left(\frac{A-B}{2}\right), & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

Solution

$$\text{L.H.S.} = \left\{\frac{2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}\right\}^n$$

$$+\left\{\frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right)}\right\}^n$$

$$= \left\{\cot\left(\frac{A-B}{2}\right)\right\}^n + \left\{-\cot\left(\frac{A-B}{2}\right)\right\}^n$$

$$= \cot^n\left(\frac{A-B}{2}\right) + (-1)^n\cot^n\left(\frac{A-B}{2}\right)$$

$$= \cot^n\left(\frac{A-B}{2}\right)[1+(-1)^n]$$

$$= \begin{cases} 2\cot^n\left(\frac{A-B}{2}\right), & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

6. If 2 tan α = 3 tan β, then prove that tan (α – β) $= \frac{\sin 2\beta}{5-\cos 2\beta}$.

Solution

$\frac{\tan\alpha}{\tan\beta} = \frac{3}{2}$ by componendo and dividendo

$$\Rightarrow \frac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta} = \frac{3+2}{3-2}$$

$$\Rightarrow \frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \frac{5}{1}$$

$$\Rightarrow \frac{\sin\{(\alpha-\beta)+2\beta\}}{\sin(\alpha-\beta)} = \frac{5}{1}$$

$$\Rightarrow \frac{\sin(\alpha-\beta)\cos 2\beta}{\sin(\alpha-\beta)} + \frac{\cos(\alpha-\beta)\sin 2\beta}{\sin(\alpha-\beta)} = \frac{5}{1}$$

$$\Rightarrow \frac{\sin 2\beta}{\tan(\alpha-\beta)} = \frac{5-\cos 2\beta}{1}$$

$$\Rightarrow \tan(\alpha-\beta) = \frac{\sin 2\beta}{5-\cos 2\beta}$$

7. If $\tan(\alpha+\theta) = n\tan(\alpha-\theta)$, show that $(n+1)\sin 2\theta = (n-1)\sin 2\alpha$.

Solution

$$\frac{\tan(\alpha+\theta)}{\tan(\alpha-\theta)} = \frac{n}{1}$$

$$\Rightarrow \frac{\tan(\alpha+\theta)+\tan(\alpha-\theta)}{\tan(\alpha+\theta)-\tan(\alpha-\theta)} = \frac{n+1}{n-1}$$

$$\Rightarrow \frac{\dfrac{\sin(\alpha+\theta)}{\cos(\alpha+\theta)}+\dfrac{\sin(\alpha-\theta)}{\cos(\alpha-\theta)}}{\dfrac{\sin(\alpha+\theta)}{\cos(\alpha+\theta)}-\dfrac{\sin(\alpha-\theta)}{\cos(\alpha-\theta)}} = \frac{n+1}{n-1}$$

$$\Rightarrow \frac{\dfrac{\sin(\alpha+\theta)\cos(\alpha-\theta)+\sin(\alpha-\theta)\cos(\alpha+\theta)}{\cos(\alpha+\theta)\cos(\alpha-\theta)}}{\dfrac{\sin(\alpha+\theta)\cos(\alpha-\theta)-\sin(\alpha-\theta)\cos(\alpha+\theta)}{\cos(\alpha+\theta)\cos(\alpha-\theta)}} = \frac{n+1}{n-1}$$

$$\Rightarrow \frac{\sin\{(\alpha+\theta)+(\alpha-\theta)\}}{\sin\{(\alpha+\theta)-(\alpha-\theta)\}} = \frac{\sin 2\alpha}{\sin 2\theta} = \frac{n+1}{n-1}$$

$$\Rightarrow (n+1)\sin 2\theta = (n-1)\sin 2\alpha$$

8. If an angle θ is divided into two parts α and β such that $\dfrac{\tan\alpha}{\tan\beta} = \dfrac{a}{b}$, then prove that $\sin(\alpha-\beta) = \left(\dfrac{a-b}{a+b}\right)\sin\theta$.

Solution

Given $\dfrac{\tan\alpha}{\tan\beta} = \dfrac{a}{b}$ by componendo and dividendo

$$\Rightarrow \frac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta} = \frac{a+b}{a-b}$$

$$\Rightarrow \frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \frac{a+b}{a-b}$$

$$\Rightarrow \frac{\sin\theta}{\sin(\alpha-\beta)} = \frac{a+b}{a-b}$$

$$\Rightarrow \sin(\alpha-\beta) = \left(\frac{a-b}{a+b}\right)\sin\theta$$

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): SOLVE THESE PROBLEMS TO GRASP THE TOPIC

EXERCISE 1

1. Simplify $\dfrac{\sin 8\theta\cos\theta-\sin 6\theta\cos 3\theta}{\cos 2\theta\cos\theta-\sin 3\theta\sin 4\theta}$.

   (**Ans:** $\tan 2\theta$)

2. Prove that $\dfrac{\sin A+2\sin 3A+\sin 5A}{\sin 3A+2\sin 5A+\sin 7A} = \dfrac{\sin 3A}{\sin 5A}$

3. Prove that $2\cos\dfrac{\pi}{13}\cos\dfrac{9\pi}{13} + \cos\dfrac{3\pi}{13}+\cos\dfrac{5\pi}{13} = 0$

   [UPSEAT-04]

4. Prove that $\dfrac{\sin A+\sin 3A+\sin 5A+\sin 7A}{\cos A+\cos 3A+\cos 5A+\cos 7A} = \tan 4A$

5. Prove that

   $\sin 10° + \sin 20° + \sin 40° + \sin 50° = \sin 70° + \sin 80°$

6. Prove that, $\cos\left(\dfrac{3\pi}{4}+x\right)-\cos\left(\dfrac{3\pi}{4}-x\right) = -\sqrt{2}\sin x$

7. Prove that, $\sin^2\left(\dfrac{\pi}{8}+\dfrac{A}{2}\right)-\sin^2\left(\dfrac{\pi}{8}-\dfrac{A}{2}\right) = \dfrac{1}{\sqrt{2}}\sin A$

8. Prove that $\sin\alpha + \sin\left(\alpha+\dfrac{2\pi}{3}\right) + \sin\left(\alpha+\dfrac{4\pi}{3}\right) = 0$

9. Prove that $\dfrac{\cos 4x+\cos 3x+\cos 2x}{\sin 4x+\sin 3x+\sin 2x}=\cot 3x$

10. Prove that $\cos 7\theta + \cos 5\theta + \cos 3\theta + \cos\theta = 4\cos\theta\cos 2\theta\cos 4\theta$

EXERCISE 2

1. Prove that $\dfrac{\sin(x-y)}{\sin(x+y)}=\dfrac{\tan x-\tan y}{\tan x+\tan y}$.

2. Prove that $\dfrac{\sin 5A-\sin 3A}{\cos 5A+\cos 3A}=\tan A$.

3. Prove that $\dfrac{\sin A+\sin B}{\cos A+\cos B}=\tan\left(\dfrac{A+B}{2}\right)$

4. Prove that $\dfrac{\cos 8A\cos 5A-\cos 12A\cos 9A}{\sin 8A\cos 5A+\cos 12A\sin 9A}$ $=\tan 4A$.

5. Prove that $\dfrac{\sin(A-C)+2\sin A+\sin(A+C)}{\sin(B-C)+2\sin B+\sin(B+C)}$ $=\dfrac{\sin A}{\sin B}$.

6. Prove that $\dfrac{\sin 11A\sin A+\sin 7A\sin 3A}{\cos 11A\sin A+\cos 7A\sin 3A}$ $=\tan 8A$.

7. Prove that $\cos A + \cos(120° - A) + \cos(120° + A) = 0$

8. Prove that $\sin((150° + x) + \sin(150° - x) = \cos x$

9. Prove that $\cos\left(\dfrac{\pi}{4}+x\right)+\cos\left(\dfrac{\pi}{4}-x\right)$ $=\sqrt{2}\cos x$

10. If $\cos(\theta + 2\alpha) = n\cos\theta$,

show that $\cot\alpha = \dfrac{1+n}{1-n}\tan(\theta+\alpha)$

11. Prove that $\dfrac{\sin 5x-2\sin 3x+\sin x}{\cos 5x-\cos x}=\tan x$

12. Prove that $\dfrac{\sin(x-y)+\sin x+\sin(x+y)}{\cos(x-y)+\cos x+\cos(x+y)}$ $=\tan x$

13. Prove that

$$\frac{\cos 2A\cos 3A-\cos 2A\cos 7A+\cos A\cos 10A}{\sin 4A\sin 3A-\sin 2A\sin 5A+\sin 4A\sin 7A}=\cot 6A\cot 5A$$

14. Prove that

$$\frac{(\sin 7x+\sin 5x)+(\sin 9x+\sin 3x)}{(\cos 7x+\cos 5x)+(\cos 9x+\cos 3x)}=\tan 6x$$

15. If $m\cos(\theta+\alpha) = n\cos(\theta-\alpha)$, then prove that

$$\tan\theta=\frac{m-n}{m+n}\cot\alpha$$

16. Prove that $\dfrac{\sin 5x+\sin 3x}{\cos 5x+\cos 3x}=\tan 4x$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. An angle θ is divided into two parts A, B such that $A - B = k$ and $\tan A : \tan B = k : 1$, then $\sin k$ is equal to ***[UPSEAT-2002]***

(a) $\dfrac{k+1}{k-1}\sin\theta$ (b) $\dfrac{k}{k+1}\sin\theta$

(c) $\dfrac{k-1}{k+1}\sin\theta$ (d) None of these

Solution

(c) $A + B = \theta$ and $A - B = k$

$\therefore \quad \dfrac{\tan A}{\tan B}=\dfrac{k}{1}$

$\Rightarrow \quad \dfrac{\tan A+\tan B}{\tan A-\tan B}=\dfrac{k+1}{k-1}$

$$\Rightarrow \quad \frac{\sin(A+B)}{\sin(A-B)} = \frac{k+1}{k-1}$$

$$\Rightarrow \quad \frac{\sin\theta}{\sin k} = \frac{k+1}{k-1}$$

$$\therefore \quad \sin k = \frac{k-1}{k+1}\sin\theta$$

2. If sin θ + sin 2θ + sin 3θ = sin α and cos θ + cos 2θ + cos 3θ = cos α, then θ is equal to ***[AMU-2001]***

(a) α/2 (b) α
(c) 2α (d) α/6

Solution

(a) sin θ + sin 3θ + sin 2θ = sin α

⇒ 2 sin 2θ cos θ + sin 2θ = sin α

⇒ sin 2θ (2 cos θ + 1) = sin α ..... (i)

Now, cos θ + cos 3θ + cos 2θ = cos α

2 cos 2θ cos θ + cos 2θ = cos α

cos 2θ (2 cos θ + 1) = cos α .... (ii)

From (i) and (ii), tan 2θ = tan α ⇒ 2θ = α ⇒ θ = α/2.

3. If cos x + cos y + cos α = 0 and sin x + sin y + sin α = 0, then $\cot\left(\frac{x+y}{2}\right) =$

(a) sin α (b) cos α
(c) cot α (d) $\sin\left(\frac{x+y}{2}\right)$

Solution

(c) Given equation cos x + cos y + cos α = 0 and sin x + sin y + sin α = 0. The given equation may be written as cos x + cos y = –cos α and sin x + sin y = –sin α.

Therefore,

$$2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = -\cos\alpha \quad \ldots\ldots \text{(i)}$$

$$2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = -\sin\alpha \quad \ldots\ldots \text{(ii)}$$

By dividing (i) by (ii), we get

$$\frac{2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}{2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)} = \frac{\cos\alpha}{\sin\alpha}$$

$$\Rightarrow \quad \cot\left(\frac{x+y}{2}\right) = \cot\alpha.$$

4. The expression $\cos\frac{10\pi}{13} + \cos\frac{8\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13}$ is equal to ***[MPPET-2006]***

(a) –1 (b) 0
(c) 1 (d) None of these

Solution

(b) Given expression $= \cos\frac{10\pi}{13} + \cos\frac{8\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13}$

$$= \left(\cos\frac{10\pi}{13} + \cos\frac{3\pi}{13}\right) + \left(\cos\frac{8\pi}{13} + \cos\frac{5\pi}{13}\right)$$

$$= 2\cos\left(\frac{13\pi}{2\times 13}\right)\cos\left(\frac{7\pi}{2\times 13}\right) + 2\cos\left(\frac{13\pi}{2\times 13}\right)\cos\left(\frac{3\pi}{2\times 13}\right)$$

$$= 2\cos\frac{\pi}{2}\left(\cos\frac{7\pi}{26} + \cos\frac{3\pi}{26}\right) = 0$$

$$\left[\because \cos\frac{\pi}{2} = 0\right]$$

5. 2 sin² β + 4 cos (α + β) sin α sin β + cos 2 (α + β) is equal to ***[IIT-1977; MNR-93]***

(a) sin 2α (b) sin 2β
(c) cos 2α (d) cos 2β

Solution

(c) Using

$2\sin^2\beta = 1 - \cos 2\beta$ and $\cos 2(\alpha+\beta) = 2\cos^2(\alpha+\beta) - 1$

Exp. $= 1 - \cos 2\beta + 4\cos(\alpha+\beta)\sin\alpha\sin\beta + 2\cos^2(\alpha+\beta) - 1$

$= -\cos 2\beta + 2\cos(\alpha+\beta)\,[2\sin\alpha\sin\beta + \cos(\alpha+\beta)]$

$= -\cos 2\beta + 2\cos(\alpha+\beta)\,[\sin\alpha\sin\beta + \cos\alpha\cos\beta]$

$= -\cos 2\beta + 2\cos(\alpha+\beta)\cos(\alpha-\beta)$

$= -\cos 2\beta + (\cos 2\alpha + \cos 2\beta)$

$= \cos 2\alpha$

6. If $\cos(\alpha-\beta)=1$ and $\cos(\alpha+\beta)=\frac{1}{e}$, $-\pi<\alpha, \beta<\pi$, then total number of ordered pair of (α, β) is ***[IIT Screening-2005]***

(a) 0 (b) 1
(c) 2 (d) 4

Solution

(d) $-2\pi<\alpha-\beta<2\pi \Rightarrow \cos(\alpha-\beta)=1$

$\Rightarrow \alpha-\beta=0 \Rightarrow \alpha=\beta$

$\cos 2\alpha=\frac{1}{e}$ and $-2\pi<2\alpha<2\pi$

Hence, there will be four solutions.

7. If θ, ϕ are acute, $\sin\theta=1/2$, $\cos\phi=1/3$ then $(\theta+\phi)\in$ ***[IIT Screening-2004]***

(a) $(\pi/3, \pi/2)$ (b) $(\pi/2, 2\pi/3)$
(c) $(2\pi/3, 5\pi/6)$ (d) $(5\pi/6, \pi)$

Solution

(b) $\sin\theta=\frac{1}{2}$

$\therefore \quad \theta=\frac{\pi}{6}$ .... (1)

$\cos\phi=\frac{1}{3}$

Now, $\frac{1}{2}>\frac{1}{3}>0$ or $\cos\frac{\pi}{3}>\cos\phi>\cos\frac{\pi}{2}$

$\therefore \quad \frac{\pi}{3}<\phi<\frac{\pi}{2}$ ...... (2)

By adding (1) and (2) $\frac{\pi}{2}<\theta+\phi<\frac{2\pi}{3}$

8. The number of integral values of k for which the equation $7\cos x+5\sin x=2k+1$ has a solution is ***[IIT. Sc.-2002]***

(a) 4 (b) 8
(c) 10 (d) 12

Solution

(b) $(7)^2+(5)^2=74=r^2$

Dividing by r on both side, we have

$\cos(x-\alpha)=\frac{2k+1}{\sqrt{74}}$ where $\tan\alpha=\frac{5}{7}$

$\therefore \quad -1<\frac{2k+1}{\sqrt{74}}<1$ or $-\sqrt{74}<2k+1<\sqrt{74}$

or $\quad -8<2k+1<8$

$\therefore \quad k=-4, -3, -2, -1, 0, 1, 2, 3$

i.e., 8 values which will satisfy the above inequality.

9. If n is a +ve integer such that

$\sin\frac{\pi}{2n}+\cos\frac{\pi}{2n}=\frac{\sqrt{n}}{2}$, then

[IIT-94]

(a) $4<n\le 8$ (b) $6\le n\le 8$
(c) $4\le n<8$ (d) $4<n<8$

Solution

(a) $\sin\frac{\pi}{2n}+\cos\frac{\pi}{2n}=\sqrt{2}\sin\left(\frac{\pi}{4}+\frac{\pi}{2n}\right)$

$\Rightarrow \quad \frac{\sqrt{n}}{2\sqrt{2}}\sin\left(\frac{\pi}{4}+\frac{\pi}{2n}\right)$

Hence, $n>1, \frac{\sqrt{n}}{2\sqrt{2}}\sin\left(\frac{\pi}{4}+\frac{\pi}{2n}\right)>\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$

$\Rightarrow \quad \frac{\sqrt{n}}{2\sqrt{2}}>\frac{1}{\sqrt{2}}=>\sqrt{n}>2=>n>4$

Also, $\sin\left(\frac{\pi}{4}+\frac{\pi}{2n}\right)\le 1\forall\, n>2$

$\Rightarrow \quad \frac{\sqrt{n}}{2\sqrt{2}}\le 1=>\sqrt{n}\le 2\sqrt{2}=>n\le 8$

10. If $\frac{\pi}{2}<\alpha<\pi, \pi<\beta<\frac{3\pi}{2}$, $\sin\alpha=\frac{15}{17}$ and $\tan\beta=\frac{12}{5}$ then $\sin(\beta-\alpha)$ is equal to

[Roorkee Screening-2000]

(a) $-\frac{17}{221}$ (b) $-\frac{21}{221}$

(c) $\frac{21}{221}$ (d) $\frac{171}{221}$

Solution

(d) $\sin\alpha=15/17 \Rightarrow \cos\alpha=-8/17$

[$\because$ $\cos\alpha<0$ then $\pi/2<\alpha<\pi$]

$\tan\beta=12/5 \Rightarrow \sin\beta=-12/13, \cos\beta=-5/13$

[$\because$ $\pi<\beta<3\pi/2 \Rightarrow \sin\beta<0, \cos\beta<0$]

Now $\sin(\beta-\alpha)=\sin\beta\cos\alpha-\sin\alpha\cos\beta$

$=(-12/13)(-8/17)-(15/17)(-5/13)$

$=171/221$

11. Find the least x (> 0) for which $\tan(x° + 100°) = \tan(x° + 50°) \tan x° \tan(x° - 50°)$. ***[IIT-1993]***

Solution

(d) Given $\tan(x° + 100°) = \tan(x° + 50°) \tan x° \tan(x° - 50°)$

$$\Rightarrow \frac{\tan(x°+100°)}{\tan x°} = \frac{\tan(x°+50°)}{\cot(x°-50°)},$$

apply componendo and dividendo to obtain

$$\frac{\tan(x°+100°)+\tan x°}{\tan(x°+100°)-\tan x°} = \frac{\tan(x°+50°)+\cot(x°-50°)}{\tan(x°+50°)-\cot(x°-50°)}$$

$$\Rightarrow \frac{\sin(x°+100°+x°)}{\sin(x°+100°-x°)} = \frac{\cos(x°+50°-x°+50°)}{-\cos(x°+50°+x°-50°)}$$

$$\Rightarrow \frac{\sin(2x°+100°)}{\sin 100°} = \frac{\cos 100°}{-\cos 2x°}$$

$\Rightarrow \sin(2x° + 100°) \cos 2x° = -\sin 100° \cos 100°$

$\Rightarrow 2\sin(2x° + 100°) \cos 2x° = -2\sin 100° \cos 100°$

$\Rightarrow \sin(2x° + 100° + 2x°) + \sin(2x° + 100° - 2x°) = -\sin 200°$

$\Rightarrow \sin(100° + 4x°) = -(\sin 200° + \sin 100°)$

$\Rightarrow \sin(100° + 4x°) = -2\sin 150° \cos 50°$

$\Rightarrow \sin(100° + 4x°) = -\cos 50° = \sin(270° - 50°)$ (note that $100° + 4x° > 100°$)

$\Rightarrow 100° + 4x° = 220°$

$\Rightarrow x° = 30°$

$\Rightarrow x = 30$

12. Prove that, ***[IIT-JEE-1978]***

$\sum \sin x.\sin y.\sin(x-y) + \sin(x-y)\sin(y-z)\sin(z-x) = 0.$

Solution

$$\sum \sin x.\sin y.\sin(x-y)$$

$$= \frac{1}{2}\sum[\cos(x-y)-\cos(x+y)]\sin(x-y)$$

$$= \frac{1}{2}[\cos(x-y)-\cos(x+y)]\sin(x-y)$$

$$+\frac{1}{2}[\cos(y-z)-\cos(y+z)]\sin(y-z)$$

$$+\frac{1}{2}[\cos(z-x)-\cos(z+x)]\sin(z-x)$$

Adding the first term of the first line and the second term of the second line and above, we get the sum,

$$= \frac{1}{4}\sum[\sin 2(x-y) - \sin 2y + \sin 2z]$$

$$= \frac{1}{4}\sum \sin 2(x-y)$$

$$= \frac{1}{4}[\sin 2(x-y) + \sin 2(y-z) + \sin 2(z-x)]$$

$$= \frac{1}{2}[\sin(x-y)\cos(x-y) + \sin(y-x) + \sin(x+y-2z)]$$

$$= \frac{1}{2}\sin(x-y)[\cos(x-y) - \cos(x+y-2z)]$$

$= \sin(x-y)\sin(x-z)\sin(y-z)$

$= -\sin(x-y)\sin(y-z)\sin(z-x)$

13. If α and β are the solutions of $a\tan\theta + b\sec\theta = c$, show that, $\tan(\alpha+\beta) = \frac{2ac}{a^2-c^2}$. ***[IIT-JEE-1973]***

Solution

$b\sec\theta = c - a\tan\theta$ gives on squaring $(a^2 - b^2)\tan^2\theta - 2ac\tan\theta + c^2 - b^2 = 0$.

The roots are $\tan\alpha$, $\tan\beta$

$$\therefore \quad \tan\alpha + \tan\beta = \frac{2ac}{a^2-b^2}$$

$$\tan\alpha\tan\beta = \frac{c^2-b^2}{a^2-b^2}$$

$$\therefore \quad \tan(\alpha+\beta) = \frac{2ac}{a^2-b^2-(c^2-b^2)} = \frac{2ac}{a^2-c^2}$$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. If $\sin A = \sin B$ and $\cos A = \cos B$, then ***[EAMCET-1994]***

(a) $\sin\frac{A-B}{2}=0$ (b) $\sin\frac{A+B}{2}=0$

(c) $\cos\frac{A-B}{2}=0$ (d) $\cos(A+B)=0$

2. $\cos A + \cos(240° + A) + \cos(240° - A)$ is equal to ***[MPPET-1991]***

(a) $\cos A$ (b) 0

(c) $\sqrt{3}\sin A$ (d) $\sqrt{3}\cos A$

3. $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$ is equal to ***[IIT-1984; WBJEE-1992]***

(a) 1/2 (b) 1/4

(c) 1/8 (d) 1/16

4. If $\cos(\alpha+\beta)\sin(\gamma+\delta) = \cos(\alpha-\beta)\sin(\gamma-\delta)$, then $\cot\alpha\cot\beta\cot\gamma$ is equal to ***[D.C.E.-1998]***

(a) $\cot\delta$ (b) $\tan\delta$

(c) $\cot\alpha$ (d) $\tan\alpha$

5. $\frac{\sin 70°+\cos 40°}{\cos 70°+\sin 40°}$ is equal to ***[Pb. CET-1986; MPPET-1999]***

(a) 1 (b) $\frac{1}{\sqrt{3}}$

(c) $\sqrt{3}$ (d) $\frac{1}{2}$

6. If $m\tan(\theta - 30°) = n\tan(\theta + 120°)$, then $\frac{m+n}{m-n}$ is equal to ***[IIT-1966]***

(a) $2\cos 2\theta$ (b) $\cos 2\theta$

(c) $2\sin 2\theta$ (d) $\sin 2\theta$

7. If $x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right) = z\cos\left(\theta+\frac{4\pi}{3}\right)$, then $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ is equal to ***[IIT-1984, Pb. CET-2003]***

(a) 1 (b) 2

(c) 0 (d) $3\cos\theta$

8. $\sin 50° - \sin 70° + \sin 10°$ is equal to ***[MNR-1979]***

(a) 1 (b) 0

(c) 1/2 (d) 2

9. Prove that $\frac{\sin x-\sin y}{\cos x+\cos y}$ is equal to

(a) $\tan\left(\frac{x+y}{2}\right)$ (b) $\tan\left(\frac{x-y}{2}\right)$

(c) $\tan\left(\frac{y-x}{2}\right)$ (d) $\tan(x-y)$

10. The expression $2\cos\frac{\pi}{13}.\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}$ is equal to ***[UPSEAT-2004]***

(a) –1 (b) 0

(c) 1 (d) None

11. $\cos 2(\theta+\varphi) - 4\cos(\theta+\varphi)\sin\theta\sin\varphi + 2\sin^2\varphi$ is equal to ***[Orissa JEE-2004]***

(a) $\cos 2\theta$ (b) $\cos 3\theta$

(c) $\sin 2\theta$ (d) $\sin 3\theta$

12. If $\sin\theta + \sin 2\theta + \sin 3\theta = \sin$ and $\cos\theta + \cos 2\theta + \cos 3\theta = \cos\alpha$, then θ is equal to ***[AMU-2001]***

(a) $\alpha/2$ (b) α

(c) 2α (d) $\alpha/6$

13. If $\sin A\sin B\sin C + \cos A\cos B = 1$, then the value of $\sin C$ is equal to ***[IIT-JEE-2006]***

(a) 1 (b) 0

(c) 1/3 (d) 1/2

HINTS AND EXPLANATIONS

1. (a) $\sin A - \sin B = 0$

$$\Rightarrow \quad 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right) = 0 \quad \ldots\ldots\ (1)$$

$\cos A - \cos B = 0$

$$\Rightarrow \quad 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right) = 0 \quad \ldots\ldots\ (2)$$

From (1) and (2) $\sin\left(\frac{A-B}{2}\right) = 0$

2. (b) $\cos A + \cos(240° + A) + \cos(240° - A)$

$= \cos A + 2\cos 240 \cos A$

$= \cos A + 2\left(-\frac{1}{2}\right)\cos A = 0 \quad \left(\because \cos 240° = \frac{-1}{2}\right)$

3. (c) $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)$

$\left(1+\cos\left(\pi-\frac{3\pi}{8}\right)\right)\left(1+\cos\left(\pi-\frac{\pi}{8}\right)\right)$

$\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)$

$\left(1-\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{\pi}{8}\right)$

$= \left(1-\cos^2\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)$

$= \left(\frac{2}{2}\sin\frac{\pi}{8}\sin\frac{3\pi}{8}\right)^2$

$= \frac{1}{4}\left(\cos\frac{\pi}{4} - \cos\frac{\pi}{2}\right)^2 = \frac{1}{8}$

4. (a) $\cos(\alpha+\beta)\sin(\gamma-\delta) = \cos(\alpha-\beta)\sin(\gamma-\delta)$

$$\frac{\sin(\gamma+\delta)}{\sin(\gamma-\delta)} = \frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)}$$

By applying componendo and dividendo

$$\frac{\sin(\gamma+\delta)+\sin(\gamma-\delta)}{\sin(\gamma+\delta)-\sin(\gamma-\delta)}$$

$$= \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{\cos(\alpha-\beta)-\cos(\alpha+\beta)}$$

$$\frac{2\sin\gamma\cos\delta}{2\sin\delta\cos\gamma} = \frac{2\cos\alpha\cos\beta}{22\sin\alpha\sin\beta}$$

$$\Rightarrow \quad \frac{\tan\gamma}{\tan\delta} = \cot\alpha\,.\,\cot\beta$$

$\Rightarrow \quad \cot\alpha\cos\beta\cos\gamma = \cot\delta$

5. (c) $\dfrac{\sin 70° + \cos 40°}{\cos 70° + \sin 40°} = \dfrac{\sin 70° + \sin 50°}{\cos 70° + \cos 50°}$

$= \dfrac{2\sin 60°\cos 10°}{2\cos 60°\cos 10°} = \tan 60° = \sqrt{3}$

6. (a) $m\tan(\theta - 30°) = n\tan(\theta + 120°)$

$$\frac{m}{n} = \frac{\tan(\theta+120°)}{\tan(\theta-30°)}$$

$$\Rightarrow \quad \frac{m+n}{m-n} = \frac{\tan(\theta+120°)+\tan(\theta-30°)}{\tan(\theta+120°)-\tan(\theta-30°)}$$

$$\frac{m+n}{m-n} = \frac{\sin(2\theta+90°)}{\sin 150°}$$

$$\frac{m+n}{m-n} = 2\cos 2\theta$$

7. (c) $x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right)$

$= z\cos\left(\theta+\frac{4\pi}{3}\right) = k$ (assume)

$$\Rightarrow \quad \frac{1}{x} = \frac{\cos\theta}{k}, \frac{1}{y} = \frac{\cos\left(\theta+\frac{2\pi}{3}\right)}{k},$$

$$\frac{1}{z} = \cos\left(\theta+\frac{4\pi}{3}\right)$$

$$\Rightarrow \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}$$

$$= \frac{1}{k}\left[\cos\theta + \cos\left(\theta+\frac{2\pi}{3}\right) + \cos\left(\theta+\frac{4\pi}{3}\right)\right]$$

$$\Rightarrow \quad \frac{1}{k}\left[\cos\theta + 2\cos(\theta+\pi)\cos\frac{\pi}{3}\right]$$

$$\Rightarrow \quad \frac{1}{k}\left[\cos\theta - 2\cos\theta\frac{1}{2}\right] = 0$$

8. (c) sin 50° – sin 70° + sin 10° = sin 50° + sin 10° – sin 70°

$$= 2\sin\frac{60°}{2}\cos\frac{40°}{2} - \sin 70°$$

$$= 2\times\frac{1}{2}\cos 20° - \sin 70°$$

$$= \sin 70° - \sin 70° = 0$$

9. (b) $\dfrac{\sin x - \sin y}{\cos x + \cos y}$

$$= \frac{2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)}{2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}$$

($\because$ $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$ and

$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$)

$\Rightarrow \quad \tan\left(\frac{x-y}{2}\right)$

10. (b) $2\cos\frac{\pi}{13}.\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13}$

$$= 2\cos\frac{\pi}{13}.\cos\frac{9\pi}{13} + 2\cos\frac{4\pi}{13}\cos\frac{\pi}{13}$$

$$= 2\cos\frac{\pi}{13}\left[\cos\frac{9\pi}{13} + \cos\frac{4\pi}{13}\right]$$

$$= 2\cos\frac{\pi}{13}\left[2\cos\frac{\pi}{2}.\cos\frac{5\pi}{26}\right] = 0,$$

$$\left[\because \cos\frac{\pi}{2} = 0\right]$$

11. (a) We have, cos 2 (θ + ϕ) – 4 cos (θ + ϕ) sin θ sin ϕ + 2 sin² ϕ

Now, Put $\theta = \phi = \frac{\pi}{4}$

$$= \cos 2\left(\frac{\pi}{2}\right) - 4\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right)$$

$$\sin\left(\frac{\pi}{4}\right) + 2\sin^2\left(\frac{2\pi}{4}\right) = 0$$

Put θ = ϕ = π/4 in option (a). Then, cos 2θ = π/2 = 0

Hence, option (a) is correct.

12. (a) sin θ + sin 3θ + sin 2θ = sin α

⇒ 2 sin 2θ cos θ + sin 2θ = sin α

⇒ sin 2θ (2 cos θ + 1) = sin α .... (i)

Now, cos θ + cos 3θ + cos 2θ = cos α

2 cos 2θ cos θ + cos 2θ = cos α

cos 2θ (2 cos θ + 1) = cos α .... (ii)

From (i) and (ii), tan 2θ = tan α

⇒ 2θ = α

⇒ θ = α/2.

13. (a) sin A sin B sin C + cos A cos B

≤ sin A sin B + cos A cos B [$\because$ sin $C \le 1$]

∴ sin A sin B sin C + cos A cos B ≤ cos (A – B)

⇒ cos (A – B) ≥ 1 [using given relation]

⇒ cos (A – B) = 1 [$\because$ max. cos (A – B) = 1]

⇒ $A - B = 0$

⇒ $A = B$

Then from given relation,

$$\sin C = \frac{1-\cos^2 A}{\sin^2 A} = 1$$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. The value of cos 52° + cos 68° + cos 172° is *[MPPET-1997; Pb, CET-1995, 99]*

(a) 0 (b) 1
(c) 2 (d) 3/2

2. What is the value of $\frac{(\cos 10° + \sin 20°)}{(\cos 20° - \sin 10°)}$? *[NDA-2007]*

(a) $\frac{1}{\sqrt{3}}$ (b) $-\frac{1}{\sqrt{3}}$
(c) $\sqrt{3}$ (d) $-\sqrt{3}$

3. If $\frac{\sin(x+y)}{\sin(x-y)} = \frac{a+b}{a-b}$, then $\frac{\tan x}{\tan y}$ is equal to *[MPPET-2009]*

(a) $\frac{a}{b}$ (b) $\frac{b}{a}$
(c) $\frac{a+b}{b}$ (d) $\frac{a+b}{a}$

4. If $\sin\alpha = \frac{1}{2}\sin(\alpha + 2\beta)$, then $\frac{\tan(\alpha+\beta)}{\tan\beta}$ is equal to

(a) $\frac{5}{3}$ (b) $\frac{1}{3}$
(c) 3 (d) $\frac{3}{5}$

5. cos 5° – sin 25° is equal to

(a) sin 20° (b) sin 30°
(c) sin 35° (d) sin 60°

6. If sin A + sin $B = x$ and cos A + cos $B = y$, then cos $(A - B)$ is equal to

(a) $x^2 + y^2 - 2$ (b) $\frac{x^2 + y^2 - 2}{2}$
(c) $\frac{2 - x^2 - y^2}{2}$ (d) None of these

7. sin 200° + cos 200° is

(a) negative (b) positive
(c) zero (d) zero or positive

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The worksheet is of 9 minutes.
3. The worksheet consists of 9 questions. The maximum marks are 27.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. $\dfrac{\tan\left(\dfrac{\pi}{4}+\theta\right)-\tan\left(\dfrac{\pi}{4}-\theta\right)}{\tan\left(\dfrac{\pi}{4}+\theta\right)+\tan\left(\dfrac{\pi}{4}-\theta\right)}$ is equal to

(a) $\sin 2\theta$ (b) $\cos 2\theta$
(c) $-\sin 2\theta$ (d) $-\cos 2\theta$

2. If $\cos\theta = 3\cos\varphi$ and $\cot\left(\dfrac{\phi+\theta}{2}\right)=k\tan\dfrac{\phi-\theta}{2}$, then the value of k is

(a) 1 (b) 1/2
(c) 2 (d) 3

3. If α and β be between 0 and $\pi/4$ and if $\cos(\alpha+\beta) = 12/13$ and $\sin(\alpha-\beta) = 3/5$, then $\sin 2\alpha$ is equal to

(a) 16/15 (b) 0
(c) 56/65 (d) 64/65

4. If $x = y\cos\dfrac{2\pi}{3} = z\cos\dfrac{4\pi}{3}$, then $xy + yz + zx$ is equal to ***[EAMCET-1994]***

(a) –1 (b) 0
(c) 1 (d) 2

5. If $x - y = (4n+1)\dfrac{\pi}{4}$ (where n is an integer) and $x + y$ is not an odd multiple of $\pi/2$, then $\dfrac{\sin 2x-\sin 2y}{\cos 2x+\cos 2y}$ is equal to ***[NDA-2004]***

(a) –1 (b) 0
(c) 1/2 (d) 1

6. In a triangle ABC, if $\cos A = \cos B \cos C$, then $\tan A - \tan B - \tan C$ is equal to ***[NDA-2006]***

(a) 0
(b) 1
(c) $1 + \tan A \tan B \tan C$
(d) $\tan A \tan B \tan C -$

7. If $\cos(\alpha+\beta)=\dfrac{4}{5}$ and $\sin(\alpha-\beta)=\dfrac{5}{13}$ where α and β lie in $\left[0,\dfrac{\pi}{4}\right]$, find the value of $\tan 2\alpha$. ***[IIT-JEE-1979]***

(a) $\dfrac{56}{33}$ (b) $\dfrac{65}{33}$
(c) $\dfrac{33}{65}$ (d) None of these

8. The value of $\cos 15° - \sin 15°$ is

(a) $\dfrac{1}{\sqrt{2}}$ (b) $-\dfrac{1}{\sqrt{2}}$
(c) $\dfrac{1}{2\sqrt{2}}$ (d) 0

9. The value of $\cos 57° + \sin 27°$ is equal to

(a) $\cos 30°$ (b) $\cos 3°$
(c) $\sin 3°$ (d) $\sin 30°$

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ
6. ⓐ ⓑ ⓒ ⓓ
7. ⓐ ⓑ ⓒ ⓓ
8. ⓐ ⓑ ⓒ ⓓ
9. ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

1. (a) Step-1: $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$,

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

Step-2: $$\frac{\tan\left(\frac{\pi}{4}+\theta\right)-\tan\left(\frac{\pi}{4}-\theta\right)}{\tan\left(\frac{\pi}{4}+\theta\right)+\tan\left(\frac{\pi}{4}-\theta\right)}$$

$$=\frac{\dfrac{1+\tan\theta}{1-\tan\theta}-\dfrac{1-\tan\theta}{1+\tan\theta}}{\dfrac{1+\tan\theta}{1-\tan\theta}+\dfrac{1-\tan\theta}{1+\tan\theta}}$$

$$\Rightarrow\ \frac{\dfrac{(1+\tan\theta)^2-(1-\tan\theta)^2}{(1-\tan\theta)(1+\tan\theta)}}{\dfrac{(1+\tan\theta)^2+(1-\tan\theta)^2}{(1-\tan\theta)(1+\tan\theta)}}$$

$$=\frac{4\tan\theta}{2(1+\tan^2\theta)}=\frac{2\tan\theta}{\sec\theta.\sec\theta}$$

$\Rightarrow$ $2\sin\theta\,.\,\cos\theta=\sin 2\theta$

or

Verification Method

Put $\theta = 15°$, then given expression is:

$$\frac{\tan 60°-\tan 30°}{\tan 60°+\tan 30°}=\frac{\sqrt{3}-\dfrac{1}{\sqrt{3}}}{\sqrt{3}+\dfrac{1}{\sqrt{3}}}=\frac{\dfrac{3-1}{\sqrt{3}}}{\dfrac{3+1}{\sqrt{3}}}=\frac{1}{2}.$$

Also put, $\theta = 15°$ in the given four options:

(a) $\sin 30°=\dfrac{1}{2}$ (b) $\dfrac{\sqrt{3}}{2}$

(c) $-\dfrac{1}{2}$ (d) $-\dfrac{\sqrt{3}}{2}$

Clearly, (a) is correct option.

3. (c) Step-1: Using $2\alpha = (\alpha + \beta) + (\alpha - \beta)$,

$$\cos(\alpha-\beta)=\frac{4}{5}$$

$$\sin(\alpha+\beta)=\frac{5}{13}$$

Setp-2: $\sin 2\alpha = \sin\{(\alpha+\beta)+(\alpha-\beta)\}$

$= \sin(\alpha+\beta)\cos(\alpha-\beta) + \cos(\alpha+\beta)\sin(\alpha-\beta)$

$$=\frac{5}{13}\times\frac{4}{5}+\frac{12}{13}\times\frac{3}{5}=\frac{20+36}{65}=\frac{56}{65}$$

5. (d) $\tan(x-y) = \tan(n\pi + \pi/4) = 1$

6. (a) $\tan A-\left[\dfrac{\sin B}{\cos B}+\dfrac{\sin C}{\cos C}\right]$

$$=\tan A-\frac{\sin(B+C)}{\cos B\cos C}$$

$$=\tan A-\frac{\sin A}{\cos A}=0\quad[\because \cos B\cos C=\cos A]$$

7. (a) $\cos(\alpha+\beta)=\dfrac{4}{5}$ $\Rightarrow \tan(\alpha+\beta)=\dfrac{3}{4}$

$\sin(\alpha-\beta)=\dfrac{5}{13}$ $\Rightarrow \tan(\alpha-\beta)=\dfrac{5}{12}$

$\tan 2\alpha = \tan(\alpha+\beta+\alpha-\beta)$

$$= \frac{\tan(\alpha+\beta)+\tan(\alpha-\beta)}{1-\tan(\alpha+\beta)\tan(\alpha-\beta)}$$

$$= \frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\cdot\frac{5}{12}} = \frac{56}{33}$$

8. cos 15° – sin 15° = cos 15° – cos 75°

$$= 2\sin\frac{75°+15°}{2}\sin\frac{75°-15°}{2}$$

9. cos 57° + sin 57° = cos 57° + cos (90° – 27°)

= cos 57° + cos 63° = 2 cos 60° cos 3°

LECTURE

6

Trigonometric Ratios of Multiple and Sub-Multiple Angles

BASIC CONCEPTS

1. FORMULAE OF DOUBLE, TRIPLE AND HALF ANGLES

(i) $\sin 2\theta = 2\sin\theta\cos\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$
$= (\sin\theta + \cos\theta)^2 - 1 = 1 - (\sin\theta - \cos\theta)^2$

(ii) $\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= \dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \dfrac{1}{\sec 2\theta}$
$\cos 2\theta - 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

(iii) $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$; $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

(iv) $\tan 2\theta \dfrac{2\tan\theta}{1-\tan^2\theta}$; $\cos 2\theta = \dfrac{\cot^2\theta - 1}{2\cot\theta}$

(v) $\cot A + \tan A = \dfrac{1}{\sin A\cos A} = \dfrac{2}{\sin 2A}$
$= 2\operatorname{cosec} 2A$

(vi) $\cot A - \tan A = 2\cot 2A$

(vii) $\sin\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1-\cos\theta}{2}}$; $\cos\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1+\cos\theta}{2}}$
$\sin\theta = 8\sin\dfrac{\theta}{8}\cos\dfrac{\theta}{2}\cos\dfrac{\theta}{4}\cos\dfrac{\theta}{8}$

(viii) $\tan\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} = \operatorname{cosec}\theta - \cot\theta$
$= \dfrac{1}{\operatorname{cosec}\theta + \cot\theta}$

(ix) $\tan\dfrac{\theta}{2} = \dfrac{1-\cos\theta}{\sin\theta}$

(x) $\cot\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}}$

(xi) $\cot\dfrac{\theta}{2} = \dfrac{1+\cos\theta}{\sin\theta} = \operatorname{cosec}\theta + \cot\theta$
$= \dfrac{1}{\operatorname{cosec}\theta - \cot\theta}$

(xii) $\dfrac{1}{\sec\theta - \tan\theta} = \tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)$

(xiii) $\dfrac{1}{\sec\theta + \tan\theta} = \tan\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)$

(xiv) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, $\sin^3\theta$
$= \dfrac{3\sin\theta - \sin 3\theta}{4}$

(xv) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, $\cos^3\theta$
$= \dfrac{\cos 3\theta + 3\cos\theta}{4}$

(xvi) $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

(xvii) $\tan 3\theta - \tan 2\theta - \tan\theta = \tan 3\theta \tan 2\theta \tan\theta$

(xviii) $4(\sin\theta\cos^3\theta) - \cos\theta\sin^3\theta) = \sin 4\theta$

(xix) $8\cos^4\theta - 8\cos^2\theta + 1 = 1 - 8\sin^2\theta\cos^2\theta = \cos^4\theta$

(xx) $8\sin^4\theta - 8\sin^2\theta + 1 = \cos 4\theta = 1 - 2\sin^2 2\theta$

(xxi) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(xxii) $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

(xxiii) coversed sin AOP = 1 – sin AOP

(xxiv) Versed sin AOP = 1 – cos AOP

2. SOME USEFUL RESULTS

(i) $\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta} = \dfrac{1+\tan\theta}{1-\tan\theta} = \tan\left(\dfrac{\pi}{4}+\theta\right)$

$= \sec 2\theta + \tan 2\theta$

(ii) $\dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta} = \dfrac{1-\tan\theta}{1+\tan\theta} = \tan\left(\dfrac{\pi}{4}-\theta\right)$

$= \sec 2\theta - \tan 2\theta$

(iii) $\tan\left(\dfrac{\pi}{4}+\theta\right)\tan\left(\dfrac{\pi}{4}-\theta\right) = 1$

(iv) $\tan\left(\dfrac{\pi}{4}+\theta\right) - \tan\left(\dfrac{\pi}{4}-\theta\right) = 2\tan 2\theta$

(v) $\tan\left(\dfrac{\pi}{4}+\theta\right) + \tan\left(\dfrac{\pi}{4}-\theta\right) = 2\sec 2\theta$

(vi) $\cos\theta \,.\, \cos 2\theta \,.\, \cos 2^2\theta \,\ldots\, \cos 2^{n-1}\theta = \dfrac{\sin 2^n\theta}{2^n \sin\theta}$

(vii) $\sin\theta \sin(60-\theta)\sin(60+\theta) = \dfrac{1}{4}\sin 3\theta$

(viii) $\cos\theta\cos(60-\theta)\cos(60+\theta) = \dfrac{1}{4}\cos 3\theta$

(ix) $\tan\theta\tan(60-\theta)\tan(60+\theta) = \tan 3\theta$

(x) $\tan\theta + \tan(\theta - \pi/3) + \tan(\theta + \pi/3) = 3\tan 3\theta$

(xi) $\tan\theta + \tan(\theta + \pi/3) + \tan(\theta + 2\pi/3) = 3\tan 3\theta$

(xii) $\cos 36° - \cos 72° = 1/2$

(xiii) $\cos 36° \,.\, \cos 72° = 1/4$

(xiv) $\sin^2\theta + \cos^4\theta = (4 - \sin^2 2\theta)/4$

(xv) $\sin^4\theta + \cos^4\theta = (2 - \sin^2 2\theta)/2$

(xvi) $\sin^6\theta + \cos^6\theta = (4 - 3\sin^2 2\theta)/4$

(xvii) $\sin^4\theta - \cos^4\theta = -\cos 2\theta$

(xviii) $\sin^4\theta + \sin^4\left(\dfrac{\pi}{2}-\theta\right) + \sin^4\left(\dfrac{\pi}{2}+\theta\right) + \sin^4(\pi-\theta) = (2 - \sin^2 2\theta)$

(xix) $\sin^6\theta + \sin^6\left(\dfrac{\pi}{2}-\theta\right) + \sin^6\left(\dfrac{\pi}{2}+\theta\right) + \sin^6(\pi-\theta) = \dfrac{4-3\sin^2 2\theta}{2}$

(xx) $\sin 18° = \dfrac{\sqrt{5}-1}{4} = \cos 72°$, $\cos 18° = \dfrac{\sqrt{10+2\sqrt{5}}}{4} = \sin 72°$

(xxi) $\sin 36° = \dfrac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54°$,

$\cos 36° = \dfrac{\sqrt{5}+1}{4} = \sin 54°$

(xxii) $\sin 22\dfrac{1°}{2} = \dfrac{1}{2}\left(\sqrt{2-\sqrt{2}}\right)$,

$\cos 22\dfrac{1°}{2} = \dfrac{1}{2}\left(\sqrt{2+\sqrt{2}}\right)$

(xxiii) $\tan 22\dfrac{1°}{2} = \sqrt{2}-1$,

$\cot 22\dfrac{1°}{2} = \sqrt{2}+1 = \tan\left(67\dfrac{1°}{2}\right)$

(xxiv) $\tan\left(7\dfrac{1}{2}^{\circ}\right) = \sqrt{2}-\sqrt{3}-\sqrt{4}+\sqrt{6} = \cot\left(82\dfrac{1}{2}^{\circ}\right)$

(xxv) $\cot\left(7\dfrac{1}{2}^{\circ}\right) = \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6} = \tan\left(82\dfrac{1}{2}^{\circ}\right)$

3. RESULTS INVOLVING SIN $\dfrac{A}{2}$ AND COS $\dfrac{A}{2}$

(i) $\left(\sin\dfrac{A}{2}+\cos\dfrac{A}{2}\right)^2 = 1+\sin A$

(ii) $\left(\sin\dfrac{A}{2}-\cos\dfrac{A}{2}\right)^2 = 1-\sin A$

(iii) $\sin\dfrac{A}{2}+\cos\dfrac{A}{2} = \pm\sqrt{1+\sin A}$

(iv) $\sin\dfrac{A}{2}-\cos\dfrac{A}{2} = \pm\sqrt{1-\sin A}$

(v) $2\sin\dfrac{A}{2} = \pm\sqrt{1+\sin A} \pm \sqrt{1-\sin A}$ ...(1)

(vi) $2\cos\dfrac{A}{2} = \pm\sqrt{1+\sin A} \mp \sqrt{1-\sin A}$ ...(2)

(vii) $\sin\dfrac{A}{2}+\cos\dfrac{A}{2} = \sqrt{2}\sin\left(\dfrac{A}{2}+\dfrac{\pi}{4}\right)$

(viii) $2\sin^2\dfrac{A}{2} = 1-\cos A$

(ix) $2\cos^2\dfrac{A}{2} = 1+\cos A$

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Prove that cot $7\frac{1°}{2}=(\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$

$$=\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$$

or $\tan 82\frac{1°}{2}=(\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$.

Solution

$$\tan 82\frac{1°}{2}=\tan\left(90°-7\frac{1°}{2}\right)=\cot 7\frac{1°}{2}=\cot A$$

say, where $A=7\frac{1°}{2}=$. Now $\cot A=\frac{\cos A}{\sin A}$

$$=\frac{2\cos^2 A}{2\sin A\cos A}$$

$$=\frac{1+\cos 2A}{\sin 2A}=\frac{1+\cos 15°}{\sin 15°}, \cot 7\frac{1°}{2}$$

$$=\frac{1+\cos(45°-30°)}{\sin(45°-30°)}$$

$$=\frac{1+\left(\frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}.\frac{1}{2}\right)}{\frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}.\frac{1}{2}},$$

$$=\frac{2\sqrt{2}+(\sqrt{3}+1)}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$=\frac{2\sqrt{2}(\sqrt{3}+1)+(\sqrt{3}+1)^2}{3-1},$$

$$=\frac{2\sqrt{6}+2\sqrt{2}+4+2\sqrt{3}}{2}$$

$$\sqrt{6}+\sqrt{2}+2+\sqrt{3}=\sqrt{2}(\sqrt{2}+1)+\sqrt{3}(\sqrt{2}+1)$$

$$=(\sqrt{2}+1)(\sqrt{3}+\sqrt{2})$$

2. Prove that cot A – tan A = 2 cot 2A. Deduct that tan α + 2 tan 2 α + 4 tan 4 α + 8 cot 8 α = cot α. ***[IIT-1988; MPPET-2006]***

Solution

Now, $\cot A-\tan A=\frac{\cos A}{\sin A}-\frac{\sin A}{\cos A}$

$$=\frac{\cos^2 A-\sin^2 A}{\sin A\cos A}=\frac{\cos 2A}{\frac{1}{2}\sin 2A}=2\cot 2A$$

i.e., cot A – tan A = 2 cot 2A ....(1)

Put A = α, 2 α and 4 α in (1) succession to obtain.

cot α – tan α = 2 cot 2 α ....(2)

cot 2α – tan 2α = 2 cot 4 α ....(3)

cot 4α – tan 4α = 2 cot 8 α ....(4)

Multiply (3) by (2), (4) by (4) and add these to (2) cot α – tan α – 2tan 2α – 4tan 4α = 8 cot 8 α

⇒ tan α + 2 tan 2 α + 4 tan 4 α + 8 cot 8 α = cot α, as desired

3. If $\cos 2\alpha=\frac{3\cos 2\beta-1}{3-\cos 2\beta}$, then $\tan\alpha=\sqrt{2}\tan\beta$.

Solution

We know that $\cos 2A=\frac{1-\tan^2 A}{1+\tan^2 A}$

Using this formula we get

$$\frac{1-\tan^2\alpha}{1+\tan^2\alpha}=\frac{3.\frac{1-\tan^2\beta}{1+\tan^2\beta}-1}{3-\frac{1-\tan^2\beta}{1+\tan^2\beta}}=\frac{1-2\tan^2\beta}{1+2\tan^2\beta}$$

Applying componendo and dividendo, we get

$$\frac{2\tan^2\alpha}{2}=\frac{4\tan^2\beta}{2}\quad \therefore \tan^2\alpha=2\tan^2\beta$$

or tan α = $\sqrt{2}\tan\beta$.

4. Prove that $\tan A+\tan(60°+A)-\tan(60°-A)$ = 3tan 3A.

Solution

We know that tan60° = $\sqrt{3}$,

$$\text{L.H.S.}=\tan A+\frac{\sqrt{3}+\tan A}{1-\sqrt{3}\tan A}-\frac{\sqrt{3}-\tan A}{1+\sqrt{3}\tan A}$$

$$= \tan A + \frac{(\sqrt{3} + \tan A)(1 + \sqrt{3}\tan A) - (\sqrt{3} - \tan A)(1 - \sqrt{3}\tan A)}{1 - 3\tan^2 A}$$

$$= \tan A + \frac{8\tan A}{1 - 3\tan^2 A} = \frac{9\tan A - 3\tan^3 A}{1 - 3\tan^2 A}$$

$$= 3.\frac{(3\tan A - \tan^3 A)}{1 - 3\tan^2 A} = 3\tan 3A$$

5. Prove that (2cosθ –1) (2cos 2θ– 1) (2cos2^2 θ – 1) ....(2cos2^{n-1} θ – 1) $= \frac{2\cos 2^n\theta + 1}{2\cos\theta + 1}$.

Solution

Take 2 cos θ + 1 from R.H.S. to L.H.S. and 4 cos² θ – 1 = 2(1 + cos 2 θ) – 1 = 2 cos 2 θ + 1 and again multiply with 2nd factor and continue like this.

6. If cos θ = $\frac{a\cos\phi + b}{a + b\cos\phi}$, prove that

tan (θ/2) $= \sqrt{[(a-b)/(a+b)]}$ tan (φ/2)

Solution

$$\cos\theta = \frac{a\cos\phi + b}{a + b\cos\phi}, \therefore \frac{1 - \tan^2(\theta/2)}{1 + \tan^2(\theta/2)}$$

$$= \frac{a\{1 - \tan^2(\phi/2)\} + b\{1 + \tan^2(\phi/2)\}}{a\{1 + \tan^2(\phi/2)\} + b\{1 - \tan^2(\phi/2)\}} \text{ or}$$

$$= \frac{1 - \tan^2(\theta/2)}{1 + \tan^2(\theta/2)} = \frac{(a+b) - (a-b)\tan^2(\phi/2)}{(a+b) + (a-b)\tan^2(\phi/2)}$$

Apply componendo and dividendo

$$\frac{2\tan^2(\theta/2)}{2} = \frac{2(a-b)\tan^2(\phi/2)}{2(a+b)}$$

$$\therefore \quad \tan(\theta/2) = \sqrt{(a-b)/(a+b)}\,\tan(\phi/2)$$

7. Prove that $\frac{1}{\cos 290°} + \frac{1}{\sqrt{3}\sin 250°} = \frac{4}{\sqrt{3}}$.

Solution

$$\text{L.H.S.} = \frac{1}{\cos(270° + 20°)} + \frac{1}{\sqrt{3}\sin(270° - 20°)}$$

$$= \frac{1}{\sin 20°} + \frac{1}{\sqrt{3}(-\cos 20°)}$$

$$= \frac{\sqrt{3}\cos 20° - \sin 20°}{\sqrt{3}\sin 20°\cos 20°}$$

$$= \frac{\frac{\sqrt{3}}{2}\cos 20° - \frac{1}{2}\sin 20°}{\frac{\sqrt{3}}{4}\sin 40°}$$

$$= \frac{\sin 60°\cos 20° - \cos 60°\sin 20°}{\frac{\sqrt{3}}{4}\sin 40°}$$

$$= \frac{\sin 40°}{\frac{\sqrt{3}}{4}\sin 40°} = \frac{4}{\sqrt{3}}$$

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): SOLVE THESE PROBLEMS TO GRASP THE TOPIC

EXERCISE 1

1. If α and β are the solutions of a cos θ + b sin θ = C, then show that

(i) $\cos(\alpha + \beta) = \frac{a^2 - b^2}{a^2 + b^2}$

(ii) $\cos(\alpha - \beta) = \frac{2C^2 - (a^2 + b^2)}{a^2 + b^2}$

2. Prove that, tan 2 α – tan α = tan α sec 2 α

3. Prove that, sin 20° sin 40° sin 60° sin 80° $= \frac{3}{16}$

4. Prove that, tan 20° tan 40° tan 80° = tan 60°

5. Prove that,

$(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\left(\frac{\alpha - \beta}{2}\right)$

6. Prove that,

$\sin 10° \sin 50° \sin 60° \sin 70° = \dfrac{\sqrt{3}}{16}$.

7. Prove that,

(i) $\dfrac{\sin 2x}{1-\cos 2x} = \cot x$

(ii) $\dfrac{1-\cos 2x}{1+\cos 2x} = \tan^2 x$

8. Prove that, $\dfrac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} = \dfrac{-\sin 2x}{\cos 10x}$

9. If $\tan x = \left(\dfrac{-3}{4}\right)$ and $\dfrac{3\pi}{2} < x < 2\pi$ find the values of

(i) $\sin 2x$

(ii) $\cos 2x$

(iii) $\tan 2x$

10. If $\tan x = \dfrac{1}{7}$ and $\tan y = \dfrac{1}{3}$, show that $\cos 2x = \sin 4y$.

11. Prove that, $\cos 4x = 1 - 8 \sin^2 x \cos^2 x$

12. Find the value of:

(i) $\sin 18°$ (ii) $\cos 18°$

(iii) $\cos 36°$ (iv) $\sin 36°$

(v) $\sin 72°$ (vi) $\cos 72°$

(vii) $\sin 54°$ (viii) $\cos 54°$

13. Prove that, $\sin\dfrac{\pi}{5}\sin\dfrac{2\pi}{5}\sin\dfrac{3\pi}{5}\sin\dfrac{4\pi}{5} = \dfrac{5}{16}$

14. Find the value of:

(i) $\sin 22° 30'$

(ii) $\cos 22° 30'$

(iii) $\tan 22° 30'$

15. If $\cos x = -1/3$ and x lies in Quadrant III, find the values of:

(i) $\sin\dfrac{x}{2}$

(ii) $\cos\dfrac{x}{2}$

(iii) $\tan\dfrac{x}{2}$

EXERCISE 2

1. Prove that, $\sin 10° \sin 30° \sin 50° \sin 70° = \dfrac{1}{16}$

2. (i) If $\sin x = \dfrac{1}{3}$, find the value of $\sin 3x$

(ii) If $\cos x = \dfrac{1}{2}$, find the value f $\cos 3x$

3. If $\cos x = \dfrac{4}{5}$ and x is acute, find the value of $\tan 2x$.

4. Prove that, $\dfrac{1-\sin 2x}{1+\sin 2x} = \tan^2\left(\dfrac{\pi}{4} - x\right)$.

5. Prove that,

$\dfrac{\cos x + \sin x}{\cos x - \sin x} - \dfrac{\cos x - \sin x}{\cos x + \sin x} = 2\tan 2x$.

6. If $\tan x = \dfrac{3}{4}$ and $\pi < x < \dfrac{3\pi}{2}$, find the values of

(i) $\sin\dfrac{x}{2}$ (ii) $\cos\dfrac{x}{2}$ (iii) $\tan\dfrac{x}{2}$

7. Prove that, $\dfrac{\cos x}{(1-\sin x)} = \tan\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)$

8. Prove that, $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$.

9. Prove that, $\dfrac{1+\sin 2\theta + \cos 2\theta}{1+\sin 2\theta - \cos 2\theta} = \cot\theta$.

10. Prove that, $\dfrac{1+\sin\theta - \cos\theta}{1+\sin\theta + \cos\theta} = \tan\left(\dfrac{\theta}{2}\right)$.

11. Prove that, $\dfrac{\cos 2\theta}{1+\sin 2\theta} = \tan\left(\dfrac{\pi}{4} - \theta\right)$.

12. Prove that, $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}} = 2\cos\theta$.

ANSWERS

EXERCISE 1

9. (i) $\frac{-24}{25}$

(ii) $\frac{7}{25}$

(iii) $\frac{-24}{7}$

12. (i) $\frac{(\sqrt{5}-1)}{4}$

(ii) $\frac{\sqrt{10+2\sqrt{5}}}{4}$

(iii) $\frac{(\sqrt{5}+1)}{4}$

(vi) $\frac{\sqrt{10-2\sqrt{5}}}{4}$

(v) $\frac{\sqrt{10+2\sqrt{5}}}{4}$

(vi) $\frac{(\sqrt{5}-1)}{4}$

(vii) $\frac{(\sqrt{5}+1)}{4}$

(viii) $\frac{\sqrt{10-2\sqrt{5}}}{4}$

14. (i) $\sqrt{\frac{(\sqrt{2}-1)}{2\sqrt{2}}}$

(ii) $\sqrt{\frac{(\sqrt{2}+1)}{2\sqrt{2}}}$

(iii) $(\sqrt{2}-1)$

15. (i) $\sqrt{6}/3$

(ii) $-\sqrt{3}/3$

(iii) $-\sqrt{2}$

EXERCISE 2

2. (i) $\frac{23}{27}$ (ii) -1

3. $\frac{24}{7}$

6. (i) $\frac{3}{\sqrt{10}}$ (ii) $\frac{-1}{\sqrt{10}}$

(iii) -3

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. If n = 1, 2, 3, ..., then $\cos A \cos 2A \cos 4A \cos 8A \ldots \cos 2^{n-1} A$ is equal to

[MP PET-2005]

(a) $\frac{1}{2^n \sin A}\sin(2^n A)$ (b) $\frac{\sin 2nA}{2n \sin A}$

(c) $\frac{\sin 2^n A}{2^n \sin 2^{n-1}} A$ (d) $\frac{\sin 4^{n-1} A}{4^{n-1}\sin A}$

Solution

(a) $\cos A \cos 2A \cos 4A \cos 8A \ldots \cos 2^{n-1} A$

Each angle being double of proceeding

Multiply above and below by $2^n \sin A$

$= \frac{2^{n-1}}{2^n \sin A}$

$[2\sin A \cos A \cos 2A \cos 4A \ldots \cos 2^{n-1} A]$

$= \frac{2^{n-2}}{2^n \sin A}$

$[2\sin 2A \cos 2A \cos 4A \ldots \cos 2^{n-1} A]$

$= \frac{2^{n-3}}{2^n \sin A}[2\sin 4A \cos 4A \ldots\ldots\ldots \cos 2^{n-1} A]$

$= \frac{1}{2^n \sin A}$

$[2\sin 2^{n-1} A \cos 2^{n-1} A]$

$= \frac{1}{2^n \sin A}$

$\sin(2.2^{n-1} A) = \frac{\sin(2^n A)}{2^n \sin(A)}$

2. $\sin\frac{\pi}{14}.\sin\frac{3\pi}{14}.\sin\frac{5\pi}{14}.\sin\frac{7\pi}{14}.\sin\frac{9\pi}{14}.$ $\sin\frac{11\pi}{14}.\sin\frac{13\pi}{14}$ is equal to

[IIT-91; MNR-92]

(a) 1/64 (b) 1/32

(c) 1/16 (d) 1/8

Solution

(a) $\sin\frac{\pi}{14}.\sin\frac{3\pi}{14}.\sin\frac{5\pi}{14}.1$

$\sin\left(\pi-\frac{5\pi}{14}\right)\sin\left(\pi-\frac{3\pi}{14}\right)\sin\left(\pi-\frac{\pi}{14}\right)$

$=\left(\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\right)^2.1$

$=\left\{\cos\left(\frac{\pi}{2}-\frac{\pi}{14}\right)\cos\left(\frac{\pi}{2}-\frac{3\pi}{14}\right)\cos\left(\frac{\pi}{2}-\frac{5\pi}{14}\right)\right\}^2$

$=\left(\cos\frac{3\pi}{7}\cos\frac{2\pi}{7}\cos\frac{\pi}{7}\right)^2$

$=\left(\cos\frac{\pi}{7}.\cos\frac{2\pi}{7}.\cos\frac{3\pi}{7}\right)^2$

$=\left(-\cos\frac{\pi}{7}.\cos\frac{2\pi}{7}.\cos\frac{4\pi}{7}\right)^2=\left[-\frac{\sin 2^3\pi/7}{2^3\sin\pi/7}\right]^2$

$\left[\because \cos\theta.\cos 2\theta.\cos 2^2\theta....\cos 2^{n-1}\theta=\frac{\sin 2^n\theta}{2^n\sin\theta}\right]$

$=\frac{1}{64}\left(\frac{\sin 8\pi/7}{\sin\pi/7}\right)^2=\frac{1}{64}$

3. If $\tan\theta+\tan\left(\theta+\frac{\pi}{3}\right)+\tan\left(\theta+\frac{2\pi}{3}\right)=3$, then

(a) $\tan 2\theta = 1$ (b) $\tan 3\theta = 1$
(c) $\tan^2\theta = 1$ (d) $\tan^3\theta = 1$

Solution

(b) From formula,

$\tan\theta+\tan\left(\theta+\frac{\pi}{3}\right)+\tan\left(\theta+\frac{2\pi}{3}\right)=3\tan 3\theta$

$\Rightarrow\ 3\tan 3\theta = 3 \Rightarrow \tan 3\theta = 1$

4. What is the minimum value of $\cos\theta + \cos 2\theta$? ***[NDA-2007]***

(a) -2 (b) $-9/8$
(c) 0 (d) $-9/16$

Solution

(b) $\cos\theta + \cos 2\theta = 2\cos^2\theta + \cos\theta - 1$

$a = 2, b = 1, c = -1$

$\therefore$ Minimum value $=\frac{4ac-b^2}{4a}=\frac{-8-1}{8}=\frac{-9}{8}$

or $2\cos^2\theta + \cos\theta - 1 =$

$2\left\{\cos^2\theta+\frac{1}{2}\cos\theta-\frac{1}{2}\right\}$

$=2\left\{\cos^2\theta+\frac{1}{2}\cos\theta+\frac{1}{16}-\frac{1}{2}-\frac{1}{16}\right\}$

$=2\left\{\left(\cos\theta+\frac{1}{4}\right)^2-\frac{1}{2}-\frac{1}{16}\right\}$

Minimum value

$=2\left\{0-\frac{1}{2}-\frac{1}{16}\right\}=2\left\{\frac{-8-1}{16}\right\}=-\frac{9}{8}.$

5. The value of $\sin\frac{\pi}{16}.\sin\frac{3\pi}{16}.\sin\frac{5\pi}{16}.\sin\frac{7\pi}{16}$ is ***[MP PET-2004]***

(a) $\frac{\sqrt{2}}{16}$ (b) $\frac{1}{8}$
(c) $\frac{1}{16}$ (d) $\frac{\sqrt{2}}{32}$

Solution

(a) $\sin\frac{\pi}{16}.\sin\frac{3\pi}{16}.\sin\frac{5\pi}{16}.\sin\frac{7\pi}{16}$

$=\frac{1}{2}\left[2\sin\frac{5\pi}{16}\sin\frac{3\pi}{16}\right]\times\frac{1}{2}\left[2\sin\frac{7\pi}{16}\sin\frac{\pi}{16}\right]$

$=\frac{1}{4}\left[\left(\cos\frac{\pi}{8}-\cos\frac{\pi}{2}\right)\left(\cos\frac{3\pi}{8}-\cos\frac{\pi}{2}\right)\right]$

$=\frac{1}{4\times 2}\left(\cos\frac{\pi}{2}+\cos\frac{\pi}{4}\right)$

$=\frac{1}{8}\left(\frac{1}{\sqrt{2}}\right)=\frac{1}{8\sqrt{2}}=\frac{\sqrt{2}}{16}$

$\left[\because \cos\frac{\pi}{2}=0\right]$

6. If $x+\frac{1}{x}=2\cos\theta$, then $x^3+\frac{1}{x^3}$ is equal to ***[MP PET-2004]***

(a) $\sin 3\theta$ (b) $2\sin 3\theta$
(c) $\cos 3\theta$ (d) $2\cos 3\theta$

Solution

(d) We have, $x+\frac{1}{x}=2\cos\theta$

or $\left(x+\frac{1}{x}\right)^3=(2\cos\theta)^3$

$$x^3+\frac{1}{x^3}+3x.\frac{1}{x}\left(x+\frac{1}{x}\right)=8\cos^3\theta$$

$$x^3+\frac{1}{x^3}+3.2\cos\theta=8\cos^3\theta$$

$$\left[\because\ x+\frac{1}{x}=2\cos\theta\ (\text{given})\right]$$

$$\Rightarrow\quad x^3+\frac{1}{x^3}=2(4\cos^3\theta-3\cos\theta)=2\cos3\theta$$

[$\because$ $4\cos^3\theta - 3\cos\theta = \cos 3\theta$]

7. If $3\sin 2\theta = 2\sin 3\theta$ and $0 < \theta < \pi$, then value of $\sin\theta$ is

[K.U.K.C.E.E.T.-1993; T.S. Rajendra-1992]

(a) $\frac{\sqrt{2}}{3}$ (b) $\frac{\sqrt{3}}{\sqrt{5}}$

(c) $\frac{\sqrt{15}}{4}$ (d) $\frac{\sqrt{2}}{\sqrt{5}}$

Solution

(c) $3\sin 2\theta = 2\sin 3\theta$

$\Rightarrow$ $6\sin\theta\cos\theta = 2(3\sin\theta - 4\sin^3\theta)$

$\Rightarrow$ $4\cos^2\theta - 3\cos\theta - 1 = 0 \Rightarrow \cos\theta = 1, \frac{1}{4}$

Since, $\cos\theta \neq 1$ ($\because$ $\theta \neq 0$)

$$\therefore\quad \cos\theta=\frac{1}{4}-\sin\theta=\frac{\sqrt{15}}{4}$$

8. $\cos^2\frac{3\pi}{5}+\cos^2\frac{4\pi}{5}$ is equal to

[E.A.M.C.E.T-1996]

(a) 4/5 (b) 5/2

(c) 5/4 (d) 3/4

Solution

(d) $\cos^2\frac{3\pi}{5}+\cos^2\frac{4\pi}{5}=\cos^2 108°+\cos^2 144°$

$= \sin^2 18° + \sin^2 36°$

$$=\left(\frac{\sqrt{5}-1}{4}\right)^2+\left(\frac{\sqrt{5}+1}{4}\right)^2$$

$$=\frac{5+1-2\sqrt{5}}{16}+\frac{5+1+2\sqrt{5}}{16}=\frac{12}{16}=\frac{3}{4}$$

9. If a is any real number, then the number of roots of $\cot x - \tan x = a$ in the first quadrant are

[EAMCET-1995]

(a) 2 (b) 0

(c) 1 (d) None

Solution

(c) $\cot x - \tan x = a$

$$\Rightarrow\quad \frac{\cos x}{\sin x}-\frac{\sin x}{\cos x}=a$$

$$\Rightarrow\quad \frac{\cos^2 x-\sin^2 x}{\sin x\cos x}=a$$

$$\Rightarrow\quad \frac{2\cos 2x}{\sin 2x}=a$$

$$\Rightarrow\quad \tan 2x=\frac{2}{a}$$

$\therefore$ x has only one value in the first quadrant.

10. The value of $\cot 70° + 4\cos 70°$ is

[Orissa. JEE-2003]

(a) $1/\sqrt{3}$ (b) $\sqrt{3}$

(c) $2\sqrt{3}$ (d) 1/2

Solution

(b) Now, $\cot 70° + 4\cos 70°$

$$=\frac{\cos 70°+4\sin 70°\cos 70°}{\sin 70°}$$

$$=\frac{\cos 70°+2\sin 140°}{\sin 70°}$$

$$=\frac{\cos 70°+2\sin(180°-40°)}{\sin 70°}$$

$$=\frac{\sin 20°+\sin 40°+\sin 40°}{\sin 70°}$$

$$=\frac{2\sin 30°\cos 10°+\sin 40°}{\sin 70°}$$

$$=\frac{\sin 80°+\sin 40°}{\sin 70°}=\frac{2\sin 60°\cos 20°}{\sin 70°}=\sqrt{3}$$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. $\cos^2 76° + \cos^2 16° - \cos 76° \cos 16°$ is equal to *[EAMCET-2002]*

(a) −1/4 (b) 1/2
(c) 0 (d) 3/4

2. $8\sin\frac{x}{8}\cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}$ is equal to

(a) $\sin x$ (b) $8 \sin x$
(c) $\cos x$ (d) $8 \cos x$

3. $\frac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta}$ is equal to

(a) $\frac{1}{2}\tan\theta$ (b) $\frac{1}{2}\cot\theta$
(c) $\tan\theta$ (d) $\cot\theta$

4. If $\cos\alpha + \cos\beta = 0 = \sin\alpha + \sin\beta$, then $\cos 2\alpha + \cos 2\beta$ is equal to *[EAMCET-1994]*

(a) $-2\sin(\alpha+\beta)$ (b) $-2\cos(\alpha+\beta)$
(c) $2\sin(\alpha+\beta)$ (d) $2\cos(\alpha+\beta)$

5. $2\cos x - \cos 3x - \cos 5x$ is equal to

(a) $16\cos^3 x \sin^2 x$ (b) $16\sin^3 x \cos^2 x$
(c) $4\cos^3 x \sin^2 x$ (d) $4\sin^3 x \sin^2 x$

6. If the solution of $a\cos 2\theta + b\sin 2\theta = c$ is α and β then the value of $\tan\alpha + \tan\beta$ is *[Kurukshetra CEE-1998]*

(a) $\frac{c+a}{2b}$ (b) $\frac{2b}{c+a}$
(c) $\frac{c-a}{2b}$ (d) $\frac{b}{c+a}$

7. If $\tan\alpha = 1/7$ and $\sin\beta = 1/\sqrt{10}$, then the value of $\tan(\alpha+2\beta)$

(a) 1 (b) 0
(c) 1/2 (d) 3/4

8. If $\sin 6\theta = 32\cos^5\theta\sin\theta - 32\cos^3\theta\sin\theta + 3x$, then x is equal to *[EAMCET-2003]*

(a) $\cos\theta$ (b) $\cos 2\theta$
(c) $\sin\theta$ (d) $\sin 2\theta$

9. cos 15° is equal to *[MPPET-1998; MNR-1978]*

(a) $\sqrt{\frac{1+\cos 30°}{2}}$ (b) $\sqrt{\frac{1-\cos 30°}{2}}$
(c) $\pm\sqrt{\frac{1+\cos 30°}{2}}$ (d) $\pm\sqrt{\frac{1-\cos 30°}{2}}$

10. If $a\tan\theta = b$, then $a\cos 2\theta + b\sin 2\theta$ is equal to *[EAMCET-1981, 82; MPPET-1996; J&K-2005]*

(a) a (b) b
(c) $-a$ (d) $-b$

11. $1 + \cos 2x + \cos 4x + \cos 6x$ is equal to *[Roorkee-1974]*

(a) $2\cos x \cos 2x \cos 3x$
(b) $4\sin x \cos 2x \cos 3x$
(c) $4\cos x \cos 2x \cos 3x$
(d) None of these

12. If $\tan A = \frac{1-\cos B}{\sin B}$, then find $\tan 2A$ in terms of $\tan B$ and show that *[IIT-1983; MPPET-94]*

(a) $\tan 2A = \tan B$
(b) $\tan 2A = \tan^2 B$
(c) $\tan 2A = \tan^2 B + 2\tan B$
(d) None of these

13. $\sqrt{3}\,\text{cosec}\,20° - \sec 20°$ is equal to *[IIT-1988; NDA-2006]*

(a) 2 (b) $\frac{2\sin 20°}{\sin 40°}$
(c) 4 (d) $\frac{4\sin 20°}{\sin 40°}$

14. $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}$ is equal to *[MPPET-1998]*

(a) 0 (b) $\frac{1}{2}$
(c) $\frac{1}{4}$ (d) $-\frac{1}{8}$

15. If $\cos\theta = \frac{1}{2}\left(a+\frac{1}{a}\right)$, then the value of $\cos 3\theta$ will be *[MPPET-2001; Pb. CET-2002]*

(a) $\frac{1}{8}\left(a^3+\frac{1}{a^3}\right)$ (b) $\frac{3}{2}\left(a+\frac{1}{a}\right)$
(c) $\frac{1}{2}\left(a^3+\frac{1}{a^3}\right)$ (d) $\frac{1}{3}\left(a^3+\frac{1}{a^3}\right)$

16. If $A = \sin^2\theta + \cos^4\theta$, then for all values of

[IIT-1980; Roorkee-1992; EAMCET-1994; Pb. CET-1999; DCE-1996, 2000, 2001; MPPET-2004; Orissa JEE-2004]

(a) $1 \le A \le 2$ (b) $3/4 \le A \le 1$
(c) $13/16 \le A \le 1$ (d) $3/4 \le A \le 13/16$

17. $\cos\frac{\pi}{5}\cos\frac{2\pi}{5}\cos\frac{4\pi}{5}\cos\frac{8\pi}{5}$ is equal to

(a) 1/16 (b) 0
(c) −1/8 (d) −1/16

18. The minimum value of $\sin\theta\cos\theta$ is ***[Pb. CET-90]***

(a) 1 (b) 0
(c) −1/2 (d) 1/2

19. $(\sin 3A + \sin A)\sin A + (\cos 3A - \cos A)\cos A$ is equal to

(a) 0 (b) 1
(c) 2 (d) None

20. If $\sin A + \sin B = C$, $\cos A + \cos B = D$, then the value of $\sin(A + B)$ is equal to ***[MPET-1986]***

(a) CD (b) $\frac{CD}{C^2+D^2}$
(c) $\frac{C^2+D^2}{2CD}$ (d) $\frac{2CD}{C^2+D^2}$

21. If $\sin\theta + \sin\varphi = a$ and $\cos\theta + \cos\varphi = b$, then $\tan\frac{\theta-\phi}{2}$ is equal to ***[MPPET-93]***

(a) $\sqrt{\frac{a^2+b^2}{4-a^2-b^2}}$ (b) $\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$
(c) $\sqrt{\frac{a^2+b^2}{4+a^2+b^2}}$ (d) $\sqrt{\frac{4+a^2+b^2}{a^2+b^2}}$

22. Which of the following pairs is correctly matched? ***[N.D.A. Sept.-1998]***

(a) If $x = \frac{1+\sin 60° - \cos 60°}{1+\sin 60° + \cos 60°}$, then $x = \tan 60°$

(b) If $x = \frac{1+\sin 90° - \cos 90°}{1+\sin 90° + \cos 90°}$, then $x = \tan 30°$

(c) If $x = \frac{2\tan 30°}{1-\tan^2 30°}$, then $x = \tan 60°$

(d) If $x = \frac{1+\tan^2 30°}{1-\tan^2 30°}$, then $x = \cos 60°$

23. $\sin\left(\frac{\pi}{10}\right)\sin\left(\frac{3\pi}{10}\right)$ is equal to ***[MNR-1984]***

(a) 1/2 (b) −1/2
(c) 1/4 (d) 1

24. $\sin 47° + \sin 61° - \sin 11° - \sin 25°$ is equal to ***[MPPET-2001; EAMCET-2003]***

(a) sin 36° (b) cos 36°
(c) sin 7° (d) cos 7°

25. $\sin 12° \sin 48° \sin 54°$ is equal to ***[IIT-1982; Kerala (Engg.)-2001]***

(a) 1/16 (b) 1/32
(c) 1/8 (d) 1/4

26. The value of $\cot 22\frac{1°}{2}$ is equal to ***[MPPET-2010]***

(a) $1+\frac{1}{\sqrt{2}}$ (b) $1+\sqrt{2}$
(c) $\sqrt{2}-1$ (d) None of these

HINTS AND EXPLANATIONS

1. (d) $\cos^2 76° + \cos^2 16° - \cos 76° \cos 16°$

$= \cos^2 76° + 1 - \sin^2 16° - \frac{1}{2}(2\cos 76° \cos 16°)$

$= \cos(76° - 16°)\cos(76° + 16°) + 1 - \frac{1}{2}(\cos 60° - \cos 92°)$

$= \frac{1}{2}\cos 92° + 1 - \frac{1}{4} + \frac{\cos 92°}{2} = \frac{3}{4}$

2. (a) $8\sin\frac{x}{8}\cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}$

$= 4\left(2\sin\frac{x}{8}\cos\frac{x}{8}\right)\cos\frac{x}{2}\cos\frac{x}{4}$

$= 4\sin\frac{x}{4}\cos\frac{x}{4}\cos\frac{x}{2} = 2\sin\frac{x}{2}\cos\frac{x}{2} = \sin x$

3. (c) $\dfrac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta}$

$= \dfrac{\sin\theta+2\sin\theta\cos\theta}{2\cos^2\theta+\cos\theta} = \dfrac{\sin\theta(1+2\cos\theta)}{\cos\theta(2\cos\theta+1)}$

$= \tan\theta$

4. (b) If $\cos\alpha + \cos\beta = 0$, $\sin\alpha + \sin\beta = 0$

$\therefore \quad \alpha = \pi + \beta \ \Rightarrow \alpha - \beta = \pi$

$\cos 2\alpha + \cos 2\beta = 2\cos(\alpha+\beta)\cos(\alpha-\beta) = -2\cos(\alpha+\beta)$

5. (a) $2\cos x - (\cos 3x + \cos 5x) = 2\cos x - 2\cos 4x \cos x$

$= 2\cos x(1 - \cos 4x) = 2\cos x(2\sin^2 2x)$

$= 4\cos x(2\sin x\cos x)^2 = 16\cos^3 x\sin^2 x$

6. (b) $a\cos 2\theta + b\sin 2\theta$

$= c, a\left(\dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right) + b\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right) = c$

$(c+a)\tan^2\theta - 2b\tan\theta + c - a = 0$, roots are

$\tan\alpha, \tan\beta$ $\tan\alpha + \tan\beta = \dfrac{2b}{c+a}$

7. (a) $\tan\alpha = \dfrac{1}{7}, \sin\beta = \dfrac{1}{\sqrt{10}}$ or $\tan\beta = \dfrac{1}{3}$

also $\tan 2\beta = \dfrac{2\tan\beta}{1-\tan^2\beta} = \dfrac{2\cdot\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2} = \dfrac{3}{4}$

$\tan(\alpha+2\beta) = \dfrac{\tan\alpha+\tan 2\beta}{1-\tan\alpha\cdot\tan 2\beta} = \dfrac{\frac{1}{7}+\frac{3}{4}}{1-\frac{1}{7}\cdot\frac{3}{4}} = 1$

8. (d) $\sin 6\theta = 32\cos^5\theta\sin\theta - 32\cos^3\theta\sin\theta + 3x$

$\sin 6\theta = 2\sin 3\theta\cos 3\theta = 2(3\sin\theta - 4\sin^3\theta)(4\cos^3\theta - 3\cos\theta)$

$= 2\sin\theta\cos\theta[3 - 4\sin^2\theta][4\cos^2\theta - 3]$

$= 2\sin\theta\cos\theta[4\cos^2\theta - 1][4\cos^2\theta - 3]$

$= 2\sin\theta\cos\theta[16\cos^4\theta - 16\cos^2\theta + 3]$

$= 32\sin\theta\cos^5\theta - 32\cos^3\theta\sin\theta + 3(2\sin\theta\cos\theta)$

$\therefore \quad x = 2\sin\theta\cos\theta = \sin 2\theta$

9. $\cos 15° = \sqrt{\dfrac{1+\cos 30°}{2}}$,

$\because$ 15° in 1st quadrant.

