Problems and Solutions in Plane Trigonometry

For the use of Colleges and Schools

By

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LaTeX Edition

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Preface

This book collects together the problems set out at end of each chapter in the author's Textbook of Plane Trigonometry (for the use of Colleges and Schools) along with the possible solutions, which are linked with an explanation of the sort of reasoning used in order to arrive at one of the answers. In many cases, several answers are given for one question. The result is a book which can be used independently of the main volume. This book helps in acquiring a better understanding of the basic principles of Plane Trigonometry and in revising a large amount of the subject matter quickly.

The Keys already issued to some of the Author's works have been found very useful by affording assistance to private students, and by saving the labour and time of teachers ; and this has led to the issue of the present volume. Care has been taken, as in the former Keys, to present the solutions in a simple natural manner, in order to meet the difficulties which are most likely to arise, and to render the work intelligible and instructive.

St. John's College, Isaac Todhunter.
September, 1874.

LaTeX Edition

In this edition, the entire manuscript was typeset using the LaTeX 2_ε document processing system originally developed by *Leslie Lamport*, based on TeX typesetting system created by *Donald Knuth*. The typesetting software used the XeLaTeX distribution.

It is also to be noticed, that each Example, or Problem (1000+) is here *enunciated* at the head of its Solution as well as all the relevant articles (100+) are part of the appendix; so that the book, though a fitting *Companion* to the textbook, is not inseparable from it, but may be used, as a *Book of Exercises*, with any other treatise on *Plane Trigonometry.*

I am grateful for this opportunity to put the materials into a consistent format, and to correct errors in the original publication that have come to my attention. I am highly indebted to *Chandra Shekhar Kumar* for the fruitful discussions which led to the idea of masterminding this entire project. He helped me put hundreds of pages of typographically difficult material into a consistent digital format.

The process of compiling this book has given me an incentive to improve the layout, to double-check almost all of the mathematical rendering, to correct all known errors, to improve the original illustrations by redrawing them with Till Tantau's marvellous Ti*k*Z. Thus the book now appears in a form that we hope will remain useful for at least another generation.

May, 2016. Neeru Singh.

List of Chapters

CHAPTER I
Measurement of Angles by Degrees or Grades

Problem 1. *The difference of two angles is 10 grades and their sum is 45 degrees; find each angle.*

Solution. Let x denote the number of degrees in the larger angle, and y the number of degrees in the smaller angle. Then, since 10 grades are equal to 9 degrees, $x - y = 9$; also $x + y = 45$: hence we obtain $x = 27$ and $y = 18$.

Problem 2. *Divide two-thirds of a right angle into two parts, such that the number of degrees in one part may be to the number of grades in the other part as 3 is to 10.*

Solution. In two-thirds of a right angle there are 60 degrees; let x denote the number of degrees in one part, then $60 - x$ denotes the number of degrees in the other part, therefore the number of grades in this part is $\frac{10}{9}(60 - x)$.
Hence

$$x : \frac{10}{9}(60 - x) :: 3 : 10; \quad \text{therefore } 10x = \frac{30}{9}(60 - x);$$

therefore $$9x = 3(60 - x); \text{ therefore } 12x = 180; \text{ therefore } x = 15.$$

Problem 3. *Divide half a right angle into two parts, such that the number of degrees in one part may be to the number of grades in the other part as 9 is to 5.*

Solution. In half a right angle there are 45 degrees; let x denote the number of degrees in one part, then $45 - x$ denotes the number of degrees in the other part, therefore the number of grades in this part is $\frac{10}{9}(45 - x)$. Hence

$$x : \frac{10}{9}(45 - x) :: 9 : 5; \quad \text{therefore } 5x = 10(45 - x);$$

therefore $$15x = 450; \quad \text{therefore } x = 30.$$

Problem 4. *Find the measure of $1'5''$ in decimals of a degree.*

Solution. $1'5'' = {\cdot}0105$ of a grade; $\frac{9}{10}$ of ${\cdot}0105 = {\cdot}00945$.

Problem 5. *Divide an angle of n degrees into two parts, one of which contains as many English minutes as the other does French.*

Solution. Let x denote the number of degrees in one part; then $n - x$ denotes the number of degrees in the other part. In x degrees there are $60x$ English minutes. In $n - x$ degrees there are $\frac{10}{9}(n - x)$ grades, and therefore $\frac{10}{9} \times 100(n - x)$ French minutes. Therefore

$$60x = \frac{1000}{9}(n - x);$$

therefore $$1540x = 1000n; \quad \text{therefore } 77x = 50n;$$

therefore $$x = \frac{50n}{77}, \quad \text{and } n - x = \frac{27n}{77}.$$

Problem 6. *If one-third of a right angle be assumed as the unit of angular measure, what number will represent 75° ?*

Solution. In one-third of a right angle there are 30 degrees; if this be taken as the unit of measurement an angle of 75 degrees must be denoted by $\frac{75}{30}$, that is by $\frac{5}{2}$, that is by $2\frac{1}{2}$.

Problem 7. *Determine the number of degrees in the unit of angular measure when an angle of $66\frac{2}{3}$ grades is represented by 20.*

Solution. Let x denote the number of grades in the unit. Then an angle of $66\frac{2}{3}$ grades is denoted by $\frac{66\frac{2}{3}}{x}$; and this is equal to 20. Therefore

$$20x = 66\frac{2}{3} = \frac{200}{3}; \quad \text{therefore } x = \frac{10}{3}.$$

Hence the number of degrees in the unit is $\frac{9}{10} \times \frac{10}{3}$, that is 3.

Problem 8. *The number of the sides of one equiangular polygon is two-thirds of the number of the sides of another; and the number of grades in an angle of the first equals the number of degrees in an angle of the second : find the angles.*

Solution. Let $3x$ denote the number of sides in the equiangular polygon which has the greater number of sides; then $2x$ denotes the number of sides in the other equiangular polygon. All the angles of the polygon of $2x$ sides are equal to $(4x - 4)$ right angles, that is to $(4x - 4)100$ grades; therefore each angle contains $\frac{(4x-4)100}{2x}$ grades. All the angles of the polygon of $3x$ sides are equal to $(6x - 4)$ right angles, that is to $(6x - 4)90$ degrees ; therefore each angle contains $\frac{(6x-4)90}{3x}$ degrees; therefore

$$\frac{(4x-4)100}{2x} = \frac{(6x-4)90}{3x};$$

therefore $(4x - 4)5 = (6x - 4)3$; therefore $2x = 8$; therefore $x = 4$. Thus one polygon has 8 sides and the other polygon has 12 sides.

Problem 9. *Show that an angle expressed in centesimal seconds will be transformed to sexagesimal by multiplying by the factor ·324.*

Solution. It is shown in *Art.* 9 (page 395) that an angle expressed in centesimal seconds is transformed to English seconds by multiplying by $\frac{81}{250}$; and $\frac{81}{250} = \frac{324}{1000}$.

Problem 10. *Compare the angles which contain the same number of English seconds as of French minutes.*

Solution. Suppose one angle to contain x English seconds, and another to contain x French minutes. The second angle then contains $100x$ French seconds, and therefore $\frac{81}{250} \times 100x$ English seconds. Hence the ratio of the former angle to the latter is that of 1 to $\frac{8100}{250}$, or of 1 to $\frac{162}{5}$, or of 5 to 162.

Problem 11. *Express in the French method $35°\,10'\,3''$.*

Solution.

$$\begin{array}{r|l} 60 & \ \ 3{\cdot}00 \\ \hline 60 & 10{\cdot}05 \\ \hline & \ \ \ {\cdot}1675 \end{array}$$

Thus $$35°10'3'' = 35° \cdot 1675.$$

$$\begin{array}{r} 35{\cdot}1675 \\ 3{\cdot}9075 \\ \hline 39{\cdot}0750 \end{array}$$

And $$39^g \cdot 0750 = 39^g 7' 50''.$$

Problem 12. *Express in the English method $69^g 22' 50''$.*

Solution. $69^g 22' 50'' = 69^g \cdot 225$.

$$\begin{array}{r} 69 \cdot 225\ \\ 6 \cdot 9225 \\ \hline 62 \cdot 3025 \\ 60 \\ \hline 18 \cdot 1500 \\ 60 \\ \hline 9 \cdot 00 \end{array}$$

CHAPTER II
Circular Measure of an Angle

Problem 1. *If D, G, C be respectively the number of degrees, grades, and units of circular measure in an angle, show that*

$$\frac{D}{90} = \frac{G}{100} = \frac{2C}{\pi}.$$

Solution. It is shown in *Art.* 8 (page 395) that $\dfrac{D}{90} = \dfrac{G}{100}$; and it is shown in *Art.* 22 (page 395) that $\dfrac{D}{180} = \dfrac{C}{\pi}$, so that $\dfrac{D}{90} = \dfrac{2C}{\pi}$. Therefore $\dfrac{D}{90} = \dfrac{G}{100} = \dfrac{2C}{\pi}$.

In fact the three expressions denote the same thing, namely the ratio of the angle considered to a right angle.

Problem 2. *Find the number of degrees in the angle subtended at the centre of a circle whose radius is* 10 *feet by an arc* 9 *inches long.*

Solution. The circular measure of the angle is $\dfrac{9}{10 \times 12}$, that is $\dfrac{3}{40}$. Therefore by *Art.* 22 (page 395), the number of degrees in the angle is $\dfrac{3}{40}$ of $\dfrac{180}{\pi}$.

Problem 3. *Find the circular measure of $5^\circ 37' 30''$.*

Solution. $5°37'30' = 337\frac{1}{2}$ minutes.

Thus the circular measure

$$= \frac{337\frac{1}{2}}{180 \times 60}\pi = \frac{675}{180 \times 60 \times 2}\pi = \frac{135}{180 \times 12 \times 2}\pi$$
$$= \frac{27}{36 \times 12 \times 2}\pi = \frac{\pi}{32}.$$

Problem 4. *Find the circular measure of $1^g 1'$.*

Solution. The angle contains 1·01 grades; therefore by *Art.* 24 (page 396), the circular measure is $\dfrac{1{\cdot}01}{200}\pi$, that is $\pi \times {\cdot}00505$.

Problem 5. *There are three angles; the circular measure of the first exceeds that of the second by $\dfrac{\pi}{10}$, the sum of the second and the third is* 30 *grades, and the sum of the first and the second is* 36 *degrees. Determine the three angles.*

Solution. Let x denote the number of degrees in the first angle, y the number in the second, and z the number in the third.

The circular measure of the first angle is $\frac{x\pi}{180}$, and the circular measure of the second is $\frac{y\pi}{180}$; therefore $\frac{x\pi}{180} - \frac{y\pi}{180} = \frac{\pi}{10}$; therefore $x - y = 18$.

The number of grades in the second angle is $\frac{10y}{9}$, and the number of grades in the third is $\frac{10z}{9}$; therefore $\frac{10y}{9} + \frac{10z}{9} = 30$; therefore $y + z = 27$.

Also $x + y = 36$.

From these three equations we have $x = 27,\ y = 9,\ z = 18$.

Problem 6. *Express five-sixteenths of a right angle in circular measure, in degrees and decimals of a degree, and in grades and decimals of a grade.*

Solution. The circular measure of a right angle is $\frac{\pi}{2}$; and therefore the circular measure of five-sixteenths of a right angle is $\frac{5}{16}$ of $\frac{\pi}{2}$, that is $\frac{5\pi}{32.}$

The number of degrees is $\frac{5}{16}$ of 90, that is $\frac{450}{16}$, that is 28·125.

The number of grades is $\frac{5}{16}$ of 100, that is $\frac{500}{16}$, that is 31·25.

Problem 7. *The angles of a triangle are in arithmetical progression, and the greatest is double the least : express the angles in degrees, in grades, and in circular measure.*

Solution. Let the numbers of degrees in the three angles be denoted respectively by $x - y,\ x$, and $x + y$. Then $x - y + x + x + y = 180$, that is $3x = 180$; therefore $x = 60$.

Also $x + y = 2(x - y)$; therefore $3y = x = 60$; therefore $y = 20$.

Hence in degrees the angles are denoted by 40, 60, and 80. Therefore in grades they will be denoted by $\frac{400}{9}$, $\frac{600}{9}$, and $\frac{800}{9}$. And in circular measure they will be denoted by $\frac{40\pi}{180}$, $\frac{60\pi}{180}$, and $\frac{80\pi}{180}$; that is by $\frac{2\pi}{9}$, $\frac{\pi}{3}$, and $\frac{4\pi}{9}$.

Problem 8. *The angles of a triangle are in arithmetical progression, and the number of degrees in the least is to the circular measure of the greatest as* 60 *is to* π : *find the angles.*

Solution. Let the numbers o degrees in the three angles be denoted respectively by $x - y,\ x$, and $x + y$. Then $x - y + x + x + y = 180$, that is $3x = 180$; therefore $x = 60$.

The circular measure of the greatest angle is $\frac{(x+y)\pi}{180}$; thus

$$x - y : \frac{(x+y)\pi}{180} :: 60 : \pi; \quad \text{therefore } (x-y)\pi = \frac{(x+y)\pi}{3};$$

therefore $$3(x - y) = x + y; \quad \text{therefore } y = \frac{x}{2} = 30.$$

Thus the angles are 30°, 60°, and 90°.

Problem 9. *Find the circular measure of an angle of an equiangular polygon of n sides.*

Solution. All the angles of the polygon are equal to $(2n-4)$ right angles, that is to $(2n-4)\frac{\pi}{2}$ in circular measures, that is to $(n-2)\pi$. Hence the circular measure of each angle is $\frac{(n-2)\pi}{n}$.

Problem 10. *Express in each system of angular measurement the angle between the long hand and the short hand of a watch at a quarter past twelve.*

Solution. During the quarter of an hour since twelve the long hand has described one-fourth of four right angles, that is a right angle. The short hand has described one-twelfth of this, that is $\frac{1}{12}$ of a right angle. Hence the angle between the hands at a quarter past twelve is $\frac{11}{12}$ of a right angle.

The measure in degrees $= \frac{11}{12}$ of $90 = \frac{11 \times 15}{2} = \frac{165}{2} = 82\frac{1}{2}$.

The measure in grades $= \frac{11}{12}$ of $100 = \frac{11 \times 25}{3} = \frac{275}{3} = 91\frac{2}{3}$.

The circular measure $= \frac{11}{12}$ of $\frac{\pi}{2} = \frac{11\pi}{24}$.

CHAPTER III
Trigonometrical Ratios

Problem 1. *The sine of a certain angle is* $\frac{3}{5}$*; find the other Trigonometrical Ratios of the angle.*

Solution. Let $\sin A = \frac{3}{5}$. Then we have

$$\cos A = \sqrt{1-\sin^2 A} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}.$$
$$\tan A = \frac{\sin A}{\cos A} = \frac{3}{5} \div \frac{4}{5} = \frac{3}{5} \times \frac{5}{4} = \frac{3}{4};$$
$$\cot A = \frac{1}{\tan A} = \frac{4}{3};$$
$$\sec A = \frac{1}{\cos A} = \frac{5}{4}; \quad \text{cosec } A = \frac{1}{\sin A} = \frac{5}{3};$$
$$\text{vers } A = 1 - \cos A = 1 - \frac{4}{5} = \frac{1}{5}.$$

Problem 2. *The tangent of a certain angle is* $\frac{4}{3}$*; find the other Trigonometrical Ratios of the angle.*

Solution. Let $\tan A = \frac{4}{3}$. Then we have

$$\sin A = \frac{\tan A}{\sqrt{1+\tan^2 A}} = \frac{\frac{4}{3}}{\sqrt{1+\frac{16}{9}}} = \frac{4}{3} \div \frac{5}{3} = \frac{4}{5};$$
$$\cos A = \frac{1}{\sqrt{1+\tan^2 A}} = \frac{1}{\sqrt{1+\frac{16}{9}}} = 1 \div \frac{5}{3} = \frac{3}{5};$$
$$\sec A = \frac{1}{\cos A} = \frac{5}{3}; \quad \text{cosec } A = \frac{1}{\sin A} = \frac{5}{4};$$
$$\text{vers } A = 1 - \cos A = 1 - \frac{3}{5} = \frac{2}{5}.$$

Problem 3. *The cosine of a certain angle is* $\sqrt{\frac{2}{3}}$*; find the other Trigonometrical Ratios of the angle.*

Solution. Let $\cos A = \sqrt{\frac{2}{3}}$. Then we have

$$\sin A = \sqrt{1-\cos^2 A} = \sqrt{1-\frac{2}{3}} = \sqrt{\frac{1}{3}};$$

$$\tan A = \frac{\sin A}{\cos A} = \sqrt{\frac{1}{3}} \div \sqrt{\frac{2}{3}} = \frac{1}{\sqrt{2}};$$
$$\cot A = \frac{1}{\tan A} = \sqrt{2};$$
$$\sec A = \frac{1}{\cos A} = \sqrt{\frac{3}{2}}; \quad \operatorname{cosec} A = \frac{1}{\sin A} = \sqrt{3};$$
$$\operatorname{vers} A = 1 - \cos A = 1 - \sqrt{\frac{2}{3}}.$$

Problem 4. *Show that* $\sec^2\theta \operatorname{cosec}^2\theta = \tan^2\theta + \cot^2\theta + 2$.

Solution.

$$\sec^2\theta \operatorname{cosec}^2\theta = \left(1+\tan^2\theta\right)\left(1+\cot^2\theta\right) = 1 + \tan^2\theta + \cot^2\theta + (\tan\theta\cot\theta)^2$$
$$= 1 + \tan^2\theta + \cot^2\theta + 1 = \tan^2\theta + \cot^2\theta + 2.$$

Problem 5. *Show that*

$$\sin^2\theta\tan\theta + \cos^2\theta\cot\theta + 2\sin\theta\cos\theta = \tan\theta + \cot\theta.$$

Solution.

$$\sin^2\theta\tan\theta + \cos^2\theta\cot\theta + 2\sin\theta\cos\theta = \frac{\sin^3\theta}{\cos\theta} + \frac{\cos^3\theta}{\sin\theta} + 2\sin\theta\cos\theta$$
$$= \frac{\sin^4\theta + \cos^4\theta + 2\sin^2\theta\cos^2\theta}{\sin\theta\cos\theta} = \frac{\left(\sin^2\theta + \cos^2\theta\right)^2}{\sin\theta\cos\theta} = \frac{1}{\sin\theta\cos\theta}$$
$$= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \tan\theta + \cot\theta.$$

Problem 6. *Show that* $2\left(\sin^6\theta + \cos^6\theta\right) - 3\left(\sin^4\theta + \cos^4\theta\right) + 1 = 0$.

Solution.

$$2\left(\sin^6\theta + \cos^6\theta\right) = 2\left(\sin^2\theta + \cos^2\theta\right)\left(\sin^4\theta - \sin^2\theta\cos^2\theta + \cos^4\theta\right)$$
$$= 2\left(\sin^4\theta - \sin^2\theta\cos^2\theta + \cos^4\theta\right);$$

therefore
$$2\left(\sin^6\theta + \cos^6\theta\right) - 3\left(\sin^4\theta + \cos^4\theta\right) + 1$$
$$= -2\sin^2\theta\cos^2\theta - \sin^4\theta - \cos^4\theta + 1$$
$$= 1 - \left(\sin^2\theta + \cos^2\theta\right)^2 = 1 - 1 = 0.$$

Obtain solutions of the following equations :

Problem 7. $2\sin^2\theta = 3\cos\theta$.

Solution.

$$\sin^2\theta = \frac{3}{2}\cos\theta; \quad \text{therefore } 1 - \cos^2\theta = \frac{3}{2}\cos\theta;$$

therefore
$$\cos^2\theta + \frac{3}{2}\cos\theta = 1.$$

By solving this quadratic in the usual way we obtain $\cos\theta = \frac{1}{2}$ or -2; but only the former value is applicable, for $\cos\theta$ cannot be numerically greater than unity. Hence $\cos\theta = \frac{1}{2}$, and therefore $\theta = \frac{\pi}{3}$.

Problem 8. $\sin\theta + \cos\theta = 1$.

Solution. $\sin\theta+\cos\theta = 1$; therefore $\cos\theta = 1-\sin\theta$; therefore $\cos^2\theta = (1-\sin\theta)^2$, therefore $1-\sin^2\theta = (1-\sin\theta)^2$, that is $(1-\sin\theta)(1+\sin\theta) = (1-\sin\theta)^2$. Therefore either $1-\sin\theta = 0$, or $1+\sin\theta = 1-\sin\theta$.

Take $1-\sin\theta = 0$; thus $\sin\theta = 1$, therefore $\theta = \frac{\pi}{2}$.

Next take $1+\sin\theta = 1-\sin\theta$; thus $\sin\theta = 0$, therefore $\theta = 0$.

Problem 9. $\cot\theta = 2\cos\theta$.

Solution.

$$\cot\theta = 2\cos\theta; \quad \text{therefore } \frac{\cos\theta}{\sin\theta} = 2\cos\theta.$$

Therefore either $$\cos\theta = 0, \text{ or } \frac{1}{\sin\theta} = 2.$$

Take $\cos\theta = 0$; then $\theta = \frac{\pi}{2}$. Next take $\frac{1}{\sin\theta} = 2$; thus $\sin\theta = \frac{1}{2}$; therefore $\theta = \frac{\pi}{6}$.

Problem 10. $\sin^2\theta - 2\cos\theta + \frac{1}{4} = 0$.

Solution. $\sin^2\theta - 2\cos\theta + \frac{1}{4} = 0$; therefore $1-\cos^2\theta - 2\cos\theta + \frac{1}{4} = 0$; therefore $\cos^2\theta + 2\cos\theta = \frac{5}{4}$. By solving this quadratic in the ordinary way we obtain $\cos\theta = \frac{1}{2}$, or $-\frac{5}{\cdot 2}$; but only the former value is applicable; therefore $\theta = \frac{\pi}{3}$.

Problem 11. $3\sec^2\theta + 8 = 10\sec^2\theta$.

Solution. $3\sec^4\theta + 8 = 10\sec^2\theta$; therefore $3\sec^4\theta - 10\sec^2\theta + 8 = 0$. By solving this quadratic in the ordinary way we obtain $\sec^2\theta = 2$ or $\frac{4}{3}$; therefore $\sec\theta = \sqrt{2}$ or $\frac{2}{\sqrt{3}}$; therefore $\theta = \frac{\pi}{4}$ or $\frac{\pi}{6}$.

Problem 12. $\tan\theta + \cot\theta = 2$.

Solution.

$$\tan\theta + \cot\theta = 2; \quad \text{therefore } \tan\theta + \frac{1}{\tan\theta} = 2;$$

therefore $$\tan^2\theta - 2\tan\theta + 1 = 0, \quad \text{that is } (\tan\theta - 1)^2 = 0;$$

therefore $$\tan\theta = 1, \quad \text{therefore } \theta = \frac{\pi}{4}.$$

Problem 13. *Given* $\sin(A - B) = \frac{1}{2}$, *and* $\cos(A + B) = \frac{1}{2}$, *find A and B.*

Solution.

$$\sin(A - B) = \frac{1}{2}; \quad \text{therefore } A - B = 30°,$$
$$\cos(A + B) = \frac{1}{2}; \quad \text{therefore } A + B = 60°;$$

from these two equations we obtain $A = 45°$, and $B = 15°$.

Problem 14. *Given* $\tan(A + B) = \sqrt{3}$, *and* $\tan(A - B) = 1$, *find A and B.*

Solution.

$$\tan(A + B) = \sqrt{3}; \quad \text{therefore } A + B = 60°,$$
$$\tan(A - B) = 1; \quad \text{therefore } A - B = 45°;$$

from these two equations we obtain $A = 52\frac{1}{2}°$, $B = 7\frac{1}{2}°$.

CHAPTER IV
Application of Algebraical Signs

Problem 1. *Determine the values of the Trigonometrical Ratios for an angle of 585°.*

Solution. $585° = 360° + 225°$. Thus the Trigonometrical Ratios are the same as for an angle of 225°.

$$\sin 225° = \sin(180° + 45°) = -\sin 45° = -\frac{1}{\sqrt{2}},$$
$$\cos 225° = \cos(180° + 45°) = -\cos 45° = -\frac{1}{\sqrt{2}}.$$

Problem 2. *Also for an angle of 690°.*

Solution. $690° = 360° + 330°$. Thus the Trigonometrical Ratios are the same as for an angle of 330°.

$$\sin 330° = \sin(180° + 150°) = -\sin 150° = -\sin 30° = -\frac{1}{2},$$
$$\cos 330° = \cos(180° + 150°) = -\cos 150° = \cos 30° = \frac{\sqrt{3}}{2}.$$

Problem 3. *Also for an angle of 930°.*

Solution. $930° = 720° + 210°$. Thus the Trigonometrical Ratios are the same as for an angle of 210°.

$$\sin 210° = \sin(180° + 30°) = -\sin 30° = -\frac{1}{2},$$
$$\cos 210° = \cos(180° + 30°) = -\cos 30° = -\frac{\sqrt{3}}{2}.$$

Problem 4. *Also for an angle of 6420°.*

Solution. $6420° = 17 \times 360° + 300°$. Thus the Trigonometrical Ratios are the same as for an angle of 300°.

$$\sin 300° = \sin(180° + 120°) = -\sin 120° = -\sin 60° = -\frac{\sqrt{3}}{2},$$
$$\cos 300° = \cos(180° + 120°) = -\cos 120° = \cos 60° = \frac{1}{2}.$$

Problem 5. *Find all the angles between 0 and 900° which satisfy the relation* $\tan\theta = 1$.

Solution. The smallest angle is 45°; the other angles are found by increasing successively by 180° : thus all the angles are 45°, 225°, 405°, 585°, 765°.

Problem 6. *Find all the angles between 0 and 900° which satisfy the relation* $\cos^2\theta = \frac{1}{2}$.

Solution. Since $\cos^2\theta = \frac{1}{2}$, we have $\cos\theta = \pm\frac{1}{\sqrt{2}}$.

Take the upper sign ; then the smallest value is 45°, and the others are $360° - 45°,\ 360° + 45°,\ 720° - 45°,\ 720° + 45°$.

Take the lower sign ; then the smallest value is 135°, and the others are $360° - 135°,\ 360° + 135°,\ 720° - 135°,\ 720° + 135°$.

Problem 7. *Find all the values of versin* $\frac{n\pi}{4}$ *where n is any integer.*

Solution. $\textit{vers}\,\frac{n\pi}{4} = 1 - \cos\frac{n\pi}{4}$.

Suppose $n = 0$; then we have $1 - \cos 0$, that is $1 - 1$, that is 0; next suppose $n = 1$, then we have $1 - \cos\frac{\pi}{4}$, that is $1 - \frac{1}{\sqrt{2}}$; next suppose $n = 2$, then we have $1 - \cos\frac{\pi}{2}$, that is $1 - 0$, that is 1; next suppose $n = 3$, then we have $1 - \cos\frac{3\pi}{4}$; that is $1 + \frac{1}{\sqrt{2}}$; next suppose $n = 4$, then we have $1 - \cos\pi$, that is $1 + 1$, that is 2. Then the values begin to recur in the inverse order; for $\cos\frac{5\pi}{4} = \cos\frac{3\pi}{4}$, $\cos\frac{6\pi}{4} = \cos\frac{2\pi}{4}$, $\cos\frac{7\pi}{4} = \cos\frac{\pi}{4}$, $\cos\frac{8\pi}{4} = \cos 2\pi = \cos 0$.

Then the whole series recurs. For $\cos\frac{9\pi}{4} = \cos\frac{\pi}{4}$, and so on.

Problem 8. *Find all the values of* $\sin\left\{\frac{n\pi}{2} + (-1)^n\frac{\pi}{6}\right\}$ *where n is any integer.*

Solution. Suppose $n = 0$, then we have $\sin\frac{\pi}{6}$, that is $\frac{1}{2}$; next suppose $n = 1$, then we have $\sin\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$, that is $\sin\frac{\pi}{3}$, that is $\frac{\sqrt{3}}{2}$; next suppose $n = 2$, then we have $\sin\left(\pi + \frac{\pi}{6}\right)$, that is $-\sin\frac{\pi}{6}$, that is $-\frac{1}{2}$; next suppose $n = 3$, then we have $\sin\left(\frac{3\pi}{2} - \frac{\pi}{6}\right)$, that is $-\sin\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$, that is $-\sin\frac{\pi}{3}$, that is $-\frac{\sqrt{3}}{2}$.

Then the values recur; for suppose $n = 4$; then we have $\sin\left(2\pi + \frac{\pi}{6}\right)$, that is $\sin\frac{\pi}{6}$, and so on.

Problem 9. *Solve* $\sin^3\theta + \cos^3\theta = 0$.

Solution. $\sin^3\theta = -\cos^3\theta$. Extract the cube root of both sides; thus $\sin\theta = -\cos\theta$, therefore $\frac{\sin\theta}{\cos\theta} = -1$, that is $\tan\theta = -1$; therefore $\theta = \frac{3\pi}{4}$.

Problem 10. *Solve* $2\sin^2\theta - 5\cos\theta - 4 = 0$.

Solution. $2\sin^2\theta - 5\cos\theta - 4 = 0$; therefore $2(1-\cos^2\theta) - 5\cos\theta - 4 = 0$; therefore $2\cos^2\theta + 5\cos\theta + 2 = 0$. By solving this quadratic in the usual way we obtain $\cos\theta = -\dfrac{1}{2}$ or -2; but only the former value is applicable; therefore $\theta = \dfrac{2\pi}{3}$.

Problem 11. *Trace the changes in the sign and value of* $\cos\theta - \sin\theta$ *as* θ *changes from* 0 *to* 2π.

Solution. When $\theta = 0$ we have $\cos\theta = 1$ and $\sin\theta = 0$, so that $\cos\theta - \sin\theta = 1$. Let θ change from 0 to $\dfrac{\pi}{2}$, then $\cos\theta$ changes from 1 to 0, and $\sin\theta$ from 0 to 1; therefore $\cos\theta - \sin\theta$ changes from 1 to -1, vanishing when $\theta = \dfrac{\pi}{4}$.

Let θ change from $\dfrac{\pi}{2}$ to π, then $\cos\theta$ changes from 0 to -1 and $\sin\theta$ from 1 to 0; thus $\cos\theta - \sin\theta$ remains negative. It has its greatest numerical value, namely $-\sqrt{2}$, when $\theta = \dfrac{3\pi}{4}$. For we have

$$(\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2 = 2\left(\cos^2\theta + \sin^2\theta\right) = 2;$$

and thus $(\cos\theta - \sin\theta)^2$ has its greatest value when $\cos\theta + \sin\theta$ vanishes, that is when $\tan\theta = -1$, that is when $\theta = \dfrac{3\pi}{4}$.

Let θ change from π to $\dfrac{3\pi}{2}$; then $\cos\theta - \sin\theta$ goes through the same numerical values, with a *contrary* sign, as when θ changes from 0 to $\dfrac{\pi}{2}$: this follows from *Art.* 50 (page 400).

Let θ change from $\dfrac{3\pi}{2}$ to 2π; then $\cos\theta - \sin\theta$ goes through the same numerical values, with a *contrary* sign, as when θ changes from $\dfrac{\pi}{2}$ to π : this follows from *Art.* 50 (page 400).

Problem 12. *Also of* $\cos^2\theta - \sin^2\theta$.

Solution. Let θ change from 0 to $\dfrac{\pi}{2}$; then $\cos^2\theta$ changes from 1 to 0, and $\sin^2\theta$ from 0 to 1; therefore $\cos^2\theta - \sin^2\theta$ changes from 1 to -1.

Let θ change from $\dfrac{\pi}{2}$ to π; then $\cos^2\theta - \sin^2\theta$ changes from -1 to 1.

Let θ change from π to $\dfrac{3\pi}{2}$; then $\cos^2\theta - \sin^2\theta$ goes through the same values as when θ changes from 0 to $\dfrac{\pi}{2}$.

Let θ change from $\dfrac{3\pi}{2}$ to 2π; then $\cos^2\theta - \sin^2\theta$ goes through the same values as when θ changes from $\dfrac{\pi}{2}$ to π.

Problem 13. *Also of* $\tan\theta + \cot\theta$.

Solution. $\tan\theta + \cot\theta = \tan\theta + \dfrac{1}{\tan\theta}$. Let θ change from 0 to $\dfrac{\pi}{2}$; then $\tan\theta$

changes from 0 to infinity. Thus $\tan\theta + \dfrac{1}{\tan\theta}$ is always positive, and is infinite both when $\theta = 0$, and when $\theta = \dfrac{\pi}{2}$. The least value is when $\theta = \dfrac{\pi}{4}$; for we have

$$\left(\tan\theta + \frac{1}{\tan\theta}\right)^2 = \left(\tan\theta - \frac{1}{\tan\theta}\right)^2 + 4,$$

and thus the least value is when $\tan\theta - \dfrac{1}{\tan\theta}$ vanishes, that is when $\tan^2\theta = 1$. Thus $\tan\theta + \cot\theta$ diminishes from infinity to 2, as θ changes from 0 to $\dfrac{\pi}{4}$; and then increases from 2 to infinity, as θ changes from $\dfrac{\pi}{4}$ to $\dfrac{\pi}{2}$.

Let θ change from $\dfrac{\pi}{2}$ to π; then $\tan\theta + \cot\theta$ goes in reverse order through the same numerical values, with a *contrary* sign, as when θ changes from 0 to $\dfrac{\pi}{2}$: this follows from *Art.* 48 (page 397).

Let θ change from π to 2π; then $\tan\theta + \cot\theta$ goes through the same values as when θ changes from 0 to π : this follows from *Art.* 50 (page 400).

Problem 14. *Is* $\sec^2\theta = \dfrac{4ab}{(a+b)^2}$ *a possible equation if a and b are unequal ?*

Solution. We know by Algebra that if a and b are unequal $2ab$ is less than a^2+b^2, and therefore $4ab$ is less than a^2+b^2+2ab, that is $4ab$ is less than $(a+b)^2$. Therefore $\dfrac{4ab}{(a+b)^2}$ is less than unity; and cannot be equal to the secant of any angle, for a secant is never less than unity.

Problem 15. *Show that*

$$\tan(A+90^\circ) = -\cot A, \qquad \cot(A+90^\circ) = -\tan A,$$
$$\sec(A+90^\circ) = -\operatorname{cosec} A, \qquad \operatorname{cosec}(A+90^\circ) = \sec A,$$
$$\operatorname{vers}(A+90^\circ) = 1+\sin A.$$

Solution.

$$\tan(A+90^\circ) = \frac{\sin(A+90^\circ)}{\cos(A+90^\circ)} = \frac{\cos A}{-\sin A}, \text{ by } \textit{Art. } 52 \text{ (page 400) }, = -\cot A,$$
$$\cot(A+90^\circ) = \frac{1}{\tan(A+90^\circ)} = -\frac{1}{\cot A} = -\tan A,$$
$$\sec(A+90^\circ) = \frac{1}{\cos(A+90^\circ)} = \frac{1}{-\sin A}, \text{ by } \textit{Art. } 52 \text{ (page 400) }, = -\operatorname{cosec} A,$$
$$\operatorname{cosec}(A+90^\circ) = \frac{1}{\sin(A+90^\circ)} = \frac{1}{\cos A}, \text{ by } \textit{Art. } 52 \text{ (page 400) }, = \sec A,$$
$$\operatorname{vers}(A+90^\circ) = 1-\cos(A+90^\circ) = 1+\sin A, \text{ by } \textit{Art. } 52 \text{ (page 400) }.$$

Problem 16. *Show that* $\sin(270^\circ - A) = -\cos A, \quad \cos(270^\circ - A) = -\sin A.$

Solution.

$$\sin(270^\circ - A) = -\sin(90^\circ - A), \qquad \text{by } \textit{Art. } 50 \text{ (page 400) }, = -\cos A.$$
$$\cos(270^\circ - A) = -\cos(90^\circ - A), \qquad \text{by } \textit{Art. } 50 \text{ (page 400) }, = -\sin A.$$

Problem 17. *Show that* $\sin(270^\circ + A) = -\cos A, \quad \cos(270^\circ + A) = \sin A.$

Solution.

$$\sin(270^\circ + A) = -\sin(90^\circ + A), \text{ by } Art.\ 50 \text{ (page 400)},$$
$$= -\cos A, \text{ by } Art.\ 52 \text{ (page 400)}.$$
$$\cos(270^\circ + A) = -\cos(90^\circ + A), \text{ by } Art.\ 50 \text{ (page 400)},$$
$$= -(-\sin A), \text{ by } Art.\ 52 \text{ (page 400)}, = \sin A.$$

Problem 18. *Show that* $\sin(360^\circ - A) = -\sin A, \quad \cos(360^\circ - A) = \cos A.$

Solution.

$$\sin(360^\circ - A) = -\sin(180^\circ - A), \text{ by } Art.\ 50 \text{ (page 400)},$$
$$= -\sin A, \text{ by } Art.\ 48 \text{ (page 397)}.$$
$$\cos(360^\circ - A) = -\cos(180^\circ - A), \text{ by } Art.\ 50 \text{ (page 400)},$$
$$= -(-\cos A), \text{ by } Art.\ 48 \text{ (page 397)}, = \cos A.$$

CHAPTER V
Angles with Given Trigonometrical Ratios

Problem 1. *Write down the general value of θ when* $\tan\theta = 1$.

Solution. $\tan\theta = 1$; the smallest value of θ is $\frac{\theta}{4}$, and the general value is $n\pi + \frac{\pi}{4}$, by *Art.* 68 (page 402).

Problem 2. *Write down the general value of θ when* $\sin\theta = 1$.

Solution. $\sin\theta = 1$; the smallest value of θ is $\frac{\pi}{2}$, and the general value is $n\pi + (-1)^n\frac{\pi}{2}$, by *Art.* 66 (page 401). This expression may be simplified; for first suppose n even, denote it by $2m$, so that we have $2m\pi + \frac{\pi}{2}$; next suppose n odd, denote it by $2m+1$, so that we have $(2m+1)\pi - \frac{\pi}{2}$, that is $2m\pi + \frac{\pi}{2}$. Hence both cases are included in the expression $2m\pi + \frac{\pi}{2}$, that is $(4m+1)\frac{\pi}{2}$.

Problem 3. *Write down the general value of θ when* $\cos\theta = 1$.

Solution. $\cos\theta = 1$; the smallest value of θ is 0, and the general value is $2n\pi$, by *Art.* 67 (page 402).

Problem 4. *Write down the general value of θ when* $\cos\theta = -\frac{1}{2}$.

Solution. $\cos\theta = -\frac{1}{2}$; the smallest value of θ is $\frac{2\pi}{3}$, and the general value is $2n\pi \pm \frac{2\pi}{3}$, by *Art.* 67 (page 402).

Problem 5. *Find all the values of θ which satisfy* $\sin^2\theta = \sin^2\alpha$.

Solution. $\sin^2\theta = \sin^2\alpha$; therefore $\sin\theta = \pm\sin\alpha$. Take the upper sign, then simplest solution is $\theta = \alpha$, and the general solution is $\theta = n\pi + (-1)^n\alpha$. Take the lower sign, then the simplest solution is $\theta = -\alpha$, and the general solution is $\theta = n\pi - (-1)^n\alpha$. The two expressions are included in the single expression $\theta = n\pi \pm \alpha$.

This might also be obtained from a diagram in the manner of *Arts.* 66 (page 401), 67 (page 402), and 68 (page 402).

Problem 6. *Write down the general value of θ when* $\operatorname{cosec}^2\theta = \frac{4}{3}$.

Solution. Since $\operatorname{cosec}^2\theta = \dfrac{4}{3}$ we have $\sin^2\theta = \dfrac{3}{4} = \sin^2\dfrac{\pi}{3}$; hence, by *Problem* 5, the general solution is $\theta = n\pi \pm \dfrac{\pi}{3}$.

Problem 7. *Find all the values of θ which satisfy $\cos^2\theta = \cos^2\alpha$.*

Solution. $\cos^2\theta = \cos^2\alpha$; therefore $\cos\theta = \pm\cos\alpha$. Take the upper sign, then the simplest solution is $\theta = \alpha$, and the general solution is $\theta = 2n\pi \pm \alpha$. Take the lower sign, then the simplest solution is $\theta = \pi - \alpha$, and the general solution is $\theta = 2n\pi \pm (\pi - \alpha)$. The two expressions are included in the single expression $\theta = m\pi \pm \alpha$.

It will be seen that the result is the same as for *Problem* 5, and this should be the case; for if $\cos^2\theta = \cos^2\alpha$, then $1 - \cos^2\theta = 1 - cos^2\alpha$, that is $\sin^2\theta = \sin^2\alpha$.

Problem 8. *Write down the general value of θ when $\sec^2\theta = 2$.*

Solution. Since $\sec^2\theta = 2$, we have $\cos^2\theta = \dfrac{1}{2} = \cos^2\dfrac{\pi}{4}$; hence, by *Problem* 7, the general solution is $\theta = n\pi \pm \dfrac{\pi}{4}$.

Problem 9. *Find all the values of θ which satisfy $\tan^2\theta = \tan^2\alpha$.*

Solution. $\tan^2\theta = \tan^2\alpha$; therefore $\tan\theta = \pm\tan\alpha$. Take the upper sign, then the simplest solution is $\theta = \alpha$, and the general solution is $\theta = n\pi + \alpha$. Take the lower sign, then the simplest solution is $\theta = -\alpha$, and the general solution is $\theta = n\pi - \alpha$. The two expressions are included in the single expression $\theta = n\pi \pm \alpha$.

The result is the same as for *Problem* 7, and this should be the case ; for if $\tan^2\theta = \tan^2\alpha$ then $1 + \tan^2\theta = 1 + \tan^2\alpha$; therefore $\sec^2\theta = \sec^2\alpha$, by *Art.* 34 (page 396); therefore $\cos^2\theta = \cos^2\alpha$.

Problem 10. *Write down the general value of θ when $\tan^2\theta = \dfrac{1}{3}$.*

Solution. $\tan^2\theta = \dfrac{1}{3} = \tan^2\dfrac{\pi}{6}$; hence, by *Problem* 9, the general solution is $\theta = n\pi \pm \dfrac{\pi}{6}$.

Problem 11. *Show that all the angles which have both the same sine and the same cosine as α, are included in the formula $2n\pi + \alpha$.*

Solution. All the angles included in the expression $2n\pi \pm \alpha$ have the same cosine as α, by *Art.* 67 (page 402).

Now by *Art.* 45 (page 397) $\sin(2n\pi + \alpha) = \sin\alpha$; and $\sin(2n\pi - \alpha) = \sin(-\alpha) = -\sin\alpha$. Thus the angles which have both the same sine and the same cosine as α are all comprised in the expression $2n\pi + \alpha$.

Problem 12. *Write down the general value of θ which satisfies both*

$$\sin\theta = -\frac{1}{2} \text{ and } \cos\theta = -\frac{\sqrt{3}}{2}.$$

Solution. $-\frac{1}{2} = \sin\left(\pi + \frac{\pi}{6}\right) = \sin\frac{7\pi}{6}$, and $-\frac{\sqrt{3}}{2} = \cos\left(\pi + \frac{\pi}{6}\right) = \cos\frac{7\pi}{6}$;

hence, by *Problem* 11, the required general value is $\theta = 2n\pi + \frac{7\pi}{6}$.

CHAPTER VI
Trigonometrical Ratios of Two Angles

Prove the following identities :

Problem 1. $\dfrac{\cos A + \sin A}{\cos A - \sin A} = \tan 2A + \sec 2A.$

Solution.

$$\begin{aligned}\frac{\cos A + \sin A}{\cos A - \sin A} &= \frac{(\cos A + \sin A)^2}{(\cos A - \sin A)(\cos A + \sin A)} \\ &= \frac{\cos^2 A + \sin^2 A + 2\sin A\cos A}{\cos^2 A - \sin^2 A} = \frac{1 + \sin 2A}{\cos 2A} \\ &= \frac{\sin 2A}{\cos 2A} + \frac{1}{\cos 2A} = \tan 2A + \sec 2A.\end{aligned}$$

Problem 2. $2\sin^2 A\sin^2 B + 2\cos^2 A\cos^2 B = 1 + \cos 2A\cos 2B.$

Solution.

$$\begin{aligned}&2\sin^2 A\sin^2 B + 2\cos^2 A\cos^2 B \\ &= \frac{(1-\cos 2A)(1-\cos 2B)}{2} + \frac{(1+\cos 2A)(1+\cos 2B)}{2} \\ &= \frac{1-\cos 2A-\cos 2B+\cos 2A\cos 2B}{2} + \frac{1+\cos 2A+\cos 2B+\cos 2A\cos 2B}{2} \\ &= 1 + \cos 2A\cos 2B.\end{aligned}$$

Problem 3. $\tan(45^\circ + A) - \tan(45^\circ - A) = 2\tan 2A.$

Solution.

$$\begin{aligned}&\tan(45^\circ + A) - \tan(45^\circ - A) \\ &= \frac{\tan 45^\circ + \tan A}{1 - \tan 45^\circ\tan A} - \frac{\tan 45^\circ - \tan A}{1 + \tan 45^\circ\tan A} = \frac{1+\tan A}{1-\tan A} - \frac{1-\tan A}{1+\tan A} \\ &= \frac{(1+\tan A)^2 - (1-\tan A)^2}{1-\tan^2 A} = \frac{4\tan A}{1-\tan^2 A} = 2\tan 2A.\end{aligned}$$

Problem 4. $\sin 3A \operatorname{cosec} A - \cos 3A\sec A = 2.$

Solution.

$$\begin{aligned}&\sin 3A\operatorname{cosec} A - \cos 3A\sec A \\ &= \frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A} = \frac{3\sin A - 4\sin^3 A}{\sin A} - \frac{4\cos^3 A - 3\cos A}{\cos A} \\ &= 3 - 4\sin^2 A - (4\cos^2 A - 3) = 6 - 4(\sin^2 A + \cos^2 A) = 6 - 4 = 2.\end{aligned}$$

Problem 5. $3\sin A - \sin 3A = 2\sin A(1 - \cos 2A).$

Solution.

$$3\sin A - \sin 3A = 3\sin A - (3\sin A - 4\sin^3 A)$$
$$= 4\sin^3 A = 2\sin A \times 2\sin^2 A = 2\sin A(1-\cos 2A).$$

Problem 6. $\dfrac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \dfrac{\sin 3A}{\sin 5A}.$

Solution.

$$\frac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \frac{\sin A + \sin 5A + 2\sin 3A}{\sin 3A + \sin 7A + 2\sin 5A}$$
$$= \frac{2\sin 3A\cos 2A + 2\sin 3A}{2\sin 5A\cos 2A + 2\sin 5A}, \text{ by } Art.\ 84 \text{ (page 405)},$$
$$= \frac{2\sin 3A(1+\cos 2A)}{2\sin 5A(1+\cos 2A)} = \frac{\sin 3A}{\sin 5A}.$$

Problem 7. $\dfrac{\sin B}{\sin A} = \dfrac{\sin(2A+B)}{\sin A} - 2\cos(A+B).$

Solution.

$$\frac{\sin(2A+B)}{\sin A} - 2\cos(A+B) = \frac{\sin(A+B+A) - 2\sin A\cos(A+B)}{\sin A}$$
$$= \frac{\sin(A+B)\cos A + \cos(A+B)\sin A - 2\sin A\cos(A+B)}{\sin A}$$
$$= \frac{\sin(A+B)\cos A - \cos(A+B)\sin A}{\sin A} = \frac{\sin(A+B-A)}{\sin A} = \frac{\sin B}{\sin A}.$$

Problem 8. $\sin 4A = 4\sin A\cos^3 A - 4\cos A\sin^3 A.$

Solution.

$$4\sin A\cos^3 A - 4\cos A\sin^3 A = 4\sin A\cos A(\cos^2 A - \sin^2 A)$$
$$= 2\sin 2A\cos 2A = \sin 4A.$$

Problem 9. $\dfrac{\cos A - \cos 3A}{\sin 3A - \sin A} = \tan 2A.$

Solution.

$$\frac{\cos A - \cos 3A}{\sin 3A - \sin A} = \frac{2\sin 2A\sin A}{2\cos 2A\sin A}, \text{ by } Art.\ 84 \text{ (page 405)},$$
$$= \frac{\sin 2A}{\cos 2A} = \tan 2A.$$

Problem 10. $\dfrac{\cos 2A - \cos 4A}{\sin 4A - \sin 2A} = \tan 3A.$

Solution.

$$\frac{\cos 2A - \cos 4A}{\sin 4A - \sin 2A} = \frac{2\sin 3A\sin A}{2\cos 3A\sin A}, \text{ by } Art.\ 84 \text{ (page 405)},$$

$$= \frac{\sin 3A}{\cos 3A} = \tan 3A.$$

Problem 11. cosec $2A + \cot 4A = \cot A -$ cosec $4A$.

Solution.

$$\begin{aligned}
\text{cosec } 2A + \cot 4A &= \frac{1}{\sin 2A} + \frac{\cos 4A}{\sin 4A} \\
&= \frac{2\cos 2A}{2\cos 2A \sin 2A} + \frac{\cos 2A}{\sin 4A} = \frac{2\cos 2A + \cos 4A}{\sin 4A} \\
&= \frac{2\cos 2A + 2\cos^2 2A - 1}{\sin 4A} = \frac{2\cos 2A(1+\cos 2A) - 1}{\sin 4A} \\
&= \frac{2\cos 2A(1+\cos 2A)}{2\sin 2A \cos 2A} - \frac{1}{\sin 4A} = \frac{1+\cos 2A}{\sin 2A} - \frac{1}{\sin 4A} \\
&= \frac{2\cos^2 A}{2\sin A\cos A} - \frac{1}{\sin 4A} \\
&= \frac{\cos A}{\sin A} - \frac{1}{\sin 4A} = \cot A - \text{cosec } 4A.
\end{aligned}$$

Problem 12. $\cos^2(A-B) + \cos^2 B - 2\cos(A-B)\cos A\cos B = \sin^2 A$.

Solution.

$$\begin{aligned}
&\cos^2(A-B) + \cos^2 B - 2\cos(A-B)\cos A\cos B \\
&= \cos(A-B)\{\cos(A-B) - \cos A\cos B\} \\
&\qquad + \cos B\{\cos B - \cos(A-B)\cos A\} \\
&= \cos(A-B)\sin A\sin B \\
&\qquad + \cos B\left\{\cos\left(A - \overline{A-B}\right) - \cos(A-B)\cos A\right\} \\
&= \cos(A-B)\sin A\sin B + \cos B\sin A\sin(A-B) \\
&= \sin A\{\cos(A-B)\sin B + \sin(A-B)\cos B\} \\
&= \sin A\sin(A-B+B) = \sin A\sin A = \sin^2 A.
\end{aligned}$$

Problem 13. $\sin^2(A-B) + \sin^2 B + 2\sin(A-B)\sin B\cos A = \sin^2 A$.

Solution.

$$\begin{aligned}
&\sin^2(A-B) + \sin^2 B + 2\sin(A-B)\sin B\cos A \\
&= \sin(A-B)\{\sin(A-B) + \sin B\cos A\} \\
&\qquad + \sin B\{\sin B + \sin(A-B)\cos A\} \\
&= \sin(A-B)\sin A\cos B \\
&\qquad + \sin B\left\{\sin\left(A - \overline{A-B}\right) + \sin(A-B)\cos A\right\} \\
&= \sin(A-B)\sin A\cos B + \sin B\sin A\cos(A-B) \\
&= \sin A\{\sin(A-B)\cos B + \cos(A-B)\sin B\} \\
&= \sin A\sin(A-B+B) = \sin A\sin A = \sin^2 A.
\end{aligned}$$

Problem 14. $\dfrac{1 - \tan^2(45^\circ - A)}{1 + \tan^2(45^\circ - A)} = \sin 2A$.

Solution.

$$\frac{1-\tan^2(45^\circ-A)}{1+\tan^2(45^\circ-A)} = \frac{1-\dfrac{\sin^2(45^\circ-A)}{\cos^2(45^\circ-A)}}{1+\dfrac{\sin^2(45^\circ-A)}{\cos^2(45^\circ-A)}}$$

$$= \frac{\cos^2(45^\circ-A)-\sin^2(45^\circ-A)}{\cos^2(45^\circ-A)+\sin^2(45^\circ-A)} = \frac{\cos 2(45^\circ-A)}{1}$$

$$= \cos(90^\circ-2A) = \sin 2A.$$

Problem 15. $\dfrac{4\tan A(1-\tan^2 A)}{(1+\tan^2 A)^2} = \sin 4A.$

Solution.

$$\frac{4\tan A(1-\tan^2 A)}{(1+\tan^2 A)^2} = \frac{\dfrac{4\sin A}{\cos A}\left(1-\dfrac{\sin^2 A}{\cos^2 A}\right)}{\left(1+\dfrac{\sin^2 A}{\cos^2 A}\right)^2}$$

$$= \frac{4\sin A\cos A(\cos^2 A-\sin^2 A)}{(\cos^2 A+\sin^2 A)^2} = 2\sin 2A\cos 2A$$

$$= \sin 4A.$$

Problem 16. $\sin A(1+\tan A)+\cos A(1+\cot A) = \sec A + \operatorname{cosec} A.$

Solution.

$$\sin A(1+\tan A)+\cos A(1+\cot A)$$

$$= \sin A\left(1+\frac{\sin A}{\cos A}\right)+\cos A\left(1+\frac{\cos A}{\sin A}\right)$$

$$= \sin A+\frac{\sin^2 A}{\cos A}+\cos A+\frac{\cos^2 A}{\sin A}$$

$$= \sin A+\frac{1-\cos^2 A}{\cos A}+\cos A+\frac{1-\sin^2 A}{\sin A}$$

$$= \sin A+\frac{1}{\cos A}-\cos A+\cos A+\frac{1}{\sin A}-\sin A$$

$$= \frac{1}{\cos A}+\frac{1}{\sin A} = \sec A+\operatorname{cosec} A.$$

Problem 17. $\dfrac{\sin 3A+\cos 3A}{\sin 3A-\cos 3A} = \dfrac{1+2\sin 2A}{1-2\sin 2A}\tan(A-45^\circ).$

Solution.

$$\frac{\sin 3A+\cos 3A}{\sin 3A-\cos 3A} = \frac{3\sin A-4\sin^3 A+4\cos^3 A-3\cos A}{3\sin A-4\sin^3 A-4\cos^3 A+3\cos A}$$

$$= \frac{3(\sin A-\cos A)-4(\sin^3 A-\cos^3 A)}{3(\sin A+\cos A)-4(\sin^3 A+\cos^3 A)}$$

$$
\begin{aligned}
&= \frac{\sin A - \cos A}{\sin A + \cos A} \times \frac{3 - 4(\sin^2 A + \cos^2 A + \sin A \cos A)}{3 - 4(\sin^2 A + \cos^2 A - \sin A \cos A)} \\
&= \frac{\sin A - \cos A}{\sin A + \cos A} \times \frac{-1 - 4 \sin A \cos A}{-1 + 4 \sin A \cos A} \\
&= \frac{\frac{\sin A}{\cos A} - 1}{\frac{\sin A}{\cos A} + 1} \frac{1 + 2\sin 2A}{1 - 2\sin 2A} \\
&= \frac{\tan A - 1}{\tan A + 1} \frac{1 + 2\sin 2A}{1 - 2\sin 2A} \\
&= \tan(A - 45^\circ) \frac{1 + 2\sin 2A}{1 - 2\sin 2A}.
\end{aligned}
$$

Problem 18. $\cos A + \cos(120^\circ - A) + \cos(120^\circ + A) = 0$.

Solution.

$$
\begin{aligned}
&\cos A + \cos(120^\circ - A) + \cos(120^\circ + A) \\
&= \cos A + \cos 120^\circ \cos A + \sin 120^\circ \sin A + \cos 120^\circ \cos A - \sin 120^\circ \sin A \\
&= \cos A + 2\cos 120^\circ \cos A = \cos A - \cos A = 0.
\end{aligned}
$$

Problem 19. $4 \sin A \sin(60^\circ - A) \sin(60^\circ + A) = \sin 3A$.

Solution.

$$
\begin{aligned}
&4 \sin A \sin(60^\circ - A) \sin(60^\circ + A) \\
&\quad = 4 \sin A(\sin^2 60^\circ - \sin^2 A), \text{ by } Art.\ 83 \text{ (page 404)}, \\
&\quad = 4 \sin A \left(\frac{3}{4} - \sin^2 A \right) \\
&\quad = 3 \sin A - 4 \sin^3 A = \sin 3A.
\end{aligned}
$$

Problem 20. $4 \cos A \cos(60^\circ - A) \cos(60^\circ + A) = \cos 3A$.

Solution.

$$
\begin{aligned}
&4 \cos A \cos(60^\circ - A) \cos(60^\circ + A) \\
&\quad = 4 \cos A(\cos^2 A - \sin^2 60^\circ), \text{ by } Art.\ 83 \text{ (page 404)}, \\
&\quad = 4 \cos A \left(\cos^2 A - \frac{3}{4} \right) \\
&\quad = 4 \cos^3 A - 3 \cos A = \cos 3A.
\end{aligned}
$$

Problem 21. $\tan A \tan(60^\circ + A) \tan(120^\circ + A) = -\tan 3A$.

Solution.

$$
\begin{aligned}
&\tan A \tan(60^\circ + A) \tan(120^\circ + A) \\
&\quad = \frac{\sin A \sin(60^\circ + A) \sin(120^\circ + A)}{\cos A \cos(60^\circ + A) \cos(120^\circ + A)}
\end{aligned}
$$

$$= -\frac{\sin A \sin(60^\circ + A)\sin(60^\circ + A)}{\cos A \cos(60^\circ + A)\cos(60^\circ + A)}, \text{ by } Art. \text{ 48 (page 397)},$$

$$= -\frac{\sin 3A}{\cos 3A}, \text{ by } Problems \text{ 19 and 20}, = -\tan 3A.$$

Problem 22. $\tan A + \tan(\mathit{60}^\circ + A) + \tan(\mathit{120}^\circ + A) = \mathit{3} \tan \mathit{3}A.$

Solution.

$$\tan A + \tan(60^\circ + A) + \tan(120^\circ + A)$$
$$= \tan A + \tan(60^\circ + A) - \tan(60^\circ - A), \text{ by } Art. \text{ 48 (page 397)},$$
$$= \tan A + \frac{\tan 60^\circ + \tan A}{1 - \tan 60^\circ \tan A} - \frac{\tan 60^\circ - \tan A}{1 + \tan 60^\circ \tan A}$$
$$= \tan A + \frac{(\tan 60^\circ + \tan A)(1 + \tan 60^\circ \tan A) - (\tan 60^\circ - \tan A)(1 - \tan 60^\circ \tan A)}{1 - \tan^2 60^\circ \tan^2 A}$$
$$= \tan A + \frac{2\tan^2 60^\circ \tan A + 2\tan A}{1 - \tan^2 60^\circ \tan^2 A}$$
$$= \tan A + \frac{8\tan A}{1 - 3\tan^2 A} = \frac{9\tan A - 3\tan^3 A}{1 - 3\tan^2 A}$$
$$= 3\tan 3A.$$

Problem 23. $\cot A + \cot(\mathit{60}^\circ + A) + \cot(\mathit{120}^\circ + A) = \mathit{3} \cot \mathit{3}A.$

Solution.

$$\cot A + \cot(60^\circ + A) + \cot(120^\circ + A)$$
$$= \frac{1}{\tan A} + \frac{1}{\tan(60^\circ + A)} - \frac{1}{\tan(60^\circ - A)}$$
$$= \frac{1}{\tan A} + \frac{1 - \tan 60^\circ \tan A}{\tan 60^\circ + \tan A} - \frac{1 + \tan 60^\circ \tan A}{\tan 60^\circ - \tan A}$$
$$= \frac{1}{\tan A} + \frac{(1 - \tan 60^\circ \tan A)(\tan 60^\circ - \tan A) - (1 + \tan 60^\circ \tan A)(\tan 60^\circ + \tan A)}{\tan^2 60^\circ - \tan^2 A}$$
$$= \frac{1}{\tan A} - \frac{2\tan^2 60^\circ \tan A + 2\tan A}{\tan^2 60^\circ - \tan^2 A}$$
$$= \frac{1}{\tan A} - \frac{8\tan A}{3 - \tan^2 A} = \frac{3 - 9\tan^2 A}{3\tan A - \tan^3 A}$$
$$= \frac{3}{\tan 3A} = 3\cot 3A.$$

Problem 24.

$$\cot A \cot(60^\circ + A) + \cot(60^\circ + A)\cot(120^\circ + A) + \cot(120^\circ + A)\cot A = -3.$$

Solution.

$$\cot A \cot(60^\circ + A) + \cot(60^\circ + A)\cot(120^\circ + A) + \cot(120^\circ + A)\cot A$$

$$= \frac{1}{\tan A \tan(60° + A)} + \frac{1}{\tan(60° + A)\tan(120° + A)} + \frac{1}{\tan(120° + A)\tan A}$$
$$= \frac{\tan(120° + A) + \tan A + \tan(60° + A)}{\tan A \tan(60° + A)\tan(120° + A)}$$
$$= \frac{3\tan 3A}{-\tan 3A}, \text{ by } \textit{Problems } 21 \text{ and } 22, = -3.$$

Problem 25. $\sin^3 A + \sin^3(120° + A) + \sin^3(240° + A) = -\frac{3}{4}\sin 3A.$

Solution.

$$\sin^3 A = \frac{1}{4}\{3\sin A - \sin 3A\},$$
$$\sin^3(120° + A) = \frac{1}{4}\{3\sin(120° + A) - \sin 3(120° + A)\}$$
$$= \frac{1}{4}\{3\sin(120° + A) - \sin 3A\},$$
$$\sin^3(240° + A) = \frac{1}{4}\{3\sin(240° + A) - \sin 3(240° + A)\}$$
$$= \frac{1}{4}\{3\sin(240° + A) - \sin 3A\},$$

By addition we obtain

$$\frac{3}{4}\{\sin A + \sin(120° + A) + \sin(240° + A)\} - \frac{3}{4}\sin 3A,$$

that is $-\frac{3}{4}\sin 3A$; for

$$\sin A + \sin(120° + A) + \sin(240° + A)$$
$$= \sin A + \sin(60° - A) - \sin(60° + A)$$
$$= \sin A + \sin 60° \cos A - \cos 60° \sin A - \sin 60° \cos A - \cos 60° \sin A$$
$$= \sin A - 2\cos 60° \sin A = \sin A - \sin A = 0.$$

Problem 26. $\sin 3A \sin^3 A + \cos 3A \cos^3 A = \cos^3 2A.$

Solution.

$$\sin 3A \sin^3 A + \cos 3A \cos^3 A$$
$$= (3\sin A - 4\sin^3 A)\sin^3 A + (4\cos^3 A - 3\cos A)\cos^3 A$$
$$= 3(\sin^4 A - \cos^4 A) - 4\sin^6 A + 4\cos^6 A$$
$$= 3(\sin^4 A - \cos^4 A)(\sin^2 A + \cos^2 A) - 4\sin^6 A + 4\cos^6 A$$
$$= \cos^6 A - 3\cos^4 A \sin^2 A + 3\cos^2 A \sin^4 A - \sin^6 A$$
$$= (\cos^2 A - \sin^2 A)^3 = \cos^3 2A.$$

Problem 27. $\cos^3 A \frac{\sin 3A}{3} + \sin^3 A \frac{\cos 3A}{3} = \frac{\sin 4A}{4}.$

Solution.

$$\cos^3 A \frac{\sin 3A}{3} + \sin^3 A \frac{\cos 3A}{3}$$
$$= \frac{1}{12}(3\cos A + \cos 3A)\sin 3A + \frac{1}{12}(3\sin A - \sin 3A)\cos 3A$$

$$= \frac{1}{4}(\sin 3A \cos A + \cos 3A \sin A)$$
$$= \frac{1}{4}\sin(3A + A) = \frac{1}{4}\sin 4A.$$

Problem 28. $\cos nA \cos(n+2)A - \cos^2(n+1)A + \sin^2 A = 0.$

Solution.

$$\cos nA \cos(n+2)A = \cos\{(n+1)A - A\}\cos\{(n+1)A + A\}$$
$$= \cos^2(n+1)A - \sin^2 A, \text{ by } Art.\ 83 \text{ (page 404);}$$

therefore $\cos nA \cos(n+2)A - \cos^2(n+1)A + \sin^2 A = 0.$

Problem 29. $\dfrac{\sin A \pm \sin nA + \sin(2n-1)A}{\cos A \pm \cos nA + \cos(2n-1)A} = \tan nA.$

Solution.

$$\frac{\sin A \pm \sin nA + \sin(2n-1)A}{\cos A \pm \cos nA + \cos(2n-1)A}$$
$$= \frac{\sin A + \sin(2n-1)A \pm \sin nA}{\cos A + \cos(2n-1)A \pm \cos nA}$$
$$= \frac{2\sin nA \cos(n-1)A \pm \sin nA}{2\cos nA \cos(n-1)A \pm \cos nA}, \text{ by } Art.\ 84 \text{ (page 405),}$$
$$= \frac{\sin nA\{2\cos(n-1)A \pm 1\}}{\cos nA\{2\cos(n-1)A \pm 1\}} = \frac{\sin nA}{\cos nA} = \tan nA.$$

Problem 30.
$\sin nA \operatorname{cosec}^2 A \sec A - \cos nA \sec^2 A \operatorname{cosec} A = 4\sin(n-1)A \operatorname{cosec}^2 2A.$

Solution.

$$\sin nA \operatorname{cosec}^2 A \sec A - \cos nA \sec^2 A \operatorname{cosec} A$$
$$= \frac{\sin nA}{\cos A \sin^2 A} - \frac{\cos nA}{\cos^2 A \sin A}$$
$$= \frac{\sin nA \cos A - \cos nA \sin A}{\sin^2 A \cos^2 A} = \frac{4\sin(nA - A)}{4\sin^2 A \cos A}$$
$$= \frac{4\sin(nA - A)}{\sin^2 2A} = 4\sin(nA - A)\operatorname{cosec}^2 2A.$$

Problem 31. $\cos 10A + \cos 8A + 3\cos 4A + 3\cos 2A = 8\cos A \cos^3 3A.$

Solution.

$$\cos 10A + \cos 8A + 3\cos 4A + 3\cos 2A$$
$$= 2\cos 9A \cos A + 6\cos 3A \cos A, \text{ by } Art.\ 84 \text{ (page 405),}$$
$$= 2\cos A(\cos 9A + 3\cos 3A)$$
$$= 2\cos A(4\cos^3 3A - 3\cos 3A + 3\cos 3A)$$
$$= 8\cos A \cos^3 3A.$$

Problem 32. $\cot A + \cot 2A + \cot 4A = \operatorname{cosec} 4A(2 + 2\cos 2A + 3\cos 4A)$.

Solution.

$$\begin{aligned}\cot A + \cot 2A + \cot 4A &= \frac{\cos A}{\sin A} + \frac{\cos 2A}{\sin 2A} + \frac{\cos 4A}{\sin 4A}\\ &= \frac{2\cos^2 A}{2\sin A\cos A} + \frac{\cos 2A}{\sin 2A} + \frac{\cos 4A}{\sin 4A}\\ &= \frac{1+2\cos 2A}{\sin 2A} + \frac{\cos 4A}{\sin 4A}\\ &= \frac{2\cos 2A(1+2\cos 2A)}{2\sin 2A\cos 2A} + \frac{\cos 4A}{\sin 4A}\\ &= \frac{1}{\sin 4A}\left\{2\cos 2A + 4\cos^2 2A + \cos 4A\right\}\\ &= \frac{1}{\sin 4A}\{2\cos 2A + 2(1+\cos 4A) + \cos 4A\}\\ &= \operatorname{cosec} 4A\{2 + 2\cos 2A + 3\cos 4A\}.\end{aligned}$$

Problem 33. $\operatorname{cosec} A = \dfrac{2\sin 2A + 2\cos 2A}{\cos A - \sin A - \cos 3A + \sin 3A}$.

Solution.

$$\frac{2\sin 2A + 2\cos 2A}{\cos A - \sin A - \cos 3A + \sin 3A} = \frac{2(\sin 2A + \cos 2A)}{\cos A - \cos 3A + \sin 3A - \sin A}$$

$$= \frac{2(\sin 2A + \cos 2A)}{2\sin 2A\sin A + 2\cos 2A\sin A},$$ by *Art.* 84 (page 405),

$$= \frac{2(\sin 2A + \cos 2A)}{2(\sin 2A + \cos 2A)\sin A} = \frac{1}{\sin A}.$$

Problem 34. $\cos^2 2A = (\cos A - \sin 3A)^2 + 2\cos A\sin 3A(\cos A - \sin A)^2$.

Solution.

$$\begin{aligned}&(\cos A - \sin 3A)^2 + 2\cos A\sin 3A(\cos A - \sin A)^2\\ &= \cos^2 A + \sin^2 3A - 2\cos A\sin 3A + 2\cos A\sin 3A(1 - 2\sin A\cos A)\\ &= \cos^2 A + \sin^2 3A - 2\cos A\sin 3A\sin 2A\\ &= \cos A\{\cos A - \sin 3A\sin 2A\} + \sin 3A\{\sin 3A - \cos A\sin 2A\}\\ &= \cos A\{\cos(3A - 2A) - \sin 3A\sin 2A\}\\ &\qquad + \sin 3A\{\sin(2A + A) - \cos A\sin 2A\}\\ &= \cos A\cos 3A\cos 2A + \sin 3A\sin A\cos 2A\\ &= \cos 2A\{\cos 3A\cos A + \sin 3A\sin A\}\\ &= \cos 2A\cos(3A - A) = \cos 2A\cos 2A = \cos^2 2A.\end{aligned}$$

Problem 35. $\cos^6 A - \sin^6 A = \cos 2A\left(1 - \frac{1}{4}\sin^2 2A\right)$.

Solution.

$$\begin{aligned}\cos^6 A - \sin^6 A &= \left(\cos^2 A - \sin^2 A\right)\left(\cos^4 A + \sin^4 A + \sin^2 A\cos^2 A\right)\\ &= \cos 2A\left(\cos^4 A + \sin^4 A + \sin^2 A\cos^2 A\right)\end{aligned}$$

$$= \cos 2A \left\{ \left(\cos^2 A + \sin^2 A\right)^2 - \sin^2 A \cos^2 A \right\}$$
$$= \cos 2A \left\{1 - \sin^2 A \cos^2 A\right\} = \cos 2A \left\{1 - \frac{\sin^2 2A}{4}\right\}.$$

Problem 36. $\sin 5A = 5 \sin A - 20 \sin^3 A + 16 \sin^5 A.$

Solution.

$$\begin{aligned}\sin 5A &= \sin(3A + 2A) = \sin 3A \cos 2A + \cos 3A \sin 2A \\ &= \left(3 \sin A - 4 \sin^3 A\right)\left(1 - 2\sin^2 A\right) + \left(4\cos^3 A - 3\cos A\right) 2 \sin A \cos A \\ &= \left(3 \sin A - 4 \sin^3 A\right)\left(1 - 2\sin^2 A\right) + \left(4\cos^2 A - 3\right) 2 \sin A \cos^2 A \\ &= \left(3 \sin A - 4 \sin^3 A\right)\left(1 - 2\sin^2 A\right) + \left(1 - 4\sin^2 A\right) 2 \sin A (1 - \sin^2 A) \\ &= 5 \sin A - 20 \sin^3 A + 16 \sin^5 A.\end{aligned}$$

Solve the following equations :

Problem 37. $\tan\left(\frac{\pi}{4} - \theta\right) + \cot\left(\frac{\pi}{4} - \theta\right) = 4.$

Solution.

$$\tan\left(\frac{\pi}{4} - \theta\right) + \cot\left(\frac{\pi}{4} - \theta\right) = 4;$$

therefore
$$\frac{\sin\left(\frac{\pi}{4} - \theta\right)}{\cos\left(\frac{\pi}{4} - \theta\right)} + \frac{\cos\left(\frac{\pi}{4} - \theta\right)}{\sin\left(\frac{\pi}{4} - \theta\right)} = 4;$$

therefore
$$\sin^2\left(\frac{\pi}{4} - \theta\right) + \cos^2\left(\frac{\pi}{4} - \theta\right) = 4 \sin\left(\frac{\pi}{4} - \theta\right) \cos\left(\frac{\pi}{4} - \theta\right);$$

therefore
$$1 = 2 \sin\left(\frac{\pi}{2} - 2\theta\right) = 2 \cos 2\theta;$$

therefore
$$\cos 2\theta = \frac{1}{2};$$

therefore
$$2\theta = 2n\pi \pm \frac{\pi}{3},$$

therefore
$$\theta = n\pi \pm \frac{\pi}{6}.$$

Problem 38. $\sin 4\theta + \sin \theta = 0.$

Solution.

$$\sin 4\theta + \sin \theta = 0,$$

therefore
$$2 \sin \frac{5\theta}{2} \cos \frac{3\theta}{2} = 0$$ by *Art.* 84 (page 405);

therefore either
$$\sin \frac{5\theta}{2} = 0, \text{ or } \cos \frac{3\theta}{2} = 0.$$

The former gives $\frac{5\theta}{2} = n\pi$; and the latter gives $\frac{3\theta}{2} = 2n\pi \pm \frac{\pi}{2}$, which may be expressed more simply as $\frac{3\theta}{2} = m\pi + \frac{\pi}{2}$.

Alternative Solution : Or we may proceed thus :

$$\sin 4\theta = -\sin\theta,$$

therefore $$\sin 4\theta = \sin(\pi + \theta).$$

Thus 4θ and $\pi + \theta$ must be angles which have the same sine; and therefore all the solutions are contained in $4\theta = n\pi + (-1)^n(\pi + \theta)$.

Problem 39. $\sin 7\theta - \sin\theta = \sin 3\theta$.

Solution.

$$\sin 7\theta - \sin\theta = \sin 3\theta;$$

therefore $$2\sin 3\theta \cos 4\theta = \sin 3\theta;$$

therefore either $$\sin 3\theta = 0, \text{ or } 2\cos 4\theta = 1.$$

The former gives $3\theta = n\pi$; and the latter gives $4\theta = 2n\pi \pm \frac{\pi}{3}$.

Problem 40. $\sin\theta + \cos\theta = \frac{1}{\sqrt{2}}$.

Solution.

$$\sin\theta + \cos\theta = \frac{1}{\sqrt{2}};$$

therefore $$\frac{\cos\theta}{\sqrt{2}} + \frac{\sin\theta}{\sqrt{2}} = \frac{1}{2};$$

therefore $$\cos\left(\theta - \frac{\pi}{4}\right) = \frac{1}{2};$$

therefore $$\theta - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{3}.$$

Problem 41. $\sin 5\theta = 16\sin^5\theta$.

Solution. By *Problem* 36 we have

$$\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta.$$

Thus $$5\sin\theta - 20\sin^3\theta + 16\sin^5\theta = 16\sin^5\theta,$$

therefore $$5\sin\theta - 20\sin^3\theta = 0,$$

therefore either $$\sin\theta = 0 \text{ or } \sin^2\theta = \frac{1}{4}.$$

The former gives $\theta = n\pi$; the latter gives $\sin^2\theta = \sin^2\frac{\pi}{6}$; and therefore $\theta = n\pi \pm \frac{\pi}{6}$, by *Chapter* V : *Problem* 5.

Problem 42. $\cos 3\theta + \cos 2\theta + \cos\theta = 0$.

Solution.

$$\cos 3\theta + \cos 2\theta + \cos\theta = 0,$$

therefore $$\cos 2\theta + 2\cos 2\theta\cos\theta = 0,$$

therefore either $\cos 2\theta = 0, \text{ or } \cos\theta = -\dfrac{1}{2}.$

The former gives $2\theta = n\pi + \dfrac{\pi}{2}$, as in *Problem* 38; and the latter gives $\theta = 2n\pi \pm \dfrac{2\pi}{3}$.

Problem 43. $\sin 3\theta + \sin 2\theta + \sin\theta = 0.$

Solution.

$$\sin 3\theta + \sin 2\theta + \sin\theta = 0,$$

therefore $$\sin 2\theta + 2\sin 2\theta\cos\theta = 0,$$

therefore either $\sin 2\theta = 0, \text{ or } \cos\theta = -\dfrac{1}{2}.$

The former gives $2\theta = n\pi$; and the latter gives $\theta = 2n\pi \pm \dfrac{2\pi}{3}$.

Problem 44. $\tan\theta + \tan\left(\dfrac{\pi}{4} + \theta\right) = 2.$

Solution.

$$\tan\theta + \tan\left(\frac{\pi}{4} + \theta\right) = 2;$$

therefore $$\tan\theta + \frac{1 + \tan\theta}{1 - \tan\theta} = 2,$$

therefore $$\tan\theta - \tan^2\theta + 1 + \tan\theta = 2 - 2\tan\theta,$$

therefore $$\tan^2\theta - 4\tan\theta + 1 = 0,$$

therefore $$\frac{\sin^2\theta}{\cos^2\theta} - \frac{4\sin\theta}{\cos\theta} + 1 = 0,$$

therefore $$\sin^2\theta + \cos^2\theta = 4\sin\theta\cos\theta,$$

$$1 = 4\sin\theta\cos\theta = 2\sin 2\theta,$$

therefore $$\sin 2\theta = \frac{1}{2}, \text{ therefore } 2\theta = n\pi + (-1)^n\frac{\pi}{6}.$$

Problem 45. $\tan 2\theta = 8\cos^2\theta - \cot\theta.$

Solution.

$$\tan 2\theta + \cot\theta = 8\cot^2\theta;$$

therefore $$\frac{\sin 2\theta}{\cos 2\theta} + \frac{\cos\theta}{\sin\theta} = 8\cos^2\theta,$$

therefore $$\sin 2\theta\sin\theta + \cos 2\theta\cos\theta = 8\cos^2\theta\sin\theta\cos 2\theta,$$

therefore $$\cos(2\theta - \theta) = 8\cos^2\theta\sin\theta\cos 2\theta;$$

therefore either $\cos\theta = 0, \text{ or } 1 = 8\cos\theta\sin\theta\cos 2\theta.$

The former gives $\theta = n\pi + \dfrac{\pi}{2}$; the latter gives

$$1 = 4\sin 2\theta\cos 2\theta = 2\sin 4\theta,$$

so that $$\sin 4\theta = \frac{1}{2}, \text{ and } 4\theta = n\pi + (-1)^n\frac{\pi}{6}.$$

Problem 46. $\tan\left(\frac{\pi}{4}+\theta\right) = 3\tan\left(\frac{\pi}{4}-\theta\right)$.

Solution.

$$\tan\left(\frac{\pi}{4}+\theta\right) = 3\tan\left(\frac{\pi}{4}-\theta\right),$$

therefore $$\tan\left(\frac{\pi}{4}+\theta\right) = 3\cot\left(\frac{\pi}{4}+\theta\right) = \frac{3}{\tan\left(\frac{\pi}{4}+\theta\right)},$$

therefore $$\tan^2\left(\frac{\pi}{4}+\theta\right) = 3 = \tan^2\frac{\pi}{3},$$

therefore $$\frac{\pi}{4}+\theta = n\pi \pm \frac{\pi}{3},$$ by *Chapter* V : *Problem* 9.

CHAPTER VII
Formulae for the Division of Angles

Problem 1. *Show that* $2\sin\frac{A}{2} = -\sqrt{1+\sin A} - \sqrt{1-\sin A}$, *when* A *lies between* $450°$ *and* $630°$.

Solution. Here $\frac{A}{2}$ lies between 225° and 315°; thus $\sin\frac{A}{2}$ is negative, and is numerically greater than $\cos\frac{A}{2}$; hence

$$\sin\frac{A}{2} + \cos\frac{A}{2} = -\sqrt{1+\sin A}, \quad \sin\frac{A}{2} - \cos\frac{A}{2} = -\sqrt{1-\sin A};$$

therefore
$$2\sin\frac{A}{2} = -\sqrt{1+\sin A} - \sqrt{1-\sin A}.$$

Problem 2. *Obtain* $\cos\frac{A}{2}$ *in terms of* $\sin A$ *when* $\frac{A}{2}$ *lies between* $405°$ *and* $495°$.

Solution. Here $\frac{A}{2}$ lies between 405° and 495°; thus $\sin\frac{A}{2}$ is negative, and is numerically greater than $\cos\frac{A}{2}$; hence

$$\sin\frac{A}{2} + \cos\frac{A}{2} = \sqrt{1+\sin A}, \quad \sin\frac{A}{2} - \cos\frac{A}{2} = \sqrt{1-\sin A};$$

therefore
$$2\cos\frac{A}{2} = \sqrt{1+\sin A} - \sqrt{1-\sin A}.$$

Problem 3. *Obtain* $\cos\frac{A}{2}$ *in terms of* $\sin A$ *when* $\frac{A}{2}$ *lies between* $-45°$ *and* $-135°$.

Solution. Here $\frac{A}{2}$ lies between $-45°$ and $-135°$; thus $\sin\frac{A}{2}$ is negative, and is numerically greater than $\cos\frac{A}{2}$; hence

$$\sin\frac{A}{2} + \cos\frac{A}{2} = -\sqrt{1+\sin A}, \quad \sin\frac{A}{2} - \cos\frac{A}{2} = -\sqrt{1-\sin A};$$

therefore
$$2\sin\frac{A}{2} = -\sqrt{1+\sin A} - \sqrt{1-\sin A}.$$

Problem 4. *Determine the limits between which* A *must lie in order that*

$$2\sin A = -\sqrt{1+\sin 2A} + \sqrt{1-\sin 2A}.$$

Solution. The proposed formula must have arisen from

$$\sin A + \cos A = -\sqrt{1+\sin 2A}, \quad \sin A - \cos A = \sqrt{1-\sin 2A};$$

the former shows that A must lie between $2n\pi + \frac{3\pi}{4}$ and $2n\pi + \frac{7\pi}{4}$, and the latter shows that A must lie between $2m\pi + \dfrac{\pi}{4}$ and $2m\pi + \dfrac{5\pi}{4}$; hence, by combining these results, it follows that A must lie between $2n\pi + \dfrac{3\pi}{4}$ and $2n\pi + \dfrac{5\pi}{4}$. See *Art.* 101 (page 408).

Problem 5. *Determine the limits between which A must lie in order that*

$$2\cos A = -\sqrt{1 + \sin 2A} + \sqrt{1 - \sin 2A}.$$

Solution. The proposed formula must have arisen from

$$\sin A + \cos A = -\sqrt{1 + \sin 2A}, \quad \sin A - \cos A = -\sqrt{1 - \sin 2A};$$

the former shows that A must lie between $2n\pi + \dfrac{3\pi}{4}$ and $2n\pi + \dfrac{7\pi}{4}$, and the latter shows that A must lie between $2m\pi + \dfrac{5\pi}{4}$ and $2m\pi + \dfrac{9\pi}{4}$; hence, by combining these results, it follows that A must lie between $2n\pi + \dfrac{5\pi}{4}$ and $2n\pi + \dfrac{7\pi}{4}$.

Problem 6. *Determine the limits between which A must lie in order that*

$$2\cos A = \sqrt{1 + \sin 2A} - \sqrt{1 - \sin 2A}.$$

Solution. The proposed formula must have arisen from

$$\sin A + \cos A = \sqrt{1 + \sin 2A}, \quad \sin A - \cos A = -\sqrt{1 - \sin 2A};$$

the former shows that A must lie between $2n\pi - \dfrac{\pi}{4}$ and $2n\pi + \dfrac{3\pi}{4}$, and the latter shows that A must lie between $2m\pi + \dfrac{5\pi}{4}$ and $2m\pi + \dfrac{9\pi}{4}$, that is, between $2(m+1)\pi - \dfrac{3\pi}{4}$ and $2(m+1)\pi + \dfrac{\pi}{4}$: hence, by combining these results, it follows that A must lie between $2n\pi - \dfrac{\pi}{4}$ and $2n\pi + \dfrac{\pi}{4}$.

Problem 7. *Divide a given angle into two parts whose sines shall be in a given ratio.*

Solution. Let A denote the given angle and m the given ratio. Let x denote one of the two parts and therefore $A - x$ the other. Then

$$\sin x = m\sin(A - x);$$

thus

$$\sin x = m(\sin A\cos x - \cos A\sin x).$$

Divide by $\cos x$; thus

$$\tan x = m(\sin A - \cos A\tan x),$$

therefore

$$\tan x = \frac{m\sin A}{1 + m\cos A}.$$

Thus $\tan x$ is known, and therefore x is known.

Problem 8. *Divide a given angle into two parts whose cosines shall be in a given ratio.*

Solution. Let A denote the given angle and m the given ratio. Let x denote one of the two parts and therefore $A - x$ the other. Then

$$\cos x = m\cos(A - x);$$

thus
$$\cos x = m(\cos A\cos x + \sin A\sin x).$$

Divide by $\cos x$; thus

$$1 = m(\cos A + \sin A\tan x),$$

therefore
$$\tan x = \frac{1 - m\cos A}{m\sin A}.$$

Thus $\tan x$ is known, and therefore x is known.

Problem 9. *Divide a given angle into two parts whose tangents shall be in a given ratio.*

Solution. Let A denote the given angle and m the given ratio. Let x denote one of the two parts and therefore $A - x$ the other. Then

$$\tan x = m\tan(A - x);$$

thus
$$\tan x = \frac{m(\tan A - \tan x)}{1 + \tan A\tan x};$$

therefore
$$\tan x(1 + \tan A\tan x) = m(\tan A - \tan x).$$

Thus we have a quadratic equation from which the value of $\tan x$ may be found.

Alternative Solution : Or we may proceed thus,

$$\tan x = m\tan(A - x),$$

therefore
$$\frac{\sin x}{\cos x} = \frac{m\sin(A - x)}{\cos(A - x)},$$

therefore
$$2\sin x\cos(A - x) = 2m\sin(A - x)\cos x,$$

therefore
$$\sin A + \sin(2x - A) = m\{\sin A + \sin(A - 2x)\}$$
$$= m\{\sin A - \sin(2x - A)\},$$

therefore
$$(m + 1)\sin(2x - A) = (m - 1)\sin A.$$

Thus $\sin(2x - A)$ is known, and therefore $2x - A$ is known, and therefore x is known.

Problem 10. *Given* $\tan\frac{A}{2} = 2 - \sqrt{3}$, *find* $\sin A$.

Solution. By *Art*. 87 (page 405),

$$\sin A = \frac{2\tan\frac{A}{2}}{1 + \tan^2\frac{A}{2}} = \frac{2(2 - \sqrt{3})}{1 + (2 - \sqrt{3})^2}$$
$$= \frac{2(2 - \sqrt{3})}{1 + 4 + 3 - 4\sqrt{3}} = \frac{2(2 - \sqrt{3})}{4(2 - \sqrt{3})} = \frac{1}{2}.$$

Problem 11. *Given* $\sin 210^\circ = -\frac{1}{2}$, *find* $\cos 105^\circ$.

Solution.

$$\sin 105^\circ + \cos 105^\circ = \sqrt{1 + \sin 210^\circ},$$

and $\sin 105^\circ - \cos 105^\circ = \sqrt{1 - \sin 210^\circ}$,

therefore $2\cos 105^\circ = \sqrt{1 + \sin 210^\circ} - \sqrt{1 - \sin 210^\circ}$

$$= \sqrt{1 - \frac{1}{2}} - \sqrt{1 + \frac{1}{2}} = \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{\sqrt{2}};$$

thus $2\cos 105^\circ = \dfrac{1 - \sqrt{3}}{\sqrt{2}}$,

and $\cos 105^\circ = \dfrac{1 - \sqrt{3}}{2\sqrt{2}} = -\dfrac{\sqrt{3} - 1}{2\sqrt{2}}$.

