TRIGONOMETRY

A CLEVER STUDY GUIDE

MAA PRESS

PROBLEM BOOK SERIES

JAMES TANTON

Trigonometry

A Clever Study Guide

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by

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MAA

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Part II: Solutions

About these Study Guides

The Mathematical Association of America's American Mathematics Competitions' website, www.maa.org/math-competitions, announces loud and clear:

> Teachers and schools can benefit from the chance to challenge students with interesting mathematical questions that are aligned with curriculum standards at all levels of difficulty.

For over six decades dedicated and clever folk of the MAA have been creating and collating marvelous, stand-alone mathematical tidbits and sharing them with the world of students and teachers through mathematics competitions. Each question devised serves as a portal for deep mathematical mulling and exploration. Each is an invitation to revel in the mathematical experience.

And more! In bringing together all the questions that link to one topic a coherent mathematical landscape, ripe for a guided journey of study, emerges. The goal of this series is to showcase the landscapes that lie within the MAA's competition resources and to invite students, teachers, and all life-long learners, to engage in the mathematical explorations they invite. Learners will not only deepen their understanding of curriculum topics, but also gain the confidence to play with ideas and work to become agile intellectual thinkers.

I was recently asked by some fellow mathematics educators what my greatest wish is for our next generation of students. I responded:

> . . . a personal sense of curiosity coupled with the confidence to wonder, explore, try, get it wrong, flail, go on tangents, make connections, be flummoxed, try some more, wait for epiphanies, lay groundwork for epiphanies, go down false leads, find moments of success nonetheless, savor the "ahas," revel in success, and yearn for more.

Our complex society demands of our next generation not only mastery of quantitative skills, but also the confidence to ask new questions, to innovate, and to succeed. Innovation comes only from bending and pushing ideas and being willing to flail. One must rely on one's wits and on one's common sense. And one must persevere. Relying on memorized answers to previously asked—and answered!—questions does not push the frontiers of business research and science research.

The MAA competition resources provide today's mathematics thinkers, teachers, and doers:

- the opportunity to learn and to teach problem-solving, and
- the opportunity to review the curriculum from the perspective of understanding and clever thinking, letting go of memorization and rote doing.

Each of these study guides

- runs through the entire standard curriculum content of a particular mathematics topic from a sophisticated and mathematically honest point of view,
- illustrates in concrete ways how to implement problem-solving strategies for problems related to the particular mathematics topic, and
- provides a slew of practice problems from the MAA competition resources along with their solutions.

As such, these guides invite you to

- review and deeply understand mathematics topics,
- practice problem-solving,
- gain incredible intellectual confidence,

and, above all,

- to enjoy mathematics!

This Guide and Mathematics Competitions

Whether you enjoy the competition experience and are motivated by it and delighted by it, or you, like me, tend to shy away from it, this guide is for you!

We all have our different styles and proclivities for mathematics thinking, doing, and sharing, and they are all good. The point, in the end, lies with the enjoyment of the mathematics itself. Whether you like to solve problems under the time pressure of a clock or while mulling on a stroll, problem-solving is a valuable art that will serve you well in all aspects of life.

This guide teaches how to think about content and how to solve challenges. It serves both the competition doers and the competition non-doers. That is, it serves the budding and growing mathematicians we all are.

On Competition Names

This guide pulls together problems from the history of the MAA's American competition resources.

The competitions began in 1950 with the Metropolitan New York Section of the MAA offering a "Mathematical Contest" each year for regional high-school students. They became national endeavors in 1957 and adopted the name "Annual High School Mathematics Examination" in 1959. This was changed to the "American High School Mathematics Examination" in 1983.

In this guide, the code "#22, AHSME, 1972," for example, refers to problem number 22 from the 1972 AHSME, Annual/American High School Mathematics Examination.

In 1985 a contest for middle school students was created, the "American Junior High School Mathematics Examination," and shortly thereafter the contests collectively became known as the "American Mathematics Competitions," the AMC for short. In the year 2000 competitions limited to high-school students in grades 10 and below were created and the different levels of competitions were renamed the AMC 8, the AMC 10, and the AMC 12.

In this guide, "#13, AMC 12, 2000," for instance, refers to problem number 13 from the 2000 AMC 12 examination.

In 2002, and ever since, two versions of the AMC 10 and the AMC 12 are administered, about two weeks apart, and these are referred to as the AMC 10A, AMC 10B, AMC 12A, and AMC 12B.

In this guide, "#24, AMC 10A, 2013," for instance, refers to problem number 24 from the 2013 AMC 10A examination.

On Competition Success

Let's be clear:

> "I am using this guide for competition practice. Does this guide promise me 100% success on all mathematics competitions, each and every time?"
>
> Of course not!

But this guide does offer, if worked through with care

- Feelings of increased confidence when taking part in competitions.
- Clear improvement on how you might handle competition problems.
- Clear improvement on how you might handle your emotional reactions to particularly outlandish-looking competition problems.

Mathematics is an intensely human enterprise and one cannot underestimate the effect of emotions when doing mathematics and attempting to solve challenges. This guide gives the human story that lies behind the mathematics content and discusses the human reactions to problem-solving.

As we shall learn, the first and the most important, effective step in solving a posed problem is

STEP 1: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

This guide provides practical content knowledge, problem-solving solving tools and techniques, and concrete discussion on getting over the barriers of emotional blocks. Even though its goal is not necessarily to improve competition scores, these are the tools that nonetheless lead to that outcome!

This Guide and the Craft of Solving Problems

Success in mathematics—however you wish to define it—comes from a strong sense of self-confidence: the confidence to acknowledge one's emotions and to calm them down, the confidence to pause over ideas and come to educated guesses or conclusions, the confidence to rely on one's wits to navigate through unfamiliar terrain, the confidence to choose understanding over impulsive rote doing, and the confidence to persevere.

Success and joy in science, business, and in life doesn't come from programmed responses to pre-set situations. It comes from agile and adaptive thinking coupled with reflection, assessment, and further adaptation.

Students—and adults too—are often under the impression that one should simply be able to leap into a mathematics challenge and make instant progress of some kind. This not how mathematics works! It is okay to fumble, and flail, and to try out ideas that turn out not help in the end. In fact, this *is* the problem-solving process and making multiple false starts should not at all be dismissed! (Think of how we solve problems in everyday life.)

It is also a natural part of the problem-solving process to react to a problem.

"This looks scary."
"This looks fun."
"I don't have a clue what the question is even asking!"
"Wow. Weird! Could that really be true?"
"Who cares?"
"I don't get it."
"Is this too easy? I am suspicious."

We are each human, and the first step to solving a problem is to come to terms with our emotional reaction to it—especially if that reaction is one of being overwhelmed. Step 1 to problem-solving mentioned in the previous section in vital.

Once we have nerves in check, at least to some degree, there are a number of techniques one could try in order to make some progress with the problem.

The ten strategies we briefly outline in the appendix are discussed in full detail on the MAA's CURRICULUM INSPIRATIONS webpage, www.maa.org/ci. There you will find essays and videos explaining each technique in full, with worked examples and slews of further practice examples and their solutions.

This guide also contains worked examples. Look for the FEATURED PROBLEMS in sections 1, 3, 5, 6, 10, 12, 13, 14, 16, 17, and 18 where I share with you my own personal thoughts, emotions, and eventual approach in solving a given problem using one of the ten problem-solving strategies.

This Guide and Mathematics Content: Trigonometry

This guide covers the story of trigonometry. It is a swift overview, but it is complete in the context of the content discussed in beginning and advanced high-school courses. The purpose of these notes is to supplement and put into perspective the material of any course on the subject you may have taken or are currently taking. (These notes will be tough going for those encountering trigonometry for the very first time!)

In reading and working through the material presented here you will

- see the story in of trigonometry in a new light,
- see the reasons why we, mankind, developed the subject in the way we did,
- begin to move away from memorization and half-understanding to deep understanding, and thereby
- be equipped for agile, clever thinking in the subject.

These notes will guide you through to sound mathematical doing in trigonometry and, of course, to sound problem-solving skills as well.

For Educators: This Guide and the Common Core State Standards

The very first Standard for Mathematical Practice asks—requires!—that we educators pay explicit attention to teaching problem-solving:

MP1 Make sense of problems and persevere in solving them.

And one can argue that several, if not all, of the remaining seven Standards for Mathematical Practice can play prominent roles in supporting this first standard. For example, when solving a problem, students will likely be engaging in the activities of these standards too:

MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure.

These guides on *Clever Studying through the MAA AMC* align directly with the Standards for Mathematical Practice.

And each individual guide directly addresses content standards too! This volume on trigonometry attends to the following standards. (The section numbers refer to the sections of this text in which the standards appear.)

8.G.6 Explain a proof of the Pythagorean Theorem and its converse. (Sections 1 and 14.)

8.G.7 Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions. (Section 1.)

8.G.8 Apply the Pythagorean Theorem to find the distance between two points in a coordinate system. (Section 1.)

F-TF.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle. (Section 4.)

F-TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle. (Sections 2, 3, 4, and 11.)

F-TF.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$, and use the unit circle to express the values of sines, cosines, and tangents for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number. (Sections 3 and 7.)

F-TF.4 (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions. (Sections 3, 5, 6, and 7.)

F-TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline. (Section 18.)

F-TF.6 (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed. (Section 12.)

F-TF.7 (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context. (Section 12.)

F-TF.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle. (Section 7.)

F-TF.9 (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems. (Section 13.)

G-SRT.6 Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles. (Section 10.)

G-SRT.7 Explain and use the relationship between the sine and cosine of complementary angles. (Section 10.)

G-SRT.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems. (Sections 8, 9, and 10.)

G-SRT.9 (+) Derive the formula $A = 1/2\,ab\sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side. (Sections 15 and 17.)

G-SRT.10 (+) Prove the Laws of Sines and Cosines and use them to solve problems. (Sections 14, 16, and 17.)

G-SRT.11 (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces). (Sections 14 and 16.)

Part I

Trigonometry

1

The Backbone Theorem: The Pythagorean Theorem

Common Core State Standards

8.G.6 Explain a proof of the Pythagorean Theorem and its converse.

8.G.7 Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.

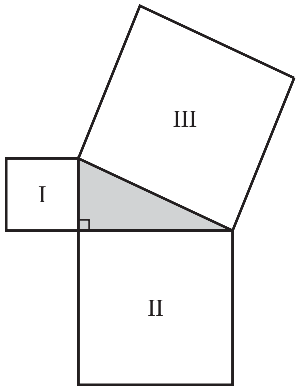
8.G.8 Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.

What is the most famous theorem in all of mathematics?

Many would argue that the Pythagorean Theorem is the best known mathematical result. If you ask your friends and relations the above question I bet most will respond with this theorem. It is a fundamentally important result, key to many deep mathematical explorations, and this theorem will make many appearances throughout our thinking in this guide.

So let's start by being clear on what the theorem is and how to prove it.

The Pythagorean Theorem: *Draw a square on each side of a right triangle (that is, a triangle with one angle of measure* $90°$*) and label these squares I, II, and III as shown.*



Then:

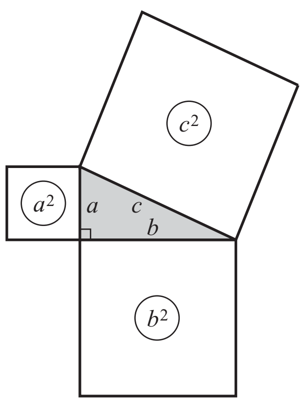
$$\textbf{Area I} + \textbf{Area II} = \textbf{Area III}$$

Two comments:

1. I personally find this result very difficult to believe on a gut level! Look at the large square in the diagram. Does it look possible to you that its area really does equal the sum of the areas of the two smaller squares—on the nose?
2. The Pythagorean Theorem is a statement from geometry. When asked to state the Pythagorean Theorem, most people rattle off "*a* squared plus *b* square equals *c* squared," which sounds like a statement of algebra.

Of course, if we label the sides of the right triangle a, b, and c as shown, then Area I does indeed have value a^2, Area II value b^2, and Area III c^2. The statement "Area I + Area II = Area III" then translates to $a^2 + b^2 = c^2$.

$a^2 + b^2 = c^2$



But one need not label the sides of the right triangle with the letters a, b, and c. A statement of the Pythagorean Theorem could read $p^2 + q^2 = r^2$ (if the two legs of the right triangle are labeled p and q, and the hypotenuse is labeled r), or as $w^2 + x^2 = h^2$, or even as $a^2 + c^2 = b^2$! (Label the legs as a and c and the hypotenuse as b.)

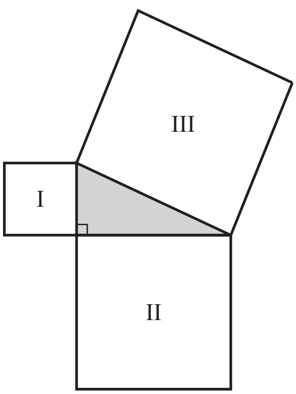
It is important not to be locked into specific notation. If you see the Pythagorean Theorem for what it is—a statement of geometry—you won't be led astray by the specific notation being used.

Proving the Pythagorean Theorem

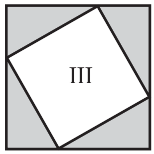
The Pythagorean Theorem, of course, was known to Pythagoras and his Greek contemporaries of 500 BCE. It was also known to Egyptian and Chinese scholars many centuries before the Greeks.

Here's a lovely, purely geometric, proof of the Pythagorean Theorem, believed to have originated in China possibly as early as 1100 BCE. This proof is today called "The Chinese Proof." We'll present it as a physical demonstration.

We wish to show that **Area I + Area II = Area III** in the picture



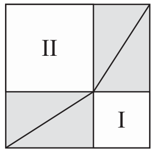
Cut out four copies of the same right triangle and arrange them in a large square as shown.



The white space in the figure is precisely Area III:

White Space = Area III.

Now arrange the triangles this way to see both Areas I and II.



Here we see:

White Space = Area I + Area II.

The area of the white space has not changed by rearranging the four triangles in the large square. We thus conclude that

Area I + Area II = Area III.

Comment. See www.jamestanton.com/?p=1219 for more discussion and twists on the Pythagorean Theorem.

MAA PROBLEMS

In each of these featured problem sections I give an account of my personal path to solving the given problem, sharing with you my human reactions and thoughts along the way. You, no doubt, will have a different set of reactions to each of these challenges and will develop alternative ways to solve them. That is, you will have your own human mathematical experience!

Featured Problem

(#24, AMC 10, 2001)

In trapezoid $ABCD$, $\overline{AB}$ and $\overline{CD}$ are perpendicular to $\overline{AD}$, with $AB + CD = BC$, $AB < CD$, and $AD = 7$. What is $AB \cdot CD$?

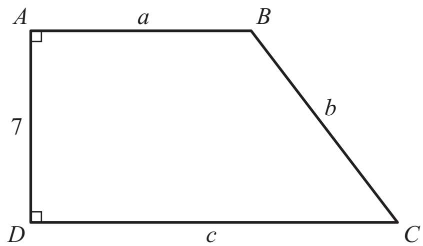
(A) 12 (B) 12.25 (C) 12.5 (D) 12.75 (E) 13

A Personal account of solving this problem

Curriculum Inspirations Strategies (www.maa.org/ci):

Strategy 1: **Engage in Successful Flailing**
Strategy 4: **Draw a Picture**

I feel a little overwhelmed as I read through this question. There are many pieces of information expressed solely in terms of letters. It is hard to read. Sketching a diagram for the problem will no doubt help.



I've made $\overline{AB}$ and $\overline{DC}$ each perpendicular to $\overline{AD}$, as required. I've marked in the length $AD = 7$. And I've also given the other three side-lengths names because I find all the capital letters in geometry visually confusing. (By the way, I should let you know it took me three tries to draw this sketch correctly!)

Let me translate the conditions of the problem into my notation:

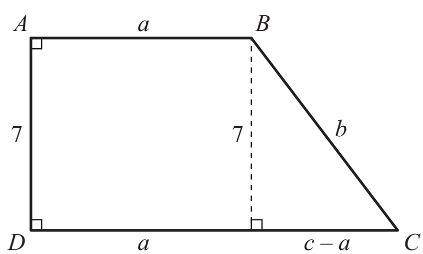
$AB + CD = BC$ translates to $a + c = b$.
$AB < CD$ translates to $a < c$.

(I have this in my picture. I didn't in my first sketches!)

And the question wants the value of $AB \cdot CD$, that is, the value of ac.

Hmm. How am I going to find this product?

Well, in looking at the diagram I feel compelled to draw in an extra line to make a rectangle and a right triangle. I don't know if this will help, but I don't really know what else to do.



What does the Pythagorean Theorem tell me?

$$(c-a)^2 + 49 = b^2$$
$$c^2 - 2ac + a^2 + 49 = b^2.$$

Hmm. I do at least see the product ac in this equation.

Oh, I forgot: $b = a + c$! Let's put that in.

$$c^2 - 2ac + a^2 + 49 = (a+c)^2$$
$$c^2 - 2ac + a^2 + 49 = a^2 + 2ac + c^2.$$

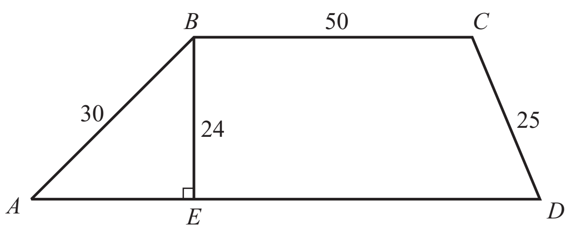
This simplifies to $-2ac + 49 = 2ac$ and so we see

$$4ac = 49$$
$$ac = \frac{49}{4} = 12.25.$$

The answer is (B). Super!

Additional Problems

1. (#19, AMC 8, 2005) What is the perimeter of trapezoid $ABCD$?



(A) 180 (B) 188 (C) 196 (D) 200 (E) 204

2. (#19, AHSME, 1986) A park is in the shape of a regular hexagon 2 km on a side. Starting at a corner, Alice walks along the perimeter of the park for a distance of 5 km. How many kilometers is she from her starting point?

(A) $\sqrt{13}$ (B) $\sqrt{14}$ (C) $\sqrt{15}$ (D) $\sqrt{16}$ (E) $\sqrt{17}$

3. (#11, AMC 12B, 2011) A frog located at (x, y), with both x and y integers, makes successive jumps of length 5 and always lands on points with integer coordinates. Suppose that the frog starts at $(0, 0)$ and ends at $(1, 0)$. What is the smallest possible number of jumps the frog makes?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

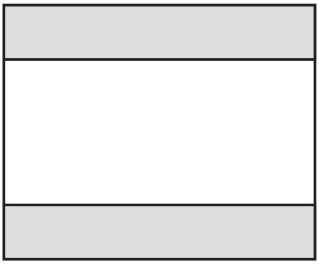
4. (#18, AMC 10A, 2008) A right triangle has perimeter 32 and area 20. What is the length of its hypotenuse?

(A) $\frac{57}{4}$ (B) $\frac{59}{4}$ (C) $\frac{61}{4}$ (D) $\frac{63}{4}$ (E) $\frac{65}{4}$

5. (#11, AMC 10A, 2011) Square $EFGH$ has one vertex on each side of square $ABCD$. Point E is on $\overline{AB}$ with $AE = 7 \cdot EB$. What is the ratio of the area of $EFGH$ to the area of$ABCD$?

(A) $\frac{49}{64}$ (B) $\frac{25}{32}$ (C) $\frac{7}{8}$ (D) $\frac{5\sqrt{2}}{8}$ (E) $\frac{\sqrt{14}}{4}$

6. (#14, AMC 10A, 2008) Older television screens have an aspect ratio of 4 : 3. That is, the ratio of the width to the height is 4 : 3. The aspect ratio of many movies is not 4 : 3, so they are sometimes shown on a television screen by "letterboxing" – darkening strips of equal height at the top and bottom of the screen, as shown. Suppose a movie has an aspect ratio of 2 : 1 and is shown on an older television screen with a 27-inch diagonal. What is the height, in inches, of each darkened strip?



(A) 2 (B) 2.25 (C) 2.5 (D) 2.7 (E) 3

7. (#26, AHSME, 1998) In quadrilateral $ABCD$, it is given that $\angle A = 120^\circ$, angles B and D are right angles, $AB = 13$, and $AD = 46$. Then $AC =$

(A) 60 (B) 62 (C) 64 (D) 65 (E) 72

8. (#15, AMC 10B, 2008) How many right triangles have integer leg lengths a and b and a hypotenuse of length $b + 1$, where $b < 100$?

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

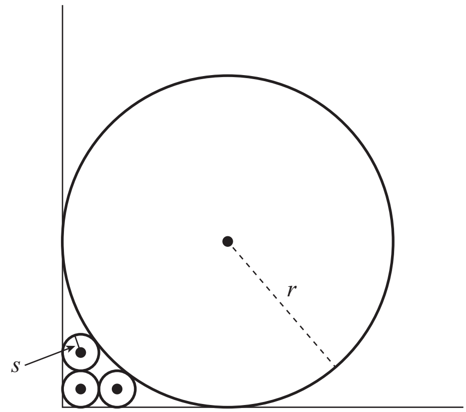
9. (#23, AMC 12B, 2007) How many non-congruent right triangles with positive integer leg lengths have areas that are numerically equal to 3 times their perimeters?

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

10. (#10, AMC 10B, 2008) Points A and B are on a circle of radius 5 and $AB = 6$. Point C is the midpoint of the minor arc AB. What is the length of the line segment AC?

(A) $\sqrt{10}$ (B) $\frac{7}{2}$ (C) $\sqrt{14}$ (D) $\sqrt{15}$ (E) 4

11. (#16, AMC 12A, 2005) Three circles of radius s are drawn in the first quadrant of the xy-plane. The first circle is tangent to both axes, the second is tangent to the first circle and the x-axis, and the third is tangent to the first circle and the y-axis. A circle of radius $r > s$ is tangent to both axes and to the second and third circles. What is r/s?



(A) 5 (B) 6 (C) 8 (D) 9 (E) 10

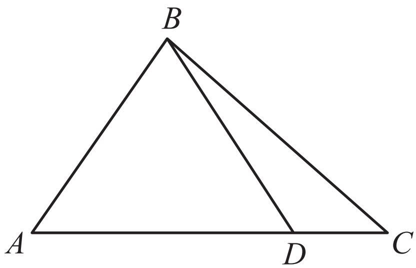
12. (#22, AMC 10A, 2013) Six spheres of radius 1 are positioned so that their centers are at the vertices of a regular hexagon of side length 2. The six spheres are internally tangent to a larger sphere whose center is the center of the hexagon. An eighth sphere is externally tangent to the six smaller spheres and internally tangent to the larger sphere. What is the radius of this eighth sphere?

(A) $\sqrt{2}$ (B) $\frac{3}{2}$ (C) $\frac{5}{3}$ (D) $\sqrt{3}$ (E) 2

13. (#29, AHSME, 1993) Which of the following sets could NOT be the lengths of the external diagonals of a right rectangular prism [a "box"]? (An *external diagonal* is a diagonal of one of the rectangular faces of the box.)

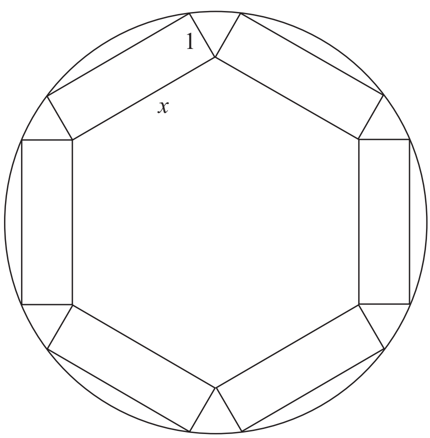
(A) $\{4, 5, 6\}$ (B) $\{4, 5, 7\}$ (C) $\{4, 6, 7\}$ (D) $\{5, 6, 7\}$
(E) $\{5, 7, 8\}$

14. (#15, AHSME, 1989) In ΔABC, $AB = 5$, $BC = 7$, $AC = 9$, and D is on $\overline{AC}$ with $BD = 5$. Find the ratio $AD : DC$.



(A) $4 : 3$ (B) $7 : 5$ (C) $11 : 6$ (D) $13 : 5$ (E) $19 : 8$

15. (#25, AMC 10A, 2008) A round table has radius 4. Six rectangular place mats are placed on the table. Each place mat has width 1 and length x as shown. They are positioned so that each mat has two corners on the edge of the table, these two corners being endpoints of the same side of length x. Further, the mats are positioned so that the inner corners each touch an inner corner of an adjacent mat. What is x?



(A) $2\sqrt{5} - \sqrt{3}$ (B) 3 (C) $\frac{3\sqrt{7}-\sqrt{3}}{2}$ (D) $2\sqrt{3}$ (E) $\frac{5+2\sqrt{3}}{2}$

16. (#12, AHSME, 1980) The equations of $L1$ and $L2$ are $y = mx$ and $y = nx$, respectively. Suppose $L1$ makes twice as large an angle with the horizontal (measured counterclockwise from the positive x-axis) as does $L2$, and that $L1$ has 4 times the slope of $L2$. If $L1$ is not horizontal, then mn is:

(A) $\frac{\sqrt{2}}{2}$ (B) $-\frac{\sqrt{2}}{2}$ (C) 2 (D) -2
(E) not uniquely determined by the given information

2
Some Surprisingly Helpful Background History

Common Core State Standards

The background to . . . F-TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

Mankind is on a perpetual scientific and intellectual quest, to answer the fundamental question:

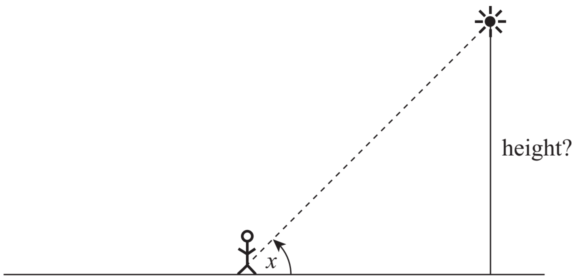
What is this universe we find ourselves in?

Our need to understand our existence and our place and role in the universe, and the nature of the universe itself, has propelled grand scientific, psychological, theological, social, and creative musings since the dawn of time. The study of astronomy was one of the earliest fields of scientific pursuit.

Imagine a human back at the dawn of time, sitting on the ground, observing the universe around her. She notices the Sun, the Moon, and the stars, and their motion. Each body seems to move in arcs across the day or night sky. It is natural to wonder what these objects are, how high or far away they are, what their influence on us might be, and so on. The mathematics to begin understanding the heavenly motions dates back to the ancient Babylonians (ca. 2000 BCE), if not earlier.

Let's address one particular natural question: How high is the Sun?

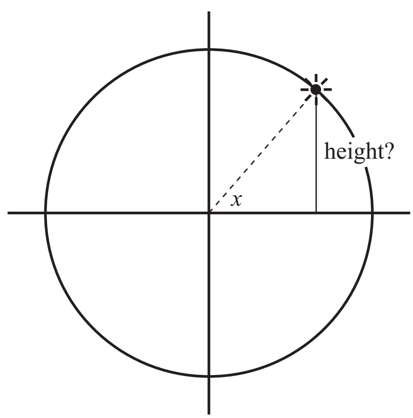
Each day, the Sun rises in the east, moves across the day sky in a large arc, sets in the west, and then returns to rise in the east again the next day (on average twelve hours later). It seems natural to suspect that the Sun stays in motion during the night, moving perhaps below us on the other side of the ground. Can we determine the height of the Sun at any desired time of day?



Unfortunately it is not possible to climb up to the Sun, drop a rope back down to the ground, and measure its length. From our vantage point, here on the ground, there is only one measurement we can make: the measure of the angle of elevation at which we observe the Sun.

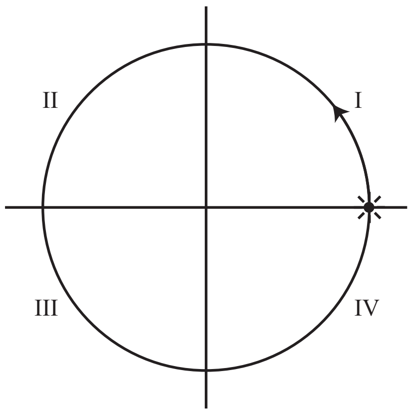
Can we determine the height of the Sun from one angle measurement?

Scholars of ancient times were fully aware that the Sun does not move on a perfectly circular arc across the day sky. But as an attempt to make some headway on this question, to develop some clever thinking that helps towards getting some kind of answer, it is natural to approximate the Sun's motion as along a perfectly circular path (with our location at the center of the circle, of course).



The study of "circle-ometry" is now born: How do you determine the height of a point on a circle knowing only the angle of elevation to that point?

Comment. The Sun rises in the east and sets in the west. If we set the direction east to the right in our maps and diagrams (as indeed became the convention over the centuries), then the Sun's motion is in a counter-clockwise direction. This explains why counter-clockwise is the preferred direction of turning in mathematics today.

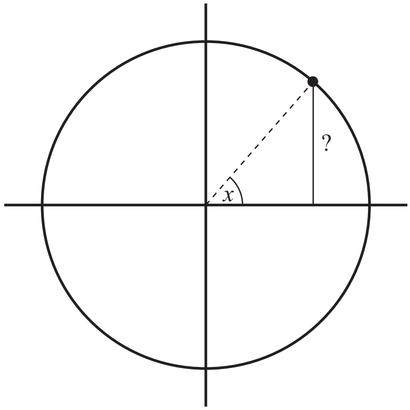


It explains why, in a coordinate system, we always measure angles from the positive x-axis (the location corresponding to the rise of the Sun), in a counter-clockwise direction, and why we number the four quadrants of the coordinate plane in the order that we do—the numbers correspond to the order of the quadrants through which the Sun moves (assuming that the Sun continues to move beneath us during the night!).

A Curious Story About Names

Indian scholars of the fifth century CE took on the challenge of "circle-ometry" with gusto and developed a significant amount of mathematics on the heights of points in circular motion.

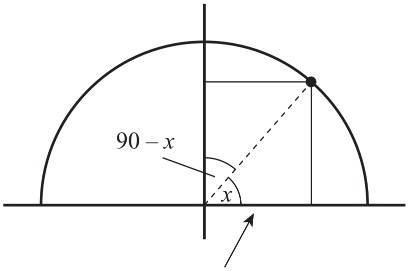
They gave a name to the line segment in diagrams whose length represents the unknown height. They called it the *jyā-ardha* (which literally means "half-chord"), which they abbreviated to *jyā* in their written work. Thus each



angle of elevation has, within a circle of a given radius, an associated *jyā* value.

In the tenth century, Islamic scholars read the Indian texts on "circle-ometry" and translated that work into Arabic. The Sanskrit word *jyā* was strange to them and they simply transcribed the word into Arabic letter by letter. The height of a half-chord in a circle became, in Arabic, *jiba*.

In the twelfth century, mathematics started to flourish in Europe. Western scholars studied the Arabic texts and worked to translate them into Latin. They came across the word *jiba*, which did not exist as a proper Arabic word. They thought it to be a mis-scribed version of the word *jaib*, which exists in the language and means "a cove or a bay," that is, a little place to put boats into safe harbor. They thought this a strange name for the length of a half chord in a circle, but they nonetheless faithfully translated *jaib* into the Latin word for "bay," namely *sinus*. In English, this word then became *sine*. Thus when we talk about the sine of an angle, we are literally saying "the cove" of the angle!



The horizontal displacement of the Sun equals the height of the Sun at the complementary angle.

Later scholars decided to give a name to the horizontal line segment adjacent to the angle, which is also the sine of the complementary angle. They called this length the "companion length to sine," which later was shortened to *cosine*.

The Story Doesn't Quite End Here

In the mid-1500s George Jaochim Rheticus wrote a book that explained how to define sine and cosine in terms of triangles without ever mentioning circles. (One does see right triangles in our circle pictures.) The study of "circle-ometry" became a study of right triangles. The name *trigonometry* was coined for this new version of the subject in 1595 by Bartholomeo Pitiscus, and it is this right-triangle version that is first taught to high-school students across the globe some 400 years later.

But let's see just how easy trigonometry is back in its original "circle-ometry" setting.

3
The Basics of "Circle-ometry"

Common Core State Standards

More background to ... F-TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

Movement towards ...
F-TF.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.

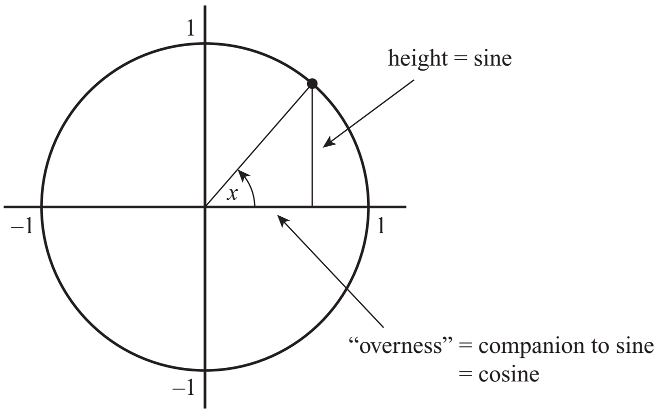
F-TF.4 (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions

Let's imagine a point—the Sun, say—moving counterclockwise about a circle centered at the origin, always rising in the east and setting in the west.

A problem: What radius circle do we consider?

Just to get the mathematics going, let's work with a circle of radius of 1 unit. For the Sun moving about the Earth (from our perspective) this is one very large unit: the number of miles from the Earth to the Sun. But let's call it one unit nonetheless.

Comment. Astronomers call the mean distance of the Earth from the Sun *one astronomical unit*, so we are right in line with the astronomical origins of this subject doing this!



If the Sun has risen x degrees from the positive horizontal axis (east), we define

> $\sin(x)$, read "sine of x," is the height of the Sun at that angle of elevation.
>
> $\cos(x)$, read as "cosine of x," is the "overness" of the Sun at that angle of elevation.

That's it!

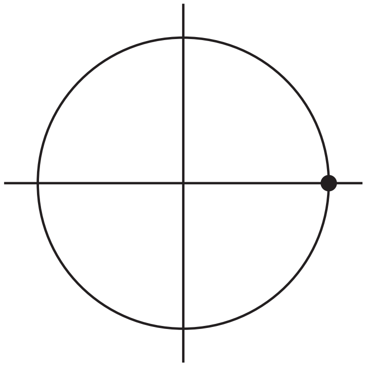
Comment. Indian scholars did not think to set a convention of using a circle of radius 1. In their work the value of sine for a given angle of elevation also depended on the radius of the circle being examined.

We'll see later on how to change the radius of the circle in our considerations too.

SOME EXAMPLES:

$\boxed{x = 0^\circ}$

Where is the Sun at an angle of elevation of 0°? It's right on the eastern horizon:

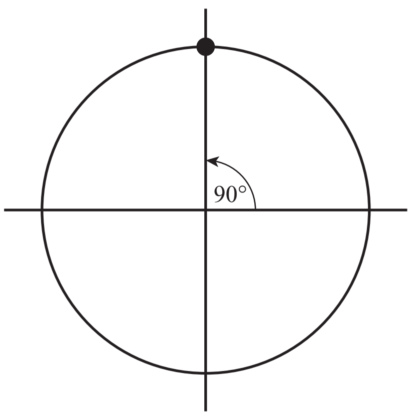


We see that it has a height of 0, and its “overness” is 1 unit to the right:

$$\sin(0^\circ) = 0$$
$$\cos(0^\circ) = 1.$$

$\boxed{x = 90^\circ}$

When the Sun is at an angle of elevation of 90°, it is positioned directly above the origin.

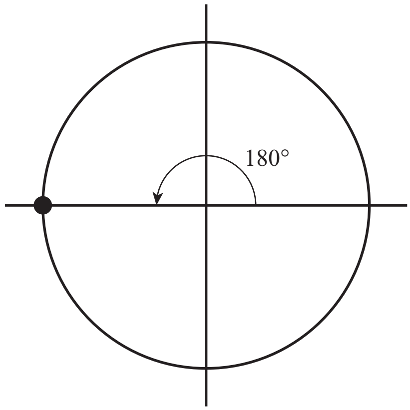


Its height is 1 and its overness is 0. (It doesn’t lie to the left or to the right of the origin.)

$$\sin(90^\circ) = 1$$
$$\cos(90^\circ) = 0.$$

$\boxed{x = 180^\circ}$

When the Sun is at an angle of elevation of 180°, it is positioned directly on the western horizon.

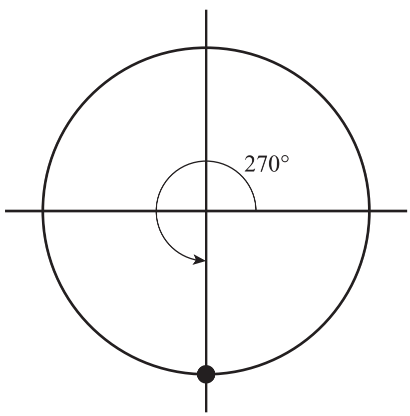


Its height is 0 and its overness is one unit in the negative direction.

$$\sin(180^\circ) = 0$$
$$\cos(180^\circ) = -1.$$

$\boxed{x = 270^\circ}$

At an angle of elevation of 270°, the Sun is positioned directly below the origin. (The other side of the Earth?)

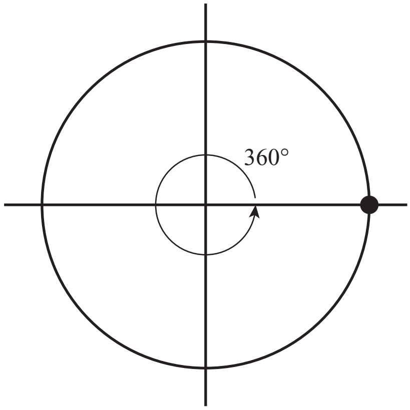


Its height is -1 and its overness is 0.

$$\sin(270^\circ) = -1$$
$$\cos(270^\circ) = 0.$$

$\boxed{x = 360^\circ}$

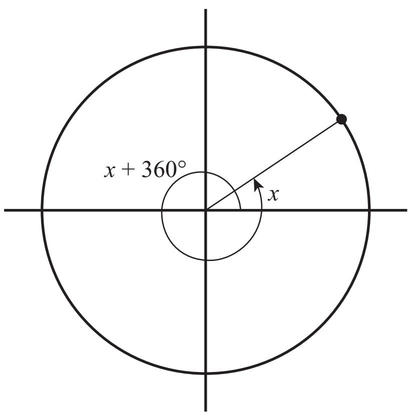
We see:



$$\sin(360^\circ) = 0$$
$$\cos(360^\circ) = 1.$$

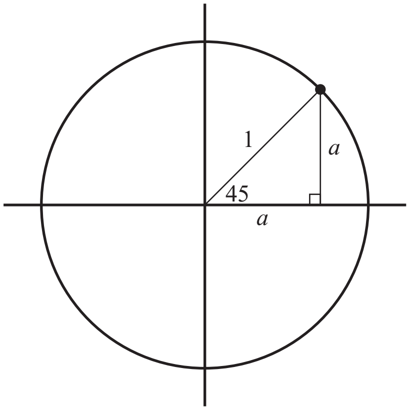
Comment. We have:

$$\boxed{\begin{aligned} \sin(x + 360^\circ) &= \sin(x) \\ \cos(x + 360^\circ) &= \cos(x) \end{aligned}}$$



Let's now work out the sine and cosine of some more interesting angles.

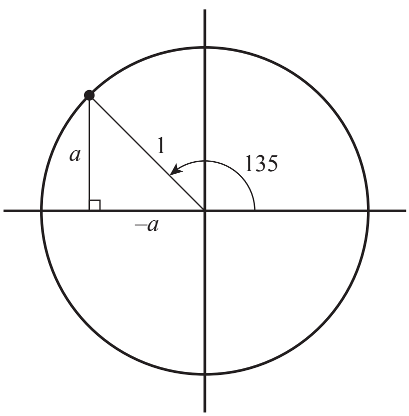
$\boxed{x = 45^\circ}$



We have an isosceles right triangle and the Pythagorean Theorem gives $a^2 + a^2 = 1$ and so $a = \frac{1}{\sqrt{2}}$. Thus

$$\sin(45^\circ) = \frac{1}{\sqrt{2}} \approx 0.707$$
$$\cos(45^\circ) = \frac{1}{\sqrt{2}} \approx 0.707.$$

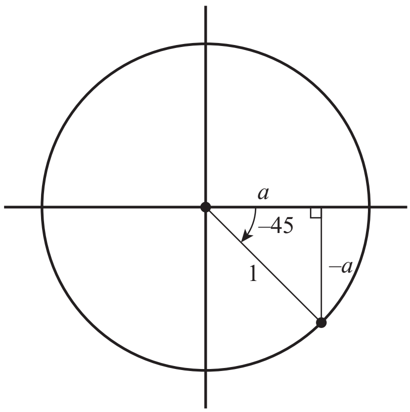
$\boxed{x = 135^\circ}$



Again we have an isosceles right triangle and the Pythagorean Theorem gives $a = \frac{1}{\sqrt{2}}$. Here the overness is in the negative direction and we have

$$\sin(135^\circ) = \frac{1}{\sqrt{2}}$$
$$\cos(135^\circ) = -\frac{1}{\sqrt{2}}.$$

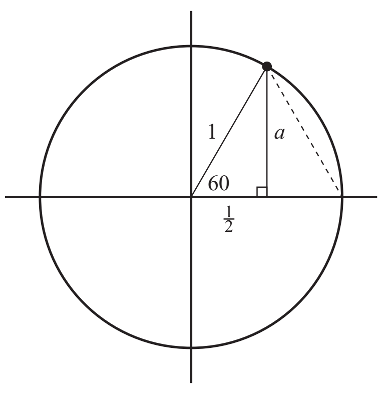
$\boxed{x = -45^\circ}$



Positive angles are measured counterclockwise from the positive x-axis, negative angles in a clockwise direction.

$$\sin(-45^\circ) = -\frac{1}{\sqrt{2}}$$
$$\cos(-45^\circ) = \frac{1}{\sqrt{2}}.$$

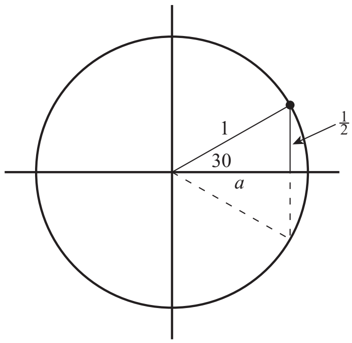
$\boxed{x = 60^\circ}$



We see half an equilateral triangle of side length 1. Thus $\cos(60^\circ) = \frac{1}{2}$. (Ooh! Is the picture accurate?) The Pythagorean Theorem gives $a = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$ and so $\sin(60^\circ) = \frac{\sqrt{3}}{2}$.

$$\sin(60^\circ) = \frac{\sqrt{3}}{2} \approx 0.866$$
$$\cos(60^\circ) = \frac{1}{2}.$$

$\boxed{x = 30^\circ}$



Again we have half an equilateral triangle.

$$\sin(30^\circ) = \frac{1}{2}$$
$$\cos(30^\circ) = \frac{\sqrt{3}}{2}.$$

Comment. For practice, compute $\sin(-30^\circ)$, $\sin(-135^\circ)$, $\cos(390^\circ)$, and $\cos(120^\circ)$.

IN SUMMARY:

We have

$$\sin(0^\circ) = 0 \;\; \sin(30^\circ) = \tfrac{1}{2} \;\; \sin(45^\circ) = \tfrac{1}{\sqrt{2}} \;\; \sin(60^\circ) = \tfrac{\sqrt{3}}{2} \;\; \sin(90^\circ) = 1$$
$$\cos(0^\circ) = 1 \;\; \cos(30^\circ) = \tfrac{\sqrt{3}}{2} \;\; \cos(45^\circ) = \tfrac{1}{\sqrt{2}} \;\; \cos(60^\circ) = \tfrac{1}{2} \;\; \sin(90^\circ) = 0$$

These are often considered standard results to be committed to memory. (See www.jamestanton.com/?p=875 for a HANDY MNEMONIC.)

The sine and cosine values for angles of elevation with the Sun positioned in other quadrants can be deduced by quickly sketching a diagram and using your wits just as in this section.

Comment. It is deemed permissible to drop the parenthesis in the notations $\sin(x)$ and $\cos(x)$ if no confusion comes of it. For example, one might write $\cos 60^\circ = \frac{1}{2}$ or $(\sin 60^\circ)^2 = 0.75$.

School curricula tend to insist that the parentheses be used. Mathematicians tend to drop the parentheses if the context is clear.

MAA PROBLEMS

Featured Problem

(Modified version of #15, AMC 12A, 2006):

Suppose $\cos(x) = 0$ and $\cos(x+z) = \frac{1}{2}$. What is the smallest possible positive value of z?

(A) 30° (B) 60° (C) 90° (D) 150° (E) 210°

Comment. This question was originally given in units of radians.

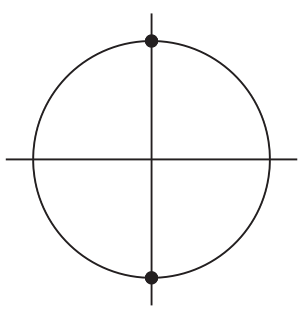
A Personal account of solving this problem

Curriculum Inspirations Strategy (www.maa.org/ci):

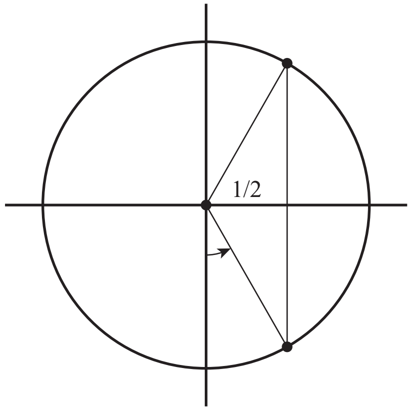
Strategy 4: **Draw a Picture**

This question looks a bit strange to me. But if I think in terms of the location of the Sun matters might fall into place. Let's see.