10. (b) $a\tan\theta = b \Rightarrow \tan\theta = \dfrac{b}{a}$

$a\cos 2\theta + b\sin 2\theta$

$= \dfrac{a(1-\tan^2\theta)}{1+\tan^2\theta} + \dfrac{b(2\tan\theta)}{1+\tan^2\theta} = b$

11. (c) $1 + \cos 2x + \cos 4x + \cos 6x = 2\cos^2 3x + 2\cos 2x\cos x$

$= 2\cos 3x(\cos 3x + \cos x) = 2\cos 3x(2\cos 2x\cos x) = a\cos x\cos 2x\cos 3x$

12. (a) If $\tan A = \dfrac{1-\cos B}{\sin B}$

$\Rightarrow \quad \tan A = \dfrac{2\sin^2\frac{B}{2}}{2\sin\frac{B}{2}\cos\frac{B}{2}}$

$\tan A = \tan\dfrac{B}{2} \ \Rightarrow\ A = \dfrac{B}{2}$ or $2A = B$

$\tan 2A = \tan B$

13. (c) $\sqrt{3}\,\text{cosec}\,20° - \sec 20°$

$= \dfrac{\sqrt{3}\cos 20° - \sin 20°}{\sin 20°\cos 20°}$

$= \dfrac{4(\cos 20°\sin 60° - \sin 20°\cos 60°)}{2\sin 20°\cos 20°}$

$= \dfrac{4\sin(60°-20°)}{\sin 40°} = \dfrac{4\sin 40°}{\sin 40°} = 4$

14. (d) $\cos\dfrac{\pi}{7}\cos\dfrac{2\pi}{7}\cos\dfrac{4\pi}{7} = \dfrac{\sin 8\pi/7}{8\sin\pi/7}$

$\dfrac{\sin\left(\pi+\frac{\pi}{7}\right)}{8\sin\frac{\pi}{7}} = -\dfrac{1}{8}$

15. (c) $\cos\theta = \dfrac{1}{2}\left(a+\dfrac{1}{a}\right)$

By cubing we get

$\cos^3\theta = \dfrac{1}{8}\left[a^3 + \dfrac{1}{a^3} + 3.a.\dfrac{1}{a}\left(a+\dfrac{1}{a}\right)\right]$

$8\cos^3\theta = a^3 + \dfrac{1}{a^3} + 3(2\cos\theta)$

$\left(\because a + \dfrac{1}{a} = 2\cos\theta\right)$

$$8\cos^3\theta - 6\cos\theta = a^3 + \frac{1}{a^3}$$

$$\Rightarrow \quad \frac{1}{2}\left(a^3 + \frac{1}{a^3}\right) = \cos 3\theta$$

16. (b) $A = \sin^2\theta + \cos^4\theta = 1 - \cos^2\theta + \cos^4\theta$

$= 1 - \cos^2\theta\,(1 - \cos^2\theta) = 1 - \cos^2\theta \sin^2\theta$

$= 1 - \frac{\sin^2 2\theta}{4} \quad \therefore \quad \frac{3}{4} \le A \le 1 \; (\because \sin^2 2\theta \in [0, 1])$

17. (d) $\cos\frac{\pi}{5}\cos\frac{2\pi}{5}\cos\frac{4\pi}{5}\cos\frac{8\pi}{5}$

$$= \frac{\sin\left(\frac{16\pi}{5}\right)}{16\sin\frac{\pi}{5}} = \frac{-1}{16}$$

$$\left(\because \sin\frac{16\pi}{5} = \sin\left(3\pi + \frac{\pi}{5}\right) = -\sin\frac{\pi}{5}\right)$$

18. (c) $\sin\theta\,\cos\theta = \frac{1}{2}\sin 2\theta$

$\therefore$ minimum value $= \frac{-1}{2}$ $(\because \sin 2\theta \ge -1)$

19. (a) $(\sin 3A + \sin A)\sin A + (\cos 3A - \cos A)\cos A$

$= \sin 3A \sin A + \cos 3A \cos A + \sin^2 A - \cos^2 A$

$= \cos(3A - A) - (\cos^2 A - \sin^2 A) = \cos 2A - \cos 2A = 0$

20. (d) $\sin A + \sin B = C$, $\cos A + \cos B = D$

$$2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) = C \quad \text{......... (i)}$$

$$2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) = D \quad \text{........ (ii)}$$

$\therefore$ By dividing (i) from (ii)

$\tan\left(\frac{A+B}{2}\right) = \frac{C}{D} \Rightarrow \sin(A+B)$

$$= \frac{2\tan\left(\frac{A+B}{2}\right)}{1 + \tan^2\left(\frac{A+B}{2}\right)}$$

$$\sin(A+B) = \frac{2\frac{C}{D}}{1 + \left(\frac{C}{D}\right)^2} = \frac{2DC}{D^2 + C^2}$$

21. (b) $\sin\theta + \sin\varphi = a$; $\cos\theta + \cos\varphi = b$

By squaring and adding both equation

$(\sin\theta + \sin\varphi)^2 + (\cos\theta + \cos\varphi) = a^2 + b^2$

$\sin^2\theta + \sin^2\varphi + \cos^2\theta + \cos^2\varphi + 2(\sin\theta\sin\varphi + \cos\theta\cos\varphi)$

$= a^2 + b^2$

$$\cos(\theta - \varphi) = \frac{a^2 + b^2 - 2}{2}$$

$$\frac{1 - \tan^2\left(\frac{\theta - \phi}{2}\right)}{1 + \tan^2\left(\frac{\theta - \phi}{2}\right)} = \frac{a^2 + b^2 - 2}{2}$$

Applying componendo and dividendo

$$\frac{1}{-\tan^2\left(\frac{\theta - \phi}{2}\right)} = \frac{a^2 + b^2}{a^2 + b^2 - 4}$$

$$\Rightarrow \quad \tan\left(\frac{\theta - \phi}{2}\right) = \sqrt{\frac{4 - a^2 - b^2}{a^2 + b^2}}$$

22. (c) Since, $\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$

$$\therefore \quad \tan 60° = \frac{2\tan 30°}{1 - \tan^2 30°}$$

$$\therefore \quad x = \frac{2\tan 30°}{1 - \tan^2 30°} = \tan 60°$$

$\therefore$ (c) is true.

23. (c) By dividing and multiplying by 2

$$\frac{2}{2}\sin\left(\frac{\pi}{10}\right)\sin\left(\frac{3\pi}{10}\right)$$

$$\Rightarrow \quad \frac{1}{2}\left[\cos\left(\frac{2\pi}{10}\right) - \cos\left(\frac{4\pi}{10}\right)\right]$$

$$\Rightarrow \quad \frac{1}{2}\left[\cos 36° - \cos\frac{2\pi}{5}\right]$$

$$\Rightarrow \quad \frac{1}{2}[\cos 36° - \cos 72°]$$

$$\Rightarrow \quad \frac{1}{2}\left[\frac{\sqrt{5}+1}{4} - \frac{\sqrt{5}-1}{4}\right]$$

$$\Rightarrow \quad \frac{1}{2}\left[\frac{2}{4}\right] = \frac{1}{4}$$

24. (d) $(\sin 47° + \sin 61°) - (\sin 11° + \sin 25°)$

$= 2\sin 54°\cos 7° - 2\sin 18°\cos 7°$

$= 2\cos 7° (\sin 54° - \sin 18°)$

$= 2\cos 7°\left(\frac{\sqrt{5}+1}{4} - \frac{\sqrt{5}-1}{4}\right) = \cos 7°$

25. (c) $\sin 12° \sin 48° \sin 54° = \frac{1}{2}$ (2sin 12° sin 48°) sin 54°

$= \frac{1}{2}(\cos 36° - \cos 60°) \sin 54°$

$= \frac{1}{2}\left(\frac{\sqrt{5}+1}{4} - \frac{1}{2}\right)\left(\frac{\sqrt{5}+1}{4}\right) = \frac{1}{8}$

26. (b) $\cot\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$

Put $\theta = 45°$

$\cot 22\frac{1°}{2} = \sqrt{\frac{1+\cos 45°}{1-\cot 45°}} = \sqrt{\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}}$

$\Rightarrow \sqrt{\frac{\sqrt{2}+1}{\sqrt{2}-1} \times \frac{(\sqrt{2}+1)}{\sqrt{2}+1}} = \sqrt{2}+1$

(rationalizing)

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. If $\sin x + \sin y = 3(\cos y - \cos x)$, then the value of $\frac{\sin 3x}{\sin 3y}$ is

(a) 1 (b) –1
(c) 0 (d) None

2. $\left(\frac{\sin 2A}{1+\cos A}\right)\left(\frac{\cos A}{1+\cos A}\right)$ is equal to

(a) $\tan A/2$ (b) $\cot A/2$
(c) $\sec A/2$ (d) $\operatorname{cosec} A/2$

3. $\operatorname{cosec} A - 2\cot 2A \cos A$ is equal to

(a) $2\sin A$ (b) $\sec A$
(c) $2\cos A \cot A$ (d) None of these

4. If $\cos\theta = \frac{3}{5}$ and $\cos\phi = \frac{4}{5}$, where θ and φ are positive acute angles, then $\cos\frac{\theta-\phi}{2}$ is equal to *[MPPET-1988]*

(a) $\frac{7}{\sqrt{2}}$ (b) $\frac{7}{5\sqrt{2}}$
(c) $\frac{7}{\sqrt{5}}$ (d) $\frac{7}{2\sqrt{5}}$

5. cos 20° cos 40° cos 80° is equal to *[MPPET-1989]*

(a) 1/2 (b) 1/4
(c) 1/6 (d) 1/8

6. $\frac{\sin^2 A - \sin^2 B}{\sin A\cos A - \sin B\cos B}$ is equal to *[MPPET-1993]*

(a) $\tan(A - B)$ (b) $\tan(A + B)$
(c) $\cot(A - B)$ (d) $\cot(A + B)$

7. $\frac{\cot^2 15° - 1}{\cot^2 15° + 1}$ is equal to *[MPPET-1998; AIEEE-2002]*

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$
(c) $\frac{3\sqrt{3}}{4}$ (d) $\sqrt{3}$

8. If $\tan\theta = t$, then $\tan 2\theta + \sec 2\theta$ is equal to *[MPPET-1999]*

(a) $\frac{1+t}{1-t}$ (b) $\frac{1-t}{1+t}$
(c) $\frac{2t}{1-t}$ (d) $\frac{2t}{1+t}$

9. If $\sin 2\theta + \sin 2\varphi = 1/2$ and $\cos 2\theta + \cos 2\varphi = 3/2$, then $\cos^2(\theta - \varphi)$ is equal to *[MPPET-2000; Pb. CET-2000]*

(a) 3/8 (b) 5/8
(c) 3/4 (d) 5/4

10. $\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8}$ is equal to

(a) 1/2 (b) 1/4
(c) 3/2 (d) 3/4

11. If $\tan\frac{A}{2}=\frac{3}{2}$, then $\frac{1+\cos A}{1-\cos A}$ is equal to

(a) –5 (b) 5
(c) 9/4 (d) 4/9

12. $\sin A \sin(60° - A)\sin(60° + A)$ is equal to

(a) sin 3A (b) $\frac{\sin 3A}{2}$
(c) $\frac{\sin 3A}{4}$ (d) None

13. $\sqrt{2+\sqrt{2+2\cos 4\theta}}$ is equal to

(a) cos θ (b) sin θ
(c) 2cos θ (d) 2sin θ

14. Which of the following number(s) is/are rational? *[IIT-1998]*

(a) sin 15° (b) cos 15°
(c) sin 15° cos 15° (d) sin 15° cos 75°

15. $\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8}$ is equal to *[Roorkee-1980]*

(a) 1/2 (b) 1/4
(c) 3/2 (d) 3/4

16. tan 20° tan 40° tan 60° tan 80° is equal to *[IIT-1974]*

(a) 1 (b) 2
(c) 3 (d) $\sqrt{3}/2$

17. If $\cos\theta=\frac{1}{2}\left(x+\frac{1}{x}\right)$, then $\frac{1}{2}\left(x^2+\frac{1}{x^2}\right)$ is equal to *[AMU-1998]*

(a) sin 2θ (b) cos 2θ
(c) tan 2θ (d) sec 2θ

18. tan 9° – tan 27° – tan 63° + tan 81° is equal to *[Roorkee-1989]*

(a) 1/2 (b) 2
(c) 4 (d) 8

19. $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}+\cos\frac{7\pi}{7}$ is equal to *[DCE-98]*

(a) 1 (b) –1
(c) –3/2 (d) –1/2

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The worksheet is of 15 minutes.
3. The worksheet consists of 15 questions. The maximum marks are 45.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. If $k = \sin\frac{\pi}{18}.\sin\frac{5\pi}{18}.\sin\frac{7\pi}{18}$, then the value of k is ***[IIT-1993; MNR-74]***

(a) 1/4 (b) 1/8
(c) 1/16 (d) None of these

2. sin 12°sin 24° sin 48° sin 84° is equal to ***[EAMCET-1989]***

(a) cos 20° cos 40° cos 60° cos 80°
(b) sin 20° sin 40° sin 60° sin 80°
(c) 3/15
(d) None of these

3. If x = cos10° cos 20° cos 40°, then the value of x is ***[Roorkee-95]***

(a) $\frac{1}{4}\tan 10°$ (b) $\frac{1}{8}\cot 10°$
(c) $\frac{1}{8}\text{cosec}\,10°$ (d) $\frac{1}{8}\sec 10°$

4. If A lies in the third quadrant and 3 tan A – 4 = 0, then 5 sin 2A + 3 sin A + 4 cos A is equal to ***[EAMCET-1994]***

(a) 0 (b) –24/5
(c) 24/5 (d) 48/5

5. If $\tan\frac{\theta}{2} = t$, then $\frac{1-t^2}{1+t^2}$ is equal to ***[Kerala (Engg.)-2002]***

(a) cos θ (b) sin θ
(c) sec θ (d) cos2 θ

6. $\frac{1+\cos\theta}{\sin\theta}$ is equal to ***[MPPET-88]***

(a) $\tan\frac{\theta}{2}$ (b) $\cot\frac{\theta}{2}$
(c) tan θ (d) cot θ

7. 2 sin A cos³ A – 2 sin³ A cos A is equal to ***[Roorkee-1975; Kerala (Engg.)-2001]***

(a) sin 4A (b) $\frac{1}{2}\sin 4A$
(c) $\frac{1}{4}\sin 4A$ (d) None of these

8. $\frac{1}{\sin 10°} - \frac{\sqrt{3}}{\cos 10°}$ is equal to ***[IIT-1974]***

(a) 0 (b) 1
(c) 2 (d) 4

9. (cos α + cos β)² + (sin α + sin β)² is equal to

(a) $4\cos^2\frac{\alpha-\beta}{2}$ (b) $4\sin^2\frac{\alpha-\beta}{2}$
(c) $4\cos^2\frac{\alpha+\beta}{2}$ (d) $4\sin^2\frac{\alpha+\beta}{2}$

10. If cos3 θ = α cos θ + β cos³ θ, then (α, β) is equal to

(a) (3, 4) (b) (4, 3)
(c) (–3, 4) (d) (3, –4)

11. If sec A + tan A = 3/2, then

(a) $\sin A = \frac{5}{13}$ (b) $\sin 2A = \frac{5}{13}$
(c) $\sin A = \frac{12}{13}$ (d) $\sin 2A = \frac{12}{13}$

12. If cos A = $\frac{3}{4}$, then 32 sin $\left(\frac{A}{2}\right)\sin\left(\frac{5A}{2}\right)$ is equal to ***[DCE-1996]***

(a) 7 (b) 8
(c) 11 (d) None of these

13. If tan² θ = 2 tan² φ + 1, then cos2 θ + sin² φ is equal to ***[MPPET-1986]***

(a) –1 (b) 0
(c) 1 (d) None of these

14. $\cot x - \tan x$ is equal to

(a) $\cot 2x$ (b) $2\cot^2 x$
(c) $2\cot 2x$ (d) $\cot^2 2x$

15. If $\sin\alpha = \frac{-3}{5}$, where $\pi < \alpha < \frac{3\pi}{2}$, then $\cos \frac{1}{2}\alpha$ is equal to

(a) $\frac{-1}{\sqrt{10}}$ (b) $\frac{1}{\sqrt{10}}$
(c) $\frac{3}{\sqrt{10}}$ (d) $\frac{-3}{\sqrt{10}}$

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	6. ⓐ ⓑ ⓒ ⓓ	11. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	7. ⓐ ⓑ ⓒ ⓓ	12. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	8. ⓐ ⓑ ⓒ ⓓ	13. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	9. ⓐ ⓑ ⓒ ⓓ	14. ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ	10. ⓐ ⓑ ⓒ ⓓ	15. ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

2. (d) $\frac{1}{4}(2\sin 12° \sin 48°)(2\sin 24° \sin 84°)$

$$\frac{1}{4}\left(\frac{\sqrt{5}-1}{4}-\frac{1}{2}\right)\left(\frac{1}{2}+\frac{\sqrt{5}-1}{4}\right)$$

$$=\frac{1}{4}\left(\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4}\right)=\frac{1}{16}$$

8. (d) $\frac{1}{\sin 10°}-\frac{\sqrt{3}}{\cos 10°}=$

$$\frac{2(\cos 10°-\sqrt{3}\sin 10°)}{2\sin 10°\cos 10°}=\frac{4\sin(30°-10°)}{\sin 20°}=4$$

13. $\cos 2\theta+\sin^2\phi=\frac{1-\sin^2\theta}{1+\tan^2\theta}+\sin^2\phi$

$$=\frac{1-(2\tan^2\phi+1)}{1+2\tan^2\phi+1}+\sin^2\phi=0$$

LECTURE

7

Identities of Trigonometric Functions

BASIC CONCEPTS

1. IF $A + B + C = 180°$, THEN

(i) $\sin(A+B) = \sin C$, $\cos(A+B) = -\cos C$, $\tan(A+B) = -\tan C$

(ii) $\sin\left(\frac{A+B}{2}\right) = \cos\frac{C}{2}$,

$\cos\left(\frac{A+B}{2}\right) = \sin\frac{C}{2}$,=

$\tan\left(\frac{A+B}{2}\right) = \cot\frac{C}{2}$

$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

(iii) $\sin(A+B-C) = \sin 2C$, $\cos(A+B-C) = -\cos 2C$ $\tan(A+B-C) = -\tan 2C$

(iv) If $A + B + C = \pi$, then

(a) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(b) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}$

$\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

2. IF $A + B + C = 2\pi$, THEN

$\sin(A+B) = -\sin C$, $\cos(A+B) = \cos C$, $\tan(A+B) = -\tan C$

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. If $A + B + C = \pi$, then prove that

$$\cos\frac{A}{2} + \cos\frac{B}{2} - \cos\frac{C}{2}$$

$$= 4\cos\frac{\pi+A}{4}\cos\frac{\pi+B}{4}\cos\frac{\pi-C}{4}$$

Solution

$$\text{L.H.S} = \left(\cos\frac{A}{2} + \cos\frac{B}{2}\right) - \cos\frac{C}{2}$$

$$= 2\cos\frac{A+B}{4}\cos\frac{A-B}{4} - \cos\frac{C}{2}$$

$$= 2\cos\frac{\pi - C}{4}\cos\frac{A-B}{4} - \sin\left(\frac{\pi}{2} - \frac{C}{4}\right)$$

$$= 2\cos\frac{\pi - C}{4}\cos\frac{A-B}{4} - 2\sin\left(\frac{\pi - C}{4}\right)\cos\frac{\pi - C}{4}$$

$$= 2\cos\frac{\pi - C}{4}\left\{\cos\left(\frac{A-B}{4}\right) - \sin\left(\frac{\pi - C}{4}\right)\right\}$$

$$= 2\cos\frac{\pi - C}{4}\left\{\cos\left(\frac{A-B}{4}\right) - \sin\left(\frac{A+B}{4}\right)\right\}$$

$$= 2\cos\frac{\pi - C}{4}\left\{\cos\frac{A-B}{4} + \cos\left(\frac{\pi}{2} + \frac{A+B}{4}\right)\right\}$$

$$= 2\cos\frac{\pi - C}{4}\left\{\cos\frac{A-B}{4} + \cos\left(\frac{2\pi + A + B}{2}\right)\right\}$$

$$= 2\cos\frac{\pi - C}{4} 2\cos\left(\frac{A - B + 2\pi + A + B}{4\times 2}\right)$$

$$\cos\left(\frac{2\pi + A + B - A + B}{4\times 2}\right)$$

$$= 2\cos\frac{\pi - C}{4}\times 2\cos\frac{\pi + A}{4}\cos\frac{\pi + B}{4}$$

$$= 4\cos\frac{\pi + A}{4}\cos\frac{\pi + B}{4}\cos\frac{\pi - C}{4}$$

2. $\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}$

$$= 1 + 4\sin\frac{\pi - A}{4}\sin\frac{\pi - B}{4}\sin\frac{\pi - C}{4}$$

Solution

$$\text{L.H.S.} = \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}$$

$$= 2\sin\frac{A+B}{4}\cos\frac{A-B}{4}\cos\frac{A+B}{2}$$

$$= 2\sin\frac{A+B}{4}\cos\frac{A-B}{4} + 1 - 2\sin^2\frac{A+B}{4}$$

$$= 1 + 2\sin\left(\frac{A+B}{4}\right)\left\{\cos\frac{A-B}{4} - \sin\frac{A+B}{4}\right\}$$

$$= 1 + 2\sin\left(\frac{A+B}{4}\right)$$

$$\left\{\cos\frac{A-B}{4} - \cos\left(\frac{\pi}{2} - \frac{A+B}{4}\right)\right\}$$

$$= 1 + 2\sin\left(\frac{A+B}{4}\right)$$

$$\left\{2\sin\left(\frac{A - B + 2\pi - A - B}{4\times 2}\right)\right.$$

$$\left.\sin\left(\frac{2\pi - A - B - A + B}{2\times 4}\right)\right\}$$

$$= 1 + 4\sin\left(\frac{A+B}{4}\right)\sin\left(\frac{\pi - B}{4}\right)\sin\left(\frac{\pi - A}{4}\right)$$

$$= 1 + 4\sin\left(\frac{\pi - A}{4}\right)\sin\left(\frac{\pi - B}{4}\right)\sin\left(\frac{\pi - C}{4}\right)$$

3. $\sin(B + 2C) + \sin(C + 2A) + \sin(A + 2B)$

$$= 4\sin\frac{B-C}{2}\sin\frac{C-A}{2}\sin\frac{A-B}{2}$$

Solution

L.H.S. = $\sin(B + 2C) + \sin(C + 2A) + \sin(A + 2B)$

= $\sin(B + C + C) + \sin(C + A + A) + \sin(A + B + B)$

= $\sin[\pi - (A - C)] + \sin[\pi - (B - A)] + \sin[\pi - (C - B)]$

= $\sin(A - C) + \sin(B - A) + \sin(C - B)$

$$= 2\sin\left(\frac{A - C + B - A}{2}\right).$$

$$\cos\left(\frac{A - C - B + A}{2}\right) - \sin(B - C)$$

$$= 2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{2A - B - C}{2}\right)$$

$$-2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B-C}{2}\right)$$

$$= 2\sin\frac{B-C}{2}.$$

$$\left\{\cos\left(\frac{2A - B - C}{2}\right) - \cos\left(\frac{B-C}{2}\right)\right\}$$

$$= 2\sin\left(\frac{B-C}{2}\right)$$

$$\left\{2\sin\left(\frac{2A-B-C+B-C}{2\times 2}\right)\right.$$

$$\left.\sin\left(\frac{B-C-2A+B+C}{2\times 2}\right)\right\}$$

$$=4\sin\left(\frac{B-C}{2}\right)\sin\left(\frac{A-C}{2}\right)\sin\left(\frac{B-A}{2}\right)$$

$$=4\sin\left(\frac{B-C}{2}\right)\sin\left(\frac{C-A}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

4. If $A+B+C=\pi$, then prove that $\cos 4A+\cos 4B+\cos 4C=-1+4\cos 2A\cos 2B\cos 2C$.

Solution

LHS $=\cos 4A+\cos 4B+\cos 4C$

$=2\cos(2A+2B)\cos(2A-2B)+\cos 4C$

$=2\cos(2\pi-2C)\cos(2A-2B)+(2\cos^2 2C-1)$

$[\because A+B+C=\pi \Rightarrow (2A+2B)=(2\pi-2C)]$

$=2\cos 2C\cos(2A-2B)+2\cos^2 2C-1$

$=2\cos 2C[\cos(2A-2B)+\cos 2C]-1$

$=2\cos 2C[\cos(2A-2B)+\cos\{2\pi-(2A+2B)\}]-1$

$[\because 2A+2B+2C=2\pi \Rightarrow 2C=2\pi-(2A+2B)]$

$=2\cos 2C[\cos(2A-2B)+\cos(2A+2B)]-1$

$=2\cos 2C[2\cos 2A\cos 2B]-1$

$=-1+4\cos 2A\cos 2B\cos 2C=$ RHS.

$\therefore\quad \cos 4A+\cos 4B+\cos 4C$

$\quad=-1+4\cos 2A\cos 2B\cos 2C$

5. If $A+B+C=\pi$, prove that

$$\cos^2\frac{A}{2}+\cos^2\frac{B}{2}-\cos^2\frac{C}{2}=2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

Solution

$$\text{LHS}=\cos^2\frac{A}{2}+\cos^2\frac{B}{2}-\cos^2\frac{C}{2}$$

$$=\frac{1}{2}(1+\cos A)+\frac{1}{2}(1+\cos B)-\frac{1}{2}(1+\cos C)$$

$$=\frac{1}{2}+\frac{1}{2}(\cos A+\cos B-\cos C)$$

$$=\frac{1}{2}+\frac{1}{2}\left[\left(4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}\right)-1\right]$$

$$=2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}=\text{RHS}.$$

$$\therefore\quad \cos^2\frac{A}{2}+\cos^2\frac{B}{2}-\cos^2\frac{C}{2}$$

$$=2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): SOLVE THESE PROBLEMS TO GRASP THE TOPIC

EXERCISE 1

1. If A, B, C, D are angles of a cyclic quadrilateral, then prove that $\cos A+\cos B+\cos C+\cos D=0$

2. In any ΔABC, then prove that $\tan\left(\frac{B+C-A}{2}\right)=\cot A$

3. If $A+B+C=\pi$, then prove that $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

4. If $A+B+C=\pi$, then prove that

$$\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

5. If $A+B+C=\pi$, then prove that $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

6. If $A+B+C=\pi$, then prove that

$$\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$$

7. If $A + B + C = 2S$, then prove that
$\cos^2 S + \cos^2(S - A) + \cos^2(S - B) + \cos^2(S - C)$
$= 2 + 2 \cos A \cos B \cos C$

EXERCISE 2

1. In any quadrilateral $ABCD$, prove that
(i) $\sin (A + B) + \sin (C + D) = 0$
(ii) $\cos (A + B) = \cos (C + D)$

2. In a cyclic quadrilateral, prove that
(i) $\cos (180° - A) + \cos (180° - B) + \cos (180° - C) - \sin (90° + D) = 0$
(ii) $\sin A + \sin B - \sin C - \sin D = 0$
(iii) $\cos A + \cos C = 0$
(iv) $\cos B + \cos D = 0$
(v) $\tan A + \tan B + \tan C + \tan D = 0$

3. If $A + B + C = \pi$, then prove that
$\sin 2A - \sin 2B + \sin 2C = 4\cos A \sin B \cos C$

4. If $A + B + C = \pi$, then prove that
$\sin A + \sin B - \sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$

5. If $A + B + C = \pi$, then prove that $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2 \sin A \sin B \cos C$

6. If $A + B + C = \pi$, then prove that $\cos A + \cos B - \cos C$

$$= \left(4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}\right) - 1$$

7. If $A + B + C = \pi$, then prove that

$$\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2}$$

$$= 2\left(1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$$

8. If $A + B + C = \pi$, then prove that $\sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C$

9. If $A + B + C = 2S$, then prove that $\sin (S - A) + \sin (S - B) + \sin (S - C) - \sin S$

$$= 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. If in a triangle ABC, $\cos 3A + \cos 3B + \cos 3C = 1$, then one angle must be exactly equal to
(a) 30° (b) 60°
(c) 90° (d) 120°

Solution

(d) We have,

$\cos 3A + \cos 3B + \cos 3C = 1$

$$\Rightarrow \quad 2\cos 3\left(\frac{A+B}{2}\right)\cos 3\left(\frac{A-B}{2}\right) - \cos 3(A + B) = 1$$

$[\because C = \pi - (A + B)]$

$$\Rightarrow \quad 2\cos 3\left(\frac{A+B}{2}\right)\cos 3\left(\frac{A-B}{2}\right) - 2\cos^2 3\left(\frac{A+B}{2}\right) = 0$$

$$\Rightarrow \quad 2\cos 3\left(\frac{A+B}{2}\right)\left\{\cos 3\left(\frac{A-B}{2}\right) - \cos 3\left(\frac{A+B}{2}\right)\right\} = 0$$

$$\Rightarrow \quad \cos 3\left(\frac{A+B}{2}\right) = 0 \text{ or}$$

$$\cos 3\left(\frac{A-B}{2}\right) = \cos 3\left(\frac{A+B}{2}\right)$$

Now, $\Rightarrow \cos 3\left(\frac{A+B}{2}\right) = 0$

$$\Rightarrow \quad A + B = \frac{\pi}{3} \Rightarrow C = \frac{2\pi}{3}$$

and $\cos 3\left(\frac{A-B}{2}\right) = \cos 3\left(\frac{A+B}{2}\right) \Rightarrow B = 0$, which is not possible.

2. In an acute-angled triangle ABC, $\tan A + \tan B + \tan C$ is
(a) ≥ 3 (b) $\geq \sqrt{3}$
(c) $\geq 3\sqrt{3}$ (d) None

Solution

(c) In ΔABC, we have

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$ .. (i)

But, A.M. $\geq$ G.M.

$\Rightarrow \dfrac{\tan A+\tan B+\tan C}{3} \geq (\tan A \tan B \tan C)^{1/3}$

$\Rightarrow \left(\dfrac{\tan A \tan B \tan C}{3}\right) \geq (\tan A \tan B \tan C)^{1/3}$ [using (i)]

$\Rightarrow \left(\dfrac{\tan A \tan B \tan C}{3}\right)^{2/3} \geq 1 \Rightarrow \tan A \tan B \tan C \geq 3^{3/2}$

$\Rightarrow \tan A \tan B \tan C \geq 3\sqrt{3}$

3. In a ΔABC, the value of $\sin A \sin B \sin C$ is

(a) $\geq \dfrac{3\sqrt{3}}{8}$ (b) $\leq \dfrac{3\sqrt{3}}{8}$

(c) $\leq \dfrac{\sqrt{3}}{8}$ (d) $\leq \dfrac{3}{8}$

Solution

(b) We have,

$\sin^2 A + \sin^2 B + \sin^2 C$

$= \dfrac{1-\cos 2A}{2} + \dfrac{1-\cos 2B}{2} + \sin^2 C$

$= 1 - \dfrac{1}{2}[\cos 2A + \cos 2B] + \sin^2 C$

$= 1 - \cos(A+B)\cos(A-B) + \sin^2 C$

$= 1 + \cos C \cos(A-B) + 1 - \cos^2 C$

$= 2 - \cos^2 C + \cos C \cos(A-B)$

$\leq 2 - \cos^2 C + \cos C \quad [\because \cos(A-B) < 1]$

$= 2 - (\cos^2 C - \cos C) = 2 - \left(\cos C - \dfrac{1}{2}\right)^2 + \dfrac{1}{4}$

$= \dfrac{9}{4} - \left(\cos C - \dfrac{1}{2}\right)^2 \leq \dfrac{9}{4}$

$\therefore \sin^2 A + \sin^2 B + \sin^2 C \leq \dfrac{9}{4}$ .... (i)

Now, A.M. $\geq$ G.M.

$\Rightarrow \dfrac{\sin^2 A+\sin^2 B+\sin^2 C}{3} \geq (\sin^2 A \sin^2 B \sin^2 C)^{1/3}$

$\Rightarrow (\sin^2 A \sin^2 B \sin^2 C)^{1/3} \leq \dfrac{3}{4}$ [using (i)]

$\Rightarrow \sin A \sin B \sin C \leq \left(\dfrac{3}{4}\right)^{3/2}$

$\Rightarrow \sin A \sin B \sin C \leq \dfrac{3\sqrt{3}}{8}$

4. In a ΔABC, $\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$ is

(a) $\leq \dfrac{3\sqrt{3}}{16}$ (b) $\leq \dfrac{3\sqrt{3}}{8}$

(c) $\leq \dfrac{3\sqrt{3}}{32}$ (d) $\dfrac{3\sqrt{3}}{64}$

Solution

(b) We know that

$\cos A + \cos B + \cos C \leq \dfrac{3}{2}$

$\Rightarrow (1 + \cos A) + (1 + \cos B) + (1 + \cos C) \leq \dfrac{9}{2}$

$\Rightarrow \cos^2\dfrac{A}{2} + \cos^2\dfrac{B}{2} + \cos^2\dfrac{C}{2} \leq \dfrac{9}{4}$

$\Rightarrow \dfrac{3}{4} \geq \dfrac{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}{3} \geq \left(\cos^2\dfrac{A}{2}\cos^2\dfrac{B}{2}\cos^2\dfrac{C}{2}\right)^{1/3}$

$\Rightarrow \left(\dfrac{3}{4}\right)^{3/2} \geq \cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

5. If $A+B+C=180°$, then $\dfrac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}$ is equal to

(a) $8\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

(b) $8\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

(c) $\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

(d) None of these

Solution

(a) $\dfrac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}$

$= \dfrac{4\sin A+\sin B+\sin C}{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$

$= \dfrac{2\sin\frac{A}{2}\cos\frac{A}{2}\left(2\sin\frac{B}{2}\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$

$= 8\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. If $A + B + C = 180°$,
$(\cot B + \cot C)(\cot C + \cot A)(\cot A + \cot B)$ is equal to *[UPSEAT-1999]*
(a) $\sec A \sec B \sec$
(b) $\operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C$
(c) $\tan A \tan B \tan C$
(d) 1

2. If $\alpha + \beta - \gamma = \pi$, then $\sin^2\alpha + \sin^2\beta - \sin^2\gamma$ is equal to *[IIT-1980; Pb. CET-2003]*
(a) $2\sin\alpha \sin\beta \cos\gamma$
(b) $2\cos\alpha \cos\beta \cos\gamma$
(c) $2\sin\alpha \sin\beta \sin\gamma$
(d) None of these

3. If $A+B+C=180°$, then $\dfrac{\sin 2A + \sin 2B + \sin 2C}{\cos A + \cos B + \cos C - 1}$ is equal to
(a) $8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
(b) $8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
(c) $8\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
(d) $8\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

4. If $A + B + C = \pi$, then $\sin 2A + \sin 2B + \sin 2C$ is equal to *[MP PET-2004]*
(a) $4\sin A \sin B \sin C$
(b) $4\cos A \cos B \cos C$
(c) $2\cos A \cos B \cos C$
(d) $2\sin A \sin B \sin C$

5. If $A + B + C = 180°$, then $\cos^2 A + \cos^2 B + \cos^2 C$ is equal to
(a) $1 - 2\cos A \cos B \cos C$
(b) $1 - 2\sin A \sin B \sin C$
(c) $1 - 2\tan A \tan B \tan C$
(d) None of these

6. If $A + B + C = 180°$, then $\cot A + \cot B + \cot C$ is equal to
(a) $\tan A \tan B \tan C$
(b) $\cot A \cot B \cot C$
(c) $\sin A \sin B \sin C$
(d) $\cos A \cos B \cos C$

7. In a ΔABC, the value of $\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$ is
(a) $\geq 3\sqrt{3}$ (b) ≥ 9
(c) $\geq 6\sqrt{3}$ (d) None

8. The minimum value of the expression $\sin\alpha + \sin\beta + \sin\gamma$, where α, β, γ are real numbers satisfying $\alpha + \beta + \gamma = \pi$ is
(a) positive (b) zero
(c) negative (d) -3

9. If $A + B + C = \pi$ $(A, B, C > 0)$ and the angle C is obtuse, then
(a) $\tan A \tan B > 1$ (b) $\tan A \tan B < 1$
(c) $\tan A \tan B = 1$ (d) None of these

10. In a ΔABC, $\cos A + \cos B + \cos C$ belongs to the interval.
(a) $(1/2, 3/2)$ (b) $(1, 3/2)$
(c) $(3/2, 2)$ (d) None

HINTS AND EXPLANATIONS

1. (b) $\cot A + \cot B = \dfrac{\sin(A+B)}{\sin A \sin B} = \dfrac{\sin(\pi - C)}{\sin A \sin B}$

$$= \frac{\sin C}{\sin A \sin B}$$

Similarly,

$$\cot C + \cot A = \frac{\sin B}{\sin A \sin C},$$

$$\cot C + \cot B = \frac{\sin A}{\sin B \sin C}$$

$\therefore$ (cot B + cot C) (cot C + cot A) (cot A + cot B) $= \dfrac{\sin A \sin B \sin C}{(\sin A \sin B \sin C)^2}$

= cosec A cosec B cosec C

2. (a) $\alpha + \beta - \gamma = \pi$

and $\sin^2 \alpha + \sin^2 \beta - \sin^2 \gamma = \sin^2 \alpha + \sin(\beta - \gamma)\sin(\beta + \gamma)$

$= \sin^2 \alpha + \sin(\pi - \alpha)\sin(\beta + \gamma) = \sin \alpha (\sin \alpha + \sin(\beta + \gamma))$

$$= \sin\alpha\left[2\sin\left(\frac{\alpha+\beta+\gamma}{2}\right)\cos\left(\frac{\alpha-\beta-\gamma}{2}\right)\right]$$

$$= \sin\alpha\left[2\sin\left(\frac{\pi+\gamma+\gamma}{2}\right)\cos\left(\frac{\pi-\beta-\beta}{2}\right)\right]$$

$= 2\sin\alpha\cos\gamma\sin\beta$

3. (b) If,

$A + B + C = 180°$, $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$

$\cos A + \cos B + \cos C$

$$= 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\therefore \quad \frac{\sin 2A + \sin 2B + \sin 2C}{\cos A + \cos B + \cos C - 1}$$

$$= \frac{4\sin A \sin B \sin C}{4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = 8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$\left(\because \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}, \frac{\sin A}{\sin\frac{A}{2}} = 2\cos\frac{A}{2}\right)$$

4. (a) $\sin 2A + \sin 2B + \sin 2C = 2\sin(A+B)\cos(A-B) + 2\sin C\cos C = 2\sin C(\cos(A-B) + \cos(\pi - (A+B)))$

$(\because \sin(A+B) = \sin(\pi - C) = \sin C)$

$= 2\sin C(\cos(A-B) - \cos(A+B))$

$= 2\sin C \times 2\sin A\sin B = 4\sin A\sin B\sin C$

5. (a) $\cos^2 A + \cos^2 B + \cos^2 C = 1 - \sin^2 A + \cos^2 B + \cos^2 C$

$= 1 - [-\cos^2 B + \sin^2 A] + \cos^2 C$

$= 1 - [\cos(A+B)\cos(A-B)] + \cos^2 C$

$= 1 - [\cos C\cos(A-B)] + \cos C$

$(\because A + B + C = 180°)$

$= 1 - \cos C[\cos(A-B) + \cos(A+B)]$

$= 1 - 2\cos A\cos B\cos C$

6. (b) $\because A + B + C = 180°$

$\therefore \quad A + B = 180° - C$

$\Rightarrow \quad \tan(A+B) = \tan(180° - C)$

$$\Rightarrow \quad \frac{\tan A + \tan B}{1 - \tan A\tan B} = -\tan C$$

$\Rightarrow \quad \tan A + \tan B = -\tan C + \tan A\tan B\tan C$

$\Rightarrow \quad \tan A + \tan B + \tan C = \tan A + \tan B + \tan C$

Dividing by $\tan A\tan B\tan C$

$\cot A + \cot B + \cot C = \cot A\cot B\cot C$

7. (a) In ΔABC we have,

$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$$

$$\Rightarrow \quad \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$$

Now, A.M. $\geq$ G.M.

$$\Rightarrow \quad \left\{\frac{\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}}{3}\right\} \geq \left\{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\right\}^{1/3}$$

$$\Rightarrow \quad \left\{\frac{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}{3}\right\} \geq \left\{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\right\}^{1/3}$$

$$\Rightarrow \quad \left\{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\right\}^{2/3} \geq 3$$

$$\Rightarrow \quad \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \geq 3\sqrt{3}$$

8. (a) We have $\alpha + \beta + \gamma = \pi$

$$\therefore \quad \sin\alpha + \sin\beta + \sin\gamma = 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$

Clearly, $\cos\frac{\alpha}{2} > 0, \cos\frac{\beta}{2} > 0, \cos\frac{\gamma}{2} > 0$

$$\therefore \quad 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} > 0$$

$\Rightarrow$ Minimum value of $\sin\alpha + \sin\beta + \sin\gamma$ is positive.

9. (b) We have,

$A + B + C = \pi$

$\Rightarrow \quad A + B = \pi - C$

$\Rightarrow \quad \tan(A + B) = \tan(\pi - C)$

$$\Rightarrow \quad \frac{\tan A + \tan B}{1 - \tan A \tan B} = \tan(\pi - C)$$

$$\Rightarrow \quad \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$$

Now, C is an obtuse angle

$\Rightarrow \quad \tan C < 0 \quad \Rightarrow -\tan C > 0$

$$\Rightarrow \quad \frac{\tan A + \tan B}{1 - \tan A \tan B} > 0$$

$\Rightarrow \quad 1 - \tan A \tan B > 0$

$\Rightarrow \quad \tan A \tan B < 1$

$$\left[\begin{array}{l} \because A, B \text{ are acute angles} \\ \therefore \tan A > 0, \tan B > 0 \end{array}\right]$$

10. (b) We have,

$\cos A + \cos B + \cos C$

$$= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} + 1 - 2\sin^2\frac{C}{2}$$

$$= 2\sin\frac{C}{2}\left\{\cos\frac{A-B}{2} - \sin\frac{C}{2}\right\} + 1$$

$$\leq 2\sin\frac{C}{2}\left\{1 - \sin\frac{C}{2}\right\} + 1 \qquad \ldots\ldots\ldots\ldots (1)$$

$$\left[\because, \text{ max. value of } \cos\frac{A-B}{2} \text{ is } 1\right]$$

$$= 1 - \left\{\sin^2\frac{C}{2} - \sin\frac{C}{2}\right\}$$

$$= 1 - 2\left\{\left(\sin\frac{C}{2} - \frac{1}{2}\right)^2 - \frac{1}{4}\right\}$$

$$= \frac{3}{2} - 2\left(\sin\frac{C}{2} - \frac{1}{2}\right)^2$$

Thus, we have

$$\cos A + \cos B + \cos C \leq \frac{3}{2} - 2\left(\sin\frac{C}{2} - \frac{1}{2}\right)^2 \qquad \ldots\ldots\ldots (ii)$$

$$\Rightarrow \quad \cos A + \cos B + \cos C \leq \frac{3}{2}$$

It is evident from (i) and (ii) that

$$\cos A + \cos B + \cos C = \frac{3}{2},$$

If $\cos\frac{A-B}{2} = 1$ and $\sin\frac{C}{2} - \frac{1}{2} = 0$

$\Rightarrow \quad A - B = 0$ and $C = 60°$

$\Rightarrow \quad A = B$ and $C = 60°$

$\Rightarrow \quad A = B = C = 60°$

$\Rightarrow \quad \Delta ABC$ is equilateral

$\because \quad \cos A + \cos B + \cos C$

$$= 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} + 1$$

and $4 \sin A/2 \sin B/2 \sin C/2 > 0$

$\therefore \quad \cos A + \cos B + \cos C > 1$

Thus, $1 < \cos A + \cos B + \cos C \leq \frac{3}{2}$

Hence, option (b) is correct.

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. If $A + B + C = \pi$, then

$$\frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin C \sin A} + \frac{\cos C}{\sin A \sin B} \text{ is}$$

equal to

(a) 0 (b) 1

(c) 2 (d) 3

2. If $A + B + C = 180°$, then $\sin 2A + \sin 2B - \sin 2C$ is equal to

(a) $4 \cos A \cos B \sin C$

(b) $4 \sin A \sin B \cos C$

(c) $4 \cos A \cos B \cos C$

(d) None of these

3. If $A + B + C = 180°$, then $\sin A + \sin B + \sin C$ is equal to

(a) $4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(b) $4\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(c) $4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(d) None of these

4. In triangle ABC, $\cos 2A + \cos 2B + \cos 2C$ is equal to
(a) $1 - 4\cos A \cos B \cos C$
(b) $1 - 4\sin A \sin B \sin C$
(c) $-1 - 4\cos A \cos B \cos C$
(d) None of these

5. In triangle ABC, $\sin^2 A - \sin^2 B + \sin^2 C$ is equal to
(a) $2\sin A \cos B \sin C$
(b) $4\sin A \cos B \sin C$
(c) $4\cos A \cos B \cos C$
(d) None of these

6. If $A + B + C = 2S$, then $\sin(S - A) + \sin(S - B) + \sin(S - C) - \sin S$ is equal to

(a) $4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(b) $4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(c) $4\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$

(d) None of these

7. In a triangle ABC, $\sin A - \cos B = \cos C$, then angle B is
(a) $\pi/2$ (b) $\pi/3$
(c) $\pi/4$ (d) $\pi/6$

8. If $A + B + C = \frac{3\pi}{2}$, then $\cos 2A + \cos 2B + \cos 2C$ is equal to
(a) $1 - 4\cos A \cos B \cos C$
(b) $4\sin A \sin B \sin C$
(c) $1 + 2\cos A \cos B \cos C$
(d) $1 - 4\sin A \sin B \sin C$

9. If $A + C = B$, then $\tan A \tan B \tan C$ is equal to
(a) $\tan A \tan B + \tan C$
(b) $\tan B - \tan C - \tan A$
(c) $\tan A + \tan C - \tan B$
(d) $-(\tan A \tan B + \tan C)$

10. If $\cos\alpha + \cos\beta = 0 = \sin\alpha + \sin\beta$, then $\cos 2\alpha + \cos 2\beta$ is equal to
(a) $-2\sin(\alpha + \beta)$ (b) $-2\cos(\alpha + \beta)$
(c) $2\sin(\alpha + \beta)$ (d) $2\cos(\alpha + \beta)$

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The worksheet is of 9 minutes.
3. The worksheet consists of 9 questions. The maximum marks are 27.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. In triangle ABC, the value of $\sin A + \sin B + \sin C$ is

(a) $4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(b) $4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(c) $4\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(d) $4\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$

2. If $x + y + z = 180°$, then $\cos 2x + \cos 2y - \cos 2z$ is equal to

(a) $4 \sin x \sin y \sin z$

(b) $1 - 4 \sin x \sin y \cos z$

(c) $4 \sin x \sin y \sin z - 1$

(d) $\cos A \cos B \cos C$

3. If $A + B + C = 270°$, then $\cos 2A + \cos 2B + \cos 2C + 4\sin A \sin B \sin C$ is equal to ***[EAMCET-2003]***

(a) 0 (b) 1

(c) 2 (d) 3

4. If $A + B + C = 180°$, then the value of $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}$ ***[UPSEAT-1999]***

(a) $2\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

(b) $4\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

(c) $\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

(d) $8\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

5. If $A + B + C = 180°$, then $\cos B + \cos C$ equals

(a) $2 \cos (A/2) \cos \{(B - C)/2\}$

(b) $2 \sin (A/2) \cos \{(B - C)/2\}$

(c) $2 \cos (C/2) \cos \{A - B)/2\}$

(d) $2 \cos (B/2) \cos (C/2)$

6. If $A + B + C = \pi$, then $\sin 2A + \sin 2B$ equals

(a) $\cos A \cos (B - C)$

(b) $\sin C \cos (A - B)$

(b) $2 \sin C \cos (A - B)$

(d) None of these

7. If $A + B + C = \pi$, then $\cos\frac{B}{2} + \cos\frac{C}{2}$ equals

(a) $\cos\frac{1}{4}(\pi - C)\cos\frac{1}{4}(B - C)$

(b) $2\cos (A/4) \cos \{(B - C)/3\}$

(c) $2\cos \{(\pi - A)/4\} \cos \{(B - C)/4\}$

(d) None of these

8. If $A + B + C = \frac{3\pi}{2}$, then $\cos 2A + \cos 2B - \cos 2C$ is equal to

(a) $1 - 4 \cos A \cos B \cos C$

(b) $4 \sin A \sin B \sin C$

(c) $1 - 2 \cos A \cos B \cos C$

(d) None of these

9. If ABC are the angles of triangle, then $\sin^2 A + \sin^2 B + \sin^2 C - 2 \cos A \cos B \cos C$ is equal to ***[CET-1989]***

(a) 1 (b) 2

(c) 3 (d) 4

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ
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7. ⓐ ⓑ ⓒ ⓓ
8. ⓐ ⓑ ⓒ ⓓ
9. ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

2. $\cos 2x + \cos 2y - \cos 2z = 2\cos(x+y) - (2\cos^2 z - 1)$

$= 1 + 2\cos(x+y)\cos(x-y) - 2\cos^2(\pi - (x+y))$

$= 1 + 2\cos(x+y)[\cos(x-y) - \cos(x+y)]$

$= 1 + 2\cos(\pi - z)(2\sin x \sin y)$

$= 1 - 4\sin x \sin y \sin z$

7. (c) $\cos\frac{B}{2} + \cos\frac{C}{2} = 2\cos\left(\frac{B+C}{4}\right)\cos\left(\frac{B-C}{4}\right)$

$= 2\cos\left(\frac{\pi - A}{4}\right)\cos\left(\frac{B-C}{4}\right)$

8. (d) $\cos 2A + \cos 2B - \cos 2C$

$= 2\cos(A+B)\cos(A-B) - (1 - 2\sin^2 C)$

$= 2\cos\left(\frac{3\pi}{2} - C\right)\cos(A-B) - 1 + 2\sin^2 C$

$= -2\sin C\cos(A-B) - 1 + 2\sin^2 C$

$= -1 + 2\sin C(\sin C - \cos(A-B))$

$= -1 - 2\sin C(\cos(A+B) + \cos(A-B))$

$= -1 - 2\cos A\cos B\sin C$

LECTURE

8

Graphs of Trigonometric Functions

BASIC CONCEPTS

1. PERIODICITY OF TRIGONOMETRIC FUNCTIONS

A function $f(x)$ is called periodic, if there exists a positive number T such that, $f(x + T) = f(x)$ for all x in the domain. Least positive value of such T is called period (or fundamental period). It should be noted that T is independent of x.

It is easy to see that all trigonometric functions are periodic and period of $\sin x$, $\cos x$, $\sec x$, $\operatorname{cosec} x$ is 2π and that of $\tan x$ and $\cot x$ is π as $\tan(x + \pi) = \tan x$, $\sin(x + 2\pi) = \sin x$ etc.

We illustrate the method of finding period by an example.

Example

Test the periodicity of the function $f(x) = \sin 2\pi x$.

Solution

We know that period of $\sin x$ is 2π so $f(x) = \sin(2\pi x) = \sin(2\pi + 2\pi x) = \sin 2\pi(x + 1) = f(x + 1)$ $\therefore f(x) = f(x + 1)$ for all x.

Hence, f is periodic with period 1.

Aliter Let $f(x)$ be periodic and T its period. Then $f(x + T) = f(x)$.

$\therefore$ $\sin[2\pi(x + T)] = \sin 2\pi x$

or $2\cos[2\pi x + (T/2)] \sin \pi T = 0$ for all x.

$\therefore$ $\sin \pi T = 0$ or $T\pi = \pi, 2\pi, 3\pi, \ldots\ldots\ldots$ or $T = 1, 2, 3, \ldots..$

Least positive value of T is 1 so $f(x)$ is periodic with period 1.

NOTES

It is interesting to note that if $f(x)$ is periodic having fundamental period p, then $f(nx)$ is also periodic having period (p/n). This can be exhibited as follows

Let $g(x) = f(nx) = f(nx + p) = f[n(x + (p/n)] = g[x + (p/n)] \forall x$.

$\therefore$ $g(x)$ i.e., $f(nx)$ is periodic with period (p/n).

2. GRAPHS OF TRIGONOMETRICAL FUNCTIONS

To draw the graph of the trigonometrical functions, we first prepare the table of values of the given trigonometrical function substituting some well-known values of the angle. Then we draw two mutually perpendicular lines $X'OX$ and $Y'OY$ as coordinate axes. By taking suitable unit, the values of the angle are marked along the line $X'OX$ (i.e., x-axis) and by taking suitable unit the corresponding values of the function are marked parallel to $Y'OY$ on the above marked points. Then the graph of the function is obtained by joining the above marked points (coordinates) by free hand.

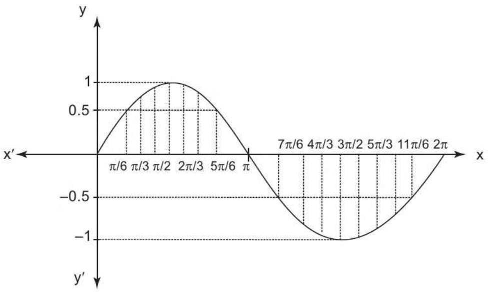
To draw the graphs, we shall use the following approximate values $\sqrt{2} = 1.4$, $\sqrt{3} = 1.8$, $1/\sqrt{2} = 0.7$, $1/\sqrt{3} = 0.6$, $\sqrt{3}/2 = 0.9$, $2/\sqrt{3} = 1.2$, $\pi = 3.1$, $\pi/2 = 1.6$, $\pi/3 = 1$.

2.1 Graph of sin *x*, for $0 \le x \le 2\pi$

Let $y = \sin x$

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	
y	0	0.5	0.7	0.9	1	0.9	0.7	0.5	
x	π	$7\pi/6$	$5\pi/4$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$7\pi/4$	$11\pi/6$	2π
y	0	–0.5	–0.7	–0.9	–1	–0.9	–0.7	–0.5	0

Now taking 1 cm on OX to represent 30° ($\pi/6$) and 1 cm on OY to represent 0.5, the graph of $\sin x$ is as shown below



NOTE

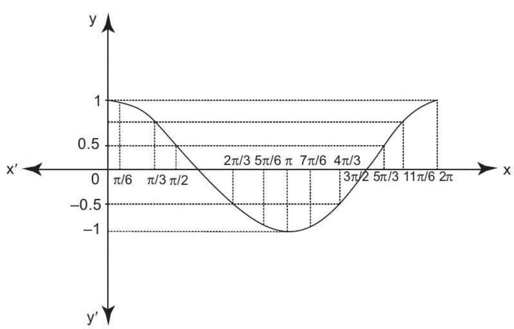
We know that $\sin(\pi + x) = -\sin x$. From this as well as from the above table we see that the values of y for $x = \pi$ to $x = 2\pi$ are the negative of the values of y for $x = 0$ to $x = \pi$. Thus the graph for $x = \pi$ to $x = 2\pi$ is identical as that from $x = 0$ to $x = \pi$ but on opposite side of the axis OX.

2.2 Graph of cos *x*, for $0 \le x \le 2\pi$

Let $y = \cos x$

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	
y	1	0.9	0.7	0.5	0	–0.5	–0.7	–0.9	
x	π	$7\pi/6$	$5\pi/4$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$7\pi/4$	$11\pi/6$	2π
y	–1	–0.9	–0.7	–0.5	0	0.5	0.7	0.9	1

Now taking 1 cm on OX to represent 30° ($\pi/6$) and 1 cm on OY to represent 0.5, the graph of $\cos x$ is as shown below



NOTE

By looking at the graphs of both the Sin and Cos functions, we find that both of them lie between the maximum ordinate 1, and the minimum ordinate-1. When a periodic function attains a maximum value M, and a minimum value m, then half the difference of the two, that is $\frac{M-m}{2}$ is called the amplitude of the function.

Thus, the amplitude of both the sine and the cosine functions is $\frac{1-(-1)}{2}$, or 1. It may be pointed $\frac{(2n-1)\pi}{4}$ out that the amplitude need not be a maximum ordinate.

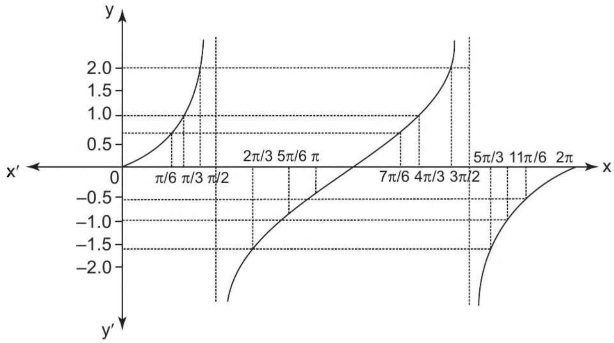
The maximum and minimum points are called turning points. They are $\left(\frac{\pi}{2}, 1\right)$ and $\left(\frac{3\pi}{2}, -1\right)$ for the sine and (0, 1), (2π, 1) and (π, –1) for the cosine functions.

2.3 Graph of tan *x*, for $0 \le x \le 2\pi$

Let $y = \cos x$

x	0	π/6	π/4	π/3	(π/2)–0	(π/2) + 0	2π/3	3π/4	5π/6	
y	0	0.6	1	1.7	+∞	–∞	–1.7	–1	–0.6	
x	π	7π/6	5π/4	4π/3	(3π/2)–0	(3π/2) + 0	5π/3	7π/4	11π/6	2π
y	0	0.6	1	1.7	+∞	–∞	–1.7	–1	–0.6	0

Now taking 1 cm on *OX* to represent 30° (π/6) and 1 cm on *OY* to represent 0.5, the graph of tan *x* is as shown below

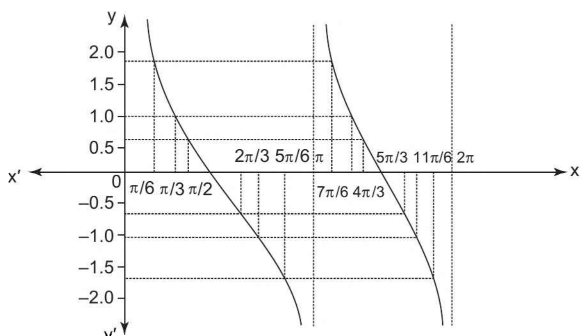


2.4 Graph of cot *x*, for $0 \le x \le 2\pi$

Let $y = \cot x$

x	0/0	π/6	π/4	π/3	π/2	2π/3	3π/4	5π/6	π–0
y	+∞	1.7	1	0.6	0	–0.6	–1	–1.7	–∞
x	π + 0	7π/6	5π/4	4π/3	3π/2	5π/3	7π/4	11π/6	2π–0
y	+∞	1.7	1	0.6	0	–0.6	–1	–1.7	–∞

Now taking 1 cm on *OX* to represent 30° (π/6) and 1 cm on *OY* to represent 0.5, the graph of cot *x* is as shown below

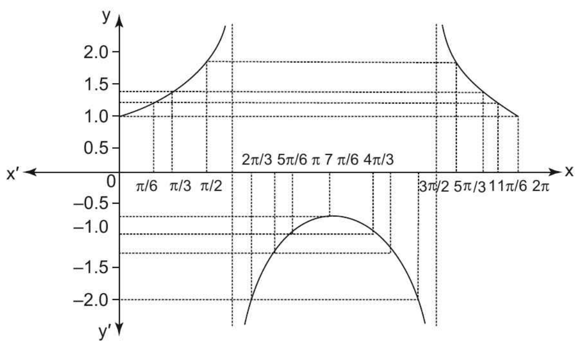


2.5 Graph of sec *x*, for $0 \le x \le 2\pi$

Let $y = \sec x$

x	0	π/6	π/4	π/3	(π/2) – 0	(π/2) + 0	2π/3	3π/4	5π/6
y	1	1.2	1.4	2	+∞	–∞	–2	–1.4	– 1.2
x	π	7π/6	5π/4	4π/3	(3π/2) –0 (3π/2) +0 5π/3	7π/4	11π/6	2π	
y	–1	–1.2	–1.4	–2	–∞	+∞	2	1.4	1.2

Now taking 1 cm on *OX* to represent 30° (π/6) and 1 cm on *OY* to represent 0.5, the graph of sec *x* is as shown below

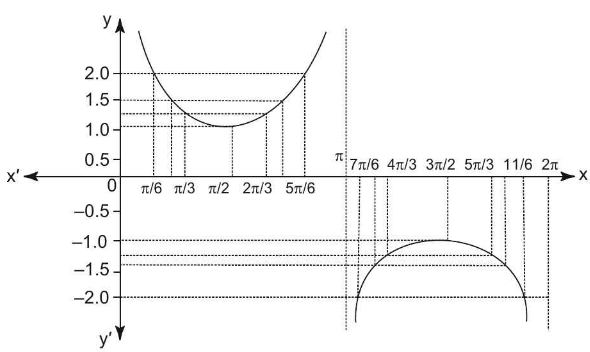


2.6 Graph of cosec *x*, for $0 \le x \le 2\pi$

Let $y = \text{cosec } x$

x	0 + 0	π/6	π/4	π/3	π/2	2π/3	3π/4	5π/6	π – 0
y	+∞	2	1.4	1.2	1	1.2	1.4	2	∞
x	π + 0	7π/6	5π/4	4π/3	3π/2	5π/3	7π/4	11π/6	2π – 0
y	+∞	–2	–1.4	–1.2	–1	–1.2	–1.4	–2	–∞

Now taking 1 cm on *OX* to represent 30° (π/6) and 1 cm on *OY* to represent 0.5, the graph of cosec *x* is as shown



3. DOMAIN AND RANGE OF TRIGONOMETRIC FUNCTIONS

Trigonometrical Function	Domain	Range
$\sin x$	R or $(-\infty, +\infty)$	$[-1, 1]$
$\cos x$	R or $(-\infty, +\infty)$	$[-1, 1]$
$\tan x$	$R-\left\{(2n+1)\frac{\pi}{2}, n \in Z\right\}$	R or $(-\infty, +\infty)$
$\cot x$	$R - \{n\pi, n \in Z\}$	R or $(-\infty, +\infty)$
$\sec x$	$R-\left\{(2n+1)\frac{\pi}{2}, n \in Z\right\}$	$R-(-1, 1)$
$\operatorname{cosec} x$	$R - \{n\pi, n \in Z\}$	$R-(-1, 1)$

4.1 Graphs of $y = a \sin bx$ and $y = a \cos bx$

Now, let us compare the graph of $y = \sin bx$, $b \neq 0$ with the graph of $y = \sin x$. Similar to $\sin x$, $\sin bx$ has values between -1 and $+1$ inclusive. Also, $\sin(bx + 2\pi) = \sin bx$, just as $\sin(x + 2\pi) = \sin x$. However, if we write $bx + 2\pi$ as $b(x + 2\pi/b)$, we see that the function defined by $y = \sin bx$ is periodic with period $\frac{2\pi}{|b|}$. We use $|b|$ instead of b to ensure a positive number for the period. $\frac{2\pi}{|b|}$ being the fundamental period of the function $y = \sin bx$, its graph is a sine wave with amplitude 1 and it completes one cycle over the interval $0 \le x \le \frac{2\pi}{|b|}$

4.2 Graph of $y = \sin(x + c)$ and $y = \cos(x + c)$,

The graph of $y = \sin(x + c)$ leads the graph of $y = \sin x$ by c.

The number c, itself, is called the phase shift of the wave.

If $c < 0$, then the graph of $y = \sin(x + c)$ is shifted $|c|$ units of the right of the graph of $y = \sin x$ and we say that it lags the graph of $y = \sin x$.

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Sketch the graph of $f(x) = \sin(x + \pi/4)$ and state the period and amplitude.

Solution

By inspecting this function, we note the following about the graph

1. It is a sin wave.

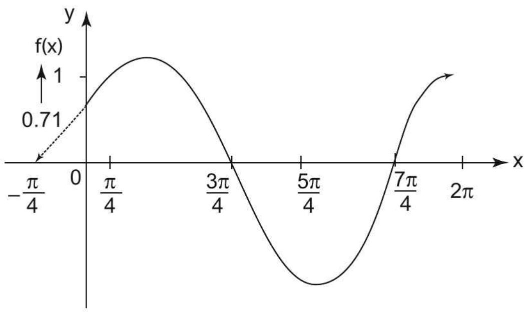
2. It has an amplitude equal to 1.

3. It has a period equal to $2\pi/1$ or 2π,

4. It leads the graph of $f(x) = \sin x$ by $\pi/4$.

With these facts and with the help of a few selected replacement values for x given in the table we can sketch the graph, as shown in figure.

x	$-\pi/4$	0	$\pi/4$	$3\pi/4$	$5\pi/4$	$7\pi/4$	2π
$x + \pi/4$	0	$\pi/4$	$\pi/2$	π	$3\pi/2$	2π	$7\pi/4$
$\sin(x + \pi/4)$	0	0.71	1	0	-1	-1	0.71



2. Sketch the graph of $h(x) = \sin(2x - \pi/3)$ and state amplitude the period.

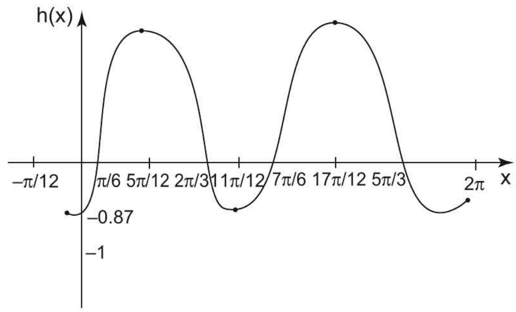
Solution

Rewrite the given function by taking 2 as a factor of the expression in the parentheses. Thus, $h(x) = \sin 2(x - \pi/6)$.

By inspecting the function, we note the following about the graph

1. It is a sine wave.
2. It has an amplitude equal to 1.
3. It has a period equal to $2\pi/2$ or π.
4. It leads the graph of $f(x) = \sin 2x$ by $\pi/6$.

With these facts and using a few replacements for x as given in the table, we can sketch the graph as shown in figure



x	0	$\pi/6$	$5\pi/12$	$2\pi/3$	$11\pi/12$	$7\pi/6$	$-\pi/12$
$2x$	0	$\pi/3$	$5\pi/6$	$4\pi/3$	$11\pi/6$	$7\pi/3$	$-\pi/6$
$2x - \pi/3$	$-\pi/3$	0	$\pi/2$	π	$3\pi/2$	2π	$-\pi/2$
$\sin(2x - \pi/3)$	−0.87	0	1	0	−1	0	−1

3. Sketch the graph of $y = 2\cos(2x - \pi/2)$.

Solution

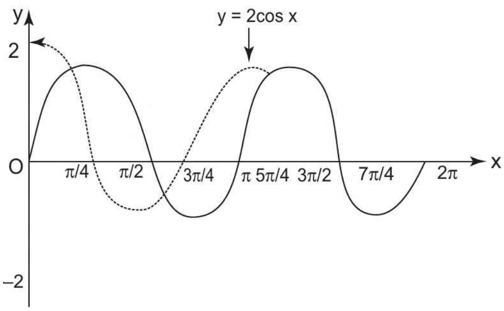
Rewrite the equation in the form $y = 2\cos 2(x - \pi/4)$

An examination of the equation reveals that

1. It is sine wave.
2. It has an amplitude equal 2.
3. It has a period equal to $2\pi/2$ or π.
4. It lags the graph of $y = 2\cos 2x$ by $\pi/4$. i.e., it is displaced $\pi/4$ to the right.

With these facts and using the replacement values given in the following table, we draw the graph as shown in figure:

x	0	$\pi/4$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
$2x - \pi/2$	$-\pi/2$	0	π	$3\pi/2$	2π	$5\pi/2$	3π	$7\pi/2$
$\cos(2x - \pi/2)$	0	1	−1	0	1	0	−1	0
$2\cos(2x - \pi/2)$	0	2	−2	0	2	0	−2	0



4. Sketch the graph of function $y = \frac{1}{2} \sin \pi x/2$.

Solution

By inspection, we note the following facts about the graph

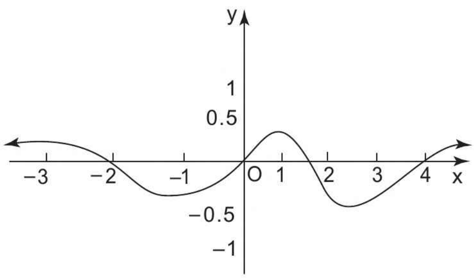
1. It is a sine wave.

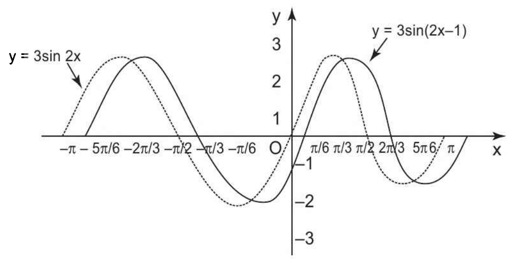
2. It has an amplitude equal to 1/2.

3. It has a period equal to $\frac{2\pi}{\pi/2}$ or 4.

Since the period is 4, we use integers as elements of the domain in sketching the graph. For our convenience, we list a few replacement values in the table.

x	0	0.5	1	1.5	2	2.5	3	3.5	4
$\pi x/2$	0	π/4	π/2	3 π/4	π	5 π/4	3 π/2	7 π/2	2 π
$\sin \pi x/2$	0	.71	1	0.71	0	–.71	–1	–71	0
$\frac{1}{2} \sin \pi x/2$	0	.36	0.5	.36	0	–.36	–0.5	–.36	0





5. Draw the graph of the function $y = 3 \sin (2x - 1)$

Solution

$$y = 3 \sin (2x - 1) = 3 \sin\left[2\left(x - \frac{1}{2}\right)\right] = 3 \sin 2u,$$

whose $x - \frac{1}{2} = u$; $= 3 \sin 2u$, $x = u + \frac{1}{2}$

This suggests that the graph of $y = 3 \sin (2x - 1)$ can be obtained by shifting the graph of $y = 3 \sin 2x$ by $\frac{1}{2}$ horizontally the forward direction.

NOTES

1. The graph of $y = \sin\left(x + \frac{\pi}{4}\right)$ can be obtained from that of $y = \sin x$ by shifting it backward horizontally through $\frac{\pi}{4}$

2. The graph of $y = \tan\left(x - \frac{\pi}{4}\right)$ from that of $y = \tan x$ by shifting it horizontally forward through $\frac{\pi}{4}$

Part B

Trigonometric Equations and Inequalities

LECTURE 1

Trigonometric Equations

BASIC CONCEPTS

1. TRIGONOMETRIC EQUATION

An equation involving one or more trigonometric ratios of unknown angles is called a trigonometric equation.
For example, $2 \cos \theta + 3 \cos 2\theta = 0$,

$$\cos^2 \theta + \sin \theta = \frac{1}{3}$$

etc., are trigonometric equations in an unknown angle θ.

2. KINDS OF TRIGONOMETRIC EQUATION

We mainly consider the three types of equations:

(i) One equation in one variable
(ii) Two equations in one variable
($\sin \theta = 1/2$, $\cos \theta = \sqrt{3}/2$)
(iii) Two equations in two variables

1. $\left(x + y = \frac{2\pi}{3}, \cos x + \cos y = \frac{3}{2}\right)$

2. $\left(x + y = \frac{\pi}{4}, \tan x + \tan y = 1\right)$

NOTES

1. Equations based on quadratic equation.
2. Equations based on maximum and minimum values of a functions.
3. Equations based on verification method.

3. SOLUTION OF A TRIGONOMETRIC EQUATION

A value of the unknown angle which satisfies the given equation is called a solution of the equation.
For example: consider the equation $\sqrt{2} \sin \theta = 1$.

The value of the angle, $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$ satisfy this equation.

Therefore, $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ are solutions of the given equation $\sqrt{2} \sin \theta = 1$.

4. GENERAL SOLUTIONS AND PRINCIPAL SOLUTIONS

Since the trigonometric functions are periodic, a solution generalized by means of periodicity is known as the general solution. The solution in the interval $[0, 2\pi]$ are called principal solutions.

5. GENERAL SOLUTIONS OF TRIGONOMETRIC EQUATIONS

5.1 Based on sin θ and cosec θ in Their Domains

$\sin \theta = 0 \Rightarrow n\pi, n \in I$, $\sin \theta = 1 \Rightarrow \theta = (4n + 1)\frac{\pi}{2}$

$\sin \theta = -1 \Rightarrow \theta = (4n - 1)\frac{\pi}{2}$

$\sin \theta = \sin \alpha \Rightarrow \theta = n\pi + (-1)^n \alpha$ or $\alpha = n\pi + (-1)^n \theta$

$\sin\theta = -\sin\alpha = \sin(-\alpha) \Rightarrow \theta = n\pi + (-1)^n\alpha$
$|\sin\theta| = \sin\alpha \Rightarrow \theta = n\pi \pm \alpha$
$\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha$

5.2 Based on cos θ and sec θ in Their Domain

$\cos\theta = 0 \Rightarrow \theta = (2n \pm 1)\frac{\pi}{2}$,
$\cos\theta = 1 \Rightarrow \theta = 2n\pi$,
$\cos\theta = -1 \Rightarrow \theta = (2n \pm 1)\pi$
$\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$
$\cos\theta = -\cos\alpha = \cos(\pi - \alpha) \Rightarrow \theta = 2n\pi \pm (\pi - \alpha)$
$|\cos\theta| = \cos\alpha \Rightarrow \theta = n\pi \pm \alpha$
$\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha$

5.3 Based on tan θ and cot θ in Their Domain

$\tan\theta = 0 \Rightarrow \theta = n\pi$
$\tan\theta = 1 \Rightarrow \theta = n\pi + \frac{\pi}{4}$,
$\tan\theta = -1 \Rightarrow \theta = n\pi - \frac{\pi}{4}$
$\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$
$\tan\theta = -\tan\alpha = \tan(-\alpha) \Rightarrow \theta = n\pi - \alpha$
$|\tan\theta| = \tan\alpha \Rightarrow \theta = n\pi \pm \alpha$
$\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$

5.4 Based on Two Equations in One Variable

(i) $\sin\theta = \sin\alpha$, $\cos\theta = \cos\alpha$ then $\theta = 2n\pi + \alpha$
(ii) $\sin\theta = \sin\alpha$, $\tan\theta = \tan\alpha$ then $\theta = 2n\pi + \alpha$
(iii) $\cos\theta = \cos\alpha$, $\tan\theta = \tan\alpha$ then $\theta = 2n\pi + \alpha$
(iv) $\cos\theta = \cos\alpha$, $\tan\theta = -\tan\alpha$ then $\theta = 2n\pi - \alpha$

Method: Find the common values of θ between 0 and 2π and then add $2n\pi$ to this common value.

5.5 Solution of the Equations of the Form of *a* cos θ ± *b* sin θ = *c*

$\theta = 2n\pi \pm \beta \pm \alpha$

where $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$, $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$ and cos

$\beta = \frac{c}{\sqrt{a^2+b^2}}$, $c \not> \sqrt{a^2+b^2}$

NOTE

If real solution of the equation exists then $|c| \le \sqrt{a^2+b^2}$.

6. SOME USEFUL HINTS FOR SOLVING TRIGONOMETRIC EQUATIONS

6.1 Squaring should be avoided as far as possible. If squaring is done, then check for extra solutions

For example, consider the equating $\sin\theta + \cos\theta = 1$

On squaring, we get

$1 + \sin 2\theta = 1$ or $\sin 2\theta = 0 => \theta = \frac{n\pi}{2}$, $n = 0 \pm 1, \pm 2...$

The values of the angle, $\theta = \pi$ and $\theta = \frac{3\pi}{2}$ do not satisfy the given equation. So we get extra solutions. Thus, if squaring is must, verify each of the solutions.

6.2 Never cancel a common factor containing 'θ' from the two sides of an equation.

For example, consider the equations $\tan\theta = \sqrt{2}\sin\theta$. If we divide both sides by sin θ, we get $\cos\theta = \frac{1}{\sqrt{2}}$, which is clearly not equivalent to the given equation as the solutions obtained by $\sin x = 0$ are lost. Thus, instead of dividing an equation by a common factor, take this factor out as a common factor from all terms of the equation.

6.3 Make sure that the answer should not contain any value of unknown 'θ' which makes any of the terms undefined.

6.4 If tan θ or sec θ is involved in the equation, θ should not be an odd multiple of π/2. ($\cos\theta \neq 0$)

6.5 If cot θ or cosec θ is involved in the equation, θ should not be a multiple of π or 0. ($\sin\theta \neq 0$)

6.6 The value of $\sqrt{f(\theta)}$ is always positive. For example,

$\sqrt{\cos^2\theta} = |\cos\theta|$ and not $\pm\cos\theta$.

6.7 All the solutions should satisfy the given equation and lie in the domain of the variable of the given equation.

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Find the general solutions of the following equation

(i) $\sin\frac{3\theta}{2} = 0$

Solution

We have,

$\sin\frac{3\theta}{2} = 0$

$\Rightarrow \quad \frac{3\theta}{2} = n\pi, n \in Z \quad [\sin\theta = 0 \Rightarrow \theta = n\pi]$

$\Rightarrow \quad \theta = \frac{2n\pi}{3}, n \in Z$

(ii) $\sin^2 2\theta = 0$

Solution

$\sin 2\theta = 0 \Rightarrow 2\theta = n\pi,\ n \in Z \Rightarrow \theta = \frac{n\pi}{2},$ $n \in Z$

2. Solve the equation: $\cos\theta + \cos 3\theta - 2\cos 2\theta = 0$.

Solution

$\cos\theta + \cos 3\theta - 2\cos 2\theta = 0$

$\Leftrightarrow \quad 2\cos 2\theta\cos\theta - 2\cos 2\theta = 0$

$\Leftrightarrow \quad 2\cos 2\theta(\cos\theta - 1) = 0$

$\Leftrightarrow \quad \cos 2\theta = 0$ or $\cos\theta - 1 = 0$

Now, $\cos 2\theta = 0 \Rightarrow 2\theta = (2n+1)\frac{\pi}{2}, n \in Z$

$\Rightarrow \quad \theta = (2n+1)\frac{\pi}{4}, n \in N$

And, $\cos\theta - 1 = 0$

$\Rightarrow \quad \cos\theta = 1$

$\Rightarrow \quad \cos\theta = \cos 0°$

$\Rightarrow \quad \theta = 2m\pi \pm 0, m \in Z$

$\Rightarrow \quad \theta = 2m\pi, m \in Z$

Hence, $\theta = (2n+1)\frac{\pi}{4}$ or, $\theta = 2m\pi$, where $m, n \in Z$.

3. Solve the equation: $\sin m\theta + \sin n\theta = 0$.

Solution

We have $\sin m\theta + \sin n\theta = 0$

$\Rightarrow \quad 2\sin\left(\frac{m+n}{2}\right)\cos\left(\frac{m-n}{2}\right)\theta = 0$

$\Rightarrow \quad \sin\left(\frac{m+n}{2}\right)\theta = 0$ or $\cos\left(\frac{m-n}{2}\right)\theta = 0$

Now, $\sin\left(\frac{m+n}{2}\right)\theta = 0$

$\Rightarrow \quad \left(\frac{m+n}{2}\right)\theta = r\pi, r \in Z$

$\Rightarrow \quad \theta = \frac{2r\pi}{m+n}, r \in Z$

And, $\cos\left(\frac{m-n}{2}\right)\theta = 0$

$\Rightarrow \quad \left(\frac{m-n}{2}\right)\theta = (2s+1)\frac{\pi}{2}, s \in Z$

$\Rightarrow \quad \theta = \frac{(2s+1)\pi}{m-n}, s \in Z$

Hence, $\theta = \frac{2r\pi}{m+n}$ or $\theta = \frac{(2s+1)\pi}{m-n}$ where $m, n \in Z$.

4. Solve the equation: $\tan^2\theta + (1-\sqrt{3})\tan\theta - \sqrt{3} = 0$.

Solution

$\tan^2\theta + (1-\sqrt{3})\tan\theta - \sqrt{3} = 0$

$\Rightarrow \quad \tan^2\theta + \tan\theta - \sqrt{3}\tan\theta - \sqrt{3} = 0$

$\Rightarrow \quad \tan(\tan\theta + 1) - \sqrt{3}(\tan\theta + 1) = 0$

$\Rightarrow \quad (\tan\theta + 1)(\tan\theta - \sqrt{3}) = 0$

$\Rightarrow \quad \tan\theta + 1 = 0$ or $\tan\theta - \sqrt{3} = 0$

$\Rightarrow \quad \tan\theta = -1$ or $\tan\theta = \sqrt{3}$

Now, $\tan\theta = -1 \Rightarrow \tan\theta = \tan\theta\left(\frac{-\pi}{4}\right)$

$\Rightarrow \quad \theta = n\pi - \frac{\pi}{4}, n \in Z$

And, $\tan\theta = \sqrt{3} \Rightarrow \tan\theta = \tan\frac{\pi}{3}$

$\Rightarrow \quad \theta = m\pi + \frac{\pi}{3}, n \in Z$

Hence, $\theta = n\pi - \frac{\pi}{4}$ or $\theta = m\pi + \frac{\pi}{3}$, where m, $n \in Z$.

5. Solve the equation: $\tan\theta + \tan 2\theta + \tan\theta \tan 2\theta = 1$

Solution

We have,

$\tan\theta + \tan 2\theta + \tan\theta\tan 2\theta = 1$

$\Rightarrow \quad \tan\theta + \tan 2\theta = 1 - \tan\theta\tan 2\theta$

$\Rightarrow \quad \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta} = 1 \Rightarrow \tan 3\theta = 1$

$\Rightarrow \quad \tan 3\theta = \tan 45°$

$\Rightarrow \quad 3\theta = n\pi + \frac{\pi}{4}, n \in Z$

$\Rightarrow \quad \theta = \frac{n\pi}{3} + \frac{\pi}{12}, n \in Z$

6. Solve: $4\sin x \sin 2x \sin 4x = \sin 3x$

Solution

We have

$4\sin x \sin 2x \sin 4x = \sin 3x$

$\Rightarrow \quad 4\sin x \sin(3x - x).\sin(3x + x) = \sin 3x$

$\Rightarrow \quad 4[\sin x(\sin^2 3x - \sin^2 x)] = 3\sin x - 4\sin^3 x$

$\Rightarrow \quad 4\sin x \sin^2 3x - 4\sin^3 x = 3\sin x - 4\sin^3 x$

$\Rightarrow \quad 4\sin x \sin^2 3x = 3\sin x$

$\Rightarrow \quad \sin x\,(4\sin^2 3x - 3) = 0$

$\Rightarrow \quad \sin x = 0$ or $4\sin^2 3x - 3 = 0$

$\Rightarrow \quad \sin x = 0$ or $\sin^2 3x = \frac{3}{4}$

Now, $\sin x = 0 \Rightarrow x = n\pi, n \in Z$

And, $\sin^2 3x = \frac{3}{4}$

$\Rightarrow \quad \sin^2 3x = \left(\frac{\sqrt{3}}{2}\right)^2 \Rightarrow \sin^2 3x = \sin^2\frac{\pi}{3}$

$\Rightarrow \quad 3x = m\pi \pm \frac{\pi}{3}, m \in Z \quad \Rightarrow \quad x = \frac{m\pi}{3} \pm \frac{\pi}{9}$

Hence, $x = n\pi$ or $x = m\pi \pm \frac{\pi}{3}$, where $m, n \in Z$

7. Solve: $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$

Solution

We have,

$\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$ .... (i)

This is of the form $a\cos\theta + b\sin\theta = c$, where,

$a = \sqrt{3}$, $b = 1$ and $c = \sqrt{2}$

Let, $a = r\cos\alpha$ and $1 = r\sin\alpha$. Then,

$\sqrt{3} = r\cos\alpha$ and $1 = r\sin\alpha$

$\Rightarrow \quad r = \sqrt{a^2 + b^2} = \sqrt{(\sqrt{3})^2 + 1^2} = 2$

and $\tan\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}$

Substituting $a = \sqrt{3} = r\cos\alpha$ and $b = 1 = r\sin\alpha$ in the equation (i) it reduce to

$r\cos\alpha\cos\theta + r\sin\alpha\sin\theta = \sqrt{2}$

$\Rightarrow \quad r\cos(\theta - \alpha) = \sqrt{2}$

$\Rightarrow \quad 2\cos\left(\theta - \frac{\pi}{6}\right) = \sqrt{2}$

$\Rightarrow \quad \cos\left(\theta - \frac{\pi}{6}\right) = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \cos\left(\theta - \frac{\pi}{6}\right) = \cos\frac{\pi}{4}$

$\Rightarrow \quad \theta - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4}, n \in Z$

$\Rightarrow \quad \theta = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{6}, n \in Z$

$\Rightarrow \quad \theta = 2n\pi + \frac{\pi}{4} + \frac{\pi}{6}$ or $\theta = 2n\pi - \frac{\pi}{4} + \frac{\pi}{6}$

$\Rightarrow \quad \theta = 2n\pi + \frac{5\pi}{12}$ or, $\theta = 2n\pi - \frac{\pi}{12}$

Hence, $\theta = 2n\pi + \frac{5\pi}{12}$ or $\theta = 2n\pi - \frac{\pi}{12}$, where $n \in Z$

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): SOLVE THESE PROBLEMS TO GRASP THE TOPIC

EXERCISE 1

1. Solve the following equations

(i) $\sin 3\theta + \cos 2\theta = 0$

(ii) $\cot^2\theta + \dfrac{3}{\sin\theta} + 3 = 0$

(iii) $-2\tan\theta - \cot\theta = -1$

(iv) $\tan\theta + \tan\left(\theta + \dfrac{\pi}{3}\right) + \tan\left(\theta + \dfrac{2\pi}{3}\right) = 3$

(v) $\sin 3\alpha = 4\sin\alpha \sin(x+\alpha)\sin(x-\alpha)$, where $\alpha \neq n\pi$, $n \in Z$

(vi) $\cot\theta + \operatorname{cosec}\theta = \sqrt{3}$

(vii) $\sin^2\theta - \cos\theta = \dfrac{1}{4}$

(viii) $\sin 3\theta - \sin\theta = 4\cos^2\theta - 2$

(ix) $\sin\theta + \cos\theta = 1$

2. Find the general solutions of the following equations:

(i) $\cos\theta = -\dfrac{\sqrt{3}}{2}$

(ii) $\operatorname{cosec}\theta = -\sqrt{2}$

(iii) $\sec\theta = \sqrt{2}$

(iv) $\tan\theta = -\dfrac{1}{\sqrt{3}}$

(v) $\sin 9\theta = \sin\theta$

ANSWERS

EXERCISE 1

1. **(i)** $\theta = \dfrac{2n\pi}{5} - \dfrac{\pi}{10}$

(ii) $\theta = m\pi + (-1)^{m+1}\dfrac{\pi}{2}$, $m, n \in Z$

(iii) $\theta = n\pi - \dfrac{\pi}{4}$

(iv) $\theta = \dfrac{n\pi}{3} + \dfrac{\pi}{12}, n \in Z$

(v) $x = n\pi \pm \dfrac{\pi}{3}, n \in Z$

(vi) $\theta = 2n\pi + \dfrac{\pi}{3}, n \in Z$

2. **(i)** $\theta = 2n\pi \pm \dfrac{5\pi}{6}, n \in Z$

(ii) $\theta = n\pi + (-1)^{m-1}\dfrac{\pi}{4}$, $n \in Z$

(iii) $\theta = 2n\pi \pm \dfrac{\pi}{4}, n \in Z$

(iv) $\theta = n\pi - \dfrac{\pi}{6}, n \in Z$

(v) $\theta = \dfrac{r\pi}{4}$ or $\theta = (2r+1)\dfrac{\pi}{10}$ where $r \in Z$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. The equation $3\sin^2 x + 10\cos x - 6 = 0$ is satisfied if *[UPSEAT-2001]*

(a) $x = n\pi \pm \cos^{-1}(1/3)$

(b) $x = 2n\pi \pm \cos^{-1}(1/3)$

(c) $x = n\pi \pm \cos^{-1}(1/6)$

(d) $x = 2n\pi \pm \cos^{-1}(1/6)$

Solution

(b) $3\sin^2 x + 10\cos x - 6 = 0$

$3(1-\cos^2 x) + 10\cos x - 6 = 0$

On solving, $(\cos x - 3)(3\cos x - 1) = 0$

Either, $\cos x = 3$, (which is not possible) or $\cos x = 1/3$

$\Rightarrow \quad x = 2n\pi \pm \cos^{-1}(1/3)$

2. If $4\sin^4 x + \cos^4 x = 1$, then x is equal to **[Roorkee-1989]**

(a) $n\pi$ (b) $n\pi \pm \sin^{-1}\frac{2}{5}$

(c) $n\pi + \frac{\pi}{6}$ (d) None of these

Solution

(a) The given equation can be put in the form

$4\sin^4 x = 1 - \cos^4 x = (1 - \cos^2 x)(1 + \cos^2 x)$

$\Rightarrow \quad \sin^2 x\,[4\sin^2 x - 1 - (1 - \sin^2 x)] = 0$

$\Rightarrow \quad \sin^2 x\,[5\sin^2 x - 2] = 0 \Rightarrow \sin x = 0$ or $\sin x = \pm\sqrt{2/5}$

Hence, $x = n\pi$ is the required answer.