Problem 12. *Given* $\tan 2A = -\dfrac{24}{7}$, *find* $\sin A$ *and* $\cos A$.

Solution.

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}; \text{ thus } \frac{2\tan A}{1 - \tan^2 A} = -\frac{24}{7};$$

therefore $14\tan A = -24(1 - \tan^2 A)$;

therefore $24\tan^2 A - 14\tan A = 24$.

By solving this quadratic in the ordinary way we obtain

$$\tan A = \frac{4}{3}, \text{ or } -\frac{3}{4}.$$

Also $\sin A = \dfrac{\tan A}{\sqrt{1 + \tan^2 A}}$, and $\cos A = \dfrac{1}{\sqrt{1 + \tan^2 A}}$.

If $\tan A = \frac{4}{3}$ we get $\sin A = \pm\frac{4}{5}$, and $\cos A = \pm\frac{3}{5}$.

If $\tan A = -\frac{3}{4}$ we get $\sin A = \pm\frac{3}{5}$, and $\cos A = \mp\frac{4}{5}$.

Problem 13. *Find* $\tan 165^\circ$ *from the known value of* $\tan 330^\circ$.

Solution.

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}. \quad \text{Let } 2A = 330^\circ, \text{ then } \tan 2A = -\frac{1}{\sqrt{3}}.$$

therefore $-\dfrac{1}{\sqrt{3}} = \dfrac{2\tan A}{1 - \tan^2 A}$,

therefore $-1 + \tan^2 A = 2\sqrt{3}\tan A$, therefore $\tan^2 A - 2\sqrt{3}\tan A = 1$.

By solving this quadratic in the ordinary way we obtain $\tan A = \sqrt{3} \pm 2$.
But $\tan 165^\circ$ must be a negative quantity, and is therefore equal to $\sqrt{3} - 2$.

Problem 14. *Show that* $\tan^2 \dfrac{A}{2} = \dfrac{2\sin A - \sin 2A}{2\sin A + \sin 2A}$.

Solution.

$$\frac{2\sin A - \sin 2A}{2\sin A + \sin 2A} = \frac{2\sin A - 2\sin A\cos A}{2\sin A + 2\sin A\cos A} = \frac{2\sin A(1 - \cos A)}{2\sin A(1 + \cos A)}$$

$$= \frac{1-\cos A}{1+\cos A} = \frac{2\sin^2 \dfrac{A}{2}}{2\cos^2 \dfrac{A}{2}} = \tan^2 \frac{A}{2}.$$

Problem 15. $\text{vers}\,(180^\circ - A) = 2\,\text{vers}\,\dfrac{180^\circ + A}{2}\,\text{vers}\,\dfrac{180^\circ - A}{2}$.

Solution.

$$\begin{aligned} &2\,\text{vers}\,\frac{1}{2}(180^\circ + A)\,\text{vers}\,\frac{1}{2}(180^\circ - A) \\ &\qquad = 2\left\{1-\cos\left(90^\circ + \frac{A}{2}\right)\right\}\left\{1-\cos\left(90^\circ - \frac{A}{2}\right)\right\} \\ &\qquad = 2\left(1+\sin\frac{A}{2}\right)\left(1-\sin\frac{A}{2}\right) \\ &\qquad = 2\left(1-\sin^2\frac{A}{2}\right) = 2\cos^2\frac{A}{2}; \end{aligned}$$

and $\qquad \text{vers}\,(180^\circ - A) = 1-\cos(180^\circ - A) = 1+\cos A = 2\cos^2 \dfrac{A}{2}.$

Thus the proposed expressions are equal.

Problem 16. $(\cos A + \cos B)^2 + (\sin A + \sin B)^2 = 4\cos^2 \dfrac{A-B}{2}$.

Solution.

$$\begin{aligned} &(\cos A + \cos B)^2 + (\sin A + \sin B)^2 \\ &= \cos^2 A + \cos^2 B + 2\cos A\cos B + \sin^2 A + \sin^2 B + 2\sin A\sin B \\ &= 2 + 2(\cos A\cos B + \sin A\sin B) = 2 + 2\cos(A-B) \\ &= 2\{1+\cos(A-B)\} = 4\cos^2 \frac{1}{2}(A-B). \end{aligned}$$

Problem 17. $(\cos A - \cos B)^2 + (\sin A - \sin B)^2 = 4\sin^2 \dfrac{A-B}{2}$.

Solution.

$$\begin{aligned} &(\cos A - \cos B)^2 + (\sin A - \sin B)^2 \\ &= \cos^2 A + \cos^2 B - 2\cos A\cos B + \sin^2 A + \sin^2 B - 2\sin A\sin B \\ &= 2 - 2(\cos A\cos B + \sin A\sin B) = 2 - 2\cos(A-B) \\ &= 2\{1-\cos(A-B)\} = 4\sin^2 \frac{1}{2}(A-B). \end{aligned}$$

Problem 18. *Show that*

$$\sin 22\frac{1}{2}^\circ = \frac{\sqrt{2-\sqrt{2}}}{2},$$
$$\cos 22\frac{1}{2}^\circ = \frac{\sqrt{2+\sqrt{2}}}{2},$$

and $$\tan 22\frac{1}{2}^\circ = \sqrt{2}-1.$$

Solution.

$$2\sin^2 22\frac{1}{2}^\circ = 1-\cos 45^\circ;\ \text{therefore}$$
$$4\sin^2 22\frac{1}{2}^\circ = 2-2\cos 45^\circ = 2-\frac{2}{\sqrt{2}} = 2-\sqrt{2},$$

therefore $$2\sin 22\frac{1}{2}^\circ = \sqrt{2-\sqrt{2}}.$$

And $$2\cos^2 22\frac{1}{2}^\circ = 1+\cos 45^\circ;\text{therefore}$$
$$4\cos^2 22\frac{1}{2}^\circ = 2+2\cos 45^\circ = 2+\frac{2}{\sqrt{2}} = 2+\sqrt{2},$$

therefore $$2\cos 22\frac{1}{2}^\circ = \sqrt{2+\sqrt{2}}.$$

Hence $$\frac{\sin 22\frac{1}{2}^\circ}{\cos 22\frac{1}{2}^\circ} = \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}} = \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}\cdot\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2-\sqrt{2}}}$$
$$= \frac{2-\sqrt{2}}{\sqrt{4-2}} = \frac{2-\sqrt{2}}{\sqrt{2}} = \sqrt{2}-1,$$

that is $$\tan 22\frac{1}{2}^\circ = \sqrt{2}-1.$$

Problem 19. $(\tan A+\cot A)\,2\tan\frac{A}{2}\left(1-\tan^2\frac{A}{2}\right) = \left(1+\tan^2\frac{A}{2}\right)^2.$

Solution.

$$(\tan A+\cot A)2\tan\frac{A}{2}\left(1-\tan^2\frac{A}{2}\right)$$
$$= \left(\frac{\sin A}{\cos A}+\frac{\cos A}{\sin A}\right)2\tan\frac{A}{2}\left(1-\tan^2\frac{A}{2}\right)$$
$$= \frac{\sin^2 A+\cos^2 A}{\sin A\cos A}\cdot\frac{2\sin\frac{A}{2}}{\cos\frac{A}{2}}\cdot\frac{\cos^2\frac{A}{2}-\sin^2\frac{A}{2}}{\cos^2\frac{A}{2}}$$
$$= \frac{1}{\sin A\cos A}\cdot\frac{2\sin\frac{A}{2}}{\cos\frac{A}{2}}\cdot\frac{\cos A}{\cos^2\frac{A}{2}}$$
$$= \frac{2\sin\frac{A}{2}}{\sin A\cos^3\frac{A}{2}} = \frac{2\sin\frac{A}{2}}{2\sin\frac{A}{2}\cos^4\frac{A}{2}} = \frac{1}{\cos^4\frac{A}{2}}$$
$$= \left\{\frac{1}{\cos^2\frac{A}{2}}\right\}^2 = \left(1+\tan^2\frac{A}{2}\right)^2.$$

Problem 20. $\tan^2\left(\frac{\pi}{4}+\frac{A}{2}\right)=\frac{\sec A+\tan A}{\sec A-\tan A}$.

Solution.

$$\tan^2\left(\frac{\pi}{4}+\frac{A}{2}\right)=\left\{\frac{1+\tan\frac{A}{2}}{1-\tan\frac{A}{2}}\right\}^2=\left\{\frac{\cos\frac{A}{2}+\sin\frac{A}{2}}{\cos\frac{A}{2}-\sin\frac{A}{2}}\right\}^2$$

$$=\frac{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}+2\sin\frac{A}{2}\cos\frac{A}{2}}{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}-2\sin\frac{A}{2}\cos\frac{A}{2}}=\frac{1+\sin A}{1-\sin A}$$

$$=\frac{\frac{1}{\cos A}+\frac{\sin A}{\cos A}}{\frac{1}{\cos A}-\frac{\sin A}{\cos A}}=\frac{\sec A+\tan A}{\sec A-\tan A}.$$

Problem 21. $\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)=\frac{\sin\theta}{\sqrt{\operatorname{vers}\theta}}$.

Solution.

$$\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$=\sin\frac{\pi}{4}\cos\frac{\theta}{2}-\cos\frac{\pi}{4}\sin\frac{\theta}{2}+\cos\frac{\pi}{4}\cos\frac{\theta}{2}+\sin\frac{\pi}{4}\sin\frac{\theta}{2}$$

$$=2\sin\frac{\pi}{4}\cos\frac{\theta}{2}=\frac{2\cos\frac{\theta}{2}}{\sqrt{2}}=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\sqrt{2}\sin\frac{\theta}{2}}$$

$$=\frac{\sin\theta}{\sqrt{2\sin^2\frac{\theta}{2}}}=\frac{\sin\theta}{\sqrt{1-\cos\theta}}=\frac{\sin\theta}{\sqrt{\operatorname{vers}\theta}}.$$

Problem 22. *Show that* $4\sin^2\frac{\theta}{4}\left(1-\sin\frac{\theta}{2}\right)=\left\{1-\sqrt{1+\sin\theta}\right\}^2$.

Solution.

$$4\sin^2\frac{\theta}{4}\left(1-\sin\frac{\theta}{2}\right)=4\sin^2\frac{\theta}{4}\left(\sin^2\frac{\theta}{4}+\cos^2\frac{\theta}{4}-2\sin\frac{\theta}{4}\cos\frac{\theta}{4}\right)$$

$$=4\sin^2\frac{\theta}{4}\left(\sin\frac{\theta}{4}-\cos\frac{\theta}{4}\right)^2$$

$$=\left(2\sin^2\frac{\theta}{4}-2\sin\frac{\theta}{4}\cos\frac{\theta}{4}\right)^2$$

$$=\left(1-\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2.$$

And $\sqrt{1+\sin\theta}=\sqrt{\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}=\sin\frac{\theta}{2}+\cos\frac{\theta}{2}$;

therefore $$\left\{1-\sqrt{1+\sin\theta}\right\}^2 = \left(1-\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right)^2.$$

Problem 23. $\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8}=\frac{3}{2}.$

Solution.

$$\begin{aligned}2\cos^2\frac{\theta}{2} &= 1+\cos\theta;\ \text{therefore}\\ 4\cos^4\frac{\theta}{2} &= (1+\cos\theta)^2 = 1+2\cos\theta+\cos^2\theta\\ &= 1+2\cos\theta+\frac{1+\cos 2\theta}{2} = \frac{1}{2}(3+4\cos\theta+\cos 2\theta);\end{aligned}$$

therefore $$\cos^4\frac{\theta}{2} = \frac{1}{8}(3+4\cos\theta+\cos 2\theta).$$

Use this formula for each of the terms; thus

$$\begin{aligned}&\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8}\\ &= \frac{12}{8}+\frac{1}{2}\left(\cos\frac{\pi}{4}+\cos\frac{3\pi}{4}+\cos\frac{5\pi}{4}+\cos\frac{7\pi}{4}\right)\\ &\quad+\frac{1}{8}\left(\cos\frac{\pi}{2}+\cos\frac{3\pi}{2}+\cos\frac{5\pi}{2}+\cos\frac{7\pi}{2}\right)\\ &= \frac{3}{2}:\ \text{see } Art.\ 50\ (\text{page } 400).\end{aligned}$$

Problem 24. $\tan 7\frac{1}{2}^\circ = \sqrt{6}-\sqrt{3}+\sqrt{2}-2.$

Solution.

$$\begin{aligned}\tan 7\frac{1}{2}^\circ &= \frac{\sin 15^\circ}{1+\cos 15^\circ} = \frac{\frac{\sqrt{3}-1}{2\sqrt{2}}}{1+\frac{\sqrt{3}+1}{2\sqrt{2}}} = \frac{\sqrt{3}-1}{2\sqrt{2}+\sqrt{3}+1}\\ &= \frac{(\sqrt{3}-1)(2\sqrt{2}+1-\sqrt{3})}{(2\sqrt{2}+\sqrt{3}+1)(2\sqrt{2}+1-\sqrt{3})} = \frac{2\sqrt{6}-2\sqrt{2}-4+2\sqrt{3}}{6+4\sqrt{2}}\\ &= \frac{\sqrt{6}-\sqrt{2}-2+\sqrt{3}}{3+2\sqrt{2}}.\end{aligned}$$

Multiply both numerator and denominator by $3-2\sqrt{2}$; then we obtain unity for denominator, and for numerator $\sqrt{6}-\sqrt{3}+\sqrt{2}-2$.

Problem 25. $\tan 142\frac{1}{2}^\circ = 2+\sqrt{2}-\sqrt{3}-\sqrt{6}.$

Solution.

$$\tan 142\frac{1}{2}^\circ = \frac{\sin 285^\circ}{1+\cos 285^\circ} = -\frac{\sin 105^\circ}{1-\cos 105^\circ} = -\frac{\cos 15^\circ}{1+\sin 15^\circ}$$

$$= -\frac{\dfrac{\sqrt{3}+1}{2\sqrt{2}}}{1+\dfrac{\sqrt{3}-1}{2\sqrt{2}}} = -\frac{\sqrt{3}+1}{2\sqrt{2}-1+\sqrt{3}}$$
$$= -\frac{(\sqrt{3}+1)(2\sqrt{2}-1-\sqrt{3})}{(2\sqrt{2}-1+\sqrt{3})(2\sqrt{2}-1-\sqrt{3})}$$
$$= -\frac{\sqrt{6}+\sqrt{2}-2-\sqrt{3}}{3-2\sqrt{2}} = \frac{2+\sqrt{3}-\sqrt{2}-\sqrt{6}}{3-2\sqrt{2}}.$$

Multiply both numerator and denominator by $3+2\sqrt{2}$; then we obtain unity for denominator, and for numerator $2+\sqrt{2}-\sqrt{3}-\sqrt{6}$.

Problem 26. *If* $\tan x = (2+\sqrt{3})\tan\frac{x}{3}$, *find the value of* $\tan x$.

Solution.

$$\tan x = \frac{3\tan\frac{x}{3} - \tan^3\frac{x}{3}}{1-3\tan^2\frac{x}{3}};$$

and since this is equal to $(2+\sqrt{3}\tan\frac{x}{3}$ we obtain

$$\frac{3-\tan^2\frac{x}{3}}{1-3\tan^2\frac{x}{3}} = 2+\sqrt{3};$$

therefore $$3-\tan^2\frac{x}{3} = (2+\sqrt{3})\left(1-3\tan^2\frac{x}{3}\right);$$

therefore $$(6+3\sqrt{3}-1)\tan^2\frac{x}{3} = 2+\sqrt{3}-3;$$

therefore $$\tan^2\frac{x}{3} = \frac{\sqrt{3}-1}{5+3\sqrt{3}} = \frac{(\sqrt{3}-1)(5-3\sqrt{3})}{(5+3\sqrt{3})(5-3\sqrt{3})}$$
$$= \frac{8\sqrt{3}-14}{25-27} = 7-4\sqrt{3};$$

therefore $$\tan\frac{x}{3} = \sqrt{7-4\sqrt{3}} = \pm(2-\sqrt{3}).$$

Hence $$\tan x = \pm(2+\sqrt{3})(2-\sqrt{3}) = \pm 1.$$

Problem 27. *If* $\alpha = \left(n+\frac{1}{4}\pm\frac{1}{6}\right)\pi$, *where n is any integer, find the value of* $\tan\alpha + \cot\alpha$.

Solution.

$$\tan\alpha+\cot\alpha = \frac{\sin\alpha}{\cos\alpha}+\frac{\cos\alpha}{\sin\alpha} = \frac{\sin^2\alpha+\cos^2\alpha}{\sin\alpha\cos\alpha} = \frac{1}{\sin\alpha\cos\alpha}$$
$$= \frac{2}{2\sin\alpha\cos\alpha} = \frac{2}{\sin 2\alpha}.$$

Put for α its value; then the expression

$$= \frac{2}{\sin 2\left(n+\frac{1}{4}\pm\frac{1}{6}\right)\pi} = \frac{2}{\sin\left(\frac{\pi}{2}\pm\frac{\pi}{3}\right)} = \frac{2}{\cos\frac{\pi}{3}}$$

$$= 2 \div \frac{1}{2} = 4.$$

Problem 28. *If* $\alpha = \frac{\pi}{17}$, *find the value of* $\frac{\cos\alpha\cos 13\alpha}{\cos 3\alpha + \cos 5\alpha}$.

Solution.

$$\frac{\cos\alpha\cos 13\alpha}{\cos 3\alpha+\cos 5\alpha} = \frac{\cos\alpha\cos 13\alpha}{2\cos\alpha\cos 4\alpha} = \frac{\cos 13\alpha}{2\cos 4\alpha} = -\frac{1}{2};$$

for $13\alpha + 4\alpha = \pi$, and therefore $\cos 13\alpha = -\cos 4\alpha$.

Problem 29. *If* $\sec(\phi+\alpha)+\sec(\phi-\alpha) = 2\sec\phi$, *then* $\cos\phi = \sqrt{2}\cos\frac{\alpha}{2}$.

Solution.

$$\sec(\phi+\alpha)+\sec(\phi-\alpha) = 2\sec\phi,$$

therefore $$\frac{1}{\cos(\phi+\alpha)} + \frac{1}{\cos(\phi-\alpha)} = \frac{2}{\cos\phi};$$

therefore $$\frac{\cos(\phi-\alpha)+\cos(\phi+\alpha)}{\cos(\phi+\alpha)\cos(\phi-\alpha)} = \frac{2}{\cos\phi};$$

therefore $$\frac{2\cos\phi\cos\alpha}{\cos^2\phi-\sin^2\alpha} = \frac{2}{\cos\phi};$$

therefore $$\cos^2\phi\cos\alpha = \cos^2\phi - \sin^2\alpha;$$

therefore $$\cos^2\phi = \frac{\sin^2\alpha}{1-\cos\alpha} = \frac{1-\cos^2\alpha}{1-\cos\alpha} = 1+\cos\alpha = 2\cos^2\frac{\alpha}{2};$$

therefore $$\cos\phi = \sqrt{2}\cos\frac{\alpha}{2}.$$

Problem 30. *If* $\tan\frac{\theta}{2} = \left(\frac{1+c}{1-c}\right)^{\frac{1}{2}}\tan\frac{\phi}{2}$, *show that* $\cos\theta = \frac{\cos\phi - c}{1-c\cos\phi}$.

Solution.

$$\tan^2\frac{\theta}{2} = \frac{1+c}{1-c}\tan^2\frac{\phi}{2};$$

therefore $$\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{1-\frac{1+c}{1-c}\cdot\frac{\sin^2\frac{\phi}{2}}{\cos^2\frac{\phi}{2}}}{1+\frac{1+c}{1-c}\cdot\frac{\sin^2\frac{\phi}{2}}{\cos^2\frac{\phi}{2}}} = \frac{(1-c)\cos^2\frac{\phi}{2}-(1+c)\sin^2\frac{\phi}{2}}{(1-c)\cos^2\frac{\phi}{2}+(1+c)\sin^2\frac{\phi}{2}};$$

therefore, by *Art.* 87 (page 405),

$$\cos\theta = \frac{\cos^2\dfrac{\phi}{2} - \sin^2\dfrac{\phi}{2} - c\left(\cos^2\dfrac{\phi}{2} + \sin^2\dfrac{\phi}{2}\right)}{\cos^2\dfrac{\phi}{2} + \sin^2\dfrac{\phi}{2} - c\left(\cos^2\dfrac{\phi}{2} - \sin^2\dfrac{\phi}{2}\right)} = \frac{\cos\phi - c}{1 - c\cos\phi}.$$

CHAPTER VIII
Miscellaneous Propositions

Prove the following formulae :

Problem 1. $\dfrac{\cos(\alpha+\beta+\gamma)}{\cos\alpha\cos\beta\cos\gamma} = 1 - \tan\beta\tan\gamma - \tan\gamma\tan\alpha - \tan\alpha\tan\beta.$

Solution. By *Art.* 113 (page 409) we have

$$\cos(\alpha+\beta+\gamma) = \cos\alpha\cos\beta\cos\gamma - \cos\alpha\sin\beta\sin\gamma - \cos\beta\sin\gamma\sin\alpha - \cos\gamma\sin\alpha\sin\beta;$$

divide both sides by $\cos\alpha\cos\beta\cos\gamma$; thus

$$\frac{\cos(\alpha+\beta+\gamma)}{\cos\alpha\cos\beta\cos\gamma} = 1 - \tan\beta\tan\gamma - \tan\gamma\tan\alpha - \tan\alpha\tan\beta.$$

Problem 2. $\dfrac{\sin(\alpha+\beta+\gamma)}{\cos\alpha\cos\beta\cos\gamma} = \tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma.$

Solution. By *Art.* 113 (page 409) we have

$$\sin(\alpha+\beta+\gamma) = \sin\alpha\cos\beta\cos\gamma + \sin\beta\cos\gamma\cos\alpha + \sin\gamma\cos\alpha\cos\beta - \sin\alpha\sin\beta\sin\gamma;$$

divide both sides by $\cos\alpha\cos\beta\cos\gamma$; thus

$$\frac{\sin(\alpha+\beta+\gamma)}{\cos\alpha\cos\beta\cos\gamma} = \tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma.$$

Problem 3.

$$\sin(\alpha-\beta) + \sin(\beta-\gamma) + \sin(\gamma-\alpha) + 4\sin\frac{\alpha-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2} = 0.$$

Solution.

$$\sin(\alpha-\beta) + \sin(\beta-\gamma) = 2\sin\frac{\alpha-\gamma}{2}\cos\frac{\alpha-2\beta+\gamma}{2} = -2\sin\frac{\gamma-\alpha}{2}\cos\frac{\alpha-2\beta+\gamma}{2};$$

$$\sin(\gamma-\alpha) = 2\sin\frac{\gamma-\alpha}{2}\cos\frac{\gamma-\alpha}{2};$$

therefore $\sin(\alpha-\beta) + \sin(\beta-\gamma) + \sin(\gamma-\alpha)$

$$= 2\sin\frac{\gamma-\alpha}{2}\left\{\cos\frac{\gamma-\alpha}{2} - \cos\frac{\alpha-2\beta+\gamma}{2}\right\}$$

$$= 2\sin\frac{\gamma-\alpha}{2}2\sin\frac{\gamma-\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$= -4\sin\frac{\alpha-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2};$$

therefore $\sin(\alpha-\beta) + \sin(\beta-\gamma) + \sin(\gamma-\alpha)$

$$+4\sin\frac{\alpha-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}=0$$

Problem 4.

$$4\sin(\theta-\alpha)\sin(m\theta-\alpha)\cos(\theta-m\theta)$$
$$=1+\cos(2\theta-2m\theta)-\cos(2\theta-2\alpha)-\cos(2m\theta-2\alpha).$$

Solution.

$$\begin{aligned}
&4\sin(\theta-\alpha)\sin(m\theta-\alpha)\cos(\theta-m\theta)\\
&\qquad \text{by } Art.\ 84 \text{ (page 405)},\\
&\qquad =2\cos(\theta-m\theta)\left\{\cos(\theta-m\theta)-\cos(\theta+m\theta-2\alpha)\right\},\\
&\qquad =2\cos^2(\theta-m\theta)-2\cos(\theta-m\theta)\cos(\theta+m\theta-2\alpha)\\
&\qquad =1+\cos 2(\theta-m\theta)-\{\cos(2\theta-2\alpha)+\cos(2m\theta-2\alpha)\}\\
&\qquad =1+\cos 2(\theta-m\theta)-\cos(2\theta-2\alpha)-\cos(2m\theta-2\alpha).
\end{aligned}$$

Problem 5. $\sin(\alpha+\beta)\cos\beta-\sin(\alpha+\gamma)\cos\gamma=\sin(\beta-\gamma)\cos(\alpha+\beta+\gamma)$.

Solution.

$$\begin{aligned}
\sin(\alpha+\beta)\cos\beta&=\sin(\alpha+\beta+\gamma-\gamma)\cos\beta\\
&=\{\sin(\alpha+\beta+\gamma)\cos\gamma-\cos(\alpha+\beta+\gamma)\sin\gamma\}\cos\beta,\\
\sin(\alpha+\gamma)\cos\gamma&=\sin(\alpha+\beta+\gamma-\beta)\cos\gamma\\
&=\{\sin(\alpha+\beta+\gamma)\cos\beta-\cos(\alpha+\beta+\gamma)\sin\beta\}\cos\gamma;\\
\therefore\ \sin(\alpha+\beta)\cos\beta&-\sin(\alpha+\gamma)\cos\gamma\\
&=\cos(\alpha+\beta+\gamma)\{\sin\beta\cos\gamma-\sin\gamma\cos\beta\}\\
&=\cos(\alpha+\beta+\gamma)\sin(\beta-\gamma).
\end{aligned}$$

Problem 6.

$$\cos(\alpha+\beta+\gamma)+\cos(\alpha+\beta-\gamma)+\cos(\alpha+\gamma-\beta)$$
$$+\cos(\beta+\gamma-\alpha)=4\cos\alpha\cos\beta\cos\gamma.$$

Solution.

$$\begin{aligned}
\cos(\alpha+\beta+\gamma)+\cos(\alpha+\beta-\gamma)&=2\cos(\alpha+\beta)\cos\gamma,\\
\cos(\alpha-\beta+\gamma)+\cos(\beta+\gamma-\alpha)&=2\cos(\alpha-\beta)\cos\gamma;\\
\text{hence the sum}\qquad &=2\cos\gamma\{\cos(\alpha+\beta)+\cos(\alpha-\beta)\}\\
&=4\cos\alpha\cos\beta\cos\gamma.
\end{aligned}$$

Problem 7.

$$\cos 2\alpha+\cos 2\beta+\cos 2\gamma+\cos 2(\alpha+\beta+\gamma)$$
$$=4\cos(\alpha+\beta)\cos(\beta+\gamma)\cos(\gamma+\alpha).$$

Solution.

$$\begin{aligned}
\cos 2\alpha+\cos 2\beta&=2\cos(\alpha+\beta)\cos(\alpha-\beta),\\
\cos 2\gamma+\cos 2(\alpha+\beta+\gamma)&=2\cos(2\gamma+\alpha+\beta)\cos(\alpha+\beta);\\
\text{hence the sum}\qquad &=2\cos(\alpha+\beta)\{\cos(\alpha-\beta)+\cos(2\gamma+\alpha+\beta)\}
\end{aligned}$$

$$= 2\cos(\alpha+\beta)2\cos(\alpha+\gamma)\cos(\beta+\gamma)$$
$$= 4\cos(\alpha+\beta)\cos(\beta+\gamma)\cos(\gamma+\alpha).$$

Problem 8.

$$\frac{\sin\alpha}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}+\frac{\sin\beta}{\sin(\beta-\gamma)\sin(\beta-\alpha)}+\frac{\sin\gamma}{\sin(\gamma-\alpha)\sin(\gamma-\beta)}=0.$$

Solution. Reduce the three fractions to have the common denominator

$$\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha);$$

then the whole numerator

$$\begin{aligned}
&= -\sin\alpha\sin(\beta-\gamma)-\sin\beta\sin(\gamma-\alpha)-\sin\gamma\sin(\alpha-\beta)\\
&= -\frac{1}{2}\{\cos(\alpha-\beta+\gamma)-\cos(\alpha+\beta-\gamma)\}\\
&\quad -\frac{1}{2}\{\cos(\beta+\alpha-\gamma)-\cos(\beta+\gamma-\alpha)\}\\
&\quad -\frac{1}{2}\{\cos(\gamma-\alpha+\beta)-\cos(\gamma+\alpha-\beta)\}=0.
\end{aligned}$$

Problem 9.

$$\cos(\alpha+\beta)\sin\beta-\cos(\alpha+\gamma)\sin\gamma$$
$$=\sin(\alpha+\beta)\cos\beta-\sin(\alpha+\gamma)\cos\gamma.$$

Solution.

$$\begin{aligned}
&\cos(\alpha+\beta)\sin\beta-\cos(\alpha+\gamma)\sin\gamma\\
&=\frac{1}{2}\{\sin(\alpha+\beta+\beta)-\sin(\alpha+\beta-\beta)\}-\frac{1}{2}\{\sin(\alpha+\gamma+\gamma)-\sin(\alpha+\gamma-\gamma)\}\\
&=\frac{1}{2}\sin(\alpha+2\beta)-\frac{1}{2}\sin(\alpha+2\gamma);\\
&\sin(\alpha+\beta)\cos\beta-\sin(\alpha+\gamma)\cos\gamma\\
&=\frac{1}{2}\{\sin(\alpha+\beta+\beta)+\sin(\alpha+\beta-\beta)\}-\frac{1}{2}\{\sin(\alpha+\gamma+\gamma)+\sin(\alpha+\gamma-\gamma)\}\\
&=\frac{1}{2}\sin(\alpha+2\beta)-\frac{1}{2}\sin(\alpha+2\gamma);
\end{aligned}$$

Thus the two expressions are equal.

Problem 10.

$$\sin(\alpha+\beta-2\gamma)\cos\beta-\sin(\alpha+\gamma-2\beta)\cos\gamma$$
$$=\sin(\beta-\gamma)\{\cos(\beta+\gamma-\alpha)+\cos(\alpha+\gamma-\beta)+\cos(\alpha+\beta-\gamma)\}.$$

Solution.

$$\begin{aligned}
&\sin(\alpha+\beta-2\gamma)\cos\beta-\sin(\alpha+\gamma-2\beta)\cos\gamma\\
&=\frac{1}{2}\{\sin(\alpha+2\beta-2\gamma)+\sin(\alpha-2\gamma)-\sin(\alpha+2\gamma-2\beta)-\sin(\alpha-2\beta)\};\\
&\sin(\beta-\gamma)\{\cos(\beta+\gamma-\alpha)+\cos(\alpha+\gamma-\beta)+\cos(\alpha+\beta-\gamma)\}
\end{aligned}$$

$$= \frac{1}{2}\{\sin(2\beta-\alpha)+\sin(\alpha-2\gamma)\}+\frac{1}{2}\{\sin\alpha+\sin(2\beta-2\gamma-\alpha)\}$$
$$+\frac{1}{2}\{-\sin\alpha+\sin(2\beta-2\gamma+\alpha)\}$$
$$= \frac{1}{2}\{\sin(2\beta-\alpha)+\sin(\alpha-2\gamma)+\sin(2\beta-2\gamma-\alpha)+\sin(2\beta-2\gamma+\alpha)\}.$$

Thus the two expressions are equal.

Problem 11. $\sin(\alpha+\beta+\gamma)\sin\beta=\sin(\alpha+\beta)\sin(\beta+\gamma)-\sin\alpha\sin\gamma$.

Solution.

$$\sin(\alpha+\beta+\gamma)\sin\beta=\frac{1}{2}\{\cos(\alpha+\gamma)-\cos(\alpha+2\beta+\gamma)\},$$
$$\sin(\alpha+\beta)\sin(\beta+\gamma)=\frac{1}{2}\{\cos(\alpha-\gamma)-\cos(\alpha+2\beta+\gamma)\},$$
$$\sin\alpha\sin\gamma=\frac{1}{2}\{\cos(\alpha-\gamma)-\cos(\alpha+\gamma)\}.$$

Hence $\quad \sin(\alpha+\beta)\sin(\beta+\gamma)-\sin\alpha\sin\gamma$

$$=\frac{1}{2}\{\cos(\alpha+\gamma)-\cos(\alpha+2\beta+\gamma)\}$$
$$=\sin(\alpha+\beta+\gamma)\sin\beta.$$

Problem 12.

$$\sin\alpha\sin\beta\sin(\beta-\alpha)+\sin\beta\sin\gamma\sin(\gamma-\beta)$$
$$+\sin\gamma\sin\alpha\sin(\alpha-\gamma)+\sin(\beta-\alpha)\sin(\gamma-\beta)\sin(\alpha-\gamma)=0.$$

Solution.

$$\sin\alpha\sin\beta\sin(\beta-\alpha)=\frac{1}{2}\{\cos(\alpha-\beta)-\cos(\alpha+\beta)\}\sin(\beta-\alpha)$$
$$=\frac{1}{2}\cos(\beta-\alpha)\sin(\beta-\alpha)-\frac{1}{4}\{\sin 2\beta-\sin 2\alpha\}$$
$$=\frac{1}{4}\sin 2(\beta-\alpha)-\frac{1}{4}\sin 2\beta+\frac{1}{4}\sin 2\alpha.$$

Similarly we may transform $\sin\beta\sin\gamma\sin(\gamma-\beta)$ and $\sin\gamma\sin\alpha\sin(\alpha-\gamma)$.

Also, by *Problem* 3, we have

$$\sin(\beta-\alpha)\sin(\gamma-\beta)\sin(\alpha-\gamma)=\frac{1}{4}\{\sin 2(\alpha-\beta)+\sin 2(\beta-\gamma)+\sin 2(\gamma-\alpha)\}.$$

Hence the sum of the four expressions is zero.

Problem 13.

$$\cos(\alpha+\beta)\sin(\alpha-\beta)+\cos(\beta+\gamma)\sin(\beta-\gamma)$$
$$+\cos(\gamma+\delta)\sin(\gamma-\delta)+\cos(\delta+\alpha)\sin(\delta-\alpha)=0.$$

Solution.

$$\cos(\alpha+\beta)\sin(\alpha-\beta)=\frac{1}{2}(\sin 2\alpha-\sin 2\beta),$$
$$\cos(\beta+\gamma)\sin(\beta-\gamma)=\frac{1}{2}(\sin 2\beta-\sin 2\gamma),$$

$$\cos(\gamma+\delta)\sin(\gamma-\delta) = \frac{1}{2}\left(\sin 2\gamma - \sin 2\delta\right),$$
$$\cos(\delta+\alpha)\sin(\delta-\alpha) = \frac{1}{2}\left(\sin 2\delta - \sin 2\alpha\right)$$

hence the sum of the four expressions is zero.

Problem 14.

$$\sin(\delta-\beta)\sin(\alpha-\gamma)+\sin(\beta-\gamma)\sin(\alpha-\delta)+\sin(\gamma-\delta)\sin(\alpha-\beta)=0.$$

Solution.

$$\sin(\delta-\beta)\sin(\alpha-\gamma) = \frac{1}{2}\left\{\cos(\alpha+\beta-\gamma-\delta)-\cos(\alpha-\beta-\gamma+\delta)\right\}$$
$$\sin(\beta-\gamma)\sin(\alpha-\delta) = \frac{1}{2}\left\{\cos(\alpha-\beta+\gamma-\delta)-\cos(\alpha+\beta-\gamma-\delta)\right\},$$
$$\sin(\gamma-\delta)\sin(\alpha-\beta) = \frac{1}{2}\left\{\cos(\alpha-\beta-\gamma+\delta)-\cos(\alpha-\beta+\gamma-\delta)\right\};$$

hence the sum of the three expressions is zero.

If $A+B+C=\pi$, prove the following formulae contained in the problems from 15 to 35 inclusive.

Problem 15. $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$

Solution.

$$\cot\frac{A}{2}+\cot\frac{B}{2} = \frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}+\frac{\cos\frac{B}{2}}{\sin\frac{B}{2}} = \frac{\sin\frac{B}{2}\cos\frac{A}{2}+\sin\frac{A}{2}\cos\frac{B}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}}$$
$$= \frac{\sin\frac{1}{2}(A+B)}{\sin\frac{A}{2}\sin\frac{B}{2}} = \frac{\cos\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}};$$

$$\frac{\cos\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}}+\cot\frac{C}{2} = \cos\frac{C}{2}\left\{\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}}\right\}$$
$$= \frac{\cos\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\left\{\sin\frac{C}{2}+\sin\frac{A}{2}\sin\frac{B}{2}\right\}$$
$$= \frac{\cos\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\left\{\cos\frac{1}{2}(A+B)+\sin\frac{A}{2}\sin\frac{B}{2}\right\}$$
$$= \frac{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$$

Problem 16. $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$.

Solution.

$$\sin A + \sin B = 2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B) = 2 \cos \frac{C}{2} \cos \frac{1}{2}(A-B)$$

$$\sin C = 2 \sin \frac{C}{2} \cos \frac{C}{2} = 2 \cos \frac{C}{2} \cos \frac{1}{2}(A+B);$$

$$\begin{aligned} \therefore\ \sin A + \sin B + \sin C &= 2 \cos \frac{C}{2} \left\{ \cos \frac{1}{2}(A-B) + \cos \frac{1}{2}(A+B) \right\} \\ &= 2 \cos \frac{C}{2} 2 \cos \frac{A}{2} \cos \frac{B}{2} \\ &= 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}. \end{aligned}$$

Problem 17. $\sin A - \sin B + \sin C = 4 \sin \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}$.

Solution.

$$\sin A + \sin C = 2 \sin \frac{1}{2}(A+C) \cos \frac{1}{2}(A-C) = 2 \cos \frac{B}{2} \cos \frac{1}{2}(A-C),$$

$$\sin B = 2 \sin \frac{B}{2} \cos \frac{B}{2} = 2 \cos \frac{B}{2} \cos \frac{1}{2}(A+C);$$

Solution.

$$\begin{aligned} \therefore\ \sin A - \sin B + \sin C &= 2 \cos \frac{B}{2} \left\{ \cos \frac{1}{2}(A-C) - \cos \frac{1}{2}(A+C) \right\} \\ &= 2 \cos \frac{B}{2} 2 \sin \frac{A}{2} \sin \frac{C}{2} \\ &= 4 \sin \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}. \end{aligned}$$

Problem 18. $\cos 2A + \cos 2B + \cos 2C + 4 \cos A \cos B \cos C + 1 = 0$.

Solution. $\cos 2A + \cos 2B = 2\cos(A+B)\cos(A-B) = -2\cos C \cos(A-B)$

$$\cos 2C = 2\cos^2 C - 1 = -2\cos C \cos(A+B) - 1;$$

$$\begin{aligned} \therefore\quad \cos 2A + \cos 2B + \cos 2C &= -2\cos C \{\cos(A-B) + \cos(A+B)\} - 1 \\ &= -2\cos C \cdot 2\cos A \cos B - 1 \\ &= -4\cos A \cos B \cos C - 1, \end{aligned}$$

therefore $\cos 2A + \cos 2B + \cos 2C + 4\cos A \cos B \cos C + 1 = 0.$

Problem 19. $\cos 4A + \cos 4B + \cos 4C + 1 = 4 \cos 2A \cos 2B \cos 2C$.

Solution. $\cos 4A + \cos 4B = 2\cos 2(A+B) \cos 2(A-B) = 2\cos 2C \cos 2(A-B),$

$$\cos 4C = 2\cos^2 2C - 1 = 2\cos 2C \cos 2(A+B) - 1;$$

$$\begin{aligned} \therefore\quad \cos 4A + \cos 4B + \cos 4C &= 2\cos 2C \{\cos 2(A-B) + \cos 2(A+B)\} - 1 \\ &= 2\cos 2C \cdot 2\cos 2A \cos 2B - 1 \\ &= 4\cos 2A \cos 2B \cos 2C - 1; \end{aligned}$$

therefore $\cos 4A + \cos 4B + \cos 4C + 1 = 4\cos 2A \cos 2B \cos 2C.$

Problem 20. $\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}=4\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}$.

Solution.

Let $$\alpha=\frac{1}{2}(\pi-A),\quad \beta=\frac{1}{2}(\pi-B),\quad \gamma=\frac{1}{2}(\pi-C);$$

therefore $$\alpha+\beta+\gamma=\frac{1}{2}(3\pi-A-B-C)=\frac{1}{2}2\pi=\pi;$$

hence, by *Problem* 16,

$$\sin\alpha+\sin\beta+\sin\gamma=4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2},$$

that is $$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}=4\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}.$$

Problem 21. $\cos\frac{A}{2}-\cos\frac{B}{2}+\cos\frac{C}{2}=4\cos\frac{\pi+A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi+C}{4}$.

Solution.

Let $$\alpha=\frac{1}{2}(\pi-A),\quad \beta=\frac{1}{2}(\pi-B),\quad \gamma=\frac{1}{2}(\pi-C);$$

therefore $$\alpha+\beta+\gamma=\frac{1}{2}(3\pi-A-B-C)=\frac{1}{2}2\pi=\pi;$$

hence, by *Problem* 17,

$$\sin\alpha-\sin\beta+\sin\gamma=4\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2},$$

that is $$\cos\frac{A}{2}-\cos\frac{B}{2}+\cos\frac{C}{2}=4\sin\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\sin\frac{\pi-C}{4}.$$

Problem 22. $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}-1=4\sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4}$.

Solution.

Let $$\alpha=\frac{1}{2}(\pi-A),\quad \beta=\frac{1}{2}(\pi-B),\quad \gamma=\frac{1}{2}(\pi-C);$$

therefore $$\alpha+\beta+\gamma=\frac{1}{2}(3\pi-A-B-C)=\frac{1}{2}2\pi=\pi;$$

hence, by *Art* 114 (page 409),

$$\cos\alpha+\cos\beta+\cos\gamma-1=4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2},$$

that is $$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}-1=4\sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4}.$$

Problem 23. $\sin^2 A+\sin^2 B+\sin^2 C-2\cos A\cos B\cos C=2$.

Solution.

$$\begin{aligned}\sin^2 A+\sin^2 B+\sin^2 C&=\frac{1}{2}\{1-\cos 2A+1-\cos 2B+1-\cos 2C\}\\&=\frac{3}{2}-\frac{1}{2}\{\cos 2A+\cos 2B+\cos 2C\}\\&=\frac{3}{2}+\frac{1}{2}\{1+4\cos A\cos B\cos C\},\ \text{by } \textit{Problem } 18,\end{aligned}$$

$$= 2 + 2\cos A\cos B\cos C;$$

therefore $\quad \sin^2 A + \sin^2 B + \sin^2 C - 2\cos A\cos B\cos C = 2.$

Problem 24. $\sin^2 2A + \sin^2 2B + \sin^2 2C + 2\cos 2A\cos 2B\cos 2C = 2.$

Solution.

$$\sin^2 2A + \sin^2 2B + \sin^2 2C = \frac{1}{2}\{3 - \cos 4A - \cos 4B - \cos 4C\}$$

by *Problem 19*,

$$= \frac{3}{2} - \frac{1}{2}\{4\cos 2A\cos 2B\cos 2C - 1\};$$

$$= 2 - 2\cos 2A\cos 2B\cos 2C;$$

therefore $\quad \sin^2 2A + \sin^2 2B + \sin^2 2C + 2\cos 2A\cos 2B\cos 2C = 2$

Problem 25. $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1.$

Solution.

$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2}$$

$$= \frac{1}{\cot\frac{A}{2}\cot\frac{B}{2}} + \frac{1}{\cot\frac{B}{2}\cot\frac{C}{2}} + \frac{1}{\cot\frac{C}{2}\cot\frac{A}{2}}$$

$$= \frac{\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}}{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}} = 1, \text{ by } \textit{Problem } 15.$$

Problem 26. $\dfrac{\sin A + \sin B - \sin C}{\sin A + \sin B + \sin C} = \tan\frac{A}{2}\tan\frac{B}{2}.$

Solution.

$$\sin A + \sin B - \sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}, \text{ by } \textit{Problem } 17;$$

$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}, \text{ by } \textit{Problem } 16;$$

therefore, by division,

$$\frac{\sin A + \sin B - \sin C}{\sin A + \sin B + \sin C} = \frac{\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}} = \tan\frac{A}{2}\tan\frac{B}{2}.$$

Problem 27.

$$1 + \cos A\cos B\cos C = \cos A\sin B\sin C + \cos B\sin A\sin C + \cos C\sin A\sin B.$$

Solution.

$$\cos A\sin B\sin C + \cos B\sin A\sin C + \cos C\sin A\sin B$$

$$
\begin{aligned}
&= \sin C(\cos A \sin B + \cos B \sin A) + \cos C \sin A \sin B \\
&= \sin C \sin(A + B) + \cos C \sin A \sin B \\
&= \sin^2 C + \cos C \sin A \sin B \\
&= 1 - \cos^2 C + \cos C \sin A \sin B \\
&= 1 + \cos C \{\cos(A + B) + \sin A \sin B\} \\
&= 1 + \cos C \cos A \cos B.
\end{aligned}
$$

Problem 28.

$$\cot A + \cot B + \cot C = \cot A \cot B \cot C + \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C.$$

Solution. Take *Problem* 27, and divide by $\sin A \sin B \sin C$;

therefore $\dfrac{1}{\sin A \sin B \sin C} + \dfrac{\cos A \cos B \cos C}{\sin A \sin B \sin C} = \dfrac{\cos A}{\sin A} + \dfrac{\cos B}{\sin B} + \dfrac{\cos C}{\sin C}$:
thus we obtain the required result.

Problem 29. $\sin^2 \dfrac{C}{2} = \dfrac{(\sin B + \sin C - \sin A)(\sin C + \sin A - \sin B)}{4 \sin A \sin B}$.

Solution. By *Problem* 17 we have

$$
\begin{aligned}
&\frac{(\sin B + \sin C - \sin A)(\sin C + \sin A - \sin B)}{4 \sin A \sin B} \\
&= \frac{16 \sin \frac{B}{2} \sin \frac{C}{2} \cos \frac{A}{2} \sin \frac{C}{2} \sin \frac{A}{2} \cos \frac{B}{2}}{16 \sin \frac{A}{2} \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{B}{2}} = \sin^2 \frac{C}{2}.
\end{aligned}
$$

Problem 30. *The expression* $\cot A + \dfrac{\sin A}{\sin B \sin C}$ *will retain the same value if any two of the quantities* A, B, C, *be interchanged.*

Solution.

$$
\begin{aligned}
\cot A + \frac{\sin A}{\sin B \sin C} &= \frac{\cos A}{\sin A} + \frac{\sin A}{\sin B \sin C} \\
&= \frac{\cos A \sin B \sin C + \sin^2 A}{\sin A \sin B \sin C} = \frac{1 - \cos^2 A + \cos A \sin B \sin C}{\sin A \sin B \sin C} \\
&= \frac{1 + \cos A \{\cos(B + C) + \sin B \sin C\}}{\sin A \sin B \sin C} = \frac{1 + \cos A \cos B \cos C}{\sin A \sin B \sin C}.
\end{aligned}
$$

We have thus an expression which involves A, B, and C symmetrically; and we shall in the same manner obtain the same result if in the original expression any two of the quantities A, B, C be interchanged.

Problem 31. $\dfrac{\tan A + \tan B + \tan C}{(\sin A + \sin B + \sin C)^2} = \dfrac{\tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}}{2 \cos A \cos B \cos C}$.

Solution.

By *Art.* 114 (page 409), $\tan A + \tan B + \tan C = \tan A \tan B \tan C$;

by *Problem* 16, $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$;

therefore, by division,

$$\frac{\tan A + \tan B + \tan C}{(\sin A + \sin B + \sin C)^2} = \frac{\tan A \tan B \tan C}{16\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}}$$

$$= \frac{8\sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}}{16\cos A\cos B\cos C\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}} = \frac{\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}{2\cos A\cos B\cos C}.$$

Problem 32. $\sin nA + \sin nB + \sin nC = 4\sin\frac{n\pi}{2}\cos\frac{nA}{2}\cos\frac{nB}{2}\cos\frac{nC}{2}$, *if n be an integer of the form* $4m + 1$ *or* $4m + 3$.

Solution.

$$\begin{aligned}\sin nA + \sin nB &= 2\sin\frac{n}{2}(A+B)\cos\frac{n}{2}(A-B)\\ &= 2\sin\frac{n}{2}(\pi - C)\cos\frac{n}{2}(A-B)\\ &= 2\left\{\sin\frac{n\pi}{2}\cos\frac{nC}{2} - \cos\frac{n\pi}{2}\sin\frac{nC}{2}\right\}\cos\frac{n}{2}(A-B)\\ &= 2\sin\frac{n\pi}{2}\cos\frac{nC}{2}\cos\frac{n}{2}(A-B);\ \text{since}\ \cos\frac{n\pi}{2} = 0.\end{aligned}$$

Also

$$\begin{aligned}\sin nC &= 2\sin\frac{nC}{2}\cos\frac{nC}{2} = 2\sin\frac{n}{2}(\pi - A - B)\cos\frac{nC}{2}\\ &= 2\left\{\sin\frac{n\pi}{2}\cos\frac{n}{2}(A+B) - \cos\frac{n\pi}{2}\sin\frac{n}{2}(A+B)\right\}\cos\frac{nC}{2}\\ &= 2\sin\frac{n\pi}{2}\cos\frac{n}{2}(A+B)\cos\frac{nC}{2}.\end{aligned}$$

Therefore $\sin nA + \sin nB + \sin nC$

$$\begin{aligned}&= 2\sin\frac{n\pi}{2}\cos\frac{nC}{2}\left\{\cos\frac{n}{2}(A-B) + \cos\frac{n}{2}(A+B)\right\}\\ &= 4\sin\frac{n\pi}{2}\cos\frac{nA}{2}\cos\frac{nB}{2}\cos\frac{nC}{2}.\end{aligned}$$

Problem 33. $\sin nA + \sin nB + \sin nC = -4\cos\frac{n\pi}{2}\sin\frac{nA}{2}\sin\frac{nB}{2}\sin\frac{nC}{2}$, *if n be an integer of the form* $4m$ *or* $4m + 2$.

Solution. Proceed as in *Problem* 32. Thus

$$\begin{aligned}\sin nA + \sin nB &= 2\left\{\sin\frac{n\pi}{2}\cos\frac{nC}{2} - \cos\frac{n\pi}{2}\sin\frac{nC}{2}\right\}\cos\frac{n}{2}(A-B)\\ &= -2\cos\frac{n\pi}{2}\sin\frac{nC}{2}\cos\frac{n}{2}(A-B);\ \text{since}\ \sin\frac{n\pi}{2} = 0.\end{aligned}$$

Also

$$\begin{aligned}\sin nC &= 2\sin\frac{nC}{2}\cos\frac{nC}{2} = 2\cos\frac{n}{2}(\pi - A - B)\sin\frac{nC}{2}\\ &= 2\left\{\cos\frac{n\pi}{2}\cos\frac{n}{2}(A+B) + \sin\frac{n\pi}{2}\sin\frac{n}{2}(A+B)\right\}\sin\frac{nC}{2}\end{aligned}$$

$$= 2\cos\frac{n\pi}{2}\sin\frac{nC}{2}\cos\frac{n}{2}(A+B).$$

Therefore $\sin nA + \sin nB + \sin nC$

$$= -2\cos\frac{n\pi}{2}\sin\frac{nC}{2}\left\{\cos\frac{n}{2}(A-B) - \cos\frac{n}{2}(A+B)\right\}$$
$$= -4\cos\frac{n\pi}{2}\sin\frac{nC}{2}\sin\frac{nA}{2}\sin\frac{nB}{2}.$$

Problem 34. $\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} = 4\cos\frac{B+C}{4}\cos\frac{A+C}{4}\cos\frac{A+B}{4}$.

Solution. By *Problem* 20,

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} = 4\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}$$
$$= 4\cos\frac{B+C}{4}\cos\frac{C+A}{4}\cos\frac{A+B}{4}.$$

Problem 35.

$$\frac{\tan A}{\tan B} + \frac{\tan B}{\tan C} + \frac{\tan C}{\tan A} + \frac{\tan A}{\tan C} + \frac{\tan B}{\tan A} + \frac{\tan C}{\tan B}$$
$$= \sec A \sec B \sec C - 2.$$

Solution.

$$\frac{\tan B}{\tan A} + \frac{\tan C}{\tan A} = \frac{1}{\tan A}\left(\frac{\sin B}{\cos B} + \frac{\sin C}{\cos C}\right) = \frac{\sin(B+C)}{\tan A\cos B\cos C}$$
$$= \frac{\sin A}{\tan A\cos B\cos C} = \frac{\cos A}{\cos B\cos C}.$$

In this was we see that the given expression

$$= \frac{\cos A}{\cos B\cos C} + \frac{\cos B}{\cos C\cos A} + \frac{\cos C}{\cos A\cos B}$$
$$= \frac{\cos^2 A + \cos^2 B + \cos^2 C}{\cos A\cos B\cos C} = \frac{3 - \sin^2 A - \sin^2 B - \sin^2 C}{\cos A\cos B\cos C}$$
$$= \frac{1 - 2\cos A\cos B\cos C}{\cos A\cos B\cos C}, \text{ by } Problem\ 23,$$
$$= \sec A\sec B\sec C - 2.$$

Problem 36. *If the sum of four angles be two right angles, the sum of their tangents is equal to the sum of the products of the tangents taken three and three.*

Solution.

Suppose $A + B + C + D = 180°$; then $A + B = 180° - C - D$;

therefore $\tan(A+B) = -\tan(C+D)$, by *Art*. 48 page 397);

$$\therefore\ \frac{\tan A + \tan B}{1 - \tan A\tan B} = -\frac{\tan C + \tan D}{1 - \tan C\tan D};$$

$$\therefore\quad (\tan A + \tan B)(1 - \tan C\tan D) = -(\tan C + \tan D)(1 - \tan A\tan B),$$

therefore $\tan A + \tan B + \tan C + \tan D$

$$= (\tan A + \tan B)\tan C\tan D + (\tan C + \tan D)\tan A\tan B$$
$$= \tan B\tan C\tan D + \tan A\tan C\tan D + \tan A\tan B\tan D + \tan A\tan B\tan C.$$

Problem 37. *If* $\dfrac{\tan(A-B)}{\tan A} + \dfrac{\sin^2 C}{\sin^2 A} = 1$, *prove that* $\tan A \tan B = \tan^2 C$.

Solution.

$$\frac{\sin^2 C}{\sin^2 A} = 1 - \frac{\tan(A-B)}{\tan A} = 1 - \frac{\sin(A-B)\cos A}{\cos(A-B)\sin A}$$
$$= \frac{\sin A\cos(A-B) - \cos A\sin(A-B)}{\cos(A-B)\sin A} = \frac{\sin\{A-(A-B)\}}{\cos(A-B)\sin A}$$
$$= \frac{\sin B}{\cos(A-B)\sin A}; \quad \text{therefore } \sin^2 C = \frac{\sin A\sin B}{\cos(A-B)}.$$

Hence $$\cos^2 C = 1 - \sin^2 C = 1 - \frac{\sin A\sin B}{\cos(A-B)}$$
$$= \frac{\cos(A-B) - \sin A\sin B}{\cos(A-B)} = \frac{\cos A\cos B}{\cos(A-B)}.$$

Therefore $$\frac{\sin^2 C}{\cos^2 C} = \frac{\sin A\sin B}{\cos(A-B)} \div \frac{\cos A\cos B}{\cos(A-B)} = \frac{\sin A\sin B}{\cos A\cos B},$$

that is $$\tan^2 C = \tan A\tan B.$$

Problem 38. *Given* $\dfrac{\tan^2\alpha}{\tan^2\beta} = \dfrac{\cos\beta(\cos x - \cos\alpha)}{\cos\alpha(\cos x - \cos\beta)}$,

show that $$\tan^2\frac{x}{2} = \tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}.$$

Solution.

$$\frac{\tan^2\alpha}{\tan^2\beta} = \frac{\cos\beta(\cos x - \cos\alpha)}{\cos\alpha(\cos x - \cos\beta)},$$

therefore $$\frac{\cos x - \cos\alpha}{\cos x - \cos\beta} = \frac{\tan^2\alpha\cos\alpha}{\tan^2\beta\cos\beta} = \frac{\sin^2\alpha\cos\beta}{\sin^2\beta\cos\alpha};$$

therefore $$\cos x = \frac{\sin^2\beta\cos^2\alpha - \sin^2\alpha\cos^2\beta}{\sin^2\beta\cos\alpha - \sin^2\alpha\cos\beta}$$
$$= \frac{\left(1-\cos^2\beta\right)\cos^2\alpha - \left(1-\cos^2\alpha\right)\cos^2\beta}{(1-\cos^2\beta)\cos\alpha - (1-\cos^2\alpha)\cos\beta}$$
$$= \frac{\cos^2\alpha - \cos^2\beta}{(\cos\alpha - \cos\beta)(1+\cos\alpha\cos\beta)} = \frac{\cos\alpha + \cos\beta}{1+\cos\alpha\cos\beta}.$$

Hence $$\frac{1-\cos x}{1+\cos x} = \frac{1+\cos\alpha\cos\beta - \cos\alpha - \cos\beta}{1+\cos\alpha\cos\beta + \cos\alpha + \cos\beta}$$
$$= \frac{(1-\cos\alpha)(1-\cos\beta)}{(1+\cos\alpha)(1+\cos\beta)};$$

therefore $$\tan^2\frac{x}{2} = \tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}, \quad \text{by } \textit{Art. } 82 \text{ (page 404)}.$$

Problem 39. *If* $\cos^2\theta = \dfrac{\cos\alpha}{\cos\beta}$, $\quad \cos^2\theta' = \dfrac{\cos\alpha'}{\cos\beta'}$, $\quad$ *and* $\dfrac{\tan\theta}{\tan\theta'} = \dfrac{\tan\alpha}{\tan\alpha'}$,

show that $$\tan^2\frac{\alpha}{2}\tan^2\frac{\alpha'}{2} = \tan^2\frac{\beta}{2}.$$

Solution.
$$\frac{\tan^2\theta}{\tan^2\theta'} = \frac{\tan^2\alpha}{\tan^2\alpha'}; \quad \text{but } \tan^2\theta = \frac{1-\cos^2\theta}{\cos^2\theta} = \frac{\cos\beta-\cos\alpha}{\cos\alpha},$$

and
$$\tan^2\theta' = \frac{1-\cos^2\theta'}{\cos^2\theta'} = \frac{\cos\beta-\cos\alpha'}{\cos\alpha'};$$

therefore
$$\frac{\cos\beta-\cos\alpha}{\cos\beta-\cos\alpha'}\cdot\frac{\cos\alpha'}{\cos\alpha} = \frac{\tan^2\alpha}{\tan^2\alpha'};$$

therefore
$$\frac{\cos\beta-\cos\alpha}{\cos\beta-\cos\alpha'} = \frac{\sin^2\alpha\cos\alpha'}{\sin^2\alpha'\cos\alpha},$$

$$\therefore\ \cos\beta = \frac{\sin^2\alpha'\cos^2\alpha - \sin^2\alpha\cos^2\alpha'}{\sin^2\alpha'\cos\alpha - \sin^2\alpha\cos\alpha'}$$
$$= \frac{\left(1-\cos^2\alpha'\right)\cos^2\alpha - \left(1-\cos^2\alpha\right)\cos^2\alpha'}{(1-\cos^2\alpha')\cos\alpha - (1-\cos^2\alpha)\cos\alpha'}$$
$$= \frac{\cos^2\alpha-\cos^2\alpha'}{(\cos\alpha-\cos\alpha')(1+\cos\alpha\cos\alpha')} = \frac{\cos\alpha+\cos\alpha'}{1+\cos\alpha\cos\alpha'}.$$

Hence
$$\frac{1-\cos\beta}{1+\cos\beta} = \frac{1+\cos\alpha\cos\alpha'-\cos\alpha-\cos\alpha'}{1+\cos\alpha\cos\alpha'+\cos\alpha+\cos\alpha'} = \frac{(1-\cos\alpha)(1-\cos\alpha')}{(1+\cos\alpha)(1+\cos\alpha')};$$

therefore
$$\tan^2\frac{\beta}{2} = \tan^2\frac{\alpha}{2}\tan^2\frac{\alpha'}{2}.$$

Problem 40. *If* $\cos\alpha = \cos\beta\cos\phi = \cos\beta'\cos\phi'$, *and*
$$\sin\alpha = 2\sin\frac{\phi}{2}\sin\frac{\phi'}{2}, \quad \textit{show that } \tan^2\frac{\alpha}{2} = \tan^2\frac{\beta}{2}\tan^2\frac{\beta'}{2}.$$

Solution. $\cos\phi = \dfrac{\cos\alpha}{\cos\beta}, \quad \cos\phi' = \dfrac{\cos\alpha}{\cos\beta'};$

therefore
$$1-\cos\phi = \frac{\cos\beta-\cos\alpha}{\cos\beta}, \quad 1-\cos\phi' = \frac{\cos\beta'-\cos\alpha}{\cos\beta'};$$

therefore
$$2\sin^2\frac{\phi}{2} = \frac{\cos\beta-\cos\alpha}{\cos\beta}, \quad 2\sin^2\frac{\phi'}{2} = \frac{\cos\beta'-\cos\alpha}{\cos\beta'};$$

therefore
$$4\sin^2\frac{\phi}{2}\sin^2\frac{\phi'}{2} = \frac{(\cos\beta-\cos\alpha)(\cos\beta'-\cos\alpha)}{\cos\beta\cos\beta'}.$$

Thus
$$\sin^2\alpha = \frac{(\cos\beta-\cos\alpha)(\cos\beta'-\cos\alpha)}{\cos\beta\cos\beta'};$$

therefore
$$\cos\beta\cos\beta'\sin^2\alpha = \cos\beta\cos\beta' - \cos\alpha(\cos\beta+\cos\beta') + \cos^2\alpha,$$

therefore
$$\cos\beta\cos\beta'\cos^2\alpha = \cos\alpha(\cos\beta+\cos\beta') - \cos^2\alpha;$$

therefore
$$\cos\alpha(1+\cos\beta\cos\beta') = \cos\beta+\cos\beta';$$

therefore
$$\cos\alpha = \frac{\cos\beta+\cos\beta'}{1+\cos\beta\cos\beta'}.$$

Hence
$$\frac{1-\cos\alpha}{1+\cos\alpha} = \frac{(1-\cos\beta)(1-\cos\beta')}{(1+\cos\beta)(1+\cos\beta')};$$

therefore
$$\tan^2\frac{\alpha}{2} = \tan^2\frac{\beta}{2}\tan^2\frac{\beta'}{2}.$$

Problem 41. *If* $\dfrac{\sin(\alpha-\beta)}{\sin\beta}=\dfrac{\sin(\alpha+\theta)}{\sin\theta}$, *show that*

$$\cot\beta-\cot\theta=\cot(\alpha+\theta)+\cot(\alpha-\beta).$$

Solution. The proposed result is true if

$$\cot\beta-\cot(\alpha+\theta)=\cot\theta+\cot(\alpha-\beta),$$

that is if

$$\frac{\cos\beta}{\sin\beta}-\frac{\cos(\alpha+\theta)}{\sin(\alpha+\theta)}=\frac{\cos\theta}{\sin\theta}+\frac{\cos(\alpha-\beta)}{\sin(\alpha-\beta)},$$

that is if

$$\frac{\sin(\alpha+\theta)\cos\beta-\cos(\alpha+\theta)\sin\beta}{\sin\beta\sin(\alpha+\theta)}=\frac{\sin(\alpha-\beta)\cos\theta+\cos(\alpha-\beta)\sin\theta}{\sin\theta\sin(\alpha-\beta)},$$

that is if

$$\frac{\sin(\alpha+\theta-\beta)}{\sin\beta\sin(\alpha+\theta)}=\frac{\sin(\alpha-\beta+\theta)}{\sin\theta\sin(\alpha-\beta)},$$

that is if

$$\sin\theta\sin(\alpha-\beta)=\sin\beta\sin(\alpha+\theta);$$

and this is true by supposition.

Problem 42. *If* $\left(\dfrac{\tan\alpha}{\sin\theta}-\dfrac{\tan\beta}{\tan\theta}\right)^2=\tan^2\alpha-\tan^2\beta$, *then* $\cos\theta=\dfrac{\tan\beta}{\tan\alpha}$.

Solution.

$$\left(\frac{\tan\alpha-\cos\theta\tan\beta}{\sin\theta}\right)^2=\tan^2\alpha-\tan^2\beta;\quad\text{therefore}$$

$$(\tan\alpha-\cos\theta\tan\beta)^2=\left(1-\cos^2\theta\right)\left(\tan^2\alpha-\tan^2\beta\right);$$

therefore

$$\tan^2\alpha-2\cos\theta\tan\alpha\tan\beta+\cos^2\theta\tan^2\beta=\left(1-\cos^2\theta\right)\left(\tan^2\alpha-\tan^2\beta\right);$$

therefore $$\tan^2\beta-2\cos\theta\tan\alpha\tan\beta+\cos^2\theta\tan^2\alpha=0,$$

that is $$(\tan\beta-\cos\theta\tan\alpha)^2=0;$$

therefore $$\tan\beta-\cos\theta\tan\alpha=0;\quad\text{therefore }\cos\theta=\frac{\tan\beta}{\tan\alpha}.$$

Problem 43. *If* $\tan\phi=\cos\theta\tan\alpha$, *and* $\tan\alpha'=\tan\theta\sin\phi$, *then one value of* $\tan^2\dfrac{\phi}{2}$ *is* $\tan\dfrac{\alpha+\alpha'}{2}\tan\dfrac{\alpha-\alpha'}{2}$.

Solution.

$$\cos\theta=\frac{\tan\phi}{\tan\alpha};\quad\text{therefore }\tan^2\theta=\frac{\tan^2\alpha-\tan^2\phi}{\tan^2\phi};$$

therefore $$\frac{\tan^2\alpha-\tan^2\phi}{\tan^2\phi}=\frac{\tan^2\alpha'}{\sin^2\phi};$$

therefore $$\frac{\cos^2\phi\tan^2\alpha-\sin^2\phi}{\sin^2\phi}=\frac{\tan^2\alpha'}{\sin^2\phi};$$

therefore $$\cos^2\phi\tan^2\alpha-\left(1-\cos^2\phi\right)=\tan^2\alpha';$$

therefore $$\cos^2\phi = \frac{1+\tan^2\alpha'}{1+\tan^2\alpha} = \frac{\cos^2\alpha}{\cos^2\alpha'};$$

therefore $$\cos\phi = \pm\frac{\cos\alpha}{\cos\alpha'}.$$

Take the upper sign; thus $\cos\phi = \dfrac{\cos\alpha}{\cos\alpha'}$; therefore

$$\frac{1-\cos\phi}{1+\cos\phi} = \frac{\cos\alpha'-\cos\alpha}{\cos\alpha'+\cos\alpha} = \frac{2\sin\frac{1}{2}(\alpha-\alpha')\sin\frac{1}{2}(\alpha+\alpha')}{2\cos\frac{1}{2}(\alpha-\alpha')\cos\frac{1}{2}(\alpha+\alpha')}$$
$$= \tan\frac{1}{2}(\alpha-\alpha')\tan\frac{1}{2}(\alpha+\alpha').$$

Problem 44. *Find the relation between the angles α, β, γ, when the cosines are connected by the relation*

$$1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma = 0.$$

Solution.

$$\begin{aligned}
&1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma\\
&= 1-(\cos\alpha-\cos\beta\cos\gamma)^2+\cos^2\beta\cos^2\gamma-\cos^2\beta-\cos^2\gamma\\
&= \left(1-\cos^2\beta\right)\left(1-\cos^2\gamma\right)-(\cos\alpha-\cos\beta\cos\gamma)^2\\
&= \sin^2\beta\sin^2\gamma-(\cos\alpha-\cos\beta\cos\gamma)^2\\
&= (\sin\beta\sin\gamma-\cos\alpha+\cos\beta\cos\gamma)(\sin\beta\sin\gamma+\cos\alpha-\cos\beta\cos\gamma)\\
&= \{-\cos\alpha+\cos(\beta-\gamma)\}\{\cos\alpha-\cos(\beta+\gamma)\}\\
&= 4\sin\frac{\alpha+\beta-\gamma}{2}\sin\frac{\alpha-\beta+\gamma}{2}\sin\frac{\alpha+\beta+\gamma}{2}\sin\frac{\beta+\gamma-\alpha}{2}
\end{aligned}$$

Hence in order that the proposed expression may be zero one of the four sines last written must be zero, and thus one of the four angles must be zero or a multiple of two right angles.

Problem 45. *If* $\dfrac{\tan(\theta+\alpha)}{x} = \dfrac{\tan(\theta+\beta)}{y} = \dfrac{\tan(\theta+\gamma)}{z}$, *then will*

$$\frac{x+y}{x-y}\sin^2(\alpha-\beta)+\frac{y+z}{y-z}\sin^2(\beta-\gamma)+\frac{z+x}{z-x}\sin^2(\gamma-\alpha) = 0.$$

Solution. Let $\dfrac{1}{k}$ denote the common value of the three fractions; so that

$$x = k\tan(\theta+\alpha),\quad y = k\tan(\theta+\beta),\quad z = k\tan(\theta+\gamma).$$

Then
$$\begin{aligned}
&\frac{x+y}{x-y}\sin^2(\alpha-\beta)\\
&= \frac{\tan(\theta+\alpha)+\tan(\theta+\beta)}{\tan(\theta+\alpha)-\tan(\theta+\beta)}\sin^2(\alpha-\beta)\\
&= \frac{\sin(\theta+\alpha)\cos(\theta+\beta)+\sin(\theta+\beta)\cos(\theta+\alpha)}{\sin(\theta+\alpha)\cos(\theta+\beta)-\sin(\theta+\beta)\cos(\theta+\alpha)}\sin^2(\alpha-\beta)\\
&= \frac{\sin(2\theta+\alpha+\beta)}{\sin(\alpha-\beta)}\sin^2(\alpha-\beta) = \sin(2\theta+\alpha+\beta)\sin(\alpha-\beta)\\
&= \frac{1}{2}\{\cos(2\theta+2\beta)-\cos(2\theta+2\alpha)\}.
\end{aligned}$$

Similarly $$\frac{y+z}{y-z}\sin^2(\beta-\gamma)=\frac{1}{2}\left\{\cos(2\theta+2\gamma)-\cos(2\theta+2\beta)\right\},$$
and $$\frac{z+x}{z-x}\sin^2(\gamma-\alpha)=\frac{1}{2}\left\{\cos(2\theta+2\alpha)-\cos(2\theta+2\gamma)\right\}.$$
Thus the sum of the three terms is zero.

Problem 46. *If* $\dfrac{\tan^2\theta}{\tan^2\alpha}+\dfrac{\tan^2\phi}{\tan^2\beta}=1$, *and* $\dfrac{\sin\theta}{\sin\alpha}=\dfrac{\sin\phi}{\sin\beta}$,

show that $$\sin\theta=\frac{\pm\sin\alpha}{\sqrt{1\pm\cos\alpha\cos\beta}}.$$

Solution. From the second given equation
$$\sin^2\phi=\frac{\sin^2\beta\sin^2\theta}{\sin^2\alpha},$$
therefore $$\tan^2\phi=\frac{\sin^2\beta\sin^2\theta}{\sin^2\alpha-\sin^2\beta\sin^2\theta}.$$
Substitute in the first given equation; thus
$$\frac{\tan^2\theta}{\tan^2\alpha}+\frac{\cos^2\beta\sin^2\theta}{\sin^2\alpha-\sin^2\beta\sin^2\theta}=1;$$
therefore $$\frac{\tan^2\theta}{\tan^2\alpha}=\frac{\sin^2\alpha-\sin^2\beta\sin^2\theta-\cos^2\beta\sin^2\theta}{\sin^2\alpha-\sin^2\beta\sin^2\theta}$$
$$=\frac{\sin^2\alpha-\sin^2\theta}{\sin^2\alpha-\sin^2\beta\sin^2\theta}$$
therefore $$\frac{\sin^2\theta\cos^2\alpha}{\left(1-\sin^2\theta\right)\sin^2\alpha}=\frac{\sin^2\alpha-\sin^2\theta}{\sin^2\alpha-\sin^2\beta\sin^2\theta};$$
therefore
$$\sin^2\theta\cos^2\alpha\left(\sin^2\alpha-\sin^2\beta\sin^2\theta\right)=\left(\sin^2\alpha-\sin^2\theta\right)\left(1-\sin^2\theta\right)\sin^2\alpha;$$
therefore
$$\sin^4\theta\left(\sin^2\alpha+\cos^2\alpha\sin^2\beta\right)$$
$$-\sin^2\theta\left(\cos^2\alpha\sin^2\alpha+\sin^2\alpha+\sin^4\alpha\right)+\sin^4\alpha=0;$$
therefore $$\sin^4\theta\left(1-\cos^2\alpha\cos^2\beta\right)-2\sin^2\theta\sin^2\alpha+\sin^4\alpha=0.$$
By solving this quadratic in the ordinary way we obtain
$$\sin^2\theta=\frac{1\pm\cos\alpha\cos\beta}{1-\cos^2\alpha\cos^2\beta}\sin^2\alpha=\frac{\sin^2\alpha}{1\mp\cos\alpha\cos\beta}.$$

Problem 47. *If* $\dfrac{\sin(\theta-\alpha)}{\sin(\theta-\beta)}=\dfrac{a}{b}$ *and* $\dfrac{\cos(\theta-\alpha)}{\cos(\theta-\beta)}=\dfrac{a'}{b'}$,

then $$\cos(\alpha-\beta)=\frac{aa'+bb'}{ab'+a'b}.$$

Solution.
$$\frac{\sin\left\{\theta-\beta-(\alpha-\beta)\right\}}{\sin(\theta-\beta)}=\frac{a}{b},$$
therefore $$\frac{\sin(\theta-\beta)\cos(\alpha-\beta)-\cos(\theta-\beta)\sin(\alpha-\beta)}{\sin(\theta-\beta)}=\frac{a}{b};$$

therefore $$\cos(\alpha-\beta)-\sin(\alpha-\beta)\cot(\theta-\beta)=\frac{a}{b}.$$

Again $$\frac{\cos\{\theta-\beta-(\alpha-\beta)\}}{\cos(\theta-\beta)}=\frac{a'}{b'},$$

therefore $$\frac{\cos(\theta-\beta)\cos(\alpha-\beta)+\sin(\theta-\beta)\sin(\alpha-\beta)}{\cos(\theta-\beta)}=\frac{a'}{b'};$$

therefore $$\cos(\alpha-\beta)+\tan(\theta-\beta)\sin(\alpha-\beta)=\frac{a'}{b'}.$$

Hence $$\sin(\alpha-\beta)\cot(\theta-\beta)\sin(\alpha-\beta)\tan(\theta-\beta)$$
$$=\left\{\cos(\alpha-\beta)-\frac{a}{b}\right\}\left\{\frac{a'}{b'}-\cos(\alpha-\beta)\right\};$$

therefore $$\sin^2(\alpha-\beta)=-\frac{aa'}{bb'}+\left(\frac{a}{b}+\frac{a'}{b'}\right)\cos(\alpha-\beta)-\cos^2(\alpha-\beta);$$

therefore $$1+\frac{aa'}{bb'}=\left(\frac{a}{b}+\frac{a'}{b'}\right)\cos(\alpha-\beta);$$

therefore $$\cos(\alpha-\beta)=\frac{aa'+bb'}{ab'+a'b}.$$

Problem 48. *Having given* $\tan\phi=\dfrac{\sin\theta\cos\theta'}{\sin\theta'+\cos\theta}$, *show that one of the values of* $\tan\dfrac{\phi}{2}$ *is* $\tan\dfrac{\theta}{2}\tan\left(\dfrac{\pi}{4}-\dfrac{\theta'}{2}\right)$.

Solution.

$$\tan\phi=\frac{2\tan\frac{\phi}{2}}{1-\tan^2\frac{\phi}{2}};\quad\text{thus}$$

$$\frac{2\tan\frac{\phi}{2}}{1-\tan^2\frac{\phi}{2}}=\frac{\sin\theta\cos\theta'}{\sin\theta'+\cos\theta};$$

therefore $$2\tan\frac{\phi}{2}(\sin\theta'+\cos\theta)=\left(1-\tan^2\frac{\phi}{2}\right)\sin\theta\cos\theta';$$

therefore $$\sin\theta\cos\theta'\tan^2\frac{\phi}{2}+2\tan\frac{\phi}{2}(\sin\theta'+\cos\theta)=\sin\theta\cos\theta';$$

By solving this quadratic in the ordinary way we obtain
$$\tan\frac{\phi}{2}=\frac{-(\sin\theta'+\cos\theta)\pm(1+\sin\theta'\cos\theta)}{\sin\theta\cos\theta'}$$

Take the upper sign; thus
$$\tan\frac{\phi}{2}=\frac{(1-\sin\theta')(1-\cos\theta)}{\sin\theta\cos\theta'}.$$

Now $$\frac{1-\cos\theta}{\sin\theta}=\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}=\tan\frac{\theta}{2},$$

and similarly $$\frac{1-\sin\theta'}{\cos\theta'} = \frac{1-\cos\left(\frac{\pi}{2}-\theta'\right)}{\sin\left(\frac{\pi}{2}-\theta'\right)} = \tan\left(\frac{\pi}{4}-\frac{\theta'}{2}\right);$$

thus $$\tan\frac{\phi}{2} = \tan\frac{\theta}{2}\tan\left(\frac{\pi}{4}-\frac{\theta'}{2}\right).$$

In like manner with the lower sign we shall find that

$$\tan\frac{\phi}{2} = -\cot\frac{\theta}{2}\cot\left(\frac{\pi}{4}-\frac{\theta'}{2}\right).$$

The product of the two values of $\tan\frac{\phi}{2}$ is -1, as it should be by the nature of quadratic equations.

Problem 49. *Given* $\cos\theta = \cos\alpha\cos\beta$, $\cos\theta' = \cos\alpha'\cos\beta$,

$$\tan\frac{\theta}{2}\tan\frac{\theta'}{2} = \tan\frac{\beta}{2}, \quad \textit{show that } \sin^2\beta = (\sec\alpha - 1)(\sec\alpha' - 1).$$

Solution. $\cos\theta = \cos\alpha\cos\beta$;

therefore $$\frac{1-\cos\theta}{1+\cos\theta} = \frac{1-\cos\alpha\cos\beta}{1+\cos\alpha\cos\beta};$$

therefore $$\tan^2\frac{\theta}{2} = \frac{1-\cos\alpha\cos\beta}{1+\cos\alpha\cos\beta}.$$

Similarly $$\tan^2\frac{\theta'}{2} = \frac{1-\cos\alpha'\cos\beta}{1+\cos\alpha'\cos\beta}.$$

Hence $$\frac{(1-\cos\alpha\cos\beta)(1-\cos\alpha'\cos\beta)}{(1+\cos\alpha\cos\beta)(1+\cos\alpha'\cos\beta)} = \tan^2\frac{\beta}{2} = \frac{1-\cos\beta}{1+\cos\beta};$$

therefore $$\frac{1-(\cos\alpha+\cos\alpha')\cos\beta+\cos\alpha\cos\alpha'\cos^2\beta}{1+(\cos\alpha+\cos\alpha')\cos\beta+\cos\alpha\cos\alpha'\cos^2\beta} = \frac{1-\cos\beta}{1+\cos\beta};$$

therefore $$\frac{(\cos\alpha+\cos\alpha')\cos\beta}{1+\cos\alpha\cos\alpha'\cos^2\beta} = \cos\beta;$$

therefore $$\cos\alpha+\cos\alpha' = 1+\cos\alpha\cos\alpha'\left(1-\sin^2\beta\right);$$

therefore $$\sin^2\beta\cos\alpha\cos\alpha' = 1-\cos\alpha-\cos\alpha'+\cos\alpha\cos\alpha'$$
$$= (1-\cos\alpha)(1-\cos\alpha');$$

therefore $$\sin^2\beta = \left(\frac{1}{\cos\alpha}-1\right)\left(\frac{1}{\cos\alpha'}-1\right)$$
$$= (\sec\alpha-1)(\sec\alpha'-1).$$

Problem 50. *Having given that* $\sin(B+C-A)$, $\sin(C+A-B)$, *and* $\sin(A+B-C)$ *are in arithmetical progression, show that* $\tan A$, $\tan B$ *and* $\tan C$, *are in arithmetical progression.*

Solution. Here

$$\sin(C+A-B)-\sin(B+C-A) = \sin(A+B-C)-\sin(C+A-B);$$

therefore $$2\sin(A-B)\cos C = 2\sin(B-C)\cos A;$$

therefore $$(\sin A\cos B-\cos A\sin B)\cos C = (\sin B\cos C-\cos B\sin C)\cos A.$$

Divide by $\cos A \cos B \cos C$; thus

$$\tan A - \tan B = \tan B - \tan C;$$

therefore $\tan A$, $\tan B$, and $\tan C$ are in Arithmetical Progression.

Problem 51. *If the sines of the angles of a triangle be in arithmetical progression, the cotangents of the half angles are also in arithmetical progression.*

Solution. Suppose $\sin A$, $\sin B$, and $\sin C$ to be in Arithmetical Progression, so that $\sin B - \sin A = \sin C - \sin B$.

Thus
$$2\sin\frac{B-A}{2}\cos\frac{B+A}{2} = 2\sin\frac{C-B}{2}\cos\frac{C+B}{2};$$
therefore
$$\sin\frac{B-A}{2}\sin\frac{C}{2} = \sin\frac{C-B}{2}\sin\frac{A}{2};$$
therefore
$$\left(\sin\frac{B}{2}\cos\frac{A}{2} - \cos\frac{B}{2}\sin\frac{A}{2}\right)\sin\frac{C}{2}$$
$$= \left(\sin\frac{C}{2}\cos\frac{B}{2} - \cos\frac{C}{2}\sin\frac{B}{2}\right)\sin\frac{A}{2}.$$

Divide by $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$; thus

$$\cot\frac{A}{2} - \cot\frac{B}{2} = \cot\frac{B}{2} - \cot\frac{C}{2};$$

thus $\cot\frac{C}{2}$, $\cot\frac{B}{2}$ and $\cot\frac{A}{2}$ are in Arithmetical Progression.

Problem 52. *If the sum of the squares of the cosines of the angles of a triangle = 1, the difference between the greatest and least angle is equal to the mean angle.*

Solution.

Suppose
$$\cos^2 A + \cos^2 B + \cos^2 C = 1;$$
therefore
$$3 - \sin^2 A - \sin^2 B - \sin^2 C = 1;$$
therefore
$$\sin^2 A + \sin^2 B + \sin^2 C = 2;$$

therefore by *Problem* 23 we have $\cos A \cos B \cos C = 0$; therefore one of the three angles is a right angle, and this will be the largest angle. Suppose it to be A, so that $A = 90°$; therefore $B + C = 90° = A$; therefore $A - C = B$.

Problem 53. *If* $A + B + C = 180°$, *and* $\sin\left(A + \frac{C}{2}\right) = n\sin\frac{C}{2}$,

show that
$$\tan\frac{A}{2}\tan\frac{B}{2} = \frac{n-1}{n+1}.$$

Solution.

$$\sin\left(A + \frac{C}{2}\right) = \sin\left(A + \frac{180° - A - B}{2}\right)$$
$$= \sin\left(90° - \frac{B-A}{2}\right) = \cos\frac{B-A}{2};$$
and
$$\sin\frac{C}{2} = \cos\frac{A+B}{2};$$

thus $$\cos\frac{B-A}{2}=n\cos\frac{A+B}{2};$$

therefore $$\cos\frac{A}{2}\cos\frac{B}{2}+\sin\frac{A}{2}\sin\frac{B}{2}=n\left(\cos\frac{A}{2}\cos\frac{B}{2}-\sin\frac{A}{2}\sin\frac{B}{2}\right);$$

therefore $$(n+1)\sin\frac{A}{2}\sin\frac{B}{2}=(n-1)\cos\frac{A}{2}\cos\frac{B}{2};$$

therefore $$\frac{\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}=\frac{n-1}{n+1},$$

therefore $$\tan\frac{A}{2}\tan\frac{B}{2}=\frac{n-1}{n+1}.$$

Problem 54. *If $A+B+C=180°$, and $\dfrac{\sin A}{x}=\dfrac{\sin B}{y}=\dfrac{\sin C}{z}$,*

then $$(x-y)\cot\frac{C}{2}+(y-z)\cot\frac{A}{2}+(z-x)\cot\frac{B}{2}=0.$$

Solution. Suppose $\dfrac{1}{k}$ to denote the value of $\dfrac{\sin A}{x}$, $\dfrac{\sin B}{y}$ and $\dfrac{\sin C}{z}$; then

$$x=k\sin A,\quad y=k\sin B,\quad z=k\sin C.$$

$$\begin{aligned}\therefore\ (x-y)\cot\frac{C}{2}&=k(\sin A-\sin B)\cot\frac{C}{2}\\&=2k\sin\frac{1}{2}(A-B)\cos\frac{1}{2}(A+B)\cot\frac{C}{2}\\&=2k\sin\frac{1}{2}(A-B)\sin\frac{C}{2}\cot\frac{C}{2}\\&=2k\sin\frac{1}{2}(A-B)\cos\frac{C}{2}\\&=2k\sin\frac{1}{2}(A-B)\sin\frac{1}{2}(A+B)\\&=2k\left\{\sin^2\frac{1}{2}A-\sin^2\frac{1}{2}B\right\},\quad\text{by } Art.\ 83\text{ page 404.}\end{aligned}$$

Similarly $$(y-z)\cot\frac{A}{2}=2k\left\{\sin^2\frac{1}{2}B-\sin^2\frac{1}{2}C\right\},$$

and $$(z-x)\cot\frac{B}{2}=2k\left\{\sin^2\frac{1}{2}C-\sin^2\frac{1}{2}A\right\}.$$

Thus the sum of the three terms is zero.