We're told that $\cos(x) = 0$. So the Sun at an angle of elevation of x degrees has zero overness. The Sun is either directly overhead or directly below.

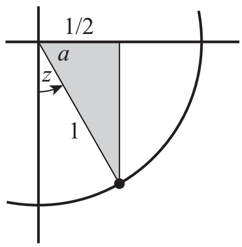


By adding an extra z degrees to the angle of elevation, we get an overness of $\frac{1}{2}$. There are two possible locations for the Sun with this overness.



The angle of smallest possible measure that shifts the Sun from one of the overhead/under-head positions to one of these new positions is the angle I've marked. All I need is the measure of that angle.

With the lengths $\frac{1}{2}$ and 1 we see we're dealing with half an equilateral triangle.



So the angle I've marked a has measure 60° and the angle z here has measure 30°.

The answer is (A).

Additional Problem

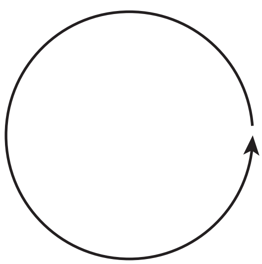
17. (#38, AHSME, 1958) Let r be the distance from the origin to a point P with coordinates x and y. Designate the ratio $\frac{y}{r}$ by s and the ratio $\frac{x}{r}$ by c. Then the values of $s^2 - c^2$ are limited to the numbers:

(A) less than -1 and greater than $+1$, both excluded
(B) less than -1 and greater than $+1$, both included
(C) between -1 and $+1$, both excluded
(D) between -1 and $+1$, both included
(E) -1 and $+1$ only

4
Radian Measure

> **Common Core State Standards**
>
> **F-TF.1** Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
>
> **F-TF.2** Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

We like to say that "there are 360° in a circle." By this, we mean that we find it convenient to divide one full rotation into 360 small parts, each called a *degree*.



1 full turn = 360°

But why the number "360"?

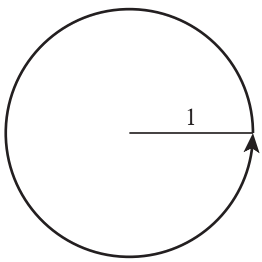
Some historians suggest that this choice of number is due to Babylonian scholars (ca 2000 BCE) who observed that one full turn of the Earth about the Sun, that is, one Earth year, takes $365\frac{1}{4}$ days. Thus the number $365\frac{1}{4}$ corresponds to one very obvious, human cycle.

But this is an awkward number with which to work and so the Babylonians rounded it to 360, a much more pleasing figure. (Why didn't they round to 365 or to 370 do you think?) To this day we continue to work with the number 360 and its fractions when thinking about rotations and cycles (including the way we measure hours, minutes, seconds).

However, the number 360 is very specific to our happenstance of having evolved on this particular planet—the Earth—revolving about our particular star—the Sun. Martians, I suspect, would naturally say that there are 660 degrees in a full turn. (How many Martian days—called *sols*—are there in a Martian year?)

The number 360 is not at all relevant to mathematics and it proves to be quite an awkward number for doing more advanced work (in the mathematics of calculus, for instance). Mathematicians soon found they needed a measure of turning that is natural to the mathematics, not to our human experience.

The simplest thing to do—as is always the best—is to work with the simplest circle possible to represent one full turn, namely, a circle of radius 1.



Walking all the way round the circumference of this circle corresponds to walking a distance of $2\pi \times 1 = 2\pi$ units. Thus it seems natural to associate the number 2π with the concept of one full turn, and fractions of this number with fractions of turns. For example, we associate the number $\frac{1}{2} \cdot 2\pi = \pi$ with half a turn and the number $\frac{3}{10} \cdot 2\pi = \frac{3\pi}{5}$ with $\frac{3}{10}$ of a turn. These numbers are the distances one physically traverses when walking these fractions of a full turn on a circle of radius one unit.

A Problem

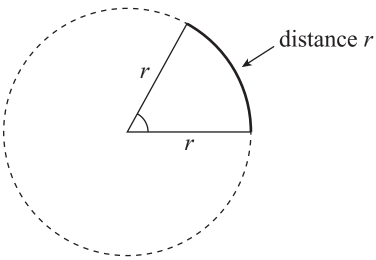
It looks like radians are measured in units of distance. Would those units be? Inches? Feet? Miles? Astronomical units? We want a measure of turning that is independent of all human choices, including human choices of unit!

The Fix

We ask: Is there a natural choice of unit associated with any given circle, independent of the units on any ruler we might pick up to measure distances on it?

The radius of the circle comes to mind!

Suppose on a given circle we walk a distance of one radius along its circumference. Let's call the fraction of turn it represents "one unit of turning." This unit of turning is natural to the circle, it is not a distance and so has no units associated with it, and it represents the same fraction of turning for all circles. Perfect!



Mathematicians give this unit of turning a name. It is called a *radian* (to match the word *radius*).

Question. How many degrees of turning is one radian? (Read on for the answer.)

A turn of two radians corresponds to walking a diameter length along a circle circumference, and a turn of 2π radians corresponds to walking a distance of 2π radii along the circumference of the circle, that is, all the way round!

We have:

A turn of 2π *radians corresponds to one full turn.*

The name radian is used whenever we want to associate a quantity with an amount of turning. For example, writing "2π" all by itself represents a number. But writing "2π radians" means "one full turn."

We have

$$\begin{aligned}
1 \text{ turn} &\leftrightarrow 360^\circ \leftrightarrow 2\pi \text{ radians} \\
\frac{1}{2} \text{ turn} &\leftrightarrow 180^\circ \leftrightarrow \frac{1}{2}\cdot 2\pi = \pi \text{ radians} \\
\frac{1}{3} \text{ turn} &\leftrightarrow 120^\circ \leftrightarrow \frac{2\pi}{3} \text{ radians} \\
\frac{3}{10} \text{ turn} &\leftrightarrow 108^\circ \leftrightarrow \frac{3}{10}\cdot 2\pi = \frac{3\pi}{5} \text{ radians}
\end{aligned}$$

and so on.

Comments

1. Many people find it helpful to remember

$$180^\circ \leftrightarrow \pi \text{ radians.}$$

This allows one to convert between radians and degrees fairly swiftly. For example

$$\begin{aligned}
\frac{2\pi}{3}\ \textit{radian} &\leftrightarrow \frac{2\times 180^\circ}{3} = 120^\circ \\
\frac{13\pi}{12}\ \textit{radian} &\leftrightarrow \frac{13\times 180^\circ}{12} = 195^\circ.
\end{aligned}$$

2. Consider again the correspondence: $180^\circ \leftrightarrow \pi$ radians.
   Dividing through by 180 gives

$$1^\circ \leftrightarrow \frac{\pi}{180} \text{ radians.}$$

This allows us to convert degrees to radians fairly swiftly.

For example, multiplying through by 20 shows $20^\circ \leftrightarrow 20 \times \frac{\pi}{180} = \frac{\pi}{9}$ radians.

Dividing through instead by π gives:

$$1 \text{ radian} \leftrightarrow \frac{180^0}{\pi} \approx 57.3^\circ.$$

Comment. Here is the MAA AMC question of the previous section in its original form:

(#15, AMC 12A, 2006):

Suppose $\cos(x) = 0$ and $\cos(x + z) = \frac{1}{2}$. What is the smallest possible positive value of z?

(A) $\dfrac{\pi}{6}$ (B) $\dfrac{\pi}{3}$ (C) $\dfrac{\pi}{2}$ (D) $\dfrac{5\pi}{6}$ (E) $\dfrac{7\pi}{6}$

Care to rethink its solution in terms of radians?

5
The Graphs of Sine and Cosine in Degrees

Common Core State Standards

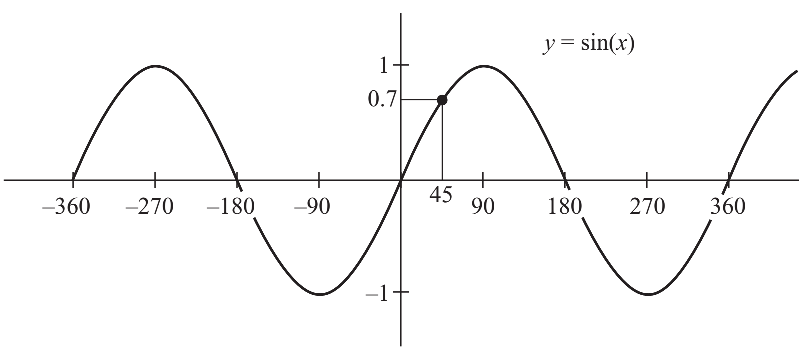
F-TF.4 (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

We have the following table of sine values:

x	$0°$	$30°$	$45°$	$60°$	$90°$	$135°$	$180°$	$225°$	$270°$	$315°$	$360°$
$\sin(x)$	0	$\frac{1}{2} = 0.5$	$\frac{1}{\sqrt{2}} \approx 0.7$	$\frac{\sqrt{3}}{2} \approx 0.9$	1	0.5	0	-0.5	-1	-0.5	0

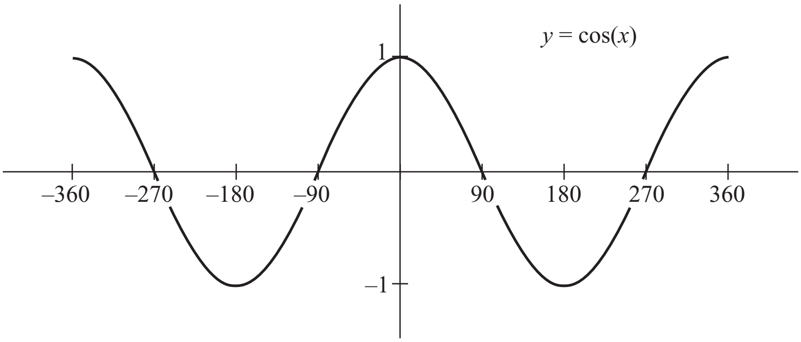
x	$0°$	$-30°$	$-45°$	$-60°$	$-90°$	$-135°$	$-180°$	$-270°$	$-360°$
$\sin(x)$	0	-0.5	-0.7	-0.9	-1	-0.5	0	1	0

Plotting points shows that the graph of $y = \sin(x)$ has the following shape:



Comment. Matters are a little deceptive here: one unit of measure on the vertical axis is physically very much longer than one unit of measure on the horizontal axis. A correctly scaled graph would have these physical lengths match. (Can you see in your mind's eye how the graph would then appear?)

A similar exercise shows that the graph of $y = \cos(x)$ appears:



Each graph is periodic with *period* 360°, meaning that the graphs repeat in values every 360°. (This, of course, is reflected in the statements $\sin(x + 360^\circ) = \sin(x)$ and $\cos(x + 360^\circ) = \cos(x)$.)

Comment. Many wave-like phenomena in nature appear as "sine curves." They are the result of looking at just the vertical component of circular motion. (If one views one object spinning about another, but from the side, all one sees is the vertical displacement of that object—a sine wave of motion!)

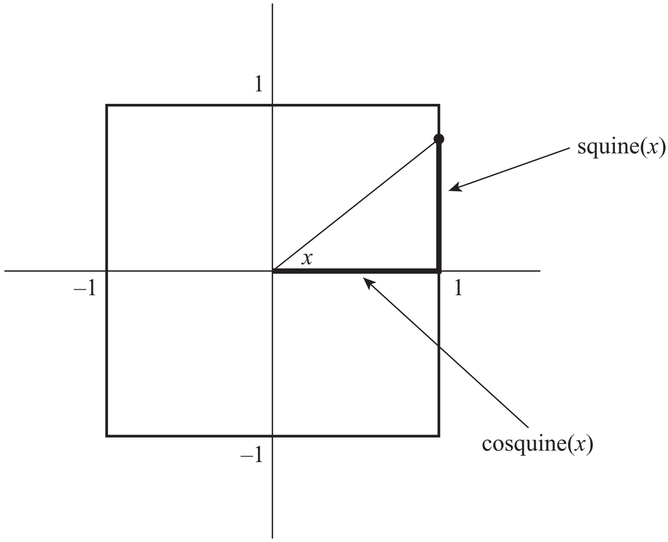
Notice that the cosine curve is a horizontal translate of the sine curve with value $x = -90^\circ$ for the cosine curve "behaving like" $x = 0^\circ$ for the sine curve. We have

$$\boxed{\cos(x) = \sin(x + 90^\circ).}$$

Comment. Why the focus on circles? Imagine an object moving about a square of "radius" 1 centered about the origin.

At each angle of elevation x, the object has a certain height and certain overness. Call these the square sine, "squine," and square cosine, "cosquine," of x. (See the diagram on the next page.)

I wonder how the plots of $y = \text{squine}(x)$ and $y = \text{cosquine}\,(x)$ appear. (See www.jamestanton.com/?p=605.)



MAA PROBLEMS

Featured Problem

(#10, AHSME, 1985):

An arbitrary circle can intersect the graph of $y = \sin x$ in

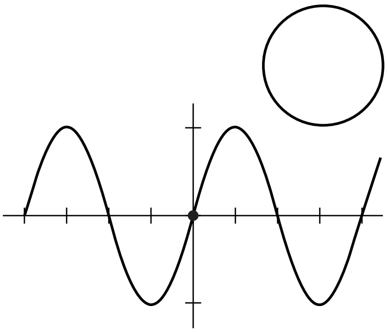
(A) at most 2 points
(B) at most 4 points
(C) at most 6 points
(D) at most 8 points
(E) more than 16 points

A Personal Account of Solving this Problem

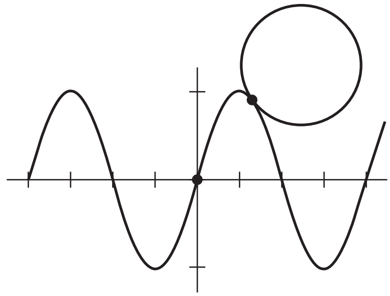
Curriculum Inspirations Strategy (www.maa.org/ci):

Strategy 10: **Go to Extremes**

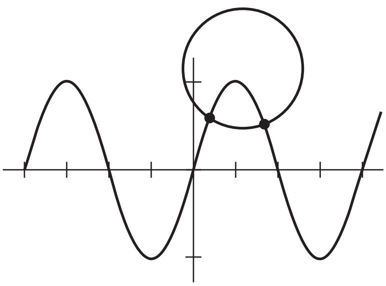
It seems that a circle can intersect a sine graph only zero, one, or two times:



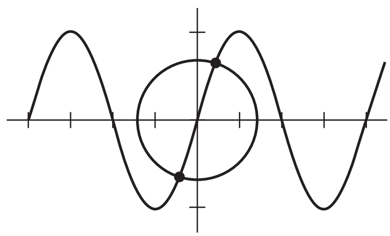
zero intersections



one intersection

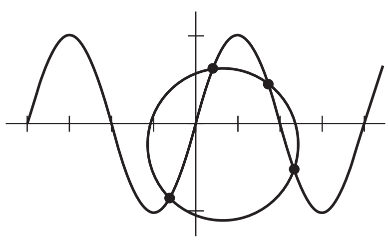


two intersections



two intersections

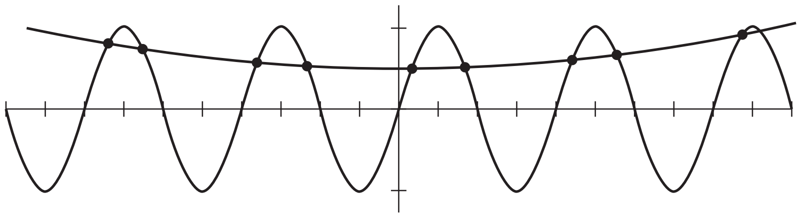
Oh. Make the circle a bit bigger and we can get to four intersection points!



Does making it bigger still help?

Hmm. How about a circle so large that any section of it is basically a straight line segment?

Ahh! We can get millions of intersection points!



The answer is (E).

Additional Problems

18. (#18, AHSME, 1999) How many zeros does $f(x) = \cos(\log(x))$ have on the interval $0 < x < 1$?

(A) 0 (B) 1 (C) 2 (D) 10 (E) infinitely many

19. (#16, AHSME, 1977) If $i^2 = -1$, then the sum

$$\cos 45^\circ + i \cos 135^\circ + \cdots + i^n \cos(45 + 90n)^\circ + \cdots + i^{40} \cos 3645^\circ$$

equals

(A) $\frac{\sqrt{2}}{2}$ (B) $-10i\sqrt{2}$ (C) $\frac{21\sqrt{2}}{2}$ (D) $\frac{\sqrt{2}}{2}(21 - 10i)$ (E) $\frac{\sqrt{2}}{2}(21 + 20i)$

6

The Graphs of Sine and Cosine in Radians

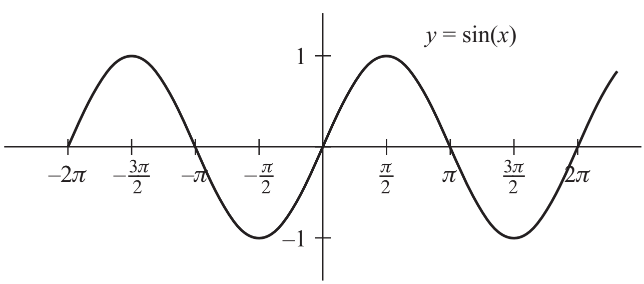
Common Core State Standards

F-TF.4 (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

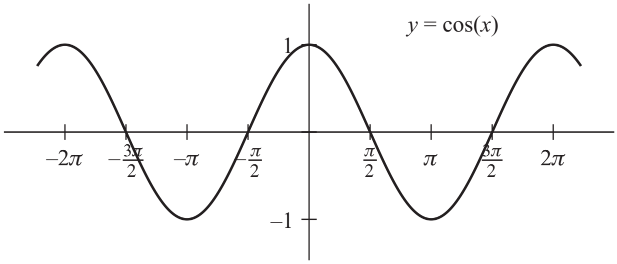
As radians are natural to mathematics, mathematicians will work solely with radian measure, particularly with regard to the circle functions sine and cosine.

Comment. One can set one's calculator into "radian mode."

If x is taken to be in radian measure, the graph of the function $y = \sin(x)$ now appears:



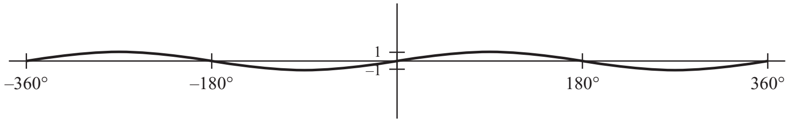
The graph for $y = \cos(x)$ is similarly adjusted.



Each graph has period 2π.

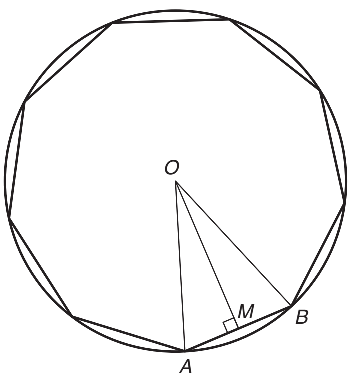
Comment. Using the scale of radian on the horizontal axis here has the lovely effect of scaling the graph of $y = \sin(x)$ so that it crosses the origin at a slope of 1. One can prove that if x is small in value, and given in radian measure, then the value of $\sin(x)$ is very close to the value of x. This means that the graphs $y = \sin(x)$ and $y = x$ look very much alike at the origin.

If x, however, is given in degrees, then $\sin(x) \approx \frac{\pi}{180}x \approx 0.017x$ and the graph crosses the origin at a very low slope of 0.017. (Our graph for $y = \sin(x)$ in the previous section was deceptive. If we truly scaled the axes so that one unit of measure along the horizontal axis represents one degree, then graph appears more as



We see the graph crossing through the origin at a very shallow slope.)

To get a sense of why $\sin(x) \approx x$ if x is close to zero (and given in radians), consider a polygon with n sides inscribed in a circle of radius 1.



Let O be the center of the circle, A and B two adjacent vertices of the polygon, and M the midpoint of the side $\overline{AB}$. We have that $m\angle AOB = \frac{2\pi}{n}$ (it's one nth of a full turn) and so $m\angle MOB = \frac{\pi}{n}$.

Looking at ΔMOB we see $BM = \sin\left(\frac{\pi}{n}\right)$, and so $AB = 2\sin\left(\frac{\pi}{n}\right)$, and the perimeter of the polygon is $2n\sin\left(\frac{\pi}{n}\right)$.

If n is very large, the polygon is very close to matching the circle. (If n equals a billion, the human eye could not detect the difference between the regular billion-gon and the circle itself.) The circle has perimeter, that is, circumference 2π. So for n very large we expect

$$2n\sin\left(\frac{\pi}{n}\right) \approx 2\pi.$$

Rearranging gives

$$\sin\left(\frac{\pi}{n}\right) \approx \frac{\pi}{n}.$$

So for very small angles of the form $x = \frac{\pi}{n}$ (x is very small if n is very large) we expect $\sin(x) \approx x$.

MAA PROBLEMS

Featured Problem

(#18, AHSME, 1981)

The number of real solutions to the equation $\frac{x}{100} = \sin x$ is

(A) 61 (B) 62 (C) 63 (D) 64 (E) 65

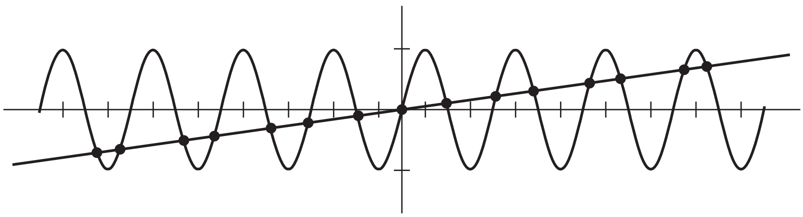
A Personal account of solving this problem

Curriculum Inspirations Strategy (www.maa.org/ci):

Strategy 4: **Draw a Picture**

This problem looks a bit scary to me. But after a deep breath, I see that is really about a line $y = \frac{1}{100}x$ and the sine function $y = \sin x$. (The author of this question chooses to omit parentheses when writing the sine function.)

Solutions to the equation $\frac{x}{100} = \sin x$ correspond to intersection points of these two graphs. Let's graph them!



(Well, my straight line should be of much shallower slope!)

The line and the sine curve certainly intersect at $x = 0$ (so that's one solution to the equation) and we see from symmetry that there will be an equal number of positive and negative solutions. This means that there is going to be an odd number of solutions. That's something! (Choices (B) and (D) are definitely out.)

Now what? Hmm.

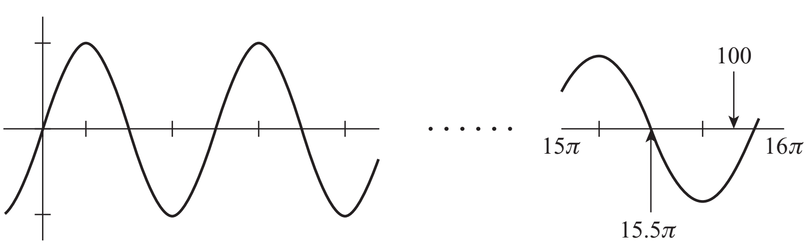
Let's focus on right half of the graph, the graph on the positive x-axis.

The straight line, eventually, will rise higher than the sine curve. If I can count how many "humps" it crosses, then I'll basically have a count of the solutions.

How am I going to count those?

Well, the sine graph never goes beyond a height of 1, and the line $y = \frac{1}{100}x$ goes above a height of 1 at $x = 100$. And since the period of the sine curve is 2π, there are $\frac{100}{2\pi} \approx 15.9$ positive humps up to $x = 100$. Hang on! That doesn't make sense. What's actually happening right out near $x = 100$?

We have that 16 cycles of 2π takes us just past 100. So $15 \cdot 2\pi$ is the start of the cycle that gets "cut off" by $x = 100$. And to be very clear, the start of the downward hump in that cycle starts at $15.5 \cdot 2\pi$. Okay, we do have the full upward hump in what would be the 16th cycle, so there are sixteen upward humps on the right side of the graph each being cut twice by the line $y = \frac{1}{100}x$.



So 32 intersection points to the right, and thus 32 to the left as well (by symmetry), making a total of 64 solutions. So (D)?

But the number of solutions has to be odd. Oh! We counted the solution $x = 0$ twice.

The answer is (C). There are precisely 63 solutions to the equation $\frac{x}{100} = \sin x$.

Additional Problem

20. (#23, AMC 12B, 2003) The number of x-intercepts on the graph of $y = \sin\left(\frac{1}{x}\right)$ in the interval $(0.0001,\ 0.001)$ is closest to

(A) 2900 (B) 3000 (C) 3100 (D) 3200 (E) 3300

7
Basic Trigonometric Identities

Common Core State Standards

F-TF.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$, and use the unit circle to express the values of sines, cosines, and tangents for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.

F-TF.4 (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

F-TF.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

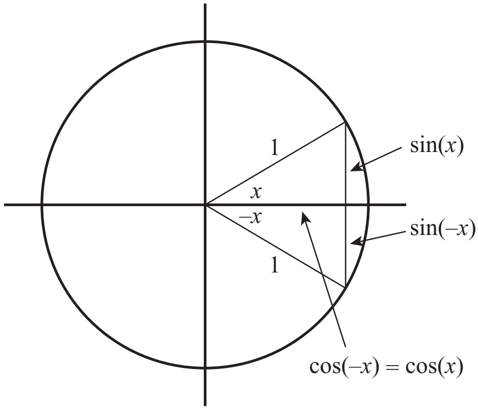
The sine and cosine functions possess a number of special properties. We have already seen, for example:

Degrees	Radians
$\sin(x + 360^\circ) = \sin(x)$	$\sin(x + 2\pi) = \sin(x)$
$\cos(x + 360^\circ) = \cos(x)$	$\cos(x + 2\pi) = \cos(x)$

and that

Degrees	Radians
$\cos(x) = \sin(x + 90^\circ)$	$\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$

Let's return to the basic diagram of the position of the Sun on a unit circle:



We see the Sun at angles of elevation x and $-x$ degrees have the same overness but opposite heights. We have

$$\boxed{\begin{aligned}\sin(-x) &= -\sin(x)\\ \cos(-x) &= \cos(x).\end{aligned}}$$

And we see these features in the graphs of sine and cosine: The cosine graph is symmetric about the y-axis (it adopts the same values on both positive and negative x-values) and the sine function adopts opposite values on positive and negative x-values.

Drawing the locations of the Sun at an angle of elevations x degrees and $x + 180$ degrees shows the following relations. (The two locations are on opposite sides of the circle of radius one.)

Degrees	Radians
$\sin(x + 180^\circ) = -\sin(x)$	$\sin(x + \pi) = -\sin(x)$
$\cos(x + 180^\circ) = -\cos(x)$	$\cos(x + \pi) = -\cos(x)$

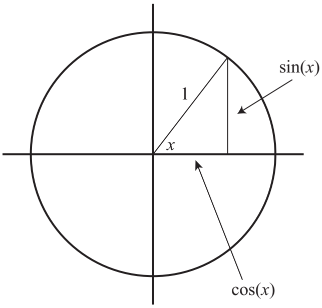
We can combine the relations we have established thus far to discover more. For example, in radians, we have

$$\sin(2\pi - x) = \sin(-x) = -\sin(x)$$

and

$$\cos(\pi - x) = -\cos(-x) = -\cos(x).$$

Consider again the defining diagram:



The Pythagorean Theorem gives

$$(\sin x)^2 + (\cos x)^2 = 1.$$

For visual ease people tend to write $\sin x$ for $\sin(x)$ and $\cos x$ for $\cos(x)$ here.

Important Comment. It has become usual to use unconventional mathematical notation for equations of this type in trigonometry. Rather than write $(\sin x)^2$, mathematicians prefer to write $\sin^2 x$. Normally, this is the notation for the composition of functions: $f^2(x)$ means $f(f(x))$, but in the context of trigonometry it is taken to mean $f(x) \times f(x)$.

Similarly, $\cos^3 x$ means $(\cos x)^3$, and not a three-fold composition of functions, and $\sin^{67} t$ means the number $\sin t$ raised to the 67th power.

> For any positive integer n . . .
>
> $\sin^n x$ means the number $\sin(x)$ raised to the nth power
> $\cos^n x$ means the number $\cos(x)$ raised to the nth power

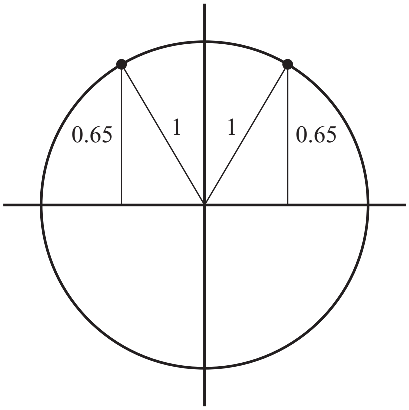
Matters are somewhat confusing for when a negative integer is used—this abuse of notation no longer applies! For instance, $\sin^{-1} x$ now means what it should: the inverse function to sine, and not $(\sin x)^{-1} = \frac{1}{\sin(x)}$. (See Section 12.)

The statement of the Pythagorean Theorem in the context of this notation is:

$$\boxed{\sin^2 x + \cos^2 x = 1}$$

Example. If $\sin(g) = 0.65$ give two possible values for $\cos(g)$.

Answer. We are being told that when the Sun is at an angle of g degrees, its height is 0.65. There are two possible locations for the Sun.



We see that $\cos(g)$ is either $\sqrt{1-0.65^2} \approx 0.76$ or the negative version of this, -0.76.

TIP: We have

$$\sin(-x) = -\sin(x)$$
$$\cos(-x) = \cos(x)$$
$$\sin(x + 90^\circ) = \cos(x).$$

If we change the input of either $\sin(x)$ or $\cos(x)$ with some combination of the addition of 90° and a change in sign the result will be equivalent to one of $\sin(x)$, $\cos(x)$, $-\sin(x)$, or $-\cos(x)$. Trying some specific values for x (in particular $x = 0^\circ$ and $x = 90^\circ$) will tell you which.

Example. Identify $\cos(x + 90^\circ)$.

Answer. For $x = 0^\circ$ this function has value $\cos(90^\circ) = 0$.
For $x = 90^\circ$ it has value $\cos(180^\circ) = -1$.
It must be the inverted version of sine.

$$\cos(x + 90^\circ) = -\sin(x)$$

Similarly, $\sin(90 - x) = \cos(x)$ and $\cos(x + 180^\circ) = -\cos(x)$.

MAA PROBLEMS

21. (#18, AHSME, 1980) If $b > 1$, $\sin x > 0$, $\cos x > 0$ and $\log_b \sin x = a$, then $\log_b \cos x$ equals

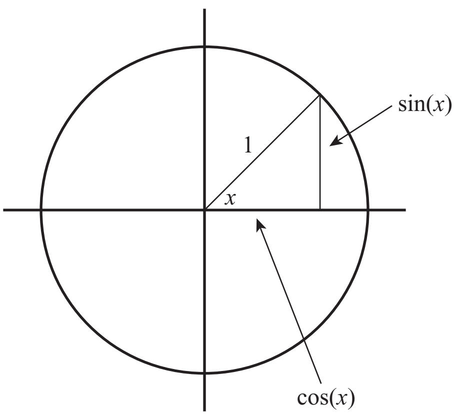
(A) $2\log_b(1 - b^{a/2})$ (B) $\sqrt{1 - a^2}$ (C) b^{a^2} (D) $\frac{1}{2}\log_b(1 - b^{2a})$
(E) none of these

8
Sine and Cosine for Circles of Different Radii

Common Core State Standards

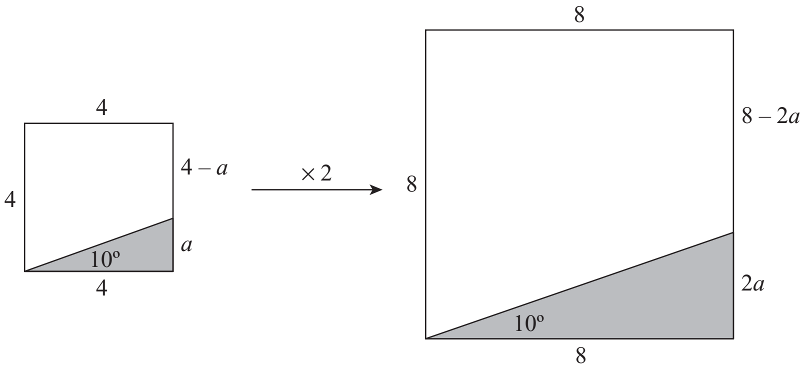
Towards . . . G-SRT.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

We have, so far, defined sine and cosine within circles only of radius 1.



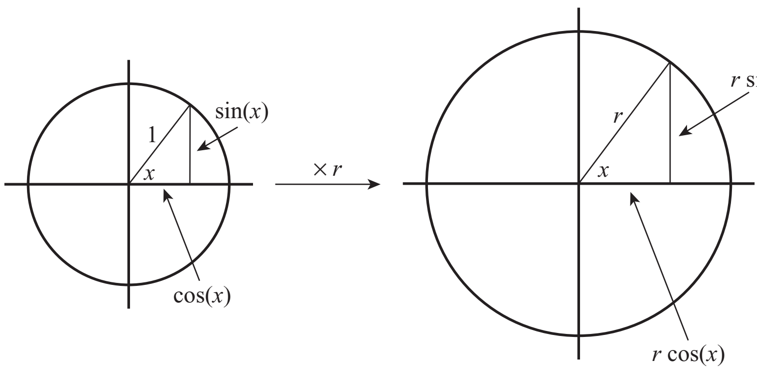
What should we do for a circle of radius different from1?

If one takes a picture to a photocopier and enlarges it by, say, a factor of two, then all lengths in that picture double while all angles in the picture are unchanged.



Scale a picture by a factor of three, all lengths triple (and all angles stay the same). Scale a picture by a factor of 0.4, then all lengths in that picture reduce to two-fifths of their original lengths (and all angles, again, remain unchanged).

So take the picture of a unit circle to a photocopier and enlarge it by a factor of r, a positive real number. All angles will remain the same, but all lengths in the picture increase by a factor of r.



We see that

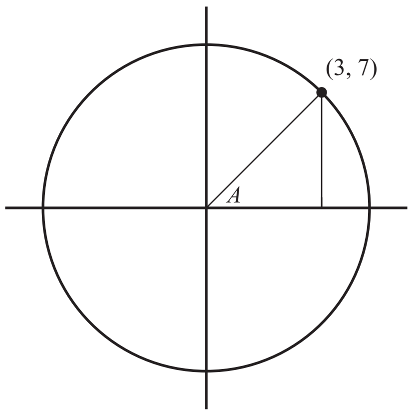
> the height and overness of the Sun traversing a circle of radius r, observed at an angle of elevation x degrees is

$$\text{height} = r\sin(x)$$

$$\text{overness} = r\cos(x).$$

There is nothing deep here.

Example. Find the sine and cosine of the angle A

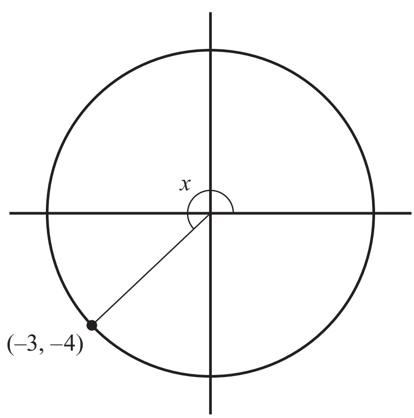


Answer. In this diagram the Sun has height 7 and overness is 3. But the Sun is on a circle of radius $\sqrt{3^2 + 7^2} = \sqrt{58}$. We have:

$$7 = \text{height} = \sqrt{58}\sin(A)$$
$$3 = \text{overness} = \sqrt{58}\cos(A)$$

So $\sin(A) = \frac{7}{\sqrt{58}}$ and $\cos(A) = \frac{3}{\sqrt{58}}$.

Example. Find the sine and cosine of the angle x indicated



Answer. The radius of the circle is 5. We have

$$-4 = \text{height} = 5\sin(x)$$
$$-3 = \text{overness} = 5\cos(x).$$

So $\sin(x) = -\frac{4}{5}$ and $\cos(x) = -\frac{3}{5}$.

MAA PROBLEMS

22. **(#19, AHSME, 1998)** How many triangles have area 10 and vertices at $(-5, 0)$, $(5, 0)$, and $(5\cos\theta, 5\sin\theta)$ for some angle θ?

(A) 0 (B) 2 (C) 4 (D) 6 (E) 8

9
A Paradigm Shift

Common Core State Standards

Still towards . . . **G-SRT.8** Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

We introduced $\sin(x)$ as the actual physical height of the Sun traversing a circle of radius 1 observed at an angle of elevation of x degrees.

For a circle of a different radius r, the physical height of the Sun observed at an angle of elevation of x degrees is now

$$\text{height} = r\sin(x).$$

The meaning of sine has now subtly changed. Solving for $\sin(x)$ we get

$$\sin(x) = \frac{\text{height}}{r},$$

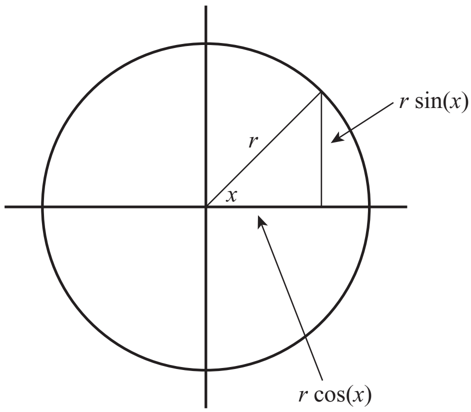
and sine is now a ratio of lengths, not an actual length in and of itself!

Comment. Another shift of thinking also occurred soon after this. In the mid 1700s, the Swiss mathematician Leonhard Euler noted that $\sin(x)$ plays the role of a function: to each angle x is assigned a number between -1 and 1. (The analogous idea holds for $\cos(x)$ too.) Euler was the first to articulate the notion of a function and seeing sine and cosine as functions provided a new mindset for thinking about them: he could graph these trigonometric functions, compose them, ask for their function inverses, and the like.

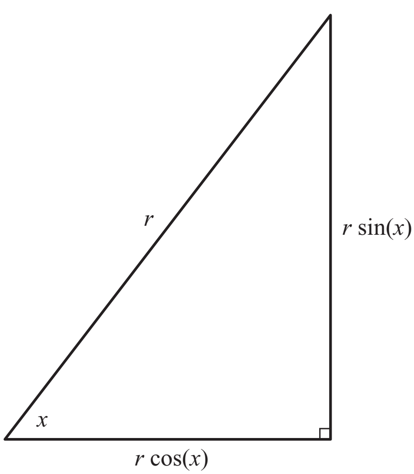
"CIRCLE-OMETRY" BECOMES TRIGONOMETRY

In the mid-1500s the scholar Jaochim Rheticus turned the study of circle-ometry into a study of right triangles. His approach is the one used today in practically all introductory texts to the subject.

Rheticus realized that in the most general case of a circle of arbitrary radius r, sine and cosine each represent a ratio of lengths in diagrams and are not themselves physical lengths (except in the case $r = 1$, perhaps).



We see this if we isolate the right triangle we see in this diagram. For example, $\sin(x)$is the ratio of the side of length $r\sin(x)$ (that is, the side opposite the angle x) to r, the length of the hypotenuse of the right triangle. We can thus focus on a study of right-triangle sides and their ratios.



Rheticus wrote an introductory text on the subject exploring the properties of ratios of sides of right triangles. His text was extraordinarily influential and, to this day, still defines how the sine and cosine functions are introduced to students for the first time in their schooling. No mention whatsoever is made of the circles from which these right triangles came.

In 1595, Bartholomeo Pitiscus coined the name "trigonometry" for Rheticus's right-triangle approach to trigonometry.

Comment. By focusing on right triangles there is now a limitation on the range of angles x that can be studied. Since the angles in a triangle sum to 180°, and a right triangle already contains an angle of measure 90°, we must now restrict x to the range $0^\circ \leq x \leq 90^\circ$. (Is $x = 0^\circ$ really allowed in a right triangle? And $x = 90^\circ$?) Asking for $\sin(120^\circ)$, for example, has no meaning in this context! This is often very confusing to students when they move from a first course in trigonometry back to a proper course on "circle-ometry" in a pre-calculus course.

10
The Basics of Trigonometry

Common Core State Standards

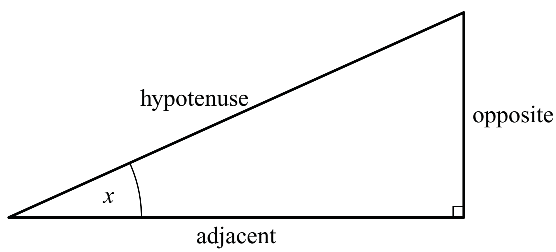
G-SRT.6 Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

G-SRT.7 Explain and use the relationship between the sine and cosine of complementary angles.

G-SRT.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

An uninspired approach to Rheticus's trigonometry begins:

Give the sides of a right triangle containing an angle x of interest the following names:

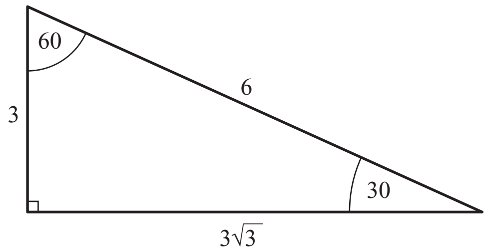


The side opposite the angle x is called the *opposite* (abbreviated "opp").

The side adjacent to the angle x that is not the hypotenuse of the triangle, is called the *adjacent* (abbreviated "adj").

The hypotenuse of the triangle is called, of course, the *hypotenuse* (abbreviated "hyp").

Example. In this triangle



the side opposite 30° has length 3,

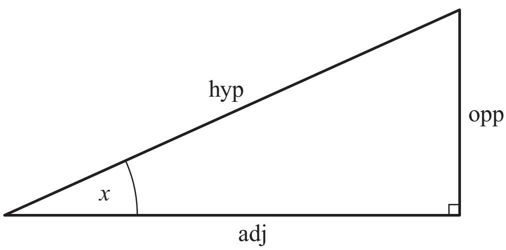
the side adjacent to 30° has length $3\sqrt{3}$,

the hypotenuse has length 6,

the side opposite 60° has length $3\sqrt{3}$,

the side adjacent 60° has length 3.

Now give special names to ratios of these sides in a right triangle.



The ratio $\frac{\text{opp}}{\text{hyp}}$ is called the *sine* of angle x. It is written $\sin(x)$.

The ratio $\frac{\text{adj}}{\text{hyp}}$ is called the *cosine* of angle x. It is written $\cos(x)$.

Question. Can you indeed see that these names are correct in the "circle-ometry" context?

In the context of right triangles there is no reason to give names only to two ratios. Rheticus suggested giving names to all six possible ratios. Here are the names for the remaining four.

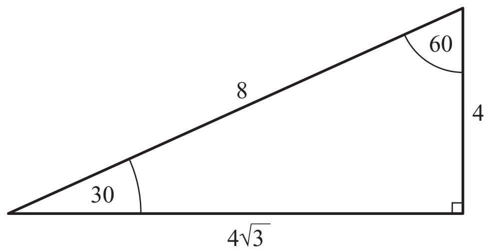
The ratio $\frac{\text{opp}}{\text{adj}}$ is called the *tangent* of angle x. It is written $\tan(x)$.

The ratio $\frac{\text{hyp}}{\text{adj}}$ is called the *secant* of x. It is written $\sec(x)$.

The ratio $\frac{\text{hyp}}{\text{opp}}$ is called the *cosecant* of x. It is written $\csc(x)$.

The ratio $\frac{\text{adj}}{\text{opp}}$ is called the *cotangent* of x. It is written $\cot(x)$.

Example. According to this triangle



$$\sin(30^\circ) = \frac{\text{opp}}{\text{hyp}} = \frac{4}{8} = \frac{1}{2} \qquad \sin(60^\circ) = \frac{\text{opp}}{\text{hyp}} = \frac{4\sqrt{3}}{8} = \frac{\sqrt{3}}{2}$$

$$\cos(30^\circ) = \frac{\text{adj}}{\text{hyp}} = \frac{4\sqrt{3}}{8} = \frac{\sqrt{3}}{2} \qquad \cos(60^\circ) = \frac{\text{adj}}{\text{hyp}} = \frac{4}{8} = \frac{1}{2}$$

$$\tan(30^\circ) = \frac{\text{opp}}{\text{adj}} = \frac{4}{4\sqrt{3}} = \frac{1}{\sqrt{3}} \qquad \tan(60^\circ) = \frac{\text{opp}}{\text{adj}} = \frac{4\sqrt{3}}{4} = \sqrt{3}.$$

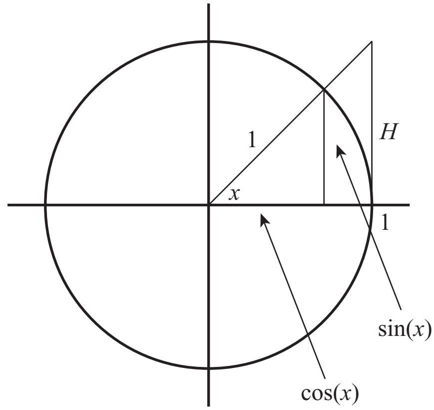
Comment. In computing the value of any of these special ratios for a particular acute angle x it does not matter which right triangle containing the angle x one draws. By the AA similarity principle for triangles, any two triangles containing a 90° angle and an angle of measure x are similar, and thus ratios of matching pairs of sides for those two triangles have the same value.