3. The roots of the equation $1 - \cos\theta = \sin\theta.\sin\frac{\theta}{2}$ is **[Orissa JEE-2004]**

(a) $k\pi, k \in I$ (b) $2k\pi, k \in I$

(c) $k\frac{\pi}{2}, k \in I$ (d) None of these

Solution

(b) We have, $1 - \cos\theta = \sin\theta\,.\,\sin\frac{\theta}{2}$

$\Rightarrow \quad 2\sin^2\theta = 2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}.\sin\frac{\theta}{2}$

$\Rightarrow \quad 2\sin^2\frac{\theta}{2}\left[\cos\frac{\theta}{2}\right] = 0 \Rightarrow \sin\frac{\theta}{2} = 0$ or $2\sin^2\frac{\theta}{4} = 0$

$\Rightarrow \quad \sin\frac{\theta}{2} = 0$ or $\sin\frac{\theta}{4} = 0 \Rightarrow \frac{\theta}{2} = k\pi$ or $\frac{\theta}{4} = k\pi$

Hence, $\theta = 2k\pi$ or $\theta = 4k\pi$, $k \in I$.

4. If $|k| = 5$ and $0° \le \theta \le 360°$, then the number of different solutions of $3\cos\theta + 4\sin\theta = k$ is

(a) Zero (b) Two

(c) One (d) Infinite

Solution

(b) $3\cos\theta + 4\sin\theta = 5\left(\frac{3}{5}\cos\theta + \frac{4}{5}\sin\theta\right) =$

$5\cos(\theta - \alpha)$ where, $\cos\alpha = \frac{3}{5}, \sin\alpha = \frac{4}{5}$.

Now $3\cos\theta + 4\sin\theta = k$

$\therefore \quad 5\cos(\theta - \alpha) = k \Rightarrow \cos(\theta - \alpha) = \pm 1$

$\Rightarrow \quad \theta - \alpha = 0°, 180° \Rightarrow \theta = \alpha, 180° + \alpha$.

5. The number of values of x in the interval $[0, 5\pi]$ satisfying the equation $3\sin^2 x - 7\sin x + 2 = 0$ is

[IIT-1998; MP PET-2000; Pb. CET-2003]

(a) 0 (b) 5

(c) 6 (d) 10

Solution

(c) $3\sin^2 x - 7\sin x + 2 = 0$

$\Rightarrow \quad 3\sin^2 x - 6\sin x - \sin x + 2 = 0$

$\Rightarrow \quad 3\sin x(\sin x - 2) - (\sin x - 2) = 0$

$\Rightarrow \quad (3\sin x - 1)(\sin x - 2) = 0 \Rightarrow \sin x = \frac{1}{3}$ or 2

$\Rightarrow \quad \sin x = \frac{1}{3}, \quad (\because \sin x \ne 2)$

Let, $\sin^{-1}\frac{1}{3} = \alpha, 0 < \alpha < \frac{\pi}{2}$ are the solutions in $[0, 5\pi]$.

Then, $\alpha, \pi - \alpha, 2\pi + \alpha, 3\pi - \alpha, 4\pi + \alpha, 5\pi - \alpha$ are the solutions in $[0, 5\pi]$.

$\therefore$ Required number of solutions = 6.

6. The equation $\sin x + \sin y + \sin z = -3$ for $0 \le x \le 2\pi, 0 \le y \le 2\pi, 0 \le z \le 2\pi$, has

[Orissa JEE-2003]

(a) One solution

(b) Two sets of solutions

(c) Four sets of solutions

(d) No solution

Solution

(a) Given $\sin x + \sin y + \sin z = -3$ is satisfied only when $x = y = z = \frac{3\pi}{2}$, for $x, y, z \in [0, 2\pi]$.

7. If $\cos 6\theta + \cos 4\theta + \cos 2\theta + 1 = 0$, where $0 < \theta < 180°$, then θ is equal to

(a) 30°, 45°

(b) 45°, 90°

(c) 135°, 150°

(d) 30°, 45°, 90°, 135°, 150°

Solution

(d) $\cos 6\theta + \cos 4\theta + \cos 2\theta + 1 = 0$

$\Rightarrow$ $2\cos^2 3\theta + 2\cos 3\theta \cdot \cos\theta = 0$

$\Rightarrow$ $4\cos 3\theta \cos 2\theta \cos\theta = 0$

$\Rightarrow$ $3\theta = (2n+1)\frac{\pi}{2};\ 2\theta = (2n+1)\frac{\pi}{2}$ and

$\theta = (2n+1)\frac{\pi}{2}$

$\Rightarrow$ $\theta = 30°, 90°, 150°, 45°, 135°$

8. $2\sin^2 x + \sin^2 2x = 2, -\pi < x < \pi$, then x is equal to ***[ISM Dhanbad-1989]***

(a) $\pm\frac{\pi}{6}$ (b) $\pm\frac{\pi}{4}$

(c) $\frac{3\pi}{2}$ (d) None of these

Solution

(b) We have $1 - \cos 2x + 1 - \cos^2 2x = 2$
or $\cos 2x(\cos 2x + 1) = 0$

$\therefore$ $\cos 2x = 0, -1$, $\therefore$ $2x = \left[n + \frac{1}{2}\right]\pi$ or $(2n + 1)\pi$

$\Rightarrow$ $x = (2n+1)\frac{\pi}{4}$ or $(2n+1)\frac{\pi}{2}$

Now put, $n = -2, -1, 0, 1, 2$

$\therefore$ $x = \frac{-3\pi}{4}, \frac{-\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$

and $\frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

Since, $-\pi \le x \le \pi$, therefore $x \pm\frac{\pi}{4}, \pm\frac{\pi}{2}, \pm\frac{3\pi}{4}$ only.

9. If $5\cos 2\theta + 2\cos^2\frac{\theta}{2} + 1 = 0$, $-\pi < \theta < \pi$, then θ is equal to

[Roorkee-1984]

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{3}, \cos^{-1}\frac{3}{5}$

(c) $\cos^{-1}\frac{3}{5}$ (d) $\frac{\pi}{3}, \pi - \cos^{-1}\frac{3}{5}$

Solution

(d) $5\cos 2\theta + 2\cos^2\frac{\theta}{2} + 1 = 0$

$\Rightarrow$ $5(2\cos^2\theta - 1) + (1 + \cos\theta) + 1 = 0$

$\Rightarrow$ $10\cos^2\theta + \cos\theta - 3 = 0$

$\Rightarrow$ $(5\cos\theta + 3)(2\cos\theta - 1) = 0$

$\Rightarrow$ $\cos\theta = \frac{1}{2}, \cos\theta = -\frac{3}{5}$

$\Rightarrow$ $\theta = \frac{\pi}{3}, \pi - \cos^{-1}\left(\frac{3}{5}\right)$

10. If $0 \le x \le \pi$ and $81^{\sin^2 x} + 81^{\cos^2 x} = 30$, then x is equal to ***[Karnataka CET-2004]***

(a) $\pi/6$ (b) $\pi/2$

(c) $\pi/4$ (d) $3\pi/4$

Solution

(a) We have $81^{\sin^2 x} + 81^{\cos^2 x} = 30$

Now check by options, put $x = \frac{\pi}{6}$

then $(81)^{\sin^2 \pi/6} + (81)^{\cos^2 \pi/6} = 30$

$\Rightarrow$ $(81)^{1/4} + (81)^{3/4} = 30 \Rightarrow 30 = 30$

Hence, (a) is the correct answer.

11. In n is any integer, then the general solution of the equation $\cos x - \sin x = \frac{1}{\sqrt{2}}$ is

[J & K-2005]

(a) $x = 2n\pi - \frac{\pi}{12}$ or $x = 2n\pi + \frac{7\pi}{12}$

(b) $x = n\pi \pm \frac{\pi}{12}$

(c) $x = 2n\pi + \frac{\pi}{12}$ or $x = 2n\pi - \frac{7\pi}{12}$

(d) $x = n\pi + \frac{\pi}{12}$ or $x = n\pi - \frac{7\pi}{12}$

Solution

(c) Given equation is, $\cos x - \sin x = \frac{1}{\sqrt{2}}$

Dividing equation by $\sqrt{2}, \frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x = \frac{1}{2}$

$\cos\left(\frac{\pi}{4} + x\right) = \cos\frac{\pi}{3}$. Hence, $\frac{\pi}{4} + x = 2n\pi \pm \frac{\pi}{3}$

$x = 2n\pi + \frac{\pi}{3} - \frac{\pi}{4} = 2n\pi + \frac{\pi}{12}$

or $x = 2n\pi - \frac{\pi}{3} - \frac{\pi}{4} = 2n\pi - \frac{7\pi}{12}$

12. Find real values of x for which $27^{\cos 2x} . 81^{\sin 2x}$ is minimum. Also find this minimum value. ***[Roorkee-2000]***

Solution

$E = 3^{3\cos 2x + 4\sin 2x}$ [by using formula]

$-\sqrt{a^2+b^2} \le a\cos x + b\sin x \le \sqrt{a^2+b^2}$

minimum value of given E is $3^{-\sqrt{a^2+b^2}}$

put $a = 3$ and $b = 4$ we get minimum value of $E = 3^{-5} = 1/243$.

13. The number of solutions of the equation $\tan x + \sec x = 2\cos x$ lying in the interval $[0, 2\pi]$ is ***[IIT-1993]***

(a) 0 (b) 1
(c) 2 (d) 3

Solution

(c) $\dfrac{1+\sin x}{\cos x} = 2\cos x \ (\cos x \neq 0)$

$\therefore$ $1 + \sin x = 2(1 + \sin x)(1 - \sin x)$ or $1 = 2(1 - \sin x)$

$\therefore$ $\sin x = \dfrac{1}{2}$ $\therefore$ $x = \dfrac{\pi}{6}$ or $\pi - \dfrac{\pi}{6} = \dfrac{5\pi}{6}$

$1 + \sin x \neq 0$ because if $1 + \sin x = 0$ i.e., $\sin x = -1$, then, $\cos x = 0$ which is not true.

14. If $2(\sin x - \cos 2x) - \sin 2x(1 + 2\sin x)\ 2\cos x = 0$ then ***[Karnataka CET-02]***

(a) $x = \dfrac{\pi}{6}(4n+1)$ or $x = \dfrac{\pi}{2}(4n-1)$

(b) $x = \dfrac{\pi}{6}(4n-1)$ or $x = \dfrac{\pi}{2}(4n-1)$

(c) $x = \dfrac{\pi}{6}(4n+1)$ or $x = \dfrac{\pi}{2}(4n+1)$

(d) None of these

Solution

(a) $2(\sin x - \cos 2x) - \sin 2x(1 + 2\sin x) + 2\cos x = 0$

$\Rightarrow$ $2\sin x - 2 + 4\sin^2 x - 2\sin x\cos x - 4\sin^2 x\cos x + 2\cos x = 0$

$\Rightarrow$ $4\sin^2 x + 2\sin x - 2 - \cos x[4\sin^2 x + 2\sin x - 2] = 0$

$\Rightarrow$ $(1 - \cos x)(\sin x + 1)(4\sin x - 2) = 0$

Hence, $\sin x = -1$ or $\cos x = 1$ or $\sin x = \dfrac{1}{2}$

$\Rightarrow$ $x = (4n-1)\dfrac{\pi}{2}$ and $x = (4n+1)\dfrac{\pi}{6}$

15. If θ and ϕ are acute satisfying $\sin\theta = \dfrac{1}{2}, \cos\phi = \dfrac{1}{3}$, then $\theta + \phi \in$ ***[IIT Screening-2004]***

(a) $\left[\dfrac{\pi}{3}, \dfrac{\pi}{2}\right]$ (b) $\left[\dfrac{\pi}{2}, \dfrac{2\pi}{3}\right]$

(c) $\left[\dfrac{2\pi}{3}, \dfrac{5\pi}{6}\right]$ (d) $\left[\dfrac{5\pi}{6}, \pi\right]$

Solution

(b) $\sin\theta = \dfrac{1}{2}$ $\Rightarrow$ $\theta = \dfrac{\pi}{6}$ $\cos\phi = \dfrac{1}{3}$

$\Rightarrow$ $\dfrac{\pi}{3} < \phi < \dfrac{\pi}{2}$. Thus, $\dfrac{\pi}{2} < (\theta + \phi) < \dfrac{2\pi}{3}$.

16. If $\sin^2\theta = \dfrac{x^2+y^2+1}{2x}$, then x must be ***[MPPET-2006]***

(a) -3 (b) -2
(c) 1 (d) None

Solution

(d) Here is one equation and two variables, therefore, it is impossible to find the value of x.

17. The solution of equation $\cos^2\theta + \sin\theta + 1 = 0$ lies in the interval ***[MPPET-2006]***

(a) $\left(-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right)$ (b) $\left(\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right)$

(c) $\left(\dfrac{3\pi}{4}, \dfrac{5\pi}{4}\right)$ (d) $\left(\dfrac{5\pi}{4}, \dfrac{7\pi}{4}\right)$

Solution

(d) The equation is, $\cos^2\theta + \sin\theta + 1 = 0$

$\Rightarrow$ $1 - \sin^2\theta + \sin\theta + 1 = 0$

$\Rightarrow$ $\sin^2\theta - \sin\theta - 2 = 0$

$\Rightarrow$ $(\sin\theta + 1)(\sin\theta - 2) = 0$

$\Rightarrow$ $\sin\theta + 1 = 0$ $(\because \sin\theta \not\geq 1)$

$\Rightarrow$ $\sin\theta = -1 = \sin\dfrac{3\pi}{2}$

$\therefore$ $\theta = \dfrac{3\pi}{2} \in \left[\dfrac{5\pi}{4}, \dfrac{7\pi}{4}\right]$

18. The general solution of sin x – 3 sin 2x + sin 3x = cos x – 3 cos 2x + cos 3x is **[IIT-89]**

(a) $n\pi + \frac{\pi}{8}$ (b) $\frac{n\pi}{2} + \frac{\pi}{8}$

(c) $(-1)^n \frac{n\pi}{2} + \frac{\pi}{8}$ (d) $2n\pi + \cos^{-1}\frac{2}{3}$

Solution

sin x – 3 sin 2x + sin 3x = cos x – 3 cos 2x + cos 3x

$\Rightarrow$ 2sin 2x cos x –3 sin 2x – 2 cos 2x cos x + 3 cos 2x = 0

$\Rightarrow$ sin 2x (2cos x – 3) – cos 2x (2 cos x – 3) = 0

$\Rightarrow$ (sin 2x – cos 2x) (2 cos x – 3) = 0

$\Rightarrow$ $\sin 2x = \cos 2x \Rightarrow \tan 2x = 1 = \tan\frac{\pi}{4}$

$\Rightarrow$ $2x = n\pi + \frac{\pi}{4} \Rightarrow x = \frac{n\pi}{2} + \frac{\pi}{8}$

19. The equation cos 2x + b sin x = 2b – 7 possesses a solution, if

(a) $b < 2$ (b) $2 \le b \le 6$
(c) $b > 2$ (d) b is any integer

Solution

$1 - 2\sin^2 x + b \sin x = 2b - 7 \Rightarrow 2\sin^2 x - b \sin x + 2b - 8 = 0$

$\Rightarrow$ $\sin x = \frac{b \pm \sqrt{b^2 - 8(2b-8)}}{4}$

For sin x to be the real value $b^2 - 8(2b - 8) \ge 0$

and $-1 \le \frac{b \pm \sqrt{b^2 - 8(2b-8)}}{4} \le 1$

For $2 \le b \le 6$, all inequalities hold good.

20. If $\left(\frac{\sin\theta}{\sin\phi}\right)^2 = \frac{\tan\theta}{\tan\phi} = 3$ then the value of θ and ϕ are

(a) $\theta = n\pi \pm \frac{\pi}{3}, \phi = n\pi \pm \frac{\pi}{6}$

(b) $\theta = n\pi - \frac{\pi}{3}, \phi = n\pi - \frac{\pi}{6}$

(c) $\theta = n\pi \pm \frac{\pi}{2}, \phi = n\pi + \frac{\pi}{3}$

(d) None of these

Solution

$\left(\frac{\sin\theta}{\sin\phi}\right)^2 = \frac{\tan\theta}{\tan\phi}$

$\Rightarrow$ $\sin\theta\cos\theta = \sin\phi\cos\phi \Rightarrow \sin 2\theta = \sin 2\phi$

$2\theta = \pi - 2\phi \Rightarrow \theta = \frac{\pi}{2} - \phi$

But, $\frac{\tan\theta}{\tan\phi} = 3 \Rightarrow \frac{\tan\theta}{\cot\theta} = 3 \Rightarrow \tan^2\theta = 3$

$\Rightarrow$ $\theta = n\pi \pm \frac{\pi}{3}$, so that $\phi = n\pi \pm \frac{\pi}{6}$

21. The solution of the equation sec θ – cosec θ $= \frac{4}{3}$ is **[Roorkee-1994]**

(a) $\frac{1}{2}[n\pi + (-1)^n \sin^{-1}(3/4)]$

(b) $n\pi + (-1)^n \sin^{-1}(3/4)$

(c) $\frac{n\pi}{2} + (-1)^n \sin^{-1}(3/4)$

(d) None of these

Solution

(a) $3(\sin\theta - \cos\theta) = 4\sin\theta\cos\theta = 2\sin 2\theta$

Squaring both sides, we get $9(1 - S) = 4S^2$, where $S = \sin 2\theta$ or $4S^2 + 9S - 9 = 0$

$\therefore$ $(S + 3)(4S - 3) = 0$ or $S = \frac{3}{4}$ as $S \ne -3$

or $\sin 2\theta = \frac{3}{4} = \sin\alpha \;\therefore\; 2\theta = n\pi + (-1)^n \alpha$

or $\frac{1}{2}\left(2\pi + (-1)^n \sin^{-1}\frac{3}{4}\right)$

22. If $\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta\tan 2\theta = \sqrt{3}$, then **[UPSEAT-2001]**

(a) θ = (6n + 1) π/18 ∀ $n \in I$
(b) θ = (6n + 1) π/9 ∀ $n \in I$
(c) θ = (3n + 1) π/9 ∀ $n \in I$
(d) None of these

Solution

Given relation is

$\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta\tan 2\theta = \sqrt{3}$

$\Rightarrow$ $\tan\theta + \tan 2\theta = \sqrt{3}(1 - \tan\theta\tan 2\theta)$

$\Rightarrow$ $\frac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta} = \sqrt{3} \Rightarrow \tan 3\theta = \tan(\pi/3)$

$\Rightarrow$ $3\theta = n\pi + \pi/3 \Rightarrow \theta = (3n + 1)\pi/9$

23. The solution of the equation $\cos^2 x - 2\cos x = 4\sin x - \sin 2x$, $(0 \le x \le \pi)$ is: ***[DCE-2001]***

(a) $\pi - \cot^{-1}(1/2)$ (b) $\pi - \tan^{-1}(2)$
(c) $\pi + \tan^{-1}(-1/2)$ (d) None of these

Solution

Given equation is $\cos^2 x - 2\cos x = 4\sin x - \sin 2x$

$\Rightarrow \cos^2 x - 2\cos x = 4\sin x - 2\sin x\cos x$

$\Rightarrow \cos x(\cos x - 2) = 2\sin x(2 - \cos x)$

$\Rightarrow (\cos x - 2)(\cos x + 2\sin x) = 0$

$\Rightarrow \cos x + 2\sin x = 0 \quad (\because \cos x \ne 2)$

$\Rightarrow \tan x = -\frac{1}{2} \Rightarrow x = n\pi + \tan^{-1}(-1/2), n \in I$

As $0 \le x \le \pi$, therefore, $x = \pi + \tan^{-1}(-1/2)$

24. The number of integral values of k, for which the equation $7\cos x + 5\sin x = 2k + 1$ has a solution, is ***[IIT Screening-2002]***

(a) 4 (b) 8
(c) 10 (d) 12

Solution

$-\sqrt{7^2+5^2} \le (7\cos x + 5\sin x) \le \sqrt{7^2+5^2}$

So, for solution $-\sqrt{74} \le (2k+1) \le \sqrt{74}$ or $-8.6 \le 2k + 1 \le 8.6$ or $-9.6 \le 2k \le 7.6$ or $-4.8 \le k \le 3.8$. So, integral values of k are $-4, -3, -2, -1, 0, 1, 2, 3$ (eight values).

25. If $\cos\theta + \cos 2\theta + \cos 3\theta = 0$, then the general value of θ is ***[UPSEAT-2003]***

(a) $\theta = 2m\pi \pm 2\pi/3$ (b) $\theta = 2m\pi \pm \pi/4$
(c) $\theta = m\pi + (-1)^m 2\pi/3$ (d) $\theta = m\pi(-1)^m \pi/3$

Solution

(a, b) $\cos\theta + \cos 2\theta + \cos 3\theta = 0$

$(\cos\theta + \cos 3\theta) + \cos 2\theta = 0$

$\Rightarrow 2\cos 2\theta\cos\theta + \cos 2\theta = 0 \Rightarrow \cos 2\theta(2\cos\theta + 1) = 0$

$\Rightarrow \cos 2\theta = 0 = \cos\pi/2 \Rightarrow \theta = \pi/4 \Rightarrow \theta = 2m\pi \pm \pi/4$

or $\cos\theta = \frac{-1}{2} = \cos\frac{2\pi}{3} \Rightarrow \theta = 2m\pi \pm \pi/3.$

26. If $\cot(\alpha + \beta) = 0$, the $\sin(\alpha + 2\beta)$ is equal to ***[Kerala (Engg.)-2001]***

(a) $\sin\alpha$ (b) $\cos\alpha$
(c) $\sin\beta$ (d) $\cos 2\beta$

Solution

Given $\cot(\alpha + \beta) = 0 \Rightarrow \cos(\alpha + \beta) = 0$

$\Rightarrow \alpha + \beta = (2n+1)\frac{\pi}{2}, n \in I$

$\therefore \sin(\alpha + 2\beta) = \sin(2\alpha + 2\beta - \alpha)$

$= \sin((2n+1)\pi - \alpha) = \sin(2n\pi + \pi - \alpha)$

$= \sin(\pi - \alpha) = \sin\alpha$

27. If $\frac{\tan 3\theta - 1}{\tan 3\theta + 1} = \sqrt{3}$, then the general value of θ is ***[MP PET-2004]***

(a) $\frac{n\pi}{3} - \frac{\pi}{12}$ (b) $n\pi + \frac{7\pi}{12}$
(c) $\frac{n\pi}{3} + \frac{7\pi}{36}$ (d) $n\pi + \frac{\pi}{12}$

Solution

(c) Given that $\frac{\tan 3\theta - 1}{\tan 3\theta + 1} = \sqrt{3}$ or $\tan 3\theta - 1 = \sqrt{3}\tan 3\theta + \sqrt{3}$

$\Rightarrow \tan 3\theta - 1 - \sqrt{3}\tan 3\theta - \sqrt{3} = 0$

$\Rightarrow \tan 3\theta(1 - \sqrt{3}) - (1 + \sqrt{3}) = 0$

$\Rightarrow \tan 3\theta = \frac{1+\sqrt{3}}{1-\sqrt{3}}$

$\Rightarrow \tan 3\theta = \tan 105° = \tan\frac{7\pi}{12}$

$\therefore 3\theta = n\pi + \frac{7\pi}{12} \Rightarrow \theta = \frac{n\pi}{3} + \frac{7\pi}{36}$

28. If $0 < \theta < 2\pi$, then the intervals of values of θ for which $2\sin^2\theta - 5\sin\theta + 2 > 0$ is ***[IIT-JEE-2006]***

(a) $0, \pi/6 \cup 5\pi/6\pi, 2\pi$ (b) $0, \pi/8 \cup \pi/6, 5\pi/6$
(c) $\pi/8, 5\pi/6$ (d) $41\pi/48, \pi$

Solution

(a) $2\sin^2\theta - 5\sin\theta + 2 > 0 \Rightarrow (\sin\theta - 2)(2\sin\theta - 1) > 0$

$\Rightarrow \sin\theta < 1/2 \Rightarrow \theta \in 0, \pi/6 \cup 5\pi/6, 2\pi$

29. The number of values of θ in the interval $[-\pi, \pi]$ satisfying the equation $\cos\theta + \sin 2\theta = 0$ is ***[MPPET-2005]***

(a) 1 (b) 2
(c) 3 (d) 4

Solution

(d) $\cos\theta + \sin 2\theta = 0$

$\Rightarrow$ $\cos\theta(1 + 2\sin\theta) = 0 \Rightarrow \cos\theta = 0$ or sin $\Rightarrow = -1/2$

$\therefore$ The number of values of θ in the interval $[-\pi, \pi]$ satisfying the equation is 4.

30. The number of solution of the equation $2\cos(e^x) = 5^x + 5^{-x}$, are ***[UPSEAT-2004; IIT-1992; MPPET-2006]***

(a) No solution
(b) One solution
(c) Two solutions
(d) Infinitely many solutions

Solution

(a) We have $\begin{array}{c} \cos(e^x) \le 1 \\ \therefore 2\cos(e^x) \le 2 \end{array} \left| 5^x + \frac{1}{5^x} \ge 2, \right.$

At $x = 0$, $5^x + \frac{1}{5^x} = 2$

But at $x = 0$, $2\cos(e^x) \ne 2$

$\therefore$ The given equation has no solution.

31. The expression $(1 + \tan x + \tan^2 x)(1 - \cot x + \cot^2 x)$ has the positive values of x, given by

(a) $0 \le x \le \pi/2$ (b) $0 \le x \le \pi$
(c) For all $x \in R$ (d) $x \ge 0$

Solution

(c) The expression is

$$\frac{(1+\tan x+\tan^2 x)(1+\tan^2 x-\tan x)}{\tan^2 x}$$

$$= \frac{(1+\tan^2 x)^2 - \tan^2 x}{\tan^2 x}$$

Obviously, $1 + \tan^2 x \ge \tan^2 x$, $\forall x$.

Hence, it is positive for all value of x.

32. If $\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 0$, then θ is equal to ***[Dhanbd Engg.-72]***

(a) $n\pi/4$ (b) $n\pi/2$
(c) $n\pi/8$ (d) None of these

Solution

$$\frac{\cos\left(\frac{\theta+7\theta}{2}\right)\sin 4(\theta)}{\sin(\theta)} = 0 \Rightarrow \frac{2}{2}\cos 4\theta \sin 4\theta = 0$$

$\sin 8\theta = 0 \Rightarrow 8\theta = n\pi$

$\Rightarrow \theta = \frac{n\pi}{8}$

33. The general solution of $a\cos x + b\sin x = c$, where a, b, c are constants

(a) $x = n\pi + \cos^{-1}\left(\frac{c}{\sqrt{a^2+b^2}}\right)$

(b) $x = 2n\pi - \tan^{-1}\left(\frac{b}{a}\right)$

(c) $x = 2n\pi - \tan^{-1}\left(\frac{b}{a}\right) \pm \cos^{-1}\left(\frac{c}{\sqrt{a^2+b^2}}\right)$

(d) $x = 2n\pi + \tan^{-1}\left(\frac{b}{a}\right) \pm \cos^{-1}\left(\frac{c}{\sqrt{a^2+b^2}}\right)$

Solution

(d) $\frac{a}{\sqrt{a^2+b^2}}\cos x + \frac{b}{\sqrt{a^2+b^2}}\sin x$

$= \frac{c}{\sqrt{a^2+b^2}}$

$\Rightarrow$ $\cos x - \left(\cos^{-1}\frac{a}{\sqrt{a^2+b^2}}\right) = \frac{c}{\sqrt{a^2+b^2}}$

$\Rightarrow$ $x - \cos^{-1}\frac{a}{\sqrt{a^2+b^2}} = \cos^{-1}\frac{c}{\sqrt{a^2+b^2}}$

General solution is,

$$x - \cos^{-1}\frac{a}{\sqrt{a^2+b^2}} = 2n\pi \pm \cos^{-1}\frac{c}{\sqrt{a^2+b^2}}$$

or $x = 2n\pi \pm \cos^{-1}\frac{c}{\sqrt{a^2+b^2}} + \cos^{-1}\frac{a}{\sqrt{a^2+b^2}}$

$$x = 2n\pi + \tan^{-1}\frac{b}{a} \pm \cos^{-1}\frac{c}{\sqrt{a^2+b^2}}$$

Trick: Put $a = b = c = 1$, then

$$\left(\cos x - \frac{\pi}{4}\right) = \cos\frac{\pi}{4}$$

$\Rightarrow$ $x = 2n\pi + \frac{\pi}{4} \pm \frac{\pi}{4}$ which is given by option (d).

34. Number of solution $\sec x \,.\, \cos 5x + 1 = 0$, $0 < x < 2\pi$ is

(a) 1 (b) 2
(c) 8 (d) 4

Solution

(c) $\sec x \cos 5x = -1 \Rightarrow \cos 5x = -\cos x$

$\Rightarrow$ $5x = 2n\pi \pm (\pi - x) \Rightarrow x = \frac{(2n+1)\pi}{6}$ or $\frac{(2n-1)\pi}{4}$

Hence, $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{6}, \frac{5\pi}{4}, \frac{7\pi}{6}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{6}$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. The solution set of $(5 + 4 \cos \theta) (2 \cos \theta + 1) = 0$ in the interval $[0, 2\pi]$ is

(a) $\{\pi/3, 2\pi/3\}$ (b) $\{\pi/3, \pi\}$
(c) $\{2\pi/3, 4\pi/3\}$ (d) $\{2\pi/3, 5\pi/3\}$

2. General solution of $\tan 5\theta = \cot 2\theta$ is ***[Karnataka CET-2000; Pb CET-01]***

(a) $\theta = \frac{n\pi}{7} + \frac{\pi}{14}$

(b) $\theta = \frac{n\pi}{7} + \frac{\pi}{5}$

(c) $\theta = \frac{n\pi}{7} + \frac{\pi}{2}$

(d) $\theta = \frac{n\pi}{7} + \frac{\pi}{3}, n \in Z$

3. If $\cos\theta = -1/\sqrt{2}$ and $\tan \theta = 1$, then the general value of θ is

(a) $2n\pi + \frac{\pi}{4}$ (b) $(2n+1)\pi + \frac{\pi}{4}$

(c) $n\pi + \frac{\pi}{4}$ (d) $n\pi \pm \frac{\pi}{4}$

4. The general value of θ is obtained from the equation $\cos 2\theta = \sin \alpha$ is ***[MPPET-96]***

(a) $2\theta = \frac{\pi}{2} - \alpha$

(b) $\theta = 2n\pi\left(\frac{\pi}{2} - \alpha\right)$

(c) $\theta = \frac{n\pi + (-1)^n\alpha}{2}$

(d) $\theta = n\pi \pm \left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$

5. General value of θ satisfying the equation $\tan^2 \theta + \sec 2\theta = 1$ is ***[IIT-96]***

(a) $m\pi, n\pi + \frac{\pi}{3}, m,n \in z$

(b) $m\pi, n\pi \pm \frac{\pi}{3}, m,n \in z$

(c) $m\pi, n\pi \pm \frac{\pi}{6}, m,n \in z$

(d) None of these

6. The smallest positive values of x and y which satisfy $\tan (x - y) = 1$, $\sec (x + y)\ 2/\sqrt{3}$ are

(a) $x = 25\pi/24, y = 19\pi/24$
(b) $x = 37\pi/24, y = 7\pi/24$
(c) $x = \pi/4, y = \pi/2$
(d) $x = \pi/3, y = 7\pi/12$

7. If $(1 + \tan \theta) (1 + \tan \phi) = 2$, then $\theta + \phi$ is equal to ***[Karnataka CET-93]***

(a) 30° (b) 45°
(c) 60° (d) 75°

8. The general solution of the equation $(\sqrt{3}-1)\sin\theta + (\sqrt{3}+1)\cos\theta = 2$ is ***[Roorkee-92]***

(a) $2n\pi \pm \frac{\pi}{4} + \frac{\pi}{12}$

(b) $n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{12}$

(c) $2n\pi \pm \frac{\pi}{4} - \frac{\pi}{12}$

(d) $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{12}$

9. The general value of θ satisfying the equation $2\sin^2 \theta - 3\sin \theta - 2 = 0$ is ***[Roorkee-93]***

(a) $n\pi + (-1)^n \frac{\pi}{6}$ (b) $n\pi + (-1)^n \frac{\pi}{2}$

(c) $n\pi + (-1)^n \frac{5\pi}{6}$ (d) $n\pi + (-1)^n \frac{7\pi}{6}$

10. If $\left[\sin\frac{\pi}{4}\cos\theta\right] = \left[\cos\frac{\pi}{4}\tan\theta\right]$, then θ is equal to ***[Pb. CET-88]***

(a) $n\pi + \frac{\pi}{4}$ (b) $2n\pi \pm \frac{\pi}{4}$

(c) $n\pi - \frac{\pi}{4}$ (d) $2n\pi \pm \frac{\pi}{6}$

11. If $r \sin \theta = 3$, $r = 4 (1 + \sin \theta)$, $0 \le \theta \le 2\pi$, then θ is equal to ***[Roorkee-74]***

(a) $\pi/6, \pi/3$ (b) $\pi/6, 5\pi/6$
(c) $\pi/3, \pi/4$ (d) $\pi/2, \pi$

12. If tan (cot x) = cot (tan x), then sin $2x$ is equal to **[MPPET-99; Pb CET-01]**

(a) $(2n+1)\frac{\pi}{4}$ (b) $4/(2n+1)\pi$

(c) $4\pi(2n+1)$ (d) None of these

13. If cos 2θ $=\sqrt{2}+1=0\left(\theta-\frac{1}{\sqrt{2}}\right)$, then the value of θ is **[Roorkee-77]**

(a) $2n\pi+\frac{\pi}{4}$ (b) $2n\pi\pm\frac{\pi}{4}$

(c) $2n\pi-\frac{\pi}{4}$ (d) None of these

14. If $\tan 2x=\tan\frac{2}{x}$, then value of x is

(a) $\frac{n\pi\pm\sqrt{n^2\pi^2+16}}{4}$ (b) $\frac{n\pi}{4}$

(c) $\frac{n\pi\pm\sqrt{n^2\pi^2-16}}{4}$ (d) None of these

15. The smallest positive angle which satisfies the equation $2\sin^2\theta+\sqrt{3}\cos\theta+1=0$ is

(a) 5 π/6 (b) 2 π/3

(c) π/3 (d) π/6

16. If tan θ + tan 2θ + tan 3θ = tan θ tan 2θ tan 3θ, then the general value of θ is

(a) $\frac{n\pi}{3}$ (b) $\frac{n\pi}{6}$

(c) $n\pi\pm\frac{\pi}{3}$ (d) $\frac{n\pi}{3}$

17. If $\sin^2\theta = 1/4$, then the most general value of θ is

(a) $2n\pi\pm(-1)^n\frac{\pi}{6}$ (b) $\frac{n\pi}{2}\pm(-1)^n\frac{\pi}{6}$

(c) $n\pi\pm\frac{\pi}{6}$ (d) $2n\pi\pm\frac{\pi}{6}$

18. If sec 4 θ – sec 2θ = 2, then the general value of θ is

(a) $(2n+1)\frac{\pi}{4}$ (b) $(2n+1)\frac{\pi}{10}$

(c) $n\pi+\frac{\pi}{2}$ or $\frac{n\pi}{5}-\frac{\pi}{10}$ (d) None of these

19. If $4\sin^2\theta+2(\sqrt{3}+1)\cos\theta=4+\sqrt{3}$, then the general value θ is

(a) $2n\pi\pm\frac{\pi}{3}$ (b) $2n\pi+\frac{\pi}{4}$

(c) $n\pi\pm\frac{\pi}{3}$ (d) $n\pi-\frac{\pi}{3}$

20. If $\cot\theta+\cot\left(\frac{\pi}{4}+\theta\right)=2$, then the general value of θ is

(a) $2n\pi\pm\frac{\pi}{6}$ (b) $2n\pi\pm\frac{\pi}{3}$

(c) $n\pi\pm\frac{\pi}{3}$ (d) $n\pi\pm\frac{\pi}{6}$

21. If $\tan^2\theta-(1+\sqrt{3})\tan\theta+\sqrt{3}=0$, then the general value of θ is

(a) $n\pi+\frac{\pi}{4}, n\pi+\frac{\pi}{3}$

(b) $n\pi-\frac{\pi}{4}, n\pi+\frac{\pi}{3}$

(c) $n\pi+\frac{\pi}{4}, n\pi-\frac{\pi}{3}$

(d) $n\pi-\frac{\pi}{4}, n\pi-\frac{\pi}{3}$

22. If cot θ + tan θ = 2 cosec θ, the general value of θ is **[Roorkee-71]**

(a) $n\pi\pm\frac{\pi}{3}$ (b) $n\pi\pm\frac{\pi}{6}$

(c) $2n\pi\pm\frac{\pi}{3}$ (d) $2n\pi\pm\frac{\pi}{6}$

23. If $\sqrt{3}\tan 2\theta+\sqrt{3}\tan 3\theta$ + tan 2 θ tan 3 θ = 1, then the general value of θ is

(a) $n\pi+\frac{\pi}{5}$ (b) $\left(n+\frac{1}{6}\right)\frac{\pi}{5}$

(c) $\left(2n\pm\frac{1}{6}\right)\frac{\pi}{5}$ (d) $\left(n+\frac{1}{3}\right)\frac{\pi}{5}$

24. The number of solutions of the given equation $a\sin x+b\cos x=c$ where $|c|>\sqrt{a^2+b^2}$, is **[DCE-98; AIEEE-02]**

(a) 1 (b) 2

(c) infinite (d) None of these

25. If $\sqrt{2}\sec\theta + \tan\theta = 1,$ then the general value of θ is

(a) $n\pi + \frac{3\pi}{4}$ (b) $2n\pi + \frac{\pi}{4}$

(c) $2n\pi - \frac{\pi}{4}$ (d) $2n\pi \pm \frac{\pi}{4}$

26. If $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2},$ then the most general value of θ is

(a) $n\pi + (-1)^n \frac{\pi}{4}$

(b) $(-1)^n \frac{\pi}{4} - \frac{\pi}{3}$

(c) $n\pi + \frac{\pi}{4} - \frac{\pi}{3}$

(d) $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{3}$

27. If sin θ + cos θ = 1, then the general value of θ is

(a) $2n\pi$

(b) $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}$

(c) $2n\pi + \frac{\pi}{2}$

(d) None of these

28. If cos 7θ = cos θ – sin4 θ, then the general solution of θ is

(a) $\frac{n\pi}{4}.\frac{n\pi}{3} + \frac{\pi}{18}$

(b) $\frac{n\pi}{3}.\frac{n\pi}{3} + (-1)^n \frac{\pi}{18}$

(c) $\frac{n\pi}{4}.\frac{n\pi}{3} + (-1)^n \frac{\pi}{18}$

(d) $\frac{n\pi}{6}.\frac{n\pi}{3} + (-1)^n \frac{\pi}{18}$

29. If $\sec\theta + 1 = (2+\sqrt{3})\tan\theta,$ then

(a) $\theta = n\pi \pm \frac{\pi}{2}$ (b) $\theta = 2n\pi + \frac{\pi}{6}$

(c) $\theta = n\pi \pm \frac{\pi}{4}$ (d) None of these

30. The set of value of x for which the expression $\frac{\tan 3x - \tan 2x}{1 + \tan 3x \tan 2x} = 1$ is

[MPPET-92; MNR-93; UPSEAT-02]

(a) ϕ

(b) $\frac{\pi}{4}$

(c) $\left(n\pi + \frac{\pi}{4}; n = 1,2,3...\right)$

(d) $\left(2n\pi + \frac{\pi}{4}; n = 1,2,3,...\right)$

HINTS AND EXPLANATIONS

1. (c) (5 + 4 cos θ) (2 cos θ + 1) = 0

$\Rightarrow \cos\theta = \frac{-5}{4}, \cos\theta = \frac{-1}{2}$

$\therefore$ possible value of $\cos\theta = \frac{-1}{2}$

$(\because -1 \le \cos\theta \le 1)$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ (when cos θ ∈ [0, 2π])

2. (a) tan 5θ = cot 2θ $\Rightarrow 5\theta = n\pi + \frac{\pi}{2} - 2\theta$

$\theta = \frac{n\pi}{7} + \frac{\pi}{14}$

3. (b) $\cos\theta = \frac{-1}{\sqrt{2}}, \tan\theta = 1$

$\therefore$ θ lies in III quadrant

$\theta = 2n\pi + \frac{5\pi}{4}$

4. (d) cos 2 θ = sin α

$\Rightarrow 2\theta = 2n\pi \pm \left(\frac{\pi}{2} - \alpha\right)$

$\theta = n\pi \pm \left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$

5. (b) $\tan^2\theta + \dfrac{1+\tan^2\theta}{1-\tan^2\theta} = 1$

$\Rightarrow$ $2\tan^2\theta - \tan^4\theta + 1 = 1 - \tan^2\theta$

$\tan^2\theta(\tan^2\theta - 3) = 0,$

$\theta = m\pi, n\pi \pm \dfrac{\pi}{3}, m, n \in I$

6. (a, b) $\tan(x-y) = 1$ $\quad \sec(x+y) = \dfrac{2}{3}$

Here, $x + y > x - y$ (i) ($\because$ x, y are positive values)

$x - y = \dfrac{\pi}{4}, \dfrac{5\pi}{4}; \; x + y = \dfrac{\pi}{6}, \dfrac{11\pi}{6}$

from (i) $x + y = \dfrac{11\pi}{6}, x - y = \dfrac{\pi}{4}$ or $\dfrac{5\pi}{4}$

$x = \dfrac{37\pi}{24}, y = \dfrac{7\pi}{24}$ or $x = \dfrac{25\pi}{24}, \dfrac{19\pi}{24}$

7. (b) $(1+\tan\theta)(1+\tan\phi) = 2$

$\Rightarrow$ $1 + \tan\theta + \tan\phi + \tan\theta\tan\phi = 2$

$\tan\theta + \tan\phi = 1 - \tan\theta\tan\phi \Rightarrow \tan(\theta+\phi) = 1$

$\theta + \phi = 45°$

8. (d) $(\sqrt{3}-1)\sin\theta + (\sqrt{3}+1)\cos\theta$ is equal to

dividing by $\sqrt{(\sqrt{3}-1)^2 + (\sqrt{3}+1)^2} = 2\sqrt{2}$

$\dfrac{\sqrt{3}-1}{\sqrt{3}+1}\sin\theta + \dfrac{\sqrt{3}+1}{2\sqrt{2}}\cos\theta = \dfrac{2}{2\sqrt{2}}$

$\sin\theta\sin\dfrac{\pi}{12} + \cos\theta\cos\dfrac{\pi}{12} = \dfrac{1}{\sqrt{2}}$

$\left(\sin\theta + \dfrac{\pi}{12}\right) = \sin\dfrac{\pi}{4}$

$\theta + \dfrac{\pi}{12} = n\pi + (-1)^n\dfrac{\pi}{4}$

$\theta = n\pi + (-1)^n\dfrac{\pi}{4} - \dfrac{\pi}{12}$

9. (d) $2\sin^2\theta - 3\sin\theta = 0$; $(2\sin\theta + 1)(\sin\theta - 2) = 0$

$\sin\theta = \dfrac{-1}{2}, 2 \Rightarrow$ possible value of $\sin\theta = \dfrac{-1}{2}$

$\theta = n\pi - (-1)^n\dfrac{\pi}{6}$ or $\theta = n\pi + (-1)^n\dfrac{7\pi}{6}$

10. (a) $\dfrac{\pi}{4}\cot\theta = \dfrac{\pi}{2} - \dfrac{\pi}{4}\tan\theta$

$\Rightarrow$ $\cot\theta + \tan\theta = 2$

$\therefore$ $\tan\theta = 1 \Rightarrow \theta = n\pi + \dfrac{\pi}{4}$

11. (b) $r = \dfrac{3}{\sin\theta}$; $r = 4(1+\sin\theta)$

$\dfrac{3}{\sin\theta} = 4 + 4\sin\theta \Rightarrow 4\sin^2\theta + 4\sin\theta - 3 = 0$

$4\sin^2\theta + 6\sin\theta - 2\sin\theta - 3 = 0$

$\Rightarrow$ $(2\sin\theta - 1)(2\sin\theta + 3) = 0$

$\sin\theta = \dfrac{1}{2}, \dfrac{-3}{2} \Rightarrow \theta = n\pi + (-1)^n\dfrac{\pi}{6}$

or $\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

12. (b) $\tan(\cot x) = \tan\left(\dfrac{\pi}{2} - \tan x\right)$

$\cot x = n\pi + \dfrac{\pi}{2} - \tan x$

$\Rightarrow$ $\dfrac{\cos x}{\sin x} + \dfrac{\sin x}{\cos x} = (2n+1)\dfrac{\pi}{2}$

$\dfrac{2}{2\sin x\cos} = (2n+1)\dfrac{\pi}{2} \Rightarrow \sin 2x = \dfrac{4}{(2n+1)\pi}$

13. (b) $\cos 2\theta = (\sqrt{2}+1)\left(\cos\theta - \dfrac{1}{\sqrt{2}}\right)$

$2\cos^2\theta - 1 = (\sqrt{2}+1)\left(\cos\theta - \dfrac{1}{\sqrt{2}}\right)$

$2\left(\cos\theta - \dfrac{1}{\sqrt{2}}\right)\left(\cos\theta + \dfrac{1}{\sqrt{2}}\right) - (\sqrt{2}+1)$

$\left(\cos\theta - \dfrac{1}{\sqrt{2}}\right) = 0$

$\left(\cos\theta - \dfrac{1}{\sqrt{2}}\right)\left(2\cos\theta + \dfrac{1}{\sqrt{2}}\right) - (\sqrt{2}+1) = 0$

$\left(\cos\theta - \dfrac{1}{\sqrt{2}}\right)(2\cos\theta - 1) = 0 \Rightarrow \cos\theta = \dfrac{1}{\sqrt{2}}$ or

$\cos\theta = \dfrac{1}{\sqrt{2}}$

$\theta = 2n\pi \pm \dfrac{\pi}{4}, \; 2n\pi \pm \dfrac{\pi}{3}$

14. $\tan 2x = \tan\frac{2}{x} \Rightarrow 2x = n\pi + \frac{2}{x}$

$2x^2 - (n\pi)x - 2 = 0$

$x = \frac{n\pi \pm \sqrt{n^2\pi^2 + 16}}{4}$

15. (b) $2\sin^2\theta + \sqrt{3}\cos\theta + 1 = 0,$

$2(1-\cos^2\theta) + \sqrt{3}\cos\theta + 1 = 0$

$2\cos^2\theta - \sqrt{3}\cos\theta - 3 = 0$

$\frac{\sqrt{3} \pm \sqrt{3-4(-6)}}{4} = \frac{\sqrt{3} \pm 3\sqrt{3}}{4}$

$\cos\theta = \frac{\sqrt{3}(4)}{4}, \frac{-\sqrt{3}}{2}$

possible value $\cos\theta = -\cos\frac{-\sqrt{3}}{2}$

$\cos\theta = -\cos\frac{\pi}{6}$

$\cos\theta = -\cos\frac{2\pi}{3}$

$\theta = 2n\pi \pm \frac{2\pi}{3} \quad \theta = \frac{2\pi}{3}$

16. (a) $\tan\theta + \tan 2\theta + \tan 3\theta = \tan\theta \ \tan 2\theta \tan 3\theta$ is possible if $\theta + 2\theta + 3\theta = n\pi \Rightarrow \theta = \frac{n\pi}{6}$

if $\theta \in$ off odd $\Rightarrow \tan 3\theta$ is not defined,

for $\theta \in$ even solution exist $\theta = \frac{2m\pi}{6} = \frac{m\pi}{3}$

17. (c) $\sin^2\theta = \frac{1}{4} \Rightarrow \theta = n\pi \pm \frac{\pi}{6}$

18. (b) $\sec 4\theta - \sec 2\theta = 2$

$\Rightarrow \frac{1}{\cos 4\theta} - \frac{1}{\cos 2\theta} = 2$

$\cos 2\theta - \cos 4\theta = 2\cos 2\theta \cos 4\theta$

$\Rightarrow \cos 2\theta - \cos 4\theta = \cos 2\theta + \cos 6\theta$

$\cos 6\theta + \cos 4\theta = 0 \Rightarrow 2\cos 5\theta \ \cos\theta = 0$

$\therefore \cos 5\theta = 0$ or $\cos\theta = 0$

$5\theta = (2n \pm 1)\frac{\pi}{2}$ or $\theta = (2n+1)\frac{\pi}{2}$

$\theta = (2n\pm)\frac{\pi}{10}$

19. (a) $4\sin^2\theta + 2(\sqrt{3}+1)\cos\theta = 4 + \sqrt{3}$

$4(1-\cos^2\theta) + 2(\sqrt{3}+1)\cos\theta = 4 + \sqrt{3}$

$\Rightarrow 4\cos^2\theta - 2(\sqrt{3}+1)\cos\theta + \sqrt{3} = 0$

$(2\cos\theta - \sqrt{3})(2\cos\theta - 1) = 0; \cos\theta = \frac{\sqrt{3}}{2}$ or $\frac{1}{2}$

$\theta = 2n\pi \pm \frac{\pi}{3}$ or $\theta = 2n\pi \pm \frac{\pi}{6}$

20. (d) $\cot\theta + \cot\left(\frac{\pi}{4} + \theta\right) = 2$

$\Rightarrow \cot\theta + \frac{\cot\theta - 1}{\cot\theta + 1} = 2$

$\cot^2\theta + 2\cot\theta - 1 = 2\cot\theta + 2 \Rightarrow \cot^2\theta = 3$

$\theta = n\pi \pm \frac{\pi}{6}$

21. (a) $\tan^2\theta - (1+\sqrt{3})\tan\theta + \sqrt{3} = 0$

$\Rightarrow (\tan\theta - \sqrt{3})(\tan\theta - 1) = 0$

$\tan\theta = \sqrt{3}$ or $1 \Rightarrow \theta = n\pi + \frac{\pi}{3}$ or $n\pi + \frac{\pi}{4}$

22. (c) $\cot\theta + \tan\theta = 2\operatorname{cosec}\theta$

$\Rightarrow \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} = \frac{2}{\sin\theta}$

$\cot\theta = \frac{1}{2} \Rightarrow \theta = 2n\pi \pm \frac{\pi}{3}$

23. $\sqrt{3}\tan 2\theta + \sqrt{3}\tan 2\theta = 1 - \tan 2\theta \ \tan 3\theta$

$\frac{\tan 2\theta + \tan 3\theta}{1 - \tan\theta\tan 3\theta} = \frac{1}{\sqrt{3}} \Rightarrow \tan(2\theta + 3\theta) = \frac{1}{\sqrt{3}}$

$5\theta = n\pi + \frac{\pi}{6} \Rightarrow \theta = \left(n + \frac{1}{6}\right)\frac{\pi}{5}$

24. (d) $a\sin x + b\cos x \in \left(-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}\right)$

$\therefore a\sin x + b\cos x = c$ will have no solution

25. (c) $\sqrt{2}\sec\theta + \tan\theta = 1 \Rightarrow \sqrt{2} + \sin\theta = \cos\theta$

$\cos\theta - \sin\theta = \sqrt{2} \Rightarrow \cos\left(\theta + \frac{\pi}{4}\right) = 1$

$\theta + \frac{\pi}{4} = 2m\pi \Rightarrow \theta = 2m\pi - \frac{\pi}{4}$

26. (d) $\left(\frac{\sqrt{3}}{2}\cos\theta+\frac{1}{2}\sin\theta=\frac{\sqrt{2}}{2}\right)$

$\Rightarrow \sin\left(\frac{\pi}{3}+\theta\right)=\frac{1}{\sqrt{2}}$

$\Rightarrow \frac{\pi}{3}+\theta=n\pi+(-1)^n\frac{\pi}{4}$

or $\theta=n\pi+(-1)^n\frac{\pi}{4}-\frac{\pi}{3}$

27. (b) sin θ + cos θ = 1 ⇒ $\sqrt{2}\sin\left(\theta+\frac{\pi}{4}=1\right)$

$\sin\left(\theta+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}\Rightarrow\theta+\frac{\pi}{4}=n\pi+(-1)^n\frac{\pi}{4}$

$\theta=n\pi+(-1)^n\frac{\pi}{4}-\frac{\pi}{4}$

28. (c) cos 7θ = cos θ – sin 4θ ⇒ sin 4θ = cos θ – cos 7θ

sin 4θ = 2sin 4θ sin 3θ = 1 ⇒ sin 4θ(1 – 2sin 3θ) = 0

sin 4θ = 0 or $\sin 3\theta=\frac{1}{3}$

$\theta=\frac{n\pi}{4}$

or $\theta=\frac{n\pi}{3}+(-1)^n\frac{\pi}{18}$

29. (b) $\sec\theta+1=(2+\sqrt{3})\tan\theta$

$\Rightarrow \frac{1+\cos\theta}{\cos\theta}=(2+\sqrt{3})\frac{\sin\theta}{\cos\theta}$

$2\cos^2\frac{\theta}{2}=(2+\sqrt{3})\left(\frac{2\sin\theta}{2},\frac{\cos\theta}{2}\right)$

$\cot\frac{\theta}{2}=2+\sqrt{3}\Rightarrow\frac{\theta}{2}=n\pi+\frac{\pi}{12}$

$\theta=2n\pi+\frac{\pi}{6}$

30. (a) $\frac{\tan 3x-\tan 2x}{1+\tan 3x\tan 2x}=1\Rightarrow\tan(3x-2x)=1$

or tan x =1

$\therefore\ x=nx+\frac{\pi}{4}$ but at this values tan 2x is not defined.

$\therefore\ x\in\phi$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. The general value of θ satisfying sin²θ + sin θ = 2 is ***[AMU-96, 99]***

(a) $n\pi+(-1)^n\frac{\pi}{6}$ (b) $2n\pi+\frac{\pi}{4}$

(c) $n\pi+(-1)^n\frac{\pi}{2}$ (d) $n\pi+(-1)^n\frac{\pi}{3}$

2. The general solution of

$\sin^2\theta\sec\theta+\sqrt{3}\tan\theta=0$ is

(a) $\theta=n\pi+(-1)^{n+1}\frac{\pi}{3},\theta=n\pi,n\in Z$

(b) θ = $n\pi$, $n \in Z$

(c) $\theta=n\pi+(-1)^{n+1}\frac{\pi}{3},\ n\in Z$

(d) $\theta=\frac{n\pi}{2},\ n\in Z$

3. The value of θ satisfying sin 7θ = sin 4θ – sin θ and $0<\theta<\frac{\pi}{2}$ are

(a) π/9, π/4 (b) π/3, π/9

(c) π/6, π/9 (d) π/3, π/4

4. If sin 2x + sin 4x = 2sin 3x, then x is equal to ***[EAMCET-89]***

(a) $n\pi/3$ (b) $n\pi/2$

(c) $2n\pi \pm \pi/3$ (d) None of these

5. The general solution of the trigonometric equation tan π = cot π is

[MPPET-94]

(a) $\theta=n\pi+\frac{\pi}{2}-\alpha$

(b) $\theta=n\pi-\frac{\pi}{2}+\alpha$

(c) $\theta = n\pi + \frac{\pi}{2} + \alpha$

(d) $\theta = n\pi - \frac{\pi}{2} - \alpha$

6. The solution of the equation 4 $\cos^2 x$ + 6 $\sin^2 x$ = 5 *[AICBSE-83]*

(a) $x = n\pi \pm \frac{\pi}{2}$

(b) $x = n\pi \pm \frac{\pi}{4}$

(c) $x = nx \pm \frac{3\pi}{2}$

(d) None of these

7. The general value of θ satisfying the equation $\tan\theta + \tan\left(\frac{\pi}{2} - \theta\right) = 2$, is *[MNR-74]*

(a) $n\pi \pm \frac{\pi}{4}$ (b) $n\pi + \frac{\pi}{4}$

(c) $2n\pi \pm \frac{\pi}{4}$ (d) $n\pi + (-1)^n \frac{\pi}{4}$

8. If sin 2θ = cos 3θ and θ is an acute angle, then sin θ is equal to *[EAMCET-80]*

(a) $\frac{\sqrt{5}-1}{4}$ (b) $\frac{-\sqrt{5}-1}{4}$

(c) 0 (d) None of these

9. The solution of 3 tan (*A* – 15°) = tan (*A* + 15°)

(a) $n\pi + \frac{\pi}{4}$ (b) $2n\pi + \frac{\pi}{4}$

(c) $2n\pi - \frac{\pi}{4}$ (d) $\frac{n\pi}{2} + (-1)^n \frac{\pi}{2}$

10. The number of solution of the equation $8\tan^2\theta + 9 = 6\sec\theta$ in the interval (–π/2, π/2) is

(a) 2 (b) 4

(c) 0 (d) None of these

11. If sin 2θ + cos 2θ = 1, then the general value of θ is

(a) $n\pi + \frac{\pi}{4}, n\pi$ (b) $n\pi - \frac{\pi}{4}, n\pi$

(c) $n\pi + \frac{\pi}{2}, \frac{n\pi}{2}$ (d) $n\pi - \frac{\pi}{2}, \frac{n\pi}{2}$

12. If $\sin^2\theta - 2\cos\theta + \frac{1}{4} = 0$, then the general value of θ is

(a) $n\pi \pm \frac{\pi}{3}$ (b) $2n\pi \pm \frac{\pi}{3}$

(c) $2n\pi \pm \frac{\pi}{6}$ (d) $n\pi \pm \frac{\pi}{6}$

13. The number of value of θ in [0, 2π] satisfying the equation $2\sin^2\theta = 4 + 3\cos\theta$ are *[MPPET-89]*

(a) 0 (b) 1

(c) 2 (d) 3

14. The number of solution of the given equation $\tan\theta + \sec\theta = \sqrt{3}$, where 0 < θ < 2 π is

(a) 0 (b) 1

(c) 2 (d) 3

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The worksheet is of 30 minutes.
3. The worksheet consists of 30 questions. The maximum marks are 90.
4. Use Blue/Black Ball point pen only for writing particulars/making responses. Use of pencil is strictly prohibited.

1. The general value of θ in the equation $2\sqrt{3}\cos\theta = \tan\theta$ is ***[MPPET-2003]***

(a) $2n\pi \pm \frac{\pi}{6}$ (b) $2n\pi \pm \frac{\pi}{4}$

(c) $n\pi + (-1)^n \frac{\pi}{3}$ (d) $n\pi + (-1)^n \frac{\pi}{4}$

2. The equation $\sqrt{3}$ sin x + cos x = 4 has ***[EAMCET-2001]***

(a) Only on solution
(b) Two solution
(c) Infinitely many solution
(d) No solution

3. If cos θ = −1/2 and 0° < θ < 360°, then the value of θ are ***[Karnataka CET-2001]***

(a) 120° and 300° (b) 60° and 120°
(c) 120° and 240° (d) 60° and 240°

4. If 2cos θ + 1 = 0, then the general value of θ is

(a) $2n\pi \pm \frac{\pi}{3}$ (b) $2n\pi \pm \frac{2\pi}{3}$

(c) $n\pi \pm \frac{2\pi}{3}$ (d) $n\pi \pm \frac{\pi}{3}$

5. The general solution of tan 3x =1 is ***[Karnataka CET-91]***

(a) $n\pi + \frac{\pi}{4}$ (b) $\frac{n\pi}{3} + \frac{\pi}{12}$

(c) $n\pi$ (d) $n\pi \pm \frac{\pi}{4}$

6. If cos $p\theta$ = cos $q\theta$ $p \neq q$, then ***[MPPET-95]***

(a) θ = 2$n\pi$ (b) $\theta = \frac{2n\pi}{p \pm q}$

(c) $\theta = \frac{n\pi}{p \pm q}$ (d) None of these

7. If $\tan\theta = -\frac{1}{\sqrt{3}}$ and $\sin\theta = \frac{1}{2}$, $\cos\theta = -\frac{\sqrt{3}}{2}$, then the principal value of θ will be ***[MPPET-83, 84]***

(a) π/6 (b) 5π/6
(c) 7π/6 (d) −π/6

8. If cot² θ = cosec² θ, then the general value of θ is

(a) $n\pi$ (b) $2n\pi \pm \frac{\pi}{4}$

(c) $n\pi + (-1)^n \frac{\pi}{4}$ (d) $2n\pi \pm \frac{\pi}{2}$

9. If tan x = 3 cot x, then the general value of x is

(a) $n\pi + \frac{\pi}{6}$ (b) $n\pi + \frac{\pi}{3}$

(c) $n\pi \pm \frac{\pi}{6}$ (d) $n\pi \pm \frac{\pi}{3}$

10. If $\frac{1-\tan^2\theta}{\sec^2\theta} = \frac{1}{2}$, then the general value of θ is

(a) $n\pi \pm \frac{\pi}{6}$

(b) $n\pi + \frac{\pi}{6}$

(c) $2n\pi \pm \frac{\pi}{6}$

(d) None of these

11. If 2tan² θ = sec² θ, then the general value of θ is

(a) $n\pi + \frac{\pi}{4}$ (b) $n\pi - \frac{\pi}{4}$

(c) $n\pi \pm \frac{\pi}{4}$ (d) $2n\pi \pm \frac{\pi}{4}$

12. If sec² θ = 4/3, then the general value of θ is

(a) $2n\pi \pm \frac{\pi}{6}$ (b) $n\pi \pm \frac{\pi}{6}$

(c) $2n\pi \pm \frac{\pi}{3}$ (d) $n\pi \pm \frac{\pi}{3}$

13. If $\sin\theta = \sin\alpha$, then
(a) $\theta = n\pi \pm \alpha$
(b) $\theta = 2n\pi + (-1)^n \alpha$
(c) $\alpha = n\pi + (-1)^n \theta$
(d) $\theta = (2n+1)\pi + \alpha$

14. If $\sin\theta = \sqrt{3}\cos\theta, \; -\pi < \theta < 0$, then θ is equal to
(a) $-5\pi/6$ (b) $-4\pi/6$
(c) $4\pi/6$ (d) $5\pi/6$

15. If $\sin A = \sin B$ and $\cos A = \cos B$, then which one of the following is correct ***[NDA-07]***
(a) $B = n\pi + A$
(b) $A = 2n\pi - B$
(c) $A = 2n\pi + B$
(d) $B = n\pi - A$ (n is an integer)

16. If $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$, then the value of $\cos(\theta - \pi/4)$ is equal to
[AMU-99]
(a) $1/2\sqrt{2}$ (b) $1/\sqrt{2}$
(c) $1/3\sqrt{2}$ (d) $1/4\sqrt{2}$

17. If $f(\theta) = a\sin\theta + b\cos\theta$ then the maximum value is
[Orrisa JEE-2007]
(a) $\sqrt{a^2+b^2}$ (b) $\sqrt{a^2}+\sqrt{b^2}$
(c) a^2+b^2 (d) None of these

18. The most general value of θ satisfying the equation $\sin\theta = \sin\alpha$ and $\cos\theta = \cos\alpha$ is
[IIT-71; Karnataka-93; DCE-99]
(a) $2n\pi + \alpha$ (b) $2n\pi - \alpha$
(c) $n\pi + \alpha$ (d) $n\pi - \alpha$

19. If $3(\sec^2\theta + \tan^2\theta) = 5$, then the general value of θ is
(a) $2n\pi + \frac{\pi}{6}$ (b) $2n\pi \pm \frac{\pi}{6}$
(c) $n\pi \pm \frac{\pi}{6}$ (d) $n\pi \pm \frac{\pi}{3}$

20. If $\frac{1-\cos 2\theta}{1+\cos 2\theta} = 3$, then the general value of θ is
(a) $2n\pi \pm \frac{\pi}{6}$ (b) $n\pi \pm \frac{\pi}{6}$
(c) $2n\pi \pm \frac{\pi}{3}$ (d) $n\pi \pm \frac{\pi}{3}$

21. If $\tan 2\theta \tan\theta = 1$, then the general value of θ is
(a) $\left(n+\frac{1}{2}\right)\frac{\pi}{3}$ (b) $\left(n+\frac{1}{2}\right)\pi$
(c) $\left(2n\pm\frac{1}{2}\right)\frac{\pi}{3}$ (d) None of these

22. If $(2\cos x - 1)(3 + 2\cos x) = 0, \; 0 \le x \le 2\pi$, then x is equal to
[MNR-88; UPSEAT-2000]
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{3}, \frac{5\pi}{3}$
(c) $\frac{\pi}{2}, \frac{5\pi}{3}, \cos^{-1}(-3/2)$ (d) $\frac{5\pi}{3}$

23. The equation $\sin x + \cos x = 2$ has
[EAMCET-86; MPPET-98; PB CET-93]
(a) One solution
(b) Two solution
(c) Infinite number of solutions
(d) No solution

24. If $2\cos^2 x + 3\sin x - 3 = 0, \; 0 \le x \le 180°$, then x is equal to
[MPPET-86]
(a) 30°, 90°, 150°
(b) 60°, 120°, 180°
(c) 0°, 30°, 150°
(d) 45°, 90°, 135°

25. The equation $3\cos x + 4\sin x = 6$ has
[Orrisa-JEE-2002]
(a) Finite solution
(b) Infinite solution
(c) One solution
(d) No solution

26. The number of solutions of the equation $\sin x \cos x \cos 2x = -1/2$ is
(a) 0 (b) 1
(c) 2 (d) 3

27. The number of solutions of the equation $\tan^4\theta - 4\tan^2\theta + 3 = 0$ between 0° and 360° is
(a) 4 (b) 6
(c) 8 (d) 10

28. If $-\sqrt{3}\sin\theta - \cos\theta = 1$, then θ has one of the two possible values
[SCRA-2007]
(a) 180°, 300° (b) 180°, 200°
(c) 150°, 300° (d) 150°, 200°

29. The general solution of the equation $\tan x + \tan 2x + \tan x \tan 2x = 1$ is *[SCRA-2007]*

(a) $n\pi - \dfrac{\pi}{4}$ (b) $n\pi + \dfrac{\pi}{4}$

(c) $\dfrac{n\pi}{3} - \dfrac{\pi}{12}$ (d) $\dfrac{n\pi}{3} + \dfrac{\pi}{12}$

30. The equation $\cos x + \sin x = 1$ has at least one of the following solution *[MPPET-2007]*

(a) $x = \pi$
(b) $x = \pi/2$
(c) $x = \pi/4$
(d) $x = \pi/3$

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	**11.** ⓐ ⓑ ⓒ ⓓ	**21.** ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	**12.** ⓐ ⓑ ⓒ ⓓ	**22.** ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	**13.** ⓐ ⓑ ⓒ ⓓ	**23.** ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	**14.** ⓐ ⓑ ⓒ ⓓ	**24.** ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ	**15.** ⓐ ⓑ ⓒ ⓓ	**25.** ⓐ ⓑ ⓒ ⓓ
6. ⓐ ⓑ ⓒ ⓓ	**16.** ⓐ ⓑ ⓒ ⓓ	**26.** ⓐ ⓑ ⓒ ⓓ
7. ⓐ ⓑ ⓒ ⓓ	**17.** ⓐ ⓑ ⓒ ⓓ	**27.** ⓐ ⓑ ⓒ ⓓ
8. ⓐ ⓑ ⓒ ⓓ	**18.** ⓐ ⓑ ⓒ ⓓ	**28.** ⓐ ⓑ ⓒ ⓓ
9. ⓐ ⓑ ⓒ ⓓ	**19.** ⓐ ⓑ ⓒ ⓓ	**29.** ⓐ ⓑ ⓒ ⓓ
10. ⓐ ⓑ ⓒ ⓓ	**20.** ⓐ ⓑ ⓒ ⓓ	**30.** ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

6. (b) $\cos p\theta = \cos q\theta \Rightarrow p\theta = 2n\pi \pm q\theta$

$$\theta = (p \mp q) = 2n\pi \Rightarrow \theta = \frac{2n\pi}{p \pm q}$$

15. (c) $\sin A = \sin B$ and $\cos A = \cos B$

$$2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right) = 0$$

$$2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) = 0$$

comparing both $\sin\left(\dfrac{A-B}{2}\right) = 0$

$$\Rightarrow \frac{A-B}{2} = n\pi \qquad A = 2n\pi + B$$

21. (a) $\tan 2\theta \tan\theta = 1$

$$\Rightarrow \tan 2\theta = \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$$

$$2\theta = \frac{\pi}{2} - \theta + n\pi$$

$$\Rightarrow \theta = \frac{n\pi}{3} + \frac{\pi}{6}$$

26. (a) $\dfrac{1}{2}(2\sin x\cos x)\cos x = \dfrac{-1}{2}$

$$\Rightarrow \frac{1}{2}(2\sin 2x\cos 2x) = -1$$

$\Rightarrow \sin 4x = -2$ not possible

Part C

Properties and Solutions of Triangles

LECTURE 1

Properties of Triangles-I

BASIC CONCEPTS

1. PROPERTIES OF TRIANGLES

In a triangle ABC, we shall denote the angles BAC, CBA and ACB by A, B and C respectively and the corresponding sides, i.e., the sides opposite to them by a, b and c respectively.

2. LAW OF SINES OR SINE RULE

The side, of a triangle are proportional to the sines of the opposite angles i.e., $a : b : c : : \sin A : \sin B : \sin C$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

3. LAW OF TANGENTS OR TANGENT RULE (NAPIERS ANALOGY)

In any ΔABC,

(a) $\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\frac{A}{2}$; If

$\angle B = 90°$, $\tan^2\frac{A}{2} = \frac{b-c}{b+c}$

(b) $\tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right)\cot\frac{C}{2}$; If

$\angle A = 90°$, $\tan^2\frac{C}{2} = \frac{a-b}{a+b}$

(c) $\tan\left(\frac{C-A}{2}\right) = \left(\frac{c-a}{c+a}\right)\cot\frac{B}{2}$;

If $\angle C = 90°$, $\tan^2\frac{B}{2} = \frac{c-a}{c+a}$

4. LAW OF COSINES OR COSINE FORMULA

In any ΔABC,

(a) $\cos A = \frac{b^2+c^2-a^2}{2bc}$ or $a^2 = b^2 + c^2 - 2bc\cos A$

(b) $\cos B = \frac{c^2+a^2-b^2}{2ac}$

(c) $\cos C = \frac{a^2+b^2-c^2}{2ab}$

$b^2 = c^2 + a^2 - 2ca\cos B$

$c^2 = a^2 + b^2 - 2ab\cos C$

5. PROJECTION FORMULA

In any ΔABC,

(a) $a = b\cos C + c\cos B$

(b) $b = a\cos C + c\cos A$

(c) $c = a\cos B + b\cos A$

6. HALF ANGLE FORMULAE OR SEMI-SUM FORMULAE

In any triangle ABC, if $a + b + c = 2s$, then

(a) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$,

$\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ac}}$

$\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(b) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$,

$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}$, $\cos\frac{C}{2} = \sqrt{\frac{(s-c)}{ab}}$,

(c) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$,

$\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$

$\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

(d) $\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{bc}$,

$\sin B = \frac{2\Delta}{ca}$, $\sin C = \frac{2\Delta}{ab}$

7. (I) AREA OF A TRIANGLE

Two sides and angle between them are given, then area of Δ is

$$\Delta = \frac{1}{2}bc\sin A \text{ or } \frac{1}{2}ca\sin B \text{ or } \frac{1}{2}ab\sin C$$

(II) HERO'S FORMULA

In a $\Delta\, ABC$, if $a + b + c = 2s$, then its area is given by, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$. (when three sides are known)

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. In any $\Delta\, ABC$, $\angle A = 75°$, $\angle B = 30°$ and $b = \sqrt{8}$ value of idea

Solution

By sin rule $\frac{a}{\sin A} = \frac{b}{\sin B}$

$\Rightarrow \frac{a}{\sin 75°} = \frac{\sqrt{8}}{\sin 30°}$

$\Rightarrow a = \frac{\sqrt{8}\sin 75°}{\sin 30°} = \frac{\sqrt{8}\sin(45°+30°)}{\left(\frac{1}{2}\right)}$

$\Rightarrow a = 2\sqrt{8}$ [sin 45° cos 30° + cos 45° sin 30°]

$\Rightarrow a = 2(2\sqrt{2})\left[\frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}.\frac{1}{2}\right] = 2(\sqrt{3}+1)$

2. In any $\Delta\, ABC$, prove that

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2+b^2+c^2}{2abc}$$

[MP-1995]

Solution

Putting the value of $\cos A$, $\cos B$, $\cos C$

$$\text{LHS} = \frac{1}{a}\left(\frac{b^2+c^2-a^2}{2bc}\right) + \frac{1}{b}\left(\frac{a^2+c^2-b^2}{2ac}\right) + \frac{1}{c}\left(\frac{a^2+b^2-c^2}{2ab}\right)$$

$$= \frac{b^2+c^2-a^2}{2abc} + \frac{a^2+c^2-b^2}{2abc} + \frac{a^2+b^2-c^2}{2abc} = \frac{a^2+b^2+c^2}{2abc} = \text{RHS}$$

3. In any triangle ABC, prove that $a(b\cos C - c\cos B) = b^2 - c^2$.

Solution

$$\text{LHS} = a\left[b.\frac{a^2+b^2-c^2}{2ab} - c.\frac{c^2+a^2-b^2}{2ca}\right]$$

$$= a\left[\frac{a^2+b^2-c^2}{2a} - \frac{c^2+a^2-b^2}{2a}\right]$$

$$=\frac{1}{2}[a^2+b^2-c^2-c^2-a^2+b^2]$$

$$=\frac{1}{2}[2b^2-2c^2]=b^2-c^2=\text{RHS}$$

4. If in $\Delta ABC, \frac{1}{a+b}+\frac{1}{a+c}=\frac{3}{a+b+c}$, then prove that $\angle A = 60°$.

Solution

$\because \quad \frac{1}{a+b}+\frac{1}{a+c}=\frac{3}{a+b+c}$

$\Rightarrow \quad \frac{(a+c)+(a+b)}{(a+b)(a+c)}=\frac{3}{a+b+c}$

$\Rightarrow \quad 3(a+b)(a+c)=(2a+b+c)(a+b+c)$

$\Rightarrow \quad 3a^2+3ac+3ab+3bc=2a^2+b^2+c^2+3ab+3ac+2bc$

$\Rightarrow \quad \frac{b^2+c^2-a^2}{2bc}=\frac{1}{2}$

$\Rightarrow \quad \cos A=\frac{1}{2}=\cos 60° \quad \Rightarrow \quad \angle A=60°$

5. If in $\Delta ABC, \frac{b+c}{11}=\frac{c+a}{12}=\frac{a+b}{13}$ then prove that $\frac{\cos A}{7}=\frac{\cos B}{19}=\frac{\cos C}{25}$.

[MP-1994]

Solution

$\because \quad \frac{b+c}{11}=\frac{c+a}{12}=\frac{a+b}{13}$

$$=\frac{2(a+b+c)}{11+12+13}=\frac{a+b+c}{18}=k\text{ (say)}$$

$\Rightarrow \quad b+c=11k,\ c+a=12k,\ a+b=13k,\ a+b+c=18k$

Solve $a=7k, b=6k, c=5k$

$$\cos A=\frac{b^2+c^2-a^2}{2bc}$$

$$=\frac{36k^2+25k^2-49k^2}{2.6k.5k}=\frac{12k^2}{60k^2}=\frac{1}{5}$$

$$\cos B=\frac{c^2+a^2-b^2}{2ca}$$

$$=\frac{25k^2+49k^2-36k^2}{2.5k.7k}=\frac{38k^2}{70k^2}=\frac{19}{35}$$

$$\cos C=\frac{a^2+b^2-c^2}{2ab}$$

$$=\frac{49k^2+36k^2-25k^2}{2.7k.6k}$$

$$=\frac{60k^2}{84k^2}=\frac{5}{7}$$

$\therefore \quad \cos A : \cos B : \cos C$

$$=\frac{1}{5}:\frac{19}{35}:\frac{5}{7}$$

$$=\frac{35}{5}:\frac{19}{1}:\frac{35\times 5}{7}=7:19:25$$

$\therefore \quad \frac{\cos A}{7}=\frac{\cos B}{19}=\frac{\cos C}{25}$

6. If in any triangle ABC, prove that

$$(a-b)^2\cos^2\frac{C}{2}+(\text{a}+\text{b})^2\sin^2\frac{C}{2}=\text{c}^2$$

Solution

$$\text{LHS}=(a-b)^2\cos^2\frac{C}{2}+(a+b)^2\sin^2\frac{C}{2}$$

$$=(a^2-2ab+b^2)\cos^2\frac{C}{2}$$

$$+(a^2+2ab+b^2)\sin^2\frac{C}{2}$$

$$=(a^2+b^2)\left(\cos^2\frac{C}{2}+\sin^2\frac{C}{2}\right)$$

$$-2ab\left(\cos^2\frac{C}{2}-\sin^2\frac{C}{2}\right)$$

$$=(a^2+b^2)\ 1-2ab$$

$\cos C=c^2=\text{RHS}$

7. The angles of a triangle are in the ratios 1 : 2 : 7, find the ratio of the greatest side to least side.

Solution

Let in $\Delta ABC, A=x°, B=2x°, C=7x°$,

then $x°+2x°+7x°=180°$

$\Rightarrow \quad x=18°$

$\therefore \quad A=18°, B=36°, C=126°$

$\therefore \quad \frac{\text{Greatest side}}{\text{least side}}=\frac{\sin 126°}{\sin 18°}$

$$=\frac{\sin(180°-54°)}{\sin 18°}$$

$$= \frac{\sin 54°}{\sin 18°} = \frac{\cos 36°}{\sin 18°}$$

$$= \frac{\sqrt{5}+1}{4} \times \frac{4}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{\sqrt{5}-1}$$

Hence, required ratio $\sqrt{5}+1 : \sqrt{5}-1$.

8. If in triangle ABC, $a = 13, b = 14, c = 15$, then the value of $\sin\frac{A}{2}\cos\frac{A}{2}, \sin\frac{B}{2}\cos\frac{B}{2}, \sin\frac{C}{2}\cos\frac{C}{2}$.

Solution

Given $a = 13$, $b = 14$, $c = 15$

$$\therefore \quad s = \frac{a+b+c}{2} = \frac{13+14+15}{2} = \frac{42}{2} = 21$$

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$= \sqrt{\frac{(21-14)(21-15)}{14\times 15}}$$

$$= \sqrt{\frac{7\times 6}{14\times 15}} = \frac{1}{\sqrt{5}}$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{21(21-13)}{14\times 15}}$$

$$= \sqrt{\frac{21\times 8}{14\times 15}} = \frac{2}{\sqrt{5}}$$

Similarly, by using the formula

$$\sin\frac{B}{2} = \frac{4}{\sqrt{(65)}}, \quad \cos\frac{B}{2} = \frac{7}{\sqrt{(65)}},$$

$$\sin\frac{C}{2} = \frac{2}{\sqrt{(13)}}, \cos\frac{C}{2} = \frac{3}{\sqrt{13}}$$

9. In any triangle ABC, prove that

$$(a+b+c)\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right) = 2c\cot\frac{C}{2}$$

Solution

$$\text{LHS} = 2s\left[\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} + \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}\right]$$

$$= 2s\sqrt{\frac{s-c}{s}}\left[\sqrt{\frac{s-b}{s-a}} + \sqrt{\frac{s-a}{s-b}}\right]$$

$$= 2\sqrt{s(s-c)}\left[\frac{s-b+s-a}{\sqrt{(s-a)(s-b)}}\right]$$

$$= \frac{2\sqrt{s(s-c)}.c}{\sqrt{(s-a)(s-b)}} \quad [\because 2s = a+b+c]$$

$$= 2c\cot\frac{C}{2}.$$

10. In any triangle ABC, prove that

$$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \frac{a+b+c}{b+c-a}\cot\frac{A}{2}$$

[MP-1995]

Solution

$$\text{LHS} = \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}$$

$$= \sqrt{\left[\frac{s(s-a)}{(s-b)(s-c)}\right]}$$

$$+ \sqrt{\left[\frac{s(s-b)}{(s-c)(s-a)}\right]} + \sqrt{\left[\frac{s(s-c)}{(s-a)(s-b)}\right]}$$

$$= \frac{\sqrt{s}[(s-a)+(s-b)+(s-c)]}{\sqrt{[(s-a)(s-b)(s-c)]}}$$

$$= \frac{\sqrt{s}[3s-(a+b+c)]}{\sqrt{[(s-a)(s-b)(s-c)]}}$$

$$= \frac{\sqrt{s}}{\sqrt{[(s-b)(s-c)]}} \times \frac{(3s-2s)}{\sqrt{(s-a)}}$$

$$= \sqrt{\left[\frac{s(s-a)}{(s-b)(s-c)}\right]} \times \frac{s}{s-a}$$

$$= \cot\frac{A}{2} \times \frac{2s}{2s-2a} = \cot\frac{A}{2}$$

$$\times \frac{a+b+c}{(a+b+c)-2a} = \frac{a+b+c}{b+c-a}\cot\frac{A}{2}$$

11. In an isosceles right angled triangle, a straight line is drawn from the middle point of one of the equal sides to the opposite angle. Show that it divides the angle into two parts whose cotangents are 2 and 3.

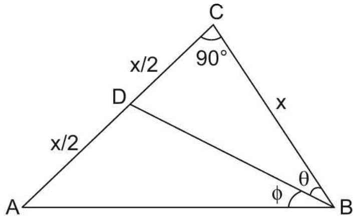
Solution

Let ABC be the triangle, right angled at C, and D be the mid-point of AC. Join DB.

Since, $AC = BC = x$, we have

$$DC = \frac{1}{2}AC = \frac{1}{2}BC = \frac{x}{2}$$

Also, $\angle CAB = \angle CBA = 45°$



If, $\angle DBC = \theta$ and $\angle DBA = \phi$

$$\tan\theta = \frac{DC}{BC} = \frac{x/2}{x} = \frac{1}{2}$$

$$\tan\phi = \tan(45° - \theta) = \frac{1-\tan\theta}{1+\tan\theta}$$

or $$\tan\phi = \frac{1-1/2}{1+1/2} = \frac{1}{3}$$

$\therefore$ $\cot\theta = 2, \cot\phi = 3.$

12. If in a triangle $a = 5, b = 4$ and $\cos(A - B) = \frac{31}{32}$, then prove that the third side c will be 6.

Solution

$$\tan\frac{\alpha}{2} = \left(\frac{1-\cos\alpha}{1+\cos\alpha}\right)^{1/2} = \left\{\frac{1-(31/32)}{1+(31/32)}\right\}^{1/2}$$

$$= \frac{1}{\sqrt{63}} = \frac{1}{3\sqrt{7}} \text{ where } \alpha = A - B$$

Now, $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$ (Napier Analogy)

put $a = 5, b = 4$

$\therefore$ $$\frac{1}{3\sqrt{7}} = \frac{1}{9}\cot\frac{C}{2} \text{ or } \tan\frac{C}{2} = \frac{\sqrt{7}}{3}$$

Now, $$\cos C = \frac{1-\tan^2(C/2)}{1+\tan^2(C/2)} = \frac{1-7/9}{1+7/9} = \frac{1}{8}$$

13. In the ambiguous case, if two triangles are formed with a, b and A, then prove that the sum of the areas of these triangles is $\frac{1}{2}b^2\sin 2A$.