Problem 55. *If $A+B+C=m\pi$ where m is any integer, then*

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C.$$

Solution. $\tan(A+B+C)=\tan m\pi=0$; and therefore, by *Art.* 113 (page 409),

$$\tan A+\tan B+\tan C-\tan A\tan B\tan C=0.$$

Problem 56. *Show that if α, β, γ, and x are any angles*

$$\sin(2\alpha+x)+\sin(2\beta+x)+\sin(2\gamma+x)-\sin(2\alpha+2\beta+2\gamma+3x)$$

$$= 4\sin(\alpha+\beta+x)\sin(\beta+\gamma+x)\sin(\gamma+\alpha+x).$$

Solution.

$$\sin(2\alpha+x)+\sin(2\beta+x) = 2\sin(\alpha+\beta+x)\cos(\alpha-\beta),$$

$$\sin(2\gamma+x)-\sin(\alpha+2\beta+2\gamma+3x) = -2\sin(\alpha+\beta+x)\cos(\alpha+\beta+2\gamma+2x);$$

$$2\sin(\alpha+\beta+x)\left\{\cos(\alpha-\beta)-\cos(\alpha+\beta+2\gamma+2x)\right\}$$
$$= 2\sin(\alpha+\beta+x)2\sin(\beta+\gamma+x)\sin(\alpha+\gamma+x)$$
$$= 4\sin(\alpha+\beta+x)\sin(\beta+\gamma+x)\sin(\gamma+\alpha+x).$$

Problem 57. *From the preceding result deduce two special cases by supposing respectively that $x = 0$ and that $x = \frac{\pi}{2}$; and from these cases obtain the first two relations of Art. 114 (page 409).*

Solution. If $x = 0$ we have

$$\sin 2\alpha+\sin 2\beta+\sin 2\gamma-\sin(2\alpha+2\beta+2\gamma) = 4\sin(\alpha+\beta)\sin(\beta+\gamma)\sin(\gamma+\alpha).$$

If then $\alpha+\beta+\gamma = \pi$ we have $\sin(2\alpha+2\beta+2\gamma) = 0$;

also $\quad \sin(\alpha+\beta) = \sin\gamma, \quad \sin(\beta+\gamma) = \sin\alpha, \quad \sin(\gamma+\alpha) = \sin\beta,$

so that $\quad \sin 2\alpha+\sin 2\beta+\sin 2\gamma = 4\sin\gamma\sin\alpha\sin\beta.$

If $x = \dfrac{\pi}{2}$ we have

$$\cos 2\alpha+\cos 2\beta+\cos 2\gamma+\cos(2\alpha+2\beta+2\gamma)$$
$$= 4\cos(\alpha+\beta)\cos(\beta+\gamma)\cos(\gamma+\alpha).$$

If then $\alpha+\beta+\gamma = \dfrac{\pi}{2}$ we have $\cos(2\alpha+2\beta+2\gamma) = -1$,

also $\quad \cos(\alpha+\beta) = \sin\gamma, \quad \cos(\beta+\gamma) = \sin\alpha, \quad \cos(\gamma+\alpha) = \sin\beta,$

so that $\quad \cos 2\alpha+\cos 2\beta+\cos 2\gamma-1 = 4\sin\alpha\sin\beta\sin\gamma.$

Problem 58. *If α, β, γ be any angles, show that*

$$\sin\alpha+\sin\beta+\sin\gamma-4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$
$$= 2\sin\frac{\alpha+\beta+\gamma-\pi}{4}\left\{\cos\frac{3\alpha-\beta-\gamma+\pi}{4}+\cos\frac{3\beta-\alpha-\gamma+\pi}{4}\right.$$
$$\left.+\cos\frac{3\gamma-\alpha-\beta+\pi}{4}+\cos\frac{\alpha+\beta+\gamma-\pi}{4}\right\}.$$

Solution.

$$4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} = 2\cos\frac{\gamma}{2}\left\{\cos\frac{1}{2}(\alpha-\beta)+\cos\frac{1}{2}(\alpha+\beta)\right\}$$
$$= \cos\frac{1}{2}(\gamma+\alpha-\beta)+\cos\frac{1}{2}(\gamma+\beta-\alpha)+\cos\frac{1}{2}(\alpha+\beta+\gamma)+\cos\frac{1}{2}(\alpha+\beta-\gamma).$$

Thus the left-hand member of the proposed expression

$$= \sin\alpha+\sin\beta+\sin\gamma-\cos\frac{1}{2}(\alpha+\beta+\gamma)-\cos\frac{1}{2}(\beta+\gamma-\alpha)$$
$$-\cos\frac{1}{2}(\alpha+\gamma-\beta)-\cos\frac{1}{2}(\alpha+\beta-\gamma).$$

Again

$$2\sin\frac{\alpha+\beta+\gamma-\pi}{4}\cos\frac{3\alpha-\beta-\gamma+\pi}{4} = \sin\alpha+\sin\frac{\beta+\gamma-\alpha-\pi}{2} = \sin\alpha-\cos\frac{\beta+\gamma-\alpha}{2};$$

so also

$$2\sin\frac{\alpha+\beta+\gamma-\pi}{4}\cos\frac{3\beta-\alpha-\gamma+\pi}{4} = \sin\beta-\cos\frac{\alpha+\gamma-\beta}{2},$$

$$2\sin\frac{\alpha+\beta+\gamma-\pi}{4}\cos\frac{3\gamma-\alpha-\beta+\pi}{4} = \sin\gamma-\cos\frac{\alpha+\beta-\gamma}{2},$$

and

$$2\sin\frac{\alpha+\beta+\gamma-\pi}{4}\cos\frac{\alpha+\beta+\gamma-\pi}{4} = \sin\frac{\alpha+\beta+\gamma-\pi}{2} = -\cos\frac{\alpha+\beta+\gamma}{2}.$$

Thus the result is established.

Problem 59. *Express* $\cos 5\theta$ *in terms of* $\cos\theta$.

Solution.

$$\begin{aligned}\cos 5\theta = \cos(3\theta+2\theta) &= \cos 3\theta\cos 2\theta - \sin 3\theta\sin 2\theta\\ &= \left(4\cos^3\theta-3\cos\theta\right)\left(2\cos^2\theta-1\right)-\left(3\sin\theta-4\sin^3\theta\right)2\sin\theta\cos\theta\\ &= \left(4\cos^3\theta-3\cos\theta\right)\left(2\cos^2\theta-1\right)-2\sin^2\theta\left(3-4\sin^2\theta\right)\cos\theta\\ &= \left(4\cos^3\theta-3\cos\theta\right)\left(2\cos^2\theta-1\right)-2\left(1-\cos^2\theta\right)\left(4\cos^2\theta-1\right)\cos\theta\\ &= 8\cos^5\theta-10\cos^3\theta+3\cos\theta-2\left(-4\cos^4\theta+5\cos^2\theta-1\right)\cos\theta\\ &= 16\cos^5\theta-20\cos^3\theta+5\cos\theta.\end{aligned}$$

Problem 60. *Show that* $\sin 6\theta = 2\sin\theta\left(16\cos^5\theta-16\cos^3\theta+3\cos\theta\right)$.

Solution.

$$\begin{aligned}\sin 6\theta &= 2\sin 3\theta\cos 3\theta = 2\left(3\sin\theta-4\sin^3\theta\right)\left(4\cos^3\theta-3\cos\theta\right)\\ &= 2\sin\theta\left(3-4\sin^2\theta\right)\left(4\cos^3\theta-3\cos\theta\right)\\ &= 2\sin\theta\left(4\cos^2\theta-1\right)\left(4\cos^3\theta-3\cos\theta\right)\\ &= 2\sin\theta\left(16\cos^5\theta-16\cos^3\theta+3\cos\theta\right).\end{aligned}$$

CHAPTER IX
Construction of Trigonometrical Tables

Problem 1. *Let P be any point in a semicircle whose diameter is AB and centre C; draw PM perpendicular to AB, and draw PA, PB; from this construction, observing that the angles BPM and PAM are each equal to half of PCB, deduce the formula*

$$\frac{1-\cos A}{1+\cos A}=\tan^2\frac{A}{2}.$$

Solution. Let $PCB=A$, so that $BPM=\frac{1}{2}A$ and $PAM=\frac{1}{2}A$. Then

$$\frac{MB}{PM}=\tan\frac{1}{2}A,\quad\text{and}\quad\frac{PM}{AM}=\tan\frac{1}{2}A,$$

so that

$$\tan^2\frac{1}{2}A=\frac{MB}{PM}\cdot\frac{PM}{AM}=\frac{MB}{AM}=\frac{CB-CM}{CA+CM}$$

$$=\frac{CP-CM}{CP+CM}=\frac{1-\dfrac{CM}{CP}}{1+\dfrac{CM}{CP}}=\frac{1-\cos A}{1+\cos A}.$$

Problem 2. *If* $\cos\theta=\dfrac{a\cos\phi-b}{a-b\cos\phi}$, *then* $\dfrac{\tan\frac{\theta}{2}}{\sqrt{a+b}}=\dfrac{\tan\frac{\phi}{2}}{\sqrt{a-b}}$.

Solution. $\cos\theta=\dfrac{a\cos\phi-b}{a-b\cos\phi}$; therefore

$$\frac{1-\cos\theta}{1+\cos\theta}=\frac{a-b\cos\phi-a\cos\phi+b}{a-b\cos\phi+a\cos\phi-b}=\frac{(a+b)(1-\cos\phi)}{(a-b)(1+\cos\phi)};$$

therefore

$$\tan^2\frac{\theta}{2}=\frac{a+b}{a-b}\tan^2\frac{\phi}{2};$$

therefore

$$\frac{\tan^2\frac{\theta}{2}}{a+b}=\frac{\tan^2\frac{\phi}{2}}{a-b}.$$

Problem 3. *If* $\tan^2\theta=2\tan^2\phi+1$, *then* $\cos 2\theta+\sin^2\phi=0$.

Solution.

$$\cos^2\theta=\frac{1}{1+\tan^2\theta}=\frac{1}{1+2\tan^2\phi+1}=\frac{1}{2(1+\tan^2\phi)}=\frac{1}{2}\cos^2\phi;$$

and

$$\cos 2\theta=2\cos^2\theta-1=\cos^2\phi-1=-\sin^2\phi,$$

therefore

$$\cos 2\theta+\sin^2\phi=0.$$

Problem 4. *If* $\sec 2\theta=2\sec\theta\operatorname{cosec}\theta$, *then* $\operatorname{cosec}2\theta=\operatorname{cosec}^2\theta-\sec^2\theta$.

Solution.

$$\sec 2\theta = 2\sec\theta \operatorname{cosec}\theta;$$

therefore
$$\frac{1}{\cos 2\theta} = \frac{2}{\cos\theta\sin\theta};$$

therefore
$$1 = \frac{2\cos 2\theta}{\cos\theta\sin\theta};$$

therefore
$$\frac{1}{\sin 2\theta} = \frac{2\cos 2\theta}{\sin 2\theta\cos\theta\sin\theta} = \frac{\cos 2\theta}{\sin^2\theta\cos^2\theta}$$
$$= \frac{\cos^2\theta - \sin^2\theta}{\sin^2\theta\cos^2\theta} = \frac{1}{\sin^2\theta} - \frac{1}{\cos^2\theta}.$$

Thus
$$\operatorname{cosec} 2\theta = \operatorname{cosec}^2\theta - \sec^2\theta.$$

Problem 5. *If* $\tan\theta = n\tan\phi$, *show that* $\tan^2(\theta - \phi)$ *cannot exceed* $\dfrac{(n-1)^2}{4n}$.

Solution.

$$\tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} = \frac{(n-1)\tan\phi}{1 + n\tan^2\phi} = \frac{n-1}{\cot\phi + n\tan\phi};$$

therefore
$$\tan^2(\theta - \phi) = \frac{(n-1)^2}{\cot^2\phi + 2n + n^2\tan^2\phi} = \frac{(n-1)^2}{(n\tan\phi - \cot\phi)^2 + 4n}.$$

The greatest value of this fraction is when the denominator is least, that is when the term $n\tan\phi - \cot\phi$ vanishes.

Problem 6. *Reduce* $\sin\theta + \sin\phi - \cos\theta\sin(\theta + \phi)$ *to a single term.*

Solution.

$$\sin\theta + \sin\phi - \cos\theta\sin(\theta + \phi)$$
$$= 2\sin\frac{1}{2}(\theta + \phi)\cos\frac{1}{2}(\theta - \phi) - 2\cos\theta\sin\frac{1}{2}(\theta + \phi)\cos\frac{1}{2}(\theta + \phi)$$
$$= 2\sin\frac{1}{2}(\theta + \phi)\left\{\cos\frac{1}{2}(\theta - \phi) - \cos\theta\cos\frac{1}{2}(\theta + \phi)\right\}$$
$$= 2\sin\frac{1}{2}(\theta + \phi)\left\{\cos\left(\theta - \frac{\theta + \phi}{2}\right) - \cos\theta\cos\frac{1}{2}(\theta + \phi)\right\}$$
$$= 2\sin\frac{1}{2}(\theta + \phi)\sin\theta\sin\frac{1}{2}(\theta + \phi) = 2\sin\theta\sin^2\frac{1}{2}(\theta + \phi).$$

Problem 7. *Show that* $\dfrac{\sin\beta\cos\alpha(\tan\alpha + \tan\beta)}{1 - \cos(\alpha + \beta)} + \dfrac{\sin\frac{1}{2}(\alpha - \beta)}{\cos\beta\sin\frac{1}{2}(\alpha + \beta)} = 1.$

Solution.

$$\frac{\sin\beta\cos\alpha(\tan\alpha + \tan\beta)}{1 - \cos(\alpha + \beta)} = \frac{\sin\beta\cos\alpha}{2\sin^2\frac{1}{2}(\alpha + \beta)}\left\{\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}\right\}$$
$$= \frac{\sin\beta\cos\alpha}{2\sin^2\frac{1}{2}(\alpha + \beta)} \cdot \frac{\sin(\alpha + \beta)}{\cos\alpha\cos\beta}$$

$$= \frac{\sin\beta \cdot 2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha+\beta)}{2\sin^2\frac{1}{2}(\alpha+\beta)\cos\beta}$$

$$= \frac{\sin\beta\cos\frac{1}{2}(\alpha+\beta)}{\sin\frac{1}{2}(\alpha+\beta)\cos\beta};$$

and
$$\frac{\sin\beta\cos\frac{1}{2}(\alpha+\beta)}{\sin\frac{1}{2}(\alpha+\beta)\cos\beta} + \frac{\sin\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}(\alpha+\beta)\cos\beta}$$

$$= \frac{\sin\left(\frac{\alpha+\beta}{2}-\beta\right) + \sin\beta\cos\frac{1}{2}(\alpha+\beta)}{\sin\frac{1}{2}(\alpha+\beta)\cos\beta} = \frac{\sin\frac{1}{2}(\alpha+\beta)\cos\beta}{\sin\frac{1}{2}(\alpha+\beta)\cos\beta} = 1.$$

Problem 8. *Find approximately the height of an object which at the distance of a mile subtends at the eye an angle of one minute.*

Solution. Let x denote the height in yards; then $\frac{x}{1760} = \tan 1'$, therefore $x = 1760\tan 1'$. The value of $\tan 1'$ is approximately equal to the circular measure of $1'$, that is to $\frac{\pi}{180\times 60}$; therefore $x = \frac{1760\pi}{180\times 60}$ approximately.

Problem 9. *Find approximately the distance at which a circular plate of six inches diameter must be placed so as just to conceal the Moon, supposing the apparent diameter of the Moon to be half a degree.*

Solution. Let x denote the distance in inches; then $\frac{3}{x} = \tan\frac{1^\circ}{4}$; and taking the tangent as approximately equal to the circular measure we have

$$\frac{3}{x} = \frac{\pi}{180\times 4}; \quad \text{therefore } x = \frac{12\times 180}{\pi}.$$

Problem 10. *If* $\sin 3A = n\sin A$ *be true for any value of* A *besides zero, or two right angles, or a multiple of two right angles, show that* n *must lie between 3 and −1; solve the equation when* $n = 2$.

Solution. We have $3\sin A - 4\sin^3 A = n\sin A$; as we suppose that A is not zero nor a multiple of two right angles we may divide by $\sin A$; thus $3 - 4\sin^2 A = n$; therefore $\sin^2 A = \frac{3-n}{4}$, and as this must lie between zero and unity, n must lie between 3 and -1.

If $n = 2$ we have $\sin^2 A = \frac{1}{4} = \sin^2\frac{\pi}{6}$; therefore $A = m\pi \pm \frac{\pi}{6}$, where m is zero or any integer.

Problem 11. *If* $\tan\beta = \frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}$, *show that* $\tan(\alpha-\beta) = (1-n)\tan\alpha$.

Solution.

$$\begin{aligned}\tan(\alpha-\beta) &= \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}\\ &= \frac{\tan\alpha-\dfrac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}}{1+\tan\alpha\cdot\dfrac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}}\\ &= \frac{\sin\alpha(1-n\sin^2\alpha)-n\sin\alpha\cos^2\alpha}{\cos\alpha(1-n\sin^2\alpha)+n\sin^2\alpha\cos\alpha}\\ &= \frac{\sin\alpha-n\sin\alpha}{\cos\alpha}=\frac{(1-n)\sin\alpha}{\cos\alpha}=(1-n)\tan\alpha.\end{aligned}$$

Problem 12. *If* $\sin 3\theta$ *be given, determine the number of values of* $\tan\theta$.

Solution. All the angles which have the same sine as 3θ are included in the formula $n\pi+(-1)^n 3\theta$. Therefore any expression which gives the value of $\tan\theta$ in terms of $\sin 3\theta$ may be expected to give the value of the tangent of every angle included in the formula $\tan\frac{1}{3}\{n\pi+(-1)^n 3\theta\}$.

Now n must be of one of the following forms :

$$6m,\ 6m+1,\ 6m+2,\ 6m+3,\ 6m+4,\ 6m+5.$$

The corresponding values of $\tan\frac{1}{3}\{n\pi+(-1)^n 3\theta\}$ are, by *Art.* 45 (page 397),

$$\tan\theta,\quad \tan\left(\frac{\pi}{3}-\theta\right),\quad \tan\left(\frac{2\pi}{3}+\theta\right),\quad \tan(\pi-\theta),$$
$$\tan\left(\pi+\frac{\pi}{3}+\theta\right),\quad \tan\left(\pi+\frac{2\pi}{3}-\theta\right).$$

Thus we have six distinct values. They may also by *Arts.* 48 (page 397) and 50 (page 400) be expressed thus:

$$\pm\tan\theta,\quad \pm\tan\left(\frac{\pi}{3}+\theta\right),\quad \pm\tan\left(\frac{2\pi}{3}+\theta\right).$$

Problem 13. *Prove that* $64\left(\cos^8 A+\sin^8 A\right)=\cos 8A+28\cos 4A+35$.

Solution. $\cos^2 A=\frac{1}{2}(1+\cos 2A)$; therefore

$$\begin{aligned}\cos^4 A &= \frac{1}{4}\left(1+2\cos 2A+\cos^2 2A\right)\\ &= \frac{1}{4}+\frac{1}{2}\cos 2A+\frac{1+\cos 4A}{8}\\ &= \frac{3}{8}+\frac{1}{2}\cos 2A+\frac{1}{8}\cos 4A.\end{aligned}$$

Similarly $$\sin^2 A=\frac{1}{2}(1-\cos 2A);$$

$$\begin{aligned}\sin^4 A &= \frac{1}{4}\left(1-2\cos 2A+\cos^2 2A\right)\\ &= \frac{3}{8}-\frac{1}{2}\cos 2A+\frac{1}{8}\cos 4A.\end{aligned}$$

$\therefore\ \cos^8 A+\sin^8 A$

$$
\begin{aligned}
&= \left(\frac{3}{8} + \frac{1}{2}\cos 2A + \frac{1}{8}\cos 4A\right)^2 + \left(\frac{3}{8} - \frac{1}{2}\cos 2A + \frac{1}{8}\cos 4A\right)^2 \\
&= 2\left\{\left(\frac{3}{8}\right)^2 + \left(\frac{1}{2}\cos 2A\right)^2 + \left(\frac{1}{8}\cos 4A\right)^2 + 2\cdot\frac{3}{8}\cdot\frac{1}{8}\cos 4A\right\} \\
&= 2\left\{\frac{9}{64} + \frac{1}{4}\cos^2 2A + \frac{1}{64}\cos^2 4A + \frac{3}{32}\cos 4A\right\} \\
&= \frac{9}{32} + \frac{1}{4}(1+\cos 4A) + \frac{1}{64}(1+\cos 8A) + \frac{3}{16}\cos 4A \\
&= \frac{1}{64}\left\{\cos 8A + 28\cos 4A + 35\right\}.
\end{aligned}
$$

Problem 14. *Find all the values of θ and ϕ which satisfy*

$$\cos\theta\cos\phi + 1 = 0.$$

Solution. $\cos\theta\cos\phi = -1$.

As the cosine of an angle is never numerically greater than unity, we must have $\cos\theta$ and $\cos\phi$ both numerically equal to unity, one being positive and the other negative. Hence one of the angles must be zero or an even multiple of π, and the other must be an odd multiple of π.

Problem 15. *If $n^2\sin^2(\alpha+\beta) = \sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta\cos(\alpha-\beta)$, show that* $\tan\alpha = \dfrac{1\pm n}{1\mp n}\tan\beta$.

Solution.

$$
\begin{aligned}
&\sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta\cos(\alpha-\beta) \\
&= \sin\alpha\{\sin\alpha - \sin\beta\cos(\alpha-\beta)\} + \sin\beta\{\sin\beta\cdot\sin\alpha\cos(\alpha-\beta)\} \\
&= \sin\alpha\{\sin(\alpha-\beta+\beta) - \sin\beta\cos(\alpha-\beta)\} \\
&\qquad + \sin\beta\{\sin\left(\alpha - \overline{\alpha-\beta}\right) - \sin\alpha\cos(\alpha-\beta)\} \\
&= \sin\alpha\sin(\alpha-\beta)\cos\beta - \sin\beta\cos\alpha\sin(\alpha-\beta) \\
&= \sin(\alpha-\beta)\{\sin\alpha\cos\beta - \sin\beta\cos\alpha\} = \sin^2(\alpha-\beta).
\end{aligned}
$$

Thus $\sin^2(\alpha-\beta) = n^2\sin^2(\alpha+\beta)$;

therefore $\sin(\alpha-\beta) = \pm n\sin(\alpha+\beta)$;

therefore $\sin\alpha\cos\beta - \cos\alpha\sin\beta = \pm n(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$;

divide by $\cos\alpha\cos\beta$; thus $\tan\alpha - \tan\beta = \pm n(\tan\alpha + \tan\beta)$;

therefore $(1\mp n)\tan\alpha = (1\pm n)\tan\beta$;

therefore $\tan\alpha = \dfrac{1\pm n}{1\mp n}\tan\beta$.

Problem 16. *Find the limit of $\dfrac{\sin 4\theta\cot\theta}{\text{vers}\, 2\theta\cot^2 2\theta}$, when θ is indefinitely diminished.*

Solution.

$$\frac{\sin 4\theta\cot\theta}{\text{vers}\, 2\theta\cot^2 2\theta} = \frac{\sin 4\theta\sin^2 2\theta\cos\theta}{(1-\cos 2\theta)\cos^2 2\theta\sin\theta} = \frac{2\sin^3 2\theta\cos 2\theta\cos\theta}{2\sin^3\theta\cos^2 2\theta}$$

$$= \frac{2(2\sin\theta\cos\theta)^3\cos\theta}{2\sin^3\theta\cos 2\theta} = \frac{8\cos^4\theta}{\cos 2\theta}.$$

When $\theta = 0$ the value is therefore 8.

Solve the following equations :

Problem 17. $\sin\theta + \cos\theta = \sqrt{2}$.

Solution.

$$\sin\theta + \cos\theta = \sqrt{2}; \quad \text{therefore } \frac{\sin\theta}{\sqrt{2}} + \frac{\cos\theta}{\sqrt{2}} = 1;$$

therefore $$\cos\left(\theta - \frac{\pi}{4}\right) = 1; \quad \text{therefore } \theta - \frac{\pi}{4} = 2n\pi.$$

Problem 18. $\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$.

Solution.

$$\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}; \quad \text{therefore } \frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta = \frac{1}{\sqrt{2}};$$

therefore $$\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta = -\frac{1}{\sqrt{2}};$$

therefore $$\cos\left(\theta + \frac{\pi}{3}\right) = -\frac{1}{\sqrt{2}};$$

therefore $$\theta + \frac{\pi}{3} = 2n\pi \pm \frac{3\pi}{4}.$$

Problem 19. $\sin 2\theta = \cos\theta$.

Solution. $\sin 2\theta = \cos\theta$; therefore $\cos\left(\frac{\pi}{2} - 2\theta\right) = \cos\theta$;

Therefore $\frac{\pi}{2} - 2\theta$ and θ are angles having the same cosine; therefore all the solutions are contained in $\frac{\pi}{2} - 2\theta = 2n\pi \pm \theta$.

Problem 20. $\cos\theta - \cos 2\theta = \sin 3\theta$.

Solution. $\cos\theta - \cos 2\theta = \sin 3\theta$; therefore

$$2\sin\frac{3\theta}{2}\sin\frac{\theta}{2} = 2\sin\frac{3\theta}{2}\cos\frac{3\theta}{2};$$

Therefore either $\sin\frac{3\theta}{2} = 0$, or $\sin\frac{\theta}{2} = \cos\frac{3\theta}{2}$.

If $$\sin\frac{3\theta}{2} = 0, \text{ then } \frac{3\theta}{2} = n\pi.$$

If $$\sin\frac{\theta}{2} = \cos\frac{3\theta}{2}, \text{ then } \cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \cos\frac{3\theta}{2};$$

and therefore $$\frac{\pi}{2} - \frac{\theta}{2} = 2n\pi \pm \frac{3\theta}{2}.$$

Problem 21. $(4-\sqrt{3})(\sec\theta+\operatorname{cosec}\theta)=4(\sin\theta\tan\theta+\cos\theta\cot\theta)$.

Solution.

$$(4-\sqrt{3})(\sec\theta+\operatorname{cosec}\theta)=4(\sin\theta\tan\theta+\cos\theta\cot\theta);$$

therefore $$(4-\sqrt{3})\left(\frac{1}{\cos\theta}+\frac{1}{\sin\theta}\right)=4\left(\frac{\sin^2\theta}{\cos\theta}+\frac{\cos^2\theta}{\sin\theta}\right);$$

therefore $$(4-\sqrt{3})(\sin\theta+\cos\theta)=4\left(\sin^3\theta+\cos^3\theta\right)$$
$$=4(\sin\theta+\cos\theta)\left(\sin^2\theta+\cos^2\theta-\sin\theta\cos\theta\right);$$

therefore either $$\sin\theta+\cos\theta=0,$$

or $$4-\sqrt{3}=4(1-\sin\theta\cos\theta).$$

If $\sin\theta+\cos\theta=0$, then $\sin\theta=-\cos\theta$; therefore $\tan\theta=-1$;

therefore $$\theta=n\pi+\frac{3\pi}{4}.$$

If $4-\sqrt{3}=4(1-\sin\theta\cos\theta)$, then $\sqrt{3}=4\sin\theta\cos\theta=2\sin 2\theta$;

therefore $$\sin 2\theta=\frac{\sqrt{3}}{2};\ \text{therefore}\ 2\theta=n\pi+(-1)^n\frac{\pi}{3}.$$

Problem 22. $\cot\theta-\tan\theta=\cos\theta+sin\theta$.

Solution.

$$\cot\theta-\tan\theta=\cos\theta+\sin\theta;\ \text{therefore}\ \frac{\cos\theta}{\sin\theta}-\frac{\sin\theta}{\cos\theta}=\cos\theta+\sin\theta;$$

therefore $$\cos^2\theta-\sin^2\theta=\sin\theta\cos\theta(\cos\theta+\sin\theta);$$

therefore either $\cos\theta+\sin\theta=0$, or $\cos\theta-\sin\theta=\sin\theta\cos\theta$.

If $\sin\theta+\cos\theta=0$, then $\sin\theta=-\cos\theta$; therefore $\tan\theta=-1$;

therefore $$\theta=n\pi+\frac{3\pi}{4}.$$

If $\cos\theta-\sin\theta=\sin\theta\cos\theta$, then by squaring

$$1-2\sin\theta\cos\theta=\sin^2\theta\cos^2\theta;$$

therefore $$1-\sin 2\theta=\frac{\sin^2 2\theta}{4}.$$

By solving this quadratic in the usual way we obtain $\sin 2\theta=-2\pm2\sqrt{2}$; the upper sign must be taken, for the lower sign would make $\sin 2\theta$ numerically greater than unity.

Problem 23. $2\sin^2\theta+\sin^2 2\theta=2$.

Solution.

$$2\sin^2\theta+\sin^2 2\theta=2;\ \text{therefore}\ \sin^2 2\theta=2-2\sin^2\theta=2(1-\sin^2\theta);$$

therefore $$4\sin^2\theta\cos^2\theta=2\cos^2\theta;$$

therefore either $$\cos^2\theta=0,\ \text{or}\ \sin^2\theta=\frac{1}{2}.$$

If $\cos^2\theta=0$, then $$\theta=n\pi+\frac{\pi}{2}.$$

If $\sin^2\theta=\frac{1}{2}$, then $$\sin^2\theta=\sin^2\frac{\pi}{4};$$

therefore $$\theta = n\pi \pm \frac{\pi}{4}.$$

Problem 24. $\tan\theta + 2\cot 2\theta = \sin\theta\left(1 + \tan\theta\tan\frac{\theta}{2}\right).$

Solution.

$$\tan\theta + 2\cot 2\theta = \sin\theta\left(1 + \tan\theta\tan\frac{\theta}{2}\right)$$

therefore $$\frac{\sin\theta}{\cos\theta} + \frac{2\cos 2\theta}{\sin 2\theta} = \sin\theta\left(1 + \frac{\sin\theta\sin\frac{\theta}{2}}{\cos\theta\cos\frac{\theta}{2}}\right);$$

therefore $$\frac{\sin^2\theta + \cos 2\theta}{\sin\theta\cos\theta} = \sin\theta\cdot\frac{\cos\left(\theta - \frac{\theta}{2}\right)}{\cos\theta\cos\frac{\theta}{2}} = \frac{\sin\theta}{\cos\theta};$$

therefore $\sin^2\theta + \cos 2\theta = \sin^2\theta$; therefore $\cos 2\theta = 0$;

therefore $$2\theta = n\pi + \frac{\pi}{2}.$$

Problem 25. $\sin^2 2\theta - \sin^2\theta = \sin^2\frac{\pi}{6}.$

Solution.

$$\sin^2 2\theta - \sin^2\theta = \sin^2\frac{\pi}{6} = \frac{1}{4};\ \text{therefore}$$

$$4\sin^2\theta\cos^2\theta - \sin^2\theta = \frac{1}{4};$$

therefore $$4\sin^2\theta(1 - \sin^2\theta) - \sin^2\theta = \frac{1}{4};$$

therefore $$4\sin^4\theta - 3\sin^2\theta + \frac{1}{4} = 0.$$

By solving this quadratic in the usual way we obtain

$$\sin^2\theta = \frac{3 \pm \sqrt{5}}{8}.$$

Taking the upper sign we have $\sin^2\theta = \sin^2\frac{3\pi}{10}$, and therefore

$$\theta = n\pi \pm \frac{3\pi}{10}.$$

Taking the lower sign we have $\sin^2\theta = \sin^2\frac{\pi}{10}$, and therefore

$$\theta = n\pi \pm \frac{\pi}{10}.$$

Problem 26. $\operatorname{cosec}\theta = \operatorname{cosec}\frac{\theta}{2}.$

Solution.

$$\operatorname{cosec}\theta = \operatorname{cosec}\frac{\theta}{2}; \quad \text{therefore } \frac{1}{\sin\theta} = \frac{1}{\sin\frac{\theta}{2}};$$

therefore $\sin\frac{\theta}{2} = \sin\theta$; therefore $\sin\frac{\theta}{2} = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$;

therefore either $\sin\frac{\theta}{2} = 0$, or $\cos\frac{\theta}{2} = \frac{1}{2}$.

If $\sin\frac{\theta}{2} = 0$, then $\frac{\theta}{2} = n\pi$.

If $\cos\frac{\theta}{2} = \frac{1}{2}$, then $\frac{\theta}{2} = 2m\pi \pm \frac{\pi}{3}$.

Problem 27. $\cos\theta\cos 3\theta = \cos 5\theta\cos 7\theta$.

Solution.

$$\cos\theta\cos 3\theta = \cos 5\theta\cos 7\theta;$$

therefore $\cos 4\theta + \cos 2\theta = \cos 12\theta + \cos 2\theta$;

therefore $\cos 4\theta = \cos 12\theta$; therefore $12\theta = 2n\pi \pm 4\theta$;

taking the upper sign we obtain $\theta = \frac{2n\pi}{8} = \frac{n\pi}{4}$,

and taking the lower sign we obtain $\theta = \frac{2n\pi}{16} = \frac{n\pi}{8}$.

It is obvious however that the second expression includes the first.

Problem 28. $\sin\theta\sin 3\theta = \frac{1}{2}$.

Solution.

$$\sin\theta\sin 3\theta = \frac{1}{2}; \quad \text{therefore } \sin\theta(3\sin\theta - 4\sin^3\theta) = \frac{1}{2};$$

therefore $4\sin^4\theta - 3\sin^2\theta + \frac{1}{2} = 0$.

By solving this quadratic in the usual way we obtain

$$\sin^2\theta = \frac{3\pm 1}{8} = \frac{1}{2} \text{ or } \frac{1}{4}.$$

If $\sin^2\theta = \frac{1}{2}$, then $\sin^2\theta = \sin^2\frac{\pi}{4}$, and $\theta = n\pi \pm \frac{\pi}{4}$.

If $\sin^2\theta = \frac{1}{4}$, then $\sin^2\theta = \sin^2\frac{\pi}{6}$, and $\theta = n\pi \pm \frac{\pi}{6}$.

See *Problem* 5 of *Chapter* V.

Problem 29. $4\sin^2\theta + \sin^2 2\theta = 3$.

Solution.

$$4\sin^2\theta + \sin^2 2\theta = 3; \quad \text{therefore } 4\sin^2\theta + 4\sin^2\theta(1-\sin^2\theta) = 3;$$

therefore $4\sin^4\theta - 8\sin^2\theta + 3 = 0$.

By solving this quadratic in the usual way we obtain $\sin^2\theta = \frac{1}{2}$ or $\frac{3}{2}$; and only the former value is admissible. Thus $\sin^2\theta = \sin^2\frac{\pi}{4}$; therefore $\theta = n\pi \pm \frac{\pi}{4}$.

Problem 30. $(1-\tan\theta)(1+\sin 2\theta) = 1+\tan\theta.$

Solution.

$$(1-\tan\theta)(1+\sin 2\theta) = 1+\tan\theta;$$

therefore $$\left(1-\frac{\sin\theta}{\cos\theta}\right)(\sin\theta+\cos\theta)^2 = 1+\frac{\sin\theta}{\cos\theta}.$$

therefore $$(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)^2 = \cos\theta+\sin\theta;$$

therefore either $\cos\theta+\sin\theta = 0$, or $(\cos\theta-\sin\theta)(\cos\theta+\sin\theta) = 1$;

If $\cos\theta+\sin\theta = 0$, then $\sin\theta = -\cos\theta$;

therefore $$\tan\theta = -1;$$

therefore $$\theta = n\pi + \frac{3\pi}{4}.$$

If $(\cos\theta-\sin\theta)(\cos\theta+\sin\theta) = 1$, then $\cos^2\theta - \sin^2\theta = 1$;

therefore $$\cos 2\theta = 1;$$

therefore $$2\theta = 2n\pi.$$

Problem 31. $\sin\theta+\sin 2\theta+\sin 3\theta+\sin 4\theta = 0.$

Solution.

$$\sin\theta+\sin 2\theta+\sin 3\theta+\sin 4\theta = 0;$$

therefore $$\sin\theta+\sin 4\theta+\sin 2\theta+\sin 3\theta = 0;$$

therefore $$2\sin\frac{5\theta}{2}\cos\frac{3\theta}{2} + 2\sin\frac{5\theta}{2}\cos\frac{\theta}{2} = 0;$$

therefore $$2\sin\frac{5\theta}{2}\left(\cos\frac{3\theta}{2}+\cos\frac{\theta}{2}\right) = 0;$$

therefore $$4\sin\frac{5\theta}{2}\cos\frac{\theta}{2}\cos\theta = 0.$$

Thus there are three cases :

If $\sin\frac{5\theta}{2} = 0$, then $\frac{5\theta}{2} = n\pi$,

If $\cos\frac{\theta}{2} = 0$, then $\frac{\theta}{2} = n\pi + \frac{\pi}{2}$,

If $\cos\theta = 0$, then $\theta = n\pi + \frac{\pi}{2}$.

Problem 32. $\sin\theta - \cos\theta = 4\sin\theta\cos^2\theta.$

Solution.

$$\sin\theta-\cos\theta = 4\sin\theta\cos^2\theta;$$

therefore $$\sin\theta - 4\sin\theta(1-\sin^2\theta) = \cos\theta;$$

therefore $$4\sin^3\theta - 3\sin\theta = \cos\theta;$$

therefore $$\cos\theta = -\sin 3\theta = \cos\left(3\theta + \frac{\pi}{2}\right);$$

therefore $$3\theta + \frac{\pi}{2} = 2n\pi \pm \theta.$$

Problem 33. $(\cot\theta - \tan\theta)^2(2-\sqrt{3}) = 4(2+\sqrt{3})$.

Solution.

$$(\cot\theta - \tan\theta)^2(2-\sqrt{3}) = 4(2+\sqrt{3});$$

therefore $$\left(\frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}\right)^2 = \frac{4(2+\sqrt{3})}{2-\sqrt{3}};$$

therefore $$\left(\frac{\cos^2\theta - \sin^2\theta}{2\sin\theta\cos\theta}\right)^2 = \frac{2+\sqrt{3}}{2-\sqrt{3}} = \frac{(2+\sqrt{3})^2}{(2-\sqrt{3})(2+\sqrt{3})};$$

therefore $$\left(\frac{\cos 2\theta}{\sin 2\theta}\right)^2 = (2+\sqrt{3})^2;$$

therefore $$\cot^2 2\theta = \cot^2\frac{\pi}{12};$$

therefore $$2\theta = n\pi \pm \frac{\pi}{12}.$$

Problem 34. $2\sqrt{2}\cos\left(\frac{\pi}{4} - \theta\right)(1+\sin\theta) = 1 + \cos 2\theta$.

Solution.

$$2\sqrt{2}\cos\left(\frac{\pi}{4} - \theta\right)(1+\sin\theta) = 1 + \cos 2\theta;$$

therefore $$2\sqrt{2}\cos\left(\frac{\pi}{4} - \theta\right)(1+\sin\theta) = 2\cos^2\theta = 2(1-\sin^2\theta);$$

therefore either $1 + \sin\theta = 0$, or $\sqrt{2}\cos\left(\frac{\pi}{4} - \theta\right) = 1 - \sin\theta$.

If $1+\sin\theta = 0$, then $\sin\theta = -1$; therefore $\theta = n\pi + (-1)^n\frac{3\pi}{2}$, which may be expressed more simply as $(4m+3)\frac{\pi}{2}$.

If $\sqrt{2}\cos\left(\frac{\pi}{4} - \theta\right) = 1 - \sin\theta$, then $\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right) = 1 - \sin\theta$;

therefore $$2\sin\theta = 1 - \cos\theta;$$

therefore $$4\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\sin^2\frac{\theta}{2};$$

therefore either $\sin\frac{\theta}{2} = 0$, or $\tan\frac{\theta}{2} = 2$.

If $\sin\frac{\theta}{2} = 0$, then $\frac{\theta}{2} = n\pi$.

If $\tan\frac{\theta}{2} = 2$, then $\frac{\theta}{2} = n\pi + \alpha$, where α is such that $\tan\alpha = 2$.

Problem 35. $\sin 9\theta + \sin 5\theta + 2\sin^2\theta = 1$.

Solution. $\sin 9\theta + \sin 5\theta + 2\sin^2\theta = 1$; therefore

$$2\sin 7\theta\cos 2\theta = 1 - 2\sin^2\theta = \cos 2\theta;$$

therefore either $\cos 2\theta = 0$, or $\sin 7\theta = \frac{1}{2}$.

If $\cos 2\theta = 0$, then $2\theta = n\pi + \frac{\pi}{2}$.

If $\sin 7\theta = \frac{1}{2}$, then $7\theta = n\pi + (-1)^n\frac{\pi}{6}$.

CHAPTER X
Logarithms and Logarithmic Series.

Problem 1. *Find the logarithm of 128 to the base* $\sqrt[3]{4}$.

Solution. Let x denote the required logarithm; then

$$128 = \left(\sqrt[3]{4}\right)^x, \text{ that is } 2^7 = 4^{\frac{x}{3}} = 2^{\frac{2x}{3}};$$

therefore $\dfrac{2x}{3} = 7$; therefore $x = \dfrac{21}{2}$.

Problem 2. *Find the logarithm of* $243\sqrt[3]{9}$ *to the base* $\sqrt{3}$.

Solution. Let x denote the required logarithm; then

$$243\sqrt[3]{9} = \left(\sqrt{3}\right)^x, \text{ that is } 3^5\sqrt[3]{9} = 3^{\frac{x}{2}}, \text{ that is } 3^{5+\frac{2}{3}} = 3^{\frac{x}{2}};$$

therefore $\dfrac{x}{2} = \dfrac{17}{3}$; therefore $x = \dfrac{34}{3}$.

Problem 3. *Find the following logarithms,* $\log_3 2187$, $\log_{10} {\cdot}0001$, $\log_2 \cos 45°$.

Solution. Let x denote the logarithm of 2187 to the base 3; then $2187 = 3^x$, that is $3^7 = 3^x$; therefore $x = 7$.

Let x denote the logarithm of ·0001 to the base 10; then $\cdot 0001 = 10^x$, that is $\dfrac{1}{10^4} = 10^x$, that is $10^{-4} = 10^x$; therefore x = -4.

Let x denote the logarithm of $\cos 45°$ to the base 2; then $\cos 45° = 2^x$, that is $\dfrac{1}{\sqrt{2}} = 2^x$, that is $2^{-\frac{1}{2}} = 2^x$; therefore $x = -\dfrac{1}{2}$.

Problem 4. *Find approximately the value of x from the equation* $5^{6-4x} = 2^{x+3}$, *having given* $\log 2 = \cdot 301030$.

Solution. $5^{6-4x} = 2^{x+3}$; therefore $(6-4x)\log 5 = (x+3)\log 2$;

therefore $$(6-4x)\log\frac{10}{2} = (x+3)\log 2;$$

therefore $$(6-4x)(1-\log 2) = (x+3)\log 2;$$

therefore $$x(4-3\log 2) = 6 - 9\log 2;$$

therefore $$x = \frac{6-9\log 2}{4-3\log 2} = \frac{3{\cdot}29073}{3{\cdot}09691} = 1{\cdot}06\ldots$$

Problem 5. Given $\log \cdot 224 = a$ and $\log 125 = b$, find $\log 2$ and $\log 7$.

Solution.

Here $$a = \log \cdot 224 = \log\frac{224}{1000} = \log\frac{7\times 32}{1000} = \log 7 + 5\log 2 - 3;$$

$$b = \log 125 = \log\frac{1000}{8} = 3 - 3\log 2.$$

From the second equation we have $\log 2 = \frac{1}{3}(3 - b)$; and then substituting in the first equation we have $\log 7 = a + 3 - \frac{5}{3}(3 - b.)$

Problem 6. *Required the characteristics of* $\log_6 725$, *and of* $\log_6 \sqrt[5]{\cdot 0725}$.

Solution. 725 lies between 6^3 and 6^4; and therefore the characteristic of the logarithm of 725 to the base 6 is 3.

Then $$\log \sqrt[5]{(\cdot 0725)} = \frac{1}{5}\log \cdot 0725 = \frac{1}{5}\log \frac{725}{10000};$$
and $\frac{725}{10000}$ lies between $\frac{1}{6}$ and $\frac{1}{36}$, that is between 6^{-1} and $6^{-2.}$ Hence $\frac{1}{5}\log\frac{725}{10000}$ to the base 6 lies between $-\frac{1}{5}$ and $-\frac{2}{5}$; and thus the characteristic will be -1, since by supposition the decimal part of a logarithm is positive.

Problem 7. *Given* $\log 2 = \cdot 301030$, $\log 405 = 2\cdot 607455$, *find* $\log \cdot 003$.

Solution. $\log 405 = \log(81 \times 5) = \log\left(81 \times \frac{10}{2}\right) = \log\frac{3^4 \times 10}{2} = 4\log 3 + 1 - \log 2$;

therefore $$4\log 3 = \log 405 + \log 2 - 1 = 8\cdot 908485;$$
therefore $$\log 3 = \cdot 477121.$$

Problem 8. *Given* $\log 2 = \cdot 301030$, $\log 7 = \cdot 845098$, *find* $\log 98$ *and* $\log\left(\frac{4}{343}\right)^{\frac{1}{2}}$.

Solution. $\log 98 = \log\left(2 \times 7^2\right) = \log 2 + 2\log 7 = \cdot 301030 + 1\cdot 690196 = 1\cdot 991226;$
$$\log\left(\frac{4}{343}\right)^{\frac{1}{2}} = \frac{1}{2}\log\frac{4}{343} = \frac{1}{2}\log\frac{2^2}{7^3} = \frac{1}{2}(2\log 2 - 3\log 7)$$
$$= -\cdot 966617 = \bar{1}\cdot 033383.$$

Problem 9. *Given* $\log 2 = \cdot 30103$, $\log 3 = \cdot 47712$, *find* $\log(\cdot 0020736)^{\frac{1}{3}}$.

Solution.
$$\log(\cdot 0020736)^{\frac{1}{3}} = \frac{1}{3}\log \cdot 0020736 = \frac{1}{3}\log\frac{20736}{10^7}$$
$$= \frac{1}{3}\log\frac{3^4 \times 2^8}{10^7} = \frac{1}{3}\{4\log 3 + 8\log 2 - 7\}$$
$$= -\cdot 89443 = \bar{1}\cdot 10557.$$

Problem 10. *Determine the sum of the series*
$$\frac{2}{\lfloor 3} + \frac{4}{\lfloor 5} + \frac{6}{\lfloor 7} + \dots \textit{ ad inf.}$$

Solution. $\dfrac{2}{\underline{|3}} = \dfrac{1}{\underline{|2}} - \dfrac{1}{\underline{|3}}, \quad \dfrac{4}{\underline{|5}} = \dfrac{1}{\underline{|4}} - \dfrac{1}{\underline{|5}}, \quad \dfrac{6}{\underline{|7}} = \dfrac{1}{\underline{|6}} - \dfrac{1}{\underline{|7}}, \ \ldots$
thus we see that the series $= \dfrac{1}{\underline{|2}} - \dfrac{1}{\underline{|3}} + \dfrac{1}{\underline{|4}} - \dfrac{1}{\underline{|5}} + \ldots = e^{-1}$.

Problem 11. *Show that*

$$\frac{e}{2} = \frac{1}{\underline{|2}} + \frac{1+2}{\underline{|3}} + \frac{1+2+3}{\underline{|4}} + \frac{1+2+3+4}{\underline{|5}} + \ldots \ \textit{ad inf.}$$

Solution.

$$\frac{1}{\underline{|2}} = \frac{1}{2} \cdot \frac{1 \cdot 2}{\underline{|2}} = \frac{1}{2},$$
$$\frac{1+2}{\underline{|3}} = \frac{1}{2} \cdot \frac{2 \cdot 3}{\underline{|3}} = \frac{1}{2} \cdot \frac{1}{1},$$
$$\frac{1+2+3}{\underline{|4}} = \frac{1}{2} \cdot \frac{3 \cdot 4}{\underline{|4}} = \frac{1}{2} \cdot \frac{1}{\underline{|2}},$$
$$\frac{1+2+3+4}{\underline{|5}} = \frac{1}{2} \cdot \frac{4 \cdot 5}{\underline{|5}} = \frac{1}{2} \cdot \frac{1}{\underline{|3}},$$

and generally $\dfrac{1+2+3+\ldots+n}{\underline{|n+1}} = \dfrac{1}{2} \cdot \dfrac{n(n+1)}{\underline{|n+1}} = \dfrac{1}{2} \cdot \dfrac{1}{\underline{|n-1}}$.
Thus we see that the series $= \dfrac{1}{2}\left\{1 + \dfrac{1}{1} + \dfrac{1}{\underline{|2}} + \dfrac{1}{\underline{|3}} + \ldots\right\} = \dfrac{e}{2}$.

Find x ***from the following six equations :***

Problem 12. $4 \sin x \sin(x - \alpha) = 2 \cos \alpha - 1$.

Solution. $4 \sin x \sin(x - a) = 2 \cos \alpha - 1$;
therefore $\quad 2\{\cos \alpha - \cos(2x - \alpha)\} = 2 \cos \alpha - 1;$
therefore $\quad \cos(2x - \alpha) = \dfrac{1}{2};$
therefore $\quad 2x - \alpha = 2n\pi \pm \dfrac{\pi}{3}.$

Problem 13. $\cos \beta \sqrt{a^2 - x^2} + a \sin \alpha = x \sin \beta$.

Solution. $\cos \beta \sqrt{(a^2 - x^2)} = x \sin \beta - a \sin \alpha;$
therefore $\quad \cos^2 \beta (a^2 - x^2) = x^2 \sin^2 \beta - 2xa \sin \beta \sin \alpha + a^2 \sin^2 \alpha;$
therefore $\quad x^2 - 2xa \sin \beta \sin \alpha = a^2 \cos^2 \beta - a^2 \sin^2 \alpha;$

therefore $\quad (x - a \sin \beta \sin \alpha)^2 = a^2 \cos^2 \beta - a^2 \sin^2 \alpha + a^2 \sin^2 \beta \sin^2 \alpha$
$\quad = a^2 \cos^2 \beta - a^2 \sin^2 \alpha \cos^2 \beta = a^2 \cos^2 \beta \cos^2 \alpha;$
therefore $\quad x - a \sin \beta \sin \alpha = \pm a \cos \beta \cos \alpha;$
therefore $\quad x = a(\sin \beta \sin \alpha \pm \cos \beta \cos \alpha) = a \cos(\beta - \alpha)$ or $-a \cos(\beta + \alpha)$.

Problem 14. $\sin\alpha + \sin(x-\alpha) + \sin(2x+\alpha) = \sin(x+\alpha) + \sin(2x-\alpha)$.

Solution. $\sin\alpha + \sin(x-\alpha) + \sin(2x+\alpha) = \sin(x+\alpha) + \sin(2x-\alpha)$;

therefore $\sin\alpha = \sin(x+\alpha) - \sin(x-\alpha) + \sin(2x-\alpha) - \sin(2x+\alpha)$

$$= 2\sin\alpha\cos x - 2\sin\alpha\cos 2x;$$

therefore $1 = 2\cos x - 2\cos 2x = 2\cos x - 2(2\cos^2 x - 1)$;

therefore $4\cos^2 x - 2\cos x - 1 = 0$.

By solving this quadratic in the usual way we obtain $\cos x = \dfrac{1\pm\sqrt{5}}{4}$.

Taking the upper sign we have $\cos x = \cos\dfrac{\pi}{5}$, and therefore $x = 2n\pi \pm \dfrac{\pi}{5}$.

Taking the lower sign we have $\cos x = \cos\dfrac{3\pi}{5}$, and therefore $x = 2n\pi \pm \dfrac{3\pi}{5}$.

Problem 15. $\cos\left(x+\dfrac{3}{2}\right)\alpha + \cos\left(x+\dfrac{1}{2}\right)\alpha = \sin\alpha$.

Solution. $\cos\left(x+\dfrac{3}{2}\right)\alpha + \cos\left(x+\dfrac{1}{2}\right)\alpha = \sin\alpha$;

therefore $2\cos(x+1)\alpha\cos\dfrac{\alpha}{2} = \sin\alpha = 2\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}$;

therefore $\cos(x+1)\alpha = \sin\dfrac{\alpha}{2} = \cos\left(\dfrac{\pi}{2}-\dfrac{\alpha}{2}\right)$.

Hence all the solutions are contained in

$$(x+1)\alpha = 2n\pi \pm \left(\frac{\pi}{2}-\frac{\alpha}{2}\right).$$

Problem 16. $x^2\cos\alpha\cos\left(\alpha-\dfrac{\beta}{2}\right) + x\cos(\alpha-\beta) = 2\cos\dfrac{\beta}{2}$.

Solution. $x^2\cos\alpha\cos\left(\alpha-\dfrac{\beta}{2}\right) + x\cos(\alpha-\beta) = 2\cos\dfrac{\beta}{2}$;

therefore
$$x^2 + \frac{x\cos(\alpha-\beta)}{\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)} = \frac{2\cos\frac{\beta}{2}}{\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)};$$

$$\therefore \left\{x + \frac{\cos(\alpha-\beta)}{2\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)}\right\}^2 = \frac{2\cos\frac{\beta}{2}}{\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)} + \frac{\cos^2(\alpha-\beta)}{4\cos^2\alpha\cos^2\left(\alpha-\frac{\beta}{2}\right)}$$

$$= \frac{\cos^2(\alpha-\beta) + 8\cos\alpha\cos\frac{\beta}{2}\cos\left(\alpha-\frac{\beta}{2}\right)}{4\cos^2\alpha\cos^2\left(\alpha-\frac{\beta}{2}\right)}$$

$$= \frac{\cos^2(\alpha-\beta)+4\cos\alpha\{\cos\alpha+\cos(\alpha-\beta)\}}{4\cos^2\alpha\cos^2\left(\alpha-\dfrac{\beta}{2}\right)}$$

$$= \frac{\{\cos(\alpha-\beta)+2\cos\alpha\}^2}{4\cos^2\alpha\cos^2\left(\alpha-\dfrac{\beta}{2}\right)};$$

therefore $$x+\frac{\cos(\alpha-\beta)}{2\cos\alpha\cos\left(\alpha-\dfrac{\beta}{2}\right)} = \pm\frac{\cos(\alpha-\beta)+2\cos\alpha}{2\cos\alpha\cos\left(\alpha-\dfrac{\beta}{2}\right)}.$$

Taking the upper sign we have

$$x = \frac{2\cos\alpha}{2\cos\alpha\cos\left(\alpha-\dfrac{\beta}{2}\right)} = \sec\left(\alpha-\frac{\beta}{2}\right).$$

Taking the lower sign we have

$$x = -\frac{\cos\alpha+\cos(\alpha-\beta)}{\cos\alpha\cos\left(\alpha-\dfrac{\beta}{2}\right)}$$

$$= -\frac{2\cos\left(\alpha-\dfrac{\beta}{2}\right)\cos\dfrac{\beta}{2}}{\cos\alpha\cos\left(\alpha-\dfrac{\beta}{2}\right)} = -2\cos\frac{\beta}{2}\sec\alpha.$$

Or we may write the proposed equation in this form

$$x\cos\alpha\left\{x\cos\left(\alpha-\frac{\beta}{2}\right)-1\right\}+2\left\{x\cos\left(\alpha-\frac{\beta}{2}\right)-1\right\}\cos\frac{\beta}{2}=0;$$

and then the two values of x which satisfy it are obvious.

Problem 17. $\cot 2^{x^{-1}}\alpha - \cot 2^x\alpha = \operatorname{cosec} 3\alpha$.

Solution. $\cot 2^{x-1}\alpha - \cot 2^x\alpha = \operatorname{cosec} 3\alpha$;
put y for $2^{x-1}\alpha$; thus $\cot y - \cot 2y = \operatorname{cosec} 3\alpha$;

therefore $$\frac{\cos y}{\sin y} - \frac{\cos 2y}{\sin 2y} = \operatorname{cosec} 3\alpha;$$

therefore $$\frac{\sin(2y-y)}{\sin y\sin 2y} = \operatorname{cosec} 3\alpha = \frac{1}{\sin 3\alpha};$$

therefore $$\sin 2y = \sin 3\alpha, \text{ that is } \sin 2^x\alpha = \sin 3\alpha.$$

Thus the general solution is $2^x\alpha = n\pi + (-1)^n 3\alpha$.

Problem 18. *Solve the equation* $m \operatorname{vers}\theta = n\operatorname{vers}(\alpha-\theta)$.

Solution. $m\operatorname{vers}\theta = n\operatorname{vers}(\alpha-\theta)$;

therefore $$m(1-\cos\theta) = n\{1-\cos(\alpha-\theta)\};$$

therefore $$2m\sin^2\frac{\theta}{2} = 2n\sin^2\frac{\alpha-\theta}{2};$$

therefore $$\sin\frac{\alpha-\theta}{2} = \left(\frac{m}{n}\right)^{\frac{1}{2}}\sin\frac{\theta}{2};$$

therefore $$\sin\frac{\alpha}{2}\cos\frac{\theta}{2}-\cos\frac{\alpha}{2}\sin\frac{\theta}{2}=\left(\frac{m}{n}\right)^{\frac{1}{2}}\sin\frac{\theta}{2};$$

Divide by $\cos\frac{\theta}{2}$; thus we obtain a simple equation for finding $\tan\frac{\theta}{2}$.

Problem 19. *Solve the equation* $\cos n\theta+\cos(n-2)\theta=\cos\theta$.

Solution. $\cos n\theta+\cos(n-2)\theta=\cos\theta$;

therefore $$2\cos(n-1)\theta\cos\theta=\cos\theta;$$

therefore either $\cos\theta=0$, or $\cos(n-1)\theta=\frac{1}{2}$.

If $\cos\theta=0$, then $\theta=m\pi+\frac{\pi}{2}$.

If $\cos(n-1)\theta=\frac{1}{2}$, then $(n-1)\theta=2m\pi\pm\frac{\pi}{3}$.

Problem 20. *Solve the following equation, and show that there are seven positive values of θ greater than 0 and less than 2π,*

$$\sin\theta+\sin 3\theta=\sin 2\theta+\sin 4\theta.$$

Solution. $\sin\theta+\sin 3\theta=\sin 2\theta+\sin 4\theta$;

therefore $$2\sin 2\theta\cos\theta=2\sin 3\theta\cos\theta;$$

therefore either $\cos\theta=0$, or $\sin 2\theta=\sin 3\theta$.

If $\cos\theta=0$, then $\theta=n\pi+\frac{\pi}{2}$.

If $\sin 2\theta=\sin 3\theta$, then $\sin 2\theta-\sin 3\theta=0$; therefore $2\sin\frac{\theta}{2}\cos\frac{5\theta}{2}=0$; therefore either $\sin\frac{\theta}{2}=0$, or $\cos\frac{5\theta}{2}=0$: taking $\sin\frac{\theta}{2}=0$ we have $\frac{\theta}{2}=n\pi$, and taking $\cos\frac{5\theta}{2}=0$ we have $\frac{5\theta}{2}=n\pi+\frac{\pi}{2}$.

The seven values greater than 0 and less than 2π are

$$\frac{\pi}{5},\ \frac{3\pi}{5},\ \frac{5\pi}{5},\ \frac{7\pi}{5},\ \frac{9\pi}{5},\ \frac{\pi}{2}\text{ and }\frac{3\pi}{2}.$$

Problem 21. *Find* $\tan x$ *from the equation* $\tan x=\tan\beta\tan(\alpha+x)$; *and show that in order that* $\tan x$ *may be real,* $\tan\beta$ *must not lie between* $(\sec\alpha-\tan\alpha)^2$ *and* $(\sec\alpha+\tan\alpha)^2$.

Solution. $\tan x=\tan\beta\tan(x+\alpha)=\dfrac{\tan\beta(\tan x+\tan\alpha)}{1-\tan x\tan\alpha}$;

therefore $$\tan x(1-\tan x\tan\alpha)=\tan\beta(\tan x+\tan\alpha);$$

therefore $$\tan^2 x\tan\alpha+(\tan\beta-1)\tan x+\tan\alpha\tan\beta=0.$$

By solving this quadratic in the usual way we obtain the values of $\tan x$. It is known by the theory of quadratic equations that for the values to be real we must have $(\tan\beta-1)^2-4\tan^2\alpha\tan\beta$ positive or zero.

And $(\tan\beta-1)^2-4\tan^2\alpha\tan\beta$.

$$=\tan^2\beta-2\tan\beta-4\tan^2\alpha\tan\beta+1$$

$$=\left\{\tan\beta-(1+2\tan^2\alpha)\right\}^2+1-\left(1+2\tan^2\alpha\right)^2$$

$$= \left\{\tan\beta - \frac{1+\sin^2\alpha}{\cos^2\alpha}\right\}^2 - \frac{4\sin^2\alpha}{\cos^4\alpha}$$
$$= \left\{\tan\beta - \frac{1+\sin^2\alpha}{\cos^2\alpha} - \frac{2\sin\alpha}{\cos^2\alpha}\right\}\left\{\tan\beta - \frac{1+\sin^2\alpha}{\cos^2\alpha} + \frac{2\sin\alpha}{\cos^2\alpha}\right\}$$
$$= \left\{\tan\beta - \left(\frac{1+\sin\alpha}{\cos\alpha}\right)^2\right\}\left\{\tan\beta - \left(\frac{1-\sin\alpha}{\cos\alpha}\right)^2\right\}.$$

This expression then must be positive or zero, and therefore $\tan\beta$ must not lie between $\left(\frac{1-\sin\alpha}{\cos\alpha}\right)^2$ and $\left(\frac{1+\sin\alpha}{\cos\alpha}\right)^2$.

Problem 22. *Find the least value of θ which satisfies*

$$\tan\left(\frac{\pi}{4}-\theta\right) + \tan\left(\frac{\pi}{4}+\theta\right) = \left(\frac{8\sqrt{2}}{1+\sqrt{2}}\right)^{\frac{1}{2}}.$$

Solution.

$$\tan\left(\frac{\pi}{4}-\theta\right) + \tan\left(\frac{\pi}{4}+\theta\right)$$
$$= \frac{\sin\left(\frac{\pi}{4}-\theta\right)}{\cos\left(\frac{\pi}{4}-\theta\right)} + \frac{\sin\left(\frac{\pi}{4}+\theta\right)}{\cos\left(\frac{\pi}{4}+\theta\right)}$$
$$= \frac{\sin\left(\frac{\pi}{4}-\theta\right)\cos\left(\frac{\pi}{4}+\theta\right) + \sin\left(\frac{\pi}{4}+\theta\right)\cos\left(\frac{\pi}{4}-\theta\right)}{\cos\left(\frac{\pi}{4}-\theta\right)\cos\left(\frac{\pi}{4}+\theta\right)}$$
$$= \frac{\sin\frac{\pi}{2}}{\cos\left(\frac{\pi}{4}-\theta\right)\cos\left(\frac{\pi}{4}+\theta\right)} = \frac{1}{\sin\left(\frac{\pi}{4}+\theta\right)\cos\left(\frac{\pi}{4}+\theta\right)}$$
$$= \frac{2}{\sin\left(\frac{\pi}{2}+2\theta\right)} = \frac{2}{\cos 2\theta}.$$

Thus $$\frac{2}{\cos 2\theta} = \left(\frac{8\sqrt{2}}{1+\sqrt{2}}\right)^{\frac{1}{2}};$$

therefore $$\frac{\cos 2\theta}{2} = \left(\frac{1+\sqrt{2}}{8\sqrt{2}}\right)^{\frac{1}{2}};$$

therefore $$\cos^2 2\theta = \frac{1+\sqrt{2}}{2\sqrt{2}};$$

therefore $$2\cos^2 2\theta - 1 = \frac{1+\sqrt{2}}{\sqrt{2}} - 1 = \frac{1}{\sqrt{2}};$$

therefore $$\cos 4\theta = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4};$$

therefore the least value of θ is given by $4\theta = \frac{\pi}{4}$.

Problem 23. *Given $\sin^2(n+1)\theta = \sin^2 n\theta + \sin^2(n-1)\theta$ where $(n+1)\theta$, $n\theta$, and $(n-1)\theta$ are the angles of a triangle, find an integral value of n.*

Solution.

$$\sin^2(n+1)\theta = \sin^2 n\theta + \sin^2(n-1)\theta;$$

therefore $$\sin^2(n+1)\theta - \sin^2(n-1)\theta = \sin^2 n\theta;$$

therefore $$\sin 2n\theta \sin 2\theta = \sin^2 n\theta.$$ (*Art.* 83) (page 404)

But $$(n+1)\theta + (n-1)\theta + n\theta = \pi;$$

therefore $$3n\theta = \pi; \qquad \text{therefore } n\theta = \frac{\pi}{3};$$

therefore $$\sin 2\theta \sin\frac{2\pi}{3} = \sin^2\frac{\pi}{3};$$

therefore $$\sin 2\theta = \sin\frac{\pi}{3};$$

thus $2\theta = \frac{\pi}{3}$; therefore $\theta = \frac{\pi}{6}$. But $n\theta = \frac{\pi}{3}$; and therefore $n = 2$.

Problem 24. *Reduce to its simplest form and solve the equation*

$$\cos^2\theta - \cos^2\alpha = 2\cos^3\theta(\cos\theta - \cos\alpha) - 2\sin^3\theta(\sin\theta - \sin\alpha).$$

Solution. $\cos^2\theta - \cos^2\alpha = 2\cos^3\theta(\cos\theta - \cos\alpha) - 2\sin^3\theta(\sin\theta - \sin\alpha)$;

therefore $$\cos^2\theta - \cos^2\alpha = \frac{\cos 3\theta + 3\cos\theta}{2}(\cos\theta - \cos\alpha) - \frac{3\sin\theta - \sin 3\theta}{2}(\sin\theta - \sin\alpha);$$

therefore $$\begin{aligned} 2(\cos^2\theta - \cos^2\alpha) &= \cos 3\theta\cos\theta + \sin 3\theta\sin\theta \\ &\quad - \cos 3\theta\cos\alpha - \sin 3\theta\sin\alpha \\ &\quad + 3\cos^2\theta - 3\sin^2\theta - 3\cos\theta\cos\alpha \\ &\quad + 3\sin\theta\sin\alpha; \end{aligned}$$

$$\therefore \quad \cos(3\theta - \theta) - \cos(3\theta - \alpha) - 3\cos(\theta + \alpha) = 3\sin^2\theta - \cos^2\theta - 2\cos^2\alpha;$$

$$\therefore \quad \begin{aligned} \cos 2\theta - \cos(3\theta - \alpha) - 3\cos(\theta + \alpha) &= 3 - 4\cos^2\theta - 2\cos^2\alpha \\ &= 3 - 2(1 + \cos 2\theta) - (1 + \cos 2\alpha) \\ &= -2\cos 2\theta - \cos 2\alpha; \end{aligned}$$

therefore $$3\cos 2\theta - 3\cos(\theta + \alpha) - \cos(3\theta - \alpha) + \cos 2\alpha = 0;$$

therefore $$3\sin\frac{3\theta + \alpha}{2}\sin\frac{\alpha - \theta}{2} + \sin\frac{(3\theta + \alpha)}{2}\sin\frac{3\theta - 3\alpha}{2} = 0;$$

therefore $$\sin\frac{3\theta + \alpha}{2}\left\{\sin\frac{3(\theta - \alpha)}{2} - 3\sin\frac{\theta - \alpha}{2}\right\} = 0;$$

therefore $$4\sin\frac{3\theta + \alpha}{2}\sin^3\frac{\theta - \alpha}{2} = 0.$$

Hence either $\sin\frac{3\theta + \alpha}{2} = 0$, or $\sin\frac{\theta - \alpha}{2} = 0$; the former gives $\frac{3\theta + \alpha}{2} = n\pi$, and the latter gives $\frac{\theta - \alpha}{2} = n\pi$.

Problem 25. *Show that all the angles which have the same sine as* α *are included in the formula* $\left(2n+\frac{1}{2}\right)\pi \pm \left(\frac{\pi}{2}-\alpha\right)$.

Solution. Let θ denote an angle having the same sine as α, so that $\sin\theta = \sin\alpha$; thus $\cos\left(\theta-\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}-\alpha\right)$; therefore all the solutions are comprised in $\theta - \frac{\pi}{2} = 2n\pi \pm \left(\frac{\pi}{2}-\alpha\right)$.

Problem 26. *Show that all the angles which have the same cosine as* α *are included in the formula* $\left(n+\frac{1}{2}\right)\pi + (-1)^n\left(\alpha-\frac{\pi}{2}\right)$.

Solution. Let θ denote an angle having the same *cosine* as α, so that $\cos\theta = \cos\alpha$; thus $\sin\left(\theta-\frac{\pi}{2}\right) = \sin\left(\alpha-\frac{\pi}{2}\right)$; therefore all the solutions are comprised in $\theta-\frac{\pi}{2} = n\pi + (-1)^n\left(\alpha-\frac{\pi}{2}\right)$.

Problem 27. *In the formula* $\cos\frac{A}{2} - \sin\frac{A}{2} = \pm\sqrt{1-\sin A}$ *the ambiguous* $\pm$ *may be replaced by* $(-1)^m$, *where* m *is the greatest integer contained in* $\frac{270+A}{360}$, *the angle* A *being expressed in degrees.*

Solution. By *Art.* 101 (page 408) it follows that the upper sign ought to be taken if $\frac{A}{2}$ lies between $n\,360° + 225°$ and $n\,360° + 405°$; in this case A lies between $2n\,360° + 450°$, and $2n\,360° + 810°$, and $A + 270°$ lies between $2n\,360° + 720°$ and $2n\,360° + 1080°$, and therefore $\frac{A+270°}{360°}$ lies between $2n+2$ and $2n+3$: thus the integral part of this fraction is an *even* number, so that denoting it by m we have $(-1)^m$ positive.

In precisely the same manner we find that the present problem agrees with *Art.* 101 (page 408) for the case in which m is *odd.*

Problem 28. *In the formula* $\tan\frac{A}{2} = \frac{\pm\sqrt{1+\tan^2 A}-1}{\tan A}$ *the ambiguity* $\pm$ *may be replaced by* $(-1)^m$, *where* m *is the greatest integer contained in* $\frac{90+A}{180}$, *the angle* A *being expressed in degrees.*

Solution. First suppose the number of degrees in A to lie between $n\,360$ and $n\,360 + 90$; then $\tan A$ and $\tan\frac{A}{2}$ are both positive, and therefore the upper sign must be taken in the ambiguity. Also in this case $\frac{A+90}{180}$ lies between $\frac{n\,360+90}{180}$ and $\frac{n\,360+180}{180}$, that is between $2n+\frac{1}{2}$ and $2n+1$; so that m is even.

Next suppose the number of degrees in A to lie between $n\,360{+}90$ and $n\,360{+}180$; then $\tan A$ is negative, and $\tan\frac{A}{2}$ is positive; and therefore the lower sign must be taken in the ambiguity. Also in this case $\dfrac{A+90}{180}$ lies between $2n+1$ and $2n+2$, so that m is odd.

Similarly we may proceed if the number of degrees in A lies between $n\,360+180$ and $n\,360+270$, or between $n\,360+270$ and $n\,360+360$.

It will be observed that in this and the preceding problem the *greatest integer* in a certain expression means that integer which with a *positive* proper fraction constitutes the whole expression.

Or we might treat the example thus :

$$\pm\sqrt{(1+\tan^2 A)} = \pm\sqrt{\frac{1}{\cos^2 A}} = \pm\frac{1}{\cos A};$$

but
$$\tan\frac{A}{2} = \frac{1-\cos A}{\sin A} = \frac{\dfrac{1}{\cos A}-1}{\tan A}:$$

hence the ambiguity in $\pm\sqrt{(1+\tan^2 A)}$ must be so taken as to ensure that the *sign is the same as the sign* of $\cos A$, and it is easy to show that $(-1)^m$ is of the same sign as $\cos A$, when m has the prescribed value.

Problem 29. *If* $\tan(\cot x) = \cot(\tan x)$, *show that the real values of* x *are given by* $\sin 2x = \dfrac{4}{(2n+1)\pi}$, *where n is any integer except −1.*

Solution.

$$\tan(\cot x) = \cot(\tan x);$$

therefore
$$\tan(\cot x) = \tan\left\{\frac{\pi}{2} - \tan x\right\};$$

therefore, by *Art.* 68 (page 402), all the possible solutions are comprised in

$$\cot x = n\pi + \frac{\pi}{2} - \tan x;$$

therefore
$$\cot x + \tan x = n\pi + \frac{\pi}{2};$$

therefore
$$\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} = n\pi + \frac{\pi}{2};$$

therefore
$$\frac{1}{\sin x\cos x} = \frac{(2n+1)\pi}{2};$$

therefore
$$\sin x\cos x = \frac{2}{(2n+1)\pi};$$

therefore
$$\sin 2x = \frac{4}{(2n+1)\pi}.$$

The value $n=-1$ would make $\sin 2x$ greater than unity.

Problem 30. *Show how to express* $\cos\dfrac{A}{2^n}$ *in terms of* $\cos A$, *where n is any positive integer.*

Solution.

$$2\cos^2\frac{A}{2} = 1 + \cos A;$$

therefore
$$4\cos^2\frac{A}{2} = 2 + 2\cos A;$$

therefore $\quad 2\cos\dfrac{A}{2} = \sqrt{(2+2\cos A)}.$

Again $\quad 2\cos^2\dfrac{A}{4} = 1+\cos\dfrac{A}{2};$

therefore $\quad 4\cos^2\dfrac{A}{4} = 2+2\cos\dfrac{A}{2};$

therefore $\quad 2\cos\dfrac{A}{4} = \sqrt{\left(2+2\cos\dfrac{A}{2})\right)} = \sqrt{\left\{2+\sqrt{(2+2\cos A)}\right\}}.$

Similarly $\quad 2\cos\dfrac{A}{8} = \sqrt{\left[2+\sqrt{\left\{2+\sqrt{(2+2\cos A)}\right\}}\right]}.$

and this process may be continued to any extent.

Problem 31. *From the equation* $\cos x = \pm\sqrt{\dfrac{1+\cos 2x}{2}}$ *deduce the formula for* $\sin x$ *in terms of* $\sin 2x$*, and show how the proper signs for the radicals may be determined.*

Solution. Change x successively to $\dfrac{\pi}{4}-x$ and $\dfrac{\pi}{4}+x$; thus

$$\cos\left(x-\frac{\pi}{4}\right) = \pm\sqrt{\frac{1}{2}\left\{1+\cos\left(2x-\frac{\pi}{2}\right)\right\}} = \pm\sqrt{\frac{1+\sin 2x}{2}},$$

and $$\cos\left(\frac{\pi}{4}+x\right) = \pm\sqrt{\frac{1}{2}\left\{1+\cos\left(\frac{\pi}{2}+2x\right)\right\}} = \pm\sqrt{\frac{1-\sin 2x}{2}}.$$

Then putting for $\cos\left(\dfrac{\pi}{4}-x\right)$ and $\cos\left(\dfrac{\pi}{4}+x\right)$ their values we have

$$\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x = \pm\sqrt{\frac{1+\sin 2x}{2}} \tag{1}$$

and $$\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x = \pm\sqrt{\frac{1-\sin 2x}{2}} \tag{2}$$

Hence by subtraction we find the required expression for $\sin x$. In (1) the upper or lower sign must be taken according as $\cos\left(x-\dfrac{\pi}{4}\right)$ is positive or negative, that is according as $x-\dfrac{\pi}{4}$ lies between $2n\pi-\dfrac{1}{2}\pi$ and $2n\pi+\dfrac{1}{2}\pi$, or between $2n\pi+\dfrac{1}{2}\pi$ and $2n\pi+\dfrac{3\pi}{2}$. Similarly we can determine the sign to be taken in (2) .

Problem 32. *If the expression* $\dfrac{A\cos(\theta+\alpha)+B\sin(\theta+\beta)}{A'\sin(\theta+\alpha)+B'\cos(\theta+\beta)}$ *retain the same value for all values of* θ*, then will*

$$AA'-BB' = (A'B-AB')\sin(\alpha-\beta).$$

Solution. Let k denote the value which the expression retains for all values of θ, so that

$$\frac{A\cos(\theta+\alpha)+B\sin(\theta+\beta)}{A'\sin(\theta+\alpha)+B'\cos(\theta+\beta)} = k;$$

then $\quad A\cos(\theta+\alpha)+B\sin(\theta+\beta) = k\{A'\sin(\theta+\alpha)+B'\cos(\theta+\beta)\};$

therefore $$\cos\theta(A\cos\alpha + B\sin\beta) + \sin\theta(B\cos\beta - A\sin\alpha)$$
$$= k\cos\theta(A'\sin\alpha + B'\cos\beta) + k\sin\theta(A'\cos\alpha - B'\sin\beta)\}$$
therefore $$\cos\theta\,\{A\cos\alpha + B\sin\beta - k(A'\sin\alpha + B'\cos\beta)\}$$
$$+\sin\theta\,\{B\cos\beta - A\sin\alpha - k(A'\cos\alpha - B'\sin\beta)\} = 0.$$

Now this is to be true for *all values* of θ. Put for θ in succession 0 and $\frac{\pi}{2}$; thus we obtain the following two results:

$$A\cos\alpha + B\sin\beta = k(A'\sin\alpha + B'\cos\beta),$$
$$B\cos\beta - A\sin\alpha = k(A'\cos\alpha - B'\sin\beta);$$

and it is obvious that if these hold the original expression does always retain the same value.

By cross multiplication we obtain

$$(A\cos\alpha + B\sin\beta)(A'\cos\alpha - B'\sin\beta) = (A'\sin\alpha + B'\cos\beta)(B\cos\beta - A\sin\alpha);$$

therefore $$AA'\cos^2\alpha - BB'\sin^2\beta + (A'B - AB')\cos\alpha\sin\beta$$
$$= BB'\cos^2\beta - AA'\sin^2\alpha + (A'B - AB')\sin\alpha\cos\beta;$$
therefore $$AA' - BB' = (A'B - AB')\sin(\alpha - \beta).$$

Problem 33. *If the sum of two angles is given, show that the sum of their sines is numerically greatest when the angles are equal. If the cosine of the given sum is positive show that the sum of the tangents is numerically least when the angles are equal.*

Solution. Let A denote the sum of the two angles x and y. Then

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2} = 2\sin\frac{A}{2}\cos\frac{x-y}{2};$$

and the numerically greatest value of this expression is when $\cos\frac{x-y}{2}$ is greatest, that is when $x - y = 0$, that is when $x = y$.

Again $$\tan x + \tan y = \frac{\sin x}{\cos x} + \frac{\sin y}{\cos y} = \frac{\sin(x+y)}{\cos x\cos y}$$
$$= \frac{\sin A}{\cos x\cos y} = \frac{2\sin A}{2\cos x\cos y}$$
$$= \frac{2\sin A}{\cos(x-y) + \cos(x+y)} = \frac{2\sin A}{\cos(x-y) + \cos A};$$

and if $\cos A$ is positive the numerically least value of this is when

$$\cos(x-y) = 1, \text{ that is when } x = y.$$

Problem 34. *If* $A + B + C = 90°$, *show that unity is the least value of* $\tan^2 A + \tan^2 B + \tan^2 C$.

Solution. By *Art*. 114 (page 409) we have

$$\tan A\tan B + \tan B\tan C + \tan C\tan A = 1;$$
$$\therefore\ \tan^2 A + \tan^2 B + \tan^2 C = 1 + \frac{1}{2}(\tan A - \tan B)^2$$
$$+ \frac{1}{2}(\tan B - \tan C)^2 + \frac{1}{2}(\tan C - \tan A)^2.$$

Hence the least value of the expression is when $\tan A - \tan B$, $\tan B - \tan C$, and $\tan C - \tan A$ all vanish; and the value is then unity.

Problem 35. *If $A+B+C=180°$, show that unity is the least value of* $\cot^2 A+\cot^2 B+\cot^2 C$.

Solution. By *Art*. 114 (page 409) we have

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C;$$

therefore $$\frac{1}{\cot A}+\frac{1}{\cot B}+\frac{1}{\cot C}=\frac{1}{\cot A\cot B\cot C};$$

therefore $$\cot B\cot C+\cot A\cot C+\cot A\cot B=1;$$

therefore $$\cot^2 A+\cot^2 B+\cot^2 C$$
$$=1+\frac{1}{2}(\cot A-\cot B)^2+\frac{1}{2}(\cot B-\cot C)^2+\frac{1}{2}(\cot C-\cot A)^2.$$

Hence the least value of the expression is when $\cot A-\cot B$, $\cot B-\cot C$, and $\cot C-\cot A$ all vanish; and the value is then unity.

Problem 36. *If A, B, C are the angles of a triangle show that* $2\cot A+2\cot B+2\cot C$ *is never less than* $\operatorname{cosec} A+\operatorname{cosec} B+\operatorname{cosec} C$.

Solution. $\cot B+\cot C-\operatorname{cosec} A=\dfrac{\cos B}{\sin B}+\dfrac{\cos C}{\sin C}-\dfrac{1}{\sin A}$
$$=\frac{\sin(B+C)}{\sin B\sin C}-\frac{1}{\sin A}=\frac{\sin A}{\sin B\sin C}-\frac{1}{\sin A}=\frac{\sin^2 A-\sin B\sin C}{\sin A\sin B\sin C}.$$

Proceeding in this way we find that the difference of the two given expressions is equivalent to a fraction with the denominator $\sin A\sin B\sin C$, while the numerator is

$$\sin^2 A+\sin^2 B+\sin^2 C-\sin B\sin C-\sin C\sin A-\sin A\sin B,$$

that is $$\frac{1}{2}(\sin A-\sin B)^2+\frac{1}{2}(\sin B-\sin C)^2+\frac{1}{2}(\sin C-\sin A)^2.$$

This expression is never negative.

Problem 37. *Show that the sum of the three acute angles which satisfy the equation* $\cos^2 A+\cos^2 B+\cos^2 C=1$ *is less than 180°*.

Solution. Suppose A, B, C to be three acute angles such that

$$\cos^2 A+\cos^2 B+\cos^2 C=1,$$

then $$\cos^2 A=1-\cos^2 C-\cos^2 B=\sin^2 C-\cos^2 B$$
$$=-\cos(C-B)\cos(C+B).$$

This shows that $C+B$ must be greater than a right angle. Now if we take $A'=180°-C-B$ we shall have $\cos^2 A'$ numerically equal to $\cos^2(B+C)$, and therefore numerically *less* than $\cos(C-B)\cos(C+B)$; for we may suppose C not less than B, and then $C-B$ is less than $180°-C-B$. Hence $\cos^2 A$ is greater than $\cos^2 A'$, and A is less than A', and therefore $A+B+C$ is less than 180°.

Problem 38. *If each of the angles A, B, C be less than 90°, then* $\sin(A+B+C)$ *is less than* $\sin A+\sin B+\sin C$.

Solution. By *Art*. 113 (page 409) we have $\sin A+\sin B+\sin C-\sin(A+B+C)$
$$=\sin A(1-\cos B\cos C)+\sin B(1-\cos C\cos A)+\sin C(1-\cos A\cos B)$$

$$+\sin A \sin B \sin C;$$

and as A, B, and C are acute this expression is necessarily positive.

Problem 39. *Find the limit of* $\left(\cos\frac{\alpha}{n}\right)^{n^2}$ *when n is increased indefinitely.*

Solution. Let $u = \left(\cos\frac{\alpha}{n}\right)^{n^2}$;

therefore $$\log u = n^2 \log\cos\frac{\alpha}{n} = \frac{n^2}{2}\log\left(1-\sin^2\frac{\alpha}{n}\right)$$
$$= -\frac{n^2}{2}\left\{\sin^2\frac{\alpha}{n} + \frac{1}{2}\sin^4\frac{\alpha}{n} + \frac{1}{3}\sin^6\frac{\alpha}{n} + \dots\right\}.$$

Now $n\sin\frac{\alpha}{n} = \alpha\frac{\sin\frac{\alpha}{n}}{\frac{\alpha}{n}}$, and this is equal to α when n is definitely increased; and therefore $n^2\sin^2\frac{\alpha}{n}$ is equal to α^2.

Then $n^2\sin^4\frac{\alpha}{n} = n^2\sin^2\frac{\alpha}{n}\times\sin^2\frac{\alpha}{n}$; and this vanishes when n is indefinitely increased. Similarly the other terms in $\log u$ vanish, and as in *Art.* 150 (page 416) their sum vanishes also; and thus $\log u = -\frac{\alpha^2}{2}$ ultimately. Therefore $u = e^{-\frac{\alpha^2}{2}}$.

Problem 40. *Find the limit of* $\left(\cos\frac{\alpha}{n}\right)^{n^3}$ *when n is increased indefinitely.*

Solution. Let $u = \left(\cos\frac{\alpha}{n}\right)^{n^3}$; therefore
$$\log u = n^3 \log\cos\frac{\alpha}{n} = \frac{n^3}{2}\log\left(1-\sin^2\frac{\alpha}{n}\right)$$
$$= -\frac{n^3}{2}\left\{\sin^2\frac{\alpha}{n} + \frac{1}{2}\sin^4\frac{\alpha}{n} + \frac{1}{3}\sin^6\frac{\alpha}{n} + \dots\right\}.$$

Now we have shown in solving the preceding Problem that $n^2\sin^2\frac{\alpha}{n} = \alpha^2$ ultimately; hence $n^3\sin^2\frac{\alpha}{n} = n\alpha^2$, and so becomes infinite. Thus the logarithm of u is negative infinity, and therefore u vanishes ultimately.

Problem 41. *Show that* $\sin\theta$ *is greater than* $\tan\theta - \frac{\tan^3\theta}{2}$ *if* θ *is positive and less than* $\frac{\pi}{2}$.

Solution. $\sin\theta - (\tan\theta - \frac{1}{2}\tan^3\theta) = \sin\theta - \tan\theta + \frac{1}{2}\tan^3\theta$
$$= \sin\theta - \frac{\sin\theta}{\cos\theta} + \frac{1\sin^3\theta}{2\cos^3\theta} = \frac{\sin\theta}{\cos^3\theta}\left\{\cos^3\theta - \cos^2\theta + \frac{1}{2}\sin^2\theta\right\}$$
$$= \frac{\sin\theta}{2\cos^3\theta}\left\{2\cos^3\theta - 2\cos^2\theta + 1 - \cos^2\theta\right\}$$

$$= \frac{\sin\theta(1-\cos\theta)}{2\cos^3\theta}\left\{1+\cos\theta-2\cos^2\theta\right\}$$
$$= \frac{\sin\theta(1-\cos\theta)(1-\cos\theta)(1+2\cos\theta)}{2\cos^3\theta}$$
$$= \frac{\sin\theta(1-\cos\theta)^2(1+2\cos\theta)}{2\cos^3\theta}, \text{ which is positive.}$$

Problem 42. *Show that* $\left(\frac{x-1}{x}\right)^x$ *continually increases as* x *increases from unity to infinity; and find the limit of the expression when* x *is increased indefinitely.*

Solution. Let $u = \left(\frac{x-1}{x}\right)^x$; then

$$\log u = x\log\frac{x-1}{x} = x\log\left(1-\frac{1}{x}\right)$$
$$= -x\left\{\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{3x^3}+\dots\right\}$$
$$= -\left\{1+\frac{1}{2x}+\frac{1}{3x^2}+\dots\right\}.$$

Thus the logarithm is always negative, and as x increases the logarithm diminishes numerically, and so u increases; when x is infinite $\log u = -1$; and therefore $u = e^{-1}$.

CHAPTER XI
Use of Logarithmic And Trigonometrical Tables.

Problem 1.

Given $\log 12440 = 4{\cdot}0948204,$

$\log 12441 = 4{\cdot}0948553,$

find $\log 12440{\cdot}35.$

Solution.

$$\begin{array}{r} 4{\cdot}0948553 \\ \underline{4{\cdot}0948204} \\ {\cdot}0000349 \end{array} \qquad 1 : {\cdot}35 :: {\cdot}0000349 : x;$$

this gives $x = {\cdot}0000122;$

therefore $\log 12440{\cdot}35 = 4{\cdot}0948326.$

Problem 2.

Given $\log 1{\cdot}0686 = {\cdot}0288152,$

$\log 1{\cdot}0687 = {\cdot}0288558,$

find the number of which the logarithm is ·0288355.

Solution.

$$\begin{array}{r} {\cdot}0288558 \\ \underline{{\cdot}0288152} \\ {\cdot}0000406 \end{array} \qquad \begin{array}{r} {\cdot}0288355 \\ \underline{{\cdot}0288152} \\ {\cdot}0000203 \end{array} \qquad {\cdot}0000406 : {\cdot}0000203 :: {\cdot}0001 : x;$$

this gives $x = {\cdot}00005;$

therefore $\log 1{\cdot}06865 = {\cdot}0288355.$

Problem 3.