A Comment on Names

We've seen the origin of the words *sine* and *cosine*. Why did Rheticus give the remaining four rations the names "tangent," "secant," "cosecant," and "cotangent"?

The ratio $\frac{\text{opp}}{\text{adj}}$

Rheticus realized that this ratio is related to a specific tangent line on the circle of radius one. Call the length of the tangent segment shown H.



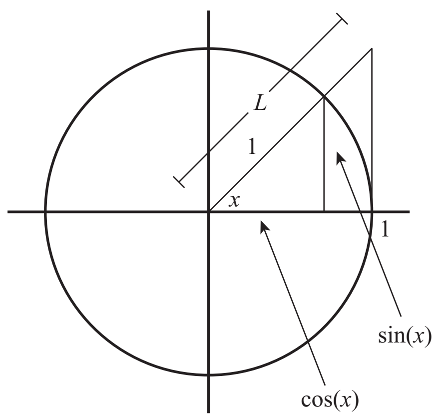
There are two similar triangles in this diagram, one with height H and base 1, and one with height $\sin(x)$ and base $\cos(x)$. (Here we are going back to our "circle-ometry" understanding.) Sides in similar triangles come in the same ratio, and so

$$\frac{H}{1} = \frac{\sin(x)}{\cos(x)} = \frac{\text{opp}/\text{hyp}}{\text{adj}/\text{hyp}} = \frac{\text{opp}}{\text{adj}}.$$

We see that $\frac{\text{opp}}{\text{adj}}$ equals H, the length of the tangent line segment. He called this ratio *tangent*. As a bonus, we also see that $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

The ratio $\frac{\text{hyp}}{\text{adj}}$

In geometry, a secant is a line that cuts through a circle.



Consider the secant line in our previous picture. Suppose its length is L. Using similar triangles we see:

$$\frac{L}{1} = \frac{1}{\cos(x)} = \frac{1}{\mathrm{adj}/\mathrm{hyp}} = \frac{\mathrm{hyp}}{\mathrm{adj}}.$$

So the ratio $\frac{\mathrm{hyp}}{\mathrm{adj}}$ represents the length of a secant line.

Rheticus noticed that if we change our focus to the complementary angle in a right triangle, what we call the opposite and the adjacent sides of the triangle interchange. So if $\sec(x)$ is $\frac{\mathrm{hyp}}{\mathrm{adj}}$ in one right triangle, then $\sec(90^\circ - x) = \frac{\mathrm{hyp}}{\mathrm{opp}}$ in that triangle. For this reason he decided to call the ratio $\frac{\mathrm{hyp}}{\mathrm{opp}}$ *cosecant*. Similarly, it is appropriate to call the ratio $\frac{\mathrm{adj}}{\mathrm{opp}}$ *cotangent*.

We write $\frac{\mathrm{hyp}}{\mathrm{opp}} = \csc(x)$ and $\frac{\mathrm{adj}}{\mathrm{opp}} = \cot(x)$.

NOTE: All six ratios can be expressed solely in terms of sine and cosine. We have, for an acute angle of measure x

$$\sin(x) = \frac{\mathrm{opp}}{\mathrm{hyp}} \quad \cos(x) = \frac{\mathrm{adj}}{\mathrm{hyp}} \quad \tan(x) = \frac{\mathrm{opp}}{\mathrm{adj}}$$

$$\csc(x) = \frac{\mathrm{hyp}}{\mathrm{opp}} \quad \sec(x) = \frac{\mathrm{hyp}}{\mathrm{adj}} \quad \cot(x) = \frac{\mathrm{adj}}{\mathrm{opp}}$$

and so we see

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad \cot(x) = \frac{\cos(x)}{\sin(x)}$$

$$\sec(x) = \frac{1}{\cos(x)} \qquad \csc(x) = \frac{1}{\sin(x)}.$$

The second set of relations shows us how to extend the functions tangent, secant, cotangent, and cosecant functions to real values x representing measures of non-acute angles. For example, we can declare $\csc(120^\circ)$ to have value $\frac{1}{\sin(120^\circ)} = \frac{2}{\sqrt{3}}$. (But notice, for example, that $\csc(180^\circ)$ is undefined!)

Comment. The printing press was invented in the fifteenth-century. It operated by arranging wooden or metal tiles, one for each letter of the alphabet, each punctuation mark, and so on, in grooved lines on a wooden plate. These tiles were then covered with ink and paper was laid across the plate of inked tiles.

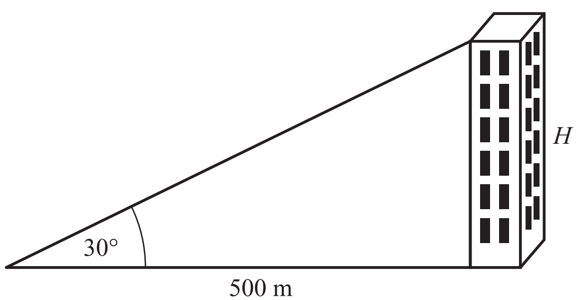
Printing symbols that required half-line positions, fractional quantities like $\frac{1}{\cos(x)}$ for instance, was extraordinarily inconvenient. Thus it became the practice of all scholars of the time to invent in-line names for all quantities in their work: $\sec(x)$ was much easier to print than $\frac{1}{\cos(x)}$. Today, with the ease of printing on our computers, we might not feel the need to give the quantity $\frac{1}{\cos(x)}$ a special in-line name.

Comment. Students beginning trigonometry are usually required to memorize ratios for sine, cosine, and tangent. The following mnemonics are offered to help:

> **SOHCAHTOA** (most popular today)
>
> *Oscar had a heap of apples. Sally counted them.* (Apparently popular in the 1950s.)
>
> *Tommy on a ship of his caught a herring*. (Popular in Britain.)
>
> *Once hero and heroine over acted.* (Popular in India.)
>
> *Studying our homework can always help towards our achievement*. (Probably invented by a teacher!)
>
> *Some old hippie caught another hippie tripping on acid.* (Probably not officially offered in classrooms—though outside of the classroom students seem to know this one!)

Can you see how each of these mnemonics help?

Example. Standing 500 meters from a skyscraper engineers measure the angle of elevation to the top of the building to be 30°. How tall is the building?



Answer. We see a right triangle in this picture, and two particular sides of the triangle seem relevant to the question: the length of 500 meters (the side *adjacent* to 30°) and the side that represents the height of the building (the side *opposite* to 30°). SOHCAHTOA suggests we focus on the ratio of *tangent*:

$$\tan(30^\circ) = \frac{\text{opp}}{\text{adj}} = \frac{H}{500}.$$

On a calculator we see that $\tan(30^\circ) \approx 0.577$ so:

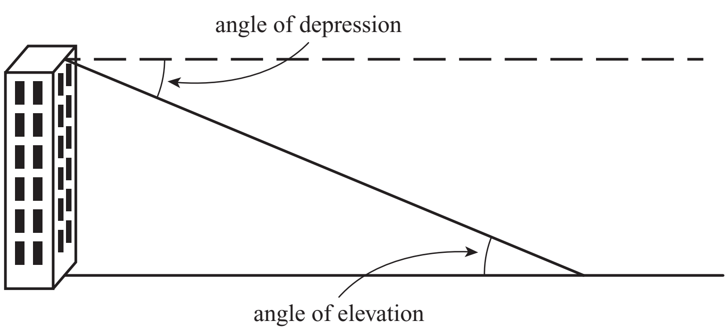
$$0.577 \approx \frac{H}{500}.$$

This gives: $H = 500 \times 0.577 \approx 288.7$ meters.

Comment. This answer is approximate. The exact answer is $H = \frac{500}{\sqrt{3}}$ because we have a $30^\circ - 60^\circ - 90^\circ$ triangle (half an equilateral triangle).

When standing at the top of a building looking down to a person on the ground, the angle you spy from, down from a horizontal, is called the *angle of depression* to the person on the ground.

The angle at which the person on the ground looks up, from the horizontal, to see you is called the *angle of elevation* to the person on the building.



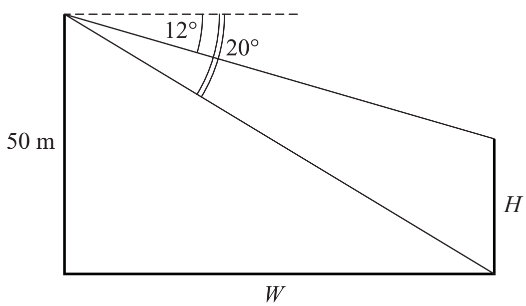
The angle of depression is always congruent in the angle of elevation.

The words "angle of elevation" and "angle of depression" are used in many contexts. For example, one can speak of the angle of elevation at which one spies an airplane, or the angle of elevation of a star, or the angle of elevation of a rope tied to the top of a tree. Standing at the rim of a crater, one might speak of the angle of depression to the bottom of the hole, or one might talk about the angle of depression of a straight section of railway track going down a hill.

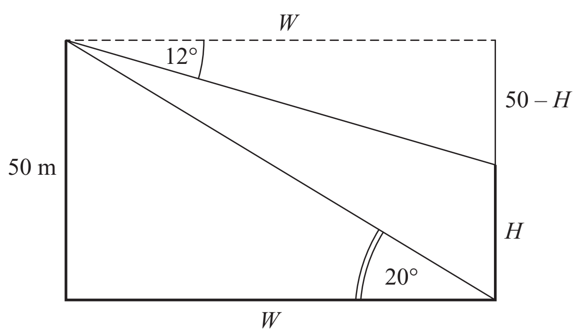
Example. Two trees are situated directly opposite one another on the banks of a river. One tree is 50 meters tall and the angle of depression from the top of this tree to the top of the tree across the river is 12° and to its base is 20°. How tall is the second tree?

Answer. Let's sketch the situation. (Assume the trees are perpendicular to the ground!)

Here H denotes the height of the second tree, and W the width of the river.



Let's draw in some additional lines and look at congruent angles:



We see $\tan(20^\circ) = \frac{50}{W}$ and so $W = \frac{50}{\tan(20^\circ)}$.

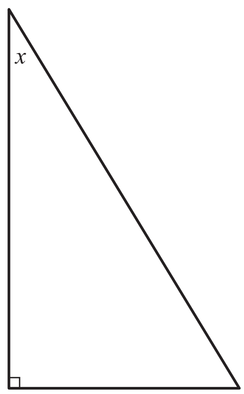
Also $\tan(12^\circ) = \frac{50-H}{W}$ and so

$$H = 50 - W\tan(12^\circ) = 50\left(1 - \frac{\tan(12^\circ)}{\tan(20^\circ)}\right).$$

On a calculator, we see $H \approx 20.8$m.

Question. Look at the right triangle below. Do you see the following relations?

$$\sin(90^\circ - x) = \cos(x)$$
$$\cos(90^\circ - x) = \sin(x)$$
$$\tan(90^\circ - x) = \frac{1}{\tan(x)}.$$



Comment. In radian these relations read: $\sin(\frac{\pi}{2} - x) = \cos(x)$, $\cos(\frac{\pi}{2} - x) = \sin(x)$, and $\tan(\frac{\pi}{2} - x) = \frac{1}{\tan(x)}$.

Comment. Dividing the identity $\sin^2 x + \cos^2 x = 1$ through by $\cos^2 x$ gives the identity:

$$\boxed{1 + \tan^2 x = \sec^2 x.}$$

This holds for all real numbers x that represent angle measures in right triangles, that is, acute angles.

(But using the relations $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and $\sec(x) = \frac{1}{\cos(x)}$ we can extend this statement to a real number x representing the measure of a non-acute angle too, provided $\cos(x) \neq 0$ for that angle measure.)

MAA PROBLEMS

Featured Problem

(#17, AMC 12B, 2012)

Square $PQRS$ lies in the first quadrant. Points (3, 0), (5, 0), (7, 0), and (13, 0) lie on lines SP, RQ, PQ, and SR, respectively. What is the sum of the coordinates of the center of the square $PQRS$?

(A) 6 (B) 6.2 (C) 6.4 (D) 6.6 (E) 6.8

A Personal account of solving this problem

Curriculum Inspirations Strategy (www.maa.org/ci):

Strategy 2: **Do Something!**

I don't think I can fully understand this question until I draw a picture.

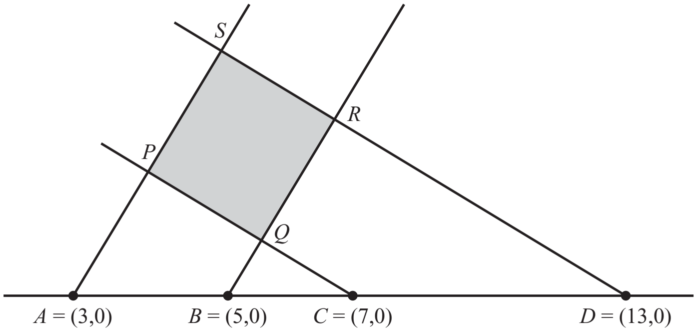
We have four points (3, 0), (5, 0), (7, 0), and (13, 0) that all lie on the x-axis, yet these points lie on the four sides of a square?

Oh, if I read the question in a very literal manner, these points only need to be on the lines that contain the sides of the square. That's makes a big difference!

Okay, here are the four points on the x-axis. I've called them A, B, C, and D. Where must $PQRS$ be in relation to them?

The point A is meant to be on the line through S and P, the point B on the line through R and Q, C on the line through P and Q, and D on the line through S and R.

Okay . . . the lines through A and B contain two opposite sides of the square, and the lines through C and D contain the other opposite pair of sides. We must have a picture something like



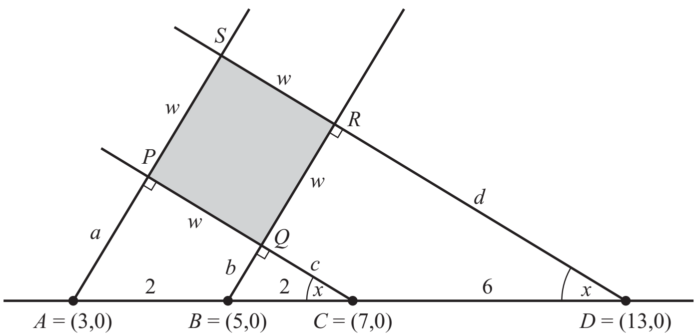
(My square isn't very square-like. Oh well!).

Now what?

What is the sum of the coordinates of the center of the square $PQRS$?

How can I find the center of the square?

I really can't think of anything to do. While I am waiting for an epiphany, let me label some lengths and angles. (I am just doing something!)



Hmm. I think that has made the picture worse.

Oh. Look at triangles CQB and CPA. They share angles and so are similar. Their scale factor is clearly $k = 2$ and so we have $w = c$ and $a = 2b$.

In fact there are four right triangles in this picture each containing the angle x. That makes four similar triangles and lots of relations like these. I am going to write a few more of them. (I really don't know what else to do!)

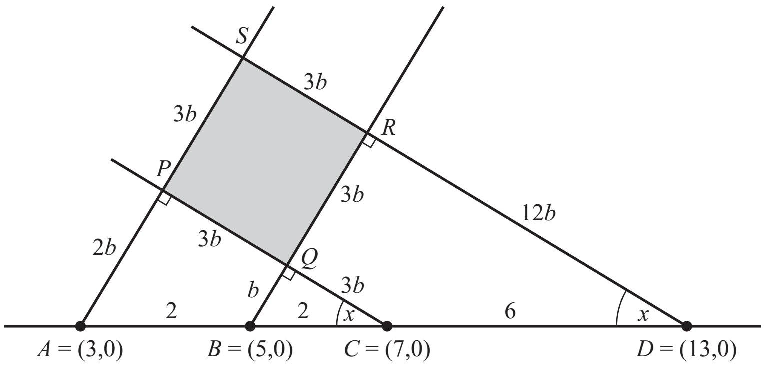
$$\frac{b}{b+w} = \frac{2}{8} \text{ and so } w = 3b.$$

$$d = 4c.$$

Ooh! I think I can now write everything in terms of b.

$$a = 2b, \quad b = b, \quad c = w = 3b, \quad d = 4c = 12b, \quad w = 3b.$$

Does that help? Let me re-label the picture:



I know what b is! By the Pythagorean Theorem, $b^2 + (3b)^2 = 4$, and so $b = \sqrt{0.4}$.

Maybe that's not helpful.

Now I am groping for ideas. Can trigonometry help?

$$\tan(x) = \frac{b}{3b} = \frac{1}{3}.$$

Does that tell me anything?

Well ... $\tan(x) = \frac{\text{opp}}{\text{adj}} = \frac{\text{rise}}{\text{run}} =$ slope. So the line through C has slope $\frac{1}{3}$.

Oops, make that $-\frac{1}{3}$ since my picture has the angle x in an awkward position for computing slope. And the perpendicular lines thus have slope 3. So I know the equations of all four lines:

Line through A:	$y - 0 = 3(x - 3)$. That is: $y = 3(x - 3)$.
Line through B:	$y = 3(x - 5)$.
Line through C:	$y = -\frac{1}{3}(x - 7)$.
Line through D:	$y = -\frac{1}{3}(x - 13)$.

I can get the coordinates of P, Q, R, and S by computing where these lines intersect. Fabulous!

Actually, let me just get the coordinates of P and R, and then find the midpoint of $\overline{PR}$ as that is the center of the square.

The lines through A and C:

$$y = 3x - 9 \;\rightarrow\; 3y = 9x - 27$$
$$3y = -x + 7 \;\rightarrow\; 3y = -x + 7$$

Subtracting gives $x = 3.4$ and $y = 1.2$.

P has coordinates $(3.4,\ 1.2)$.

The lines through B and D:

$$y = 3x - 15 \;\rightarrow\; 3y = 9x - 45$$
$$3y = -x + 13 \;\rightarrow\; 3y = -x + 13$$

R has coordinates $(5.8,\ 2.4)$.

The center of the square is thus $(\frac{9.2}{2}, \frac{3.6}{2}) = (4.6, 1.8)$. Its coordinates sum to 6.4 and so the answer is (C).

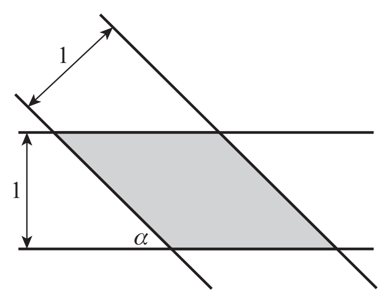
Comment. On reflection I didn't need trigonometry for this question: if I drew in some lines for "rise" and "run" I would have seen even more similar triangles. But when one is groping for ideas, the advice is to go with whatever inspires! One can, and should, later refine one's argument and streamline it.

Additional Problems

23. (#5, AHSME, 1983) Triangle ABC has a right angle at C. If $\sin A = \frac{2}{3}$, then $\tan B$ is

(A) $\frac{3}{5}$ (B) $\frac{\sqrt{5}}{3}$ (C) $\frac{2}{\sqrt{5}}$ (D) $\frac{\sqrt{5}}{2}$ (E) $\frac{5}{3}$

24. (#13, AHSME, 1989) Two strips of width 1 overlap at an angle of α as shown. The area of the overlap (shown shaded) is



(A) $\sin\alpha$ (B) $\frac{1}{\sin\alpha}$ (C) $\frac{1}{1-\cos\alpha}$ (D) $\frac{1}{\sin^2\alpha}$ (E) $\frac{1}{(1-\cos\alpha)^2}$

25. (#20, AHSME, 1972) If $\tan x = \frac{2ab}{a^2-b^2}$, where $a > b > 0$ and $0^\circ < x < 90^\circ$, then $\sin x$ is equal to

(A) $\frac{a}{b}$ (B) $\frac{b}{a}$ (C) $\frac{\sqrt{a^2-b^2}}{2a}$ (D) $\frac{\sqrt{a^2-b^2}}{2ab}$ (E) $\frac{2ab}{a^2+b^2}$

26. (#19, AMC 12B, 2007) Rhombus $ABCD$, with side length 6, is rolled to form a cylinder of volume 6 by taping $\overline{AB}$ to $\overline{DC}$. What is $\sin(\angle ABC)$?

(A) $\frac{\pi}{9}$ (B) $\frac{1}{2}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{4}$ (E) $\frac{\sqrt{3}}{2}$

27. (#15, AHSME, 1999) Let x be a real number such that $\sec x - \tan x = 2$. Then $\sec x + \tan x =$

(A) 0.1 (B) 0.2 (C) 0.3 (D) 0.4 (E) 0.5

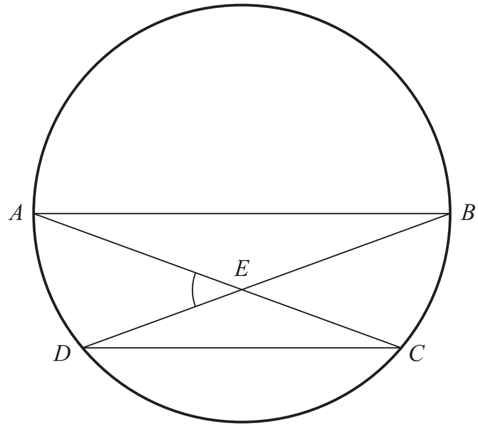
28. (#15, AHSME, 1973) A sector with acute central angle θ is cut from a circle of radius 6. The radius of the circle circumscribed about the sector is

(A) $3\cos\theta$ (B) $3\sec\theta$ (C) $3\cos\frac{\theta}{2}$ (D) $3\sec\frac{\theta}{2}$ (E) 3

29. (#21, AHSME, 1991) If $f(\frac{x}{x-1}) = \frac{1}{x}$ for all $x \neq 0,1$ and $0 < \theta < \frac{\pi}{2}$, then $f(\sec^2\theta) =$

(A) $\sin^2\theta$ (B) $\cos^2\theta$ (C) $\tan^2\theta$ (D) $\cot^2\theta$ (E) $\csc^2\theta$

30. (#27, AHSME, 1986) In the adjoining figure, $\overline{AB}$ is a diameter of the circle, $\overline{CD}$ is a chord parallel to $\overline{AB}$, and $\overline{AC}$ intersects $\overline{BD}$ at E, with $\angle AED = \alpha$. The ratio of the area of ΔCDE to that of ΔABE is



(A) $\cos\alpha$ (B) $\sin\alpha$ (C) $\cos^2\alpha$ (D) $\sin^2\alpha$ (E) $1-\sin\alpha$

31. (#20, AHSME, 1983) If $\tan\alpha$ and $\tan\beta$ are the roots of $x^2 - px + q = 0$, and $\cot\alpha$ and $\cot\beta$ are the roots of $x^2 - rx + s = 0$, then rs is necessarily

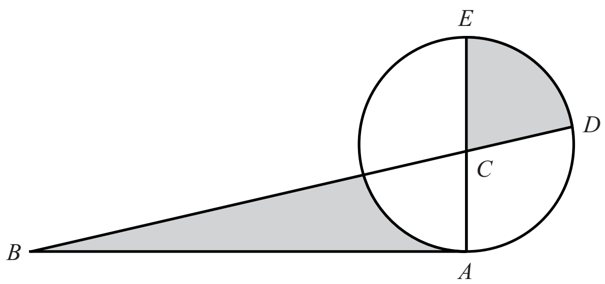
(A) pq (B) $\frac{1}{pq}$ (C) $\frac{p}{q^2}$ (D) $\frac{q}{p^2}$ (E) $\frac{p}{q}$

32. (#20, AHSME, 1987) Evaluate

$$\log_{10}(\tan 1^\circ) + \log_{10}(\tan 2^\circ) + \log_{10}(\tan 3^\circ) + \cdots + \log_{10}(\tan 88^\circ) + \log_{10}(\tan 89^\circ)$$

(A) 0 (B) $\frac{1}{2}\log_{10}(\frac{1}{2}\sqrt{3})$ (C) $\frac{1}{2}\log_{10} 2$ (D) 1 (E) None of these

33. (#21, AHSME, 1986) In the configuration below, $\angle ABC = \theta$ is measured in radians, C is the center of the circle, segments $\overline{BD}$ and $\overline{AE}$ contain C, and $\overline{AB}$ is tangent to the circle at A.



A necessary and sufficient condition for the equality of the two shaded areas, given $0 < \theta < \frac{\pi}{2}$, is

(A) $\tan\theta = 2\theta$ (B) $\tan 2\theta = 2\theta$ (C) $\tan\theta = \theta$ (D) $\tan 2\theta = \theta$ (E) $\tan\frac{\theta}{2} = \theta$

34. (#15, AHSME, 1984) If $\sin 2x \cdot \sin 3x = \cos 2x \cdot \cos 3x$, then one value for x is

(A) 18° (B) 30° (C) 36° (D) 45° (E) 60°

35. (#13, AHSME, 1988) If $\sin x = 3\cos x$ then what is $\sin x \cdot \cos x$?

(A) $\frac{1}{6}$ (B) $\frac{1}{5}$ (C) $\frac{2}{9}$ (D) $\frac{1}{4}$ (E) $\frac{3}{10}$

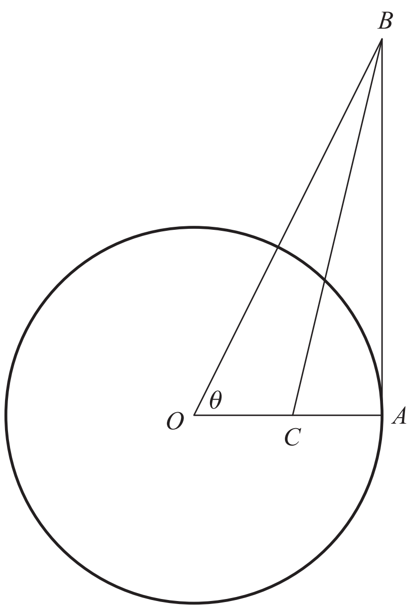
36. (#24, AHSME, 1979) Sides AB, BC and CD of (simple) quadrilateral $ABCD$ have lengths 4, 5 and 20, respectively. If vertex angles B and C are obtuse and $\sin C = -\cos B = \frac{3}{5}$, then side AD has length

(A) 24 (B) 24.5 (C) 24.6 (D) 24.8 (E) 25

37. (#23, AHSME, 1980) Line segments drawn from the vertex opposite the hypotenuse of a right triangle to the points trisecting the hypotenuse have lengths $\sin x$ and $\cos x$, where x is a real number such that $0 < x < \frac{\pi}{2}$. The length of the hypotenuse is

(A) $\frac{4}{3}$ (B) $\frac{3}{2}$ (C) $\frac{3\sqrt{5}}{5}$ (D) $\frac{2\sqrt{5}}{3}$
(E) not uniquely determined by the given information

38. (#17, AMC 12, 2000) A circle centered at O has radius 1 and contains the point A. Segment AB is tangent to the circle at A and $\angle ABO = \theta$. If point C lies on $\overline{OA}$ and $\overline{BC}$ bisects $\angle AOB$, then $OC =$

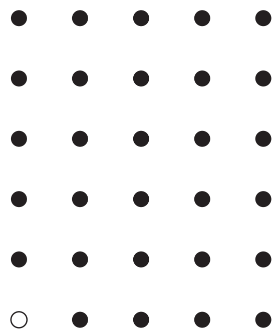


(A) $\sec^2\theta - \tan\theta$ (B) $\frac{1}{2}$ (C) $\frac{\cos^2\theta}{1+\sin\theta}$ (D) $\frac{1}{1+\sin\theta}$ (E) $\frac{\sin\theta}{\cos^2\theta}$

39. (#25, AMC 12B, 2012)
Let $S = \{(x, y) : x \in \{0, 1, 2, 3, 4\},\ y \in \{0, 1, 2, 3, 4, 5\}$, and $(x, y) \neq (0, 0)\}$.

Let T be the set of all right triangles whose vertices are in S. For every right triangle $t = \Delta ABC$ with vertices A, B, and C in counter-clockwise order and right angle at A, let $f(t) = \tan(\angle CBA)$. What is $\prod_{t \in T} f(t)$?

S



(A) 1 (B) $\frac{625}{144}$ (C) $\frac{125}{24}$ (D) 6 (E) $\frac{625}{24}$

11

The Tangent, Cotangent, Secant, and Cosecant Graphs

Common Core State Standards

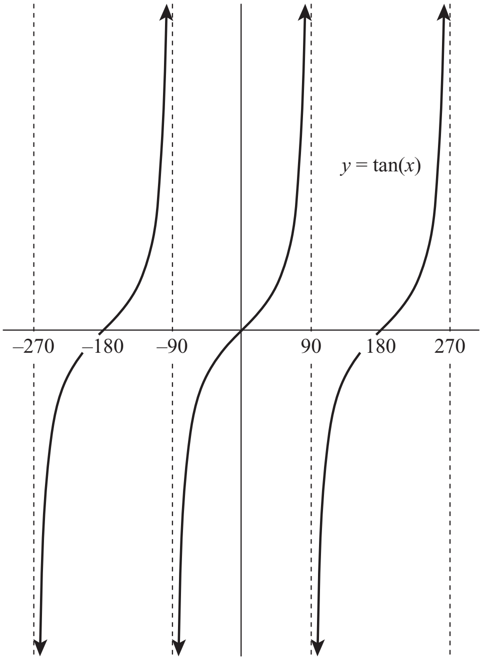
F-TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

In our study of "circle-ometry" we sketched, very swiftly, the graphs of $y = \sin(x)$ and $y = \cos(x)$. The remaining four trigonometric functions were defined only in the theory of right triangles, where all angles in consideration are assumed to be acute, that is, less than 90°. But we can extend the definitions of these four functions to ones valid for all (or almost all) angles by using the relations

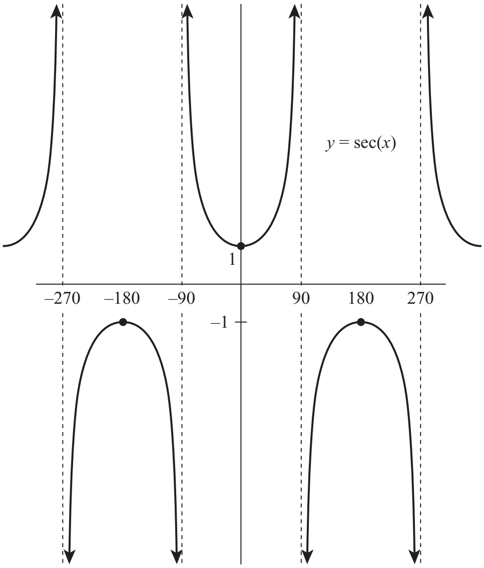
$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \quad \cot(x) = \frac{\cos(x)}{\sin(x)}, \quad \sec(x) = \frac{1}{\cos(x)}, \quad \csc(x) = \frac{1}{\sin(x)}.$$

These functions are undefined at values x which give a denominator of zero. This means that the graphs of these functions have vertical asymptotes at these values. For example, $y = \tan(x)$, has a vertical asymptote at every location for which $\cos(x)$ is zero, namely, at $\pm 90^\circ$, $\pm 270^\circ$, $\pm 450^\circ$, ... (or, in radians, at all $n\frac{\pi}{2}$ with n an odd integer).

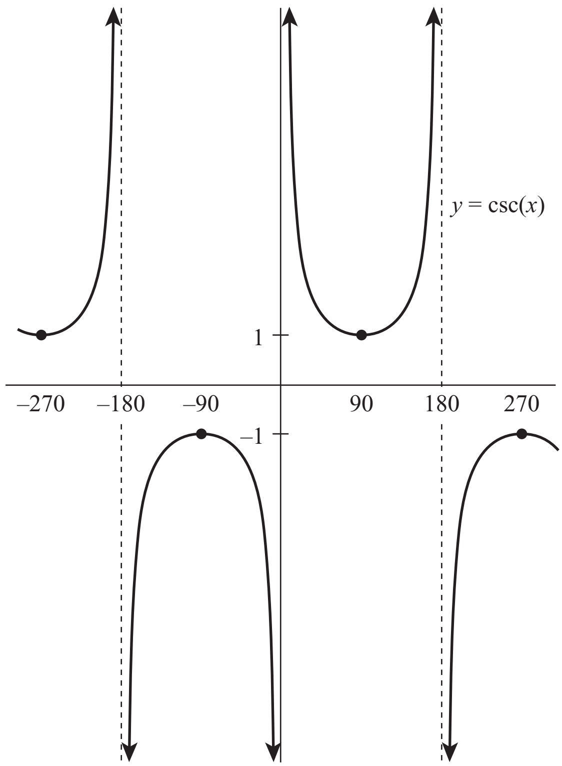
By plotting points one can see that the graph of the tangent function appears as:



Here's the graph of $y = \sec(x)$:



And here's the graph of $y = \csc(x)$:



Comment. Since $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$ we have

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin(x)}{\cos(x)} = -\tan(x)$$
$$\cot(-x) = -\cot(x)$$
$$\sec(-x) = \sec(x)$$
$$\csc(-x) = -\csc(x).$$

These algebraic symmetries appear as symmetries in their graphs.

12
Inverse Trigonometric Functions

Common Core State Standards

F-TF.6 (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.

F-TF.7 (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.

Recall that the notation $y = \sin^{-1}(x)$ refers to the inverse function of sine—going through the "sine machine" backwards.

Forward sine:	Put in an angle x and out comes a value between -1 and 1, the sine of that angle.
Inverse sine:	Put in a value between -1 and 1 and out comes an angle with that value for its sine.

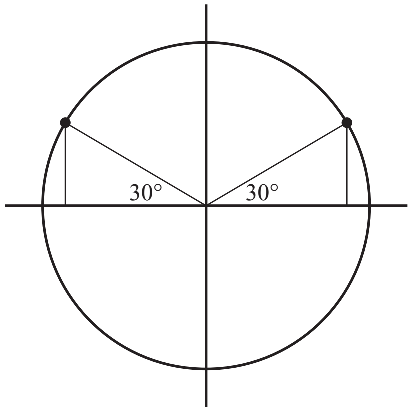
For example, $\sin^{-1}(\frac{1}{2})$ is the angle whose sine is $\frac{1}{2}$. That would be 30°:

$$\sin^{-1}\left(\frac{1}{2}\right) = 30^\circ.$$

In the context of right triangles, this statement is correct. In the context of circle-ometry, however, it is not: there is more than one angle whose sine has value one half:

$$\sin^{-1}\left(\frac{1}{2}\right) = 30^\circ \text{ or } 150^\circ \text{ or } 390^\circ \text{ or } 510^\circ \text{ or } -210^\circ \text{ or } -330^\circ \text{ or } \cdots$$

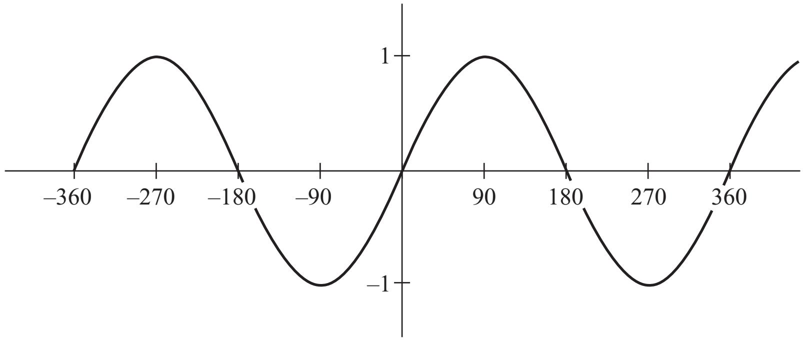
We have two "basic" values, 30° and 150° (and it is clear from the sketch that there should be two fundamental situations) and then all the variations of these angles given by adding and subtracting multiples of 360°.



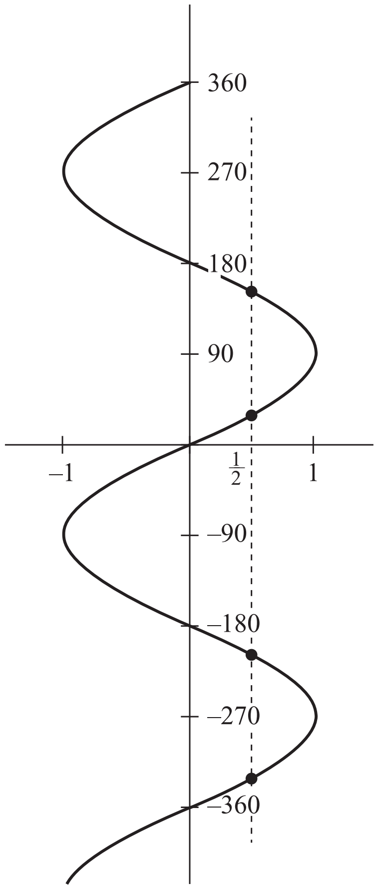
Example. $\cos^{-1}(-\frac{1}{\sqrt{2}}) = 135^\circ$ or -135° plus multiples of 360°.

$\tan^{-1}(\sqrt{3}) = 60^\circ$ or 240° plus multiples of 360°.

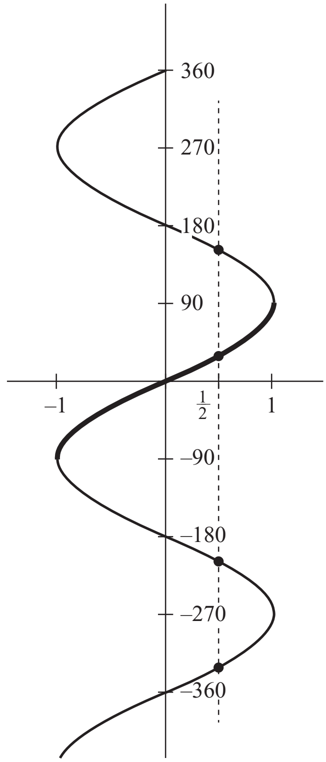
Recall that the graph of an inverse function is the graph of the original function reflected about the diagonal line $y = x$. The graph of $y = \sin(x)$ is



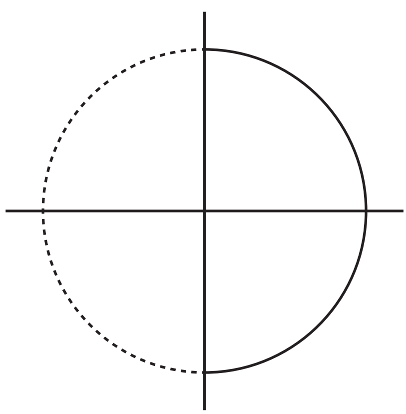
Thus the graph of $y = \sin^{-1}(x)$ is:



Of course, this is not a function in the traditional sense: it is multi-valued. For example, we see that $\sin^{-1}(\frac{1}{2})$ does indeed have multiple values. For this reason, folk might sometimes restrict this picture to the only the highlighted portion shown. (This is sometimes called the *principal branch* of inverse sine.) In this way, by agreeing that all angles are to lie between -90° and 90°, each value between -1 and 1 has a unique angle associated to it.

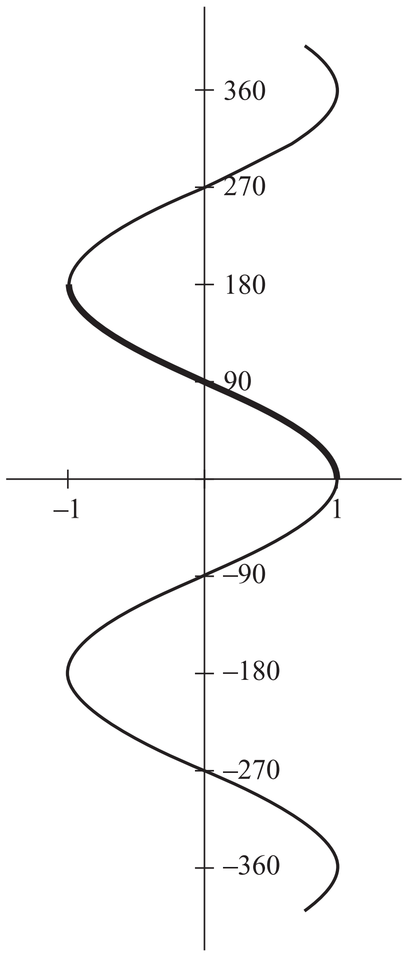


Thus, given a value x between -1 and 1, the *principal value* of $\sin^{-1}(x)$ refers to the angle of elevation of the Sun when it has height x and is located on the right-half portion of the unit circle:

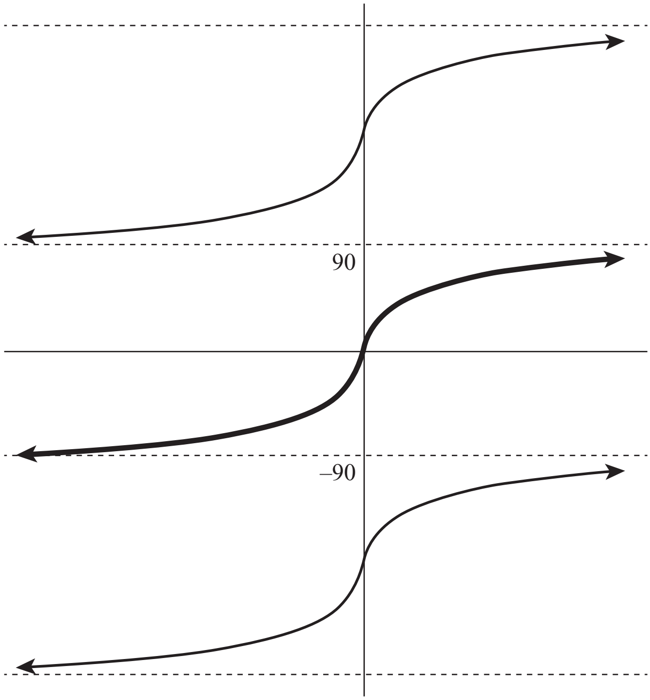


This choice for a principal branch is somewhat arbitrary. It is conceptually easier to keep in mind that there are many admissible angles and to choose only those that are appropriate for the problem at hand.

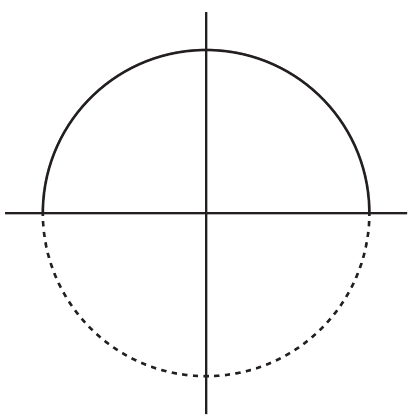
The graph of $y = \cos^{-1}(x)$ appears as (with the portion that folk typically deem the principal branch highlighted)



and the graph of $y = \tan^{-1}(x)$ as:



Given a value x between -1 and 1, the principal value of $\cos^{-1}(x)$ refers to the angle of elevation of the Sun when it has height x and is located in the upper portion of the unit circle:



The principal value of $\tan^{-1}(x)$ refers to the angle of elevation of the Sun in the right half of the unit circle.

Example. Compute $\sin^{-1}(\cos(30^\circ))$ using the principal branch of inverse sine.

Answer 1: $\cos(30^\circ) = \frac{\sqrt{3}}{2}$ and so $\sin^{-1}(\cos(30^\circ)) = \sin^{-1}(\frac{\sqrt{3}}{2}) = 60^\circ$.
Answer 2: $\sin^{-1}(\cos(30^\circ)) = \sin^{-1}(\sin(90^\circ - 30^\circ)) = 60^\circ$.

Example. Suppose $0^\circ < x < 90^\circ$. Make $\cos(\sin^{-1}(\cos(x)))$ look friendlier using the principal branch of the inverse sine.

Answer. $\cos(\sin^{-1}(\cos(x))) = \cos(\sin^{-1}(\sin(90^\circ - x))) = \cos(90^\circ - x) = \sin(x)$.
(In general, show for all angles x we have $\cos(\sin^{-1}(\cos x)) = |\sin(x)|$.)

Comment. Calculators offer the means to compute inverse sines, cosines, and tangents. They only give one answer for each, the principal value. One must use common sense to determine which variation of this answer is appropriate for the problem at hand.

Comment. Because of the fifteenth-century printing press, mathematicians sometimes use the terms $\arcsin(x)$ for $\sin^{-1}(x)$, $\arccos(x)$ for $\cos^{-1}(x)$, and so on. (Which part of the circle arc must we be on for a given sine value?)

MAA PROBLEMS

Featured Problem

(#18, AMC 12B, 2009)

For how many values of x in $[0, \pi]$ is $\sin^{-1}(\sin 6x) = \cos^{-1}(\cos x)$?

Note: The functions $\sin^{-1} = \arcsin$ and $\cos^{-1} = \arccos$ denote inverse trigonometric functions.

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

A Personal account of solving this problem

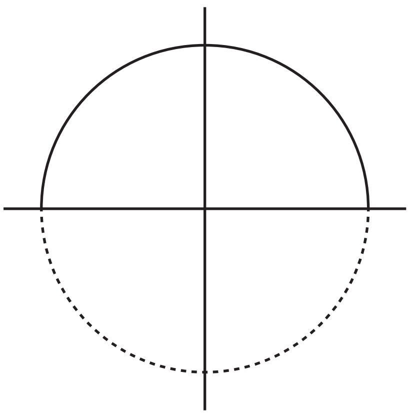
Curriculum Inspirations Strategy (www.maa.org/ci):

Strategy 3: **Engage in Wishful Thinking**

This problem looks scary to me! I understand all the concepts in the question, but working with inverse functions is not at all intuitive. I need to take this slowly—very slowly—and build things up from what I know, with absolute care and patience!

Firstly, let me just collect some relevant thoughts:

$\cos^{-1}$ gives an angle in the "top half of the circle" (assuming we are working with the principal branch).