Solution

Sum of the areas of the two triangles

$$= \frac{1}{2}ab\ \sin C_1 + \frac{1}{2}ab\ \sin C_2 = \frac{1}{2}ab$$

$$(\sin C_1 + \sin C_2) \quad .....(1)$$

Now, from part (b), we have $c_1 + c_2 = 2b \cos A$

or $k(\sin C_1 + \sin C_2) = 2k \sin B \cos A$

or $\sin C_1 + \sin C_2 = 2 \sin B \cos A$

Hence, from (1),

Sum of the areas of the two triangles

$$= \frac{1}{2}ab.\ 2\sin B\cos A$$

$$= b^2\left(\frac{a\sin B}{b}\right)\cos A = b^2 \sin A \cos A$$

$$= \frac{1}{2}b^2.2\sin A\cos A$$

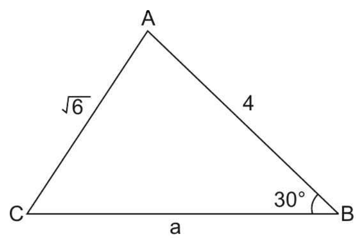
$$= \frac{1}{2}b^2\sin 2A.$$

14. Two sides of a triangle are of lengths $\sqrt{6}$ and 4 and the angle opposite to smaller side is 30°. How many such triangles are possible? Find the length of their third side.

[Roorkee-98]

Solution

$$\frac{4}{\sin C} = \frac{\sqrt{6}}{\sin 30°} \quad \therefore \quad \sin C = \frac{2}{\sqrt{6}} = \sqrt{\frac{2}{3}}$$



There will be two values of C which will be supplementary.

Hence, there will be two such triangles.

Again by cosine rule

$$(\sqrt{6})^2 = a^2 + 16 - 2a.4\cos 30°$$

or $$a^2 - 4\sqrt{3}a + 10 = 0$$

$\therefore$ $$a = \frac{4\sqrt{3} \pm \sqrt{48-40}}{2}$$

$$= 2\sqrt{3} \pm \sqrt{2}$$

Above gives the values of two sides.

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): SOLVE THESE PROBLEMS TO GRASP THE TOPIC

EXERCISE 1

1. In any ΔABC, $\angle A = 30°$, $\angle C = 45°$, find the value of $a : c$.
2. In any ΔABC, $a = 7$, $b = 8$ and $c = 5$, then prove that $A = 60°$. **[MP-1994]**
3. If in ΔABC, $\angle C = 90°$, then prove that $\tan\frac{A-B}{2} = \frac{a-b}{a+b}$.
4. If in triangle ABC, $a = 25$, $b = 52$, $c = 63$, then the value of $\tan\frac{A}{2}, \tan\frac{B}{2}$ and $\tan\frac{C}{2}$ will be
5. In ΔABC, if $(a + b + c)(b + c - a) = 3bc$, then prove that $\angle A = 60°$
6. In ΔABC, $a = 5$, $b = 4$ and $\tan\frac{C}{2} = \sqrt{\frac{7}{9}}$, then find the value of c.
7. In ΔABC, if $2s = a + b + c$, then prove that $\frac{s(s-a)}{bc} - \frac{(s-b)(s-c)}{bc} = \cos A$.
8. In ΔABC, prove that $\frac{a+b+c}{a-b+c} = \cot\frac{A}{2}\cot\frac{C}{2}$.
9. Find the area of triangle whose sides are 4 cm, 5 cm and the angle included by them is $\cot^{-1}\frac{4}{3}$.

EXERCISE 2

1. In any ΔABC, $a = 2$, $b = 3$, $c = 4$ then the value of $\cos A$, $\cos B$ will be **[MP-1994]**
2. In triangle ABC, prove that $(b + c)\cos A + (c + a)\cos B + (a + b)\cos C = a + b + c$.
3. If in triangle ABC, $a\cos A = b\cos B$, then prove that $a = b$ or $\angle C = 90°$.
4. In any triangle ABC, $a = 18$, $b = 24$, $c = 30$, then find the value of $\sin A$, $\sin B$ and $\sin C$.
5. In any triangle ABC, prove that $(b+c-a)\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right) = 2a\cot\frac{A}{2}$. **[MP-1995]**
6. Find the area of the triangle whose adjacent sides are 32 cm and $15\sqrt{3}$ cm and angle between them is 60°.
7. Find the area of triangle if $a = 25$, $b = 60$ and $c = 65$.
8. In any ΔABC, $a = 16$, $b = 24$, $c = 20$, then find $\cos\frac{B}{2}$.

ANSWERS

EXERCISE 1

1. $a:c = 1:\sqrt{2}$
4. $\tan\frac{A}{2} = \frac{1}{5}$; $\tan\frac{B}{2} = \frac{1}{2}$; $\tan\frac{C}{2} = \frac{9}{7}$
6. $c = 6$.
9. 6 cm^2.

EXERCISE 2

1. $\cos A = \frac{7}{8}, \cos B = \frac{11}{16}$
4. $\sin A = 3/5$, $\sin B = 4/5$, $\sin C = 1$
6. 360 cm^2
7. 750 Square unit.
8. 3/4

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

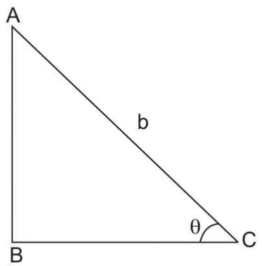
1. In ΔABC, $\angle B = 90°$ and $b + a = 4$. The area of the triangle is the maximum when $\angle C$ is **[DCE-1996]**

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

Solution

(c) From the figure $a = b\cos\theta$



$\therefore$ $b\cos\theta + b = 4$ or $b = \frac{4}{1+\cos\theta}$ and

similarly, $a = \frac{4\cos\theta}{1+\cos\theta}$

Required area of $\Delta = \frac{1}{2}ab$

$\sin\theta = \frac{1}{2}\times\frac{16\cos\theta\sin\theta}{(1+\cos\theta)^2}$

$\frac{d\Delta}{d\theta} = 4$

$$\left[\frac{2\cos 2\theta.(1+\cos\theta)^2 + \sin 2\theta.2(1+\cos\theta)\sin\theta}{(1+\cos\theta)^4}\right]$$

$\therefore$ $\frac{d\Delta}{d\theta} = 0$

$\Rightarrow$ $\cos 2\theta(1 + \cos\theta) + \sin 2\theta.\ \sin\theta = 0$

$\Rightarrow$ $\theta = \frac{\pi}{3}$

2. If the angles of a triangle are 30° and 45° and the included side is $(\sqrt{3}+1)$ cm, then the area of triangle is **[DCE-1998: PET-1997]**

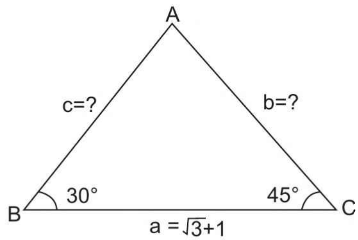
(a) $\frac{1}{\sqrt{3}-1}$ (b) $\sqrt{3}+1$

(c) $\frac{1}{\sqrt{3}+1}$ (d) None of these

Solution

(a) $\angle A = 180° - 30° - 45° = 105°$

$\sin(105°) = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin A$



Area of triangle $ABC = \frac{1}{2}a^2\frac{\sin B\sin C}{\sin(B+C)}$

$= \frac{1}{2}bc\sin A = \frac{1}{2}\sqrt{2}\times 2\times\frac{(\sqrt{3}+1)}{2\sqrt{2}}$

$= \frac{1}{2}\times\frac{(\sqrt{3}+1)^2\times\frac{1}{2}\times\frac{1}{2}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{2}{2(\sqrt{3}-1)} = \frac{1}{(\sqrt{3}-1)}$

3. In ΔABC, $A = \pi/3$ and $b : c = 2 : 3$

If $\tan\theta\frac{\sqrt{3}}{5}$, $0 < \theta < \frac{\pi}{2}$ then **[DCE-2002]**

(a) $B = 60° + \theta$ (b) $C = 60° + \theta$

(c) $B = 60° - \theta$ (d) $C = 60° - \theta$

Solution

(b) $A = \frac{\pi}{3}$, $\tan\frac{C-B}{2} = \frac{c-b}{c+b}\cot\frac{A}{2}$

$\tan\left(\frac{C-B}{2}\right) = \frac{1}{5}\cot 30° = \frac{\sqrt{3}}{5} = \tan\theta$

$\therefore$ $C - B = 2\theta$, $C + B = 180° - A = 120°$

$\therefore$ $2C = 120° + 2\theta$, $C = 60° + \theta$

4. In a ΔABC, a, c, A are given and b_1, b_2 are two values, if the third side b is such that $b_2 = 2b_1$ then $\sin A$ is equal to

[DCE-2006]

(a) $\frac{\sqrt{9a^2 - c^2}}{8a^2}$ (b) $\sqrt{\frac{9a^2 - c^2}{8c^2}}$

(c) $\sqrt{\frac{9a^2 + c^2}{8c^2}}$ (d) None

Solution

(b) We have $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$\Rightarrow b^2 - 2bc\cos A + (c^2 - a^2) = 0$

It is given that b_1 and b_2 are roots of this equation

Therefore,

$b_1 + b_2 = 2c\cos A$ and $b_1 b_2 = c^2 - a^2$

$\Rightarrow 3b_1 = 2c\cos A$ and $2b_1^2 = c^2 - a^2$ $[\because b_2 = 2b_1]$

$$\Rightarrow \left(\frac{2c}{3}\cos A\right)^2 = c^2 - a^2$$

$\Rightarrow 8c^2(1 - \sin^2 A) = 9c^2 - 9a^2$

$$\Rightarrow \sin A = \sqrt{\frac{9a^2 - c^2}{8c^2}}$$

5. In triangle ABC, if $A + C = 2B$, then $\frac{a + c}{\sqrt{a^2 - ac + c^2}}$ is equal to **[UPSEAT-1999]**

(a) $2\cos\frac{A - C}{2}$ (b) $\sin\frac{A + C}{2}$

(c) $\sin\frac{A}{2}$ (d) None of these

Solution

(a) $A + C = 2B$

$\Rightarrow B = 60°$ $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$

Since, $B = 60°$

$\Rightarrow ac = a^2 + c^2 - b^2$

$\Rightarrow b^2 = a^2 + c^2 - ac$

Therefore,

$$\frac{a + c}{\sqrt{a^2 - ac + c^2}} = \frac{a + c}{b} = \frac{\sin A + \sin C}{\sin B}$$

$$= \frac{2\sin\frac{A + C}{2}\cos\frac{A - C}{2}}{2\sin\frac{B}{2}\sin\frac{A + C}{2}} = \frac{\cos\frac{A - C}{2}}{\sin\frac{B}{2}}$$

$$= \frac{\cos\frac{A - C}{2}}{\sin 30°}$$

$$\Rightarrow 2\cos\frac{A - C}{2}$$

6. In a ΔABC, $2a\sin\left(\frac{A - B + C}{2}\right)$ is equal to

[IIT Screening-2000]

(a) $a^2 + b^2 - c^2$ (b) $c^2 + a^2 - b^2$

(b) $b^2 - c^2 - a^2$ (d) $c^2 - a^2 - b^2$

Solution

(b) $2ac\sin\frac{A - B + C}{2} = 2ac$

$$\sin\frac{\pi - 2B}{2} = 2ac\cos B$$

$$= 2ac\frac{c^2 + a^2 - b^2}{2ca} = c^2 + a^2 - b^2$$

7. Let D be the middle point of the side BC of a triangle ABC. If the triangle ADC is equilateral, then $a^2 : b^2 : c^2$ is equal to

[Pb.CET-2004]

(a) 1 : 4 : 3 (b) 4 : 1 : 3

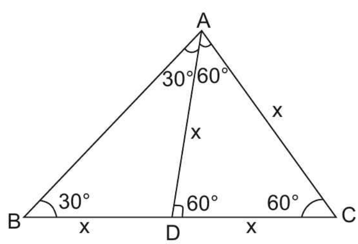
(c) 4 : 3 : 1 (d) 3 : 4 : 1

Solution

(b) $\cos 120° = \frac{x^2 + x^2 - AB^2}{2x^2}$

$$\Rightarrow \frac{2x^2 - AB^2}{2x^2} = \frac{-1}{2}$$

$\Rightarrow 4x^2 - 2AB^2 = -2x^2$



$\Rightarrow \quad 3x^2 = AB^2$

$\Rightarrow \quad AB = x\sqrt{3}$

$\Rightarrow a^2 : b^2 : c^2 = (2x)^2 : x^2 : (x\sqrt{3})^2$

$= 4x^2 : x^2 : 3x^2$

$= 4 : 1 : 3$

8. If α, β, γ are angles of a triangle, then $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma - 2 \cos \alpha \cos \beta \cos \gamma$ is

[Orissa JEE-2004]

(a) 2 (b) –1
(c) –2 (d) 0

Solution

(a) We have, $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma - 2 \cos \alpha \cos \beta \cos \gamma$

$= 3 - [\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma] - 2\cos \alpha \cos \beta \cos \gamma$

$$= 3 - \left[\frac{1+\cos 2\alpha}{2} + \frac{1+\cos 2\beta}{2} + \frac{1+\cos 2\gamma}{2}\right]$$
$$-2\cos\alpha\cos\beta\cos\gamma$$

$$= 3 - \frac{1}{2}[3 + \cos 2\alpha + \cos 2\beta + \cos 2\gamma]$$
$$-2\cos\alpha\cos\beta\cos\gamma$$

$$= 3 - \frac{3}{2} - \frac{1}{2}(\cos 2\alpha + \cos 2\beta)$$
$$-\frac{1}{2}\cos 2\gamma - 2\cos\alpha\cos\beta\cos\gamma$$

$$= \frac{3}{2} - \frac{1}{2}[-2\cos\gamma\cos(\alpha-\beta)]$$
$$-\frac{1}{2}[2\cos^2\gamma - 1]$$
$$-2\cos\alpha\cos\beta\cos\gamma$$

$$= \frac{3}{2} + \cos\gamma\cos(\alpha-\beta) - \cos^2\gamma$$
$$+\frac{1}{2} - 2\cos\alpha\cos\beta\cos\gamma = 2$$

9. If b, c and $\sin B$ are such that B is an acute angle and $b < c \sin B$, then in this case

[CET (Karnataka)-93]

(a) No triangle is possible.
(b) One triangle is possible.
(c) Two triangles are possible.
(d) One right angled triangle is possible.

Solution

(a) $\dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

$\Rightarrow \sin C = \dfrac{c}{b}\sin B > 1$

$[\because b < c \sin B]$

This is not possible, so no triangle is possible.

10. In a triangle ABC, $a^3 \cos (B - C) + b^3 \cos (C - A) + c^3 \cos (A - B)$ is equal to

[IIT-70; EAMCET-89; UPSEAT-99; Kerala Engg.-2002]

(a) 0 (b) $a + b + c$
(c) 3 abc (d) abc

Solution

(c) $\sum a^3 \cos(B - C)$

$= \sum k^3 \sin^3 A \cos(B - C)$

[by sin formula]

$= k^3 \sum \sin^2 A(\sin A)\cos(B - C)$

$= k^3 \sum \sin^2 A \sin(B + C)\cos(B - C)$

$= \frac{1}{2}k^3 \sum \sin^2 A(\sin 2B + \sin 2C)$

$= k^3 \sum \sin^2 A(\sin B\cos B + \sin C\cos C)$

$= k^3 [\sin A \sin B (\sin A \cos B + \cos A \sin B) + ... + ..]$

$= k^3 [\sin A \sin B \sin C + \sin B \sin C \sin A + \sin C \sin A \sin B]$

$= 3k^3 \sin A \sin B \sin C = 3\,abc$

11. In a triangle ABC, if $(a + b + c)(b + c - a) = \lambda\, bc$, then

[CET (Pb.)-97; CET (Karnataka)-98]

(a) $\lambda > 0$ (b) $\lambda < 0$
(c) $0 < \lambda < 4$ (d) $\lambda > 4$

Solution

(c) $2s(2s - 2a) = \lambda bc$

$\Rightarrow \dfrac{s(s-a)}{bc} = \dfrac{\lambda}{4}$

$\Rightarrow \cos^2\frac{A}{2} = \frac{\lambda}{4}$

$\Rightarrow \lambda = 4\cos^2\frac{A}{2}$

$\Rightarrow 0 < \lambda < 4.$

12. In an ambiguous case if a, b and A are given and if there are two possible values of third side which are c_1 and c_2, then

[UPSEAT-1999]

(a) $c_1 - c_2 = 2\sqrt{(a^2 + b^2\sin^2 A)}$

(b) $c_1 - c_2 = 2\sqrt{(a^2 - b^2\sin^2 A)}$

(c) $c_1 - c_2 = 4\sqrt{(a^2 + b^2\sin^2 A)}$

(d) $c_1 - c_2 = 3\sqrt{(a^2 - b^2\sin^2 A)}$

Solution

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

or $c^2 - (2b\cos A)\,c + (b^2 - a^2) = 0$

which is quadratic equation in c,

Let there be two roots c_1 and c_2 of above quadratic equation then $c_1 + c_2 = 2b\cos A$ and $c_1 c_2 = b^2 - a^2$

$$\therefore \quad c_1 - c_2 = \sqrt{[(c_1 + c_2)^2 - 4c_1c_2]}$$

$$= \sqrt{[(2b\cos A)^2 - 4(b^2 - a^2)]}$$

$$= \sqrt{[4a^2 - 4b^2(1 - \cos^2 A)]}$$

$$= 2\sqrt{(a^2 - b^2\sin^2 A)}$$

OBJECTIVE PROBLEMS: IMPORTANT QUESTION WITH SOLUTIONS

1. In a triangle ABC, $a = 5$, $b = 7$ and $\sin A = 3/4$, how many such triangle are possible? ***[Roorkee-90]***

(a) 1 (b) 2
(c) 0 (d) ∞

2. In a ΔABC, if $2s = a + b + c$ and $(s - b)(s - c) = x\sin^2 A/2$, then x is equal to ***[PET-92]***

(a) bc (b) ca
(c) ab (d) abc

3. If the angles of ΔABC be in A.P., then

(a) $c^2 = a^2 + b^2 - ab$
(b) $b^2 = a^2 + c^2 - ac$
(c) $a^2 = b^2 + c^2 - ac$
(d) $b^2 = a^2 + c^2$

4. In triangle ABC, $(b + c)\cos A + (c + a)\cos B + (a + b)\cos C$ is equal to ***[PET-85]***

(a) 0 (b) 1
(c) $a + b + c$ (d) $2(a + b + c)$

5. In $\Delta ABC, \frac{\sin(A - B)}{\sin(A + B)}$ is equal to ***[PET-86]***

(a) $\frac{a^2 - b^2}{c^2}$ (b) $\frac{a^2 + b^2}{c^2}$
(c) $\frac{c^2}{a^2 - b^2}$ (d) $\frac{c^2}{a^2 + b^2}$

6. In ΔABC, if $b^2 + c^2 = 3a^2$, then $\cot B + \cot C - \cot A$ is equal to ***[PET-91]***

(a) 1 (b) $ab/4\,\Delta$
(c) 0 (d) $ac/4\,\Delta$

7. In ΔABC, if $c^2 + a^2 - b^2 = ac$, then $\angle B$ is equal to ***[PET-83, 89, 90]***

(a) $\pi/6$ (b) $\pi/4$
(c) $\pi/3$ (d) None of these

8. In ΔABC, if $3a = b + c$, then the value of $\cot B/2 \cot C/2$ is equal to ***[PET-90, 97, 98, 03]***

(a) 1 (b) 2
(c) $\sqrt{3}$ (d) $\sqrt{2}$

9. In ΔABC, if $a = 2x$, $b = 2y$ and $\angle C = 120°$, then the area of the triangle is ***[PET-86, 02]***

(a) xy (b) $xy\sqrt{3}$
(c) $3xy$ (d) $2xy$

10. In ΔABC, if $a = 16$, $b = 24$, $c = 20$, then $\cos B/2$ is equal to ***[PET-88]***

(a) 3/4 (b) 1/4
(c) 1/2 (d) 1/3

11. In ΔABC, if $\cot A$, $\cot B$, $\cot C$ be in A.P., then a^2, b^2, c^2 are in ***[PET-97]***

(a) H.P. (b) G.P.
(c) A.P. (d) None of these

12. In ΔABC, if $(a+b+c)(a-b+c)=3ac$, then ***[AMU-96]***

(a) $\angle B = 60°$ (b) $\angle B = 30°$
(c) $\angle C = 60°$ (d) $\angle A + \angle C = 90°$

13. In ΔABC, $\operatorname{cosec} A(\sin B\cos C+\cos B\sin C)$ is equal to ***[PET-86, 95]***

(a) c/a (b) a/c
(c) 1 (d) c/ab

14. If $\cos^2 A+\cos^2 C=\sin^2 B$, then ΔABC is ***[PET-91]***

(a) Equilateral (b) Right angled
(c) Isosceles (d) None of these

15. In ΔABC, $\dfrac{b-c\cos A}{c-b\cos A}$ is equal to

(a) $\sin B/\sin C$ (b) $\cos C/\cos B$
(c) $\cos B/\cos C$ (d) None of these

16. In ΔABC, $\dfrac{\cos 2A}{a^2}-\dfrac{\cos 2B}{b^2}$ is equal to

(a) $\dfrac{c^2}{a^2b^2}$ (b) $\dfrac{1}{a^2}-\dfrac{1}{b^2}$
(c) $\dfrac{1}{ab}$ (d) None of these

17. If in a triangle, $a\cos^2\dfrac{C}{2}+c\cos^2\dfrac{A}{2}=\dfrac{3b}{2}$, then its sides will be in ***[PET-82; AIEEE-2003]***

(a) A.P. (b) G.P.
(c) H.P. (d) A.G.

18. If the angles A, B, C of a triangle are in A.P. and the sides a, b, c opposite to these angles are in G.P., then a^2, b^2, c^2 are in ***[MP PET-1998]***

(a) A.P. (b) H.P.
(c) G.P. (d) None of these

19. In ΔABC, if $\dfrac{\cos A}{a}=\dfrac{\cos B}{b}=\dfrac{\cos C}{c}$ and the side $a=2$, then area of the triangle is ***[IIT-93; PET-2000]***

(a) 1 (b) 2
(c) $\sqrt{3}/2$ (d) $\sqrt{3}$

20. If angles of a triangles are in the ratio of 2 : 3 : 7, then the sides are in the ratio of ***[PET-96]***

(a) $\sqrt{2}:2:(\sqrt{3}+1)$ (b) $2:\sqrt{2}:(\sqrt{3}+1)$
(c) $\sqrt{2}:(\sqrt{3}\sqrt{2}+1):2$ (d) $2:(\sqrt{3}+1):\sqrt{2}$

21. If in a ΔABC, the altitudes from the vertices A, B, C on opposite sides are in H.P., then $\sin A$, $\sin B$, $\sin C$ are in ***[AIEEE-2005]***

(a) H.P.
(b) Arithmetic-Geometric progression
(c) A.P.
(d) G.P.

22. In ΔABC, $(a+b+c)\left(\tan\dfrac{A}{2}+\tan\dfrac{B}{2}\right)$ is equal to ***[EAMCET-2007]***

(a) $2c\cot\dfrac{C}{2}$ (b) $2a\cot\dfrac{A}{2}$
(c) $2b\cot\dfrac{B}{2}$ (d) $\tan\dfrac{C}{2}$

23. The area of the triangle whose sides are 6, $5\sqrt{13}$ (in square units) is ***[Kerala PET-2008]***

(a) $5\sqrt{2}$ (b) 9
(c) $6\sqrt{2}$ (d) 11

24. If the sides of a triangle are 3, 5, 7 then ***[PET-96]***

(a) All its angles are acute
(b) One angle is obtuse
(c) Triangle is right angled
(d) None of these

HINTS AND EXPLANATIONS

1. (b) $\frac{\sin A}{a}=\frac{\sin B}{b} \Rightarrow \frac{3}{4\times 5}=\frac{\sin B}{7} \Rightarrow \sin B$ $=\frac{21}{20}$
Hence, not possible

2. (a) $\sin\frac{A}{2}=\frac{\sqrt{(s-b)(s-c)}}{bc} \Rightarrow bc\sin^2\frac{A}{2}$
$=(s-b)(s-c)$
Hence, $x = bc$

3. (b) A, B, C are in A.P. then $B = 60°$, $\cos B$ $=\frac{a^2+c^2-b^2}{2ac}$
$\{\because A + B + C = 180°$ and $A + C = 2B$ $\Rightarrow B = 60°\}$
$\Rightarrow \frac{1}{2}=\frac{a^2+c^2-b^2}{2ac} \Rightarrow a^2 + c^2 - b^2 = ac$

4. (c) $(b + c)\cos A + (c + a)\cos B + (a + b)\cos C = a + b + c$
{By Projection Formula $a = b\cos C + c\cos B$}

5. (a) $\frac{\sin(A-B)}{\sin(A+B)}=\frac{\sin A\cos B-\sin B\cos A}{\sin C}$
$=\frac{a}{c}\cos B-\frac{b}{c}\cos A.$
But, $\cos B=\frac{a^2+c^2-b^2}{2ac}$, $\cos A=\frac{b^2+c^2-a^2}{2bc}$
$\Rightarrow \frac{a}{c}\cos B-\frac{b}{c}\cos A=\frac{1}{2c^2}$
$\{a^2+c^2-b^2-b^2-c^2+a^2\}$
$=\frac{a^2-b^2}{c^2}$

6. (c) $\cot B+\cot C-\cot A=\frac{\cos B}{\sin B}+\frac{\cos C}{\sin C}$ $-\cot A$
$=\frac{\sin(B+C)}{\sin B\ \sin C}-\frac{\cos A}{\sin A}$
$=\frac{\sin^2 A-\sin B\ \sin C\ \cos A}{\sin A\ \sin B\ \sin C}$
$=\frac{a^2-bc\cos A}{k(abc)}$
$\left\{\because \frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}=k(\text{Let}) \text{ and } \cos A=\frac{b^2+c^2-a^2}{2bc}\right\}$
$=\frac{a^2-\frac{bc(b^2+c^2-a^2)}{2bc}}{(abc)k}=\frac{(a^2-a^2)}{abck}=0$
$\left\{\because \frac{b^2+c^2-a^2}{2}=\frac{3a^2-a^2}{2}=\frac{2a^2}{2}=a^2\right\}$

7. (c) $\cos B=\frac{c^2+a^2-b^2}{2ac} \Rightarrow \cos B=\frac{1}{2} \Rightarrow$ $B=\frac{\pi}{3}$

8. (b) $\cot\frac{B}{2}\cot\frac{C}{2}$
$=\sqrt{\frac{s(s-b)}{(s-a)(s-c)}\cdot\frac{s(s-c)}{(s-a)(s-b)}}$
$=\frac{s}{s-a}$ $\{\because 3a = b + c$ or $a + b + c = 2s = 4a\}$
$=\frac{2a}{a}=2$

9. (b) $\Delta=\frac{1}{2}ab\sin c=\frac{1}{2}\times 2x\times 2y\times\frac{\sqrt{3}}{2}=xy\sqrt{3}$

10. (a) $2s = a + b + c$, $\cos\frac{B}{2}=\frac{\sqrt{30\times 6}}{320}=\frac{3}{4}$

11. (c) $\cot A + \cot B$ and $\cot C$ are in A.P.
$\Rightarrow \cot A + \cot C = 2\cot B$
$\Rightarrow \frac{\cos A}{\sin A}+\frac{\cos C}{\sin C}=\frac{2\cos B}{\sin B}$
$\Rightarrow \frac{b^2+c^2-a^2}{2bca}+\frac{a^2+b^2-c^2}{2abc}=\frac{a^2+c^2-b^2}{2abc}$
$\Rightarrow a^2 + c^2 = 2b^2$, Hence, a^2, b^2, c^2 are in A.P.

12. (a) $(a + c)^2 - b^2 = 3ac \Rightarrow a^2 + c^2 - b^2 = ac$

But $\cos B = \dfrac{a^2+c^2-b^2}{2ac} = \dfrac{1}{2} \Rightarrow B = \dfrac{\pi}{3}$

13. (c) $\left(\dfrac{b}{a}\cos c + \dfrac{c}{a}\cos B\right) = 1$, (by proejection formula)

14. (b) Obvious

Trick: Clearly, it is not an equilateral triangle because $A = B = C$ 60° does not satisfy the given condition

But if $B = 90°$ then $\sin^2 B = 1$ and $\cos^2 A +$

$\cos^2 C = \cos^2 A + \cos^2\left(\dfrac{\pi}{2} - A\right)$

$= \cos^2 A + \sin^2 A = 1.$

Hence, this equation is satisfied when given triangle is right angled but it is not necessary that it is iscosceles.

15. (b) $\dfrac{b - c\cos A}{c - b\cos A} = \dfrac{b - \dfrac{b^2+c^2-a^2}{2b}}{c - \dfrac{b^2+c^2-a^2}{2c}}$

$= \left(\dfrac{b^2+a^2-c^2}{c^2+a^2-b^2}\right) = \dfrac{c}{b}$

$= \dfrac{b^2+a^2-c^2}{2ab} \cdot \dfrac{2ac}{c^2+a^2-b^2} = \dfrac{\cos C}{\cos B}$

16. $\dfrac{1-2\sin^2 A}{a^2} - \dfrac{1-2\sin^2 B}{b^2} = \dfrac{1}{a^2} - \dfrac{1}{b^2} - 2k^2(1-1)$

$= \dfrac{1}{a^2} - \dfrac{1}{b^2}$

17. (a) $a.\dfrac{s(s-c)}{ab} + c.\dfrac{s(s-a)}{bc} = \dfrac{3b}{2}$

$\Rightarrow \quad 2s(s - c + s - a) = 3b^2$

$\Rightarrow \quad 2s(b) = 3b^2 \Rightarrow 2s = 3b \Rightarrow a + b + c = 3b$

$\Rightarrow \quad a + c = 2b \Rightarrow a, b, c$ are in A.P.

18. (a)$\because$ A, B and C are in A.P.

Hence, $B = 60°$ and $b^2 = ac$

$\cos B = \dfrac{a^2+c^2-b^2}{2ac}$

$\Rightarrow \quad \dfrac{1}{2} = \dfrac{a^2+c^2-b^2}{2ac}$

$\Rightarrow \quad b^2 = a^2 + c^2 - b^2$

$\Rightarrow \quad a^2 + c^2 = 2b^2$ are in A.P.

19. (d) $\dfrac{\cos A}{a} = \dfrac{\cos B}{b} = \dfrac{\cos C}{c}$

$\Rightarrow \quad \dfrac{\cos A}{k\sin A} = \dfrac{\cos B}{k\sin B} = \dfrac{\cos C}{k\sin C}$

$\Rightarrow \quad \cot A = \cot B = \cot C$

$\Rightarrow \quad A = B = C = 60°$

$\Rightarrow \quad \Delta ABC$ is an equilateral triangle.

$\therefore \quad \Delta = \dfrac{\sqrt{3}}{4}a^2 = \sqrt{3}$

20. (a) Given angles are 30°, 45°, 105°

$\therefore \quad a : b : c = \sin 30° : \sin 45° : \sin 105°$

$= \dfrac{1}{2} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

$= \sqrt{2} : 2 : (\sqrt{3}+1)$

{Multiply by $2\sqrt{2}$ }

21. (c) $\Delta = \dfrac{1}{2}p_1 a = \dfrac{1}{2}p_2 b = \dfrac{1}{2}p_3 b$

p_1, p_2, p_3 are in H.P.

$\dfrac{2\Delta}{a}, \dfrac{2\Delta}{b}, \dfrac{2\Delta}{c}$ are in H.P.

$\Rightarrow \quad \dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ are in H.P.

$\Rightarrow \quad k\sin A, k\sin B, k\sin C$ are in A.P.

$\Rightarrow \quad \sin A, \sin B, \sin C$ are in A.P.

22. (a) $\tan\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}} = \dfrac{\Delta}{s(s-a)}$,

Similarly, $\tan\dfrac{B}{2} = \dfrac{\Delta}{s(s-b)}$

Put in equation to get

$2s.\left(\dfrac{\Delta}{s(s-a)} + \dfrac{\Delta}{s(s-b)}\right) = \dfrac{2c\Delta}{(s-a)(s-b)}$

$= 2c\cot\dfrac{C}{2}$

23. (b) Let $a = 6$, $b = 5$, $c = \sqrt{13}$

$\cos C = \dfrac{6^2+5^2-13}{2\times6\times5} = \dfrac{4}{5}, \sin C = \dfrac{3}{5}$

Area $= \dfrac{1}{2}ab\sin C = \dfrac{1}{2}\times6\times5\times\dfrac{3}{5} = 9$

24. (b) Greatest angle is opposite to greatest side $a = 3$, $b = 5$, $c = 7$ greatest angle = C.

$$\because \quad \cos C = \frac{3^2 + 5^2 - 7^2}{2 \times 3 \times 5} = \frac{-15}{30} = \frac{-1}{2}$$

$$\Rightarrow C = \frac{2\pi}{3}$$

$$\Rightarrow \quad C = 120°$$

One angle is obtuse

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. The area of a ΔABC is equal to ***[PET-84]***

(a) $\frac{1}{2} ab \sin A$ (b) $\frac{1}{2} bc \sin A$

(c) $\frac{1}{2} ca \sin A$ (d) $\frac{1}{2} bc \sin A$

2. If the sides of a triangle are in the ratio $2 : \sqrt{6} : (\sqrt{3} + 1)$, then the largest angle of the triangle will be ***[PET-90]***

(a) 60° (b) 75°

(c) 90° (d) 120°

3. If the lengths of the sides of a triangle be 7 cm, $4\sqrt{3}$ cm and $\sqrt{13}$ cm, then the smallest angle is ***[MNR-85]***

(a) 15° (b) 30°

(c) 60° (d) 45°

4. If the sides of a triangle be 6 cm, 10 cm and 14 cm, then the triangle is ***[PET-82]***

(a) Obtuse angled (b) Acute angled

(c) Right angled (d) Equilateral

5. In a ΔABC, side b is equal to

(a) $c \cos A + a \cos C$ (b) $c \cos B + \cos A$

(c) $c \cos C + a \cos B$ (d) None of these

6. In a $\Delta ABC, \frac{1 + \cos(A - B)\cos C}{1 + \cot(A - C)\cos B}$ is equal to

(a) $\frac{a - b}{a - c}$ (b) $\frac{a + b}{a + c}$

(c) $\frac{a^2 - b^2}{a^2 - c^2}$ (d) $\frac{a^2 + b^2}{a^2 + c^2}$

7. If the angles of a triangle are in the ratio 1: 2: 3, then their corresponding sides are in the ratio ***[PET-93]***

(a) 1 : 2 : 3 (b) $1 : \sqrt{3} : 2$

(c) $\sqrt{2} : \sqrt{3} : 3$ (d) $1 : \sqrt{3} : 3$

8. In any ΔABC, if $a \cos B = b \cos A$, then the triangle is ***[PET-84]***

(a) Equilateral triangle

(b) Isosceles triangle

(c) Scalene

(d) Right angled

9. If $a = 9$, $b = 8$ and $c = x$ satisfies $3 \cos C = 2$, then ***[PET-84]***

(a) $x = 5$ (b) $x = 6$

(c) $x = 4$ (d) $x = 7$

10. The sides of a triangle are in the ratio of $1 : \sqrt{3} : 2$, then the angles of the triangle are in the ratio ***[IIT-2004]***

(a) 1 : 3 : 5 (b) 2 : 3

(c) 3 : 2 : 1 (d) 1 : 2 : 3

11. In a ΔABC a, b, c, are the length of its sides and A, B, C are the angles of triangle ABC then the correct relation is

(a) $(b + c)\cos\frac{A}{2} = a\sin\left(\frac{B + C}{2}\right)$

(b) $(b + c)\cos\left(\frac{B + C}{2}\right) = a\sin\frac{A}{2}$

(c) $(b - c)\cos\left(\frac{B - C}{2}\right) = a\cos\left(\frac{A}{2}\right)$

(d) $(b - c)\cos\frac{A}{2} = a\sin\left(\frac{B - C}{2}\right)$

12. The area of the triangle ABC, in which $a = 1$, $b = 2$, $\angle C = 60°$, is **[MPPET-2004]**

(a) 4 sq. units (b) 1/2 sq. units

(c) $\sqrt{3}/2$ sq.units (d) $\sqrt{3}$ sq. units

13. In any ΔABC, $a \cos B + b \cos A$ is equal to

(a) c (b) $\frac{a^2+b^2}{c}$

(c) $\frac{a^2+b^2}{2c}$ (d) $\frac{2c^2-a^2-b^2}{c}$

14. In a triangle ABC, if $A = 30°$, $b = 2$, $c = \sqrt{3}+1$, then $\frac{C-B}{2}$ is equal to

(a) 15° (b) 30°

(c) 45° (d) None of these

15. If in a triangle ABC, $\frac{\sin A}{4} = \frac{\sin B}{5} = \frac{\sin C}{6}$, then the value of $\cos A + \cos B + \cos C$ is equal to

(a) 69/48 (b) 96/48

(c) 48/69 (d) None

16. If in a ΔABC, $(s - a)(s - b) = s(s - c)$, then angle C is equal to **[PET-86]**

(a) 90° (b) 45°

(c) 30° (d) 60°

17. In ΔABC, $\sin A : \sin B : \sin C = 1 : 2 : 3$. If $b = 4$ cm, then the perimeter of the triangle is **[PET-86]**

(a) 6 cm (b) 24 cm

(c) 12 cm (d) 8 cm

18. The area of an isosceles triangle is 9 cm^2. If the equal sides are 6 cm in length, the angle between them is **[PET-86]**

(a) 60° (b) 30°

(c) 90° (d) 45°

19. In a ΔABC, $(c + a + b)(a + b - c) = ab$ then $\angle c$ is **[DCE-2002]**

(a) $\pi/3$ (b) $\pi/6$

(c) $2\pi/3$ (d) $\pi/2$

20. In a triangle, the lengths of two larger sides are 10 cm and 9 cm respectively. If the angles of the triangle are in A.P., then the length of the third side in cm can be **[PET-90, 01; DCE-01]**

(a) $5-\sqrt{6}$ only

(b) $5+\sqrt{6}$ only

(c) $5-\sqrt{6}$ or $5+\sqrt{6}$

(d) Neither $5-\sqrt{6}$ nor $5+\sqrt{6}$

21. The ratios, of the sides in a triangle are 5 : 12 : 13 and its area is 270 cm^2. The sides of a triangle in cm are **[PET-89]**

(a) 5, 12, 13 (b) 10, 24, 26

(c) 15, 36, 39 (d) 20, 48, 52

22. In ΔABC, if $a = 3$, $b = 4$, $c = 5$, then $\sin 2B$ is equal to **[PET-83]**

(a) 4/5 (b) 3/20

(c) 24/25 (d) 1/50

23. If in a triangle ABC, $b = \sqrt{3}, c = 1$ and $B - C = 90°$, then $\angle A$ is **[PET-83]**

(a) 30° (b) 45°

(c) 75° (d) 15°

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet
2. The worksheet is of 15 minutes.
3. The worksheet consists of 15 questions. The maximum marks are 45.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. If a, b, c are the sides and A, B, C are the angles of a triangle ABC, then $\tan(A/2)$ is equal to **[PET-99]**

(a) $\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$ (b) $\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

(c) $\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$ (d) $\sqrt{\frac{(s-a)}{(s-b)(s-c)}}$

2. In a Δ vertex angle A, B, C and side BC are given, the area of ΔABC is **[DCE-2003]**

(a) $\frac{s(s-a)(s-b)(s-c)}{2}$

(b) $\frac{b^2 \sin C \sin A}{\sin B}$

(c) $ab \sin C$

(d) $\frac{1}{2}\frac{a^2 \sin B \sin C}{\sin A}$

3. In a triangle $a = \sqrt{3}+1, B = 30°$, $C = 45°$. What is the value of side c? **[NDA-2005]**

(a) $(\sqrt{3}+1)/2$ (b) 1

(c) $\sqrt{2}$ (d) 2

4. In a ΔABC, $b \sin B = c \sin C$, then which one of the following is correct? **[NDA-2005]**

(a) The triangle is right-angled
(b) The triangle is isosceles
(c) The triangle is equilateral
(d) The triangle is scalene

5. In a triangle ABC, $b^2 = c^2 + a^2$, then what is the value of $\tan A + \tan C$? **[NDA-2004]**

(a) $\tan B$ (b) $\tan A . \tan C$

(c) $\frac{b}{ac}$ (d) $\frac{b^2}{ac}$

6. If in a ΔABC, $a + c = 2b$, then the value of $\cot\frac{A}{2}.\cot\frac{C}{2}$ is equal to **[NDA-2001]**

(a) 4.5 (b) 3
(c) 1.5 (d) 1

7. In a triangle ABC, $a = 2b$ and $\angle A = 3\angle B$. Then ΔABC **[NDA-2006]**

(a) is isosceles (b) is equilateral
(c) is right angled (d) does not exist

8. In a ΔABC, $A : B : C = 3 : 5 : 4$. The $a+b+c\sqrt{2}$ is equal to **[DCE-2001]**

(a) $2b$ (b) $2c$
(c) $3b$ (d) $3a$

9. The sides of a triangle are 4 cm, 5 cm and 6 cm. The area of the triangle is equal to **[MPPET-2006]**

(a) $\frac{15}{4}\text{cm}^2$ (b) $\frac{15}{4}\sqrt{7}\text{ cm}^2$

(c) $\frac{4}{15}\sqrt{7}\text{ cm}^2$ (d) None of these

10. In a triangle ABC, if $b + c = 2a$ and $\angle A = 60°$, then ΔABC is **[MPPET-2004]**

(a) equilateral (b) right angled
(c) isosecles (d) scalene

11. If in a triangle ABC, $2\cos A = \sin B \operatorname{cosec} C$, then **[MPPET-96]**

(a) $a = b$ (b) $b = c$
(c) $c = a$ (d) $2a = bc$

12. If the sides of a triangle area a, b and $\sqrt{a^2+ab+b^2}$, then the biggest angle is **[Kerala Engg.-2005]**

(a) 105° (b) 120°
(c) 150° (d) 135°

13. If in a triangle ABC, the sides AB and AC are perpendicular, then the true equation is

(a) $\tan A + \tan B = 0$ (b) $\tan B + \tan C = 0$
(c) $\tan A + 2\tan C = 0$ (d) $\tan B . \tan C = 1$

14. If the sides of a triangle are in A.P., then the contangent of its half the angles will be in ***[MPPET-93]***

(a) H.P.
(b) G.P.
(c) A.P.
(d) No particular order

15. In a triangle ABC, $b = \sqrt{3}$ cm, $c = 1$ cm, $\angle A = 30°$. What is the value of a? ***[NDA-2008]***

(a) $\sqrt{2}$ cm
(b) 2 cm
(c) 1 cm
(d) 1/2 cm

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	**6.** ⓐ ⓑ ⓒ ⓓ	**11.** ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	**7.** ⓐ ⓑ ⓒ ⓓ	**12.** ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	**8.** ⓐ ⓑ ⓒ ⓓ	**13.** ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	**9.** ⓐ ⓑ ⓒ ⓓ	**14.** ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ	**10.** ⓐ ⓑ ⓒ ⓓ	**15.** ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

2. (d) Area $= \frac{1}{2} bc \sin A$

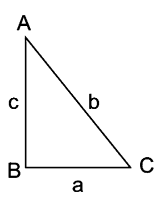
$= \frac{1}{2}(2R\sin B)(2R\sin C)\sin A$

$= \frac{1}{2} \times \frac{a}{\sin A}\sin B . \frac{a}{\sin A}\sin c \ \ \sin A$

$= \frac{a^2}{2}\frac{\sin B \ \sin C}{\sin A}$

4. (b) $b \sin B = c \sin C \Rightarrow 2R \sin B \ \sin B = 2R \sin C \sin C$
$\sin^2 B = \sin^2 C \Rightarrow B = C$
$\therefore$ Δ is isosceles.

5. (d) $\tan A + \tan C = \frac{a}{c} + \frac{c}{a} = \frac{a^2+c^2}{ac} = \frac{b^2}{ac}$



8. $A : B : C = 3 : 5 : 4 \Rightarrow 2x + 5x + 4x = 180$
$x = 15°$, $\angle A = 45°$, $\angle B = 45°$, $\angle C = 60°$

$a + b + c\sqrt{2} = 2R(\sin 45° + \sin 75° + \sin 60°. \sqrt{2})$

$= 2R\left(\frac{1}{\sqrt{2}} + \frac{\sqrt{3}+1}{2\sqrt{2}} + \frac{\sqrt{3}}{\sqrt{2}}\right)$

$= 2R = \left(\frac{3(\sqrt{3}+1)}{2\sqrt{2}}\right) = 3(2R\sin 75°)$

$= 32R \sin B = 3b$

10. (c) $b + c = 2a \Rightarrow 2R(\sin B + \sin C) = 2(2R \sin A)$

$2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right) = 2.\sin A$

$\cos\frac{A}{2}\cos\left(\frac{B-C}{2}\right) = \sin A$

$\Rightarrow \cos 30° \cos\left(\frac{B-C}{2}\right) = \sin 60°$

$$\Rightarrow \frac{B-C}{2} = 0$$

$$\Rightarrow B = C$$

12. (b) Here, biggest side is $\sqrt{a^2+ab+b^2}$

$$\cos\theta = \frac{a^2+b^2-(\sqrt{a^2+ab+b^2})^2}{2ab} = \frac{-ab}{2ab}$$

$$= \frac{-1}{2}$$

$$\theta = \frac{2\pi}{3} \text{ or } 120°$$

15. (c) $a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 3 + 1 - 2\sqrt{3}.\cos 30°$

$a = 1$ cm

LECTURE

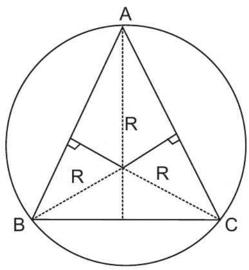
2

Properties of Triangles-II

BASIC CONCEPTS

1. CIRCUMCIRCLE OF A TRIANGLE

The circle which passes through three vertices of a triangle is called the circumcircle of the triangle. The centre of the circle is called circum-centre, usually denoted as O and its radius is called circumradius, usually denoted by R.



The circumcentre is the point of intersection of right bisectors of the sides of a triangle.
In any triangle ABC,

(i) $R = \dfrac{a}{2\sin A} = \dfrac{b}{2\sin B} = \dfrac{c}{2\sin C}$

(ii) $R = \dfrac{abc}{4\Delta}$

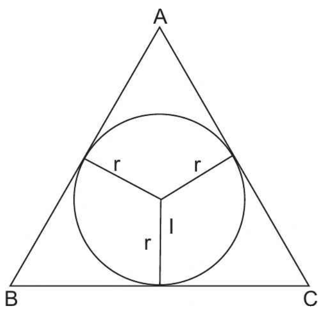
2. INCIRCLE OF A TRIANGLE

The circle which is drawn within a triangle such that the three sides touch this circle is called the **incircle**.

The centre of this circle is called **incentre**, usually denoted by I, and its radius is called **inradius**, usually denoted by r.

The incentre is the point of concurrence of the bisectors of the three (internal) angles of the triangle.
In any triangle ABC,

(i) $r = \dfrac{\Delta}{s}$ = In radius



(ii) $r = (s-a)\tan\dfrac{A}{2} = (s-b)\tan\dfrac{B}{2} = (s-c)\tan\dfrac{C}{2}$

(iii) $r = \dfrac{a\sin\dfrac{B}{2}\sin\dfrac{C}{2}}{\cos\dfrac{A}{2}}$,

$r = \dfrac{b\sin\dfrac{A}{2}\sin\dfrac{C}{2}}{\cos\dfrac{B}{2}}$,

$r = \dfrac{c\sin\dfrac{B}{2}\sin\dfrac{A}{2}}{\cos\dfrac{C}{2}}$

(iv) $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(v) Length of the angle bisector

$$AD = \frac{2bc}{b+c}\cos\frac{A}{2} = \frac{bc\sin A}{(b+c)\sin A/2}$$

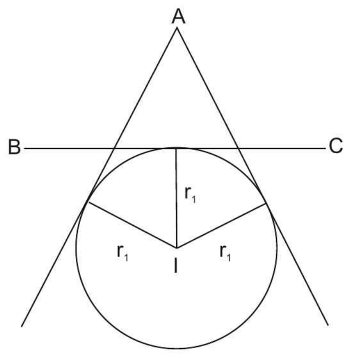
$$BE = \frac{2ac}{a+c}\cos\frac{B}{2}, \qquad CF = \frac{2ab}{a+b}\cos\frac{C}{2}$$

(vi) $\frac{R}{r} = \frac{abc}{4(s-a)(s-b)(s-c)}$

3. ESCRIBED CIRCLES OF A TRIANGLE

The circle which lies outside the triangle and touches the side BC and also the sides AB and AC produced is called **escribed circle** or **excircle** opposite to angle A. Its centre is called **excentre** and its radius is called **exradius**. Similarly, there are two more excircles, one opposite to angle B and one opposite to angle C.

The three excentre are usually denoted as I_1, I_2, I_3 and the three ex-radii are usually denoted as r_1, r_2, r_3. Excentre is the point of concurrence of internal bi-sector of angle A and external bisectors of angles B and C. In any ΔABC



(i) $r_1 = \frac{\Delta}{s-a}, r_2 = \frac{\Delta}{s-b}, r_3 = \frac{\Delta}{s-c}$

(ii) $r_1 = s\tan\frac{A}{2}, r_2 = s\tan\frac{B}{2}, r_3 = s\tan\frac{C}{2}$

(iii) $r_1 = a\frac{\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}, r_2$

$$= b\frac{\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}}, r_3 = c\frac{\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$$

(iv) $r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}, r_2$

$$= 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2},$$

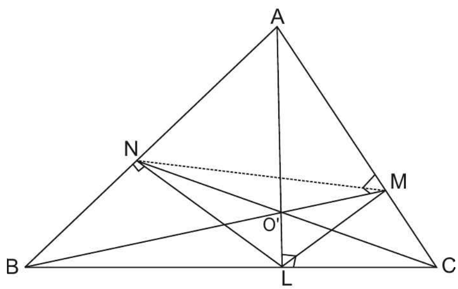
$$r_3 = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

(v) $r_1 + r_2 + r_3 - 4R = r$

4. ORTHOCENTRE OF A TRIANGLE

Let the perpendiculars AL, BM and CN from the vertices A, B and C on the opposite sides BC, CA and AB of ΔABC, respectively, meet at O'. Then O' is the orthocentre of the ΔABC.

The triangle LMN is called the **pedal triangle** of the ΔABC. $O'A = 2R\cos A = a\cot A$, $O'L + 2R\cos B\cos C$



4.1 The Distances of the Orthocentre From the Vertices

The distances of the orthocentre of the triangle from the vertices are $2R\cos A$, $2R\cos B$, $2R\cos C$ and its distances from the sides are $2R\cos B\cos C$, $2R\cos C\cos A$, $2R\cos A\cos B$.

Important Results

(i) Circumradius of the pedal triangle $= \frac{R}{2}$

(ii) Area of the pedal triangle = $2\Delta\cos A\cos B\cos C$.

(iii) Lengths of the medians AL, BM and CN of ΔABC are given by

$$AL = \frac{1}{2}\sqrt{b^2 + c^2 + 2bc\cos A}$$

$$= \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2} ?$$

$$BM = \frac{1}{2}\sqrt{c^2 + a^2 + 2ac\cos B}$$

$$= \frac{1}{2}\sqrt{2c^2 + 2a^2 - b^2}$$

$$CN = \frac{1}{2}\sqrt{a^2 + b^2 + 2ab\cos C}$$

$$= \frac{1}{2}\sqrt{2a^2 + 2ab^2 - c^2}$$

(iv) Circumcentre, Centroid and orthocentre are collinear and G divides OO' in the ratio 1 : 2.

(v) Distance between the circumcentre O and the incentre I is $OI = R\sqrt{1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$
$= \sqrt{R(R-2r)}$

(vi) Distance between the circumcentre and orthocentre $OO' = R\sqrt{1 - 8\cos A\cos B\cos C}$

(vii) Distance between circumcentre and centroid =
$O'G = \frac{1}{3}OO' = \frac{R}{3}\sqrt{1 - 8\cos A\cos B\cos C}$

(viii) Distance between orthocentre and centroid
$= O'G = \frac{2}{3}OO' = \frac{2}{3}R\sqrt{1 - 8\cos A\cos B\cos C}$

4.2 Area, Side and Angle of Pedal

(i) Area of pedal $\Delta = 2\,\Delta\cos A\cos B\cos C$

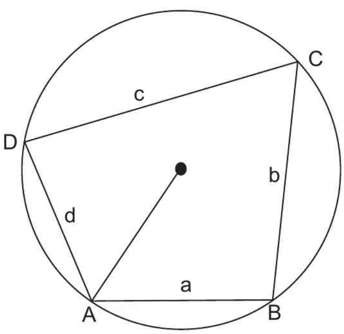
(ii) Sides of Pedal Δ : MN = $a\cos A$, $\angle M = c\cos C$,
$\angle N = b\cos B$.

(iii) Angles of pedal Δ : $\angle L = 180° - 2A$,
$\angle M = 180° - 2B$, $\angle N = 180° - 2C$,

(iv) Circum radius of pedal triangle = R/Z

5. CYCLIC QUADRILATERAL

A quadrilateral is a cyclic quadrilateral if its vertices lie on a circle.



$$\cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$$

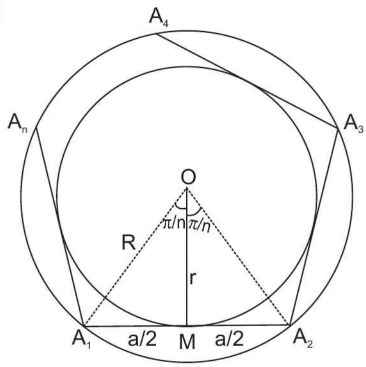
(i) Area of cyclic quadrilateral $ABCD$ is
$= \sqrt{(s-a)(s-b)(s-c)(s-d)}$
where, $2s = a + b + c + d$.

(ii) Circumradius of cyclic quadrilateral
$$R = \frac{1}{4}\sqrt{\frac{(ab+cd)(ad+bc)(ac+bd)}{(s-a)(s-b)(s-c)(s-d)}}$$

(iii) $\cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab+cd)}$ and similarly other angles.

(iv) **Ptolemy's Theorem** If $ABCD$ is a cyclic quadrilateral, then $AC \,.\, BD = AB.\, CD + BC \,.\, AD$ i.e., in a cyclic quadrilateral the product of the diagonals is equal to the sum of the products of the lengths of opposite sides.

6. REGULAR POLYGON



Area of $\Delta OA_1A_2 = \frac{1}{2} \times a \times r, A_1A_2 = a,$

A polygon is called a regular polygon if all its sides are equal and its angles are equal.

NOTES

(i) If a polygon has 'n' sides, sum of its internal angles is $(n-2)\,\pi$ and each angle is $(n-2)\frac{\pi}{n}$.

(ii) In a regular polygon the centroid, the circumcentre and the incentre are same.

6.1 Radius of Circumcribing Circle

$$R = \frac{a}{2\sin\frac{\pi}{n}} = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n},$$

where, a is the length of each side of regular polygon of n sides.

6.2 Radius of Inscribed Circle

$$r = \frac{a}{2}\cot\frac{\pi}{n}$$

6.3 Area of the Polygon

($n \times$ Area of $\Delta\, OA_1A_2$)

$$\Delta = \frac{1}{4}na^2\cot\frac{\pi}{n} = \frac{1}{2}nR^2\sin\frac{2\pi}{n} = nr^2\tan\frac{\pi}{n}$$

where a is the length of each side, R is the circumradius and r is the inradius of the regular polygon of n sides.

Important Result

(i) If $\tan\frac{A}{2}\tan\frac{C}{2} = \frac{1}{3}$, then $a + c = 2b$

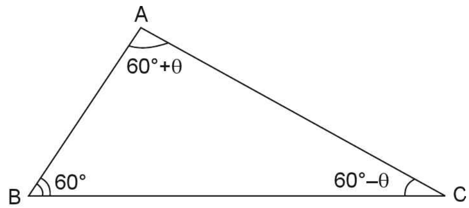
i.e., a, b, c are in A.P.

(ii) If $a = 4$, $b = 5$, $c = 6$, then

$$\cos A = \frac{3}{4}, \cos B = \frac{9}{16}, \cos C = \frac{1}{8}.$$

(iii) If $a = 18$, $b = 24$, $c = 30$, then $r = 6$, $R = 15$ $r_1 = 12$, $r_2 = 18$, $r_3 = 36$, $\Delta = 216$, $\angle C = 90°$.

(iv) $\angle A, \angle B, \angle C$ are in A.P. $\angle B = 60°$



(v) $a + b > c$; $b + c > a$, $c + a > b$

(vi) $a - b < c$; $b - c < a$, $c - a < b$

Example

If the sides of a Δ are in A.P. and the greatest angle exceeds the least by 90°, the show that the sides are as

$$\theta, 90 - 2\theta, 90 + \theta;\ \sqrt{7}-1 : \sqrt{7} : \sqrt{7}+1$$

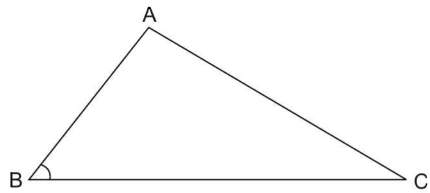
Solution

$$\frac{x-y}{\sin\theta} = \frac{x}{\cos 2\theta} = \frac{x+y}{\cos\theta} = \frac{\sqrt{4xy}}{\sqrt{\cos 2\theta}} = \frac{2\sqrt{xy}}{\sqrt{\cos 2\theta}}$$

$$x = 2\sqrt{xy}\sqrt{\cos 2\theta}$$

$$\sqrt{x} = 2\sqrt{y\cos 2\theta}$$

$$x = 4\,y\cos 2\theta \qquad \ldots\ldots\ldots\ldots(1)$$



$$\frac{x-y}{\sin\theta} = \frac{4y}{1} = \frac{x+y}{\cos\theta}$$

$x - y = 4y\sin\theta$

$x + y = 4y\sin\theta$

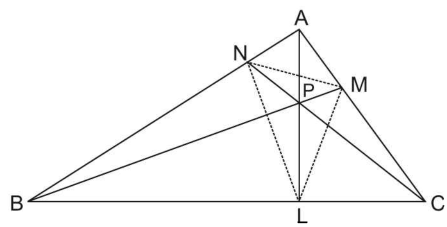
$2\{x^2 + y^2\} = 16y^2\{1\}$; $2x^2 = 14y^2$

$x^2 = 7y^2$

$\Rightarrow x = y\sqrt{7}$

7. ORTHOCENTRE

Let AL, BM and CN be the perpendiculars from A, B, C on opposite sides in a ΔABC. Then from geometry, we know that these perpendiculars are concurrent. Their point of intersection P is called the orthocentre of the triangle ABC. The ΔLMN is called the pedal triangle of ΔABC. The distance of the orthocentres from the vertices.



From ΔAPM, we have

$AP = AM \sec PAM = AM \sec LAC$

$= AM \sec(90° - C) = AM \operatorname{cosec} C$

$= AB\cos A \operatorname{cosec} C = c\cos A/\sin C$

$= \dfrac{2R\sin C\cos A}{\sin C}$ $\qquad$ [$\because c = 2R\sin C$]

$= 2R\cos A$

Thus, $\quad AP = 2R\cos A$

Similarly, $BP = 2R\cos B$ and $CP = 2R\cos C$

Again, $\quad PL = BL\tan LBP = BL\tan(90° - C)$

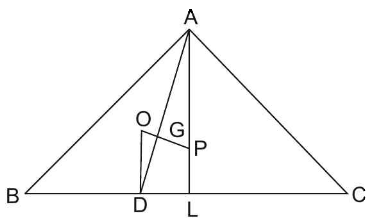
$= AB\cos B\cot C = c\cos B\cot C$

$= 2R\sin C\cos B(\cos C/\sin C) = 2R\cos B\cos C$

Similarly, $PM = 2R\cos C\cos A$ and $PN = 2R\cos A\cos B$.

8. CIRCUM-CENTRE, CENTROID AND ORTHOCENTRE ARE COLLINEAR

Let O be the circum-centre and P the orthocentre in a ΔABC. If OD is perpendicular to BC, then D will be its mid-point. Let the median AD meet OP in G. Then it is clear that $\Delta\, OGD$ and ΔPGA are similar.



$\therefore \quad \dfrac{DG}{AG} = \dfrac{OG}{PG} = \dfrac{OD}{PA}$

But, $\quad OD = R\cos A$ and $PA = 2R\cos A$

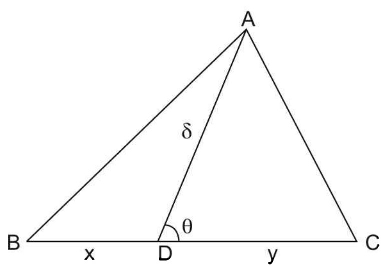
Hence, $\quad \dfrac{OD}{PA} = \dfrac{R\cos A}{2R\cos A} = \dfrac{1}{2}$

Thus, $\quad \dfrac{DG}{AG} = \dfrac{OG}{PG} = \dfrac{OD}{PA} = \dfrac{1}{2}$

It follows that G is the centroid of the ΔABC and is situated on the line OP and divides it in the ratio 1 : 2.

9. BISECTORS OF THE ANGLES

Let AD be the bisector of an angle A and suppose AD divides the base BC into two parts x and y. Then by geometry, $\dfrac{x}{y} = \dfrac{AB}{AC} = \dfrac{c}{b}$ ....(1)



$\therefore \quad \dfrac{x}{c} = \dfrac{y}{b} = \dfrac{x+y}{c+b} = \dfrac{a}{b+c}$

Again if δ be the length of AD and θ the angle it makes with BC, then $\Delta\, ABD + \Delta\, ACD = \Delta\, ABC$

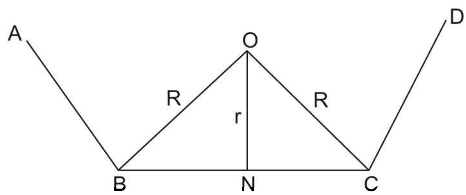
or $\quad \dfrac{1}{2}c\delta\sin\dfrac{A}{2} + \dfrac{1}{2}b\delta\sin\dfrac{A}{2} = \dfrac{1}{2}bc\sin A$

or $\quad \delta \dfrac{bc}{b+c}\cdot\dfrac{\sin A}{\sin(A/2)} = \dfrac{2bc}{b+c}\cos\dfrac{A}{2}$ ......(2)

Also, $\theta = \angle DAB + \angle B = (A/2) + B$ ....(3)

10. RADII OF THE INSCRIBED AND CIRCUMSCRIBING CIRCLES OF A REGULAR POLYGON

Let AB, BC and CD be three successive sides of the polygon and let n be the number of its sides. Let the angle ABC and BCD be bisected by the lines BO and CO meeting at O. Draw ON perpendicular to BC. Then it is clear that O is the centre of both the in-circles and the circumcircle of the polygon.



Also $BN = CN$

Hence, $\quad OB = OC = R$, the radius of the circumcircle and $ON = r$, the radius of in-circle.

Now, $\angle BOC = (2\,\pi/n)$ radians

Hence, $\quad \angle BON = \angle CON = \dfrac{1}{2}\angle BOC = \dfrac{\pi}{n}$.

If a be the length of a side of the polygon, then $a = BC = 2BN = 2R\sin BON$

$= 2R\sin(\pi/n)$ or $R = \dfrac{a}{2}\operatorname{cosec}\dfrac{\pi}{n}$.

Again, $\quad a = 2BN = 2ON\tan BON = 2r\tan(\pi/n)$

$\therefore \quad r = \dfrac{a}{2}\cot\dfrac{\pi}{n}$.

11. AREA OF A REGULAR POLYGON

Area of the polygon is n times the area of $\Delta\, BOC$. Hence, the area of the polygon

$= n \times \dfrac{1}{2} ON.BC = n.ON.BN$

$= n.BN\cot\dfrac{\pi}{n}.BN = n.\left(\dfrac{a}{2}\right)^2 \cot\dfrac{\pi}{n}$

$[\because BN = a/2]$

$= \dfrac{1}{4}na^2\cot\dfrac{\pi}{n}$ ....(1)

The above expression is for the area in terms of the sides.

Also the area $= n.ON.BN = n.OB\cos\frac{\pi}{n}.OB\sin\frac{\pi}{n}$

$$= n.R^2\sin\frac{\pi}{n}\cos\frac{\pi}{n} = \frac{1}{2}.nR^2\sin\frac{2\pi}{n} \quad \ldots\ldots\ldots\ldots\ldots(2)$$

Again, the area $= n\ .\ ON\ .\ BN = n\ .\ ON\ .\ ON\tan(\pi/n)$

$$= nr^2\tan(\pi/n) \quad \ldots\ldots\ldots\ldots\ldots(3)$$

The formulae (2) and (3) give the area in terms of the radius of the circumscribed and inscribed circles.

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. If in ΔABC, $a = 60$, $b = 22$ and $c = 50$, then find the value of R, r_1, r_2 and r_3.

Solution

$$s = \frac{60+22+50}{2} = \frac{132}{2} = 66$$

$$\therefore \quad \Delta = \sqrt{[66(66-60)(66-22)(66-50)]}$$

$$= \sqrt{(66\times 6\times 44\times 16)} = \sqrt{528}$$

(i) $R = \frac{abc}{4\Delta} = \frac{60.22.50}{4.528} = \frac{125}{4} = 31.25$

(ii) $r = \frac{\Delta}{s} = \frac{528}{66} = 8$

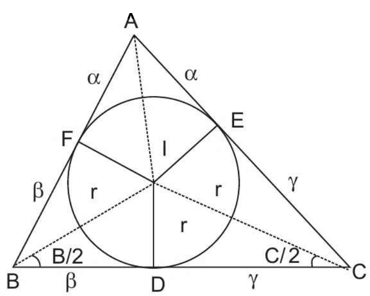
(iii) $r_1 = \frac{\Delta}{s-a} = \frac{528}{66-60} = 88$

(iv) $r_2 = \frac{\Delta}{s-b} = \frac{528}{66-22} = 12$

(v) $r_3 = \frac{\Delta}{s-c} = \frac{528}{66-50} = 33$

2. Prove that the distance of the in-centre of ΔABC from A is $4R\sin(B/2)\sin(C/2)$.

Solution



The bisectors of the angles of ΔABC meet in I. Draw ID, IE, IF perpendicular to the sides from I.

We have $\frac{IF}{IA} = \sin\frac{A}{2}$ or $r = IA\sin(A/2)$ $\ldots\ldots\ldots\ldots\ldots\ldots(1)$

or $\quad 4R\sin(A/2)\sin(B/2)\sin(C/2)$.

$= IA\sin(A/2)$

$\therefore \quad IA = 4R\sin(B/2)\sin(C/2)$.

3. In any triangle ABC, prove that

$$\frac{b^2-c^2}{a^2}\sin 2A + \frac{c^2-a^2}{b^2}\sin 2B + \frac{a^2-b^2}{c^2}$$

$\sin 2C = 0$.

Solution

Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$ (say)

$\therefore \quad a = k\sin A,\ b = k\sin B,\ c = k\sin C$

First term

$$= \frac{b^2-c^2}{a^2}\sin 2A$$

$$= \frac{k^2\sin^2 B - k^2\sin^2 C}{k^2\sin^2 A}\sin 2A$$

$$= \frac{\sin^2 B - \sin^2 C}{\sin^2 A}.2\sin A\cos A$$

$= \sin(B+C)\sin(B-C) - 2\cos A$

$= 2\sin(B-C)\{-\cos(B+C)\}$

$[\because \sin(B+C) = \sin A$ and $\cos A = -\cos(B+C)]$

$= -2\cos(B+C)\sin(B-C) = \cos 2C - \cos 2B$

Similarly, second term = cos2A – cos2C
and third term = cos2B – cos2A

Hence, adding

LHS = (cos2C – cos2B) + (cos2A – cos2C) + (cos2B – cos2A)
= 0 = RHS

4. In any triangle ABC, prove that $(b^2 - c^2)\cot A + (c^2 - a^2)\cot B + (a^2 - b^2)\cot C = 0$

Solution

LHS

$$(b^2-c^2).\frac{\cos A}{\sin A}+(c^2-a^2).\frac{\cos B}{\sin B}+(a^2-b^2).\frac{\cos C}{\sin C}$$

$$=\frac{(b^2-c^2)(b^2+c^2-a^2)}{(ak).2bc}+\frac{(c^2-a^2)(a^2+c^2-b^2)}{(bk).2ac}+\frac{(a^2-b^2)(a^2+b^2-c^2)}{(ck).2ab}$$

putting the values of cos$A$, sin$A$ etc.

$$=\frac{1}{2abck}\Big[(b^4-c^4-a^2b^2+a^2c^2)+(c^4-a^4-b^2c^2+a^2b^2)+(a^4-b^4-a^2c^2+b^2c^2))\Big]$$

$$=\frac{1}{2abck}[0]=0=\text{RHS}$$

5. In any triangle ABC, prove that

$$\frac{a^2\sin(B-C)}{\sin B+\sin C}+\frac{b^2\sin(C-A)}{\sin C+\sin A}+\frac{c^2\sin(A-B)}{\sin A+\sin B}=0$$

Solution

First term of LHS

$$=\frac{a^2\sin(B-C)}{\sin B+\sin C}$$

$$=\frac{a^2(\sin B\cos C-\cos B\sin C)}{\sin B+\sin C}$$

$$=\frac{a^2\left[\frac{2\Delta}{ca}.\frac{a^2+b^2-c^2}{2ca}-\frac{c^2+a^2-b^2}{2ca}.\frac{2\Delta}{ab}\right]}{\frac{2\Delta}{ac}+\frac{2\Delta}{ab}}$$

[where $\Delta=\sqrt{s(s-a)(s-b)(s-c)}$]

= Area of ΔABC

$$=\frac{\frac{\Delta}{bc}\left[a^2+b^2-c^2-a^2+b^2\right]}{\frac{2\Delta}{a}\left[\frac{1}{c}+\frac{1}{b}\right]}$$

$$=\frac{\frac{2\Delta}{bc}\left[b^2-c^2\right]}{\frac{2\Delta}{a}.\left[\frac{b+c}{bc}\right]}=a(b-c)$$

Similarly, second term of LHS = $b(c-a)$
and third term of LHS = $c(a-b)$

Hence, LHS = $a(b-c)+b(c-a)+c(a-b)$ = 0 = RHS

6. The sides of a triangle are in A.P. and the greatest angle exceeds the least by 90°, prove that the sides are proportional $\sqrt{7}+1$, $\sqrt{7}$ and $\sqrt{7}-1$.

Solution

Let A be the greatest and C the least angle of ΔABC.

It is given that a, b, c are in A.P. so that $a + c = 2b$ ....(1)

Also, $A - C = 90°$ From (1)

$\sin A + \sin C = 2\sin B = 2\sin(A + C)$ ....(2)

$$\therefore\quad 2\sin\frac{A+C}{2}\cos\frac{A-C}{2}=4\sin\frac{A+C}{2}\cos\frac{A+C}{2}$$

$$\text{or}\quad \cos\frac{A-C}{2}=2\cos\frac{A+C}{2}=2\sin\frac{B}{2}$$

$\therefore$ $2\sin(B/2)=\cos(90°/2)=\cos 45° = 1/\sqrt{2}$, by (2)

Hence, $\sin\frac{B}{2}=\frac{1}{2\sqrt{(2)}}$ and $\cos\frac{B}{2}$

$$=\sqrt{\left(1-\frac{1}{8}\right)}=\frac{\sqrt{7}}{2\sqrt{(2)}}$$

$\therefore \quad \sin B = 2\sin\frac{B}{2}\cos\frac{B}{2}$

$$= 2.\frac{1}{2\sqrt{(2)}}.\frac{\sqrt{7}}{2\sqrt{(2)}} = \frac{\sqrt{7}}{4}$$

$\therefore \quad \sin A + \sin C = 2 \sin B = \sqrt{7}/2$ ....(3)

Also $\sin A - \sin C = 2 \cos [(A + C)/2] . \sin [(A - C)/2]$

$= 2 \sin (B/2) \sin 45°$

$$= 2.\ \frac{1}{2\sqrt{(2)}}.\frac{1}{\sqrt{(2)}} = \frac{1}{2} \qquad \text{................(4)}$$

Adding and subtracting (3) and (4), we get

$$2\sin A = (\sqrt{7}+1)/2$$

i.e., $\sin A = (\sqrt{7}+1)/4$ and

$$2\sin C = (\sqrt{7}-1)/2$$

i.e., $\sin C = (\sqrt{7}-1)/4$

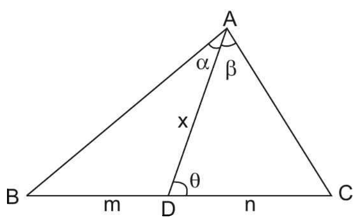
Hence, $a : b : c = \sin A : \sin B : \sin$

$$C = \sqrt{7}+1 : \sqrt{7} : \sqrt{7}-1$$

7. In a triangle ABC if D be any point of the base BC, such that $BD : DC :: m : n$, if $\angle BAD = \alpha$, $\angle DAC = \beta$
$\angle CDA = \theta$, and $AD = x$, then prove that (i) $(m + n) \cot \theta = m \cot \alpha - n \cot \beta = n \cot B - m \cot C$ (This formula is also known as $m : n$ rule) and (ii) $(m + n)^2x^2 = (m + n)(mb^2 + nc^2) - mna^2$.

Solution

(i) We are given $\frac{m}{n} = \frac{BD}{DC}$ or $m.DC = n.BD$ .... (1)



Now, $\frac{m}{n} = \frac{BD}{DC} = \frac{BD}{AD}.\frac{AD}{DC}$

$$= \frac{\sin\alpha}{\sin(\theta-\alpha)}.\frac{\sin[\pi-(\theta+\beta)]}{\sin\beta}$$

[Note]

or $m \sin \beta \sin (\theta - \alpha) = n \sin \alpha \sin (\theta + \beta)$

or $m \sin \beta \sin \theta \cos \alpha - m \sin \beta \cos \theta \sin \alpha$

$= n \sin \alpha \sin \theta \cos \beta + n \sin \alpha \cos \theta \sin \beta$

Dividing by $\sin \alpha \sin \beta \sin \theta$, we get

$m \cot \alpha - m \cot \theta = n \cot \beta + n \cot \theta$

or $m \cot \alpha - n \cot \beta = (m + n) \cot \theta$

Similarly, second result can be proved.

(ii) From (1), we have

$$\frac{BD}{m} = \frac{DC}{n} = \frac{BD+DC}{m+n} = \frac{BC}{m+n} = \frac{a}{m+n}$$

$$\therefore \quad BD = \frac{am}{m+n}, DC = \frac{an}{m+n}$$ .... (1)

Now, $AC^2 = AD^2 + DC^2 - 2AD . DC \cos \theta$

or $b^2 = x^2 + DC^2 - 2x\, DC \cos\theta$ ....(2)

Similarly, $c^2 = x^2 + BD^2 - 2x.\ BD \cos (\pi - \theta)$

or $c^2 = x^2 + BD^2 + 2x .BD \cos\theta$ ....(3)

Multiplying (2) by m and (3) by n and adding, we get

$mb^2 + nc^2 = (m + n)x^2 + m. DC^2 + n . BD^2$

$- 2x \cos \theta . (m . DC - n . BD)$

$= (m + n) x^2 + m . DC^2 + n. BD^2$ from (1)

Hence, substituting the values of BD and DC, we get $mb^2 + nc^2$

$$= (m+n)x^2 + m.\frac{a^2n^2}{(m+n)^2} + n.\frac{a^2m^2}{(m+n)^2}$$

$$= (m+n)x^2 + \frac{a^2mn}{m+n}$$

or $(m + n)(mb^2 + nc^2) = (m + n)^2x^2 + a^2 mn$

or $(m + n)^2x^2 = (m + n)(mb^2 + nc^2) - a^2 mn$

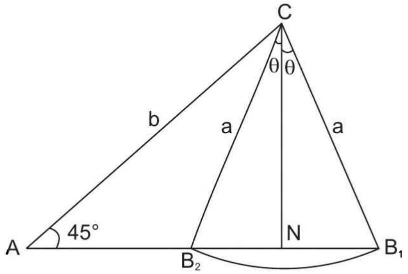
8. In a ΔABC, $A = 45°$ and c_1, c_2 are the two values of side c in the ambiguous case, show that $\cos B_1CB_2 = \frac{2c_1c_2}{c_1^2 + c_2^2}$

Solution

Let the two triangles formed be AB_1C and AB_2C. Draw $CN \perp$ to AB_1. Then, since ΔB_1CB_2 is isosceles, we have $\angle B_1CN = \angle B_2CN = \theta$, say.

Since, $\angle A = 45°$,

We have, $CN = b \sin 45° = b/\sqrt{2}$.



Now, $a^2 = b^2 + c^2 - 2bc \cos A$

$= b^2 + c^2 - 2bc \cos 45°$

$= b^2 + c^2 - \sqrt{2}bc$

or $c^2 - \sqrt{2}bc + b^2 - a^2 = 0$

If, c_1, c_2 are the two values of c, then

$c_1 + c_2 = \sqrt{2}b$ and $c_1, c_2 = b^2 - a^2$.

$\therefore \quad c_1^2 + c_2^2 = (c_1 + c_2)^2 - 2c_1c_2$

$= 2b^2 - 2(b^2 - a^2) = 2a^2$ .... (2)

Now $\cos B_1CB_2 = \cos 2\theta = 2\cos^2\theta - 1$.

$$= 2.\left(\frac{CN}{a}\right)^2 - 1 = 2.\frac{b^2}{2a^2} - 1 \quad (\because \ CN = b/\sqrt{2})$$

$$\frac{b^2 - a^2}{a^2} = \frac{c_1c_2}{\frac{1}{2}(c_1^2 + c_2^2)} = \frac{2c_1c_2}{c_1^2 + c_2^2}.$$

9. In the ambiguous case, if b and A are given and c_1, c_2 are the two values of third side, then prove that

(i) $c_1 + c_2 = 2b \cos A$ and $c_1 c_2 = b^2 - a^2$

(ii) $c_1^2 - 2c_1c_2 \cos 2A + c_2^2 = 4a^2\cos^2 A$

(iii) $(c_1 - c_2)^2 + (c_1 + c_2)^2 \tan^2 A = 4a^2$

Solution

The quadratic giving the two values c_1, c_2 of c is $c^2 - 2bc \cos A + (b^2 - a^2) = 0$

Then we have (i) $c_1 + c_2 = 2b \cos A$ ....(1)

and $c_1 c_2 = b^2 - a^2$ ....(2)

(ii) From (1) and (2)

$(c_1 + c_2)^2 = 4b^2 \cos^2 A = 4(c_1 c_2 + a^2) \cos^2 A$

or $c_1^2 + 2c_1c_2 + c_2^2 - 4 c_1 c_2 \cos^2 A = 4a^2\cos^2 A$

or $c_1^2 - 2c_1c_2(2\cos^2 A - 1) + c_2^2 = 4a^2\cos^2 A$

or $c_1^2 - 2c_1c_2 \cos 2A + c_2^2 = 4a^2\cos^2 A$

10. In the ambiguous case, if the remaining angles of the triangles formed with a, b and A be B_1, C_1 and B_2, C_2, then prove that

$$\frac{\sin C_1}{\sin B_1} + \frac{\sin C_2}{\sin B_2} = 2\cos A$$

Solution

c_1, c_2 are two values of c, we have

$c_1 + c_2 = 2b \cos A$. .... (1)

Also, B_1, B_2 are supplementary angles, that is $B_2 = 180° - B_1$ so that $\sin B_2 = \sin B_1$

Hence, $$\frac{\sin C_1}{\sin B_1} + \frac{\sin C_2}{\sin B_2} = \frac{\sin C_1}{\sin B_1} + \frac{\sin C_2}{\sin B_1}$$

$$= \frac{\sin C_1 + \sin C_2}{\sin B_1}$$

$$= \frac{kc_1 + kc_2}{kb} = \frac{c_1 + c_2}{b} = 2\cos A, \text{ from (1)}$$

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): SOLVE THESE PROBLEMS TO GRASP THE TOPIC

EXERCISE 1

1. Prove that, $\sin(B - C) = \dfrac{b^2 - c^2}{a^2}\sin A$.

2. In any triangle ABC, prove that

$$\frac{1 + \cos(A - B)\cos C}{1 + \cos(A - C)\cos B} = \frac{a^2 + b^2}{a^2 + c^2}.$$

3. In any triangle ABC, prove that

$b \sin B - c \sin C = a \sin (B - C)$

4. In any triangle ABC, prove that

$$\frac{c + a}{b} = \frac{\cos\frac{C - A}{2}}{\sin\frac{B}{2}}$$

5. In any ΔABC prove that

$$\frac{a-b}{c}=\frac{\tan\frac{A}{2}-\tan\frac{B}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}}$$

6. In any ΔABC prove that

$$b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}=\frac{a+b+c}{2}$$

7. In any ΔABC, prove that

$$\frac{1}{(s-a)}+\frac{1}{(s-b)}+\frac{1}{(s-c)}-\frac{1}{s}=\frac{4R}{\Delta}$$ ***[MP-1989]***

8. If $8R^2 = a^2 + b^2 + c^2$, then prove that triangle is right angled.

EXERCISE 2

1. If sides of any triangle are 5 cm, 12 cm, 13 cm, then find the circumradius.

2. In any triangle ABC, prove that $a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0$ ***[MP-1995]***

3. In any triangle ABC, prove that

$$a\cos\frac{1}{2}(B-C)=(b+c)\sin\frac{1}{2}A$$

4. If $a\cos^2\frac{C}{2}+c\cos^2\frac{A}{2}=\frac{3b}{2}$ then prove that $\frac{1}{a},\frac{1}{b},\frac{1}{c}$ are in H.P.

5. The length of the base of the triangle and angle of base are 30° and 60° respectively, then find the area of triangle.

6. In a ΔABC, if $2b^2 = a^2 + c^2$, then prove that

$$\frac{\sin 3B}{\sin B}=\left(\frac{a^2-c^2}{2ac}\right)^2$$

[MP-1995]

7. In ΔABC, prove that $\sin A + \sin B + \sin C = \frac{s}{R}$

ANSWERS

EXERCISE 2

1. 65 cm.

5. $8\sqrt{3}$ cm^2

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. Internal bisector of $\angle A$ of triangle ABC meets side BC at D, a line drawn through D perpendicular to AD intersects AC at E and AB at F. Then ***[IIT-2006]***

(a) AE is HM of b, c

(b) $AD=\frac{2bc}{b+c}\cos\frac{A}{2}$

(c) $EF=\frac{4bc}{b+c}\sin\frac{A}{2}$

(d) All the above

Solution

(d) $\Delta ABC = \Delta ABD + \Delta ACD$

$$\Rightarrow \frac{1}{2}bc\sin A=\frac{1}{2}c(AD)\sin\frac{A}{2}+\frac{1}{2}b(AD)\sin\frac{A}{2}$$

$$\Rightarrow 2bc\sin\frac{A}{2}\cos\frac{A}{2}=c(AD)\sin$$

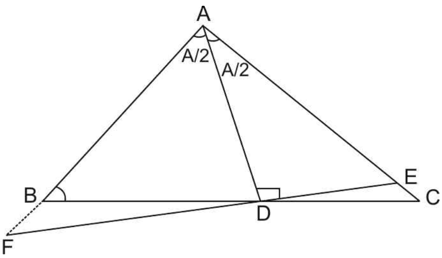
$\frac{A}{2} + b(AD)\sin\frac{A}{2}$

$\Rightarrow \quad 2bc\cos\frac{A}{2} = c(AD) + b(AD)$

$\Rightarrow \quad AD = \frac{2bc}{b+c}\cos\frac{A}{2}$ ....(i)

$\Rightarrow$ Again, $AE = AD\sec\frac{A}{2} = \frac{2bc}{b+c}$ [by (i)]

$\Rightarrow$ AE is HM of b and c



Also, $EF = ED + FD = 2DE = 2AD \tan A/2$

$$= \frac{4bc}{b+c}\cos\frac{A}{2}.\tan\frac{A}{2} = \frac{4bc}{b+c}\sin\frac{A}{2}$$

Hence, (a), (b) and (c) all are correct.