Given $\log 23456 = 4{\cdot}3702540,$

$\log 23457 = 4{\cdot}3702725,$

form a table of proportional parts for the intermediate numbers, and find log *·2345638.*

Solution.

$$\begin{array}{r} 4{\cdot}3702725 \\ \underline{4{\cdot}3702540} \\ {\cdot}0000185 \end{array}$$

1	185
2	370
3	555
4	740
5	925
6	1110
7	1295
8	1480
9	1665

$$\begin{array}{ll} \log 23456 = 4{\cdot}3702540 & \\ \text{add for } 3 & 555 \\ \quad 8 & 1480 \\ \hline & = 4{\cdot}370261030 \end{array}$$

therefore retaining 7 places of decimals

$$\log 23456{\cdot}38 = 4{\cdot}3702610, \text{ and } \log {\cdot}2345638 = \bar{1}{\cdot}3702610.$$

Problem 4. *Find the number whose logarithm is $-(1{\cdot}8753145)$, having given*

$$\log 1{\cdot}3325 = {\cdot}1246672, \quad \log 1{\cdot}3326 = {\cdot}1246998.$$

Solution. $-(1{\cdot}8753145) = \bar{2}{\cdot}1246855$.

$$\begin{array}{r} {\cdot}1246998 \\ {\cdot}1246672 \\ \hline {\cdot}0000326 \end{array} \qquad \begin{array}{r} {\cdot}1246855 \\ {\cdot}1246672 \\ \hline {\cdot}0000183 \end{array} \qquad {\cdot}0000326 : {\cdot}0000183 :: {\cdot}0001 : x;$$

this gives $\qquad x = {\cdot}000056;$

therefore $\qquad \log 1{\cdot}332556 = {\cdot}1246855,$

therefore $\qquad \log {\cdot}01332556 = \bar{2}{\cdot}1246855.$

Problem 5.

Given
$$\log 3{\cdot}855 = {\cdot}5860244,$$
$$\log 3{\cdot}8551 = {\cdot}5860356,$$

find
$$\log({\cdot}00385504)^{\frac{1}{4}}.$$

Solution.

$$\begin{array}{r} {\cdot}5860356 \\ {\cdot}5860244 \\ \hline {\cdot}0000112 \end{array} \qquad {\cdot}0001 : {\cdot}00004 :: {\cdot}0000112 : x;$$

this gives $\qquad x = {\cdot}0000045;$

therefore $\qquad \log 3{\cdot}85504 = {\cdot}5860289;$

therefore $\qquad \log {\cdot}00385504 = \bar{3}{\cdot}5860289;$

therefore $\log({\cdot}00385504)^{\frac{1}{4}} = \frac{1}{4}(\bar{3}{\cdot}5860289) = \frac{1}{4}(-4 + 1{\cdot}5860289) = \bar{1}{\cdot}3965072.$

Problem 6.

Given
$$\log 24 = 1{\cdot}3802112,$$
$$\log 4{\cdot}8989 = {\cdot}6900986,$$
$$\log 4{\cdot}8990 = {\cdot}6901074,$$

find $(24)^{\frac{1}{2}}$ to six places of decimals.

Solution.

$$\log(24)^{\frac{1}{2}} = \frac{1}{2}\log 24 = .6901056.$$

$$\begin{array}{r} {\cdot}6901074 \\ {\cdot}6900986 \\ \hline {\cdot}0000088 \end{array} \qquad \begin{array}{r} {\cdot}6901056 \\ {\cdot}6900986 \\ \hline {\cdot}0000070 \end{array} \qquad {\cdot}0000088 : {\cdot}0000070 :: {\cdot}0001 : x;$$

this gives $x = \cdot 000079;$

therefore $\log 4\cdot 898979 = \cdot 6901056;$

therefore $(24)^{\frac{1}{2}} = 4\cdot 898979.$

Problem 7.

Given

$$\log 14271 = 4\cdot 1544544,$$
$$\log 20313 = 4\cdot 3077741,$$
$$\log 20314 = 4\cdot 3077954,$$

find $(142\cdot 71)^{\frac{1}{2}}$.

Solution.

$$\log(142\cdot 71)^{\frac{1}{7}} = \frac{1}{7} \times 2\cdot 1544544 = \cdot 3077792.$$

$\cdot 3077954$	$\cdot 3077792$	
$\cdot 3077741$	$\cdot 3077741$	$\cdot 0000213 : \cdot 0000051 :: 1 : x;$
$\cdot 0000213$	$\cdot 0000051$	

this gives $x = \cdot 24;$

therefore $\log 20313\cdot 24 = 4\cdot 3077792;$

therefore $\log 2\cdot 031324 = \cdot 3077792;$

therefore $(142\cdot 71)^{\frac{1}{7}} = 2\cdot 031324.$

Problem 8.

Given

$$\log 7 = \cdot 8450980,$$
$$\log 58751 = 4\cdot 7690153,$$
$$\log 58752 = 4\cdot 7690227,$$

find $(\cdot 07)^{\frac{1}{5}}$ *to seven significant figures.*

Solution.

$$\log(\cdot 07)^{\frac{1}{5}} = \frac{1}{5}\log\cdot 07 = \frac{1}{5}(\bar{2}\cdot 8450980) = \frac{1}{5}(-5 + 3\cdot 8450980) = \bar{1}\cdot 7690196.$$

$\cdot 7690227$	$\cdot 7690196$	
$\cdot 7690153$	$\cdot 7690153$	$\cdot 0000074 : \cdot 0000043 :: 1 : x;$
$\cdot 0000074$	$\cdot 0000043$	

this gives $x = \cdot 58;$

therefore $\log 58751\cdot 58 = 4\cdot 7690196;$

therefore $\log \cdot 5875158 = \bar{1}\cdot 7690196;$

therefore $(\cdot 07)^{\frac{1}{5}} = \cdot 5875158.$

Problem 9. *Given* $\log 2 = \cdot 3010300$, $\log 5\cdot 743491 = \cdot 7591760$, *find the fifth root of* $\cdot 0625$.

Solution.

$$\log(\cdot 0625)^{\frac{1}{5}} = \log\left(\frac{625}{10000}\right)^{\frac{1}{5}} = \log\left(\frac{125}{2000}\right)^{\frac{1}{5}} = \log\left(\frac{25}{400}\right)^{\frac{1}{5}}$$
$$= \log\left(\frac{1}{16}\right)^{\frac{1}{5}} = -\frac{1}{5}\log 16 = -\frac{4}{5}\log 2 = -\cdot 2408240$$
$$= \bar{1}\cdot 7591760 = \log \cdot 5743491;$$

therefore $(\cdot 0625)^{\frac{1}{5}} = \cdot 5743491.$

Problem 10. *Given* $\log 2\cdot 7 = \cdot 4313638$, $\log 5\cdot 172818 = \cdot 7137272$, *find the value of* $27^{-\frac{1}{5}}$.

Solution.

$$\log(27)^{-\frac{1}{5}} = -\frac{1}{5}\log 27 = -\frac{1}{5}(1\cdot 4313638) = -\cdot 2862728$$
$$= \bar{1}\cdot 7137272 = \log \cdot 5172818;$$

therefore $(27)^{-\frac{1}{5}} = \cdot 5172818.$

Problem 11. *Given* $\log 71968 = 4\cdot 8571394$, *diff. for* $1 = \cdot 0000060$, *find the value of* $\sqrt[8]{\cdot 0719686}$.

Solution. $\log 71968\cdot 6 = 4\cdot 8571394 + \dfrac{6}{10}$ of $\cdot 0000060 = 4\cdot 8571430;$

$$\log(\cdot 0719686)^{\frac{1}{8}} = \frac{1}{8}(\bar{2}\cdot 8571430) = \frac{1}{8}(-8 + 6\cdot 8571430) = \bar{1}\cdot 8571429.$$

But $\log \cdot 719686 = \bar{1}\cdot 8571430$; therefore $(\cdot 0719686)^{\frac{1}{8}} = \cdot 719686.$

Problem 12. *Given* $\log 103 = 2\cdot 0128372$, $\log 7440942 = 6\cdot 871628$, *find* $(1\cdot 03)^{-10}$.

Solution. $\log(1\cdot 03)^{-10} = -10 \times \cdot 0128372 = -\cdot 128372 = \bar{1}\cdot 871628 = \log \cdot 7440942;$

therefore $(1\cdot 03)^{-10} = \cdot 7440942.$

Problem 13. *Find the value of* $64\left\{1 - (1\cdot 05)^{-20}\right\}$, *having given*

$$\log 105 = 2\cdot 0211893, \quad \log 37689 = 4\cdot 5762140.$$

Solution. $\log(1\cdot 05)^{-20} = -20 \times \cdot 0211893 = -\cdot 423786 = \bar{1}\cdot 576214 = \log \cdot 37689;$

therefore $(1\cdot 05)^{-20} = \cdot 37689;$

therefore $64\left\{1 - (1\cdot 05)^{-20}\right\} = 64\{1 - \cdot 37689\}$

$$= 64 \times \cdot 62311 = 39\cdot 87904.$$

Problem 14. *Find approximately* $5^{\sqrt{5}}$, *having given*

$$\begin{array}{llll} \log 2 = \cdot 301030, & \log 1{\cdot}562944 = \cdot 193943, \\ \log 349485 = 5{\cdot}543428, & \log 3{\cdot}655 = \cdot 562887, \\ & \log 3{\cdot}656 = \cdot 563006. \end{array}$$

Solution. Denote it by u; then $\log u = \sqrt{5}\log 5 = 2\sqrt{5}\log\sqrt{5}$;
therefore $$\log(\log u) = \log 2 + \log\sqrt{5} + \log\bigl(\log\sqrt{5}\bigr).$$

Now $$\log\sqrt{5} = \frac{1}{2}\log 5 = \frac{1}{2}\log\frac{10}{2} = \frac{1}{2}(1-\log 2)$$
$$= \frac{1}{2}(1-\cdot 301030) = \frac{1}{2}(\cdot 698970) = \cdot 349485,$$

$$\log\left(\log\sqrt{5}\right) = \log \cdot 349485 = \bar{1}{\cdot}543428.$$

Therefore $$\log(\log u) = \cdot 301030 + \cdot 349485 + \bar{1}{\cdot}543428 = \cdot 193943.$$
Therefore $$\log u = 1{\cdot}562944.$$

$$\begin{array}{r} \cdot 563006 \\ \underline{\cdot 562887} \\ \cdot 000119 \end{array} \qquad \begin{array}{r} \cdot 562944 \\ \underline{\cdot 562887} \\ \cdot 000057 \end{array} \qquad \cdot 000119 : \cdot 000057 :: \cdot 001 : x;$$

this gives $x = \cdot 00048$; therefore $u = 36{\cdot}5548$.

Problem 15. *Having given*

$$\begin{array}{ll} \log 12 = 1{\cdot}0791812, & \log 1{\cdot}257915 = \cdot 0996512, \\ \log 1{\cdot}121568 = \cdot 0498256, & \end{array}$$

find the value of

$$(1{\cdot}44)^{-6} - (1{\cdot}44)^{-12}.$$

Solution. $\log 144 = \log 12^2 = 2\log 12 = 2{\cdot}1583624$;
$$\log(1{\cdot}44)^{-6} = -6\log 1{\cdot}44 = -6(\cdot 1583624) = -\cdot 9501744$$
$$= \bar{1}{\cdot}0498256 = \log \cdot 1121568;$$

therefore $$(1{\cdot}44)^{-6} = \cdot 1121568.$$

$$\log(1{\cdot}44)^{-12} = -12\log 1{\cdot}44 = -12(\cdot 1583624) = -1{\cdot}9003488$$
$$= \bar{2}{\cdot}0996512 = \log \cdot 01257915;$$

therefore $$(1{\cdot}44)^{-12} = \cdot 01257915;$$
therefore $$(1{\cdot}44)^{-6} - (1{\cdot}44)^{-12} = \cdot 1121568 - \cdot 01257915 = \cdot 09957765.$$

Problem 16. *Having given*

$$\begin{array}{ll} \log 105 = 2{\cdot}0211893, & \log 5303214 = 6{\cdot}7245391, \\ \log 3768894 = 6{\cdot}576214, & \end{array}$$

find the value of

$$\frac{1}{\cdot 05}\left\{\frac{1}{(1{\cdot}05)^{13}} - \frac{1}{(1{\cdot}05)^{20}}\right\}.$$

Solution.

$$\log\frac{1}{(1{\cdot}05)^{13}} = -13\log 1{\cdot}05 = -13({\cdot}0211893) = -{\cdot}2754609$$
$$= \bar{1}{\cdot}7245391 = \log{\cdot}5303214;$$

therefore $$\frac{1}{(1{\cdot}05)^{13}} = {\cdot}5303214;$$

$$\log\frac{1}{(1{\cdot}05)^{20}} = -20\log 1{\cdot}05 = -20({\cdot}0211893) = -{\cdot}423786$$
$$= \bar{1}{\cdot}576214 = \log{\cdot}3768894;$$

therefore $$\frac{1}{(1{\cdot}05)^{20}} = {\cdot}3768894;$$

therefore $$\frac{1}{{\cdot}05}\left\{\frac{1}{(1{\cdot}05)^{13}} - \frac{1}{(1{\cdot}05)^{20}}\right\} = 20\{{\cdot}5303214 - {\cdot}3768894\}$$
$$= 20\times{\cdot}153432 = 3{\cdot}06864.$$

Problem 17.

Given
$$\sin 47^\circ = {\cdot}7313537,$$
$$\sin 48^\circ = {\cdot}7431448,$$
$$\sin 47^\circ 1'.$$

Solution.

$$\begin{array}{r} {\cdot}7431448 \\ {\cdot}7313537 \\ \hline {\cdot}0117911 \end{array} \qquad 60' : 1' :: {\cdot}0117911 : x;$$

this gives $$x = {\cdot}0001965;$$

therefore $$\sin 47^\circ 1' = {\cdot}7313537 + {\cdot}0001965 = {\cdot}7315502.$$

Problem 18.

Given
$$\sin 7^\circ 17' = {\cdot}1267761,$$
$$\sin 7^\circ 18' = {\cdot}1270646,$$

find $$\sin 7^\circ 17' 25''.$$

Solution.

$$\begin{array}{r} {\cdot}1270646 \\ {\cdot}1267761 \\ \hline {\cdot}0002885 \end{array} \qquad 60'' : 25'' :: {\cdot}0002885 : x;$$

this gives $$x = {\cdot}0001202;$$

therefore $$\sin 7^\circ 17' 25'' = {\cdot}1267761 + {\cdot}0001202 = {\cdot}1268963.$$

Problem 19.

Given
$$L\sin 17^\circ 1' = 9{\cdot}4663483,$$
$$L\sin 17^\circ = 9{\cdot}4659353,$$

find $L\sin 17°0'12''$.

Solution.

$$\begin{array}{r} 9\cdot4663483 \\ 9\cdot4659353 \\ \hline \cdot0004130 \end{array} \qquad 60'' : 12'' :: \cdot0004130 : x;$$

this gives $x = \cdot0000826;$

therefore $L\sin 17°0'12'' = 9{\cdot}4659353 + \cdot0000826 = 9{\cdot}4660179.$

Problem 20.

Given $L\sin 26°24' = 9{\cdot}6480038,$

$L\sin 26°25' = 9{\cdot}6482582,$

find $L\sin 26°24'12''.$

Solution.

$$\begin{array}{r} 9\cdot6482582 \\ 9\cdot6480038 \\ \hline \cdot0002544 \end{array} \qquad 60'' : 12'' :: \cdot0002544 : x;$$

this gives $x = \cdot0000509;$

therefore $L\sin 26°24'12'' = 9{\cdot}6480038 + \cdot0000509 = 9{\cdot}6480547.$

Problem 21.

Given $L\cot 72°15' = 9{\cdot}5052891,$

$L\cot 72°16' = 9{\cdot}5048538,$

find $L\cot 72°15'35''.$

Solution.

$$\begin{array}{r} 9\cdot5052891 \\ 9\cdot5048538 \\ \hline \cdot0004353 \end{array} \qquad 60'' : 35'' :: \cdot0004353 : x;$$

this gives $x = \cdot0002539;$

therefore $L\cot 72°15'35'' = 9{\cdot}5052891 - \cdot0002539 = 9{\cdot}5050352.$

Problem 22. *Given $L\cot 81°46' = 9\cdot1604569$, diff. for $10'' = \cdot0001486$, find the angle whose $L\cot$ is $9\cdot1603493$.*

Solution.

$$\begin{array}{r} 9\cdot1604569 \\ 9\cdot1603493 \\ \hline \cdot0001076 \end{array} \qquad \cdot0001486 : \cdot0001076 :: 10 : x;$$

this gives $x = 7$; therefore the required angle is $81°46'7''$.

Problem 23. *Given $L\cos 20°35'20'' = 9\cdot9713351$, diff. for $10'' = \cdot0000079$, find the angle whose $L\cos$ is $9\cdot9713383$.*

Solution.

$$\begin{array}{r} 9{\cdot}9713383 \\ 9{\cdot}9713351 \\ \hline {\cdot}0000032 \end{array} \qquad {\cdot}0000079 : {\cdot}0000032 :: 10 : x;$$

this gives $x = 4$; therefore the required angle is $20°35'20'' - 4''$, that is $20°35'16''$. For as the L cosine increases the angle diminishes.

Problem 24. *Given $L\cos 34°24' = 9{\cdot}9165137$, diff. for $1' = {\cdot}0000865$, find $L\cos 34°24'26''$, and also the angle whose $L\cos$ is $9{\cdot}9165646$.*

Solution.

$$60'' : 26'' :: {\cdot}0000865 : x;$$

this gives $x = {\cdot}0000375;$

therefore $L\cos 34°24'26'' = 9{\cdot}9165137 - {\cdot}0000375 = 9{\cdot}9164762.$

Again

$$\begin{array}{r} 9{\cdot}9165646 \\ 9{\cdot}9165137 \\ \hline {\cdot}0000509 \end{array} \qquad {\cdot}0000865 : {\cdot}0000509 :: 60 : x;$$

this gives $x = 35$; therefore the required angle is $34°24' - 35''$, that is $34°23'25''$.

Problem 25.

Given $L\sin 37°19' = 9{\cdot}7826301$, diff. for $1' = {\cdot}0001657$,
$L\cos 37°19' = 9{\cdot}9005294$, diff. for $1' = {\cdot}0000963$,

find $L\sec 37°19'47''$, and $L\tan 37°19'47''$.

Solution. Since $\sec\theta \times \cos\theta = 1$, we have $\log\sec\theta + \log\cos\theta = 0$;

therefore $L\sec\theta + L\cos\theta - 20 = 0$; therefore $L\sec\theta = 20 - L\cos\theta$.

We shall first find $L\cos 37°19'47''$.

$$60'' : 47'' :: {\cdot}0000963 : x;$$

this gives $x = {\cdot}0000754;$

therefore $L\cos 37°19'47'' = 9{\cdot}9005294 - {\cdot}0000754 = 9{\cdot}9004540.$

Then $L\sec 37°19'47'' = 20 - 9{\cdot}9004540 = 10{\cdot}0995460.$

Next find $L\sin 37°19'47''$.

$$60'' : 47'' :: {\cdot}0001657 : x;$$

this gives $x = {\cdot}0001298;$

therefore $L\sin 37°19'47'' = 9{\cdot}7826301 + {\cdot}0001298 = 9{\cdot}7827599.$

Then $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$; therefore $\log\tan\theta = \log\sin\theta - \log\cos\theta$;

therefore $L\tan\theta - 10 = L\sin\theta - 10 - (L\cos\theta - 10) = L\sin\theta - L\cos\theta$;

therefore $L\tan\theta = 10 + L\sin\theta - L\cos\theta.$

Thus $L\tan 37°19'47'' = 10 + 9{\cdot}7827599 - 9{\cdot}9004540 = 9{\cdot}8823059.$

Problem 26.

Given $L\sin 32°18' = 9{\cdot}7278277$, diff. for $1' = {\cdot}0001998$,
$L\cos 32°18' = 9{\cdot}9269913$, diff. for $1' = {\cdot}0000799$,

find L sine, L cosine, and L tangent of $32°18'24''{\cdot}6$.

Solution.

$$60'' : 24''.6 :: {\cdot}0001998 : x;$$

this gives $x = {\cdot}0000819;$

therefore $L \sin 32°18'24''{\cdot}6 = 9{\cdot}7278277 + {\cdot}0000819 = 9{\cdot}7279096.$

$$60'' : 24''{\cdot}6 :: {\cdot}0000799 : x;$$

this gives $x = {\cdot}0000328;$

therefore $L \cos 32°18'24''{\cdot}6 = 9{\cdot}9269913 - {\cdot}0000328 = 9{\cdot}9269585.$

And $L \tan 32°18'24''{\cdot}6 = 10 + L \sin 32°18'24''{\cdot}6 - L \cos 32°18'24''{\cdot}6$

$= 9{\cdot}8009511.$

CHAPTER XII
Theory of Proportional Parts

Problem 1. From one of the angles of a rectangle a perpendicular is drawn to its diagonal, and from the point of their intersection lines are drawn perpendicular to the sides which contain the opposite angle; show that if p and p' be the lengths of the perpendiculars last drawn, and c the diagonal of the rectangle,

$$p^{\frac{2}{3}} + p'^{\frac{2}{3}} = c^{\frac{2}{3}}.$$

Solution. Let $ABCD$ denote the rectangle. From A draw AP perpendicular to the diagonal BD; and from P draw PM perpendicular to BC, and PN perpendicular to CD.

Let the angle DBA be denoted by α; then

$$AB = c\cos\alpha,\ BP = AB\cos\alpha = c\cos^2\alpha,$$

$$PM = BP\cos BPM = BP\cos\alpha = c\cos^3\alpha.$$

Thus denoting PM by p we have $p = c\cos^3\alpha$.

Similarly

$$AD = c\sin\alpha,\ PD = AD\sin PAD = AD\sin\alpha = c\sin^2\alpha,$$

$$PN = PD\sin PDN = PD\sin\alpha = c\sin^3\alpha.$$

Thus $q = c\sin^3\alpha$.

$$\therefore p^{\frac{2}{3}} + q^{\frac{2}{3}} = \left(c\cos^3\alpha\right)^{\frac{2}{3}} + \left(c\sin^3\alpha\right)^{\frac{2}{3}} = c^{\frac{2}{3}}\left(\cos^2\alpha + \sin^2\alpha\right) = c^{\frac{2}{3}}.$$

Problem 2. If two circles whose radii are a and b touch each other externally, and if θ be the angle contained by the two common tangents to these circles, show that

$$\sin\theta = \frac{4(a-b)\sqrt{ab}}{(a+b)^2}.$$

Solution. Let a denote the radius of the larger circle, and b the radius of the smaller circle. Let x denote the distance of the point of intersection of the two common tangents from the centre of the larger circle; therefore $x-a-b$ denotes the distance of this point from the centre of the smaller circle.

$$\text{Then}\ \sin\frac{\theta}{2} = \frac{a}{x},\ \text{and also}\ \sin\frac{\theta}{2} = \frac{b}{x-a-b};$$

$$\text{therefore}\ x = \frac{a}{\sin\frac{\theta}{2}},\ \text{and}\ x-a-b = \frac{b}{\sin\frac{\theta}{2}};$$

Therefore, by subtraction, $a+b = \dfrac{a-b}{\sin\frac{\theta}{2}}$;

$$\therefore \sin\frac{\theta}{2} = \frac{a-b}{a+b};\ \therefore \cos\frac{\theta}{2} = \frac{2\sqrt{ab}}{a+b};$$

$$\therefore \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{4(a-b)\sqrt{ab}}{(a+b)^2}.$$

Problem 3. Given $\sec\alpha\sec\theta+\tan\alpha\tan\theta=\sec\beta$, find $\tan\theta$.

Solution.

$$\sec\alpha\sec\theta+\tan\alpha\tan\theta=\sec\beta$$
$$\therefore\sec\alpha\sec\theta=\sec\beta-\tan\alpha\tan\theta;$$
$$\therefore\sec^2\alpha\sec^2\theta=(\sec\beta-\tan\alpha\tan\theta)^2;$$
$$\therefore\sec^2\alpha\left(1+\tan^2\theta\right)=\sec^2\beta-2\sec\beta\tan\alpha\tan\theta+\tan^2\alpha\tan^2\theta;$$
$$\therefore\left(\sec^2\alpha-\tan^2\alpha\right)\tan^2\theta+2\sec\beta\tan\alpha\tan\theta=\sec^2\beta-\sec^2\alpha;$$
$$\therefore\tan^2\theta+2\sec\beta\tan\alpha\tan\theta=\sec^2\beta-\sec^2\alpha;$$
$$\begin{aligned}\therefore(\tan\theta+\tan\alpha\sec\beta)^2&=\sec^2\beta-\sec^2\alpha+\tan^2\alpha\sec^2\beta\\&=\sec^2\beta\sec^2\alpha-\sec^2\alpha=\tan^2\beta\sec^2\alpha;\end{aligned}$$
$$\therefore\tan\theta+\tan\alpha\sec\beta=\pm\tan\beta\sec\alpha;$$
$$\begin{aligned}\therefore\tan\theta&=-\tan\alpha\sec\beta\pm\tan\beta\sec\alpha\\&=-\frac{\sin\alpha}{\cos\alpha\cos\beta}\pm\frac{\sin\beta}{\cos\beta\cos\alpha}=\frac{-\sin\alpha\pm\sin\beta}{\cos\alpha\cos\beta}.\end{aligned}$$

Problem 4. Find the limit when $\theta=0$ of $\dfrac{\sin\frac{\theta}{2}\cos 2\theta}{\operatorname{vers}\theta\cot\theta}$, and of $\dfrac{\tan^2\theta}{\sec 2\theta-1}$. Kindly note that $\operatorname{vers}\theta=1-\cos\theta$.

Solution.

$$\begin{aligned}\frac{\sin\frac{\theta}{2}\cos 2\theta}{\operatorname{vers}\theta\cot\theta}&=\frac{\sin\frac{\theta}{2}\cos 2\theta\sin\theta}{\operatorname{vers}\theta\cos\theta}=\frac{2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\cos 2\theta}{(1-\cos\theta)\cos\theta}\\&=\frac{2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\cos 2\theta}{2\sin^2\frac{\theta}{2}\cos\theta}=\frac{\cos\frac{\theta}{2}\cos 2\theta}{\cos\theta}\end{aligned}$$

and the value of this is unity when $\theta=0$.

$$\begin{aligned}\frac{\tan^2\theta}{\sec 2\theta-1}&=\frac{\sin^2\theta}{\cos^2\theta(\sec 2\theta-1)}=\frac{\sin^2\theta\cos 2\theta}{\cos^2\theta(1-\cos 2\theta)}\\&=\frac{\sin^2\theta\cos 2\theta}{2\cos^2\theta\sin^2\theta}=\frac{\cos 2\theta}{2\cos^2\theta};\end{aligned}$$

and the value of this is $\frac{1}{2}$ when $\theta=0$.

Problem 5. Show that $\cot\frac{\theta}{2}$ is greater than $1+\cot\theta$ for all values of θ between 0 and π.

Solution.

$$\cot\frac{\theta}{2}-(1+\cot\theta)=\cot\frac{\theta}{2}-\cot\theta-1=\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}-\frac{\cos\theta}{\sin\theta}-1$$

$$= \frac{\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}\sin\theta} - 1 = \frac{\sin\left(\theta - \frac{\theta}{2}\right)}{\sin\frac{\theta}{2}\sin\theta} - 1$$

$$= \frac{1}{\sin\theta} - 1.$$

Now this is always positive as θ changes from 0 to π, except when $\theta = \frac{\pi}{2}$, and when it is zero.

Problem 6. If $\tan\frac{\theta}{2} = \frac{\tan\theta + c - 1}{\tan\theta + c + 1}$, find $\tan\frac{\theta}{2}$.

Solution.

$$\tan\frac{\theta}{2} = \frac{\tan\theta + c - 1}{\tan\theta + c + 1}$$

$$\therefore \tan\frac{\theta}{2}(\tan\theta + c + 1) = \tan\theta + c - 1$$

$$\therefore \tan\frac{\theta}{2}\left(\frac{2\tan\frac{\theta}{2}}{1 - \tan^2\frac{\theta}{2}} + c + 1\right) = \frac{2\tan\frac{\theta}{2}}{1 - \tan^2\frac{\theta}{2}} + c - 1$$

$$\therefore 2\tan^2\frac{\theta}{2} + (c+1)\left(1 - \tan^2\frac{\theta}{2}\right)\tan\frac{\theta}{2} = 2\tan\frac{\theta}{2} + (c-1)\left(1 - \tan^2\frac{\theta}{2}\right)$$

$$\therefore (c+1)\tan^3\frac{\theta}{2} - (1+c)\tan^2\frac{\theta}{2} + (1-c)\tan\frac{\theta}{2} + (c-1) = 0$$

$$\therefore (c+1)\tan^2\frac{\theta}{2}\left(\tan\frac{\theta}{2} - 1\right) = (c-1)\left(\tan\frac{\theta}{2} - 1\right)$$

$$\therefore \tan\frac{\theta}{2} - 1 = 0, \text{ or } (c+1)\tan^2\frac{\theta}{2} = c - 1$$

$$\therefore \tan\frac{\theta}{2} = 1, \text{ or } \pm\sqrt{\frac{c-1}{c+1}}.$$

Problem 7. Find the condition necessary that the same value of θ may satisfy both the equations

$$a\sec^2\theta - b\sec\theta = 2a,$$

$$b\cos^2\theta - a\sec\theta = 2b.$$

Solution.

$$a\sec^2\theta - b\sec\theta = 2a$$

$$\therefore a - b\cos^3\theta = 2a\cos^2\theta$$

$$\therefore b\cos^3\theta = a - 2a\cos^2\theta. \tag{3}$$

Again

$$b\cos^2\theta - a\sec\theta = 2b$$

$$\therefore b\cos^3\theta = 2b\cos\theta + a. \tag{4}$$

From (3) and (4):

$$a - 2a\cos^2\theta = 2b\cos\theta + a$$

$$\therefore -a\cos\theta = b; \ \therefore \cos\theta = -\frac{b}{a}.$$

Substitute this value of $\cos\theta$ in either of the gien equations, for instance the first; thus

$$\frac{a^3}{b^2} + \frac{b^2}{a} = 2a$$
$$\therefore a^4 + b^4 - 2a^2b^2 = 0$$
$$\therefore a^2 = b^2.$$

Problem 8. Eliminate α and β from the equations

$$a = \sin\alpha\cos\beta\sin\theta + \cos\alpha\cos\theta,$$
$$b = \sin\alpha\cos\beta\cos\theta - \cos\alpha\sin\theta,$$
$$c = \sin\alpha\sin\beta\sin\theta.$$

Solution.

$$a^2 + b^2 = (\sin\alpha\cos\beta\sin\theta + \cos\alpha\cos\theta)^2 + (\sin\alpha\cos\beta\cos\theta - \cos\alpha\sin\theta)^2$$
$$= \sin^2\alpha\cos^2\beta + \cos^2\alpha$$

and

$$\frac{c^2}{\sin^2\theta} = \sin^2\alpha\sin^2\beta.$$

$$\therefore a^2 + b^2 + \frac{c^2}{\sin^2\theta} = \sin^2\alpha\cos^2\beta + \sin^2\alpha\sin^2\beta + \cos^2\alpha$$
$$= \sin^2\alpha + \cos^2\alpha = 1.$$

Problem 9. Eliminate α and β from the equations

$$b + c\cos\alpha = u\cos(\alpha - \theta),$$
$$b + c\cos\beta = u\cos(\beta - \theta),$$
$$\alpha - \beta = 2\delta.$$

and show that

$$u^2 - 2uc\cos\theta + c^2 = b^2\sec^2\delta.$$

Solution. Adding the first two equations

$$2b + c(\cos\alpha + \cos\beta) = u\cos(\alpha - \theta) + u\cos(\beta - \theta)$$
$$\therefore b + c\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = u\cos\left(\frac{\alpha+\beta}{2} - \theta\right)\cos\frac{\alpha-\beta}{2}$$
$$b\sec\frac{\alpha-\beta}{2} = u\cos\left(\frac{\alpha+\beta}{2} - \theta\right) - c\cos\frac{\alpha+\beta}{2} \qquad (5)$$

Again from the first two equations, by subtraction,

$$c(\cos\alpha - \cos\beta) = u\cos(\alpha - \theta) - u\cos(\beta - \theta)$$
$$\therefore c\sin\frac{\beta-\alpha}{2}\sin\frac{\alpha+\beta}{2} = u\sin\frac{\beta-\alpha}{2}\sin\left(\frac{\alpha+\beta}{2} - \theta\right)$$
$$\therefore 0 = u\sin\left(\frac{\alpha+\beta}{2} - \theta\right) - c\sin\frac{\alpha+\beta}{2} \qquad (6)$$

Square and add (5) and (6); thus

$b^2\sec^2\delta = u^2 + c^2$

$$- 2uc\left\{\cos\left(\frac{\alpha+\beta}{2}-\theta\right)\cos\frac{\alpha+\beta}{2}+\sin\left(\frac{\alpha+\beta}{2}-\theta\right)\sin\frac{\alpha+\beta}{2}\right\}$$
$$= u^2 + c^2 - 2uc\cos\theta.$$

Problem 10. Eliminate x from the equations

$$\frac{a\tan^2\theta - x}{\tan 2\alpha\tan 2\alpha'} = \frac{2a\tan\theta}{\tan 2\alpha + \tan 2\alpha'} = a - x;$$

and show that $\theta = \alpha + \alpha'$, or $\frac{\pi}{2} + \alpha + \alpha'$.

Solution.

$$a\tan^2\theta - x = \frac{2a\tan\theta\tan 2\alpha\tan 2\alpha'}{\tan 2\alpha + \tan 2\alpha'}$$
$$a - x = \frac{2a\tan\theta}{\tan 2\alpha + \tan 2\alpha'}$$

therefore, by subtraction,

$$a\left(1 - \tan^2\theta\right) = \frac{2a\tan\theta(1 - \tan 2\alpha\tan 2\alpha')}{\tan 2\alpha + \tan 2\alpha'}$$
$$\therefore \frac{\tan 2\alpha + \tan 2\alpha'}{1 - \tan 2\alpha\tan 2\alpha'} = \frac{2\tan\theta}{1 - \tan^2\theta}$$
$$\therefore \tan(2\alpha + 2\alpha') = \tan 2\theta.$$

Problem 11. Eliminate θ and ϕ from the equations

$$\sin\theta + \sin\phi = a,$$
$$\cos\theta + \cos\phi = b,$$
$$\cos(\theta - \phi) = c.$$

Solution. Square and add the first two equations

$$2 + 2(\cos\theta\cos\phi + \sin\theta\sin\phi) = a^2 + b^2$$
$$\therefore 2 + 2\cos(\theta - \phi) = a^2 + b^2$$
$$\therefore 2 + 2c = a^2 + b^2.$$

Problem 12. Eliminate θ and ϕ from the equations

$$x\cos\theta + y\sin\theta = a,$$
$$x\cos(\theta + 2\phi) - y\sin(\theta + 2\phi) = a,$$
$$b\sin(\theta + \phi) = a\sin\phi.$$

Subtracting the second equation from the first one,

$$x\left\{\cos\theta - \cos(\theta + 2\phi)\right\} + y\left\{\sin\theta + \sin(\theta + 2\phi)\right\} = 0$$
$$\therefore x\sin(\theta + \phi)\sin\phi + y\sin(\theta + \phi)\cos\phi = 0$$
$$\therefore x\sin\phi + y\cos\phi = 0 \qquad (7)$$

Again, by addition,

$$x\left\{\cos\theta + \cos(\theta + 2\phi)\right\} + y\left\{\sin\theta - \sin(\theta + 2\phi)\right\} = 2a$$
$$\therefore x\cos(\theta + \phi)\cos\phi - y\cos(\theta + \phi)\sin\phi = 0$$
$$\therefore x\cos\phi - y\sin\phi = \frac{a}{\cos(\theta + \phi)} \qquad (8)$$

Square and add (7) and (8),

$$x^2+y^2=\frac{a^2}{\cos^(\theta+\phi)}=\frac{a^2}{1-\sin^2(\theta+\phi)}=\frac{a^2}{1-\dfrac{a^2}{b^2}\sin^2\phi}$$

$$\therefore (x^2+y^2)\left(1-\frac{a^2}{b^2}\sin^2\phi\right)=a^2. \tag{9}$$

But from (7)

$$x^2\sin^2\phi=y^2\cos^2\phi=y^2\left(1-\sin^2\phi\right)$$

$$\therefore \sin^2\phi=\frac{y^2}{x^2+y^2}.$$

Substituting this in (9),

$$(x^2+y^2)\left(1-\frac{a^2y^2}{b^2(x^2+y^2)}\right)=a^2$$

$$\therefore x^2+y^2=a^2+\frac{a^2y^2}{b^2}.$$

Problem 13. Eliminate x and y from the equations

$$\tan x+\tan y=a,$$
$$\cot x+\cot y=b,$$
$$x+y=c.$$

Solution.

$$\tan c=\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}.$$

Now

$$\tan x+\tan y=a,\ \text{and}\ \cot x+\cot y=b$$

$$\therefore \frac{1}{\tan x}+\frac{1}{\tan y}=b$$

$$\therefore \tan x+\tan y=b\tan x\tan y$$

$$\therefore a=b\tan x\tan y;\ \therefore \tan x\tan y=\frac{a}{b}$$

$$\therefore \tan c=\frac{a}{1-\dfrac{a}{b}}=\frac{ab}{b-a}$$

$$\therefore \cot c=\frac{b-a}{ab}=\frac{1}{a}-\frac{1}{b}.$$

Problem 14. Eliminate θ from the equations

$$\frac{x}{a}=\frac{\sec^2\theta-\cos^2\theta}{\sec^2\theta+\cos^2\theta},$$

$$\frac{2b}{y}=\sec^2\theta+\cos^2\theta.$$

Solution.

$$\frac{x}{a}=\frac{\sec^2\theta-\cos^2\theta}{\sec^2\theta+\cos^2\theta}=\frac{1-\cos^4\theta}{1+\cos^4\theta},$$

$$\frac{2b}{y} = \sec^2\theta + \cos^2\theta = \frac{1+\cos^4\theta}{\cos^2\theta}$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{1-\cos^4\theta}{1+\cos^4\theta}\right)^2 + \frac{4\cos^4\theta}{(1+\cos^4\theta)^2} = \left(\frac{1+\cos^4\theta}{1+\cos^4\theta}\right)^2 = 1.$$

Problem 15. Eliminate θ from the equations

$$(a+b)\tan(\theta-\phi) = (a-b)\tan(\theta+\phi),$$
$$a\cos 2\phi + b\cos 2\theta = c.$$

Solution.

$$(a+b)\tan(\theta-\phi) = (a-b)\tan(\theta+\phi),$$
$$\therefore (a+b)\sin(\theta-\phi)\cos(\theta+\phi) = (a-b)\sin(\theta+\phi)\cos(\theta-\phi)$$
$$\therefore b\left\{\sin(\theta+\phi)\cos(\theta-\phi) + \sin(\theta-\phi)\cos(\theta+\phi)\right\}$$
$$= a\left\{\sin(\theta+\phi)\cos(\theta-\phi) - \sin(\theta-\phi)\cos(\theta+\phi)\right\}$$
$$\therefore b\sin 2\theta = a\sin 2\phi, \text{ and}$$
$$b\cos 2\theta = c - a\cos 2\phi.$$

Square and add; thus $b^2 = c^2 + a^2 - 2ac\cos 2\phi$.

Problem 16. Given

$$\frac{x^2}{a^2}\cos\theta = \frac{y^2}{a^2}\cos\theta + \frac{z^2}{b^2}\cos\theta',$$
$$\text{and } \frac{x}{\sin(\theta+\theta')} = \frac{y}{\sin(\theta-\theta')} = \frac{z}{\sin 2\theta}.$$

Show that

$$\frac{\sin\theta}{\sin\theta'} = \frac{b^2}{a^2}.$$

Solution.

$$x = \frac{z\sin(\theta+\theta')}{\sin 2\theta},\ y = \frac{z\sin(\theta-\theta')}{\sin 2\theta}.$$

Square and substitute in the first given equation; thus

$$\frac{z^2\sin^2(\theta+\theta')}{a^2\sin^2 2\theta}\cos\theta = \frac{z^2\sin^2(\theta-\theta')}{a^2\sin^2 2\theta}\cos\theta + \frac{z^2}{b^2}\cos\theta'$$
$$\therefore \frac{\sin^2(\theta+\theta')\cos\theta - \sin^2(\theta-\theta')\cos\theta}{a^2\sin^2 2\theta} = \frac{\cos\theta'}{b^2}$$
$$\therefore \frac{(\sin\theta\cos\theta' + \cos\theta\sin\theta')^2 - (\sin\theta\cos\theta' - \cos\theta\sin\theta')^2}{4a^2\sin^2\theta\cos^2\theta}\cos\theta = \frac{\cos\theta'}{b^2}$$
$$\therefore \frac{4\sin\theta\cos^2\theta\sin\theta'\cos\theta'}{4a^2\sin^2\theta\cos^2\theta} = \frac{\cos\theta'}{b^2}$$
$$\therefore \frac{\sin\theta'}{\sin\theta} = \frac{a^2}{b^2}.$$

Problem 17. Eliminate ϕ from the equations

$$y\cos\phi - x\sin\phi = a\cos 2\phi,$$
$$y\sin\phi + x\cos\phi = 2a\sin 2\phi.$$

And show that

$$(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=2a^{\frac{2}{3}}.$$

Solution. Given equations are

$$y\cos\phi - x\sin\phi = a\cos 2\phi, \qquad (10)$$

$$y\sin\phi + x\cos\phi = 2a\sin 2\phi. \qquad (11)$$

Multiply (10) by $\cos\phi$, and (11) by $\sin\phi$, and add; thus

$$\begin{aligned} y &= a\cos 2\phi\cos\phi + 2a\sin 2\phi\sin\phi \\ &= a\cos\phi\left(\cos 2\phi + 4\sin^2\phi\right) = a\cos\phi\left(\cos^2\phi + 3\sin^2\phi\right). \end{aligned}$$

Again multiply (11) by $\cos\phi$, and (10) by $\sin\phi$, and subtract : thus

$$\begin{aligned} x &= 2a\sin 2\phi\cos\phi - a\cos 2\phi\sin\phi \\ &= a\sin\phi\left(4\cos^2\phi - \cos 2\phi\right) = a\sin\phi\left(3\cos^2\phi + \sin^2\phi\right). \end{aligned}$$

$$\begin{aligned} \therefore x+y &= a\left(\sin^3\phi + \cos^3\phi + 3\sin^2\phi\cos\phi + 3\cos^2\phi\sin\phi\right) \\ &= a(\sin\phi + \cos\phi)^3; \\ \therefore x-y &= a\left(\sin^3\phi - 3\sin^2\phi\cos\phi + 3\cos^2\phi\sin\phi - \cos^3\phi\right) \\ &= a(\sin\phi - \cos\phi)^3; \end{aligned}$$

$$\therefore (x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}} = a^{\frac{2}{3}}\left\{(\sin\phi+\cos\phi)^2 + (\sin\phi-\cos\phi)^2\right\} = 2a^{\frac{2}{3}}.$$

Problem 18. Eliminate θ and ϕ from the equations

$$\cos\theta = \frac{\sin\beta}{\sin\alpha},$$

$$\cos\phi = \frac{\sin\gamma}{\sin\alpha},$$

$$\cos(\theta-\phi) = \sin\beta\sin\gamma.$$

And show that

$$\tan^2\alpha = \tan^2\beta + \tan^2\gamma.$$

Solution.

$$\cos\theta\cos\phi + \sin\theta\sin\phi = \sin\beta\sin\gamma$$

$$\therefore \sin^2\theta\sin^2\phi = (\sin\beta\sin\gamma - \cos\theta\cos\phi)^2$$

$$\therefore \left(1-\cos^2\theta\right)\left(1-\cos^2\phi\right) = (\sin\beta\sin\gamma - \cos\theta\cos\phi)^2$$

$$\therefore \left(1-\frac{\sin^2\beta}{\sin^2\alpha}\right)\left(1-\frac{\sin^2\gamma}{\sin^2\alpha}\right) = \left(\sin\beta\sin\gamma - \frac{\sin\beta\sin\gamma}{\sin^2\alpha}\right)^2$$

$$\therefore \left(\sin^2\alpha - \sin^2\beta\right)\left(\sin^2\alpha - \sin^2\gamma\right) = \sin^2\beta\sin^2\gamma\left(\sin^2\alpha - 1\right)^2$$

$$\therefore \sin^4\alpha - \sin^2\alpha\left(\sin^2\beta + \sin^2\gamma\right) = \sin^2\beta\sin^2\gamma\left(\sin^4\alpha - 2\sin^2\alpha\right)$$

$$\therefore \sin^2\alpha - \sin^2\beta - \sin^2\gamma = \sin^2\beta\sin^2\gamma\left(\sin^2\alpha - 2\right)$$

$$\therefore \sin^2\alpha\left(1 - \sin^2\beta\sin^2\gamma\right) = \sin^2\beta\cos^2\gamma + \cos^2\beta\sin^2\gamma$$

$$\therefore \sin^2\alpha = \frac{\sin^2\beta\cos^2\gamma + \cos^2\beta\sin^2\gamma}{1-\sin^2\beta\sin^2\gamma}$$

$$\therefore \cos^2\alpha = \frac{1-\sin^2\beta\sin^2\gamma - \sin^2\beta\cos^2\gamma - \cos^2\beta\sin^2\gamma}{1-\sin^2\beta\sin^2\gamma}$$

$$= \frac{\left(\sin^2\beta+\cos^2\beta\right)\left(\sin^2\gamma+\cos^2\gamma\right) - \sin^2\beta\sin^2\gamma - \sin^2\beta\cos^2\gamma - \cos^2\beta\sin^2\gamma}{1-\sin^2\beta\sin^2\gamma}$$

$$= \frac{\cos^2\beta\cos^2\gamma}{1-\sin^2\beta\sin^2\gamma}.$$

$$\therefore \tan^2\alpha = \frac{\sin^2\beta\cos^2\gamma+\cos^2\beta\sin^2\gamma}{\cos^2\beta\cos^2\gamma}$$

$$= \frac{\sin^2\beta}{\cos^2\beta}+\frac{\sin^2\gamma}{\cos^2\gamma} = \tan^2\beta+\tan^2\gamma.$$

Problem 19. Eliminate θ from the equations

$$m = \operatorname{cosec}\theta - \sin\theta,$$
$$n = \sec\theta - \cos\theta.$$

Solution.

$$m = \operatorname{cosec}\theta - \sin\theta = \frac{1}{\sin\theta} - \sin\theta = \frac{1-\sin^2\theta}{\sin\theta} = \frac{\cos^2\theta}{\sin\theta}$$

$$n = \sec\theta - \cos\theta = \frac{1}{\cos\theta} - \cos\theta = \frac{1-\cos^2\theta}{\cos\theta} = \frac{\sin^2\theta}{\cos\theta}$$

$$\therefore mn = \frac{\cos^2\theta\sin^2\theta}{\sin\theta\cos\theta} = \cos\theta\sin\theta$$

$$\therefore \sin\theta = \frac{mn}{\cos\theta}, \text{ and } \cos\theta = \frac{mn}{\cos\theta}$$

$$\therefore m = \frac{\cos^3\theta}{mn}, \text{ and } n = \frac{\sin^3\theta}{mn}$$

$$\therefore \cos\theta = \left(m^2n\right)^{\frac{1}{3}}, \text{ and } \sin\theta = \left(mn^2\right)^{\frac{1}{3}}$$

$$\therefore \cos^2\theta+\sin^2\theta = \left(m^2n\right)^{\frac{2}{3}} + \left(mn^2\right)^{\frac{2}{3}}$$

$$\therefore 1 = (mn)^{\frac{2}{3}}\left\{m^{\frac{2}{3}}+n^{\frac{2}{3}}\right\}.$$

Problem 20. Eliminate θ from the equations

$$x\sin\theta - y\cos\theta = \sqrt{x^2+y^2},$$
$$\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2} = \frac{1}{x^2+y^2}.$$

Solution.

$$(x\sin\theta - y\cos\theta)^2 = x^2+y^2$$

$$\therefore x^2+y^2-(x\sin\theta-y\cos\theta)^2 = 0$$

$$\therefore x^2\cos^2\theta + 2xy\sin\theta\cos\theta + y^2\sin^2\theta = 0$$

$$\therefore (x\cos\theta+y\sin\theta)^2 = 0$$

$$\therefore x\cos\theta + y\sin\theta = 0$$

$$\therefore \tan\theta = -\frac{x}{y}.$$

Hence we obtain $\cos^2\theta = \dfrac{y^2}{x^2+y^2}$ and $\sin^2\theta = \dfrac{x^2}{x^2+y^2}$.
Substitute in the second given equation : thus

$$\frac{1}{x^2+y^2}\left(\frac{y^2}{a^2}+\frac{x^2}{b^2}\right) = \frac{1}{x^2+y^2};$$

$$\therefore \frac{y^2}{a^2} + \frac{x^2}{b^2} = 1.$$

Problem 21. Eliminate θ and θ' from the equations

$$a \sin^2 \theta + a' \cos^2 \theta = b,$$
$$a' \sin^2 \theta' + a \cos^2 \theta' = b',$$
$$a \tan \theta = a' \tan \theta'.$$

And show that

$$\frac{1}{b} + \frac{1}{b'} = \frac{1}{a} + \frac{1}{a'}.$$

Solution.

$$a \sin^2 \theta + a' \cos^2 \theta = b$$
$$\therefore a \sin^2 \theta + a' \left(1 - \sin^2 \theta\right) = b$$
$$\therefore \sin^2 \theta = \frac{b - a'}{a - a'}$$
$$\therefore \cos^2 \theta = \frac{a - b}{a - a'}$$
$$\therefore \tan^2 \theta = \frac{b - a'}{a - b}.$$

Similarly we find $\tan^2 \theta' = \dfrac{b' - a}{a' - b'}$.

$$\because a^2 \tan^2 \theta = a'^2 \tan^2 \theta'$$
$$\therefore a^2 \frac{b - a'}{a - b} = a'^2 \frac{b' - a}{a' - b'}$$
$$\therefore a^2 (b - a')(b' - a') = a'^2 (b' - a)(b - a)$$
$$\therefore a^2 \left\{bb' - a'(b + b')\right\} = a'^2 \left\{bb' - a(b + b')\right\}$$
$$\therefore bb' \left(a^2 - a'^2\right) = aa'(a - a')(b + b')$$
$$\therefore bb'(a + a') = aa'(b + b').$$

Divide by $aa'bb'$; thus $\dfrac{1}{a'} + \dfrac{1}{a} = \dfrac{1}{b'} + \dfrac{1}{b}$.

Problem 22. Given

$$x^2 + y^2 = a^2 + b^2,$$
$$xy = ab \sin \alpha,$$
$$\frac{\cos^2 \theta}{x^2} + \frac{\sin^2 \theta}{y^2} = \frac{1}{a^2},$$

show that

$$\pm \cot 2\theta = \cot 2\alpha + \frac{a^2}{b^2} \operatorname{cosec} 2\alpha.$$

Solution.

$$y^2 \cos^2 \theta + x^2 \sin^2 \theta = \frac{x^2 y^2}{a^2} = \frac{a^2 b^2 \sin^2 \alpha}{a^2} = b^2 \sin^2 \alpha$$
$$\therefore \frac{y^2}{2}(1 + \cos 2\theta) + \frac{x^2}{2}(1 - \cos 2\theta) = b^2 \sin^2 \alpha$$
$$\therefore x^2 + y^2 + \left(y^2 - x^2\right) \cos 2\theta = 2b^2 \sin^2 \alpha$$

$$\therefore a^2 + b^2 + \left\{(y^2 + x^2)^2 - 4y^2x^2\right\}^{\frac{1}{2}} \cos 2\theta = 2b^2 \sin^2 \alpha$$

$$\therefore a^2 + b^2 + \left\{(a^2 + b^2)^2 - 4a^2b^2 \sin^2 \alpha\right\}^{\frac{1}{2}} \cos 2\theta = 2b^2 \sin^2 \alpha$$

$$\therefore \cos 2\theta = \frac{2b^2 \sin^2 \alpha - b^2 - a^2}{\left\{(a^2 + b^2)^2 - 4a^2b^2 \sin^2 \alpha\right\}^{\frac{1}{2}}}$$

$$\therefore \sin^2 2\theta = \frac{-4a^2b^2 \sin^2 \alpha + 4b^2 \sin^2 \alpha (b^2 + a^2) - 4b^4 \sin^4 \alpha}{(a^2 + b^2)^2 - 4a^2b^2 \sin^2 \alpha}$$

$$= \frac{4b^4 \sin^2 \alpha (1 - \sin^2 \alpha)}{(a^2 + b^2)^2 - 4a^2b^2 \sin^2 \alpha}$$

$$\therefore \pm \sin 2\theta = \frac{2b^2 \sin \alpha \cos \alpha}{\left\{(a^2 + b^2)^2 - 4a^2b^2 \sin^2 \alpha\right\}^{\frac{1}{2}}}$$

Hence by division

$$\pm \cot 2\theta = \frac{2b^2 \sin^2 \alpha - b^2 - a^2}{2b^2 \sin \alpha \cos \alpha} = -\frac{a^2 + b^2 \cos 2\alpha}{b^2 \sin 2\alpha}$$

$$= -\cot 2\alpha - \frac{a^2}{b^2} \operatorname{cosec} 2\alpha,$$

which we may also express thus

$$\pm \cot 2\theta = \cot 2\alpha + \frac{a^2}{b^2} \operatorname{cosec} 2\alpha.$$

Problem 23. If $\dfrac{\cos x}{a_1} = \dfrac{\cos 2x}{a_2} = \dfrac{\cos 3x}{a_3}$, show that

$$\sin^2 \frac{x}{2} = \frac{2a_2 - a_1 - a_3}{4a_2}.$$

Solution. Let $\dfrac{\cos x}{a_1}$, $\dfrac{\cos 2x}{a_2}$, and $\dfrac{\cos 3x}{a_3}$ each be equal to $\dfrac{1}{k}$; then

$$a_1 = k \cos x, \; a_2 = k \cos 2x, \text{ and } a_3 = k \cos 3x.$$

$$\therefore \frac{2a_2 - a_1 - a_3}{4a_2} = \frac{2\cos 2x - \cos x - \cos 3x}{4\cos 2x}$$

$$= \frac{2\cos 2x - 2\cos 2x \cos x}{4\cos 2x} = \frac{1 - \cos x}{2} = \sin^2 \frac{x}{2}.$$

Problem 24. If $\dfrac{\sin x}{a_1} = \dfrac{\sin 3x}{a_3} = \dfrac{\sin 5x}{a_5}$, show that

$$\frac{a_1 - 2a_3 + a_5}{a_3} = \frac{a_3 - 3a_1}{a_1}.$$

Solution. Let $\dfrac{\sin x}{a_1}$, $\dfrac{\sin 3x}{a_3}$, and $\dfrac{\sin 5x}{a_5}$ each be equal to $\dfrac{1}{k}$; then

$$a_1 = k \sin x, \; a_3 = k \sin 3x, \text{ and } a_5 = k \sin 5x.$$

$$\therefore \frac{a_1 - 2a_3 + a_5}{a_3} = \frac{\sin x - 2\sin 3x + \sin 5x}{\sin 3x} = \frac{2\sin 3x \cos 2x - 2\sin 3x}{\sin 3x}$$

$$= 2(\cos 2x - 1) = -4\sin^2 x.$$

$$\text{and } \frac{a_3 - 3a_1}{a_1} = \frac{\sin 3x - 3\sin x}{\sin x} = \frac{3\sin x - 4\sin^3 x - 3\sin x}{\sin x} = -4\sin^2 x.$$

$$\therefore \frac{a_1 - 2a_3 + a_5}{a_3} = \frac{a_3 - 3a_1}{a_1}.$$

Problem 25. Given

$$\frac{\cos x}{a_1} = \frac{\cos(x+\theta)}{a_2} = \frac{\cos(x+2\theta)}{a_3} = \frac{\cos(x+3\theta)}{a_4},$$

show that

$$\frac{a_1 + a_3}{a_2} = \frac{a_2 + a_4}{a_3}.$$

Solution. Let $\dfrac{1}{k}$ denote the value of the fractions which are given equal; thus

$$a_1 = k\cos x,\ a_2 = k\cos(x+\theta),\ a_3 = k\cos(x+2\theta),\ a_4 = k\cos(x+3\theta).$$

$$\therefore \frac{a_1 + a_3}{a_2} = \frac{\cos x + \cos(x+2\theta)}{\cos(x+\theta)} = \frac{2\cos(x+\theta)\cos\theta}{\cos(x+\theta)} = 2\cos\theta,\ \text{and}$$

$$\frac{a_2 + a_4}{a_3} = \frac{\cos(x+\theta) + \cos(x+3\theta)}{\cos(x+2\theta)} = \frac{2\cos(x+2\theta)\cos\theta}{\cos(x+2\theta)} = 2\cos\theta;$$

thus the required result is established.

Problem 26. If $\sin^2\phi = \dfrac{\cos 2\alpha \cos 2\alpha'}{\cos^2(\alpha+\alpha')}$, then $\tan^2\dfrac{\phi}{2} = \dfrac{\tan\left(\dfrac{\pi}{4} \pm \alpha\right)}{\tan\left(\dfrac{\pi}{4} \pm \alpha'\right)}$.

Solution.

$$\sin^2\phi = \frac{\cos 2\alpha \cos 2\alpha'}{\cos^2(\alpha+\alpha')}$$

$$\therefore \cos^2\phi = \frac{\cos^2(\alpha+\alpha') - \cos 2\alpha\cos 2\alpha'}{\cos^2(\alpha+\alpha')}$$

$$= \frac{1 + \cos 2(\alpha+\alpha') - \cos 2(\alpha+\alpha') - \cos 2(\alpha-\alpha')}{2\cos^2(\alpha+\alpha')} = \frac{\sin^2(\alpha-\alpha')}{\cos^2(\alpha+\alpha')}$$

$$\therefore \cos\phi = \pm\frac{\sin(\alpha-\alpha')}{\cos(\alpha+\alpha')}$$

Take the upper sign ; then $\cos\phi = \dfrac{\sin(\alpha-\alpha')}{\cos(\alpha+\alpha')}$;

$$\therefore \frac{1-\cos\phi}{1+\cos\phi} = \frac{\cos(\alpha+\alpha') - \sin(\alpha-\alpha')}{\cos(\alpha+\alpha') + \sin(\alpha-\alpha')} = \frac{\sin\left(\frac{\pi}{2} - \alpha - \alpha'\right) - \sin(\alpha-\alpha')}{\sin\left(\frac{\pi}{2} - \alpha - \alpha'\right) + \sin(\alpha-\alpha')}$$

$$= \frac{2\sin\left(\frac{\pi}{4} - \alpha\right)\cos\left(\frac{\pi}{4} - \alpha'\right)}{2\sin\left(\frac{\pi}{4} - \alpha'\right)\cos\left(\frac{\pi}{4} - \alpha\right)} = \frac{\tan\left(\frac{\pi}{4} - \alpha\right)}{\tan\left(\frac{\pi}{4} - \alpha'\right)}$$

$$\therefore \tan^2\frac{\phi}{2} = \frac{\tan\left(\frac{\pi}{4} - \alpha\right)}{\tan\left(\frac{\pi}{4} - \alpha'\right)}.$$

Take the lower sign ; then $\cos\phi = -\dfrac{\sin(\alpha-\alpha')}{\cos(\alpha+\alpha')}$;

$$\therefore \frac{1-\cos\phi}{1+\cos\phi} = \frac{\cos(\alpha+\alpha')+\sin(\alpha-\alpha')}{\cos(\alpha+\alpha')-\sin(\alpha-\alpha')} = \frac{\sin\left(\frac{\pi}{2}-\alpha-\alpha'\right)+\sin(\alpha-\alpha')}{\sin\left(\frac{\pi}{2}-\alpha-\alpha'\right)-\sin(\alpha-\alpha')}$$

$$= \frac{2\sin\left(\frac{\pi}{4}-\alpha'\right)\cos\left(\frac{\pi}{4}-\alpha\right)}{2\sin\left(\frac{\pi}{4}-\alpha\right)\cos\left(\frac{\pi}{4}-\alpha'\right)} = \frac{\cot\left(\frac{\pi}{4}-\alpha\right)}{\cot\left(\frac{\pi}{4}-\alpha'\right)} = \frac{\tan\left(\frac{\pi}{4}+\alpha\right)}{\tan\left(\frac{\pi}{4}+\alpha'\right)}$$

$$\therefore \tan^2\frac{\phi}{2} = \frac{\tan\left(\frac{\pi}{4}+\alpha\right)}{\tan\left(\frac{\pi}{4}+\alpha'\right)}.$$

Problem 27.
If

$$\frac{\sin(\theta-\beta)\cos\alpha}{\sin(\phi-\alpha)\cos\beta} + \frac{\cos(\alpha+\theta)\sin\beta}{\cos(\phi-\beta)\sin\alpha} = 0$$

and

$$\frac{\tan\theta\tan\alpha}{\tan\phi\tan\beta} + \frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)} = 0,$$

show that

$$\tan\theta = \frac{1}{2}(\tan\beta+\cot\alpha),\ \tan\phi = \frac{1}{2}(\tan\alpha-\cot\beta).$$

Solution.

$$\because \frac{\sin(\theta-\beta)\cos\alpha}{\sin(\phi-\alpha)\cos\beta} + \frac{\cos(\alpha+\theta)\sin\beta}{\cos(\phi-\beta)\sin\alpha} = 0$$

$$\therefore \frac{\sin(\theta-\beta)\cos\alpha}{\cos(\alpha+\theta)\cos\beta} + \frac{\sin(\phi-\alpha)\sin\beta}{\cos(\phi-\beta)\sin\alpha} = 0$$

$$\therefore \frac{(\sin\theta\cos\beta-\cos\theta\sin\beta)\cos\alpha}{(\cos\alpha\cos\theta-\sin\alpha\sin\theta)\cos\beta} + \frac{(\sin\phi\cos\alpha-\cos\phi\sin\alpha)\sin\beta}{(\cos\phi\cos\beta+\sin\phi\sin\beta)\sin\alpha} = 0$$

$$\therefore \frac{(\tan\theta\cos\beta-\sin\beta)\cos\alpha}{(\cos\alpha-\sin\alpha\tan\theta)\cos\beta} + \frac{(\tan\phi\cos\alpha-\sin\alpha)\sin\beta}{(\cos\beta+\tan\phi\sin\beta)\sin\alpha} = 0$$

$$\therefore \frac{\tan\theta-\tan\beta}{1-\tan\alpha\tan\theta} + \frac{\tan\phi\cot\alpha-1}{\cot\beta+\tan\phi} = 0$$

$$\therefore (\tan\theta-\tan\beta)(\cot\beta+\tan\phi) + (\tan\phi\cot\alpha-1)(1-\tan\alpha\tan\theta) = 0$$

$$\therefore \tan\theta(\cot\beta+\tan\alpha) + \tan\phi(\cot\alpha-\tan\beta) = 2.$$

But

$$\tan\theta = -\tan\phi\cdot\frac{\tan\beta}{\tan\alpha}\frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)};$$

$$\therefore -\tan\phi\cdot(\cot\beta+\tan\alpha)\frac{\tan\beta}{\tan\alpha}\frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)} + \tan\phi(\cot\alpha-\tan\beta) = 2$$

$$\therefore -\tan\phi(\cot\alpha+\tan\beta)\cos(\alpha-\beta) + \tan\phi(\cot\alpha-\tan\beta)\cos(\alpha+\beta) = 2\cos(\alpha+\beta)$$

$$\therefore \tan\phi\left\{\cot\alpha\left[\cos(\alpha+\beta)-\cos(\alpha-\beta)\right]-\tan\beta\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]\right\} = 2\cos(\alpha+\beta)$$

$$\therefore \tan\phi\,\{\cot\alpha\sin\alpha\sin\beta+\tan\beta\cos\alpha\cos\beta\}=-\cos(\alpha+\beta).$$

$$\therefore \tan\phi=-\frac{\cos(\alpha+\beta)}{2\cos\alpha\sin\beta}=\frac{1}{2}(\tan\alpha-\cot\beta),\ \text{and}$$

$$\tan\theta=-\tan\phi\frac{\tan\beta}{\tan\alpha}\frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)}$$

$$=\frac{\cos(\alpha-\beta)}{2\sin\alpha\cos\beta}=\frac{1}{2}(\cot\alpha+\tan\beta).$$

Problem 28.
If

$$\frac{2}{1+x}=\frac{\sin\beta\sin\theta}{\cos(\beta-\theta)}=\frac{\tan(\theta-\alpha)}{\cot\beta},$$

prove that

$$x^2=\left(\cot\frac{\alpha}{2}-2\cot\beta\right)\left(\tan\frac{\alpha}{2}+2\cot\beta\right).$$

Solution. If $\dfrac{2}{1+x}=\dfrac{\sin\beta\sin\theta}{\cos(\beta-\theta)}=\dfrac{\sin\beta\sin\theta}{\cos\beta\cos\theta+\sin\beta\sin\theta}=\dfrac{1}{\cot\beta\cot\theta+1}$.

$$\therefore \cot\beta\cot\theta+1=\frac{1+x}{2}$$

$$\therefore \cot\beta\cot\theta=\frac{1+x}{2}-1=\frac{x-1}{2} \tag{12}$$

Again

$$\frac{2}{1+x}=\frac{\tan(\theta-\alpha)}{\cot\beta}=\frac{(\tan\theta-\tan\alpha)\tan\beta}{1+\tan\theta\tan\alpha}$$

$$\therefore 2(1+\tan\theta\tan\alpha)=(1+x)(\tan\theta-\tan\alpha)\tan\beta;$$

$$\therefore \tan\theta=\frac{2+(1+x)\tan\alpha\tan\beta}{(1+x)\tan\beta-2\tan\alpha} \tag{13}$$

From (12) and (13) by multiplication

$$\cot\beta=\frac{2+(1+x)\tan\alpha\tan\beta}{(1+x)\tan\beta-2\tan\alpha}\cdot\frac{x-1}{2}$$

$$\therefore 2\cot\beta\,\{(1+x)\tan\beta-2\tan\alpha\}=2(x-1)+\left(x^2-1\right)\tan\alpha\tan\beta$$

$$\therefore 2(1+x)-4\cot\beta\tan\alpha=2(x-1)+\left(x^2-1\right)\tan\alpha\tan\beta$$

$$\therefore x^2\tan\alpha\tan\beta=4-4\cot\beta\tan\alpha+\tan\alpha\tan\beta$$

$$x^2=4\cot\alpha\cot\beta-4\cot^2\beta+1$$

$$=2\left(\cot\frac{\alpha}{2}-\tan\frac{\alpha}{2}\right)\cot\beta-4\cot^2\beta+1$$

$$=\left(\cot\frac{\alpha}{2}-2\cot\beta\right)\left(\tan\frac{\alpha}{2}+2\cot\beta\right).$$

Problem 29. Given $\sin\theta\sin\phi=\sin\alpha\sin\beta,\ \tan\phi\cos\beta=\cot\dfrac{\alpha}{2}$, prove that one of the values of $\sin\dfrac{\theta}{2}$ is $\sin\dfrac{\alpha}{2}\sin\beta$.

Solution.

$$\because \sin\theta\sin\phi = \sin\alpha\sin\beta;\ \therefore 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{\sin\alpha\sin\beta}{\sin\phi}$$

$$\therefore 4\sin^2\frac{\theta}{2} - 4\sin^4\frac{\theta}{2} = \frac{\sin^2\alpha\sin^2\beta}{\sin^2\phi}$$

$$\therefore 4\sin^4\frac{\theta}{2} - 4\sin^2\frac{\theta}{2} + 1 = 1 - \frac{\sin^2\alpha\sin^2\beta}{\sin^2\phi}$$

$$\because \sin^2\phi = \frac{\cot^2\frac{\alpha}{2}}{\cot^2\frac{\alpha}{2} + \cos^2\beta}$$

$$\therefore 4\sin^4\frac{\theta}{2} - 4\sin^2\frac{\theta}{2} + 1 = 1 - \frac{\sin^2\alpha\left(\cot^2\frac{\alpha}{2} + \cos^2\beta\right)\sin^2\beta}{\cot^2\frac{\alpha}{2}}$$

$$= 1 - 4\sin^4\frac{\alpha}{2}\left(\cot^2\frac{\alpha}{2} + \cos^2\beta\right)\sin^2\beta$$

$$= 1 - 4\sin^4\frac{\alpha}{2}\left(\cot^2\frac{\alpha}{2} + 1 - \sin^2\beta\right)\sin^2\beta$$

$$= 1 - 4\sin^2\frac{\alpha}{2}\sin^2\beta + 4\sin^4\frac{\alpha}{2}\sin^4\frac{\beta}{2}$$

$$\therefore 2\sin^2\frac{\theta}{2} - 1 = \pm\left(1 - 2\sin^2\frac{\alpha}{2}\sin^2\beta\right).$$

Taking the lower sign we have $\sin^2\frac{\theta}{2} = \sin^2\frac{\alpha}{2}\sin^2\beta$.

Problem 30. Given $\sin\phi = n\sin\theta$, $\tan\phi = 2\tan\theta$, find the limiting values of n that these equations may coexist.

Solution. $\sin\phi = n\sin\theta;\ \therefore \cos\phi = \sqrt{1 - n^2\sin^2\theta}$.

$$\therefore \tan\phi = \frac{n\sin\theta}{\sqrt{1 - n^2\sin^2\theta}}.$$

$$\therefore \frac{n\sin\theta}{\sqrt{1 - n^2\sin^2\theta}} = 2\tan\theta = \frac{2\sin\theta}{\cos\theta}$$

$$\therefore n\cos\theta = 2\sqrt{1 - n^2\sin^2\theta}$$

$$\therefore n^2\left(1 - \sin^2\theta\right) = 4\left(1 - n^2\sin^2\theta\right)$$

$$\therefore 3n^2\sin^2\theta = 4 - n^2;\ \therefore \sin^2\theta = \frac{4 - n^2}{3n^2}.$$

This must lie between 0 and 1, so that $4 - n^2$ must lie between 0 and $3n^2$, therefore 4 must lie between n^2 and $4n^2$; therefore n^2 must lie between 1 and 4.

Problem 31. Show by means of a Trigonometrical formula that if

$$x + y + z = xyz,$$

then

$$\frac{2x}{1 - x^2} + \frac{2y}{1 - y^2} + \frac{2z}{1 - z^2} = \frac{2x}{1 - x^2}\cdot\frac{2y}{1 - y^2}\cdot\frac{2z}{1 - z^2}.$$

Solution. Assume $x = \tan A$ and $y = \tan B$; then by *Art.* 114 (page 409), we have $z = \tan C$, where $A + B + C = 180°$.

Therefore $2A + 2B + 2C = 360°$; therefore $\tan(2A + 2B + 2C) = 0$; and therefore, as in *Art.* 114, (page 409)

$$\tan 2A + \tan 2B + \tan 2C = \tan 2A \tan 2B \tan 2C;$$

$$\therefore \frac{2\tan A}{1-\tan^2 A} + \frac{2\tan B}{1-\tan^2 B} + \frac{2\tan C}{1-\tan^2 C} = \frac{2\tan A}{1-\tan^2 A}\cdot\frac{2\tan B}{1-\tan^2 B}\cdot\frac{2\tan C}{1-\tan^2 C}.$$

Problem 32. Find the values of v, x, y, z from the equations

$$v = \frac{\sin x}{\sin a} = \frac{\sin y}{\sin b} = \frac{\sin z}{\sin c}; \ x + y + z = 2\pi.$$

Solution.

$$v \sin c = \sin z = \sin(2\pi - x - y) = -\sin(x + y)$$

$$= -\sin x \cos y - \cos x \sin y = -v \sin a \cos y - v \sin b \cos x.$$

$$\therefore v = 0; \text{ or } \sin c = -\sin a \cos y - \sin b \cos x.$$

Take the latter, thus $\sin a \cos y = -\sin c - \sin b \cos x$;

but $\sin a \sin y = \sin b \sin x$;

square and add, thus

$$\sin^2 a = \sin^2 b + \sin^2 c + 2 \sin b \sin c \cos x;$$

$$\therefore \cos x = \frac{\sin^2 a - \sin^2 b - \sin^2 c}{2 \sin b \sin c}.$$

Similarly $\cos y$ and $\cos z$ may be found, and then v.

If $v = 0$, we have $\sin x = 0$, $\sin y = 0$, and $\sin z = 0$. This will give us three solutions; $x = 0$, $y = \pi$, $z = \pi$; $x = \pi$, $y = 0$, $z = \pi$; $x = \pi$, $y = \pi$, $z = 0$: and also three solutions, $x = 0$, $y = 0$, $z = 2\pi$; $x = 0$, $y = 2\pi$, $z = 0$; $x = 2\pi$, $y = 0$, $z = 0$.

Problem 33. Find the limit of $(\cos \alpha x)^{\operatorname{cosec}^2 \beta x}$ when x is zero.

Solution. Let $u = (\cos \alpha x)^{\operatorname{cosec}^2 \beta x}$; therefore

$$\log u = \operatorname{cosec}^2 \beta x \log \cos \alpha x = \frac{1}{2}\operatorname{cosec}^2 \beta x \log\left(1 - \sin^2 \alpha x\right)$$

$$= -\frac{1}{2\sin^2 \beta x}\left\{\sin^2 \alpha x + \frac{1}{2}\sin^4 \alpha x + \frac{1}{3}\sin^6 \alpha x + \dots\right\}.$$

Now $\dfrac{\sin \alpha x}{\sin \beta x} = \dfrac{\alpha}{\beta} \cdot \dfrac{\sin \alpha x}{\alpha x} \cdot \dfrac{\beta x}{\sin \beta x}$;

when x is zero the value of $\dfrac{\sin \alpha x}{\alpha x}$ is unity, and so also if the value of $\dfrac{\beta x}{\sin \beta x}$; thus $\dfrac{\sin \alpha x}{\sin \beta x} = \dfrac{\alpha}{\beta}$; therefore $\dfrac{\sin^2 \alpha x}{\sin^2 \beta x} = \dfrac{\alpha^2}{\beta^2}$.

The limit of $\dfrac{\sin^4 \alpha x}{\sin^4 \beta x}$ is zero, and so also the other terms in $\log u$ vanish, and as in *Art.* 150 (page 416), their sum vanishes also. Hence $\log u = -\dfrac{\alpha^2}{2\beta^2}$, and therefore $u = e^{-\frac{\alpha^2}{2\beta^2}}$.

Problem 34. From a table of natural tangents which goes to 7 places of decimals, show that an angle may be determined within about $\frac{1}{200}$th part of a second when the angle is nearly 60°.

Solution. By *Art.* 188 (page 416), if h is very small we have $\tan(\theta+h)-\tan\theta = h\sec^2\theta$; thus if θ be nearly equivalent to 60° we have approximately

$$\tan(\theta+h)-\tan\theta = 4h.$$

Since the tables extend to 7 places of decimals it follows that we can discriminate angles which are near 60°, by means of their tangents, when the circular measure h of the difference is such that $4h = .0000001$. Thus $h = \frac{1}{4}$ of $\frac{1}{10^7}$; the corresponding value in seconds is $\frac{1}{4}\times\frac{1}{10^7}\times\frac{180}{\pi}\times 60\times 60$, that is $\frac{18\times 9}{10000\pi}$, that is about $\frac{1}{200}$.

Problem 35. When an angle is very nearly equal to 64°36′, show that the angle can be determined from its L sine within about $\frac{1}{10}$th of a second; having given $\log_e 10\cdot\tan 64°36' = 4.8492$, and the tables going to 7 places of decimals.

Solution. By *Art.* 196 (page 417), if h is very small we have

$$L\sin(\theta+h)-L\sin\theta = \mu h\cot\theta = \frac{h}{(\log_e 10)\tan\theta};$$

thus if θ be nearly equivalent to 64°36′, we have approximately

$$L\sin(\theta+h)-L\sin\theta = \frac{h}{4.8492};$$

Since the tables extend to 7 places of decimals it follows that we can discriminate angles which are near 64°36′, by means of their L sines, when the circular measure of the difference is such that $\frac{h}{4.8492} = .0000001$. Thus $h = \frac{4.8492}{10^7}$; the corresponding value in seconds is

$$\frac{4.8492}{10^7}\times\frac{180}{\pi}\times 60\times 60;$$

this will be found to be about $\frac{1}{10}$.

Problem 36. Show that

$$\left(1-\tan^2\frac{\alpha}{2}\right)\left(1-\tan^2\frac{\alpha}{2^2}\right)\left(1-\tan^2\frac{\alpha}{2^3}\right)\ldots\ \textit{ad inf.} = \frac{\alpha}{\tan\alpha}.$$

Solution.

$$1-\tan^2\frac{\alpha}{2} = \frac{\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}} = \frac{\cos\alpha}{\cos^2\frac{\alpha}{2}},$$

$$1-\tan^2\frac{\alpha}{4} = \frac{\cos^2\frac{\alpha}{4}-\sin^2\frac{\alpha}{4}}{\cos^2\frac{\alpha}{4}} = \frac{\cos\frac{\alpha}{2}}{\cos^2\frac{\alpha}{4}},$$

and so on.

In this way we find that the proposed expression

$$= \frac{\cos\alpha\cos\frac{\alpha}{2}\cos\frac{\alpha}{2^2}\cos\frac{\alpha}{2^3}\ldots}{\cos^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2^2}\cos^2\frac{\alpha}{2^3}\ldots}$$

$$= \frac{\cos\alpha}{\cos\frac{\alpha}{2}\cos\frac{\alpha}{2^2}\cos\frac{\alpha}{2^3}\cdots}$$

$$= \cos\alpha \div \frac{\sin\alpha}{\alpha} = \frac{\alpha}{\tan\alpha}.$$ See *Art.* 129 (page 412).

Problem 37. If A, B, C be positive angles which satisfy the equation

$$\sin^2 A + \sin^2 B + \sin^2 C = 1,$$

prove that $A + B + C$ is greater than 90°.

Solution. We have universally

$$\begin{aligned}\sin^2(A+B) &= (\sin A\cos B + \cos A\sin B)^2\\ &= \sin^2 A\cos^2 B + \cos^2 A\sin^2 B + 2\sin A\cos A\sin B\cos B\\ &= \sin^2 A\left(1-\sin^2 B\right) + \sin^2 B\left(1-\sin^2 A\right) + 2\sin A\cos A\sin B\cos B\\ &= \sin^2 A + \sin^2 B + 2\sin A\sin B\left(\cos A\cos B - \sin A\sin B\right)\end{aligned}$$

$$\therefore \sin^2(A+B) = \sin^2 A + \sin^2 B + 2\sin A\sin B\cos(A+B) \tag{14}$$

Also in the present case

$$\sin^2 A + \sin^2 B = 1 - \sin^2 C = \cos^2 C \tag{15}$$

If $A + B$ is greater than 90°, then *a fortiori* $A + B + C$ is greater than 90°.

If $A + B$ is less than 90°, then $\sin^2(A + B)$ is greater than $\sin^2 A + \sin^2 B$ by (14), and therefore greater than $\cos^2 C$ by (15); and therefore $A + B$ is greater than $90° - C$, so that $A + B + C$ is greater than 90°.

Problem 38. A circle is drawn touching the tangent and secant of a given angle α, as well as the corresponding arc; find its radius and explain the double value. If one value be equal to the radius of the original circle, show that $\alpha = \frac{\pi}{3}$.

Solution. Take the diagram of *Art.* 71 (page 403). Let α be the angle PAB. Suppose a circle having its centre O within the space bounded by PB, BT and TP; let it touch the arc PB, the tangent BT, and the secant APT. Let ρ denote the radius of this circle, and r the radius of the original circle.

OT will bisect the angle ATB, and OA will pass through the point of contact of the circles. Let N be the point of contact of the secant APT and the circle with centre O. Then

$$NT = \rho\cot\frac{1}{2}\left(\frac{\pi}{2}-\alpha\right);\ OA = r + \rho;$$

$$\therefore AN = \sqrt{(r+\rho)^2-\rho^2} = \sqrt{r^2+2r\rho}.$$

$$\therefore \sqrt{r^2+2r\rho} + \rho\cot\left(\frac{\pi}{4}-\frac{\alpha}{2}\right) = AT = r\sec\alpha;$$

$$\therefore \sqrt{r^2+2r\rho} = r\sec\alpha - \rho\cot\left(\frac{\pi}{4}-\frac{\alpha}{2}\right).$$

By squaring we obtain a quadratic equation for determining ρ. The reason why we have a quadratic equation is that another circle can also be drawn, which may be said to fulfil the conditions. For produce PA through A to meet the original circle again at p; then we may have a circle outside the arc Bp, touching this arc, touching TB produced through B, and touching Tp produced through p. The corresponding equation would be

$$\rho \cot\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) - \sqrt{r^2 + 2r\rho} = r \sec\alpha.$$

This differs from the former only in the sign of the radical, and therefore leads to the same quadratic equation.

Suppose $\rho = r$; then

$$\begin{aligned}
\pm\sqrt{3} &= \frac{1}{\cos\alpha} - \frac{\cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)} \\
&= \frac{1}{\cos\alpha} - \frac{\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}} \\
&= \frac{1}{\cos\alpha} - \frac{\left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\right)^2}{\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}} \\
&= \frac{1}{\cos\alpha} - \frac{1 + \sin\alpha}{\cos\alpha} = -\tan\alpha.
\end{aligned}$$

Hence taking $\sqrt{3} = \tan\alpha$, we have $\alpha = \frac{\pi}{3}$.

Problem 39.

If

$$l\cos(\theta - \beta) - m\cos(\theta - \alpha) = n,$$

show that

$$l\sin(\theta - \beta) - m\sin(\theta - \alpha) = \sqrt{l^2 + m^2 - n^2 - 2lm\cos(\alpha - \beta)}.$$

Solution. Let x denote the value of $l\sin(\theta - \beta) - m\sin(\theta - \alpha)$; so that

$$l\cos(\theta - \beta) - m\cos(\theta - \alpha) = n,\ l\sin(\theta - \beta) - m\sin(\theta - \alpha) = x.$$

Square and add; thus

$$l^2 + m^2 - 2lm\left\{\cos(\theta - \beta)\cos(\theta - \alpha) + \sin(\theta - \beta)\sin(\theta - \alpha)\right\} = n^2 + x^2;$$

$$\therefore l^2 + m^2 - 2lm\cos(\alpha - \beta) = n^2 + x^2;$$

$$\therefore x = \sqrt{l^2 + m^2 - n^2 - 2lm\cos(\alpha - \beta)}.$$

Problem 40. Show that $\theta - \sin\theta$ is less than $\tan\theta - \theta$ if θ lies between 0 and $\frac{\pi}{2}$.

Solution. $\theta - \sin\theta$ is less than $\tan\theta - \theta$ if 2θ is less than $\sin\theta + \tan\theta$, that is if 2θ is less than $\tan\theta(1 + \cos\theta)$, that is if 2θ is less than

$$\frac{2\tan\frac{\theta}{2}}{1 - \tan^2\frac{\theta}{2}} \times \frac{2}{1 + \tan^2\frac{\theta}{2}},$$

that is if $\frac{\theta}{2}$ is less than $\dfrac{\tan\frac{\theta}{2}}{1 - \tan^4\frac{\theta}{2}}$, and this is obviously the case, because $\frac{\theta}{2}$ is less than $\tan\dfrac{\theta}{2}$.

CHAPTER XIII
Relations between the Sides of a Triangle and the Trigonometrical Functions of the Angles

Problem 1. The sides of a triangle are $x^2+x+1,\ 2x+1,$ and x^2-1, show that the greatest angle is 120°.

Solution. The greatest angle is opposite to the greatest side; thus the cosine

$$= \frac{\left(x^2-1\right)^2+(2x+1)^2-\left(x^2+x+1\right)^2}{2\left(x^2-1\right)(2x+1)}$$

$$= \frac{x^4-2x^2+1+4x^2+4x+1-\left(x^4+x^2+1+2x^3+2x^2+2x\right)}{2\left(x^2-1\right)(2x+1)}$$

$$= \frac{-2x^3-x^2+2x+1}{2\left(2x^3+x^2-2x-1\right)} = -\frac{1}{2}.$$

Therefore the angle is 120°.

Problem 2. If $\cos B = \dfrac{\sin A}{2\sin C}$, show that the triangle is isosceles.

Solution.

$$2\sin C\cos B = \sin A = \sin(B+C) = \sin B\cos C + \cos B\sin C;$$

$$\therefore \sin C\cos B = \sin B\cos C;$$

$$\therefore \sin(C-B) = 0;\ \therefore B = C.$$

Problem 3. In a right-angled triangle of which C is the right angle,

$$\cot\frac{A}{2} = \frac{b+c}{a}.$$

Solution. We have $\cos A = \dfrac{b}{c}$, $\sin A = \dfrac{a}{c}$;

$$\therefore \frac{1+\cos A}{\sin A} = \frac{c+b}{a};\ \therefore \cot\frac{A}{2} = \frac{b+c}{a}.$$

Problem 4. If in a triangle $a\tan A + b\tan B = (a+b)\tan\dfrac{A+B}{2}$, show that $A = B$.

Solution.

$$a\tan A + b\tan B = (a+b)\tan\frac{A+B}{2}$$

$$\therefore a\left(\tan A - \tan\frac{A+B}{2}\right) = b\left(\tan\frac{A+B}{2} - \tan B\right)$$

$$\therefore \frac{a\left(\sin A\cos\dfrac{A+B}{2}-\cos A\sin\dfrac{A+B}{2}\right)}{\cos A\cos\dfrac{A+B}{2}}=\frac{b\left(\sin\dfrac{A+B}{2}\cos B-\cos\dfrac{A+B}{2}\sin B\right)}{\cos B\cos\dfrac{A+B}{2}}$$

$$\therefore \frac{a\sin\dfrac{A-B}{2}}{\cos A}=\frac{b\sin\dfrac{A-B}{2}}{\cos B}$$

$$\therefore \frac{a}{b}=\frac{\cos A}{\cos B}.$$

$$\because \frac{a}{b}=\frac{\sin A}{\sin B}$$

$$\therefore \frac{\sin A}{\sin B}=\frac{\cos A}{\cos B}$$

$$\therefore \tan A=\tan B;\ \therefore A=B.$$

Problem 5. The angles of a plane triangle form a geometrical progression of which the common ratio is $\frac{1}{2}$; show that the greatest side is to the perimeter as $2\sin\frac{\pi}{14}$ is to unity.

Solution. Let 2α denote the least angle; then the other angles are 4α and 8α respectively: therefore $2\alpha+4\alpha+8\alpha=\pi$; therefore $\alpha=\frac{\pi}{14}$.