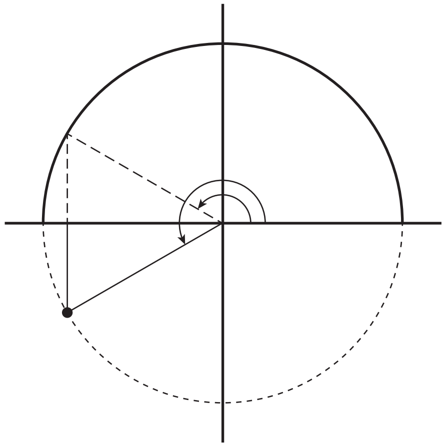


So $\cos^{-1}$ (anything) is an angle between 0 and π. In particular, $\cos^{-1}(\cos(x))$ is an angle between 0 and π whose cosine is $\cos x$ even if the angle x isn't in this range!

So if x does happen to be "in the top half of the circle," that is, if $0 < x < \pi$, then x is the correct angle for $\cos^{-1}(\cos(x))$ and we have $\cos^{-1}(\cos(x)) = x$.

If x happens to be "in the bottom half," that is, $\pi < x < 2\pi$, then x is the incorrect angle for $\cos^{-1}(\cos(x))$. But its matching "correct" angle in the top half of the circle is $2\pi - x$, and so in this case $\cos^{-1}(\cos(x)) = 2\pi - x$.

If x is outside of these ranges, then we would need to adjust x by some multiple of 2π and therefore adjust our answers by multiples of 2π too.



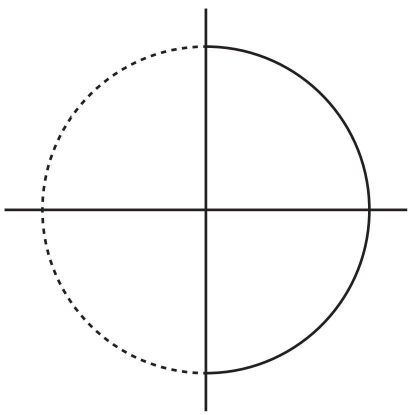
Overall, we see

$$\cos^{-1}(\cos(x)) = \pm x \ \ + 2n\pi$$

and we must use the multiple of 2π that produces an answer in the range $[0,\ \pi]$. (I've written just "$-x$" instead of "$2\pi - x$," absorbing the 2π that appears there.)

For $\sin^{-1}$...

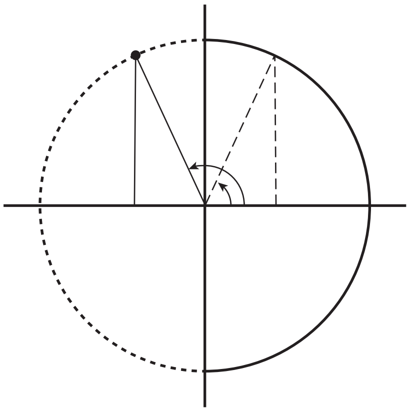
$\sin^{-1}$ gives an angle in the "right half of the circle."



$\sin^{-1}(\sin(x))$ is an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with sine value equal to $\sin(x)$ even if x isn't in this range.

So $\sin^{-1}(\sin(x)) = x$ if we already happen to have $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

If x lies in the left half of the circle, then the matching angle with the same height in the right half is $\pi - x$, and $\sin^{-1}(\sin(x)) = \pi - x$.



Again we might need to adjust these answers by multiples of 2π if x is an angle outside of these ranges.

All right, back to the question. Umm. What was the question?

For how many values of x in $[0, \pi]$ is $\sin^{-1}(\sin 6x) = \cos^{-1}(\cos x)$?

Well,

$$\cos^{-1}(\cos x) = x \ \textit{or} \ -x \text{ adjusted by } 2n\pi \text{ if necessary.}$$
$$\sin^{-1}(\sin 6x) = 6x \ \textit{or} \ \pi - 6x \text{ adjusted by } 2n\pi \text{ if necessary.}$$

Bringing all the multiples of 2π together, we basically have four cases to consider for equality.

$$\begin{aligned} 6x &= x + 2n\pi, \\ 6x &= -x + 2n\pi, \\ \pi - 6x &= x + 2n\pi, \\ \pi - 6x &= -x + 2n\pi. \end{aligned}$$

These give

$$x = \frac{2n\pi}{5}, \ \frac{2n\pi}{7}, \ \frac{(2n+1)\pi}{7}, \quad \text{or} \quad \frac{(2n+1)\pi}{5},$$

which are just the even and odd multiples of $\frac{\pi}{5}$ and of $\frac{\pi}{7}$. Since we're looking for solutions in the range $[0,\ \pi]$, then we have:

$$x = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}, \frac{4\pi}{7}, \frac{5\pi}{7}, \frac{6\pi}{7}, \quad \text{or} \quad \pi.$$

But that makes for more solutions than the question suggests!

Actually, I am not surprised. I've been quite careless with keeping track of which multiples of 2π to use when. I just slapped all the multiples of 2π together and hoped for the best!

But this exercise in wishful thinking has given me a range of angles to consider. Not all of them will likely be solutions, but the solutions to the equation that do exist must be among these candidates. Let's just go through them one at a time and see which work!

$x = 0$:

$$\sin^{-1}(\sin 0) = 0$$

$$\cos^{-1}(\cos 0) = 0.$$

We have a solution.

$x = \frac{\pi}{5}$:

$$\sin^{-1}\left(\sin\frac{6\pi}{5}\right) = \pi - \frac{6\pi}{5} = -\frac{\pi}{5}$$

$$\cos^{-1}\left(\cos\frac{\pi}{5}\right) = \frac{\pi}{5}.$$

Not a solution.

$x = \frac{2\pi}{5}$:

$$\sin^{-1}\left(\sin\frac{12\pi}{5}\right) = \sin^{-1}\left(\sin\frac{2\pi}{5}\right) = \frac{2\pi}{5}$$

$$\cos^{-1}\left(\cos\frac{2\pi}{5}\right) = \frac{2\pi}{5}.$$

We have a solution.

Oh! $\cos^{-1}(\cos x) = x$ for all the angles we are considering! So we need to only check if $\sin^{-1}(\sin 6x) = x$ for our angles.

$x = \frac{3\pi}{5}$:

$$\sin^{-1}\left(\sin\frac{18\pi}{5}\right) = \sin^{-1}\left(\sin\frac{8\pi}{5}\right) = \pi - \frac{8\pi}{5} = -\frac{3\pi}{5}.$$

Not a solution.

$x = \frac{4\pi}{5}$:

$$\sin^{-1}\left(\sin\frac{24\pi}{5}\right) = \sin^{-1}\left(\sin\frac{4\pi}{5}\right) = \pi - \frac{4\pi}{5} = \frac{\pi}{5}.$$

Not a solution.

$x = \frac{\pi}{7}$:

$$\sin^{-1}\left(\sin\frac{6\pi}{7}\right) = \pi - \frac{6\pi}{7} = \frac{\pi}{7}.$$

A solution!

$x = \frac{2\pi}{7}$:

$$\sin^{-1}\left(\sin\frac{12\pi}{7}\right) = -\frac{5\pi}{7}.$$

Not a solution.

$x = \frac{3\pi}{7}$:

$$\sin^{-1}\left(\sin\frac{18\pi}{7}\right) = \sin^{-1}\left(\sin\frac{4\pi}{7}\right) = \frac{3\pi}{7}.$$

A solution.

$x = \frac{4\pi}{7}$:

$$\sin^{-1}\left(\sin\frac{14\pi}{7}\right) = \sin^{-1}\left(\sin\frac{10\pi}{7}\right) = -\frac{3\pi}{7}.$$

Not a solution.

$x = \frac{5\pi}{7}$:

$$\sin^{-1}\left(\sin\frac{30\pi}{7}\right) = \sin^{-1}\left(\sin\frac{2\pi}{7}\right) = \frac{2\pi}{7}.$$

Not a solution.

$x = \frac{6\pi}{7}$:

$$\sin^{-1}\left(\sin\frac{36\pi}{7}\right) = \sin^{-1}\left(\sin\frac{8\pi}{7}\right) = -\frac{\pi}{7}.$$

Not a solution.

$x = \pi$:

$$\sin^{-1}(\sin\pi) = 0.$$

Not a solution.

There are four solutions, $x = 0$, $\frac{2\pi}{5}$, $\frac{\pi}{7}$, and $\frac{3\pi}{7}$, and the answer to the question is (B).

Phew!

(This problem really got me thinking about how inverse sine and inverse cosine operate. Great question!)

Additional Problem

40. (#24, AHSME, 1988) An isosceles trapezoid is circumscribed around a circle. The longer base of the trapezoid is 16, and one of the base angles is arcsin(.8). Find the area of the trapezoid.

(A) 72 (B) 75 (C) 80 (D) 90 (E) not uniquely determined

13

Addition and Subtraction Formulas; Double and Half Angle Formulas

Common Core State Standards

F-TF.9 (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

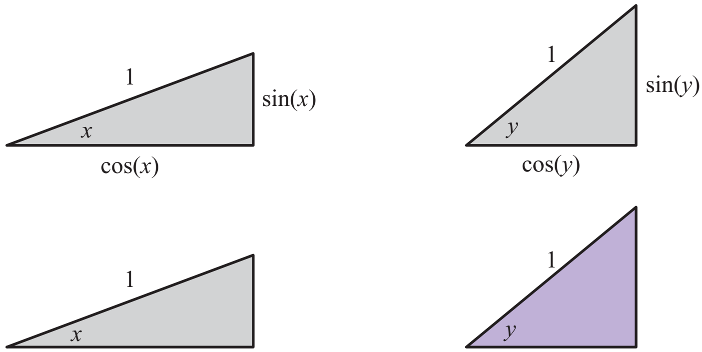
It would be wonderfully convenient if the following two claims were true:

$$\sin(x+y) = \sin(x) + \sin(y)$$
$$\cos(x+y) = \cos(x) + \cos(y)\,.$$

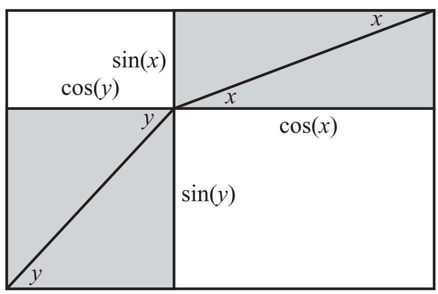
As experimentation on a calculator shows (or just put in $x = 30^\circ$ and $y = 60^\circ$) these dream formulas do not hold.

Can we find expressions for $\sin(x+y)$ and $\cos(x+y)$ nonetheless? You bet! We can use an approach similar to the one we used to prove the Pythagorean Theorem.

Draw two copies each of two right triangles, each with hypotenuse 1.



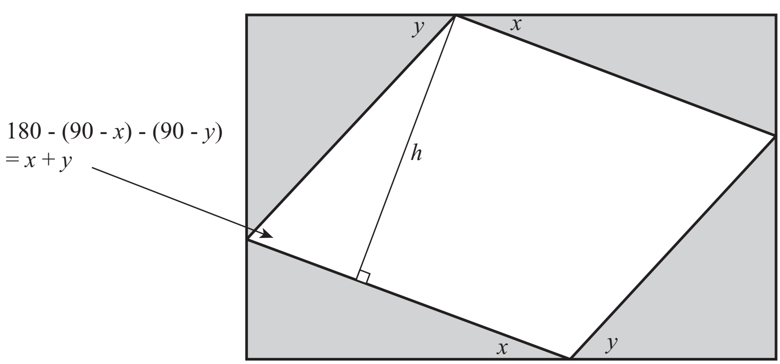
Arrange them into a rectangle as shown:



The area of the white space is the sum of the areas of two small rectangles:

$$\text{White Space} = \sin(x)\cos(y) + \cos(x)\sin(y).$$

We can also rearrange the four triangles within the large rectangle as



The white space is now a rhombus with side length 1. The area of a rhombus (in fact, of any parallelogram) is "base times height." The base length is 1 and the height is the length h indicated. We see that h is the opposite edge of a right triangle of hypotenuse 1 and angle $x + y$. Thus:

$$\text{White Space} = 1 \times h = 1 \times \sin(x + y) = \sin(x + y).$$

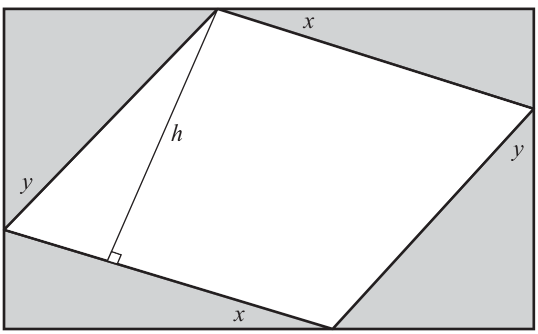
It is the same white space. Thus we have proved:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y,$$

at least for angles x and y that lie between 0° and 90°.

(For visual ease, we've omitted displaying all the parentheses in this equation.)

In the same way, we can consider this variation of the proof. (Do you see the change on which angle is called y?)



It establishes

$$\cos(x - y) = \cos x \cos y + \sin x \sin y,$$

at least for angles x and y that lie between 0° and 90°.

Two questions naturally arise:

1. Do the formulas we established for $\sin(x + y)$ and $\cos(x - y)$ also hold for real numbers x and y representing the measures of non-acute angles, and even negative angles? (If you play on a calculator, it seems they do!)
2. Are there formulas for $\sin(x - y)$ and $\cos(x + y)$?

If the answer to question 1 is yes, then we can replace y with $-y$ in the two formulas we have so far to obtain

$$\begin{aligned}\sin(x-y) &= \sin(x+(-y))\\ &= \sin(x)\cos(-y)+\cos(x)\sin(-y)\\ &= \sin x\cos y-\cos x\sin y\end{aligned}$$

and

$$\begin{aligned}\cos(x+y) &= \cos(x-(-y))\\ &= \cos(x)\cos(-y)+\sin(x)\sin(-y)\\ &= \cos x\cos y-\sin x\sin y.\end{aligned}$$

Again, playing on a calculator, these formulas do seem to hold.

Further, we obtained our formula for $\cos(x-y)$ by changing which angle we labeled y in our triangles. That is, we chose to replace y with $90^\circ - y$ in our formula for $\sin(x+y)$. This gives

$$\sin(x+(90^\circ-y)) = \cos(x)\cos(90^\circ-y)+\sin(x)\sin(90^\circ-y).$$

Using the identities

$$\begin{aligned}\sin(w-90^\circ) &= \cos(w)\\ \sin(w+90^\circ) &= -\cos(w)\\ \cos(w-90^\circ) &= \sin(w)\\ \cos(w+90^\circ) &= -\sin(w)\\ \sin(-w) &= -\sin(w)\\ \cos(-w) &= \cos(w)\end{aligned}$$

we can rewrite this as:

$$-\cos(x-y) = \cos x\sin y-\sin x\cos y,$$

giving our formula for $\cos(x-y)$.

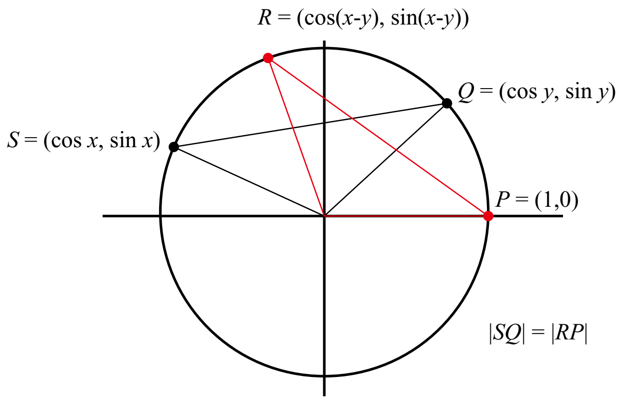
So, in summary, we have the four potential angle addition and subtraction formulas

$$\begin{aligned}\cos(x+y) &= \cos x\cos y-\sin x\sin y\\ \sin(x+y) &= \sin x\cos y+\cos x\sin y\\ \cos(x-y) &= \cos x\cos y+\sin x\sin y\\ \sin(x-y) &= \sin x\cos y-\cos x\sin y.\end{aligned}$$

and the work shows that if we can establish any one as valid for all real values x and y, then all four formulas are valid. Our proof mimicking the Chinese Proof of the Pythagorean Theorem established two of these formulas only for x and y representing the measures to acute angles.

ANOTHER APPROACH TO $\cos(x - y)$

Consider a point S on a circle of radius 1 with angle of elevation x, and a point Q with angle of elevation y. In the diagram below I've drawn x to represent an obtuse angle and y an acute angle, but they can be angles of any type.



The coordinates of each of these points is just its overness and height as an ordered pair:

$$S = (\cos x, \sin x)$$
$$Q = (\cos y, \sin y)\,.$$

The distance d between them satisfies:

$$\begin{aligned} d^2 &= (\cos x - \cos y)^2 + (\sin x - \sin y)^2 \\ &= 2 - 2\cos x \cos y - 2\sin x \sin y. \end{aligned}$$

Now rotate S and Q clockwise about the origin through an angle of measure y. Then Q is taken to a point R with angle of elevation $x - y$, $R = (\cos(x - y), \sin(x - y))$, and Q to the point $P = (1, 0)$ on the horizontal axis. As rotations preserve distances, we have:

$$\begin{aligned} d^2 &= (\cos(x - y) - 1)^2 + (\sin(x - y))^2 \\ &= 2 - 2\cos(x - y)\,. \end{aligned}$$

It follows that $\cos(x - y) = \cos x \cos y + \sin x \sin y$ holds for all real values x and y.

We now have

> The four addition and subtraction trigonometric formulas
>
> $$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
> $$\sin(x + y) = \sin x \cos y - \cos x \sin y$$
> $$\cos(x - y) = \cos x \cos y - \sin x \sin y$$
> $$\sin(x - y) = \sin x \cos y - \cos x \sin y$$
>
> each hold for all real values x and y.

Example. Find the exact value of $\cos(15^\circ)$.

Answer.

$$\begin{aligned}\cos(15^\circ) &= \cos(60^\circ - 45^\circ)\\ &= \cos 60^\circ \cos 45^\circ + \sin 60^\circ \sin 45^\circ\\ &= \frac{1}{2}\cdot\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}\\ &= \frac{1+\sqrt{3}}{2\sqrt{2}}.\end{aligned}$$

DOUBLE ANGLE FORMULAS

Going further . . . .

Put $y = x$ into the formulas

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$\sin(x + y) = \sin x \cos y + \sin y \cos x$$

to obtain

$$\boxed{\begin{aligned}\cos(2x) &= \cos^2 x - \sin^2 x\\ \sin(2x) &= 2 \sin x \cos x.\end{aligned}}$$

Since $\sin^2 x + \cos^2 x = 1$ we can alternatively write the first formula as:

$$\cos(2x) = 2\cos^2(x) - 1$$
$$\cos(2x) = 1 - 2\sin^2(x).$$

Now $\tan(2x) = \frac{\sin(2x)}{\cos(2x)} = \frac{2\sin x \cos x}{\cos^2 x - \sin^2 x}$. Divide the numerator and denominator each by $\cos^2 x$ to obtain

$$\tan(2x) = \frac{\dfrac{2\sin x \cos x}{\cos^2 x}}{\dfrac{\cos^2 x - \sin^2 x}{\cos x}} = \frac{2\dfrac{\sin x}{\cos x}}{1 - \left(\dfrac{\sin x}{\cos x}\right)^2}.$$

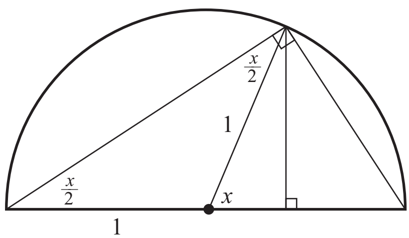
That is:

$$\boxed{\tan(2x) = \frac{2\tan x}{1 - \tan^2 x}.}$$

Similar work shows

$$\boxed{\begin{aligned} \tan(x+y) &= \frac{\tan x + \tan y}{1 - \tan x \tan y} \\ \tan(x-y) &= \frac{\tan(x) + \tan(-y)}{1 - \tan(x)\tan(-y)} = \frac{\tan x - \tan y}{1 + \tan x \tan y} \end{aligned}}$$

Question. The following diagram shows that $\tan(\frac{x}{2}) = \frac{\sin x}{1+\cos x}$. Can you see how? It also shows that $\tan(\frac{x}{2}) = \frac{1-\cos x}{\sin x}$.



Comment. There is a myriad of trigonometric identities one can look at. They can each usually be proved by plugging away with the addition and subtraction identities, or the double angle identities, and grinding through the algebra. (Simple visual proofs like the ones above are always a delight to stumble upon.) For example, can you prove the following identities?

$$\boxed{\begin{aligned} \sin x + \sin y &= 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) \\ \cos x + \cos y &= 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right). \end{aligned}}$$

Comment. We have the angle addition formulas

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

from which we derived the double angle formulas

$$\sin(2x) = 2\sin x \cos x$$
$$\cos(2x) = \cos^2 x - \sin^2 x,$$

The triple angles formulas follow too. For example,

$$\begin{aligned}
\sin(3x) &= \sin(2x+x) \\
&= \sin(2x)\cos x + \cos(2x)\sin x \\
&= 2\sin x \cos^2 x + \cos^2 x \sin x - \sin^3 x \\
&= 3\sin x \cos^2 x - \sin^3 x \\
\cos(3x) &= \cos(2x+x) \\
&= \cos(2x)\cos x - \sin(2x)\sin x \\
&= \cos^3 x - \sin^2 x \cos x - 2\sin^2 x \cos x \\
&= \cos^3 x - 3\sin^2 x \cos x.
\end{aligned}$$

With the use of the complex number i (which satisfies $i^2 = -1$) the triple angle formulas can be united as a single identity:

$$\cos(3x) + i\sin(3x) = (\cos x + i\sin x)^3.$$

One can prove, in general, by induction that for each positive integer n we have

$$\boxed{\cos(nx) + i\sin(nx) = (\cos x + i\sin x)^n.}$$

If one is aware of Euler's famous formula $e^{ix} = \cos x + i\sin x$, this is nothing more than the statement

$$e^{inx} = (e^{ix})^n.$$

For a complete discussion of the Euler's formula and role of using complex numbers to simplify trigonometry see my text *THINKING MATHEMATICS! Vol 5: Slope, e, i, pi and all that.*

SOMETHING EXTRA

If one graphs on a calculator the function $y = 6\sin x + 8\cos x$ the result appears to be another sine curve. This is surprise!

It turns out that we can write $6\sin x + 8\cos x$ in the form $c\sin(x + k)$ for suitable values of c and k. Here's how:

Writing

$$c\sin(x + k) = c\sin x\cos k + c\cos x\sin k = 6\sin x + 8\cos x$$

shows that setting

$$c\cos k = 6$$

$$c\sin k = 8$$

and solving will give us the values that we need. Now

$$6^2 + 8^2 = c^2\cos^2 k + c^2\sin^2 k = c^2(\cos^2 k + \sin^2 k) = c^2$$

gives $c = \sqrt{6^2 + 8^2} = 10$. Also,

$$\frac{c\sin k}{c\cos k} = \frac{8}{6}$$

gives $\tan k = \frac{8}{6}$ and so $k = \tan^{-1}(\frac{8}{6})$.

Thus we have

$$6\sin x + 8\cos x = 10\sin(x + \alpha)$$

where $\alpha = \tan^{-1}(4/3)$.

MAA PROBLEMS

Featured Problem

(#20, AMC 12B, 2013)

For $135^\circ < x < 180^\circ$, points $P = (\cos x, \cos^2 x)$, $Q = (\cot x, \cot^2 x)$, $R = (\sin x, \sin^2 x)$, and $S = (\tan x, \tan^2 x)$ are the vertices of a trapezoid. What is $\sin(2x)$?

(A) $2 - 2\sqrt{2}$ (B) $3\sqrt{3} - 6$ (C) $3\sqrt{2} - 5$ (D) $-\frac{3}{4}$ (E) $1 - \sqrt{3}$

A PERSONAL ACCOUNT OF SOLVING THIS PROBLEM:

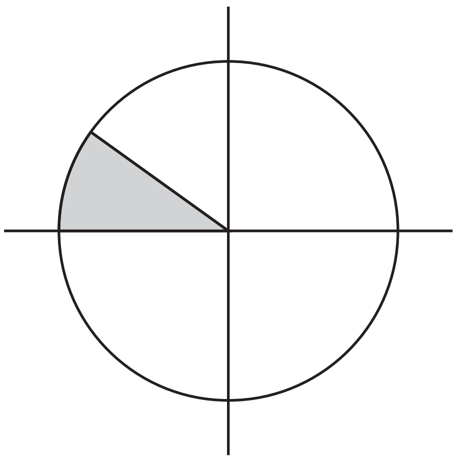
Curriculum Inspirations Strategies (www.maa.org/ci):

Strategy 1: **Engage in Successful Flailing**

This problem, to me, looks weird! I see we have four points all of the form (z, z^2), so they lie on the parabola $y = x^2$. Their coordinates are given by trigonometric functions and apparently these points form a trapezoid. But I can't imagine how a trapezoid can fit on a parabola!

Let's start by getting a sense of where these four points P, Q, R, and S sit.

We have that $135° < x < 180°$. This has to be special for the question.



In this range:

$\sin x$ is positive,

$\cos x$ is negative,

$\tan x$ and $\cot x$ are each negative.

The "size" of $\sin x$ is smaller than the "size" of $\cos x$ (that is, $\sin x < |\cos x|$).

So $|\tan x| < 1$ and $|\cot x| > 1$.

We also have of course $\sin x < 1$ and $|\cos x| < 1$.

Umm. Anything else?

The relations tell me about the horizontal coordinates of the points P, Q, R, and S. We have $\sin x$ (for R) sits on the positive part of the horizontal axes, $\cot x$ (for Q) to the left of -1 on this axis, and $\cos x$ (for P) and $\tan x$ (for S) each lie between -1 and 0 on this axis.

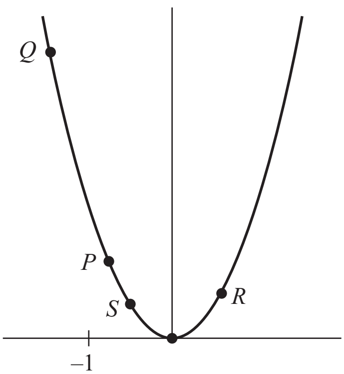
Can I determine the relative positions of these last two: cosine and tangent?

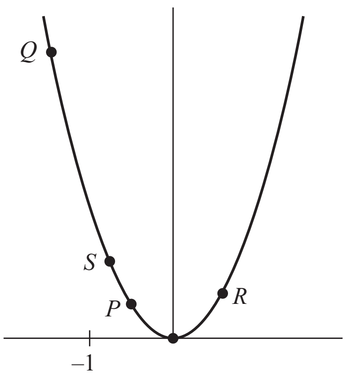
We have

$$|\tan x| = \frac{\sin x}{|\cos x|}.$$

Hmm. I don't see how this helps.

All right, but I do see that the picture has to be one of the following two options:





These make a trapezoid.

I am guessing that the sides $\overline{QR}$ and $\overline{PS}$ will be the parallel pair. I'll assume this for now and come back to this issue if a problem arises!

I can only think to compute the slopes of the segments:

$$\text{slope } \overline{QR} = \frac{\sin^2 x - \cot^2 x}{\sin x - \cot x} = \sin x + \cot x$$

$$\text{slope } \overline{PS} = \frac{\cos^2 x - \tan^2 x}{\cos x - \tan x} = \cos x + \tan x$$

for both pictures. So it doesn't matter which picture we work with, we will obtain

$$\sin x + \cot x = \cos x + \tan x$$

either way!

(Another possible problem: Could the slopes be undefined? Could $\sin x - \cot x = 0$ or $\cos x - \tan x = 0$? I'll assume not and again worry about this later if it turns out to be a problem!)

The question wants us to obtain a value for $\sin(2x) = 2\sin x \cos x$. Surely we can work with the equation we just obtained to make this expression appear?

We have

$$\begin{aligned}
\sin x + \cot x &= \cos x + \tan x, \\
\sin x + \frac{\cos x}{\sin x} &= \cos x + \frac{\sin x}{\cos x}, \\
\sin^2 x \cos x + \cos^2 x &= \sin x \cos^2 x + \sin^2 x.
\end{aligned}$$

Let's rearrange this:

$$\begin{aligned}
\sin x \cos x(\sin x - \cos x) &= \sin^2 x - \cos^2 x \\
&= (\sin x - \cos x)(\sin x + \cos x).
\end{aligned}$$

Divide through by $\sin x - \cos x$ (which is not zero because $\tan x$ is not actually 1–phew!) to get

$$\sin x \cos x = \sin x + \cos x.$$

Is this helpful? I am looking for $2\sin x \cos x$. I see it on the left, but not on the right.

Square the equation and get some helpful cross terms?

$$\sin^2 x \cos^2 x = \sin^2 x + 2\sin x \cos x + \cos^2 x = 1 + 2\sin x \cos x$$

Oh! This is:

$$\frac{1}{4}(\sin 2x)^2 = 1 + \sin 2x.$$

We have the quadratic equation $\frac{1}{4}U^2 = 1 + U$, that is, $U^2 = 4 + 4U$, giving,

$$\begin{aligned}
U^2 - 4U &= 4 \\
U^2 - 4U + 4 &= 8 \\
(U - 2)^2 &= 8 \\
U &= 2 \pm \sqrt{8}
\end{aligned}$$

Since $U = \sin 2x$ we need an answer between -1 and 1. So $\sin 2x = 2 - \sqrt{8}$ and the answer is (A)!

Comment. We have some problems to fix still in this solution.

1. We assumed $\overline{QR} \| \overline{PS}$. Why can't it be the case that the other pair of sides make the parallel pair?
2. Could the slopes of $\overline{QR}$ and $\overline{PS}$ be undefined? (Are we certain that $\sin x \neq \cot x$ and $\cos x \neq \tan x$?)

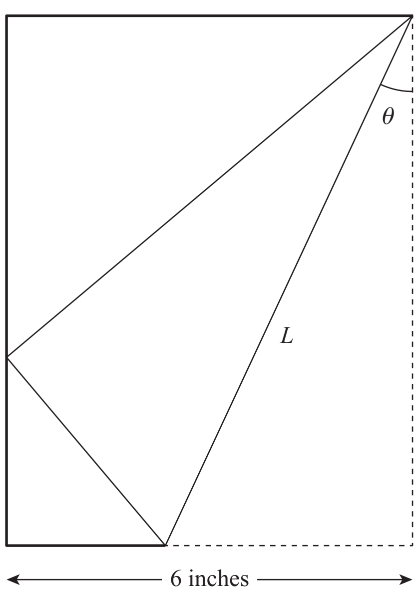
The question also assumed that this trapezoid exists!

3. Is it possible to prove that there is an angle x with measure between 135° and 180° that does indeed create a trapezoid just as the question dictates?

These are good questions to mull on!

Additional Problems

41. (#30, AHSME, 1972) A rectangular piece of paper 6 inches wide is folded as in the diagram so that one corner touches the opposite side. The length in inches of the crease L in terms of the angle θ is



(A) $3\sec^2\theta\csc\theta$ (B) $6\sin\theta\sec\theta$ (C) $3\sec\theta\csc\theta$
(D) $6\sec\theta\csc^2\theta$ (E) none of these

42. (#17, AHSME, 1976) If θ is an acute angle and $\sin 2\theta = a$, then $\sin\theta + \cos\theta$ is

(A) $\sqrt{a+1}$ (B) $(\sqrt{2}-1)a+1$ (C) $\sqrt{a+1}-\sqrt{a^2-a}$
(D) $\sqrt{a+1}+\sqrt{a^2-a}$ (E) $\sqrt{a+1}+a^2-a$

43. (#17, AMC 12A, 2007) Suppose that $\sin a + \sin b = \sqrt{\frac{5}{3}}$ and $\cos a + \cos b = 1$. What is $\cos(a-b)$?

(A) $\sqrt{\frac{5}{3}}-1$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{2}{3}$ (E) 1

44. (#17, AHSME, 1973) If θ is an acute angle and $\sin\frac{\theta}{2} = \sqrt{\frac{x-1}{2x}}$, then $\tan\theta$ equals

(A) x (B) $\frac{1}{x}$ (C) $\frac{\sqrt{x-1}}{x+1}$ (D) $\frac{\sqrt{x^2-1}}{x}$ (E) $\sqrt{x^2-1}$

45. (#30, AHSME, 1975) Let $x = \cos 36^\circ - \cos 72^\circ$. Then x equals

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) $3-\sqrt{6}$ (D) $2\sqrt{3}-3$ (E) none of these

46. (#11, AHSME, 1983) Simplify $\sin(x-y)\cos y + \cos(x-y)\sin y$.

(A) 1 (B) $\sin x$ (C) $\cos x$ (D) $\sin x \cos 2y$ (E) $\cos x \cos 2y$

47. (#15, AHSME, 1978) If $\sin x + \cos x = \frac{1}{5}$ and $0 \le x < \pi$, then $\tan x$ is

(A) $-\frac{4}{3}$ (B) $-\frac{3}{4}$ (C) $\frac{3}{4}$ (D) $\frac{4}{3}$
(E) not completely determined by the given information

48. (#22, AHSME, 1974) The minimum value of $\sin\frac{A}{2} - \sqrt{3}\cos\frac{A}{2}$ is attained when A is

(A) -180° (B) 60° (C) 120° (D) 0° (E) none of these

49. (#21, AMC 12A, 2004) If $\sum_{n=0}^{\infty} \cos^{2n}\theta = 5$, what is the value of $\cos 2\theta$?

(A) $\frac{1}{5}$ (B) $\frac{2}{5}$ (C) $\frac{\sqrt{5}}{5}$ (D) $\frac{3}{5}$ (E) $\frac{4}{5}$

50. (#16, AHSME, 1985) If $A = 20^\circ$ and $B = 25^\circ$, then the value of $(1+\tan A)(1+\tan B)$ is

(A) $\sqrt{3}$ (B) 2 (C) $1+\sqrt{2}$ (D) $2(\tan A + \tan B)$
(E) none of these

51. (#20, AHSME, 1979) If $a = \frac{1}{2}$ and $(a+1)(b+1) = 2$, then the radian measure of $\arctan a + \arctan b$ equals

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{5}$ (E) $\frac{\pi}{6}$

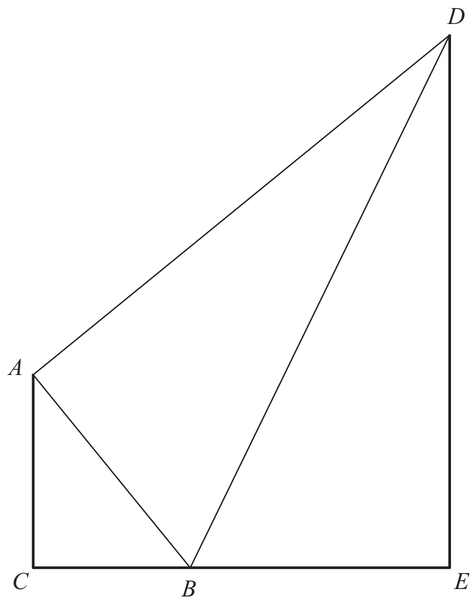
52. (#14, AHSME, 1989) $\cot 10 + \tan 5 =$

(A) $\csc 5$ (B) $\csc 10$ (C) $\sec 5$ (D) $\sec 10$ (E) $\sin 15$

53. (#21, AHSME, 1990) Consider a pyramid whose base $ABCD$ is square and whose vertex P is equidistant from A, B, C, and D. If $AB = 1$ and $\angle APB = 2\theta$, then the volume of the pyramid is

(A) $\frac{\sin\theta}{6}$ (B) $\frac{\cot\theta}{6}$ (C) $\frac{1}{6\sin\theta}$ (D) $\frac{1-\sin 2\theta}{6}$ (E) $\frac{\sqrt{\cos 2\theta}}{6\sin\theta}$

54. (#19, AHSME, 1991) Triangle ABC has a right angle at C, $AC = 3$ and $BC = 4$. Triangle ABD has a right angle at A and $AD = 12$. Points C and D are on opposite sides of AB. The line through D parallel to AC meets CB extended at E. If $\frac{DE}{DB} = \frac{m}{n}$, where m and n are relatively prime positive integers, then $m + n =$



(A) 25 (B) 128 (C) 153 (D) 243 (E) 256

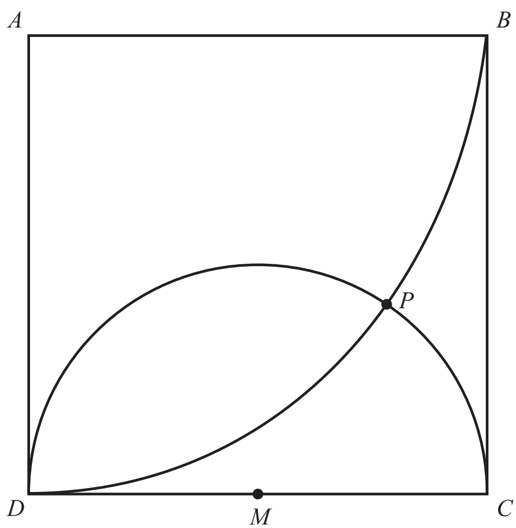
55. (#24, AMC 12A, 2008) Triangle ABC has $\angle C = 60^\circ$ and $BC = 4$. Point D is the midpoint of $\overline{BC}$. What is the largest possible value of $\tan(\angle BAD)$?

(A) $\frac{\sqrt{3}}{6}$ (B) $\frac{\sqrt{3}}{6}$ (C) $\frac{\sqrt{3}}{2\sqrt{2}}$ (D) $\frac{\sqrt{3}}{4\sqrt{2}-3}$ (E) 1

56. (#25, AMC 12A, 2009) The first two terms of a sequence are $a_1 = 1$ and $a_2 = \frac{1}{\sqrt{3}}$. For $n \geq 1$, $a_{n+2} = \frac{a_n + a_{n+1}}{1 - a_n a_{n+1}}$. What is $|a_{2009}|$?

(A) 0 (B) $2 - \sqrt{3}$ (C) $\frac{1}{\sqrt{3}}$ (D) 1 (E) $2 + \sqrt{3}$

57. (#17, AMC 12A, 2003) Square $ABCD$ has sides of length 4, and M is the midpoint of $\overline{CD}$. A circle with radius 2 and center M intersects a circle with radius 4 and center A at points P and D. What is the distance from P to $\overline{AD}$?



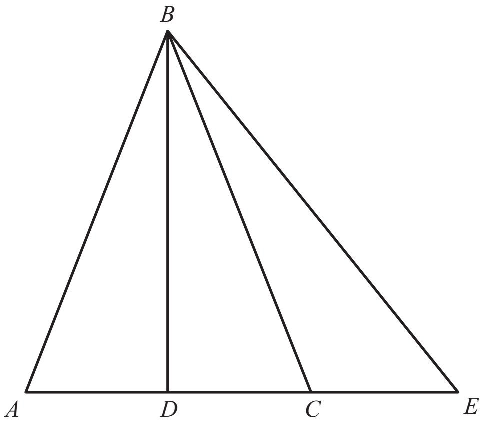
(A) 3 (B) $\frac{16}{5}$ (C) $\frac{13}{4}$ (D) $2\sqrt{3}$ (E) $\frac{7}{2}$

58. (#23, AHSME, 1984) $\frac{\sin(10^\circ)+\sin(20^\circ)}{\cos(10^\circ)+\cos(20^\circ)}$ equals

(A) $\tan(10^\circ) + \tan(20^\circ)$ (B) $\tan(30^\circ)$ (C) $\frac{1}{2}(\tan(10^\circ) + \tan(20^\circ))$
(D) $\tan(15^\circ)$ (E) $\frac{1}{4}\tan(60^\circ)$

59. (#24, AMC 12B, 2004) In ΔABC, $AB = BC$, and $\overline{BD}$ is an altitude. Point E is on the extension of $\overline{AC}$ such that $BE = 10$. The values of

$\tan \angle CBE$, $\tan \angle DBE$, and $\tan \angle ABE$ form a geometric progression, and the values of $\cot \angle DBE$, $\cot \angle CBE$, $\cot \angle DBC$ form an arithmetic progression. What is the area of ΔABC?



(A) 16 (B) $\frac{50}{3}$ (C) $10\sqrt{3}$ (D) $8\sqrt{5}$ (E) 18

60. (#24, AMC 12B, 2005) All three vertices of an equilateral triangle are on the parabola $y = x^2$, and one of its sides has a slope of 2. The x-coordinates of the three vertices have a sum of m/n, where m and n are relatively prime positive integers. What is the value of $m + n$?

(A) 14 (B) 15 (C) 16 (D) 17 (E) 18

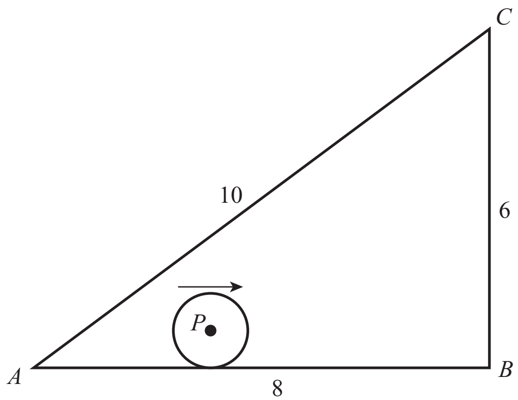
61. (#28, AHSME, 1989) Find the sum of the roots of $\tan^2 x - 9 \tan x + 1 = 0$ that are between $x = 0$ and $x = 2\pi$ radians.

(A) $\frac{\pi}{2}$ (B) π (C) $\frac{3\pi}{2}$ (D) 3π (E) 4π

62. (#29, AHSME, 1984) Find the largest value of $\frac{y}{x}$ for pairs of real numbers (x, y) which satisfy $(x - 3)^2 = (y - 3)^2 = 6$.

(A) $3 + 2\sqrt{2}$ (B) $2 + \sqrt{3}$ (C) $3\sqrt{3}$ (D) 6 (E) $6 + 2\sqrt{3}$

63. (#27, AHSME, 1993) The sides of ΔABC have lengths 6, 8, and 10. A circle with center P and radius 1 rolls around the inside of ΔABC, always remaining tangent to at least one side of the triangle. When P first returns to its original position, through what distance has P traveled?



(A) 10 (B) 12 (C) 14 (D) 15 (E) 17

64. (#22, AMC 12A, 2006) A circle of radius r is concentric with and outside a regular hexagon of side length 2. The probability that three entire sides of the hexagon are visible from a randomly chosen point on the circle is $\frac{1}{2}$. What is r?

(A) $2\sqrt{3}+2\sqrt{3}$ (B) $3\sqrt{3}+\sqrt{2}$ (C) $2\sqrt{6}+\sqrt{3}$ (D) $3\sqrt{2}+\sqrt{6}$
(E) $6\sqrt{2}-\sqrt{3}$

65. (#24, AMC 12B, 2006) Let S be the set of all points (x, y) in the coordinate plane such that $0 \le x \le \frac{\pi}{2}$ and $0 \le y \le \frac{\pi}{2}$. What is the area of the subset of S for which

$$\sin^2 x - \sin x \sin y + \sin^2 y \le \frac{3}{4}?$$

(A) $\frac{\pi^2}{9}$ (B) $\frac{\pi^2}{8}$ (C) $\frac{\pi^2}{6}$ (D) $\frac{3\pi^2}{16}$ (E) $\frac{2\pi^2}{9}$

66. (#24, AMC 12A, 2013) Three distinct segments are chosen at random among the segments whose endpoints are the vertices of a regular 12-gon. What is the probability that the lengths of these three segments are the three side lengths of a triangle with positive area?

(A) $\frac{553}{715}$ (B) $\frac{443}{572}$ (C) $\frac{111}{143}$ (D) $\frac{81}{104}$ (E) $\frac{223}{286}$

14
The Law of Cosines

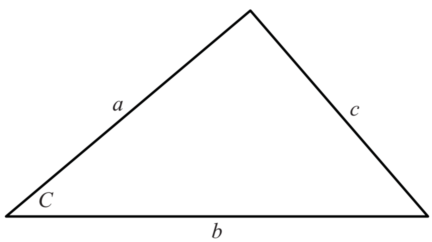
Common Core State Standards

8.G.6 Explain a proof of the Pythagorean Theorem and its converse.

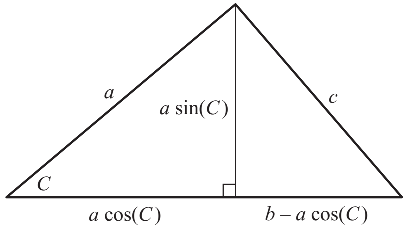
G-SRT.10 (+) Prove the Laws of Sines and Cosines and use them to solve problems.

G-SRT.11 (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Consider a triangle with sides a, b, and c that contains no right angle. Call the angle opposite side c, C.



We can introduce right triangles by drawing in an altitude:



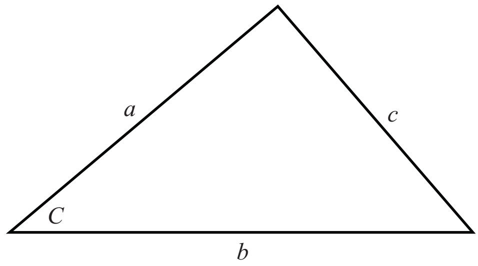
The left triangle has "radius" a and so the height and overness in this triangle are $a \sin C$ and $a \cos C$. The right-hand triangle is a right triangle with legs $a \sin C$ and $b - c \cos C$.