2. Let ABC be a triangle such that one of its sides is double the other and let the angles opposite to those sides differ by an angle of $\pi/3$, then the triangle is

[Orissa JEE-2007]

(a) obtuse triangle
(b) isosceles triangle
(c) right angled triangle
(d) equilateral triangle

Solution

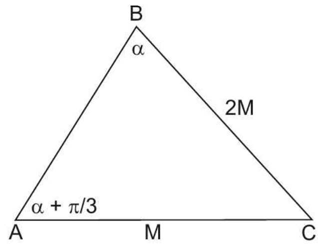
(c) Let one angle α, other be $\alpha + \pi/3$
Let the side opposite to angle α be M so the side opposite to $\alpha + \pi/3$ be $2M$ using sine formula

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$\Rightarrow \quad \frac{M}{\sin a} = \frac{2M}{\sin(a + \pi/3)}$

$\Rightarrow \quad 2\sin\alpha = \sin(\alpha + \pi/3)$

$\Rightarrow \quad 2\sin a = \frac{1}{2}\sin a + \frac{\sqrt{3}}{2}\cos a$



$\Rightarrow \quad \frac{3}{2}\sin a = \frac{\sqrt{3}}{2}\cos a$

$\Rightarrow \quad \tan a = \frac{1}{\sqrt{3}} \Rightarrow a = 30°$

So other angle $\alpha + \pi/3 = 90°$ or triangle is right angle triangle.

3. In a triangle ABC, AD is altitude from A. Given $b > c$, $\angle C = 23°$ and $AD = \frac{abc}{b^2 - c^2}$, then $\angle B$ is equal to

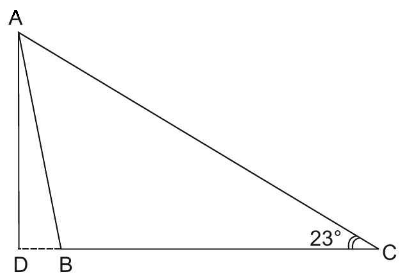
[IIT-1994; DCE-2002]

(a) 67° (b) 44°
(c) 113° (d) None of these

Solution

(c) $\sin B = \frac{AD}{AB = c}, AD = c\sin B = \frac{abc}{b^2 - c^2}$

$\sin B = \frac{ab}{b^2 - c^2}$



$$\sin B = \frac{\sin A \sin B}{\sin(B + C)\sin(B - C)}$$

$\sin(B - C) = 1 = \sin 90°$

$B - C = 90°$

$B = C + 90° = 23° + 90° = 113°$

4. The lengths of the sides of a triangle are $\alpha - \beta$, $\alpha + \beta$ and $\sqrt{3\alpha^2 + \beta^2}$, ($\alpha > \beta > 0$). Its largest angle is *[Roorkee-1999]*

(a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{2}$

(c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

Solution

(c) Let $a = \alpha - \beta$, $b = \alpha + \beta$, $c = \sqrt{3\alpha^2 + \beta^2}$

$$\therefore \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\Rightarrow \quad \cos C = \frac{\alpha^2 + \beta^2 - 2\alpha\beta + \alpha^2 + \beta^2 + 2\alpha\beta - 3\alpha^2 - \beta^2}{2(\alpha^2 - \beta^2)}$$

$$\Rightarrow \quad \cos C = -\frac{(\alpha^2 + \beta^2)}{2(\alpha^2 - \beta^2)}$$

$$= \cos\left(\frac{2\pi}{3}\right) \Rightarrow \angle C = \frac{2\pi}{3}, \text{ (largest angle).}$$

5. In triangle ABC, with general notions $r_1 + r_2 + r_3 - r$ is equal to *[UPSEAT-2001]*

(a) $4R$ (b) Δ^2

(c) Δ (d) $2R$

Solution

(a) Using corresponding formulae

$$r_1 + r_2 + r_3 - r = \frac{\Delta}{s-a} + \frac{\Delta}{s-b} + \frac{\Delta}{s-c} + \frac{\Delta}{s}$$

$$= \Delta\left[\left(\frac{1}{s-a} + \frac{1}{s-b}\right) + \left(\frac{1}{s-c} - \frac{1}{s}\right)\right]$$

$$= \Delta\left[\frac{2s-a-b}{(s-a)(s-b)} + \frac{c}{s(s-c)}\right]$$

$$= \frac{\Delta c}{\Delta^2}[s(s-c) + (s-a)(s-b)]$$

$$= \frac{c}{\Delta}[2s^2 - s(a+b+c) + ab]$$

$$= \frac{c}{\Delta}(ab) = \frac{abc}{\Delta} = 4R$$

6. In a triangle ABC, $\frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3}$ is equal to *[Kurukshetra (CEE)-1997]*

(a) 1 (b) 0

(c) abc (d) $r_1 r_2 r_3$

Solution

(b) $$\frac{b-c}{\left(\frac{\Delta}{s-a}\right)} + \frac{c-a}{\left(\frac{\Delta}{s-b}\right)} + \frac{a-b}{\left(\frac{\Delta}{s-c}\right)}$$

$$= \frac{1}{\Delta}[(b-c)(s-a) + (c-a)(s-b) + (a-b)(s-c)]$$

$$= \frac{1}{\Delta}[s(b-c+c-a+a-b) - a(b-c) - b(c-a) - c(a-b)] = 0$$

7. In a triangle ABC, if $\angle B = \pi/3$, $\angle C = \pi/4$ and D divides BC in ratio 1:3 internally, then $\frac{\sin\angle BAD}{\sin\angle CAD}$ is equal to *[IIT-95; DCE-99; UPSEAT-2001, 2003]*

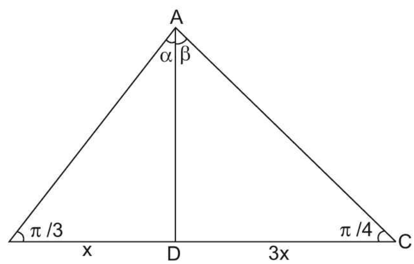
(a) $1/\sqrt{3}$ (b) $1/\sqrt{6}$

(c) $1/3$ (d) $\sqrt{2/3}$

Solution

(b) Let $\angle BAD = \alpha$, $\angle CAD = \beta$, $B = x$, $DC = 3x$. Then,

$$\frac{x}{\sin\alpha} = \frac{AD}{\sin\pi/3} \quad \text{.............(1)}$$



$$\text{and } \frac{3x}{\sin\beta} = \frac{AD}{\sin\pi/4} \quad \text{...........(2)}$$

$$(1), (2) \Rightarrow \frac{x}{\sin\alpha} \cdot \frac{\sin\beta}{3x} = \frac{AD}{\sin\pi/3} \cdot \frac{\sin\pi/4}{AD}$$

$$\Rightarrow \quad \frac{\sin\beta}{3\sin\alpha} = \frac{1/\sqrt{2}}{\sqrt{3}/2} \Rightarrow \frac{\sin\alpha}{\sin\beta} = \frac{1}{\sqrt{6}}$$

$$\therefore \quad \frac{\sin\angle BAD}{\sin\angle CAD} = \frac{1}{\sqrt{6}}$$

8. In a triangle ABC, $\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)\left(a\sin^2\frac{B}{2}+B\sin^2\frac{A}{2}\right)$ is equal to **[Roorkee-88]**

(a) $\cot\frac{C}{2}$ (b) $\cot\frac{C}{2}$

(c) $c\cot C$ (d) $\cot C$

Solution

(b) $$\left(\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}+\sqrt{\frac{s(s-b)}{(s-c)(s-a)}}\right)$$

$$\left[\frac{(s-c)(s-a)}{c}+\frac{(s-b)(s-c)}{c}\right]$$

$$=\frac{\sqrt{s}.c}{\sqrt{(s-a)(s-b)(s-c)}}.\frac{s-c}{c}.$$

$$c=c\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}=c\cot\frac{C}{2}$$

9. In a triangle, r_1, r_2, r_3 are in HP. If its area is 24 cm^2 and its perimeter is 24 cm, then lengths of its sides are **[Roorkee-99]**

(a) 3, 9, 11 (b) 4, 6, 8

(c) 6, 8, 10 (d) 5, 7, 10

Solution

(c) r_1, r_2, r_3 are in HP $\Rightarrow a, b, c$ are in AP

Now, $a + b + c = 24 \Rightarrow s = 12$

Also $b = 8$, $c = 16 - a$

$\therefore$ $\Delta = 24 \Rightarrow 12(12 - a)(4)(12 - c) = 576$

$\Rightarrow$ $a^2 - 16a + 60 = 0$ $|\Rightarrow a = 10, 6$

$\therefore$ sides are 6, 8, 10

10. If p_1, p_2, p_3 are altitudes of a triangle ABC from the vertices A, B, C and Δ the area of the triangle, then $p_1^{-2}+p_2^{-2}+p_3^{-2}$ is equal to

(a) $\frac{a+b+c}{\Delta}$ (b) $\frac{a^2+b^2+c^2}{4\Delta^2}$

(c) $\frac{a^2+b^2+c^2}{\Delta^2}$ (d) None of these

Solution

(b) We have $\frac{1}{2}ap_1=\Delta, \frac{1}{2}bp_2=\Delta, \frac{1}{2}cp_3=\Delta$

$$\Rightarrow \quad p_1=\frac{2\Delta}{a}, p_2=\frac{2\Delta}{b}, p_3=\frac{2\Delta}{c}$$

$$\therefore \quad \frac{1}{p_1^2}+\frac{1}{p_2^2}+\frac{1}{p_3^2}=\frac{a^2+b^2+c^2}{4\Delta^2}$$

11. In a triangle ABC, $a : b : c = 4 : 5 : 6$. The ratio of the radius of the circumcircle to that of the incircle is **[IIT-96]**

(a) 15 : 7 (b) 16 : 7

(c) 17 : 7 (d) 18 : 7

Solution

(b) $a = 4k$, $b = 5k$, $c = 6k$

$$s=\frac{15}{2}k, s-a=\frac{7}{2}k, s-b$$

$$=\frac{5}{2}k, s-c=\frac{3}{2}k$$

$$S^2=15\times7\times5\times3\left(\frac{k}{2}\right)^4$$

$$\therefore S=15\sqrt{7}\left(\frac{k}{2}\right)^2$$

$$r=\frac{s}{s}=15\sqrt{7}\left(\frac{k}{2}\right)^2\div\frac{15}{2}k=\sqrt{7}\frac{k}{2}$$

$$R=\frac{abc}{4S}=\frac{4.5.6k^3}{4.15\sqrt{7}k^2/4}=\frac{8}{\sqrt{7}}k$$

$$\therefore \quad \frac{R}{r}=\frac{8}{\sqrt{7}}\div\frac{\sqrt{7}}{2}=\frac{16}{7}$$

12. Which of the following pieces of data does not uniquely determine an acute angled Δ ABC (R = circumradius) **[IIT Screening-2002]**

(a) a, $\sin A$, $\sin B$ (b) a, b, c

(c) a, $\sin B$, R (d) a, $\sin A$, R

Solution

(d) $\frac{a}{\sin A}=R$ and $b = 2R\sin B$. So, two sides and

two angles are known. So, $\angle C$ is known

Therefore, two sides and included angle is known

So, Δ is uniquely known in case (a)

If a, b, c are known the Δ is uniquely known in case (b)

$b = 2R \sin B$, $\sin A = \dfrac{a}{2R}$. So, sides a, b and angle A, B are known. So $\angle C$ is known

Therefore, two sides and included angle is known

So, Δ is uniquely known in case (c)

$\dfrac{a}{\sin A} = R$. So, only a side and an angle is known

So, Δ is not uniquely known in case (d).

13. Given an isosceles triangle, whose one angle is 120° and radius of its incircle $= \sqrt{3}$. Then the area of the triangle

[IIT-JEE-2006]

(a) 4π (b) $12 + 7\sqrt{3}$

(c) $7 + 12\sqrt{3}$ (d) $12 - 7\sqrt{3}$

Solution

(b) Let ABC be given triangle with $\angle A = 120°$. Then

$$\Delta = (b \sin 60°)(b \cos 60°) = \frac{\sqrt{3}}{4}b^2 \quad \ldots\ldots\ldots\ldots(1)$$

Also, $$\frac{a}{\sin 120°} = \frac{b}{\sin 30°} \Rightarrow a = \sqrt{3}b \quad \ldots\ldots\ldots\ldots(2)$$

Further, $$r = \frac{\Delta}{s} \Rightarrow \Delta = rs = \sqrt{3}s$$

$$\Rightarrow \quad \Delta = \sqrt{3}\cdot\frac{1}{2}(a + 2b)$$

$$= \frac{\sqrt{3}}{2}(\sqrt{3} + 2)b \text{ [by (2)]} \quad \ldots\ldots\ldots\ldots(3)$$

$$(1), (3) \Rightarrow \frac{\Delta^2}{\Delta} = \frac{\frac{3}{4}(\sqrt{3}+2)^2 b^2}{(\sqrt{3}/4)b^2}$$

$$= \sqrt{3}(7 + 4\sqrt{3}) \Rightarrow \quad \Delta = 12 + 7\sqrt{3}$$

14. A triangle satisfies the condition

[IIT-86; Pb. CET-03]

(a) $b \sin A = a, A < \pi/2$

(b) $b \sin A > a, A > \pi/2$

(c) $b \sin A > a, A < \pi/2$

(d) None of these

Solution

(a) $\because \dfrac{\sin A}{a} = \dfrac{\sin B}{b} \Rightarrow b\sin A = a\sin B$

$\Rightarrow b \sin A = a$, if $B = \pi/2$, then $A < \pi/2$

15. In a triangle ABC, a, b, c are in AP and $\cos\theta_1 = \dfrac{a}{b+c}, \cos\theta_2 = \dfrac{b}{c+a}, \cos\theta_3 = \dfrac{c}{a+b}$

then $\tan^2\dfrac{\theta_1}{2} + \tan^2\dfrac{\theta_3}{2}$ is equal to

[IIT-JEE-2006]

(a) 1/3 (b) 2/3

(c) 1/2 (d) 1

Solution

(b) As given,

$$\cot\theta_1 = \frac{a}{b+c} \Rightarrow \frac{1 - \tan^2\frac{\theta_1}{2}}{1 + \tan^2\frac{\theta_1}{2}} = \frac{a}{b+c}$$

$$\Rightarrow \quad \tan^2\frac{\theta_1}{2} = \frac{b+c-a}{b+c+a}$$

Similarly, $$\tan^2\frac{\theta_3}{2} = \frac{a+b-c}{b+c+a}$$

$$\therefore \quad \tan^2\frac{\theta_1}{2} + \tan^2\frac{\theta_3}{2} = \frac{2b}{3b} = \frac{2}{3}$$

$[\because a + c = 2b]$

or

Verification Method: $(a = 3, b = 5, c = 7)$

and $\cos\theta_1 = \dfrac{3}{12} = \dfrac{1}{4}$;

$\cos\theta_2 = \dfrac{5}{10} = \dfrac{1}{2}$; $\cos\theta_3 = \dfrac{7}{8}$

$$\tan^2\left(\frac{\theta_1}{2}\right) + \tan^2\left(\frac{\theta_3}{2}\right) = \frac{1-\cos\theta_1}{1+\cos\theta_1} + \frac{1-\cos\theta_3}{1+\cos\theta_3}$$

$$\frac{1-\frac{1}{4}}{1+\frac{1}{4}} + \frac{1-\frac{7}{8}}{1+\frac{7}{8}} = \frac{3}{5} + \frac{1}{15} = \frac{10}{15} = \frac{2}{3}$$

16. If $A_1A_2A_3A_4A_5$ be a regular pentagon inscribed in a unit circle. Then, $(A_1A_2)(A_1A_3)$ is equal to

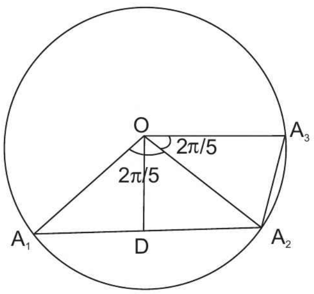
(a) 1 (b) 3

(c) 4 (d) $\sqrt{5}$

Solution

(d) Angle subtended by each side at centre $=\dfrac{2\pi}{5}$

$\therefore \quad \angle A_1OA_2 = \dfrac{2\pi}{5};\ \angle A_1OA_3 = \dfrac{4\pi}{5}$



In triangle A_2DO; $\sin\dfrac{\pi}{5} = \dfrac{A_2D}{OA_2}$

$\Rightarrow \quad A_2D = \sin\dfrac{\pi}{5}$

$A_1A_2 = 2A_2D = 2\sin\dfrac{\pi}{5}$

Similarly, $A_1A_3 = 2\sin\dfrac{2\pi}{5}$

$(A_1A_2)(A_1A_3) = 4\sin\dfrac{\pi}{5}\times\sin\dfrac{2\pi}{5}$

$= 4\sin 36° \sin 72°$

$=\dfrac{1}{4}(\sqrt{10-2\sqrt{5}})\times(\sqrt{10+2\sqrt{5}})$

$=\dfrac{4\sqrt{5}}{4} = \sqrt{5}$

17. If perpendiculars drawn from verties of a triangle ABC intersect in point O. If $OA = x$, $OB = y$, $OC = z$ then $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}$ is equal to

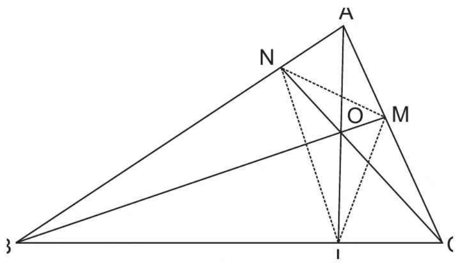
(a) $\dfrac{a+b+c}{x+y+z}$ (b) $\dfrac{ab+bc}{xy+yz}$

(c) $\dfrac{xyz}{abc}$ (d) $\dfrac{abc}{xyz}$

Solution

(d) We know distance of orthocentre O from vertex

$A = x = 2R\cos A$
$B = y = 2R\cos B$
$C = z = 2R\cos C$



therefore, $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}$

$=\dfrac{2R\sin A}{2R\cos A}+\dfrac{2R\sin B}{2R\cos B}+\dfrac{2R\sin C}{2R\cos C}$

$= \tan A + \tan B + \tan C$

In a triangle ABC, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$\therefore \quad \dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z} = \left(\dfrac{abc}{xyz}\right)$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. In triangle ABC, if $a = 13$, $b = 14$ and $c = 15$, then R is equal to

(a) 65/8

(b) 21/4

(c) 6

(d) None of these

2. If the two angles on the base of a triangle are $\left(22\dfrac{1}{2}\right)^\circ$ and $\left(112\dfrac{1}{2}\right)^\circ$, then the ratio of the height of the triangle to the length of the base is

[MPPET-1993]

(a) 1 : 2 (b) 2 : 1
(c) 2 : 3 (d) 1 : 1

3. In any triangle *ABC*, then value of a $(b^2 + c^2)\cos A + b(c^2 + a^2)\cos B + c(a^2 + b^2)\cos C$ is
(a) $3abc^2$ (b) $3a^2bc$
(c) $3abc$ (d) $3ab^2c$

4. If in a right angled triangle the hypotenuse is four times as long as the perpendicular drawn to it from opposite vertex, then one of its acute angle is ***[PET-1998, 2004]***
(a) 15° (b) 30°
(c) 45° (d) None of these

5. If in any Δ *ABC*, $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in A.P., then ***[PET-2003]***
(a) $\cot\frac{A}{2}\cot\frac{B}{2} = 4$ (b) $\cot\frac{A}{2}\cot\frac{C}{2} = 3$
(c) $\cot\frac{B}{2}\cot\frac{C}{2} = 1$ (d) $\cot\frac{B}{2}\cot\frac{C}{2} = 0$

6. If the angles of a triangle are in the ratio 4 : 1 : 1, then the ratio of the longest side to the perimeter is ***[IIT-2003]***
(a) $\sqrt{3}:(2+\sqrt{3})$ (b) 1 : 6
(c) $1:2+\sqrt{3}$ (d) 2 : 3

7. In a triangle *ABC*, $b = \sqrt{3}$, $c = 1$ and $\angle A = 30°$, then the largest angle of the triangle is ***[MPPET-2004]***
(a) 60° (b) 135°
(c) 90° (d) 120°

8. In a triangle *ABC*, $\tan\frac{A}{2} = \frac{5}{6}, \tan\frac{C}{2} = \frac{2}{5}$, then ***[AIEEE-2002]***
(a) *a, c, b* are in AP (b) *a, b, c* are in AP
(c) *b, a, c* are in AP (d) *a, b, c* are in GP

9. The sides of a triangle are $3x + 4y$, $4x + 3y$ and $5x + 5y$ units, where $x, y > 0$. The triangle is ***[AIEEE-2002]***
(a) Right angled (b) Equilateral
(c) Obtuse angled (d) None of these

10. The perimeter of a Δ *ABC* is 6 times the arithmetic mean of the sines of its angles. If the sides *a* is 1, then the angle *A* is ***[IIT (Sc.)-92; DCE-99]***
(a) $\pi/6$ (b) $\pi/3$
(c) $\pi/2$ (d) π

11. If in a triangle *ABC*, $\cos A + \cos B + \cos C = 3/2$, then the triangle is ***[IIT-84]***
(a) Isosceles (b) Equilateral
(c) Right angled (d) None of these

12. In a triangle *ABC*, angle *A* is greater than angle *B*. If the measures of angles *A* and *B* satisfy the equation $3\sin x - 4\sin^3 x - k = 0$, $0 < k < 1$, then the measure of angle *C* is ***[IIT-99; DCE-2001]***
(a) $\pi/3$ (b) $\pi/2$
(c) $2\pi/3$ (d) $5\pi/6$

13. If the perpendicular *AD* divides the base of the triangle *ABC* such that *BD*, *CD* and *AD* are in the ratio 2, 3 and 6, then angles *A* is equal to ***[MPPET-93]***
(a) $\pi/2$ (b) $\pi/3$
(c) $\pi/4$ (d) $\pi/6$

14. In Δ *ABC*, if $\angle A = 45°$, $\angle B = 75°$, then $a + c\sqrt{2} =$ ***[IIT-88]***
(a) 0 (b) 1
(c) *b* (d) 2*b*

15. In Δ ABC, $s\left|\tan\frac{A}{2} + \tan\frac{B}{2}\right|$ is equal to
(a) ab/R (b) $2ab/\Delta$
(c) $c\cot\frac{C}{2}$ (d) None of these

16. The sum of the radii of inscribed and circumscribed circles for an *n* sided regular polygon of side *a*, is ***[AIEEE-2003]***
(a) $a\cot\left(\frac{\pi}{n}\right)$ (b) $\frac{a}{2}\cot\left(\frac{\pi}{2n}\right)$
(c) $a\cot\left(\frac{\pi}{2n}\right)$ (d) $\frac{a}{4}\cot\left(\frac{\pi}{2n}\right)$

17. If in a Δ*PQR*, sin *P* sin *Q* sin *R* are in A.P., then
(a) The altitudes are in A.P.
(b) The altitudes are in H.P.
(c) The medians are in G.P.
(d) The medians are in A.P.

18. If the radius of the circumcircle of an isosceles triangle *PQR* is equal to *PQ* (= *PR*), then the angle *P* is ***[IIT (Sc.)-99; Pb. CET-04]***

(a) $\pi/6$ (b) $\pi/3$
(c) $\pi/2$ (d) $2\pi/3$

19. $\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\left(\frac{1}{r_2}+\frac{1}{r_3}\right)\left(\frac{1}{r_3}+\frac{1}{r_1}\right)$ is equal to

(a) $64R^2/abc$ (b) R^3/abc
(c) $64R^3/a^2b^2c^2$ (d) R^3/abc

20. In a Δ the lengths of two larger sides are 8 cm and 7 cm and angles are in A.P., then the length of third side can be
(a) 3 or 5 (b) 4 or 5
(c) 2 or 3 (d) 1 or 2

21. In a triangle ABC of sides a, b, c the ratio $\frac{\tan\left(\frac{A-B}{2}\right)}{\tan\left(\frac{A+B}{2}\right)}$ equals ***[MPPET-2007]***
(a) $(a-b)(a+b)$
(b) $(a+b)/(a-b)$
(c) $(a-b)/(a+b)$
(d) None of these

22. In a triangle ABC $\angle C = \pi/2$. If its inradius and circumradius be r and R respectively, then $2(r+R)$ is equal to
[IIT (Screening)-2000; AIEEE-2005]
(a) $a+b$ (b) $b+c$
(c) $c+a$ (d) $a+b+c$

23. In a triangle ABC, AD and BE are its two medians. If $AD = 4$, $\angle DAB = \pi/6$ and $\angle ABE = \pi/3$, then area of triangle ABC is
[AIEEE-2003]
(a) 64/3 (b) 32/3
(c) $32/3\sqrt{3}$ (d) None

24. In a triangle with one angle $\frac{2\pi}{3}$, the lengths of the sides form an A.P. If the length of the greatest side is 7 cm, then the radius of the circumcircle of the triangle is
[Kerala PET-2008]
(a) $\frac{7\sqrt{3}}{3}$cm (b) $\frac{5\sqrt{3}}{3}$cm
(c) $\frac{2\sqrt{3}}{3}$cm (d) $7\sqrt{3}$ cm

25. In ΔABC, if $\frac{\cos A}{a}=\frac{\cos B}{b}=\frac{\cos C}{c}$, then the triangle is ***[Pb CET-89; KCET-91]***
(a) Right angled (b) Obtuse angled
(c) Equilateral (d) Isosceles

26. In ΔABC, $\frac{\sin B}{\sin(A+B)}$ is equal to ***[PET-89]***
(a) $\frac{b}{a+b}$ (b) $\frac{b}{c}$
(c) $\frac{c}{b}$ (d) None of these

27. In a ΔABC, $\frac{\cos\frac{1}{2}(B-C)}{\sin\frac{1}{2}A}$ is equal to
[PET-1993]
(a) $\frac{b-c}{a}$ (b) $\frac{b+c}{a}$
(c) $\frac{a}{b-c}$ (d) $\frac{a}{b+c}$

28. In a triangle ABC, $\frac{\cos A}{a}=\frac{\cos B}{b}=\frac{\cos C}{c}$. If $a=\frac{1}{\sqrt{6}}$, then the area of the triangle (in square units) is ***[Kerala PET-08]***
(a) $\frac{1}{24}$ (b) $\frac{1}{8\sqrt{3}}$
(c) $\frac{1}{8}$ (d) $\frac{1}{24\sqrt{3}}$

29. In Δ ABC, if $\tan\frac{A}{2}\tan\frac{C}{2}=\frac{1}{3}$, then a, b, c are in
(a) A.P. (b) H.P.
(c) G.P. (d) None of these

30. If $\tan\frac{B-C}{2}=x\cot\frac{A}{2}$, then x is equal to
[PET-92, 2002]
(a) $\frac{c-a}{c+a}$ (b) $\frac{a-b}{a+b}$
(c) $\frac{b-c}{b+c}$ (d) None of these

31. In any triangle ABC, if $\cos A=\frac{\sin B}{2\sin C}$, then
[MPPET-04]
(a) $a=b=c$ (b) $c=a$
(c) $a=b$ (d) $b=c$

HINTS AND EXPLANATIONS

1. (a) $\because R = \dfrac{abc}{4\Delta} \Rightarrow R = \dfrac{65}{8}$

2. (a) In ΔACD, $\dfrac{b}{\sin 67.5°} = \dfrac{AC}{\sin 90°} \Rightarrow$

$\dfrac{h}{AC} = \sin 67.5°$ .....(i)

In ΔABC, $\dfrac{AC}{\sin 22.5°} = \dfrac{x}{\sin 45°} \Rightarrow$

$\dfrac{AC}{x} = \sqrt{2}\sin 22.5°$ ..(ii)

From (i) and (ii) $\dfrac{h}{x} = \dfrac{1}{2}$

3. (c) $ab^2 \cos A + ba^2 \cos B + ac^2 \cos A + ca^2 \cos C + bc^2 \cos B + b^2c \cos C$

$= ab(b \cos A + a \cos B) + ac(c \cos A + a \cos C) + bc(c \cos B + b \cos C)$

$= abc + abc + abc = 3abc$

4. (a) $\dfrac{\text{length of hypotenuse}}{\text{length of perpendicular on the hypotenuse from opposite vertex}}$

$= \dfrac{2}{\sin 2\theta} \Rightarrow \dfrac{4}{1} = \dfrac{2}{\sin 2\theta}$

$\Rightarrow \sin 2\theta = \dfrac{1}{2} = \sin 30° \Rightarrow \theta = 15°$

5. (b) Taking $A = B = C = 60°$, $\cot\dfrac{A}{2}, \cot\dfrac{B}{2}$ and $\cot\dfrac{C}{2}$ will be in AP and there common difference is zero. So option (b) is satisfied.

6. (a) Given that $A : B : C = 4 : 1 : 1$

Let $A = 4x, B = x, C = x$

But $A + B + C = 180°$

$\Rightarrow 4x + x + x = 180° \Rightarrow x = 30°$

$\therefore A = 120°, B = 30°, C = 30°$

By sine law $= \dfrac{a}{\sin 120} = \dfrac{b}{\sin 30} = \dfrac{c}{\sin 30}$

$\Rightarrow \dfrac{a}{\sqrt{3}/2} = \dfrac{b}{1/2} = \dfrac{c}{1/2} \Rightarrow a : b : c = \sqrt{3} : 1 : 1$

$\therefore$ Ratio of longest side to the perimeter

$= \sqrt{3} : 1 + 1 + \sqrt{3} = \sqrt{3} : 2 + \sqrt{3}$

7. (d) Given $b = \sqrt{3}$, $c = 1$ and $\angle A = 30°$

As b is greatest $\therefore \angle B$ will be largest

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\cos 30 = \dfrac{3 + 1 - a^2}{2\sqrt{3}} \Rightarrow \dfrac{\sqrt{3}}{2} = \dfrac{4 - a^2}{2\sqrt{3}}$

$a^2 = 1$ or $a = 1$

Side b is longest so angle B must be largest

$\therefore$ by $\dfrac{b}{\sin B} = \dfrac{a}{\sin A}$

$\dfrac{\sqrt{3}}{\sin B} = \dfrac{1}{\sin 30°}$, $\sin B = \dfrac{\sqrt{3}}{2}$, $B = 120°$

8. (b) Given $\tan\dfrac{A}{2} = \dfrac{5}{6}$ ....(1)

$\tan\dfrac{C}{2} = \dfrac{2}{5}$ ....(2)

By multiplying equation (1) and (2)

$\tan\dfrac{A}{2}\tan\dfrac{C}{2} = \dfrac{5}{6} \times \dfrac{2}{5} = \dfrac{1}{3}$

$= \left[\dfrac{(s-b)(s-c)}{s(s-a)} - \dfrac{(s-a)(s-b)}{s(s-c)}\right]^{1/2} = \dfrac{1}{3}$

$= \left[\dfrac{s-b}{s}\right] = \dfrac{1}{3} \Rightarrow 3s - 3b = s$

$\Rightarrow 2s - 3b = 0$

$\Rightarrow a + b + c - 3b = 0$ ($\because 2s = a + b + c$)

$\Rightarrow 2b = a + c$

$\therefore$ a, b, c are in A.P.

9. (b) Let $a = 3x + 4y$, $b = 4x + 3y$ and $c = 5x + 5y$ as $x, y > 0$, $c = 5x + 5y$ is the largest side

$\therefore$ C is the largest angle. Now,

$\cos C = \dfrac{(3x+4y)^2 + (4x+3y)^3 - (5x+5y)^2}{2(3x+4y)(4x+3y)^2}$

$= \dfrac{-2xy}{2(3x+4y)(4x+3y)} < 0$

$\therefore$ C is obtuse angle

$\Rightarrow$ ΔABC is obtuse angled.

10. (a) Given $a + b + c = 6\frac{(\sin A + \sin B + \sin C)}{3}$ ....(1)

$a = 1$

$\Rightarrow \quad a + b + c = 2(\sin A + \sin B + \sin C)$

But, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$

Now $\Rightarrow a + b + c = 2k(a + b + c) \Rightarrow 2k = 1$

$\Rightarrow \quad k = \frac{1}{2}$

$\frac{\sin A}{a} = k = \frac{1}{2} \Rightarrow \sin A = \frac{1}{2}a = \frac{1}{2}$

as $\quad a = 1 \Rightarrow A = 30°$

11. (b) Given that in ΔABC

$\cos A + \cos B + \cos C = 3/2$

$\Rightarrow \frac{b^2+c^2-a^2}{2bc} = \frac{a^2+c^2-b^2}{2ac} + \frac{a^2+b^2-c^2}{2ab}$

$= \frac{3}{2}$

$ab^2 + ac^2 - a^3 + a^2b + bc^2 - b^3 + ac^2 + b^2c - c^3 = 3abc$

$\Rightarrow ab^2 + ac^2 + bc^2 + ba^2 + ca^2 + cb^2 - 6abc = a^3 + b^3 + c^3 - 3abc$

$\Rightarrow a(b - c)^2 + b(c - a)^2 + c(a - b)^2$

$= \left(\frac{a+b+c}{2}\right)\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right]$

$\Rightarrow (a + b - c)(a - b)^2 + (b + c - a)(b - c)^2 + (c + a - b)(c - a)^2 = 0$ ....(1)

As we know that

$\left.\begin{array}{l} a+b>c \\ b+c>a \\ c+a>b \end{array}\right\}$ Sum of any two sides of a Δ is greater than the third side

$\therefore$ Each side on the LHS of equation (1) has positive coefficient multiplied by perfect square, each must be separately zero.

$\therefore a - b = 0; b - c = 0; c - a = 0 \Rightarrow a = b = c$

Hence, Δ is an equilateral Δ

12. (c) Given that $A > B$ and $3\sin x - 4\sin^3 x - k = 0$, $0 < k < 1$

$\Rightarrow \sin 3x = k$,

As A and B satisfy above equation (given)

$\therefore \sin 3A = k, \sin 3b = k$

$\Rightarrow \sin 3A - \sin 3B = 0$

$\Rightarrow 3\cos\frac{3A+3B}{2}\sin\frac{3A-3B}{2} = 0$

$\Rightarrow \cos\left(\frac{3A+3B}{2}\right) = 0$ or $\sin\left(\frac{3A-3B}{2}\right) = 0$

$\Rightarrow \frac{3A+3B}{2} = 90°$ or $\frac{3A-3B}{2} = 90°$

$\Rightarrow A + B = 60°$ or $A = B$

But given that, $A > B$, $\therefore A \neq B$

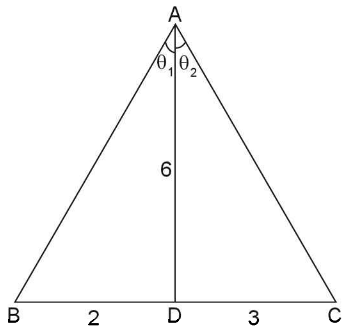
Thus, $A + B = 60°$

But, $A + B + C = 180°$

$\therefore C = 180° - 60° = 120°$

$\therefore C = 2\pi/3$

13. (c) From ΔADB $\tan\theta_1 = \frac{BD}{AD} = \frac{2}{6} = \frac{1}{3}$



From ΔADC $\tan\theta_2 = \frac{CD}{AD} = \frac{3}{6} = \frac{1}{2}$

$\tan A = \tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1\tan\theta_2} = 1$

$\therefore A\ \theta_1 + \theta_2 = 45°$

14. (d) Given A = 45°, C = 60°

$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

$\Rightarrow \frac{\sin 45°}{a} = \frac{\sin 75°}{b} = \frac{\sin 60°}{c}$

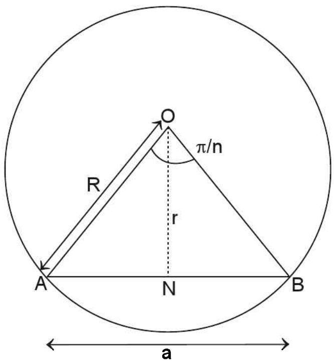
$\Rightarrow \frac{1}{\sqrt{2}a} = \frac{\sqrt{3}+1}{2\sqrt{2}b} = \frac{\sqrt{3}}{2c} \Rightarrow \frac{a}{1} = \frac{c\sqrt{2}}{\sqrt{3}} = \frac{2b}{\sqrt{3}+1}$

$\Rightarrow a + c\sqrt{2} = \frac{2b}{\sqrt{3}+1} + \frac{2b\sqrt{3}}{\sqrt{3}+1} = 2b$

15. (c) $s\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)=s\left(\frac{\Delta}{s(s-a)}+\frac{\Delta}{s(s-b)}\right)$

$=\Delta\left(\frac{1}{s-a}+\frac{1}{s-b}\right)=\frac{\Delta.c}{(s-a)(s-c)}=c\cot\frac{c}{2}$

16. (b) $\tan\left(\frac{\pi}{n}\right)=\frac{a}{2r};\sin\left(\frac{\pi}{n}\right)=\frac{a}{2R}$

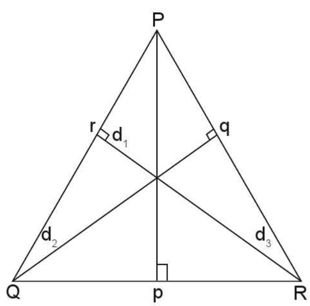


$\Rightarrow r+R=\frac{a}{2}\left[\cot\frac{\pi}{n}+\text{cosec}\frac{\pi}{n}\right]$

$\Rightarrow \frac{a}{2}\left[\frac{\cos\frac{\pi}{n}+1}{\sin\frac{\pi}{n}}\right]=\frac{a}{2}\left[\frac{2\cos^2\frac{\pi}{2n}}{2\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}}\right]$

$=\frac{a}{2}\cot\frac{\pi}{2n}$

17. (a) In ΔPQR let d_1, d_2, d_3 be the altitude on QR, RP and PQ respectively.



Then, $Ar(\Delta PQR)=\Delta=\frac{1}{2}pd_1=\frac{1}{2}qd_2=\frac{1}{2}rd_3$

$\Rightarrow d_1=\frac{2\Delta}{P}, d_2=\frac{2\Delta}{q}, d_3=\frac{2\Delta}{r}$

$\Rightarrow d_1=\frac{2\Delta}{k\sin P}, d_2=\frac{2\Delta}{k\sin Q}, d_3=\frac{2\Delta}{k\sin R}$

[Using sine law]

$\Rightarrow d_1, d_2, d_3$ are in H.P. (As given that $\sin P$, $\sin Q$, $\sin R$ in A.P.)

18. (d) In ΔPQR, radius of circumcircle is $PQ = PR$

$\therefore PQ = PR$

$\therefore PQ = PR = \frac{PQ}{2\sin R}=\frac{QR}{2\sin P}=\frac{PR}{2\sin Q}$

$\sin R = \sin Q = \frac{1}{2} \Rightarrow \angle R = \angle Q = \pi/6$

$\Rightarrow \angle P = \pi - \angle R - \angle Q = 2\pi/3$

19. (c) $\left(\frac{1}{r_1}+\frac{1}{r_2}\right)=\frac{s-a}{\Delta}+\frac{s-b}{\Delta}=\frac{c}{\Delta}$

$\therefore$ Expression $=\frac{c}{\Delta}.\frac{a}{\Delta}.\frac{b}{\Delta}$

$=\frac{abc}{\Delta^3}=\frac{abc}{\left(\frac{abc}{4R}\right)^3}=\frac{64R^3}{a^2b^2c^2}$

20. (a) Given $2B = A + C$

$\Rightarrow A + B + C = 180° \Rightarrow B = 60°$

$\cos 60° = \frac{8^2+x^2-7^2}{2.8.x} \Rightarrow x^2 - 8x + 15 = C$

$\Rightarrow \quad x = 3, 5$

21. (c) By napiers analogy in ΔABC

$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2} \quad \because \cot\frac{C}{2}=\tan\frac{A+B}{2}$

$\therefore \tan\frac{A-B}{2}=\frac{a-b}{a+b}\tan\left(\frac{A+B}{2}\right)$

or $\frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}}=\frac{a-b}{a+b}$

22. (a) We know that for the circle circumscribing a right triangle, hypotenuse is the diameter is $\angle C = 90°$

$\therefore 2R = c \Rightarrow R = \frac{c}{2}$

also $r=\frac{\Delta}{s}=\frac{\frac{1}{2}\times a\times b}{\frac{a+b+c}{2}}$

$\Rightarrow r = \dfrac{ab}{a+b+c}$ $\therefore$ $2r + 2R = \dfrac{2ab}{a+b+c} + c$

$= \dfrac{2ab+ac+bc+c^2}{a+b+c} = \dfrac{2ab+ac+bc+a^2b^2}{a+b+c}$

($\because c^2 = a^2 + b^2$)

$= \dfrac{(a+b)^2+(a+b)c}{a+b+c} = (a+b)$

Alternative

We know by $\sin C$ rule $\dfrac{c}{\sin C} = 2R$

$\Rightarrow \quad c = 2R\sin C$

$\Rightarrow \quad c = 2R$ ($\because \angle C = 90°$)

Also, $\tan\dfrac{C}{2} = \dfrac{r}{s-C}$

$\Rightarrow \quad \tan\dfrac{\pi}{4} = \dfrac{r}{s-c}$ ($\because \angle C = 90°$)

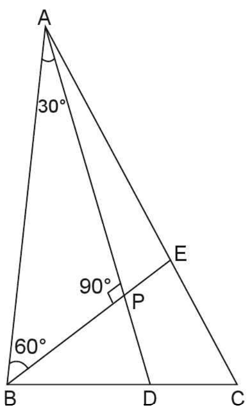
$\Rightarrow \quad r = s - c = \dfrac{a+b-c}{2}$

$\Rightarrow \quad 2r + c = a + b$

$\Rightarrow \quad 2r + 2R = a + b$ (using $c = 2R$)

23. (c) $AP = \dfrac{2}{3}AD\dfrac{8}{3}$; $PD = \dfrac{4}{3}$; Let $PB = x$

$\tan 60° = \dfrac{8/3}{x}$ or $x = \dfrac{8}{3\sqrt{3}}$



Area of $\Delta ABD = \dfrac{1}{2}\times 4\times\dfrac{8}{3\sqrt{3}} = \dfrac{16}{3\sqrt{3}}$

$\therefore$ Area of $\Delta ABC = 2\times\dfrac{16}{3\sqrt{3}} = \dfrac{32}{3\sqrt{3}}$ [$\because$ Median of a Δ divides into two Δ's of equal area]

24. (a) Radius of circumcircle $(R) = \dfrac{a}{\sin A}$

$= \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$\because$ Greatest Angle $= \dfrac{2\pi}{3}$ and side = 7

$R = \dfrac{7}{2\sin\dfrac{2\pi}{3}} = \dfrac{7\times 2}{2\sqrt{3}} = \dfrac{7\sqrt{3}}{3}$

25. (d) $\dfrac{\cos A}{a} = \dfrac{\cos B}{b} = \dfrac{\cos C}{c}$

$\Rightarrow \quad \dfrac{\cos A}{k\sin A} = \dfrac{\cos B}{k\sin B} = \dfrac{\cos C}{k\cos C}$

$\Rightarrow \quad \cot A = \cot B = \cot C = 60°$

$\Rightarrow \quad \Delta ABC$ is a isosceles.

26. (b) In ΔABC

$\Rightarrow A + B + C = 180°$

$\Rightarrow A + B = 180° - C$

$\Rightarrow \sin(A + B) = \sin(180 - C) = \sin C$

$\Rightarrow \quad \dfrac{\sin B}{\sin(A+B)} = \dfrac{\sin B}{\sin C} = \dfrac{b}{c}$

27. (b) $\dfrac{\cos\left(\dfrac{B-C}{2}\right)}{\sin\left(\dfrac{A}{2}\right)} = \dfrac{2\cos\left(\dfrac{B-C}{2}\right)\sin\dfrac{A}{2}}{2\sin\dfrac{A}{2}\cos\dfrac{A}{2}}$

$= \dfrac{2\cos\left(\dfrac{B-C}{2}\right)\sin\left(\dfrac{B+C}{2}\right)}{\sin A}$

$= \dfrac{\cos\left(\dfrac{B-C}{2}\right)}{\sin\dfrac{A}{2}} = \dfrac{\sin B+\sin C}{\sin A} = \dfrac{b+c}{a}$

28. (b) $\dfrac{\cos A}{a} = \dfrac{\cos B}{b} = \dfrac{\cos C}{c}$

$\Rightarrow \quad \dfrac{\cos A}{k\sin A} = \dfrac{\cos B}{k\sin B} = \dfrac{\cos C}{k\sin C}$

$\Rightarrow \quad \cot A = \cot B = \cot C = 60°$

$\Rightarrow \quad \Delta ABC$ is a isosceles.

$\therefore \quad \Delta\dfrac{\sqrt{3}}{4}a^2 = \dfrac{\sqrt{3}}{4}\times\dfrac{1}{6} = \dfrac{\sqrt{3}}{24} = \dfrac{\sqrt{3}}{8\times 3} = \dfrac{1}{8\sqrt{3}}$

29. (a) $\tan\left(\frac{A}{2}\right)\tan\left(\frac{C}{2}\right)=\frac{1}{3}$

$\Rightarrow \left[\frac{(s-b)(s-c)}{s(s-a)}\cdot\frac{(s-a)(s-b)}{s(s-c)}\right]^{1/2}=\frac{1}{3}$

$\Rightarrow \left[\frac{(s-b)^2}{s^2}\right]^{1/2}=\frac{1}{3}$

$\Rightarrow \frac{s-b}{s}=\frac{1}{3}$

$\Rightarrow$ $3s - 2b = s$

$\Rightarrow$ $2s = 3b$ ....(1)

$\because$ $2s = a + b + c$ ....(2)

$\Rightarrow$ $3b = a + b + c$

$\Rightarrow$ $2b = a + c$

So, a, b, c are in A.P.

30. (c) By Napier's Analogy

$\tan\frac{B-C}{2}=\left(\frac{b-c}{b+c}\right)\cot\frac{A}{2}$ ....(i)

$\therefore$ $\tan\frac{B-C}{2}=x\cot\frac{A}{2}$

On comparing above equation from equation (i) we get $x=\frac{b-c}{b+c}$

31. (b) As $\cos A=\frac{\sin B}{2\sin C}$

$\therefore$ $\cos A=\frac{k\sin B}{2k\sin C}$

or $\frac{b^2+c^2-a^2}{2bc}=\frac{b}{2c}$ or $b^2 + c^2 - a^2 = b^2$

$\therefore$ $c^2 = a^2$ or $c = a$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE) FOR IMPROVING SPEED WITH ACCURACY

1. If two angles of ΔABC are 45° and 60°, then the ratio of the smallest and the greatest sides are ***[EAMCET-2007]***

(a) $(\sqrt{3}-1):1$ (b) $\sqrt{3}:\sqrt{2}$

(c) $1:\sqrt{3}$ (d) $\sqrt{3}:1$

2. In a triangle ABC, $AB = 2BC$, then $\tan\frac{B}{2}:\cot\left(\frac{C-A}{2}\right)$ is

(a) 3 : 1 (b) 2 : 1

(c) 1 : 2 (d) 1 : 3

3. If any triangle ABC, $a\cot A + b\cot B + c\cot C$ is equal to

(a) $r + R$ (b) $r - R$

(c) $2(r + R)$ (d) $2(r - R)$

4. If in a triangle the angles are in A.P. and $b : c = \sqrt{3}:\sqrt{2}$, then $\angle A$ is equal to ***[IIT-81; Kurukshetra CEE-98; Pb CET-90]***

(a) 30° (b) 60°

(c) 15° (d) 75°

5. In a triangle ABC, $\frac{2\cos A}{a}+\frac{\cos B}{b}+\frac{2\cos C}{c}=\frac{a}{bc}+\frac{b}{ca}$, then the value of angle A is ***[IIT-93]***

(a) 45° (b) 30°

(c) 90° (d) 60°

6. In ΔABC, if $\frac{a^2-b^2}{a^2+b^2}=\frac{\sin(A-B)}{\sin(A+B)}$, then the triangle is ***[Roorkee-87]***

(a) Right angled

(b) Isosceles

(c) Right angled or Isosceles

(d) Right angled isosceles

7. In ΔABC, $b^2\cos 2A - a^2\cos 2B$ is equal to

(a) $b^2 - a^2$ (b) $b^2 - c^2$

(c) $c^2 - a^2$ (d) $a^2 + b^2 + c^2$

8. In ΔABC, $a = 2$cm, $b = 3$cm and $c = 4$ cm, then angle A is ***[MNR-73; MPPET-84, 02]***

(a) $\cos^{-1}(1/24)$ (b) $\cos^{-1}(11/16)$

(c) $\cos^{-1}(7/8)$ (d) $\cos^{-1}(-1/4)$

9. In ΔABC, if $a = 2$, $b = 4$ and $\angle C = 60°$, then $\angle A$ and $\angle B$ are equal to

(a) 90°, 30° (b) 60°, 60°
(c) 30°, 90° (d) 60°, 45°

10. The triangle PQR of which the angles P, Q, R satisfy $\cos P = \frac{\sin Q}{2\sin R}$ is

(a) Equilateral (b) Right angled
(c) Any triangle (d) Isosceles

11. If R is the radius of the circumcircle of the ΔABC and Δ is its area, then **[Karnataka CET-2000]**

(a) $R = \frac{a+b+c}{\Delta}$ (b) $R = \frac{a+b+c}{4\Delta}$
(c) $R = \frac{abc}{4\Delta}$ (d) $R = \frac{abc}{\Delta}$

12. In a triangle ABC, $a = 5$, $b = 7$, and $\sin A = 3/4$, then how many such triangles are possible? **[Roorkee-90]**

(a) 1 (b) 0
(c) 2 (d) ∞

13. In a ΔABC is $(\sqrt{3}-1)a = 2b$, $A = 3B$, then C is **[Kerala PET-2007]**

(a) 60° (b) 120°
(c) 30° (d) 45°

14. ABC is a right angled triangle with $\angle B = 90°$, $a = 6$ cm. If the radius of the circumcircle is 5 cm, then area of ΔABC is **[Kerala PET-2007]**

(a) 25 cm^2 (b) 30 cm^2
(c) 36 cm^2 (d) 24 cm^2

15. If in a triangle ABC, $a = 5$, $b = 4$, $A = \frac{\pi}{2} + B$, then C is equal to **[Kerala (CEE)-2005]**

(a) $\tan^{-1}(1/9)$ (b) $2\tan^{-1}(1/9)$
(c) $\tan^{-1}(1/40)$ (d) $2\tan^{-1}(1/40)$

16. In a ΔABC, if $a = 3$, $b = 5$, $c = 4$, then $\sin\frac{B}{2} + \cos\frac{B}{2}$ is equal to **[CET(Karnataka)-2005]**

(a) $\frac{\sqrt{3}-1}{2}$ (b) 1
(c) $\sqrt{2}$ (d) $\frac{\sqrt{3}+1}{2}$

17. In ΔABC, $a = 13$ cm, $b = 12$ cm and $c = 5$ cm. Then the distance of A from BC is **[Kerala PET-2008]**

(a) $\frac{25}{13}$cm (b) $\frac{60}{13}$cm
(c) $\frac{65}{12}$cm (d) $\frac{144}{13}$cm

18. The sides of a triangle are respectively 7 cm, $4\sqrt{3}$ cm and $\sqrt{(13)}$ cm, then the smallest angle of the triangle is **[MPPET-2008]**

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$
(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{5}$

19. If in a triangle ABC, $a = 6$ cm, $b = 8$ cm, $c = 10$ cm then the value of $\sin 2A$ is **[MPPET-2008]**

(a) 6/25 (b) 8/25
(c) 10/25 (d) 24/25

20. In a triangle ABC, $a = 4$, $b = 3$, $\angle A = 60°$, then c is the root of the equation **[AIEEE-2002]**

(a) $c^2 - 3c - 7 = 0$ (b) $c^2 + 3c + 7 = 0$
(c) $c^2 - 3c + 7 = 0$ (d) $c^2 + 3c - 7 = 0$

21. In ΔABC, $a\cos A + b\cos B + c\cos C$ is equal to

(a) $4R \sin A \sin B \sin C$
(b) $3R \sin A \sin B \sin C$
(c) $\sin A \sin B \sin C$
(d) $4R \cos A \cos B \cos C$

22. In a triangle ABC, the sides b and c are the roots of the equation $x^2 - 61x + 820 = 0$ and $A = \tan^{-1}\left(\frac{4}{3}\right)$, then a^2 is equal to **[VITEEE-2008]**

(a) 1098 (b) 1096
(c) 1097 (d) 1095

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The worksheet is of 15 minutes.
3. The worksheet consists of 15 questions. The maximum marks are 45.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. If $0 < x < \pi/2$, then the largest angle of a triangle whose series are 1, $\sin x$, $\cos x$ is

(a) $\pi/3$ (b) $\pi/2$

(c) x (d) $\pi/2 - x$

2. If A, B and C are angles of a triangle such that $\tan A = 1$, $\tan B = 2$, then what is the value of $\tan C$ ***[NDA-07]***

(a) 2 (b) 3

(c) 0 (d) 1

3. In a triangle ABC,
$(b + c)(bc)\cos A + (a + c)(ac)\cos B + (a + b)(ab)\cos C$ is ***[Kerala PET-2007]***

(a) $a^2 + b^2 + c^2$

(b) $a^3 = b^3 + c^3$

(c) $(a + b + c)(a^2 + b^2 + c^2)$

(d) $(a + b + c)(ab + bc + ca)$

4. If the sides of a right-angle triangle form an A.P., then the 'Sin' of the acute angles are ***[VITEEE-2008]***

(a) $\left(\frac{3}{5}, \frac{4}{5}\right)$

(b) $\left(\sqrt{3}, \frac{1}{\sqrt{3}}\right)$

(c) $\left(\sqrt{\frac{\sqrt{5}-1}{2}}, \sqrt{\frac{\sqrt{5}-1}{2}}\right)$

(d) $\left(\sqrt{\frac{\sqrt{3}-1}{2}}, \sqrt{\frac{\sqrt{3}-1}{2}}\right)$

5. The sides of a triangle are $\sin\alpha$, $\cos\alpha$ and $\sqrt{1+\sin\alpha\cos\alpha}$ for some $0<\alpha<\frac{\pi}{2}$. Then the greatest angle of the triangle is ***[AIEEE-2004]***

(a) 60° (b) 90°

(c) 120° (d) 150°

6. In ΔABC, with usual rotation, observe the two statements given below ***[EAMCET-2007]***

(I) $rr_1r_2r_3 = \Delta^2$ (II) $r_1r_2 + r_2r_3 + r_3r_1 = s^2$

Which of the following is correct?

(a) both I and II are true

(b) I is true, II is false

(c) I is false, II is true

(d) both I and II are false

7. If the sides of a triangle are in ratio 3 : 7 : 8, then $R : r$ is equal to

(a) 2 : 7 (b) 7 : 2

(c) 3 : 7 (d) 7 : 3

8. In ΔABC, if $\sin^2 A = \sin^2 B = \sin^2 C$, then the triangle is

(a) Equilateral

(b) Isosceles

(c) Right angled

(d) None of these

9. If in a ΔABC, $(s - a)(s - b) = s(s - c)$, then angle C is equal to ***[MPPET-86]***

(a) 90° (b) 45°

(c) 30° (d) 60°

10. In a triangle ABC $r_1 < r_2 < r_3$, then ***[EAMCET-2003]***

(a) $a < b < c$ (b) $a > b > c$

(c) $b < a < c$ (d) $a < c < b$

11. In triangle ABC if area is denoted by Δ, then Δ equals

(a) $ab \sin c$

(b) $\frac{1}{2}a\left(\frac{\sin B \sin C}{\sin(B+C)}\right)$

(c) $\frac{1}{4}\sqrt{4(b^2c^2 + c^2a^2 + a^2b^2 - (a^2 + b^2 + c^2)}$

(d) $\frac{1}{2}(a + b - c)r$

12. In a ΔABC, $a^2 \sin 2C + c^2 \sin 2A$ is equal to ***[EAMCET-2001]***

(a) Δ (b) 2Δ

(c) 3Δ (d) 4Δ

13. If P is a point on the altitude AD of the ΔABC, such that $\angle CBP = B/3$, then AP is equal to

(a) $2a \sin(C/3)$ (b) $2b \sin(A/3)$
(c) $2c \sin(B/3)$ (d) $2c \sin(C/3)$

14. If in ΔABC, $\sin\frac{A}{2}\sin\frac{C}{2} = \sin\frac{B}{2}$ and $2s$ is the perimeter of the triangle, then s is **[Kerala PET-2007]**

(a) $2b$ (b) b
(c) $3b$ (d) $4b$

15. In any triangle ABC, $c^2 \sin 2B + b^2 \sin 2C$ is equal to **[Kerala PET-2008]**

(a) $\frac{\Delta}{2}$ (b) Δ
(c) 2Δ (d) 4Δ

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	**6.** ⓐ ⓑ ⓒ ⓓ	**11.** ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	**7.** ⓐ ⓑ ⓒ ⓓ	**12.** ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	**8.** ⓐ ⓑ ⓒ ⓓ	**13.** ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	**9.** ⓐ ⓑ ⓒ ⓓ	**14.** ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ	**10.** ⓐ ⓑ ⓒ ⓓ	**15.** ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

1. (b) Largest angle is opposite to 1

$$\cos\theta = \frac{\sin^2 x + \cos^2 x - 1}{2\sin x \cos x} = 0$$

$\theta = 90°$ or $\pi/2$

2. (b) $\because$ $\tan A = 1$, $\tan B = 2$

We know that

$\tan(A + B + C)$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

$$\tan 180° = \frac{1 + 2 + \tan C - 2\tan C}{1 - 2 - 2\tan C - \tan C}$$

$3 - \tan C = 0 \Rightarrow \tan C = 3$

5. (c) $a = \sin\alpha$, $b = \cos\alpha$, $c = \sqrt{1 + \sin\alpha\cos\alpha}$

then $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

$$\cos C = \frac{\sin^2\alpha + \cos^2\alpha - 1 - \sin\alpha\cos\alpha}{2\sin\alpha\cos\alpha}$$

$$\cos C = \frac{-\sin\alpha\cos\alpha}{2\sin\alpha\cos\alpha}$$

$$\cos C = -\frac{1}{2} = \cos 120° \Rightarrow \angle C = 120°$$

6. (a) I. $rr_1r_2r_3 = \frac{\Delta}{s}.\frac{\Delta}{s-a}.\frac{\Delta}{s-b}.\frac{\Delta}{s-c} = \Delta^2$

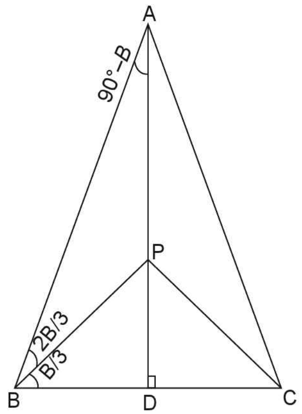
II. $r_1r_2 + r_2r_3 + r_3r_1 = \frac{\Delta}{s-a}\frac{\Delta}{s-b} + \frac{\Delta}{s-b}\frac{\Delta}{s-c}$

$$+ \frac{\Delta}{s-c}\frac{\Delta}{s-a}$$

$$= \frac{\Delta^2(s-c) + \Delta^2(s-a) + \Delta^2(s-b)}{(s-a)(s-b)(s-c)}$$

$$= \frac{\Delta^2(s-c+s-a+s-b)}{\Delta^2/s} = s^2$$

15. (c) $\angle APB = 180° - (\angle BAP + \angle PBA)$



$$= 180° - \left(90 - B + \frac{2B}{3}\right) = 90 - \frac{B}{3}$$

$\therefore$ In $\Delta\, APB$ applying sine rule

$$\frac{AP}{\sin 2B} = \frac{AB}{\sin APB}$$

$$\Rightarrow \quad AP = \left(2\sin\frac{B}{2}\cos\frac{B}{2}\right) \times \frac{c}{\sin\left(90 + \frac{B}{3}\right)}$$

$$= 2c\sin\frac{B}{3}$$

PART D

Inverse Trigonometric Functions

LECTURE

1

Inverse Trigonometric Functions

BASIC CONCEPTS

1. INVERSE FUNCTIONS

If $f: X \to Y$ is a function which is both one-one and onto, then we define its inverse function $f^{-1}: Y \to X$ is defined as:

$y = f(x) \Leftrightarrow f^{-1}(y) = x, \forall\, x \in X, \forall\, y \in Y.$

1.1 Inverse Trigonometric Functions

Consider the sine function with domain R and range $[-1, 1]$. This function is many-one and onto. So, its inverse does not exist. If we restrict its domain to the interval $[-\pi/2, \pi/2]$, then the function

sin: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ given by $\sin\theta = x$ is one-one and onto and therefore it is invertible.

The inverse of sine function is defined as

$\sin^{-1}: [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

such that $\sin^{-1} x = \theta \Leftrightarrow \sin\theta = x$.

Thus, if x is a real number between -1 and 1, then

$\sin^{-1} x$ is an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is x,

i.e., $\sin^{-1} x = \theta \Leftrightarrow x = \sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $-1 \le x \le 1$.

The least numerical value among all the values of the angle whose sine is x, is called the principal value of $\sin^{-1} x$. Similar definition for $\cos^{-1} x$, $\tan^{-1} x$ etc., can be given.

1.2 If $\sin\theta = x$, then the angle θ is called the "sin inverse x" and is denoted by the notation $\sin^{-1} x$ or "are sin x". Thus, the notation $\sin^{-1}x$ means the angle whose sine is x.

NOTES

1. $\sin^{-1}x \neq (\sin x)^{-1}$, since (-1) is a notation in the first and exponent in the second.
2. $\sin(\sin^{-1} x) \neq \sin^{-1}(\sin x)$ since, x is a number in the first and angle in the second.

2. DOMAIN AND RANGE OF INVERSE TRIGONOMETRIC FUNCTIONS

Function	Domain	Range (Principal Value)
1. $y = \sin^{-1}x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
2. $y = \cos^{-1}x$	$-1 \le x \le 1$	$0 \le y \le \pi$
3. $y = \tan^{-1}x$	$(-\infty, \infty)$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
4. $y = \text{cosec}^{-1} x$	$x \ge 1$ or $x \le -1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \neq 0$
5. $y = \sec^{-1} x$	$x \ge 1$ or $x \le -1$	$0 \le y \le \pi$, $y \neq \frac{\pi}{2}$
6. $y = \cot^{-1} x$	$(-\infty, \infty)$	$0 < y < \pi$

3. SOME IMPORTANT FORMULAE

3.1 (a) $\sin^{-1}(\sin\theta) = \theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

(b) $\cos^{-1}(\cos\theta) = \theta,\ 0 \le \theta \le \pi$

(c) $\tan^{-1}(\tan\theta) = \theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

(d) $\cot^{-1}(\cot\theta) = \theta,\ 0 < \theta < \pi$

(e) $\sec^{-1}(\sec\theta) = \theta,\ 0 \le \theta \le \pi,\ \theta \ne \frac{\pi}{2}$

(f) $\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \theta \ne 0$

3.2 (a) $\sin(\sin^{-1} x) = x,\ -1 \le x \le 1$

(b) $\cos(\cos^{-1} x) = x,\ -1 \le x \le 1$

(c) $\tan(\tan^{-1} x) = x,\ -\infty < x < \infty$

(d) $\cot(\cot^{-1} x) = x,\ -\infty < x < \infty$

(e) $\sec(\sec^{-1} x) = x,\ x \le -1 \text{ or } x \ge 1$

(f) $\text{cosec}(\text{cosec}^{-1} x) = x,\ x \le -1 \text{ or } x \ge 1$

3.3 (a) $\sin^{-1}(-x) = -\sin^{-1}x,\ -1 \le x \le 1$

(b) $\cos^{-1}(-x) = \pi - \cos^{-1}x,\ -1 \le x \le 1$

(c) $\tan^{-1}(-x) = -\tan^{-1}x,\ -\infty < x < \infty$

(d) $\cot^{-1}(-x) = \pi - \cot^{-1}x,\ -\infty < x < \infty$

(e) $\sec^{-1}(-x) = \pi - \sec^{-1}x,\ x \le -1 \text{ or } x \ge 1$

(f) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x,\ x \le -1 \text{ or } x \ge 1$

3.4 (a) $\text{cosec}^{-1} x = \sin^{-1}\frac{1}{x},\ x \le -1 \text{ or } x \ge 1$

(b) $\sec^{-1} x = \cos^{-1}\frac{1}{x},\ x \le -1 \text{ or } x \ge 1$

(c) $\cot^{-1} x = \begin{cases} \tan^{-1}\frac{1}{x}, & x > 0 \\ \pi + \tan^{-1}\frac{1}{x}, & x < 0 \end{cases}$

3.5 (a) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, -1 \le x \le 1$

(b) $\sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2}, x \le -1 \text{ or } x \ge 1$

(c) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, -\infty < x < \infty$

3.6 (a) If $xy < 1$, $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}$

(b) If $xy > 1$, $\tan^{-1} x + \tan^{-1} y$

$$= \begin{cases} \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), & x > 0, y > 0 \\ -\pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), & x < 0, y < 0 \end{cases}$$

(c) If $xy = 1$, $\tan^{-1} x + \tan^{-1}$

$$y = \begin{cases} \frac{\pi}{2}, & x > 0, y > 0 \\ -\frac{\pi}{2}, & x < 0, y < 0 \end{cases}$$

(d) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\frac{x-y}{1+xy}$

3.7 (a) If $x, y \ge 0, x^2 + y^2 \le 1$ $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}$ $(x\sqrt{1-y^2} \pm y\sqrt{1-x^2})$

(b) If $x, y \ge 0, x^2 + y^2 > 1$ $\sin^{-1} x \pm \sin^{-1} y = \pi -$ $\sin^{-1}(x\sqrt{1-y^2} \pm y\sqrt{1-x^2})$

(c) If $x, y \ge 0$, $x^2 + y^2 \le 1$ $\cos^{-1} x \pm \cos^{-1} y =$ $\cos^{-1}(xy \mp \sqrt{1-x^2}\sqrt{1-y^2})$

(d) If $x, y \ge 0, x^2 + y^2 > 1$ $(xy \mp \sqrt{1-x^2}\sqrt{1-y^2})$ $\cos^{-1} x \pm \cos^{-1} y = \pi$ $-\cos^{-1}$

3.8 To convert one inverse circular function into other

(a) $\sin^{-1} = \cos^{-1}\sqrt{1-x^2}$

$$= \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x}$$

$$= \sec^{-1}\frac{1}{\sqrt{1-x^2}} = \text{cosec}^{-1}\frac{1}{x}$$

(b) $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) =$ $\sec^{-1}\frac{1}{x}$

$$= \text{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

(c) $\tan^{-1} = \sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

$$= \cot^{-1}\frac{1}{x}$$

$$= \sec^{-1}\sqrt{1-x^2} = \text{cosec}^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

NOTE

$$\sin^{-1}\left(\frac{p}{h}\right) = \cos^{-1}\left(\frac{b}{h}\right) = \tan^{-1}\left(\frac{p}{b}\right)$$

3.9 (a) $2\tan^{-1}x = \sin^{-1}\dfrac{2x}{1+x^2}, |x| \le 1$

(b) $2\tan^{-1}x = \cos^{-1}\dfrac{1-x^2}{1+x^2}, x \ge 0$

(c) $2\tan^{-1}x = \tan^{-1}\dfrac{2x}{1-x^2}, |x| < 1$

(d) $\pi - 2\tan^{-1}x = \sin^{-1}\dfrac{2x}{1-x^2}, |x| > 1$

(e) $2\tan^{-1}x = \pi + \tan^{-1}\dfrac{2x}{1+x^2}$ if $x > 1$

(f) $2\tan^{-1}x = -\pi + \tan^{-1}\dfrac{2x}{1+x^2}, x < -1$

(g) $2\tan^{-1}x = \dfrac{\pi}{2}, x = 1$

(h) $2\tan^{-1}x = -\dfrac{\pi}{2}, x = -1$

(i) $2\tan^{-1}\left[\sqrt{\dfrac{a-b}{a+b}}\tan\dfrac{\theta}{2}\right]$

$= \cos^{-1}\left[\dfrac{a\cos\theta + b}{a + b\cos\theta}\right]$

3.10 $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}$

$\left[\dfrac{x+y+z-xyz}{1-xy-xz-yz}\right]$

3.11 (a) $\cot^{-1}x + \cot^{-1}y = \cot^{-1}\dfrac{xy-1}{y+x}$

(b) $\cot^{-1}x - \cot^{-1}y = \cot^{-1}\dfrac{xy+1}{y-x}$

3.12 (a) $2\sin^{-1}x = \sin^{-1} 2x\sqrt{1-x^2} = \cos^{-1}(1-2x^2)$

(b) $2\cos^{-1}x = \cos^{-1}(2x^2 - 1)$

(c) $\dfrac{1}{2}\tan^{-1}x = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$

(d) $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}+1}{x}\right)$

$= \dfrac{\pi}{2} - \dfrac{1}{2}\tan^{-1}x$

(e) $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$

(f) $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$

(g) $3\tan^{-1}x = \tan^{-1}\dfrac{3x-x^3}{1-3x^2}$

(h) $4\cos^{-1}x = \cos^{-1}(8x^4 - 8x^2 + 1)$

(i) $5\sin^{-1}x = \sin^{-1}(16x^5 - 20x^3 + 5x)$

(j) $5\cos^{-1}x = \cos^{-1}(16x^5 - 20x^3 + 5x)$

4. (a) $\tan^{-1}\left(\dfrac{a\sin x \pm b\cos x}{a\cos x \mp b\sin x}\right)$

$= x \pm \tan^{-1}\left(\dfrac{b}{a}\right)$

(b) $\tan^{-1}\left(\dfrac{a\cos x \pm b\sin x}{b\cos x \mp a\sin x}\right)$

$= \tan^{-1}\left(\dfrac{a}{b}\right) \pm x$

(c) $\tan^{-1}\left[\dfrac{\sqrt{1+x} \pm \sqrt{1-x}}{\sqrt{1+x} \mp \sqrt{1-x}}\right] = \dfrac{\pi}{4} \pm \dfrac{1}{2}\cos^{-1}(x)$

(d) $\tan^{-1}\left[\dfrac{\sqrt{1+x^2} \pm \sqrt{1-x^2}}{\sqrt{1+x^2} \mp \sqrt{1-x^2}}\right]$

$= \dfrac{\pi}{4} \pm \dfrac{1}{2}\cos^{-1}(x^2)$

(e) $\tan^{-1}\left(\dfrac{3a^2x - x^3}{a(a^2 - 3x^2)}\right) = 3\tan^{-1}\left(\dfrac{x}{a}\right)$

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. If $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$, then find the value of x.

[NCERT; CBSE-2008]

Solution

Given equation is $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right) = \frac{\pi}{4}$$

$$\left\{\begin{array}{l} \because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) \text{for } xy < 1 \\ \text{and for } |x| < 1, \left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right) = \frac{1-x^2}{4-x^2} < 1 \end{array}\right\}$$

$$\Rightarrow \frac{(x-1)(x+2)+(x+1)(x-2)}{(x-2)(x+2)-(x-1)(x+1)} = \tan\frac{\pi}{4}$$

$$\Rightarrow \frac{2x^2-4}{(x^2-4)-(x^2-1)} = 1$$

$$\Rightarrow 2x^2 - 4 = -3$$

$$\Rightarrow 2x^2 = 1 \Rightarrow x^2 = \frac{1}{2}$$

$$\Rightarrow x = \pm\frac{1}{\sqrt{2}}$$

2. Prove that $\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$.

[NCERT; CBSE-2008]

Solution

We know that $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ for $xy < 1$, therefore,

$$\text{L.H.S.} = \left(\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7}\right) + \left(\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8}\right)$$

$$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\cdot\frac{1}{7}}\right) + \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\cdot\frac{1}{8}}\right)$$

$$= \tan^{-1}\left(\frac{12}{34}\right) + \tan^{-1}\left(\frac{11}{23}\right)$$

$$= \tan^{-1}\left(\frac{6}{17}\right) + \tan^{-1}\left(\frac{11}{23}\right)$$

$$= \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\cdot\frac{11}{23}}\right)$$

$$= \tan^{-1}\left(\frac{6\times 23 + 11\times 17}{17\times 23 - 6\times 11}\right)$$

$$= \tan^{-1}\left(\frac{325}{325}\right) = \tan^{-1}1$$

$$= \frac{\pi}{4} = R.H.S$$

3. Prove that $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}, x \in \left(0, \frac{\pi}{2}\right)$

[NCERT]

Solution

$$\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$$

$$= \cot^{-1}$$

$$\left\{\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2} + \sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2} - \sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right\}$$

$$\left(\begin{array}{l}\because 1=\cos^2\left(\frac{x}{2}\right)+\sin^2\left(\frac{x}{2}\right)\\ \text{and } \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}\end{array}\right)$$

$$=\cot^{-1}\left\{\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right\}$$

$$\left(\begin{array}{l}\because 0<x<\frac{\pi}{2}, \text{therefore, } 0<\frac{\pi}{2}<\frac{\pi}{4}\\ \Rightarrow \cos\frac{x}{2}>\sin\frac{x}{2}\end{array}\right)$$

$$=\cot^{-1}\left(\cot\left(\frac{x}{2}\right)\right)=\frac{x}{2}$$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. The equation $\sin^{-1}x - \cos^{-1}x = \cos^{-1}\frac{\sqrt{3}}{2}$ has

[DCE-2006]

(a) no solution
(b) unique solution
(c) infinite number of solution
(d) None of these

Solution

(b) We have $\sin^{-1}x - \cos^{-1}x = \cos^{-1}\frac{\sqrt{3}}{2}$

$\sin^{-1}x - \cos^{-1}x = \frac{\pi}{6}$ But, $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

Therefore, $\sin^{-1}x = \frac{\pi}{3}$ and $\cos^{-1}x = \frac{\pi}{6}$

$\Rightarrow \quad x = \frac{\sqrt{3}}{2}$ is the unique solutions.

2. If $\tan^{-1}\frac{1-x}{1+x}=\frac{1}{2}\tan^{-1}x$, then the value of x is

[DCE-2006; NCERT]

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{3}}$
(c) $\sqrt{3}$ (d) 2

Solution

(b) $\therefore \quad \tan^{-1}\frac{1-x}{1+x}=\frac{1}{2}\tan^{-1}x$

Let $x=\tan\theta$

$\therefore \quad \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)=\frac{1}{2}\tan^{-1}(\tan\theta)$

$\Rightarrow \quad \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\theta\right)\right)=\frac{1}{2}\tan^{-1}(\tan\theta)$

$\Rightarrow \quad \frac{\pi}{4}-\theta=\frac{\theta}{2} \Rightarrow \frac{3\theta}{2}=\frac{\pi}{4}\Rightarrow \theta=\frac{\pi}{6}$

$\therefore \quad x=\tan\theta=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$

3. If p, q, r are in G.P. and $\tan^{-1}p$, $\tan^{-1}q$, $\tan^{-1}r$ are in A.P. then p, q, r satisfies the relation

[DCE-2005]

(a) $p=q=r$ (b) $p\neq q\neq r$
(c) $p+q=r$ (d) None of these

Solution

(a) Since $p, q, r,$ in G.P.

$\Rightarrow \quad q^2=pr$ and $\tan^{-1}p$, $\tan^{-1}q$, $\tan^{-1}r$ in A.P.

$\Rightarrow \quad \tan^{-1}q=\frac{\tan^{-1}p+\tan^{-1}r}{2}$

$\Rightarrow \quad 2\tan^{-1}q=\tan^{-1}p+\tan^{-1}r$

$\Rightarrow \quad \tan^{-1}\frac{2q}{1-q^2}=\tan^{-1}\frac{(p+r)}{1-pr}$

$\Rightarrow \quad \frac{2q}{1-pr}=\frac{p+r}{1-pr}(\because q^2=pr)$

$\Rightarrow \quad 2q=p+r \Rightarrow$ p, q, r are in A.P.

But p, q, r in G.P. $\quad \therefore p=q=r$

4. The value of x satisfying $\sin^{-1}x + \sin^{-1}(1-x) = \cos^{-1}x$ are

[DCE-2003]

(a) 0 (b) 1, –1
(c) 0, 1/2 (d) None of these

Solution

(c) Given equation can be written as

$\sin^{-1}x - \cos^{-1}x = -\sin^{-1}(1-x)$

$\Rightarrow \sin^{-1}x + \sin^{-1}x = \cos^{-1}(1-x)$

$\Rightarrow 2\sin^{-1}x = \cos^{-1}(1-x)$

$\Rightarrow \cos(2\sin^{-1}x) = 1-x$

$\Rightarrow 1 - 2\sin^2 x\,(\sin^{-1}x) = 1-x$

$\Rightarrow 1-2x^2 = 1-x$

$\Rightarrow 2x^2 - x = 0$

$\therefore x = 0, 1/2$

5. $\tan^{-1}\left(\frac{c_1x-y}{c_1y+x}\right)+\tan^{-1}\left(\frac{c_2-c_1}{1+c_2c_1}\right)+\tan^{-1}\left(\frac{c_3-c_2}{1+c_3c_2}\right)+....+\tan^{-1}\left(\frac{1}{c_n}\right)$ is equal to

[DCE-2002]

(a) $\tan^{-1}\left(\frac{y}{x}\right)$ (b) $\tan^{-1}\left(\frac{x}{y}\right)$

(c) $-\tan^{-1}\left(\frac{x}{y}\right)$ (d) None

Solution

(b) We have, $\tan^{-1}\left(\frac{c_1x-y}{c_1y+x}\right)+\tan^{-1}\left(\frac{c_2-c_1}{1+c_2c_1}\right)$

$$+\tan^{-1}\left(\frac{c_3-c_2}{1+c_3c_2}\right)+....\tan^{-1}\left(\frac{1}{c_n}\right)$$

$$=\tan^{-1}\left|\frac{\frac{x}{y}-\frac{1}{c_1}}{1+\frac{x}{y}\frac{1}{c_1}}\right|+.....+\tan^{-1}\left(\frac{\frac{1}{c_1}-\frac{1}{c_2}}{1+\frac{1}{c_1c_2}}\right)$$

$$+\tan^{-1}\left[\frac{\frac{1}{c_2}-\frac{1}{c_3}}{1+\frac{1}{c_2c_3}}\right]+....+\tan^{-1}\left(\frac{1}{c_n}\right)$$

$$=\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{1}{c_1}\right)+\tan^{-1}\left(\frac{1}{c_1}\right)$$

$$-\tan^{-1}\left(\frac{1}{c_2}\right)+\tan^{-1}\left(\frac{1}{c_2}\right)-\tan^{-1}\left(\frac{1}{c_3}\right)$$

$$+\ldots-\tan^{-1}\left(\frac{1}{c_n}\right)+\tan^{-1}\left(\frac{1}{c_n}\right)=\tan^{-1}\left(\frac{x}{y}\right)$$

6. $\tan^{-1}\left(\frac{1}{11}\right)+\tan^{-1}\left(\frac{2}{12}\right)$ is equal to

[DCE-1999]

(a) $\tan^{-1}\left(\frac{33}{132}\right)$ (b) $\tan^{-1}\left(\frac{1}{2}\right)$

(c) $\tan^{-1}\left(\frac{132}{33}\right)$ (d) None

Solution

(d) $$=\tan^{-1}\frac{1}{11}=\tan^{-1}\frac{1}{6}=\tan^{-1}\frac{\frac{1}{11}+\frac{1}{6}}{1-\frac{1}{11}\times\frac{1}{6}}$$

$$=\tan^{-1}\frac{\frac{17}{66}}{\frac{65}{66}}=\tan^{-1}\left(\frac{17}{65}\right)$$

7. If $\cos^{-1}x/2 + \cos^{-1}y/3 = \theta$. Then $9x^2 - 12xy\cos\theta + 4y^2$ is equal to *[DCE-1997]*

(a) 36 (b) $-36\sin^2\theta$
(c) $36\sin^2\theta$ (d) $36\cos^2\theta$

Solution

(c) Given that $\cos^{-1}\frac{x}{2}+\cos^{-1}\frac{y}{3}=\theta$

$$\Rightarrow \cos^{-1}\left[\frac{x}{2}.\frac{y}{3}-\sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{9}}\right]=\theta$$

$$\Rightarrow \frac{xy}{6}-\frac{1}{6}\sqrt{4-x^2}\sqrt{9-y^2}=\cos\theta$$

$$\Rightarrow xy-6\cos\theta=\sqrt{4-x^2}\sqrt{9-y^2}$$

Squaring both sides

$x^2y^2 + 36\cos^2\theta - 12xy\cos\theta = 36 - 9x^2 - 4y^2 + x^2y^2$

$\Rightarrow 9x^2 - 12xy\cos\theta + 4y^2 = 36 - 36\cos^2\theta = 36(1-\cos^2\theta)$

$\Rightarrow \quad 9x^2 - 12xy\cos\theta + 4y^2 = 36\sin^2\theta$

OR

Verify for $x = 1,\ y = \frac{3\sqrt{2}}{2}$.

8. If $\sum_{i=1}^{\infty}\tan^{-1}\left(\frac{1}{2i^2}\right) = t$, then t is equal to

[IIT-2006]

(a) 2 (b) 1

(c) 0 (d) ∞

Solution

(b) $t = \sum_{i=1}^{\infty}\tan^{-1}\left(\frac{1}{2i^2}\right) = \sum_{i=1}^{\infty}\tan^{-1}\left(\frac{2}{4i^2-1+1}\right)$

$= \sum_{i=1}^{\infty}\tan^{-1}\left(\frac{(2i+1)-(2i-1)}{1+(2i+1)(2i-1)}\right)$

$= \sum_{i=1}^{\infty}\tan^{-1}(2i+1) - \tan^{-1}(2i-1)$

$= (\tan^{-1}3 - \tan^{-1}1) + (\tan^{-1}5 - \tan^{-1}3) + \ldots. + (\tan^{-1}(2n+1) - \tan^{-1}(2n-1)) + \ldots..\infty$

$\therefore \quad t = \lim_{n\to\infty}[\tan^{-1}(2n+1) - \tan^{-1}1]$

$= \pi/2 - \pi/4 = \pi/4$

$\Rightarrow \quad \tan t = \tan\frac{\pi}{4} = 1$

9. If a, b, c be positive real numbers and the value of

$$\theta = \tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}} + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$$

than $\tan\theta$ is equal to

[IIT-1981]

(a) 0 (b) 1

(c) $\frac{a+b+c}{abc}$ (d) None of these

Solution

$$\theta = \tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}}.$$

$$+\tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}} + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$$

Let $s^2 = \frac{a+b+c}{abc}$

Hence $\theta = \tan^{-1}\sqrt{a^2s^2} + \tan^{-1}\sqrt{b^2s^2}$

$+\tan^{-1}\sqrt{b^2s^2} + \tan^{-1}\sqrt{c^2s^2}$

$= \tan^{-1}(as) + \tan^{-1}(bs) + \tan^{-1}(cs)$

$= \tan^{-1}\left[\frac{as+bs+cs-abcs^3}{1-(ab+bc+ca)s^2}\right]$

$= \left[\frac{s[(a+b+c)-(a+b+c)]}{1-s^2(ab+bc+ca)}\right] = 0$

{Since $s^2abc = (a+b+c)$}

Trick: Since it is an identity, so it will be true for any value of a, b, c Let $a = b = c = 1$, then

$\theta = \tan^{-1}\sqrt{3} + \tan^{-1}\sqrt{3}$

$+\tan^{-1}\sqrt{3} = \pi \Rightarrow \tan\theta = 0$

10. The greatest and the least values of $(\sin^{-1}x)^3 + (\cos^{-1}x)^3$ are

(a) $-\frac{\pi}{2}, \frac{\pi}{2}$ (b) $-\frac{\pi^3}{8}, \frac{\pi^3}{8}$

(c) $\frac{\pi^3}{32}, \frac{7\pi^3}{8}$ (d) None of these

Solution

We have $(\sin^{-1}x)^3 + (\cos^{-1}x)^3$

$(\sin^{-1}x + \cos^{-1}x)^3 - 3\sin^{-1}x\cos^{-1}x(\sin^{-1}x + \cos^{-1}x)$

$= \frac{\pi^3}{8} - 3(\sin^{-1}x\cos^{-1}x)\frac{\pi}{2}$

$= \frac{\pi^3}{8} - \frac{3\pi}{2}\sin^{-1}x\left(\frac{\pi}{2} - \sin^{-1}x\right)$

$= \frac{\pi^3}{8} - \frac{3\pi^3}{4}\sin^{-1}x + \frac{3\pi}{2}(\sin^{-1}x)^2$

$= \frac{\pi^3}{8} + \frac{3\pi}{2}\left[(\sin^{-1}x)^2 - \frac{\pi}{2}\sin^{-1}x\right]$

$= \frac{\pi^3}{8} + \frac{3\pi}{2}\left[\left(\sin^{-1}x - \frac{\pi}{4}\right)^2\right] - \frac{3\pi^3}{32}$

$= \frac{\pi^3}{32} + \frac{3\pi}{2}\left(\sin^{-1}x - \frac{\pi}{4}\right)^2$

$\therefore$ The least value is $\frac{\pi^3}{32}$ and since

$$\left(\sin^{-1}x-\frac{\pi}{4}\right)^2\le\left(\frac{3\pi}{4}\right)^2$$

$\therefore$ The greatest value is $\frac{\pi^3}{32}+\frac{9\pi^2}{16}\times\frac{3\pi}{2}=\frac{7\pi^3}{8}$

11. If $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\frac{5\pi^2}{8}$, then x equals

[Kerala PET-2008]

(a) -1 (b) 1
(c) 0 (d) None

Solution

(a) $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\frac{5\pi^2}{8}$

$$\Rightarrow(\tan^{-1}x+\cot^{-1}x)^2$$

$$-2\tan^{-1}x\left(\frac{x}{2}-\tan^{-1}x\right)=\frac{5\pi^2}{8}$$

$$\Rightarrow\frac{\pi^2}{4}-2\times\frac{\pi}{2}\tan^{-1}x+2(\tan^{-1}x)^2=\frac{5\pi^2}{8}$$

$$\Rightarrow 2(\tan^{-1}x)^2-\pi\tan^{-1}x-\frac{3\pi^2}{8}=0$$

$$\Rightarrow\tan^{-1}x=-\frac{\pi}{4},\frac{3\pi}{4}\Rightarrow\tan^{-1}x=-\frac{\pi}{4}\Rightarrow x=-1$$

12. The value of $\cos^{-1}\left(\cos\frac{5\pi}{3}\right)+\sin^{-1}\left(\sin\frac{5\pi}{3}\right)$ is

[Roorkee-2000]

(a) 0 (b) $\frac{\pi}{2}$

(c) $\frac{2\pi}{3}$ (d) $\frac{10\pi}{3}$

Solution

(a) $\cos^{-1}\left(\cos\frac{5\pi}{3}\right)+\sin^{-1}\left(\sin\frac{5\pi}{3}\right)$

$$=\cos^{-1}\left[\cos\left(2\pi-\frac{\pi}{3}\right)\right]$$

$$+\sin^{-1}\left[\sin\left(2\pi-\frac{\pi}{3}\right)\right]$$

$$=\frac{\pi}{3}-\frac{\pi}{3}=0$$

13. If $u=\cot^{-1}\left[\sqrt{\cos2\theta}\right]-\tan^{-1}\left[\sqrt{\cos2\theta}\right]$, then prove that $\sin u=\tan^2\theta$.