Then by *Art.* 214 (page 418), the ratio of the greatest side to the perimeter

$$=\frac{\sin 8\alpha}{\sin 2\alpha+\sin 4\alpha+\sin 8\alpha}$$

$$=\frac{\sin 8\alpha}{\sin 2\alpha+\sin 4\alpha+\sin 6\alpha}=\frac{2\sin 4\alpha\cos 4\alpha}{2\sin 3\alpha\cos\alpha+2\sin 3\alpha\cos 3\alpha};$$

but $4\alpha+3\alpha=\frac{\pi}{2}$, $\therefore \cos 4\alpha=\sin 3\alpha$; hence this expression

$$=\frac{\sin 4\alpha}{\cos\alpha+\cos 3\alpha}=\frac{2\sin 2\alpha\cos 2\alpha}{2\cos\alpha\cos 2\alpha}=\frac{\sin 2\alpha}{\cos\alpha}=2\sin\alpha.$$

Problem 6. If A', B', C' are the *external* angles of a triangle, show that

$$2bc\,\mathrm{vers}\,A'+2ca\,\mathrm{vers}\,B'+2ab\,\mathrm{vers}\,C'=(a+b+c)^2.$$

Solution.

$$\begin{aligned}2bc\,\mathrm{vers}\,A'+2ca\,\mathrm{vers}B'+2ab\,\mathrm{vers}\,C'\\ &=2bc(1-\cos A')+2ca(1-\cos B')+2ab(1-\cos C')\\ &=2bc(1+\cos A)+2ca(1+\cos B)+2ab(1+\cos C)\\ &=4bc\cos^2\frac{A}{2}+4ca\cos^2\frac{B}{2}+4ab\cos^2\frac{C}{2}\\ &=4s(s-a)+4s(s-b)+4s(s-c)\\ &=4s(3s-a-b-c)=4s^2=(2s)^2=(a+b+c)^2.\end{aligned}$$

Problem 7. From the angle A of any triangle ABC a perpendicular AD is drawn to the base, and from D perpendiculars DE, DF are drawn to AB, AC respectively

: show that

$$AE \cdot EB \cdot \cos^2 C = AF \cdot FC \cdot \cos^2 B.$$

Solution. Let $AD = p$. Suppose the angles B and C to be acute, as in the left-handed diagram of *Art.* 214 (page 418). Then

$$AE = p\cos(90^\circ - B) = p\sin B,$$
$$DE = p\sin(90^\circ - B) = p\cos B,$$
$$EB = DE\cot B = p\cos B\cot B;$$
$$AE \cdot EB = p^2\cos^2 B.$$
$$\text{Similarly } AF \cdot FC = p^2\cos^2 C.$$
$$\therefore AE \cdot EB\cos^2 C = AF \cdot FC\cos^2 B.$$

Next suppose one of the angles B and C to be obtuse, say the angle C, as in the right-hand diagram of *Art.* 214 (page 418).
Then

$$AE \cdot EB = p^2\cos^2 B \text{ as before,}$$
$$AF = p\cos(C - 90^\circ) = p\sin C,$$
$$DF = p\sin(C - 90^\circ) = -p\cos C,$$
$$FC = DF\cot(180^\circ - C) = -DF\cot C = p\cos C\cot C;$$
$$\therefore AF \cdot FC = p^2\cos^2 C, \text{ as before.}$$

Problem 8. Let a, b, c be the sides of a triangle and the opposite angles be 2θ, 3θ, 4θ.
Show that

$$\tan^2\theta = \left(\frac{2b}{a+c}\right)^2 - 1.$$

Solution.

$$\frac{\sin 2\theta + \sin 4\theta}{\sin 3\theta} = \frac{a+c}{b};$$
$$\therefore 2\cos\theta = \frac{a+c}{b}; \therefore \cos\theta = \frac{a+c}{2b};$$
$$\therefore \tan^2\theta = \frac{1}{\cos^2\theta} - 1 = \left(\frac{2b}{a+c}\right)^2 - 1.$$

Problem 9. ABC is a triangle of which C is an obtuse angle. Show that $\tan A\tan B$ is less than unity.

Solution. Since C is obtuse, $A + B$ is less than 90°; therefore $\cos(A + B)$ is positive, therefore $\cos A\cos B - \sin A\sin B$ is positive; therefore $\sin A\sin B$ is less than $\cos A\cos B$; therefore $\dfrac{\sin A}{\cos A}\dfrac{\sin B}{\cos B}$ is less than unity, that is $\tan A\tan B$ is less than unity.

Problem 10. If the sides a, b, c of a triangle be in arithmetical progression, show that

$$\cos\frac{A-C}{2} = 2\sin\frac{B}{2}, \text{ and}$$
$$a\cos^2\frac{C}{2} + c\cos^2\frac{A}{2} = \frac{3b}{2}.$$

Solution. Since a, b, c are in Arithmetical Progression, so are $\sin A$, $\sin B$, $\sin C$.

$$\therefore \sin A + \sin C = 2\sin B;$$
$$\therefore \sin\frac{A+C}{2}\cos\frac{A-C}{2} = 2\sin\frac{B}{2}\cos\frac{B}{2} = 2\sin\frac{B}{2}\sin\frac{A+C}{2};$$
$$\therefore \cos\frac{A-C}{2} = 2\sin\frac{B}{2}.$$

Again

$$a\cos^2\frac{C}{2} + c\cos^2\frac{A}{2}$$
$$= \frac{a}{2}(1+\cos C) + \frac{c}{2}(1+\cos A)$$
$$= \frac{1}{2}(a+c) + \frac{1}{2}(a\cos C + c\cos A) = \frac{1}{2}(a+c) + \frac{b}{2}, \text{ by } Art.\,216 \text{ (page 419)},$$
$$= b + \frac{b}{2}, \text{ by hypothesis,}$$
$$= \frac{3b}{2}.$$

Problem 11. If D be the middle point of the side BC of a triangle

$$\cot BAD - \cot B = 2\cot A.$$

Solution. From the triangle ABD we have

$$\frac{\sin ADB}{\sin BAD} = \frac{AB}{BD} = \frac{2c}{a}.$$

Put ϕ for BAD; thus

$$\frac{\sin(\phi+B)}{\sin\phi} = \frac{2c}{a} = \frac{2\sin C}{\sin A};$$
$$\therefore \frac{\sin\phi\cos B + \cos\phi\sin B}{\sin\phi} = \frac{2\sin C}{\sin A};$$
$$\therefore \cot B + \cot\phi = \frac{2\sin C}{\sin A\sin B} = \frac{2\sin(A+B)}{\sin A\sin B}$$
$$= 2\cot A + 2\cot B;$$
$$\therefore \cot\phi - \cot B = 2\cot A.$$

Problem 12. If an angle of a triangle be divided into two parts such that the sines are in the ratio of the sides adjacent to them respectively, show that the difference of their cotangents is equal to the difference of the cotangents of the angles opposite to their sides.

Solution. Let the angle A of a triangle be divided into two parts by a straight line

AD; denote BAD by ϕ and CAD by ψ, and suppose that $\dfrac{\sin\phi}{\sin\psi} = \dfrac{c}{b}$.

$$\therefore \frac{\sin(A-\psi)}{\sin\psi} = \frac{c}{b} = \frac{\sin C}{\sin B};$$
$$\therefore \sin A \cot\psi - \cos A = \frac{\sin C}{\sin B};$$
$$\therefore \cot\psi = \cot A + \frac{\sin(A+B)}{\sin A \sin B} = 2\cot A + \cot B.$$

Similarly

$$\cot\phi = 2\cot A + \cot C.$$
$$\therefore \cot\psi - \cot\phi = \cot B - \cot C.$$

Problem 13. If the cotangents of the angles of a triangle be in arithmetic progression, the squares of the sides will also be in arithmetic progression.

Solution. Suppose $\cot A + \cot B = 2\cot B$;

$$\therefore \frac{\cos A}{\sin A} + \frac{\cos C}{\sin C} = \frac{2\cos B}{\sin B};$$
$$\therefore \frac{\sin(A+C)}{\sin A \sin B} = \frac{2\cos B}{\sin B};$$
$$\therefore \frac{\sin^2 B}{\sin A \sin C} = 2\cos B;$$
$$\therefore \frac{b^2}{ac} = \frac{a^2+c^2-b^2}{ac};$$
$$\therefore 2b^2 = a^2 + c^2.$$

Thus a^2, b^2, c^2 are in Arithmetical Progression.

Problem 14. Given the vertical angle and the ratio between the base and altitude of a triangle, find the tangents of the angles into which the vertical angle is divided by the perpendicular drawn from it to the base.

Solution. Let a perpendicular AD be drawn from the angle A of a triangle on the base BC. Let $BAD = \phi$, and $CAD = \psi$. Let m denote the ratio of the base BC to the perpendicular AD.

Then in the case of the left-hand diagram of *Art.* 214 (page 418), we have

$$\tan\phi = \frac{BD}{AD}, \ \tan\psi = \frac{CD}{AD};$$
$$\therefore \tan\phi + \tan\psi = \frac{BD+CD}{AD} = \frac{BC}{AD} = m; \qquad (16)$$

Also $\phi + \psi = A$; thus

$$\tan A = \tan(\phi+\psi) = \frac{\tan\phi + \tan\psi}{1 - \tan\phi\tan\psi} \qquad (17)$$

Hence from (16) and (17), we can find $\tan\phi$ and $\tan\psi$.

Similarly in the case of the right-hand diagram of *Art.* 214 (page 418), we have

$$\tan\phi - \tan\psi = m, \text{and}$$
$$\tan A = \tan(\phi-\psi) = \frac{\tan\phi - \tan\psi}{1 + \tan\phi\tan\psi}.$$

Problem 15. If the base of a triangle be divided into three equal parts, and t_1, t_2, t_3 be the tangents of the angles which they subtend at the vertex

$$\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\left(\frac{1}{t_2}+\frac{1}{t_3}\right)=4\left(1+\frac{1}{t_2^2}\right).$$

Solution. Let the base BC of a triangle be divided at D and E, so that $BD = DE = EC$. Let the angle BAD be denoted by ϕ_1, the angle DAE by ϕ_2, and the angle EAC by ϕ_3.

Then from the triangle AEB, we have

$$\frac{\sin(\phi_1+\phi_2)}{\sin AEB}=\frac{BE}{AB}=\frac{2}{3}\cdot\frac{a}{c},$$

and from the triangle AEC, we have

$$\frac{\sin\phi_3}{\sin AEC}=\frac{EC}{AC}=\frac{1}{3}\cdot\frac{a}{b};$$

therefore by division

$$\frac{\sin(\phi_1+\phi_2)}{\sin\phi_3}=\frac{2b}{c}.$$

In the same manner we wee that

$$\frac{\sin(\phi_3+\phi_2)}{\sin\phi_1}=\frac{2c}{b}.$$

$$\therefore\ \frac{\sin(\phi_1+\phi_2)\sin(\phi_3+\phi_2)}{\sin\phi_1\sin\phi_3}=4=4\left(\sin^2\phi_2+\cos^2\phi_2\right);$$

$$\therefore (\cos\phi_2+\sin\phi_2\cot\phi_1)(\cos\phi_2+\sin\phi_2\cot\phi_3)=4\left(\sin^2\phi_2+\cos^2\phi_2\right);$$

$$\therefore (\cot\phi_2+\cot\phi_1)(\cot\phi_2+\cot\phi_3)=4\left(1+\cot^2\phi_2\right).$$

Problem 16. If the sines of the angles of a triangle be in arithmetical progression, the product of the tangents of half the greatest and half the least is $\frac{1}{3}$.

Solution. Suppose that $\sin A+\sin C=2\sin B$,

$$2\sin\frac{A+C}{2}\cos\frac{A-C}{2}=4\sin\frac{B}{2}\cos\frac{B}{2}=4\cos\frac{A+C}{2}\sin\frac{A+C}{2};$$

$$\therefore \cos\frac{A-C}{2}=2\cos\frac{A+C}{2};$$

$$\therefore \cos\frac{A}{2}\cos\frac{C}{2}+\sin\frac{A}{2}\sin\frac{C}{2}=2\cos\frac{A}{2}\cos\frac{C}{2}-2\sin\frac{A}{2}\sin\frac{C}{2};$$

$$\therefore 3\sin\frac{A}{2}\sin\frac{C}{2}=\cos\frac{A}{2}\cos\frac{C}{2};$$

$$\therefore \tan\frac{A}{2}\tan\frac{C}{2}=\frac{1}{3}.$$

Problem 17. If the side BC of a triangle be bisected at D and AD be drawn, show that

$$\tan ADB=\frac{2bc\sin A}{b^2-c^2}.$$

Solution. Denote ADB by ϕ. From the triangle ABD we have

$$\frac{\sin BAD}{\sin ADB}=\frac{BD}{AB}=\frac{a}{2c};$$

$$\therefore \frac{\sin(\phi + B)}{\sin\phi} = \frac{a}{2c};$$
$$\therefore \cos B + \sin B \cot\phi = \frac{a}{2c};$$
$$\therefore \cot\phi = \frac{\frac{a}{2c} - \cos B}{\sin B};$$
$$\tan\phi = \frac{2c\sin B}{a - 2c\cos B} = \frac{2ac\sin B}{a^2 - (a^2 + c^2 - b^2)}$$
$$= \frac{2ac\sin B}{b^2 - c^2} = \frac{2bc\sin A}{b^2 - c^2}.$$

Problem 18. If A, B, C be the angles of a triangle and $\cot\frac{A}{2}$, $\cot\frac{B}{2}$, $\cot\frac{C}{2}$ in arithmetical progression, show that

$$\cot\frac{A}{2}\cot\frac{C}{2} = 3.$$

Solution. Here $\cot\frac{A}{2} + \cot\frac{C}{2} = 2\cot\frac{B}{2}$;

$$\therefore \frac{\cos\frac{A}{2}}{\sin\frac{A}{2}} + \frac{\cos\frac{C}{2}}{\sin\frac{C}{2}} = \frac{2\cos\frac{B}{2}}{\sin\frac{B}{2}} = \frac{2\sin\frac{A+C}{2}}{\cos\frac{A+C}{2}};$$
$$\therefore \frac{\sin\frac{A+C}{2}}{\sin\frac{A}{2}\sin\frac{C}{2}} = \frac{2\sin\frac{A+C}{2}}{\cos\frac{A+C}{2}};$$
$$\therefore \cos\frac{A+C}{2} = 2\sin\frac{A}{2}\sin\frac{C}{2};$$
$$\therefore \cos\frac{A}{2}\cos\frac{C}{2} - \sin\frac{A}{2}\sin\frac{C}{2} = 2\sin\frac{A}{2}\sin\frac{C}{2};$$
$$\therefore \cos\frac{A}{2}\cos\frac{C}{2} = 3\sin\frac{A}{2}\sin\frac{C}{2};$$
$$\therefore \cot\frac{A}{2}\cot\frac{C}{2} = 3.$$

Problem 19. Straight lines are drawn from the angles A and B of a triangle dividing the angles respectively into parts whose sines are in the ratio of 1 to n; these straight lines intersect at D. Show that DC either bisects the angle C or divides it into parts whose sines are in the ratio of 1 to n^2.

Solution. First suppose that $\frac{\sin DAC}{\sin DAB} = \frac{1}{n}$, and that $\frac{\sin DBC}{\sin DBA} = \frac{1}{n}$.
We have

$$\frac{\sin DCB}{\sin DBC} = \frac{BD}{DC}, \text{ and}$$
$$\frac{\sin DBC}{\sin DBA} = \frac{1}{n};$$
$$\therefore \frac{\sin DCB}{\sin DBC} = \frac{BD}{DC}\cdot\frac{1}{n}.$$

Similarly

$$\frac{\sin DCA}{\sin DAB} = \frac{AD}{DC} \cdot \frac{1}{n}.$$
$$\therefore \frac{\sin DCB}{\sin DCA} \cdot \frac{\sin DAB}{\sin DBA} = \frac{BD}{DA};$$
$$\therefore \frac{\sin DCB}{\sin DCA} \cdot \frac{DB}{DA} = \frac{BD}{DA};$$
$$\therefore \frac{\sin DCB}{\sin DCA} = 1.$$

In this case the angle C is bisected by DC.

Next suppose that $\frac{\sin DAC}{\sin DAB} = \frac{1}{n}$, and that $\frac{\sin DBA}{\sin DBC} = \frac{1}{n}$; thus the angle B is divided into two parts equal to the two former, but differently situated.

Then proceeding as before we have

$$\frac{\sin DCB}{\sin DBC} = \frac{BD}{DC}, \text{ and}$$
$$\frac{\sin DBC}{\sin DBA} = n;$$
$$\therefore \frac{\sin DCB}{\sin DBA} = \frac{n \cdot BD}{DC}.$$
$$\text{Also } \frac{\sin DCA}{\sin DAB} = \frac{AD}{DC}\frac{1}{n}.$$

Hence we find that

$$\frac{\sin DCB}{\sin DCA} = n^2.$$

Problem 20. If l be the length of the straight line which bisects the angle A of a triangle and is terminated by the base, θ the angle which it makes with the base, $2s$ the perimeter of the triangle, show that

$$s\left(\sin\theta - \sin\frac{A}{2}\right) = l\cos\frac{A}{2}\sin\theta.$$

Solution. Let the straight line which bisects the angle A of a triangle meet the base at D.

Then

$$\angle ADC = \angle B + \angle BAD;$$
$$\therefore \sin\theta = \sin\left(B + \frac{A}{2}\right).$$
$$\therefore s\left(\sin\theta - \sin\frac{A}{2}\right) = s\left\{\sin\left(B + \frac{A}{2}\right) - \sin\frac{A}{2}\right\}$$
$$= 2s\cos\frac{B+A}{2}\sin\frac{B}{2} = 2s\sin\frac{C}{2}\sin\frac{B}{2};$$

Put for $\sin\frac{C}{2}$ and $\sin\frac{B}{2}$ their values by $Art.$ 217 (page 419); thus we have

$$2s\sin\frac{C}{2}\sin\frac{B}{2} = \frac{2s}{a}(s-a)\sqrt{\frac{(s-b)(s-c)}{bc}}$$
$$= \frac{2s(s-a)}{a}\sin\frac{A}{2} = \frac{2bc}{a}\cos^2\frac{A}{2}\sin\frac{A}{2}$$

$$= \frac{bc}{a} \cos \frac{A}{2} \sin A.$$

$$\text{Again } l \sin \theta = \text{ the perpendicular from } A \text{ on } BC$$
$$= b \sin C.$$

$$\therefore l \sin \theta \cos \frac{A}{2} = b \sin C \cos \frac{A}{2}$$
$$= \frac{bc}{a} \sin A \cos \frac{A}{2}, \text{ by } Art. 214 \text{ (page 418).}$$

$$\therefore s \left(\sin \theta - \sin \frac{A}{2} \right) = l \sin \theta \cos \frac{A}{2}.$$

Problem 21. If θ and ϕ be the greatest and least angles of a triangle, the sides of which are in arithmetical progression, show that

$$4(1 - \cos \theta)(1 - \cos \phi) = \cos \theta + \cos \phi.$$

Solution. The third angle of the triangle will be $\pi - \theta - \phi$; and as the sines of the angles must be in Arithmetical Progression, we have

$$\sin \theta + \sin \phi = 2 \sin(\pi - \theta - \phi) = 2 \sin(\theta + \phi);$$

$$\therefore 2 \sin \frac{\theta + \phi}{2} \cos \frac{\theta - \phi}{2} = 4 \sin \frac{\theta + \phi}{2} \cos \frac{\theta + \phi}{2};$$

$$\therefore 2 \cos \frac{\theta + \phi}{2} = \cos \frac{\theta - \phi}{2};$$

$$\therefore 2 \left(\cos \frac{\theta}{2} \cos \frac{\phi}{2} - \sin \frac{\theta}{2} \cos \frac{\phi}{2} \right) = \cos \frac{\theta}{2} \cos \frac{\phi}{2} + \sin \frac{\theta}{2} \cos \frac{\phi}{2};$$

$$\therefore \cos \frac{\theta}{2} \cos \frac{\phi}{2} = 3 \sin \frac{\theta}{2} \sin \frac{\phi}{2};$$

$$\therefore \cos^2 \frac{\theta}{2} \cos^2 \frac{\phi}{2} = 9 \sin^2 \frac{\theta}{2} \sin^2 \frac{\phi}{2};$$
$$= 9 \left(1 - \cos^2 \frac{\theta}{2} \right) \left(1 - \cos^2 \frac{\phi}{2} \right);$$

$$\therefore 8 \left(1 - \cos^2 \frac{\theta}{2} \right) \left(1 - \cos^2 \frac{\phi}{2} \right) = \cos^2 \frac{\theta}{2} \cos^2 \frac{\phi}{2} - \left(1 - \cos^2 \frac{\theta}{2} \right) \left(1 - \cos^2 \frac{\phi}{2} \right)$$
$$= \cos^2 \frac{\theta}{2} + \cos^2 \frac{\phi}{2} - 1 = \cos^2 \frac{\theta}{2} - \sin^2 \frac{\phi}{2};$$

$$\therefore 8 \sin^2 \frac{\theta}{2} \sin^2 \frac{\phi}{2} = \cos \frac{\theta + \phi}{2} \cos \frac{\theta - \phi}{2};$$

$$\therefore 4(1 - \cos \theta)(1 - \cos \phi) = 2 \cos \frac{\theta + \phi}{2} \cos \frac{\theta - \phi}{2};$$
$$= \cos \theta + \cos \phi.$$

$$\text{Or thus, } \cos \theta = \frac{a^2 + b^2 - c^2}{2ab}, \text{ and } b = \frac{a + c}{2};$$

$$\therefore \cos \theta = \frac{a - c}{a} + \frac{b}{2a} = \frac{a - c}{a} + \frac{a + c}{4a} = \frac{5a - 3c}{4a}.$$

Similarly

$$\cos \phi = \frac{5c - 3a}{4c}.$$

$$\therefore 4(1-\cos\theta)(1-\cos\phi)=\frac{(3c-a)(3a-c)}{4ac}=\frac{10ac-3a^2-3c^2}{4ac};\ \text{and}$$
$$\cos\theta+\cos\phi=\frac{5a-3c}{4a}+\frac{5c-3a}{4c}=\frac{10ac-3a^2-3c^2}{4ac}.$$

Problem 22. From the angular points of a triangle ABC straight lines are drawn making each the same angle α towards the same parts with the sides of the triangle taken in order. Show that these straight lines will form another triangle similar to the former, and that the linear dimensions of the two triangles are in the ratio of

$$\cos\alpha-\sin\alpha(\cot A+\cot B+\cot C)\ \text{to}\ 1.$$

Solution. Draw from A, B, C respectively straight lines to meet the opposite sides at D, E, F, so that the $\angle BAD=\angle CBE=\angle ACF=\alpha$.

Let LMN be the triangle formed by the straight lines thus drawn : so that A, L, M, D are in one straight line; B, M, N, E on another; and C, N, L, F on a third. Then will the triangle LMN be similar to the triangle ABC.

For the $\angle MLN=\angle MAC+\angle LCA=A-\alpha+\alpha=A$; similarly $\angle NML=B$, and $\angle LNM=C$. Thus the triangle LMN is equivalent to the original triangle, and therefore similar to it.

Again

$$\frac{BN}{BC}=\frac{\sin BCN}{\sin BNC}=\frac{\sin(C-\alpha)}{\sin(\pi-C)}=\frac{\sin(C-\alpha)}{\sin C};$$
$$\therefore BN=\frac{a\sin(C-\alpha)}{\sin C},$$

and

$$\frac{BM}{BA}=\frac{\sin BAM}{\sin BMA}=\frac{\sin\alpha}{\sin(\pi-B)}=\frac{\sin\alpha}{\sin B};$$
$$\therefore BM=\frac{c\sin\alpha}{\sin B}.$$

$$\begin{aligned}\therefore MN&=\frac{a\sin(C-\alpha)}{\sin C}-\frac{c\sin\alpha}{\sin B}\\&=a\cos\alpha-a\cot C\sin\alpha-\frac{a\sin C}{\sin A\sin B}\sin\alpha\\&=a\cos\alpha-a\cot C\sin\alpha-\frac{a\sin(A+B)}{\sin A\sin B}\sin\alpha\\&=a\cos\alpha-a\sin\alpha(\cot C+\cot B+\cot A).\end{aligned}$$

The ratio of this to a is the same as the ratio of

$$\cos\alpha-\sin\alpha(\cot A+\cot B+\cot C)\ \text{to unity.}$$

Show that in any triangle the relations given in the following Examples, from 23 to 42, hold:

Problem 23. $a(b\cos C-c\cos B)=b^2-c^2.$

Solution. $ab\cos C-ac\cos B=\dfrac{a^2+b^2-c^2}{2}-\dfrac{a^2+c^2-b^2}{2}=b^2-c^2.$

Problem 24. $a(\cos B\cos C+\cos A)=b(\cos A\cos C+\cos B)=c(\cos A\cos B+\cos C)$.

Solution.

$$a(\cos B\cos C+\cos A)=a\{\cos B\cos C-\cos(B+C)\}$$
$$=a\sin B\sin C=\frac{a}{\sin A}\sin A\sin B\sin C.$$

Similarly

$$b(\cos A\cos C+\cos B)=\frac{b}{\sin B}\sin A\sin B\sin C;\ \text{and}$$
$$c(\cos A\cos B+\cos C)=\frac{c}{\sin C}\sin A\sin B\sin C.$$

Thus the three expressions are equal by *Art.* 214 (page 418).

Problem 25. $(b+c-a)\tan\frac{A}{2}=(c+a-b)\tan\frac{B}{2}=(a+b-c)\tan\frac{C}{2}$.

Solution.

$$(b+c-a)\tan\frac{A}{2}=2(s-a)\tan\frac{A}{2}=2(s-a)\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$
$$=\frac{2\sqrt{(s-a)(s-b)(s-c)}}{\sqrt{s}}.$$

Similarly the other two proposed expressions reduce to the same symmetrical form.

Problem 26. $b\cos B+c\cos C=a\cos(B-C)$.

Solution.

$$b\cos B+c\cos C=\frac{a\sin B}{\sin A}\cos B+\frac{a\sin C}{\sin A}\cos C$$
$$=\frac{a}{2\sin A}(\sin 2B+\sin 2C)$$
$$=\frac{2a\sin(B+C)\cos(B-C)}{2\sin(B+C)}=a\cos(B-C).$$

Problem 27. $(a+b)\cos C+(b+c)\cos A+(c+a)\cos B=a+b+c$.

Solution. By *Art.* 216 (page 419),

$$c\cos B+b\cos C=a,\ a\cos C+c\cos A=b,\ b\cos A+a\cos B=c;$$

therefore by addition

$$c(\cos B+\cos A)+b(\cos A+\cos C)+a(\cos C+\cos B)=a+b+c.$$

Problem 28. $(a^2-b^2)\cot C+(b^2-c^2)\cot A+(c^2-a^2)\cot B=0$.

Solution. Let k stand for $\frac{a}{\sin A}$, $\frac{b}{\sin B}$, and $\frac{c}{\sin C}$ which we know are all equal. Then

$(a^2-b^2)\cot C+(b^2-c^2)\cot A+(c^2-a^2)\cot B$

$= k^2\left\{\left(\sin^2 A - \sin^2 B\right)\cot C + \left(\sin^2 B - \sin^2 C\right)\cot A + \left(\sin^2 C - \sin^2 A\right)\cot B\right\}$
$= k^2\{\sin(A+B)\sin(A-B)\cot C + \sin(B+C)\sin(B-C)\cot A$
$+ \sin(C+A)\sin(C-A)\cot B\}$
$= k^2\{\sin(A-B)\cos C + \sin(B-C)\cos A + \sin(C-A)\cos B\}$
$= -k^2\{\sin(A-B)\cos(A+B) + \sin(B-C)\cos(B+C) + \sin(C-A)\cos(C+A)\}$
$= -\frac{k^2}{2}\{\sin 2A - \sin 2B + \sin 2B - \sin 2C + \sin 2C - \sin 2A\}$
$= 0.$

Problem 29. $(a-b)\cot\frac{C}{2} + (c-a)\cot\frac{B}{2} + (b-c)\cot\frac{A}{2} = 0.$

Solution. Let k have the same meaning as in the preceding solution; then

$$(a-b)\cot\frac{C}{2} + (c-a)\cot\frac{B}{2} + (b-c)\cot\frac{A}{2}$$
$$= k\left\{(\sin A - \sin B)\cot\frac{C}{2} + (\sin C - \sin A)\cot\frac{B}{2} + (\sin B - \sin C)\cot\frac{A}{2}\right\}$$
$$= 2k\left\{\sin\frac{A-B}{2}\sin\frac{A+B}{2} + \sin\frac{C-A}{2}\sin\frac{C+A}{2} + \sin\frac{B-C}{2}\sin\frac{B+C}{2}\right\}$$
$$= 2k\left\{\sin^2\frac{A}{2} - \sin^2\frac{B}{2} + \sin^2\frac{C}{2} - \sin^2\frac{A}{2} + \sin^2\frac{B}{2} - \sin^2\frac{C}{2}\right\}$$
$$= 0.$$

Problem 30. $1 - \tan\frac{A}{2}\tan\frac{B}{2} = \frac{2c}{a+b+c}.$

Solution.

$$1 - \tan\frac{A}{2}\tan\frac{B}{2} = 1 - \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \times \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$
$$= 1 - \frac{s-c}{s} = 1 - \frac{a+b-c}{a+b+c} = \frac{2c}{a+b+c}.$$

Problem 31.

$$(a+b+c)(\cos A + \cos B + \cos C)$$
$$= 2a\cos^2\frac{A}{2} + 2b\cos^2\frac{B}{2} + 2c\cos^2\frac{C}{2}.$$

Solution.

$(a+b+c)(\cos A + \cos B + \cos C)$
$= a\cos A + b\cos B + c\cos C$
$+ a\cos B + b\cos A + a\cos C + c\cos A + b\cos C + c\cos B$
$= a\cos A + b\cos B + c\cos C + c + b + a$, by *Art.* 216 (page 419),
$= a(1+\cos A) + b(1+\cos B) + c(1+\cos C)$
$= 2a\cos^2\frac{A}{2} + 2b\cos^2\frac{B}{2} + 2c\cos^2\frac{C}{2}.$

Problem 32. $\dfrac{\sin^2 A}{a^2} = \dfrac{\cos A \cos B}{ab} + \dfrac{\cos A \cos C}{ac} + \dfrac{\cos B \cos C}{bc}$.

Solution. Let k have the same meaning as in the solution of **Problem 28**; then

$$\begin{aligned}
&\frac{\cos A \cos B}{ab} + \frac{\cos A \cos C}{ac} + \frac{\cos B \cos C}{bc} \\
&\qquad = \frac{1}{k^2}\left\{\frac{\cos A \cos B}{\sin A \sin B} + \frac{\cos A \cos C}{\sin A \sin C} + \frac{\cos B \cos C}{\sin B \sin C}\right\} \\
&\qquad = \frac{1}{k^2}\{\cot A \cot B + \cot A \cot C + \cot B \cot C\} \\
&\qquad = \frac{1}{k^2}\frac{\tan A + \tan B + \tan C}{\tan A \tan B \tan C} = \frac{1}{k^2}, \text{ by } \textit{Art.}\ 114 (\text{page } 409), \\
&\qquad = \frac{\sin^2 A}{a^2}.
\end{aligned}$$

Problem 33. $a \cos A + b \cos B + c \cos C = 2a \sin B \sin C$.

Solution.

$$\begin{aligned}
&a \cos A + b \cos B + c \cos C \\
&\qquad = a \cos A + \frac{a \sin B}{\sin A}\cos B + \frac{a \sin C}{\sin A}\cos C \\
&\qquad = a \cos A + \frac{a(\sin 2B + \sin 2C)}{2 \sin A} = a \cos A + \frac{2a \sin(B + C)\cos(B - C)}{2 \sin A} \\
&\qquad = a \cos A + a \cos(B - C) = -a \cos(B + C) + a \cos(B - C) \\
&\qquad = 2a \sin B \sin C.
\end{aligned}$$

Problem 34. $\cos A + \cos B + \cos C = 1 + \dfrac{2a \sin B \sin C}{a + b + c}$.

Solution.

$$\begin{aligned}
\frac{2a \sin B \sin C}{a + b + c} &= \frac{2 \sin B \sin C}{1 + \dfrac{b}{a} + \dfrac{c}{a}} = \frac{2 \sin B \sin C}{1 + \dfrac{\sin B}{\sin A} + \dfrac{\sin C}{\sin A}} \\
&= \frac{2 \sin A \sin B \sin C}{\sin A + \sin B + \sin C} \\
&= \frac{2 \sin A \sin B \sin C}{4 \cos \dfrac{A}{2} \cos \dfrac{B}{2} \cos \dfrac{C}{2}}, \text{ by Chapter VIII. Problem 16,} \\
&= 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \\
&= \cos A + \cos B + \cos C - 1, \text{ by } \textit{Art.}\ 114 \text{ (page 409)};
\end{aligned}$$

$$\therefore \cos A + \cos B + \cos C = 1 + \frac{2a \sin B \sin C}{a + b + c}.$$

Problem 35. $a^2 - 2ab \cos(60° + C) = c^2 - 2bc \cos(60° + A)$.

$$\begin{aligned}
a^2 - 2ab \cos(60° + C) &= a^2 - 2ab\,(\cos 60° \cos C - \sin 60° \sin C) \\
&= a^2 - ab \cos C + 2ab \sin 60° \sin C
\end{aligned}$$

$$\begin{aligned}
&= a^2 - \frac{a^2+b^2-c^2}{2} + 2cb\sin 60^\circ \sin A \\
&= c^2 - \frac{c^2+b^2-a^2}{2} + 2bc\sin 60^\circ \sin A \\
&= c^2 - bc\cos A + 2bc\sin 60^\circ \sin A \\
&= c^2 - 2bc\cos(60^\circ + A)\,.
\end{aligned}$$

Problem 36. $\cot\frac{A}{4} - \operatorname{cosec}\frac{A}{2} : \cot\frac{B}{2} + \cot\frac{C}{2} :: b+c-a : 2a.$

$$\begin{aligned}
\frac{b+c-a}{2a} &= \frac{\sin B + \sin C - \sin A}{2\sin A} \\
&= \frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2} - 2\sin\frac{A}{2}\cos\frac{A}{2}}{4\sin\frac{A}{2}\cos\frac{A}{2}} \\
&= \frac{\cos\frac{B-C}{2} - \sin\frac{A}{2}}{2\sin\frac{A}{2}} = \frac{\cos\frac{B-C}{2} - \cos\frac{B+C}{2}}{2\sin\frac{A}{2}} = \frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}}.
\end{aligned}$$

Again

$$\cot\frac{A}{4} - \operatorname{cosec}\frac{A}{2} = \frac{\cos\frac{A}{4}}{\sin\frac{A}{4}} - \frac{1}{\sin\frac{A}{2}} = \frac{2\cos^2\frac{A}{4} - 1}{\sin\frac{A}{2}} = \frac{\cos\frac{A}{2}}{\sin\frac{A}{2}};\ \text{and}$$

$$\cot\frac{B}{2} + \cot\frac{C}{2} = \frac{\cos\frac{B}{2}}{\sin\frac{B}{2}} + \frac{\cos\frac{C}{2}}{\sin\frac{C}{2}} = \frac{\sin\frac{B+C}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}};$$

$$\therefore \frac{\cot\frac{A}{4} - \operatorname{cosec}\frac{A}{2}}{\cot\frac{B}{2} + \cot\frac{C}{2}} = \frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}} = \frac{b+c-a}{2a}.$$

Problem 37.

$$\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2} = 4\sum\left(\Sigma - \cos\frac{A}{2}\right)\left(\Sigma - \cos\frac{B}{2}\right)\left(\Sigma - \cos\frac{C}{2}\right),$$

where

$$2\Sigma = \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}.$$

Solution.

$$4\sum\left(\Sigma - \cos\frac{A}{2}\right)\left(\Sigma - \cos\frac{B}{2}\right)\left(\Sigma - \cos\frac{C}{2}\right) = \text{ the product of}$$

$$\frac{1}{4}\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right)\left(\cos\frac{B}{2} + \cos\frac{C}{2} - \cos\frac{A}{2}\right)$$

$$\text{into}\ \left(\cos\frac{A}{2} + \cos\frac{C}{2} - \cos\frac{B}{2}\right)\left(\cos\frac{A}{2} + \cos\frac{B}{2} - \cos\frac{C}{2}\right).$$

Now substitute for the trinomial expressions results given by Chapter VIII. Problems

20 and 21; thus we obtain

$$\left\{8\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}\cos\frac{\pi+A}{4}\cos\frac{\pi+B}{4}\cos\frac{\pi+C}{4}\right\}^2,$$

that is $\left\{8\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}\sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4}\right\}^2,$

that is $\left\{\sin\frac{\pi-A}{2}\sin\frac{\pi-B}{2}\sin\frac{\pi-C}{2}\right\}^2,$

that is $\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}.$

Problem 38. The perimeter of any triangle is $2c\cos\frac{A}{2}\cos\frac{B}{2}\sec\frac{A+B}{2}$.

Solution. The perimeter $= a+b+c = \frac{c\sin A}{\sin C}+\frac{c\sin B}{\sin C}+c$

$$= \frac{c(\sin A+\sin B+\sin C)}{\sin C} = \frac{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\sin C}, \text{ by Example VIII. 16,}$$

$$= \frac{2\cos\frac{A}{2}\cos\frac{B}{2}}{\sin\frac{C}{2}} = \frac{2\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{A+B}{2}}$$

$$= 2\cos\frac{A}{2}\cos\frac{B}{2}\sec\frac{A+B}{2}.$$

Problem 39. If $y\sin^2 A+x\sin^2 B = z\sin^2 B+y\sin^2 C = x\sin^2 C+z\sin^2 A$, then $x:y:z :: \sin 2A : \sin 2B : \sin 2C$.

Solution. Let $h = y\sin^2 A+x\sin^2 B = z\sin^2 B+y\sin^2 C = x\sin^2 C+z\sin^2 A$.

$$\therefore h\left(\sin^2 C-\sin^2 A\right) = x\sin^2 B\sin^2 C - z\sin^2 A\sin^2 B, \text{ and}$$

$$h = x\sin^2 C+z\sin^2 A;$$

$$\therefore h\left(\sin^2 C-\sin^2 A\right)+h\sin^2 B = 2x\sin^2 B\sin^2 C,$$

$$\therefore h\sin(C-A)\sin(C+A)+h\sin^2 B = 2x\sin^2 B\sin^2 C;$$

$$\therefore h\sin(C-A)+h\sin(C+A) = 2x\sin B\sin^2 C;$$

$$\therefore x\sin B\sin^2 C = h\sin C\cos A.$$

$$\therefore x = \frac{h\cos A}{\sin B\sin C} = \frac{h\sin 2A}{2\sin A\sin B\sin C}.$$

Similarly

$$y = \frac{h\sin 2B}{2\sin A\sin B\sin C}, \text{ and}$$

$$z = \frac{h\sin 2C}{2\sin A\sin B\sin C}.$$

Problem 40. $8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ is less than 1, except when $A=B=C$.

Solution. Since $A+B+C=\pi$, we may show that $8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ has its greatest value when A, B and C are all equal.

$$\text{For } \sin\frac{A}{2}\sin\frac{B}{2}=\sin\left(\frac{A+B}{4}+\sin\frac{A-B}{4}\right)\sin\left(\frac{A+B}{4}-\sin\frac{A-B}{4}\right)$$
$$=\sin^2\frac{A+B}{4}-\sin^2\frac{A-B}{4};$$

thus whatever may be the value of C, it follows that $\sin\frac{A}{2}\sin\frac{B}{2}$ has its greatest value when $A=B$; for $\sin\frac{A+B}{4}$ does not change while A and B change in such a manner as to leave C unchanged. In this way we see that the greatest value of the expression is when all the angles are equal, and the value then is $8\sin^3\frac{\pi}{6}$, that is 1.

Problem 41.

$$a\sin(B-C)\cos(B+C-A)+b\sin(C-A)\cos(C+A-B)$$
$$+c\sin(A-B)\cos(A+B-C)=0.$$

Solution. Let k have the same meaning as in the solution of Example 28; then

$$\begin{aligned}a\sin(B-C)\cos(B+C-A)&=k\sin A\sin(B-C)\cos\left(180^\circ-2A\right)\\&=-k\sin(B+C)\sin(B-C)\cos 2A\\&=k\left(\sin^2 B-\sin^2 C\right)\left(2\sin^2 A-1\right)\\&=2k\sin^2 A\left(\sin^2 B-\sin^2 C\right)-k\left(\sin^2 B-\sin^2 C\right).\end{aligned}$$

Similarly the other two terms of the proposed expression may be transformed; and then the whole vanishes because

$$\sin^2 A\left(\sin^2 B-\sin^2 C\right)+\sin^2 B\left(\sin^2 C-\sin^2 A\right)+\sin^2 C\left(\sin^2 A-\sin^2 B\right)=0,$$
$$\text{and } \sin^2 B-\sin^2 C+\sin^2 C-\sin^2 A+\sin^2 A-\sin^2 B=0.$$

Problem 42.

$$\frac{\sin A}{\cos B}+\frac{\sin B}{\cos C}+\frac{\sin C}{\cos A}+\frac{\sin A}{\cos C}+\frac{\sin B}{\cos A}+\frac{\sin C}{\cos B}$$
$$=\sin A+\sin B+\sin C+(\cos A+\cos B+\cos C)\tan A\tan B\tan C.$$

Solution.

$$\begin{aligned}\frac{\sin A}{\cos B}+\frac{\sin B}{\cos A}&=\frac{\sin A\cos A+\sin B\cos B}{\cos A\cos B}=\frac{\sin 2A+\sin 2B}{2\cos A\cos B}\\&=\frac{2\sin(A+B)\cos(A-B)}{2\cos A\cos B}=\frac{\sin C}{\cos A\cos B}(\cos A\cos B+\sin A\sin B)\\&=\sin C+\cos C\tan A\tan B\tan C.\end{aligned}$$

Similarly

$$\frac{\sin B}{\cos C}+\frac{\sin C}{\cos B}=\sin A+\cos A\tan A\tan B\tan C,$$
$$\text{and } \frac{\sin C}{\cos A}+\frac{\sin A}{\cos C}=\sin B+\cos B\tan A\tan B\tan C.$$

Hence by addition we obtain the required result.

CHAPTER XIV
Solution of triangles

Problem 1. *Find the values of the angle A having given $\sin B = \cdot 25$, $a = 5$, $b = 2\cdot 5$.*

Solution. $\sin A = \dfrac{a}{b}\sin B = \dfrac{5}{2\cdot 5}\cdot 25 = \dfrac{1}{2}$; therefore $A = 30°$ or $150°$.

Problem 2. *One side of a triangle is half another and the included angle is 60° : find the other angles.*

Solution. Suppose $c = \dfrac{1}{2}b$, and $A = 60°$; then, by *Art.* 229 (page 421),

$$\tan\frac{1}{2}(B-C) = \frac{b-c}{b+c}\cot\frac{A}{2} = \frac{1-\frac{1}{2}}{1+\frac{1}{2}}\cot 30° = \frac{1}{3}\cdot\sqrt{3} = \frac{1}{\sqrt{3}};$$

therefore $$\frac{1}{2}(B-C) = 30°; \text{ and } \frac{1}{2}(B+C) = 60°.$$

Hence $$B = 90° \text{ and } C = 30°.$$

Problem 3. *The sides of a triangle are in the ratio of $2 : \sqrt{6} : 1+\sqrt{3}$: determine the angles.*

Solution. Let a, b, c denote these sides in order. Then

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{6+(1+\sqrt{3})^2-4}{2(1+\sqrt{3})\sqrt{6}} = \frac{6+2\sqrt{3}}{2(1+\sqrt{3})\sqrt{6}}$$
$$= \frac{\sqrt{3}(1+\sqrt{3})}{(1+\sqrt{3})\sqrt{6}} = \frac{1}{\sqrt{2}}; \text{ therefore } A = 45°.$$
$$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{4+(1+\sqrt{3})^2-6}{4(1+\sqrt{3})} = \frac{2+2\sqrt{3}}{4(1+\sqrt{3})} = \frac{1}{2};$$

therefore $B = 60°$.

$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{4+6-(1+\sqrt{3})^2}{4\sqrt{6}} = \frac{6-2\sqrt{3}}{4\sqrt{6}} = \frac{3-\sqrt{3}}{2\sqrt{6}}$$
$$= \frac{\sqrt{3}-1}{2\sqrt{2}}; \text{ therefore } C = 75°.$$

Problem 4. *If $A = 30°$, $b = 100$, $a = 40$, is there any ambiguity ?*

Solution. $\sin B = \dfrac{b}{a}\sin A = \dfrac{100}{40}\cdot\dfrac{1}{2} = \dfrac{5}{4}$; but this is impossible, for a sine cannot be greater than unity.

Problem 5. *Having given $A = 18°$, $a = 4$, $b = 4 + \sqrt{80}$, solve the triangle.*

Solution.

$$\sin B = \frac{b}{a}\sin A = \frac{4+\sqrt{(80)}}{4}\sin 18° = (1+\sqrt{5})\sin 18°$$

$$= \frac{(1+\sqrt{5})(\sqrt{5}-1)}{4} = 1;\ \text{therefore } B = 90°.$$

Thus $$C = 72°;\ \text{and } c^2 = b^2 - a^2 = \left\{4+\sqrt{(80)}\right\}^2 - 16$$

$$= 80 + 8\sqrt{(80)} = 16(5+2\sqrt{5});$$

therefore $$c = 4\sqrt{(5+2\sqrt{5})}.$$

Problem 6. *Having given $A = 15°$, $a = 4$, $b = 4 + \sqrt{48}$, solve the triangle.*

Solution. $$\sin B = \frac{b}{a}\sin A = \frac{4+\sqrt{48}}{4}\sin 15° = (1+\sqrt{3})\frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{1}{\sqrt{2}};$$

therefore $$B = 45° \text{ or } 135°.$$

$$\text{If } B = 45°,\ \text{then } C = 120°;\ \text{and } c = \frac{a\sin C}{\sin A} = 4\cdot\frac{2\sqrt{2}}{\sqrt{3}-1}\cdot\frac{\sqrt{3}}{2}$$

$$= \frac{4\sqrt{6}}{\sqrt{3}-1} = \frac{4\sqrt{6}(\sqrt{3}+1)}{2} = 2\sqrt{6}(\sqrt{3}+1).$$

$$\text{If } B = 135°,\ \text{then } C = 30°;\ \text{and } c = \frac{a\sin C}{\sin A} = 4\cdot\frac{2\sqrt{2}}{\sqrt{3}-1}\cdot\frac{1}{2}$$

$$= \frac{4\sqrt{2}}{\sqrt{3}-1} = \frac{4\sqrt{2}(\sqrt{3}+1)}{2} = 2\sqrt{2}(\sqrt{3}+1).$$

Problem 7. *If a, b, A be given, and a be less than b, and if c, c' be the two values found for the third side of the triangle, then*

$$c^2 - 2cc'\cos 2A + c'^2 = 4a^2\cos^2 A.$$

Solution. With the first diagram of *Art.* 234 (page 422) we may put $c = AB$ and $c' = AB'$; thus

$$c = b\cos A - a\cos CBB',\ \text{and } c' = b\cos A + a\cos CBB';$$

therefore $$c + c' = 2b\cos A,$$

and $$cc' = b^2\cos^2 A - a^2\cos^2 CBB' = b^2\cos^2 A - a^2\cos^2 B$$

$$= b^2(1-\sin^2 A) - a^2(1-\sin^2 B) = b^2 - a^2.$$

Hence $$(c+c')^2 = 4b^2\cos^2 A,$$

$$4cc'\cos^2 A = 4(b^2-a^2)\cos^2 A;$$

therefore $$c^2 + 2cc' + c'^2 - 4cc'\cos^2 A = 4a^2\cos^2 A,$$

that is $$c^2 - 2cc'\cos 2A + c'^2 = 4a^2\cos^2 A.$$

Problem 8. *Find the sum of the areas of the two triangles which satisfy the conditions of the problem in the ambiguous case. See Art. 247 (page 425).*

Solution. With the notation of the preceding solution the area of the smaller triangle is $\frac{c}{2}b\sin A$, and the area of the larger triangle is $\frac{c'}{2}b\sin A$; hence the sum of the areas $= \frac{1}{2}(c+c')b\sin A = b^2\sin A\cos A$.

Problem 9. *If B_1, C_1, and B_2, C_2 are the angles of the two triangles in the ambiguous case, then*

$$\frac{\sin C_1}{\sin B_1} + \frac{\sin C_2}{\sin B_2} = 2\cos A.$$

Solution. With the notation of the two preceding solutions we have

$$\frac{\sin C_1}{\sin B_1} = \frac{c}{b} \text{ and } \frac{\sin C_2}{\sin B_2} = \frac{c'}{b};$$

therefore
$$\frac{\sin C_1}{\sin B_1} + \frac{\sin C_2}{\sin B_2} = \frac{c+c'}{b} = \frac{2b\cos A}{b} = 2\cos A.$$

Problem 10. As in the solution of *Problem* 8, we have

$$\frac{1}{2}c'b\sin A = \frac{n}{2}cb\sin A;$$

therefore $c' = nc$.

And as in the solution of *Problem* 7,

$$\frac{c'+c}{c'-c} = \frac{2b\cos A}{2a\cos CBB'};$$

therefore
$$\frac{b}{a} = \frac{n+1}{n-1}\cdot\frac{\cos CBB'}{\cos A};$$

but the angle CBB' is greater than A, and therefore $\frac{\cos CBB'}{\cos A}$ is less than unity. Hence $\frac{b}{a}$ is less than $\frac{n+1}{n-1}$.

Problem 11. *If* $\log a + 10 = \log b + L\sin A$, *can the triangle be ambiguous ?*

Solution. $\sin B = \frac{b}{a}\sin A$; therefore $L\sin B - 10 = \log b + L\sin A - 10 - \log a$.

Thus if $\log a + 10 = \log b + L\sin A$ we have $L\sin B - 10 = 0$; therefore $L\sin B = 10$, therefore $\log\sin B = 0$, therefore $\sin B = 1$, therefore $B = 90°$, and the triangle is not ambiguous.

Problem 12. *If θ be an angle determined from the equation $\cos\theta = \frac{a-b}{c}$, prove that in any triangle*

$$\cos\frac{A-B}{2} = \frac{(a+b)\sin\theta}{2\sqrt{ab}}, \quad \cos\frac{A+B}{2} = \frac{c\sin\theta}{2\sqrt{ab}}.$$

Solution.

$$\frac{a+b}{c} = \frac{\sin A + \sin B}{\sin C} = \frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\sin\frac{C}{2}\cos\frac{C}{2}}$$

$$= \frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C};$$

therefore $$\cos\frac{1}{2}(A-B) = \frac{(a+b)\sin\frac{C}{2}}{c},$$

Now assume $\cos\theta = \dfrac{a-b}{c}$; therefore

$$\sin^2\theta = \frac{c^2-(a-b)^2}{c^2} = \frac{a^2+b^2-2ab\cos C-(a-b)^2}{c^2}$$

$$= \frac{2ab(1-\cos C)}{c^2} = \frac{4ab}{c^2}\sin^2\frac{C}{2};$$

therefore $$\sin\theta = \frac{2\sqrt{(ab)}}{c}\sin\frac{C}{2};$$

therefore $$\cos\frac{1}{2}(A-B) = \frac{(a+b)\sin\theta}{2\sqrt{(ab)}}.$$

And $$\sin\theta = \frac{2\sqrt{(ab)}}{c}\sin\frac{C}{2} = \frac{2\sqrt{(ab)}}{c}\cos\frac{1}{2}(A+B);$$

therefore $$\cos\frac{1}{2}(A+B) = \frac{c\sin\theta}{2\sqrt{(ab)}}.$$

Problem 13. *If* $\tan\phi = \dfrac{2\sqrt{ab}}{a-b}\sin\dfrac{C}{2}$, *then* $c = (a-b)\sec\phi$.

Solution.

$$c^2 = a^2+b^2-2ab\cos C = a^2+b^2-2ab\left(1-2\sin^2\frac{C}{2}\right)$$

$$= (a-b)^2 + 4ab\sin^2\frac{C}{2} = (a-b)^2 + (a-b)^2\tan^2\phi$$

$$= (a-b)^2\left\{1+\tan^2\phi\right\} = (a-b)^2\sec^2\phi.$$

Problem 14. *In a triangle ABC in which* $a = 18$, $b = 20$, $c = 22$, *find* $L\tan\dfrac{A}{2}$, *having given* $\log 2 = \cdot 3010300$, $\log 3 = \cdot 4771213$.

Solution. Here $s = 30$, $s-a = 12$, $s-b = 10$, $s-c = 8$.

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{10\times 8}{30\times 12}} = \sqrt{\frac{8}{36}} = \sqrt{\frac{2}{9}};$$

therefore $$L\tan\frac{A}{2} = 10 + \log\sqrt{\frac{2}{9}} = 10 + \frac{1}{2}\log 2 - \log 3 = 9\cdot 6733937.$$

Problem 15. *The sides of a triangle are 32, 40, 66 : find the greatest angle, having given*

$$\log 207 = 2{\cdot}3159703, \quad \log 1073 = 3{\cdot}0305997,$$
$$L\cot 66^\circ 18' = 9{\cdot}6424342, \quad \text{diff. for } 1' = {\cdot}0003431.$$

Solution. The greatest angle is opposite to the side 66; denote this angle by C. Then

$$\cot\frac{C}{2} = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}.$$

Here $\quad s = 69, \quad s-a = 37, \quad s-b = 29, \quad s-c = 3;$

therefore $$\cot\frac{C}{2} = \sqrt{\frac{69\times 3}{37\times 29}} = \sqrt{\frac{207}{1073}};$$

therefore $$L\cot\frac{C}{2} = 10 + \log\sqrt{\frac{207}{1073}}$$
$$= 10 + \frac{1}{2}(\log 207 - \log 1073) = 9{\cdot}6426853.$$

$$\begin{array}{r} 9{\cdot}6426853 \\ 9{\cdot}6424342 \\ \hline {\cdot}0002511 \end{array} \qquad {\cdot}0003431 : {\cdot}0002511 :: 60'' : x'';$$

this gives $\quad x = 44; \quad$ therefore $\dfrac{C}{2} = 66^\circ 18' - 44'' = 66^\circ 17' 16'';$

therefore $$C = 132^\circ 34' 32''.$$

Problem 16. *The sides of a triangle are 4, 5, 6 : find B, having given* $\log 2 = {\cdot}3010300$, $L\cos 27^\circ 53' = 9{\cdot}9464040$, *diff. for* $1' = {\cdot}0000669$.

Solution. Here $\quad s = \dfrac{15}{2}, \quad s-a = \dfrac{7}{2}, \quad s-b = \dfrac{5}{2}, \quad s-c = \dfrac{3}{2}.$

$$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}} = \sqrt{\frac{15\times 5}{8\times 12}} = \sqrt{\frac{25}{32}} = \sqrt{\frac{100}{2^7}};$$

therefore $$L\cos\frac{B}{2} = 10 + \log\sqrt{\frac{100}{2^7}} = 10 + \frac{1}{2}(\log 100 - \log 2^7)$$
$$= 10 + 1 - \frac{7}{2}\log 2 = 9{\cdot}9463950.$$

$$\begin{array}{r} 9{\cdot}9464040 \\ 9{\cdot}9463950 \\ \hline {\cdot}0000090 \end{array} \qquad {\cdot}0000669 : {\cdot}0000090 :: 60'' : x'';$$

this gives $x = 8$; therefore $\dfrac{B}{2} = 27^\circ 53' 8''$; therefore $B = 55^\circ 46' 16''$.

Problem 17. *Apply the formula* $\cos\dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$ *to find the greatest angle in a triangle whose sides are 5, 6, 7 feet respectively, having given*

$$\log 6 = {\cdot}7781513,$$
$$L\cos 39^\circ 14' = 9{\cdot}8890644, \quad \text{diff. for } 60'' = {\cdot}0001032.$$

Solution. Here $a = 7,\ s = 9,\ s - a = 2$; therefore

$$\cos\frac{A}{2} = \sqrt{\frac{9\times 2}{5\times 6}} = \sqrt{\frac{3}{5}} = \sqrt{\frac{6}{10}};$$

therefore $$L\cos\frac{A}{2} = 10 + \log\sqrt{\frac{6}{10}} = 10 + \frac{1}{2}(\log 6 - \log 10)$$
$$= 10 + \frac{1}{2}\log 6 - \frac{1}{2} = 9{\cdot}8890756.$$

$$\begin{array}{r} 9{\cdot}8890756 \\ 9{\cdot}8890644 \\ \hline {\cdot}0000112 \end{array} \qquad {\cdot}0001032 : {\cdot}0000112 :: 60'' : x'';$$

this gives $x = 6.5$; therefore $\frac{A}{2} = 39°14' - 6''{\cdot}5 = 39°13'53''{\cdot}5$; therefore $A = 78°27'47''$.

Problem 18. *Two sides of a triangle are 18 and 2 feet respectively, and the included angle is 55° : find the other angles, having given*

$$\log 2 = {\cdot}3010300, \quad L\cot 27°30' = 10{\cdot}2835233,$$
$$L\tan 56°56' = 10{\cdot}1863769, \quad \text{diff. for } 1' = {\cdot}0002763.$$

Solution. As in *Art*. 229 (page 421) we have

$$\tan\frac{1}{2}(B-C) = \frac{18-2}{18+2}\cot\frac{A}{2} = \frac{8}{10}\cot 27°30';$$

therefore $$L\tan\frac{1}{2}(B-C) = L\cot 27°30' + \log 8 - \log 10$$
$$= L\cot 27°30' + 3\log 2 - 1 = 10{\cdot}1866133.$$

$$\begin{array}{r} 10{\cdot}1866133 \\ 10{\cdot}1863769 \\ \hline {\cdot}0002364 \end{array} \qquad {\cdot}0002763 : {\cdot}0002364 :: 60'' : x'';$$

this gives $x = 51$; therefore $\frac{1}{2}(B-C) = 56°56'51''$.

And $\frac{1}{2}(B+C) = 62°30'$; therefore $B = 119°26'51'',\ C = 5°33'9''$.

Problem 19. *Two sides of a triangle are in the ratio of 9 to 7, and the included angle is 64°12' : find the other angles, having given*

$$\log 2 = {\cdot}3010300, \quad L\tan 57°54' = 10{\cdot}2025255,$$
$$L\tan 11°16' = 9{\cdot}2993216, \quad L\tan 11°17' = 9{\cdot}2999804.$$

Solution.

$$\tan\frac{1}{2}(B-C) = \frac{9-7}{9+7}\cot\frac{A}{2} = \frac{1}{8}\cot 32°6'';$$

therefore $$L\tan\frac{1}{2}(B-C) = L\cot 32°6' - \log 8$$
$$= L\tan 57°54' - 3\log 2 = 9.2994355.$$

$$\begin{array}{r} 9{\cdot}2999804 \\ 9{\cdot}2993216 \\ \hline {\cdot}0006588 \end{array} \qquad \begin{array}{r} 9{\cdot}2994355 \\ 9{\cdot}2993216 \\ \hline {\cdot}0001139 \end{array} \qquad {\cdot}0006588 : {\cdot}0001139 :: 60'' : x'';$$

this gives $x = 10$; therefore $\frac{1}{2}(B-C) = 11°16'10''$.

And $\frac{1}{2}(B+C) = 57°54'$; therefore $B = 69°10'10''$, $C = 46°37'50''$.

Problem 20. *If $a = 70$, $b = 35$, $C = 36°52'12''$, find the other angles, having given* $\log 3 = {\cdot}4771213$, $L\cot 18°26'6'' = 10{\cdot}4771213$.

Solution. $\tan\frac{1}{2}(A-B) = \frac{a-b}{a+b}\cot\frac{C}{2} = \frac{70-35}{70+35}\cot\frac{C}{2} = \frac{1}{3}\cot 18°26'6''$;

therefore $$L\tan\frac{1}{2}(A-B) = L\cot 18°26'6'' - \log 3 = 10;$$

therefore $$\log\tan\frac{1}{2}(A-B) = 0;$$

therefore $\tan\frac{1}{2}(A-B) = 1$; therefore $\frac{1}{2}(A-B) = 45°$.

And $\frac{1}{2}(A+B) = 71°33'54''$; therefore $A = 116°33'54''$, $B = 26°33'54''$.

Problem 21. *The ratio of two sides of a triangle is 9 to 7, and the included angle is $47°25'$: find the other angles, having given*

$$\log 2 = {\cdot}3010300, \quad L\tan 66°17'30'' = 10{\cdot}3573942,$$
$$L\tan 15°53' = 9{\cdot}4541479, \quad \text{diff. for } 1' = {\cdot}0004797.$$

Solution. $\tan\frac{1}{2}(B-C) = \frac{9-7}{9+7}\cot\frac{A}{2} = \frac{1}{8}\cot 23°42'30''$;

therefore $$L\tan\frac{1}{2}(B-C) = L\cot 23°42'30'' - \log 8$$
$$= L\tan 66°17'30'' - 3\log 2 = 9{\cdot}4543042.$$

$$\begin{array}{r} 9{\cdot}4543042 \\ 9{\cdot}4541479 \\ \hline {\cdot}0001563 \end{array} \qquad {\cdot}0004797 : {\cdot}0001563 :: 60'' : x'';$$

this gives $x = 20''$; therefore $\frac{1}{2}(B-C) = 15°53'20''$.

And $\frac{1}{2}(B+C) = 66°17'30''$; therefore $B = 82°10'50''$, $C = 50°24'10''$.

Problem 22. *In a triangle $a = 30$, $b = 20$, and the contained angle $= 22°$: find the other angles, having given*

$$L\cot 11° = 10{\cdot}7113477, \quad L\tan 45°48' = 10.0121294,$$
$$L\tan 45°49' = 10{\cdot}0123821, \quad \log 2 = {\cdot}3010300.$$

Solution. $\tan\frac{1}{2}(A-B) = \frac{a-b}{a+b}\cot\frac{C}{2} = \frac{30-20}{30+20}\cot\frac{C}{2} = \frac{2}{10}\cot 11°$;

therefore $$L\tan\frac{1}{2}(A-B) = L\cot 11° + \log 2 - \log 10$$

$$= L\cot 11^\circ + \log 2 - 1 = 10{\cdot}0123777.$$

$$\begin{array}{r} 10{\cdot}0123821 \\ 10{\cdot}0121294 \\ \hline {\cdot}0002527 \end{array} \qquad \begin{array}{r} 10{\cdot}0123777 \\ 10{\cdot}0121294 \\ \hline {\cdot}0002483 \end{array} \qquad {\cdot}0002527 : {\cdot}0002483 :: 60'' : x'';$$

this gives $x = 59$; therefore $\frac{1}{2}(A - B) = 45^\circ 48' 59''$.

And $\frac{1}{2}(A + B) = 79^\circ$; therefore $A = 124^\circ 48' 59''$, $B = 33^\circ 11' 1''$.

Problem 23. *Given $b = 14$, $c = 11$, $A = 60^\circ$, show that $B = 71^\circ 44' 29''$, having given*

$$L\tan 11^\circ 44' 29'' = 9.31774, \quad \log 2 = {\cdot}30103, \quad \log 3 = {\cdot}47712.$$

Solution.

$$\tan\frac{1}{2}(B - C) = \frac{b-c}{b+c}\cot\frac{A}{2} = \frac{3}{25}\cot 30^\circ = \frac{3\sqrt{3}}{25};$$

therefore $$L\tan\frac{1}{2}(B - C) = 10 + \log\frac{3\sqrt{3}}{25} = 10 + \frac{3}{2}\log 3 - \log 25$$

$$= 10 + \frac{3}{2}\log 3 - \log\frac{100}{4} = 10 + \frac{3}{2}\log 3 - 2 + 2\log 2 = 9{\cdot}31774;$$

therefore $$\frac{1}{2}(B - C) = 11^\circ 44' 29''.$$

And $\frac{1}{2}(B + C) = 60^\circ$; therefore $B = 71^\circ 44' 29''$.

Problem 24. *The sides of a triangle are 7, 8, 9 : determine all the angles, having given*

$$\log 2 = {\cdot}3010300,$$
$$L\tan 24^\circ 5' 40'' = 9{\cdot}6505069, \quad L\tan 24^\circ 5' 50'' = 9{\cdot}6505634,$$
$$L\tan 29^\circ 12' 20'' = 9{\cdot}7474183, \quad L\tan 29^\circ 12' 30'' = 9{\cdot}7474677.$$

Solution. Let $a = 7$, $b = 8$, $c = 9$; then $s = 12$, $s - a = 5$, $s - b = 4$, $s - c = 3$.

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{4 \times 3}{12 \times 5}} = \sqrt{\frac{1}{5}} = \sqrt{\frac{2}{10}}.$$

$$L\tan\frac{A}{2} = 10 + \log\sqrt{\frac{2}{10}} = 10 + \frac{1}{2}(\log 2 - \log 10)$$

$$= 10 + \frac{1}{2}(\log 2 - 1) = 9{\cdot}6505150.$$

$$\begin{array}{r} 9{\cdot}6505634 \\ 9{\cdot}6505069 \\ \hline {\cdot}0000565 \end{array} \qquad \begin{array}{r} 9{\cdot}6505150 \\ 9{\cdot}6505069 \\ \hline {\cdot}0000081 \end{array} \qquad {\cdot}0000565 : {\cdot}0000081 :: 10'' : x'';$$

this gives $x = 1.5$; therefore $\frac{A}{2} = 24^\circ 5' 41''{\cdot}5$; therefore $A = 48^\circ 11' 23''$.

$$\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{5 \times 3}{12 \times 4}} = \sqrt{\frac{5}{16}} = \sqrt{\frac{10}{32}}.$$

$$L\tan\frac{B}{2} = 10 + \log\sqrt{\frac{10}{32}} = 10 + \frac{1}{2}(\log 10 - \log 32)$$
$$= 10 + \frac{1}{2} - \frac{5}{2}\log 2 = 9{\cdot}7474250.$$

9·7474677	9·7474250	
9·7474183	9·7474183	·0000494 : ·0000067 :: 10″ : x'';
·0000494	·0000067	

this gives $x = 1{\cdot}5$; therefore $\frac{B}{2} = 29°12'21''{\cdot}5$; therefore $B = 58°24'43''$.

Hence $$C = 180° - 48°11'23'' - 58°24'43'' = 73°23'54''.$$

Problem 25. *In a right-angled triangle the hypotenuse $c = 6953$ and $b = 3$: find B, having given*

$$\log 3{\cdot}475 = {\cdot}5409548, \quad \log 6{\cdot}953 = {\cdot}8421722,$$
$$L\sin 44°59'15'' = 9{\cdot}8493902, \quad \text{diff. for } 1'' = {\cdot}0000021.$$

Solution. As in *Art.* 238 (page 423) we have

$$\sin\left(45° - \frac{B}{2}\right) = \sqrt{\frac{1-\sin B}{2}} = \sqrt{\frac{1}{2}\left(1 - \frac{3}{6953}\right)}$$
$$= \sqrt{\frac{1}{2}\times\frac{6950}{6953}} = \sqrt{\frac{3475}{6953}};$$

therefore $$L\sin\left(45° - \frac{B}{2}\right) = 10 + \log\sqrt{\frac{3475}{6953}}$$
$$= 10 + \frac{1}{2}\log(3475 - \log 6953)$$
$$= 10 - \frac{1}{2}({\cdot}3012174) = 9{\cdot}8493913.$$

9·8493913	
9·8493902	·0000021 : ·0000011 :: 1″ : x'';
·0000011	

this gives $x = {\cdot}5$; therefore $45° - \frac{B}{2} = 44°59'15''{\cdot}5$; therefore $\frac{B}{2} = 44''{\cdot}5$; therefore $B = 1'29''$.

Problem 26. *Two sides are 80 and 100 feet, and the included angle 60° : find the other angles, having given*

$$\log 3 = {\cdot}47712, \quad L\tan 10°53'36'' = 9{\cdot}28432.$$

Solution.

Let $$b = 100,\ c = 80;$$

$$\tan\frac{1}{2}(B-C) = \frac{b-c}{b+c}\cot\frac{A}{2} = \frac{1}{9}\cot 30° = \frac{\sqrt{3}}{9} = 3^{-\frac{3}{2}};$$

therefore $$L\tan\frac{1}{2}(B-C) = 10 + \log 3^{-\frac{3}{2}} = 10 - \frac{3}{2}\log 3 = 9{\cdot}28432;$$

therefore $$\frac{1}{2}(B-C) = 10°53'36''.$$

And $\frac{1}{2}(B+C) = 60°$; therefore $B = 70°53'36''$, $C = 49°6'24''$.

Problem 27. *Two sides of a triangle are 3 and 5 feet, and the included angle is 120°: find the other angles, having given*

$$\log 4.8 = \cdot 6812412,$$
$$L\tan 8°12' = 9{\cdot}1586706, \quad \text{diff. for } 60'' = \cdot 0008940.$$

Solution.

Let $$b = 5,\ c = 3;$$

$$\tan\frac{1}{2}(B-C) = \frac{b-c}{b+c}\cot\frac{A}{2} = \frac{1}{4}\cot 60° = \frac{1}{4\sqrt{3}} = \frac{1}{\sqrt{(48)}};$$

therefore $$L\tan\frac{1}{2}(B-C) = 10 + \log\frac{1}{\sqrt{48}} = 10 - \frac{1}{2}\log 48$$
$$= 10 - \frac{1}{2}(1.6812412) = 9{\cdot}1593794.$$

$$\begin{array}{r} 9{\cdot}1593794 \\ 9{\cdot}1586706 \\ \hline \cdot 0007088 \end{array} \qquad \cdot 0008940 : \cdot 0007088 :: 60'' : x'';$$

this gives $x = 48$; therefore $\frac{1}{2}(B-C) = 8°12'48''$.

And $\frac{1}{2}(B+C) = 30°$; therefore $B = 38°12'48''$ and $C = 21°47'12''$.

Problem 28. *A side of a base of a square pyramid is 200 feet and each edge is 150 feet: find the slope of each face, having given*

$$\log 2 = \cdot 30103, \quad L\tan 26°33' = 9{\cdot}69868, \quad L\tan 26°34' = 9{\cdot}69900.$$

Solution. Let $ABCD$ denote the square base, P the vertex. From P suppose a perpendicular PQ drawn to the ground, and from Q draw QR perpendicular to AB. Let ϕ denote the required inclination; then $\tan\phi = \frac{PQ}{QR}$.

Now $QR = 100$. Also $PQ^2 + QR^2 = PR^2$, and $PR^2 + AR^2 = AP^2$; thus $PQ^2 = PR^2 - QR^2 = AP^2 - AR^2 - QR^2 = (150)^2 - (100)^2 - (100)^2 = 2500$; therefore $PQ = 50$. Therefore $\tan\phi = \frac{50}{100} = \frac{1}{2}$.

Hence $L\tan\phi = 10 + \log\frac{1}{2} = 10 - \log 2 = 9{\cdot}69897$.

$$\begin{array}{r} 9{\cdot}69900 \\ 9{\cdot}69868 \\ \hline \cdot 00032 \end{array} \qquad \begin{array}{r} 9{\cdot}69897 \\ 9{\cdot}69868 \\ \hline \cdot 00029 \end{array} \qquad \cdot 00032 : \cdot 00029 :: 60'' : x'';$$

this gives $x = 54$; therefore $\phi = 26°33'54''$.

Problem 29. *Find the other angles, having given*

$$\frac{a}{b} = 1{\cdot}\dot{2},\ C = 60°,\ \log 3 = \cdot 4771213,$$

$$L\cot 9^\circ 49' = 10{\cdot}7618797, \quad \text{diff. for } 1' = {\cdot}0007514.$$

Solution. $\tan\dfrac{1}{2}(A-B) = \dfrac{a-b}{a+b}\cot\dfrac{C}{2} = \dfrac{1\frac{2}{9}-1}{1\frac{2}{9}+1}\cot 30^\circ = \dfrac{1}{10}\cot 30^\circ = \dfrac{\sqrt{3}}{10}$;

therefore $L\tan\dfrac{1}{2}(A-B) = 10 + \log\dfrac{\sqrt{3}}{10} = 10 + \dfrac{1}{2}\log 3 - 1 = 9{\cdot}2385606.$

Now $L\cot 9^\circ 49' = 10{\cdot}7618797$; and as $\tan\theta \times \cot\theta = 1$, we have

$$\log\tan\theta + \log\cot\theta = 0;$$

therefore
$$L\tan\theta - 10 + L\cot\theta - 10 = 0;$$

therefore
$$L\tan\theta = 20 - L\cot\theta.$$

Thus
$$L\tan 9^\circ 49' = 9{\cdot}2381203.$$

$$\begin{array}{r} 9{\cdot}2385606 \\ 9{\cdot}2381203 \\ \hline {\cdot}0004403 \end{array} \qquad {\cdot}0007514 : {\cdot}0004403 :: 60'' : x'';$$

this gives $x = 35$; therefore $\dfrac{1}{2}(A-B) = 9^\circ 49' 35''$.

And $\dfrac{1}{2}(A+B) = 60^\circ$; therefore $A = 69^\circ 49' 35''$, $B = 50^\circ 10' 25''$.

Problem 30. *If $a = 2$, $c = 3$, $L\sin A = 9{\cdot}5228787$, find C; $\log 3$ being ${\cdot}4771213$.*

Solution.

$$\sin C = \frac{c}{a}\sin A; \quad L\sin C = L\sin A + \log c - \log a$$
$$= L\sin A + \log 3 - \log 2$$
$$= 9{\cdot}5228787 + {\cdot}4771213 - \log 2 = 10 - \log 2;$$

therefore
$$\log\sin C = -\log 2 = \log\frac{1}{2};$$

therefore $\sin C = \dfrac{1}{2}$; therefore $C = 30^\circ$ or 150°.

Problem 31. *Show how to solve a triangle having given the base, the height, and the difference of the angles at the base; these angles being supposed both acute.*

Solution. Let c be the given base and let h denote the given height. With the left-hand diagram of *Art*. 214 (page 418) we have

$$\cot B = \frac{BD}{h} \text{ and } \cot C = \frac{CD}{h};$$

therefore
$$\cot B + \cot C = \frac{BD + CD}{h} = \frac{c}{h} \tag{18}$$

Also $B - C$ is supposed given, so that $\cot(B-C)$ is known; call it m : thus

$$\frac{\cot B - \cot C}{1 + \cot B \cot C} = m \tag{19}$$

From (18) and (19) we can find $\cot A$ and $\cot B$.

Problem 32. *Show how to solve a triangle having given the three perpendiculars from the angles on the opposite sides.*

Solution. Let $a,\ b,\ c$ denote the sides; and $l,\ m,\ n$ the perpendiculars on them respectively from the opposite angles. Then $al = bm = cn$; for each of these expressions denotes twice the area of the triangles. Hence the sides $a,\ b,\ c$ are respectively inversely proportional to $l,\ m,\ n$. Thus the *ratios* of the sides are known; and hence the angles of the triangle can be calculated by *Art.* 217 (page 419). Then the actual lengths of the sides can be found; for $l = c\sin B$, and l and B are known, so that c can be found; and then a and b can be deducted as the ratios of the sides are already known.

CHAPTER XV
Measurement of Heights and Distances

Problem 1. *A person standing on the bank of a river observes the angle subtended by a tree on the opposite bank to be 60°, and when he retires 40 feet from the river's bank he finds the angle to be 30° : determine the height of the tree and the breadth of the river.*

Solution. Take the diagram of *Art.* 240 (page 423). The angle $PBC = 60°$, the angle $PAC = 30°$; therefore the angle $APB = 30°$. Also $AB = 40$ feet.

Since the angle $PAB =$ the angle APB, we have $BP = AB = 40$. Then

$$PC = BP\sin 60° = 40\frac{\sqrt{3}}{2} = 20\sqrt{3};$$

and

$$BC = BP\cos 60° = 40 \cdot \frac{1}{2} = 20.$$

Problem 2. *From a station B at the base of a mountain its summit A is seen at an elevation of 60°; after walking one mile towards the summit up a plane making an angle of 30° with the horizon to another station C, the angle BCA is observed to be 135°. Find the height of the mountain in yards.*

Solution. Let AC produced through C meet the horizontal plane which contains B at D. Then the angle $ABD = 60°$, and the angle $CBD = 30°$; therefore the angle $ABC = 30°$. The angle $ACB = 135°$. Hence

$$\text{the angle } BAC = 180° - 30° - 135° = 15°.$$

$$\frac{AB}{BC} = \frac{\sin ACB}{\sin BAC} = \frac{\sin 135°}{\sin 15°} = \frac{1}{\sqrt{2}} \div \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{2}{\sqrt{3}-1};$$

therefore

$$AB = \frac{2 \times 1760}{\sqrt{3}-1} \text{ yards.}$$

$$\begin{aligned}\text{The height of the mountain} &= AB\sin 60° = AB\frac{\sqrt{3}}{2} \\ &= \frac{1760\sqrt{3}}{\sqrt{3}-1} = \frac{1760\sqrt{3}(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} \\ &= \frac{1760\sqrt{3}(\sqrt{3}+1)}{2} = 880(3+\sqrt{3}).\end{aligned}$$

Problem 3. *The altitude of a tower is observed to be 30° at the end of a horizontal base of 100 yards measured from its foot. Find the height of the tower.*

Solution. Let h denote the height of the tower in yards; then

$$\frac{h}{100} = \tan 30° = \frac{1}{\sqrt{3}}; \text{ therefore } h = \frac{100}{\sqrt{3}}.$$

Problem 4. *The angular elevation of a tower at a place A due south of it is 30°;*

and at a place B, due west of A, and at the distance a from it, the elevation is 18° : show that the height of the tower is

$$\frac{a}{\sqrt{2+2\sqrt{5}}}.$$

Solution. Let h denote the height of the tower, x the distance of the foot from A, and y the distance of the foot from B. Then

$$x = h\cot 30°,\ \text{and}\ y = h\cot 18°.$$

But $\quad y^2 - x^2 = a^2;$ therefore $h^2\left(\cot^2 18° - \cot^2 30°\right) = a^2;$

therefore $$h^2\left\{\frac{10+2\sqrt{5}}{(\sqrt{5}-1)^2} - 3\right\} = a^2;$$

therefore $$h^2\left\{\frac{5+\sqrt{5}}{3-\sqrt{5}} - 3\right\} = a^2;$$

therefore $$4h^2(\sqrt{5}-1) = a^2(3-\sqrt{5});$$

therefore $$h^2 = \frac{3-\sqrt{5}}{4(\sqrt{5}-1)}a^2 = \frac{(3-\sqrt{5})(3+\sqrt{5})}{4(\sqrt{5}-1)(3+\sqrt{5})}$$
$$= \frac{4a^2}{4(2+2\sqrt{5})} = \frac{a^2}{2+2\sqrt{5}}.$$

Problem 5. *A spherical balloon whose radius is r feet subtends at an observer's eye an angle α, when the angular elevation of its centre is β : determine the height of the centre of the balloon.*

Solution. Let A denote the eye of the spectator, and B the centre of the balloon. The angle α is formed by straight lines drawn from A in the vertical plane which contains B, so as to touch the balloon. Hence

$$\frac{r}{AB} = \sin\frac{\alpha}{2};\ \text{therefore}\ AB = r\operatorname{cosec}\frac{\alpha}{2}.$$

And the height of the centre of the balloon $= AB\sin\beta = r\sin\beta\operatorname{cosec}\frac{\alpha}{2}$.

Problem 6. *A person wishing to ascertain the distances between three inaccessible objects A, B, C, places himself in a straight line with A and B; he then measures the distances along which he must walk in a direction at right angles to AB until A, C and B, C respectively are in a straight line with him, and also observes in those positions their angular bearings : show how he can find the distances between A, B, C.*

Solution. Let O denote the station which is in the same straight line as A and B; let P be the station which is in the same straight line as A and C; and let Q be the station which is in the same straight line as B and C. Then O, P and Q are in a straight line which is at right angles to AB. Let $OP = p$, $OQ = q$; let $APO = \alpha$, and $BQO = \beta$. Then $OA = p\tan\alpha$, and $OB = q\tan\beta$. Thus $AB = q\tan\beta - p\tan\alpha$. And the angles of the triangle ABC are known; for $ABQ = \frac{\pi}{2} - \beta$, and $OAP = \frac{\pi}{2} - \alpha$. Hence AC and BC can be found.

Problem 7. *Two posts AB and CD are placed at the edge of a river at a distance $AC = AB$, the height of CD being such that AB and CD subtend equal angles at E, a point on the other bank exactly opposite to A : show that the square of the breadth of the river is equal to $\dfrac{AB^4}{CD^2 - AB^2}$, and that AD and BC subtend equal angles at E.*

Solution. The tangent of the angle which AB subtends E is $\dfrac{AB}{AE}$; and the tangent of the angle which CD subtends at E is $\dfrac{CD}{CE}$; therefore $\dfrac{AB}{AE} = \dfrac{CD}{CE}$;

therefore $$CE = \frac{AE \cdot CD}{AB}; \quad \text{therefore } CE^2 = \frac{AE^2 \cdot CD^2}{AB^2};$$

therefore $$CA^2 + AE^2 = \frac{AE^2 \cdot CD^2}{AB^2} :$$

but $$CA^2 = AB^2; \quad \text{therefore } AE^2 = \frac{AB^4}{CD^2 - AB^2}.$$

Again $$\cos DEA = \frac{EA}{ED}; \text{ and } \cos BEC = \frac{EB^2 + EC^2 - BC^2}{2EB \cdot EC}$$
$$= \frac{EA^2 + AB^2 + EA^2 + AC^2 - (AB^2 + AC^2)}{2EB \cdot EC} = \frac{EA^2}{EB \cdot EC}.$$

But by hypothesis the cosine of BEA is equal to the cosine of DEC, that is $\dfrac{EA}{EB} = \dfrac{EC}{ED}$; therefore $EA \cdot ED = EB \cdot EC$; therefore $\dfrac{EA^2}{EB \cdot EC} = \dfrac{EA}{ED}$.

Problem 8. *A flag-staff a feet high stands on the top of a tower b feet high. Find at what point on a horizontal plane passing through the base of the tower an observer must place himself so that the tower and the flag-staff may subtend equal angles, the height of the eye being h.*

Solution. Let A be the top of the flag staff, B the top of the tower, C the foot of the tower, E the eye. From E draw a perpendicular ED on the horizontal plane which contains C. Then the angle BEC is to equal to the angle BEA.

Now $$\frac{\sin BEC}{\sin EBC} = \frac{BC}{EC}, \text{ and } \frac{\sin BEA}{\sin EBA} = \frac{AB}{AE};$$

therefore $$\frac{BC}{EC} = \frac{AB}{AE}.$$

This coincides with *Euclid* VI. 3.

Let $CD = x$; then $EC = \sqrt{(h^2 + x^2)}$, $\quad EA = \sqrt{(a + b - h)^2 + x^2}$;

therefore $$\frac{b}{\sqrt{h^2 + x^2}} = \frac{a}{\sqrt{(a + b - h)^2 + x^2}};$$

therefore $$\{(a + b - h)^2 + x^2\}b^2 = (h^2 + x^2)a^2;$$

therefore $$x^2 = \frac{b^2(a + b - h)^2 - h^2a^2}{a^2 - b^2};$$

therefore $$EC^2 = \frac{h^2(a^2 - b^2) + b^2(a + b - h)^2 - h^2a^2}{a^2 - b^2}$$
$$= \frac{b^2\{(a + b - h)^2 - h^2\}}{a^2 - b^2} = \frac{b^2(a + b)(a + b - 2h)}{a^2 - b^2} = \frac{b^2(a + b - 2h)}{a - b};$$

therefore $$EC = b\left(\sqrt{\frac{a+b-2h}{a-b}}\right).$$

Problem 9. *A tower situated on a horizontal plane leans towards the North; at two points due South and distant a, b, respectively from the base, the angular altitudes of the tower are α and β. Show that if θ be the inclination of the tower, and h the perpendicular height,*
$$\tan\theta = \frac{b-a}{b\cot\alpha - a\cot\beta}, \quad h = \frac{b-a}{\cot\beta - \cot\alpha}.$$

Solution. Let P denote the top of the tower; from P draw PQ perpendicular to the ground; then $PQ = h$. Let x denote the distance of Q from the base of the tower; $x + a$ is the distance of Q from one point of observation, and $x + b$ is the distance of Q from the other point of observation.

Thus $$\cot\theta = \frac{x}{h}, \quad \cot\alpha = \frac{x+a}{h}, \quad \cot\beta = \frac{x+b}{h};$$

therefore $$h\cot\alpha = x + a, \quad h\cot\beta = x + b;$$

therefore $$h = \frac{b-a}{\cot\beta - \cot\alpha};$$

and $$x = h\cot\alpha - a = \frac{(b-a)\cot\alpha}{\cot\beta - \cot\alpha} - a = \frac{b\cot\alpha - a\cot\beta}{\cot\beta - \cot\alpha}.$$

Thus $$\tan\theta = \frac{h}{x} = \frac{b-a}{b\cot\alpha - a\cot\beta}.$$

Problem 10. *An object a feet high placed on the top of a tower subtends an angle γ at a place whose horizontal distance from the foot of the tower is b feet : determine the height of the tower.*

Solution. Let x denote the required height; and suppose θ the angle which the tower subtends: then
$$x = b\tan\theta, \quad x + a = b\tan(\theta + \gamma);$$

therefore $$x + a = \frac{b(\tan\theta + \tan\gamma)}{1 - \tan\theta\tan\gamma} = \frac{x + b\tan\gamma}{1 - \dfrac{x\tan\gamma}{b}}$$

thus we have a quadratic equation for finding x.

Problem 11. *On the bank of a river there is a column 200 feet high supporting a statue 30 feet high; the statue to an observer on the opposite bank subtends an equal angle with a man 6 feet high standing at the base of the column : required the breadth of the river.*

Solution. Let x denote the breadth of the river in feet; let α denote the angle subtended by the column, and β the angle subtended by the column and statue.

Thus $$\tan\alpha = \frac{200}{x}, \text{ and } \tan\beta = \frac{230}{x};$$

therefore $$\tan(\beta - \alpha) = \frac{\dfrac{230}{x} - \dfrac{200}{x}}{1 + \dfrac{200 \times 230}{x^2}} = \frac{30x}{x^2 + 46000}.$$

But, by hypotheses, $\tan(\beta - \alpha) = \dfrac{6}{x}$; therefore

$$\frac{6}{x} = \frac{30x}{x^2 + 46000}; \text{ therefore } x^2 + 46000 = 5x^2;$$

$$\text{therefore } x^2 = 11500; \text{ therefore } x = 10\sqrt{115}.$$

Problem 12. *The height of a house subtends a right angle at an opposite window, the top being 60° above a horizontal straight line : find the height of the house, taking the breadth of the street to be 30 feet.*

Solution. The part of the house above the horizontal straight line subtends an angle of 60°, and thus the height of the top of the house above the window is $30 \tan 60°$ feet. The part of the house below the horizontal straight line subtends an angle of 30°, and thus the depth of the foot of the house below the window is $30 \tan 30°$ feet. Hence the distance from the foot of the house to the top of the house in feet

$$= 30(\tan 60° + \tan 30°) = 30\left(\sqrt{3} + \frac{1}{\sqrt{3}}\right) = \frac{4}{\sqrt{3}}30 = 40\sqrt{3}.$$

Problem 13. *Two chimneys are of equal height. A person standing between them in the straight line joining their bases observes the elevation of the nearer one to him to be 60°. After walking 80 feet in a direction at right angles to the straight line joining their bases he observes the elevations of the two to be respectively 45° and 30°. Find their height and the distance between them.*

Solution. Let x denote the height of each chimney in feet, and y the distance between them. The distance of the first point of observation from the nearer chimney is $x \cot 60°$, and therefore the distance of the second point of observation is $\sqrt{(80)^2 + x^2 \cot^2 60°}$. Thus

$$\frac{x}{\sqrt{(80)^2 + x^2 \cot^2 60°}} = \tan 45° = 1;$$

$$\therefore\ x^2 = (80)^2 + x^2 \cot^2 60° = (80)^2 + \frac{x^2}{3}; \text{ therefore } 2x^2 = 3(80)^2;$$

$$\therefore\ x^2 = 6(40)^2; \text{ therefore } x = 40\sqrt{6}.$$

The distance of the first point of observation from the further chimney is $y - x \cot 60°$, and therefore the distance of the second point of observation is $\sqrt{(80)^2 + (y - x \cot 60°)^2}$. Thus

$$\frac{x}{\sqrt{(80)^2 + (y - x \cot 60°)^2}} = \tan 30° = \frac{1}{\sqrt{3}};$$

$$\therefore\ 3x^2 = (80)^2 + (y - x \cot 60°)^2; \text{ therefore } 14(40)^2 = (y - x \cot 60°)^2;$$

therefore $$y = x \cot 60° + 40\sqrt{14} = 40(\sqrt{2} + \sqrt{14}).$$

Problem 14. *An object is observed at three points A, B, C lying in a horizontal straight line which passes directly underneath the object ; the angular elevation at B*

is twice that at A, and at C is three times that at A; $AB = a$, $BC = b$: show that the height of the object is

$$\frac{a}{2b}\sqrt{\{(a+b)(3b-a)\}}.$$

If the tangent of the angle of elevation at A be $\frac{1}{3}$, show that $5a = 13b$.