Apply the Pythagorean Theorem to the right-hand triangle:

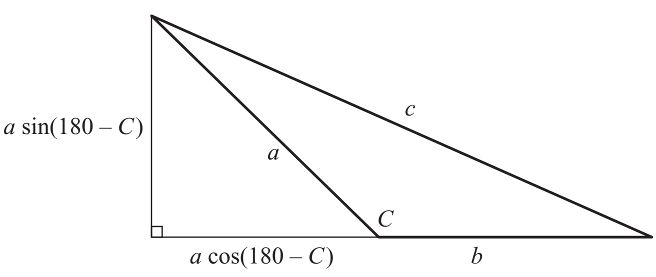
$$\begin{aligned}(a \sin C)^2 + (b - a \cos C)^2 &= c^2,\\ a^2 \sin^2 C + b^2 - 2ab \cos C + a^2 \cos^2 C &= c^2,\\ a^2(\sin^2 C + \cos^2 C) + b^2 - 2ab \cos C &= c^2,\\ a^2 + b^2 - 2ab \cos C &= c^2.\end{aligned}$$

This result is called the *law of cosines*.

Law of Cosines. *For any triangle with sides and angles labeled as shown we have* $c^2 = a^2 + b^2 - 2ab \cos C$.



Comment. The Law of Cosines holds even if angle C is obtuse. Can you see how the following diagram establishes this?



The law of cosines gives a sense of the degree to which the Pythagorean Theorem fails to hold. We have $c^2 = a^2 + b^2$ but with an "error term" of $-2ab \cos C$ introduced.

Notice that if $C = 90°$, then $\cos C = 0$ giving

$$c^2 = a^2 + b^2 - 2ab \cdot 0 = a^2 + b^2$$

and this is exactly the statement of the Pythagorean Theorem (as it should be). And conversely, if a triangle has side lengths that satisfy the famous relation $a^2 + b^2 = c^2$, it must be a right triangle.

Example. Find the measure of the largest angle in a triangle with sides of lengths 5, 10, and 11.

Answer. The largest angle lies opposite the side of the longest length. Call this angle x.

By the law of cosines we have $11^2 = 5^2 + 10^2 - 100\cos x$ giving $x = \cos^{-1}\left(\frac{4}{100}\right) \approx 87.7°$.

Comment. An angle C in a triangle is acute precisely when $\cos > 0$ and obtuse when $\cos C < 0$. By the law of cosines this means, in a triangle with sides a, b, and c, the angle between sides of lengths a and b is

acute precisely when $c^2 < a^2 + b^2$,
obtuse precisely when $c^2 > a^2 + b^2$.

As the largest angle in a triangle is opposite its largest side of the triangle, we can apply this test once and determine whether or not this largest angle is acute. If so, then all three angles in the triangle are acute and we have an acute triangle. If not, the triangle is an obtuse triangle.

MAA PROBLEMS

Featured Problem

(#14, AMC 12B, 2011)

A segment though the focus F of a parabola with vertex V is perpendicular to $\overline{FV}$ and intersects the parabola in points A and B. What is $\cos(\angle AVB)$?

(A) $-\frac{3\sqrt{5}}{7}$ (B) $-\frac{2\sqrt{5}}{5}$ (C) $-\frac{4}{5}$ (D) $-\frac{3}{5}$ (E) $-\frac{1}{2}$

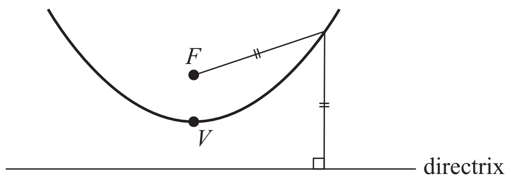
A Personal account of solving this problem

Curriculum Inspirations Strategies (www.maa.org/ci):

Strategy 4: **Draw a Picture**

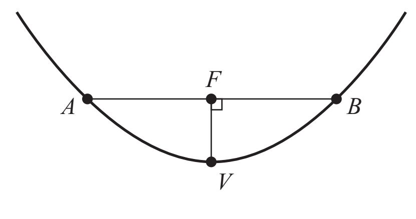
This question gives a lot to digest! I am going to have to draw a picture and attempt to recall all I know about parabolas, their vertices, and their foci.

A basic picture:

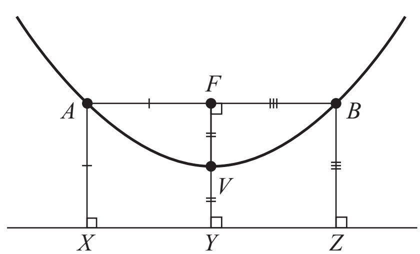


My diagram isn't accurate, but I remember that a parabola is defined as the set of all points whose perpendicular distance to the directrix (a line) equals its distance to the focus (a point). Points on parabolas provide equal distances. (I didn't draw that well here.)

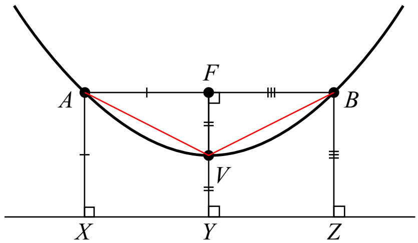
For the question we have a segment though F that is perpendicular to the line segment $\overline{FV}$. I can draw that.



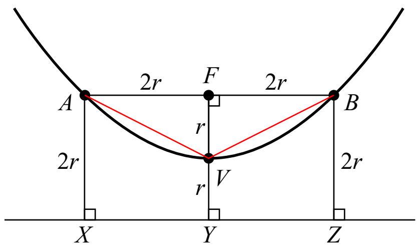
It seems irresistible to draw perpendicular lines to the directrix and note equal distances.



Hmm. $FBZY$ is a rectangle. So is $AFYX$.
Okay . . . What are we meant to be doing?
What is $\cos(\angle AVB)$?
Let's draw in that angle.



This is confusing. Actually, it is all the little markings of congruent segments that are visually messy. Since we have rectangles, and opposite sides of rectangles are congruent, let's give length FV a name, say r, and write in all the lengths in terms of r.



(Ooh! Our rectangles are squares!)

We want $\cos(\angle AVB)$. Well. . . $\angle AVB$ is part of an isosceles triangle with one side of length $4r$ and two legs of length $\sqrt{r^2 + (2r)^2} = \sqrt{5}r$. Law of cosines?

$$(4r)^2 = (\sqrt{5}r)^2 + (\sqrt{5}r)^2 - 2(\sqrt{5}r)(\sqrt{5}r)\cos(\angle AVB).$$

This gives

$$\cos(\angle AVB) = \frac{5r^2 + 5r^2 - 16r^2}{10r^2} = -\frac{3}{5}.$$

Whoa! We have it. (And a negative answer makes sense since it looks like $\angle AVB$ could be larger than 90° in measure.)

The answer is option (D).

Additional Problems

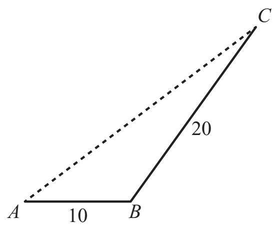
67. (#22, AHSME, 1988) For how many integers x does a triangle with side lengths 10, 24 and x have all its angles acute?

(A) 4 (B) 5 (C) 6 (D) 7 (E) more than 7

68. (#16, AMC 12A, 2012) Circle C_1 has its center O lying on circle C_2. The two circles meet at X and Y. Point Z in the exterior of C_1 lies on circle C_2 and $XZ = 13$, $OZ = 11$, and $YZ = 7$. What is the radius of circle C_1?

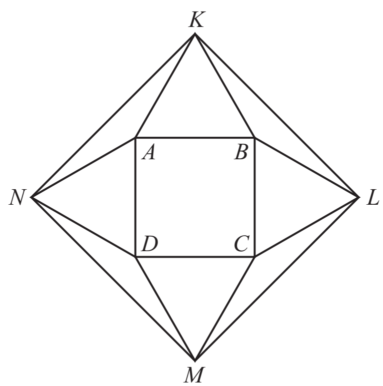
(A) 5 (B) $\sqrt{26}$ (C) $3\sqrt{3}$ (D) $2\sqrt{7}$ (E) $\sqrt{30}$

69. (#13, AMC 12A, 2009) A ship sails 10 miles in a straight line from A to B, turns through an angle between 45° and 60°, and then sails another 20 miles to C. Let AC be measured in miles. Which of the following intervals contains AC^2?



(A) [400, 500] (B) [500, 600] (C) [600, 700] (D) [700, 800]
(E) [800, 900]

70. (#14, AMC 12A, 2003) Points K, L, M, and N lie in the plane of the square $ABCD$ so that AKB, BLC, CMD, and DNA are equilateral triangles. If $ABCD$ has an area of 16, find the area of $KLMN$.



(A) 32 (B) $16 + 16\sqrt{3}$ (C) 48 (D) $32 + 16\sqrt{3}$ (E) 64

71. (#21, AMC 12B, 2003) An object moves 8 cm in a straight line from A to B, turns at an angle α, measured in radians and chosen at random from the interval $(0, \pi)$, and moves 5 cm in a straight line to C. What is the probability that $AC < 7$?

(A) $\frac{1}{6}$ (B) $\frac{1}{5}$ (C) $\frac{1}{4}$ (D) $\frac{1}{3}$ (E) $\frac{1}{2}$

72. (#15, AHSME, 1952) The sides of a triangle are in the ratio 6 : 8 : 9. Then

(A) the triangle is obtuse
(B) the angles are in the ratio 6 : 8 : 9
(C) the triangle is acute
(D) the angle opposite the largest side is double the angle opposite the smallest side
(E) none of these

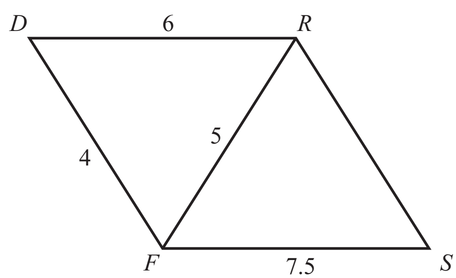
73. (#21, AHSME, 1981) In a triangle with sides of lengths a, b and c,

$$(a + b + c)(a - b - c) = 3ab.$$

The measure of the angle opposite the side of length c is

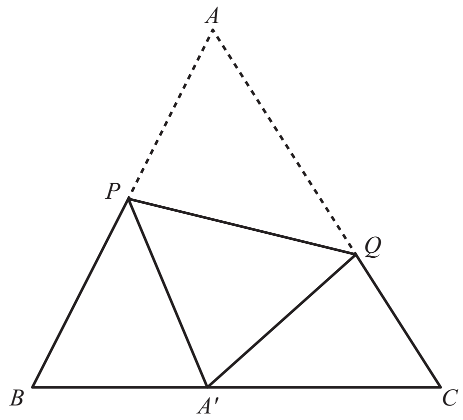
(A) 15° (B) 30° (C) 45° (D) 60° (E) 150°

74. (#29, AHSME, 1964) In this figure $\angle RFS = \angle FDR$, $\overline{FD} = 4$ inches, $\overline{DR} = 6$ inches, $\overline{FR} = 5$ inches, $\overline{FS} = 7\frac{1}{2}$ inches. The length of $\overline{RS}$ in inches is



(A) undetermined (B) 4 (C) $5\frac{1}{5}$ (D) 6 (E) $6\frac{1}{4}$

75. (#29, AHSME, 1991) Equilateral triangle ABC has been creased and folded so that vertex A now rests at A' on $\overline{BC}$ as shown. If $BA' = 1$ and $A'C = 2$ then the length of crease $\overline{PQ}$ is



(A) $\frac{8}{5}$ (B) $\frac{7}{20}\sqrt{21}$ (C) $\frac{1+\sqrt{5}}{2}$ (D) $\frac{13}{8}$ (E) $\sqrt{3}$

76. (#25, AHSME, 1972) Inscribed in a circle is a quadrilateral having sides of lengths 25, 39, 52, and 60 taken consecutively. The diameter of this circle has length

(A) 62 (B) 63 (C) 65 (D) 66 (E) 69

77. (#24, AMC 12B, 2013) Let ABC be a triangle where M is the midpoint of $\overline{AC}$, and $\overline{CN}$ is the angle bisector of $\angle ACB$ with N on $\overline{AB}$. Let X be the intersection of the median $\overline{BM}$ and the bisector $\overline{CN}$. In addition $\triangle BXN$ is equilateral and $AC = 2$. What is BN^2?

(A) $\frac{10-6\sqrt{2}}{7}$ (B) $\frac{2}{9}$ (C) $\frac{5\sqrt{2}-3\sqrt{3}}{8}$ (D) $\frac{\sqrt{2}}{6}$ (E) $\frac{3\sqrt{3}-4}{5}$

78. (#23, AMC 12B, 2002) In $\triangle ABC$, we have $AB = 1$ and $AC = 2$. Side $\overline{BC}$ and the median from A to $\overline{BC}$ have the same length. What is BC?

(A) $\frac{1+\sqrt{2}}{2}$ (B) $\frac{1+\sqrt{3}}{2}$ (C) $\sqrt{2}$ (D) $\frac{3}{2}$ (E) $\sqrt{3}$

79. (#19, AHSME, 1983) Point D is on side CB of triangle ABC.
If $\angle CAD = \angle DAB = 60°$, $AC = 3$ and $AB = 6$, then the length of AD is

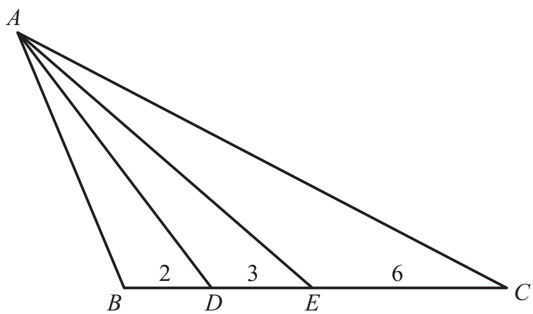
(A) (B) 2.5 (C) 3 (D) 3.5 (E) 4

80. (#30, AHSME, 1996) A hexagon inscribed in a circle has three consecutive sides each of length 3 and three consecutive sides each of length 5.

The chord of the circle that divides the hexagon into two trapezoids, one with three sides each of length 3 and the other with three sides each of length 5, has length equal to $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

(A) 309 (B) 349 (C) 369 (D) 389 (E) 409

81. (#25, AHSME, 1981) In triangle ABC in the adjoining figure, AD and AE trisect $\angle BAC$. The lengths of BD, DE and EC are 2, 3 and 6, respectively. The length of the shortest side of $\triangle ABC$ is



(A) $2\sqrt{10}$ (B) 11 (C) $6\sqrt{6}$ (D) 6
(E) not uniquely determined by the given information

82. (#12, AMC 12A, 2013) The angles in a particular triangle are in arithmetic progression, and the side lengths are 4, 5, and x. The sum of the possible values of x equals $a + \sqrt{b} + \sqrt{c}$, where a, b, and c are positive integers. What is $a + b + c$?

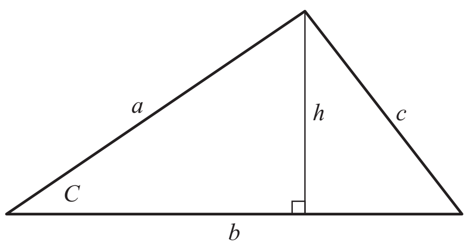
(A) 36 (B) 38 (C) 40 (D) 42 (E) 44

15
The Area of a Triangle

Common Core State Standards

G-SRT.9 (+) Derive the formula $A = 1/2\ ab\ \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

The area of a triangle is one-half base times height.



Thus in this picture

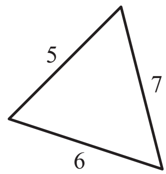
$$\text{Area} = \frac{1}{2} \cdot bh.$$

But $h = a \sin C$.

Area of a Triangle. *The area of the triangle is* $\frac{1}{2}ab \sin C$ *where* C *is the angle between two sides of lengths* a *and* b.

Comment. This formula also valid if C is an obtuse angle. Can you see why? (Use $\sin(180^\circ - C) = \sin C$.)

Example. Find, to one decimal place, the area of the following triangle:



Answer. Let x be the angle between sides of lengths 5 and 6. Then, by the law of cosines

$$x = \cos^{-1}\left(\frac{5^2 + 6^2 - 7^2}{2 \cdot 5 \cdot 6}\right) \approx 78.5^\circ.$$

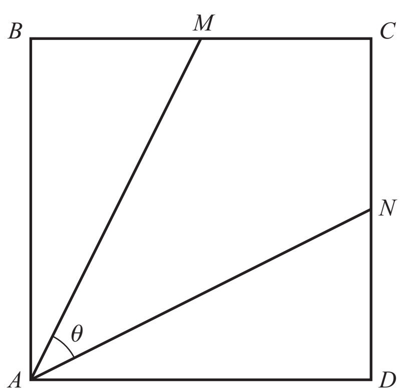
Thus

$$\text{area} = \tfrac{1}{2} \cdot 5 \cdot 6 \cdot \sin(78.5) \approx 14.7.$$

Question. We rounded our value of x to one decimal place and then used that value to compute the area, allegedly, to one decimal place. Did the initial rounding affect our final answer adversely?

MAA PROBLEMS

83. (#14, AHSME, 1987) $ABCD$ is a square and M and N are the midpoints of $\overline{BC}$ and $\overline{CD}$ respectively. Then $\sin\theta =$



(A) $\frac{\sqrt{5}}{5}$ (B) $\frac{3}{5}$ (C) $\frac{\sqrt{10}}{5}$ (D) $\frac{4}{5}$ (E) none of these

84. (#10, AMC 12A, 2012) A triangle has area 30, one side of length 10, and the median to that side of length 9. Let θ be the acute angle formed by that side and the median. What is $\sin\theta$?

(A) $\frac{3}{10}$ (B) $\frac{1}{3}$ (C) $\frac{9}{20}$ (D) $\frac{2}{3}$ (E) $\frac{9}{10}$

85. (#26, AHSME, 1977) Let a, b, c, and d be the lengths of sides $\overline{MN}$, $\overline{NP}$, $\overline{PQ}$ and $\overline{QM}$, respectively, of quadrilateral *MNPQ*. If A is the area of *MNPQ*, then

(A) $A = \left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$ if and only if *MNPQ* is convex.

(B) $A = \left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$ if and only if *MNPQ* is a rectangle.

(C) $A \leq \left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$ if and only if *MNPQ* is a rectangle.

(D) $A \leq \left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$ if and only if *MNPQ* is a parallelogram.

(E) $A \geq \left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$ if and only if *MNPQ* is a parallelogram.

86. (#32, AHSME, 1961) A regular polygon of n sides is inscribed in a circle of radius R. The area of the polygon is $3R^2$. Then n equals

(A) 8 (B) 10 (C) 12 (D) 15 (E) 18

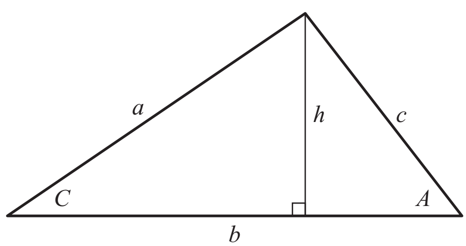
16
The Law of Sines

Common Core State Standards

G-SRT.10 (+) Prove the Laws of Sines and Cosines and use them to solve problems.

G-SRT.11 (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Return to the following diagram of a triangle with an altitude drawn in:



Label angles C and A as shown. (The third angle will be called B.)

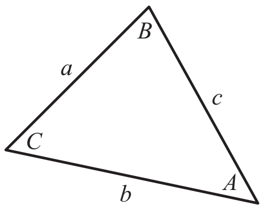
> Looking at the left right triangle we have $h = a \sin C$.
>
> Looking at the right right triangle have $h = c \sin A$.
>
> We must have $a \sin C = c \sin A$ yielding $\frac{a}{\sin A} = \frac{c}{\sin C}$.

If, instead, we focused on the altitude that cuts angle A we'd obtain $\frac{b}{\sin B} = \frac{c}{\sin C}$.

Stringing these equalities together yields the *law of sines*:

Law of Sines. *In a triangle with sides* a, b, and c *and angles* A, B, and C *as shown we have*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

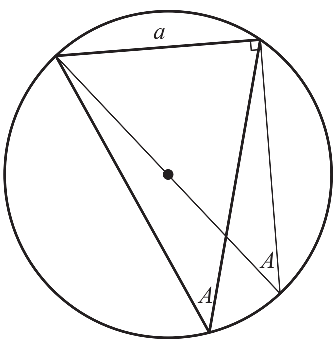


Comment. Our derivation of the law of sines relied on the diagram of an acute triangle. Can you see that the law of sines is valid for obtuse triangles too? (If the angle C is obtuse in our opening diagram, then $h = a\sin(180^\circ - C) = -a\sin(-C) = a\sin C$.)

A Little Known Fact

The number given by the common value $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ has a geometric interpretation.

One learns in a geometry course that any triangle can be circumscribed by a circle.



One also learns that all inscribed angles from the same chord have the same measure.

Circumscribe our given triangle in a circle and slide the vertex at angle A along the circumference of the circle until one side of the triangle passes through the center of the circle. The measure A of the angle does not change.

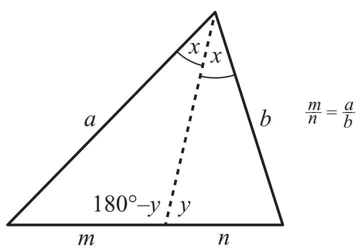
This produces a right triangle containing angle A, and we see that $\sin A$ equals a divided by the diameter of the circle:

$$\sin A = \frac{a}{\text{diameter}}.$$

Thus:

> The common value of the ratio $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ equals the diameter of the circle that circumscribes the triangle.

Comment. One can use the law of sines to prove the "angle bisector theorem" in geometry: an angle bisector in a triangle divides the side it meets into two sections of lengths in proportion to the ratio of the remaining two sides of the triangle.



MAA PROBLEMS

Featured Problem

#23, AHSME, 1982

The lengths of the sides of a triangle are consecutive integers, and the largest angle is twice the smallest angle. The cosine of the smallest angle is

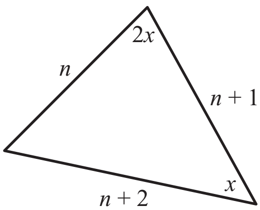
(A) $\frac{3}{4}$ (B) $\frac{7}{10}$ (C) $\frac{2}{3}$ (D) $\frac{9}{14}$ (E) none of these

A Personal account of solving this problem

Curriculum Inspirations Strategies (www.maa.org/ci):

Strategy 7: **Perseverance is Key**

I can't tell if this question is straightforward or if it is going to be tricky. Here's a picture of the triangle we're working with:



Here n is a positive integer and the smallest angle x is opposite the smallest side and the largest angle $2x$ is opposite the largest side.

We've been told about two angles in the triangles and their opposite sides, so I can't help to wonder what the law of sines might say here:

$$\frac{n}{\sin x} = \frac{n+2}{\sin(2x)}.$$

So

$$\frac{\sin(2x)}{\sin x} = \frac{2\sin x\cos x}{\sin x} = \frac{n+2}{n}.$$

We get

$$\cos x = \frac{n+2}{2n}.$$

It looks like (D) fits this bill with $n = 7$. Is that the answer?

I am a little bit nervous to leap to this conclusion as fractions can be expressed in many forms. For example, I see that $n = 4$ gives $\cos x = \frac{6}{8} = \frac{3}{4}$, which is option (A).

Since we want $\cos x$ I can't help but wonder what too the law of cosines has to say:

$$n^2 = (n+1)^2 + (n+2)^2 - 2(n+1)(n+2)\cos x.$$

Putting in our expression for $\cos x$:

$$n^2 = (n+1)^2 + (n+2)^2 - \frac{(n+1)(n+2)^2}{n}.$$

Let's just grind our way through some algebra and see what value of n we get.

$$n^2 = n^2 + 2n + 1 + n^2 + 4n + 4 - \frac{n^3 + 5n^2 + 8n + 4}{n},$$
$$n^2 + 5n + 8 + \frac{4}{n} = n^2 + 6n + 5,$$
$$3 + \frac{4}{n} = n,$$
$$n^2 = 3n + 4.$$

This gives $n = 4$ or $n = -1$. Only $n = 4$ is relevant for the question and the answer is (A) after all!

Additional Problems

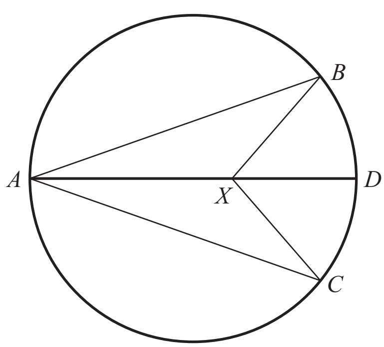
87. (#18, AHSME, 1995) Two rays with common endpoint O form a 30° angle. Point A lies on one ray, point B on the other ray, and $AB = 1$. The maximum possible length of OB is

(A) 1 (B) $\frac{1+\sqrt{3}}{\sqrt{2}}$ (C) $\sqrt{3}$ (D) 2 (E) $\frac{4}{\sqrt{3}}$

88. (#14, AHSME, 1972) A triangle has angles of 30° and 45°. If the side opposite the 45° angle has length 8, then the side opposite the 30° angle has length

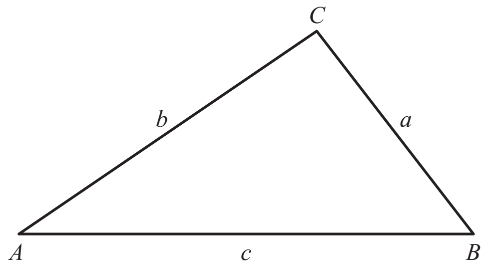
(A) 4 (B) $4\sqrt{2}$ (C) $4\sqrt{3}$ (D) $4\sqrt{6}$ (E) 6

89. (#23, AHSME, 1993) Points A, B, C and D are on a circle of diameter 1, and X is on diameter $\overline{AD}$. If $BX = CX$ and $3\angle BAC = \angle BXC = 36^\circ$, then $AX =$



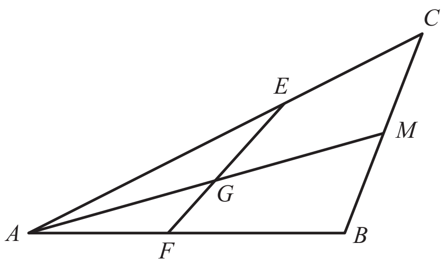
(A) $\cos 6^\circ \cos 12^\circ \sec 18^\circ$ (B) $\cos 6^\circ \sin 12^\circ \csc 18^\circ$
(C) $\cos 6^\circ \sin 12^\circ \sec 18^\circ$ (D) $\sin 6^\circ \sin 12^\circ \csc 18^\circ$
(E) $\sin 6^\circ \sin 12^\circ \sec 18^\circ$

90. (#28, AHSME, 1985) In ΔABC, we have $\angle C = 3\angle A$, $a = 27$ and $c = 48$. What is b?



(A) 33 (B) 35 (C) 37 (D) 39 (E) not uniquely determined

91. (#28, AHSME, 1975) In the triangle ABC shown, M is the midpoint of side BC, $AB = 12$ and $AC = 16$. Points E and F are taken on AC and AB respectively, and lines EF and AM intersect at G. If $AE = 2AF$ then $\frac{EG}{GF}$ equals



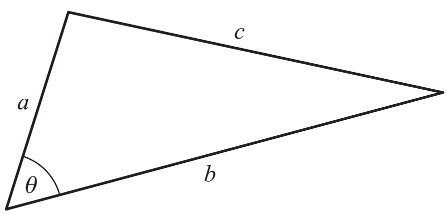
(A) $\frac{3}{2}$ (B) $\frac{4}{3}$ (C) $\frac{5}{4}$ (D) $\frac{6}{5}$
(E) not enough information to solve the problem.

92. (#28, AHSME, 1998) In triangle ABC, angle C is a right angle and $CB > CA$. Point D is located on $\overline{BC}$ so that angle CAD is twice angle DAB. If $AC/AD = 2/3$, then $CD/BD = m/n$, where m and n are relatively prime positive integers. Find $m + n$.
(A) 10 (B) 14 (C) 18 (D) 22 (E) 26

17
Heron's Formula for the Area of a Triangle

> **Common Core State Standards**
>
> **G-SRT.9** (+) Derive the formula A = 1/2 ab sin(C) for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
>
> **G-SRT.10** (+) Prove the Laws of Sines and Cosines and use them to solve problems.

Consider a triangle with side lengths a, b, and c. Let θ be the angle between the sides of lengths $\sin a$ and $\cos b$.



The area of the triangle is $\frac{1}{2}ab\sin\theta$.

The law of cosines states that $a^2 + b^2 - 2ab\cos\theta = c^2$.

If we solve for $\sin\theta$ in the first equation and for $\cos\theta$ in the second and substitute each into the equation:

$$\cos^2\theta + \sin^2\theta = 1$$

an expression that relates the area of the triangle to its three side-lengths must result.

If you are gung-ho for the algebra, rearranging this expression (eventually) yields:

$$\text{area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{a+b+c}{2}$ is the *semi-perimeter* of the triangle.

This is known as *Heron's Formula*.

Comment. See the essay at www.jamestanton.com/wp-content/uploads/2012/03/Cool-Math-Essay_April-2014_On-Cyclic-Quadrilaterals.pdf for a generalization of Heron's formula to cyclic quadrilaterals (Brahmagupta's formula) and then to quadrilaterals in general (Bretschneider's formula). This material will be of tremendous help to answering question 100 at the end of this section.

MAA PROBLEMS

Featured Problem

(#39, AHSME, 1965)

A foreman noticed an inspector checking a $3''$-hole with a $2''$-plug and a $1''$-plug and suggested that two more gauges be inserted to be sure that the fit was snug. If the new gauges are alike, then the diameter, d, of each, to the nearest hundredth of an inch, is

(A) 0.87 (B) 0.86 (C) 0.83 (D) 0.75 (E) 0.71

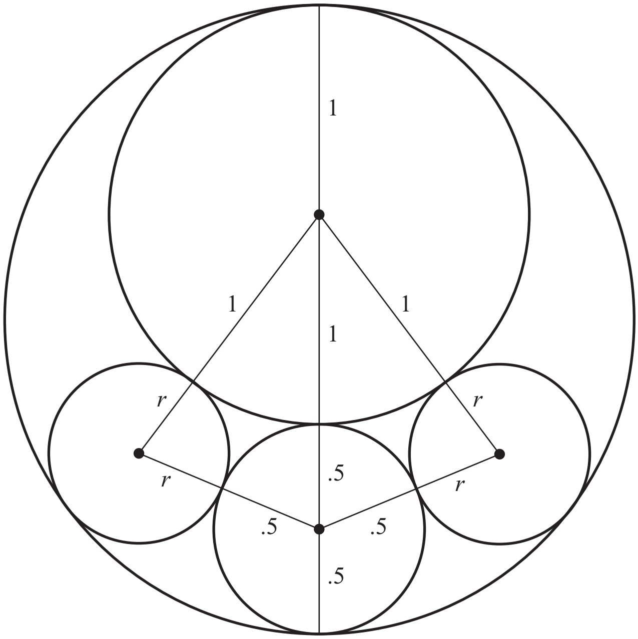
A Personal account of solving this problem

Curriculum Inspirations Strategies (www.maa.org/ci):

Strategy 2: **Do Something!**

I need to draw a picture of the situation before I can assess my reaction to this question!

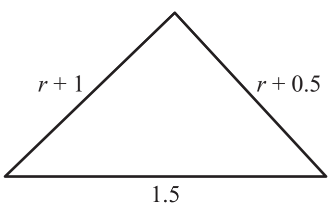
We have something like



It's a complicated picture and I am now feeling nervous about the question!

I've marked in all the radii I can, including the unknown radius r of each of the two congruent additional circles. If I can figure out r I can double it and get its diameter.

I see in the picture two triangles each with sides 1.5, $r + 5$, and $r + 1$.



I don't know what to do with this.

Heron's formula?

The triangle has semi-perimeter

$$\frac{r + 1 + r + \frac{1}{2} + \frac{3}{2}}{2} = r + \frac{3}{2}.$$

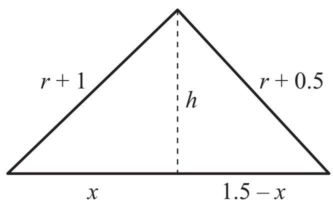
By Heron's formula, the area of the triangle is:

$$\begin{aligned}\text{area} &= \sqrt{\left(r+\frac{3}{2}\right)\left(r+\frac{3}{2}-r-1\right)\left(r+\frac{3}{2}-r-\frac{1}{2}\right)\left(r+\frac{3}{2}-\frac{3}{2}\right)} \\ &= \sqrt{\left(r+\frac{3}{2}\right)\cdot\frac{1}{2}\cdot 1\cdot r} \\ &= \sqrt{\frac{r^2}{2}+\frac{3}{4}}.\end{aligned}$$

I don't think this is doing anything for me. Hmm.

I suppose I could try to work out the area of this triangle a second way, and then compare results. I might get an equation that must be true for r.

Let h be the height of the triangle. Then area $= \frac{1}{2}\cdot\frac{3}{2}\cdot h = \frac{3}{4}h$. Can I work out h?



We have

$$x^2+h^2=(r+1)^2$$

and

$$\left(\frac{3}{2}-x\right)^2+h^2=\left(r+\frac{1}{2}\right)^2.$$

Let's expand and manipulate the second equation and use the first one along the way:

$$\begin{aligned}\frac{9}{4}-3x+x^2+h^2 &= \left(r+\frac{1}{2}\right)^2, \\ \frac{9}{4}-3x+(r+1)^2 &= \left(r+\frac{1}{2}\right)^2, \\ \frac{9}{4}-3x+2r+1 &= r+\frac{1}{4}, \\ r &= 3x-3.\end{aligned}$$

Hang on! I want h.

$$\begin{aligned} h^2 &= (r+1)^2 - x^2 \\ &= (r+1)^2 - \left(\frac{r}{3}+1\right)^2 \\ &= \frac{8}{9}r^2 + \frac{4}{3}r. \end{aligned}$$

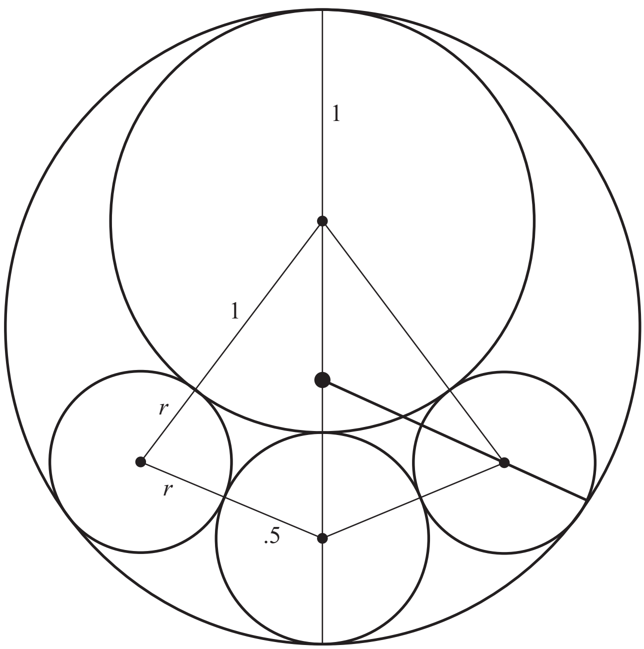
So the area of the triangle is

$$\text{area} = \frac{3}{4}h = \frac{3}{4}\sqrt{\frac{8r^2}{9}+\frac{4}{3}r} = \frac{1}{4}\sqrt{8r^2+12r} = \frac{1}{2}\sqrt{2r^2+3r},$$

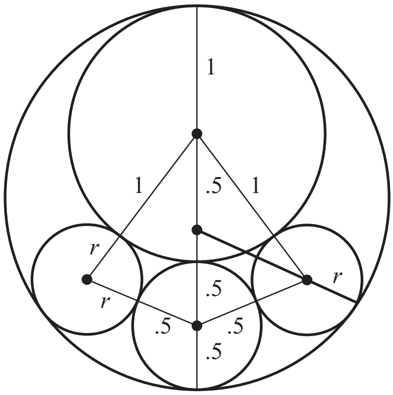
which is the same formula as before and I have just gone in a circle. Bother!

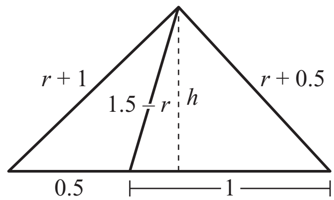
Hmm. What else is there about this question?

I haven't used the center of the big circle in any way. The large circle has radius 1.5.

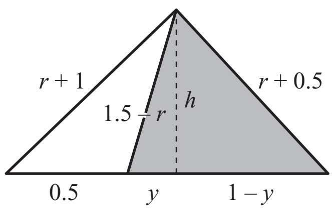


Let me draw a radius for the large circle that cuts through the apex of my triangle. That radius has length 1.5. Oh! I can mark in a great number of lengths:





Actually, maybe I can find h by doing the same work I did before but for this part of the diagram:



We have

$$y^2 + h^2 = \left(\frac{3}{2} - r\right)^2$$

and

$$\begin{aligned}(1-y)^2 + h^2 &= \left(r + \frac{1}{2}\right)^2,\\ 1 - 2y + \left(\frac{3}{2} - r\right)^2 &= \left(r + \frac{1}{2}\right)^2,\\ 1 - 2y + \frac{9}{4} - 3r &= r + \frac{1}{4},\\ y &= \frac{3}{2} - 2r,\end{aligned}$$

so

$$h^2 = \left(\frac{3}{2} - r\right)^2 - y^2 = \left(\frac{3}{2} - r\right)^2 - \left(\frac{3}{2} - 2r\right)^2 = 3r - 3r^2$$

giving

$$A = \frac{3}{4}h = \frac{3}{4}\sqrt{3r - 3r^2}.$$

This is a different formula. Let's now equate the two area formulas and solve for r:

$$\begin{aligned}
\frac{3}{4}\sqrt{3r - 3r^2} &= \sqrt{\frac{r^2}{2} + \frac{3r}{4}}, \\
\frac{9}{16}(3r - r^2) &= \frac{r^2}{2} + \frac{3r}{4}, \\
9(3r - 3r^2) &= 8r^2 + 12r, \\
27r - 27r^2 &= 8r^2 + 12r, \\
35r^2 &= 15r, \\
r &= \frac{3}{7}.
\end{aligned}$$

So the diameter is $2 \times \frac{3}{7} = \frac{6}{7} \approx 0.86$. The answer is (B).

Whoa!

Additional Problems

93. (#18, AMC 12A, 2012) Triangle ABC has $AB = 27$, $AC = 26$, and $BC = 25$. Let I denote the intersection of the internal angle bisectors of ΔABC. What is BI?

(A) 15 (B) $5 + \sqrt{26} + 3\sqrt{3}$ (C) $3\sqrt{26}$ (D) $\frac{2}{3}\sqrt{546}$ (E) $9\sqrt{3}$

94. (#50, AHSME, 1953) One of the sides of a triangle is divided into segments of 6 and 8 units by the point of tangency of the inscribed circle. If the radius of the circle is 4, then the length of the shortest side of the triangle is:

(A) 12 units (B) 13 units (C) 14 units (D) 15 units (E) 16 units

95. (#20, AMC 12B, 2012) A trapezoid has side lengths 3, 5, 7, and 11. The sum of all the possible areas of the trapezoid can be written in the form of $r_1\sqrt{n_1} + r_2\sqrt{n_2} + r_3$, where r_1, r_2, and r_3 are rational numbers and

n_1 and n_2 are positive integers not divisible by the square of a prime. What is the greatest integer less than or equal to

$$r_1 + r_2 + r_3 + n_1 + n_2?$$

(A) 57 (B) 59 (C) 61 (D) 63 (E) 65

96. (#19, AMC 12, 2000) In triangle ABC, $AB = 13$, $BC = 14$, and $AC = 15$. Let D denote the midpoint of $\overline{BC}$ and let E denote the intersection of $\overline{BC}$ with the bisector of angle BAC. Which of the following is closest to the area of the triangle ADE?

(A) 2 (B) 2.5 (C) 3 (D) 3.5 (E) 4

97. (#23, AMC 12A, 2002) In triangle ABC, side $\overline{AC}$ and the perpendicular bisector of $\overline{BC}$ meet in point D and $\overline{BD}$ bisects $\angle ABC$. If $AD = 9$ and $DC = 7$, what is the area of triangle ABD?

(A) 14 (B) 21 (C) 28 (D) $14\sqrt{5}$ (E) $28\sqrt{5}$

98. (#20, AMC 12B, 2011) Triangle ABC has $AB = 13$, $BC = 14$, and $AC = 15$. The points D, E, and F are the midpoints of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ respectively. Let $X \neq E$ be the intersection of the circumcircles of ΔBDE and ΔCEF. What is $XA + XB + XC$?

(A) 24 (B) $14\sqrt{3}$ (C) $\frac{195}{8}$ (D) $\frac{129\sqrt{7}}{14}$ (E) $\frac{69\sqrt{2}}{4}$

99. (#18, AMC 12B, 2008) A pyramid has a square base $ABCD$ and vertex E. The area of square $ABCD$ is 196, and the areas of ΔABE and ΔCDE are 105 and 91, respectively. What is the volume of the pyramid?

(A) 392 (B) $196\sqrt{6}$ (C) $392\sqrt{2}$ (D) $392\sqrt{3}$ (E) 784

100. (#24, AMC 12A, 2011) Consider all quadrilaterals $ABCD$ such that $AB = 14$, $BC = 9$, $CD = 7$, and $DA = 12$. What is the radius of the largest possible circle that fits inside or on the boundary of such a quadrilateral?

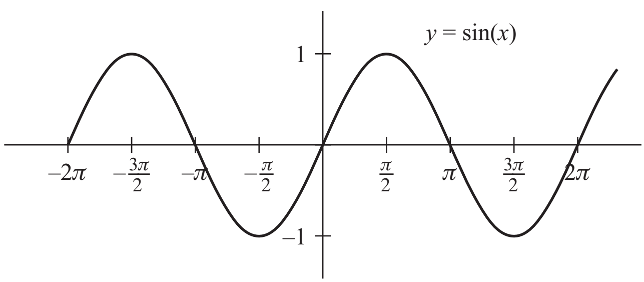
(A) $\sqrt{15}$ (B) $\sqrt{21}$ (C) $2\sqrt{6}$ (D) 5 (E) $2\sqrt{7}$

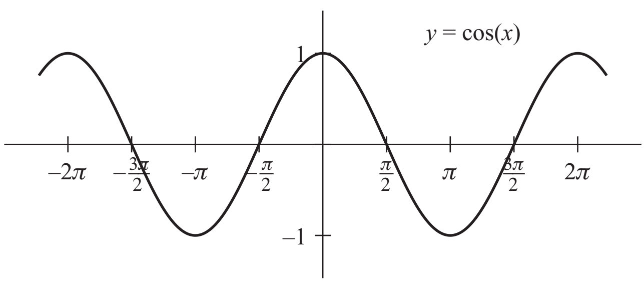
18
Fitting Trigonometric Functions to Periodic Data

Common Core State Standards

F-TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

Recall that the sine and cosine curves each represent a linear displacement of a point moving about a circle. They each provide a wave-like curve that cycles every 360 degrees, or every 2π radians if one is thinking in radians.



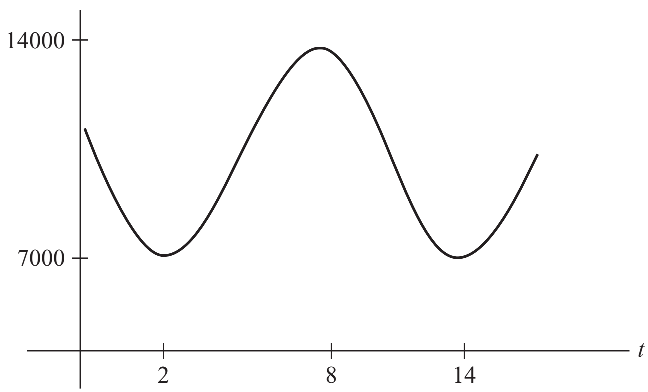


Many phenomena in nature follow cyclical patterns and it often is desirous to model data from them via trigonometric functions. Here's a ludicrously contrived example:

Example. The population of wombats in Adelaide, Australia, seems to rise and fall in a cyclic fashion throughout the year. At its minimum in February, the peak of summer, there are approximately 7000 wombats. By the time of August, the middle of the green winter months, the population usually doubles. In another six months' time it falls back to 7000.

Use a trigonometric function to model the wombat population.

Answer. Let $P(t)$ be the count of wombats at time t measured in months with $t = 1$ corresponding to January. We have $P(2) = 7000$, $P(8) = 14000$, $P(14) = 7000$, $p(20) = 14000$ and so forth. This suggests a trigonometric function of the type



This looks like a cosine curve with "new zero" at $t = 8$, but its period is 12 months rather than 2π. Also, the curve is raised 10500 places and it "stretches" up and down 3500 places. This suggests

$$P(t) = 3500\cos(k(t-8)) + 10500$$

for some number k that is going to change the period of the function appropriately. (Can you see that $3500\cos(k(t-8) + 10500$ oscillates above and below the value 10500, with a range 3500 up and 3500 down? Can you see that putting $t = 8$ does indeed give the high value of 14000?)

To find k we can simply plug in a value for t. Let's try $t = 2$, say. We have:

$$7000 = 3500\cos(-6k) + 10500$$

giving

$$\cos(-6k) = -1.$$

This suggests $-6k = \pi$ or $-\pi$ or 3π or -3π and so on. For ease, let's go with $k = \frac{\pi}{6}$, going with the smallest positive value for k. (The other choices are valid too.)