Solution

We have $u=\cot^{-1}\left[\sqrt{\cos2\theta}\right]$

$$-\tan^{-1}\left[\sqrt{\cos2\theta}\right]$$

$$=\tan^{-1}(1/\sqrt{\cos2\theta})-\tan^{-1}\sqrt{\cos2\theta}$$

$$=\tan^{-1}\frac{1/(\sqrt{\cos2\theta})-\sqrt{\cos2\theta}}{1+[1/\sqrt{\cos2\theta}]\sqrt{(\cos2\theta)}}$$

$$=\tan^{-1}\frac{1-\cos2\theta}{2\sqrt{(\cos2\theta)}}\ \therefore\ \tan u=\frac{1-\cos2\theta}{2\sqrt{(\cos2\theta)}}$$

or $\cot u=\frac{2\sqrt{\cos2\theta}}{1-\cos2\theta}$

Hence, $\operatorname{cosec}^2u=1+\cot^2u$

$$=1+\frac{4\cos2\theta}{(1-\cos2\theta)^2}=\frac{(1+\cos2\theta)^2}{(1-\cos2\theta)^2}$$

or $\operatorname{cosec}u=\frac{1+\cos2\theta}{1-\cos2\theta}=\frac{2\cos^2\theta}{2\sin^2\theta}=\cot^2\theta$

or $\sin u=\tan^2\theta$

14. If $\tan^{-1}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{(1-x^2)}+\sqrt{(1-x^2)}}=\alpha$, then prove that $x^2=\sin2\alpha$.

Solution

(a) From the given question

$$\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}=\tan\alpha=\frac{\sin\alpha}{\cos\alpha}$$

Apply componendo and dividendo

$$\frac{2\sqrt{1+x^2}}{2\sqrt{(1-x^2)}}=\frac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha}$$

Square $\frac{1+x^2}{1-x^2}=\frac{1+\sin 2\alpha}{1-\sin 2\alpha}$

$\because \cos^2\alpha + \sin^2\alpha = 1$ and $2\sin\alpha\cos\alpha = \sin 2\alpha$.

Hence $x^2 = \sin 2\alpha$ from above.

Alternative Put $x^2 = \cos\theta$. Rest do yourself or put $x^2 = \sin 2\alpha$

$\therefore \sqrt{1+x^2} = \sqrt{1+\sin 2\alpha} = \cos\alpha + \sin\alpha$ etc.

15. Solve the equation $\cos^{-1}(\sqrt{6}x)$

$+\cos^{-1}(3\sqrt{3}x^2)=\frac{\pi}{2}$.

[Roorkee-2001]

Solution

$$\cos^{-1}3\sqrt{3}x^2=\frac{\pi}{2}-\cos^{-1}(\sqrt{6}x)$$

$$=\sin^{-1}(\sqrt{6}x)=\cos^{-1}(1-6x^2)$$

$$\therefore \quad 3\sqrt{3}x^2=1-6x^2 \text{ or } x^2(6+3\sqrt{3})=1$$

$$\therefore \quad x^2=\frac{1}{6+3\sqrt{3}}=\frac{6-3\sqrt{3}}{9}=\frac{2-\sqrt{3}}{3}$$

$$=\frac{4-2\sqrt{3}}{6}$$

$$\text{or} \quad x^2=\frac{(2-\sqrt{3})^2}{6} \quad \therefore x=\frac{2-\sqrt{3}}{\sqrt{6}}$$

16. Prove that $\cos^{-1}\left(\frac{2+3\cos x}{3+2\cos x}\right)$

$=2\tan^{-1}\left(\frac{1}{\sqrt{5}}\tan\frac{x}{2}\right)$.

Solution

Let $\theta=\cos^{-1}\frac{2+3\cos x}{3+2\cos x}$

$$\therefore \quad \frac{\cos\theta}{1}=\frac{2+3\cos x}{3+2\cos x} \quad \text{[Apply C and D]}$$

$$\frac{1-\cos\theta}{1+\cos\theta}=\frac{1-\cos x}{5(1+\cos x)}$$

$$\therefore \quad \tan^2\frac{\theta}{2}=\frac{1}{5}\tan^2\frac{x}{2}$$

$$\therefore \quad \theta=2\tan^{-1}\left[\frac{1}{\sqrt{5}}\tan\frac{x}{2}\right]=\text{R.H.S.}$$

17. Prove that $\tan -1\left(\frac{1}{2}\tan 2A\right)$ + $\tan^{-1}(\cot A)$ + $\tan^{-1}(\cot^3 A)$

= 0 if $\pi/4 < A < \pi/2$ and = π if $0 < A < \pi/4$.

Solution

First note that

$\cot A > 1$ if $0 < A < \pi/4$

and $\cot A < 1$ if $\pi/4 < A < \pi/2$. Hence

$\therefore \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A)$

$$=\pi+\tan^{-1}\frac{\cot A+\cot^3 A}{1-\cot^4 A} \text{ if } 0<A<\frac{\pi}{4}$$

$$\text{and} \quad =\tan^{-1}\frac{\cot A+\cot^3 A}{1-\cot^4 A} \text{ if } \frac{\pi}{4}<A<\frac{\pi}{4}$$

$$\text{Also} \quad \frac{\cot A+\cot^3 A}{1-\cot^4 A}=\frac{\cot A}{1-\cot^2 A}$$

$$=\frac{\cos A\sin A}{(\sin^2 A-\cos^2 A)}=-\frac{\sin 2A}{2\cos 2A}$$

$$=-\frac{1}{2}\tan 2A$$

Hence $\tan^{-1}\left(\frac{1}{2}\tan 2A\right)$ + $\tan^{-1}(\cot A)$ + $\tan^{-1}(\cot^3 A)$

= π in the first case.

= 0 in the 2nd case because $\tan^{-1}(-x) = -\tan^{-1}x$

18. $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{2}{9}+\ldots.+\tan^{-1}\frac{2^{n-1}}{1+2^{2n-1}}$

Solution

$$T_n=\tan^{-1}\frac{2^{n-1}(2-1)}{1+2^n.2^{n-1}}=\tan^{-1}2^n-\tan^{-1}2^{n-1}$$

Now put $n = 1, 2, 3, \ldots\ldots$ N and add.

$S_n = \tan^{-1}2^n - \tan^{-1}1$

$$S_\infty=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}.$$

19. The value of $\cot\left(\operatorname{cosec}^{-1}\frac{5}{3}+\tan^{-1}\frac{2}{3}\right)$ is

[AIEEE-2008]

(a) $\frac{5}{17}$ (b) $\frac{6}{17}$

(c) $\frac{3}{17}$ (d) $\frac{4}{17}$

Solution

(b) $\cot\left(\operatorname{cosec}^{-1}\frac{5}{3}+\tan^{-1}\frac{2}{3}\right)$

$$=\cot\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}\right)$$

$$=\cot\left(\tan^{-1}\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right)=\cot\left(\tan^{-1}\frac{17}{6}\right)=\frac{6}{17}$$

20. The value of $\sin^{-1}(\sin 10)$ is

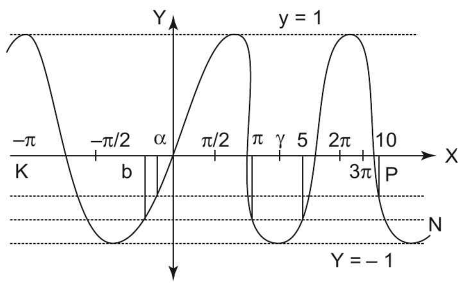
[DCE-2002]

(a) 10
(b) $3\pi - 10$
(c) $10 - 3\pi$
(d) None of these

Solution

(b) By definition $\alpha = \arc\sin(\sin 10)$ is an angle that satisfies two conditions
$\sin\alpha = \sin 10$ and
$-\pi/2 \le \alpha \le \pi/2$ or $\alpha \in \left(\frac{-\pi}{2},\frac{\pi}{2}\right)$

We find α with help of graph let $y = \sin x$ plot the number 10 on x-axis and find sin 10 geometrically.



It is easy to see that the points α and 10 are symmetric about the point $3\pi/2$, so that

$$10-\frac{3\pi}{2}=\frac{3\pi}{2}-\alpha$$

$\Rightarrow \alpha = 3\pi - 10$
Hence are $\sin^{-1}(\sin 10) = 3\pi - 10$

21. The value of $\cos(\tan^{-1}(\tan 2))$ is

[AMU-2002]

(a) $1/\sqrt{5}$ (b) $-1/\sqrt{5}$
(c) cos2 (d) − cos2

Solution

(d) $\cos[\tan^{-1}(\tan 2)] = \cos(\pi - 2) = -\cos 2$.

22. The value of $\cos^{-1}(\cos 12) - \sin^{-1}(\sin 14)$ is

[J & K-2005]

(a) − 2 (b) $8\pi - 26$
(c) $4\pi + 2$ (d) None of these

Solution

(b) $\cos^{-1}(\cos 12) - \sin^{-1}(\sin 14)$
$\cos^{-1}(\cos(4\pi - 12)) - \sin^{-1}(\sin(14 - 4\pi))$
$= 8\pi - 26$

23. If $0 < x < 1$, then
$\sqrt{1+x^2}\left[\{x\cos(\cot^{-1}x)+\sin(\cot^{-1}x)\}^2-1\right]^{1/2}$
is equal to ***[IIT JEE-2008]***

(a) $\frac{x}{\sqrt{1+x^2}}$ (b) x
(c) $x\sqrt{1+x^2}$ (d) $\sqrt{1+x^2}$

Solution

(c) $=\sqrt{1+x^2}\left[\begin{matrix}\{x\cos(\cot^{-1}x)\\ +\sin(\cot^{-1}x)\}^2-1\end{matrix}\right]^{1/2}$

$$=\sqrt{1+x^2}\left[\left\{\begin{matrix}x\cos\left(\cos^{-1}\frac{x}{\sqrt{1+x^2}}\right)\\ +\sin\left(\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right)\end{matrix}\right\}^2-1\right]^{1/2}$$

$$=\sqrt{1+x^2}\left[\left\{\frac{x^2}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+x^2}}\right\}^2-1\right]^{1/2}$$

$$=\sqrt{1+x^2}\left[\{\sqrt{1+x^2}\}^2-1\right]^{1/2}=\sqrt{1+x^2}\,(x^2)^{1/2}$$

$$=\sqrt{1+x^2}.|x|=x\sqrt{1+x^2}$$

(as $0<x<1$, $|x|=x$)

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. If $\cos^{-1}(1/x) = \theta$, then $\tan\theta$ is equal to **[PET-1989]**

(a) $\frac{1}{\sqrt{x^2-1}}$ (b) $\sqrt{x^2+1}$

(c) $\sqrt{1-x^2}$ (d) $\sqrt{x^2-1}$

2. $\tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)\right]$ is equal to

(a) $\frac{2a}{1+a^2}$ (b) $\frac{1-a^2}{1+a^2}$

(c) $\frac{2a}{1-a^2}$ (d) None of these

3. $\tan\left[\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right]$ is equal to **[NDA-2003]**

(a) 6/17 (b) 17/6

(c) 7/16 (d) 16/7

4. $\sin(\cot^{-1} x)$ is equal to **[PET-2001; DCE-2002]**

(a) $\sqrt{1+x^2}$

(b) x

(c) $(1+x^2)^{-3/2}$

(d) $(1+x^2)^{-1/2}$

5. $\tan\left[\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right]+\tan\left[\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right]$ is equal to

(a) $\frac{2a}{b}$ (b) $\frac{2b}{a}$

(c) $\frac{a}{b}$ (d) $\frac{b}{a}$

6. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, then

(a) $x^2 + y^2 + z^2 + xyz = 0$

(b) $x^2 + y^2 + z^2 + 2xyz = 0$

(c) $x^2 + y^2 + z^2 + xyz = 1$

(d) $x^2 + y^2 + z^2 + 2xyz = 1$

7. If $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$, then x is equal to

(a) $3\pi/4$ (b) $\pi/4$

(c) $\pi/3$ (d) None of these

8. $\sin\left[\frac{\pi}{2}-\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$ is equal to

(a) $\frac{\sqrt{3}}{2}$ (b) $-\frac{\sqrt{3}}{2}$

(c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

9. $\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right]$ is equal to **[IIT-84]**

(a) 17/7 (b) –17/7

(c) 7/17 (d) –7/17

10. $\sec^{-1}[\sec(-30^\circ)]$ is equal to **[PET-92]**

(a) -60° (b) -30°

(c) 30° (d) 150°

11. $\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]$ is equal to

(a) $\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$ (b) $\frac{\pi}{4}+\cos^{-1}x^2$

(c) $\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x$ (d) $\frac{\pi}{4}-\frac{1}{2}+\cos^{-1}x^2$

12. $\sec^2(\tan^{-1}2) + \operatorname{cosec}^2(\cot^{-1}3)$ is equal to **[EAMCET-2001]**

(a) 5 (b) 13

(c) 15 (d) 6

13. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, then $\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}$ is equal to **[PET-1991]**

(a) 0 (b) 1

(c) $1/xyz$ (d) xyz

14. $3\tan^{-1} a$ is equal to **[PET-93]**

(a) $\tan^{-1}\frac{3a+a^3}{1+3a^2}$ (b) $\tan^{-1}\frac{3a-a^3}{1+3a^2}$

(c) $\tan^{-1}\frac{3a+a^3}{1-3a^2}$ (d) $\tan^{-1}\frac{3a-a^3}{1-3a^2}$

15. If $\cot^{-1}\alpha + \cot^{-1}\beta = \cot^{-1} x$, then x is equal to **[PET-92]**

(a) $\alpha+\beta$ (b) $\alpha-\beta$

(c) $\frac{1+\alpha\beta}{\alpha+\beta}$ (d) $\frac{\alpha\beta-1}{\alpha+\beta}$

16. If $\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \sin^{-1}\left(\frac{2b}{1+b^2}\right) = 2\tan^{-1}x$, then x is equal to ***[MNR-84; Pb. CET-04]***

(a) $\frac{a-b}{1+ab}$ (b) $\frac{b}{1+ab}$

(c) $\frac{b}{1-ab}$ (d) $\frac{a+b}{1-ab}$

17. $\cos^{-1}\frac{1}{2} + 2\sin^{-1}\frac{1}{2}$ is equal to ***[PET-98, 2006]***

(a) $\pi/4$ (b) $\pi/6$

(c) $\pi/3$ (d) $2\pi/3$

18. If $\sin^{-1} x = \frac{\pi}{5}$ for some $x \in (-1, 1)$ then the value of $\cos^{-1} x$ is

(a) $3\pi/10$ (b) $5\pi/10$

(c) $7\pi/10$ (d) $9\pi/10$

19. If $\sin^{-1} x + \sin^{-1} y = \frac{2\pi}{3}$, then $\cos^{-1}x + \cos^{-1} y$ is equal to

(a) $2\pi/3$ (b) $\pi/3$

(c) $\pi/6$ (d) π

20. $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{4}$ or $-\frac{3\pi}{4}$

21. The principal value of $\sin^{-1}\frac{5\pi}{3}$ is ***[PET-96]***

(a) $\frac{5\pi}{3}$ (b) $\frac{-5\pi}{3}$

(c) $\frac{\pi}{3}$ (d) $\frac{4\pi}{3}$

22. If $4\sin^{-1} x + \cos^{-1} x = \pi$, then x is equal to

(a) 0 (b) 1/2

(c) $-\sqrt{3}/2$ (d) $1/\sqrt{2}$

23. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = 3\pi$, then $xy + yz + zx$ is equal to ***[ECET-2003]***

(a) 0 (b) 1

(c) 3 (d) –3

24. $\sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right)$ is equal to

(a) $1/\sqrt{10}$ (b) $-1/\sqrt{10}$

(c) 1/10 (d) –1/10

25. If $\cot^{-1}(\sqrt{\cos\alpha}) - \tan^{-1}(\sqrt{\cos\alpha}) = u$, then the $\sin u$ equals ***[DCE-1997; AIEEE-2002]***

(a) $\tan^2 \alpha$ (b) $\tan 2\alpha$

(c) $\tan^2\frac{\alpha}{2}$ (d) $\cot^2\frac{\alpha}{2}$

26. The value of x for which $\sin(\cot^{-1}(1 + x)) = \cos(\tan^{-1} x)$ is ***[IIT-2004]***

(a) 1/2 (b) 1

(c) 0 (d) –1/2

27. If $\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4} - ....\right) + \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - ....\right) = \frac{\pi}{2}$, for $0 < |x| < \sqrt{2}$, then x equals ***[IIT-2001; MPPET-96]***

(a) 1/2 (b) 1

(c) –1/2 (d) –1

28. The number of real solutions of $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{x}{2}$ is ***[IIT-1999]***

(a) 0 (b) 1

(c) 2 (d) ∞

29. If $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$, then $4x^2 - 4xy\cos\alpha + y^2$ is equal to ***[AIEEE-2005]***

(a) $-4\sin^2\alpha$ (b) $4\sin^2\alpha$

(c) 4 (d) $2\sin^2\alpha$

30. The trigonometric equation $\sin^{-1} x = 2\sin^{-1} a$, has a solution for ***[AIEEE-2003]***

(a) $\frac{1}{2} < |a| < \frac{1}{\sqrt{2}}$

(b) all real values of a

(c) $|a| < \frac{1}{2}$

(d) $|a| \geq \frac{1}{\sqrt{2}}$

31. $\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{2}{9}\right)$ is equal to ***[AIEEE-2002]***

(a) $\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$ (b) $\frac{1}{2}\sin^{-1}\left(\frac{3}{5}\right)$

(c) $\frac{1}{2}\tan^{-1}\left(\frac{3}{5}\right)$ (d) $\tan^{-1}\left(\frac{1}{2}\right)$

32. If $\sin^{-1}(x/5) + \text{cosec}^{-1}(5/4) = \pi/2$ then x is equal to

[EAMCET-83; Karnataka CET-04; Orissa JEE-07; AIEEE-07]

(a) –3 (b) 3

(c) 1 (d) None of these

33. If $\cos(2\sin^{-1} x) = 1/9$, then x is equal to ***[Roorkee-75]***

(a) Only 2/3

(b) Only –2/3

(c) 2/3, –2/3

(d) Neither 2/3 nor –2/3

34. $\tan\left(2\cos^{-1}\frac{3}{5}\right)$ is equal to

(a) 7/25

(b) 24/25

(c) –24/7

(d) 8/3

HINTS AND EXPLANATIONS

1. (d) Given that $\cos^{-1}\left(\frac{1}{x}\right) = \theta \Rightarrow \cos\theta = \frac{1}{x}$

Now, $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{1-\left(\frac{1}{x}\right)^2}}{\frac{1}{x}} = \sqrt{x^2-1}$

2. (c) $\tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)\right]$

$= \tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\right.$

$\left.+\frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)\right]$ {Let $a = \tan\theta$}

$= \tan\left[\frac{1}{2}\sin^{-1}(\sin 2\theta)+\frac{1}{2}\cos^{-1}(\cos 2\theta)\right]$

$= \tan(2\theta)$

$= \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2a}{1-a^2}$

3. (b) $\tan\left[\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right]$

$= \tan\left[\tan^{-1}\frac{\sqrt{\left(1-\frac{16}{25}\right)}}{\frac{4}{5}}+\tan^{-1}\frac{2}{3}\right]$

$= \tan\left[\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}.\frac{2}{3}}\right)\right] = \tan\tan^{-1}\frac{17}{6} = \frac{17}{6}$

4. (d) Let $\cot^{-1} x = \theta \Rightarrow x = \cot\theta$

Now $\text{cosec}\,\theta = \sqrt{1+\cot^2\theta} = \sqrt{1+x^2}$

$\therefore \quad \sin\theta = \frac{1}{\text{cosec}\,\theta} = \frac{1}{\sqrt{1+x^2}}$

$\Rightarrow \quad \theta = \sin^{-1}\frac{1}{\sqrt{1+x^2}}$

Hence $\sin(\cot^{-1}x), \sin\left(\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right)$

$= \frac{1}{\sqrt{1+x^2}} = (1+x^2)^{-1/2}$

5. (b) $\tan\left[\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right]+\tan\left[\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right]$

Let $\frac{1}{2}\cos^{-1}\frac{a}{b}=\theta$ and using formula,

$\tan\left[\frac{\pi}{4}+\theta\right]+\tan\left[\frac{\pi}{4}-\theta\right]=2\sec 2\theta \Rightarrow \frac{2b}{a}$

6. (d) Given that $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$

$\Rightarrow \cos^{-1}(x)+\cos^{-1}(y)+\cos^{-1}(z)=\cos^{-1}(-1)$

$\Rightarrow \cos^{-1}(x)+\cos^{-1}(y)=\cos^{-1}(-1)-\cos^{-1}(z)$

$\Rightarrow \cos^{-1}(xy-\sqrt{1-x^2}\sqrt{1-y^2})$

$=\cos^{-1}\{(-1)(z)\}$

$\Rightarrow xy-(\sqrt{(1-x^2)(1-y^2)})=-z$

$\Rightarrow (xy+z)=(\sqrt{(1-x^2)(1-y^2)})$

Squaring both sides, we get, $x^2+y^2+z^2+2xyz=1$

7. (b) We have, $2\tan^{-1}(\cos x)=\tan^{-1}(2\text{cosec}x)$

$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{(1-\cos x)^2}\right)=\tan^{-1}(2\text{cosec}x)$

$\Rightarrow \frac{2\cos x}{\sin^2 x}=2\text{cosec}x$

$\Rightarrow 2\cos x=2\sin x$ or $\sin x=\cos x\Rightarrow x=\frac{\pi}{4}$

8. (c) $\sin\left[\frac{\pi}{2}-\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)\right]=\cos\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

$=\cos\cos^{-1}\sqrt{1-\frac{3}{4}}=\frac{1}{2}$

9. (d) $\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right]$

$=\tan\left[\tan^{-1}\frac{\frac{2}{5}}{1-\frac{1}{25}}-\tan^{-1}(1)\right]$

$=\tan\left[\tan^{-1}\frac{5}{12}-\tan^{-1}(1)\right]$

$=\tan\tan^{-1}\left(\frac{\frac{5}{12}-1}{1+\frac{5}{12}}\right)=\frac{-7}{17}$

10. (c) $\sec^{-1}[\sec(-30^\circ)]=\sec^{-1}(\sec 30^\circ)=30^\circ$

11. (a) $\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]=\tan^{-1}$

$\left[\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right]$

Putting $x^2=\cos^2\theta\Rightarrow\theta=\frac{1}{2}\cos^{-1}x^2$

$=\tan^{-1}\left[\frac{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}\right]$

$=\tan^{-1}\left[\frac{1+\tan\theta}{1-\tan\theta}\right]$

$=\tan^{-1}\left[\frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\tan\theta}\right]$

$=\tan^{-1}\tan\left(\frac{\pi}{4}+\theta\right)=\frac{\pi}{4}+\theta$

$=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$

12. (c) $\sec^2(\tan^{-1}2)+\text{cosec}^2(\cot^{-1}3)$

Let $\tan^{1}2=\theta_1$ and $\cot^{-1}3=\theta_2\Rightarrow\sec^2\theta_1$

$=1+\tan^2\theta_1$

$=1+4=5\Rightarrow\theta_1=\sec^{-2}(5)$ and $\text{cosec}^2\theta_2$

$=1\cot^2\theta_2$

$=1+9=10\Rightarrow\theta_2=\text{cosec}^{-2}(10)$

Hence, $\sec^{-2}(\tan^{-1}2)+\text{cosec}^2(\cot^{-1}3)$

$=\sec^{-2}[\sec^{-2}(5)+\text{cosec}^2[\text{cosec}^2(\cot^{-1}3)]]$

$=5+10=15$

13. (b) $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$

$\Rightarrow \tan^{-1}x+\tan^{-1}y=\pi-\tan^{-1}z\Rightarrow\frac{x+y}{1-xy}=-z$

$\Rightarrow x+y=-z+xyz \Rightarrow x+y+z=xyz$

Dividing by xyz, we get $\frac{1}{yz}+\frac{1}{xz}+\frac{1}{xy}=1$

14. (d) $3\tan^{-1}a=\tan^{-1}\left(\frac{3a-a^3}{1-3a^2}\right)$

15. (d) Given that, $\cot^{-1}\alpha+\cot^{-1}\beta=\cot^{-1}x$

$\Rightarrow \cot^{-1}\left(\frac{\alpha\beta-1}{\alpha+\beta}\right)=\cot^{-1}x\Rightarrow x=\frac{\alpha\beta-1}{\alpha+\beta}$

16. (d) $\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\sin^{-1}\left(\frac{2b}{1+b^2}\right) = 2\tan^{-1}x$

Putting $a = \tan\theta$ and $b = \tan\phi$

so, $\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)+\sin^{-1}\left(\frac{2\tan\phi}{1+\tan^2\phi}\right)$

$= 2\tan^{-1}x$

$\Rightarrow$ $\sin^{-1}\sin(2\theta) + \sin^{-1}\sin(2\phi) = 2\tan^{-1}x$

$\Rightarrow$ $2(\theta+\phi) = 2\tan^{-1}x$, Hence x

$= \tan(\theta+\phi)$

$\Rightarrow$ $x = \frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$

substituting these value we get $x = \frac{a+b}{1-ab}$

17. (d) $\cos^{-1}\frac{1}{2}+2\sin^{-1}\frac{1}{2} = \frac{\pi}{3}+\frac{2\pi}{6} \Rightarrow \frac{2\pi}{3}$

18. (a) $\sin^{-1} + \cos^{-1} = \frac{\pi}{2}$ $\Rightarrow \cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x$

$= \frac{\pi}{2}-\frac{\pi}{5}=\frac{3\pi}{10}$

19. (b) $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$ $\Rightarrow$ $\frac{\pi}{2} - \cos^{-1} + \frac{\pi}{2} -$ $\cos^{-1}y = \frac{2\pi}{3}$

$\Rightarrow$ $\cos^{-1}x + \cos^{-1}y$

$= \pi - \frac{2\pi}{3} = \frac{\pi}{3}$

20. (c) $\tan^{-1}\frac{x}{y} - \tan^{-1}\left(\frac{x-y}{x+y}\right) = \tan^{-1}\frac{x}{y} - \tan^{-1}\left(\frac{1-\frac{y}{x}}{1+\frac{y}{x}}\right)$

$= \tan^{-1}\frac{x}{y} - \left(\tan^{-1}1 - \tan^{-1}\frac{y}{x}\right)$

$= \tan^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x} - \frac{\pi}{4}$

$= \tan^{-1}\frac{x}{y} + \cot^{-1}\frac{x}{y} - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

21. (c) $\sin^{-1}\left(\sin\frac{5\pi}{3}\right) = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{-\pi}{3}$

22. (b) we know that $4\sin^{-1}x + \cos^{-1}x = \pi$

$\Rightarrow$ $3\sin^{-1}x + \sin^{-1}x + \cos^{-1}x = \pi$

$\Rightarrow$ $3\sin^{-1}x = \pi - \frac{\pi}{2} = \frac{\pi}{2} \Rightarrow \sin^{-1}x = \frac{\pi}{6}$

$\Rightarrow$ $x = \sin\frac{\pi}{6} = \frac{1}{2}$

23. (c) Given $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$

$\because$ $0 \le \cos^{-1}x \le \pi$ $\therefore$ $0 \le \cos^{-1}y \le \pi$

and $0 \le \cos^{-1}z \le \pi$

here $\cos^{-1}x = \cos^{-1}y = \cos^{-1}z = \pi$

$\Rightarrow$ $x = y = z = \cos\pi = -1$

$\therefore$ $xy + yz + zx = (-1)(-1) + (-1)(-1)$ $+ (-1)(-1)$

$= 1 + 1 + 1 = 3$

24. (a) Let $\cos^{-1}\frac{4}{5} = x \Rightarrow \cos x = \frac{4}{5}$ ....(i)

Now $\sin(\frac{1}{2}\cos^{-1}\frac{4}{5}) = \sin\left(\frac{x}{2}\right)$ ....(ii)

from (i) $\cos x = \frac{4}{5}$ $\Rightarrow$ $1 - 2\sin^2\frac{x}{2} = \frac{4}{5}$

$\Rightarrow$ $2\sin^2\frac{x}{2} = 1 - \frac{4}{5} = \frac{1}{5}$ $\Rightarrow$ $\sin\frac{x}{2} = \sqrt{\frac{1}{10}}$

25. (c) Given equation can be rewritten as follows

$\Rightarrow$ $\tan^{-1}\frac{1}{\sqrt{\cos\alpha}} - \tan^{-1}\sqrt{\cos\alpha} = u$

$\Rightarrow$ $\tan^{-1}\frac{\frac{1}{\sqrt{\cos\alpha}} - \sqrt{\cos\alpha}}{1+\frac{1}{\sqrt{\cos\alpha}}\times\sqrt{\cos\alpha}} = u$

$\Rightarrow$ $\tan^{-1}\frac{1-\cos\alpha}{2\sqrt{\cos\alpha}} = u$ i.e.,

$\Rightarrow$ $\tan u = \frac{1-\cos\alpha}{2\sqrt{\cos\alpha}} = \frac{\text{perpendicular}}{\text{base}}$

$\therefore$ $\sin u = \frac{\text{perpendicular}}{\text{hypotenuse}}$

$= \frac{1-\cos\alpha}{\sqrt{(1-\cos\alpha)^2+4\cos\alpha}}$

or $\sin u = \frac{1-\cos\alpha}{1+\cos\alpha} = \tan^2\frac{\alpha}{2}$

26. (d) $\sin(\cot^{-1}(1+x)) = \cos(\tan^{-1} x)$

$$\Rightarrow \sin\left(\sin^{-1}\frac{1}{\sqrt{x^2+2x+2}}\right)$$

$$= \cos\left(\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right)$$

$$\Rightarrow \frac{1}{\sqrt{x^2+2x+2}} = \frac{1}{\sqrt{1+x^2}}$$

$\Rightarrow x^2 + 2x + 2 = 1 + x^2$

$\Rightarrow x = -1/2.$

27. (b) We know that $\sin^{-1}(\alpha) + \cos^{-1}(\alpha) = \frac{\pi}{2}$

Therefore, α should be equal in both functions

$$\Rightarrow x - \frac{x^2}{2} + \frac{x^3}{4} - .... = x^2 - \frac{x^4}{2} + \frac{x^6}{4} ....$$

$$\Rightarrow \frac{x}{1+\frac{x}{2}} = \frac{x^2}{1+\frac{x^2}{2}} \Rightarrow \frac{x}{\frac{2+x}{2}} = \frac{x^2}{\frac{2+x^2}{2}}$$

$$\Rightarrow \frac{2x}{2+x} = \frac{2x^2}{2+x^2}$$

$\Rightarrow 2x(2+x^2) = 2x^2(2+x)$

$\Rightarrow 4x + 2x^3 = 4x^2 + 2x^3$

$\Rightarrow x[4 + 2x^2 - 4x - 2x^2] = 0$

$\Rightarrow$ either $x = 0$ or $4 - 4x = 0$

$\Rightarrow x = 0$ or $x = 1$

$\because 0 < |x| < \sqrt{2}, x \neq 0, \qquad \therefore x = 1$

Therefore, (b) is the answer.

28. (c) From function it is clear that

(1) $x(x+1) \geq 0 \because$ Domain of square root function.

(2) $x^2 + x + 1 \geq 0 \because$ Domain of square root function.

(3) $x^2 + x + 1 \leq 1 \because \sqrt{x^2+x+1} \leq 1$ Domain of $\sin^{-1}$ function from (2) and (3), $0 \leq x^2 + x + 1 \cap x^2 + x \geq 0$

$\Rightarrow 0 \leq x^2 + x + 1 \leq 1 \cap x^2 + x + 1 \geq 1$

$\Rightarrow x^2 + x + 1 = 1$

$\Rightarrow x^2 + x = 0$

$\Rightarrow x(x+1) = 0$

$\Rightarrow x = 0, x = -1$, Therefore, (c) is the answer.

29. (b) $\cos^{-1} x - \cos^{-1}\frac{y}{2} = \alpha$

$$\Rightarrow \cos^{-1}\left(\frac{xy}{2} + \sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}}\right) = \alpha$$

$\Rightarrow xy + \sqrt{1-x^2}\sqrt{4-y^2} = 2\cos\alpha$

$\Rightarrow (xy - 2\cos\alpha)^2 = 4 - y^2 - 4x^2 + x^2y^2$

$\Rightarrow 4x^2 - 4xy\cos\alpha + y^2 = 4\sin^2\alpha.$

30. (d) The range of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

We have $\sin^{-1} x = 2\sin^{-1} a$

$$\Rightarrow -\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}$$

$$\Rightarrow -\frac{\pi}{2} \leq 2\sin^{-1} a \leq \frac{\pi}{2}$$

$$\Rightarrow -\frac{\pi}{4} \leq \sin^{-1} a \leq \frac{\pi}{4}$$

$$\Rightarrow \sin\left(-\frac{\pi}{4}\right) \leq a \leq \sin\left(\frac{\pi}{4}\right)$$

$$\Rightarrow -\frac{1}{\sqrt{2}} \leq a \leq \frac{1}{\sqrt{2}}$$

$$\Rightarrow |a| \leq \frac{1}{\sqrt{2}}$$

31. (d) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$$\therefore \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9}$$

$$= \tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\times\frac{2}{9}}\right) = \tan^{-1}\left(\frac{1}{2}\right)$$

32. (b) Given $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$

$$\sin^{-1}\left(\frac{x}{5}\right) + \sin^{-1}\left(\frac{4}{5}\right) = \frac{\pi}{2}$$

$$\sin^{-1}\left(\frac{4}{5}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{x}{5}\right)$$

$\Rightarrow \quad \sin^{-1}\left(\frac{4}{5}\right) = \cos^{-1}\left(\frac{x}{5}\right)$

$\Rightarrow \quad \cos^{-1}\left(\frac{3}{5}\right) = \cos^{-1}\left(\frac{x}{5}\right)$

$\Rightarrow \frac{3}{5} = \frac{x}{5} \Rightarrow x = 3$

33. (c) If $\cos(2\sin^{-1}x) = \frac{1}{9}$

$\Rightarrow \quad \cos(\sin^{-1} 2x\sqrt{1-x^2}) = \frac{1}{9}$

$\Rightarrow \quad \cos(\cos^{-1}\sqrt{1-4x^2+4x^4}) = \frac{1}{9}$

$\Rightarrow \quad 1-2x^2 = \frac{1}{9}$

$\Rightarrow \quad 2x^2 = 1-\frac{1}{9} \Rightarrow 2x^2 = \frac{8}{9}$

$\Rightarrow \quad x^2 = \frac{4}{9} \quad \Rightarrow x = \pm\frac{2}{3}$

34. (c) $\tan(2\cos^{-1}\frac{3}{5}) = \tan\left[\cos^{-1}\left(2.\frac{9}{25}-1\right)\right]$

{since $2\cos^{-1}x = \cos^{-1}(2x^2-1)$}

$= \tan\cos^{-1}\left(\frac{-7}{25}\right) = \tan\tan^{-1}\left[\frac{\sqrt{1-\frac{49}{625}}}{\frac{-7}{25}}\right]$

$= \frac{-24}{7} \qquad \therefore \tan(2\cos^{-1}\frac{3}{5}) = \frac{-24}{7}$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. $\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{3}{5}$ is equal to

(a) $\tan^{-1}\frac{27}{11}$ (b) $\sin^{-1}\frac{11}{27}$

(c) $\cos^{-1}\frac{11}{27}$ (d) None of these

2. $\tan\left(90° - \cot^{-1}\frac{1}{2}\right)$ is equal to

(a) 3 (b) 2/3

(c) 1/3 (d) $1/\sqrt{10}$

3. If $A = \tan^{-1}x$, then $\sin 2A =$ ***[MNR-88; UPSEAT-2000]***

(a) $\frac{2x}{\sqrt{1-x^2}}$ (b) $\frac{2x}{1-x^2}$

(c) $\frac{2x}{1+x^2}$ (d) None of these

4. $\sin(2\sin^{-1}0.8) =$

(a) 0.96 (b) 0.48

(c) 0.64 (d) None of these

5. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi/2$, then

(a) $x + y + z - xyz = 0$

(b) $x + y + z + xyz = 0$

(c) $xy + yz + zx + 1 = 0$

(d) $xy + yz + zx - 1 = 0$

6. $\cos\left[\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2}\right]$ is equal to ***[PET-91]***

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{\sqrt{3}}{2}$

(c) $\frac{1}{2}$ (d) $\frac{\pi}{4}$

7. $\tan^{-1}\frac{a-b}{1+ab} + \tan^{-1}\frac{b-c}{1+bc}$ is equal to

(a) $\tan^{-1}a - \tan^{-1}b$ (b) $\tan^{-1}a - \tan^{-1}c$

(c) $\tan^{-1}b - \tan^{-1}c$ (d) $\tan^{-1}c - \tan^{-1}a$

8. $\sin^{-1}\frac{2a}{1+a^2} - \cos^{-1}\frac{1-b^2}{1+b^2} = \tan^{-1}\frac{2x}{1-x^2}$, then x is equal to ***[EAMCET-89]***

(a) a (b) b

(c) $\frac{a+b}{1-ab}$ (d) $\frac{a-b}{a+ab}$

9. tan ($\cos^{-1}x$) is equal to **[PET-93; NDA-2008]**

(a) $\frac{\sqrt{1-x^2}}{x}$ (b) $\frac{x}{1+x^2}$

(c) $\frac{\sqrt{1+x^2}}{x}$ (d) $\sqrt{1-x^2}$

10. The value of x which satisfies the equation $\tan^{-1}x = \sin\,(3\sqrt{10})$ is

(a) 3 (b) –3

(c) 1/3 (d) – 1/3

11. $\sin\left\{\tan^{-1}\left(\frac{1-x^2}{2x}\right)+\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right\}$ is equal to

(a) 0 (b) 1

(c) $\sqrt{2}$ (d) $1\sqrt{2}$

12. cos [$\cos^{-1}(-1/7) + \sin^{-1}(-1/7)$] is equal to **[EAMCET-2003]**

(a) –1/3 (b) 0

(c) 1/3 (d) 4/9

13. The value of $\sin\left(\sin^{-1}\frac{1}{3}+\sec^{-1}3\right)$

$+\cos\left(\tan^{-1}\frac{1}{2}+\tan^{-1}2\right)$ is **[MPPET-2006]**

(a) 1 (b) 2

(c) 3 (d) 4

14. Which one of the following is true? **[MPPET-2005]**

(a) $\sin(\cos^{-1}x) = \cos(\sin^{-1}x)$

(b) $\sin(\tan^{-1}x) = \tan(\sec^{-1}x)$

(c) $\cos(\tan^{-1}x) = \tan(\cos^{-1}x)$

(d) $\tan(\sin^{-1}x) = \sin(\tan^{-1}x)$

15. If $\tan^{-1}a + \tan^{-1}b = \sin^{-1}1 - \tan^{-1}c$, then **[MPPET-2005]**

(a) $a + bc = abc$

(b) $ab + bc + ca = abc$

(c) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{1}{abc}=0$

(d) $ab + bc + ca = a + b + c$

16. The value of x where $x > 0$ and $\tan\left(\sec^{-1}\left(\frac{1}{x}\right)\right)$ $= \sin(\tan^{-1}2)$ is **[EAMCET-2007]**

(a) $\sqrt{5}$ (b) $\frac{\sqrt{5}}{3}$

(c) 1 (d) $\frac{2}{3}$

17. The value of $\sin\left[2\cos^{-1}\frac{\sqrt{5}}{3}\right]$ is **[Karnataka CET-2007]**

(a) $\frac{\sqrt{5}}{3}$ (b) $\frac{2\sqrt{5}}{3}$

(c) $\frac{4\sqrt{5}}{9}$ (d) $\frac{2\sqrt{5}}{9}$

18. $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$ is equal to **[PET-97]**

(a) 0 (b) $\pi/4$

(c) $\pi/2$ (d) π

19. The value of $\tan\left[\frac{1}{2}\cos^{-1}\left(\frac{\sqrt{5}}{3}\right)\right]$ is equal to **[NDA-02]**

(a) $\frac{2}{5-\sqrt{5}}$ (b) $\frac{2}{5+\sqrt{5}}$

(c) $\frac{3+\sqrt{5}}{2}$ (d) $\frac{3-\sqrt{5}}{2}$

20. What is the value of cos [$\tan^{-1}\{\tan(15\pi/4)\}$] **[NDA-07]**

(a) $-\frac{1}{\sqrt{2}}$ (b) 0

(c) $\frac{1}{\sqrt{2}}$ (d) $\frac{1}{2\sqrt{2}}$

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The worksheet is of 22 Minutes.
3. The worksheet consists of 22 questions. The maximum marks are 66.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. $\cos^{-1}(-1)$ is equal to

(a) $\pi/2$ (b) 0
(c) π (d) 2π

2. $\cos\left(\sin^{-1}\frac{5}{13}\right)$ is equal to

(a) 12/13 (b) – 12/13
(c) 5/12 (d) None of these

3. If $\tan^{-1}x - \tan^{-1}y = \tan^{-1}A$, then A is equal to

(a) $x - y$ (b) $x + y$
(c) $\frac{x-y}{1+xy}$ (d) $\frac{x+y}{1-xy}$

4. $\sin^{-1}\frac{1}{\sqrt{5}} + \cot^{-1}3$ is equal to **[PET-93]**

(a) $\pi/6$ (b) $\pi/4$
(c) $\pi/3$ (d) $\pi/2$

5. The domain of $\sin^{-1}x$ is **[Roorkee-93]**

(a) $(-\pi, \pi)$ (b) $[-1, 1]$
(c) $0, 2\pi$ (d) $(-\infty, \infty)$

6. The principal value of $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$ is **[Roorkee-92]**

(a) $-2\pi/3$ (b) $-\pi/3$
(c) $4\pi/3$ (d) $5\pi/3$

7. $\left[\sin\left(\tan^{-1}\frac{3}{4}\right)\right]^2$ is equal to **[EAMCET-83]**

(a) 3/5 (b) 5/3
(c) 9/25 (d) 25/9

8. $\sin\left\{\sin^{-1}\frac{1}{2} + \cos^{-1}\frac{1}{2}\right\}$ is equal to **[EAMCET-85]**

(a) 0 (b) –1
(c) 2 (d) 1

9. $\sin^{-1}x + \cos^{-1}x$ is equal to

(a) $\pi/4$ (b) $\pi/2$
(c) – 1 (d) 1

10. The value of $\sin^{-1}(\sqrt{3}/2) - \sin^{-1}(1/2)$ **[PET-2003]**

(a) 45° (b) 90°
(c) 15° (d) 30°

11. Principal value of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ is **[NDA-2006]**

(a) $-2\pi/3$ (b) $2\pi/3$
(c) $4\pi/3$ (d) $\pi/3$

12. $\tan(\tan^{-1}x + \tan^{-1}y + \tan^{-1}z) - \cot(\cot^{-1}x + \cot^{-1}y + \cot^{-1}z)$ is equal to **[NDA-2006]**

(a) 0 (b) $3\pi/2$
(c) $2(x + y + z)$ (d) $\frac{3\pi}{2} + x + y + z$

13. The value of x which satisfies the equation $\cos^{-1}x = 2\sin^{-1}x$ is **[NDA-2006]**

(a) 1 (b) 1/2
(c) – 1 (d) – 1/2

14. If $\cos^{-1}x + \cos^{-1}y = \pi$, what is the value of $\sin^{-1}x + \sin^{-1}y$? **[NDA-2005]**

(a) 0 (b) $\pi/2$
(c) π (d) 2π

15. $\sin\left(2\sin^{-1}\sqrt{\frac{63}{65}}\right)$ is equal to **[Karnataka CET-2008]**

(a) $\frac{8\sqrt{63}}{65}$ (b) $\frac{\sqrt{63}}{65}$
(c) $\frac{2\sqrt{126}}{65}$ (d) $\frac{4\sqrt{65}}{65}$

16. For $x = 2$ and $y = 3$ the angle $\tan^{-1}x + \tan^{-1}y$ equals *[MPPET-2007]*

(a) 135° (b) 45°
(c) 90° (d) 180°

17. If $\cos^{-1}\left(\frac{1}{\sqrt{5}}\right) = \theta$, then what is the value of $\text{cosec}^{-1}(\sqrt{5})$? *[NDA-2007]*

(a) $(\pi/2) + \theta$ (b) $(\pi/2) - \theta$
(c) $\pi/2$ (d) $-\theta$

18. If $5\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) + 7\sin^{-1}\left(\frac{2x}{1-x^2}\right)$ $-4\tan^{-1}\left(\frac{2x}{1-x^2}\right) - \tan^{-1}x = 5\pi$, then x is equal to *[Kerala PET-2008]*

(a) 3 (b) $\frac{1}{3}$
(c) $\frac{1}{\sqrt{3}}$ (d) $\sqrt{3}$

19. The simplified expression of $\sin(\tan^{-1}x)$, for any real number x is given by *[VITEEE-2008]*

(a) $\frac{1}{\sqrt{1+x^2}}$ (b) $\frac{x}{\sqrt{1+x^2}}$
(c) $-\frac{1}{\sqrt{1+x^2}}$ (d) $-\frac{x}{\sqrt{1+x^2}}$

20. If $\sin^{-1}x - \cos^{-1}x = \frac{\pi}{6}$, then what is the value of x? *[NDA-2008]*

(a) $x = -\frac{1}{2}$ (b) $x = 1$
(c) $x = \frac{1}{2}$ (d) $x = \frac{\sqrt{3}}{2}$

21. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, then the value of $x + y + z$ is *[MPPET-2008]*

(a) $-xyz$ (b) xyz
(c) $\frac{1}{xyz}$ (d) 0

22. If $\tan^{-1} 1/3 + \tan^{-1} 3/4 - \tan^{-1} x/3 = 0$, then x is equal to *[MPPET-2008]*

(a) 7/3 (b) 3
(c) 11/3 (d) 13/3

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ
6. ⓐ ⓑ ⓒ ⓓ
7. ⓐ ⓑ ⓒ ⓓ
8. ⓐ ⓑ ⓒ ⓓ
9. ⓐ ⓑ ⓒ ⓓ
10. ⓐ ⓑ ⓒ ⓓ
11. ⓐ ⓑ ⓒ ⓓ
12. ⓐ ⓑ ⓒ ⓓ
13. ⓐ ⓑ ⓒ ⓓ
14. ⓐ ⓑ ⓒ ⓓ
15. ⓐ ⓑ ⓒ ⓓ
16. ⓐ ⓑ ⓒ ⓓ
17. ⓐ ⓑ ⓒ ⓓ
18. ⓐ ⓑ ⓒ ⓓ
19. ⓐ ⓑ ⓒ ⓓ
20. ⓐ ⓑ ⓒ ⓓ
21. ⓐ ⓑ ⓒ ⓓ
22. ⓐ ⓑ ⓒ ⓓ

Part E

Heights and Distances

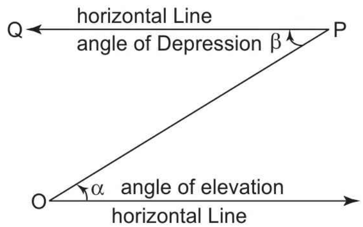
LECTURE 1

Heights and Distances

BASIC CONCEPTS

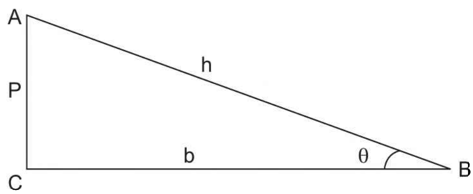
FORMULA

1. Angle of Elevation and Depression ($0<\alpha> 90$, $0 < \beta < 90$)



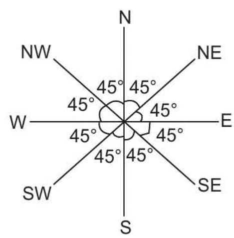
$\alpha = \beta$ = alternate angle

2. Problems based on Pythagoras Theorem

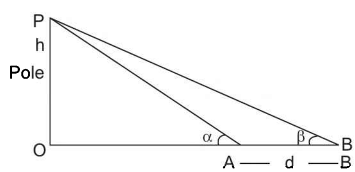


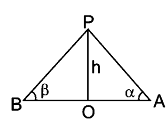
$\sin\theta = \frac{P}{h}, \cos\theta = \frac{b}{h}. \tan\theta = \frac{P}{b}$ etc.

3. In any triangle other than right angled Δ, we can use ‘the sine rule’ or cosine formula.
4. Angles in the same segment of a circle are equal.
5. Angle subtended at the centre = twice the angle subtended on the circumference.
6. If a line is perpendicular to a plane, then it is perpendicular to every line in that plane.
7. Concept of Direction.



The angle of elevation of the top of a pole at point *B* on the ground is β. If on walking *d* metres (from *B* to *A*) towards the tower, the angle of elevation becomes α, then the height of the pole is given by formula.
$d = h(\cot\beta - \cot\alpha)$, where $AB = d$.

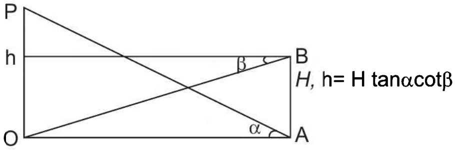




NOTE

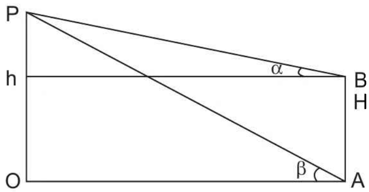
If *A* and *B* are on either side of the pole, then the height of the pole is given by $d = h(\cot\beta + \cot\alpha)$.

8. A tower subtends an angle α at a point A in the plane of its base and the angle of depression of the foot of the tower at a point H metres just above A is β, then the height of the tower is $h = H \tan \alpha \cot \beta$. α = angle of elevation of the top of h.



β = angle of depression of the bottom of h.

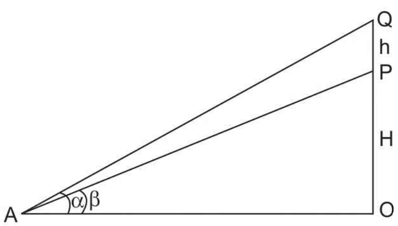
9. From the top and bottom of a house H metre high, the angles of elevation of the top of a tower are α and β. The height of the tower (h) is



$$h = \frac{H \tan\beta}{\tan\beta - \tan\alpha}$$

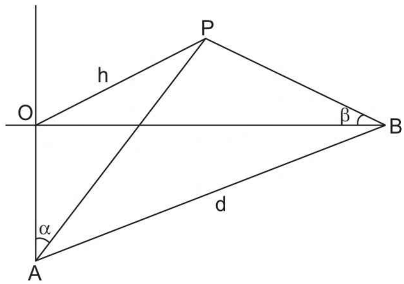
$$\text{h} = \frac{H \cot\alpha}{\cot\alpha - \cot\beta} \text{ or H} = \frac{h \sin(\beta - \alpha)}{\cos\alpha \sin\beta}$$

10. An observer in a boat finds that the angle of elevation of the top of a tower is α and that of the top of cliff is β. If the height of the tower be h, then the height of the cliff.



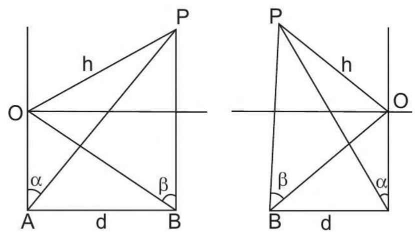
$$H = \frac{h \tan\beta}{\tan\alpha - \tan\beta}$$

11. The angle of elevation of the top of a tower from a point A due south of the tower is α and from a point B due east of the tower is β. If $AB = d$, then the height of the tower (h) is given by



$$h = \frac{d}{\sqrt{\cot^2\alpha + \cot^2\beta}}$$

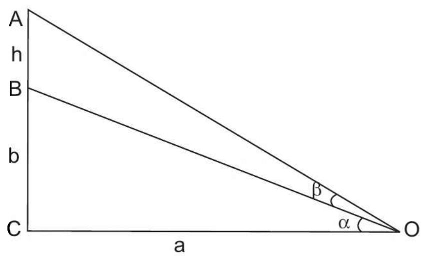
12. The angle of elevation of the top of a tower from a point A due south of the tower is α and from a point B due east (or west) or A is β. If $AB = d$, then the height of the tower is given by



$$h = \frac{AB(= d)}{\sqrt{\cot^2\beta - \cot^2\alpha}}$$

13. A tower of height b subtends an angle α at a point O on the level of the foot of the tower and at a distance a from the foot of the tower. If a pole mounted on the tower subtends angle β at O, then the height (h) of the pole is

$h = a\{\tan(\alpha + \beta) - \tan\alpha\}$ or $h + b = a \tan(\alpha + \beta)$



NOTE

If α = β, then $h = \dfrac{b(a^2 + b^2)}{a^2 - b^2}$

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. A tower stands vertically in a field. The field is in the shape of an equilateral triangle of side 100 metres. The tower subtends angles of 45°, 60° and 60° at the vertices of the triangle. Find the height of tower.

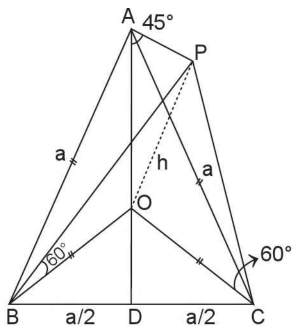
[Roorkee-1998]

Solution

Let the tower stand at O and its height $OP = h$ which subtends an angle of 45°, 60°, 60° at A, B, C respectively.

$\therefore \quad OA = h \cot 45° = h$

$OB = h \cot 60° = \dfrac{h}{\sqrt{3}} = OC \quad \therefore OB = OC$



Also, $AB = AC = BC = A$ say.

If D be the mid-point of BC then OD and AD both are perpendicular to base BC.

$\therefore$ AD is median as well as altitude. In an isosceles or equilateral Δ, both centroid and orthocentre coincide, then

$$OA = \frac{2}{3}AD = \frac{2}{3}\sqrt{a^2 + \frac{a^2}{4}}$$

or $\quad h = OA = \dfrac{2}{3}\cdot\dfrac{1}{2}\sqrt{5}a = \dfrac{100\sqrt{5}}{3}$

$\therefore \quad a = 100$, given.

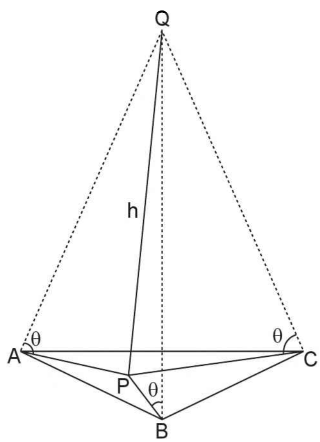
2. PQ is a vertical tower. P is the foot, Q the top of the tower, A, B, C are three points in the horizontal plane through P. The angles of elevation of Q from A, B, C are equal and each is equal to θ. The sides of the triangle ABC are a, b, c and the area of the triangle ABC is Δ. Show that the height of the tower is $(abc \tan\theta)/4\Delta$

[IIT-1980]

Solution

Let the height PQ of the tower be h. Since, the angles of elevation of Q from each of the points A, B, C is θ, we have

$PA = PB = PC = h \cot\theta \qquad \ldots (1)$



Since, P is equidistant from A, B and C it is the circum-centre of the ΔABC. $\therefore PA = PB = PC$

$= \text{circumradius of } \Delta ABC = \dfrac{abc}{4\Delta} \qquad \ldots (2)$

Hence, from (1) and (2), we obtain

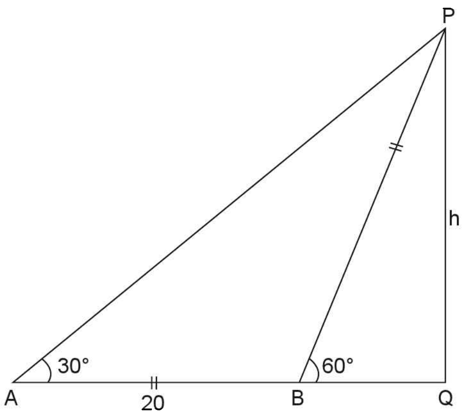
$$h\cot\theta = \frac{abc}{4\Delta} \text{ or } h = \frac{abc\tan\theta}{4\Delta}$$

3. Angle of elevation of top of a tower at a point on the ground is 30°. After moving 20 metres towards the tower, the angle of elevation becomes 60°, then find the height of the tower?

Solution

Let height of the tower h, then from the figure $PB = 20$ cm.

Hence, $\dfrac{h}{20} = \sin 60°$



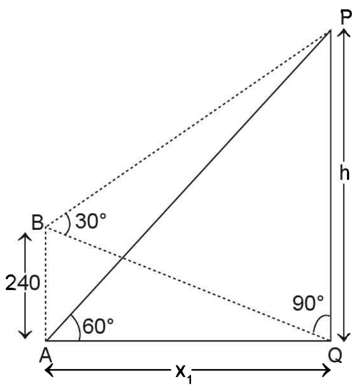
$\Rightarrow \quad h = 10\sqrt{3}$ m.

4. Angle of elevation of top of the temple towards the east of a man is 60°. After moving 240 metres towards the north, angle of elevation becomes 30°, then find the height of the temple? ***[MP-2003]***

Solution

Let PQ is temple, whose height is $= h$
A, B are two given position of the man.

Let $AQ = x$, then $x = h \cot 60° = h/\sqrt{3}$



$\because \quad AB = 240$, so $BQ = \sqrt{x^2 + (240)^2}$

Now, in right angled triangle

$$\Delta PQB, \frac{PQ}{BQ} = \tan 30°$$

$$\Rightarrow \quad \frac{h}{\sqrt{x^2 + (240)^2}} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad 3h^2 = \frac{h^2}{3} + (240)^2 [\because x = h/\sqrt{3}]$$

$$\Rightarrow \quad h^2 = 3 \times 30 \times 240 \Rightarrow h = 60\sqrt{6} \text{ meter}$$

5. A balloon is analysed from three points A, B and C on the road. Angle of elevation at B is twice of the angle at A and angle of elevation at C is thrice of the angle at A. If distance between A and B is 200 metre and B and C is 100 metre, then height of balloon from the road is

Solution

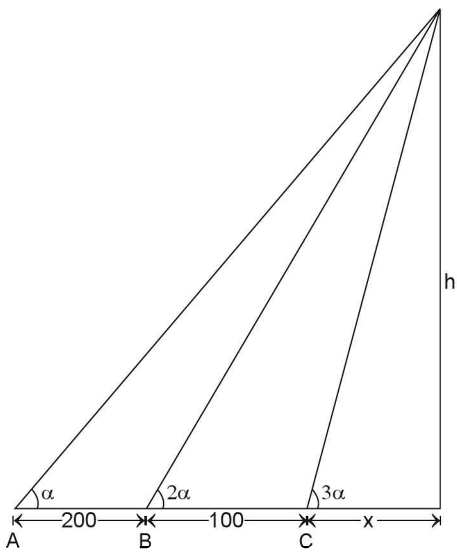
$x = h \cot 3\alpha$ ....(i)

$(x + 100) = h \cot 2\alpha$ ....(ii)

$(x + 300) = h \cot \alpha$ ....(iii)

from equation (i) and (ii), $100 = h(\cot 3\alpha - \cot 2\alpha)$,

from equation (ii) and (iii), $200 = h(\cot 2\alpha - \cot \alpha)$,



$$100 = h\left(\frac{\sin\alpha}{\sin 3\alpha \sin 2\alpha}\right) \text{ and}$$

$$200 = h\left(\frac{\sin\alpha}{\sin 2\alpha \sin\alpha}\right)$$

$$\text{or } \frac{\sin 3\alpha}{\sin\alpha} = \frac{200}{100} \Rightarrow \frac{\sin 3\alpha}{\sin\alpha} = 2$$

$\Rightarrow \quad 3 \sin \alpha - 4 \sin^3 \alpha - 2\sin \alpha = 0$

$\Rightarrow \quad 4\sin^3 \alpha - \sin \alpha = 0 \Rightarrow \sin \alpha = 0$

or $\sin^2 \alpha = \frac{1}{4} = \sin^2\left(\frac{\pi}{6}\right) \Rightarrow \alpha = \frac{\pi}{6}$

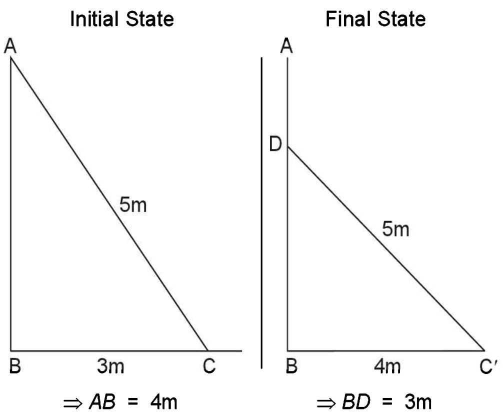
Hence, $h = 200 \sin\frac{\pi}{3} = 200\frac{\sqrt{3}}{2} = 100\sqrt{3}$

{from (ii)}

6. A ladder of length 5 metre is inclined at a vertical wall at some angle. Foot of the ladder is at 3 metres from the wall. If foot of the ladder is moved 1 metre away from the wall, then by what length top of the ladder will slide downwards.

Solution

(a)



$\therefore \quad AD = 4 - 3 = 1$ metre

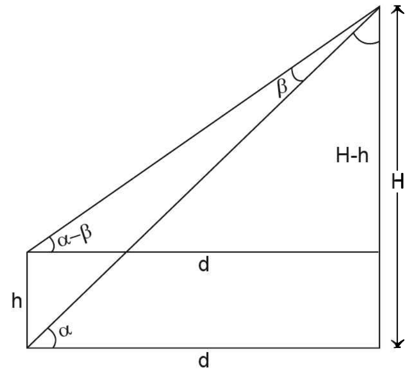
7. From the foot of h metre high pole angle of elevation of top of the tower is α and pole subtends angle β on the top of the tower, then prove that height of the tower is

$$\frac{h\cot(\alpha-\beta)}{\cot(\alpha-\beta)-\cot\alpha}.$$

Solution

$d = H \cot \alpha$

$d = (H - h) \cot (\alpha - \beta) \Rightarrow H \cot \alpha$



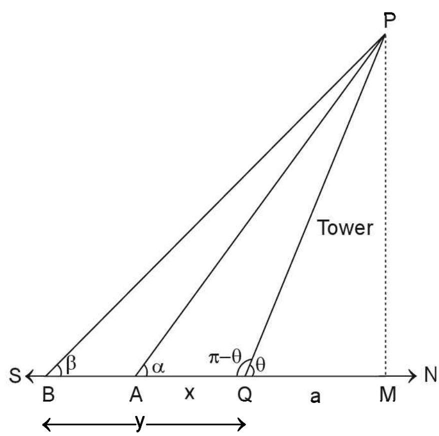
$= (H - h) \cot (\alpha - \beta)$ or

$$H = \frac{h\cot(\alpha-\beta)}{\cot(\alpha-\beta)-\cot\alpha}$$

8. Due south of a tower which is leaning towards north, there are two stations at distances x, y respectively from its foot. If α, β respectively be the angles of elevation of the top of the tower at these stations, show that the inclination of the tower to the horizon is $\cot^{-1}\frac{y\cot\alpha - x\cot\beta}{y-x}$.

Solution

$AQ = x$, $BQ = y$ and PQ is a tower leaning towards North at an angle θ. Apply $m - n$ theorem on ΔPBQ.



$(y - x + x) \cot \alpha = x \cot \beta - (y - x) \cot (\pi - \theta)$

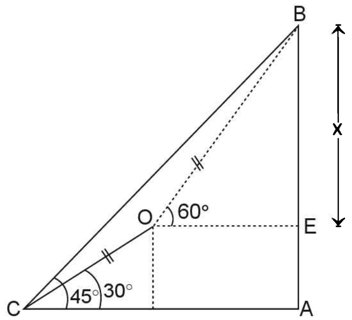
or $y \cot \alpha - x \cot \beta = (y - x) \cot \theta$

$$\therefore \quad \theta = \cot^{-1} \frac{y \cot \alpha - x \cot \beta}{y - x}$$

9. From the foot of the hill angle of elevation of its top is 45°. After climbing 1000 metres at the slope of 30°, angle of elevation of the top becomes 60°. Obtained the height of hill.

Solution

Let, height of the hill $AB = h$ and distance OC = 1000 metre



Now, from figure,

$\angle OCB = 45° - 30° = 15°$

and $\angle OBE = 90° - 60° = 30°$

$\therefore \quad \angle OCB = 45° - 30° = 15°$

$[\because \angle ABC = 45°]$

$OC = OB = 1000$ metre

Now, $AB = AE + EB$

$= OF + BE$

$= OC \sin \angle OCF + OB \sin \angle BOE$

$= 1000 \sin 30° + 1000 \sin 60°$

$= 500 + 500\sqrt{3}$

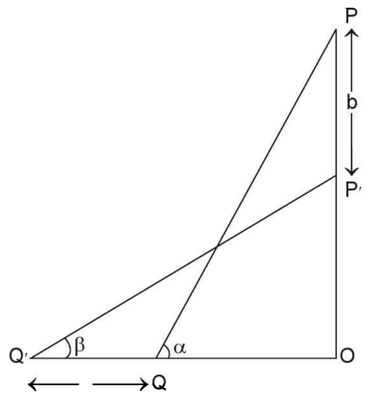
$= 500 (\sqrt{3} + 1)$ metre

10. A ladder slide from the wall horizontally at a distance α, so that its slides by distance b from the wall and makes angle β with the horizon, then prove that $a = b \tan \dfrac{\alpha + \beta}{2}$

Solution

If PQ and $P'Q'$ are two positions of ladder, whose height is l then $a = OQ' - OQ = 1 (\cos\beta - \cos\alpha)$ ....(1)

$b = OP - OP' = 1 (\sin\alpha - \sin\beta)$ ...(2)



from (1) and (2),

$$\frac{a}{b} = \frac{\cos\beta - \cos\alpha}{\sin\alpha - \sin\beta} = \frac{2\sin\frac{\alpha+\beta}{2} \cdot \sin\frac{\alpha-\beta}{2}}{2\cos\frac{\alpha+\beta}{2} \cdot \sin\frac{\alpha-\beta}{2}}$$

$$= \tan\left(\frac{\alpha+\beta}{2}\right)$$

11. The angle of elevation of the top of a tower from a point A due south of the tower is α and from a point B due east of the tower is β. If $AB = d$, then the height of the tower is

[Roorkee-79; Kurukshetra CEE-98]

(a) $\dfrac{d}{\sqrt{\tan^2 \alpha - \tan^2 \beta}}$

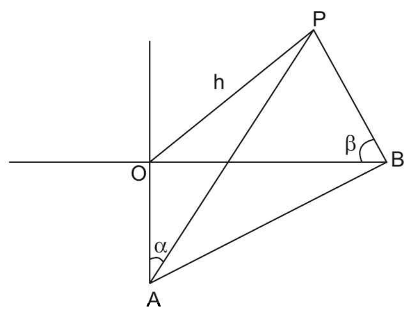
(b) $\dfrac{d}{\sqrt{\tan^2 \alpha + \tan^2 \beta}}$

(c) $\dfrac{d}{\sqrt{\cot^2 \alpha + \cot^2 \beta}}$

(d) $\dfrac{d}{\sqrt{\cot^2 \alpha - \cot^2 \beta}}$

Solution

(c) $OB = h \cot \beta$, $OA = h \cot \alpha$



$$h^2 = \frac{d^2}{\cot^2\beta + \cot^2\alpha}$$

$$\Rightarrow \quad h = \frac{d}{\sqrt{\cot^2\beta + \cot^2\alpha}}$$

12. An object is observed from three points A, B, C in the same horizontal line. Passing through the base of the object. The angle of elevation at B is twice and at C thrice that at A if $AB = a$, $BC = b$ Prove that the height of the object is

(a) $\frac{a}{2b}\sqrt{(a+b)(3b-a)}$

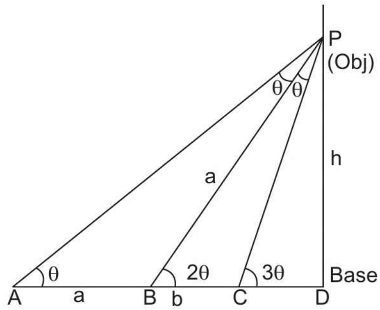
(b) $\frac{2b}{a}\sqrt{(b+a)(3a-b)}$

(c) $2ab\sqrt{(3a+b)(b-a)}$

(d) $2a^2b\sqrt{(3b-a)(2a-b)}$

Solution

(a)



$h = a \sin 2\theta = 2a \sin\theta \cos\theta$ ....(i)

In ΔPBC, by sine rule

$$\frac{a}{\sin(\pi - 3\theta)} = \frac{b}{\sin\theta} \quad \therefore \frac{a}{b} = \frac{3\sin\theta - 4\sin^3\theta}{\sin\theta}$$

$$\therefore \sin^2\theta = \frac{3b-a}{4b} \ \& \ \cos^2\theta = \frac{a+b}{4b}$$

Putting the value $\sin\theta$ and $\cos\theta$ in equation (i)

$$\frac{a}{2b}\sqrt{(a+b)(3b-a)}$$

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): SOLVE THESE PROBLEMS TO GRASP THE TOPIC

EXERCISE 1

1. The angle of depression of two ships from the top of a minar are 45° and 30° respectively. Both the ships are on one side of the minar and are in a line passing through the base of the minar. If distance between the ships is 100 metre, then find the height of the minar.

2. The angle of elevation of a vertical pillar standing on a horizontal plane is θ, going a distance 'a' towards the pillar the angle of elevation is 45°, again proceeding a distance 'b' towards the pillar the angle becomes (90° – θ). Find the altitude of the plane.

3. A Statue whose height is 10 metre stands on 30 metre high column. On a man standing on the horizontal plane, angle made by the statue and column are equal. Find the distance of the man from the top of the statue.

4. The shadow of the pole is equal to the height of the pole. Find the elevation of the sun.

5. The angle of elevation of the top of a tower from a point which is at a distance of 20 metres from bottom of the tower is 45°. Find the height of the tower.

EXERCISE 2

1. The angle of elevation of the top of a tower from a point on a ground level is 30°. On walking 20 metre towards the tower the angle of elevation becomes 45°. Find the height of the tower.

2. The angle of elevation of the top of a tower (which is yet incomplete) at a point 120 metre from its base is 45°. How much higher should it be raised, so that the elevation at the same point may become 60°?

3. From the top of a lighthouse angle of depression of two ships are 45° and 30° respectively towards east. If distance between the ships is 60 metre, find height of the lighthouse.

4. From an aeroplane vertically over a straight horizontal road, the angles of depression of two consecutive milestones opposite sides of the aeroplane are observed to be α and β. Show that the height in miles of aeroplane above the road is given by $\dfrac{\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$.

5. A flagstaff standing on a tower of height 10 metre subtends an angle whose tangent is $\dfrac{1}{8}$ at a point on horizontal ground the distance of the point from the tower is 24 metre. Find the height of the flagstaff.

ANSWERS

EXERCISE 1

1. 136.60 metre
2. $\dfrac{ab}{a-b}$
3. $10\sqrt{2}$ metre
4. 45°
5. 20 metre

EXERCISE 2

1. 27.32 metre
2. $120\ (\sqrt{3}-1)$ metre
3. $30\ (\sqrt{3}+1)$ metre
5. $3\frac{5}{7}$ metre

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

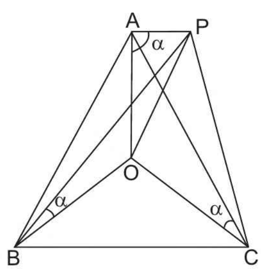
1. A pole stands vertically inside a triangular park Δ *ABC*. If the angle of elevation of the top of the pole from each corner of the park is same, then in Δ *ABC* the pole is at the **[IIT-2000]**

(a) centroid
(b) circumcentre
(c) incentre
(d) orthocentre

Solution

(b) Let $OP \equiv$ Pole, $\angle PAO = \angle PBO = \angle PCO = \alpha$

$\dfrac{OP}{OB} = \tan\alpha \Rightarrow OB = OP\cot\alpha$ ....(i)



Similarly, $OA = OP\cot\alpha$ ....(ii)

and $OC = OP\cot\alpha$ ...(iii)

From (i), (ii) and (iii), $OA = OB = OC$

$\Rightarrow$ *O* is the mid-point of circumcentre of the triangle *ABC*.

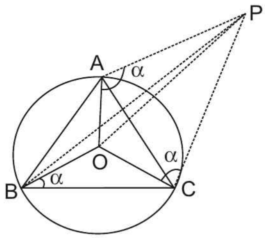
2. The angle of elevation of the tower observed from each of the three point A,B,C on the ground forming a triangle is the same angle α. If R is the circum-radius of the triangle ABC, then the height of the tower is

[DEC-2006]

(a) $R \sin \alpha$ (b) $R \cos \alpha$
(c) $R \cot \alpha$ (d) $R \tan \alpha$

Solution

(d) Since the tower makes equal angle at the vertices of the triangle, therefore foot of the tower is the circumcentre.
From ΔOAP, we have



$$\tan\alpha = \frac{OP}{OA} \Rightarrow OP = OA\tan\alpha$$

$$\Rightarrow \quad OP = R \tan \alpha$$

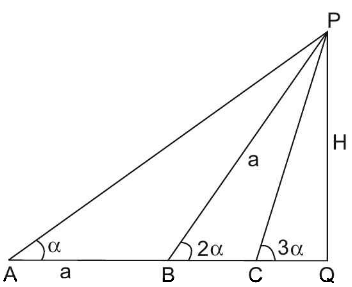
3. Angles of elevation of the top of a TV tower from three points A, B, C lying in the base plane of the tower are α, 2α, 3α respectively, If $AB = a$, then height of tower is equal to?

[DCE-2002; KCET-2008]

(a) $a \tan \alpha$ (b) $a \tan \alpha$
(c) $a \sin 2\alpha$ (d) $a \sin 3\alpha$

Solution

(c) $\therefore \angle APB = \angle BAP = \alpha$
$\therefore \quad AB = BP = \alpha$
Exterior angle $2\alpha = \alpha + \angle APB$
$\Rightarrow \quad \angle APB = \alpha$
$AB = BP = a$
In $H = a \sin 2\alpha$

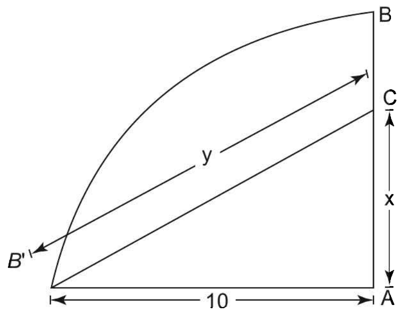


4. A tree is broken by wind, its upper part touches the ground at a point 10 m from the foot of the tree and makes an angle of 60° with the ground the entire length of the tree is ***[DCE-1998]***

(a) 15 m
(b) 20 m
(c) $\sqrt{3}\left(10+\frac{20}{\sqrt{3}}\right)$ m
(d) $\left(1+\frac{\sqrt{3}}{2}\right)$ m

Solution

(c) Let AB the tree when it broken at point C. It touches the ground at B'.



In $\Delta ACB''$ $\quad \tan 60° = \frac{x}{10} \Rightarrow x = 10\sqrt{3}$ m

$$\cos 60° = \frac{10}{y} \Rightarrow y = \frac{10}{\cos 60°}, \; y = 20 \text{ m}$$

Therefore, length of tree.

$$AB = x + y = 10\sqrt{3} + 20 = \sqrt{3}\left(10+\frac{20}{\sqrt{3}}\right)\text{m}$$

5. The angle of elevation of an object from a point P on the level ground is α. Moving d metres on the ground towards the object, the angle of elevation is found to be β. Then the height in metres of the object is

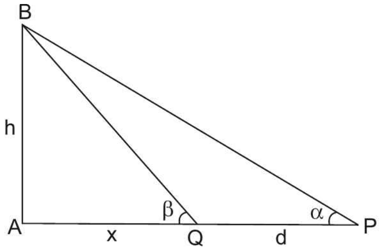
[EAMCET-2007]

(a) $d \tan \alpha$ (b) $d \cot \beta$

(c) $\dfrac{d}{\cot\alpha + \cot\beta}$ (d) $\dfrac{d}{\cot\alpha - \cot\beta}$

Solution

(d) From ΔABQ, $x = h \cot \beta$, From ΔABP, $x + d = h \cot \alpha$



∵ $d = h(\cot\alpha - \cot\beta)$

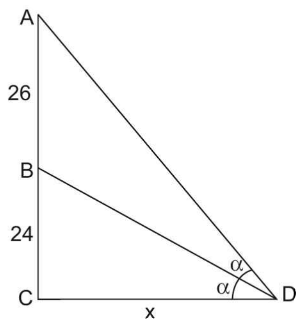
or $h = \dfrac{d}{\cot\alpha - \cot\beta}$ is the height of the object.

6. The lower 24 m portion of a 50 m tall tower is painted green and the remaining portion red. What is the distance of a point on the ground from the base of the tower where the two different portions of the tower subtend equal angles **[NDA-2007]**

(a) 90 m (b) 120 m
(c) 60 m (d) 72 m

Solution

(b) The lower 24 m portion



$\tan\alpha = \dfrac{24}{x}$ ....(1)

$\tan 2\alpha = \dfrac{50}{x}$ ....(2)

∵ $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

⇒ $\dfrac{50}{x} = \dfrac{2\times 24/x}{1-(24/x)^2}$

⇒ $2x^2 = 576 \times 50$

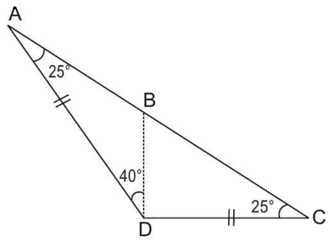
⇒ $x = 120$ m.

7. The relative positions of four ships A, B, C, D in a sea area as follows:
B is on line segment AC; B is north to D and D is just west to C, $BD = 2$ km. If $\angle BDA = 40°$, $\angle BCD = 25°$, then distance between A and D is (here $\sin 25° = 0.423$) **[IIT-83]**

(a) 3.28 km (b) 3.46 km
(c) 4.28 km (d) 4.83 km

Solution

(c) $\angle BDC = 90°, \angle DAC$
$= 180° - (130° + 25°) = 25°$



$AD = DC = BD \cot 25°$

$= 2\dfrac{\sqrt{1-\sin^2 25°}}{\sin 25°} = 2\dfrac{\sqrt{1-(0.423)^2}}{0.423} = 4.28$ km

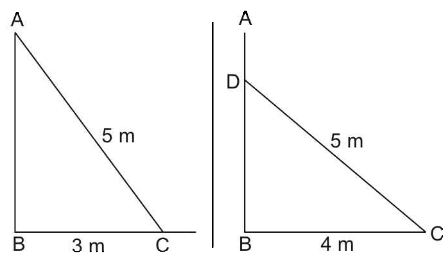
8. A ladder 5 metre long leans against a vertical wall. The bottom of the ladder is 3 metre from the wall. If the bottom of the ladder is pulled 1 metre farther from the wall, how much does the top of the ladder slide down the wall? **[AMU-2000]**

(a) 1 m (b) 7 m
(c) 2 m (d) None of these

Solution

(a)

From first case, From second case,



⇒ *AB* = 4 m ⇒ *BD* = 3 m

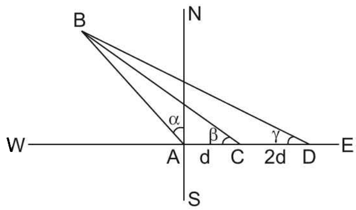
∴ AD = 4 – 3 = 1 m

9. A tower AB leans towards west making an angle α with the vertical. The angular elevation of B, the top most point of the tower is β as observed from a point C due east of A at a distance d from A. If the angular elevation of B from a point D due east of C at a distance $2d$ from C is γ, then $2 \tan \alpha$ can be given as **[IIT-1994]**

(a) $3 \cot \beta - 2 \cot \gamma$ (b) $3 \cot \gamma - 2 \cot \beta$
(c) $3 \cot \beta - \cot \gamma$ (d) $\cot \beta - 3 \cot \gamma$

Solution

(c)



By $m - n$ theorem at C

$$(d + 2d) \cot \beta = d \cot \gamma - 2d \cot (90° + \alpha)$$
$$3d \cot \beta = d \cot \gamma + 2d \tan \alpha$$
$$\Rightarrow \quad 3 \cot \beta = \cot \gamma + 2 \tan \alpha$$
$$\therefore \quad 2 \tan \alpha = 3 \cot \beta - \cot \gamma.$$

10. A towers stands vertically in a field. The field is in the shape of an equilateral triangle of side 240 metres. The tower subtends angles of 45°, 60° and 60° at the vertices of the triangle. Find the height of tower. **[Roorkee-1998]**

(a) $80\sqrt{5}$ m (b) $70\sqrt{5}$ m
(c) $60\sqrt{5}$ m (d) $50\sqrt{5}$ m

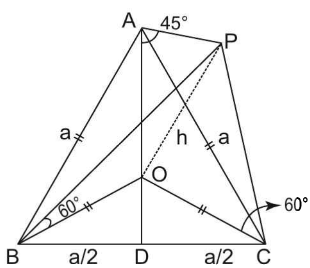
Solution

(a) Let the tower stand at O and its height $OP = h$ which subtends an angle of 45°, 60°, 60° at A, B, C respectively.

$\therefore \quad OA = h \cot 45° = h$

$$OB = h \cot 60° = \frac{h}{\sqrt{3}} = OC$$

$\therefore \quad OB = OC$



Also $AB = AC = BC = a$ say.

If D be the mid-point of BC then OD and AD both are perpendicular to base BC.