Solution. Let P be the object, PQ the perpendicular from P on the horizontal plane which contains A, B, and C.

Let $PQ = x$, $CQ = y$. Suppose θ the angle PAQ, then $PBQ = 2\theta$, and $PCQ = 3\theta$. Thus

$$\tan\theta = \frac{x}{y+a+b}, \qquad \tan 2\theta = \frac{x}{y+b}, \qquad \tan 3\theta = \frac{x}{y};$$

therefore $$y+a+b = x\cot\theta, \quad y+b = x\cot 2\theta, \quad y = x\cot 3\theta;$$

therefore $$a = x(\cot\theta - \cot 2\theta), \quad b = x(\cot 2\theta - \cot 3\theta);$$

therefore $$a = x\left(\frac{\cos\theta}{\sin\theta} - \frac{\cos 2\theta}{\sin 2\theta}\right) = \frac{x\sin(2\theta-\theta)}{\sin\theta\sin 2\theta} = \frac{x}{\sin 2\theta},$$

and $$b = x\left(\frac{\cos 2\theta}{\sin 2\theta} - \frac{\cos 3\theta}{\sin 3\theta}\right) = \frac{x\sin(3\theta-2\theta)}{\sin 2\theta\sin 3\theta} = \frac{x\sin\theta}{\sin 2\theta\sin 3\theta}$$

$$= \frac{x}{\sin 2\theta(3-4\sin^2\theta)}.$$

Thus $$\sin 2\theta = \frac{x}{a}, \text{ and } 3 - 4\sin^2\theta = \frac{x}{b\sin 2\theta} = \frac{a}{b};$$

therefore $$3 - 2(1-\cos 2\theta) = \frac{a}{b}; \text{ therefore } \cos 2\theta = \frac{1}{2}\left(\frac{a}{b} - 1\right).$$

Hence $$\frac{x^2}{a^2} + \frac{1}{4}\left(\frac{a}{b}-1\right)^2 = 1;$$

therefore $$\frac{x^2}{a^2} = 1 - \frac{1}{4}\left(\frac{a}{b}-1\right)^2 = \frac{4b^2-(a-b)^2}{4b^2}$$

$$= \frac{3b^2+2ab-a^2}{4b^2} = \frac{(3b-a)(a+b)}{4b^2};$$

therefore $$x = \frac{a}{2b}\sqrt{(a+b)(3b-a)}.$$

If $\tan\theta = \frac{1}{3}$, then $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta} = \dfrac{\frac{2}{3}}{1+\frac{1}{9}} = \dfrac{3}{5}$, and $\sin 2\theta = \dfrac{x}{a}$;

thus $$\frac{3}{5} = \frac{\sqrt{(a+b)(3b-a)}}{2b};$$

therefore $$36b^2 = 25(a+b)(3b-a) = 25(3b^2+2ab-a^2);$$

therefore $$39b^2 + 50ab - 25a^2 = 0;$$

therefore $$(13b-5a)(3b+5a) = 0; \text{ therefore } 13b - 5a = 0.$$

Problem 15. *A vertical tower whose base is in the same horizontal plane with the observer, is observed from a station A to bear directly North and to subtend an angle of 15°; the observer then walks 100 yards so that the tower always subtends the same angle, and then it bears North-East : find its height and distance from A.*

Solution. Let x denote the height of the tower in yards; then the distance from A

to the foot of the tower is $x\cot 15°$. The observer moves so that the tower always subtends the same angle, hence he must describe the arc of a circle having its centre at the foot of the tower; and as the bearing of the tower changes from north to north-east he must describe one-eighth part of the circumference; therefore

$$\frac{2\pi x\cot 15°}{8}=100;\ \text{therefore } x=\frac{400\tan 15°}{\pi}.$$

Problem 16. *A person walking along a straight road observes that the greatest angle which two objects subtend is α; from the spot where this is the case he walks a distance c, and the objects now appear as one, their direction making an angle β with the road. Show that the distance between the objects is*

$$\frac{2c\sin\alpha\sin\beta}{\cos\alpha+\cos\beta}.$$

Solution. Let A denote the object which is further from the road, B that which is nearer to the road, C the point where AB subtends the greatest angle, D the second point of observation.

It is known that the point C is such that a circle described round A, B, and C will touch CD at C. Therefore the angle BCD is equal to the angle BAC; denote it by θ. Then the angle $ABC=\theta+\beta$, and also $=\pi-\theta-\alpha$; therefore $2\theta=\pi-\alpha-\beta$.

$$\text{Now } \frac{BC}{CD}=\frac{\sin\beta}{\sin(\theta+\beta)},\ \text{therefore } BC=\frac{c\sin\beta}{\sin(\theta+\beta)},\ \text{and } \frac{AB}{BC}=\frac{\sin\alpha}{\sin\theta};$$

therefore

$$AB=\frac{c\sin\alpha\sin\beta}{\sin\theta\sin(\theta+\beta)}=\frac{2c\sin\alpha\sin\beta}{\cos\beta-\cos(2\theta+\beta)}$$
$$=\frac{2c\sin\alpha\sin\beta}{\cos\beta-\cos(\pi-\alpha)}=\frac{2c\sin\alpha\sin\beta}{\cos\beta+\cos\alpha}.$$

Problem 17. *A fortress was observed by a ship at sea to bear East-north-east, and after sailing 4 miles to the East it was observed to bear North-north-east : show that the distance of the ship from the fortress at the first and second observation was $\sqrt{\left(16+8\sqrt{2}\right)}$ and $\sqrt{\left(16-8\sqrt{2}\right)}$ miles respectively.*

Solution. Let A denote the fortress, B the first position of the ship, C the second; produce BC through C to any point E. Then the angle $ABC=22\frac{1}{2}°$, and the angle $ACE=67\frac{1}{2}°$; therefore the angle $BAC=45°$.

$$\frac{AB}{BC}=\frac{\sin ACB}{\sin BAC}=\frac{\sin\left(180°-67\frac{1}{2}°\right)}{45°}$$
$$=\frac{\sqrt{(2+\sqrt{2})}}{2}\div\frac{1}{\sqrt{2}}=\frac{\sqrt{(2+\sqrt{2})}}{\sqrt{2}}=\sqrt{\frac{2+\sqrt{2}}{2}};$$

therefore

$$AB=4\sqrt{\frac{2+\sqrt{2}}{2}}=\sqrt{16+8\sqrt{2}}.$$

And

$$\frac{AC}{BC}=\frac{\sin ABC}{\sin BAC}=\frac{\sin 22\frac{1}{2}°}{\sin 45°}=\frac{\sqrt{(2-\sqrt{2})}}{2}\div\frac{1}{\sqrt{2}}$$
$$=\sqrt{\frac{2-\sqrt{2}}{2}};$$

therefore $AC = 4\sqrt{\dfrac{2-\sqrt{2}}{2}} = \sqrt{(16-8\sqrt{2})}$. See *Chapter* VIII : *Problem* 18.

Problem 18*. A ship sailing towards the North observes two light-houses in a line due West; and after an hour's sailing the bearings of the light-houses are observed to be South-west and South-south-west. The distance between the light-houses being 8 miles, find the rate at which the ship is sailing.*

Solution. Let P be the first position of the ship, A the nearer lighthouse, and B the further lighthouse; let Q be the second position of the ship. Then the angle $BQP = 45°$, and the angle $AQP = 22\frac{1}{2}^\circ$; therefore the angle $QAP = 67\frac{1}{2}^\circ$.

$$\frac{BQ}{BA} = \frac{\sin BAQ}{\sin BQA} = \frac{\sin\left(180° - 67\frac{1}{2}^\circ\right)}{\sin 22\frac{1}{2}^\circ} = \frac{\sin 67\frac{1}{2}^\circ}{\sin 22\frac{1}{2}^\circ} = \frac{\cos 22\frac{1}{2}^\circ}{\sin 22\frac{1}{2}^\circ} = \cot 22\frac{1}{2}^\circ$$

$= \sqrt{2}+1$, by *Chapter* VIII : *Problem* 18; therefore $BQ = 8(\sqrt{2}+1)$.

And
$$PQ = BQ\sin 45° = \frac{8(\sqrt{2}+1)}{\sqrt{2}} = 8+4\sqrt{2}.$$

Problem 19*. From the top of the mast of a ship 64 feet above the level of the sea the light of a distant lighthouse is just seen in the horizon; and after the ship has sailed directly towards the light for 30 minutes it is seen from the deck of the ship, which is 16 feet above the sea. Find the rate at which the ship is sailing, considering the earth as a sphere of 4000 miles radius.*

Solution. Let A denote the top of the lighthouse, P the top of the mast at the first observation, C the centre of the earth. Draw a straight line from P to A and let it touch the earth at B.

Let r denote the radius of the earth in feet; then
$PB = \sqrt{PC^2 - BC^2} = \sqrt{(r+64)^2 - r^2} = \sqrt{2r \times 64 + (64)^2} = \sqrt{2r \times 64}$ very nearly, for r is very large compared with $(64)^2$.

In precisely the same manner if Q denote the deck of the ship at the second observation, $QB = \sqrt{2r \times 16}$.

Now, since PCB is a very small angle, we may, by the principle that $\tan\theta$ is nearly equal to θ when θ is very small, consider the straight line PB to be equal to the arc which measures the distance of the ship from B at the first observation; and similarly we may consider QB to be equal to the arc which measures the distance of the ship from B at the second observation. Thus between the two observations the ship has sailed over $\sqrt{2r \times 64} - \sqrt{2r \times 16}$, that is, $4\sqrt{2r}$; that is, in half-an-hour it has sailed over $4\sqrt{8000 \times 5280}$ feet, so that the rate is $8\sqrt{8000 \times 5280}$ feet per hour, that is, $\dfrac{8\sqrt{8000 \times 5280}}{5280}$ miles per hour, that is $8\sqrt{\dfrac{800}{528}}$ miles per hour, that is, $8\sqrt{\dfrac{50}{33}}$ miles per hour; this is very nearly $8\sqrt{\dfrac{3}{2}}$ miles per hour.

Problem 20*. A man ascends a mountain by a path which is the shortest distance between the base and the vertex. The inclination of the path to the horizon at first is α, but afterwards suddenly increases to β, and then continues the same. On* *reaching the vertex he finds by the barometer he has ascended n feet in altitude, and observes the angle of depression γ of the point from which he started. Show that the distance he traveled in the ascent is*

$$\frac{n\cos\left(\dfrac{\alpha+\beta}{2}-\gamma\right)}{\cos\dfrac{\beta-\alpha}{2}\sin\gamma}.$$

Solution. Let A denote the summit of the mountain, B the base, BC the first part of the path, CA the second part. From A draw AE perpendicular to the horizontal plane which contains B; then $AE = n$.

The following are the angles:

$$BAE = \frac{\pi}{2} - \gamma, \qquad CBE = \alpha, \qquad CAE = \frac{\pi}{2} - \beta;$$

therefore $\quad BAC = \beta - \gamma, \qquad ABC = \gamma - \alpha, \qquad ACB = \pi + \alpha - \beta.$

$$AB = \frac{AE}{\sin\gamma} = \frac{n}{\sin\gamma},$$

$$\frac{BC}{AB} = \frac{\sin BAC}{\sin ACB} = \frac{\sin(\beta-\gamma)}{\sin(\beta-\alpha)},$$

$$\frac{AC}{AB} = \frac{\sin ABC}{\sin ACB} = \frac{\sin(\gamma-\alpha)}{\sin(\beta-\alpha)};$$

$$\therefore \quad \frac{BC+AC}{AB} = \frac{\sin(\beta-\gamma)+\sin(\gamma-\alpha)}{\sin(\beta-\alpha)}$$

$$= \frac{2\sin\dfrac{\beta-\alpha}{2}\cos\left(\dfrac{\alpha+\beta}{2}-\gamma\right)}{\sin(\beta-\alpha)} = \frac{\cos\left(\dfrac{\alpha+\beta}{2}-\gamma\right)}{\cos\dfrac{\beta-\alpha}{2}};$$

therefore

$$BC + AC = \frac{n}{\sin\gamma}\cdot\frac{\cos\left(\dfrac{\alpha+\beta}{2}-\gamma\right)}{\cos\dfrac{\beta-\alpha}{2}}.$$

Problem 21. *If from two points in a horizontal plane an object be seen at angles of elevation α, β, and if from a third point between the two points and in the straight line joining them and at distances a, b from them respectively the object be seen at an angle of elevation γ, show that the height of the object above the horizontal plane is*

$$\frac{\sin\alpha\sin\beta\sin\gamma\,\{ab(a+b)\}^{\frac{1}{2}}}{\left\{a\sin^2\alpha\left(\sin^2\gamma-\sin^2\beta\right)+b\sin^2\beta\left(\sin^2\gamma-\sin^2\alpha\right)\right\}^{\frac{1}{2}}}.$$

Solution. Let O denote the foot of the object; and let A, B, and C denote the three points of observation. Let x denote the height of the object; then $OA = x\cot\alpha$, $OB = x\cot\beta$, and $OC = x\cot\gamma$.

From the triangle AOC we have

$$x^2\cot^2\alpha = x^2\cot^2\gamma + a^2 - 2ax\cot\gamma\cos ACO,$$

and from the triangle BOC we have

$$x^2\cot^2\beta = x^2\cot^2\gamma + b^2 - 2bx\cot\gamma\cos BCO.$$

Multiply the first equation by b and the second by a, and add; thus

$$x^2\left(b\cot^2\alpha + a\cot^2\beta\right) = ab(a+b) + x^2(a+b)\cot^2\gamma;$$

$$\therefore\ x^2 = \frac{ab(a+b)\sin^2\alpha\sin^2\beta\sin^2\gamma}{a\left(\cos^2\beta\sin^2\gamma - \cos^2\gamma\sin^2\beta\right)\sin^2\alpha + b\left(\cos^2\alpha\sin^2\gamma - \cos^2\gamma\sin^2\alpha\right)\sin^2\beta}$$

$$= \frac{ab(a+b)\sin^2\alpha\sin^2\beta\sin^2\gamma}{a\left(\sin^2\gamma - \sin^2\beta\right)\sin^2\alpha + b\left(\sin^2\gamma - \sin^2\alpha\right)\sin^2\beta}.$$

Problem 22*. A person walking along a straight road observes the angles of elevation α, α' of the summits of two hills in ront of him, one behind and partially hid by the other. After walking c miles the farther hill becomes entirely hidden, and on observing the elevation of the lower hill after walking another mile he finds it to be β. Find the heights of the two hills.*

Solution. Let P be the summit of the lower hill, Q the summit of the higher hill; let A be the first point of observation, B the second, C the third. From P and Q draw PM and QN, respectively perpendicular to the horizontal plane which contains A, B, and C.

Let $PM = h$, and $QN = h'$.

Then $AM = h\cot\alpha$, and $AM = AB + BC + CM = c + 1 + h\cot\beta$; therefore $h\cot\alpha = c + 1 + h\cot\beta$; therefore $h(\cot\alpha - \cot\beta) = c + 1$;

therefore $$h = \frac{(c+1)\sin\alpha\sin\beta}{\sin(\beta-\alpha)}.$$

And by similar triangles

$$\frac{h'}{h} = \frac{QN}{PM} = \frac{BN}{BM} = \frac{AN - AB}{AM - AB} = \frac{h'\cot\alpha' - c}{h\cot\alpha - c};$$

thus since h is known we can find h'.

Problem 23*. A tower is surrounded by a circular moat. At noon on a certain day the shadow of the top of the tower is observed to project 45 feet beyond the edge of the moat. When the sun is due West on the same day the shadow projects 120 feet beyond the moat. The distance between the extremities of the shadow is 375 feet. The angle of elevation of the top of the tower from any point of the edge of the moat is 60°. Find the height of the tower and the altitude of the sun at noon.*

Solution. Let h be the height of the tower in feet, α the altitude of the sun at noon. The distance between the foot of the tower and the edge of the moat is $h\cot 60°$; hence the distance between the foot of the tower and the extremity of the shadow is $h\cot 60° + 45$ at noon, and $h\cot 60° + 120$ when the sun is due west. The directions of the shadows include a right angle;

therefore $$(h\cot 60° + 45)^2 + (h\cot 60° + 120)^2 = (375)^2.$$

Therefore $$\frac{2h^2}{3} + \frac{2h}{\sqrt{3}}\cdot 165 + (45)^2 + (120)^2 = (375)^2;$$

therefore $$\frac{2h^2}{3} + \frac{2h}{\sqrt{3}}165 = 124200.$$

By solving this quadratic in the usual way we obtain $h = 180\sqrt{3}$ or $-345\sqrt{3}$; only the positive value is applicable. Then $h\cot\alpha - h\cot 60° = 45$;

therefore $$\cot\alpha = \cot 60° + \frac{45}{h} = \frac{1}{\sqrt{3}} + \frac{45}{180\sqrt{3}} = \frac{1}{\sqrt{3}} + \frac{1}{4\sqrt{3}} = \frac{5}{4\sqrt{3}};$$

therefore $$\tan\alpha = \frac{4\sqrt{3}}{5}.$$

Problem 24*. A tower stands upon an inclined plane, meeting it at a point A; at a point C in the plane the tower is observed to subtend an angle α; on proceeding to a point D in the straight line AC such that $CD = AC$, the tower is observed to subtend an angle β : if ϕ be the angle between the tower and AC, show that*

$$\cot\phi = 2\cot\alpha - \cot\beta.$$

Also if similar observations be made in another straight line $AC'D'$, it is found that $\tan\alpha' = 2\tan\beta'$; the angle $CAC' = \gamma$: prove that if θ be the inclination of the plane to the horizon,

$$\sin\theta\sin\gamma = \cos\phi.$$

Solution. Let P denote the top of the tower. Then ϕ is the angle between PA and CA produced through A. Thus the angle $CPA = \phi - \alpha$, and the angle $DPC = \alpha - \beta$.

Then $$\frac{DC}{CP} = \frac{\sin DPC}{\sin CDP} = \frac{\sin(\alpha-\beta)}{\sin\beta},$$

$$\frac{CA}{CP} = \frac{\sin CPA}{\sin CAP} = \frac{\sin(\phi-\alpha)}{\sin(\pi-\phi)} = \frac{\sin(\phi-\alpha)}{\sin\phi};$$

therefore $$\frac{\sin(\alpha-\beta)}{\sin\beta} = \frac{\sin(\phi-\alpha)}{\sin\phi};$$

therefore $$\sin\alpha\cot\beta - \cos\alpha = \cos\alpha - \sin\alpha\cot\phi;$$

therefore $$\cot\phi = 2\cot\alpha - \cot\beta.$$

Now let α', β', and ϕ' correspond to observations made in another straight line $AC'D'$; then $\cot\phi' = 2\cot\alpha' - \cot\beta'$; but by supposition $2\tan\beta' = \tan\alpha'$; therefore $\cot\phi' = 0$; therefore $\phi' = \frac{\pi}{2}$. Thus $AC'D'$ makes a right angle with AP; and therefore $AC'D'$ is a horizontal straight line.

From D draw DM perpendicular to AD', and from M draw MN perpendicular to the horizontal plane which contains D; and produce PA through A to meet the same plane at Q.

Then $$\sin\theta = \frac{MN}{MD},\quad \sin\gamma = \frac{DM}{DA},\quad \cos\phi = \frac{AQ}{AD} = \frac{MN}{AD};$$

therefore $$\cos\phi = \sin\theta\sin\gamma.$$

Problem 25*. In a triangle ABC having given $A = 30°$, $b = 3\sqrt{3}$, $a = 3$, solve the triangle; and supposing that an error of $2''$ is made in observing the angle A, find approximately the corresponding error in the angle B.*

Solution. $$\sin B = \frac{b}{a}\sin A = \frac{3\sqrt{3}}{3}\sin A = \sqrt{3}\sin A;$$

thus if $A = \frac{\pi}{6}$ we have $\sin B = \sqrt{3}\cdot\frac{1}{2}$; therefore $B = \frac{\pi}{3}$ or $\frac{2\pi}{3}$.

Suppose however that $A = \frac{\pi}{6} \pm h$, where h is the circular measure of $2''$; then $\sin B = \sqrt{3}\sin\left(\frac{\pi}{6} \pm h\right) = \sqrt{3}\left\{\sin\frac{\pi}{6} \pm h\cos\frac{\pi}{6}\right\}$ very nearly. Suppose that $B = \frac{\pi}{3} \pm k$; then approximately $\sin\frac{\pi}{3} \pm k\cos\frac{\pi}{3} = \sqrt{3}\left\{\sin\frac{\pi}{6} \pm h\cos\frac{\pi}{6}\right\}$; therefore $\pm k\cos\frac{\pi}{3} = \pm h\sqrt{3}\cos\frac{\pi}{6}$; therefore $k = h\sqrt{3}\cdot\cot\frac{\pi}{6} = 3h$.

In the same way if $B = \frac{2\pi}{3} \pm k$ we find that $k = -3h$. Thus the approximate error in B is 6 seconds.

Problem 26. *The distance between two objects on the opposite bank of a river is known to be c. An equal distance is taken anywhere along the bank on this side and the angles subtended by c at the extremities of this distance are α and β. Find the breadth of the river, the sides being parallel.*

Solution. Let A and B be the two objects on the opposite bank of the river; and suppose P and Q two points on this bank, such that $PQ = AB$; and let P correspond to A and Q to B, so that AP is equal and parallel to BQ. Let AQ and BP intersect at C.

Then $\alpha =$ the angle APB, and $\beta =$ the angle $AQB =$ the angle PAQ.

Therefore
$$\frac{PC}{PA} = \frac{\sin\beta}{\sin(\alpha+\beta)}, \quad \frac{AC}{PA} = \frac{\sin\alpha}{\sin(\alpha+\beta)};$$
but
$$PQ^2 = PC^2 + QC^2 - 2PC\cdot QC\cdot\cos PCQ; \text{ and } QC = AC;$$
therefore
$$c^2 = PA^2\frac{\sin^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta\cos(\alpha+\beta)}{\sin^2(\alpha+\beta)}.$$

Let x denote the breadth of the river; then the area of the triangle $APB = \frac{1}{2}xc$; and this area is also equal to
$$\frac{1}{2}PA\cdot PB\sin APB = PA\cdot PC\sin\alpha = \frac{PA^2\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$$
$$= \frac{c^2\sin\alpha\sin\beta\sin(\alpha+\beta)}{\sin^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta\cos(\alpha+\beta)}.$$

Problem 27. *A person wishing to obtain the breadth of a square fort on a distant hill, observes that when he is due South of one corner, the face towards him subtends an angle α. He then walks due West, and at a distance of a feet from his first position, finds that the face subtends the same angle as before. On walking b feet further, he is due South of the other corner of the face. Show that the breadth of the fort is*
$$(a+b)\sec\phi \textit{ feet,} \quad \textit{where } \tan\phi = \frac{b\tan\alpha}{a+b}.$$

Solution. Let AB denote a side of the fort, C the position due south of A; let D be the second position, so that $CD = a$, and the angle $ACD = 90°$; also A, B, D, and C will lie on the circumference of a circle. Let E be the third position, so that E is on CD produced through D, and $DE = b$; and the angle BED is a right angle.

Let ϕ be the angle between AB produced through B and CE produced through E. Then $a + b = AB\cos\phi$; therefore $AB = (a+b)\sec\phi$.

And
$$BE = EC\tan BCE, \text{ and } = ED\tan BDE;$$

$$\therefore\ (a+b)\tan(90^\circ-\alpha) = b\tan BAC = b\tan(90^\circ-\phi).\ (\textit{Euclid}\ \text{III}.\ 22)$$

Problem 28. *A and A' are the peaks of two mountains, and BC is a straight horizontal road; show that if the nearer of the two peaks just conceals the more distant at some point of the road, then $\sin\alpha\sin\beta' = \sin\alpha'\sin\beta$, where α is the altitude of A as seen from any point B of the road, β is the angle ABC, and α', β' are similar quantities for the peak A' as seen from any point B' of the road.*

Solution. From A draw AM perpendicular to the horizontal plane which contains the road, and draw AN perpendicular to the straight road.

Then $$\sin\alpha = \frac{AM}{AB}, \text{ and } \sin\beta = \frac{AN}{AB}.$$

Similarly from A' draw $A'M'$ perpendicular to the horizontal plane, and $A'N'$ perpendicular to the straight road.

Then $$\sin\alpha' = \frac{A'M'}{A'B'} \text{ and } \sin\beta' = \frac{A'N'}{A'B'}.$$

Thus we have to show that $\dfrac{AM}{AB}\cdot\dfrac{A'N'}{A'B'} = \dfrac{A'M'}{A'B'}\cdot\dfrac{AN}{AB}$,

or that $$AM\cdot A'N' = A'M'\cdot AN, \text{ or that } \frac{AM}{AN} = \frac{A'M'}{A'N'}.$$

Now if A is just hidden by A' at some point of the road, the straight line $A'A$ if produced through A will intersect the road; and then AA' and the road will lie in one plane; the sine of the inclination of this plane to the horizontal plane is expressed by $\dfrac{AM}{AN}$ and also by $\dfrac{A'M'}{A'N'}$; so that these are equal.

Problem 29. *A and B are two objects in the same horizontal plane, P is a point in the same plane at which the angle α subtended by AB is observed; from P two persons walk in this plane in directions at right angles to PA, PB respectively, to points Q, R, at each of which the angle subtended by AB is α; the distanced PQ, PR are a, b. Find the length of AB.*

Solution. There are two cases. Suppose the angles APQ and BPR to be on the same sides of AP and BP respectively; then the angle $QPR =$ the angle $APB = \alpha$. Suppose the angles APQ and BPR not to fall on the same sides of AP and BP respectively; then the angle $RPQ = \pi-\alpha$. In both cases $AB = RQ$; for the diameter of the circle which goes round the five points A, B, P, Q and $R = \dfrac{AB}{\sin APB}$ and also $= \dfrac{RQ}{\sin RPQ}$.

In the former case $AB = \sqrt{(a^2+b^2-2ab\cos\alpha)}$, and in the latter case
$$AB = \sqrt{(a^2+b^2+2ab\cos\alpha)}.$$

Problem 30. *A, C, B are three objects in the same plane as an observer; $AC = CB$, and AC, CB are at right angles to each other. At the point O, AC, CB subtend angles α, β respectively. The observer moves from O in the direction OO' at right angles to CO through a space $OO' = d$; here he finds that AC, CB subtend angles α', β' respectively. Find the distance AB.*

Solution. Suppose both straight lines OC and $O'C$ to fall within the angle ACB. Let $AC = \alpha, ACO = \phi$; then from the triangles ACO and BCO we get

$$OC = \frac{a\sin(\phi+\alpha)}{\sin\alpha} \text{ and } OC = \frac{a\cos(\phi-\beta)}{\sin\beta};$$

therefore

$$OC\sin\alpha = a(\sin\phi\cos\alpha + \cos\phi\sin\alpha),$$
$$OC\sin\beta = a(\cos\phi\cos\beta + \sin\phi\sin\beta).$$

Hence

$$a\sin\phi = \frac{OC\sin\alpha(\cos\beta - \sin\beta)}{\cos(\alpha+\beta)},$$
$$a\cos\phi = \frac{OC\sin\beta(\cos\alpha - \sin\alpha)}{\cos(\alpha+\beta)}$$

Square and add; thus

$$a^2\cos^2(\alpha+\beta) = OC^2\left\{\sin^2\alpha(\cos\beta-\sin\beta)^2 + \sin^2\beta(\cos\alpha-\sin\alpha)^2\right\}$$
$$= OC^2\{\sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta\sin(\alpha+\beta)\}.$$

Thus

$$OC^2 = \frac{a^2\cos^2(\alpha+\beta)}{\sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta\sin(\alpha+\beta)}.$$

A similar expression will be found for $O'C^2$ in terms of α' and β'. Then $O'C^2 = OC^2 + d^2$. This finds a; and then $AB = a\sqrt{2}$.

Similarly the problem may be solved for any other positions of the lines OC, $O'C$.

Problem 31. *A tower 150 feet high throws a shadow 75 feet long upon the horizontal plane upon which it stands. Find the Sun's altitude, having given*

$$\log 2 = \cdot 3010300, \quad L\ \tan 63°26' = 10\cdot 3009994,$$
$$L\ \tan 63°27' = 10\cdot 3013153.$$

Solution. Let α denote the Sun's altitude; then $\tan\alpha = \dfrac{150}{75} = 2$;

therefore $L\ \tan\alpha = 10 + \log 2 = 10\cdot 3010300.$

$$\begin{array}{rr} 10\cdot 3013153 & 10\cdot 3010300 \\ 10\cdot 3009994 & 10\cdot 3009994 \\ \hline \cdot 0003159 & \cdot 0000306 \end{array} \qquad \cdot 0003159 : \cdot 0000306 :: 60'' : x'';$$

this gives $x = 6$; therefore $\alpha = 63°26'6''$.

Problem 32. *A person standing at the edge of a river observes that the top of a tower on the edge of the opposite side subtends an angle of 55° with a horizontal straight line drawn from his eye; receding backwards 30 feet he then finds it to subtend an angle of 48°. Determine the breadth of the river, having given*

$$L\ \sin 7° = 9\cdot 08589, \quad L\ \sin 35° = 9\cdot 75859,$$
$$L\ \sin 48° = 9\cdot 87107, \quad \log 3 = \cdot 47712,$$
$$\log 1\cdot 0493 = \cdot 02089.$$

Solution. Take the diagram of *Art.* 240 (page 423). Here $PBC = 55°$, $PAC = 48°$, $AB = 30$ feet.

$$\frac{PB}{BA} = \frac{\sin PAB}{\sin APB} = \frac{\sin 48°}{\sin 7°}; \text{ therefore } PB = \frac{30\sin 48°}{\sin 7°};$$

$$BC = BP\cos PBC = BP\cos 55° = BP\sin 35° = \frac{30\sin 48°\sin 35°}{\sin 7°};$$

$$\log BC = \log 30 + L\sin 48^\circ - 10 + L\sin 35^\circ - 10 - (L\sin 7^\circ - 10)$$
$$= 1.47712 + 9.87107 + 9.75859 - 9.08589 - 10 = 2.02089;$$

therefore $BC = 104.93$.

Problem 33. *A rope-dancer wishes to ascend a tower 100 feet high, by means of a rope 196 feet long. If he can do so, find at what inclination he must be able to walk up the rope, having given*

$$\log 2 = \cdot30103, \quad L\sin 30^\circ 40' = 9\cdot70761,$$
$$\log 7 = \cdot84510, \quad L\,30^\circ 41' = 9\cdot70782.$$

Solution. Let α denote the inclination; then $\sin\alpha = \dfrac{100}{196} = \dfrac{100}{4\times 49}$;
therefore $L\sin\alpha = 10 + \log 100 - \log(4\times 49) = 12 - 2\log 2 - 2\log 7 = 9.70774$.

$$\begin{array}{r} 9\cdot70782 \\ 9\cdot70761 \\ \hline \cdot00021 \end{array} \qquad \begin{array}{r} 9\cdot70774 \\ 9\cdot70761 \\ \hline \cdot00013 \end{array} \qquad \cdot00021 : \cdot00013 :: 60'' : x'';$$

this gives $x = 37$; therefore $\alpha = 30^\circ 40' 37''$.

Problem 34. *Two hills rise at the same point, with inclinations of 60° and 40° to the horizon. At a distance of 64 feet from the base of the latter hill the angles of elevation of the bottom and top of a vertical object on the former hill are 40° and 70°. Find the height of the object, having given*

$$L\tan 20^\circ = 9\cdot5610659, \quad L\cos 40^\circ = 9\cdot8842540,$$
$$\log 2 = \cdot3010300; \quad \log 26940031 = 7\cdot4303981.$$

Solution. Let A be the point of intersection of the hills, B the point of observation on the hill, P the top of the object, C the bottom. Produce PB through B to meet at D the horizontal straight line which contains A; produce DA through A to any point E. Then $AB = 64$ feet; and the following are the given angles:

$$CAE = 60^\circ,\ BAD = 40^\circ,\ BDA = 70^\circ,\ BPC = 90^\circ - 70^\circ = 20^\circ.$$

Therefore $\qquad BAC = 80^\circ,\ BCA = 20^\circ,\ PBC = 30^\circ.$

$$\frac{BC}{BA} = \frac{\sin BAC}{\sin BCA} = \frac{\sin 80^\circ}{\sin 20^\circ},$$
$$\frac{PC}{BC} = \frac{\sin PBC}{\sin BPC} = \frac{\sin 30^\circ}{\sin 20^\circ};\ \text{therefore}\ \frac{PC}{BA} = \frac{\sin 80^\circ \sin 30^\circ}{\sin^2 20^\circ};$$

therefore $\qquad PC = \dfrac{64\sin 80^\circ \sin 30^\circ}{\sin^2 20^\circ} = \dfrac{64\sin 40^\circ \cos 40^\circ}{\sin^2 20^\circ} = \dfrac{128\cos 40^\circ}{\tan 20^\circ};$

therefore $\qquad \log PC = 7\log 2 + L\cos 40^\circ - L\tan 20^\circ = 2.4303981;$

therefore $\qquad PC = 269.40031.$

Problem 35. *A vessel observed another α° from the North sailing in a direction parallel to its own. After an hour's sailing its bearing was β°, and after another hour γ° from the North. Find in what direction the vessels were sailing.*

Solution. Let A, B, C be the three successive positions of the ship from which the observations are made; let P, Q, R be the corresponding positions of the other ship.

Then the straight line ABC is parallel to the straight line PQR; also $AB = BC$, and $PQ = QR$.

Let θ be the angle between the North direction and the direction of sailing.

From B draw a straight line parallel to AP, meeting PQ at M; then

$$\frac{QM}{BM} = \frac{\sin QBM}{\sin BQM} = \frac{\sin QBM}{\sin BQP} = \frac{\sin(\beta - \alpha)}{\sin(\theta - \beta)}.$$

Again, from C draw a straight line parallel to AP, meeting QR at N; then

$$\frac{RN}{CN} = \frac{\sin RCN}{\sin CRN} = \frac{\sin RCN}{\sin CRP} = \frac{\sin(\gamma - \alpha)}{\sin(\theta - \gamma)}.$$

But $BM = CN$; and $RN = 2QM$, for RN is the difference of the paths of the ships in two hours, and QM is the difference in one hour.

Therefore $$\frac{2\sin(\beta - \alpha)}{\sin(\theta - \beta)} = \frac{\sin(\gamma - \alpha)}{\sin(\theta - \gamma)};$$

therefore $$2\sin(\theta - \gamma)\sin(\beta - \alpha) = \sin(\gamma - \alpha)\sin(\theta - \beta),$$

therefore

$$2(\sin\theta\cos\gamma - \cos\theta\sin\gamma)\sin(\beta - \alpha) = (\sin\theta\cos\beta - \cos\theta\sin\beta)\sin(\gamma - \alpha).$$

Divide by $\cos\theta$; thus we obtain the value of $\tan\theta$.

Problem 36. *In the problem discussed in Art. 242 (page 424), show that if*

$$\alpha + \beta + C = \pi, \quad \text{then } \phi = \frac{\pi}{4},$$

and the solution cannot be obtained from the data.

Solution. If $\alpha + \beta + C = \pi$, then $x + y = \pi$; therefore $\sin x = \sin y$; therefore $\tan\phi = 1$. We might as in *Art*. 242 (page 424) say that

$$\frac{\sin x - \sin y}{\sin x + \sin y} = \tan\left(\phi - \frac{\pi}{4}\right),$$

that is $$\frac{2\sin\frac{x-y}{2}\cos\frac{x+y}{2}}{2\sin\frac{x+y}{2}\cos\frac{x-y}{2}} = \tan\left(\phi - \frac{\pi}{4}\right).$$

But as $\cos\frac{x+y}{2}$ is now zero we cannot divide both numerator and denominator of the last fraction by it, and thus we cannot proceed further. In fact in this case a circle would go round P, A, C, and B, and P may be at any point of the arc between A and B.

CHAPTER XVI
Properties of Triangles

Problem 1. *The sides of a plane triangle are 24, 30, 18 : find the area.*

Solution. Here $s = 36,\ s - a = 12,\ s - b = 6,\ s - c = 18.$

The area of the triangle $= \sqrt{36 \times 12 \times 6 \times 18} = \sqrt{36 \times 36 \times 36} = 6^3 = 216.$

Problem 2. *Two angles of a triangle are 15° and 45°, and the included side is 10 feet : find the area.*

Solution. The third angle of the triangle $= 180° - 60° = 120°$.

One of the containing sides $= \dfrac{10 \times \sin 15°}{\sin 120°}$, and the other $= \dfrac{10 \times \sin 45°}{\sin 120°}$.

Hence the area

$$= \frac{1}{2}\frac{(10)^2 \sin 15° \sin 45°}{\sin^2 120°} \sin 120° = \frac{50 \sin 15° \sin 45°}{\sin 120°} = \frac{50(\sqrt{3}-1)}{2\sqrt{2}} \times \frac{1}{\sqrt{2}} \times \frac{2}{\sqrt{3}}$$

$$= \frac{25(\sqrt{3}-1)}{\sqrt{3}}.$$

Problem 3. *The sides of a triangle are equal to 3 and 12 respectively, and the contained angle is 30° : find the hypotenuse of an equal right-angled isosceles triangle.*

Solution. The area of the triangle $= \dfrac{1}{2} \times 3 \times 12 \times \sin 30° = \dfrac{36}{4} = 9.$

Let x denote the hypotenuse of the right-angled triangle; then each of the equal sides is $\dfrac{x}{\sqrt{2}}$, and the area is $\dfrac{1}{2} \times \left(\dfrac{x}{\sqrt{2}}\right)^2$, that is $\dfrac{x^2}{4}$.

Hence $\dfrac{x^2}{4} = 9;\ \therefore x^2 = 36;\ \therefore x = 6.$

Problem 4. *The area of a triangle* $= \dfrac{1}{4}\left(a^2 \sin 2B + b^2 \sin 2A\right).$

Solution. From the angle C of a triangle draw a perpendicular CD to the side AB, or AB produced.

First suppose A and B acute, so that D is between A and B. Then $CD = b \sin A,\ AD = b \cos A$; thus the area of $ACD = \dfrac{1}{2}b^2 \sin A \cos A = \dfrac{1}{4}b^2 \sin 2A.$

Similarly the area of $BCD = \dfrac{1}{2}a^2 \sin B \cos B = \dfrac{1}{4}a^2 \sin 2B.$

Therefore the area of the whole triangle $= \dfrac{1}{4}\left(a^2 \sin 2B + b^2 \sin 2A\right).$

Next suppose the angle B obtuse, so that D falls on AB produced through D. Then as before the area of $ACD = \dfrac{1}{4}b^2 \sin 2A.$

And the area of CBD

$$= \frac{1}{2}a^2 \sin(180^\circ - B)\cos(180^\circ - B)$$
$$= \frac{a^2}{4}\sin(360^\circ - 2B).$$

Therefore the area of ABC

$$= \frac{1}{4}\left\{b^2 \sin 2A - a^2 \sin(360^\circ - 2B)\right\} = \frac{1}{4}\left(b^2 \sin 2A + a^2 \sin 2B\right).$$

This mode of solution shows the geometrical meaning of the two parts of the expression. We may proceed more briefly thus :

$$\frac{1}{4}\left(a^2 \sin 2B + b^2 \sin 2A\right)$$
$$= \frac{1}{2}(a \sin B\, a\cos B + b \sin A\, b\cos A)$$
$$= \frac{1}{2}a \sin B(a\cos B + b\cos A),\ \text{by } Art.\ 214 \text{ (page 418)},$$
$$= \frac{1}{2}ac \sin B,\ \text{by } Art.\ 216 \text{ (page 419)},$$
$$= \text{ the area of the triangle by } Art.\ 247 \text{ (page 425)},$$

Problem 5. *The area of a triangle* $= \dfrac{a^2 - b^2}{2}\dfrac{\sin A \sin B}{\sin(A-B)}$.

Solution.

$$\frac{a^2 - b^2}{2}\frac{\sin A \sin B}{\sin(A-B)} = \frac{\sin A \sin B}{2\sin(A-B)}\left\{\frac{c^2 \sin^2 A}{\sin^2 C} - \frac{c^2 \sin^2 B}{\sin^2 C}\right\}$$
$$= \frac{c^2 \sin A \sin B\left(\sin^2 A - \sin^2 B\right)}{2\sin(A-B)\sin^2 C}$$
$$= \frac{c^2 \sin A \sin B \sin(A+B)\sin(A-B)}{2\sin(A-B)\sin^2 C}$$
$$= \frac{c^2 \sin A \sin B}{2\sin C}$$
$$= \text{ area of the triangle, by } Art.\ 247 \text{ (page 425)}.$$

Problem 6. *The area of a triangle* $= \dfrac{2abc}{a+b+c}\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$.

Solution.

$$\frac{2abc}{a+b+c}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{2abc}{2s}\sqrt{\frac{s(s-a)}{bc}} \times \sqrt{\frac{s(s-b)}{ac}} \times \sqrt{\frac{s(s-c)}{ab}}$$
$$= \sqrt{s(s-a)(s-b)(s-c)}$$
$$= S = \text{ the area of the triangle.}$$

Problem 7. *Show that the triangle whose sides are proportional to*

$$gh\left(k^2 + l^2\right),\ kl\left(g^2 + h^2\right),\ (hk + gl)(hl - gk).$$

has its area and the trigonometry ratios of its angles rational.

Solution.

Here
$$\begin{aligned} 2s &= gh\left(k^2+l^2\right)+kl\left(g^2+h^2\right)+(hk+gl)(hl-gk) \\ &= gh\left(k^2+l^2\right)+kl\left(g^2+h^2\right)+h^2kl+ghl^2-hgk^2-klg^2 \\ &= 2ghl^2+2h^2kl. \end{aligned}$$
$$\therefore s = ghl^2+klh^2 = hl(gl+hk);$$
$$\begin{aligned} \therefore s-a &= klh^2-ghk^2 = kh(lh-gk), \\ s-b &= ghl^2-klg^2 = gl(lh-gk), \\ s-c &= hgk^2+klg^2 = kg(hk+lg). \end{aligned}$$

Thus
$$s(s-a)(s-b)(s-c) = g^2h^2k^2l^2(lh-gk)^2(hk+lg)^2;$$
$$\therefore S = ghkl(lh-gk)(hk+lg).$$

Therefore by *Art.* 218 (page 420) the sines of the angles of the triangle are rational quantities; and by *Art.* 215 (page 419) the cosines of the angles are rational quantities.

Problem 8. *The sides of a triangle are in arithmetical progression, and its area is to that of an equilateral triangle of the same perimeter as 3 is to 5. Find the ratio of the sides and the value of the largest angle.*

Solution. Let a, b, c be in Arithmetical Progression; then $2b = a+c$. Thus the perimeter $= 3b$, and the side of an equilateral triangle of equal perimeter is b.

Thus
$$\sqrt{s(s-a)(s-b)(s-c)} = \frac{3}{5}\cdot\frac{1}{2}b^2\sin 60^\circ = \frac{3\sqrt{3}}{20}b^2,$$
$$\therefore \sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)} = \frac{3\sqrt{3}}{5}b^2,$$
$$\therefore \sqrt{3b^2(b+c-a)(a+b-c)} = \frac{3\sqrt{3}}{5}b^2,$$
$$\therefore \sqrt{(b+c-a)(b+a-c)} = \frac{3}{5}b,$$
$$\therefore \sqrt{\frac{3c-a}{2}\times\frac{3a-c}{2}} = \frac{3}{10}(a+c);$$
$$\therefore (3c-a(3a-c) = \frac{9}{25}(a+c)^2;$$
$$\therefore 10ac-3\left(a^2+c^2\right) = \frac{9}{25}\left(a^2+2ac+c^2\right);$$
$$\therefore 84\left(a^2+c^2\right)-232ac = 0,$$
$$\therefore 21\left(\frac{a^2}{c^2}+1\right) = 58\frac{a}{c}.$$

By solving this quadratic in the usual way, we obtain $\dfrac{a}{c} = \dfrac{7}{3}$ or $\dfrac{3}{7}$.

Take $\dfrac{a}{c} = \dfrac{7}{3}$; thus a, b, and c are proportional to 7, 5, and 3 respectively.

Then
$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{5^2+3^2-7^2}{2\times5\times3} = -\frac{1}{2};$$
$$\therefore A = 120^\circ.$$

Problem 9. *If the alternate angles of a regular hexagon be joined so as to form another regular hexagon, and again the alternate angles of the latter hexagon be joined, and so on, show that the sum of the areas of all the figures so formed =* $\dfrac{S}{2}$*, where* S *is the area of the original figure.*

Solution. Let A, B, C, D, E be five consecutive angles of the hexagon; draw AC, BD, CE; let AC and BD intersect at P, and let BD and CE intersect at Q. Then PQ is the side of the second regular hexagon.

The angle DBC is half of the angle which DC would subtend at the centre of the circle circumscribing the regular hexagon, and is therefore $\dfrac{\pi}{6}$. Similarly the angle ACB is $\dfrac{\pi}{6}$.

Then $$\frac{PC}{BC} = \frac{\sin\frac{\pi}{6}}{\sin\left(\pi - \frac{2\pi}{6}\right)} = \frac{\sin\frac{\pi}{6}}{\sin\frac{2\pi}{6}} = \frac{1}{2\cos\frac{\pi}{6}};\ \therefore PC = \frac{BC}{2\cos\frac{\pi}{6}}.$$

And $$PQ = 2PC\sin\frac{1}{2}PCQ = 2PC\sin\frac{\pi}{6} = BC\tan\frac{\pi}{6}.$$

Thus $PQ = \dfrac{BC}{\sqrt{3}}$. And the areas of similar polygons are as the squares of their homologus sides; so that if S denote the area of the first hexagon the area of the second is $\dfrac{S}{3}$. In like manner the area of the next hexagon is $\dfrac{1}{3}$ of $\dfrac{S}{3}$, that is $\dfrac{S}{9}$; and so on. Hence the sum of the areas of all the derived figures is $\dfrac{S}{3} + \dfrac{S}{9} + \dfrac{S}{27} + \dots$, that is $\dfrac{1}{3}\dfrac{S}{1-\frac{1}{3}}$, that is $\dfrac{S}{2}$.

Problem 10. *If we proceed with a regular figure of n sides, and of area* S*, as with the hexagon of Example 9, and* Σ *denote the sum of the areas so formed, show that*
$$\Sigma\sin\frac{3\pi}{n}\sin\frac{\pi}{n} = S\cos^2\frac{2\pi}{n}.$$
Explain the cases where $n = 3$ *or* 4*.*

Solution. Suppose that the original figure instead of being a hexagon is a regular polygon of n sides. Proceed as before and we have
$$\frac{PC}{BC} = \frac{\sin\frac{\pi}{n}}{\sin\left(\pi - \frac{2\pi}{n}\right)} = \frac{\sin\frac{\pi}{n}}{\sin\frac{2\pi}{n}} = \frac{1}{\cos\frac{\pi}{n}}.$$

Then $$PQ = 2PC\sin\frac{1}{2}PCQ;$$

and the angle $$PCQ = (n-4)\frac{\pi}{n};\ \therefore PQ = 2PC\sin(n-4)\frac{\pi}{2n}$$
$$= 2PC\sin\left(\frac{\pi}{2} - \frac{2\pi}{n}\right) = 2PC\cos\frac{2\pi}{n} = BC\frac{\cos\frac{2\pi}{n}}{\cos\frac{\pi}{n}}.$$

Thus the area of the second polygon is $\dfrac{S\cos^2\dfrac{2\pi}{n}}{\cos^2\dfrac{\pi}{n}}$;

and $\qquad \Sigma = S\left(1+m^2+m^3+\ldots\right)$ where m stands for $\dfrac{\cos^2\dfrac{2\pi}{n}}{\cos^2\dfrac{\pi}{n}}$;

thus $$\Sigma = \frac{Sm}{1-m} = \frac{S\cos^2\dfrac{2\pi}{n}}{\cos^2\dfrac{\pi}{n}-\cos^2\dfrac{2\pi}{n}} = \frac{S\cos^2\dfrac{2\pi}{n}}{\sin^2\dfrac{2\pi}{n}-\sin^2\dfrac{\pi}{n}} = \frac{S\cos^2\dfrac{2\pi}{n}}{\sin\dfrac{3\pi}{n}\sin\dfrac{\pi}{n}}.$$

If $n = 3$ this becomes infinite; for $\sin\pi = 0$; in this case the original figure is a triangle, and the second figure is the *same* triangle, and so on : thus the sum of the areas is infinite.

If $n = 4$ the expression vanishes; for $\cos\dfrac{2\pi}{4} = 0$; in this case the original figure is a square, and the second figure is only a point, and so on : thus the sum of the areas is zero.

Problem 11. *If an equilateral triangle be described with its angular points on the sides of a given right-angled isosceles triangle, and one side parallel to the hypotenuse, its area will be* $2a^2\sin 60°\,(\sin 15°)^2$ *, where a is a side of the given triangle.*

Solution. Let ABC denote the right-angled isosceles triangle where C is the right angle. Let F be the middle point of AB; let D be on BC, and E on AC, such that DE is parallel to AB, and the triangle DEF is equilateral.

Then the angle $DEC = 45°$, and the angle $DEF = 60°$; therefore the angle $AEF = 75°$. Now $\dfrac{FE}{FA} = \dfrac{\sin FAE}{\sin FEA} = \dfrac{\sin 45°}{\sin 75°}$;

$$\therefore FE = \frac{FA\sin 45°}{\sin 75°} = \frac{a}{\sqrt{2}}\cdot\frac{\sin 45°}{\cos 15°} = \frac{a}{2}\frac{1}{\cos 15°} = \frac{a\sin 15°}{2\cos 15°\sin 15°}$$
$$= \frac{a\sin 15°}{\sin 30°} = 2a\sin 15°.$$

Therefore the area of the equilateral triangle
$$= \frac{1}{2}\,(2a\sin 15°)^2\sin 60° = 2a^2\sin^2 15°\sin 60°.$$

Problem 12. *Show that, with the notation of Arts. 248 (page 426) and 250 (page 427),*

$$r_1r_2r_3 = r^3\cot^2\frac{A}{2}\cot^2\frac{B}{2}\cot^2\frac{C}{2}.$$

Solution. $$r_1r_2r_3 = \frac{S^3}{(s-a)(s-b)(s-c)},\qquad r^3 = \frac{S^3}{s^3};$$

therefore $$\frac{r_1r_2r_3}{r^3} = \frac{s^3}{(s-a)(s-b)(s-c)};$$

and $$\cot^2\frac{A}{2}\cot^2\frac{B}{2}\cot^2\frac{C}{2} = \frac{s(s-a)}{(s-b)(s-c)}\times\frac{s(s-b)}{(s-a)(s-c)}\times\frac{s(s-c)}{(s-a)(s-b)}$$

$$= \frac{s^3}{(s-a)(s-b)(s-c)};$$

therefore $$\frac{r_1 r_2 r_3}{r^3} = \cot^2\frac{A}{2}\cot^2\frac{B}{2}\cot^2\frac{C}{2}.$$

Problem 13. *The straight lines which bisect the angles A, C of a triangle ABC meet the circumference of the circumscribing circle at the points A', C'. Show that $A'C'$ is divided by CB, BA into three parts, which are in the proportion*

$$\sin^2\frac{A}{2} : 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} : \sin^2\frac{C}{2}.$$

Solution. Let $C'A'$ intersect AB at E and CB at F.

The angle $A'FC$ is equal to the sum of the angles $FC'C$ and FCC', that is to the sum of the angles $A'AC$ and FCC', that is to $\frac{1}{2}A + \frac{1}{2}C$; the angle $BCA' = \angle BAA' = \frac{1}{2}A$.

Thus $$\frac{FA'}{CA'} = \frac{\sin\frac{A}{2}}{\sin\frac{(A+C)}{2}} = \frac{\sin\frac{A}{2}}{\cos\frac{B}{2}}.$$

Let R be the radius of the circle; then

$$A'C = 2R\sin\frac{A}{2}; \therefore FA' = \frac{2R\sin^2\frac{A}{2}}{\cos\frac{B}{2}}.$$

In the same manner $$EC' = \frac{2R\sin^2\frac{C}{2}}{\cos\frac{B}{2}}.$$

And $$A'C' = 2R\sin\frac{(A+C)}{2} = 2R\cos\frac{B}{2}.$$

$$\therefore EF = 2R\cos\frac{B}{2} - \frac{2R\left(\sin^2\frac{A}{2} + \sin^2\frac{C}{2}\right)}{\cos\frac{B}{2}}$$

$$= \frac{2R}{\cos\frac{B}{2}}\left\{\cos^2\frac{B}{2} - \sin^2\frac{A}{2} - \sin^2\frac{C}{2}\right\}$$

$$= \frac{R}{\cos\frac{B}{2}}\{1 + \cos B - (1-\cos A) - (1-\cos C)\}$$

$$= \frac{R}{\cos\frac{B}{2}}(\cos A + \cos B + \cos C - 1)$$

$$= \frac{2R}{\cos\frac{B}{2}} \times 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$ by *Art.114* (page 409).

Problem 14. *If h be the difference between the sides containing the right angle of* *a right-angled triangle, and S its area, the diameter of the circumscribing circle is equal to* $\sqrt{h^2 + 4S}$.

Solution. Let a denote one side of the right-angled triangle, and $a + h$ the other side; then the hypotenuse $= \sqrt{a^2 + (a+h)^2} = \sqrt{h^2 + 2a(a+h)}$.

But $S =$ half the product of the sides $= \frac{1}{2}a(a+h)$; therefore $4S = 2a(a+h)$.

Thus the hypotenuse $= \sqrt{h^2 + 4S}$; and the hypotenuse is a diameter of the circumscribing circle.

Problem 15. *The sides of a plane triangle are 3, 5, 6. Compare the radii of the inscribed and circumscribed circles.*

Solution. $r = \frac{S}{s}, \quad R = \frac{abc}{4S}; \quad \therefore \frac{R}{r} = \frac{sabc}{4S^2}.$

Now $s = 7,\ s - a = 4,\ s - b = 2,\ s - c = 1$; $\therefore S = \sqrt{7 \times 4 \times 2}$; thus

$$\frac{R}{r} = \frac{7 \times 3 \times 5 \times 6}{4 \times 7 \times 4 \times 2} = \frac{45}{16}.$$

Problem 16. *O is the centre of the circle circumscribed round an acute-angled triangle, and AO is produced to meet BC at D. Show that*

$$DO\cos(B - C) = AO\cos A.$$

Solution. The angle $ABO =$ the angle $BAO = \frac{\pi}{2} - C$; and therefore the angle $BOD = \pi - 2C$; the angle $OBD = \frac{\pi}{2} - A$;

therefore the angle $BDO = 2C + A - \frac{\pi}{2} = A + C + B + C - B - \frac{\pi}{2} = \frac{\pi}{2} + C - B.$

Thus $$\frac{DO}{BO} = \frac{\sin DBO}{\sin BDO} = \frac{\sin\left(\frac{\pi}{2} - A\right)}{\sin\left(\frac{\pi}{2} + C - B\right)} = \frac{\cos A}{\cos(C - B)},$$

$$\text{and } BO = AO; \quad \therefore DO\cos(B - C) = AO\cos A.$$

Problem 17. *A circle is inscribed within a given triangle, and another triangle formed by joining the points of contact; within this latter triangle a circle is inscribed, and another triangle is formed as before, and so on continually. Show that the triangles thus formed ultimately become equilateral.*

Solution. Take the diagram of *Art.* 248 (page 426); draw FD, DE, and EF.

The angle $FDB = \frac{1}{2}(\pi - B)$, the angle $EDC = \frac{1}{2}(\pi - C)$; therefore the angle $FDE = \frac{1}{2}(B + C)$. Similarly the angle $DEF = \frac{1}{2}(C + A)$, and the angle $EFD = \frac{1}{2}(A + B)$.

Suppose A, B, C in ascending order of magnitude; then

$$\frac{1}{2}(A + B), \quad \frac{1}{2}(A + C), \quad \frac{1}{2}(B + C),$$

are in ascending order of magnitude; and

$$\frac{1}{2}(B+C) - \frac{1}{2}(A+B) = \frac{1}{2}(C-A).$$

Thus the difference between the greatest and least angles of the first derived triangle is *half* the difference between the greatest and least angles of the original triangle. In like manner the difference between the greatest and least angles of the second derived triangle is *half* the difference between the greatest and least angles of the first derived triangle, and therefore a *fourth* of the difference between the greatest and least angles of the original triangle. Proceeding in this way we see that the triangles thus formed ultimately become equilateral.

Problem 18. *The sum of the diameters of the inscribed and circumscribed circles of any plane triangle is equal to*

$$a\cot A + b\cot B + c\cot C.$$

Solution.

$$\begin{aligned} a\cot A &+ b\cot B + c\cot C \\ &= \frac{a}{\sin A}\cos A + \frac{b}{\sin B}\cos B + \frac{c}{\sin C}\cos C \\ &= 2R(\cos A + \cos B + \cos C) = 2R + 2R(\cos A + \cos B + \cos C - 1) \\ &= 2R + 8R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}, \text{ by } Art.\ 114, \text{ (page 409)}, \\ &= 2R + 8R\cdot\sqrt{\frac{(s-b)(s-c)}{bc}} \times \sqrt{\frac{(s-a)(s-c)}{ac}} \times \sqrt{\frac{(s-a)(s-b)}{ab}} \\ &= 2R + \frac{8R}{abc}\frac{S^2}{s} = 2R + \frac{2S}{s} = 2R + 2r. \end{aligned}$$

Problem 19. *Perpendiculars are drawn from the angles A, B, C of an acute-angled triangle on the opposite sides, and produced to meet the circumscribing circle; if those produced parts be α, β, γ respectively, show that*

$$\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 2(\tan A + \tan B + \tan C).$$

Solution. From A draw AD perpendicular to BC, and produce AD to meet the circumference of the circle at L.

Then the angle $ALB =$ the angle $ACB = C$;

$$\alpha = DL = BD\cot ALB = BD\cot C = \frac{c\cos B\cos C}{\sin C} = \frac{a\cos B\cos C}{\sin A};$$

therefore $\quad \dfrac{a}{\alpha} = \dfrac{\sin A}{\cos B\cos C} = \dfrac{\sin(B+C)}{\cos B\cos C} = \tan B + \tan C.$

Similarly $\quad \dfrac{b}{\beta} = \tan A + \tan C$, and $\dfrac{c}{\gamma} = \tan C + \tan A.$

Therefore $\quad \dfrac{a}{\alpha} + \dfrac{b}{\beta} + \dfrac{c}{\gamma} = 2(\tan A + \tan B + \tan C).$

Problem 20. *In any triangle the area of the inscribed circle is to the area of the* *triangle as* π *is to* $\cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}$.

Solution. The area of the inscribed circle is to the area of the triangle as πr^2 is to S, that is, as π is to $\dfrac{S}{r^2}$. Thus we have to show that

$$\frac{S}{r^2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$$

Now

$$\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}=\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}\times\sqrt{\frac{s(s-b)}{(s-a)(s-c)}}\times\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$=\frac{s\sqrt{s}}{\sqrt{(s-a)(s-b)(s-c)}}=\frac{s^2}{S}=S\times\frac{s^2}{S^2}=\frac{S}{r^2}.$$

Problem 21. *On each side of an acute-angled triangle as base an isosceles triangle is constructed externally, the sides of each being equal to the radius of the circumscribed circle : if the vertices of these be joined a triangle will be formed equal and similar to the original.*

Solution. Let the triangle constructed on BC have its vertex at L, let that constructed on CA have its vertex at M, and that constructed on AB have its vertex at N.

Take the diagram of *Art.* 252 (page 428). The triangle CLB will be equal to the triangle COB in all respects; therefore the angle $BCL=$ the angle $OCB=\dfrac{\pi}{2}-A$.

In the same manner the angle $ACM=\dfrac{\pi}{2}-B$;

therefore the angle $LCM=\dfrac{\pi}{2}-A+\dfrac{\pi}{2}-B+C=2C$.

Then $(LM)^2=R^2+R^2-2R^2\cos 2C=2R^2(1-\cos 2C)=4R^2\sin^2 C$;

therefore $$LM=2R\sin C=c.$$

In a similar manner we find that $MN=a$, and $NL=b$. Thus the triangle LMN is in all respects equal to the triangle ABC.

Problem 22. *If R be the radius of the circumscribed circle of a triangle,*

$$a\cos A+b\cos B+c\cos C=4R\sin A\sin B\sin C.$$

Solution.

$$\begin{aligned}a\cos A+b\cos B+c\cos C&=2R\sin A\cos A+2R\sin B\cos B+2R\sin C\cos C\\&=R(\sin 2A+\sin 2B+\sin 2C)\\&=4R\sin A\sin B\sin C,\ \text{by } \textit{Art. } 114 \text{ (page 409)}.\end{aligned}$$

Problem 23. *O is the centre of the circle circumscribed about a triangle ABC; from O the perpendiculars OD, OE, OF are drawn to the sides. Show that*

$$4\left(OD^2+OE^2+OF^2\right)=a^2\cot^2 A+b^2\cot^2 B+c^2\cot^2 C.$$

Solution. $$OD^2=R^2\cos^2 A=\frac{a^2}{4\sin^2 A}\cos^2 A=\frac{a^2}{4}\cot^2 A,$$

$$OE^2 = R^2\cos^2 B = \frac{b^2}{4\sin^2 B}\cos^2 B = \frac{b^2}{4}\cot^2 B,$$
$$OF^2 = R^2\cos^2 C = \frac{c^2}{4\sin^2 C}\cos^2 C = \frac{c^2}{4}\cot^2 C.$$
$$\therefore\ 4\left(OD^2 + OE^2 + OF^2\right) = a^2\cot^2 A + b^2\cot^2 B + c^2\cot^2 C.$$

Problem 24. *If r be the radius of the circle inscribed in a triangle, and r_a the radius of the circle inscribed between this circle and the sides containing the angle A, show that*

$$r_a = r\frac{1-\sin\dfrac{A}{2}}{1+\sin\dfrac{A}{2}} = r\frac{\left(\cos\dfrac{A}{4} - \sin\dfrac{A}{4}\right)^2}{\left(\cos\dfrac{A}{4} + \sin\dfrac{A}{4}\right)^2}.$$

Solution. Take the diagram of *Art.* 248 (page 426). The circle which is to be drawn will have its centre, and its point of contact with the circle already drawn, on the straight line OA. Thus the length of $OA = r + r_a + r_a \operatorname{cosec}\dfrac{A}{2}$; and this distance also $= r\operatorname{cosec}\dfrac{A}{2}$; therefore

$$r_a\left(1+\operatorname{cosec}\frac{A}{2}\right) = r\left(\operatorname{cosec}\frac{A}{2} - 1\right);$$
$$\therefore r_a = \frac{r\left(1-\sin\dfrac{A}{2}\right)}{1+\sin\dfrac{A}{2}} = \frac{r\left(\cos\dfrac{A}{4} - \sin\dfrac{A}{4}\right)^2}{\left(\cos\dfrac{A}{4} + \sin\dfrac{A}{4}\right)^2}.$$

Problem 25. *If r be the radius of the circle inscribed in a triangle, and r_a, r_b, r_c, the radii of the circles inscribed between this circle and the sides containing the angles A, B, C respectively. Show that*

$$\sqrt{r_a r_b} + \sqrt{r_b r_c} + \sqrt{r_c r_a} = r.$$

Solution. By *Problem* 24 we have

$$r_a r_b = \frac{r^2\left(\cos\dfrac{A}{4} - \sin\dfrac{A}{4}\right)^2\left(\cos\dfrac{B}{4} - \sin\dfrac{B}{4}\right)^2}{\left(\cos\dfrac{A}{4} + \sin\dfrac{A}{4}\right)^2\left(\cos\dfrac{B}{4} + \sin\dfrac{B}{4}\right)^2};$$
$$\therefore \sqrt{r_a r_b} = \frac{r\left(\cos\dfrac{A}{4} - \sin\dfrac{A}{4}\right)\left(\cos\dfrac{B}{4} - \sin\dfrac{B}{4}\right)}{\left(\cos\dfrac{A}{4} + \sin\dfrac{A}{4}\right)\left(\cos\dfrac{B}{4} + \sin\dfrac{B}{4}\right)};$$
$$= \frac{r\left(\cos\dfrac{A}{4} - \sin\dfrac{A}{4}\right)\left(\cos\dfrac{B}{4} - \sin\dfrac{B}{4}\right)\left(\cos\dfrac{C}{4} + \sin\dfrac{C}{4}\right)}{\left(\cos\dfrac{A}{4} + \sin\dfrac{A}{4}\right)\left(\cos\dfrac{B}{4} + \sin\dfrac{B}{4}\right)\left(\cos\dfrac{C}{4} + \sin\dfrac{C}{4}\right)};$$

$$= \frac{r\cos\frac{A+\pi}{4}\cos\frac{B+\pi}{4}\cos\frac{C-\pi}{4}}{\cos\frac{A-\pi}{4}\cos\frac{B-\pi}{4}\cos\frac{C-\pi}{4}}$$

$$= \frac{r\left(\cos\frac{A}{2}+\cos\frac{B}{2}-\cos\frac{C}{2}\right)}{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}, \text{ by Examples VIII. 20 and 21.}$$

Similar expressions can be found for $\sqrt{r_b r_c}$ and $\sqrt{r_c r_a}$; and the sum of the three expressions $= r$.

Problem 26. *If a triangle $A'B'C'$ be formed by joining the feet of the perpendiculars let fall from A, B, C on the opposite sides, show that $B'C'$ is numerically equal to $R\sin 2A$, where R is the radius of the circle circumscribed about ABC.*

Solution. Suppose A to be acute; then $AB' = c\cos A$, $AC' = b\cos A$, and

$$\begin{aligned}(B'C')^2 &= AB'^2 + AC'^2 - 2AC'\cdot AB'\cos A\\ &= \cos^2 A\left(c^2+b^2-2bc\cos A\right)\\ &= a^2\cos^2 A;\\ \therefore B'C' &= a\cos A = 2R\sin A\cos A = R\sin 2A.\end{aligned}$$

If A is obtuse we find that $AB' = c\cos(\pi - A)$, $AC' = b\cos(\pi - A)$, and $(B'C')^2 = a^2\cos^2 A$ as before.

Problem 27. *Perpendiculars drawn from the angular points of an acute-angled triangle to the opposite sides meet those sides at the points D, E, F. Show that if R and R_1 be the radii of the circles described about the triangles ABC and DEF respectively, and r_1 the radius of the circle inscribed in the latter triangle,*

$$R_1 = \frac{1}{2}R, \text{ and } r_1 = 2R\cos A\cos B\cos C.$$

Solution. Let P denote the point of intersection of AD and BE.

Then since PEC and PDC are right angles a circle would go round $PECD$; therefore the angle $PDE =$ the angle $PCE = \frac{\pi}{2} - A$. Similarly $PDF = \frac{\pi}{2} - A$. Therefore $FDE = \pi - 2A$.

$$R_1 = \frac{FE}{2\sin FDE} = \frac{FE}{2\sin 2A} = \frac{R\sin 2A}{2\sin 2A}, \text{ by Example 26,} = \frac{1}{2}R.$$

And

$$\begin{aligned}r_1 &= \frac{\text{area of } FDE}{\text{semi-perimeter of } FDE} = \frac{FD\cdot ED\sin 2A}{R(\sin 2A+\sin 2B+\sin 2C)}\\ &= \frac{R\sin 2A\sin 2B\sin 2C}{\sin 2A+\sin 2B+\sin 2C}, \text{ by } \textit{Problem } 26,\\ &= \frac{R\sin 2A\sin 2B\sin 2C}{4\sin A\sin B\sin C}, \text{ by } \textit{Art. } 114 \text{ (page 409)}, = 2R\cos A\cos B\cos C.\end{aligned}$$

Problem 28. *If r, r_1, r_2, r_3 denote the radii of the inscribed and escribed circles* *of a triangle, show that*

$$\tan^2\frac{A}{2}=\frac{rr_1}{r_2r_3}.$$

Solution.

$$\frac{rr_1}{r_2r_3}=\frac{S^2}{s(s-a)}\div\frac{S^2}{(s-b)(s-c)}=\frac{(s-b)(s-c)}{s(s-a)}=\tan^2\frac{A}{2}.$$

Problem 29. *If A be the area of the circle inscribed in a triangle, A_1, A_2, A_3 the areas of the escribed circles, then*

$$\frac{1}{\sqrt{A}}=\frac{1}{\sqrt{A_1}}+\frac{1}{\sqrt{A_2}}+\frac{1}{\sqrt{A_3}}.$$

Solution.

$$\frac{1}{\sqrt{A}}=\frac{1}{\sqrt{\pi r^2}}=\frac{1}{\sqrt{\pi}}\cdot\frac{1}{r}=\frac{1}{\sqrt{\pi}}\cdot\frac{s}{S}.$$

Similarly $\dfrac{1}{\sqrt{A_1}}=\dfrac{1}{\sqrt{\pi}}\cdot\dfrac{s-a}{S}$, $\quad\dfrac{1}{\sqrt{A_2}}=\dfrac{1}{\sqrt{\pi}}\cdot\dfrac{s-b}{S}$, $\quad\dfrac{1}{\sqrt{A_3}}=\dfrac{1}{\sqrt{\pi}}\cdot\dfrac{s-c}{S}$,

$$\therefore\ \frac{1}{\sqrt{A_1}}+\frac{1}{\sqrt{A_2}}+\frac{1}{\sqrt{A_3}}=\frac{1}{\sqrt{\pi}}\left(\frac{s-a}{S}+\frac{s-b}{S}+\frac{s-c}{S}\right)$$
$$=\frac{1}{\sqrt{\pi}}\cdot\frac{3s-a-b-c}{S}=\frac{1}{\sqrt{\pi}}\cdot\frac{s}{S}.$$

Problem 30. *If the sides of a triangle be in arithmetical progression, the perpendicular on the mean side from the opposite angle, and the radius of the circle which touches the mean side and the other two sides produced, are each equal to three times the radius of the inscribed circle.*

Solution. Suppose a, b, c to be in Arithmetic Progression; so that $2b=a+c$.

The perpendicular on the mean side from the opposite angle

$$=a\sin C=\frac{ab\sin C}{b}=\frac{2S}{b}.$$

The radius of the circle which touches the mean side and the other two sides produced $=\dfrac{S}{s-b}=\dfrac{2S}{a+c-b}=\dfrac{2S}{b}$.

The radius of the inscribed circle $=\dfrac{S}{s}=\dfrac{2S}{a+b+c}=\dfrac{2S}{3b}$.

The first and the second of these are each three times the third.

Problem 31. *The distances of the centre of the circle inscribed in a triangle from the centres of the three escribed circles are respectively equal to*

$$a\sec\frac{A}{2},\ b\sec\frac{B}{2},\ c\sec\frac{C}{2}.$$

Solution. Let O denote the centre of the inscribed circle, and P the centre of the escribed circle which is opposite to the angle A. Then O and P are both on the straight line which bisects the angle A.

$$\angle OBP=\frac{1}{2}B+\frac{1}{2}(\pi-B)=\frac{\pi}{2}.$$

Thus $$OP = \frac{OB}{\cos BOP} = \frac{OB}{\cos \frac{1}{2}(A+B)} = \frac{OB}{\sin \frac{C}{2}};$$

and $$OB = \frac{AB \sin \frac{1}{2}A}{\sin\left(\pi - \frac{A}{2} - \frac{B}{2}\right)} = \frac{c \sin \frac{1}{2}A}{\sin \frac{A+B}{2}} = \frac{c \sin \frac{1}{2}A}{\cos \frac{C}{2}}.$$

$$\therefore OP = \frac{c \sin \frac{1}{2}A}{\sin \frac{C}{2} \cos \frac{C}{2}} = \frac{2c \sin \frac{1}{2}A}{\sin C} = \frac{2a \sin \frac{1}{2}A}{\sin A} = \frac{a}{\cos \frac{A}{2}}.$$

Problem 32. *Two similar triangles have a common escribed circle touching sides not homologous a_1, b_2. Show that*

$$a_1 : a_2 = \sin B + \sin C - \sin A : \sin A + \sin C - \sin B.$$

Solution. Let a_1, b_1 c_1 be the sides of one triangle, S_1 its area; let a_2, b_2, c_2 be the sides of the other triangle, S_2 its area.

Then, by hypothesis, $$\frac{S_1}{b_1 + c_1 - a_1} = \frac{S_2}{a_2 + c_2 - b_2};$$

$$\therefore \frac{S_1}{S_2} = \frac{b_1 + c_1 - a_1}{a_2 + c_2 - b_2} = \frac{a_1 \frac{\sin B}{\sin A} + a_1 \frac{\sin C}{\sin A} - a_1}{a_2 + a_2 \frac{\sin C}{\sin A} - a_2 \frac{\sin B}{\sin A}}$$
$$= \frac{a_1}{a_2} \cdot \frac{\sin B + \sin C - \sin A}{\sin A + \sin C - \sin B}.$$

But the areas of similar triangles are as the square of their homologous sides; thus $\frac{S_1}{S_2} = \frac{a_1^2}{a_2^2}$; therefore, finally,

$$\frac{a_1}{a_2} = \frac{\sin B + \sin C - \sin A}{\sin A + \sin C - \sin B}.$$

Problem 33. *If O_1, O_2, O_3 are the centres of the escribed circles of a triangle, then the area of the triangle $O_1 O_2 O_3$*

$$= \text{area of triangle } ABC \left\{1 + \frac{a}{b+c-a} + \frac{b}{a+c-b} + \frac{c}{a+b-c}\right\}.$$

Solution. The points O_2, O_3, and A are in a straight line; similarly O_3, O_1, and B are in a straight line; and O_1, O_2, and C are in a straight line.

The triangle $O_1O_2O_3$ consists of four parts; namely ABC, O_1BC, O_2CA, and O_3AB.

The area of $O_1BC = \frac{1}{2}ar_1 = \frac{aS}{2(s-a)} = \frac{aS}{b+c-a}$.

Similar expressions hold for the areas of O_2CA and O_3AB.

Thus the area of $O_1O_2O_3 = S\left(1 + \frac{a}{b+c-a} + \frac{b}{a+c-b} + \frac{c}{a+b-c}\right)$.

Problem 34. *The centres of the three escribed circles of a triangle are joined. Show that the area of the triangle thus formed is* $\frac{abc}{2r}$*, where r is the radius of the inscribed circle of the original triangle.*

Solution. Here we have another expression for the area of the triangle considered in the preceding solution.

We have
$$\frac{O_1C}{BC} = \frac{\sin\frac{1}{2}(\pi - B)}{\sin\frac{1}{2}(B+C)} = \frac{\cos\frac{B}{2}}{\cos\frac{A}{2}};$$
$$\therefore O_1C = \frac{a\cos\frac{B}{2}}{\cos\frac{A}{2}} = \frac{2R\sin A\cos\frac{B}{2}}{\cos\frac{A}{2}} = 4R\sin\frac{A}{2}\cos\frac{B}{2}.$$
Similarly
$$O_2C = 4R\sin\frac{B}{2}\cos\frac{A}{2};$$
$$\therefore O_1O_2 = 4R\left(\sin\frac{A}{2}\cos\frac{B}{2} + \sin\frac{B}{2}\cos\frac{A}{2}\right)$$
$$= 4R\sin\frac{A+B}{2} = 4R\cos\frac{C}{2}.$$
In like manner
$$O_1O_3 = 4R\cos\frac{B}{2}.$$
Then area of
$$O_1O_2O_3 = \frac{1}{2}O_1O_2 \times O_1O_3 \times \sin O_2O_1O_3$$
$$= 8R^2\cos\frac{C}{2}\cos\frac{B}{2}\sin\frac{B+C}{2}$$
$$= 8R^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$
$$= 8R^2\sqrt{\frac{s(s-a)}{bc}} \times \sqrt{\frac{s(s-b)}{ac}} \times \sqrt{\frac{s(s-c)}{ab}}$$
$$= \frac{8R^2sS}{abc} = \frac{abcs}{2S} = \frac{abc}{2r}.$$

Problem 35. *A', B', C' are the centres of the escribed circles of a triangle; A', B', C' are joined so as to form a triangle. If r and r' be the radii of the circles inscribed in ABC and $A'B'C'$ respectively,*
$$\frac{r'}{r} = \frac{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}{\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}}.$$

Solution. We have
$$r' = \frac{\text{area of } A'B'C'}{\text{semi-perimeter of } A'B'C'} = \frac{\frac{abc}{2r}}{2R\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right)},$$

by the solution of the preceding Problem,

$$= \frac{abc}{4Rr\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)} = \frac{S}{r\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)}$$

$$= \frac{s}{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}.$$

Also $$\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} = \frac{s^2}{S} = \frac{s}{r};$$

(see the solution of Problem 20 : page 180),

$$\therefore r' = \frac{r\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}.$$

Problem 36. *If r be the radius of the circle inscribed in a triangle ABC, $2s$ the sum of the sides, r', $2s'$ similar quantities for the triangle which is formed by joining the centres of the escribed circles, show that*

$$\frac{rs}{r's'} = 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Solution.

We have $$r' = \frac{\text{area of } A'B'C'}{s'};$$

$$\therefore r's' = \text{ area of } A'B'C' = \frac{abc}{2r}, \text{ by } Problem \text{ 34 (page 185).}$$

Again, $$rs = \text{ area of } ABC = S.$$

$$\therefore \frac{rs}{r's'} = \frac{2rS}{abc} = \frac{2S^2}{abcs}.$$

And

$$2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 2\sqrt{\frac{(s-b)(s-c)}{bc}\times\frac{(s-c)(s-a)}{ac}\times\frac{(s-a)(s-b)}{ab}}$$

$$= \frac{2S^2}{abcs} = \frac{rs}{r's'}.$$

Problem 37. *Let α, α_1 be the distances of the angle A of a triangle from the centres of the inscribed circle, and the circle touching the side a and the other two sides produced; β, β_1 similar quantities for the angle B; γ, γ_1 similar quantities for the angle C. Show that*

$$\alpha\beta\gamma\alpha_1\beta_1\gamma_1 = (abc)^2.$$

Solution. We have

$$\alpha = r\operatorname{cosec}\frac{A}{2}, \qquad \alpha_1 = r_1\operatorname{cosec}\frac{A}{2},$$

$$\beta = r\operatorname{cosec}\frac{B}{2}, \qquad \beta_1 = r_2\operatorname{cosec}\frac{B}{2},$$

$$\gamma = r\operatorname{cosec}\frac{C}{2}, \qquad \gamma_1 = r_3\operatorname{cosec}\frac{C}{2},$$

$$\therefore \alpha\beta\gamma\alpha_1\beta_1\gamma_1$$
$$= r^3 r_1 r_2 r_3 \operatorname{cosec}^2 \frac{A}{2} \operatorname{cosec}^2 \frac{B}{2} \operatorname{cosec}^2 \frac{C}{2}$$
$$= \frac{S^3}{s^3} \times \frac{S^3}{(s-a)(s-b)(s-c)}$$
$$\times \frac{bc}{(s-c)(s-b)} \times \frac{ca}{(s-a)(s-c)} \times \frac{ab}{(s-a)(s-b)}$$
$$= \frac{S^6 a^2 b^2 c^2}{S^6} = a^2 b^2 c^2.$$

Problem 38. *Show also that* $\frac{bc}{\alpha_1^2} + \frac{ca}{\beta_1^2} + \frac{ab}{\gamma_1^2} = 1$.

Solution.

$$\frac{bc}{\alpha_1^2} + \frac{ca}{\beta_1^2} + \frac{ab}{\gamma_1^2} = \frac{bc\sin^2\frac{A}{2}}{r_1^2} + \frac{ca\sin^2\frac{B}{2}}{r_2^2} + \frac{ab\sin^2\frac{C}{2}}{r_3^2}$$
$$= \frac{1}{s^2}\left\{bc\cos^2\frac{A}{2} + ca\cos^2\frac{B}{2} + ab\cos^2\frac{C}{2}\right\}, \text{ by } \textit{Art. } 251 \text{ (page 427)},$$
$$= \frac{1}{s^2}\{s(s-a) + s(s-b) + s(s-c)\} = \frac{1}{s}(s-a+s-b+s-c)$$
$$= \frac{1}{s}(3s-a-b-c) = 1.$$

Problem 39. *Show also that* $\alpha^2\left(\frac{1}{c}-\frac{1}{b}\right) + \beta^2\left(\frac{1}{a}-\frac{1}{c}\right) + \gamma^2\left(\frac{1}{b}-\frac{1}{a}\right) = 0$.

Solution.

$$\alpha^2\left(\frac{1}{c}-\frac{1}{b}\right) + \beta^2\left(\frac{1}{a}-\frac{1}{c}\right) + \gamma^2\left(\frac{1}{b}-\frac{1}{a}\right)$$
$$= \frac{r^2(b-c)}{bc\sin^2\frac{A}{2}} + \frac{r^2(c-a)}{ca\sin^2\frac{B}{2}} + \frac{r^2(a-b)}{ab\sin^2\frac{C}{2}}$$
$$= \frac{r^2}{abc}\left\{\frac{a(b-c)}{\sin^2\frac{A}{2}} + \frac{b(c-a)}{\sin^2\frac{B}{2}} + \frac{c(a-b)}{\sin^2\frac{C}{2}}\right\}$$
$$= \frac{4Rr^2}{abc}\left\{(b-c)\cot\frac{A}{2} + (c-a)\cot\frac{B}{2} + (a-b)\cot\frac{C}{2}\right\}$$
$$= 0, \text{ by Chapter XIII : } \textit{Problem } 29 \text{ (page 140)}.$$

Problem 40. *Show also that* $\frac{b-c}{a\alpha_1^2} + \frac{c-a}{b\beta_1^2} + \frac{a-b}{c\gamma_1^2} = 0$.

Solution. $\dfrac{b-c}{a\alpha_1^2}+\dfrac{c-a}{b\beta_1^2}+\dfrac{a-b}{c\gamma_1^2}$

$= \dfrac{b-c}{ar_1^2}\sin^2\dfrac{A}{2}+\dfrac{c-a}{br_2^2}\sin^2\dfrac{B}{2}+\dfrac{a-b}{cr_3^2}\sin^2\dfrac{C}{2}$

$= \dfrac{1}{s^2}\left\{\dfrac{b-c}{a}\cos^2\dfrac{A}{2}+\dfrac{c-a}{b}\cos^2\dfrac{B}{2}+\dfrac{a-b}{c}\cos^2\dfrac{C}{2}\right\}$, by *Art.* 251 (page 427),

$= \dfrac{1}{4Rs^2}\left\{(b-c)\cot\dfrac{A}{2}+(c-a)\cot\dfrac{B}{2}+(a-b)\cot\dfrac{C}{2}\right\}$

$= 0$, by Chapter XIII : *Problem* 29 (page 140).

Problem 41. *There is only one point within a triangle, such that if perpendiculars be drawn from it to the sides, circles can be inscribed in each of the three resulting quadrilaterals : prove this, and if ρ_1, ρ_2, ρ_3 be the radii of those circles, and ρ that of the inscribed circle of the triangle, then*

$$\left(\frac{1}{\rho_1}-\frac{1}{\rho}\right)\left(\frac{1}{\rho_2}-\frac{1}{\rho}\right)+\left(\frac{1}{\rho_2}-\frac{1}{\rho}\right)\left(\frac{1}{\rho_3}-\frac{1}{\rho}\right)+\left(\frac{1}{\rho_3}-\frac{1}{\rho}\right)\left(\frac{1}{\rho_1}-\frac{1}{\rho}\right)=\frac{1}{\rho^2}.$$

Solution. In order that it may be possible to inscribe a circle within a quadrilateral the sum of one pair of opposite sides must be equal to the sum of the other pair. Now if we take the point O of the diagram of *Art.* 248 (page 426), we see that the condition is satisfied for $OFAE$, $OECD$, and $ODBF$; since $OE+AF=OF+AE$, and so on. We have then to show that no other point but O can be taken.

Take any other point P; from it draw PM perpendicular to AC and PN perpendicular to AB. The centre of a circle inscribed within $PMAN$ must be on the straight line which bisects the angle A; and also on the straight line which bisects the angle NPM; but unless P is on AO, the latter straight line will be *parallel* to AO, the former straight line, and therefore cannot meet it. Thus P must be on AO; similarly it must be on BO and on CO.