So we seem to have

$$P(t) = 3500 \cos\left(\frac{\pi}{6}(t-8)\right) + 10500$$

To check,

$$\begin{aligned} P(2) &= 3500\cos(-\pi) + 10500 = 3500(-1) + 10500 = 7000 \\ P(8) &= 3500\cos(0) + 10500 = 14000 \\ P(14) &= 3500\cos(\pi) + 10500 = 7000. \end{aligned}$$

Looking good!

And we can be sure that we have the correct period by noting that the cosine curve completes one full cycle from 0 radians to 2π and we have

$$\frac{\pi}{6}(t-8) = 0 \text{ when } t = 8$$

and

$$\frac{\pi}{6}(t-8) = 2\pi \text{ when } t = 20.$$

Thus we do have a twelve-month cycle. We're in good shape!

The work here illustrates a standard practice in applied mathematics and in physics, using a trigonometric function to model a periodic phenomenon. The trickiest part is adjusting the period of the trigonometric function.

To be clear . . .

$y = \sin x$, for instance, completes one full cycle between $x = 0$ and $x = 2\pi$. That is, it is "done" when $x = 2\pi$.

Thus

$$y = \sin(5x) \text{ is done when } 5x = 2\pi, \text{ that is, when } x = \frac{2\pi}{5}.$$

$$y = \cos(3x) \text{ is done when } 3x = 2\pi, \text{ that is, when } x = \frac{2\pi}{3}.$$

$$y = \sin(2\pi x) \text{ is done when } 2\pi x = 2\pi, \text{ that is, when } x = 1.$$

$$y = \sin\left(\tfrac{x}{3}\right) \text{ is done when } \tfrac{x}{3} = 2\pi, \text{ that is, when } x = 6\pi.$$

In general:

$$y = \sin(kx) \text{ or } y = \cos(kx) \text{ is done when } kx = 2\pi.$$

Example. Find an example of a sine function with period 7.

Answer. Consider $y = \sin(kx)$. We need it to be done when $x = 7$. That is, we need $k \cdot 7 = 2\pi$. This gives $k = \frac{2\pi}{7}$ and our function is $y = \sin(\frac{2\pi x}{7})$. (Check by putting in $x = 7$.)

In general

$$\boxed{\text{the functions } y = \sin\left(\frac{2\pi x}{T}\right) \text{ and } y = \cos\left(\frac{2\pi x}{T}\right) \text{ have period } T.}$$

[Do check that these are each indeed done when $x = T$.]

Example. What is the period of the function $y = \sin(\frac{\pi}{6}(x - 67))$?

Answer. This is the function $y = \sin(\frac{\pi}{6}x)$ with $c = 67$ behaving as zero. Its period is the same as the period of $y = \sin(\frac{\pi}{6}x)$.

Now $y = \sin(\frac{\pi}{6}x)$ is done when $\frac{\pi}{6}x = 2\pi$, that is, when $x = 12$. The period of $y = \sin(\frac{\pi}{6}x)$, and of $y = \sin(\frac{\pi}{6}(x - 67))$, is thus 12.

In general

$$\boxed{\text{the functions } y = \sin\left(\frac{2\pi(x-a)}{T}\right) \text{ and } y = \cos\left(\frac{2\pi(x-a)}{T}\right) \text{ also have period } T.}$$

Comment. In the context of degree measure, this statement reads

$$\boxed{\text{the functions } y = \sin\left(\frac{360(x-a)}{T}\right) \text{ and } y = \cos\left(\frac{360(x-a)}{T}\right) \text{ have period } T.}$$

MAA PROBLEMS

Featured Problem

(#24, AMC 12A, 2007)

For each integer $n > 1$, let $F(n)$ be the number of solutions of the equation $\sin x = \sin nx$ on the interval $[0, \pi]$. What is $\sum_{n=2}^{2007} F(n)$?

(A) 2,014,524 (B) 2,015,028 (C) 2,015,033 (D) 2,016,532
(E) 2,017,033

A Personal account of solving this problem

Curriculum Inspirations Strategies (www.maa.org/ci):

Strategy 5: **Solve a smaller version of the same problem.**

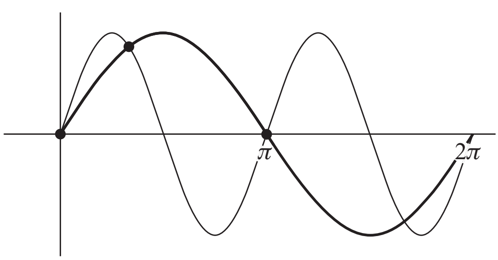
This problem looks positively frightful!

Deep breath!

What's this question basically about? Answer: The graphs of $\sin x$ and and $\sin nx$. Specifically, about where they intersect.

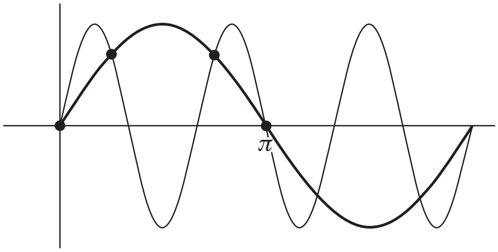
That's too abstract. How about I just solve where $\sin x$ and $\sin 2x$ intersect.

I know what the graph of $y = \sin x$ looks like. And I know what $y = \sin 2x$ looks like too: it's the same graph but with double the period.



These graphs intersect three times in the interval $[0, \pi]$.

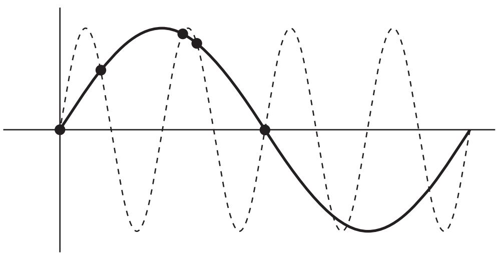
How about $y = \sin x$ and $y = \sin 3x$? We have that $\sin 3x$ has period $\frac{2\pi}{3}$.



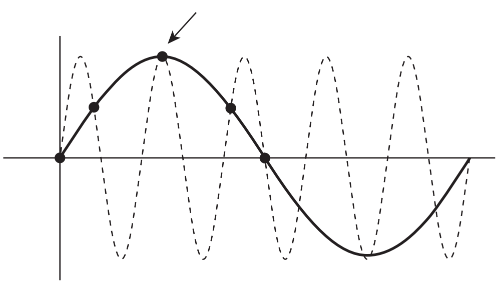
These intersect four times in the interval $[0, \pi]$.

I don't yet have a feel for what happens in general, so let me just keep looking at these small number cases.

For $y = \sin x$ and $y = \sin 4x$ (with period $\frac{2\pi}{4}$) there are five intersection points, two for each positive "hump" of $\sin 4x$ and one for $x = \pi$.



For $y = \sin x$ and $y = \sin 5x$ (with period $\frac{2\pi}{5}$):



Ooh. This one is interesting. Each positive hump of the $y = \sin nx$ graph gives two intersection points, except in this case where the middle hump lands right at $x = \frac{\pi}{2}$. This hump has only one intersection point.

I am seeing the following:

> If $y = \sin nx$ has k positive humps in the interval $[0, \pi]$ then there are $2k$ intersection points, two for each hump, unless we have $\sin nx = 1$ at $x = \frac{\pi}{2}$, in which case there is one less intersection point for that hump.

Now $\sin\left(n \cdot \frac{\pi}{2}\right) = 1$ if $n\frac{\pi}{2} = \frac{\pi}{2} + 2m\pi$ for some integer m, that is, if $n = 4m + 1$. I can add to my thoughts then:

> This exceptional case happens when n is one more than a multiple of 4.

Ooh. I don't quite have it right. There is another detail I can't forget.

> For the graph $y = \sin nx$ with n even we have two intersection points for each hump, plus an extra intersection at $x = \pi$.

I think that's all the pieces.

So now the question is: How many positive humps does $y = \sin nx$ have in the interval $[0, \pi]$?

Looking at my pictures:

$\sin 2x$ has 1 hump
$\sin 3x$ has 2 humps
$\sin 4x$ has 2 humps
$\sin 5x$ has 3 humps

and I can see in my mind's eye that

> $\sin 6x$ will have 3 humps.

We have the counts $1, 2, 2, 3, 3, \ldots$. A pattern!

In fact, I can see that whenever we draw the graph of $y = \sin(n + 1)x$ from looking at the graph of $y = \sin x$ all we need to do is "squeeze in" another upward or downward hump in the interval $[0, \pi]$.

Actually, in going from $y = \sin nx$ to $y = \sin(n + 1)x$ with n odd we squeeze in one more downward hump and the number of upward humps won't change. In going from $y = \sin nx$ to $y = \sin(n + 1)x$ with n even we squeeze in one more upward hump and the count increases by one. I am convinced then that the pattern above persists.

Looking at the pattern we see:

> $y = \sin nx$ has $\frac{n}{2}$ upward humps in the interval $[0, \pi]$ if n is even, and $\frac{n+1}{2}$ if n is odd.

I've got to pull all this information together.

Each positive hump gives 2 intersection points, except for the exceptional middle hump for n one more than a multiple of four (it has one fewer intersection point).

For n even there are $\frac{n}{2}$ upward humps. For n odd there are $\frac{n+1}{2}$ upward humps.

For n even there is an addition intersection point at $x = \pi$.

Okay, in terms of $F(n)$ we have

$$\text{For } n \text{ even: } F(n) = 2 \times \frac{n}{2} + 1 = n + 1.$$

$$\text{For } n = 4m + 1 : F(n) = 2 \times \frac{n+1}{2} - 1 = n.$$

$$\text{For } n = 4m + 3 : F(n) = 2 \times \frac{n+1}{2} = n + 1.$$

So $F(n) = n + 1$ except when n is one more than a multiple of four, $n = 5$, $9, 13, \ldots, 2005$, at which $F(n) = n$. There are 501 of these exceptional cases.

Getting there!

$$\begin{aligned}
\sum_{n=2}^{2007} F(n) &= \left(\sum_{n=2}^{2007} (n+1)\right) - 501 \\
&= \left(\sum_{n=2}^{2007} n\right) + 2006 - 501 \\
&= \left(\sum_{n=1}^{2007} n\right) - 1 + 2006 - 501 \\
&= \frac{2007 \times 2008}{2} - 1 + 2006 - 501 \\
&= 2007 \times 1000 + 2007 \times 4 - 1 + 2006 - 501 \\
&= 2{,}007{,}000 + 8028 - 1 + 2006 - 501 \\
&= 2{,}016{,}532.
\end{aligned}$$

The answer is (D).

Crazy!

19
(EXTRA) Polar Coordinates

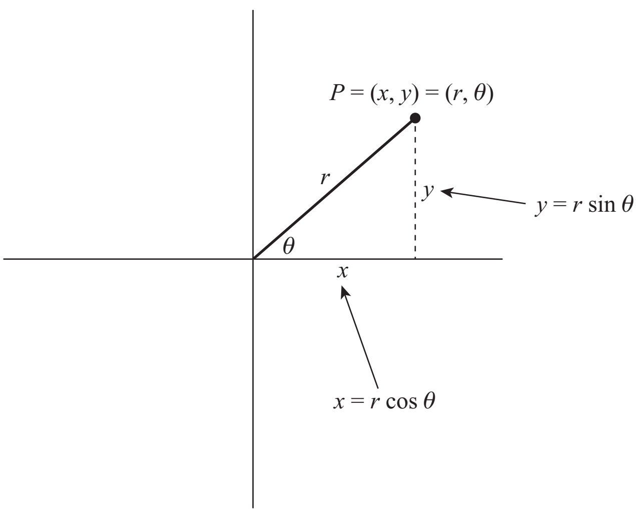
The location of a point P in the plane is specified by its x- and y-coordinates:

$$P = (x, y).$$

In the late 1600s, Sir Isaac Newton realized that one could also specify the location of a point via two other parameters:

i) The distance of the point from the origin.
ii) The angle of elevation of the point from the positive x-axis.

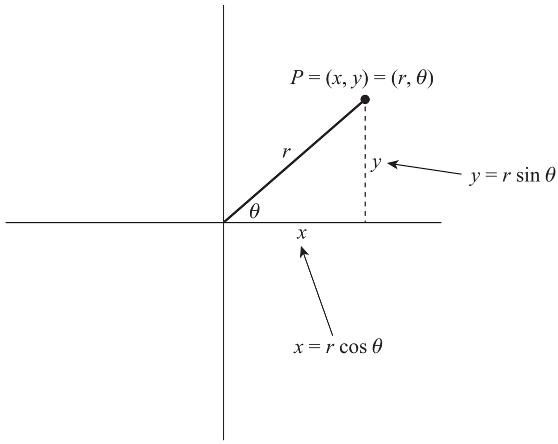
(Sound familiar?)



The distance of a point from the origin is usually denoted r and its angle of elevation θ. This angle is usually given in radians. We then say that the point P has *polar coordinates* $P = (r, \theta)$.

Comment. As we shall soon see polar coordinates are very handy when attempting to describe the location of points following circular motion—say, the orbits of planets about a star. (Again, sound familiar?)

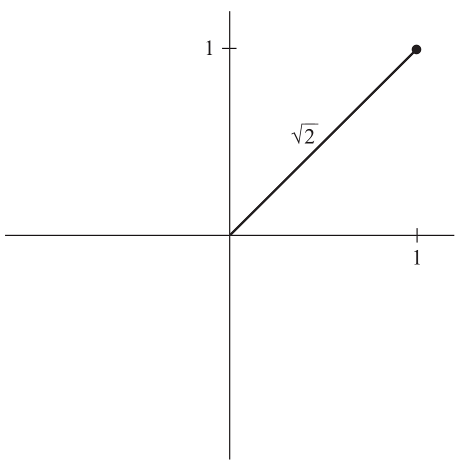
We have the following relations:



$$\boxed{\begin{array}{ll} x = r\cos\theta & r = \sqrt{x^2 + y^2} \\ y = r\cos\theta & \tan\theta = \dfrac{y}{x} \end{array}}$$

These relations give a means for converting between the Cartesian coordinates of a point and its polar coordinates.

Example. The point (1, 1) in Cartesian coordinates is $\sqrt{2}$ units from the origin and has an angle of elevation $\frac{\pi}{4}$. It has polar coordinates $(\sqrt{2}, \frac{\pi}{4})$.



Comment. One could also say that the polar coordinates of this point are $(\sqrt{2}, \frac{9\pi}{4})$. Do you see why?

Example. The point $Q = (3, \frac{\pi}{3})$, in polar coordinates, has Cartesian coordinates given by

$$x = 3\cos\left(\frac{\pi}{3}\right) = \frac{3}{2}$$
$$y = 3\sin\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}.$$

Thus $Q = (\frac{3}{2}, \frac{3\sqrt{3}}{2})$.

Example. The point $R = (2, 0)$ lies on the positive x-axis. Its distance from the origin is $r = 2$ and its angle of elevation is 0 radians. Thus, in polar coordinates, $R = (2, 0)$.

Example. The point $H = (\sqrt{5}, -\frac{\pi}{2})$ in polar coordinates has Cartesian coordinates $H = (0, -\sqrt{5})$. Do you see why?

Comment. The notation for Cartesian and polar coordinates is identical! If an author states that a point W has coordinates $(\pi, \frac{\pi}{6})$, there is nothing in the notation to indicate which coordinate system is being used. Hopefully the context of the statement makes it clear which it is.

Comment. The polar coordinates of a point are not unique. For example, the point with polar cooridinates (r, θ) also has polar coordinates $(r, \theta + 2\pi)$ and polar coordinates $(r, \theta - 2\pi)$, and so on.

Also, one can assign any angle of elevation one desires to the point with $r = 0$, that is, to the origin.

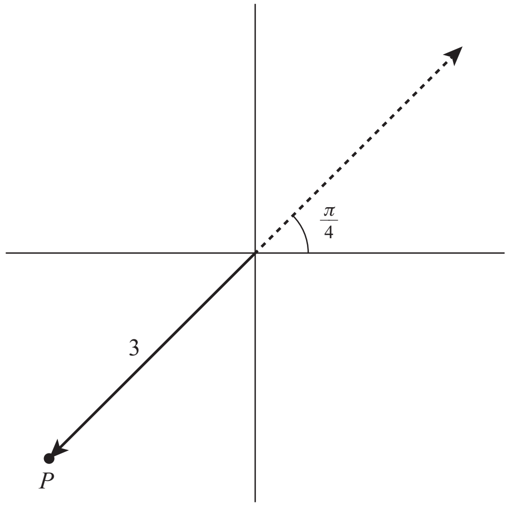
In addition, mathematicians will allow r to adopt negative values! Negative distances are to be measured in the opposite direction from what one expects. For example, the point P with polar coordinates $P = (-3, \frac{\pi}{4})$ is "-3 units out from the origin" along the ray at angle $\frac{\pi}{4}$.

This is the same as the point with polar coordinates $(3, \frac{5\pi}{4})$.

In the same way, the point with polar coordinates $(-5, -\frac{\pi}{2})$, for example, is also the point $(5, \frac{\pi}{2})$.

Recall the equations:

$$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2}$$
$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$



Example. Using approximations to one decimal place, find the polar coordinates of the point whose Cartesian coordinates are $P = (-3, 7)$.

Answer. Here $r = \sqrt{(-3)^2 + 7^2} = \sqrt{58} \approx 7.6$ and according to my calculator, set in radians, $\theta = \tan^{-1}\left(-\frac{7}{3}\right) \approx -1.2$. But there is an issue here as there are many angles that have tangent equal to $-\frac{7}{3}$. Has the calculator offered me the correct one for this problem?

It might be easier to work in degrees for the moment. In degrees, $\tan^{-1}\left(-\frac{7}{3}\right) \approx -66.8^\circ$ and we have an angle that lies in the fourth quadrant. But $P = (-3, 7)$ lies in the second quadrant. The angle in the second quadrant with the same tangent as -66.8° is $-66.8^\circ + 180 = 113.2^\circ$.

Returning to radians, this means that the angle θ appropriate for the problem is $-1.2 + \pi \approx 1.9$ radians.

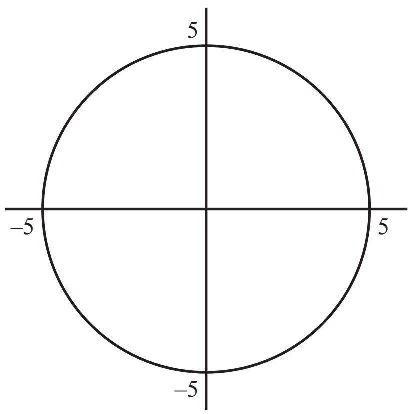
Thus, in polar coordinates, $P \approx (7.6, 1.9)$.

In General: The two angles θ and $\theta + \pi$ have the same tangent value. If the calculator provides a value for $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ that is in an inappropriate quadrant, then the angle will need to be adjusted by π.

20
(EXTRA) Polar Graphs

One advantage of polar coordinates is that they make the graphs of circles and spirals somewhat trivial.

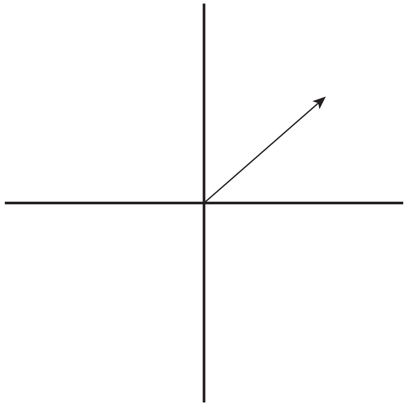
Example. Write an equation for the circle of radius 5 centered about the origin.



Answer. All points that a distance 5 from the origin constitute the circle. Thus an equation describing this set of points is

$$r = 5.$$

Example. Write the equation of the ray at angle $\frac{\pi}{4}$ from the origin.



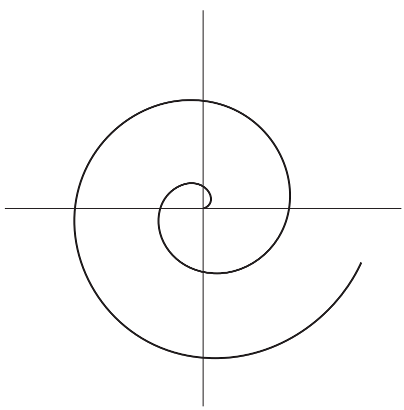
Answer. The equation $\theta = \frac{\pi}{4}$ does it.

Example. Sketch the curve given by the formula $r = \theta$.

Answer. There is never any harm in just plotting points! Let's evaluate r for different values of θ.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	3π	4π
r	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	3π	4π

Thus when $\theta = \frac{\pi}{4}$ we have a point a distance $r = \frac{\pi}{4}$ out from the origin at this angle. When $\theta = \frac{\pi}{2}$ we have point a distance $r = \frac{\pi}{2}$ out from the origin at this angle (on the vertical axis); when $\theta = 2\pi$ we have a point a distance 2π from the origin (on the horizontal axis); and so on. We obtain a spiral:

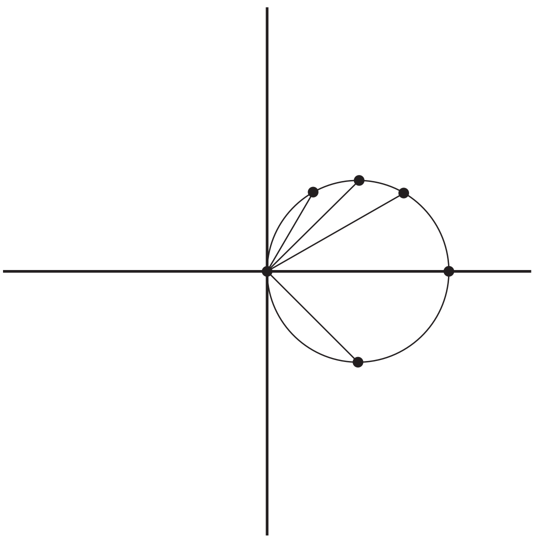


Example. Sketch the curve given by the formula: $r = \cos\theta$.

Answer. Again let's plot points, evaluating r for different values of θ.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$	2π
r	1	$\frac{\sqrt{3}}{2} \approx 0.9$	$\frac{1}{\sqrt{2}} \approx 0.7$	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{2}} \approx -0.7$	-1	0	1

If one plots these points carefully (being careful to interpret the negative distances appropriately) it seems we have a circle.



Do we?

Recall that for polar coordinates, $x = r\cos\theta$ and $r = \sqrt{x^2 + y^2}$, and so our equation $r = \cos\theta$ reads

$$\sqrt{x^2 + y^2} = \frac{x}{r}$$

That is,

$$\sqrt{x^2 + y^2} = \frac{x}{\sqrt{x^2 + y^2}}$$

giving

$$
\begin{aligned}
x^2 + y^2 &= x \\
x^2 - x + y^2 &= 0 \\
x^2 - x + \frac{1}{4} + y^2 &= \frac{1}{4} \\
\left(x - \frac{1}{2}\right)^2 + y^2 &= \frac{1}{4}.
\end{aligned}
$$

This is indeed the equation of a circle. The center is $\left(\frac{1}{2}, 0\right)$ and the radius is $\frac{1}{2}$.

Practice. Sketch the curve given in polar coordinates by $r = 1 - \cos\theta$. (This curve is called a cardioid.) Show that in Cartesian coordinates the equation of this curve is

$$(x^2 + y^2 + x)^2 = x^2 + y^2.$$

Comment. It is standard practice in the curriculum to have students explore the shapes of the polar graphs of $r = \cos(n\theta)$ and $r = \sin(n\theta)$ for different integer values of n, and predict the number of "petals" the graphs will possess.

Part II

Solutions

Solutions

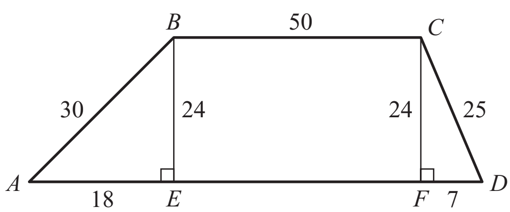
Here are previously published solutions to the competition problems as they appear at www.edfinity.com. One can also find these solutions in the MAA's published texts on the AMC competitions (go to www.maa.org/publications/books and click on "Book Categories" to find their problem-solving books).

Warning: Each solution presented here is fast paced and to the point, simply working through the mathematics of the task at hand to get the job done. These solutions are written by a variety of authors.

For an account of the problem-solving practices behind each solution guiding the mathematical steps presented—along with discussion on its connections to the Common Core State Standards and further, deeper, queries and possible explorations—see the Curriculum Bursts at www.maa.org/ci.

Comment. Sometimes multiple solutions to the same problem were published, with the alternate solutions using mathematical techniques different from those discussed in the sections of this text. We present here only the solutions featuring the tools of the Pythagorean Theorem and trigonometry. We've also made some minor editorial changes to these published solutions.

1. (A)

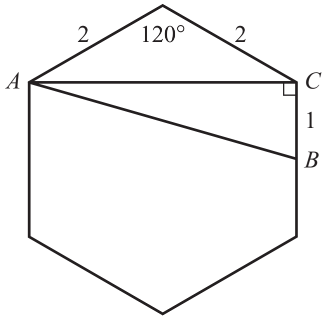


By the Pythagorean Theorem, $AE = \sqrt{30^2 - 24^2} = \sqrt{324} = 18$. (Or note that triangle AEB is similar to a 3-4-5 right triangle, so

$AE = 3 \times 6 = 18$.) It follows that $CF = 24$ and $FD = \sqrt{25^2 - 24^2} = \sqrt{49} = 7$. The perimeter of the trapezoid is $50 + 30 + 18 + 50 + 7 + 25 = 180$.

2. (A)

Assume that Alice starts at A in the figure and ends at B. In ΔABC, $\angle ACB$ is right and $AC = 2\sqrt{3}$. The Pythagorean Theorem shows that $(AB)^2 = 13$.



OR

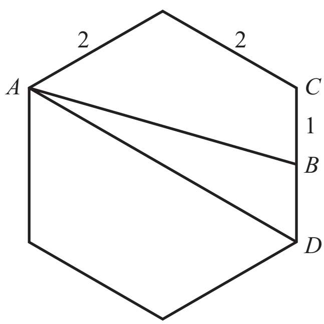
In ΔABD in the second figure,

1) $\angle ADB = 60^\circ$,
2) $AD = 4$, and
3) $BD = 1$.

By the law of cosines,

$$(AB)^2 = 1^2 + 4^2 - 2(1)(4)(1/2) = 13.$$

Therefore $AB = \sqrt{13}$.



3. (B)

Because $AB = 1$, the smallest number of jumps is at least 2. The perpendicular bisector of $\overline{AB}$ is the line with equation $x = \frac{1}{2}$, which has no points with integer coordinates, so two jumps are not possible. A sequence of three jumps is possible; one such sequence is $(0, 0)$ to $(3, 4)$ to $(6, 0)$ to $(1, 0)$.

4. (B)

Let x be the length of the hypotenuse, and let y and z be the lengths of the legs. The given conditions imply that

$$y^2 + z^2 = x^2, y + z = 32 - x, \text{ and } yz = 40.$$

Thus

$$(32 - x)^2 = (y + z)^2 = y^2 + z^2 + 2yz = x^2 + 80,$$

from which $1024 - 64x = 80$, and $x = \frac{59}{4}$.

Note: Solving the system of equations yields leg lengths of

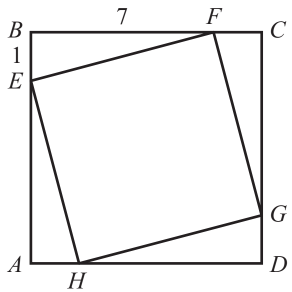
$$\frac{1}{8}(69 + \sqrt{2201}) \text{ and } \frac{1}{8}(69 - \sqrt{2201})$$

so a triangle satisfying the given conditions does in fact exist.

5. (B)

Without loss of generality, assume that F lies on $\overline{BC}$ and that $EB = 1$. Then $AE = 7$ and $AB = 8$. Because $EFGH$ is a square, $BF = AE = 7$, so the hypotenuse $\overline{EF}$ of ΔEBF has length $\sqrt{1^2 + 7^2} = \sqrt{50}$. The ratio of the area of $EFGH$ to that of $ABCD$ is therefore

$$\frac{EF^2}{AB^2} = \frac{50}{64} = \frac{25}{32}.$$



6. (D)

Let h and w be the height and width of the screen, respectively, in inches. By the Pythagorean Theorem, $h : w : 27 = 3 : 4 : 5$, so

$$h = \frac{3}{5} \cdot 27 = 16.2 \quad \text{and} \quad w = \frac{4}{5} \cdot 27 = 21.6.$$

The height of the non-darkened portion of the screen is half the width, or 10.8 inches. Therefore the height of each darkened strip is

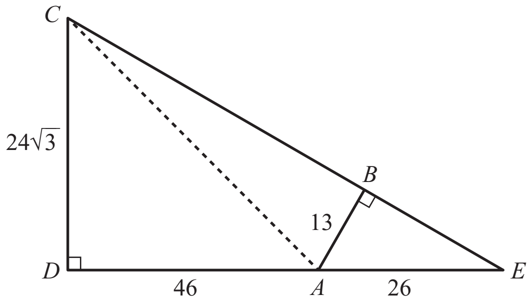
$$\frac{1}{2}(16.2 - 10.8) = 2.7 \text{ inches.}$$

OR

The screen has dimensions $4a \times 3a$ for some a. The portion of the screen not covered by the darkened strips has aspect ratio $2 : 1$, so it has dimensions $4a \times 2a$. Thus the darkened strips each have height $\frac{a}{2}$. By the Pythagorean Theorem, the diagonal of the screen is $5a = 27$ inches. Hence the height of each darkened strip is $\frac{27}{10} = 2.7$ inches.

7. (B)

Extend $\overline{DA}$ through A and $\overline{CB}$ through B and denote the intersection by E. Triangle ABE is a 30–60–90 triangle with $AB = 13$, so $AE = 26$. Triangle CDE is also a 30–60–90 triangle, from which it follows that $CD = \frac{(26+46)}{\sqrt{3}} = 24\sqrt{3}$. Now apply the Pythagorean Theorem to triangle CDA to find that $AC = \sqrt{46^2 + (24\sqrt{3})^2} = 62$.



OR

Since the opposite angles sum to a straight angle, the quadrilateral is cyclic, and AC is the diameter of the circumscribed circle. Thus AC is the diameter of the circumcircle of triangle ABD. By the law of sines,

$$AB = \frac{BD}{\sin 120^\circ} = \frac{BD}{\sqrt{3}/2}.$$

We determine BD by the law of cosines:

$$BD^2 = 13^2 + 46^2 + 2 \cdot 13 \cdot 46 \cdot \frac{1}{2} = 2883 = 3 \cdot 31^2$$

so $BD = 31\sqrt{3}$. Hence $AC = 62$.

8. (A)

By the Pythagorean Theorem we have $a^2 + b^2 = (b+1)^2$, so

$$a^2 = (b+1)^2 - b^2 = 2b + 1.$$

Because b is an integer with $b < 100$, a^2 is an odd perfect square between 1 and 201 and there are six of these, namely, 9, 25, 49, 81, 121, and 169. Hence a must be 3, 5, 7, 9, 11, or 13, and there are six triangles that satisfy the given conditions.

9. (A)

Let the triangle have leg lengths a and b, with $a \le b$. The given condition implies that

$$\frac{1}{2}ab = 3(a + b + \sqrt{a^2 + b^2}),$$

so

$$ab - 6a - 6b = 6\sqrt{a^2 + b^2}.$$

Squaring both sides and simplifying yields

$$ab(ab - 12a - 12b + 72) = 0,$$

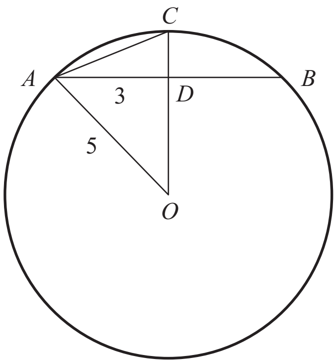
from which

$$(a - 12)(b - 12) = 72.$$

The positive integer solutions of the last equation are $(a, b) = (3, 4), (14, 48), (15, 36), (18, 24)$, and $(20, 21)$. However, the solution $(3, 4)$ is extraneous, and there are six right triangles with the required property.

10. (A)

Let O be the center of the circle, and let D be the intersection of $\overline{OC}$ and $\overline{AB}$. Because $\overline{OC}$ bisects minor arc AB, $\overline{OD}$ is a perpendicular bisector of chord $\overline{AB}$. Hence $AD = 3$, and applying the Pythagorean Theorem to ΔADO yields $OD = \sqrt{5^2 - 3^2} = 4$. Therefore $DC = 1$, and applying the Pythagorean Theorem to ΔADC yields $AC = \sqrt{3^2 + 1^2} = \sqrt{10}$.



11. (D)

Consider a right triangle as shown. By the Pythagorean Theorem,

$$(r+s)^2 = (r-3s)^2 + (r-s)^2$$

so

$$r^2 + 2rs + s^2 = r^2 - 6rs + 9s^2 + r^2 - 2rs + s^2$$

and

$$0 = r^2 - 10rs + 9s^2 = (r-9s)(r-s).$$

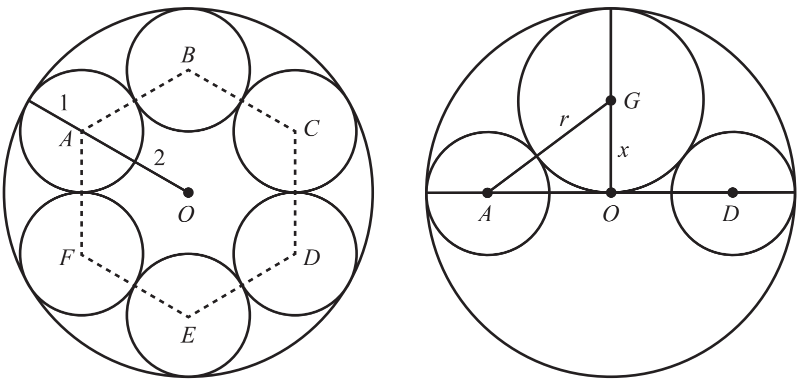
But $r \neq s$, so $r = 9s$ and $r/s = 9$.

OR

Because the ratio r/s is independent of the value of s, assume that $s = 1$ and proceed as in the previous solution.

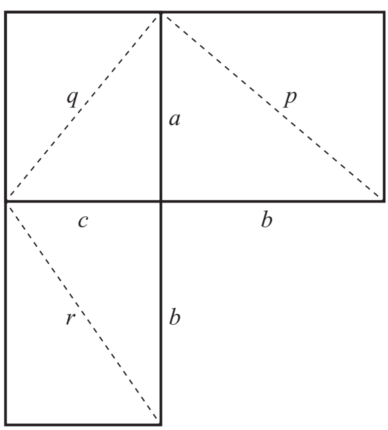
12. (B)

Let the vertices of the regular hexagon be labeled in order A, B, C, D, E, and F. Let O be the center of the hexagon, which is also the center of the largest sphere. Let the eighth sphere have center G and radius r. Because the centers of the six small spheres are each a distance 2 from O and the small spheres have radius 1, the radius of the largest sphere is 3. Because G is equidistant from A and D, the segments $\overline{GO}$ and $\overline{AO}$ are perpendicular. Let x be the distance from G to O. Then $x + r = 3$. The Pythagorean Theorem applied to ΔAOG gives $(r+1)^2 = 2^2 + x^2 = 4 + (3-r)^2$, which simplifies to $2r + 1 = 13 - 6r$, so $r = \frac{3}{2}$. This shows that the eighth sphere is tangent to $\overline{AD}$ at O.



13. (B)

Let a, b, and c, with $a \le b \le c$, be the lengths of the edges of the box; and let p, q, and r, with $p \le q \le r$, be the lengths of its external diagonals. (The diagram shows three faces adjacent at a vertex "flattened out" as in the net of a solid.)



The Pythagorean Theorem implies that

$$\begin{aligned} p^2 &= a^2 + b^2 \\ q^2 &= a^2 + \ +c^2 \\ r^2 &= b^2 + c^2 \\ \text{so } r^2 &= p^2 + q^2 - 2a^2 < p^2 + q^2 \end{aligned}$$

is a necessary condition for a set $\{p, q, r\}$ to represent the lengths of the diagonals. Only choice (B) fails this test:

(A) $6^2 = 36 < 41 = 4^2 + 5^2$
(B) $7^2 = 49 \not< 41 = 4^2 + 5^2$
(C) $7^2 = 49 < 52 = 4^2 + 6^2$
(D) $7^2 = 49 < 61 = 5^2 + 6^2$
(E) $8^2 = 64 < 74 = 5^2 + 7^2$.

The other four choices do correspond to actual prisms because the condition $r^2 < q^2 + q^2$ is also sufficient. To see this, just solve the equations for a, b, and c:

$$\begin{aligned} a^2 &= \frac{p^2 + q^2 - r^2}{2} \\ b^2 &= \frac{p^2 - q^2 + r^2}{2} \\ c^2 &= \frac{-p^2 + q^2 + r^2}{2}. \end{aligned}$$

OR

The angle θ, formed at the vertex of the parallelepiped by diagonals of two adjacent faces, is less than the 90° dihedral angle formed by the two faces. It follows that a triangle formed by choosing one diagonal in each of three faces must be an acute triangle. Therefore, if p, q and r are the lengths of the face diagonals, then from the law of cosines if follows that

$$r^2 = p^2 + q^2 - 2pq\cos\theta < P^2 + q^2$$

since $\cos\theta > 0$.

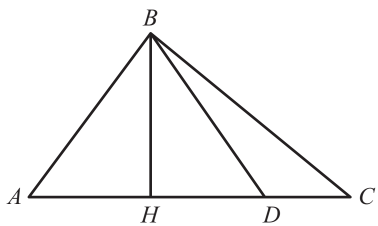
14. (E)

Apply the law of cosines to ΔBAC:

$$BC^2 = BA^2 + AC^2 - 2(BA)(AC)\cos A$$
$$49 = 25 + 81 - 2(5)(9)\cos A.$$

Thus $\cos A = \frac{19}{30}$.

Let H be the foot of the altitude from B.



Then

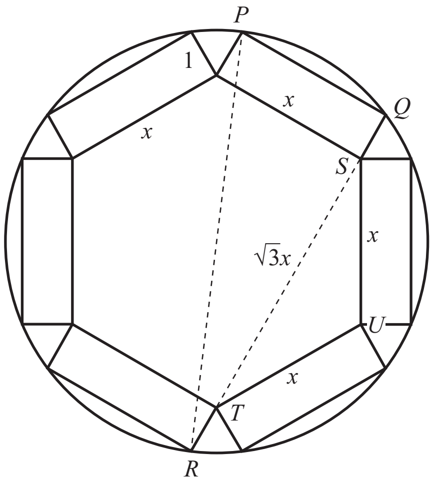
$$AD = 2 \cdot AH = 2(AB)\cos A = \frac{19}{3}$$
$$DC = AC - AD - \frac{8}{3}$$

and

$$AD : DC = 19 : 8.$$

15. (C)

Select one of the mats. Let P and Q be the two corners of the mat that are on the edge of the table, and let R be the point on the edge of the table that is diametrically opposite P as shown. Then R is also a corner of a mat and ΔPQR is a right triangle with hypotenuse $PR = 8$. Let S be the inner corner of the chosen mat that lies on $\overline{QR}$, T the analogous point on the mat with corner R, and U the corner common to the other mat with corner S and the other mat with corner T. Then ΔSTU is an isosceles triangle with two sides of length x and vertex angle 120°. It follows that $ST = \sqrt{3x}$, so $QR = QS + ST + TR = \sqrt{3}x + 2$.



Apply the Pythagorean Theorem to ΔPQR to obtain $(\sqrt{3}x + 2)^2 + x^2 = 8^2$, from which $x^2 + \sqrt{3}x - 15 = 0$. Solve for x and ignore the negative root to obtain

$$x = \frac{3\sqrt{7} - \sqrt{3}}{2}.$$

16. (C)

Let θ_1 and θ_2 be the angles of inclination of lines L_1 and L_2, respectively. Then $m = \tan\theta_1$ and $n = \tan\theta_2 \cdot \theta_1 = 2\theta_2$, and $m = 4n$,

$$4n = m = \tan\theta_1 = \tan 2\theta_2 = \frac{2\tan\theta_2}{1 - \tan^2\theta_2} = \frac{2n}{1 - n^2}.$$

Thus

$$4n = \frac{2n}{1 - n^2} \text{ and } 4n(1 - n^2) = 2n.$$

Since $n \neq 0$, $2n^2 = 1$, and mn, which equals $4n^2$, is 2.

17. (D)

Notice $-r \leq x, y \leq r$ and so each of $c = \frac{x}{r}$ and $s = \frac{y}{r}$ lie between -1 and 1.

Now $s^2 - c^2 \leq s^2 \leq 1$ with equality when $c = 0$, that is, when $x = 0$ (forcing $y = \pm r$).

Also, $s^2 - c^2 \geq -c^2 \geq -1$ with equality when $s = 0$, that is, when $y = 0$ and $x = \pm r$.

18. (E)

Note that the range of $\log(x)$ on the interval $(0, 1)$ is the set of all negative numbers, infinitely many of which are zeros of the cosine function. In fact since $\cos x = 0$ for all x of the form $\frac{\pi}{2} \pm n\pi$,

$$\begin{aligned} f\left(10^{\frac{\pi}{2} \pm n\pi}\right) &= \cos\left(\log\left(10^{\frac{\pi}{2} \pm n\pi}\right)\right) \\ &= \cos\left(\frac{\pi}{2} \pm n\pi\right) \\ &= 0 \end{aligned}$$

for all positive integers n.

19. (D)

Since

$$\begin{aligned} i^{n+2}\cos(45 + 90(n+2))^\circ &= -i^n(-\cos(45 + 90n)^\circ) \\ &= i^n(\cos(45 + 90n)^\circ) \end{aligned}$$

every other term has the same value. The first is $\sqrt{2}/2$, and there are 21 terms with this value ($n = 0, 2, 4 \ldots, 40$). The second term is $i\cos 135^\circ = -i\sqrt{2}/2$, and there are 20 terms with this value ($n = 1, 3, \ldots, 39$). Thus the sum is

$$\frac{\sqrt{2}}{2}(21 - 20i).$$

20. (A)

The intercepts occur where $\sin(\frac{1}{x}) = 0$, that is, where $x = 1/(k\pi)$ and k is a nonzero integer. Solving

$$0.0001 < \frac{1}{k\pi} < 0.001$$

yields

$$\frac{1000}{\pi} < k < \frac{10{,}000}{\pi}.$$

Thus the number of x intercepts in $(0.0001, 0.001)$ is

$$\left\lfloor \frac{10{,}000}{\pi} \right\rfloor - \left\lfloor \frac{1000}{\pi} \right\rfloor = 3183 - 318 = 2865,$$

which is closest to 2900.

21. (D)

$$\log_b \sin x = a$$
$$\sin x = b^a$$
$$\sin^2 x = b^{2a}$$
$$\cos x = \sqrt{1 - b^{2a}}$$
$$\log_b \cos x = \frac{1}{2}\log_b(1 - b^{2a}).$$

22. (C)

The area of the triangle is $\frac{1}{2}$(base)(height) $= \frac{1}{2}(5 - (-5)) \cdot |5\sin\theta| = 25|\sin\theta|$. There are four values of θ between 0 and 2π for which $|\sin\theta| = 0.4$, and each value corresponds to a distinct triangle with area 10.

OR

The vertex $(5\cos\theta, 5\sin\theta)$ lies on a circle of diameter 10 centered at the origin. In order that the triangle have area 10, the altitude from that vertex must be 2. There are four points on the circle that are 2 units from the x-axis.

23. (E)

From the identity $1 + \tan^2 x = \sec^2 x$ it follows that

$$1 = \sec^2 x - \tan^2 x = (\sec x - \tan x)(\sec x + \tan x) = 2(\sec x + \tan x).$$

So $\sec x + \tan x = 0.5$.

OR

The given relation can be written as $\frac{1-\sin x}{\cos x} = 2$. Squaring both sides yields $\frac{(1-\sin x)^2}{1-\sin^2 x} = 4$, hence $\frac{1-\sin x}{1+\sin x} = 4$. It follows that $\sin x = -\frac{3}{5}$ and that

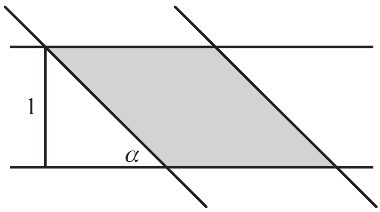
$$\cos x = \frac{1 - \sin x}{2} = \frac{1 - (-3/5)}{2} = \frac{4}{5}.$$
$$\sec x + \tan x = \frac{5}{4} - \frac{3}{4} = 0.5$$

Thus

24. (B)

The shaded figure in the problem is a rhombus. Each side has length $1/\sin\alpha$, which can be observed in the right triangle indicated in the figure.

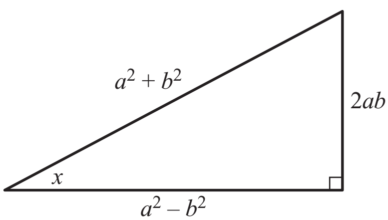
The height of the rhombus is 1, which is the width of each strip. The area of the rhombus is base $\cdot$ height $= 1 \cdot (1/\sin\alpha)$.



OR

The area of a parallelogram is the product of the lengths of two adjacent sides and the sine of the angle between them. Therefore, the area of this rhombus is $(1/\sin\alpha)^2 \sin\alpha$.