$\therefore$ AD is median as well as altitude. In an isosceles or equilateral Δ, both centroid and orthocentre coincide, then

$$OA = \frac{2}{3} AD = \frac{2}{3}\sqrt{a^2 + \frac{a^2}{4}}$$

$$h = OA = \frac{2}{3} \cdot \frac{1}{2}\sqrt{5}a = \frac{240\sqrt{5}}{3}$$

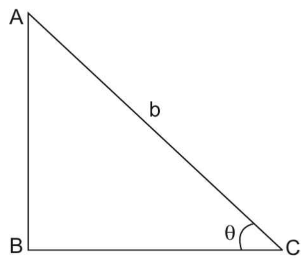
$$h = 80\sqrt{5} \text{ m}$$

11. In ΔABC, $\angle B = 90°$ and $b + a = 4$. The area of the triangle is the maximum when $\angle C$ is **[DCE-1996]**

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$
(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

Solution

(c) From the figure $a = b \cos\theta$



$\therefore \quad b \cos\theta + b = 4$

or $b = \dfrac{4}{1+\cos\theta}$ and similarly $a = \dfrac{4\cos\theta}{1+\cos\theta}$

Required area

$$\Delta = \frac{1}{2}ab\sin\theta = \frac{1}{2} \times \frac{16\cos\theta\sin\theta}{(1+\cos\theta)^2} \quad \frac{d\Delta}{d\theta} = 4$$

$$\left[\frac{2\cos 2\theta.(1+\cos\theta)^2 + \sin 2\theta.2(1+\cos\theta)\sin\theta}{(1+\cos\theta)^4}\right]$$

$$\therefore \quad \frac{d\Delta}{d\theta} = 0$$

$$\Rightarrow \quad \cos 2\theta\,(1+\cos\theta) + \sin 2\theta.\sin\theta = 0$$

$$\Rightarrow \quad \theta = \frac{\pi}{3}.$$

12. If the angles of a triangle are 30° and 45° and the included side is $(\sqrt{3}+1)$ cms, then the area of triangle is ***[DCE-1998; PET-1997]***

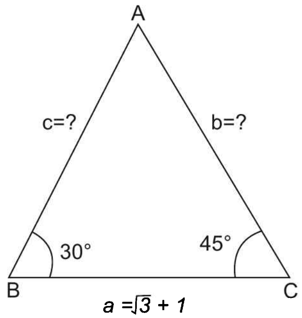
(a) $\frac{1}{\sqrt{3}-1}$ (b) $\sqrt{3}+1$

(c) $\frac{1}{\sqrt{3}+1}$ (d) None of these

Solution

(a) $\angle A = 180° - 30° - 45° = 105°$

$\sin(105°) = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin A$



Area of triangle $ABC = \frac{1}{2}a^2\frac{\sin B \sin C}{\sin(B+C)}$

$= \frac{1}{2}bc\sin A = \frac{1}{2}\sqrt{2}\times 2\times\frac{(\sqrt{3}+1)}{2\sqrt{2}}$

$= \frac{1}{2}\times\frac{(\sqrt{3}+1)^2\times\frac{1}{2}\times\frac{1}{2}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{2}{2(\sqrt{3}-1)} = \frac{1}{(\sqrt{3}-1)}$

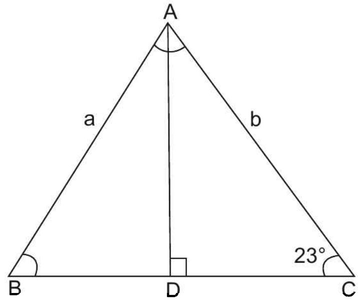
13. In a ΔABC, AD is altitude from A. Given $b > c$, $\angle C = 23°$ and $AD = \frac{abc}{b^2-c^2}$ then $\angle B$ is equal to ***[DCE-2002]***

(a) 53° (b) 113°

(c) 87° (d) None of these

Solution

(b) $\therefore\ AD = \frac{abc}{b^2-c^2} = \frac{k\sin A\times b.k\sin C}{k^2(\sin^2 B - \sin^2 C)}$



$AD = \frac{b\sin C}{\sin(B-C)} \Rightarrow b\sin C$

$\therefore\ \sin(B-C) = 1$

$B - C = 90°, B = 90° + 23 = 123°$

14. In $\Delta\ ABC$, $A = \pi/3$ and $b : c = 2 : 3$. If $\tan\theta = \frac{\sqrt{3}}{5},\ 0 < \theta < \frac{\pi}{2}$ then

[DCE-2002]

(a) $B = 60° + \theta$ (b) $C = 60° + \theta$

(c) $B = 60° - \theta$ (d) $C = 60° - \theta$

Solution

(b) $A = \frac{\pi}{3}$, $\tan\frac{C-B}{2} = \frac{c-b}{c+b}\cot\frac{A}{2}$

$\tan\left(\frac{C-B}{2}\right) = \frac{1}{5}\cot 30° = \frac{\sqrt{3}}{5} = \tan\theta$

$\therefore\ C - B = 2\theta, C + B = 180° - A = 120°$

$\therefore\ 2C = 120° + 2\theta, C = 60° + \theta$

15. In a $\Delta\ ABC$, a, c, A are given and b_1, b_2 are two values, if the third side b such that $b_2 = 2b_1$ then $\sin A$ is equal to

[DCE-2006]

(a) $\sqrt{\frac{9a^2-c^2}{8a^2}}$ (b) $\sqrt{\frac{9a^2-c^2}{8c^2}}$

(c) $\sqrt{\frac{9a^2+c^2}{8a^2}}$ (d) None of these

Solution

(b) We have $\cos A = \frac{b^2+c^2-a^2}{2bc}$

$\Rightarrow\ b^2 - 2bc\cos A + (c^2 - a^2) = 0$

It is given that b_1 and b_2 are roots of this equation.

Therefore, $b_1 + b_2 = 2c \cos A$ and $b_1 b_2 = c^2 - a^2$.

$\Rightarrow$ $3b_1 = 2c \cos A$ and $2b_1^2 = c^2 - a^2$ $[\because b_2 = 2b_1]$

$$\Rightarrow \left(\frac{2c}{3}\cos A\right)^2 = c^2 - a^2$$

$$\Rightarrow 8c^2(1 - \sin^2 A) = 9c^2 - 9a^2$$

$$\Rightarrow \sin A = \sqrt{\frac{9a^2 - c^2}{8c^2}}$$

16. In triangle ABC if $A + C = 2B$, then $\dfrac{a+c}{\sqrt{a^2 - ac + c^2}}$ is equal to ***[UPSEAT-1999]***

(a) $2\cos\dfrac{A-C}{2}$ (b) $\sin\dfrac{A+C}{2}$

(c) $\sin\dfrac{A}{2}$ (d) None of these

Solution

(a) $A + C = 2B \Rightarrow B = 60°$,

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

Since $B = 60° \Rightarrow ac = a^2 + c^2 - b^2$

$\Rightarrow b^2 = a^2 + c^2 - ac$

Therefore

$$\frac{a+c}{\sqrt{a^2 - ac + c^2}} = \frac{a+c}{b} = \frac{\sin A + \sin C}{\sin B}$$

$$= \frac{2\sin\dfrac{A+C}{2}\cos\dfrac{A-C}{2}}{2\sin\dfrac{B}{2}\sin\dfrac{A+C}{2}} = \frac{\cos\dfrac{A-C}{2}}{\sin\dfrac{B}{2}}$$

$$= \frac{\cos\dfrac{A-C}{2}}{\sin 30°} \Rightarrow 2\cos\frac{A-C}{2}.$$

17. In $a\ \Delta ABC$, $2a \sin\left(\dfrac{A-B+C}{2}\right)$ is equal to ***[IIT Screening-2000]***

(a) $a^2 + b^2 - c^2$ (b) $c^2 + a^2 - b^2$

(c) $b^2 - c^2 - a^2$ (d) $c^2 - a^2 - b^2$

Solution

(b) $$2ac\sin\frac{A-B+C}{2}$$

$$= 2ac\sin\frac{\pi - 2B}{2} = 2ac\cos B$$

$$= 2ac\frac{c^2 + a^2 - b^2}{2ca} = c^2 + a^2 - b^2.$$

18. Let D be the middle point of the side BC of a triangle ABC. If the triangle ADC is equilateral, then $a^2 : b^2 : c^2$ is equal to ***[Pb. CET-2004]***

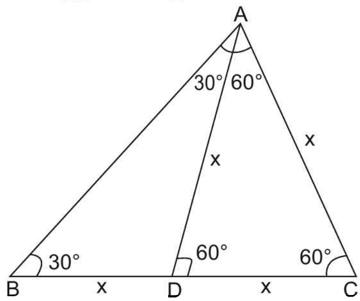
(a) 1 : 4 : 3 (b) 4 : 1 : 3

(c) 4 : 3 : 1 (d) 3 : 4 : 1

Solution

(b) $$\cos 120° = \frac{x^2 + x^2 - AB^2}{2x^2}$$

$$\Rightarrow \frac{2x^2 - AB^2}{2x^2} = \frac{-1}{2} \Rightarrow 4x^2 - 2AB^2 = -2x^2$$



$\Rightarrow 3x^2 = AB^2 \Rightarrow AB = x\sqrt{3}$

$\Rightarrow a^2 : b^2 : c^2 = (2x)^2 : x^2 : (x\sqrt{3})^2$

$= 4x^2 : x^2 : 3x^2 = 4 : 1 : 3$

19. If α, β, γ are angles of a triangle, then $\sin^2\alpha + \sin^2\beta + \sin^2\gamma - 2\cos\alpha\cos\beta\cos\gamma$ is ***[Orissa JEE-2004]***

(a) 2 (b) –1

(c) –2 (d) 0

Solution

(a) We have, $\sin^2\alpha + \sin^2\beta + \sin^2\gamma - 2\cos\alpha\cos\beta\cos\gamma = 3 - [\cos^2\alpha + \cos^2\beta + \cos^2\gamma] - 2\cos\alpha\cos\beta\cos\gamma$

$$= 3 - \left[\frac{1+\cos 2a}{2} + \frac{1+\cos 2\beta}{2} + \frac{1+\cos 2\gamma}{2}\right] - 2\cos\alpha\cos\beta\cos\gamma$$

$$= 3 - \frac{1}{2}[3 + \cos 2\alpha + \cos 2\beta + \cos 2\gamma] - 2\cos\alpha\cos\beta\cos\gamma$$

$$= 3 - \frac{3}{2} - \frac{1}{2}(\cos 2\alpha + \cos 2\beta) - \frac{1}{2}\cos 2\gamma - 2\cos\alpha\cos\beta\cos\gamma$$

$$= \frac{3}{2} - \frac{1}{2}[-2\cos\gamma\cos(\alpha-\beta)] - \frac{1}{2}[2\cos^2\gamma - 1] - 2\cos\alpha\cos\beta\cos\gamma$$

$$= \frac{3}{2} + \cos\gamma\cos(\alpha-\beta) - \cos^2\gamma$$

$$+ \frac{1}{2} - 2\cos\alpha\cos\beta\cos\gamma = 2$$

20. If b, c and $\sin B$ are such that B is an acute angle and $b < c \sin B$, then in this case

[CET (Karnataka)-93]

(a) no triangle is possible
(b) one triangle is possible
(c) two triangles are possible
(d) one right angled triangle is possible

Solution

(a) $\frac{\sin B}{b} = \frac{\sin C}{c} \Rightarrow \sin C = \frac{c}{b}\sin B > 1$

$[\because b < c \sin B]$

This is not possible, so no triangle.

21. In a triangle ABC, $a^3 \cos(B - C) + b^3 \cos(C - A) + c^3 \cos(A - B)$ is equal to

[IIT-70; EAMCET-89; UPSEAT-99; Kerala Engg-2002]

(a) 0 (b) $a + b + c$
(c) $3abc$ (d) abc

Solution

(c) Exp. $= \sum a^3 \cos(B - C)$

$= \sum k^3 \sin^3 A \cos(B-C)$

[by sin formula]

$= k^3 \sum \sin^2 A(\sin A)\cos(B-C)$

$= k^3 \sum \sin^2 A \sin(B+C)\cos(B-C)$

$= \frac{1}{2}k^3 \sum \sin^2 A(\sin 2B + \sin 2C)$

$= k^3 \sum \sin^2 A$ $(\sin B \cos B + \sin C \cos C)$

$= k^3 [\sin A \sin B (\sin A \cos B + \cos A \sin B) + \ldots. + ..]$

$= k^3 [\sin A \sin B \sin C + \sin B \sin C \sin A + \sin C \sin A \sin B]$

$= 3k^3 \sin A \sin B \sin C = 3\, abc$

22. In a triangle ABC, if $(a + b + c)(b + c - a) = \lambda\, bc$, then

[CET (Pb.) 97; CET (Karnataka)-98]

(a) $\lambda > 0$ (b) $\lambda < 0$
(c) $0 < \lambda < 4$ (d) $\lambda > 4$

Solution

(c) $2s(2s - 2a) = \lambda\, bc$

$\Rightarrow \frac{s(s-a)}{bc} = \frac{\lambda}{4}$

$\Rightarrow \cos^2\frac{A}{2} = \frac{\lambda}{4} \Rightarrow \lambda = 4\cos^2\frac{A}{2}$

$\Rightarrow 0 < \lambda < 4.$

23. Internal bisector of $\angle A$ of triangle ABC meets side BC at D a line drawn through D perpendicular to AD intersects AC at E and AB at F. Then

[IIT-2006]

(a) AE is HM of b,c
(b) $AD = \frac{2bc}{b+c}\cos\frac{A}{2}$
(c) $EF = \frac{4bc}{b+c}\sin\frac{A}{2}$
(d) all of above

Solution

(d) $\Delta ABC = \Delta ABD + \Delta ACD$

$\Rightarrow \frac{1}{2}bc\sin A = \frac{1}{2}c(AD)\sin\frac{A}{2} + \frac{1}{2}b(AD)\sin\frac{A}{2}$

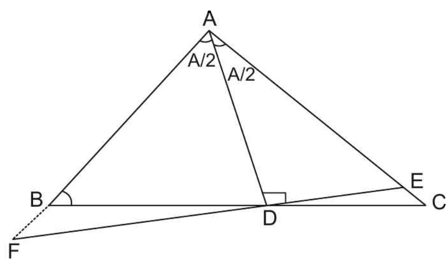
$\Rightarrow 2bc\sin\frac{A}{2}\cos\frac{A}{2} = c(AD)\sin\frac{A}{2} + b(AD)\sin\frac{A}{2}$

$\Rightarrow 2bc\cos\frac{A}{2} = c(AD) + b(AD)$

$\Rightarrow AD = \frac{2bc}{b+c}\cos\frac{A}{2}$ ....(i)

Again $AE = AD\sec\frac{A}{2} = \frac{2bc}{b+c}$ [by (i)]

$\Rightarrow$ AE is HM of b and c.



Also $EF = ED + ED = 2DE = 2AD \tan A/2$

$$= \frac{4bc}{b+c}\cos\frac{A}{2}.\tan\frac{A}{2} = \frac{4bc}{b+c}\sin\frac{A}{2}$$

Hence (a), (b) and (c) all are correct.

24. Let ABC be a triangle such that one of its sides is double the other and let the angles opposite to those sides differ by an angle of $\pi/3$, then the triangle is

[Orissa JEE-2007]

(a) obtuse triangle
(b) isosceles triangle
(c) right angled triangle
(d) equilateral triangle

Solution

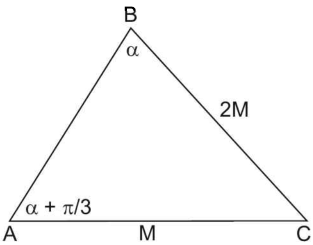
(c) Let one angle α, other be $\alpha + \pi/3$

Let the side opposite to angle α be M so the side opposite to $\alpha + \pi/3$ be $2M$ using sine formula

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad \frac{M}{\sin\alpha} = \frac{2M}{\sin(\alpha+\pi/3)}$$

$\Rightarrow \quad 2 \sin \alpha = \sin (\alpha + \pi/3)$

$\Rightarrow \quad 2\sin\alpha = \frac{1}{2}\sin\alpha + \frac{\sqrt{3}}{2}\cos\alpha$



$\Rightarrow \quad \frac{3}{2}\sin\alpha = \frac{\sqrt{3}}{2}\cos\alpha$

$\Rightarrow \quad \tan\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = 30°.$

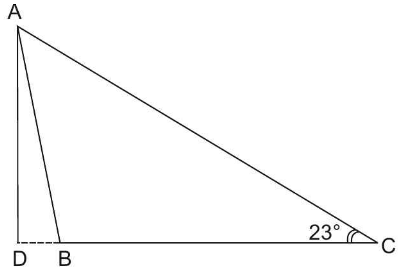
So other angle $\alpha + \pi/3 = 90°$ or triangle is right angle triangle.

25. In a triangle ABC, AD is altitude from A. Given $b > c$, $\angle C = 23°$ and $AD = \frac{abc}{b^2 - c^2}$, then $\angle B =$ ***[IIT-1994; DCE-2002]***

(a) 67°
(b) 44°
(c) 113°
(d) None of these

Solution

$$\sin B = \frac{AD}{AB = c}, AD = c\sin B = \frac{abc}{b^2 - c^2}$$



$$\sin B = \frac{ab}{b^2 - c^2}$$

$$\sin B = \frac{\sin A \sin B}{\sin(B+C)\sin(B-C)}$$

$\sin (B - C) = 1 = \sin 90°$

$B - C = 90°$

$B = C + 90° = 23° + 90° = 113°$

26. The lengths of the sides of a triangle are $\alpha - \beta$, $\alpha + \beta$ and $\sqrt{3\alpha^2 + \beta^2}$, $(\alpha > \beta > 0)$. Its largest angle is

[Roorkee-1999]

(a) $\frac{3\pi}{4}$
(b) $\frac{\pi}{2}$
(c) $\frac{2\pi}{3}$
(d) $\frac{5\pi}{6}$

Solution

(c) Let $a = \alpha - \beta$, $b = \alpha + \beta$, $c = \sqrt{3\alpha^2 + \beta^2}$

$\therefore \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$

$\Rightarrow \quad \cos C =$

$$\frac{\alpha^2 + \beta^2 - 2\alpha\beta + \alpha^2 + \beta^2 + 2\alpha\beta - 3\alpha^2 - \beta^2}{2(\alpha^2 - \beta^2)}$$

$\Rightarrow \quad \cos C = -\frac{(\alpha^2 - \beta^2)}{2(\alpha^2 - \beta^2)} = \cos\left(\frac{2\pi}{3}\right)$

$\Rightarrow \quad \angle C = \frac{2\pi}{3}$, (largest angle).

27. In triangle ABC, with general notions $r_1 + r_2 + r_2 - r$ is equal to

[UPSEAT-2001]

(a) $4R$ (b) Δ^2
(c) Δ (d) $2R$

Solution

(a) Using corresponding formulae

$$r_1 + r_2 + r_3 - r = \frac{\Delta}{s-a} + \frac{\Delta}{s-b} + \frac{\Delta}{s-c} + \frac{\Delta}{s}$$

$$= \Delta\left[\left(\frac{1}{s-a} + \frac{1}{s-b}\right) + \left(\frac{1}{s-c} - \frac{1}{s}\right)\right]$$

$$= \ddot{A}\left[\frac{2s-a-b}{(s-a)(s-b)} + \frac{c}{s(s-c)}\right]$$

$$= \frac{\Delta c}{\Delta^2}[s(s-c) + (s-a)(s-b)]$$

$$= \frac{c}{\Delta}[2s^2 - s(a+b+c) + ab]$$

$$= \frac{c}{\Delta}(ab) = \frac{abc}{\Delta} = 4R$$

28. In a triangle ABC, $\frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3}$ is equal to **[Kurukshetra (CEE)-1997]**

(a) 1 (b) 0
(c) abc (d) $r_1 r_2 r_3$

Solution

(b) Exp. $= \frac{b-c}{\left(\frac{\Delta}{s-a}\right)} + \frac{c-a}{\left(\frac{\Delta}{s-b}\right)} + \frac{a-b}{\left(\frac{\Delta}{s-c}\right)}$

$= \frac{1}{\Delta}$ $[(b-c)(s-a) + (c-a)(s-b) + (a-b)(s-c)]$

$= \frac{1}{\Delta}$ $[s(b-c+c-a+a-b) - a(b-c) - b(c-a) - c(a-b)] = 0$

29. In a triangle ABC, if $\angle B = \pi/3$, $\angle C = \pi/4$ and D divides BC in ratio 1 : 3 internally, then $\frac{\sin\angle BAD}{\sin\angle CAD}$ is equal to

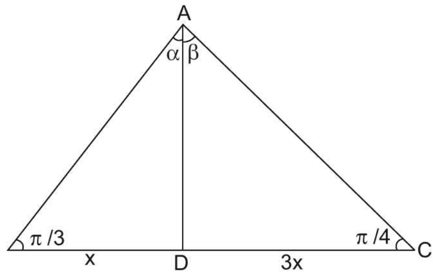
[IIT-95; DCE-99; UPSEAT-2001, 2003]

(a) $1/\sqrt{3}$ (b) $1/\sqrt{6}$
(c) $1/3$ (d) $\sqrt{2/3}$

Solution

(b) Let $\angle BAD = \alpha$, $\angle CAD = \beta$, $BD = x$, $DC = 3x$. Then

$$\frac{x}{\sin\alpha} = \frac{AD}{\sin\pi/3} \quad \text{.........(1)}$$



$$\text{and } \frac{3x}{\sin\beta} = \frac{AD}{\sin\pi/4} \quad \text{.........(2)}$$

$$(1), (2) \Rightarrow \frac{x}{\sin\alpha}\cdot\frac{\sin\beta}{3x} = \frac{AD}{\sin\pi/3}\cdot\frac{\sin\pi/4}{AD}$$

$$\Rightarrow \frac{\sin\beta}{3\sin\alpha} = \frac{1/\sqrt{2}}{\sqrt{3}/2} \Rightarrow \frac{\sin\alpha}{\sin\beta} = \frac{1}{\sqrt{6}}$$

$$\therefore \frac{\sin\angle BAD}{\sin\angle CAD} = \frac{1}{\sqrt{6}}$$

30. In a triangle ABC,

$\left(\cot\frac{A}{2} + \cot\frac{B}{2}\right)\left(a\sin^2\frac{B}{2} + b\sin^2\frac{A}{2}\right)$ is equal to **[Roorkee-88]**

(a) $\cot\frac{C}{2}$ (b) $\cot\frac{C}{2}$
(c) $c \cot C$ (d) $\cot C$

Solution

(b) Exp.

$$= \left(\sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-b)}{(s-c)(s-a)}}\right)$$

$$\left[\frac{(s-c)(s-a)}{c} + \frac{(s-b)(s-c)}{c}\right]$$

$$= \frac{\sqrt{s}.c}{\sqrt{(s-a)(s-b)(s-c)}}\cdot\frac{s-c}{c}.c$$

$$= c\sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = c\cot\frac{C}{2}$$

31. In a triangle, r_1, r_2, r_3 are in *HP*. If its area is 24 cm^2 and its perimeter is 24 cm, then lengths of its sides are

[Roorkee-99]

(a) 3, 9,11 (b) 4, 6, 8
(c) 6, 8, 10 (d) 5, 7, 10

Solution

(c) $r_1, r_2 r_3$ are in $HP \Rightarrow a, b, c$ are in AP.

Now $a + b + c = 24 \Rightarrow s = 12$.

Also $b = 8$, $c = 16 - a$

$\therefore \quad \Delta = 24 \Rightarrow 12(12 - a)(4)(12 - c) = 576$

$\Rightarrow \quad a^2 - 16a + 60 = 0 \Rightarrow a = 10, 6$

$\therefore$ sides are 6, 8, 10.

32. If p_1, p_2, p_3 are altitudes of a triangle *ABC* from the vertices *A*, *B*, *C* and Δ the area of the triangle, then $p_1^{-2} + p_2^{-2} + p_3^{-2}$ is equal to

(a) $\dfrac{a+b+c}{\Delta}$ (b) $\dfrac{a^2+b^2+c^2}{4\Delta^2}$

(c) $\dfrac{a^2+b^2+c^2}{\Delta^2}$ (d) None

Solution

(b) We have $\frac{1}{2}ap_1 = \Delta, \frac{1}{2}bp_2 = \Delta, \frac{1}{2}cp_3 = \Delta$

$$\Rightarrow \quad p_1 = \frac{2\Delta}{a}, p_2 = \frac{2\Delta}{b}, p_3 = \frac{2\Delta}{c}$$

$$\therefore \quad \frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} = \frac{a^2+b^2+c^2}{4\Delta^2}$$

33. In a triangle *ABC*, $a : b : c = 4 : 5 : 6$. The ratio of the radius of the circumcircle to that of the incircle is........

[IIT-96]

Solution

$\frac{16}{7}$, $a = 4k, b = 5k, c = 6k$

$$s = \frac{15}{2}k, s - a = \frac{7}{2}k,$$

$$s - b = \frac{5}{2}k, s - c = \frac{3}{2}k$$

$$S^2 = 15 \times 7 \times 5 \times 3\left(\frac{k}{2}\right)^2$$

$$\therefore \quad S = 15\sqrt{7}\left(\frac{k}{2}\right)^2$$

$$r = \frac{S}{s} = 15\sqrt{7}\left(\frac{k}{2}\right)^2 \div \frac{15}{2}k = \sqrt{7}\frac{k}{2}$$

$$R = \frac{abc}{4S} = \frac{4.5.6k^3}{4.15\sqrt{7}k^2/4} = \frac{8}{\sqrt{7}}k$$

$$\therefore \quad \frac{R}{r} = \frac{8}{\sqrt{7}} \div \frac{\sqrt{7}}{2} = \frac{16}{7}.$$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. The angle of elevation of a tower at a point distant *d* metres from its base is 30°. If the tower is 20 metres high, then the value of *d* is

[PET-82, 88]

(a) $10/\sqrt{3}$ m (b) $20/\sqrt{3}$ m
(c) $20\sqrt{3}$ m (d) 10 m

2. From the top of a light house 60 metres high with its base at the sea level, the angle of depression of a boat is 15°. The distance of the boat from the foot of light house is

[PET-94, 01]

(a) $\left(\dfrac{\sqrt{3}-1}{\sqrt{3}+1}\right)60$ m (b) $\left(\dfrac{\sqrt{3}+1}{\sqrt{3}-1}\right)60$ m

(c) $\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$ m (d) None

3. A tower subtends an angle α at a point *A* in the plane of its base and the angle of depression of the foot of the tower at a point l metres just above A is β. The height of the tower is **[PET-90]**

(a) $l \tan \beta \cot \alpha$ (b) $l \tan \alpha \cot \beta$
(c) $l \tan \alpha \tan \beta$ (d) $l \cot \alpha \cot \beta$

4. The angle of elevation of the sun, when the shadow of the pole is $\sqrt{3}$ times the height of the pole, is **[PET-91, 96]**

(a) 60° (b) 30°
(c) 45° (d) 15°

5. A tower of height b subtends an angle at a point O on the level of the foot of the tower and at a distance a from the foot of the tower. If a pole mounted on the tower also subtends an equal angle at O, the height of the pole is **[PET-93, 2004]**

(a) $b\left(\frac{a^2-b^2}{a^2+b^2}\right)$ (b) $b\left(\frac{a^2+b^2}{a^2-b^2}\right)$
(c) $a\left(\frac{a^2-b^2}{a^2+b^2}\right)$ (d) $a\left(\frac{a^2+b^2}{a^2-b^2}\right)$

6. A person standing on the bank of a river finds that the angle of elevation of the top of a tower on the opposite bank is 45°. Then which of the following statements is correct. **[PET-94]**

(a) Breadth of the river is twice the height of the tower.
(b) Breadth of the river and the height of the tower are the same.
(c) Breadth of the river is half of the height of the tower
(d) None of these

7. A tower is situated on horizontal plane. From two points, the line joining these points passes through the base and which are a and b distance from the base. The angle of elevation of the top are α and $(90° - \alpha)$ and θ is that angle which two points joining the line makes at the top, the height of tower will be **[UPSEAT-99]**

(a) $a + b/a - b$ (b) $a - b/a + b$
(c) $\sqrt{ab}$ (d) $(ab)^{1/3}$

8. The angle of elevation of the top of a pillar at any point A on the ground is 15°. On walking 40 metres towards the piller, the angle becomes 30°. The height of the pillar is **[PET-2001; IIT-95]**

(a) 40 m (b) 20 m
(c) $20\sqrt{3}$ m (d) $\frac{40}{3}\sqrt{3}$ m

9. The upper 3/4th portion of a vertical pole subtends an angle $\tan^{-1}(3/5)$ at a point in the horizontal plane through its foot and at a distance 40 m from the foot. A possible height of the vertical pole is **[AIEEE-03]**

(a) 20 m (b) 40 m
(c) 60 m (d) 80 m

10. The upper part of a tree broken over by the wind makes an angle of 30° with the ground and the distance from the root to the point where the top of the tree touches the ground is 10 m; what was the height of the tree

(a) 8.66 m (b) 15 m
(c) 17.32 m (d) 25.98

11. The angle of elevation of a cliff at a point A on the ground and a point B, 100 m vertically at A are α and β respectively. The height of the cliff is **[EAMCET-86]**

(a) $\frac{100\cot\alpha}{\cot\alpha-\cot\beta}$ (b) $\frac{100\cot\beta}{\cot a-\cot\beta}$
(c) $\frac{100\cot\beta}{\cot\beta-\cot\alpha}$ (d) $\frac{100\cot\beta}{\cot\beta+\cot\alpha}$

12. If a flag-staff of 6 metres high placed on the top of a tower throws a shadow of $2\sqrt{3}$ metres along the ground, then the angle (in degrees) that the sun makes with the ground is **[EAMCET-90]**

(a) 60° (b) 80°
(c) 75° (d) None of these

13. A flag-post 20 m high standing on the top of a house subtends an angle whose tangent is 1/6 at a distance 70 m from the foot of the house. The height of the house is

(a) 30 m (b) 60 m
(c) 50 m (d) None of these

14. A vertical pole (more than 100 m high) consists of two portions, the lower being one-third of the whole. If the upper portion subtends an angle of $\tan^{-1}(1/2)$ at a point in a horizontal plane through the foot of the pole and distance 40 ft from it, then the height of the pole is

(a) 100 ft (b) 120 ft
(c) 150 ft (d) None of these

15. From an aeroplane vertically over a straight horizontally road, the angle of depression of two consecutive mile stones on opposite sides of the aeroplane are observed to be α and β, then the height in miles of aeroplane above the road is

(a) $\dfrac{\tan\alpha\tan\beta}{\cot\alpha+\cot\beta}$ (b) $\dfrac{\tan\alpha+\tan\beta}{\tan\alpha.\tan\beta}$

(c) $\dfrac{\cot\alpha+\cot\beta}{\tan\alpha.\tan\beta}$ (d) $\dfrac{\tan\alpha.\tan\beta}{\tan\alpha+\tan\beta}$

16. The angular depressions of the top and the foot of a chimney as seen from the top of a second chimney, which is 150 m high and standing on the same level as the first are θ and ϕ respectively, then the distance between their tops when tan θ = 4/3 and tan ϕ = 5/2, is ***[IIT-65]***

(a) $150/\sqrt{3}$ m (b) $100\sqrt{3}$ m

(c) 150 m (d) 100 m

17. A man whose eye level is 1.5 metres above the ground observes the angle of elevation of a tower to be 60°. If the distance of the man from the tower be 10 metres, the height of the tower is

(a) $(1.5+10\sqrt{3})$ m (b) $10\sqrt{3}$ m

(c) $(1.5+10/\sqrt{3})$ m (d) None

18. From the bottom and top of a house *h* metre high, the angles of elevation of the top of a tower are α and β. The height of the tower is

(a) $\dfrac{h\sin\beta}{\cos\beta-\sin\alpha}$ (b) $\dfrac{h\cos\beta}{\cos\beta-\cos\alpha}$

(c) $\dfrac{h\tan\beta}{\tan\beta-\tan\alpha}$ (d) $\dfrac{h\cot\beta}{\cot\beta-\cot\alpha}$

19. A spherical balloon of radius *r* subtends an angle α at the eye of an observer. If the angle of elevation of the centre of the balloon be β, the height of the centre of the balloon is ***[IIT-1970]***

(a) *r* cosec (α/2) sin β
(b) *r* cosec α sin (β/2)
(c) *r* sin (α/2) cosec β
(d) *r* sin α cosec (β/2)

20. A house subtends a right angle at the window of an opposite house and the angle of elevation of the window from the bottom of the first house is 60°. If the distance between the two houses be 6 metres, then the height of the first house is

(a) $6\sqrt{3}$ m (b) $8\sqrt{3}$ m

(c) $4\sqrt{3}$ m (d) None of these

21. An aeroplane flying horizontally 1 km above the ground is observed at an elevation of 60° and after 10 seconds the elevation is observed to be 30°. The uniform speed of the aeroplane in km/h is ***[IIT-65]***

(a) 240 (b) $240\sqrt{3}$

(c) $60\sqrt{3}$ (d) None of these

22. *AB* is a vertical tower. The point *A* is on the ground and *C* is the middle point of *AB*. The part *CB* subtend an angle α at a point *P* on the ground. If *AP* = *nAB*, then the correct relation is ***[IIT-80]***

(a) $n(n^2 + 1)$ tan α
(b) $n = (2n^2 - 1)$ tan α
(c) $n^2 = (2n^2 + 1)$ tan α
(d) $n = (2n^2 + 1)$ tan α

HINTS AND EXPLANATIONS

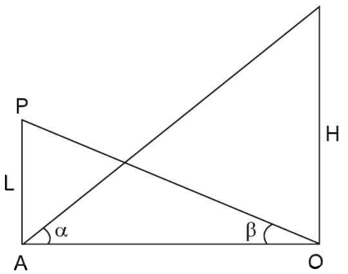
1. (c) 20 cot 30° = $d \Rightarrow 20 \times \dfrac{1}{\sqrt{3}} = d$

$\Rightarrow \quad d = \dfrac{20}{\sqrt{3}}.$

2. (b) Required distance = 60 cot 15° = 60

$\left(\dfrac{\sqrt{3}+1}{\sqrt{3}-1}\right)$

3. (b) From fig., we can deduce $H = l \tan \alpha \cot \beta$.

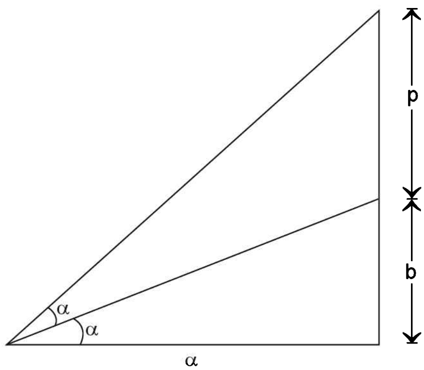


4. (b) $\tan \alpha = \dfrac{h}{\sqrt{3}h} = \dfrac{1}{\sqrt{3}} \Rightarrow \alpha = 30°$

5. (b) $\tan \alpha = \dfrac{b}{a}$, $\tan 2\alpha = \dfrac{2(b/a)}{1-(b/a)^2} = \dfrac{p+b}{a}$

$$\Rightarrow \frac{2ba}{a^2-b^2} = \frac{p+b}{a} \Rightarrow \frac{2ba^2-a^2b+b^3}{a^2-b^2} = p$$

$$\Rightarrow p = \frac{b(a^2+b^2)}{(a^2-b^2)}$$



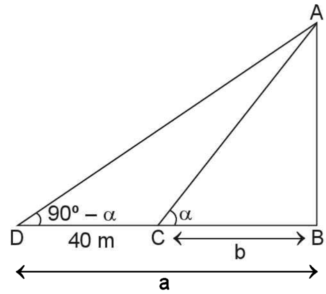
6. (b) $\tan 45° = \dfrac{h}{x} \Rightarrow x = h$ (Breadth of the river and the height of the tower are the same).

7. (c) Let there are two points C and D on horizontal line passing from point B of the base of tower AB.
The distance of these point are b and a from B respectively. $\therefore BD = a$ and $BC = b$

$\because$ line CD, on the top of tower A subtends and angle θ,

$\therefore \angle CAD = \theta$.

According to question on point C and D, the elevation of top are α and $90° - \alpha$.



$\therefore \angle BCA = \alpha$ and $\angle BDA = 90° - \alpha$

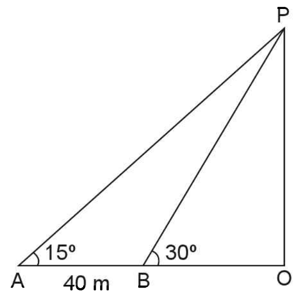
in ΔABC, $AB = BC \tan \alpha = b \tan \alpha$ ....(i)

and in ΔABD, $AB = BD \tan(90° - \alpha) = a \cot \alpha$ ......(ii)

Multiply equation (i) and (ii)

$(AB)^2 = (b \tan \alpha)(a \cot \alpha) = ab \therefore AB = \sqrt{ab}$

8. (b) Let h be the height of pillar

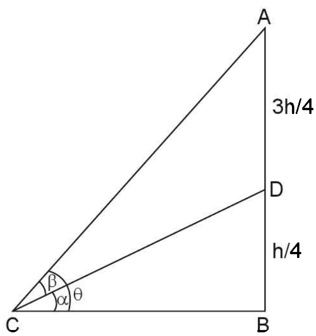


$OB = h \cot 30°$ and $OA = h \cot 15°$

$\Rightarrow AB = OA - OB = h(\cot 15° - \cot 30°)$

$$\Rightarrow h = \frac{40}{\cot 15° - \cot 30°} = 20 \text{ m}.$$

9. (b) $\theta = \alpha + \beta \Rightarrow \beta = \theta - \alpha$; $\tan \beta = \dfrac{\tan\theta - \alpha}{1+\tan\theta\tan\alpha}s$



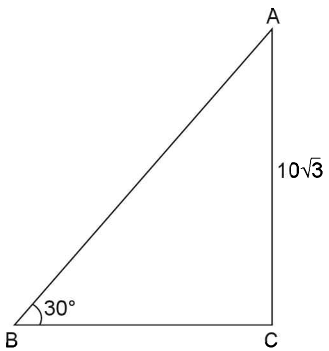
$$\Rightarrow \quad \frac{3}{5} = \frac{\frac{h}{40} - \frac{h}{160}}{1 + \frac{h}{40}\frac{h}{160}}$$

$\Rightarrow \quad h^2 - 200h + 6400 = 0$

$\Rightarrow \quad h = 40$ or 160 m.

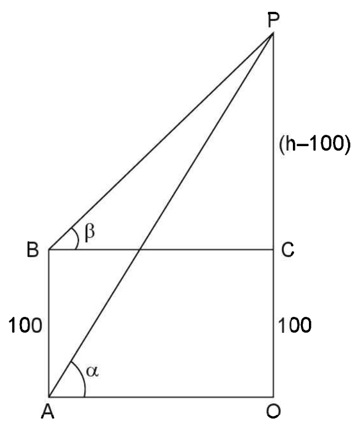
$\therefore \quad$ Possible height = 40 metre

10. (c) Height of tree is



$$AB + AC = \frac{10}{\sqrt{3}} + \frac{20}{\sqrt{3}} = \frac{30}{\sqrt{3}} = 10\sqrt{3} = 17.32 \text{ m.}$$

11. (c) If $OP = h$, then $CP = h - 100$

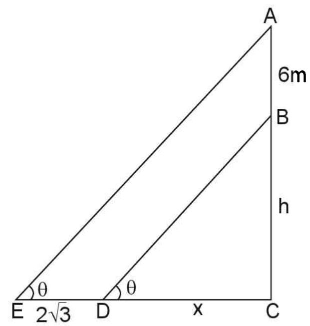


Now equate the values of OA and BC
$h \cot \alpha = (h - 100) \cot \beta$.

$$\therefore \quad h = \frac{100 \cot \beta}{\cot \beta - \cot \alpha}$$

12. (a) Accordingly, $\tan\theta = \frac{h}{x} = \frac{h+6}{x + 2\sqrt{3}} = \frac{6}{2\sqrt{3}}$

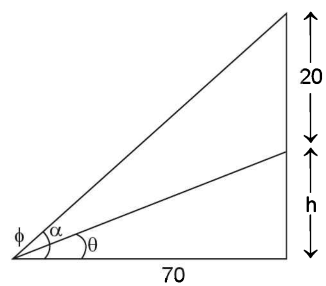
$\Rightarrow \theta = 60°$



[since the triangle AEC and BDC are similar]

13. (c) $\tan\alpha = \tan(\phi - \theta)$, $\tan\alpha$

$$= \frac{1}{6} = \frac{\frac{20+h}{70} - \frac{h}{70}}{1 + \frac{(20+h)h}{(70)^2}}$$



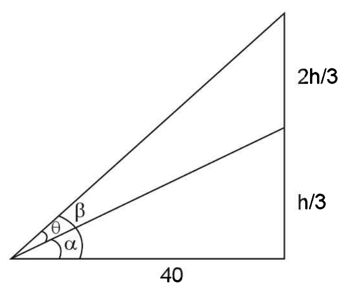
$\Rightarrow \quad (70)^2 + 20h + h^2 = (6)(70)(20)$

$\Rightarrow \quad h^2 + 20h + 70(70 - 120) = 0$

$\Rightarrow \quad h^2 + 20h - (50)(70) = 0$

$$\Rightarrow \quad h = \frac{-20 \pm \sqrt{400 + (4)(50)(70)}}{2} = 50 \text{ m}$$

14. Obviously from fig.



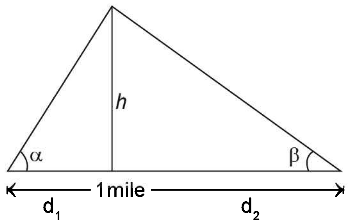
$\tan\alpha = \frac{h}{120}$ ....(i), $\tan\beta = \frac{3h}{120}$ ....(ii)

therefore, $\tan\theta = \tan(\beta - \alpha)$

$$\Rightarrow \quad \frac{1}{2} = \frac{\frac{3h}{120} - \frac{h}{120}}{1 + \frac{3h^2}{14400}} \quad \Rightarrow h = 120.40$$

But h = 40 cannot be taken accordingly to the condition, therefore h = 120 ft.

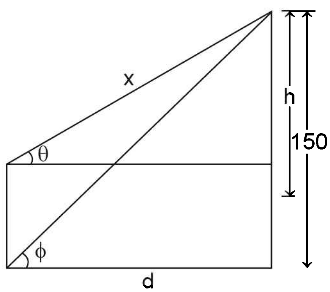
15. (d) $d_1 = h \cot \alpha$ and $d_2 = h \cot \beta$; $d_1 + d_2$ = 1 mile = $h(\cot \alpha + \cot \beta)$



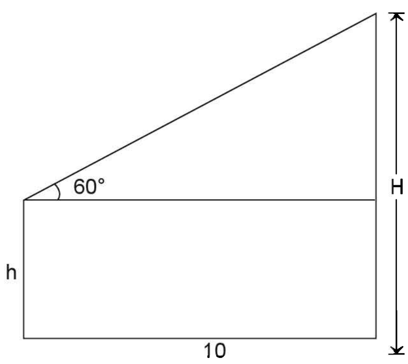
$$\Rightarrow h = \frac{1}{(\cot \alpha + \cot \beta)} = \frac{\tan \alpha \tan \beta}{\tan \alpha + \tan \beta}$$

16. (d) $d = 150 \cot \phi$ = 60 m. Also, $h = 60 \tan \theta$ = 80 m

Hence $x = \sqrt{80^2 + 60^2} = 100$ m

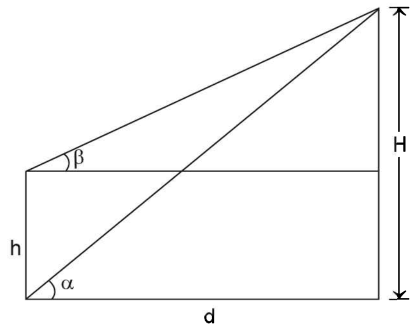


17. (a) $H = (10 \tan 60° + 1.5) = (10\sqrt{3} + 1.5)$ m



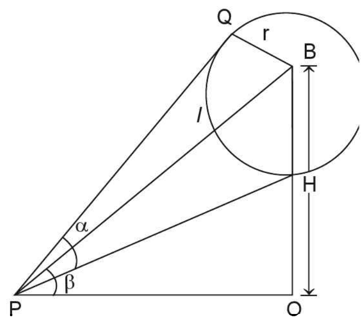
18. (d) $H \cot \alpha = (H - h) \cot \beta$

$$\Rightarrow \quad H = \frac{h \cot \beta}{\cot \beta - \cot \alpha}$$

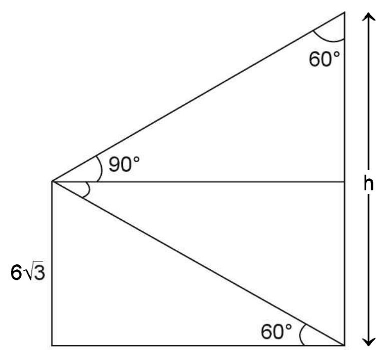


19. (a) In ΔPQB, $l = \frac{r}{\sin \frac{\alpha}{2}}$ and

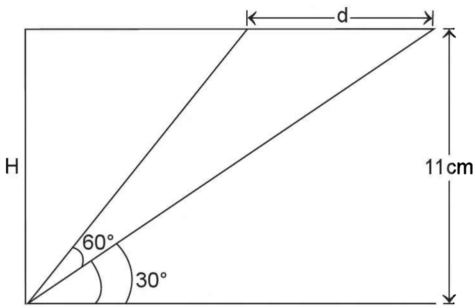
In ΔPOB, $H = l \sin \beta \Rightarrow H = r \sin \beta \operatorname{cosec} \frac{\alpha}{2}$



20. (b) $\frac{12}{h} = \sin 60°$ $\quad \Rightarrow \quad h = 8\sqrt{3}$

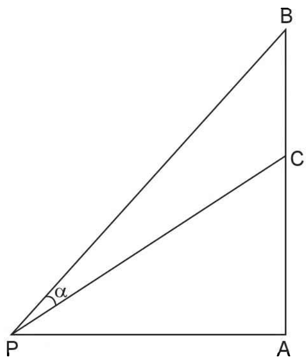


21. (b) $d = H \cot 30° - H \cot 60°$; Time taken = 10 sec.



Speed $= \dfrac{\cot 30° - \cot 60°}{10} \times 60 \times 60 = 240\sqrt{3}.$

22. (d) $\tan\alpha = \dfrac{-\dfrac{AC}{AP} + \dfrac{AB}{AP}}{1 + \dfrac{AC}{AP}\cdot\dfrac{AB}{AP}}$



$\{AP = n(AB) \Rightarrow AP = 2n(AC)\}$

$\tan\alpha = \dfrac{-\dfrac{1}{2n} + \dfrac{1}{n}}{1 + \dfrac{1}{2n^2}} \Rightarrow \dfrac{A}{(2n^2+1)} = \tan\alpha$

$\Rightarrow \quad n = (2n^2 + 1)\tan\alpha$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE) FOR IMPROVING SPEED WITH ACCURACY

1. The angle of elevation of the top of a tower from a point 20 metres away from its base is 45°. The height of the tower is ***[PET-84, 89]***

(a) 10 m (b) 20 m
(c) 40 m (d) $20\sqrt{3}$ m

2. A tower subtends an angle of 30° at a point distant *d* from the foot of the tower and on the same level as the foot of the tower. At a second point *h* vertically above the first, the depression of the foot of the tower is 60°. The height of the tower is ***[PET-93, 2004]***

(a) $h/3$ (b) $h/3d$
(c) $3h$ (d) $3h/d$

3. From the roof of a 15 metre high house the angle of elevation of a point located 15 metres distant to the base of the house is ***[PET-88]***

(a) 45° (b) 30°
(c) 60° (d) 90°

4. The angle of depression of a ship from the top of a tower 30 metre high is 60°, then the distance of ship from the base of tower is

(a) 30 m (b) $30\sqrt{3}$ m
(c) $10\sqrt{3}$ m (d) 10 m

5. 20 metre high flag pole is fixed on a 80 metre high piller, 50 metre away from it, on a point on the base of piller the flag pole makes an angle α, then the value of tan α, is ***[PET-2003]***

(a) 2/11 (b) 2/21
(c) 21/2 (d) 21/4

6. If the angle of elevation of the top of a tower at a distance 500 m from its foot is 30°, then height of the tower is

(a) $1/\sqrt{3}$ (b) $500/\sqrt{3}$
(c) $\sqrt{3}$ (d) 1/500

7. A man from the top of a 100 metres high tower sees a car moving towards the tower at an angle of depression of 30°. After some time, the angle of depression of becomes 60°. The distance (in metres) travelled by the car during the time is

(a) $100\sqrt{3}$ (b) $200\sqrt{3}/3$

(c) $100\sqrt{3}/3$ (d) $200\sqrt{3}$

8. Two men are on the opposite side of a tower. They measure the angles of elevation of the top of the tower 45° and 30° respectively. If the height of the tower is 40 m, find the distance between the men

[KCET-98]

(a) 40 m (b) $40\sqrt{3}$ m

(c) 68.280 m (d) 109.28 m

9. The angular elevation of a tower *CD* at a point *A* due south of it is 60° and at a point *B* due west of *A*, the elevation is 30°. If *AB* = 3 km, the height of the tower is

[PET-98]

(a) $2\sqrt{3}$ km (b) $2\sqrt{6}$ km

(c) $3\sqrt{3}/2$ km (d) $3\sqrt{6}/4$ km

10. On walking 50 m towards the base of a tower, the angle of elevation of the top of the tower changes from 30° to 45°. The height of the tower is

(a) 25 m (b) 50 m

(c) $25(\sqrt{3}-1)$ m (d) $25(\sqrt{3}+1)$ m

11. A flag-staff of 5 m high stands on a building of 25 m high. At an observer at a height of 30 m. The flag-staff and the building subtend equal angles. The distance of the observer from the top of the flag-staff is ***[EAMCET-93]***

(a) $5\sqrt{3}/2$ (b) $5\sqrt{3/2}$

(c) $5\sqrt{2/3}$ (d) None of these

12. *AB* is a vertical pole resting at the end *A* on the level ground. *P* is a point on the level ground such that *AP* = 3 *AB*. If *C* is the mid-point of *AB* and *CB* subtends an angle β at *P*, the value of tan β is

(a) 18/19 (b) 3/19

(c) 1/6 (d) None of these

13. An observer standing on a 300 m high tower observers two boats in the same directions, their angles of depression are 60° and 30° respectively. The distance between two boats is

(a) 173.2 m (b) 346.4 m

(c) 25 m (d) 72 m

14. A tree is broken by wind, its upper part touches the ground at a point 10 metres from the foot of the tree and makes an angle of 45° with the ground. The total length of tree is

(a) 15 m

(b) 20 m

(c) $10(1+\sqrt{2})$ m

(d) $10(1+\sqrt{3}/2)$ m

15. The angle of elevation of the top of a tower from two points *A* and *B* on the ground distant *a* and *b* from the tower are complimentary. If the line *AB* passes through the foot of the tower, the height of the tower is

(a) *ab* (b) *a/b*

(c) $\sqrt{ab}$ (d) $\sqrt{a/b}$

16. The angle of elevation of the top of a tower from the top of a house is 60° and angle of depression of its base is 30°. If the horizontal distance between the house and the tower be 12 m, then the height of the tower is

(a) $48\sqrt{3}$ m (b) $16\sqrt{3}$ m

(c) $24\sqrt{3}$ m (d) $16/\sqrt{3}$ m

17. Some portion of a 20 metres long tree is broken by the wind and the top struck the ground at an angle of 30°. The height of the point where the tree is broken is

(a) 10 m

(b) $(2-\sqrt{3}-3)20$ m

(c) 20/3 m

(d) None of these

18. If the angles of elevation of two towers from the middle point of the line joining their feet be 60° and 30° respectively, then the ratio of their heights is

(a) 2 : 1 (b) $1:\sqrt{2}$

(c) 3 : 1 (d) $1:\sqrt{3}$

19. A kite is flying at an inclination of 60° with the horizontal. If the length of the thread is 120 m, then the height of the kite is

(a) $60\sqrt{3}$ m (b) 60 m

(c) $60/\sqrt{3}$ m (d) 120 m

20. The angle of elevation of the top of a tower from the top and bottom of a house are 30° and 60° respectively. If the height of the house be 25 m, then the height of the tower is

(a) 25 m (b) 50 m

(c) 37.5 m (d) 75 m

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet
2. The test is of 10 minutes.
3. The worksheet consists of 10 questions. The maximum marks are 30.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. A vertical pole with heights more than 100 m consists of two parts, the lower being one-third of the whole. At a point on a horizontal plane through the foot and 40 m from it, the upper part subtends an angle whose tangent is 1/2. The height of the pole is *[NDA-2006]*

(a) 110 m (b) 200 m
(c) 120 m (d) 150 m

2. The length of the shadow of a tree is $10\sqrt{3}$ m, when the angle of elevation of the sun is 60°. What is the length of the shadow of the tree when the angle of elevation of the sun is 30°? *[NDA-2005]*

(a) $30\sqrt{3}$ m (b) $10\sqrt{3}$ m
(c) $5\sqrt{3}$ m (d) $4\sqrt{3}$ m

3. From the top of a tower 60 metres high, the angles of depression of two objects which are on the horizontal plane and in a line with the foot of the tower are α and β with $\beta > \alpha$. What is the distance between the two objects in metres? *[NDA-2004]*

(a) $60 \sin(\beta - \alpha) \operatorname{cosec} \alpha \operatorname{cosec} \beta$
(b) $60 \cos(\beta - \alpha) \sec \alpha \sec \beta$
(c) $60 (\cot \alpha + \cot \beta)$
(d) $60 (\tan \beta - \tan \alpha)$

4. An observer on the top of a tree, finds the angle of depression of a car moving towards the tree to be 30°. After 10 minutes this angle becomes 60°. After how much more time, the car will reach the tree? *[NDA-2004]*

(a) 20 minutes (b) 15 minutes
(c) 5 minutes (d) 10 minutes

5. A person at the top of a hill observes that the angles of depression of two consecutive kilometer stones on a road leading to the foot of the hill are 30° and 60°. The height of the hill is *[NDA-2003]*

(a) $\frac{\sqrt{3}}{2}$km (b) $\frac{\sqrt{5}}{2}$km
(c) $\frac{\sqrt{6}}{2}$km (d) $\frac{\sqrt{7}}{2}$km

6. The angle of elevation of the top of a temple situated on east side of a person is 60°. After walking 240 m on north side it become 30°. The height of the temple is *[MPPET-2003]*

(a) $60\sqrt{6}$ (b) 60 m
(c) $50\sqrt{3}$ (d) $30\sqrt{6}$

7. Angles of elevation of the top of the towers as observed from the bottom and top of a building of height 60 m are 60° and 45° respectively. The distance of the base of the tower from the base of the building is *[NDA-2003]*

(a) $30(\sqrt{3}-1)$m (b) $30(\sqrt{3}+3)$m
(c) $30(3-\sqrt{3})$m (d) $30(\sqrt{3}+1)$m

8. If from the top of a light-house, 100 metres high, the angle of depression of a boat is $\tan^{-1}\left(\frac{5}{12}\right)$, then the distance in metres, between the boat and the light-house, is equal to *[NDA-2002]*

(a) 125/3 m (b) 120 m
(c) 240 m (d) 206 m

9. An observer measures angles of elevation of two towers of equal heights from a point between the towers. If the angles of elevation are 60° and 30° and distance of the nearer tower is 100 m, then the height of each tower and the distance between the towers, respectively are *[NDA-2002]*

(a) $\frac{100}{\sqrt{3}}$ m and 400 m

(b) $\frac{100}{\sqrt{3}}$ m and 300 m

(c) $100\sqrt{3}$ m and 400 m

(d) $100\sqrt{3}$ m and 300 m

10. A flag-staff 6 m high is placed on the top of a tower. The flag-staff casts a shadow, which is $2\sqrt{3}$ m long when measured along the ground. The angle, in degrees, that the sun-rays make with the ground is

[NDA-2001]

(a) 60° (b) 45°
(c) 30° (d) 15°

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ
6. ⓐ ⓑ ⓒ ⓓ
7. ⓐ ⓑ ⓒ ⓓ
8. ⓐ ⓑ ⓒ ⓓ
9. ⓐ ⓑ ⓒ ⓓ
10. ⓐ ⓑ ⓒ ⓓ

PART F

Test Your Skills

LECTURE 1

Test Your Skills

ASSERTION/REASONING 1

ASSERTION AND REASONING TYPE QUESTIONS

Each questions has 4 choices (a), (b), (c) and (d), out of which only one is correct in which Statement-1 is Assertion and Statement-2 is Reason.

(a) Statement-1 is True, Statement-2 is True and Statement-2 is correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True and Statement-2 is not correct explanation for Statement-1
(c) Statement-1 is True and Statement-2 is False
(d) Statement-1 is False and Statement-2 is True

1. Statement-1
$\tan 5\theta - \tan 3\theta - \tan 2\theta = \tan 5\theta \tan 3\theta \tan 2\theta$
Statement-2: $x = y + z$
$\Rightarrow \quad \tan x - \tan y - \tan z = \tan x \tan y \tan z$

2. Statement-1: The maximum value of $\sin\theta + \cos\theta$ is 2
Statement-2: The maximum value of $\sin\theta$ is 1 and that of $\cos\theta$ is also 1.

3. Statement-1: The maximum value of $\prod_{i=1}^{n} \cos\alpha_i$ under the restriction $0 \le \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \le \frac{\pi}{2}$ is $\frac{1}{n^{n/2}}$
Statement-2: $\prod_{i=1}^{n} \cot\alpha_i = 1$,

4. Statement-1: If $A + B + C = \pi$, then the minimum value of $\Pi \tan A$ is $3\sqrt{3}$.
Statement-2: $AM \ge GM$

5. Statement-1: If a, b, $c \in$ R and not all equal, then $\sec\theta = \frac{(bc + ca + ab)}{(a^2 + b^2 + c^2)}$,
Statement-2: $\sec\theta \le -1$ and $\sec\theta \ge 1$

6. Statement-1: If A is obtuse angle in ΔABC, then $\tan B \tan C > 1$.
Statement-2: In ΔABC,
$$\tan A = \frac{\tan B + \tan C}{\tan B \tan C - 1}$$

7. Statement-1: $\sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{8\pi}{7}\right) = -\frac{1}{2}$
Statement-2: $\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$ is complex 7th root of unity.

8. Statement-1:
$\cos^3\alpha + \cos^3\left(\alpha + \frac{2\pi}{3}\right) + \cos^3\left(\alpha + \frac{4\pi}{3}\right)$
$= 3\cos\alpha \cos\left(\alpha + \frac{2\pi}{2}\right)\cos\left(\alpha + \frac{4\pi}{3}\right)$
Statement-2: If $a + b + c = 0 \Leftrightarrow a^3 + b^3 + c^3 = 3abc$

9. Statement-1: If $xy + yz + zx = 1$, then

$$\sum\frac{x}{(1+x^2)}=\frac{2}{\sqrt{\Pi(1+x^2)}}$$

Statement-2: In a ΔABC

$\sin 2A + \sin 2B - \sin 2C = 4\cos A\cos B\sin C$

10. Statement-1: If α and β are two distinct solutions of the equations $a\cos x + b\sin x = c$, then $\tan\left(\frac{\alpha+\beta}{2}\right)$ is independent of c.

Statement-2: Solution of $a\cos x + b\sin x = c$ is possible, if $-\sqrt{(a^2+b^2)}\le c\le\sqrt{(a^2+b^2)}$

11. Statement-1:

The number of real solutions of the equation $\sin x = 2^x + 2^{-x}$ is zero.

Statement-2: Since $|\sin x| \le 1$

12. Statement-1: If $\tan\left(\frac{\pi}{2}\sin\theta\right)=\cot\left(\frac{\pi}{2}\cos\theta\right)$, then $\sin\theta + \cos\theta = \pm\sqrt{2}$

Statement-2: $-\sqrt{2}\le\sin\theta+\cos\theta\le\sqrt{2}$

13. Statement-1: In $(0, \pi)$, the number of solutions of the equation $\tan\theta + \tan 2\theta + \tan 3\theta = \tan\theta \tan 2\theta \tan 3\theta$ is 2

Statement-2: $\tan 6\theta$ is not defined at $\theta = (2n+1)\frac{\pi}{12}, n\in I.$

14. Statement-1: The equation $\sin(\cos x) = \cos(\sin x)$ does not possess real roots.

Statement-2: If $\sin x > 0$, then $2n\pi < x < (2n + 1), n \in I.$

15. Statement-1: If $\sin^2 A = \sin^2 B$ and $\cos^2 A = \cos^2 B$, then $A = n\pi + B, n \in I.$

Statement-2: If $\sin A = \sin B$ and $\cos A = \cos B$, then $A = n\pi + B, n \in I.$

16. Statement-1: $\frac{\sin 2\theta}{1+\cos 2\theta}$ is equal to $\tan\theta$.

Statement-2: $\cos 2\theta = \cos^2\theta - 1$ and $\sin 2\theta = 2\sin\theta \,.\, \cos\theta$

17. Statement-1: $\frac{\tan 36°+\tan 9°}{1-\tan 36°\tan 9°}=1$

Statement-2: $\tan 36°=\sqrt{5-2\sqrt{5}}$

18. Statement-1: $\sin 1^c < \sin 1°$

Statement-2: $1^c = 57°$ (nearly)

19. Statement-1: The maximum value of $7\sin 2\theta - 24\cos 2\theta$ is 25.

Statement-2:

$-\sqrt{a^2+b^2}\le(a\sin\theta\pm b\cos\theta)\le\sqrt{a^2+b^2}$

20. Statement-1: If $\sin\theta + \operatorname{cosec}\theta = 2$. Then $\sin^4\theta + \operatorname{cosec}^4\theta$ is equal to 2.

Statement-2: $X+\frac{1}{X}=2.$ when $X>0.$

21. Statement-1: S = If $\sin(e^x) = 2^x + 2^{-x}$. Then there is no solution

Statement-2: $\sin\theta$ is increasing in Ist and IVth quadrant.

22. Statement-1: If $|\tan x| \le 1$ $x \in [-\pi, \pi]$ then the solution set for x is

$$\left[-x,-\frac{3\pi}{4}\right]\cup\left[-\frac{\pi}{4},\frac{\pi}{4}\right]\cup\left[\frac{3\pi}{4},\pi\right].$$

Statement-2: If $|\tan x| = 1$, then $x=\frac{\pi}{4},\frac{3\pi}{4}$ when $x \in [-\pi, \pi]$

23. Statement-1: In a triangle ABC, $\angle B = 60°$ such that $\sin(2A+B)\frac{1}{2}$ so $\angle A$ and $\angle C$ are 45° and 75° respectively.

Statement-2: If $\sin\theta + \sin\alpha$ then general solution forθ is $n\pi + (-1)^n\alpha$ wher $n \in I.$

24. Statement-1: If $\sec\theta + \tan\theta = p$ then $\tan\theta$ is equal to $\frac{p^2-1}{2p}$

Statement-2: $\sec\theta+\tan\theta=\frac{1}{(\sec\theta-\tan\theta)}$

25. Statement-1: If $\cos A\cos B\cos C=\frac{\sqrt{3}-1}{8}$ and $\sin A\sin B\sin C=\frac{3+\sqrt{3}}{8}$. Then $\tan A + \tan B + \tan C$ is equal to $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-1}$

Statement-2: In a ΔABC, $\tan A + \tan B + \tan C = \tan A\tan B\tan C$

26. Statement-1: $\sin 44° < \cos 44°$

Statement-2: $\cos x > \sin x$ when $x\in\left(0,\frac{\pi}{4}\right)$

27. Statement-1: If $(1 + \tan A) \,.\, (1 + \tan B) = 2$

Statement-2: $\angle A + \angle B = 45°$

28. Statement-1: If tanθ + tan 4θ + tan 7θ = tanθ tan 4θ tan 7θ then θ is equal $\frac{n\pi}{12}$.

Statement-2: In a Δ*ABC*, tan *A* + tan *B* + tan *C* = tan *A* tan *B* tan *C*.

29. Statement-1: The general solution for θ when $\cos\theta = -\frac{3}{2}$ so there is no solution.

Statement-2: $-1 \le \cos\theta \le 1$.

30. Statement-1: The value of θ satisfying sin 7θ = sinθ is $\frac{n\pi}{2}$.

Statement-2: In [0, π], $\cos x = \frac{\sqrt{3}}{2}$ has exactly one solution.

31. Statement-1: The general solution of $\cos^2 = \frac{1}{2}$ is $\theta = n\pi \pm \frac{\pi}{4}$.

Statement-2: $\cos^2\theta = \cos^2\alpha$ then θ is nπ + α

32. Statement-1: If tanθ + tan 2θ + tanθ tan 2θ = 1 then general solution of θ is $\frac{n\pi}{2} + \frac{\pi}{12}$.

Statement-2: tanθ = 0, then θ is nπ.

33. Statement-1: In trigonometric equation, avoid both side squaring

Statement-2: By squaring some extraneous roots appear.

34. Statement-1: Principle value of $\tan\theta = -\sqrt{3}$ is $\frac{2\pi}{3}$.

Statement-2: tanθ is negative in IInd and IVth quadrant.

35. Statement-1: If 2 cosθ − 1 = 0 then general solution of θ is $2n\pi \pm \frac{\pi}{6}$.

Statement-2: cosθ = 0, then θ = 2*n*π ± α, where α is a principle angle and *n* is an integer.

ASSERTION/REASONING: SOLUTIONS 1

1. (a) ∵ 5θ = 3θ + 2 θ

$$\Rightarrow \tan 5\theta = \tan(3\theta + 2\theta) = \frac{\tan 3\theta + \tan 2\theta}{1 - \tan 3\theta \tan 2\theta}$$

⇒ tan 5θ − tan 5θ tan 3θ tan 2θ = tan 3θ + tan 2θ

⇒ tan 5θ − tan 3θ − tan 2θ = tan 5θ tan 3θ tan 2θ

2. (d) ∵ $-\sqrt{2} \le \sin\theta + \cos\theta \le \sqrt{2}$

∴ Maximum value of sinθ + cosθ is $\sqrt{2}$

But maximum value of sinθ is 1 and that of cosθ is also 1 which is always true.

3. (a) From Reason (R)

$$\prod_{i=1}^{n} \cos\alpha_i = \prod_{i=1}^{n} \sin\alpha_i$$

$$\Rightarrow \prod_{i=1}^{n} \cos^2\alpha_i = \prod_{i=1}^{n} \left(\frac{\sin 2\alpha_i}{2}\right) \quad \ldots\ldots\ldots\ldots \text{(i)}$$

Now,

$$\because \quad 0 \le \alpha_i \le \frac{\pi}{2}$$

$$\therefore \quad 0 \le 2\alpha_i \le \pi$$

then maximum value of sin $2\alpha_i$ is 1 for all i

$$\therefore \quad \prod_{i=1}^{n} \cos^2\alpha_i \le \frac{1}{2^n}$$

$$\therefore \quad \prod_{i=1}^{n} \cos\alpha_i \le \frac{1}{2^{n/2}}$$

4. (a) ∵ *A* + *B* + *C* = π

∴ tan *A* + tan *B* + tan *C* = tan *A* tan *B* tan *C*

⇒ *AM* ≥ *GM*

$$\therefore \quad \frac{\tan A + \tan B + \tan C}{3} \ge \tan A \tan B \tan C^{1/3}$$

$$\Rightarrow \quad \frac{\tan A \tan B \tan C}{3} \ge (\tan A \tan B \tan C)^{1/3}$$

⇒ $(\tan A \tan B \tan C)^{2/3} \ge 3$

∴ $\tan A \tan B \tan C \ge 3\sqrt{3}$

Hence, H $\tan A \ge 3\sqrt{3}$

5. (d) $\because\ a^2 + b^2 + c^2 - ab - bc - ca$

$$= \frac{1}{2}\{(a-b)^2 + (b-c)^2 + (c-a^2)\} > 0$$

$\Rightarrow\quad a^2 + b^2 + c^2 > ab + bc + ca$

or $\quad \dfrac{ab+bc+ca}{a^2+b^2+c^2} < 1$

$\Rightarrow\quad \sec\theta < 1$, which is false.

6. (d) $\because\ A + B + C = 180°$

$\Rightarrow\quad A = 180° - (B + C)$

$\therefore\quad \tan A = \tan(180° - (B + C))$

$$= -\tan(B + C) = -\left\{\frac{\tan B + \tan C}{1 - \tan B \tan C}\right\}$$

$$= \left\{\frac{\tan B + \tan C}{\tan B \tan C - 1}\right\}$$

Now, $\because$ A is obtuse

$\therefore\quad \tan A < 0$,

then $\tan B + \tan C > 0$

$\therefore\quad \tan B \tan C - 1 < 0$

$\Rightarrow\quad \tan B \tan C < 1$

7. (d) Let $S = \sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{8\pi}{7}\right)$

and $C = \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{8\pi}{7}\right)$

$\therefore\quad C + iS = \alpha + \alpha^2 + \alpha^4 \quad \ldots\ldots\ldots\ \text{(i)}$

Where, $\alpha = \cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)$ is complex 7th root of unity.

Then, $C - iS = \bar{\alpha} + \bar{\alpha}^2 + \bar{\alpha}^4 = \alpha^6 + \alpha^5 + \alpha^3 \quad \ldots\ldots\ldots\ \text{(ii)}$

By adding equations (i) and (ii), then

$2C = \alpha + \alpha^2 + \alpha^4 + \alpha^6 + \alpha^5 + \alpha^3 = -1$

($\because$ sum of 7, 7th roots of unity is zero)

$\therefore\quad C = -\dfrac{1}{2}$

Also, multiplying equations (i) and (ii), then

$C^2 + S^2 = 2$ ($\because\ \alpha^7 = 1$ and sum of 7, 7th roots of unity)

$\Rightarrow\quad S^2 = 2 - \left(\dfrac{1}{2}\right)^2 = \dfrac{7}{4}$

$\therefore\quad S = \dfrac{\sqrt{7}}{2}$

8. (a) $\because\ \cos\alpha + \cos\left(\alpha + \frac{2\pi}{3}\right) + \cos\left(\alpha + \frac{4\pi}{3}\right)$

$$= \cos\alpha + 2\cos(\alpha + \pi)\cos\frac{\pi}{3}$$

$$= \cos\alpha + (-2\cos\alpha)\left(\frac{1}{2}\right) = 0$$

$\therefore\quad \cos^3\alpha + \cos^3\left(\alpha + \frac{2\pi}{3}\right) + \cos^3\left(\alpha + \frac{4\pi}{3}\right)$

$$= 3\cos\alpha\cos\left(\alpha + \frac{2\pi}{3}\right)\cos\left(\alpha + \frac{4\pi}{3}\right)$$

9. (b) Let $x = \cot A$, $y = \cot B$, $z = \cot C$

$\therefore\quad \cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

$\therefore\quad A + B + C = 180°$

$\therefore\quad \sum\dfrac{x}{(1+x^2)} = \sum\dfrac{\cot A}{(1+\cot^2 A)}$

$$= \frac{1}{2}\sum\frac{2\tan A}{(1+\tan^2 A)}$$

$$= \frac{1}{2}\sum\sin 2A$$

$$= \frac{1}{2}(\sin 2A + \sin 2B + \sin 2C)$$

$$= \frac{1}{2}(4\sin A\sin B\sin C)$$

$$= 2\sin A\sin B\sin C$$

$$= \frac{2}{\sqrt{(1+\cot^2 A)(1+\cot^2 B)(1+\cot^2 C)}}$$

$$= \frac{2}{\sqrt{(1+x^2)(1+y^2)(1+z^2)}}$$

$$= \frac{2}{\sqrt{\Pi(1+x^2)}}$$

and $\sin A + \sin 2B - \sin 2C$

$= 2\sin(A + B)\cos(A - B) - 2\sin C\cos C$

$= 2\sin C(\cos(A - B) - \cos C)$

$= 2\sin C(\cos(A - B) + \cos(A + B))$

$= 2\sin C(2\cos A\cos B) = 4\cos A\cos B\sin C$

10. (b) $\because\ a\cos x + b\sin x = c$

$\Rightarrow\quad a\left(\dfrac{1-\tan^2 x/2}{1+\tan^2 x/2}\right) + b\left(\dfrac{2\tan x/2}{1+\tan^2 x/2}\right) = c$

$\Rightarrow \quad (a+b)\tan^2\left(\frac{x}{2}\right)-$

$2b\tan\left(\frac{x}{2}\right)+(c-a)=0$

$\therefore \quad \tan\left(\frac{\alpha}{2}\right)+\tan\left(\frac{\beta}{2}\right)=\frac{2b}{(a+c)}$

and tan (α/2) tan (β/2) $=\frac{c-a}{a+c}$

$\Rightarrow \quad \tan\left(\frac{\alpha+\beta}{2}\right)=\frac{\tan\left(\frac{\alpha}{2}\right)+\tan\left(\frac{\beta}{2}\right)}{1-\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)}$

$\Rightarrow \quad \frac{\frac{2b}{a+c}}{1-\left(\frac{c-a}{a+c}\right)}=\frac{b}{a}.$

= Independent of c

$\Rightarrow \quad -\sqrt{(a^2+b^2)}\le a\cos x+b\sin x\le \quad (a^2+b^2)$

$b\sin x\le\sqrt{(a^2+b^2)}$

$\therefore \quad -\sqrt{(a^2+b^2)}\le c\le\sqrt{(a^2+b^2)}$

11. (a) $\because\ 2^x + 2^{-x} \ge 2$

$\Rightarrow$ $\sin x \ge 2$ (impossible)

$\because$ $|\sin x| \le 1$

12. (d) $\because \tan\left(\frac{\pi}{2}\sin\theta\right)=\cot\left(\frac{\pi}{2}\cos\theta\right)$

$=\tan\left(\frac{\pi}{2}-\frac{\pi}{2}\cos\theta\right)$

$\because \quad \frac{\pi}{2}\sin\theta$

$=n\pi+\frac{\pi}{2}-\frac{\pi}{2}\cos\theta$

$\Rightarrow$ $\sin\theta + \cos\theta$

$= 2n + 1, n \in I$

$\because \quad -\sqrt{2}\le\sin\theta+\cos\theta\le\sqrt{2}$

$\therefore \quad n = 0; -1$

Then, $\sin\theta + \cos\theta = 1, -1$

13. (b) The given equation is equivalent to tan (θ + 2θ + 3θ) = 0

or $\tan 6\theta = 0$

Then $6\theta = n\pi$

$\therefore \quad \theta=\frac{n\pi}{6}, n\in I$

In (0, π) we have $\theta=\frac{\pi}{6},\frac{\pi}{3},\frac{\pi}{2},\frac{2\pi}{3},\frac{5\pi}{6}$

However, tanθ and tan 3θ are not defined at $\frac{\pi}{2},\frac{\pi}{6},\frac{5\pi}{6}$

$\therefore \quad \frac{\pi}{3},\frac{2\pi}{3}$ are the only solutions.

14. (b) $\because$ sin (cos x) = cos (sin$x$)

$\Rightarrow$ cos (sin x) $=\cos\left(\frac{\pi}{2}-\cos x\right)$

$\Rightarrow$ $\sin x = 2n\pi \pm \left(\frac{\pi}{2}-\cos x\right), n\in I$

$\Rightarrow$ $\sin x \pm \cos x = \left(2n\pm\frac{1}{2}\right)\pi$

By squaring $1 \pm \sin 2x = \left(2n\pm\frac{1}{2}\right)^2\pi^2$

$\Rightarrow$ $|\sin 2x| = \left(2n\pm\frac{1}{2}\right)^2\pi^2-1$

But, $\left(2n\pm\frac{1}{2}\right)^2\pi^2>2$ for all $n \in I$

$\therefore$ $|\sin 2x| > 1$ which is inadmissive.

Hence, the given equation does not possess real roots.

and $\because$ $\sin x > 0$ (x lies in I and II equadrant)

$\because$ $2n\pi < x < (2n + 1)\pi, n \in I$

15. (c) $\because$ $\sin^2 A = \sin^2 B$ and $\cos^2 A = \cos^2 B$

$\therefore$ $\cos 2A = \cos 2B$

$\Rightarrow$ $2A = 2n\pi \pm B, n \in I$

or $A = n\pi \pm B$

or $A = n\pi + B, n \in I$

($\because$ Both sides square given)

Now, $\sin A = \sin B$

$\Rightarrow$ $A = n\pi + (-1)^n B, n \in I$

In n is even, then $A = n\pi + B$

and $\cos A = \cos B$

$\Rightarrow$ $A = 2n\pi \pm B, n \in I$

Hence, Assertion is true but Reason is false.

16. (a) Statement-1 and Statement-2 both are correct.

$\frac{\sin 2\theta}{1+\cos 2\theta}=\frac{2\sin\theta\cos\theta}{1+2\cos^2\theta-1}=\tan\theta$

and Statement-1 follow from Statement-2.

17. (b) $\dfrac{\tan 36° + \tan 9°}{1 - \tan 36° \tan 9°} = 1$

$\tan (36° - 9°) = \tan 45° = 1$

and $\tan 36° = \sqrt{5 - 2\sqrt{5}}$

It means both Statement-1 and Statement-2 are true but there is no need to give value of tan 36°

18. (d) $1^c = 57° \, 17' 44.8''$

then, we can write $1^c = 57°$, (nearly)

so, $\sin 1^c > \sin 1°$

because, $\sin 57° > \sin 1°$

[$\because$ $\sin x$ is increasing in 1st quadrant]

It means Statement-1 is not true.

19. (a) the formula for finding maximum and minimum value of the expression

$-\sqrt{a^2 + b^2} \le (a \sin\theta \pm b\cos\theta) \le \sqrt{a^2 + b^2}$

This formulae is applicable when both the terms having same angle and both the terms are in linear power.

so,

$-\sqrt{7^2 + 24^2} \le (7 \sin 2\theta - 24 \cos 2\theta) \le \sqrt{7^2 + 24^2}$

$-25 \le (7 \sin 2\theta - 24 \cos 2\theta) \le 25$

Maximum value is 25.

It means Statement-1 and Statement-2 both are true and Statement-1 follow from Statement-2.

20. (c) If $\sin\theta + \text{cosec}\theta = 2$

So, $\sin\theta + \dfrac{1}{\sin\theta} = 2$

$(\sin^2\theta - 2 \sin\theta + 1) = 0$

$\because$ $(\sin\theta - 1)^2 = 0$ and $\sin\theta = 1$

$\text{cosec}\theta = 1$

if x is any real positive number, then $A.M. \ge G.M.$

$\dfrac{\left(X - \dfrac{1}{X}\right)}{2} \ge \sqrt{X . \dfrac{1}{X}} \quad \left(X + \dfrac{1}{X}\right) \ge 2.$

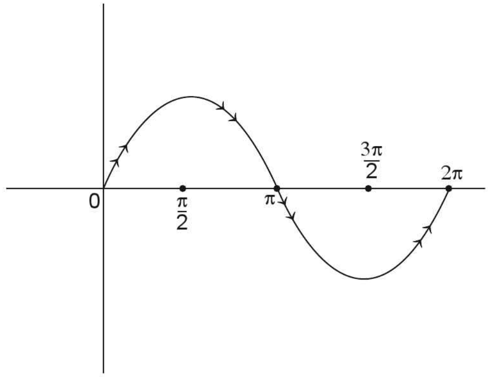
it means only Statement-1 is true.

21. (b) $\left(2^x + \dfrac{1}{2^x}\right) \ge 2$

But $\sin (e^x) \ngtr 1$

So, there is no solution.

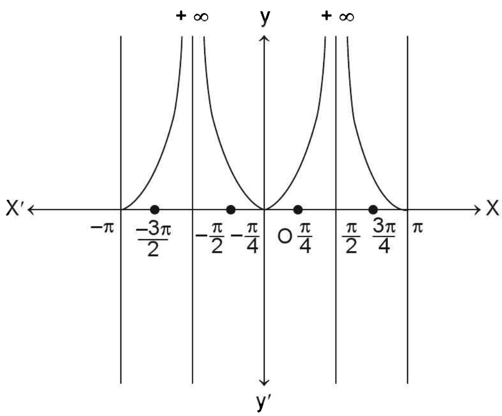
By graph of $\sin\theta$: $\sin\theta$ is increasing in Ist and IVth quadrant and $\sin\theta$ is decreasing in IInd and IIIrd quadrant.



So, Statement 1 and Statement-2 are true but Statement 1 does not follow from Statement-2.

22. (c) By graph of

$|\tan x| \le 1$



So, $x \in \left[-\pi, -\dfrac{3\pi}{4}\right] \cup \left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right] \cup \left[\dfrac{3\pi}{4}, \pi\right].$

If $|\tan x| = 1$

Then, $x = -\dfrac{3\pi}{4}, -\dfrac{\pi}{4}, \dfrac{\pi}{4}, \dfrac{3\pi}{4}$ (four solutions)

It means Statement-1 is true but Statemtent-2 is not correct.

23. (b) If $\angle A + \angle B + \angle C = \pi$

and $\angle B = 60°$ and $\sin(2A + B) = \dfrac{1}{2} = \sin 150°$

$\Rightarrow \quad 2A + B = 150° \Rightarrow 2A = 90° \Rightarrow A = 45°$

If $\angle A = 45°$ and $\angle B = 60°$ then $\angle C = 75°$

$$\sin(2A+B) = \frac{1}{2} \neq \sin 30° \quad (\text{because } \angle B = 60°)$$

It means both Statement-1 and Statement-2 are true but Statement-1 does not follow from Statement-2.

24. (a) We know that

$1 + \tan 2\theta = \sec^2\theta$

$\sec^2\theta - \tan 2\theta = 1$

$(\sec\theta - \tan\theta) . (\sec\theta + \tan\theta) = 1$

$\because \quad (\sec\theta + \tan\theta) = \dfrac{1}{(\sec\theta - \tan\theta)}$

If $\sec\theta + \tan\theta = p$ ....(i)

then $\sec\theta - \tan\theta = \dfrac{1}{p}$ .... (ii)

By (i) and (ii) $\tan\theta = \dfrac{p^2-1}{2p}$

25. (d) If $\cos A \cos B \cos C = \dfrac{\sqrt{3}-1}{8}$ .... (i)

and $\sin A \sin B \sin C = \dfrac{3+\sqrt{3}}{8}$ .... (ii)

By $\dfrac{(ii)}{(i)}$ we get

$\tan A \tan B \tan C = \dfrac{3+\sqrt{3}}{\sqrt{3}-1}$ .... (iii)

In a ΔABC, $A + B = \pi - C$

Taking tan on both sides

$\tan(A + B) = \tan(\pi - C) \Rightarrow \tan(A + B) = -\tan C$

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = -\frac{\tan C}{1}$$

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$

By (iii) equation

$\tan A + \tan B + \tan C = \dfrac{3+\sqrt{3}}{\sqrt{3}-1}$

26. (a) In the Ist quadrant sin x and cos x both are positive but $x \in \left(0, \dfrac{\pi}{4}\right)$ so, cos x is greater than sin x because cos x lies between 1 to $\dfrac{1}{\sqrt{2}}$ and sin x lies between 0 to $\dfrac{1}{\sqrt{2}}$.

So, Statement-1 and Statement-2 both true and Statement-1 follows from Statement-2.

27. (a) $(1 + \tan A) . (1 + \tan B) = 2$

$\tan A + \tan B + 1 + \tan A . \tan B = 2$

$(\tan A + \tan B) = (1 - \tan A \tan B)$

$$\left(\frac{\tan A + \tan B}{1 - \tan A \tan B}\right) = 1$$

$\tan(A + B) = 1$

$\angle A + \angle B = 45°$

It means Statement-1 and Statement-2 both are true and Statement-1 follows Statement-2.

28. (b) In a ΔABC, $A + B = \pi - C$

$\tan(A + B) = \tan(\pi - C)$ (Using formula)

get $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

But this result is use, when $\angle$A, $\angle$B, $\angle C$ are the angles of triangle.

29. (a) Maximum value of $\cos\theta = 1$

Minimum values of $\cos\theta = -1$

So, $-1 \leq \cos\theta \leq 1$

But, $\cos\theta = -\dfrac{3}{2}$ is not possible.

It means Statement-1 and Statement-2 booth are true.

30. (d) $\sin 7\theta - \sin\theta = 0$

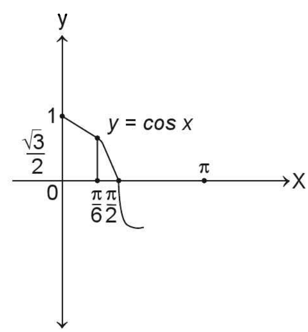
$2\cos 4\theta.\sin 3\theta = 0$

Either, $\cos 4\theta = 0$ or $\sin 3\theta = 1$

$4\theta = (2n+1)\dfrac{\pi}{2}$ or $3\theta = n\pi$

$\theta = (2n+1)\dfrac{\pi}{8}$ or $\theta = \dfrac{n\pi}{3}$

So, Statement-1 is false.



Exactly one solution, when $x = \dfrac{\pi}{6}$

So, Statement 2 is true.

31. (c) If $\cos^2\theta = \cos^2\alpha$

($\because$ Both the side are multiplied by 2)

$-2\cos^2\theta = -2\cos^2\alpha$

$2\cos^2\theta - 1 = 2\cos^2\alpha - 1$

$\cos 2\theta = \cos 2\alpha$

$2\theta = 2n\pi \pm 2\alpha$

$\theta = n\pi \pm \alpha;\ n \in I$

and α is principle value.

If $\cos^2\theta = \frac{1}{2}$

then use the above formula

$$\cos^2\theta = \left(\frac{1}{\sqrt{2}}\right)^2 = \cos^2\left(\frac{\pi}{4}\right)$$

$$\theta = n\pi \pm \frac{\pi}{4}$$

32. (d) The general solution of the equation $\tan\theta = 0$ is $\theta = n\pi$, where $n \in I$

We have, $\tan\theta + \tan 2\theta = (1 - \tan\theta + \tan 2\theta)$

$$\left(\frac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta}\right) = 1$$

$$\tan 3\theta = 1 = \tan\left(\frac{\pi}{4}\right)$$

$$3\theta = n\pi + \frac{\pi}{4} \quad \theta = \frac{n\pi}{3} + \frac{\pi}{12}$$

Statement 1 is false but Statement 2 is true.