Then take the circle inscribed in $OFAE$, and draw perpendiculars from the centre on the sides of the quadrilateral. Thus we have

$$\rho_1(AF+FO+OE+EA)=\text{ twice the area of } OFAE;$$

$$\therefore \rho_1\left\{\rho+\rho\cot\frac{A}{2}\right\}=\rho^2\cot\frac{A}{2};$$

$$\therefore \rho_1=\frac{\rho\cot\frac{A}{2}}{1+\cot\frac{A}{2}};\quad \therefore\ \frac{1}{\rho_1}=\frac{1+\cot\frac{A}{2}}{\rho\cot\frac{A}{2}}.$$

Similarly
$$\frac{1}{\rho_2}=\frac{1+\cot\frac{B}{2}}{\rho\cot\frac{B}{2}}.$$

Thus
$$\left(\frac{1}{\rho_1}-\frac{1}{\rho}\right)\left(\frac{1}{\rho_2}-\frac{1}{\rho}\right)=\frac{1}{\rho^2\cot\frac{A}{2}\cot\frac{B}{2}}=\frac{1}{\rho^2}\tan\frac{A}{2}\tan\frac{B}{2}.$$

In this manner we find that the proposed expression

$$=\frac{1}{\rho^2}\left\{\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}\right\}$$

$$= \frac{1}{\rho^2}, \text{ by Chapter VIII : } \textit{Problem } 25.$$

Problem 42. *A circle is inscribed in a plane triangle ABC. Another circle is inscribed so as to touch the two sides AB, AC, and the last circle; again, a third circle is inscribed so as to touch the same two sides AB, AC, and the second circle, and so on. Circles are also inscribed in the same way so as to touch BC, BA and CA, CB. Show that the area of the inscribed circle is to the sum of the areas of all the other circles as 1 is to*

$$\sin^4 \frac{B+C}{4} \operatorname{cosec} \frac{A}{2} + \sin^4 \frac{C+A}{4} \operatorname{cosec} \frac{B}{2} + \sin^4 \frac{A+B}{4} \operatorname{cosec} \frac{C}{2}.$$

Solution. As in *Problem* 24 (page 182) we shall find that the radii of the circles successively inscribed in the angle A are lr, l^2r, $l^3r, \ldots$ where

$$l = \frac{1 - \sin \dfrac{A}{2}}{1 + \sin \dfrac{A}{2}}.$$

Hence the sum of the areas of all these circles is

$$= \pi \left(l^2r^2 + l^4r^2 + l^6r^2 + \ldots\right);$$

$$= \frac{\pi l^2 r^2}{1 - l^2} = \frac{\pi \left(1 - \sin \dfrac{A}{2}\right)^2 r^2}{4 \sin \dfrac{A}{2}},$$

$$= \frac{\pi \left(1 - \cos \dfrac{B+C}{2}\right)^2 r^2}{4 \sin \dfrac{A}{2}} = \pi r^2 \sin^4 \frac{B+C}{4} \operatorname{cosec} \frac{A}{2}.$$

Similarly we find the areas of the circles inscribed within the angles B and C. Thus the sum of all the areas is

$$\pi r^2 \left\{ \sin^4 \frac{B+C}{4} \operatorname{cosec} \frac{A}{2} + \sin^4 \frac{C+A}{4} \operatorname{cosec} \frac{B}{2} + \sin^4 \frac{A+B}{4} \operatorname{cosec} \frac{C}{2} \right\}.$$

Problem 43. *O and O' are respectively the centres of the circles described about and inscribed in a plane triangle ABC. Join OA, OB, OC, $O'A$, $O'B$, $O'C$, and let R_a, R_b, R_c, r_a, r_b, r_c, be respectively the radii of the circles circumscribing the triangles BOC, COA, AOB, $BO'C$, $CO'A$, $AO'B$. If R be the radius of the circle circumscribing the given triangle ABC, show that*

$$\frac{r_a r_b r_c}{abc} = \frac{R}{a+b+c}, \text{ and } \frac{a}{R_a} + \frac{b}{R_b} + \frac{c}{R_c} = \frac{abc}{R^3}.$$

Solution.
$$R_a = \frac{BC}{2 \sin BOC} = \frac{a}{2 \sin 2A};$$

Similarly
$$R_b = \frac{b}{2 \sin 2B}, \text{ and } R_c = \frac{c}{2 \sin 2C}.$$

Thus
$$\frac{a}{R_a} + \frac{b}{R_b} + \frac{c}{R_c} = 2(\sin 2A + \sin 2B + \sin 2C)$$

$$= 8 \sin A \sin B \sin C, \text{ by } \textit{Art. } 114 \text{ (page 409)},$$

$$= \frac{a}{R} \times \frac{b}{R} \times \frac{c}{R} = \frac{abc}{R^3}.$$

Again,

$$r_a = \frac{BC}{2\sin BO'C} = \frac{a}{2\sin\left(\pi - \frac{B}{2} - \frac{C}{2}\right)} = \frac{A}{2\sin\frac{B+C}{2}} = \frac{a}{2\cos\frac{A}{2}};$$

Similarly, $$r_b = \frac{b}{2\cos\frac{B}{2}}, \text{ and } r_c = \frac{c}{2\cos\frac{C}{2}}.$$

$$\therefore \ \frac{r_a r_b r_c}{abc} = \frac{1}{8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$= \frac{1}{2(\sin A + \sin B + \sin C)}, \text{ by Chapter VIII : } \textit{Problem } 16,$$

$$= \frac{R}{2R\sin A + 2R\sin B + 2R\sin C} = \frac{R}{a+b+c}.$$

Problem 44. *From any point P within or without a triangle ABC, perpendiculars PA', PB', PC' are dropped on the sides BC, CA, AB; and circles are described about the triangles $PA'B'$, $PB'C$, $PC'A'$. Show that the area of the triangle formed by joining the centres of these circles is one-fourth of the area of the triangle ABC.*

Solution. Since the angles at B' and C' are right angles it will follow that A will be on the circumference of the circle which is described round $PB'C'$, and that PA is a diameter of the circle. Let O_1 denote the centre of the circle, then $PO_1 = \frac{1}{2}PA$.

In a similar manner if O_2 is the centre of the circle round $PC'A'$, and O_3 the centre of the circle round $PA'B'$, we have

$$PO_2 = \frac{1}{2}PB, \text{ and } PO_3 = \frac{1}{2}PC.$$

Then is the triangle PO_2O_3 we have

$$O_2O_3^2 = PO_2^2 + PO_3^2 - 2PO_2PO_3\cos O_2PO_3;$$

and in the triangle PBC we have

$$BC^2 = PB^2 + PC^2 - 2PB \cdot PC\cos BPC.$$

Hence $O_2O_3 = \frac{1}{2}BC$. Or this might be obtained by *Euclid* VI. 2, and VI. 4.

Similarly $O_3O_1 = \frac{1}{2}CA$, and $O_1O_2 = \frac{1}{2}AB$. Thus the area of $O_1O_2O_3$ is one-fourth of the area of ABC.

Problem 45. *Three circles touch each other externally. Prove that the square of the area of the triangle formed by joining their centres is equal to the product of the sum and product of their radii.*

Solution. Let r_1, r_2, r_3 denote the radii of the circles; then the sides of the triangle are respectively $r_2 + r_3$, $r_3 + r_1$, and $r_1 + r_2$. Thus

$$s = r_1 + r_2 + r_3, \quad s - a = r_1, \quad s - b = r_2, \quad s - c = r_3.$$

$$\therefore \ S^2 = (r_1 + r_2 + r_3)\, r_1 r_2 r_3.$$

Problem 46. *If the sides of a triangle be in geometrical progression, and the perpendiculars from the angles on the opposite sides be taken as the sides of a new triangle, then the angles of this new triangle will be equal to those of the original triangle.*

Solution. Suppose $a,\ b,\ c$ in Geometrical Progression, so that $b^2 = ac$; let $p_1,\ p_2,\ p_3$ denote the perpendiculars from the opposite angles on $a,\ b,\ c$ respectively.

Then $\frac{1}{2}p_1 a = S$, so that $p_1 = \frac{2S}{a}$; similarly $p_2 = \frac{2S}{b}$, and $p_3 = \frac{2S}{c}$.

Let $A_1,\ B_1,\ C_1$ be the angles opposite $p_1,\ p_2,\ p_3$ respectively in the new triangle.

Then
$$\cos A_1 = \frac{p_2^2 + p_3^2 - p_1^2}{2p_2p_3} = \frac{\frac{1}{b^2} + \frac{1}{c^2} - \frac{1}{a^2}}{\frac{2}{bc}} = \frac{\frac{b^2+c^2}{b^2c^2} - \frac{1}{a^2}}{\frac{2}{bc}}$$
$$= \frac{a^2(b^2+c^2) - b^2c^2}{2a^2bc} = \frac{b^2(a^2-c^2) + a^2c^2}{2a^2bc}$$
$$= \frac{a^2 - c^2 + ac}{2ab} = \frac{a^2 + b^2 - c^2}{2ab} = \cos C.$$

Thus $A_1 = C$. Similarly $C_1 = A$. Therefore $B_1 = B$.

Problem 47. *If $\alpha,\ \beta,\ \gamma$ be the ratios which the sides $a,\ b,\ c$ of a triangle bear to the perpendiculars on them from the opposite angles $A,\ B,\ C$ then*
$$\alpha^2 + \beta^2 + \gamma^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) + 4 = 0.$$

Here
$$\alpha = \frac{a}{c\sin B} = \frac{\sin A}{\sin B \sin C}.$$

Similarly
$$\beta = \frac{\sin B}{\sin C \sin A}, \quad \text{and } \gamma = \frac{\sin C}{\sin A \sin B}.$$

Therefore $2(\beta\gamma + \gamma\alpha + \alpha\beta) - \alpha^2 - \beta^2 - \gamma^2 =$ the product of $\frac{1}{\sin^2 A \sin^2 B \sin^2 C}$ into $\left\{2\sin^2 B \sin^2 C + 2\sin^2 C \sin^2 A + 2\sin^2 A \sin^2 B - \sin^4 A - \sin^4 B - \sin^4 C\right\}$.

The expression within brackets is equal to
$$(\sin A + \sin B + \sin C)(\sin A + \sin B - \sin C)$$
$$(\sin A - \sin B + \sin C)(\sin B + \sin C - \sin A),$$
as we know from a similar process in *Art.* 218 (page 420).

Then, by Chapter VIII : *Problems* 16 and 17, we obtain
$$4^4 \sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2} = 4\sin^2 A \sin^2 B \sin^2 C.$$

Hence
$$2(\beta\gamma + \gamma\alpha + \alpha\beta) - \alpha^2 - \beta^2 - \gamma^2 = 4,$$
and therefore
$$\alpha^2 + \beta^2 + \gamma^2 - 2(\beta\gamma + \gamma\alpha + \alpha\beta) + 4 = 0.$$

Problem 48. *On the sides of any triangle equilateral triangles are described externally, and their centres are joined. Show that the triangle thus formed is equilateral.*

Solution. Let $P,\ Q,\ R$ be the centres of the equilateral triangles described on $BC,\ CA,\ AB$ respectively.

Then
$$PQ^2 = PC^2 + QC^2 - 2PC \cdot QC \cos PCQ;$$

also $$PC = \frac{a}{\sqrt{3}}, \text{ and } QC = \frac{b}{\sqrt{3}}.$$

Thus
$$\begin{aligned} 3PQ^2 &= a^2 + b^2 - 2ab\cos(C + 60^\circ) \\ &= a^2 + b^2 - 2ab(\cos C\cos 60^\circ - \sin C\sin 60^\circ) \\ &= a^2 + b^2 - ab\cos C + ab\sin C\sqrt{3} \\ &= a^2 + b^2 - \frac{a^2 + b^2 - c^2}{2} + ab\sin C\sqrt{3} \\ &= \frac{a^2 + b^2 + c^2}{2} + 2S\sqrt{3}. \end{aligned}$$

We shall obtain the same symmetrical expression for $3QR^2$ and $3RP^2$.

Thus $PQ = QR = RP$.

Problem 49. *The sides of a triangle are 65 and 25, and the difference of the opposite angles is 60°. Find all the angles, having given*

$$\log 3 = .4771213, \quad \log 2 = .3010300,$$
$$L\tan 52^\circ 24' = 10.1134508, \quad L\tan 52^\circ 25' = 10.1137122.$$

Solution.

We have
$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2};$$
$$\therefore\ \cot\frac{A}{2} = \frac{65+25}{65-25}\tan 30^\circ = \frac{9}{4}\cdot\frac{1}{\sqrt{3}} = \frac{3\sqrt{3}}{4};$$
$$\therefore\ L\cot\frac{A}{2} = 10 + \frac{3}{2}\log 3 - 2\log 2 = 10.1136219.$$

10.1137122	10.1136219	
10.1134508	10.1134508	
.0002614	.0001711	$.0002614 : .0001711 :: 60'' : x''$;

This gives $x = 39$; $\therefore\ \frac{A}{2} = 37^\circ 36' - 39'' = 37^\circ 35' 21''$. Therefore $A = 75^\circ 10' 42''$. Thus $B + C = 180^\circ - 75^\circ 10' 42''$; and $B - C = 60^\circ$. Therefore $B = 82^\circ 24' 39''$ and $C = 22^\circ 24' 39''$.

Problem 50. *If perpendiculars be drawn from the angles of an acute-angled triangle to the opposite sides, show that the sides of the triangle formed by joining the feet of those perpendiculars are $a\cos A$, $b\cos B$, and $c\cos C$; and thence show that*

$$\frac{a^2\cos^2 A - b^2\cos^2 B - c^2\cos^2 C}{2bc\cos B\cos C} = \cos 2A.$$

Solution. In the solution of *Problem* 26 (page 183), it is shown that the sides of the new triangle are $a\cos A$, $b\cos B$, and $c\cos C$ respectively.

In the solution of *Problem* 27 (page 183), it is shown that the angles of the new triangle are $\pi - 2A$, $\pi - 2B$, and $\pi - 2C$ respectively. Then, by *Art.* 215 (page 419),

$$\cos(\pi - 2A) = \frac{b^2\cos^2 B + c^2\cos^2 C - a^2\cos^2 A}{2bc\cos B\cos C};$$

but $\cos(\pi - 2A) = -\cos 2A$. Therefore

$$\cos 2A = \frac{a^2\cos^2 A - b^2\cos^2 B - c^2\cos^2 C}{2bc\cos B\cos C}.$$

Problem 51. *Six circles are inscribed between the three escribed circles of a triangle and the angular points, each touching a side and a side produced. Show that the products of their radii taken alternately are equal.*

Solution. Let ρ_1 denote the radius of the circle which touches BD, BF and the arc DF in the diagram of *Art.* 250 (page 427). Let ρ_2 denote the radius of the circle which touches CD, CE, and the arc DE.

The angle $DBF = \pi - B$. Hence, by the method of *Problem* 24 (page 182), we have

$$\rho_1 = r_1\frac{1-\sin\dfrac{\pi - B}{2}}{1+\sin\dfrac{\pi - B}{2}} = r_1\frac{1-\cos\dfrac{B}{2}}{1+\cos\dfrac{B}{2}} = r_1\tan^2\frac{B}{4}.$$

Similarly
$$\rho_2 = r_1\tan^2\frac{C}{4}.$$

In this was we see that the product of three of the radii

$$= r_1\tan^2\frac{B}{4}\times r_2\tan^2\frac{C}{4}\times r_3\tan^2\frac{A}{4};$$

and the product of the other three

$$= r_1\tan^2\frac{C}{4}\times r_2\tan^2\frac{A}{4}\times r_3\tan^2\frac{B}{4}.$$

The two products are equal.

Problem 52. *Straight lines are drawn from the angles A, B, C of a triangle through any point P meeting the opposite sides of the triangle at the points A', B', C' respectively. Show that*

$$AB'\cdot BC'\cdot CA' = AC'\cdot BA'\cdot CB'.$$

Solution.

$$\frac{AB'}{AP} = \frac{\sin APB'}{\sin AB'P}; \quad \therefore\ AB' = \frac{AP\sin APB'}{\sin AB'P}.$$

Similarly
$$BC' = \frac{BP\sin BPC'}{\sin BC'P}, \text{ and } CA' = \frac{CP\sin CPA'}{\sin CA'P}.$$

Thus
$$AB'\cdot BC'\cdot CA' = \frac{AP\cdot BP\cdot CP\sin APB'\sin BPC'\sin CPA'}{\sin AB'P\cdot\sin BC'P\cdot\sin CA'P}.$$

In like manner

$$AC'\cdot BA'\cdot CB' = \frac{AP\cdot BP\cdot CP\sin APC'\sin BPA'\sin CPB'}{\sin AC'P\cdot\sin BA'P\cdot\sin CB'P}.$$

The two expressions are obviously equal; for $\sin APB' = \sin BPA'$, $\sin BPC' = \sin B'PC$, and $\sin CPA' = \sin C'PA$. Also, $\sin AB'P = \sin CB'P$, and so on.

Problem 53. *Show conversely that if the relation just expressed holds then the straight lines AA', BB', CC' meet at a point.*

Solution. Let P denote the intersection of AA' and BB'; then, if CC' does not pass through P, let a straight line be drawn from C through P, and let it meet AB at C_1.

Then, by the Example, we have

$$AB' \cdot BC_1 \cdot CA' = AC_1 \cdot BA' \cdot CB'.$$

But by hypothesis,

$$AB' \cdot BC' \cdot CA' = AC' \cdot BA' \cdot CB'.$$

$$\therefore\ \frac{BC_1}{BC'} = \frac{AC_1}{AC'};$$

$$\therefore\ \frac{BC' - C_1C'}{BC'} = \frac{AC' + C_1C'}{AC'},$$

$$\therefore\ -\frac{C_1C'}{BC'} = \frac{C_1C'}{AC'};$$

$$\therefore\ C_1C' = 0;$$

Therefore C_1 must coincide with C'.

Problem 54. *Show that the perpendiculars from the angles of a triangle on the opposite sides meet at a point.*

Solution. Let the feet of the perpendiculars from A, B, C be denoted by A', B', C' respectively. If all the angles are acute, we have

$$AB' = c\cos A, \quad BC' = a\cos B, \quad CA' = b\cos C,$$
$$AC' = b\cos A, \quad BA' = c\cos B, \quad CB' = a\cos C;$$

thus $$AB' \cdot BC' \cdot CA' = AC' \cdot BA' \cdot CB'.$$

Therefore, by *Problem* 53 (page 194), the straight lines AA', BB' and CC' meet at a point.

Next suppose one angle obtuse, say C. Then

$$CA' = b\cos(180^\circ - C)\,, \text{ and } CB' = a\cos(180^\circ - C)\,;$$

the other expressions remain as before, and the result holds as before.

Problem 55. *Show that the straight lines which bisect the internal angles of a triangle meet at a point.*

Solution. Let the straight lines which bisect the angles A, B, C respectively meet the opposite sides at A', B', C' respectively. Then

$$\frac{AB'}{BB'} = \frac{\sin\frac{B}{2}}{\sin A}, \quad \frac{BC'}{CC'} = \frac{\sin\frac{C}{2}}{\sin B}, \quad \frac{CA'}{AA'} = \frac{\sin\frac{A}{2}}{\sin C};$$

$$\therefore\ AB' \cdot BC' \cdot CA' = AA' \cdot BB' \cdot CC' \frac{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\sin A \sin B \sin C};$$

the same value may be obtained for $AC' \cdot BA' \cdot CB'$.

Therefore, by *Problem* 53 (page 194), the straight lines AA', BB', and CC' meet at a point.

Problem 56. *Show that the straight lines which join the angles of a triangle with the middle points of the opposite sides meet at a point.*

Solution. Let A', B', C' denote the middle points of BC, CA, AB respectively. Then

$$AB' \cdot BC' \cdot CA' = \frac{b}{2} \times \frac{c}{2} \times \frac{a}{2} = \frac{abc}{8}.$$

Similarly $$AC' \cdot BA' \cdot CB' = \frac{abc}{8}.$$

Therefore, by *Problem* 53 (page 194), the straight lines AA', BB', and CC' meet at a point.

Problem 57. *Show that the straight lines which join the angles of a triangle with the points where the inscribed circle touches the opposite sides respectively, meet at a point.*

Solution. Let the points of contact opposite to A, B, C respectively be denoted by A', B', C' respectively.

Then $$AB' = r \cot \frac{A}{2}, \quad BC' = r \cot \frac{B}{2}, \quad CA' = r \cot \frac{C}{2}.$$

Thus $$AB' \cdot BC' \cdot CA' = r^3 \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}.$$

Similarly $$AC' \cdot BA' \cdot CB' = r^3 \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}.$$

Therefore, by *Problem* 53 (page 194), the straight lines AA', BB' and CC' meet at a point.

Problem 58. *Let a straight line be drawn from the angle A of a triangle to the point where the escribed circle opposite to the angle A touches the side opposite to it; let similar straight lines be drawn from B and C with respect to the other escribed circles. Show that these straight lines meet at a point.*

Solution. Let the points of contact opposite to A, B, C respectively be denoted by A', B', C' respectively.

Then
$$AB' = r_2 \cot \frac{\pi - A}{2} = r_2 \tan \frac{A}{2},$$
$$BC' = r_3 \tan \frac{B}{2},$$
$$CA' = r_1 \tan \frac{C}{2};$$
$$\therefore\ AB' \cdot BC' \cdot CA' = r_1 r_2 r_3 \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}.$$

Similarly $$AC' \cdot BA' \cdot CB' = r_1 r_2 r_3 \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}.$$

Therefore, by *Problem* 53 (page 194), the straight lines AA', BB' and CC' meet at a point.

Problem 59. *In the figure of Art. 250 (page 427) show that the straight lines BE, CF and AD meet at a point.*

Solution.

Here $$AE = AF,\ CE = CD,\ BD = BF;$$

$$\therefore\ AE \cdot BF \cdot CD = AF \cdot BD \cdot CE.$$

Therefore, by *Problem* 53 (page 194), the straight lines AD, BE and CF meet at a point.

Problem 60. *A quadrilateral figure is so taken that a circle can be described about it and inscribed in it. It its sides be produced in both directions, and r_a, r_b, r_c, r_d be the radii of the circles, inscribed in the triangles formed on two sides, and escribed on the other two sides, then $r_a r_b r_c r_d = r^4$, where r is the radius of the circle inscribed in the quadrilateral.*

Solution. Let $ABCD$ be the quadrilateral figure. Then, denoting by A, B, C and D the internal angles of the figure, we have

$$r_a\left(\cot\frac{\pi - B}{2} + \cot\frac{\pi - C}{2}\right) = BC;$$

$$\therefore\ r_a\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right) = BC;$$

Again in like manner we have

$$r\left(\cot\frac{A}{2} + \cot\frac{D}{2}\right) = DA,$$

$$\therefore\ r\left(\tan\frac{C}{2} + \tan\frac{B}{2}\right) = DA,$$

for $A + C = \pi$, and $B + D = \pi$, by Euclid III. 22.

Hence
$$\frac{r_a}{r} = \frac{BC}{DA}.$$

In the same manner we can show that

$$\frac{r_b}{r} = \frac{CD}{AB}, \quad \frac{r_c}{r} = \frac{AD}{BC}, \quad \text{and}\ \frac{r_d}{r} = \frac{AB}{DC}.$$

$$\therefore\ \frac{r_a r_c}{r^2} = 1, \quad \text{and}\ \frac{r_b r_d}{r^2} = 1;$$

$$\therefore\ r_a r_b r_c r_d = r^4.$$

CHAPTER XVII
Use of Subsidiary Angles in solving Equations and in adapting Formulae to Logarithmic Computation

Problem 1. *Solve* $x^3 - 6x + 4 = 0$.

Solution. Here $r = -4,\ q = 6$; therefore

$$\cos 3\alpha = -16\left(\frac{3}{24}\right)^{\frac{3}{2}} = -16\left(\frac{1}{8}\right)^{\frac{3}{2}} = -\frac{16}{8}\left(\frac{1}{8}\right)^{\frac{1}{2}} = -\frac{2}{2\sqrt{2}} = -\frac{1}{\sqrt{2}}.$$

Therefore $3\alpha = \frac{3\pi}{4}$. Hence $\alpha = \frac{\pi}{4}$. Therefore the roots are

$$2\left(\frac{6}{3}\right)^{\frac{1}{2}} \cos\frac{\pi}{4}, \text{ and } 2\left(\frac{6}{3}\right)^{\frac{1}{2}} \cos\left(\frac{2\pi}{3} \pm \frac{\pi}{4}\right).$$

Now $$2\left(\frac{6}{3}\right)^{\frac{1}{2}} \cos\frac{\pi}{4} = 2\sqrt{2}\frac{1}{\sqrt{2}} = 2;$$

$$2\left(\frac{6}{3}\right)^{\frac{1}{2}} \cos\left(\frac{2\pi}{3} + \frac{\pi}{4}\right) = 2\sqrt{2}\cos\frac{11\pi}{12} = -2\sqrt{2}\cos\frac{\pi}{12} = -(\sqrt{3}+1);$$

$$2\left(\frac{6}{3}\right)^{\frac{1}{2}} \cos\left(\frac{2\pi}{3} - \frac{\pi}{4}\right) = 2\sqrt{2}\cos\frac{5\pi}{12} = \sqrt{3} - 1.$$

Problem 2. *Show that the roots of the equation* $x^3 - 3x - 1 = 0$ *are* $2\cos 20°$, $-2\sin 10°$, $-2\cos 40°$.

Solution. Here $r = 1,\ q = 3$; therefore $\cos 3\alpha = 4\left(\frac{3}{12}\right)^{\frac{3}{2}} = 4\left(\frac{1}{4}\right)^{\frac{3}{2}} = \frac{1}{2}$; therefore $3\alpha = 60°$; therefore $\alpha = 20°$.

Therefore the roots are $2\cos 20°$, and $2\cos(120° \pm 20°)$.

Also $$2\cos(120° + 20°) = 2\cos 140° = -2\cos 40°.$$

And $$2\cos(120° - 20°) = 2\cos 100° = -2\sin 10°.$$

Problem 3. *Show that the roots of the equation* $x^5 - px^3 + qx - r = 0$ *are*

$$2\left(\frac{p}{5}\right)^{\frac{1}{2}}\cos\alpha,\ 2\left(\frac{p}{5}\right)^{\frac{1}{2}}\cos\left(\frac{2\pi}{5} \pm \alpha\right), \text{ and } 2\left(\frac{p}{5}\right)^{\frac{1}{2}}\cos\left(\frac{4\pi}{5} \pm \alpha\right),$$

where $$\cos 5\alpha = \frac{r}{2}\left(\frac{5}{p}\right)^{\frac{5}{2}},$$

provided $p^2 = 5q$ and $\left(\frac{r}{2}\right)^2$ be not greater than $\left(\frac{p}{5}\right)^5$.

Solution. Take the equation $\quad x^5 - px^3 + qx - r = 0.$

Put $x = ny$; thus $$n^5y^5 - pn^3y^3 + qny - r = 0;$$

therefore $$y^5 - \frac{p}{n^2}y^3 + \frac{q}{n^4}y = \frac{r}{n^5}.$$

Now by *Chapter* VIII : *Problem* 59, $\cos 5\alpha = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$.

Thus $$\cos^5\alpha - \frac{5}{4}\cos^3\alpha + \frac{5}{16}\cos\alpha = \frac{\cos 5\alpha}{16}.$$

Assume $y = \cos\alpha,\ \frac{5}{4} = \frac{p}{n^2},\ \frac{5}{16} = \frac{q}{n^4}$, then $\frac{r}{n^5} = \frac{\cos 5\alpha}{16}$.

Here $\left(\frac{5}{4}\right)^2 = \frac{p^2}{n^4}$, and $\frac{5}{16} = \frac{q}{n^4}$; so that we have $n^4 = \frac{16p^2}{25}$, and $n^4 = \frac{16q}{5}$.
Thus the process will not be admissible unless $p^2 = 5q$; and this condition is satisfied by hypothesis.

Then α must be found from $\cos 5\alpha = \frac{16r}{n^5}$; put for n its value $\left(\frac{4p}{5}\right)^{\frac{1}{2}}$: thus $\cos 5\alpha = 16r \times \left(\frac{5}{4p}\right)^{\frac{5}{2}} = \frac{r}{2}\left(\frac{5}{p}\right)^{\frac{5}{2}}$. Thus process then will not be admissible if this expression is numerically greater than unity. Hence $\left(\frac{r}{2}\right)^2\left(\frac{5}{p}\right)^5$ must not be greater than unity; that is $\left(\frac{r}{2}\right)^2$ must not be greater than $\left(\frac{p}{5}\right)^5$.

Suppose this condition also to hold; then one root is $n\cos\alpha$, that is $2\left(\frac{p}{5}\right)^{\frac{1}{2}}\cos\alpha$.

Moreover we might also suppose $y = \cos\left(\frac{2\pi}{5} \pm \alpha\right)$ or $y = \cos\left(\frac{4\pi}{5} \pm \alpha\right)$, and we shall still arrive at the same value for $\cos 5\alpha$, since
$$\cos 5\left(\frac{2\pi}{5} \pm \alpha\right) = \cos\alpha \text{ and } \cos 5\left(\frac{4\pi}{5} \pm \alpha\right) = \cos 5\alpha.$$

Hence we see that the other roots of the equation are
$$2\left(\frac{p}{5}\right)^{\frac{1}{2}}\cos\left(\frac{2\pi}{5} \pm \alpha\right) \text{ and } 2\left(\frac{p}{5}\right)^{\frac{1}{2}}\cos\left(\frac{4\pi}{5} \pm \alpha\right).$$

Problem 4. *Find the roots of the equation*
$$x^5 - 10x^3 + 20x - 8 = 0.$$

Solution. Proceed as in *Problem* 3.

Here $r = 8,\ q = 20,\ p = 10$.
$$\cos 5\alpha = 4\left(\frac{1}{2}\right)^{\frac{5}{2}} = \frac{1}{\sqrt{2}};\ \text{therefore } 5\alpha = 45^\circ.$$

Hence the roots are
$$2\sqrt{2}\cos 9^\circ,\ 2\sqrt{2}\cos(72^\circ \pm 9^\circ),\ 2\sqrt{2}\cos(144^\circ \pm 9^\circ),$$
that is
$$2\sqrt{2}\cos 9^\circ,\quad 2\sqrt{2}\cos 63^\circ,\quad 2\sqrt{2}\cos 81^\circ,\quad 2\sqrt{2}\cos 153^\circ,\quad 2\sqrt{2}\cos 135^\circ;$$
the last is equal to -2.

Problem 5. *A person wishes to ascertain the side BC of a triangular field ABC, but is only able to make measurement of lines within the boundary of a circle which passes through A and touches BC : show how after measuring four straight lines he may determine BC.*

Solution. Let D denote the point of contact of the circle with BC. Let AC intersect the circumference of the circle at E, and let AB intersect the circumference at F. Then the four straight lines lines AE, ED, DF, FA can be measured. Then, by *Art.* 254 (page 429), the diagonal AD can be determined.

Then all the angles of the triangles ADE and ADF can be found; and thus the angles of the triangles ADC and ADB are known. Thus DC and BD can be found. See *Euclid* III. 32.

Problem 6. *Two men standing at the same point C observe the horizontal angle subtended by two objects A and B; they then both move away, one in the direction AC, the other in the direction BC, until each observes the horizontal angle to be half what it was before. The distance each walked being given and the horizontal angle at C, determine the distance AB.*

Solution. Let D be the point on AC produced through C such that the angle ADB is half the angle ACB; then $CD = CB$. Thus CB is known. Again, let E be the point on BC produced through C such that the angle AEB is half the angle ACB; then $CE = CA$. Thus CA is known. Then in the triangle ACB we know AC, and CB, and the angle ACB; thus AB can be found by *Art.* 215 (page 419).

Problem 7. *The altitude of a balloon is observed at three places A, B, C simultaneously to be 45°, 45° and 60° respectively; A and B are respectively West and North of C : form an equation for determining the height of the balloon.*

Solution. Let x denote the height of the balloon, and a, b, c the sides of the triangle ABC. Let O be the plane of ABC which is vertically under the balloon. Then

$$AO = x\cot 45^\circ = x,\ BO = x\cot 45^\circ = x,\ CO = x\cot 60^\circ = \frac{x}{\sqrt{3}}. \qquad \text{Therefore}$$

$$\cos ACO = \frac{b^2 + \dfrac{x^2}{3} - x^2}{\dfrac{2bx}{\sqrt{3}}} = \frac{3b^2 - 2x^2}{2bx\sqrt{3}}, \quad \cos BCO = \frac{a^2 + \dfrac{x^2}{3} - x^2}{\dfrac{2ax}{\sqrt{3}}} = \frac{3a^2 - 2x^2}{2ax\sqrt{3}}.$$

But ACB is a right angle, and therefore $\cos BCO = \sin ACO$; thus

$$\left(\frac{3b^2 - 2x^2}{2bx\sqrt{3}}\right)^2 + \left(\frac{3a^2 - 2x^2}{2ax\sqrt{3}}\right)^2 = 1;$$

therefore $$a^2\left(3b^2 - 2x^2\right)^2 + b^2\left(3a^2 - 2x^2\right)^2 = 12a^2b^2x^2;$$

therefore $$4x^4\left(a^2 + b^2\right) - 36a^2b^2x^2 + 9a^2b^2\left(a^2 + b^2\right) = 0;$$

therefore $$4c^2x^4 - 36a^2b^2x^2 + 9a^2b^2c^2 = 0.$$

Problem 8. *The distances b and c of a station A from two other stations B and C are known, and the angle BAC is required. It not being practicable to observe the angle BAC, the angle BOC (α) and the angle AOC (β) are observed at a position O situated in the plane ABC, at a small known distance n from A, such that the triangle ABC is entirely within the triangle OBC. Show that if θ be the circular* *measure of the angle $(BAC - BOC)$ then approximately*

$$\theta = n\left\{\frac{\sin(\alpha-\beta)}{b} + \frac{\sin\beta}{c}\right\}.$$

Solution. Here the angle $BAC-$ the angle $BOC =$ the sum of the angles ABO and ACO.

Now $\dfrac{\sin ACO}{\sin AOC} = \dfrac{AO}{AC}$; therefore $\sin ACO = \dfrac{n\sin\beta}{c}$, and since ACO is very small the circular measure of it is nearly equal to the sine, so that it is nearly equal to $\dfrac{n\sin\beta}{c}$.

Again $\dfrac{\sin ABO}{\sin AOB} = \dfrac{AO}{AB}$; therefore $\sin ABO = \dfrac{n\sin(\alpha-\beta)}{b}$, therefore the circular measure of ABO is nearly equal to $\dfrac{n\sin(\alpha-\beta)}{b}$.

Thus the circular measure of $BAC - BOC$ is nearly $n\left\{\dfrac{\sin(\alpha-\beta)}{b} + \dfrac{\sin\beta}{c}\right\}$.

Problem 9. *At a distance of 50 feet from the foot of a tower the elevation of its top is 45°: if the elevation and the distance be correctly measured within 1′ and 1 inch respectively, find approximately the greatest error in the height.*

Solution. If the distance is 50 feet and the elevation is $\dfrac{\pi}{4}$, the height in feet is $50\tan\dfrac{\pi}{4}$, that is 50.

But suppose the distance to be $50+h$, and the elevation to be $\dfrac{\pi}{4}+\alpha$. Then the height is $(50+h)\tan\left(\dfrac{\pi}{4}+\alpha\right)$. If α is very small this is very nearly equal to $(50+h)\left(\tan\dfrac{\pi}{4} + \alpha\sec^2\dfrac{\pi}{4}\right)$, by *Art.* 188 (page 416), that is $(50+h)(1+2\alpha)$.

If h is also very small this is very nearly $50+h+100\alpha$. Now suppose $h = \dfrac{1}{12}$ and $\alpha = \dfrac{\pi}{180\times 60}$; then we obtain $50+\dfrac{1}{12}+\dfrac{\pi}{108}$. Thus the difference between this and the former value is $\dfrac{1}{12}+\dfrac{\pi}{108}$, that is about $\dfrac{1}{12}+\dfrac{1}{36}$, that is, $1\frac{1}{3}$ inches.

Problem 10. *A person standing at a distance a from a tower surmounted by a spire, observes the tower and the spire to subtend the same angle : if b be the known height of the tower, express the height of the spire (c) in terms of b and a.*

If γ be the error in the height of the spire corresponding to a small error β in the height of the tower, show that

$$\frac{\gamma}{c} = \frac{\beta}{b}\left\{1 + \frac{4b^2a^2}{a^4-b^4}\right\}.$$

Solution. Suppose that the tower and the spire each subtend the angle α.

Then $$\tan\alpha = \frac{b}{a}, \text{ and } \tan 2\alpha = \frac{b+c}{a}.$$

Therefore $$\frac{b+c}{a} = \frac{\frac{2b}{a}}{1-\frac{b^2}{a^2}} = \frac{2ab}{a^2-b^2};$$

therefore $b+c = \dfrac{2a^2b}{a^2-b^2}$; therefore $c = \dfrac{2a^2b}{a^2-b^2} - b = \dfrac{(a^2+b^2)b}{a^2-b^2}$.

If however the height of the tower is $b+\beta$, and the height of the spire is $c+\gamma$, we have

$$c+\gamma = \frac{a^2(b+\beta)+(b+\beta)^3}{a^2-(b+\beta)^2}.$$

Hence, by subtraction,

$$\gamma = \frac{a^2(b+\beta)+(b+\beta)^3}{a^2-(b+\beta)^2} - \frac{(a^2+b^2)b}{a^2-b^2}.$$

Now $$(b+\beta)^2 = b^2+2b\beta+\beta^2,$$

and if β is very small this is very nearly $b^2+2b\beta$.

And $$(b+\beta)^3 = b^3+3b^2\beta+3b\beta^2+\beta^3,$$

and if β is very small this is very nearly $b^3+3b^2\beta$.

Thus $$\gamma = \frac{a^2b+b^3+(a^2+3b^2)\beta}{a^2-b^2-2b\beta} - \frac{a^2b+b^3}{a^2-b^2}$$

$$= \frac{(a^2+3b^2)(a^2-b^2)+2(a^2+b^2)b^2}{(a^2-b^2-2b\beta)(a^2-b^2)}\beta$$

$$= \frac{a^4+4a^2b^2-b^4}{(a^2-b^2)(a^2-b^2-2b\beta)}\beta.$$

Therefore $$\frac{\gamma}{c} = \frac{(a^4+4a^2b^2-b^4)\beta}{(a^2-b^2)(a^2-b^2-2b\beta)} \div \frac{(a^2+b^2)b}{a^2-b^2}$$

$$= \frac{\beta}{b}\cdot\frac{a^4+4a^2b^2-b^4}{(a^2+b^2)(a^2-b^2-2b\beta)}.$$

But when β is very small we may put a^2-b^2 for $a^2-b^2-2b\beta$; and thus

$$\frac{\gamma}{c} = \frac{\beta}{b}\cdot\frac{a^4+4a^2b^2-b^4}{(a^4-b^4)}.$$

Problem 11. *The side a of a triangle and the opposite angle A remain constant : show that the small variations of the other sides γ and β are connected by the relation*

$$\gamma\sec C + \beta\sec B = 0.$$

Solution. We have $a^2 = b^2+c^2-2bc\cos A$;

suppose that b is changed to $b+\beta$, and c to $c+\gamma$; thus

$$a^2 = (b+\beta)^2+(c+\gamma)^2-2(b+\beta)(c+\gamma)\cos A.$$

Therefore, by subtraction,

$$2b\beta+\beta^2+2c\gamma+\gamma^2-2(b\gamma+c\beta+\beta\gamma)\cos A = 0.$$

If β and γ are very small this becomes very nearly

$$2b\beta+2c\gamma-2(b\gamma+c\beta)\cos A = 0;$$

therefore $$\beta(b-c\cos A)+\gamma(c-b\cos A) = 0;$$

therefore $\beta a\cos C + \gamma a\cos B = 0$, by *Art*. 216 (page 419).

Therefore $$\frac{\beta}{\cos B} + \frac{\gamma}{\cos C} = 0,$$

therefore $$\beta \sec B + \gamma \sec C = 0.$$

Problem 12. *The angular altitude and breadth of a cylindrical tower on a level plane are observed to be α and β respectively; and at a point a feet nearer the tower they are observed to be α' and β' : find the height and the radius of the tower. Find also the relation existing between α, α', β, β'.*

Solution. Suppose h the height of the tower, r the radius, x the distance of the first place of observation from the centre. Then

$$\frac{x}{r} = \operatorname{cosec} \frac{\beta}{2}, \qquad \frac{x-a}{r} = \operatorname{cosec} \frac{\beta'}{2};$$
$$h = x \tan \alpha, \qquad h = (x-a) \tan \alpha'.$$

Hence $$\frac{a}{r} = \operatorname{cosec} \frac{\beta}{2} - \operatorname{cosec} \frac{\beta'}{2}.$$

This finds r.

Also $$h = x \tan \alpha' - a \tan \alpha' = \frac{h \tan \alpha'}{\tan \alpha} - a \tan \alpha';$$

therefore $$h = \frac{a \tan \alpha \tan \alpha'}{\tan \alpha' - \tan \alpha}.$$

This finds h.

Again, from the first and second equations,

$$\frac{x-a}{x} = \frac{\operatorname{cosec} \dfrac{\beta'}{2}}{\operatorname{cosec} \dfrac{\beta}{2}}.$$

And from the third and fourth equations,

$$\frac{x-a}{x} = \frac{\cot \alpha'}{\cot \alpha}.$$

Therefore $$\frac{\operatorname{cosec} \dfrac{\beta'}{2}}{\operatorname{cosec} \dfrac{\beta}{2}} = \frac{\cot \alpha'}{\cot \alpha}.$$

Problem 13. *In the preceding Problem if the observed angular breadth be subject to an error δ, and if ρ be the greatest consequent error in the calculated radius (r), show that ρ will be given by the equation*

$$\frac{2\rho}{r} = \cot \frac{1}{4} (\beta' - \beta) \left\{ \operatorname{cosec} \frac{\beta}{2} \operatorname{cosec} \frac{\beta'}{2} - \cot \frac{\beta}{2} \cot \frac{\beta'}{2} \right\} \delta.$$

If $\beta = 60°$, $\beta' = 120°$, $\delta =$ the circular measure of $6'$, find approximately the ratio of the greatest error in the calculated radius to the radius.

Solution.

We have $$\frac{a}{r} = \operatorname{cosec} \frac{\beta}{2} - \operatorname{cosec} \frac{\beta'}{2} \tag{20}$$

If we suppose an error δ of the *same* sign to be made in β and β' these errors will tend to compensate each other; the greatest possible error in r will be determined by supposing that errors of *opposite* signs are made in β and β'. Suppose then that instead of β we ought to have $\beta - \delta$, and instead of β' we ought to have $\beta' + \delta$. Then we have

$$\frac{a}{r-\rho} = \operatorname{cosec}\frac{\beta-\delta}{2} - \operatorname{cosec}\frac{\beta'+\delta}{2}.$$

Hence, by subtraction, $\dfrac{a}{r-\rho} - \dfrac{a}{r}$, that is $\dfrac{a\rho}{r(r-\rho)}$

$$= \operatorname{cosec}\frac{\beta-\delta}{2} - \operatorname{cosec}\frac{\beta}{2} - \left\{\operatorname{cosec}\frac{\beta'+\delta}{2} - \operatorname{cosec}\frac{\beta'}{2}\right\}.$$

Therefore, if δ and ρ be very small, we obtain

$$\frac{a\rho}{r^2} = \frac{\delta}{2}\left\{\frac{\cos\frac{\beta}{2}}{\sin^2\frac{\beta}{2}} + \frac{\cos\frac{\beta'}{2}}{\sin^2\frac{\beta'}{2}}\right\};$$ see *Art.* 194 (page 417).

Thus
$$\frac{a\rho}{r^2} = \frac{\delta}{2}\,\frac{\cos\frac{\beta}{2}\left(1-\cos^2\frac{\beta'}{2}\right) + \cos\frac{\beta'}{2}\left(1-\cos^2\frac{\beta}{2}\right)}{\sin^2\frac{\beta}{2}\sin^2\frac{\beta'}{2}}$$

$$= \frac{\delta}{2}\,\frac{\left(\cos\frac{\beta}{2}+\cos\frac{\beta'}{2}\right)\left(1-\cos\frac{\beta}{2}\cos\frac{\beta'}{2}\right)}{\sin^2\frac{\beta}{2}\sin^2\frac{\beta'}{2}}$$

$$= \frac{\delta\cos\frac{\beta+\beta'}{4}\cos\frac{\beta'-\beta}{4}\left(1-\cos\frac{\beta}{2}\cos\frac{\beta'}{2}\right)}{\sin^2\frac{\beta}{2}\sin^2\frac{\beta'}{2}} \tag{21}$$

Now (20) may be put in the form

$$\frac{a}{r} = \frac{\sin\frac{\beta'}{2} - \sin\frac{\beta}{2}}{\sin\frac{\beta'}{2}\sin\frac{\beta}{2}} = \frac{2\sin\frac{\beta'-\beta}{4}\cos\frac{\beta'+\beta}{4}}{\sin\frac{\beta'}{2}\sin\frac{\beta}{2}} \tag{22}$$

Divide (21) by (22) ; then

$$\frac{\rho}{r} = \frac{\delta}{2}\cot\frac{1}{4}(\beta'-\beta)\cdot\frac{1-\cos\frac{\beta}{2}\cos\frac{\beta'}{2}}{\sin\frac{\beta'}{2}\sin\frac{\beta}{2}}$$

$$= \frac{\delta}{2}\cot\frac{1}{4}(\beta'-\beta)\left\{\operatorname{cosec}\frac{\beta'}{2}\operatorname{cosec}\frac{\beta}{2} - \cot\frac{\beta'}{2}\cot\frac{\beta}{2}\right\}.$$

If $\beta = 60°$ and $\beta' = 120°$, we obtain for $\dfrac{2\rho}{r}$ the value

$$\cot 15°\{\operatorname{cosec} 30° \operatorname{cosec} 60° - \cot 30° \cot 60°\}\delta,$$

that is
$$(2+\sqrt{3})\left(\frac{4}{\sqrt{3}} - 1\right)\delta, \text{ that is } \frac{5+2\sqrt{3}}{\sqrt{3}}\delta.$$

Put for δ the circular measure of $6'$, that is $\dfrac{\pi}{1800}$.

Hence $\dfrac{2\rho}{r} = \dfrac{5+2\sqrt{3}}{\sqrt{3}} \times \dfrac{\pi}{1800}$; therefore $\dfrac{\rho}{r} = \dfrac{5+2\sqrt{3}}{\sqrt{3}} \times \dfrac{\pi}{3600}$.

Problem 14. *P, Q, R are three known positions in a straight line, and PQ, QR* *are observed to subtend equal angles at a certain point S : find the error in the calculated distance of S from Q in consequence of a small error α in the observed angles.*

Solution. Let β denote the angle PSQ, and the equal angle QSR; and let ϕ denote the angle SQR.

Then $$\frac{PQ}{SQ} = \frac{\sin PSQ}{\sin SPQ} = \frac{\sin\beta}{\sin(\phi-\beta)}.$$

and $$\frac{QR}{SQ} = \frac{\sin QSR}{\sin SRQ} = \frac{\sin\beta}{\sin(\phi+\beta)},$$

therefore $$\frac{PQ}{QR} = \frac{\sin(\phi+\beta)}{\sin(\phi-\beta)}.$$

Let $PQ = a$, and $QR = b$; thus
$$a\sin(\phi-\beta) = b\sin(\phi+\beta),$$

therefore $$a(\sin\phi\cos\beta - \cos\phi\sin\beta) = b(\sin\phi\cos\beta + \cos\phi\sin\beta);$$

therefore $$\tan\phi = \frac{(a+b)\sin\beta}{(a-b)\cos\beta} = \frac{a+b}{a-b}\tan\beta.$$

Also $$\frac{1}{SQ} = \frac{\sin\beta}{a\sin(\phi-\beta)}, \text{ and } \frac{1}{SQ} = \frac{\sin\beta}{b\sin(\phi+\beta)};$$

$$\therefore \frac{1}{SQ^2} = \frac{\sin^2\beta}{ab\sin(\phi-\beta)\sin(\phi+\beta)} = \frac{\sin^2\beta}{ab(\sin^2\phi - \sin^2\beta)}$$
$$= \frac{\tan^2\beta}{ab\{\sin^2\phi(1+\tan^2\beta) - \tan^2\beta\}}$$

But $$\sin^2\phi = \frac{(a+b)^2\tan^2\beta}{(a-b)^2 + (a+b)^2\tan^2\beta},$$

thus $$\frac{1}{SQ^2} = \frac{(a-b)^2 + (a+b)^2\tan^2\beta}{ab\{(a+b)^2 - (a-b)^2\}} = \frac{(a-b)^2 + (a+b)^2\tan^2\beta}{4a^2b^2}.$$

Suppose that instead of β we ought to have $\beta+\alpha$, and instead of SQ we ought to have $SQ+c$, where α and c are very small. Then
$$\frac{1}{(SQ+c)^2} = \frac{(a-b)^2}{4a^2b^2} + \frac{(a+b)^2}{4a^2+b^2}\tan^2(\beta+\alpha).$$

Hence, by subtraction,
$$\frac{1}{(SQ+c)^2} - \frac{1}{SQ^2} = \frac{(a+b)^2}{4a^2b^2}\{\tan^2(\beta+\alpha) - \tan^2\beta\};$$

therefore $$\frac{SQ^2 - (SQ+c)^2}{SQ^2(SQ+c)^2} = \frac{(a+b)^2}{4a^2b^2}\{(\tan\beta + a\sec^2\beta)^2 - \tan^2\beta\},$$

approximately, by *Art.* 188 (page 416)

Thus $$-\frac{2c}{SQ^3} = \frac{(a+b)^2}{4a^2b^2}2\tan\beta\sec^2\beta\alpha, \text{ nearly;}$$

$$\therefore \frac{c}{SQ^3} = -\frac{(a+b)^2\sin\beta}{4a^2b^2\cos^3\beta}\alpha, \text{ nearly.}$$

CHAPTER XVIII
Inverse Trigonometrical Functions

Problem 1. *Show that* $\tan^{-1}\dfrac{3}{4} = 2\tan^{-1}\dfrac{1}{3}$.

Solution. Let $\tan^{-1}\dfrac{1}{3} = \theta$, then $\tan\theta = \dfrac{1}{3}$; therefore

$$\tan 2\theta = \frac{\frac{2}{3}}{1-\frac{1}{9}} = \frac{6}{8} = \frac{3}{4};$$

therefore $\quad 2\theta = \tan^{-1}\dfrac{3}{4}$. Therefore $\tan^{-1}\dfrac{3}{4} = 2\tan^{-1}\dfrac{1}{3}$.

Problem 2. *Find the value of* $\sin\left(\sin^{-1}\dfrac{1}{2} + \cos^{-1}\dfrac{1}{2}\right)$.

Solution.

Let $\quad \sin^{-1}\dfrac{1}{2} = \theta, \text{ and } \cos^{-1}\dfrac{1}{2} = \phi;$

therefore $\quad \sin\theta = \dfrac{1}{2}, \text{ and } \cos\theta = \dfrac{\sqrt{3}}{2},$

and $\quad \cos\phi = \dfrac{1}{2}, \text{ and } \sin\phi = \dfrac{\sqrt{3}}{2}.$

Therefore $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi = \dfrac{1}{4} + \dfrac{3}{4} = 1.$

Problem 3. *Show that* $\sin^{-1}\dfrac{77}{85} = \sin^{-1}\dfrac{3}{5} + \sin^{-1}\dfrac{8}{17}$.

Solution. Let $\sin^{-1}\dfrac{3}{5} = \alpha$, and $\sin^{-1}\dfrac{8}{17} = \beta$;

then $\quad \sin\alpha = \dfrac{3}{5}, \qquad \cos\alpha = \sqrt{\left(1-\dfrac{9}{25}\right)} = \dfrac{4}{5};$

and $\quad \sin\beta = \dfrac{8}{17}, \qquad \cos\beta = \sqrt{\left(1-\dfrac{64}{289}\right)} = \dfrac{15}{17};$

therefore
$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\\ &= \frac{3\times 15}{5\times 17} + \frac{4\times 8}{5\times 17} = \frac{45+32}{85} = \frac{77}{85};\end{aligned}$$

therefore $\quad \alpha + \beta = \sin^{-1}\dfrac{77}{85}.$

Problem 4. *Find the value of* $\tan\left(\tan^{-1} x + \cot^{-1} x\right)$.

Solution. Let $\alpha = \tan^{-1} x$, and $\beta = \cot^{-1} x$;

then $\tan\alpha = x$, and $\cot\beta = x$; therefore $\tan\beta = \frac{1}{x}$,

and $$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{x + \frac{1}{x}}{1-1} = \frac{x + \frac{1}{x}}{0}.$$

Thus $\tan(\alpha+\beta)$ is infinite.

Problem 5. *Show that* $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$.

Solution. Let $\tan^{-1}\frac{1}{3} = \alpha$, $\tan^{-1}\frac{1}{5} = \beta$, $\tan^{-1}\frac{1}{7} = \gamma$, $\tan^{-1}\frac{1}{8} = \delta$.

Thus $$\tan\alpha = \frac{1}{3}, \text{ and } \tan\beta = \frac{1}{5};$$

therefore $$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{\frac{1}{3} + \frac{1}{5}}{1 - \frac{1}{3}\times\frac{1}{5}} = \frac{8}{14} = \frac{4}{7}.$$

And $$\tan(\gamma+\delta) = \frac{\tan\gamma + \tan\delta}{1 - \tan\gamma\tan\delta} = \frac{\frac{1}{7} + \frac{1}{8}}{1 - \frac{1}{7}\times\frac{1}{8}} = \frac{15}{55} = \frac{3}{11}.$$

Then

$$\tan(\alpha+\beta+\gamma+\delta) = \frac{\tan(\alpha+\beta) + \tan(\gamma+\delta)}{1 - \tan(\alpha+\beta)\tan(\gamma+\delta)} = \frac{\frac{4}{7} + \frac{3}{11}}{1 - \frac{4}{7}\times\frac{3}{11}} = \frac{65}{65} = 1;$$

therefore $$\alpha+\beta+\gamma+\delta = \frac{\pi}{4}.$$

Problem 6. *Show that* $\tan^{-1} a = \tan^{-1}\frac{a-b}{1+ab} + \tan^{-1}\frac{b-c}{1+bc} + \tan^{-1} c$.

Solution.

Let $$\tan^{-1} a = \theta, \text{ and } \tan^{-1} b = \phi;$$

then $$a = \tan\theta, \text{ and } b = \tan\phi;$$

and $$\tan(\theta-\phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} = \frac{a-b}{1+ab}.$$

Thus $$\tan^{-1}\frac{a-b}{1+ab} = \tan^{-1} a - \tan^{-1} b.$$

Similarly $$\tan^{-1}\frac{b-c}{1+bc} = \tan^{-1} b - \tan^{-1} c.$$

Therefore $$\tan^{-1}\frac{a-b}{1+ab} + \tan^{-1}\frac{b-c}{1+bc} = \tan^{-1} a - \tan^{-1} c,$$

and $$\tan^{-1}\frac{a-b}{1+ab} + \tan^{-1}\frac{b-c}{1+bc} + \tan^{-1} c = \tan^{-1} a.$$

Problem 7. *Find the tangent of* $3\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{26}-\frac{\pi}{4}$.

Solution.

Let $\alpha=\tan^{-1}\frac{1}{7},\qquad \beta=\tan^{-1}\frac{1}{3},\qquad \gamma=\tan^{-1}\frac{1}{26}.$

$$\tan 3\alpha=\frac{3\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha}=\frac{\frac{3}{7}-\frac{1}{7^3}}{1-\frac{3}{7^2}}=\frac{146}{322}=\frac{73}{161}.$$

$$\tan(\beta+\gamma)=\frac{\tan\beta+\tan\gamma}{1-\tan\beta\tan\gamma}=\frac{\frac{1}{3}+\frac{1}{26}}{1-\frac{1}{3\times 26}}=\frac{29}{77}.$$

$$\tan(3\alpha+\beta+\gamma)=\frac{\tan 3\alpha+\tan(\beta+\gamma)}{1-\tan 3\alpha\tan(\beta+\gamma)}=\frac{\frac{73}{161}+\frac{29}{77}}{1-\frac{73}{161}\times\frac{29}{77}}$$
$$=\frac{10290}{10280}=\frac{1029}{1028}.$$

$$\tan\left(3\alpha+\beta+\gamma-\frac{\pi}{4}\right)=\frac{\tan(3\alpha+\beta+\gamma)-1}{1+\tan(3\alpha+\beta+\gamma)}=\frac{\frac{1029}{1028}-1}{1+\frac{1029}{1028}}=\frac{1}{2057}.$$

Problem 8. *Show that*
$$\tan^{-1}\left\{\left(\sqrt{2}+1\right)\tan\alpha\right\}-\tan^{-1}\left\{\left(\sqrt{2}-1\right)\tan\alpha\right\}=\tan^{-1}(\sin 2\alpha).$$

Solution. We see as in the solution of *Problem* 6 that
$$\tan^{-1}\{(\sqrt{2}+1)\tan\alpha\}-\tan^{-1}\{(\sqrt{2}-1)\tan\alpha\}$$
$$=\tan^{-1}\frac{(\sqrt{2}+1)\tan\alpha-(\sqrt{2}-1)\tan\alpha}{1+(\sqrt{2}+1)(\sqrt{2}-1)\tan^2\alpha}$$
$$=\tan^{-1}\frac{2\tan\alpha}{1+\tan^2\alpha}=\tan^{-1}(\sin 2\alpha).$$

Problem 9. *If* $\tan(\theta-\alpha)\tan(\theta-\beta)=\tan^2\theta$; *then*
$$\theta=\frac{1}{2}\tan^{-1}\frac{2\sin\alpha\sin\beta}{\sin(\alpha+\beta)}.$$

Solution. $\tan(\theta-\alpha)\tan(\theta-\beta)=\tan^2\theta$;

therefore $$\frac{\sin(\theta-\alpha)\sin(\theta-\beta)}{\cos(\theta-\alpha)\cos(\theta-\beta)}=\frac{1-\cos 2\theta}{1+\cos 2\theta};$$

therefore $$\frac{\cos(\alpha-\beta)-\cos(2\theta-\alpha-\beta)}{\cos(\alpha-\beta)+\cos(2\theta-\alpha-\beta)}=\frac{1-\cos 2\theta}{1+\cos 2\theta};$$

therefore $$\cos(\alpha-\beta)\cos 2\theta=\cos(2\theta-\alpha-\beta)$$
$$=\cos 2\theta\cos(\alpha+\beta)+\sin 2\theta\sin(\alpha+\beta);$$

therefore $$\tan 2\theta\sin(\alpha+\beta)=\cos(\alpha-\beta)-\cos(\alpha+\beta)=2\sin\alpha\sin\beta;$$

therefore $$\tan 2\theta = \frac{2\sin\alpha\sin\beta}{\sin(\alpha+\beta)};$$

therefore $$2\theta = \tan^{-1}\frac{2\sin\alpha\sin\beta}{\sin(\alpha+\beta)}.$$

Problem 10. *Show that* $\cos^{-1}\dfrac{9}{\sqrt{82}} + \operatorname{cosec}^{-1}\dfrac{\sqrt{41}}{4} = \dfrac{\pi}{4}$.

Solution. Let $\alpha = \cos^{-1}\dfrac{9}{\sqrt{(82)}}$, and $\beta = \operatorname{cosec}^{-1}\dfrac{\sqrt{(41)}}{4}$;

then $$\cos\alpha = \frac{9}{\sqrt{(82)}},\ \text{and}\ \sin\alpha = \frac{1}{\sqrt{(82)}};$$
$$\sin\beta = \frac{4}{\sqrt{(41)}}, \qquad \cos\beta = \frac{5}{\sqrt{(41)}}.$$

Therefore $$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$= \frac{45-4}{\sqrt{(82)}\times\sqrt{(41)}} = \frac{41}{41\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

Therefore $$\alpha+\beta = \frac{\pi}{4}.$$

Problem 11. *Show that* $\sin^{-1}\dfrac{4}{5} + \sin^{-1}\dfrac{5}{13} + \sin^{-1}\dfrac{16}{65} = \dfrac{\pi}{2}$.

Solution. Let $\alpha = \sin^{-1}\dfrac{4}{5}, \qquad \beta = \sin^{-1}\dfrac{5}{13}, \qquad \gamma = \sin^{-1}\dfrac{16}{65}$;

then $$\sin\alpha = \frac{4}{5}, \qquad \sin\beta = \frac{5}{13}, \qquad \sin\gamma = \frac{16}{65},$$

and $$\cos\alpha = \frac{3}{5}, \qquad \cos\beta = \frac{12}{13}, \qquad \cos\gamma = \frac{63}{65}.$$

Then $$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha + \cos\alpha\sin\beta = \frac{48+15}{65} = \frac{63}{65};$$

thus $$\sin(\alpha+\beta) = \cos\gamma,\ \text{so that}\ \alpha+\beta+\gamma = \frac{\pi}{2}.$$

Problem 12. *Show that* $3\tan^{-1}\dfrac{1}{4} + \tan^{-1}\dfrac{1}{20} = \dfrac{\pi}{4} - \tan^{-1}\dfrac{1}{1985}$.

Solution. Let $\alpha = \tan^{-1}\dfrac{1}{4}$, and $\beta = \tan^{-1}\dfrac{1}{20}$.

$$\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha} = \frac{\dfrac{3}{4} - \dfrac{1}{4^3}}{1-\dfrac{3}{4^2}} = \frac{47}{52}.$$

$$\tan(3\alpha+\beta) = \frac{\tan 3\alpha + \tan\beta}{1-\tan 3\alpha\tan\beta} = \frac{\dfrac{47}{52}+\dfrac{1}{20}}{1-\dfrac{47}{52\times 20}} = \frac{992}{993}.$$

Again, let $\gamma = \tan^{-1}\dfrac{1}{1985}$; then

$$\tan\left(\frac{\pi}{4} - \gamma\right) = \frac{1 - \dfrac{1}{1985}}{1 + \dfrac{1}{1985}} = \frac{1984}{1986} = \frac{992}{993}.$$

Therefore $$3\alpha + \beta = \frac{\pi}{4} - \gamma.$$

Problem 13. *Show that* $\tan^{-1}\dfrac{2a-b}{b\sqrt{3}} + \tan^{-1}\dfrac{2b-a}{a\sqrt{3}} = \dfrac{\pi}{3}$.

Solution. Let $\theta = \tan^{-1}\dfrac{2a-b}{b\sqrt{3}}$, $\qquad \phi = \tan^{-1}\dfrac{2b-a}{a\sqrt{3}}$;

then $$\tan(\theta+\phi) = \frac{\dfrac{2a-b}{b\sqrt{3}} + \dfrac{2b-a}{a\sqrt{3}}}{1 - \dfrac{(2a-b)(2b-a)}{3ab}} = \frac{a(2a-b)+b(2b-a)}{3ab-(2a-b)(2b-a)}\sqrt{3}$$
$$= \frac{2(a^2+b^2)-2ab}{2(a^2+b^2)-2ab}\sqrt{3} = \sqrt{3}.$$

Thus $$\theta + \phi = \frac{\pi}{3}.$$

Problem 14. *Show that* $\tan\left(2\tan^{-1} a\right) = 2\tan\left(\tan^{-1} a + \tan^{-1} a^3\right)$.

Solution. Let $\tan^{-1} a = \theta$, and $\tan^{-1} a^3 = \phi$.

Then $$\tan\theta = a, \text{ and } \tan 2\theta = \frac{2a}{1-a^2}.$$

Also $$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = \frac{a+a^3}{1-a^4} = \frac{a}{1-a^2}.$$

Therefore $\tan\left(2\tan^{-1} a\right) = 2\tan\left(\tan^{-1} a + \tan^{-1} a^3\right)$.

Problem 15. *Show that*

$$\tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}(\cot A) + \tan^{-1}\left(\cot^3 A\right) = 0.$$

Solution. Let $\tan^{-1}\left(\dfrac{1}{2}\tan 2A\right) = \alpha$, then $\tan\alpha = \dfrac{1}{2}\tan 2A$;

let $$\tan^{-1}(\cot A) = \beta, \text{ then } \tan\beta = \cot A;$$

let $$\tan^{-1}(\cot^3 A) = \gamma, \text{ then } \tan\gamma = \cot^3 A.$$

Thus $$\tan(\beta+\gamma) = \frac{\tan\beta + \tan\gamma}{1 - \tan\beta\tan\gamma} = \frac{\cot A + \cot^3 A}{1 - \cot^4 A} = \frac{\cot A}{1 - \cot^2 A}$$
$$= \frac{\dfrac{1}{\tan A}}{1 - \dfrac{1}{\tan^2 A}} = \frac{\tan A}{\tan^2 A - 1} = -\frac{1}{2}\tan 2A = -\tan\alpha.$$

Therefore $$\tan(\alpha+\beta+\gamma) = \frac{\tan\alpha + \tan(\beta+\gamma)}{1 - \tan\alpha\tan(\beta+\gamma)} = 0.$$

Thus $$\alpha + \beta + \gamma = 0.$$

Problem 16. *Show that*
$$\frac{2b}{a} = \tan\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right) + \tan\left(\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right).$$

Solution. Let $\cos^{-1}\dfrac{a}{b} = \theta$, then $\cos\theta = \dfrac{a}{b}$.

And $$\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \frac{1 - \tan\frac{\theta}{2}}{1 + \tan\frac{\theta}{2}} + \frac{1 + \tan\frac{\theta}{2}}{1 - \tan\frac{\theta}{2}}.$$

$$= \frac{\left(1 - \tan\frac{\theta}{2}\right)^2 + \left(1 + \tan\frac{\theta}{2}\right)^2}{1 - \tan^2\frac{\theta}{2}} = 2\frac{1 + \tan^2\frac{\theta}{2}}{1 - \tan^2\frac{\theta}{2}} = 2\frac{1}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}$$

$$= \frac{2}{\cos\theta} = \frac{2b}{a}.$$

Problem 17. *Show that*
$$\frac{a^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{a}{b}\right) + \frac{b^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{b}{a}\right) = (a+b)\left(a^2+b^2\right).$$

Solution. Let $\tan^{-1}\dfrac{a}{b} = \theta$; then $\tan\theta = \dfrac{a}{b}$;

$$\operatorname{cosec}^2\frac{\theta}{2} = \frac{1}{\sin^2\frac{\theta}{2}} = \frac{2}{1 - \cos\theta} = \frac{2}{1 - \dfrac{b}{\sqrt{(a^2+b^2)}}} = \frac{2\sqrt{(a^2+b^2)}}{\sqrt{(a^2+b^2)} - b}$$

$$= \frac{2\sqrt{a^2+b^2}}{\sqrt{(a^2+b^2)} - b} \times \frac{\sqrt{(a^2+b^2)} + b}{\sqrt{(a^2+b^2)} + b} = \frac{2(a^2+b^2) + 2b\sqrt{(a^2+b^2)}}{a^2};$$

therefore $$\frac{a^3}{2}\operatorname{cosec}^2\frac{\theta}{2} = a(a^2+b^2) + ab\sqrt{(a^2+b^2)}.$$

Let $\tan^{-1}\dfrac{b}{a} = \phi$; then $\tan\phi = \dfrac{b}{a}$;

$$\sec^2\frac{\phi}{2} = \frac{1}{\cos^2\frac{\phi}{2}} = \frac{2}{1 + \cos\phi} = \frac{2}{1 + \dfrac{a}{\sqrt{(a^2+b^2)}}} = \frac{2\sqrt{(a^2+b^2)}}{\sqrt{(a^2+b^2)} + a}$$

$$= \frac{2\sqrt{a^2+b^2}}{\sqrt{(a^2+b^2)} + a} \times \frac{\sqrt{(a^2+b^2)} - a}{\sqrt{(a^2+b^2)} - a} = \frac{2(a^2+b^2) - 2a\sqrt{(a^2+b^2)}}{b^2};$$

therefore $$\frac{b^3}{2}\sec^2\frac{\phi}{2} = b(a^2+b^2) - ab\sqrt{(a^2+b^2)}.$$

Therefore $$\frac{a^3}{2}\operatorname{cosec}^2\frac{\theta}{2} + \frac{b^3}{2}\sec^2\frac{\phi}{2} = (a+b)(a^2+b^2).$$

Solve the following seven equations in x :

Problem 18. $\sin^{-1} x + \sin^{-1} \frac{x}{2} = \frac{\pi}{4}$.

Solution. $\sin^{-1} x + \sin^{-1} \frac{x}{2} = \frac{\pi}{4}$;

therefore $$\sin^{-1} \frac{x}{2} = \frac{\pi}{4} - \sin^{-1} x.$$

Take the sines of both sides; thus

$$\frac{x}{2} = \sin\left(\frac{\pi}{4} - \sin^{-1} x\right) = \sin\frac{\pi}{4} \cdot \sqrt{(1-x^2)} - \cos\frac{\pi}{4} \cdot x$$
$$= \frac{\sqrt{(1-x^2)} - x}{\sqrt{2}};$$

therefore $$x\left(\frac{1}{\sqrt{2}} + 1\right) = \sqrt{(1-x^2)};$$

therefore $$x^2\left(\frac{1}{\sqrt{2}} + 1\right)^2 = 1 - x^2;$$

therefore $$x^2\left(\frac{5}{2} + \frac{2}{\sqrt{2}}\right) = 1;$$

therefore $$x^2(5 + 2\sqrt{2}) = 2;$$

therefore $$x^2 = \frac{2}{5+2\sqrt{2}} = \frac{2}{5+2\sqrt{2}} \cdot \frac{5-2\sqrt{2}}{5-2\sqrt{2}} = \frac{2}{17}(5 - 2\sqrt{2}).$$

Problem 19. $\sin^{-1} \frac{2a}{1+a^2} + \sin^{-1} \frac{2b}{1+b^2} = 2\tan^{-1} x$.

Solution. We shall first show that $\sin^{-1} \frac{2a}{1+a^2} = 2\tan^{-1} a$.

Let $\tan^{-1} a = \theta$; then $\tan\theta = a$; and $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{2a}{1+a^2}$;

therefore $$\sin^{-1} \frac{2a}{1+a^2} = 2\theta = 2\tan^{-1} a.$$

Similarly $$\sin^{-1} \frac{2b}{1+b^2} = 2\tan^{-1} b.$$

Hence the equation may be written

$$2\tan^{-1} a + 2\tan^{-1} b = 2\tan^{-1} x;$$

therefore $$\tan^{-1} x = \tan^{-1} a + \tan^{-1} b.$$

Take the tangents of both sides; thus

$$x = \tan\left(\tan^{-1} a + \tan^{-1} b\right) = \frac{a+b}{1-ab}.$$

Problem 20. $\tan^{-1}(x-1) + \tan^{-1} x + \tan^{-1}(x+1) = \tan^{-1} 3x$.

Solution. Let $\tan^{-1}(x-1) = \alpha,\ \tan^{-1} x = \beta,\ \tan^{-1}(x+1) = \gamma$.

Thus $$\tan^{-1} 3x = \alpha + \beta + \gamma.$$

Take the tangents of both sides; thus

$$3x = \tan(\alpha+\beta+\gamma) = \frac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - \tan\beta\tan\gamma - \tan\gamma\tan\alpha - \tan\alpha\tan\beta}$$

$$= \frac{3x - x(x^2-1)}{1 - x(x+1) - (x+1)(x-1) - x(x-1)} = \frac{4x - x^3}{2 - 3x^2}.$$

Therefore either $x = 0$, or $3(2 - 3x^2) = 4 - x^2$; the latter gives $8x^2 = 2$; therefore $x^2 = \frac{1}{4}$; therefore $x = \pm\frac{1}{2}$.

Problem 21. $\sin^{-1} 2x - \sin^{-1} x\sqrt{3} = \sin^{-1} x.$

Solution. $\sin^{-1} 2x - \sin^{-1} x\sqrt{3} = \sin^{-1} x.$

Take the sines of both sides; thus

$$2x\sqrt{(1-3x^2)} - x\sqrt{3} \times \sqrt{(1-4x^2)} = x.$$

Thus either $\quad x = 0,\ \text{or}\ 2\sqrt{(1-3x^2)} - \sqrt{3} \times \sqrt{(1-4x^2)} = 1.$

Transpose, thus $\quad 2\sqrt{(1-3x^2)} = 1 + \sqrt{3} \times \sqrt{(1-4x^2)}.$

Square, $\quad 4(1-3x^2) = 1 + 2\sqrt{3} \times \sqrt{(1-4x^2)} + 3(1-4x^2);$

therefore $\quad 2\sqrt{3} \times \sqrt{(1-4x^2)} = 0;$

therefore $\quad 1 - 4x^2 = 0, \qquad \text{therefore } x = \pm\frac{1}{2}.$

Problem 22. $\tan^{-1}\frac{1}{4} + 2\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{6} + \tan^{-1}\frac{1}{x} = \frac{\pi}{4}.$

Solution. $\tan^{-1}\frac{1}{4} + 2\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{6} + \tan^{-1}\frac{1}{x} = \frac{\pi}{4}.$

Let $\quad \tan^{-1}\frac{1}{4} = \alpha,\ \tan^{-1}\frac{1}{5} = \beta,\ \tan^{-1}\frac{1}{6} = \gamma.$

Thus the equation may be written

$$\tan^{-1}\frac{1}{x} = \frac{\pi}{4} - (\alpha + 2\beta + \gamma);$$

therefore
$$\frac{1}{x} = \frac{1 - \tan(\alpha + 2\beta + \gamma)}{1 + \tan(\alpha + 2\beta + \gamma)}.$$

Now
$$\tan(\alpha + \beta) = \frac{\frac{1}{4} + \frac{1}{5}}{1 - \frac{1}{4} \times \frac{1}{5}} = \frac{9}{19},$$

$$\tan(\beta + \gamma) = \frac{\frac{1}{5} + \frac{1}{6}}{1 - \frac{1}{5} \times \frac{1}{6}} = \frac{11}{29};$$

therefore
$$\tan(\alpha + \beta + \beta + \gamma) = \frac{\frac{9}{19} + \frac{11}{29}}{1 - \frac{9}{19} \times \frac{11}{29}} = \frac{470}{452}.$$

Hence
$$\frac{1}{x} = \frac{1 - \frac{470}{452}}{1 + \frac{470}{452}} = -\frac{18}{922} = -\frac{9}{461}.$$

Problem 23. $\sin 2\cos^{-1}\cot 2\tan^{-1} x = 0$.

Solution. Let $\tan^{-1} x = \theta$, then $\tan\theta = x; \cot 2\theta = \dfrac{1}{\tan 2\theta} = \dfrac{1-x^2}{2x}$.

Thus the equation may be written

$$\sin 2\cos^{-1}\frac{1-x^2}{2x} = 0.$$

Now since $2\cos^{-1}\dfrac{1-x^2}{2x}$ has zero for its sine, the angle must be of the form $n\pi$, where n is zero or some integer.

Thus $$2\cos^{-1}\frac{1-x^2}{2x} = n\pi;\ \text{therefore}\ \cos^{-1}\frac{1-x^2}{2x} = \frac{n\pi}{2};$$

therefore $$\frac{1-x^2}{2x} = \cos\frac{n\pi}{2}.$$

Since n is zero or an integer we have $\cos\dfrac{n\pi}{2} = 0$, or 1, or -1.

If $\dfrac{1-x^2}{2x} = 0$, then $x = \pm 1$.

If $\dfrac{1-x^2}{2x} = 1$, then $x^2 + 2x = 1$; and from this we deduce $x = -1 \pm \sqrt{2}$.

If $\dfrac{1-x^2}{2x} = -1$, then $x^2 - 2x = 1$; and from this we deduce $x = 1 \pm \sqrt{2}$.

Problem 24. $\tan^{-1}\dfrac{1}{a-1} = \tan^{-1}\dfrac{1}{x} + \tan^{-1}\dfrac{1}{a^2-x+1}$.

Solution.

$$\tan^{-1}\frac{1}{a-1} = \tan^{-1}\frac{1}{x} + \tan^{-1}\frac{1}{a^2-x+1};$$

therefore $$\tan^{-1}\frac{1}{a-1} - \tan^{-1}\frac{1}{x} = \tan^{-1}\frac{1}{a^2-x+1}.$$

Take the tangents of both sides; thus

$$\frac{\dfrac{1}{a-1} - \dfrac{1}{x}}{1 + \dfrac{1}{(a-1)x}} = \frac{1}{a^2-x+1};$$

therefore $$\frac{x-a+1}{ax-x+1} = \frac{1}{a^2-x+1};$$

therefore $$(x-a+1)(a^2-x+1) = ax-x+1;$$

therefore $$-x^2 + x(a^2+a) - a^3 + a^2 - a + 1 = ax - x + 1;$$

therefore $$x^2 - x(a^2+1) + a^3 - a^2 + a = 0.$$

By solving this quadratic in the ordinary way we obtain $x = a$ or $a^2 - a + 1$.

Problem 25. *If* $\sec\theta - \operatorname{cosec}\theta = \dfrac{4}{3}$, *show that* $\theta = \dfrac{1}{2}\sin^{-1}\dfrac{3}{4}$.

Solution. $\sec\theta - \operatorname{cosec}\theta = \dfrac{4}{3}$;

therefore $$\frac{1}{\cos\theta} - \frac{1}{\sin\theta} = \frac{4}{3};$$

therefore $$\sin\theta - \cos\theta = \frac{4}{3}\sin\theta\cos\theta = \frac{2}{3}\sin 2\theta.$$

Square, thus $$1 - \sin 2\theta = \frac{4}{9}\sin^2 2\theta.$$

By solving this quadratic in the usual way we obtain $\sin 2\theta = \frac{3}{4}$, or -3; the former value is alone applicable. Thus $\sin 2\theta = \frac{3}{4}$;

therefore $$2\theta = \sin^{-1}\frac{3}{4}; \text{ therefore } \theta = \frac{1}{2}\sin^{-1}\frac{3}{4}.$$

Problem 26. *If* $\sin(\pi\cos\theta) = \cos(\pi\sin\theta)$, *show that* $\theta = \pm\frac{1}{2}\sin^{-1}\frac{3}{4}$.

Solution. $\sin(\pi\cos\theta) = \cos(\pi\sin\theta)$;

therefore $$\cos\left(\frac{\pi}{2} - \pi\cos\theta\right) = \cos(\pi\sin\theta).$$

Hence, by *Art*. 67 (page 402) the solutions are comprised in
$$\frac{\pi}{2} - \pi\cos\theta = 2n\pi \pm \pi\sin\theta;$$

therefore $$\cos\theta \pm \sin\theta = \frac{1}{2} - 2n.$$

Square, thus $$1 \pm \sin 2\theta = \left(\frac{1}{2} - 2n\right)^2.$$

If we give to n any integral value, positive or negative, the value of $\sin 2\theta$ is greater than unity. Thus we must have n zero. Then $1 \pm \sin 2\theta = \frac{1}{4}$; and therefore $\sin 2\theta = \pm\frac{3}{4}$; thus $2\theta = \pm\sin^{-1}\frac{3}{4}$, and $\theta = \pm\frac{1}{2}\sin^{-1}\frac{3}{4}$.

Problem 27. *Show that if* $\sin^2\theta + \sin^2\phi = \frac{1}{2}$, *then* $(2n+1)\frac{\pi}{2}$ *is one of the values of* ψ *which satisfy the equation*
$$\psi = \sin^{-1}(\sin\theta + \sin\phi) + \sin^{-1}(\sin\theta - \sin\phi).$$

Solution. Let $\psi = \sin^{-1}(\sin\theta + \sin\phi) + \sin^{-1}(\sin\theta - \sin\phi)$.
Take the cosines of both sides; thus

$$\begin{aligned}\cos\psi &= \sqrt{1-(\sin\theta+\sin\phi)^2}\sqrt{1-(\sin\theta-\sin\phi)^2} - (\sin\theta+\sin\phi)(\sin\theta-\sin\phi)\\ &= \sqrt{1-\frac{1}{2}-2\sin\theta\sin\phi}\sqrt{1-\frac{1}{2}+\sin\theta\sin\phi} - (\sin^2\theta - \sin^2\phi)\\ &= \sqrt{\frac{1}{4} - 4\sin^2\theta\sin^2\phi} - (\sin^2\theta - \sin^2\phi).\end{aligned}$$

Now $$\frac{1}{2} = \sin^2\theta + \sin^2\phi, \text{ therefore } \frac{1}{4} = (\sin^2\theta + \sin^2\phi)^2;$$

therefore $$\frac{1}{4} - 4\sin^2\theta\sin^2\phi = (\sin^2\theta - \sin^2\phi)^2.$$

Thus $$\cos\psi = \pm(\sin^2\theta - \sin^2\phi) - (\sin^2\theta - \sin^2\phi).$$

Taking the upper sign we have $\cos\psi = 0$, and therefore $\psi = (2n+1)\dfrac{\pi}{2}$, where n is any integer.

Problem 28. *Find x from the following equation,*

$$3\tan^{-1}\frac{1}{2+\sqrt{3}} - \tan^{-1}\frac{1}{x} = \tan^{-1}\frac{1}{3}.$$

Solution.

$$3\tan^{-1}\frac{1}{2+\sqrt{3}} - \tan^{-1}\frac{1}{x} = \tan^{-1}\frac{1}{3}.$$

Now let $\quad \tan^{-1}\dfrac{1}{2+\sqrt{3}} = \theta$, then $\tan\theta = \dfrac{1}{2+\sqrt{3}}$;

therefore
$$\tan 3\theta = \frac{\dfrac{3}{2+\sqrt{3}} - \dfrac{1}{(2+\sqrt{3})^2}}{1 - \dfrac{3}{(2+\sqrt{3})^2}} = \frac{3(2+\sqrt{3})^2 - 1}{(2+\sqrt{3})^3 - 3(2+\sqrt{3})}$$
$$= \frac{20+12\sqrt{3}}{20+12\sqrt{3}} = 1;\ \text{therefore } 3\theta = \tan^{-1}1.$$

This might also have been inferred from the fact that
$$\frac{1}{2+\sqrt{3}} = \tan 15^\circ = \tan\frac{\pi}{12},\ \text{so that } \theta = \frac{\pi}{12}.$$

The equation may now be written
$$\tan^{-1}1 - \tan^{-1}\frac{1}{3} = \tan^{-1}\frac{1}{x}.$$

Take the tangents of both sides; thus
$$\frac{1-\dfrac{1}{3}}{1+\dfrac{1}{3}} = \frac{1}{x};$$

therefore $\quad \dfrac{1}{x} = \dfrac{1}{2}$, therefore $x = 2$.

Problem 29. *Show that one of the expressions*

$$\sin^{-1}\frac{2b+a-c}{a+c} \pm 2\sin^{-1}\sqrt{\left(\frac{a+b}{a+c}\right)}$$

is an odd multiple of $\dfrac{\pi}{2}$.

Solution. Let $\sin^{-1}\sqrt{\left(\dfrac{a+b}{a+c}\right)} = \theta$, then $\sin\theta = \sqrt{\left(\dfrac{a+b}{a+c}\right)}$;

then
$$\cos 2\theta = 1 - 2\sin^2\theta = 1 - \frac{2(a+b)}{a+c} = \frac{c-a-2b}{a+c}$$

and
$$2\theta = \cos^{-1}\frac{c-a-2b}{a+c}.$$

Thus the proposed expression is
$$\sin^{-1}\frac{2b+a-c}{a+c} \pm \cos^{-1}\frac{c-a-2b}{a+c};$$

that is $$\sin^{-1} p \pm \cos^{-1}(-p);$$
when p is put for $\dfrac{2b+a-c}{a+c}$.

Now $$\cos\left\{\sin^{-1} p \pm \cos^{-1}(-p)\right\} = -p\sqrt{(1-p^2)} \mp p\sqrt{(1-p^2)};$$
thus zero is one of the values of the cosine, and the corresponding angle is an odd multiple of $\dfrac{\pi}{2}$.

Problem 30. *Find all the positive integral solutions of*
$$\tan^{-1} x + \cot^{-1} y = \tan^{-1} 3.$$

Solution.
$$\tan^{-1} x + \cot^{-1} y = \tan^{-1} 3,$$
therefore $$\tan^{-1} x + \tan^{-1}\frac{1}{y} = \tan^{-1} 3.$$

Take the tangents of both sides; thus
$$\frac{x+\dfrac{1}{y}}{1-\dfrac{x}{y}} = 3;$$
therefore $$3(y-x) = yx+1;$$
therefore $$x = \frac{3y-1}{y+3} = \frac{3y+9-10}{y+3} = 3 - \frac{10}{y+3}.$$

Thus if x and y are to be positive integers $y+3$ must be a divisor of 10. Try in succession the various cases, namely $y+3 = 1$ or 2 or 5 or 10. It will be found that the only admissible cases are $y+3=5$, and $y+3=10$. These give $y=2$ or 7, and the corresponding values of x are 1 and 2.

Problem 31. *Show that if c be a positive integer, the equation*
$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} c$$
has no positive integral solutions; while the equation
$$\cot^{-1} x + \cot^{-1} y = \cot^{-1} c$$
has as many as there are different divisors of $1+c^2$.

Solution. $\tan^{-1} x + \tan^{-1} y = \tan^{-1} c.$

Take the tangents of both sides; thus $\dfrac{x+y}{1-xy} = c;$

therefore $$x+y = c(1-xy);\ \text{therefore } x = \frac{c-y}{1+cy}.$$

It is obvious that if c and y are positive integers x is either a positive or negative proper fraction, and cannot be a positive integer.

Next take $$\cot^{-1} x + \cot^{-1} y = \cot^{-1} c.$$

Take the cotangents of both sides; thus $\dfrac{xy-1}{x+y} = c;$

therefore $$xy - 1 = c(x+y),$$
therefore $$x = \frac{cy+1}{y-c} = \frac{cy-c^2+c^2+1}{y-c}$$

$$= c + \frac{c^2+1}{y-c}.$$

Thus if α denote any divisor of c^2+1 we may put $y - c = \alpha$, so that $y = c + \alpha$; and then $x = c + \dfrac{c^2+1}{\alpha}$.

Hence we see that there are as many solutions in positive integers as there are divisors of c^2+1.

Problem 32. *Show that*

$$\tan^{-1}\frac{x}{y} = \tan^{-1}\frac{c_1x-y}{c_1y+x} + \tan^{-1}\frac{c_2-c_1}{c_2c_1+1} + \tan^{-1}\frac{c_3-c_2}{c_3c_2+1} + \ldots\ldots$$
$$+ \tan^{-1}\frac{c_n-c_{n-1}}{c_nc_{n-1}+1} + \tan^{-1}\frac{1}{c_n},$$

where c_1, c_2, $\ldots\ldots c_n$ are any quantities whatever.

Solution. $$\tan^{-1}\frac{c_1x-y}{c_1y+x} = \tan^{-1}\frac{\dfrac{x}{y}-\dfrac{1}{c_1}}{1+\dfrac{x}{c_1y}} = \tan^{-1}\frac{x}{y} - \tan^{-1}\frac{1}{c_1},$$

as in the solution of *Problem* 6.

Similarly $$\tan^{-1}\frac{c_2-c_1}{c_2c_1+1} = \tan^{-1}\frac{1}{c_1} - \tan^{-1}\frac{1}{c_2},$$
$$\tan^{-1}\frac{c_3-c_2}{c_3c_2+1} = \tan^{-1}\frac{1}{c_2} - \tan^{-1}\frac{1}{c_3},$$

and so on.

Thus the sum of the terms on the right-hand side of the proposed expression is $\tan^{-1}\dfrac{x}{y}$.

Problem 33. *Show that we can express the sum of any number of angles of the form* $\sin^{-1}\dfrac{2ab}{a^2+b^2}$, $\sin^{-1}\dfrac{2a'b'}{a'^2+b'^2}, \ldots$ *in the form* $\sin^{-1}\dfrac{2mn}{m^2+n^2}$, *where m and n are rational functions of a, b, a', b', $\ldots$*

Solution.

$$\text{Let } \sin^{-1}\frac{2ab}{a^2+b^2} = \theta, \text{ and } \sin^{-1}\frac{2a'b'}{a'^2+b'^2} = \phi;$$

then $$\sin\theta = \frac{2ab}{a^2+b^2}, \text{ and } \sin\phi = \frac{2a'b'}{a'^2+b'^2};$$

therefore $$\cos\theta = \frac{a^2-b^2}{a^2+b^2}, \text{ and } \cos\phi = \frac{a'^2-b'^2}{a'^2+b'^2};$$

therefore $$\sin(\theta+\phi) = \frac{2ab(a'^2-b'^2)+2a'b'(a^2-b^2)}{(a^2+b^2)(a'^2+b'^2)}$$

$$= \frac{2ab(a'^2-b'^2)+2a'b'(a^2-b^2)}{(ab'+a'b)^2+(aa'-bb')^2} = \frac{2(ab'+a'b)(aa'-bb')}{(ab'+a'b)^2+(aa'-bb')^2}$$

therefore $$\theta+\phi = \sin^{-1}\frac{2pq}{p^2+q^2},$$

where p and q are rational expressions.