25. (E)



The acute angle x may be taken as opposite leg of $2ab$ in a right triangle with the other leg of length $a^2 - b^2$. Then the square of the hypotenuse h is, by the Pythagorean Theorem,

$$h^2 = (2ab)^2 + (a^2 - b^2)^2 = a^4 + 2a^2b^2 + b^4 = (a^2 + b^2)^2.$$

We now see from the figure and the definition of sine that $\sin x = 2ab/(a^2 + b^2)$.

26. (A)

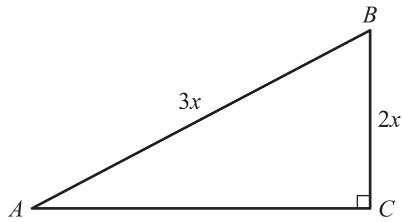
Let $\theta = \angle ABC$. The base of the cylinder is a circle with circumference 6, so the radius of the base is $\frac{6}{2\pi} = \frac{3}{\pi}$. The height of the cylinder is the altitude of the rhombus, which is $6\sin\theta$. Thus the volume of the cylinder is

$$6 = \pi\left(\frac{3}{\pi}\right)^2 (\sin\theta) = \frac{54}{\pi}\sin\theta,$$

so $\sin\theta = \pi/9$.

27. (D)

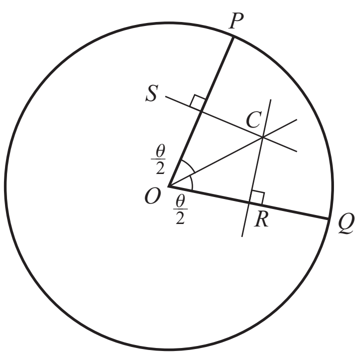
In the figure, $\sin A = \frac{BC}{AB} = \frac{2}{3}$. So for some $x > 0$, $BC = 2x$, $AB = 3x$, and $AC = \sqrt{(AB)^2 - (BC)^2} = \sqrt{5}x$. Thus $\tan B = \frac{AC}{BC} = \frac{\sqrt{5}}{2}$.



28. (D)

The center of the circle that circumscibes the sector POQ is at C, the intersection of the perpendicular bisectors SC and RC. Considering triangle ORC we see that

$$\sec\frac{\theta}{2} = \frac{OC}{3} \quad \text{or} \quad OC = 3\sec\frac{\theta}{2}.$$



29. (A)

If $\frac{x}{x-1} = \sec^2\theta$ then

$$x = x\sec^2\theta - \sec^2\theta$$
$$x(\sec^2\theta - 1) = \sec^2\theta$$
$$x\tan^2\theta = \sec^2\theta.$$

Hence

$$x = \frac{\sec^2\theta}{\tan^2\theta} = \frac{1}{\sin^2\theta}$$

and

$$f(\sec^2\theta) = \sin^2\theta.$$

OR

First solve $y = \frac{x}{x-1}$ for x to find $x = \frac{y}{y=1}$. Then $f(y) = \frac{y-1}{y}$. Hence

$$f(\sec^2\theta) = \frac{\sec^2\theta - 1}{\sec^2\theta} = 1 - \cos^2\theta = \sin^2\theta.$$

OR

Since $\frac{1}{1-\frac{1}{x}} = \frac{x}{x-1}$, $f(\frac{1}{1-t}) = t$. Thus

$$f(\sec^2\theta) = f\left(\frac{1}{\cos^2\theta}\right) = f\left(\frac{1}{1-\sin^2\theta}\right) = \sin^2\theta.$$

30. (C)

Because $\overline{AB}||\overline{DC}$, arc AD is equal to arc CB and CDE and ABE are similar isosceles triangles. Thus

$$\frac{\text{area}\, CDE}{\text{area}\, ABE} = \left(\frac{DE}{AE}\right)^2.$$

Draw $\overline{AD}$. Since $\overline{AB}$ is a diameter, $\angle ADB = 90^\circ$. Thus, considering right triangle ADE, $DE = AE\cos\alpha$, and

$$\left(\frac{DE}{AE}\right)^2 = \cos^2\alpha.$$

31. (C)

By the relationship between the roots and the coefficients of a quadratic equation, it follows that

$$p = \tan\alpha + \tan\beta, q = \tan\alpha\tan\beta$$

$$r = \cot\alpha + \cot\beta, s = \cot\alpha\cot\beta$$

Since

$$\cot\alpha + \cot\beta = \frac{1}{\tan\alpha} + \frac{1}{\tan\beta} = \frac{\tan\alpha + \tan\beta}{\tan\alpha\tan\beta}$$

and

$$\cot\alpha\cot\beta = \frac{1}{\tan\alpha\tan\beta}$$

the equalities $r = p/q$ and $s = 1/q$ follow. Thus $rs = p/q^2$.

32. (A)

Using $\log a + \log b = \log ab$ repeatedly, we find that the sum is

$$P = \log_{10}[(\tan 1^\circ)(\tan 2^\circ)\cdots\tan(45^\circ)\cdots\tan(89^\circ)].$$

Moreover, $(\tan 1^\circ)(\tan 89^\circ) = 1$, $(\tan 2^\circ)(\tan 88^\circ) = 1$, and so on, because $\tan\theta\tan(90^\circ - \theta) = \tan\theta\cot\theta = 1$ for all θ at which both $\tan\theta$ and $\cot\theta$ are defined. Thus

$$P = \log_{10}(\tan 45^\circ) = \log_{10} 1 = 0.$$

33. (A)

The area of the shaded sector is $\frac{\theta}{2}(AC)^2$. This must equal half the area of ΔABC, which is $\frac{1}{2}(AC)(AB)$. Hence the shaded regions have equal area if, and only if, $\frac{\theta}{2}(AC)^2 = \frac{1}{4}(AC)(AB)$, which is equivalent to $2\theta = \frac{AB}{AC} = \tan\theta$.

34. (A)

The following statements are equivalent:

$$\sin 2x \sin 3x = \cos 2x \cos 3x,$$

$$\cos 2x \cos 3x - \sin 2x \sin 3x = 0,$$

$$\cos(2x + 5x) = 0,$$

$$5x = 90^\circ \pm 180^\circ k,\ k = \pm 1, \pm 2, \ldots,$$

$$x = 18^\circ \pm 36^\circ k, k = \pm 1, \pm 2, \ldots.$$

The only correct value listed among the answers is 18°.

OR

By inspection of the original equation, it is sufficient that $\sin 2x = \cos 3x$ and $\sin 3x = \cos 2x$, which are both true if $2x$ and $3x$ are complementary. Thus $2x + 3x = 90^\circ$, that is, $x = 18^\circ$ is a correct value.

35. (E)

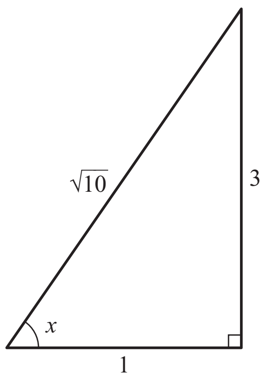
If $\sin x = 3\cos x$, then $\tan x = 3$. From the figure we conclude that

$$\sin x \cos x = \frac{3}{\sqrt{10}} \cdot \frac{1}{\sqrt{10}} = \frac{3}{10}$$

for any acute angle x. If x' is another angle with $x' = 3$, then $x' - x$ is a multiple of π. Thus

$$\sin x' = \pm\frac{3}{\sqrt{10}} \qquad \cos x' = \pm\frac{1}{\sqrt{10}}.$$

So $\sin x' \cos x'$ is still $3/10$ (since $\sin x'$ and $\cos x'$ have the same sign).



OR

Multiplying the given equation first by $\sin x$ and then by $\cos x$ yields

$$\sin^2 x = 3\sin x \cos x$$
$$\cos^2 x = (1/3)\sin x \cos x.$$

Adding gives

$$1 = (10/3)\sin x \cos x,$$

so

$$\sin x \cos x = \frac{3}{10}.$$

36. (E)

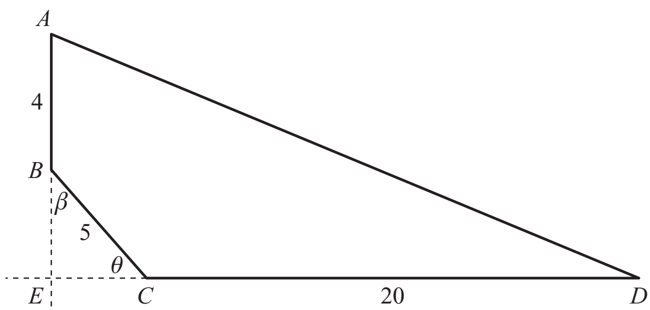
Let E be the intersection of lines AB and CD, and let β and θ be the measures of $\angle EBC$ and $\angle ECB$, respectively. Since

$$\cos\beta = -\cos B = \sin C = \sin\theta,$$

$\beta + \theta = 90^\circ$, so $\angle BEC$ is a right angle, and

$$BE = BC\sin\theta = 3, \quad CE = BC\sin\beta = 4.$$

Therefore, $AE = 7$, $DE = 24$ and AD, which is the hypotenuse of right triangle ADE, is $\sqrt{7^2 + 24^2} = 25$.



37. (C)

Applying the Pythagorean Theorem to ΔCDF and ΔCEG in the adjoining figure yields

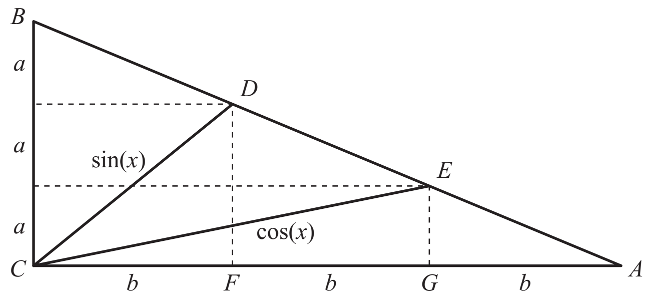
$$4a^2 + b^2 = \sin^2 x$$
$$a^2 + 4b^2 = \cos^2 x.$$

Adding these equations, we obtain

$$5(a^2 + b^2) = \sin^2 x + \cos^2 x = 1.$$

Hence

$$AB = 3\sqrt{a^2 + b^2} = 3\sqrt{\frac{1}{5}} = 3\frac{\sqrt{5}}{5}.$$



38. (D)

The fact that $OA = 1$ implies that $BA = \tan\theta$ and $BO = \sec\theta$. Since $\overline{BC}$ bisects $\angle ABO$, it follows that $\frac{OB}{BA} = \frac{OC}{CA}$, which implies $\frac{OB}{OB+BA} = \frac{OC}{OC+CA} = OC$. Substituting yields

$$OC = \frac{\sec\theta}{\sec\theta + \tan\theta} = \frac{1}{1+\sin\theta}.$$

39. (B)

First note that the isosceles right triangles t can be excluded from the product because $f(t) = 1$ for these triangles. All triangles mentioned from now on are scalene right triangles.

Let $O = (0, 0)$. First consider all triangles $t = \Delta ABC$ with vertices in $S \cup \{O\}$. Let R_1 be the reflection with respect to the line with equation $x = 2$. Let $A_1 = R_1(A)$, $B_1 = R_1(B)$, $C_1 = R_1(C)$, and $t_1 = \Delta A_1B_1C_1$. Note that $\Delta ABC \cong \Delta A_1B_1C_1$ with right angles at A and A_1, but the counterclockwise order of the vertices of t_1 is A_1, C_1, and B_1. Thus $f(t_1) = \tan(\angle A_1C_1B_1) = \tan(\angle ACB)$ and

$$f(t)f(t_1) = \tan(\angle CAB)\tan(\angle ACB) = \frac{AC}{AB}\cdot\frac{AB}{AC} = 1.$$

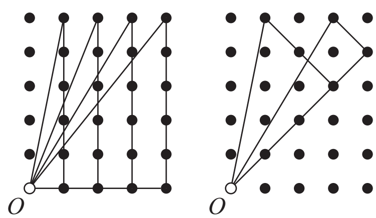
The reflection R_1 is a bijection of $S \cup \{O\}$ and it induces a partition of the triangles in pairs (t, t_1) such that $f(t)f(t_1) = 1$. Thus the product over all triangles in $S \cup \{O\}$ is equal to 1, and thus the required product is equal to the reciprocal of $\prod_{t\in T_1} f(t)$, where T_1 is the set of triangles with vertices in $S \cup \{O\}$ having O as one vertex.

Let $S_1 = \{(x, y) : x \in \{0, 1, 2, 3, 4\}$ and $y \in \{0, 1, 2, 3, 4\}\}$ and let R_2 be the reflection with respect to the line with equation $x = y$. For every right triangle $t = \Delta OBC$ with vertices B and C in S_1, let $B_2 = R_2(B)$,

$C_2 = R_2(C)$, and $t_2 = \Delta OB_2C_2$. Similarly as before, R_2 is a bijection of S_1 and it induces a partition of the triangles in pairs (t, t_2) such that $f(t)f(t_2) = 1$. Thus $\prod_{t\in T_1} f(t) = \prod_{t\in T_2} f(t)$, where T_2 is the set of triangles with vertices in $S \cup \{O\}$ with O as one vertex, and another vertex with y-coordinate equal to 5.

Next, consider the reflection R_3 with respect to the line with equation $y = \frac{5}{2}$. Let $X = (0, 5)$. For every right triangle $t = \Delta OXC$ with C in S, let $C_3 = R_3(C)$ and $t_3 = \Delta OXC_3$. As before R_3 induces a partition of these triangles in pairs (t, t_3) such that $f(t)f(t_3) = 1$. Therefore to calculate $\prod_{t\in T_2} f(t)$, the only triangles left to consider are the triangles of the form $t = \Delta OYZ$ where $Y \in \{(x, 5) : x \in \{1, 2, 3, 4\}\}$ and $Z \in S/\{X\}$.

The following argument shows that there are six such triangles. Because the y-coordinate of Y is greater than zero, the right angle of t is not at O. The slope of the line OY has the form $\frac{5}{x}$ with $1 \le x \le 4$, so if the right angle were at Y, then the vertex Z would need to be at least 5 horizontal units away from Y, which is impossible. Therefore the right angle is at Z. There are four such triangles with Z on the x-axis, with vertices O, $Z = (x, 0)$, and $Y = (x, 5)$ for $1 \le x \le 4$. There are two more triangles: those with vertices O, $Z = (3, 3)$, and $Y = (1, 5)$, and with vertices O, $Z = (4, 4)$, and $Y = (3, 5)$.



The product of the values $f(t)$ over these six triangles is equal to

$$\frac{1}{5}\cdot\frac{2}{5}\cdot\frac{3}{5}\cdot\frac{4}{5}\cdot\frac{3\sqrt{2}}{2\sqrt{2}}\cdot\frac{4\sqrt{2}}{\sqrt{2}} = \frac{144}{625}.$$

Thus the required product equals

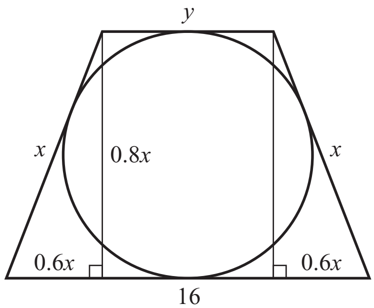
$$\prod_{t\in T} f(t) = \left(\prod_{t\in T_1} f(t)\right)^{-1} = \left(\prod_{t\in T_2} f(t)\right)^{-1} = \left(\frac{144}{625}\right)^{-1} = \frac{625}{144}.$$

40. (C)

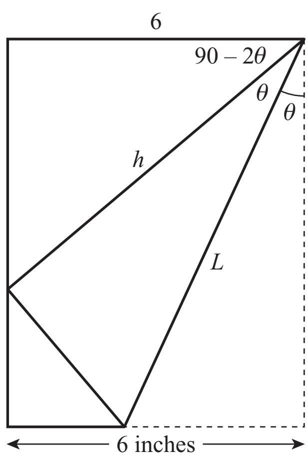
By viewing the sides of the trapezoid as tangents to the circle, we find that the sums of the lengths of opposite sides are equal. (Indeed, this is true for any circumscribed quadrilateral.) Defining x and y as shown in the figure, we have

$$2y + 1.2x = 2x$$
$$y + 1.2x = 16.$$

Then $y = 4$, $x = 10$ and the area is $\frac{1}{2}(4 + 16) \cdot 8 = 80$.



41. (A)



In the figure, where h denotes the length of the sheet,

$$\frac{6}{h} = \cos(90^\circ - 2\theta) = \sin 2\theta = 2\sin\theta\cos\theta$$

from which $h = 2/(\sin\theta\cos\theta)$. Also $L/h = \sec\theta$ and therefore

$$L = h\sec\theta = 3\sec\theta/(\sin\theta\cos\theta) = 3\sec^2\theta\csc\theta.$$

42. (A)

Using trigonometric identities, we obtain

$$\begin{aligned}(\sin\theta+\cos\theta)^2 &= \sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta\\ &= 1+\sin 2\theta\\ &= 1+a.\end{aligned}$$

Since θ is acute, $\sin\theta+\cos\theta > 0$ and $\sin\theta+\cos\theta = \sqrt{1+a}$.

43. (B)

Square both sides of both equations to obtain

$$\sin^2 a + 2\sin a\sin a + \sin^2 b = 5/3 \text{ and}$$
$$\cos^2 a + 2\cos a\cos b + \cos^2 b = 1.$$

Then add corresponding sides of the resulting equations to obtain

$$(\sin^2 a+\cos^2 a)+(\sin^2 b+\cos^2 b)+2(\sin a\sin b+\cos a\cos b) = \frac{8}{3}.$$

Because $\sin^2 a+\cos^2 a = \sin^2 b+\cos^2 b = 1$, it follows that

$$\cos(a-b) = \sin a\sin b+\cos a\cos b = \frac{1}{3}.$$

One ordered pair (a, b) that satisfies the given condition is approximately (0.296, 1.527).

44. (E)

Using the formula for the cosine of twice the angle $\frac{1}{2}\theta$, we have

$$\cos\theta = \cos 2\left(\frac{\theta}{2}\right) = 1-2\sin^2\frac{\theta}{2} = 1-2\frac{x-1}{2x} = \frac{1}{x}.$$

(Note that since $0 < \theta < 90^\circ$, $0 < \frac{1}{x} < 1$, so $x > 1$.) Now

$$\tan^2\theta = \sec^2\theta - 1 = \frac{1}{\cos^2\theta} - 1 = x^2 - 1,$$

so

$$\tan\theta = \sqrt{x^2 - 1}.$$

45. (B)

Let $w = \cos 36^\circ$ and $y = \cos 72^\circ$. Applying the identities

$$\cos 2\theta = 2\cos^2\theta - 1 \text{ and } \cos 2\theta = 1 - 2\sin^2\theta$$

with $\theta = 36^\circ$ in the first identity and $\theta = 18^\circ$ in the second identity yields

$$y = 2w^2 - 1 \quad \text{and} \quad w = 1 - 2y^2.$$

Adding the equations yields

$$w + y = 2(w^2 - y^2) = 2(w - y)(w + y)$$

and division by $w + y$ yields $2(w - y) = 1$, so

$$x = w - y = \frac{1}{2}.$$

46. (B)

Let $w = x - y$. Then the given expression is

$$\sin w \cos y + \cos w \sin y = \sin(w + y) = \sin x.$$

47. (A)

If $\sin x + \cos x = \frac{1}{5}$, then $\cos x = \frac{1}{5} - \sin x$ and

$$\cos^2 x = 1 - \sin^2 x = \left(\frac{1}{5} - \sin x\right)^2$$

so

$$25\sin^2 x - 5\sin x - 12 = 0.$$

The solutions of $25s^2 - 5s - 12 = 0$ are $s = 4/5$ and $s = -3/5$. Since $0 \le x < \pi$, $\sin x \ge 0$, so $\sin x = 4/5$ and $\cos x = (1/5) - \sin x = -3/5$. Hence $\tan x = -4/3$.

48. (E)

Write

$$\begin{aligned}\sin\frac{A}{2} - \sqrt{3}\cos\frac{A}{2} &= 2\left(\frac{1}{2}\sin\frac{A}{2} - \frac{\sqrt{3}}{2}\cos\frac{A}{2}\right)\\ &= 2\sin\left(\frac{A}{2} - 60^\circ\right).\end{aligned}$$

This expression is minimized when $\sin(\frac{A}{2} - 60^\circ) = -1$ or when $\frac{A}{2} - 60^\circ = 270^\circ + (360m)^\circ$, that is when $A = 660^\circ + (270m)^\circ$, $m = 0, \pm 1, \pm 2, \ldots$. None of (A) through (D) is an angle of this form.

49. (D)

The given series is geometric with an initial term of 1 and a common ratio of $\cos^2\theta$, so its sum is

$$5 = \sum_{n=0}^{\infty} \cos^{2n}\theta = \frac{1}{1 - \cos^2\theta} = \frac{1}{\sin^2\theta}.$$

Therefore $\sin^2\theta = \frac{1}{5}$ and

$$\cos 2\theta = 1 - 2\sin^2\theta = 1 - \frac{2}{5} = \frac{3}{5}.$$

50. (B)

We show that for any angles A and B for which the tangent function is defined and $A + B = 45^\circ$, $(1 + \tan A)(1 + \tan B) = 2$. By the addition law for tangents,

$$\begin{aligned} 1 = \tan 45^\circ = \tan(A + B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\ 1 - \tan A \tan B &= \tan A + \tan B \\ 1 &= \tan A + \tan B + \tan A \tan B. \end{aligned}$$

Thus

$$(1 + \tan A)(1 + \tan B) = 1 + \tan A + \tan B + \tan A \tan B = 1 + 1 = 2.$$

51. (C)

Let $x = \arctan a$ and $y = \arctan b$.

Then

$$\begin{aligned} (a + 1)(b + 1) &= 2 \\ (\tan x + 1)(\tan y + 1) &= 2 \\ \tan x + \tan y &= 1 - \tan x \tan y \\ \frac{\tan x + \tan y}{1 - \tan x \tan y} &= 1. \end{aligned}$$

The left hand side is $\tan(x+y)$, so

$$\tan(x+y)=1 \quad \text{and} \quad x+y=\frac{\pi}{4}=\arctan a+\arctan b.$$

OR

Substituting $a=\frac{1}{2}$ in the equation $(a+1)(b+1)=2$ and solving for b, we obtain $b=\frac{1}{3}$. Then

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}=\frac{a+b}{1-ab}=\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}}=1,$$

so $x+y=\frac{\pi}{4}$.

52. (B)

Use the definitions of the tangent and cotangent functions and the identity for the cosine of the difference of two angles to obtain

$$\begin{aligned}\cot 10+\tan 5&=\frac{\cos 10}{\sin 10}+\frac{\sin 5}{\cos 5}\\&=\frac{\cos 10\cos 5+\sin 10\sin 5}{\sin 10\cos 5}\\&=\frac{\cos(10-5)}{\sin 10\cos 5}\\&=\frac{\cos 5}{\sin 10\cos 5}\\&=\frac{1}{\sin 10}\\&=\csc 10.\end{aligned}$$

This is an instance of the identity

$$\cot 2x+\tan x=\csc 2x.$$

To prove it, imitate the above substituting $2x$ for 10 and x for 5.

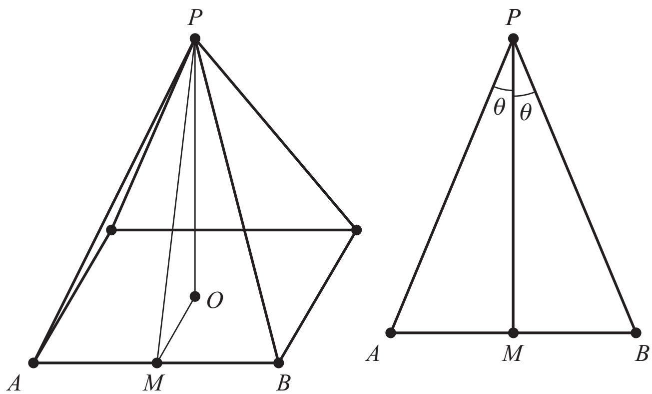
53. (E)

Let M be the midpoint of $\overline{AB}$ and O be the center of the square. Thus $AM = OM = 12$ and slant height $PM = \frac{1}{2}\cos\theta$. Hence

$$\begin{aligned} PO^2 = PM^2 - OM^2 &= \frac{1}{4}\cot^2\theta - \frac{1}{4} \\ &= \frac{\cos^2\theta - \sin^2\theta}{4\sin^2\theta} \\ &= \frac{\cos 2\theta}{4\sin^2\theta}. \end{aligned}$$

Since $0 < \theta < 45^\circ$, the volume is

$$\frac{1}{3}\cdot 1^2 \cdot PO = \frac{\sqrt{\cos 2\theta}}{6\sin\theta}.$$



54. (B)

Since $AB = \sqrt{3^2 + 4^2} = 5$ and $BD = \sqrt{5^2 + 12^2} = 13$, it follows that

$$\begin{aligned} \frac{m}{n} = \frac{DE}{DB} &= \sin\angle DBE = \sin(180^\circ - \angle DBE) \\ &= \sin\angle DBC \\ &= \sin(\angle DBA + \angle ABC) \\ &= \sin(\angle DBA)\cos(\angle ABC) + \cos(\angle DBA)\sin(\angle ABC) \\ &= \frac{12}{13}\cdot\frac{4}{5} + \frac{5}{13}\cdot\frac{3}{5} \\ &= \frac{63}{65} \end{aligned}$$

and $m + n = 128$.

55. (D)

Let $C = (0, 0)$, $B = (2, 2\sqrt{3})$, and $A = (x, 0)$ with $x > 0$. Then $D = (1, \sqrt{3})$. Let P be on the positive x-axis to the right of A.

Then $\angle BAD = \angle PAD - \angle PAB$. Provided $\angle PAD$ and $\angle PAB$ are not right angles, it follows that

$$\begin{aligned}
\tan(\angle BAD) &= \tan(\angle PAD - \angle PAB) \\
&= \frac{\tan(\angle PAD) - \tan(\angle PAB)}{1 + \tan(\angle PAD)\tan(\angle PAB)} \\
&= \frac{m_{AD} - m_{AB}}{1 + m_{AD}m_{AB}} \\
&= \frac{\dfrac{\sqrt{3}}{1-x} - \dfrac{2\sqrt{3}}{2-x}}{1 + \dfrac{\sqrt{3}}{1-x} \cdot \dfrac{2\sqrt{3}}{2-x}} \\
&= \frac{\sqrt{3}x}{x^2 - 3x + 8} \\
&= \frac{\sqrt{3}}{\left(\sqrt{x} - \dfrac{2\sqrt{2}}{\sqrt{x}}\right)^2 + (4\sqrt{2} - 3)} \le \frac{\sqrt{3}}{4\sqrt{2} - 3},
\end{aligned}$$

with equality when $x = 2\sqrt{2}$. If $\angle PAD = 90^\circ$, then

$$\tan(\angle BAD) = -\cot(\angle PAB) = \frac{1}{\sqrt{3}} < \frac{\sqrt{3}}{4\sqrt{2} - 3}.$$

Therefore the largest possible value of $\tan(\angle BAD)$ is $\frac{\sqrt{3}}{4\sqrt{2}-3}$.

56. (A)

Recognize the similarity between the recursion formula given and the trigonometric identity

$$\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}.$$

Also note that the first two terms of the sequence are tangents of familiar angles, namely $\frac{\pi}{4}$ and $\frac{\pi}{6}$.

Let $c_1 = 3$, $c_2 = 2$, and $c_{n+1} = (c_n + c_{n+1}) \bmod 12$. We claim that the sequence $\{a_n\}$ satisfies $a_n = \arctan(\frac{\pi c_n}{12})$. Note that

$$a_1 = 1 = \tan\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi c_1}{12}\right)$$

and

$$a_2 = \frac{1}{\sqrt{3}} = \tan\left(\frac{\pi}{6}\right) = \tan\left(\frac{\pi c_2}{12}\right).$$

By induction on n, the formula for the tangent of the sum of two angles, and the fact that the period of x is π,

$$\begin{aligned} a_{n+2} = \frac{a_n + a_{n-1}}{1 - a_n a_{n+1}} &= \frac{\tan\left(\frac{\pi c_n}{12}\right) + \tan\left(\frac{\pi c_{n+1}}{12}\right)}{1 - \tan\left(\frac{\pi c_n}{12}\right)\tan\left(\frac{\pi c_{n+1}}{12}\right)} \\ &= \tan\left(\frac{\pi(c_n + c_{n+1})}{12}\right) \\ &= \tan\left(\frac{\pi c_{n+2}}{12}\right). \end{aligned}$$

The first few terms of the sequence $\{c_n\}$ are

3, 2, 5, 7, 0, 7, 7, 2, 9, 11, 8, 7, 3, 10, 1, 11, 0, 11, 11, 10, 9, 7, 4, 11, 3, 2.

So the sequence $\{c_n\}$ is periodic with period 24. Because $2009 = 24 \cdot 83 + 17$, it follows that $c_{2009} = c_{17} = 0$. Thus

$$|a_{2009}| = \left|\tan\left(\frac{\pi c_{17}}{12}\right)\right| = 0.$$

57. (B)

(Triangle ADM is congruent to triangle APM by the side-side-side principle.) Let $\angle MAD = \alpha$. Then

$$\begin{aligned} PQ = (PA)\sin(\angle PAQ) &= 4\sin(2\alpha) = 8\sin\alpha\cos\alpha \\ &= 8\left(\frac{2}{\sqrt{20}}\right)\left(\frac{4}{\sqrt{20}}\right) = \frac{16}{5}. \end{aligned}$$

58. (D)

Two trigonometric identities for expressing sums as products are

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}.$$

Thus

$$\frac{\sin 10^\circ + \sin 20^\circ}{\cos 10^\circ + \sin 20^\circ} = \frac{\sin 15^\circ}{\cos 15^\circ} = \tan 15^\circ.$$

59. (B)

Let $\angle DBE = \alpha$ and $\angle DBC = \beta$. Then $\angle CBE = \alpha - \beta$ and $\angle ABE = \alpha + \beta$, so $\tan(\alpha - \beta)\tan(\alpha + \beta) = \tan^2 \alpha$. Thus

$$\frac{\tan\alpha - \tan\beta}{1 + \tan\alpha + \tan\beta} \cdot \frac{\tan\alpha - \tan\beta}{1 - \tan\alpha\tan\beta} = \tan^2\alpha,$$

from which it follows that

$$\tan^2\alpha - \tan^2\beta = \tan^2\alpha(1 - \tan^2\alpha\tan^2\beta).$$

Upon simplifying, $\tan^2\beta(\tan^4\alpha - 1) = 0$, so $\tan\alpha = 1$ and $\alpha = \frac{\pi}{4}$. Let $DC = a$ and $BD = b$. Then $\cot\angle DBC = \frac{b}{a}$. Because $\angle CBE = \frac{\pi}{4} - \beta$ and $\angle ABE = \frac{\pi}{4} + \beta$, it follows that

$$\cot\angle CBE = \tan\angle ABE = \tan\left(\frac{\pi}{4} + \beta\right) = \frac{1 + \frac{a}{b}}{1 - \frac{a}{b}} = \frac{b+a}{b-a}.$$

Thus the numbers 1, $\frac{b+a}{b-a}$, and $\frac{b}{a}$ form an arithmetic progression, so $\frac{b}{a} = \frac{b+3a}{b-a}$. Setting $b = ka$ yields $k^2 - 2k - 3 = 0$, and the only positive solution is $k = 3$. Hence $b = \frac{BE}{\sqrt{2}} = 5\sqrt{2}$, $a = \frac{5\sqrt{2}}{3}$, and the area of ΔABC is $ab = \frac{50}{3}$.

60. (A)

Suppose that the triangle has vertices $A(a, a^2)$, $B(b, b^2)$, and $C(c, c^2)$. The slope of line segment $\overline{AB}$ is

$$\frac{b^2 - a^2}{b - a} = b + a,$$

so the slopes of the three sides of the triangle have a sum

$$(b + a) + (c + b) + (a + c) = 2 \cdot \frac{m}{n}.$$

The slope of one side is $2 = \tan\theta$ for some angle θ, and the two remaining sides have slopes

$$\tan\left(\theta \pm \frac{\pi}{3}\right) = \frac{\tan\theta \pm \tan(\pi/3)}{1 \mp \tan\theta\tan(\pi/3)} = \frac{2 \pm \sqrt{3}}{1 \mp 2\sqrt{3}} = -\frac{8 \pm 5\sqrt{3}}{11}.$$

Therefore

$$\frac{m}{n} = \frac{1}{2}\left(2 - \frac{8 + 5\sqrt{3}}{11} - \frac{8 - 5\sqrt{3}}{11}\right) = \frac{3}{11},$$

and $m + n = 14$.

Such a triangle exists. The x-coordinates of its vertices are $(11 \pm 5\sqrt{3})/11$ and $-19/11$.

61. (D)

Since

$$\begin{aligned}\tan^2 x - 9\tan x + 1 &= \sec^2 x - 9\tan x \\ &= \frac{1}{\cos^2 x} - 9\frac{\sin x}{\cos x} \\ &= \frac{1 - 9\sin x\cos x}{\cos^2 x} \\ &= \frac{1 - \frac{9}{2}\sin 2x}{\cos^2 x}\end{aligned}$$

we need to sum the roots of the equation $\sin 2x = \frac{2}{9}$ between as $x = 0$ and $x = 2\pi$. The four roots are

$$x = \frac{\arcsin\frac{2}{9}}{2}, \frac{\pi - \arcsin\frac{2}{9}}{2}, \frac{2\pi + \arcsin\frac{2}{9}}{2}, \quad \text{and } \frac{3\pi - \arcsin\frac{2}{9}}{2}.$$

and their sum is 3π.

OR

For any $b > 2$ the solutions of $y^2 - by + 1 = 0$ are

$$y_1, y_2 = \frac{b \pm \sqrt{b^2 - 4}}{2},$$

which are distinct and positive. Then either use the fact that the product of the roots of a quadratic is the constant term, or note that

$$y_1 \cdot y_2 = \frac{(b + \sqrt{b^2 - 4})(b - \sqrt{b^2 - 4})}{2^2} = \frac{b^2 - (b^2 - 4)}{4} = 1$$

to see that y_1 and y_2 are reciprocals. Choose the first quadrant angles, x_1 and x_2, so $x_1 = y_1$ and $x_2 = y_2$. Then

$$\tan x_2 = y_2 = \frac{1}{y_1} = \frac{1}{\tan x_1} = \cot x_1 = \tan\left(\frac{\pi}{2} - x_1\right),$$

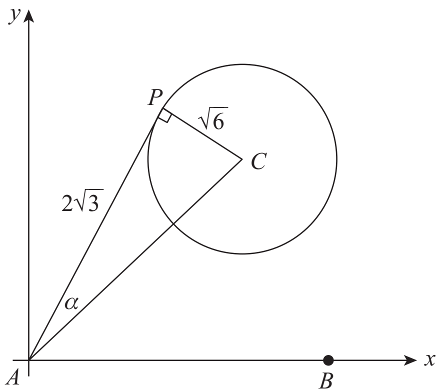
so x_1 and x_2 are complementary. Since $\tan(x + \pi) = \tan x$, there are four values of x between 0 and 2π, and all can be expressed in terms of x_1

$$x_1, \frac{\pi}{2} - x_1, \pi + x_1, \frac{3\pi}{2} - x_1.$$

Their sum is 3π.

62. (A)

Let $P = (x, y)$, $A = (0, 0)$, $C = (3, 3)$ and B be any point on the positive x-axis. The locus of P is the circle with center C and radius $\sqrt{6}$ and $\frac{y}{x}$ is the slope of segment AP. Clearly this slope is the greatest when AP is tangent to the circle on the left side, as in the adjoining figure (note: $\sqrt{6} < 3$).



Let $\alpha = \angle CAP$. Since $\angle BAC = 45°$, the answer is

$$\tan(\alpha + 45°) = \frac{\tan\alpha + 1}{1 - \tan\alpha}.$$

Since $\angle APC = 90°$, $\tan\alpha = PC/PA$. By the Pythagorean Theorem,

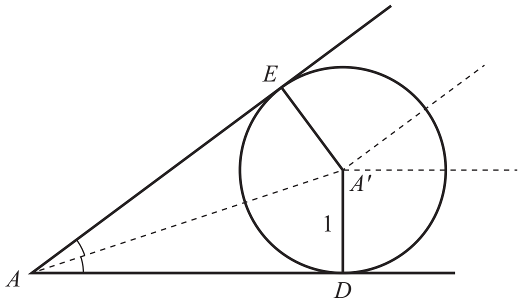
$$PA = \sqrt{(AC)^2 - (PC)^2} = 2\sqrt{3}.$$

Thus $\tan\alpha = 1/\sqrt{2}$ and the answer is $3 + 2\sqrt{2}$.

63. (B)

When the circle is closest to A with its center P at A', let its points of tangency to $\overline{AB}$ and $\overline{AC}$ be D and E, respectively. The path parallel to $\overline{AB}$ is shorter than $\overline{AB}$ by AD plus the length of a similar segment at the other end. Now $AD = AE = \cot(A/2)$. Similar reasoning at the other vertices shows that the length L of the path of P is

$$L = AB + BC + CA - 2\cot\frac{A}{2} - 2\cot\frac{B}{2} - 2\cos\frac{C}{2}.$$



Note that

$$\cot\frac{A}{2} = \left(\frac{\cos(A/2)}{\sin(A/2)}\right)\left(\frac{2\cos(A/2)}{2\cos(A/2)}\right)$$

$$= \frac{2\cos^2(A/2)}{2\sin(A/2)\cos(A/2)}$$

$$= \frac{1+\cos^2(A/2)-\sin^2(A/2)}{\sin A}$$

$$= \frac{1+\cos A}{\sin A}$$

$$= \frac{1+(4/5)}{3/5} = 3.$$

Similarly

$$\cot\frac{C}{2} = \frac{1+\cos C}{\sin C} = \frac{1+(3/5)}{4/5} = 2.$$

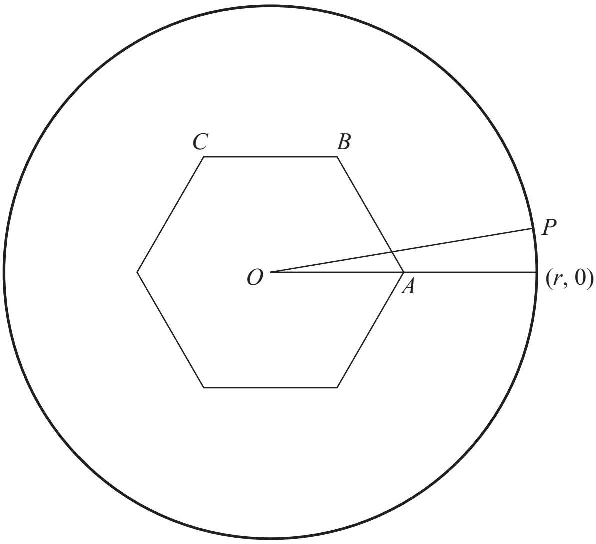
Of course, $\cot(B/2) = \cot 45^\circ = 1$. The length of the path is

$$L = 8+6+10-2(3)-2(1)-2(2) = 12.$$

Challenge. Prove that for any triangle and for any circle that rolls around inside the triangle, the perimeter of the triangle that is the locus of the center of the circle is the perimeter of the original triangle diminished by the perimeter of the similar triangle that circumscribes the circle.

64. (D)

Place the hexagon in a coordinate plane with center at the origin O and vertex A at $(2, 0)$. Let B, C, D, E and F be the other vertices in counterclockwise order.



Corresponding to each vertex of the hexagon, there is an arc on the circle from which only the two sides meeting at that vertex are visible. The given probability condition implies that those arcs have a combined degree measure of 180°, so by symmetry each is 30°. One such arc is centered at $(r, 0)$. Let P be the endpoint of this arc in the upper half-plane. Then $\angle POA = 15°$. Side $\overline{BC}$ is visible from points immediately above P, so P is collinear with B and C. Because the perpendicular distance from O to $\overline{BC}$ is $\sqrt{3}$, we have

$$\sqrt{3} = r \sin 15^\circ = r \sin(45^\circ - 30^\circ) = r(\sin 45^\circ \cos 30^\circ - \sin 30^\circ \cos 45^\circ).$$

So

$$\sqrt{3} = r \cdot \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = r \cdot \frac{\sqrt{6} - \sqrt{2}}{4}.$$

Therefore

$$r = \frac{4\sqrt{3}}{\sqrt{6} - \sqrt{2}} = \frac{4\sqrt{3}}{\sqrt{6} - \sqrt{2}} \cdot \frac{\sqrt{6} + \sqrt{2}}{\sqrt{6} + \sqrt{2}} = \sqrt{18} + \sqrt{6} = 3\sqrt{2} + 6.$$

65. (C)

For a fixed value of y, the values of $\sin x$ for which $\sin^2 x - \sin x \sin y + \sin^2 y = \frac{3}{4}$ can be determined by the quadratic formula. Namely,

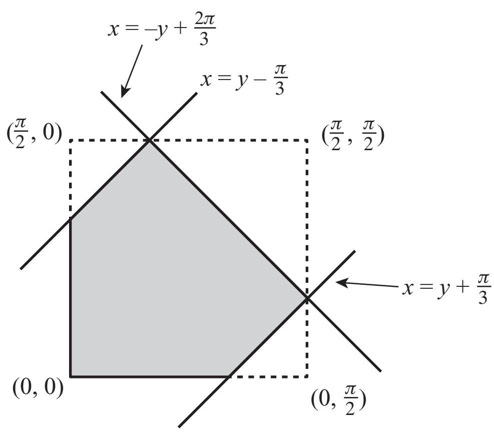
$$\sin x = \frac{\sin y \pm \sqrt{\sin^2 y - 4\left(\sin^2 y - \frac{3}{4}\right)}}{2} = \frac{1}{2}\sin y \pm \frac{\sqrt{3}}{2}\cos y.$$

Because $\cos(\frac{\pi}{3}) = \frac{1}{2}$ and $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$, this implies that

$$\sin x = \cos\left(\frac{\pi}{3}\right)\sin y \pm \sin\left(\frac{\pi}{3}\right)\cos y = \sin\left(y \pm \frac{\pi}{3}\right).$$

Within S, $\sin x = \sin(y - \frac{\pi}{3})$ implies $x = y - \frac{\pi}{3}$. However, the case $\sin x = \sin(y - \frac{\pi}{3})$ implies $x = (y + \frac{\pi}{3}$ when $y \le \frac{\pi}{6}$, and $x = -y + \frac{2\pi}{3}$ when $y \ge \frac{\pi}{6}$. Those three lines divide the region S into four subregions, within each of which the truth value of the inequality is constant. Testing the points $(0, 0)$, $(\frac{\pi}{2}, 0)$, $(0, \frac{\pi}{2})$, and $(\frac{\pi}{2}, \frac{\pi}{2})$ shows that the inequality is true only in the shaded subregion. Its area is

$$\left(\frac{\pi}{2}\right)^2 - \frac{1}{2}\cdot\left(\frac{\pi}{3}\right)^2 - 2\cdot\frac{1}{2}\cdot\left(\frac{\pi}{6}\right)^2 = \frac{\pi^2}{6}.$$



66. (E)

Assume without loss of generality that the regular 12-gon is inscribed in a circle of radius 1. Every segment with endpoints in the 12-gon subtends an angle of $\frac{360}{12}k = 30k$ degrees for some $1 \le k \le 6$. Let d_k be the length of the segments that subtend an angle of $30k$ degrees. There are 12 such segments of length d_k for every $1 \le k \le 5$ and 6 segments of length d_6.

Because $d_k = 2\sin(15k^\circ)$, it follows that

$$\begin{aligned}
d_2 &= 2\sin(30^\circ) = 1,\\
d_3 &= 2\sin(45^\circ) = \sqrt{2},\\
d_4 &= 2\sin(60^\circ) = \sqrt{3},\\
d_6 &= 2\sin(90^\circ) = 2,\\
d_1 &= 2\sin(15^\circ) = 2\sin(45^\circ - 30^\circ)\\
&= 2\sin(45^\circ)\cos(30^\circ) - 2\sin(30^\circ)\cos(45^\circ)\\
&= \frac{\sqrt{6}-\sqrt{2}}{2},\\
d_5 &= 2\sin(75^\circ) = 2\sin(45^\circ + 30^\circ)\\
&= 2\sin(45^\circ)\cos(30^\circ) + 2\sin(30^\circ)\cos(45^\circ)\\
&= \frac{\sqrt{6}+\sqrt{2}}{2}.
\end{aligned}$$

If $a \le b \le c$, then $d_a \le d_b \le d_c$ and the segments with lengths d_a, d_b, and d_c do not form a triangle with positive area if and only if $d_c \ge d_a + d_b$. Because $d_2 = 1 < \sqrt{6} - \sqrt{2} = 2d_1 < \sqrt{2} = d_3$, it follows that for $(a, b, c) \in \{(1, 1, 3)(1, 1, 4), (1, 1, 5), (1, 1, 6)\}$, the segments of lengths d_a, d_b, and d_c do not form a triangle with positive area.

Similarly,

$$\begin{aligned}
d_3 = \sqrt{2} &< \frac{\sqrt{6}-\sqrt{2}}{2} + 1 = d_1 + d_3 < \sqrt{3} = d_4,\\
d_4 < d_5 &= \frac{\sqrt{6}+\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{2} + \sqrt{2} = d_1 + d_3,
\end{aligned}$$

and

$$d_5 < d_6 = 2 = 1 + 1 = 2d_2,$$

so for $(a, b, c) \in \{(1, 2, 4), (1, 2, 5), (1, 2, 6), (1, 3, 5), (1, 3, 6), (2, 2, 6)\}$, the segments of lengths d_a, d_b, and d_c do not form a triangle with positive area.