33. (a) In the trigonometric equation, to avoid both side squaring, because they give extra solutions. Those extra solutions which cannot satisfy the given equation. It means, both Statement-1 and Statement-2 are true and Statement-1 follows Statement-2.

34. (d) The principal value of $\tan\theta = -\sqrt{3}$ is $-\frac{\pi}{3}$. because principal value is never numerically greater than π. Principal value always lies in the first circle (i.e., in first rotation) and tan θ is negative in IInd and IVth quadrant by ASTC Rule.

35. (d) If $\cos\theta = \cos\alpha$

Then, general solution of θ is $2n\pi \pm \alpha$, where $n \in I$

So, $\cos\theta = \frac{1}{2}$ (sue above for mula)

$$\theta = 2n\pi \pm \frac{\pi}{3}$$

and $\cos\theta = 0$

means, $\cos\theta = \cos\frac{\pi}{2}$

$$\theta = 2n\pi + \frac{\pi}{2}$$

or $2n\pi - \frac{\pi}{2}$

$$\theta = (2n+1)\frac{\pi}{2}$$

or $(2n-1)\frac{\pi}{2}$

ASSERTION/REASONING 2

ASSERTION AND REASONING TYPE QUESTIONS

Each question has 4 choices (a), (b), (c) and (d), out of which only one correct.

(a) Assertion is True, Reason is True and Reason is a correct explanation for Assertion.

(b) Assertion is True, Reason is True and Reason is not a correct explanation for Assertion.

(c) Assertion is True and Reason is False.

(d) Assertion is False and Reason is True.

1. **Assertion (A):** In any ΔABC, $2R^2 \sin A \sin B \sin C = \Delta$

Reason (R): In any ΔABC,

$$\frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = R$$

2. **Assertion (A):** If p_1, p_2, p_3 are the perpendiculars from the angular points of a triangle on the opposite sides, then $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r}$

Reason (R): In a triangle, $\frac{s}{\Delta} = \frac{1}{r}$

3. **Assertion (A):** In a triangle ABC, if $\tan\frac{A}{2}=\frac{5}{6}$ and $\tan\frac{C}{2}=\frac{2}{5}$, then the sides a, b, c are in A.P.

   Reason (R): In a ΔABC,

   $$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \text{ and}$$

   $$\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

4. **Assertion (A):** If A is the area and $2s$ is the sum of three sides of a triangle, then $A\le\frac{s^2}{3\sqrt{3}}$

   Reason (R): A.M. $\ge$ G.M.

5. **Assertion (A):** A circle is inscribed in an equilateral triangle of side a. The area of any square inscribed in the circle is $\frac{a^2}{6}$.

   Reason (R): Radius of the inscribed circle $r=\frac{\Delta}{s}$.

6. **Assertion (A):** If in a triangle $\tan A : \tan B : \tan C = 1 : 2 : 3$, then $A = 45°$

   Reason (R): If $\alpha : \beta : \gamma = 1 : 2 : 3$, then $\alpha = 1$

7. **Assertion (A):** If two sides of a triangle are 2 and 3, then its area cannot exceeds 3.

   Reason (R): Area of a triangle $=\frac{1}{2}$ be $\sin A$ and $\sin A \le 1$.

8. **Assertion (A):** If in a triangle, $\cos A + 2\cos B + \cos C = 2$, then a, b, c must be in AP.

   Reason (R): $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

9. **Assertion (A):** In any triangle ABC, the minimum value of $\frac{r_1+r_2+r_3}{r}$ is 9.

   Reason (R): A.M. $\ge$ G.M.

10. **Assertion (A):** In any triangle minimum value of $\frac{r_1r_2r_3}{r}$ is 27.

    Reason (R): If $a_1 + a_2 + a_3 + \ldots + a_n = k$ (constant), then the value of $a_1a_2a_3 \ldots a_n$ is minimum when $a_1 = a_2 = a_3 = \ldots = a_n$.

11. **Assertion (A):** In acute angled triangle $\tan A \tan B \tan C = 1$

    Reason (R): In obtuse angled triangle $\tan A \tan B \tan C$ is negative quantity.

12. **Assertion (A):** In any right angled triangle $\frac{a^2+b^2+c^2}{R^2}$ is always equal to 8.

    Reason (R): $a^2 = b^2 + c^2$.

13. **Assertion (A):** A man observes that when he moves up a distance c metres on a slope, the angle of depression of a point on the horizontal plane from the base of the slope is 30°, and when he moves up further a distance c metres the angle of depression of that point is 45°. The angle of inclination of the slope with the horizontal is 75°.

    Reason (R): In any ΔABC, If $BD : DC =$ m : n, $\angle BAD = \alpha$, $\angle CAD = \beta$ and $\angle ADC = \theta$, then $(m + n)\cot\theta = m\cot\alpha - n\cot\beta$

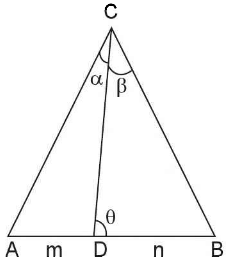
14. **Assertion (A):** A pole of length h stands inside a triangular plot ABC and subtends equal angles α at its vertices, then $2h\cos\alpha\sin A = a\sin\alpha$

    Reason (R): For circumscribed radius R of $a\,\Delta ABC$, $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$

15. **Assertion (A):** A tower leans towards west making an angle α with the vertical. The angular elevation of B, the top most point of the tower, is β as observed from a point C due east of A at a distance d from A. If the angular elevation of B from a point due east of C at a distance $2d$ from C is γ, then $2\tan\alpha = 3\cot\beta - \cot\gamma$

    Reason (R): In any ΔABC, if $BD : DC = m : n$, $\angle BAD = \alpha$, $\angle CAD = \beta$ and $\angle ADC = \theta$, then $(m + n)\cot\theta = m\cot\alpha - n\cot\beta$.

16. **Assertion (A):** In a triangle ABC, if D is a point on AB such that $AD : AB = m : n$ and CD makes an angle θ with AB. Then $(m + n)\cot\theta = n\cot\alpha - n\cot\beta$



Reason (R): The mid-point of the hypotenuse of a right angled triangle is the circum-centre of the triangle.

17. Assertion (A): If AB is a tower of height 100 m and make an angle with C is 45°. So AC is 100 m.

Reason (R): Angle BCA is called angle of elevation.

ASSERTION/REASONING: SOLUTIONS 2

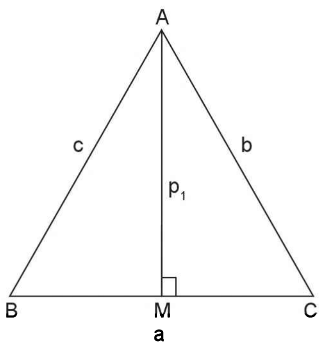
1. (a) We have, $2R^2 \sin A \sin B \sin C$

$$= 2R^2 \frac{a}{2R}.\frac{b}{2R}.\frac{c}{2R}$$

$$\left[\begin{array}{l} \because \dfrac{a}{2\sin A} = \dfrac{b}{2\sin B} = \dfrac{c}{2\sin C} = R \\ \Rightarrow \dfrac{a}{2R} = \sin A; \dfrac{b}{2R} = \sin B; \dfrac{c}{2R} = \sin C \end{array}\right]$$

$$= \frac{abc}{4R} = \Delta$$

2. (a) We have, $\Delta = \frac{1}{2} BC.AM = \frac{1}{2} ap_1$



$$\therefore \quad p_1 = \frac{2\Delta}{a}$$

Similarly, $p_2 = \frac{2\Delta}{b}, p_3 = \frac{2\Delta}{c}$

$$\therefore \quad \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{a}{2\Delta} + \frac{b}{2\Delta} + \frac{c}{2\Delta}$$

$$= \frac{a+b+c}{2\Delta} = \frac{2s}{2\Delta} = \frac{s}{\Delta} = \frac{1}{r}$$

3. (a) We have, $\tan\frac{A}{2}.\tan\frac{C}{2} = \frac{5}{6}.\frac{2}{5} = \frac{1}{3}$

$$\Rightarrow \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.\sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \frac{1}{3}$$

$$\Rightarrow \frac{s-b}{s} = \frac{1}{3} \Rightarrow 3s - 3b = s \Rightarrow 2s = 2b$$

$$\Rightarrow a + b + c = 3b \Rightarrow a + b = 2b$$

$\therefore$ a, b, c are in $A.P.$

4. (a) We have, $2s = a + b + c$

Also, $A = s(s-a)(s-b)(s-c)$

Since, $AM \geq GM$

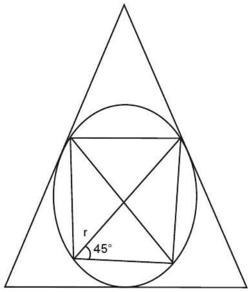
$$\Rightarrow \frac{s-a+s-b+s-c}{3} \geq [(s-a)(s-b)(s-c)]^{1/3}$$

$$\Rightarrow \frac{3s-2s}{3} \geq \frac{(A^2)^{1/3}}{s^{1/3}} \Rightarrow \frac{s^3}{27} \geq \frac{A^2}{s}$$

$\therefore A \leq \frac{s^2}{3\sqrt{3}}$ [$\because$ s > 0 always]

5. (a) Radius of the inscribed circle

$$r = \frac{\Delta}{s} = \frac{\frac{1}{2}a^2 \sin C}{\frac{a+b+c}{2}}$$



$$= \frac{a^2 \sin 60°}{3a} = \frac{a}{2\sqrt{3}}$$

Side of the square = 2 × r cos 45°

$$= 2 \times \frac{a}{2\sqrt{3}} \times \frac{1}{\sqrt{2}} = \frac{a}{\sqrt{6}}$$

Hence, area of square $= \left(\frac{a}{\sqrt{6}}\right)^2 = \frac{a^2}{6}$

6. (c) Let tan A = λ, tan B = 2λ, tan C = 3λ

∵ In triangle tan A + tan B + tan C = tan A tan B tan C

⇒ λ + 2λ + 3λ = λ : 2λ : 3λ

⇒ $6\lambda = 6\lambda^2$

∴ $\lambda = -1, 0, 1$

⇒ $\lambda = 1, \lambda \neq -1, 0$

∴ tan A = 1 ⇒ A = 45° (If λ = –1 then all angles becomes obtuse and if λ = 0 then all angles becomes zero) and reason is obviously.

7. (a) ∵ Area of a triangle $= \frac{1}{2} bc \sin A$

$$\le \frac{1}{2} bc \qquad (\because \sin A \le 1)$$

$$= \frac{1}{2}.2.3 = 3$$

Hence, area of a triangle ≤ 3
i.e., cannot exceed 3.

8. ∵ (cos A + cos C) = 2 (1 – cos B)

$$\Rightarrow \quad 2\cos\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right)$$

$$= 2.2\sin^2\left(\frac{B}{2}\right)$$

$$\Rightarrow \quad 2\sin\frac{B}{2}\cos\left(\frac{A-C}{2}\right) = 4\sin^2\left(\frac{B}{2}\right)$$

$$\Rightarrow \quad \cos\left(\frac{A-C}{2}\right) = 2\sin\left(\frac{B}{2}\right)$$

$$\Rightarrow \quad \frac{\cos\left(\frac{A-C}{2}\right)}{\cos\left(\frac{A+C}{2}\right)} = \frac{2}{1}$$

$$\Rightarrow \quad \frac{\cos\left(\frac{A+C}{2}\right)+\cos\left(\frac{A-C}{2}\right)}{\cos\left(\frac{A-C}{2}\right)-\cos\left(\frac{A+C}{2}\right)} = \frac{2+1}{2-1}$$

$$\therefore \quad \frac{2\cos\left(\frac{A}{2}\right)\cos\left(\frac{C}{2}\right)}{2\sin\left(\frac{A}{2}\right)\sin\left(\frac{C}{2}\right)} = 3$$

$$\text{or} \quad \cot\left(\frac{A}{2}\right)\cot\left(\frac{C}{2}\right) = 3$$

$$\text{or} \quad \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}.\sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = 3$$

⇒ s = 3 (s – b)

or $2s = 3b \Rightarrow a + c = 2b$

And in ΔABC

cos A + cos B + cos C

$$= 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

9. $\because \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3} = \frac{s-a}{\Delta}+\frac{s-b}{\Delta}+\frac{s-c}{\Delta}$

$$= \frac{3s-(a+b+c)}{\Delta} = \frac{3s-2s}{\Delta} = \frac{s}{\Delta} = \frac{1}{r}$$

∵ AM ≥ HM

$$\therefore \quad \frac{r_1+r_2+r_3}{r} \ge \frac{3}{\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)} = 3r$$

$$\therefore \quad \frac{r_1+r_2+r_3}{r} \ge 9$$

∴ Minimum value of $\frac{r_1+r_2+r_3}{r}$ is 9.

10. ∵ GM ≥ HM

$$\therefore \quad (r_1r_2r_3)^{1/3} \ge \frac{3}{\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}} = \frac{3}{\frac{1}{r}} = 3r$$

$\therefore \quad (r_1 r_2 r_3)^{1/3} \geq 3r$ or $\dfrac{r_1 r_2 r_3}{r^3} \geq 27$

Also, if $a_1 + a_2 + a_3 + \ldots + a_n = k$ (constant)
Then, the value $a_1 a_2 a_3 \ldots\ldots, a_n$ is greatest when $a_1 = a_2 = a_3 = \ldots = a_n$.

11. In acute angled triangle

$$0 < A < \frac{\pi}{2}, 0 < B < \frac{\pi}{2}, \quad 0 < C < \frac{\pi}{2} \text{ and}$$
$$A + B + C = \pi$$

Also, $\dfrac{\tan A + \tan B + \tan C}{3} \geq (\tan A \tan B$ $\tan C)^{1/3}$

$$\Rightarrow \quad \frac{(\tan A \tan B \tan C)}{3} \geq (\tan A \tan B \tan C)^{1/3}$$

$$\Rightarrow \quad \frac{1}{3} \geq 1 \quad \text{(impossible)}$$

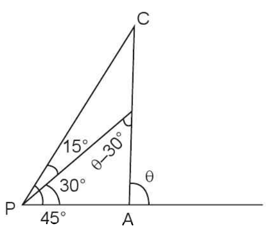
Also in obtuse angled triangle one angle lies in II quadrant and other two lie in I quadrant.

12. $a^2 = b^2 + c^2$, then $\angle A = \pi/2$

$\therefore \quad a^2 = 2R \sin A = 2R$

$$\Rightarrow \quad \frac{a^2 + b^2 + c^2}{R^2} = \frac{a^2 + a^2}{R^2} = \frac{2a^2}{R^2} = \frac{2(2R)^2}{R^2}$$

13. (a) Applying $m - n$ theorem of trigonometry, we get $(c + c)\cot(\theta - 30°) = c \cot 15° - c \cot 30°$



or $\cot(\theta - 30°) = \dfrac{1}{2} \dfrac{\sin(30° - 15°)}{\sin 15° \sin 30°}$

or $\cot(\theta - 30°) = \dfrac{1}{2} \dfrac{1}{\sin 30°} = 1 = \cot 45°$

$\Rightarrow \quad \theta - 30° = 45°$

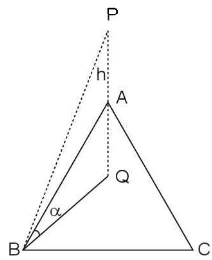
$\therefore \quad \theta = 75°$

14. (a) From the figure, (if PQ is the pole of height h),

$BQ = h \cot \alpha = CQ = AQ$

$\therefore \quad Q$ is the circumcentre of ΔABC.

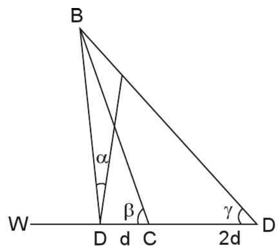
Hence, $\dfrac{a}{\sin A} = 2R = 2h \cot \alpha$



$\Rightarrow \quad a \sin \alpha = 2h \cos \alpha \sin A$.

15. (a) By $(m - n)$ theorem, we have

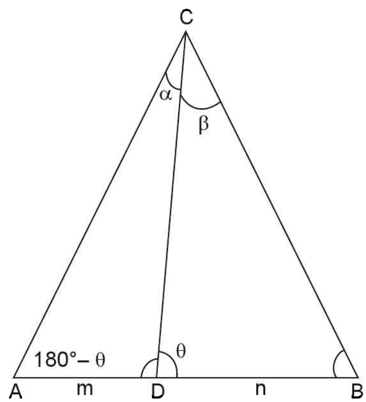
$$(d + 2d) \cot \beta = d \cot \gamma - 2d \cot\left(\frac{\pi}{2} + \alpha\right)$$



$\Rightarrow \quad 3 \cot \beta = \cot \gamma + 2 \tan \alpha$

$\therefore \quad 2 \tan \alpha = 3 \cot \beta - \cot \gamma$

16. (b)



$\angle ADC = 180° - \theta$

$\because \quad \angle CAD = 180° - (\alpha + 180° - \theta) = \theta - \alpha$

and $\angle CBD = 180° - (\theta + \beta)$

From ΔACD, $\dfrac{AD}{\sin \alpha} = \dfrac{CD}{\sin(\theta - \alpha)}$ .... (i)

From ΔBCD, $\dfrac{DB}{\sin \beta} = \dfrac{CD}{\sin[\pi - (\theta + \beta)]}$

$$\frac{DB}{\sin\beta}=\frac{CD}{\sin(\theta+\beta)} \quad \text{........ (ii)}$$

By (i) and (ii) $[AD : DB = m : n]$

$$\frac{m\sin\beta}{n\sin\alpha}=\frac{\sin\theta.\cos\beta+\cos\theta.\sin\beta}{\sin\theta.\cos\alpha-\cos\theta.\sin\alpha}$$

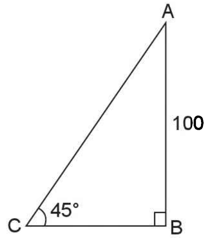
$(m + n)\cot\theta = m \cot \alpha - n \cot \beta$

Then Statement 1 and Statement 2 both are true, but Statement 1 does not follow from Statement 2.

17. (d) $\tan 45° = \frac{AB}{BC} = \frac{100}{BC}$

$\because BC = 100$

$AC^2 = BC^2 + AB^2$



$AC = 100\sqrt{2}$ m

and $\angle BCA$ is called angle of elevation.

ASSERTION/REASONING 3

ASSERTION AND REASONING TYPE QUESTIONS

Each question has 4 choices (a), (b), (c) and (d), out of which only one is correct.

(a) Assertion is True, Reason is True and Reason is a correct explanation for Assertion.
(b) Assertion is True, Reason is True and Reason is not a correct explanation for Assertion.
(c) Assertion is True and Reason is False.
(d) Assertion is False and Reason is True.

1. **Assertion (A):** $\sin^{-1}(\sin 3) = 3$

   Reason (R): For principal values $\sin^{-1}(\sin x) = x$

2. **Assertion (A):** If $x < 0$, $\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$

   Reason (R): $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\ \forall x \in R$

3. **Assertion (A):** $\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\frac{\pi}{4}$

   Reason (R): For $x > 0, y > 0$

   $$\tan^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y-x}{y+x}\right)=\frac{\pi}{4}$$

4. **Assertion (A):** If $x = \text{cosec}(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1}a)))))$ and $y = \sec(\cot^{-1}(\sin(\tan^{-1}(\text{cosec}(\cos^{-1}a)))))$, where $a \in [0, 1]$, then $x = y$

   Reason (R): $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1}x,\ \forall x \in (-\infty, -1]\cup[1, \infty)$

   $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x,\ \forall x \in (-\infty, -1]\cup[1, \infty)$

   $$\tan^{-1}\left(\frac{1}{x}\right)=\begin{cases}\cot^{-1}x, & x>0\\ -\pi+\cot^{-1}x, & x<0\end{cases}$$

5. **Assertion (A):** The solution of system of equations

   $\cos^{-1}x + (\sin^{-1} y)^2 = \frac{p\pi^2}{4}$ and $(\cos^{-1} x)(\sin^{-1}y)^2 = \frac{\pi^4}{16}$ is $x = \cos\frac{\pi^2}{4}$ and $y = \pm 1$, $\forall p \in I$.

   Reason (R): AM≥GM

6. **Assertion (A):** If $p > q > 0$ and $pr < -1 < qr$, then

   $$\tan^{-1}\left(\frac{p-q}{1+pq}\right)\tan^{-1}\left(\frac{q-r}{1+qr}\right)\tan^{-1}\left(\frac{r-p}{1+rp}\right)$$

   $$\tan^{-1}\left(\frac{r-p}{1+rp}\right)=\pi$$

   Reason (R): $\tan^{-1}x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ for all x, y

7. **Assertion (A):** $\sin^{-1}2x + \sin^{-1}3x = \frac{\pi}{3}$

$\Rightarrow \quad x = \sqrt{\left(\frac{3}{76}\right)}$ only

Reason (R): Sum of two negative angles cannot be positive.

8. **Assertion (A):** $\sum_{i=1}^{2n} \sin^{-1} x_i = n\pi, n \in N$

Then, $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i^3$

Reason (R): $-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}, \forall x \in [-1,1]$

ASSERTION/REASONING: SOLUTIONS 3

1. (d) $\because$ 3 ≈ 171° (lies in II quadrant)
$\therefore \sin^{-1} \sin 3 = 3 - \pi \ne 3$
But $\sin^{-1} \sin x = x$ for principal values

2. (d) For $x < 0$, $\tan^{-1}\left(\frac{1}{x}\right) = -\pi + \cot^{-1} x$

$\therefore \tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right) = \tan^{-1} x - \pi + \cot^{-1}x$

$= \frac{x}{2} - \pi$

$= -\frac{\pi}{2} \ne \frac{\pi}{2}$

and $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, \forall x \in R.$

3. (a) $\because \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left\{\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}\right\}$

$= \tan^{-1}(1) = \frac{\pi}{4}$

Also, $\frac{4-3}{4+3} = \frac{1}{7}$

$(\because 4 < 0, 3 > 0>$

$\therefore$ R is the correct reason of A.

4. (b) $\because x = \text{cosec}\,(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1}a)))))$

$= \text{cosec}\left(\frac{\pi}{2} - \cot^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1}a))))\right)$

$= \sec(\cot^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1}a)))))$

$\sec\left(\cot^{-1}\left(\cos\left(\frac{\pi}{2} - \tan^{-1}(\sec(\sin^{-1}a))\right)\right)\right)$

$= \sec(\cot^{-1}(\sin(\tan^{-1}(\sec(\sin^{-1}a)))))$

$= \sec\cot^{-1}\left(\sin\left(\tan^{-1}\left(\sec\left(\frac{\pi}{2} - \cos^{-1}a\right)\right)\right)\right)$

$= \sec\cot^{-1}(\sin(\tan^{-1}(\text{cosec}\,(\cos^{-1}a))))) = y$

5. (a) $\because$ AM ≥ GM

$\therefore \frac{\cos^{-1}x + (\sin^{-1}y)^2}{2} \ge \sqrt{(\cos^{-1}x)(\sin^{-1}y)^2}$

$\Rightarrow \frac{p\pi^2}{8} \ge \frac{\pi^2}{4}$

$\Rightarrow p \ge 2$

Thus, we conclude that the only value of p that satisfies all conditions is $p = 2$.

Then, $\cos^{-1} x = (\sin^{-1}y)^2$

$\Rightarrow (\cos^{-1}x)^2 = \frac{\pi^2}{16} \quad \Rightarrow \cos^{-1}x = \pm\frac{\pi^2}{4}$

$\Rightarrow x = \cos\left(\pm\frac{\pi^2}{4}\right) \quad \therefore x = \cos\left(\frac{\pi^2}{4}\right)$

Also, $(\sin^{-1}y)^4 = \frac{\pi^4}{16}$

$\Rightarrow \sin^{-1}y = \pm\frac{\pi}{2} \quad \therefore y = \sin\left(\pm\frac{\pi}{2}\right) = \pm 1$

6. (d) Since, $p, q > 0$ therefore $pq > 0$

and so, $\tan^{-1}\left(\frac{p-q}{1+pq}\right) = \tan^{-1}p - \tan^{-1}q$ ....(i)

Since, $qr > -1$

$\therefore \quad \tan^{-1}\left(\frac{q-r}{1+qr}\right) = \tan^{-1}q - \tan^{-1} r$ ....(ii)

and, since $pr < -1$ and $r < 0$

$\therefore \quad \tan^{-1}\left(\frac{r-p}{1+rp}\right) = \pi \tan^{-1}r - \tan^{-1}p$ ....(iii)

On adding equations (i), (ii) and (iii), we get

$$\tan^{-1}\left(\frac{p-q}{1+pq}\right)+\tan^{-1}\left(\frac{q-r}{1+qr}\right)+\tan^{-1}\left(\frac{r-p}{1+rp}\right)=\pi$$

$$+\tan^{-1}\left(\frac{r-p}{1+rp}\right)=\pi$$

7. (a) $\therefore \sin^{-1}2x+\sin^{-1}3x=\frac{\pi}{3}$

$$\Rightarrow \frac{\pi}{2}-\cos^{-1}2x+\frac{\pi}{2}-\cos^{-1}3x=\frac{\pi}{3}$$

$$\Rightarrow \cos^{-1}2x+\cos^{-1}3x=\frac{2\pi}{3}$$

$$\Rightarrow \cos^{-1}\{6x^2-\sqrt{(-(2x)^2)}\sqrt{1-(3x)^2}\}=\frac{2\pi}{3}$$

$$\Rightarrow 6x^2-\sqrt{(1-13x^2+36x^4)}=-\frac{1}{2}$$

$$\Rightarrow \left(6x^2+\frac{1}{2}\right)^2=1-13x^2+36x^4$$

$$\Rightarrow 19x^2=\frac{3}{4} \text{ or } x=\pm\sqrt{\frac{3}{76}}$$

But some of two –ve number cannot be $\pi/3$.

$\therefore x=\sqrt{\frac{3}{76}}$ is only solution.

8. (a) Since, maximum value of $\sin^{-1}x$ is

$\sin^{-1}x_i$ is $\frac{\pi}{2}$

$\therefore \sum_{i=1}^{2n}\sin^{-1}x_i=n\pi$ is possible, if

$x_1=x_2=x_3=$ .... $=x_{2n}=1$

$\therefore \sum_{i=1}^{n}x_i=1+1+1+$........ up to n times $=n$

$\therefore \sum_{i=1}^{n}x_i^2=1^2+1^2+1^2+1^2+$........ up to n times $=n$

and $\sum_{i=1}^{n}x_i^3=1^3+1^3+1^3+1^3+$........ up to n times $=n$

Hence, $\sum_{i=1}^{n}x_i=\sum_{i=1}^{n}x_i^2=\sum_{I=1}^{n}x_i^3=n$

TOPICWISE WARMUP TEST

1. If $\sin A \sin B \sin C + \cos A \cos B = 1$, then the value of $\sin C$ is equal to ***[IIT-JEE-2006]***

(a) 1 (b) 0
(c) 1/3 (d) 1/2

2. Let $\theta \in (0, \pi/4)$ and

$t_1 = (\tan\theta)^{\tan\theta}$, $t_2 = (\tan\theta)^{\cot\theta}$,

$t_3 = (\cot\theta)^{\tan\theta}$, $t_4 = (\cot\theta)^{\cot\theta}$; then ***[IIT-JEE-2006]***

(a) $t_1 > t_2 > t_3 > t_4$ (b) $t_4 > t_3 > t_1 > t_2$
(c) $t_2 > t_3 > t_1 > t_4$ (d) $t_3 > t_1 > t_2 > t_4$

3. Let α, β such that $\pi < \alpha - \beta < 3\pi$. If $\sin\alpha + \sin\beta = -\frac{21}{65}$ and $\cos\alpha + \cos\beta = -\frac{27}{65}$, then the value of $\cos\frac{\alpha-\beta}{2}$ is ***[AIEEE-2004]***

(a) $\frac{6}{65}$ (b) $\frac{3}{\sqrt{130}}$
(c) $-\frac{3}{\sqrt{130}}$ (d) $-\frac{6}{\sqrt{65}}$

4. If $a=\frac{\pi}{18}$ radians, then $\cos a + \cos 2a + \ldots + \cos 18a$ is equal to ***[MP PET-2005]***

(a) 0 (b) −1
(c) 1 (d) ±1

5. $\cos 2(\theta + \phi) - 4\cos(\theta + \phi)\sin\theta\sin\phi + 2\sin^2\phi$ is equal to ***[Orissa JEE-2004]***
(a) $\cos 2\theta$ (b) $\cos 3\theta$
(c) $\sin 2\theta$ (d) $\sin 3\theta$

6. If θ is an acute angle and $\sin\frac{\theta}{2} = \sqrt{\frac{x-1}{2x}}$, then $\tan\theta$ is equal to ***[Orissa JEE-2005]***
(a) $x^2 - 1$ (b) $\sqrt{x^2-1}$
(c) $\sqrt{x^2+1}$ (d) $x^2 + 1$

7. The value of $\cos\frac{\pi}{65}.\cos\frac{2\pi}{65}.\cos\frac{4\pi}{65}.\cos\frac{8\pi}{65}.\cos\frac{16\pi}{65}.\cos\frac{32\pi}{65}$ is ***[M.N.R.-1997]***
(a) 1/64 (b) 1/65
(c) 1/63 (d) 2/65

8. If $0 < x < \pi$ and $\cos x + \sin x = 1/2$, then $\tan x$ is equal to ***[AIEEE-2006]***
(a) $(1-\sqrt{7})/4$ (b) $(1+\sqrt{7})/4$
(c) $(4-\sqrt{7})/3$ (d) $-(4+\sqrt{7})/3$

9. $\log(\tan 1^0) + \log(\tan 2^0) + \log(\tan 3^0) + \ldots + \log(\tan 89^0)$ is equal to ***[CDS-95]***
(a) 1 (b) −1
(c) 0 (d) none

10. $\log(\sin 1^0).\log(\sin 2^0).\log(\sin 3^0) \ldots \log(\sin 179^0)$ is equal to ***[CDS-99]***
(a) 1 (b) −1
(c) 0 (d) none

11. If $2y\cos\theta = x\sin\theta$ and $2x\sec\theta - y\,\text{cosec}\,\theta = 3$, then $x^2 + 4y^2$ equals ***[JEE (WB)-88]***
(a) ±4 (b) 4
(c) −4 (d) none

12. For any angle θ, $\frac{3\cos 8\theta + 1}{2\cos\theta + 1}$ is equal to ***[Kerala (CEE)-2003; MPPET-2005]***
(a) $(2\cos\theta + 1)(2\cos 2\theta + 1)(2\cos 4\theta + 1)$
(b) $(\cos\theta - 1)(\cos 2\theta - 1)(\cos 4\theta - 1)$
(c) $(2\cos\theta - 1)(2\cos 2\theta - 1)(2\cos 4\theta - 1)$
(d) $(2\cos\theta - 1)(2\cos 2\theta + 1)(2\cos 4\theta + 1)$

13. If $\sin 44\theta = \cos\theta$, then $\tan 15\theta$ is equal to ***[NDA-2005]***
(a) 1/2 (b) $\sqrt{2}$
(c) $\sqrt{3}$ (d) $1/\sqrt{3}$

14. In a triangle ABC, if $\cos A = \cos B\cos C$, then $\tan A - \tan B - \tan C$ is equal to ***[NDA-2006]***
(a) 0
(b) 1
(c) $1 + \tan A\tan B\tan C$
(d) $\tan A\tan B\tan C - 1$

15. If $\sin(\pi\cos x) + \cos(\pi\sin x)$, then one value of $\sin 2x$ is equal to ***[NDA-2006]***
(a) −1/4 (b) −1/2
(c) −3/4 (d) −1

16. If $\tan^2\alpha\tan^2\beta + \tan^2\beta\tan^2\gamma + \tan^2\gamma\tan^2\alpha + 2\tan^2\alpha\tan^2\beta\tan^2\gamma = 1$, then the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is
(a) 0 (b) −1
(c) 1 (d) None of these

17. $\cos^4\theta - \sin^4\theta$ is equal to ***[MPPET-2006]***
(a) $1 + 2\sin^2\left(\frac{\theta}{2}\right)$ (b) $2\cos^2\theta - 1$
(c) $1 - 2\sin^2\left(\frac{\theta}{2}\right)$ (d) $1 + 2\cos^2\theta$

18. $3\left[\sin^4\left(\frac{3\pi}{2} - \alpha\right) + \sin^4(3\pi + \alpha)\right] - 2\left[\sin^6\left(\frac{\pi}{2} + \alpha\right) + \sin^6(5\pi - \alpha)\right]$ is equal to ***[IIT-86; NDA-03]***
(a) 3 (b) 1
(c) 0 (d) $\sin 4\alpha + \sin 6\alpha$

19. $a\cos\theta + b\sin\theta = c$, then $(a\sin\theta - b\cos\theta)^2$ is equal to ***[AMU-1995]***
(a) $a^2 + b^2$ (b) $b^2 + c^2$
(c) $c^2 + a^2$ (d) $a^2 + b^2 - c^2$

20. If $\sin B = \frac{1}{5}\sin(2A + B)$, then $\frac{\tan(A+B)}{\tan A}$ is equal to ***[T.S. Rajendra-1992]***
(a) $\frac{5}{3}$ (b) $\frac{2}{3}$
(c) $\frac{3}{2}$ (d) $\frac{3}{5}$

21. $\sin^2 17.5 + \sin^2 72.5$ is equal to ***[KCET-2007]***

(a) $\cos^2 90°$ (b) $\tan 245°$
(c) $\cos^2 30°$ (d) $\sin^2 45°$

22. If θ lies in first quadrant and 5 tanθ = 4, then $\frac{5\sin\theta - 3\cos\theta}{\sin\theta + 2\cos\theta} =$ ***[EAMCET-2007]***

(a) 5/4 (b) 3/14
(c) 1/14 (d) 0

23. If cos $(A - B) = 3/5$ and $\tan A \tan B = 2$, then which one of the following is true ***[EAMCET-2007]***

(a) $\sin(A + B) = 1/5$ (b) $\sin(A + B) = -1/5$
(c) $\cos(A - B) = 1/5$ (d) $\cos(A + B) = -1/5$

24. $\frac{\tan 80° - \tan 10°}{\tan 70°}$ is equal to ***[EAMCET-2007]***

(a) 0 (b) 1
(c) 2 (d) 3

25. sin 4A – cos 2A = cos 4A – sin 2A, then the value of tan 4A = $\left[0 < A < \frac{\pi}{4}\right]$ ***[Kerala PET-2007]***

(a) 1 (b) $\frac{1}{\sqrt{3}}$
(c) $\sqrt{3}$ (d) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$

26. If $x \sin^3\theta + y \sin^3\theta = \sin\theta \cos\theta$ and $x \sin\theta = y \sin\theta$ then $x^2 + y^2$ is equal to ***[Kerala PET-2007]***

(a) 2 (b) 0
(c) 3 (d) 4
(e) 1

27. If tan 2x = –4/3 at least one of the values of cos x is ***[MPPET-2007]***

(a) 0 (b) $-\frac{1}{\sqrt{5}}$
(c) $\frac{2}{\sqrt{5}}$ (d) $\frac{1}{\sqrt{5}}$

28. If sin 3θ = sinθ, how many solutions exist such that $-2\pi < \theta < 2\pi$ ***[Karnataka CET-2007]***

(a) 8 (b) 9
(c) 5 (d) 7

29. What is the measure of the angle 114° 35′ 30″ in radian? ***[NDA-2008]***

(a) 1 radian (b) 2 radian
(c) 3 radian (d) 4 radian

30. What is the value of $\left(\sin 22\frac{1}{2}^\circ + \cos 22\frac{1}{2}^\circ\right)^4$? ***[NDA-2008]***

(a) $\frac{3+2\sqrt{2}}{2}$ (b) $\frac{1+2\sqrt{2}}{2}$
(c) $\frac{3\sqrt{2}+2}{2}$ (d) 1

31. Which one of the following is correct? $\left(1+\cos 67+\frac{1°}{2}\right)\left(1+\cos 112\frac{1°}{2}\right)$ is ***[NDA-2008]***

(a) an irrational number and is greater than 1
(b) a rational number but not an integer
(c) an integer
(d) an irrational number and is less than 1

32. If $\sin 2A = \frac{4}{5}$, then what is the value of $\tan A \left(0 \le A \le \frac{\pi}{4}\right)$? ***[NDA-2008]***

(a) 1 (b) –1
(c) 1/2 (d) 2

33. What is the value of $\frac{\cos 10° - \sin 10°}{\cos 10° + \sin 10°}$? ***[NDA-2008]***

(a) tan 35° (b) tan 10°
(c) $\frac{1}{\sqrt{2}}$ (d) 1

34. Which one of the following pairs is not correctly matched? ***[NDA-2008]***

(a) sin 2π : sin (–2π)
(b) tna 45° : tan (–315°)
(c) cot ($\tan^{-1}$ 0.5) : tan ($\cot^{-1}$ 0.5)
(d) tan 420° : tan (–60°)

35. What is the value of $\sin\left(\frac{5\pi}{12}\right)$? ***[NDA-2008]***

(a) $\frac{\sqrt{3}+1}{2}$ (b) $\frac{\sqrt{6}+\sqrt{2}}{4}$
(c) $\frac{\sqrt{3}+\sqrt{2}}{4}$ (d) $\frac{\sqrt{6}+1}{2}$

36. What is the correct sequence of the following values?

1. $\sin\left(\frac{\pi}{12}\right)$ 2. $\cos\left(\frac{\pi}{12}\right)$

3. $\cot\left(\frac{\pi}{12}\right)$

Select the correct answer using the code given below ***[NDA-2008]***

(a) $3 > 2 > 1$ (b) $1 > 2 > 3$
(c) $1 > 3 > 2$ (d) $3 > 1 > 2$

37. For what value of x does the equation $4 \sin x + 3 \sin 2x - 2 \sin 3x + \sin 4x = 2\sqrt{3}$ hold? ***[NDA-2008]***

(a) $\pi/6$ (b) $\pi/4$
(c) $\pi/3$ (d) $\pi/2$

38. The value of $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}$ is

(a) 1/16 (b) 1/8
(c) 1/2 (d) 1

39. If $\tan\theta = b/a$, then $b \cos 2\theta + a \sin 2\theta$ is equal to

(a) a (b) b
(c) b/a (d) None

40. $\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$

(a) $\sin 2\theta + \cos 2\theta$ (b) $\tan 2\theta + \sec 2\theta$
(c) $\cot 2\theta + \text{cosec } 2\theta$ (d) $\tan 2\theta + \cot 2\theta$

41. $\sin 36° \sin 72° \sin 108° \sin 144°$ is equal to ***[IIT-1965]***

(a) 1/4 (b) 1/16
(c) 3/4 (d) 5/16

42. If $\text{cosec } \theta = \frac{p+q}{p-q}$, then $\cot\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$ is equal to ***[EAMCET-2001]***

(a) $\sqrt{p/q}$ (b) $\sqrt{q/p}$
(c) $\sqrt{pq}$ (d) pq

43. If $\alpha, \beta, \gamma, \delta$ are the smallest positive angles in ascending order of magnitude which have their sines equal to the positive quantity k, then the value of $4\sin\frac{\alpha}{2} + 3\sin\frac{\beta}{2} + 2\sin\frac{\gamma}{2} + \sin\frac{\delta}{2}$ is equal to

(a) $2\sqrt{1-k}$ (b) $2\sqrt{1+k}$
(c) $2\sqrt{k}$ (d) $2\sqrt{k+2}$

TOPICWISE WARMUP TEST: SOLUTION

1. (a) $\sin A \sin B \sin C + \cos A \cos B$
$\leq \sin A \sin B + \cos A \cos B$ [$\because \sin C \leq 1$]
$\therefore$ $\sin A \sin B \sin C + \cos A \cos B \leq \cos (A - B)$

$\Rightarrow$ $\cos (A - B) \geq 1$ [using given relation]
$\Rightarrow$ $\cos (A - B) = 1$
[$\because$ max. $\cos (A - B) = 1$]
$\Rightarrow$ $A - B = 0$ $\Rightarrow$ $A = B$

Then from given relation $\sin C = \frac{1-\cos^2 A}{\sin^2 A} = 1$

2. (b) $\theta \in (0, \pi/4) \Rightarrow \tan\theta < 1$ and $\cot\theta > 1$

So let $\tan\theta = 1 - \lambda$ and $\cot\theta = 1 + \mu$, where λ and μ are very small positive numbers.

Then

$t_1 = (1 - \lambda)^{1-\lambda}$, $t_2 = (1 - \lambda)^{1-\mu}$,

$t_3 = (1 + \mu)^{1-\lambda}$, $t_4 = (1 + \mu)^{1-\mu}$

$\Rightarrow$ $t_4 > t_3 > t_1 > t_2$

3. (c) $\cos\alpha + \cos\beta = -\frac{27}{65}$

$\Rightarrow$ $2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = -\frac{27}{65}$

$\sin\alpha + \sin\beta = -\frac{21}{65}$

$$\Rightarrow \quad 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = -\frac{21}{65}$$

By squaring and adding, we get

$$4\cos^2\frac{\alpha-\beta}{2}\left[\cos^2\frac{\alpha+\beta}{2}+\sin^2\frac{\alpha+\beta}{2}\right]$$

$$= \frac{729+441}{(65)^2}$$

$$\Rightarrow \quad 4\cos^2\frac{\alpha-\beta}{2} = \frac{1170}{65\times 65} = \frac{90}{5\times 65} = \frac{18}{65}$$

$$\Rightarrow \quad \cos^2\frac{\alpha-\beta}{2} = \frac{9}{130}$$

$$\Rightarrow \quad \cos\frac{\alpha-\beta}{2} = -\frac{3}{\sqrt{130}}$$

$\left[\because \frac{\alpha-\beta}{2} \text{ lies in the second quadrant}\right]$

4. (b) cos a + cos $2a$ + ... + cos $18a$

$$= \cos\frac{\pi}{18}+\cos\frac{2\pi}{18} + \ldots\ldots + \cos\frac{16\pi}{18}+\cos\frac{17\pi}{18}+\cos\pi$$

$$= \cos\frac{\pi}{18}+\cos\frac{2\pi}{18} + \ldots\ldots\ldots - \cos\frac{2\pi}{18}-\cos\frac{\pi}{18}+\cos\pi = \cos\pi = -1.$$

5. (a) We have, cos 2(θ + φ) – 4 cos (θ + φ) sinθ sin φ + 2 sin² φ

Now, put $\theta = \phi = \frac{\pi}{4}$

$$\cos 2\left(\frac{\pi}{2}\right) - 4\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)+2\sin^2\left(\frac{2\pi}{4}\right) = 0$$

Put θ = φ = π/4 in option (a). Then, cos 2θ = π/2 = 0

Hence, option (a) is correct.

6. (b) $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$$\Rightarrow \quad \tan\theta = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1-2\sin^2\frac{\theta}{2}} = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$$

$$\left[\begin{array}{l} U\sin g\ \sin\frac{\theta}{2} = \sqrt{\frac{x-1}{2x}} \\ \therefore\ \cos\frac{\theta}{2} = \sqrt{1-\sin^2\frac{\theta}{2}} = \sqrt{\frac{x+1}{2x}} \text{ and} \\ \tan\frac{\theta}{2} = \frac{\sqrt{x-1}}{x+1} \end{array}\right]$$

$$\therefore \quad \tan\theta = \sqrt{x^2-1}$$

7. Let $\theta = \frac{\pi}{65}, n = 6$

$$\text{L.H.S.} = \frac{\sin 2^6\theta}{2^6\sin\theta} = \frac{1}{64}\frac{\sin 64\frac{\pi}{65}}{\sin\frac{\pi}{65}}$$

$$= \frac{1}{64}\frac{\sin\left(\pi-\frac{\pi}{65}\right)}{\sin\frac{\pi}{65}} = \frac{1}{64}$$

[as sin(π – θ) = sin θ]

8. (d) By squaring both sides of given relation, we get

$1 + \sin 2x = 1/4$

$\Rightarrow \quad \sin 2x = -3/4$

$$\Rightarrow \quad \frac{2\tan x}{1+\tan^2 x} = -\frac{3}{4}$$

$\Rightarrow \quad 3\tan^2 x + 8\tan x + 3 = 0$

$$\Rightarrow \quad \tan x = \frac{-8\pm\sqrt{64-36}}{6} = -(4+\sqrt{7})/3$$

9. (c) $[\log(\tan 1°)+\log(\tan 89°)]$

$+[\log(\tan 2°)+\log(\tan 88°)]+\ldots+\log(\tan 45°)$

$=[\log(\tan 1°)+\log(\cot 1°)]$

$+[\log(\tan 2°)+\log(\cot 2°)]+\ldots+\log 1$

$= \log 1 + \log 1 + \ldots + \log 1 = 0 + 0 + \ldots.. + 0 = 0$

10. (c) ∵ log (sin90°) = 0. So exp. = 0

11. (b) $2y\cos\theta = x\sin\theta$ ....(1)

$2x\sec\theta - y\,\text{cosec}\,\theta = 3$

$$\Rightarrow \quad \frac{2x}{\cos\theta} - \frac{y}{\sin\theta} = 3$$

$\Rightarrow \quad 2x\sin\theta - y\cos\theta = 3\sin\theta\cos\theta$ .....(2)

(1), (2) $x = 2\cos\theta, y = \sin\theta$

$\therefore \quad x^2 + 4y^2 = 4\cos^2\theta + 4\sin^2\theta = 4$

12. (c) ∵ $2\cos 8\theta + 1 = 2(\cos^2 4\theta - 1) + 1 = 4\cos^2 4\theta - 1$
$= (2\cos 4\theta - 1)(2\cos 4\theta + 1)$ ....(1)

Similarly, $2\cos 4\theta + 1 = (2\cos 2\theta - 1)(2\cos 2\theta + 1)$ ...(2)

$2\cos 2\theta + 1 = (2\cos\theta - 1)(2\cos\theta + 1)$ ...(3)

(1), (2), (3) ⇒ Exp. $= (2\cos 4\theta - 1)(2\cos 2\theta - 1)(2\cos\theta - 1)$

13. (d) $\sin 44\theta = \cos\theta$
⇒ $\sin 44\theta = \sin(\pi/2 - \theta)$
⇒ $44\theta = \pi/2 - \theta$
⇒ $\theta = \pi/90$
∴ $\tan 15\theta = \tan\pi/6 = 1/\sqrt{3}$

14. (a) $\tan A - \left[\frac{\sin B}{\cos B} + \frac{\sin C}{\cos C}\right]$

$= \tan A - \frac{\sin(B+C)}{\cos B\cos C}$

$= \tan A - \frac{\sin A}{\cos A}$ [∵ $\cos B\cos C = \cos A$] = 0

15. (c) $\sin(\pi\cos x) = \sin\left(\frac{\pi}{2} - \pi\sin x\right)$

⇒ $\cos = \frac{1}{2} - \sin x$

⇒ $\sin x + \cos x = 1/2 \Rightarrow 1 + \sin 2x = 1/4$
⇒ $\sin 2x = -3/4$

16. (c) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$

$= \frac{\tan^2\alpha}{1+\tan^2\alpha} + \frac{\tan^2\beta}{1+\tan^2\beta} + \frac{\tan^2\gamma}{1+\tan^2\gamma}$

$= \frac{x}{1+x} + \frac{y}{1+y} + \frac{z}{1+z}$ ($x = \tan^2\alpha$, $y = \tan^2\beta$, $z = \tan^2\gamma$)

$= \frac{(x+y+z)+(xy+yz+zx+2xyz) + xy+yz+zx+xyz}{(1+x)(1+y)(1+z)}$

$= \frac{1+x+y+z+xy+yz+zx+xyz}{(1+x)(1+y)(1+z)} = 1$

(∵ $xy + yz + zx + 2xyz = 1$)

17. (b) $\cos^4\theta - \sin^4\theta = (\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)$
$= \cos 2\theta = 2\cos^2\theta - 1$.

18. (b) $3(\cos^4\alpha + \sin^4\alpha) - 2(\cos^6\alpha + \sin^6\alpha)$
$= 3(1 - 2\sin^2\alpha\cos^2\alpha) - 2(1 - 3\cos^2\alpha\sin^2\alpha) = 1$

19. (d) Since $(a\cos\theta + b\sin\theta)^2 + (a\sin\theta - b\cos\theta)^2 = a^2 + b^2$
∴ $(a\sin\theta - b\cos\theta)^2 = a^2 + b^2 - c^2$

20. (c) $\frac{\sin(2A+B)}{\sin B} = \frac{5}{1}$. By componendo and dividendo

we shall get $\frac{\tan(A+B)}{\tan A} = \frac{3}{2}$

21. (b) $\sin^2 17.5 + \sin^2 72.5$
$= \sin^2 17.5 + \cos^2 17.5 = 1 = \tan 245°$.

22. (a) $\frac{5\sin\theta - 3\cos\theta}{\sin\theta + 2\cos\theta} = \frac{5\tan\theta - 3}{\tan\theta + 2}$

[∵ Given that the θ = 4/5]

⇒ $\frac{5\times(4/5)-3}{4/5+2} = \frac{\frac{20}{5}-3}{\frac{14}{5}} = \frac{5}{14}$.

23. (d) $\tan A + \tan B = 2$
⇒ $\sin A\sin B = 2\cos A\cos B$
$\cos(A - B) = 3/5 = \cos A\cos B + \sin A\sin B$
⇒ $\cos A\cos B + 2\cos A\cos B = 3/5$
⇒ $\cos A\cos B = 1/5$
$\cos(A + B) = \cos A\cos B - \sin A\sin B$
$= \cos A\cos B - 2\cos A\cos B = -\cos A\cos B = -1/5$.

24. (c) $\frac{\tan 80° - \tan 10°}{\tan 70°} = \frac{\tan 80° - \tan 10°}{\tan(80° - 10°)}$

$= \frac{\tan 80° - \tan 10°}{\frac{\tan 80° - \tan 10°}{1 + \tan 80° \times \tan 10°}}$

$= 1 + \tan 80° \times \tan 10°$
$= 1 + \tan 80° \times \cot 80° = 1 + 1 = 2$.

25. (c) $\sin 4A - \cos 4A = \cos 2A - \sin 2A$
By squaring both sides $\sin^2 4A + \cos^2 4A - 2\sin 4A\cos 4A$
$= \sin^2 2A + \cos^2 2A - 2\sin 2A\cos 2A$
⇒ $2\sin 4A\cos 4A = 2\sin A\cos 2A$
⇒ $\cos 4A = 1/2$ ⇒ $\tan 4A = \sqrt{3}$

26. (e) $x\sin\theta = y\cos\theta$ ..... (1)
$x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$ ...... (2)

$\Rightarrow$ $x \sin\theta.\sin^2\theta + y\cos^3\theta = \sin\theta\cos\theta$

$\Rightarrow$ $y\cos\theta(\sin^2\theta + \cos^2\theta) = \sin\theta\cos\theta$

$\Rightarrow$ $y = \sin\theta$ from eqn. (1) $x = \cos\theta$

$\Rightarrow$ $x^2 + y^2 = \sin^2\theta + \cos^2\theta = 1.$

27. (d) $\cos 2x = -3/5 \Rightarrow 2\cos^2 x = -3/5 + 1 = 2/5$

$\Rightarrow$ $\cos^2 x = 1/5 \Rightarrow \cos x = \frac{1}{\sqrt{5}}.$

28. (d) $\sin 3\theta = \sin\theta \Rightarrow \sin 3\theta - \sin\theta = 0$

$\Rightarrow$ $2\cos\left(\frac{3\theta+\theta}{2}\right)\sin\left(\frac{3\theta-\theta}{2}\right) = 0$

$\Rightarrow$ $2\cos 2\theta.\sin\theta = 0$

$\Rightarrow$ $\cos 2\theta.\sin\theta = 0$

$\Rightarrow$ $\cos 2\theta = 0$ or $\sin\theta = 0$

$\Rightarrow$ $\cos 2\theta = \cos\left(\frac{\pi}{2}\right)$ or $\theta = 0, \pi, 2\pi$

$\Rightarrow$ $2\theta = \frac{\pi}{2}$ or $\theta = 0, \pi, 2\pi$

$\Rightarrow$ $\theta = \frac{n\pi}{4}$ or $\theta = 0, \pi, 2\pi$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ or $\theta = 0, \pi, 2\pi$

$\therefore$ Total number of solutions = 7.

29. (b) 114° 35′ 30″

$$30'' = \left(\frac{30}{60}\right)' = \left(\frac{1}{2}\right)'$$

$\therefore$ $35'\,30'' = \left(35\frac{1}{2}\right)' = \left(\frac{71}{2}\right)'$

$$= \left(\frac{71}{2}\times\frac{1}{60}\right)^\circ = \left(\frac{71}{120}\right)^\circ$$

So, $114°\,35'\,30'' = \left(114\frac{71}{120}\right)^\circ$

$$= \left(\frac{13680+71}{120}\right)^\circ$$

$$= \left(\frac{13751}{120}\times\frac{22}{7\times 180}\right)^C$$

$$= \left(\frac{302522}{151200}\right)^c$$

= 2 radian

30. (a) $\left(\left(\sin 22\frac{1}{2}^\circ + \cos 22\frac{1}{2}^\circ\right)^2\right)^2$

$$= \left(\sin^2 22\frac{1}{2}^\circ + \cos^2 22\frac{1}{2}^\circ + 2\sin 22\frac{1}{2}^\circ\cos 22\frac{1}{2}^\circ\right)^2$$

$= (1 + \sin 45°)^2$

$$= \left(1 + \frac{1}{\sqrt{2}}\right)^2$$

$$= 1 + \frac{1}{2} + 2\times\frac{1}{\sqrt{2}}$$

$$= \frac{3+2\sqrt{2}}{2}$$

31. (d) $\left(1+\cos 67\frac{1}{2}\right)\left(1+\cos\left(180-67\frac{1}{2}\right)\right)$

$$= \left(1+\cos 67\frac{1}{2}\right)\left(1-\cos 67\frac{1}{2}\right)$$

$$= \left(1-\cos^2 67\frac{1}{2}\right)$$

$$= \sin^2 67\frac{1}{2} = \left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)^2$$

32. (c) $\sin 2A = \frac{4}{5}$

$$\frac{4}{5} = \sin 2A = \frac{2\tan A}{1+\tan^2 A}$$

$4\tan 2A - 10\tan A + 4 = 0$

$4\tan 2A - 8\tan A - 2\tan A + 4 = 0$

$4\tan A(\tan A - 2) - 2(\tan A - 2) = 0$

$(4\tan A - 2)(\tan A - 2) = 0$

$\tan A = \frac{2}{4} = \frac{1}{2}$ and $\tan A = 2$

$\because$ $0 < A \le \frac{\pi}{8}$ $\because$ $\tan A = \frac{1}{2}$

33. (a) $\frac{\cos 10° - \sin 10°}{\cos 10° + \sin 10°}$

$$= \frac{1-\tan 10°}{1+\tan 10°}$$

$= \tan(45° - 10°) = \tan 35°$

34. (d)

(a) $\sin 2\pi = 0$: $\sin(-2x) = -\sin 2\pi = 0$

(b) $\tan 45 = \tan(-315°)$
$\tan 45° = -\tan 315 = -\tan(360 - 45)$
$\tan 45° = \tan 45°$

(c) $\cot(\tan^{-1} 0.5) = \tan(\cot^{-1} 0.5)$ (inverse)
$\frac{2}{1} = \frac{2}{1}$

(d) $\tan(360 + 60)$: $\tan(-60)$
$\tan 60$: $-\tan 60$

35. (b) $\sin\left(\frac{5\times 180}{12}\right) = \sin 75° = \frac{\sqrt{3}+1}{2\sqrt{2}}$

36. (a) $\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$ $\cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}+1}{2\sqrt{2}}$

$\cot\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3}$ $\therefore$ $3 > 2 > 1$

37. (a) Verify each option in the given equation to get (a).

38. (b) $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}$

$\Rightarrow$ $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}$

$= -\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}$

$= -\frac{\sin\frac{8\pi}{7}}{8\sin\frac{\pi}{7}} = -\frac{\left(\sin\frac{\pi}{7}\right)}{8\sin\frac{\pi}{7}} = \frac{1}{8}$

39. (b) $b\cos 2\theta + a\sin 2\theta$

$= b\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) + a\left(\frac{2+\tan\theta}{1+\tan 2\theta}\right)$

$= b\left(\frac{1-\frac{b^2}{a^2}}{1+\frac{b^2}{a^2}}\right) + a\left(\frac{2b/a}{1+\frac{b^2}{a^2}}\right) = b$

$\left(\because \tan\theta = \frac{b}{a}\right)$

40. (b) $\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\times\frac{\cos\theta+\sin\theta}{\cos\theta+\sin\theta}$

$= \frac{1+\sin 2\theta}{\cos 2\theta} = \sec 2\theta + \tan 2\theta$

41. (d) $\sin 36° \sin 72° \sin 108° \sin 144°$

$= \left(\frac{2}{2}\sin 36°\sin 72°\right)^2 = \frac{5}{16}$

$= \frac{1}{4}(\cos 36° - \cos 108°)^2$

$= \frac{1}{4}\left(\frac{\sqrt{5}+1}{4} + \frac{\sqrt{5}-1}{4}\right)^2 = \frac{5}{16}$

42. (b) $\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

$= \sqrt{\frac{1+\cos\left(\frac{\pi}{2}+\theta\right)}{1-\cos\left(\frac{\pi}{4}+\theta\right)}}$

$= \sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sqrt{\frac{\operatorname{cosec}\theta - 1}{\cos\theta + 1}}$

$= \sqrt{\frac{\frac{p+q}{p-q}-1}{\frac{p+q}{p-q}+1}} = \sqrt{\frac{q}{p}}$

43. (b) It is given that α, β, γ, δ are the smallest positive angles in ascending order of magnitude such that $\sin\alpha = \sin\beta = \sin\gamma = \sin\delta = k$ (positive quantity)

$\Rightarrow$ $\beta = \pi - \alpha, \gamma = 2\pi + \alpha$ and
$\delta = 3\pi - \alpha$

$\therefore$ $4\sin\frac{\alpha}{2} + 3\sin\frac{\beta}{2} + 2\sin\frac{\gamma}{2} + \sin\frac{\delta}{2}$

$= 4\sin\frac{\alpha}{2} + 3\cos\frac{\alpha}{2} - 2\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}$

$= 2\left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\right)$

$= 2\sqrt{1+\sin\alpha} = 2\sqrt{1+k}$

QUESTION BANK

1. If $\sin\theta + \sin\phi = a$ and $\cos\theta + \sin\phi = b$ then $\tan\frac{\theta-\phi}{2}$ is equal to

(a) $\sqrt{\frac{a^2+b^2}{4-a^2-b^2}}$ (b) $\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$

(c) $\sqrt{\frac{a^2+b^2}{4+a^2+b^2}}$ (d) $\sqrt{\frac{4+a^2+b^2}{a^2+b^2}}$

2. $\frac{\sin^2 A-\sin^2 B}{\sin A\cos A-\sin B\cos B}$ is equal to

(a) $\cot(A-B)$
(b) $\cot(A+B)$
(c) $\tan(A-B)$
(d) $\tan(A+B)$

3. The value of $\tan(-945°)$

(a) –1 (b) –2
(c) –3 (d) –4

4. $\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}$ is equal to

(a) 1/2 (b) 1/8
(c) 1/4 (d) None of these

5. If $\tan\theta = t$ then $\tan 2\theta + \sec 2\theta$ is equal to

(a) $\frac{1+t}{1-t}$ (b) $\frac{1-t}{1+t}$

(c) $\frac{2t}{1-t}$ (d) $\frac{2t}{1+t}$

6. $\frac{\sin 70°+\cos 40°}{\cos 70°+\sin 40°}$ is equal to

(a) $\frac{1}{\sqrt{3}}$ (b) $\sqrt{3}$

(c) $\frac{1}{2}$ (d) 1

7. sin 163° cos 347° + sin 73° sin 167° is equal to

(a) 0 (b) 1/2
(c) 1 (d) None of these

8. If $\sin 2\theta + \sin 2\phi = 1/2$, $\cos 2\theta + \cos 2\phi = 3/2$ then $\cos^2(\theta-\phi)$ is equal to

(a) 3/8 (b) 5/8
(c) 3/4 (d) 5/4

9. If the $\sin x + \sin^2 x = 1$ then the value of $\cos^{12} x + 3\cos^{10} x + 3\cos^8 x + \cos^6 x - 2$ is

(a) 0 (b) 1
(c) –1 (d) 2

10. If $\tan x = \frac{b}{a}$ then $\sqrt{\frac{a+b}{a-b}}+\sqrt{\frac{a-b}{a+b}}$ is equal to

(a) $\frac{2\sin x}{\sqrt{\sin 2x}}$ (b) $\frac{2\cos x}{\sqrt{\cos 2x}}$

(c) $\frac{2\cos x}{\sqrt{\sin 2x}}$ (d) $\frac{2\sin x}{\sqrt{\cos 2x}}$

11. In any triangle ABC, $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}$ is equal to

(a) $1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(b) $1-2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(c) $1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

(d) $1-2\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

12. If A, B, C are angles of a triangle, then $\sin 2A + \sin 2B - \sin 2C$ is equal to

(a) $4\sin A\cos B\cos C$
(b) $4\cos A\cos B\cos C$
(c) $4\cos A\cos B\sin C$
(d) $4\sin A\sin B\cos C$

13. The value of $\sin\frac{\pi}{16}.\sin\frac{3\pi}{16}.\sin\frac{5\pi}{16}.\sin\frac{7\pi}{16}$ is

(a) $\frac{\sqrt{2}}{16}$ (b) $\frac{1}{8}$

(c) $\frac{1}{16}$ (d) $\frac{\sqrt{2}}{32}$

14. If $2\cos\theta = x+\frac{1}{x}$ and $2\cos\phi = y+\frac{1}{y}$ then $2\cos(\theta+\phi)$ is equal to

(a) $\frac{x}{y}+\frac{y}{x}$ (b) $xy-\frac{1}{xy}$

(c) $xy+\frac{1}{xy}$ (d) $\frac{y}{x}-\frac{x}{y}$

15. In equation $\sin x + \cos x = 1$ value/values of x is/are

(a) 0 (b) $\frac{\pi}{2}$

(c) $0, \frac{\pi}{2}$ (d) $0, \frac{\pi}{2}, \pi$

16. $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x)$ is equal to

(a) 11 (b) 12

(c) 13 (d) 14

17. The value of the expression $\sqrt{3}$ cosec 20° – sec 20° is equal to

(a) 2 (b) 2 sin 20°/sin 40°

(c) 4 (d) 4 sin 20°/sin 40°

18. The general values of θ satisfying the equation $2\sin^2\theta - 3\sin\theta - 2 = 0$ is

(a) $n\pi + (-1)^n \pi/6$ (b) $n\pi + (-1)^n \pi/2$

(c) $n\pi + (-1)^n 5\pi/6$ (d) $n\pi + (-1)^n 7\pi/6$

19. If $\alpha + \beta = \pi/2$ and $\beta + \lambda = \alpha$ then tan α equals

(a) 2 (tan β + tan λ)

(b) tan β + tan λ

(c) tan β + 2 tan λ

(d) 2 tan β + tanλ

20. The number of integral values of k for which the equation $3 \cos x + 5 \sin x = 2k + 1$ has a solution is

(a) 4 (b) 8

(c) 6 (d) 12

21. Given both θ and ϕ are acute angles and $\sin\theta = \frac{1}{2}, \cos\phi = \frac{1}{3}$, then the value of θ + ϕ belongs to

(a) $\left(\frac{\pi}{3}, \frac{\pi}{2}\right]$ (b) $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right]$

(c) $\left(\frac{2\pi}{3}, \frac{5\pi}{6}\right]$ (d) $\left(\frac{5\pi}{6}, \pi\right]$

22. $\cos(\alpha - \beta) = 1$ and $\cos(\alpha + \beta) = 1/2$ where $\alpha, \beta \in [-\pi, \pi]$ pairs of α, β which satisfy both the equations is/are

(a) 0 (b) 1

(c) 2 (d) 4

23. The number of values of x in the interval $[0, 4\pi]$ satisfying the equation $3 \sin^2 x - 7\sin x + 2 = 0$ is

(a) 4 (b) 5

(c) 6 (d) None

24. If $\cos(\alpha + \beta) = \frac{4}{5}$, $\sin(\alpha - \beta) = \frac{5}{13}$ and α, β lies between 0 and $\frac{\pi}{4}$, then tan 2α is equal to

(a) 56/33 (b) –16/33

(c) 47/18 (d) 19/24

25. $\sin\alpha \sin\beta - \cos\alpha \cos\beta + 1 = 0$, then the value of $1 + \cot\alpha \tan\beta$ is equal to

(a) –1 (b) –2

(c) 1 (d) 0

26. If 3 tanθ tan ϕ = 1, then the value of $\frac{\cos(\theta - \varphi)}{\cos(\theta + \varphi)}$ is

(a) –2 (b) 1

(c) –1 (d) 2

27. If $(1 + \tan\alpha)(1 + \tan 4\alpha) = 2$, $\alpha \in (0, \pi/16)$ then α is equal to

(a) $\frac{\pi}{20}$ (b) $\frac{\pi}{30}$

(c) $\frac{\pi}{40}$ (d) $\frac{\pi}{60}$

28. The number value of $\tan\frac{\pi}{3} + 2\tan\frac{2\pi}{3} + 4\tan\frac{4\pi}{3} + 8\tan\frac{8\pi}{3}$ is equal to

(a) $-5\sqrt{3}$ (b) $-\frac{5}{\sqrt{3}}$

(c) $5\sqrt{3}$ (d) $\frac{5}{\sqrt{3}}$

29. If $\sec\theta - 1 = (\sqrt{2} - 1)\tan\theta$, then θ is equal to

(a) $(2n - 1)\pi$ (b) $2n\pi + \frac{\pi}{4}$

(c) $2n\pi - \frac{\pi}{4}$ (d) $2n\pi + \frac{\pi}{3}$

30. If $\frac{1}{6}$ sin θ, cos θ, tanθ are in G.P., then θ is equal to

(a) $2n\pi \pm \frac{\pi}{3}$ (b) $2n\pi \pm \frac{\pi}{6}$

(c) $n\pi + (-1)^n \frac{\pi}{3}$ (d) $n\pi + \frac{\pi}{3}$

31. General solution of the equation $\tan\theta + \tan 4\theta + \tan 7\theta = \tan\theta \tan 4\theta \tan 7\theta$ is

(a) $\theta = n\pi/12$, where $n \in Z$

(b) $\theta = n\pi/9$, where $n \in Z$

(c) $\theta = n\pi + \pi/12$, where $n \in Z$
(d) None of these

32. Total number of solutions of $\tan x + \cot x = 2 \operatorname{cosec} x$ in $[-2\pi, 2\pi]$ is equal to
(a) 2 (b) 4
(c) 6 (d) 8

33. $\tan\left(\frac{p\pi}{4}\right) = \cot\left(\frac{q\pi}{4}\right)$ if
(a) $p + q = 0$ (b) $p + q = 2n + 1$
(c) $p + q = 2n$ (d) $p + q = 2(2n + 1)$

34. A man from the top of a 100 m high tower sees a car moving towards the tower at an angle of depression of 30°. After some time, the angle of depression becomes 60°. The distance (in metres) travelled by the car daily this time is
(a) $100\sqrt{3}$ (b) $\frac{200\sqrt{3}}{3}$
(c) $\frac{100\sqrt{3}}{3}$ (d) $200\sqrt{3}$

35. A tower stands at the centre of a circular park. A and B are two points on the boundary of the park such that AB (= a) subtends an angle of 60° at the foot of the tower and the angle of elevation of the top of the tower from A or B is 30°. The height of the tower is
(a) $2a\sqrt{3}$ (b) $\frac{a}{\sqrt{3}}$
(c) $a\sqrt{3}$ (d) $\frac{2a}{\sqrt{3}}$

36. $a \cot A + b \cot B + \cot C$ is equal to
(a) $r + R$ (b) $r - R$
(c) $2(r + R)$ (d) $2(r - R)$

37. Angle of elevation of a tower from place A which is due south of the tower is α and from another place B due west of A is β then height of tower is not equal to
(a) $\frac{AB}{\sqrt{\cos^2\beta - \cos^2\alpha}}$
(b) $\frac{AB \sin\alpha \sin\beta}{\sqrt{\sin^2\alpha - \sin^2\beta}}$
(c) $\frac{AB \tan\alpha \tan\beta}{\sqrt{\tan^2\alpha - \tan^2\beta}}$
(d) $\frac{AB \cos\alpha \cos\beta}{\sqrt{\sin(\alpha+\beta)\sin(\alpha-\beta)}}$

38. If in a ΔABC, $\tan A + \tan B + \tan C$ has the value 6 then the value of $\cot A . \cot B . \cot C$ is equal to
(a) 6 (b) 1
(c) 1/6 (d) None of these

39. If $\operatorname{cosec}\theta - \cot\theta = \frac{1}{2}$, $0 < \theta < \frac{\pi}{2}$, then $\cos\theta$ is equal to
(a) 5/3 (b) 3/5
(c) –3/5 (d) –5/3

40. The roots of the equation $4x^2 - 2\sqrt{5}x + 1 = 0$ are
(a) cos 18°, cos 36° (b) sin 36°, cos 18°
(c) sin 18°, cos 36° (d) sin 36°, sin 18°

41. If $\tan A - \tan B = x$ and $\cot B - \cot A = y$, then $\cot (A - B)$ is equal to
(a) $\frac{1}{x} - \frac{1}{y}$ (b) $\frac{1}{x} + \frac{1}{y}$
(c) $\frac{1}{y} - \frac{1}{x}$ (d) None of these

42. The number of distinct value of θ satisfying $0 \le \theta \le \pi$ and satisfying the equation $\sin\theta + 5\theta = \sin 3\theta$, is
(a) 6 (b) 7
(c) 8 (d) 9

43. If H is the orthocentre of ΔABC, then AH is equal to
(a) $c \cot A$ (b) $b \cot A$
(c) $a \cot B$ (d) $a \cot A$

44. In ΔABC, $b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2}$ is equal to
(a) $3s$ (b) $2s$
(c) s (d) None of these

45. If the radius of the incircle of a triangle with its sides $5k$, $6k$ and $5k$ is 6, then k is equal to
(a) 3 (b) 4
(c) 5 (d) 6

46. The equations of the internal bisector of $\angle BAC$ of ΔABC with vertices $A(5, 2)$, $B(2, 3)$ and $C(6, 5)$ is
(a) $2x + y + 12 = 0$ (b) $x + 2y - 12 = 0$
(c) $2x + y - 12 = 0$ (d) None of these

47. If the orthocentre and centroid of a triangle are (–3, 5) and (3, 3) respectively then the circumcentre is
(a) (0, 4) (b) (6, –2)
(c) (0, 8) (d) (6, 2)

48. If a, c, b are three terms of a G.P., then the line $ax + by + c = 0$
(a) has a fixed direction
(b) always passes through a fixed point
(c) forms a triangle with the axes whose area is constant
(d) always cut intercepts on the axes such that their sum is zero.

49. A line through the point $A(2, 0)$, which makes an angle of 30° with the positive direction of x-axis is rotated about A in clockwise direction through an angle 15°. The equation of the straight line in the new position is
(a) $(2-\sqrt{3})\ x - y - 4 + 2\sqrt{3} = 0$
(b) $(2-\sqrt{3})\ x + y - 4 + 2\sqrt{3} = 0$
(c) $(2-\sqrt{3})\ x - y + 4 + 2\sqrt{3} = 0$
(d) None of these

50. A rectangle has two opposite vertices at the points (1, 2) and (5, 5). If the other vertices lie on the line $x = 3$, then the coordinates of the other vertices are
(a) (3, –1), (3, –6)
(b) (3, 1), (3, 5)
(c) (3, 2), (3, 6)
(d) (3, 1), (3, 6)

51. If the lines $ax + 12y + 1 = 0$, $bx + 13y + 1 = 0$ and $cx + 14y + 1 = 0$ are concurrent, then a, b, c are in
(a) H.P. (b) G.P.
(c) A.P. (d) None of these

52. The base of a triangle lies along the line $x = a$ and is of length a. The area of the triangle is a^2, if the vertex lies on the line
(a) $x = 0$
(b) $x = -a$
(c) $x = 3a$
(d) $x = -3a$

53. If $3\sin\theta + 5\cos\theta = 5$, then the value of $|5\sin\theta - 3\cos\theta|$ is
(a) 3 (b) –3
(c) 5 (d) –5

54. The number of value of x in $[0, 2\pi]$ satisfying the equation $|\cos x - \sin x| \geq \sqrt{2}$, is
(a) 0 (b) 1
(c) 2 (d) 3

55. If the sides of a right-angled triangle are in A.P., then tangents of the acute angles of the triangle are
(a) $\sqrt{\sqrt{3}+\frac{1}{2}}, \sqrt{\sqrt{3}-\frac{1}{2}}$
(b) $\sqrt{\sqrt{5}-\frac{1}{2}}, \sqrt{\sqrt{5}+\frac{1}{2}}$
(c) $\sqrt{3}, \frac{1}{\sqrt{3}}$
(d) $\frac{3}{4}, \frac{4}{3}$

56. In a ΔABC, let $\angle C = \frac{\pi}{2}$. If r is the inradius and R is the circumradius of the triangle, then $2(r + R)$ is equal to
(a) $a + b$ (b) $b + c$
(c) $c + a$ (d) $a + b + c$

57. If p_1, p_2, p_3 are the perpendiculars from the angular points of a triangle on the opposite sides, then $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}$ is equal to
(a) $\frac{1}{r}$ (b) $\frac{2}{r}$
(c) $\frac{3}{r}$ (d) None of these

58. An aeroplane flying at a height of 300 metres above the ground passes vertically above another plane at an instant when the angle of elevation of the two planes from the same point on the ground are 60° and 45° respectively. Then the height of the lower plane from the ground is (in metres)
(a) $100\sqrt{3}$ (b) $\frac{100}{\sqrt{3}}$
(c) 50 (d) $150(\sqrt{3}+1)$

59. If in a triangle ABC $\cos^2 A + \cos^2 B + \cos^2 C = 1$, then the triangle is always
(a) right angled
(b) acute angled
(c) obtuse angled
(d) equilateral triangle

60. Consider a triangle PQR with $P \equiv (0, 0)$, $Q \equiv (a, 0)$, $R \equiv (0, b)$. Then the centroid, orthocentre and circumcentre

(a) lie on a straight line

(b) form a scalene triangle with area $\frac{1}{2}|ab|$

(c) form a right angled triangle with area $\frac{1}{2}|ab|$

(d) None of these

61. The general value of θ satisfying $\cot\theta = -\sqrt{3}$ and $\cos\theta = \frac{\sqrt{3}}{2}$ is

(a) $n\pi + \frac{\pi}{3}$ (b) $n\pi - \frac{\pi}{3}$

(c) $2n\pi + \frac{\pi}{6}$ (d) $2n\pi - \frac{\pi}{6}$

62. The number of solutions of $\sin 2x + \cos 4x = 2$ in the interval $(0, 2\pi)$ is

(a) 0 (b) 1

(c) 2 (d) infinite

Instructions for questions 63 to 65

Each question has 4 choices (a), (b), (c) and (d), out of which only one is correct.

(a) Assertion is True, Reason is True and Reason is a correct explanation for Assertion.

(b) Assertion is True, Reason is True and Reason is not a correct explanation for Assertion.

(c) Assertion is True, Reason is False

(d) Assertion is False, Reason is True

63. Assertion: The values of θ for which $\frac{1 - i\sin\theta}{1 + i\cos\theta}$ is purely real then $\theta = 2n\pi + \frac{3\pi}{4}$ or $\theta = 2n\pi - \frac{\pi}{4}$ where $n \in Z$

Reason: If $\sin\theta \cos\theta = 2$, then there does not exist any θ.

64. Assertion: $\cos 1 < \cos 7$

Reason: $1 < 7$

65. Assertion: The minimum value of $27^{\cos 2x}\, 81^{\sin 2x}$ is $\frac{1}{243}$

Reason: The minimum value of $a\cos\theta + b\sin\theta$ is $-\sqrt{a^2 + b^2}$

ANSWERS

PART A

LECTURE 1

Worksheet: To Check The Preparation Level

1. (d)	**5.** (b)	**9.** (a)	**13.** (a)
2. (a)	**6.** (c)	**10.** (b)	**14.** (c)
3. (c)	**7.** (c)	**11.** (b)	**15.** (c)
4. (b)	**8.** (c)	**12.** (b)	

LECTURE 2

Unsolved Objective Problems (Identical Problems For Practice) For Improving Speed With Accuracy

1. (d)	**4.** (b)	**7.** (c)	**10.** (d)
2. (b)	**5.** (c)	**8.** (b)	**11.** (b)
3. (c)	**6.** (c)	**9.** (c)	

Worksheet: To Check The Preparation Level

1. (d)	**5.** (b)	**9.** (d)	**13.** (c)
2. (b)	**6.** (c)	**10.** (d)	**14.** (c)
3. (c)	**7.** (d)	**11.** (a)	**15.** (a)
4. (d)	**8.** (a)	**12.** (a)	

LECTURE 3

Unsolved Objective Problems (Identical Problems For Practice) For Improving Speed With Accuracy

1. (c)	**6.** (b)	**11.** (a)	**16.** (b)
2. (a)	**7.** (b)	**12.** (a)	**17.** (a)
3. (a)	**8.** (c)	**13.** (b)	**18.** (c)
4. (b)	**9.** (c)	**14.** (a)	
5. (c)	**10.** (a)	**15.** (d)	

Worksheet: To Check The Preparation Level

1. (a)	**5.** (c)	**9.** (a)	**13.** (c)
2. (d)	**6.** (d)	**10.** (d)	**14.** (a)
3. (d)	**7.** (b)	**11.** (d)	**15.** (c)
4. (a)	**8.** (d)	**12.** (b)	

LECTURE 4

Unsolved Objective Problems (Identical Problems For Practice) For Improving Speed With Accuracy

1. (a)	**4.** (d)	**7.** (d)	**10.** (a)
2. (c)	**5.** (a)	**8.** (d)	**11.** (a)
3. (a)	**6.** (c)	**9.** (d)	

Worksheet: To Check The Preparation Level

1. (a)	**6.** (c)	**11.** (a)	**16.** (b)
2. (a)	**7.** (c)	**12.** (b)	
3. (b)	**8.** (c)	**13.** (b)	
4. (b)	**9.** (c)	**14.** (c)	
5. (a)	**10.** (b)	**15.** (b)	

LECTURE 5

Unsolved Objective Problems (Identical Problems For Practice) For Improving Speed With Accuracy

1. (a)	**4.** (c)	**7.** (a)
2. (c)	**5.** (c)	
3. (a)	**6.** (b)	

Worksheet: To Check The Preparation Level

1. (a)	**4.** (b)	**7.** (a)
2. (c)	**5.** (d)	**8.** (a)
3. (c)	**6.** (a)	**9.** (b)

LECTURE 6

Unsolved Objective Problems (Identical Problems For Practice) For Improving Speed With Accuracy

1. (b)	**6.** (b)	**11.** (d)	**16.** (c)
2. (a)	**7.** (c)	**12.** (c)	**17.** (b)
3. (a)	**8.** (a)	**13.** (c)	**18.** (c)
4. (a)	**9.** (b)	**14.** (d)	**19.** (c)
5. (d)	**10.** (c)	**15.** (c)	

Worksheet: To Check The Preparation Level

1. (b)	**5.** (b)	**9.** (a)	**13.** (b)
2. (a)	**6.** (b)	**10.** (c)	**14.** (c)
3. (b)	**7.** (b)	**11.** (a)	**15.** (a)
4. (a)	**8.** (d)	**12.** (c)	

LECTURE 7

Unsolved Objective Problems (Identical Problems For Practice) For Improving Speed With Accuracy

1. (c)	**4.** (c)	**7.** (a)	**10.** (b)
2. (a)	**5.** (a)	**8.** (d)	
3. (a)	**6.** (a)	**9.** (b)	

Worksheet: To Check The Preparation Level

1. (b)	**4.** (c)	**7.** (a)
2. (b)	**5.** (b)	**8.** (d)
3. (b)	**6.** (c)	**9.** (b)

PART B

LECTURE 1

Unsolved Objective Problems (Identical Problems For Practice) For Improving Speed With Accuracy

1. (c)	**8.** (b)	**15.** (a)	**22.** (c)
2. (b)	**9.** (b)	**16.** (b)	**23.** (a)
3. (a)	**10.** (a)	**17.** (b)	**24.** (b)
4. (a)	**11.** (a)	**18.** (d)	**25.** (a)
5. (a)	**12.** (b)	**19.** (d)	**26.** (c)
6. (b)	**13.** (d)	**20.** (c)	
7. (a)	**14.** (c)	**21.** (c)	

Worksheet: To Check The Preparation Level

1. (c)	**9.** (d)	**17.** (a)	**25.** (d)
2. (d)	**10.** (a)	**18.** (a)	**26.** (a)
3. (c)	**11.** (c)	**19.** (c)	**27.** (c)
4. (b)	**12.** (b)	**20.** (d)	**28.** (a)
5. (b)	**13.** (c)	**21.** (a)	**29.** (d)
6. (b)	**14.** (a)	**22.** (b)	**30.** (b)
7. (b)	**15.** (c)	**23.** (d)	
8. (d)	**16.** (a)	**24.** (a)	

PART C

LECTURE 1

Unsolved Objective Problems (Identical Problems For Practice) For Improving Speed With Accuracy

1. (b)	**8.** (b)	**15.** (a)	**22.** (c)
2. (b)	**9.** (d)	**16.** (a)	**23.** (a)
3. (b)	**10.** (d)	**17.** (c)	
4. (a)	**11.** (d)	**18.** (b)	
5. (a)	**12.** (c)	**19.** (c)	
6. (d)	**13.** (a)	**20.** (c)	
7. (b)	**14.** (b)	**21.** (c)	

Worksheet: To Check The Preparation Level

1. (b)	**5.** (d)	**9.** (b)	**13.** (d)
2. (d)	**6.** (b)	**10.** (a)	**14.** (c)
3. (d)	**7.** (c)	**11.** (c)	**15.** (c)
4. (b)	**8.** (c)	**12.** (b)	

LECTURE 2

Unsolved Objective Problems (Identical Problems For Practice) For Improving Speed With Accuracy

1. (a)	**7.** (a)	**13.** (b)	**19.** (d)
2. (d)	**8.** (c)	**14.** (d)	**20.** (a)
3. (c)	**9.** (a)	**15.** (b)	**21.** (a)
4. (d)	**10.** (d)	**16.** (c)	**22.** (c)
5. (c)	**11.** (c)	**17.** (b)	
6. (c)	**12.** (a)	**18.** (a)	

Worksheet: To Check The Preparation Level

1. (b)	**5.** (c)	**9.** (a)	**13.** (c)
2. (b)	**6.** (a)	**10.** (a)	**14.** (a)
3. (b)	**7.** (b)	**11.** (c)	**15.** (d)
4. (a)	**8.** (c)	**12.** (d)	

PART E

LECTURE 1

Unsolved Objective Problems (Identical Problems For Practice) For Improving Speed With Accuracy

1. (b)	**6.** (b)	**11.** (b)	**16.** (b)
2. (a)	**7.** (b)	**12.** (b)	**17.** (c)
3. (a)	**8.** (d)	**13.** (b)	**18.** (c)
4. (c)	**9.** (d)	**14.** (c)	**19.** (a)
5. (b)	**10.** (d)	**15.** (c)	**20.** (c)

Worksheet: To Check The Preparation Level

1. (d)	**4.** (b)	**7.** (c)	**10.** (a)
2. (b)	**5.** (c)	**8.** (b)	
3. (d)	**6.** (c)	**9.** (b)	

PART F

LECTURE 1

Question Bank

1. (b)	**18.** (d)	**35.** (b)	**52.** (c)
2. (d)	**19.** (c)	**36.** (b)	**53.** (c)
3. (a)	**20.** (c)	**37.** (c)	**54.** (b)
4. (b)	**21.** (b)	**38.** (b)	**55.** (d)
5. (a)	**22.** (d)	**39.** (d)	**56.** (c)
6. (b)	**23.** (c)	**40.** (a)	**57.** (c)
7. (b)	**24.** (a)	**41.** (b)	**58.** (a)
8. (b)	**25.** (d)	**42.** (b)	**59.** (d)
9. (c)	**26.** (d)	**43.** (b)	**60.** (a)
10. (b)	**27.** (a)	**44.** (b)	**61.** (d)
11. (a)	**28.** (a)	**45.** (c)	**62.** (a)
12. (c)	**29.** (b)	**46.** (c)	**63.** (a)
13. (a)	**30.** (a)	**47.** (a)	**64.** (b)
14. (c)	**31.** (d)	**48.** (c)	**65.** (a)
15. (c)	**32.** (b)	**49.** (c)	
16. (c)	**33.** (d)	**50.** (a)	
17. (c)	**34.** (b)	**51.** (c)	