Finally, if $a \ge 2$ and $b \ge 3$, then $d_a + d_b \ge d_2 + d_3 = 1 + \sqrt{2} > 2 \ge d_c$, and also if $a \ge 3$, then $d_a + d_b \ge 2d_3 = 2\sqrt{2} > 2 \ge d_c$.

Therefore the complete list of forbidden triples (d_a, d_b, d_c) is given by

$$(a, b, c) \in \{(1,1,3), (1,1,4), (1,1,5), (1,1,6),\\ (1,2,4), (1,2,5), (1,2,6), (1,3,5), (1,3,6), (2,2,6)\}.$$

For each $(a, b, c) \in \{(1,1,3), (1,1,4), (1,1,5)\}$, there are $\binom{12}{2}$ pairs of segments of length d_a and 12 segments of length d_c. For each $(a, b, c) \in \{(1,1,6), (2,2,6)\}$, there are $\binom{12}{2}$ pairs of segments of length d_a and 6 segments of length d_c. For each $(a, b, c) \in \{(1,2,4), (1,2,5), (1,3,5)\}$, there are 12^3 triples of segments with lengths d_a, d_b, and d_c. Finally, for each $(a, b, c) \in \{(1,2,6), (1,3,6)\}$, there are 12^2 pairs of segments with lengths d_a and d_b, and 6 segments of length d_c. Because the total number of triples of segments equals $\binom{\binom{12}{2}}{3} = \binom{66}{3}$, the required probability equals

$$1 - \frac{3 \cdot 12 \cdot \binom{12}{2} + 2 \cdot 6 \cdot \binom{12}{2} + 3 \cdot 12^3 + 2 \cdot 12^2 \cdot 6}{\binom{66}{3}} = 1 - \frac{63}{286} = \frac{223}{286}.$$

67. (A)

In any triangle with sides a, b, c, the angle opposite a is acute if, and only if, $a^2 < b^2 + c^2$. This follows from the law of cosines. Applying this fact to the angles opposite x, 24, and 10, we find

$$x^2 < 10^2 + 24^2 = 26^2,$$
$$24^2 < x^2 + 10^2 \Leftrightarrow 476 < x^2,$$
$$10^2 < x^2 + 24^2.$$

The first line tells us that $x < 26$. The second tells us that $x \geq 22$ (since x is an integer). The third is satisfied for every x. Thus there are 4 integer values that meet all the conditions: 22, 23, 24, 25.

68. (E)

Let r be the radius of C_1. Because $OX = OY = r$, it follows that $\angle OZY = \angle XZO$. Applying the Law of Cosines to triangles XZO and OZY gives

$$\frac{11^2 + 13^2 - r^2}{2 \cdot 11 \cdot 13} = \cos\angle XZO = \cos\angle OZY = \frac{7^2 + 11^2 - r^2}{2 \cdot 7 \cdot 11}.$$

Solving for r^2 gives $r^2 = 30$ and so $r = \sqrt{30}$.

69. (D)

By the law of cosines,

$$AC^2 = AB^2 + BC^2 - 2 \cdot AB \cdot BC \cdot \cos$$
$$\angle ABC = 500 - 400 \cos \angle ABC.$$

Because $\cos \angle ABC$ is between $\cos 120^\circ = -\frac{1}{2}$ and $\cos 135^\circ = -\frac{\sqrt{2}}{2}$, it follows that

$$700 = 500 + 200 \leq AC^2 \leq 500 + 200\sqrt{2} < 800.$$

70. (D)

Quadrilateral $KLMN$ is a square because it has 90° rotational symmetry, which implies that each pair of adjacent sides is congruent and perpendicular.

Since $KLMN$ is a square, its area is $(NK)^2$. Note that $m(\angle NAK) = 150^\circ$. By the Law of Cosines,

$$(NK)^2 = 4^2 + 4^2 - 2(4)(4)\left(-\frac{\sqrt{3}}{2}\right) = 32 + 16\sqrt{3}.$$

71. (D)

Let $\beta = \pi - \alpha$. Apply the Law of Cosines to ΔABC to obtain

$$(AC)^2 = 8^2 + 5^2 - 2(8)(5)\cos\beta = 89 - 80\cos\beta.$$

Thus $AC < 7$ if and only if

$$89 - 80\cos\beta < 49,$$

that is, if and only if

$$\cos\beta > \frac{1}{2}.$$

Therefore we must have $0 < \beta < \frac{\pi}{3}$, and the requested probability is $\frac{\pi/3}{\pi} = \frac{1}{3}$.

72. (C)

Since $6^2 + 8^2 = 100 > 9^2$, the triangle is acute, so that (C) is a correct choice. A check of (D) by the Law of Sines or the Law of Cosines shows that it is incorrect. (B) is obviously incorrect by the Law of Sines.

73. (D)

Let θ be the angle opposite the side of length c. Now

$$\begin{aligned}(a+b+c)(a+b-c) &= 3ab \\ (a+b)^2 - c^2 &= 3ab \\ a^2+b^2-ab &= c^2.\end{aligned}$$

But

$$a^2+b^2-2ab\cos\theta = c^2,$$

so that $ab = 2ab\cos\theta$, $\cos\theta = \frac{1}{2}$, and $\theta = 60^\circ$.

74. (E)

By the law of cosines, $5^2 = 4^2 + 6^2 - 2\times 4\times 6\times\cos\angle D$, hence $\cos\angle D = 27/48 = \cos\angle RFS$. So

$$RS^2 = 5^2 + \left(7\frac{1}{2}\right)^2 - 2(5)\left(7\frac{1}{2}\right)\left(\frac{27}{48}\right)$$

giving $RS = 6\frac{1}{4}$.

75. (B)

Let $x = PA = PA'$ and $y = QA = QA'$. Apply the Law of Cosines to $\Delta PBA'$ to obtain

$$x^2 = (3-x)^2 + 1^2 - 2(3-x)\cos 60^\circ$$

which leads to $x = \frac{7}{5}$. Consider $\Delta QCA'$ in a similar fashion to find $y = \frac{7}{4}$.

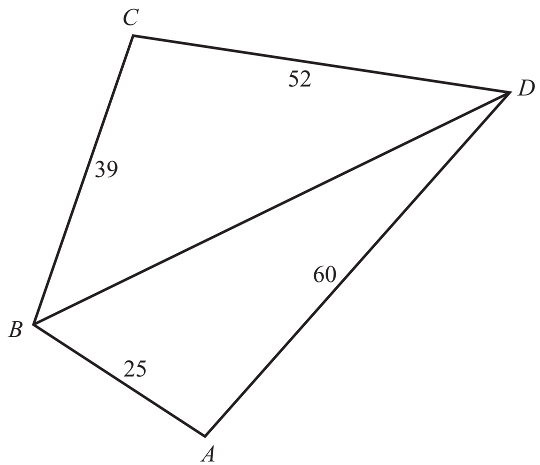
Now apply the Law of Cosines to ΔPAQ,

$$PQ^2 = x^2+y^2-2xy\cos 60^\circ = \frac{49}{25}+\frac{49}{16}-\frac{49}{20} = \frac{49\cdot 21}{400},$$

which leads to $PQ = (7/20)\sqrt{21}$.

76. (C)

As part of a cyclic quadrilateral, angles A and C are supplementary.



By the law of cosines on triangles ABD and CBD,

$$BD^2 = 39^2 + 52^2 - 2(39)(52)\cos C$$
$$BD^2 = 25^2 + 60^2 - 2(25)(60)\cos A.$$

Since C is the supplement of A, replacing $\cos C$ by its equal $-\cos A$ and subtracting gives

$$0 = 0 + (2(39)(52) + 2(25)(60))\cos A$$

so $\cos A = 0$, A is a right angle, and BD is the diameter of the circumscribing circle. Its length is $BD = \sqrt{39^2 + 52^2} = \sqrt{3^2 13^2 + 4^2 13^2} = 65$.

77. (A)

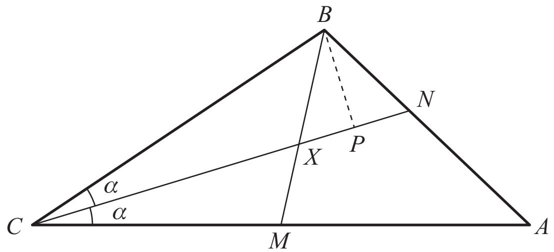
Let $\alpha = \angle ACN = \angle NCB$ and $x = BN$. Because ΔBXN is equilateral it follows that $\angle BXC = \angle CNA = 120°$, $\angle CBX = \angle BAC = 60° - \alpha$, and $\angle CBA = \angle BMC = 120° - \alpha$. Thus $\Delta ABC \sim \Delta BMC$ and $\Delta ANC \sim \Delta BXC$. Then

$$\frac{BC}{2} = \frac{BC}{AC} = \frac{MC}{BC} = \frac{1}{BC}$$

so $BC = \sqrt{2}$ and

$$\frac{CX + x}{2} = \frac{CN}{AC} = \frac{CX}{BC} = \frac{CX}{\sqrt{2}}$$

so $CX = (\sqrt{2}+1)x$.



Then the Law of Cosines applied to ΔBCX gives

$$\begin{aligned} 2 = BC^2 &= BX^2 + CX^2 - 2 \cdot BX \cdot CX \cdot \cos(120^\circ) \\ &= x^2 + (1+\sqrt{2})^2\, x^2 + (1+\sqrt{2})x^2 \\ &= (5+3\sqrt{2})x^2 \end{aligned}$$

and solving for x^2 gives the requested answer.

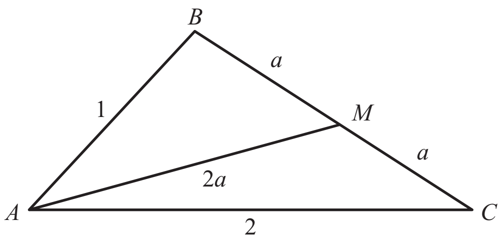
78. (C)

Let M be the midpoint of $\overline{BC}$, let $AM = 2a$ AM, and let $\theta = \angle AMB$. Then $\cos \angle AMC = -\cos\theta$. Applying the Law of Cosines to ΔABM and to ΔAMC yields, respectively,

$$a^2 + 4a^2 - 4a^2 \cos\theta = 1$$

and

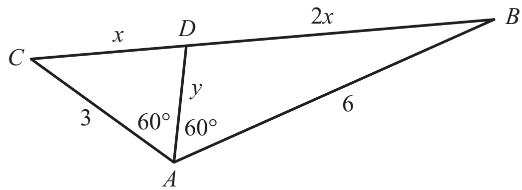
$$a^2 + 4a^2 + 4a^2 \cos\theta = 4.$$



Adding, we obtain $10a^2 = 5$, so $a = \sqrt{2}/2$ and $BC = 2a = \sqrt{2}$.

79. (A)

Let $AD = y$. Since AD bisects $\angle BAC$, we have $\frac{DB}{CD} = \frac{AB}{BC} = 2$, so we may set $CD = x$ and $DB = 2x$ as in the figure.



Applying the Law of Cosines to ΔCAD and ΔDAB, we have

$$x^2 = 3^2 + y^2 - 3y,$$
$$(2x)^2 = 6^2 + y^2 - 6y.$$

Subtracting 4 times the first equation from the second yields $0 = -3y^2 + 6y = -3y(y-2)$. Since $y \neq 0$, $y = 2$.

80. (E)

In hexagon $ABCDEF$, let $AB = BC = CD = a$ and let $DE + EF + FA = b$. Let O denote the center of the circle, and let r denote the radius. Since the arc BAF is one-third of the circle, it follows that $\angle BAF = \angle FOB = 120^\circ$. By using the Law of Cosines to compute BF two ways, we have $a^2 + ab + b^2 = 3r^2$. Let $\angle AOB = 2\theta$. Then $a = 2r\sin\theta$, and

$$\begin{aligned} AD &= 2r\sin(3\theta) \\ &= 2r\sin\theta \cdot (3 - 4\sin^2\theta) \\ &= a\left(3 - \frac{a^2}{r^2}\right) \\ &= 3a\left(1 - \frac{a^2}{a^2+ab+b^2}\right) \\ &= \frac{3ab(a+b)}{a^2+ab+b^2}. \end{aligned}$$

Substituting $a = 3$ and $b = 5$, we get $AD = \frac{360}{49}$, so $m + n = 409$.

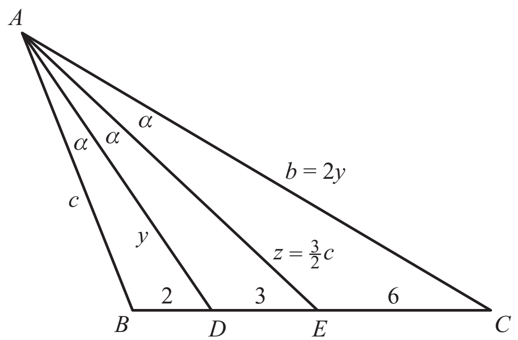
81. (A)

In the figure let $\angle BAC = 3\alpha$, $c = AB$, $y = AD$, $z = AE$, and $b = AC$. Then by the angle bisector theorem

$$\frac{c}{2} = \frac{2}{3} \quad \text{and} \quad \frac{y}{b} = \frac{1}{2}$$

so

$$z = \frac{3}{2}c, b = 2y.$$



Using the law of cosines in ΔADB, ΔAED, and ΔACE, respectively, yields

$$\frac{c^2 + y^2 - 4}{2cy} = \frac{\frac{9}{4}c^2 + y^2 - 9}{3cy} = \frac{\frac{9}{4}c^2 + 4y^2 - 36}{6cy}.$$

The equality of the first and second expressions implies

$$3c^2 - 2y^2 = 12.$$

The equality of the first and third expressions implies

$$3c^2 - 4y^2 = -96.$$

Solving these equations for c^2 and y^2 yields

$$c^2 = 40, y^2 = 54.$$

Thus the sides are

$$AB = c = 2\sqrt{10} \approx 6.3,$$

$$AC = b = 2y = 2\sqrt{54} = 6\sqrt{6} \approx 14.7,$$

$$BC = 11.$$

82. (A)

Let the angles of the triangle be $\alpha - \delta, \alpha, \alpha + \delta$. Then $3\alpha = \alpha - \delta + \alpha + \alpha + \delta = 180^\circ$, so $\alpha = 60^\circ$. There are three cases depending on which side is opposite to the 60° angle. In each case, the Law of Cosines can be used to solve for the unknown side.

If the unknown side is opposite the 60° angle, then

$$x^2 = 4^2 + 5^2 - 2 \cdot 4 \cdot 5 \cdot \cos 60^\circ = 21,$$

so $x = \sqrt{21}$.

If the side of length 5 is opposite to the 60° angle, then

$$5^2 = x^2 + 4^2 - 2 \cdot 4 \cdot x \cdot \cos 60^\circ = x^2 - 4x + 16,$$

and the positive solution is $2 + \sqrt{13}$.

If the side of length 4 is opposite the 60° angle, then

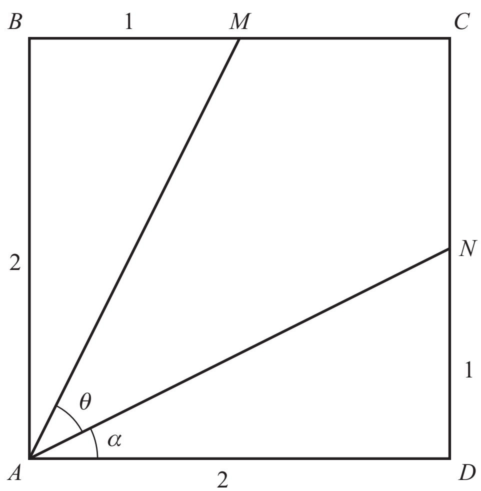
$$4^2 = x^2 + 5^2 - 2 \cdot x \cdot 5 \cdot \cos 60^\circ = x^2 - 5x + 25,$$

which has no real solutions.

The sum of all possible side lengths is $2 + \sqrt{13} + \sqrt{21}$. The requested sum is $2 + 13 + 21 = 36$.

83. (B)

We may suppose that the sides of the square have length 2, so that $BM = ND = 1$. Then



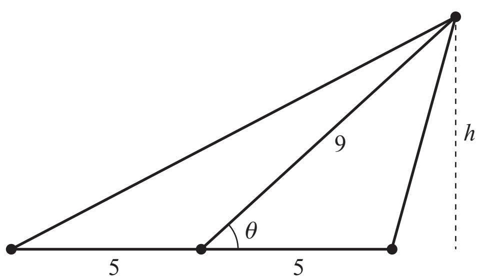
$$\begin{aligned}\sin\theta &= \sin\left(\frac{\pi}{2} - 2\alpha\right) = \cos 2\alpha \\ &= 2\cos^2\alpha - 1 \\ &= 2\left(\frac{2}{\sqrt{5}}\right)^2 - 1 \\ &= \frac{3}{5}.\end{aligned}$$

OR

One may express area(ΔAMN) in terms of $\sin\theta$, find area (ΔAMN) again numerically by subtracting the areas of other (right) triangles from the area of the square, and then solve for $\sin\theta$.

84. (D)

The area of a triangle equals one half the product of two sides and the sine of the included angle. Because the median divides the base in half, it partitions the triangle in two triangles with equal areas. Thus $\frac{1}{2} \cdot 5 \cdot 9 \sin\theta = 15$ and $\sin\theta = \frac{2 \cdot 15}{5 \cdot 9} = \frac{2}{3}$.



OR

The altitude h to the base forms a right triangle with the median as its hypotenuse, and thus $h = 9\sin\theta$. Hence the area of the original triangle is $\frac{1}{2} \cdot 10h = \frac{1}{2} \cdot 10 \cdot 9 \sin\theta = 30$ so $\sin\theta = \frac{2 \cdot 30}{10 \cdot 9} = \frac{2}{3}$.

85. (B)

If $MNPQ$ is convex, then A is the sum of the areas of the triangles into which $MNPQ$ is divided by diagonal $\overline{MP}$, so that

$$A = \frac{1}{2}ab\sin N + \frac{1}{2}cd\sin Q.$$

Similarly, dividing $MNPQ$ with the diagonal NQ yields

$$A = \frac{1}{2}ad \sin M + \frac{1}{2}bc \sin P.$$

We show below that these two equations for A hold also if $MNPQ$ is not convex. Therefore, in any case,

$$A \leq \frac{1}{4}(ab + cd + ad + bc) = \frac{a+c}{2} \cdot \frac{b+d}{2}.$$

The inequality is an equality if and only if

$$\sin M = \sin N = \sin P = \sin Q = 1,$$

i.e., if and only if $MNPQ$ is a rectangle.

If $MNPQ$ is not convex, for example if the interior angle Q of the quadrilateral $MNPQ$ is greater than 180°, then A is the difference

$$\text{Area of } \Delta MNP - \text{Area of } \Delta MQP$$

so that

$$\begin{aligned} A &= \frac{1}{2}ab \sin N - \frac{1}{2}cd \sin \angle PQM \\ &= \frac{1}{2}ab \sin N - \frac{1}{2}cd \sin(360^\circ - \angle MQP) \\ &= \frac{1}{2}ab \sin N + \frac{1}{2}cd \sin \angle MQP. \end{aligned}$$

86.(C)

The area of a regular polygon is $\frac{1}{2} \times$ perimeter $\times$ apothem. Here the apothem is $R\cos(\frac{180}{n})$ and the perimeter is $n \times 2R\sin(\frac{180}{n})$. Therefore $3R^2 = \frac{1}{2}R\cos(\frac{180}{n}) \times 2nR\sin(\frac{180}{n})$ giving $\frac{6}{n} = 2\sin(\frac{180}{n})\cos(\frac{180}{n}) = \sin(\frac{360}{n})$ where n is a positive integer equal to or greater than 3. Of the possible angles the only one whose sine is a rational number is 30° and so $n = 12$. To check, note that $\frac{6}{12} = \frac{1}{2} = \sin(\frac{360}{12})$.

87. (D)

By the Law of Sines, $\frac{OB}{\sin\angle AOB} = \frac{AB}{\sin\angle AOB} = \frac{1}{1/2}$, so $OB = 2\sin\angle OAB \leq 2\sin 90^\circ = 2$, with equality if and only if $\angle OAB = 90^\circ$.

88. (B)

Let s denote the required side. Then the Law of Sines gives

$$\frac{s}{\sin 30^\circ} = \frac{8}{\sin 45^\circ}$$

or

$$s = \frac{8 \sin 30^\circ}{\sin 45^\circ} = \frac{8(1/2)}{(\sqrt{2}/2)} = 4\sqrt{2}.$$

89. (B)

The center of the circle is not X since $2\angle BAC \neq \angle BXC$. Thus $\overline{AD}$ bisects $\angle BXC$ and $\angle BAC$. (Why?) Since $\angle ABD$ is inscribed in a semicircle, $\angle ABD = 90^\circ$, and thus

$$AB = AD \cdot \cos \angle BAD = 1 \cdot \cos\left(\frac{1}{2}\angle BAC\right) = \cos 6^\circ.$$

Also,

$$\angle AXB = 180^\circ - \angle DXB = 180^\circ - \frac{36^\circ}{2} = 162^\circ.$$

Since the sum of the angles in ΔAXB is 180°,

$$\angle AXB = 180^\circ - (162^\circ + 6^\circ) = 12^\circ.$$

By the Law of Sines,

$$\frac{AB}{\sin \angle AXB} = \frac{AX}{\sin \angle ABX}$$

so

$$\frac{\cos 6^\circ}{\sin 162^\circ} = \frac{AX}{\sin 12^\circ}.$$

Since $\sin 162^\circ = \sin 18^\circ$, we have

$$AX = \frac{\cos 6^\circ \sin 12^\circ}{\sin 18^\circ} = \cos 6^\circ \sin 12^\circ \csc 18^\circ.$$

90. (B)

By the Law of Sines,

$$\frac{27}{\sin A} = \frac{48}{\sin 3A}.$$

Using the identity

$$\sin 3A = 3 \sin A - 4 \sin^3 A$$

we have

$$\frac{48}{27} = \frac{16}{9} = \frac{\sin 3A}{\sin A} = 2 - 4 \sin^2 A.$$

Solving for $\sin A$ gives $\sin A = \sqrt{11}/6$ and $\cos A = 5/6$. ($\cos A$ cannot be negative since $0 < 3A < 180^\circ$). Again by the Law of Sines,

$$\frac{b}{\sin(180^\circ - 4A)} = \frac{27}{\sin A}$$

or

$$b = \frac{27 \sin 4A}{\sin A}.$$

Since

$$\sin 4A = 2 \sin 2A \cos 2A = 4 \sin A \cos A(\cos^2 A - \sin^2 A)$$

we have

$$b = 27 \cdot 4 \cdot \frac{5}{6} \left(\frac{25 - 11}{36} \right) = 35.$$

91. (A)

Construct line CP parallel to EF and intersecting AB at P. Then

$$\frac{AP}{AC} = \frac{AF}{AE}.$$

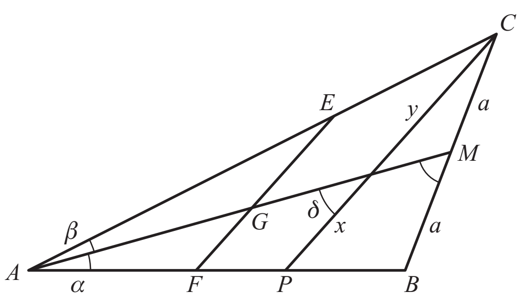
That is,

$$\frac{AP}{16} = \frac{AF}{2AF},$$

so

$$AP = 8.$$

Let a, x, y, α, β, δ, and θ be as shown in the adjoining diagram. The desired ratio EG/GF is the same as y/x which we now determine.



By the law of sines,

$$\frac{a}{\sin \alpha} = \frac{12}{\sin \theta} \text{ and } \frac{a}{\sin \beta} = \frac{16}{\sin(180^\circ - \theta)} = \frac{16}{\sin \theta}.$$

Hence

$$\frac{\sin\beta}{\sin\alpha} = \frac{3}{4}.$$

Moreover,

$$\frac{x}{\sin\alpha} = \frac{8}{\sin\delta} \quad \text{and} \quad \frac{y}{\sin\beta} = \frac{16}{\sin(180^\circ - \delta)} = \frac{16}{\sin\delta}.$$

Hence

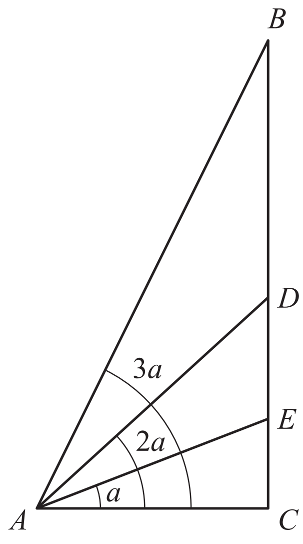
$$\frac{y}{x} = 2\frac{\sin\beta}{\sin\alpha} = \frac{3}{2}.$$

92. (B)

Let E denote the point on $\overline{BC}$ for which $\overline{AE}$ bisects $\angle CAD$. Because the answer is not changed by a similarity transformation, we may assume that $AC = 2\sqrt{5}$ and $AD = 3\sqrt{5}$. Apply the Pythagorean Theorem to triangle ACD to obtain $CD = 5$, then apply the angle bisector theorem to ΔCAD to obtain $CE = 2$ and $ED = 3$. Let $x = DB$. Apply the Pythagorean Theorem to ΔACE to obtain $AE = \sqrt{24}$, then apply the angle bisector theorem to ΔEAB to obtain $AB = (x/3)\sqrt{24}$. Now apply the Pythagorean Theorem to ΔABC to get

$$\left(2\sqrt{5}\right)^2 + (x+5)^2 = \left(\frac{x}{3}\sqrt{24}\right)^2$$

from which it follows that $x = 9$. Hence $BD/DC = 9/5$, and $m + n = 14$.



OR

Denote by a the measure of angle CAE. Let $AC = 2u$, and $AD = 3u$. It follows that $CD = \sqrt{5}u$. We may assume $BD = \sqrt{5}$ (Otherwise, we could simply modify the triangle with a similarity transformation.) Hence, the ratio CD/BD we seek is just u. Since $\cos 2a = \frac{2}{3}$, we have $\sin a = \frac{1}{\sqrt{6}}$. Applying the Law of Sines in triangle ABD yields

$$\frac{\sin D}{AB} = \frac{\sin a}{\sqrt{5}} = \frac{2/3}{\sqrt{(2u)^2 + \left(\sqrt{5}\,(1+u)\right)^2}} = \frac{1/\sqrt{6}}{\sqrt{5}}.$$

Solve this for u to get

$$\begin{aligned} 2\sqrt{5}\sqrt{6} &= 3\sqrt{4u^2 + 5(1 + 2u + u^2)} \\ 120 &= 9(9u^2 + 10u + 5) \\ 0 &= 27u^2 + 30u - 25 \\ 0 &- (9u - 5)(3u + 5) \end{aligned}$$

so $u = 5/9$ and $m + n = 14$.

93. (A)

Let $a = BC$, $b = AC$, and $c = AB$. Let D, E, and F be the feet of the perpendiculars from I to $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively. Because $\overline{BF}$ and $\overline{BD}$ are common tangent segments to the incircle of ΔABC, it follows that $BF = BD$. Similarly, $CD = CE$ and $AE = AF$. Thus

$$\begin{aligned} 2 \cdot BD = BD + BF &= (BC - CD) + (AB - AF) \\ &= BC + AB - (CE + AE) \\ &= a + c - b = 25 + 27 - 26 = 26, \end{aligned}$$

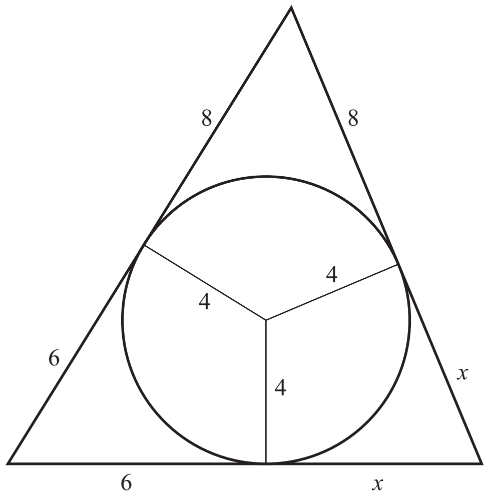
so $BD = 13$. Let $s = \frac{1}{2}(a + b + c) = 39$ be the semiperimeter of ΔABC and $r = DI$ the inradius of ΔABC. The area of ΔABC is equal to rs and also equal to $\sqrt{s(s-a)(s-b)(s-c)}$ by Heron's formula. Thus

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s} = \frac{14 \cdot 13 \cdot 12}{39} = 56.$$

Finally, by the Pythagorean Theorem applied to the right triangle BDI, it follows that $BI^2 = DI^2 + BD^2 = r^2 + BD^2 = 56 + 13^2 = 56 + 169 = 225$ so $BI = 15$.

94. (B)

Let x be as shown, and s the semi-perimeter of the triangle. Denoting the sides of the triangle by a, b, c we observe that $a = 8 + 6 = 14$, $b = 8 + x$, $c = x + 6$.



$$2s = a + b + c - 2x + 28 \text{ so } s = x + 14.$$

On the one hand, the area of the triangle is half the product of the perimeter and the radius of the inscribed circle; on the other hand, it is given in terms of s so that

$$\text{area} = rs = 4(x + 14) = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{48x(x+14)}$$

or

$$(x + 14)^2 = 3x(x + 14).$$

Therefore $x = 7$, and the shortest side is $c = 6 + 7 = 13$.

OR

$$\sin\frac{B}{2} = \frac{4}{\sqrt{4^2 + 6^2}} = \frac{2}{\sqrt{13}}, \quad \cos\frac{B}{2} = \frac{3}{\sqrt{13}},$$
$$\sin B = 2\sin\frac{B}{2}\cos\frac{B}{2} = \frac{4}{\sqrt{13}}.$$

Similarly $\sin C = \frac{4}{5}$. Using the Law of Sines

$$\frac{\sin B}{b} = \frac{\sin C}{c}, \qquad \frac{12/13}{8+x} = \frac{4/5}{6+x}.$$

Thus $x = 7$.

95. (D)

Let $ABCD$ be a trapezoid with $\overline{AB}||\overline{CD}$. Let E be the point on $\overline{CD}$ such that $CE = AB$. Then $ABCE$ is a parallelogram. Set $AB = a$, $BC = b$, $CD = c$, and $DA = d$. Then the side lengths of ΔADE are b, d, and $c - a$. If one of b or d is equal to 11, say $b = 11$ by symmetry, then $d + (c - a) \le 7 + (5 - 3) < 11 = d$, which contradicts the triangle inequality. Thus $c = 11$. There are three cases to consider, namely, $a = 3$, $a = 5$, and $a = 7$.

If $a = 3$, then ΔADE has side lengths 5, 7, and 8 and by Heron's formula its area is

$$\frac{1}{4}\sqrt{(5+7+8)(7+8-5)(8+5-7)(5+7-8)} = 10\sqrt{3}.$$

The area of ΔAEC is $3/8$ of the area of ΔADE, and triangles ABC and AEC have the same area. It follows that the area of the trapezoid is $\frac{1}{2}(35\sqrt{3})$.

If $a = 5$, then ΔADE has side lengths 3, 6, and 7, and area

$$\frac{1}{4}\sqrt{(3+6+7)(6+7-3)(7+3-6)(3+6-7)} = 4\sqrt{5}.$$

The area of ΔAEC is $5/6$ of the area of ΔADE, and triangles ABC and AEC have the same area. It follows that the area of the trapezoid is $\frac{1}{3}(32\sqrt{5})$.

If $a = 7$, then ΔADE has side lengths 3, 4, and 5. Hence we have a right trapezoid with height 3 and base lengths 7 and 11. It has area $\frac{1}{2}(3(7+11)) = 27$.

The sum of the three possible areas is $\frac{35}{2}\sqrt{3} + \frac{32}{3}$. Hence $r_1 = \frac{35}{2}$, $r_2 = \frac{32}{3}$, $r_3 = 27$, $n_1 = 3$, $n_2 = 5$, and $r_1 + r_2 + r_3 + n_1 + n_2 = \frac{35}{2} + \frac{32}{3} + 27 + 3 + 5 = 63 + \frac{1}{6}$. Thus the required integer is 63.

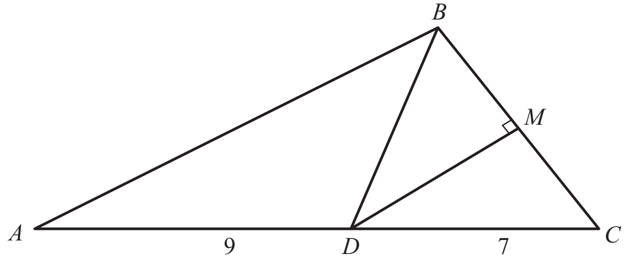
96. (C)

By Heron's formula the area of ΔABC is $\sqrt{(21)(8)(7)(6)}$, which is 84, so the altitude from vertex A is $2(84)/14 = 12$. The midpoint D divides $\overline{BC}$ into two segments of length 7, and the bisector of $\angle BAC$ divides $\overline{BC}$ into segments of length $14(13/28) = 6.5$ and $14(15/28) = 7.5$ (since the angle bisector divides the opposite side into lengths proportional to the remaining two sides). Thus the triangle ΔADE has base $DE = 7 - 6.5 = 0.5$ and altitude 12, so its area is 3.

97. (D)

By the angle bisector theorem, $AB/BC = 9/7$. Let $AB = 9x$ and $BC = 7x$, let $m\angle ABD = m\angle CBD = \theta$, and let M be the midpoint of $\overline{BC}$. Since M is on the perpendicular bisector of $\overline{BC}$, we have $BD = DC = 7$. Then

$$\cos\theta = \frac{\frac{7x}{2}}{7} = \frac{x}{2}.$$



Applying the Law of Cosines to ΔABD yields

$$9^2 = (9x)^2 + 7^2 - 2(9x)(7)\left(\frac{x}{2}\right),$$

from which $x = 4/3$ and $AB = 12$. Apply Heron's formula to obtain the area of triangle ABD as $\sqrt{14 \cdot 2 \cdot 5 \cdot 7} = 14\sqrt{5}$.

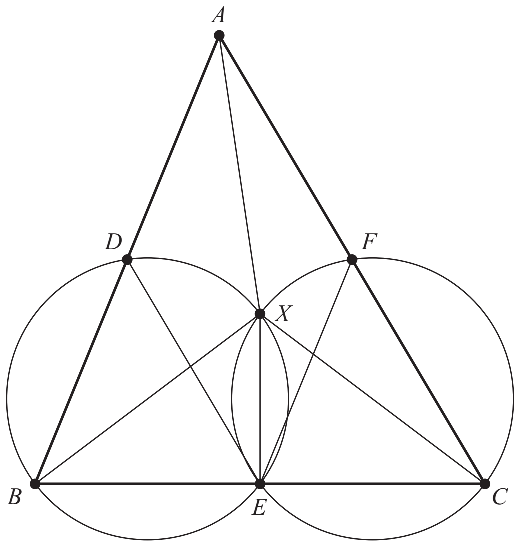
98. (C)

Because $\overline{DE}$ is parallel to $\overline{AC}$ and $\overline{EF}$ is parallel to $\overline{AB}$ it follows that $\angle BDE = \angle BAC = \angle EFC$.

By the inscribed angle theorem, $\angle BDE = \angle BXE$ and $\angle EFC = \angle EXC$. Therefore $\angle BXE = \angle EXC$. Furthermore $BE = EC$, so by the angle bisector theorem $XB = XC$.

Note that $\angle BXC = 2\angle BXE = 2\angle BDE = 2\angle BAC$.

By the inscribed angle theorem, it follows that X is the circumcenter of ΔABC, so $XA = XB = XC = R$ the circumradius of ΔABC.



Let $a = BC$, $b = AC$, and $c = AB$. The area of ΔABC equals $\frac{1}{4R}(abc)$, and by Heron's formula it also equals $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$. Thus

$$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} = \frac{13 \cdot 14 \cdot 15}{4\sqrt{21 \cdot 8 \cdot 7 \cdot 6}} = \frac{65}{8},$$

and $XA + XB + XC = 3R = \frac{195}{8}$.

99. (E)

Square $ABCD$ has side length 14. Let F and G be the feet of the altitudes from E in ΔABE and ΔCDE, respectively. Then $FG = 14$, $EF = 2 \cdot \frac{105}{14} = 15$ and $EG = 2 \cdot \frac{91}{14} = 13$. Because ΔEFG is perpendicular to the plane of $ABCD$, the altitude to $\overline{FG}$ is the altitude of the pyramid. By Heron's formula, the area of ΔEFG is $\sqrt{(21)(6)(7)(8)} = 84$, so the altitude to $\overline{FG}$ is $2 \cdot \frac{84}{14} = 12$. Therefore the volume of the pyramid is $\left(\frac{1}{3}\right)(196)(12) = 784$.

100. (C)

(See www.jamestanton.com/wp-content/uploads/2012/03/Cool-Math-Essay_April-2014_On-Cyclic-Quadrilaterals.pdf for some help with this solution.)

Because $AB + CD = 21 = BC + DA$ it follows that $ABCD$ always has an inscribed circle tangent to its four sides. Let r be the radius of the inscribed circle.

Using square brackets to denote area, we have

$$[ABCD] = \frac{1}{2}r(AB + BC + CD + DA) = 21r.$$

Thus the radius is maximum when the area is maximized.

Note that

$$[ABC] = \frac{1}{2} \cdot 14 \cdot 6 \sin B = 63 \sin B$$

and

$$[ACD] = \frac{1}{2} \cdot 12 \cdot 7 \sin D = 42 \sin D.$$

On the one hand,

$$\begin{aligned}[ABCD]^2 &= ([ABC] + [ACD])^2 \\ &= 63^2 \sin^2 B + 42^2 \sin^2 D + 2 \cdot 42 \cdot 63 \sin B \sin D.\end{aligned}$$

On the other hand, by the Law of Cosines,

$$AC^2 = 12^2 + 7^2 - 2 \cdot 7 \cdot 12 \cos D = 14^2 + 9^2 - 2 \cdot 9 \cdot 14 \cos B.$$

Thus

$$\begin{aligned}21^2 &= \left(\frac{2 \cdot 26 + 2 \cdot 16}{4}\right)^2 = \left(\frac{14^2 - 12^2 + 9^2 - 7^2}{4}\right)^2 \\ &= (63 \cos B - 42 \cos D)^2 \\ &= 63^2 \cos^2 B + 42^2 \cos^2 D - 2 \cdot 42 \cdot 63 \cos B \cos D.\end{aligned}$$

Adding these two identities yields

$$\begin{aligned}[ABCD]^2 + 21^2 = 63^2 + 42^2 - 2 \cdot 42 \cdot 63 \cos(B + D) \leq 63^2 \\ + 42^2 - 2 \cdot 42 \cdot 63 = (63 + 42)^2 = 105^2,\end{aligned}$$

with equality if and only if $B + D = \pi$ (that is, $ABCD$ is cyclic). Therefore

$$[ABCD]^2 \leq 105^2 - 21^2 = 21^2(5^2 - 1) = 42^2 \cdot 6,$$

and the required maximum $r = \frac{1}{21}[ABCD] = 2\sqrt{6}$.

OR

Establish as in the first solution that r is maximized when the area is maximized. Bretschneider's formula, which generalizes Brahmagupta's formula, states that the area of an arbitrary quadrilateral with side lengths a, b, c, and d, is given by

$$\sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\theta},$$

where $s = \frac{1}{2}(a+b+c+d)$ and θ is half the sum of either pair of opposite angles. For a, b, c, and d fixed, the area is maximized when $\cos\theta = 0$. Thus the area is maximized when $\theta = \frac{1}{2}\pi$ that is, when the quadrilateral is cyclic. In this case, the area equals $\sqrt{7 \cdot 12 \cdot 14 \cdot 9} = 42\sqrt{6}$ and the required maximum radius $r = \frac{1}{21} \cdot 42\sqrt{6} = 2\sqrt{6}$.

Appendix: Ten Problem-Solving Strategies

Here, in brief, are the ten problem solving strategies that apply particularly well to solving competition mathematics problems. Please go to www.maa.org/ci for full explanations of these strategies and further practice problems.

Remember that these strategies come after conducting the first, and most important, step in problem solving:

> **STEP 1**: Read the question, have an emotional reaction to it, take a deep breath, and then reread the question.

The strategies:

STRATEGY 1: Engage in Successful Flailing

Often one can often identify to which topic a challenge belongs—this question is about right triangles, or this question is about repeating decimals—but still have no clue as to how to start on the challenge. The thing to do then is to engage in direct flailing.

To do this, think about everything you know about right triangles or about repeating decimals. Read the question out loud and then describe it again in different words. Draw a picture. Try an example with actual numbers. Mark something on the diagram. And so on. Do everything you can think of that is relevant to the content at hand. In doing so, a step forward with the problem often emerges.

See the FEATURED PROBLEMS of sections 1 and 13 of this guide to see this strategy in action (as well as www.maa.org/ci).

STRATEGY 2: Do Something!

Innovation in research and business is not easy. Many times one is stymied and not even able to conceive of any next step to take. This can happen in mathematics problem solving too.

Perhaps the most powerful of problem solving techniques, in mathematics and in life, is to simply DO SOMETHING, no matter how irrelevant or unhelpful it may seem. Unblock the emotional or intellectual impasse by writing something—most anything—on the page. Turn the diagram upside-down and shade in a feature that now stands out to you. Underline some words in the question statement. Just do something!

See the FEATURED PROBLEMS of sections 10 and 17 of this guide to see this strategy in action (as well as www.maa.org/ci).

STRATEGY 3: Engage in Wishful Thinking

As I tell my students, if there is something in life you want, make it happen. (And then be willing to handle the consequences of your actions with care and grace!) If you wish, for example, that "+4" appeared on the left side of an equation, then make it happen by adding a 4 to the left (with the consequence of adding a 4 four to the right as well).

A beautiful problem-solving technique is to simply change the problem to what you wish it could be.

See the FEATURED PROBLEM of section 12 of this guide to see this strategy in action (as well as www.maa.org/ci).

STRATEGY 4: Draw a Picture

The upper-school mathematics curriculum tends to drill down to analytic techniques and visualization is put to the side. But visual thinking can unlock deep insight. Don't underestimate the power of drawing a picture for the problem. (For example, perhaps the best way to "prove" that $1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1 = 5^2$ is to draw in the diagonals of a 5×5 array of dots.)

See the FEATURED PROBLEMS of sections 1, 3, 6, and 14 of this guide to see this strategy in action (as well as www.maa.org/ci).

STRATEGY 5: Solve a Smaller Version of the Same Problem

A large, complex task can be made comprehensible by examining a smaller, analogous version.

For example, if there are areas to find, can I use symmetry to my advantage and work out the area of just one piece? If there is a list, instead of finding the hundredth number, can I just find the third to get a feel for things? If I make a first move, is the rest of the game just a smaller version of the original game? Is the count of things I don't want easier to compute than the count of things I do want? And so on.

See the FEATURED PROBLEM of section 18 of this guide to see this strategy in action (as well as www.maa.org/ci).

STRATEGY 6: Eliminate Incorrect Choices

Eliminating what cannot be right helps determine that which must be correct.

Must the answer be even or odd?

Must the answer be large or small?

Should the answer involve π?

Should the graph be straight or concave?

Should the numbers increase or decrease?

With how many zeros should the answer end?

And so on.

See www.maa.org/ci for some examples of this strategy in action.

STRATEGY 7: Perseverance is Key

What impression of the mathematical pursuit do we give students? Answers are pre-known (they are at the back of the book or are in the teacher's mind), all can be accomplished in a fixed amount of time (quizzes and exams are usually timed), and the goal is to follow a pre-set intellectual path (as dictated by a curriculum). Open research and problem-solving tasks, on the other hand, usually possess none of these features. Persistence and perseverance are key skills absolutely vital for any success in original intellectual endeavors.

See the FEATURED PROBLEM of sections 16 of this guide to see this strategy in action (as well as www.maa.org/ci).

STRATEGY 8: Second-Guess the Author

If a problem feels staged, then use that to your advantage!

The number 131 mentioned is prime. Coincidence?

The number $203 = 7 \times 29$ only has one two-digit factor. Is that helpful?

Why are we rolling the dice first and then tossing the coin? Is that order important?

Why are we focusing on factors of 2 and 5? Is it because $2 \times 5 = 10$?

Hmm. $2^{10} = 1024$ is very close to 1000.

Both equations involve $\sqrt{2x}$. Is that a coincidence?

Why the arc of a circle? Is that because we want the ship to stay the same distance from a certain point?

Why a parabola? Do we need equal distances of some kind?

See www.maa.org/ci for some examples of this strategy in action.

STRATEGY 9: Avoid Hard Work

No one enjoys hard computation or a tedious grind through formulas and equations. Brute-force work should be undertaken only as a last resort. Do what a mathematician does—think long and hard to devise a creative, elegant approach that avoids hard work!

Do I really need to work out $2^{21} - 1$ to see if it factors?

See www.maa.org/ci for some examples of this strategy in action.

STRATEGY 10: Go to Extremes

It is fun to be quirky and push ideas to the edge. Taking the parameters of a problem to an extreme can give insight to the workings of the situation described. And such insight can often illuminate a path for success.

If the escalator had zero velocity (that is, it wasn't moving) how many steps would I have to climb? What if it was moving really fast?

What if the number had very few factors? What if the number were prime?

What if everyone had the same age?

What if the point P was on top of point Q? What if point P was very far away?

What if cake's temperature was a billion degrees?

What if x was really close to zero?

What if the circle was so big that it practically a straight line?

See the FEATURED PROBLEM of section 5 of this guide to see this strategy in action (as well as www.maa.org/ci).