Then if there be another angle $\sin^{-1}\dfrac{2a''b''}{a''^2+b''^2}$, we may denote if by ψ; then $\sin\{(\theta+\phi)+\psi\}$ will take the form $\dfrac{2rs}{r^2+s^2}$ where r and s are rational. And so on.

Problem 34. *Write down the general value of* $\sin^{-1}\dfrac{(-1)^m}{2}$, *where m is an integer.*

Solution. We may take for the simplest value of $\sin^{-1}\dfrac{(-1)^m}{2}$ the angle $(-1)^m\dfrac{\pi}{6}$; as is evident by supposing m first even and then odd. This will be the α of *Art*. 66 (page 401) and the general solution is $n\pi+(-1)^n\alpha$, that is $n\pi+(-1)^{m+n}\dfrac{\pi}{6}$.

Or we may take the form $(m+n)\pi+(-1)^n\dfrac{\pi}{6}$.

For the sine of this angle

$$= \sin m\pi\cos\left\{n\pi+(-1)^n\frac{\pi}{6}\right\}+\cos m\pi\sin\left\{n\pi+(-1)^n\frac{\pi}{6}\right\}$$
$$= \cos m\pi\sin\left\{n\pi+(-1)^n\frac{\pi}{6}\right\}=\cos m\pi\times\frac{1}{2}=(-1)^m\times\frac{1}{2}.$$

Problem 35. *Write down the general value of* $\cos^{-1}\dfrac{(-1)^m}{2}$, *where m is an integer.*

Solution. If m be even the value is $\cos^{-1}\dfrac{1}{2}$, that is $2n\pi\pm\dfrac{\pi}{3}$.

If m be odd the value is $\cos^{-1}\left(-\dfrac{1}{2}\right)$, that is $2n\pi\pm\left(\pi+\dfrac{\pi}{3}\right)$.

Both forms may be comprised in $(2p+m)\pi\pm\dfrac{\pi}{3}$, where p is any integer.

For $2n\pi\pm\dfrac{\pi}{3}$ consists of an *even* multiple of π augmented by $\pm\dfrac{\pi}{3}$; and $2n\pi\pm\left(\pi+\dfrac{\pi}{3}\right)$ consists of an *odd* multiple of π augmented by $\pm\dfrac{\pi}{3}$.

Problem 36. *Write down the general value of* $\tan^{-1}(-1)^m$, *where m is an integer.*

Solution. If m be even the value is $\tan^{-1}1$, that is $n\pi+\dfrac{\pi}{4}$.

If m be odd the value is $\tan^{-1}(-1)$, that is $n\pi-\dfrac{\pi}{4}$.

Both forms may be comprised in $n\pi+(-1)^m\dfrac{\pi}{4}$.

CHAPTER XIX
De Moivre's Theorem

Problem 1. *Extract the square root of* $\cos 4A + \sqrt{-1}\sin 4A$.

Solution. $\left\{\cos 4A + \sqrt{(-1)}\sin 4A\right\}^{\frac{1}{2}} = \pm\left\{\cos 2A + \sqrt{(-1)}\sin 2A\right\}$ by *Art.* 267 (page 431).

Problem 2. *Find the values of* $(-1)^{\frac{1}{3}}$.

Solution. $-1 = \cos\pi = \cos\pi + \sqrt{(-1)}\sin\pi$;

therefore one value of $(-1)^{\frac{1}{3}} = \cos\dfrac{\pi}{3} + \sqrt{(-1)}\sin\dfrac{\pi}{3}$,

so we may put $-1 = \cos 3\pi$, or $\cos 5\pi$, and thus we obtain two other values for $(-1)^{\frac{1}{3}}$, namely,

$$\cos\frac{3\pi}{3} + \sqrt{(-1)}\sin\frac{3\pi}{3}, \text{ that is } -1,$$

and

$$\cos\frac{5\pi}{3} + \sqrt{(-1)}\sin\frac{5\pi}{3}.$$

Problem 3. *Obtain the six values of* $(-1)^{\frac{1}{6}}$.

Solution. We may put $-1 = \cos\pi$, or $\cos 3\pi$, or $\cos 5\pi$, or $\cos 7\pi$, or $\cos 9\pi$, or $\cos 11\pi$; and thus

$$-1 = \cos\theta + \sqrt{(-1)}\sin\theta$$

where $\theta = \pi$, or 3π, or 5π, or 7π, or 9π, or 11π.

Hence the six values of $(-1)^{\frac{1}{6}}$ are contained in

$$\cos\frac{\theta}{6} + \sqrt{(-1)}\sin\frac{\theta}{6},$$

where θ has any of the six values just specified.

Problem 4. *Find the three values of* $\left\{1 + \sqrt{-1}\right\}^{\frac{1}{3}}$.

Solution.

$$1 + \sqrt{(-1)} = \sqrt{2}\left\{\frac{1}{\sqrt{2}} + \frac{\sqrt{(-1)}}{\sqrt{2}}\right\}$$

$$= \sqrt{2}\{\cos\theta + \sqrt{(-1)}\sin\theta\},$$

where for θ we may put $\dfrac{\pi}{4} + 2n\pi$, where n is any integer.

Therefore $$\{1 + \sqrt{(-1)}\}^{\frac{1}{3}} = 2^{\frac{1}{6}}\left\{\cos\frac{\theta}{3} + \sqrt{(-1)}\sin\frac{\theta}{3}\right\};$$

and the three values will be obtained by putting for θ in succession $\frac{\pi}{4}$, $2\pi+\frac{\pi}{4}$, and $4\pi+\frac{\pi}{4}$.

Problem 5. *Given* $\frac{\sin\theta}{\theta}=\frac{2165}{2166}$, *show that* θ *is nearly the circular measure of* 3°.

Solution. Since $\frac{\sin\theta}{\theta}$ is given nearly equal to unity, we may infer that θ is a small angle. Hence we have approximately, by *Art*. 274 (page 432),

$$\sin\theta=\theta-\frac{\theta^3}{6};$$

thus $$1-\frac{\theta^2}{6}=\frac{2165}{2166};$$

therefore $$\frac{\theta^2}{6}=\frac{1}{2166},$$

therefore $$\theta^2=\frac{1}{361},$$

therefore $$\theta=\frac{1}{19}.$$

This is the circular measure of the angle; therefore the number of degrees $=\frac{1}{19}$ of $\frac{180}{\pi}=\frac{1}{19}$ of $57°.29\ldots=3°$ approximately.

Problem 6. *Given* $\sin\left(\frac{\pi}{6}+\theta\right)=\cdot 51$, *find approximately the value of* θ, *neglecting powers of* θ *above the second.*

Solution. $\sin\left(\frac{\pi}{6}+\theta\right)=.51$.

As .51 is very nearly equal to $\sin\frac{\pi}{6}$ we may infer that θ is very small. We have

$$\sin\frac{\pi}{6}\cos\theta+\cos\frac{\pi}{6}\sin\theta=.51,$$

therefore $$\frac{1}{2}\left(1-\frac{\theta^2}{2}\right)+\frac{\sqrt{3}}{2}\theta=.51 \text{ approximately.}$$

Hence, neglecting θ^2, we have $\frac{\sqrt{3}}{2}\theta=\frac{1}{100}$, and therefore $\theta=\frac{1}{50\sqrt{3}}$.

Then if we retain the term in θ^2 we have

$$\theta=\frac{1}{50\sqrt{3}}+\frac{\theta^2}{2\sqrt{3}};$$

and putting for θ^2 its approximate value, we have for a closer approximation

$$\theta=\frac{1}{50\sqrt{3}}+\frac{1}{2\sqrt{3}}\left(\frac{1}{50\sqrt{3}}\right)^2$$
$$=\frac{1}{50\sqrt{3}}+\frac{1}{15000\sqrt{3}}.$$

The same result will be obtained if we solve the quadratic equation $\theta=\frac{1}{50\sqrt{3}}+$

$\dfrac{\theta^2}{2\sqrt{3}}$ in the usual way, select the least root, and take its approximate value. See *Algebra*, *Art*. 526, Example (3) (page 450).

Problem 7. *If* $\tan x = a_1 x + \dfrac{a_3 x^3}{\underline{|3}} + \dfrac{a_5 x^5}{\underline{|5}} + \ldots\ldots$
show that

$$a_{2n+1} = \frac{(2n+1)2n}{1 \cdot 2} a_{2n-1} - \frac{(2n+1)2n(2n-1)(2n-2)}{\underline{|4}} a_{2n-3} + \ldots$$
$$+ \ldots + (-1)^{n+1}(2n+1)a_1 + (-1)^n.$$

Solution.
Suppose
$$\tan x = a_1 x + \frac{a_3 x^3}{\underline{|3}} + \frac{a_5 x^5}{\underline{|5}} + \ldots;$$
then
$$\sin x = \cos x \left\{ a_1 x + \frac{a_3 x^3}{\underline{|3}} + \frac{a_5 x^5}{\underline{|5}} + \ldots; \right\}.$$

Substitute for $\sin x$ and $\cos x$ by *Art*. 274 (page 432); thus
$$x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \ldots$$
$$= \left\{ 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \ldots \right\} \left\{ a_1 x + \frac{a_3 x^3}{\underline{|3}} + \frac{a_5 x^5}{\underline{|5}} + \ldots \right\}.$$

Then, according to the known principles of Algebra, we may equate the coefficient of any poser of x on the left-hand side to the coefficient of the same power obtained by working out the product on the right-hand side. Take, for instance, the coefficient x^{2n+1}; thus we obtain
$$\frac{(-1)^n}{\underline{|2n+1}} = \frac{a_{2n+1}}{\underline{|2n+1}} - \frac{a_{2n-1}}{2\underline{|2n-1}} + \frac{a_{2n-3}}{\underline{|4}\,\underline{|2n-3}} \ldots + (-1)^n \frac{a_1}{\underline{|2n}}.$$

Multiply by $\underline{|2n+1}$ and transpose; thus we get
$$a_{2n+1} = \frac{(2n+1)2n}{2} a_{2n-1} - \frac{(2n+1)2n(2n-1)(2n-2)}{\underline{|4}} a_{2n-3}$$
$$+ \ldots + (2n+1)(-1)^{n+1} a_1 + (-1)^n.$$

Problem 8. *If* $\theta \cot \theta = a_0 + a_2\theta^2 + a_4\theta^4 + \ldots$
show that
$$a_{2n} = \frac{a_{2n-2}}{\underline{|3}} - \frac{a_{2n-4}}{\underline{|5}} + \ldots + \frac{(-1)^{n-1}a_0}{\underline{|2n+1}} + \frac{(-1)^n}{\underline{|2n}};$$
hence find $\theta \cot \theta$ *to four terms.*

Solution. Let $\theta \cot \theta = a_0 + a_2\theta^2 + a_4\theta^4 + \ldots;$
then
$$\theta \cos \theta = \sin \theta \left\{ a_0 + a_2\theta^2 + a_4\theta^4 + \ldots \right\}.$$

Substitute for $\cos\theta$ and $\sin\theta$ by *Art*. 274 (page 432); thus
$$\theta \left(1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \frac{\theta^6}{\underline{|6}} + \ldots \right)$$
$$= \left\{ \theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} - \frac{\theta^7}{\underline{|7}} + \ldots \right\} \left\{ a_0 + a_2\theta^2 + a_4\theta^4 + \ldots \right\}$$

Equate the coefficients of θ^{2n+1}; thus

$$\frac{(-1)^n}{\underline{|2n}} = a_{2n} - \frac{a_{2n-2}}{\underline{|3}} + \frac{a_{2n-4}}{\underline{|5}} - \ldots + \frac{(-1)^n a_0}{\underline{|2n+1}}.$$

Transpose; thus we get

$$a_{2n} = \frac{a_{2n-2}}{\underline{|3}} - \frac{a_{2n-4}}{\underline{|5}} + \ldots + \frac{(-1)^{n-1} a_0}{\underline{|2n+1}} + \frac{(-1)^n}{\underline{|2n}}.$$

To find the first four terms of $\theta \cot \theta$ we have the following equations:

$$1 = a_0,$$
$$-\frac{1}{\underline{|2}} = a_2 - \frac{a_0}{\underline{|3}},$$
$$\frac{1}{\underline{|4}} = a_4 - \frac{a_2}{\underline{|3}} + \frac{a_0}{\underline{|5}},$$
$$-\frac{1}{\underline{|6}} = a_6 - \frac{a_4}{\underline{|3}} + \frac{a_2}{\underline{|5}} - \frac{a_0}{\underline{|7}}.$$

Hence we obtain

$$a_0 = 1, \qquad a_2 = \frac{1}{\underline{|3}} - \frac{1}{2} = \frac{1}{6} - \frac{1}{2} = -\frac{1}{3}, \qquad a_4 = \frac{1}{\underline{|4}} - \frac{1}{3\underline{|3}} - \frac{1}{\underline{|5}} = -\frac{1}{45};$$

$$a_6 = -\frac{1}{\underline{|6}} - \frac{1}{45\underline{|3}} + \frac{1}{3\underline{|5}} + \frac{1}{\underline{|7}} = -\frac{2}{945}.$$

Problem 9. *If* $\sec\theta = a_0 + a_2\theta^2 + a_4\theta^4 + \ldots + a_{2n}\theta^{2n} + \ldots$ *show that*

$$a_{2n} = \frac{a_{2n-2}}{\underline{|2}} - \frac{a_{2n-4}}{\underline{|4}} + \ldots + \frac{(-1)^{n+1} a_0}{\underline{|2n}}.$$

Solution. Let $\sec\theta = a_0 + a_2\theta^2 + a_4\theta^4 + \ldots;$

then $\qquad 1 = \cos\theta\left(a_0 + a_2\theta^2 + a_4\theta^4 + \ldots\right)$

$$= \left(1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \frac{\theta^6}{\underline{|6}} + \ldots\right)\left(a_0 + a_2\theta^2 + a_4\theta^4 + \ldots\right).$$

Then equating to zero the coefficient of θ^{2n} in the expression on the right-hand side we get

$$0 = a_{2n} - \frac{a_{2n-2}}{\underline{|2}} + \frac{a_{2n-4}}{\underline{|4}} - \frac{a_{2n-6}}{\underline{|6}} + \ldots + \frac{(-1)^n}{\underline{|2n}} a_0.$$

Transpose; then we obtain $a_{2n} = \dfrac{a_{2n-2}}{\underline{|2}} - \dfrac{a_{2n-4}}{\underline{|4}} + \ldots + \dfrac{(-1)^{n+1} a_0}{\underline{|2n}}.$

Problem 10. *If* $\cos 2\alpha + \sqrt{-1}\sin 2\alpha$ *be substituted for* a *in the expression* $\dfrac{bc}{(a+b)(a+c)}$, *and similar quantities for* b *and* c, *and the result reduced to the form* $A + B\sqrt{-1}$, *find the values of* A *and* B *in terms of* α, β, γ.

Solution.

$$a + b = \cos 2\alpha + \cos 2\beta + \sqrt{(-1)}\{\sin 2\alpha + \sin 2\beta\}$$
$$= 2\cos(\alpha+\beta)\cos(\alpha-\beta) + 2\sqrt{(-1)}\sin(\alpha+\beta)\cos(\alpha-\beta)$$
$$= 2\cos(\alpha-\beta)\left\{\cos(\alpha+\beta) + \sqrt{(-1)}\sin(\alpha+\beta)\right\}.$$

Thus $\dfrac{b}{a+b} = \dfrac{\cos 2\beta + \sqrt{(-1)}\sin 2\beta}{2\cos(\alpha-\beta)\left\{\cos(\alpha+\beta)+\sqrt{(-1)}\sin(\alpha+\beta)\right\}}$; multiply both numerator and denominator by $\cos(\alpha+\beta) - \sqrt{(-1)}\sin(\alpha+\beta)$;

thus we get $\dfrac{\cos(\beta-\alpha)+\sqrt{(-1)}\sin(\beta-\alpha)}{2\cos(\alpha-\beta)}$.

Similarly $\dfrac{c}{a+c} = \dfrac{\cos(\gamma-\alpha)+\sqrt{(-1)}\sin(\gamma-\alpha)}{2\cos(\alpha-\gamma)}$

Therefore $\dfrac{bc}{(a+b)(a+c)} = \dfrac{\cos(\beta+\gamma-2\alpha)+\sqrt{(-1)}\sin(\beta+\gamma-2\alpha)}{4\cos(\alpha-\beta)\cos(\alpha-\gamma)}$.

Problem 11. *Show that*

$$\begin{aligned}&\left\{\cos\theta+\cos\phi+\sqrt{-1}(\sin\theta+\sin\phi)\right\}^n\\ &+\left\{\cos\theta+\cos\phi-\sqrt{-1}(\sin\theta+\sin\phi)\right\}^n\\ &=2^{n+1}\left(\cos\frac{\theta-\phi}{2}\right)^n\cos\frac{n(\theta+\phi)}{2}.\end{aligned}$$

Solution.

$$\begin{aligned}&\left\{\cos\theta+\cos\phi+\sqrt{(-1)}(\sin\theta+\sin\phi)\right\}^n\\ &=\left\{2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}+2\sqrt{(-1)}\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\right\}^n\\ &=2^n\left(\cos\frac{\theta-\phi}{2}\right)^n\left\{\cos\frac{\theta+\phi}{2}+\sqrt{(-1)}\sin\frac{\theta+\phi}{2}\right\}^n\\ &=2^n\left(\cos\frac{\theta-\phi}{2}\right)^n\left\{\cos\frac{n}{2}(\theta+\phi)+\sqrt{(-1)}\sin\frac{n}{2}(\theta+\phi)\right\}.\end{aligned}$$

Similarly $\{\cos\theta+\cos\phi-\sqrt{(-1)}(\sin\theta+\sin\phi)\}^n$

$$=2^n\left(\cos\frac{\theta-\phi}{2}\right)^n\left\{\cos\frac{n}{2}(\theta+\phi)-\sqrt{(-1)}\sin\frac{n}{2}(\theta+\phi)\right\}.$$

Hence by addition we get $2^{n+1}\left(\cos\dfrac{\theta-\phi}{2}\right)^n\cos\dfrac{n}{2}(\theta+\phi)$.

Problem 12. *Show that if* $x = e^{\theta\sqrt{-1}}$, *and* $\sqrt{1-c^2} = nc-1$,

$$1+c\cos\theta=\frac{c}{2n}(1+nx)\left(1+\frac{n}{x}\right).$$

Solution.

$$\begin{aligned}\frac{c}{2n}(1+nx)\left(1+\frac{n}{x}\right)&=\frac{c}{2n}\left\{1+n^2+n\left(x+\frac{1}{x}\right)\right\}\\ &=\frac{c}{2n}\{1+n^2+2n\cos\theta\}=\frac{c(1+n^2)}{2n}+c\cos\theta.\end{aligned}$$

Now $\sqrt{(1-c^2)} = nc-1$; therefore $1-c^2 = (nc-1)^2$, therefore $-c^2 = n^2c^2-2nc$, therefore $2n = (n^2+1)c$.

Thus $\dfrac{c(1+n^2)}{2n} = 1$, and the expression becomes $1+c\cos\theta$.

Problem 13. *Prove the following rule for finding the length of a small circular arc : from eight times the chord of half the arc subtract the chord of the whole arc, and one-third of the remainder will give the length of the arc nearly.*

Solution. Let r denote the radius, and θ the circular measure of the angle; then the length of the arc is $r\theta$.

The chord of the arc is $2r\sin\frac{\theta}{2}$, and the chord of half the arc is $2r\sin\frac{\theta}{4}$.

Now let it be required to determine two numbers l and m, such that approximately

$$l \times 2r\sin\frac{\theta}{2} + m \times 2r\sin\frac{\theta}{4} = r\theta.$$

Expand $\sin\frac{\theta}{4}$ and $\sin\frac{\theta}{2}$ by *Art.* 274 (page 432). Thus

$$2l\left\{\frac{\theta}{2} - \frac{1}{\underline{|3}}\left(\frac{\theta}{2}\right)^3 + \dots\right\} + 2m\left\{\frac{\theta}{4} - \frac{1}{\underline{|3}}\left(\frac{\theta}{4}\right)^3 + \dots\right\} = 0.$$

Neglect all powers of θ above θ^3; then to make this formula hold we must put

$$l + \frac{m}{2} = 1, \quad \frac{l}{(2)^3} + \frac{m}{(4)^3} = 0.$$

Therefore $m = -8l$; therefore $-3l = 1$.

Thus $$l = -\frac{1}{3} \text{ and } m = \frac{8}{3}.$$

This establishes the rule.

Problem 14. *Show that the following rule for finding the length of a small circular arc is more accurate than that in the preceding problem : to 256 times the chord of one-fourth of the arc add the chord of the arc; subtract 40 times the chord of half the arc, and divide the remainder by 45.*

Solution. Proceed as in *Problem* 13.

The chord of one-fourth of the arc is $2r\sin\frac{\theta}{8}$.

Let it be required to determine the numbers l, m, n such that approximately

$$l \times 2r\sin\frac{\theta}{2} + m \times 2r\sin\frac{\theta}{4} + n \times 2r\sin\frac{\theta}{8} = r\theta.$$

In this case we can make the approximation closer than in *Problem* 13; for we shall retain θ^5 and neglect only the higher powers. Thus

$$2l\left\{\frac{\theta}{2} - \frac{1}{\underline{|3}}\left(\frac{\theta}{2}\right)^3 + \frac{1}{\underline{|5}}\left(\frac{\theta}{2}\right)^5\right\} + 2m\left\{\frac{\theta}{4} - \frac{1}{\underline{|3}}\left(\frac{\theta}{4}\right)^3 + \frac{1}{\underline{|5}}\left(\frac{\theta}{4}\right)^5\right\}$$
$$+ 2n\left\{\frac{\theta}{8} - \frac{1}{\underline{|3}}\left(\frac{\theta}{8}\right)^3 + \frac{1}{\underline{|5}}\left(\frac{\theta}{8}\right)^5\right\} = \theta.$$

Hence we must put

$$l + \frac{m}{2} + \frac{n}{4} = 1, \quad \frac{l}{(2)^3} + \frac{m}{(4)^3} + \frac{n}{(8)^3} = 0, \quad \frac{l}{(2)^5} + \frac{m}{(4)^5} + \frac{n}{(8)^5} = 0.$$

The values of l, m, n given by these equations are

$$l = \frac{1}{45}, \qquad m = -\frac{40}{45}, \qquad n = \frac{256}{45}.$$

Problem 15. *From the identical equation*

$$\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}+\frac{(x-a)(x-b)}{(c-a)(c-b)}=1,$$

deduce the following by assuming

$$x=\cos 2\theta+\sqrt{-1}\sin 2\theta,$$

and corresponding assumptions for a, b, *and* c :

$$\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}\sin 2(\theta-\alpha)+\frac{\sin(\theta-\gamma)\sin(\theta-\alpha)}{\sin(\beta-\gamma)\sin(\beta-\alpha)}\sin 2(\theta-\beta)+\frac{\sin(\theta-\alpha)\sin(\theta-\beta)}{\sin(\gamma-\alpha)\sin(\gamma-\beta)}\sin 2(\theta-\gamma)=0.$$

Solution.

$$\begin{aligned}x-b&=\cos 2\theta+\sqrt{(-1)}\sin 2\theta-\cos 2\beta-\sqrt{(-1)}\sin 2\beta\\&=2\sin(\beta-\theta)\left\{\sin(\beta+\theta)-\sqrt{(-1)}\cos(\beta+\theta)\right\}\\&=\frac{2\sin(\beta-\theta)}{\sqrt{(-1)}}\left\{\cos(\beta+\theta)+\sqrt{(-1)}\sin(\beta+\theta)\right\}.\end{aligned}$$

In like manner

$$a-b=\frac{2\sin(\beta-\alpha)}{\sqrt{(-1)}}\left\{\cos(\beta+\alpha)+\sqrt{(-1)}\sin(\beta+\alpha)\right\}.$$

Therefore $\dfrac{x-b}{a-b}=\dfrac{\sin(\beta-\theta)}{\sin(\beta-\alpha)}\cdot\dfrac{\cos(\beta+\theta)+\sqrt{(-1)}\sin(\beta+\theta)}{\cos(\beta+\alpha)+\sqrt{(-1)}\sin(\beta+\alpha)}$, multiply both numerator and denominator by $\cos(\beta+\alpha)-\sqrt{(-1)}\sin(\beta+\alpha)$;

thus we get $\quad\dfrac{\sin(\theta-\beta)}{\sin(\alpha-\beta)}\left\{\cos(\theta-\alpha)+\sqrt{(-1)}\sin(\theta-\alpha)\right\}.$

Similarly we transform $\dfrac{x-c}{a-c}$; and thus we obtain

$$\frac{(x-b)(x-c)}{(a-b)(a-c)}=\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}\left\{\cos 2(\theta-\alpha)+\sqrt{(-1)}\sin 2(\theta-\alpha)\right\}.$$

In like manner we transform $\dfrac{(x-c)(x-a)}{(b-c)(b-a)}$ and $\dfrac{(x-a)(x-b)}{(c-a)(c-b)}$. Then by equating to zero the coefficient of the imaginary part we obtain

$$\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}\sin 2(\theta-\alpha)+\frac{\sin(\theta-\gamma)\sin(\theta-\alpha)}{\sin(\beta-\gamma)\sin(\beta-\alpha)}\sin 2(\theta-\beta)+\frac{\sin(\theta-\alpha)\sin(\theta-\beta)}{\sin(\gamma-\alpha)\sin(\gamma-\beta)}\sin 2(\theta-\gamma)=0.$$

And then by equating the real parts we have

$$\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}\cos 2(\theta-\alpha)+\frac{\sin(\theta-\gamma)\sin(\theta-\alpha)}{\sin(\beta-\gamma)\sin(\beta-\alpha)}\cos 2(\theta-\beta)$$

$$+ \frac{\sin(\theta-\alpha)\sin(\theta-\beta)}{\sin(\gamma-\alpha)\sin(\gamma-\beta)}\cos 2(\theta-\gamma) = 1.$$

CHAPTER XX
Expansions of Some Trigonometrical Functions

Problem 1. *Expand* $(\sin\theta)^{4n+2}$ *in terms of cosines of multiples of* θ.

Solution. Proceed as in *Art.* 282 (page 434). Thus we obtain

$$\begin{aligned}-2^{4n+1}(\sin\theta)^{4n+2} &= \cos(4n+2)\theta-(4n+2)\cos 4n\theta\\ &+\frac{(4n+2)(4n+1)}{\lfloor\underline{2}}\cos(4n-2)\theta-\dots\\ &-\frac{(4n+2)(4n+1)\dots(2n+2)}{2\lfloor\underline{2n+1}}.\end{aligned}$$

Problem 2. *Expand* $(\sin\theta)^{4n+1}$ *in terms of sines of multiples of* θ.

Solution. Proceed as in *Art.* 283 (page 434). Thus we obtain

$$\begin{aligned}2^{4n}(\sin\theta)^{4n+1} &= \sin(4n+1)\theta-(4n+1)\sin(4n-1)\theta\\ &+\frac{(4n+1)4n}{\lfloor\underline{2}}\sin(4n-3)\theta-\dots\\ &+\frac{(4n+1)4n(4n-1)\dots(2n+2)}{\lfloor\underline{2n}}\sin\theta.\end{aligned}$$

Problem 3. *Expand* $(\cos\theta)^{2n}$ *in terms of cosines of multiples of* θ.

Solution. Proceed as in *Art.* 280 (page 433). Thus we obtain

$$\begin{aligned}2^{2n-1}\cos^{2n}\theta &= \cos 2n\theta+2n\cos(2n-2)\theta+\frac{2n(2n-1)}{\lfloor\underline{2}}\cos(2n-4)\theta\\ &+\dots\frac{2n(2n-1)\dots(n+1)}{2\lfloor\underline{n}}.\end{aligned}$$

Problem 4. *Prove that in any triangle*

$$\frac{a^2\cos\frac12(B-C)}{\cos\frac12(B+C)}+\frac{b^2\cos\frac12(C-A)}{\cos\frac12(C+A)}+\frac{c^2\cos\frac12(A-B)}{\cos\frac12(A+B)} = 2(ab+bc+ca).$$

Solution. We have

$$\begin{aligned}\frac{a^2\cos\dfrac12(B-C)}{\cos\dfrac12(B+C)} &= \frac{a^2\cos\dfrac12(B-C)}{\sin\dfrac12 A}=\frac{a^2\cos\dfrac12 A\cos\dfrac12(B-C)}{\cos\dfrac12 A\sin\dfrac12 A}\\ &=\frac{2a^2}{\sin A}\sin\frac12(B+C)\cos\frac12(B-C)=\frac{a^2}{\sin A}(\sin B+\sin C)\\ &=\frac{a^2\sin B}{\sin A}+\frac{a^2\sin C}{\sin A}=\frac{a^2b}{a}+\frac{a^2c}{a}=ab+ac.\end{aligned}$$

Similarly $$\frac{b^2\cos\frac{1}{2}(C-A)}{\cos\frac{1}{2}(C+A)} = ba+bc,$$

and $$\frac{c^2\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)} = ca+cb.$$

Then by addition we obtain the required result.

Problem 5. *From the angles of a triangle ABC, perpendiculars AD, BE, CF are let fall on the opposite sides : prove that*

$$a\sin(BAD-CAD)+b\sin(CBE-ABE)+c\sin(ACF-BCF)=0.$$

Solution. Suppose the triangle to have all its angles acute.

Then $$a\sin(BAD-CAD) = a(\sin BAD\cos CAD-\cos BAD\sin CAD)$$
$$= a(\cos B\sin C-\sin B\cos C).$$

Similarly $$b\sin(CBE-ABE) = b(\cos C\sin A-\sin C\cos A),$$

and $$c\sin(ACF-BCF) = c(\cos A\sin B-\sin A\cos B).$$

The sum of the three expressions

$$= \cos A(c\sin B-b\sin C)+\cos B(a\sin C-c\sin A)$$
$$+\cos C(b\sin A-a\sin B)=0,$$ by *Art.* 214 (page 418).

If the triangle has an obtuse angle, let it be C; then it will be found that instead of $\cos C$ we have $\cos(180°-C)$ in the preceding expressions; and the result is still zero.

Problem 6. *From A and B two acute angles of a triangle draw AD and BD at right angles respectively to AC and BC. If ρ be the radius of the circle inscribed in ABD, then*

$$AB = \rho\,(\sec A+\sec B+\tan A+\tan B)\,.$$

Solution. We have $$\rho\left\{\cot\frac{1}{2}DAB+\cot\frac{1}{2}DBA\right\} = AB.$$

Now $$\cot\frac{1}{2}DAB = \cot\frac{1}{2}\left(\frac{\pi}{2}-A\right) = \frac{\cos\frac{1}{2}\left(\frac{\pi}{2}-A\right)}{\sin\frac{1}{2}\left(\frac{\pi}{2}-A\right)}$$
$$= \frac{2\cos^2\frac{1}{2}\left(\frac{\pi}{2}-A\right)}{2\sin\frac{1}{2}\left(\frac{\pi}{2}-A\right)\cos\frac{1}{2}\left(\frac{\pi}{2}-A\right)} = \frac{1+\cos\left(\frac{\pi}{2}-A\right)}{\sin\left(\frac{\pi}{2}-A\right)}$$
$$= \frac{1+\sin A}{\cos A} = \sec A+\tan A.$$

Similarly $$\cot\frac{1}{2}DBA = \sec B+\tan B.$$

Therefore $$\rho\,\{\sec A+\tan A+\sec B+\tan B\} = AB.$$

Problem 7. *Three equal circles of radius a touch each other : show that the area of the space between them is*

$$\left(\sqrt{3} - \frac{\pi}{2}\right) a^2.$$

Solution. By joining the centers of the circles we form an equilateral triangle of which each side is $2a$; and therefore the area is $\frac{(2a)^2\sqrt{3}}{4}$, that is $a^2\sqrt{3}$. The area of each of the three sectors which are formed by the radii and arc of a circle is $\frac{a^2}{2} \times \frac{2\pi}{6}$, that is $\frac{\pi a^2}{6}$; therefore the area of the three sectors is $\frac{\pi a^2}{2}$. Hence the area of the space between the circles $= a^2\sqrt{3} - \frac{\pi a^2}{2} = a^2\left(\sqrt{3} - \frac{\pi}{2}\right)$.

Problem 8. *Let area of a regular polygon inscribed in a circle is a geometric mean between the areas of an inscribed and of a circumscribed regular polygon of half the number of sides.*

Solution. Let R be the radius of the circle. The inscribed polygon of n sides consists of n triangles; and therefore the area of the polygon is

$$n\frac{R^2}{2}\sin\frac{2\pi}{n}.$$

The area of an inscribed figure of half the number of sides is

$$\frac{n}{2}\frac{R^2}{2}\sin\frac{4\pi}{n}.$$

The area of a circumscribed polygon of $\frac{n}{2}$ sides is $\frac{n}{2}R^2\tan\frac{2\pi}{n}$.

We have to show that

$$\left(\frac{nR^2}{2}\sin\frac{2\pi}{n}\right)^2 = \frac{n}{4}R^2\sin\frac{4\pi}{n} \times \frac{n}{2}R^2\tan\frac{2\pi}{n},$$

or that

$$\sin^2\frac{2\pi}{n} = \frac{1}{2}\sin\frac{4\pi}{n}\tan\frac{2\pi}{n};$$

and this is obvious since $\sin\frac{4\pi}{n} = 2\sin\frac{2\pi}{n}\cos\frac{2\pi}{n}$.

Problem 9. *The area of a regular polygon circumscribed about a circle is an harmonic mean between the areas of an inscribed regular polygon of the same number of sides, and of a circumscribed regular polygon of half that number.*

Solution. Let R be the radius of the circle. The area of the circumscribed polygon of n sides is $nR^2\tan\frac{\pi}{n}$. The area of the inscribed polygon of n sides is $\frac{nR^2}{2}\sin\frac{2\pi}{n}$. The area of the circumscribed polygon of $\frac{n}{2}$ sides is $\frac{n}{2}R^2\tan\frac{2\pi}{n}$. We have to show that $nR^2\tan\frac{\pi}{n}$ is an harmonic mean between $\frac{nR^2}{2}\sin\frac{2\pi}{n}$ and $\frac{nR^2}{2}\tan\frac{2\pi}{n}$; or that $2\tan\frac{\pi}{n}$ is an harmonic mean between $\sin\frac{2\pi}{n}$ and $\tan\frac{2\pi}{n}$.

Now the harmonic mean between $\sin\dfrac{2\pi}{n}$ and $\tan\dfrac{2\pi}{n}$ is

$$\frac{2\sin\dfrac{2\pi}{n}\tan\dfrac{2\pi}{n}}{\sin\dfrac{2\pi}{n}+\tan\dfrac{2\pi}{n}},\ \text{that is}\ \frac{2\sin\dfrac{2\pi}{n}}{1+\cos\dfrac{2\pi}{n}},\ \text{that is}\ 2\tan\frac{\pi}{n}.$$

Problem 10. *If the side of a pentagon inscribed in a circle be c, the radius is* $\dfrac{c\sqrt{5+\sqrt{5}}}{\sqrt{10}}$.

Solution. Let r be the radius of the circle; then $c = 2r\sin\dfrac{2\pi}{10} = 2r\sin\dfrac{\pi}{5}$; therefore $r = \dfrac{c}{2\sin\dfrac{\pi}{5}} = \dfrac{2c}{\sqrt{(10-2\sqrt{5})}} = \dfrac{2c}{\sqrt{2\sqrt{(5-\sqrt{5})}}}$. Multiply both numerator and denominator by $\sqrt{(5+\sqrt{5})}$. Thus we obtain $\dfrac{2c\sqrt{(5+\sqrt{5})}}{\sqrt{2}\times\sqrt{(25-5)}}$, that is $\dfrac{2c\sqrt{(5+\sqrt{5})}}{\sqrt{(40)}}$, that is $\dfrac{c\sqrt{(5+\sqrt{5})}}{\sqrt{(10)}}$.

Problem 11. *Three circles whose radii are a, b, c touch each other externally : prove that the tangents at the points of contact meet at a point whose distance from any one of them is*

$$\left(\frac{abc}{a+b+c}\right)^{\frac{1}{2}}.$$

Solution. Let A, B, C denote the centers of the three circles. Let tangents to the arc of the first circle meet at T; then the distance of T from the point of contact is $a\tan\dfrac{A}{2}$.

Now $$\cos A = \frac{(a+c)^2+(a+b)^2-(b+c)^2}{2(a+c)(a+b)} = \frac{a^2+a(b+c)-bc}{(a+c)(a+b)};$$

therefore $$\frac{1-\cos A}{1+\cos A} = \frac{bc}{a(a+b+c)};$$

therefore $$\tan\frac{A}{2} = \sqrt{\frac{bc}{a(a+b+c)}};\ \text{therefore}\ a\tan\frac{A}{2} = \sqrt{\frac{abc}{a+b+c}}.$$

We shall obtain the same symmetrical expression for the distance of the point of intersection of any two tangents from the points of contact; and thus it follows that the three tangents meet at a common point.

Problem 12. *The sides taken in order of a quadrilateral whose opposite angles are supplementary are 3, 3, 4, 4 : find the area and the radii of the inscribed and circumscribed circles.*

Solution. Use *Article* 254 (page 429). Here $s = 7$, $s-a = 4$, $s-b = 4$, $s-c = 3$, $s-d = 3$. Thus the area $= \sqrt{4\times4\times3\times3} = 12$.

Since the sum of a pair of opposite sides is equal to the sum of the other pair, a circle may be inscribed in the quadrilateral. Let ρ denote the radius of this inscribed circle; then $\frac{\rho}{2}(a+b+c+d) =$ the area of the quadrilateral. Thus $\rho s =$ the area.

In the present case $$\rho = \frac{12}{7}$$

Also $$ab + cd = 25, \quad ac + bd = 24, \quad ad + bc = 24.$$

Hence the radius of the circumscribed circle

$$= \frac{1}{4}\sqrt{\frac{25 \times 24 \times 24}{3 \times 3 \times 4 \times 4}} = \frac{5 \times 24}{4 \times 12} = \frac{5}{2}.$$

Problem 13. *The area of a regular polygon inscribed in a circle is to that of the circumscribed polygon of the same number of sides as 3 is to 4 : find the number of sides.*

Solution. Let n be the number of the sides, R the radius of the circle.

Then $$\frac{n}{2}R^2 \sin\frac{2\pi}{n} \text{ is to } nR^2 \tan\frac{\pi}{n} \text{ as 3 is to 4.}$$

Thus $$\frac{\sin\frac{2\pi}{n}}{2\tan\frac{\pi}{n}} = \frac{3}{4};$$

therefore $$\cos^2\frac{\pi}{n} = \frac{3}{4}; \text{ therefore } \cos\frac{\pi}{n} = \frac{\sqrt{3}}{2}.$$

But $$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}; \text{ therefore } n = 6.$$

Problem 14. *If the radii of three circles which touch each other be a, b, c, and α, β, γ be the chords of the arcs between the points of contact in each, show that*

$$\frac{8}{\alpha\beta\gamma} = \left(\frac{1}{a} + \frac{1}{b}\right)\left(\frac{1}{b} + \frac{1}{c}\right)\left(\frac{1}{c} + \frac{1}{a}\right).$$

Solution. Let A, B, C denote the centers of the three circles.

Then $$\alpha = 2a\sin\frac{A}{2}, \quad \beta = 2b\sin\frac{B}{2}, \quad \gamma = 2c\sin\frac{C}{2}.$$

Now from the solution of *Problem* 11 we see that

$$\cos A = \frac{a^2 + a(b+c) - bc}{(a+b)(a+c)}; \quad \text{therefore } 1 - \cos A = \frac{2bc}{(a+b)(a+c)};$$

therefore $$\sin\frac{A}{2} = \sqrt{\frac{bc}{(a+b)(a+c)}};$$

therefore $$\alpha = 2a\sqrt{\frac{bc}{(a+b)(a+c)}}.$$

Similar expressions hold for β and γ. Thus

$$\frac{1}{\alpha\beta\gamma} = \frac{1}{8abc}\left\{\frac{(a+b)(a+c)}{bc} \times \frac{(b+a)(b+c)}{ac} \times \frac{(c+a)(c+b)}{bc}\right\}^{\frac{1}{2}}$$
$$= \frac{(a+b)(b+c)(c+a)}{8a^2b^2c^2};$$

therefore $$\frac{8}{\alpha\beta\gamma} = \left(\frac{1}{b} + \frac{1}{a}\right)\left(\frac{1}{c} + \frac{1}{b}\right)\left(\frac{1}{a} + \frac{1}{c}\right).$$

Problem 15. *Show that the limit of* $\left(\frac{\tan\theta}{\theta}\right)^{\frac{3}{\theta^2}}$*, when* θ *is indefinitely diminished, is* e*.*

Solution. Let $u = \left(\frac{\tan\theta}{\theta}\right)^{\frac{3}{\theta^2}}$; then $\log u = \frac{3}{\theta^2}\log\frac{\tan\theta}{\theta}$.

Now $\tan\theta = \theta + \frac{\theta^3}{3} +$ terms in θ^5 and higher powers of θ; see *Chapter* XIX : *Problem* 7.

Therefore $$\frac{\tan\theta}{\theta} = 1 + \frac{\theta^2}{3} + \ldots$$

Then $\log\left(1 + \frac{\theta^2}{3} + \ldots\right) = \frac{\theta^2}{3} +$ terms in θ^4 and higher powers of θ.

Therefore $\frac{3}{\theta^2}\log\frac{\tan\theta}{\theta} = 1 +$ terms in θ^2 and higher powers of θ.

Therefore when θ is indefinitely diminished $\frac{3}{\theta^2}\log\frac{\tan\theta}{\theta} = 1$, and therefore

$$\left(\frac{\tan\theta}{\theta}\right)^{\frac{3}{\theta^2}} = e.$$

Problem 16. *The two diagonals of a quadrilateral figure whose opposite angles are supplementary cannot be equal unless some one of the sides be equal to the opposite one.*

Solution. Equate the expressions for AC^2 and BD^2 given in *Art* 254 (page 429). Thus

$$\frac{(ac+bd)(ad+bc)}{ab+cd} = \frac{(ac+bd)(ab+cd)}{ad+bc}.$$

Therefore $$(ad+bc)^2 = (ab+cd)^2;$$

therefore $$ad+bc = ab+cd;$$

therefore $$(a-c)(d-b) = 0.$$

Therefore either $a = c$ or $b = d$.

Problem 17. *Two circles whose radii are* a *and* b *cut one another at an angle* γ *: show that the length of the common chord is*

$$\frac{2ab\sin\gamma}{\sqrt{a^2 + 2ab\cos\gamma + b^2}}.$$

Solution. Let A and B be the centers of the two circles, and C a point of intersection. The angle between the tangents at C is therefore γ.

$$\text{The angle} ACB = \frac{\pi}{2} + \frac{\pi}{2} - \gamma = \pi - \gamma.$$

Then $$AB^2 = a^2 + b^2 - 2ab\cos(\pi - \gamma) = a^2 + b^2 + 2ab\cos\gamma.$$

Let x denote the length of the common chord; then the area of the triangle $ABC = \frac{1}{2} \times \frac{x}{2} \times AB$; and this area also $= \frac{1}{2} AC \cdot CB \sin ACB$.

Thus
$$x = \frac{2AC \cdot CB \sin ACB}{AB}$$
$$= \frac{2ab \sin \gamma}{\sqrt{(a^2 + b^2 + 2ab \cos \gamma)}}.$$

Problem 18. *The radius of the circle inscribed in a triangle can never be greater than half the radius of the circle described about the triangle.*

Solution. We have $\frac{r}{R} = \frac{S}{s} \div \frac{abc}{4S} = \frac{4S^2}{sabc} = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$.

Now we have shown in the solution of *Chapter* XIII : *Problem* 40, that the expression $4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$ can never be greater than $\frac{1}{2}$. Hence r cannot be greater than $\frac{1}{2} R$.

CHAPTER XXI
Exponential Values of the Cosine and Sine

Problem 1. *Apply the exponential values of the sine and cosine to show that*

$$\frac{\sin 2\theta}{1-\cos 2\theta}=\cot\theta.$$

Solution.

$$\frac{\sin 2\theta}{1-\cos 2\theta}=\frac{e^{2\theta\iota}-e^{-2\theta\iota}}{2\iota\left(1-\dfrac{e^{2\theta\iota}+e^{-2\theta\iota}}{2}\right)}=\frac{e^{2\theta\iota}-e^{-2\theta\iota}}{\iota(2-e^{2\theta\iota}-e^{-2\theta\iota})}$$

$$=\frac{\iota\left(e^{2\theta\iota}-e^{-2\theta\iota}\right)}{e^{2\theta\iota}+e^{-2\theta\iota}-2}=\frac{\iota\left(e^{\theta\iota}+e^{-\theta\iota}\right)\left(e^{\theta\iota}-e^{-\theta\iota}\right)}{(e^{\theta\iota}-e^{-\theta\iota})^2}$$

$$=\frac{\iota\left(e^{\theta\iota}+e^{-\theta\iota}\right)}{e^{\theta\iota}-e^{-\theta\iota}}=\frac{e^{\theta\iota}+e^{-\theta\iota}}{2}\div\frac{e^{\theta\iota}-e^{-\theta\iota}}{2\iota}=\frac{\cos\theta}{\sin\theta}.$$

Problem 2. *If the sides of a right-angled triangle be 49 and 51, show that the angles opposite to them are 43°51′15″ and 46°8′45″ nearly.*

Solution. Let the angle opposite to the smaller side be $\dfrac{\pi}{4}-\theta$, and the angle opposite to the larger side $\dfrac{\pi}{4}+\theta$. Thus

$$\frac{\sin\left(\dfrac{\pi}{4}-\theta\right)}{\sin\left(\dfrac{\pi}{4}+\theta\right)}=\frac{49}{51};$$

therefore
$$\frac{\sin\left(\dfrac{\pi}{4}+\theta\right)-\sin\left(\dfrac{\pi}{4}-\theta\right)}{\sin\left(\dfrac{\pi}{4}+\theta\right)+\sin\left(\dfrac{\pi}{4}-\theta\right)}=\frac{51-49}{51+49}=\frac{1}{50};$$

therefore
$$\frac{2\sin\theta\cos\dfrac{\pi}{4}}{2\cos\theta\sin\dfrac{\pi}{4}}=\frac{1}{50};$$

therefore
$$\tan\theta=\frac{1}{50}.$$

But by *Art.* 293 (page 435) we have

$$\theta=\tan\theta-\frac{1}{3}\tan^3\theta+\ldots,$$

thus
$$\theta=.02-\frac{1}{3}(.02)^3+\frac{1}{5}(.02)^5-\ldots$$

If we stop at the first term we have $\theta=.02$.

Then the number of degrees in the angle $=.02\times 57.29577951...=1.14591559$; and this $=1^{0}8'45''$.

Problem 3. *If the angle C of a triangle be given, and the other two adjacent sides* *a, b be nearly equal, show that the other angles are nearly equal to*

$$90^\circ - \frac{C}{2} \pm \frac{180^\circ}{\pi}\left\{\frac{a-b}{a+b}\cot\frac{C}{2} - \frac{1}{3}\left(\frac{a-b}{a+b}\cot\frac{C}{2}\right)^3\right\}.$$

Solution. We have, as in *Art.* 229 (page 421),

$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}.$$

Hence by *Art.* 293 (page 435), the circular measure of $\frac{A-B}{2}$

$$= k - \frac{k^3}{3} + \frac{k^5}{5} - \dots$$

where k stands for $\frac{a-b}{a+b}\cot\frac{C}{2}$.

Therefore the number of degrees in $\frac{A-B}{2}$

$$= \frac{180}{\pi}\left\{k - \frac{k^3}{3} + \frac{k^5}{5} - \dots\right\}.$$

Also $\frac{A+B}{2} = 90^\circ - \frac{C}{2}$. Thus A is found by taking the upper sign, and B by taking the lower sign in

$$90^\circ - \frac{C}{2} \pm \frac{180^\circ}{\pi}\left\{k - \frac{k^3}{3} + \frac{k^5}{5} - \dots\right\}.$$

Problem 4. *In any triangle, if $A-B$ be small compared with C, show that the circular measure of $A-B$ is equal to*

$$2\frac{a-b}{c}\sin B + \left(\frac{a-b}{c}\right)^2 \sin 2B \text{ nearly.}$$

Solution. $$\frac{\sin A}{\sin C} = \frac{a}{c}, \qquad \frac{\sin B}{\sin C} = \frac{b}{c};$$

therefore $$\frac{\sin A - \sin B}{\sin C} = \frac{a-b}{c};$$

therefore $$\frac{\sin\frac{A-B}{2}\cos\frac{A+B}{2}}{\sin\frac{C}{2}\cos\frac{C}{2}} = \frac{a-b}{c};$$

therefore $$\sin\frac{A-B}{2} = \frac{a-b}{c}\cos\frac{C}{2}$$

$$= \frac{a-b}{c}\sin\frac{A+B}{2}$$

$$= \frac{a-b}{c}\sin\left(\frac{A-B}{2} + B\right).$$

Hence by *Art.* 208 (page 418) the circular measure of $\frac{A-B}{2}$

$$= n\sin B + \frac{n^2}{2}\sin 2B + \frac{n^3}{3}\sin 3B + \dots,$$

where n stands for $\frac{a-b}{c}$. Therefore the circular measure of $A-B$

$$= 2n\sin B + n^2\sin 2B \text{ nearly.}$$

Problem 5. *If a and b be the sides of a plane triangle, A and B the opposite angles, then will* $\log b - \log a$

$$= \cos 2A - \cos 2B + \frac{1}{2}(\cos 4A - \cos 4B) + \frac{1}{3}(\cos 6A - \cos 6B) + \ldots$$

Solution. $\dfrac{b}{a} = \dfrac{\sin B}{\sin A} = \dfrac{e^{B\iota} - e^{-B\iota}}{e^{A\iota} - e^{-A\iota}} = \dfrac{e^{B\iota}\left(1 - e^{-2B\iota}\right)}{e^{A\iota}\left(1 - e^{-2A\iota}\right)}.$

Take the logarithms : thus

$$\log b - \log a = B\iota - A\iota + \log\left(1 - e^{-2B\iota}\right) - \log\left(1 - e^{-2A\iota}\right)$$

$$= (B - A)\iota - \left\{e^{-2B\iota} + \frac{1}{2}e^{-4B\iota} + \frac{1}{3}e^{-6B\iota} + \ldots\right\}$$

$$+ e^{-2A\iota} + \frac{1}{2}e^{-4A\iota} + \frac{1}{3}e^{-6A\iota} + \ldots$$

Now $e^{-2B\iota} = \cos 2B - \iota \sin 2B, \quad e^{-2A\iota} = \cos 2A - \iota \sin 2A,$

and so on. Then, as the real and imaginary parts of the expression must be separately equal, we have

$$\log b - \log a = \cos 2A - \cos 2B + \frac{1}{2}(\cos 4A - \cos 4B)$$

$$+ \frac{1}{3}(\cos 6A - \cos 6B) + \ldots$$

Problem 6. *Show that* $\dfrac{\pi}{8} = \dfrac{1}{1 \cdot 3} + \dfrac{1}{5 \cdot 7} + \dfrac{1}{9 \cdot 11} + \ldots$'

Solution. By *Art.* 294 (page 435)

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots$$

$$= \frac{2}{1 \cdot 3} + \frac{2}{5 \cdot 7} + \frac{2}{9 \cdot 11} + \ldots;$$

therefore $$\frac{\pi}{8} = \frac{1}{1 \cdot 3} + \frac{1}{5 \cdot 7} + \frac{1}{9 \cdot 11} + \ldots$$

Problem 7. *If* $A + B\iota = \log(m + n\iota)$, *show that*

$$\tan B = \frac{n}{m}, \quad \text{and } 2A = \log\left(n^2 + m^2\right).$$

Solution.

Let $$A + B\iota = \log(m + n\iota);$$

therefore $$e^{A+B\iota} = m + n\iota;$$

therefore $$m + n\iota = e^A e^{B\iota} = e^A(\cos B + \iota \sin B);$$

therefore $$m = e^A \cos B,$$

and $$n = e^A \sin B.$$

By division $$\frac{n}{m} = \tan B.$$

By squaring and adding $$m^2 + n^2 = e^{2A};$$

therefore $$2A = \log\left(m^2 + n^2\right).$$

Problem 8. *Reduce* $\cos(\theta+\phi\iota)$ *to the form* $\alpha+\beta\iota$.

Solution.

$$\begin{aligned}\cos(\theta+\phi\iota) &= \cos\theta\cos\phi\iota - \sin\theta\sin\phi\iota\\ &= \cos\theta\frac{e^{-\phi}+e^{\phi}}{2} - \sin\theta\frac{e^{-\phi}-e^{\phi}}{2\iota}\\ &= \cos\theta\frac{e^{-\phi}+e^{\phi}}{2} + \iota\sin\theta\frac{e^{-\phi}-e^{\phi}}{2};\end{aligned}$$

this is of the form $\alpha+\beta\iota$ where

$$\alpha = \cos\theta\frac{e^{-\phi}+e^{\phi}}{2}, \text{ and } \beta = \sin\theta\frac{e^{-\phi}-e^{\phi}}{2}.$$

Problem 9. *Reduce* $\sin(\theta+\phi\iota)$ *to the form* $\alpha+\beta\iota$.

Solution.

$$\begin{aligned}\sin(\theta+\phi\iota) &= \sin\theta\cos\phi\iota + \cos\theta\sin\phi\iota\\ &= \sin\theta\frac{e^{-\phi}+e^{\phi}}{2} + \cos\theta\frac{e^{-\phi}-e^{\phi}}{2\iota}\\ &= \sin\theta\frac{e^{-\phi}+e^{\phi}}{2} - \iota\cos\theta\frac{e^{-\phi}-e^{\phi}}{2};\end{aligned}$$

this is of the form $\alpha+\beta\iota$ where

$$\alpha = \sin\theta\frac{e^{-\phi}+e^{\phi}}{2}, \text{ and } \beta = -\cos\theta\frac{e^{-\phi}-e^{\phi}}{2}.$$

Problem 10. *If* $u=(a+b\iota)^{p+q\iota}$, *express* $\log u$ *in the form* $\alpha+\beta\iota$.

Solution.

$$\begin{aligned}\log u &= (p+q\iota)\log(a+b\iota)\\ &= (p+q\iota)\log\sqrt{(a^2+b^2)}\left\{\frac{a}{\sqrt{(a^2+b^2)}}+\frac{b\iota}{\sqrt{(a^2+b^2)}}\right\}\\ &= (p+q\iota)\log\sqrt{(a^2+b^2)}\{\cos\gamma+\iota\sin\gamma\},\end{aligned}$$

where $$\cos\gamma = \frac{a}{\sqrt{(a^2+b^2)}}, \text{ and } \sin\gamma = \frac{b}{\sqrt{(a^2+b^2)}},$$

$$\begin{aligned}&= (p+q\iota)\log e^{\gamma\iota}\sqrt{(a^2+b^2)}\\ &= (p+q\iota)\left\{\log e^{\gamma\iota}+\log\sqrt{(a^2+b^2)}\right\}\\ &= (p+q\iota)\left\{\gamma\iota+\log\sqrt{(a^2+b^2)}\right\}\\ &= p\log\sqrt{(a^2+b^2)} - q\gamma + \left\{p\gamma+q\log\sqrt{(a^2+b^2)}\right\}\iota.\end{aligned}$$

This is of the form $\alpha+\beta\iota$, where

$$\alpha = p\log\sqrt{(a^2+b^2)} - q\gamma, \text{ and } \beta = p\gamma + q\log\sqrt{(a^2+b^2)}.$$

Problem 11. *Reduce* $(a+b\iota)^{p+q\iota}$ *to the form* $\alpha+\beta\iota$.

Solution. By *Problem* 10 we can express $\log(a+b\iota)^{p+q\iota}$ in the form $\alpha+\beta\iota$; therefore

$$(a+b\iota)^{p+q\iota}=e^{\alpha+\beta\iota}=e^{\alpha}e^{\beta\iota}=e^{\alpha}(\cos\beta+\iota\sin\beta);$$

and this is of the form $\lambda+\iota\mu$, where $\lambda=e^{\alpha}\cos\beta$, and $\mu=e^{\alpha}\sin\beta$.

Problem 12. *Prove that*

$$\left\{\sin(\alpha-\theta)+e^{\pm\alpha\iota}\sin\theta\right\}^{n}=\sin^{n-1}\alpha\left\{\sin(\alpha-n\theta)+e^{\pm\alpha\iota}\sin n\theta\right\}.$$

Solution. $\{\sin(\alpha-\theta)+e^{\alpha\iota}\sin\theta\}^{n}=\{\sin(\alpha-\theta)+(\cos\alpha+\iota\sin\alpha)\sin\theta\}^{n}$

$$=(\sin\alpha\cos\theta+\iota\sin\alpha\sin\theta)^{n}=\sin^{n}\alpha(\cos\theta+\iota\sin\theta)^{n}$$
$$=\sin^{n}\alpha(\cos n\theta+\iota\sin n\theta).$$

Again $\qquad \sin^{n-1}\alpha\{\sin(\alpha-n\theta)+e^{\alpha\iota}\sin n\theta\}$

$$=\sin^{n-1}\alpha\{\sin(\alpha-n\theta)+(\cos\alpha+\iota\sin\alpha)\sin n\theta\}$$
$$=\sin^{n-1}\alpha\{\sin\alpha\cos n\theta+\iota\sin\alpha\sin n\theta\}$$
$$=\sin^{n}\alpha(\cos n\theta+\iota\sin n\theta):$$

thus the two expressions agree.

In a similar way we may proceed when we take the lower sign in the expressions.

CHAPTER XXII
Summation of Trigonometrical series

Problem 1. *Find the sum of n terms of the series*

$$\sin^2 \alpha + \sin^2(\alpha+\beta) + \sin^2(\alpha + 2\beta) + \dots$$

Solution.

$$\sin^2 \alpha = \frac{1}{2}(1 - \cos 2\alpha),$$

$$\sin^2(\alpha+\beta) = \frac{1}{2}\{1 - \cos 2(\alpha+\beta)\},$$

$$\sin^2(\alpha+2\beta) = \frac{1}{2}\{1 - \cos 2(\alpha+2\beta)\},$$

and so on.

Hence the sum of n terms

$$= \frac{n}{2} - \frac{1}{2}\{\cos 2\alpha + \cos 2(\alpha+\beta) + \cos 2(\alpha+2\beta) + \dots\}$$

$$= \frac{n}{2} - \frac{\cos\{2\alpha + (n-1)\beta\}\sin n\beta}{2\sin\beta}.$$

Problem 2. *Find the sum of n terms of the series*

$$\sin^3 \alpha + \sin^3(\alpha+\beta) + \sin^3(\alpha + 2\beta) + \dots$$

Solution.

$$\sin^3 \alpha = \frac{1}{4}(3\sin\alpha - \sin 3\alpha),$$

$$\sin^3(\alpha+\beta) = \frac{1}{4}\{3\sin(\alpha+\beta) - \sin 3(\alpha+\beta)\},$$

$$\sin^3(\alpha+2\beta) = \frac{1}{4}\{3\sin(\alpha+2\beta) - \sin 3(\alpha+2\beta)\},$$

and so on

Hence the sum of n terms

$$= \frac{3}{4}\{\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \dots\}$$

$$- \frac{1}{4}\{\sin 3\alpha + \sin 3(\alpha+\beta) + \sin 3(\alpha+2\beta) + \dots\}$$

$$= \frac{3}{4}\frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{1}{2}\beta} - \frac{1}{4}\frac{\sin\left(3\alpha + \frac{n-1}{2}3\beta\right)\sin\frac{3n\beta}{2}}{\sin\frac{3}{2}\beta}.$$

Problem 3. *Find the sum of n terms of the series*

$$\cos^4 \alpha + \cos^4(\alpha+\beta) + \cos^4(\alpha + 2\beta) + \dots$$

Solution. We have $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$;

$$\therefore\ \cos^4\theta = \frac{1}{4}(1+\cos 2\theta)^2 = \frac{1}{4}(1 + 2\cos 2\theta + \cos^2 2\theta)$$

$$= \frac{1}{4}\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) = \frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta.$$

Apply this transformation to every term of the proposed series; thus the sum of n terms

$$= \frac{3n}{8} + \frac{1}{2}\{\cos 2\alpha + \cos 2(\alpha+\beta) + \cos 2(\alpha + 2\beta) + \ldots\}$$
$$+\frac{1}{8}\{\cos 4\alpha + \cos 4(\alpha+\beta) + \cos 4(\alpha+2\beta) + \ldots\}$$
$$= \frac{3n}{8} + \frac{\cos\{2\alpha + (n-1)\beta\}\sin n\beta}{2\sin\beta} + \frac{\cos\{4\alpha + (n-1)2\beta\}\sin 2n\beta}{8\sin 2\beta}.$$

Problem 4. *Show that* $\tan n\theta = \dfrac{\sin\theta + \sin 3\theta + \sin 5\theta + \ldots \text{ to } n \text{ terms}}{\cos\theta + \cos 3\theta + \cos 5\theta + \ldots \text{ to } n \text{ terms}}$.

Solution. $\sin\theta + \sin 3\theta + \sin 5\theta + \ldots$ to n terms

$$= \frac{\sin\{\theta + (n-1)\theta\}\sin n\theta}{\sin\theta} = \frac{\sin^2 n\theta}{\sin\theta}$$

$\cos\theta + \cos 3\theta + \cos 5\theta + \ldots$ to n terms

$$= \frac{\cos\{\theta + (n-1)\theta\}\sin n\theta}{\sin\theta} = \frac{\sin n\theta\cos n\theta}{\sin\theta}.$$

Divide the former result by the latter; thus we obtain $\tan n\theta$.

Problem 5. *Sum to n terms the series*

$$\cos\theta\cos(\theta+\alpha) + \cos(\theta+\alpha)\cos(\theta+2\alpha) + \cos(\theta+2\alpha)\cos(\theta+3\alpha) + \ldots$$

Solution. $\cos A\cos B = \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)$.

Apply this transformation to every term of the proposed series; thus the sum of n terms

$$= \frac{n}{2}\cos\alpha + \frac{1}{2}\{\cos(2\theta+\alpha) + \cos(2\theta+3\alpha) + \cos(2\theta+5\alpha) + \ldots\}$$
$$= \frac{n}{2}\cos\alpha + \frac{\cos\{2\theta + \alpha + (n-1)\alpha\}\sin n\alpha}{2\sin\alpha}$$
$$= \frac{n}{2}\cos\alpha + \frac{\cos(2\theta + n\alpha)\sin n\alpha}{2\sin\alpha}.$$

Problem 6. *Show that*

$$\frac{\sin\theta - \sin 2\theta + \sin 3\theta - \ldots \text{ to } n \text{ terms}}{\cos\theta - \cos 2\theta + \cos 3\theta - \ldots \text{ to } n \text{ terms}} = \tan\frac{n+1}{2}(\pi+\theta).$$

Solution. By *Art.* 307 (page 438) we have

$\sin\theta - \sin 2\theta + \sin 3\theta - \ldots$ to n terms

$$= \frac{\sin\left\{\theta + \dfrac{(n-1)(\theta+\pi)}{2}\right\}\sin\dfrac{n(\theta+\pi)}{2}}{\sin\dfrac{\theta+\pi}{2}}.$$

And $\cos\theta - \cos 2\theta + \cos 3\theta - \ldots$ to n terms

$$= \frac{\cos\left\{\theta + \frac{(n-1)(\theta+\pi)}{2}\right\}\sin\frac{n(\theta+\pi)}{2}}{\sin\frac{\theta+\pi}{2}}$$

Divide the former by the latter : the result

$$= \frac{\sin\left\{\theta + \frac{(n-1)(\theta+\pi)}{2}\right\}}{\cos\left\{\theta + \frac{(n-1)(\theta+\pi)}{2}\right\}} = \frac{\sin\left\{\theta + \frac{(n-1)(\theta+\pi)}{2} + \pi\right\}}{\cos\left\{\theta + \frac{(n-1)(\theta+\pi)}{2} + \pi\right\}}$$

$$= \frac{\sin\frac{n+1}{2}(\theta+\pi)}{\cos\frac{n+1}{2}(\theta+\pi)} = \tan\frac{n+1}{2}(\theta+\pi).$$

Problem 7. *Sum to n terms the series*

$$\sin(p+1)\theta\cos\theta + \sin(p+2)\theta\cos 2\theta + \dots$$

Solution. $\sin A\cos B = \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B)$.

Apply this transformation to every term of the proposed series; thus the sum of n terms

$$= \frac{n\sin p\theta}{2} + \frac{1}{2}\{\sin(p+2)\theta + \sin(p+4)\theta + \sin(p+6)\theta + \dots\}$$

$$= \frac{n\sin p\theta}{2} + \frac{\sin(p+1+n)\theta\sin n\theta}{2\sin\theta}.$$

Problem 8. *Sum to n terms the series*

$$\sin\alpha\sin 2\alpha + \sin 2\alpha\sin 3\alpha + \sin 3\alpha\sin 4\alpha + \dots$$

Solution. $\sin A\sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$.

Apply this transformation to every term of the proposed series; thus the sum of n terms

$$= \frac{n}{2}\cos\alpha - \frac{1}{2}\{\cos 3\alpha + \cos 5\alpha + \cos 7\alpha + \dots\}$$

$$= \frac{n}{2}\cos\alpha - \frac{\cos\{3\alpha + (n-1)\alpha\}\sin n\alpha}{2\sin\alpha} = \frac{n}{2}\cos\alpha - \frac{\cos(n+2)\alpha\sin n\alpha}{2\sin\alpha}.$$

Problem 9. *Deduce from the result of Problem 8 the sum to n terms of the series*

$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \dots$$

Solution. Suppose that in the preceding result we put for the sines of the angles their values from *Art.* 274 (page 432); the proposed series become an expansion in powers of α, and it is obvious that the coefficient of α^2 is

$$1\cdot 2 + 2\cdot 3 + 3\cdot 4\dots + n(n+1).$$

We must therefore find the coefficient of α^2 in the expansion of the expression found for the sum of the Trigonometrical Series, and equate it to the above.

Now $\frac{n}{2}\cos\alpha = \frac{n}{2}\left(1 - \frac{\alpha^2}{\underline{|2}} + \frac{\alpha^4}{\underline{|4}} - \ldots\right)$, so that the coefficient of α^2 in this term is $-\frac{n}{4}$;

and
$$\frac{\cos(n+2)\alpha\sin n\alpha}{2\sin\alpha} = \frac{\left\{1 - \frac{(n+2)^2}{2}\alpha^2 + \ldots\right\}\left\{n\alpha - \frac{n^3\alpha^3}{6} + \ldots\right\}}{2\left(\alpha - \frac{\alpha^3}{6} + \ldots\right)}$$
$$= n\frac{\left\{1 - \frac{(n+2)^2}{2}\alpha^2 + \ldots\right\}\left\{1 - \frac{n^2\alpha^2}{6} + \ldots\right\}}{2\left(1 - \frac{\alpha^2}{6} + \ldots\right)}$$
$$= \frac{n}{2}\left\{1 - \frac{(n+2)^2}{2}\alpha^2 + \ldots\right\}\left\{1 - \frac{n^2\alpha^2}{6} + \ldots\right\}\left\{1 - \frac{\alpha^2}{6} + \ldots\right\}^{-1}$$
$$= \frac{n}{2}\left\{1 - \frac{(n+2)^2}{2}\alpha^2 + \ldots\right\}\left\{1 - \frac{n^2\alpha^2}{6} + \ldots\right\}\left\{1 + \frac{\alpha^2}{6} + \ldots\right\}.$$

Multiply out and it will be found that the coefficient of α^2 is
$$\frac{n}{2}\left\{\frac{-(n+2)^2}{2} - \frac{n^2}{6} + \frac{1}{6}\right\}.$$

Hence the required sum
$$= -\frac{n}{4} - \frac{n}{2}\left\{-\frac{(n+2)^2}{2} - \frac{n^2}{6} + \frac{1}{6}\right\}$$
$$= -\frac{n}{4} + \frac{n(n+2)^2}{4} + \frac{n^3}{12} - \frac{n}{12}$$
$$= \frac{n}{12}\{3(n+2)^2 + n^2 - 1 - 3\}$$
$$= \frac{n}{12}\{4n^2 + 12n + 8\} = \frac{n}{3}(n^2 + 3n + 2)$$
$$= \frac{n(n+1)(n+2)}{3}.$$

Problem 10. *Sum to n terms the series*
$$\sin 3\theta\sin\theta + \sin 6\theta\sin 2\theta + \sin 12\theta\sin 4\theta + \ldots$$

Solution. $\sin A\sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$.

Apply this transformation to every term of the proposed series; hence the sum of n terms
$$= \frac{1}{2}(\cos 2\theta - \cos 4\theta) + \frac{1}{2}(\cos 4\theta - \cos 8\theta) + \frac{1}{2}(\cos 8\theta - \cos 16\theta) + \ldots$$
$$= \frac{1}{2}\left(\cos 2\theta - \cos 2^{n+1}\theta\right)$$

Sum to infinity the following series contained in the Problems from 11 to 16 inclusive:

Problem 11. $\cos\theta + \dfrac{\cos\theta}{1}\cos 2\theta + \dfrac{\cos^2\theta}{1\cdot 2}\cos 3\theta + \dfrac{\cos^3\theta}{|\underline{3}}\cos 4\theta + \dots$

Solution. Omitting the first term, we can find the sum of the rest of the series by *Art.* 311 (page 440); we must put $\cos\theta$ for c, and put $\alpha = \beta = \theta$. Hence the sum of the whole series

$$= \cos\theta + e^{\cos^2\theta}\cos(\theta + \sin\theta\cos\theta) - \cos\theta = e^{\cos^2\theta}\cos(\theta + \sin\theta\cos\theta).$$

Problem 12. $\sin\theta - \dfrac{\sin 2\theta}{1\cdot 2} + \dfrac{\sin 3\theta}{|\underline{3}} - \dots$

Solution. In the first series of *Art.* 311 (page 440) put -1 for c, and 0 for α, and θ for β, and change the sign. Thus we obtain for the required sum

$$-e^{-\cos\theta}\sin(-\sin\theta), \text{that is } e^{-\cos\theta}\sin(\sin\theta).$$

Problem 13. $1 - \dfrac{\cos 2\theta}{1\cdot 2} + \dfrac{\cos 4\theta}{|\underline{4}} - \dots$

Solution. For $\cos 2\theta$, $\cos 4\theta$, ... put the exponential values; thus denoting $e^{\iota\theta}$ by z, the proposed series becomes

$$\frac{1}{2}\left\{2 - \frac{1}{|\underline{2}}\left(z^2 + z^{-2}\right) + \frac{1}{|\underline{4}}\left(z^4 + z^{-4}\right) - \dots\right\}$$
$$= \frac{1}{2}\{\cos z + \cos z^{-1}\}$$
$$= \frac{1}{2}\{\cos\theta + \iota\sin\theta) + \cos(\cos\theta - \iota\sin\theta)\}$$
$$= \cos(\cos\theta)\cos(\iota\sin\theta) = \frac{1}{2}\cos(\cos\theta)(e^{-\sin\theta} + e^{\sin\theta}).$$

Problem 14. $2\cos\theta + \dfrac{3}{2}\cos^2\theta + \dfrac{4}{3}\cos^3\theta + \dfrac{5}{4}\cos^4\theta + \dots$

Solution. The proposed series

$$= \cos\theta + \cos^2\theta + \cos^3\theta + \cos^4\theta + \dots$$
$$+ \cos\theta + \frac{1}{2}\cos^2\theta + \frac{1}{3}\cos^3\theta + \frac{1}{4}\cos^4\theta + \dots$$
$$= \frac{\cos\theta}{1 - \cos\theta} - \log(1 - \cos\theta).$$

Problem 15. $\sin\theta\cos\theta + \dfrac{\sin 2\theta\cos^2\theta}{1\cdot 2} + \dfrac{\sin 3\theta\cos^3\theta}{|\underline{3}} + \dots$

Solution. In the first series of *Art.* 311 (page 440) put 0 for α and $\cos\theta$ for c, and θ for β. Thus the sum $= e^{\cos^2\theta}\sin(\sin\theta\cos\theta)$.

Problem 16. $\cos\theta + \dfrac{\sin\theta}{1}\cos 2\theta + \dfrac{\sin^2\theta}{1\cdot 2}\cos 3\theta + \dots$

Solution. In the second series of *Art.* 311 (page 440) put θ for α, and θ for β, and $\sin\theta$ for c. Thus

$$\sin\theta\cos 2\theta + \frac{\sin^2\theta}{2}\cos 3\theta + \frac{\sin^3\theta}{\lfloor 3}\cos 4\theta + \dots$$
$$= e^{\sin\theta\cos\theta}\cos\left(\theta+\sin^2\theta\right) - \cos\theta.$$

Therefore
$$\cos\theta + \sin\theta\cos 2\theta + \frac{\sin^2\theta}{2}\cos 3\theta + \frac{\sin^3\theta}{\lfloor 3}\cos 4\theta + \dots$$
$$= e^{\sin\theta\cos\theta}\cos\left(\theta+\sin^2\theta\right).$$

Problem 17. *Show that* $\cos\theta - \frac{1}{2}\cos 2\theta + \frac{1}{3}\cos 3\theta - \dots = \log\left(2\cos\frac{\theta}{2}\right).$

Solution. Put the exponential values for $\cos\theta$, $\cos 2\theta$, $\cos 3\theta, \dots$ Thus denoting $e^{\iota\theta}$ by z, the proposed series becomes

$$\frac{1}{2}\left\{z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \dots\right\} + \frac{1}{2}\left\{z^{-1} - \frac{1}{2}z^{-2} + \frac{1}{3}z^{-3} - \frac{1}{4}z^{-4} + \dots\right\};$$

that is $\quad \frac{1}{2}\log(1+z) + \frac{1}{2}\log\left(1+z^{-1}\right)$, that is $\frac{1}{2}\log(1+z)(1+z^{-1})$,

that is $\quad \frac{1}{2}\log\left(2+z+z^{-1}\right)$, that is $\frac{1}{2}\log(2+2\cos\theta)$,

that is $\quad \frac{1}{2}\log\left(4\cos^2\frac{\theta}{2}\right)$, that is $\log\left(2\cos\frac{\theta}{2}\right)$.

Problem 18. *Show that* $\cos 2\theta + \frac{1}{3}\cos 6\theta + \frac{1}{5}\cos 10\theta + \dots = \frac{1}{2}\log(\cot\theta).$

Solution. Proceed as in the solution of *Problem* 17. Thus the proposed series becomes

$$\frac{1}{2}\left\{z^2 + \frac{1}{3}z^6 + \frac{1}{5}z^{10} + \dots\right\} + \frac{1}{2}\left\{z^{-2} + \frac{1}{3}z^{-6} + \frac{1}{5}z^{-10} + \dots\right\}$$
$$= \frac{1}{4}\log\frac{1+z^2}{1-z^2} + \frac{1}{4}\log\frac{1+z^{-2}}{1-z^{-2}} = \frac{1}{4}\log\left(\frac{1+z^2}{1-z^2}\times\frac{1+z^{-2}}{1-z^{-2}}\right)$$
$$= \frac{1}{4}\log\frac{2+z^2+z^{-2}}{2-z^2-z^{-2}} = \frac{1}{4}\log\frac{2+2\cos 2\theta}{2-2\cos 2\theta} = \frac{1}{4}\log\frac{1+\cos 2\theta}{1-\cos 2\theta}$$
$$= \frac{1}{4}\log\cot^2\theta = \frac{1}{2}\log\cot\theta.$$

Problem 19. *Show that*

$$x\sin\theta - \frac{x^2\sin 2\theta}{2} + \frac{x^3\sin 3\theta}{3} - \dots = \cot^{-1}\left(\frac{\operatorname{cosec}\theta}{x} + \cot\theta\right).$$

Solution. Put the exponential values for $\sin\theta$, $\sin 2\theta$, $\sin 3\theta$, $\dots$ Thus, denoting $e^{\iota\theta}$ by z, the proposed series becomes

$$\frac{1}{2\iota}\left\{xz - \frac{1}{2}x^2z^2 + \frac{1}{3}x^3z^3 - \frac{1}{4}x^4z^4 + \dots\right\}$$

$$-\frac{1}{2\iota}\left\{xz^{-1}-\frac{1}{2}x^2z^{-2}+\frac{1}{3}x^3z^{-3}-\frac{1}{4}x^4z^{-4}+\ldots\right\}$$

This $$=\frac{1}{2\iota}\log(1+xs)-\frac{1}{2}\log\left(1+xz^{-1}\right)$$

$$=\frac{1}{2\iota}\log\frac{1+xz}{1+xz^{-1}}=\frac{1}{2}\log\frac{1+x(\cos\theta+\iota\sin\theta)}{1+x(\cos\theta-\iota\sin\theta)}.$$

Assume $\tan\phi=\dfrac{x\sin\theta}{1+x\cos\theta}$; thus the sum of the proposed series

$$=\frac{1}{2\iota}\log\frac{1+\iota\tan\phi}{1-\iota\tan\phi}=\frac{1}{2}\log\frac{\cos\phi+\iota\sin\phi}{\cos\phi-\iota\sin\phi}$$

$$=\frac{1}{2\iota}\log\frac{e^{\iota\phi}}{e^{-\iota\phi}}=\frac{1}{2\iota}\log e^{2\iota\phi}=\phi=\cot^{-1}\frac{1+x\cos\theta}{x\sin\theta}$$

$$=\cot^{-1}\left(\frac{\text{cosec }\theta}{x}+\cot\theta\right).$$

Problem 20. *Show that*

$$\log\cos\theta+\log\cos\frac{\theta}{2}+\log\cos\frac{\theta}{2^2}+\ldots=\log\left(\frac{\sin 2\theta}{2\theta}\right).$$

Solution. By *Art.* 129 (page 412) the limit of $\cos\dfrac{\theta}{2}\cos\dfrac{\theta}{4}\cos\dfrac{\theta}{8}\ldots$ is $\dfrac{\sin\theta}{\theta}$; therefore the limit of $\cos\theta\cos\dfrac{\theta}{2}\cos\dfrac{\theta}{4}\cos\dfrac{\theta}{8}\ldots$ is $\dfrac{\cos\theta\sin\theta}{\theta}$, that is $\dfrac{\sin 2\theta}{2\theta}$. Then take the logarithms of both sides.

Sum the following series to n terms contained in the Problems from 21 to 35 inclusive :

Problem 21. $\sin\theta\left(\sin\dfrac{\theta}{2}\right)^2+2\sin\dfrac{\theta}{2}\left(\sin\dfrac{\theta}{4}\right)^2+4\sin\dfrac{\theta}{4}\left(\sin\dfrac{\theta}{8}\right)^2+\ldots$

Solution.

$$\sin\theta\left(\sin\frac{\theta}{2}\right)^2=\frac{1}{2}\sin\theta(1-\cos\theta)=\frac{1}{2}\sin\theta-\frac{1}{4}\sin 2\theta,$$

$$2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{4}\right)^2=\sin\frac{\theta}{2}\left(1-\cos\frac{\theta}{2}\right)=\sin\frac{\theta}{2}-\frac{1}{2}\sin\theta.$$

$$4\sin\frac{\theta}{4}\left(\sin\frac{\theta}{8}\right)^2=2\sin\frac{\theta}{4}\left(1-\cos\frac{\theta}{4}\right)=2\sin\frac{\theta}{4}-\sin\frac{\theta}{2},$$

$$8\sin\frac{\theta}{8}\left(\sin\frac{\theta}{16}\right)^2=4\sin\frac{\theta}{8}\left(1-\cos\frac{\theta}{8}\right)=4\sin\frac{\theta}{8}-2\sin\frac{\theta}{4}.$$

Proceeding in this way, and adding the terms, we see that all cancel on the right-hand side except two, namely

$$2^{n-2}\sin\frac{\theta}{2^{n-1}}-\frac{1}{4}\sin 2\theta.$$

Problem 22. $\tan\frac{\theta}{2}\sec\theta+\tan\frac{\theta}{4}\sec\frac{\theta}{2}+\tan\frac{\theta}{8}\sec\frac{\theta}{4}+\dots$

Solution.

$$\tan\frac{\theta}{2}\sec\theta=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}\cos\theta}=\frac{\sin\left(\theta-\frac{\theta}{2}\right)}{\cos\frac{\theta}{2}\cos\theta}=\frac{\sin\theta\cos\frac{\theta}{2}-\cos\theta\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}\cos\theta}$$
$$=\tan\theta-\tan\frac{\theta}{2};$$

therefore
$$\tan\frac{\theta}{4}\sec\frac{\theta}{2}=\tan\frac{\theta}{2}-\tan\frac{\theta}{4},$$
$$\tan\frac{\theta}{8}\sec\frac{\theta}{4}=\tan\frac{\theta}{4}-\tan\frac{\theta}{8},$$
and so on.

Then adding the terms, we see that all cancel on the right-hand side except two, namely
$$\tan\theta-\tan\frac{\theta}{2^n}.$$

Problem 23. $\cot\theta\operatorname{cosec}\theta+2\cot 2\theta\operatorname{cosec}2\theta+2^2\cot 2^2\theta\operatorname{cosec}2^2\theta+\dots$

Solution.

$$\cot\theta\operatorname{cosec}\theta=\frac{\cos\theta}{\sin^2\theta}=\frac{2\cos^2\frac{\theta}{2}-1}{\sin^2\theta}$$
$$=\frac{2\cos^2\frac{\theta}{2}}{4\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}}-\frac{1}{\sin^2\theta}=\frac{1}{2\sin^2\frac{\theta}{2}}-\frac{1}{\sin^2\theta};$$

therefore
$$2\cot 2\theta\operatorname{cosec}2\theta=\frac{1}{\sin^2\theta}-\frac{2}{\sin^2 2\theta},$$
$$4\cot 4\theta\operatorname{cosec}4\theta=\frac{2}{\sin^2 2\theta}-\frac{4}{\sin^2 4\theta}.$$

Proceeding in this way, and adding the terms, we see that all cancel on the right-hand side except two, namely
$$\frac{1}{2\sin^2\frac{\theta}{2}}-\frac{2^{n-1}}{\sin^2 2^{n-1}\theta}.$$

Problem 24. $\frac{1}{\sin\theta\sin 2\theta}+\frac{1}{\sin 2\theta\sin 3\theta}+\frac{1}{\sin 3\theta\sin 4\theta}+\dots$

Solution.

$$\frac{1}{\sin\theta\sin 2\theta}=\frac{1}{\sin\theta}\cdot\frac{\sin(2\theta-\theta)}{\sin\theta\sin 2\theta}=\frac{1}{\sin\theta}\cdot\frac{\sin 2\theta\cos\theta-\cos 2\theta\sin\theta}{\sin\theta\sin 2\theta}$$
$$=\frac{1}{\sin\theta}(\cot\theta-\cot 2\theta).$$

Similarly
$$\frac{1}{\sin 2\theta\sin 3\theta}=\frac{1}{\sin\theta}\frac{\sin(3\theta-2\theta)}{\sin 2\theta\sin 3\theta}$$

$$= \frac{1}{\sin\theta}(\cot 2\theta - \cot 3\theta);$$

$$\frac{1}{\sin 3\theta \sin 4\theta} = \frac{1}{\sin\theta}(\cot 3\theta - \cot 4\theta).$$

Proceeding in this way, and adding the terms, we see that all cancel on the right-hand side except two, namely

$$\frac{1}{\sin\theta}\{\cot\theta - \cot(n+1)\theta\}.$$

Problem 25. $\dfrac{1}{\sin\theta\cos 2\theta} - \dfrac{1}{\cos 2\theta \sin 3\theta} + \dfrac{1}{\sin 3\theta \cos 4\theta} - \dots$

Solution. Let $\phi = \theta + \frac{\pi}{2}$; thus the proposed series becomes

$$\frac{1}{\cos\phi\cos 2\phi} + \frac{1}{\cos 2\phi\cos 3\phi} + \frac{1}{\cos 3\phi \cos 4\phi} + \dots$$

Now $$\frac{1}{\cos\phi\cos 2\phi} = \frac{1}{\sin\phi}\frac{\sin(2\phi-\phi)}{\cos\phi\cos 2\phi} = \frac{1}{\sin\phi}(\tan 2\phi - \tan\phi),$$

$$\frac{1}{\cos 2\phi\cos 3\phi} = \frac{1}{\sin\phi}\frac{\sin(3\phi-2\phi)}{\cos 2\phi\cos 3\phi} = \frac{1}{\sin\phi}(\tan 3\phi - \tan 2\phi),$$

$$\frac{1}{\cos 3\phi\cos 4\phi} = \frac{1}{\sin\phi}\frac{\sin(4\phi-3\phi)}{\cos 3\phi\cos 4\phi} = \frac{1}{\sin\phi}(\tan 4\phi - \tan 3\phi).$$

Proceeding in this way, and adding the terms, we see that all cancel on the right-hand side except two, namely

$$\frac{1}{\sin\phi}\{\tan(n+1)\phi - \tan\phi\}.$$

Problem 26. $\tan^{-1}\dfrac{1}{1+1+1^2} + \tan^{-1}\dfrac{1}{1+2+2^2} + \tan^{-1}\dfrac{1}{1+3+3^2} + \dots$

Solution. $$\tan^{-1}\frac{1}{1+m+m^2} = \tan^{-1}\frac{1}{m} - \tan^{-1}\frac{1}{1+m};$$

this is obvious, for by taking the tangent of $\tan^{-1}\frac{1}{m} - \tan^{-1}\frac{1}{1+m}$, we obtain

$$\frac{\frac{1}{m} - \frac{1}{m+1}}{1 + \frac{1}{m(m+1)}}, \text{ that is } \frac{1}{m^2+m+1}.$$

Apply this transformation to every term of the proposed series; thus we obtain

$$\tan^{-1}\frac{1}{1} - \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{3} - \tan^{-1}\frac{1}{4} + \dots,$$

that is $$\tan^{-1}1 - \tan^{-1}\frac{1}{n+1}, \text{ that is } \frac{\pi}{4} - \tan^{-1}\frac{1}{n+1}.$$

Problem 27. $\tan^{-1}x + \tan^{-1}\dfrac{x}{1+1\cdot 2\cdot x^2} + \tan^{-1}\dfrac{x}{1+2\cdot 3\cdot x^2} + \dots$

Solution. $$\tan^{-1}\frac{x}{1+m(m+1)x^2} = \tan^{-1}(m+1)x - \tan^{-1}mx;$$

this is obvious, for by taking the tangent of $\tan^{-1}(m+1)x - \tan^{-1}mx$, we obtain

$\dfrac{(m+1)x - mx}{1+m(m+1)x^2}$, that is $\dfrac{x}{1+m(m+1)x^2}$.

Apply this transformation to every term of the proposed series after the first; thus we obtain

$$\tan^{-1} x + \tan^{-1} 2x - \tan^{-1} x + \tan^{-1} 3x - \tan^{-1} 2x + \ldots,$$

that is $\tan^{-1} nx$.

Problem 28. $\sin\alpha \sin 3\alpha + \sin\dfrac{\alpha}{2}\sin\dfrac{3\alpha}{2} + \sin\dfrac{\alpha}{2^2}\sin\dfrac{3\alpha}{2^2} + \ldots$

Solution. $\sin A \sin B = \dfrac{1}{2}\cos(A-B) - \dfrac{1}{2}\cos(A+B)$.

Apply this transformation to every term of the proposed series; thus we obtain

$$\frac{1}{2}(\cos 2\alpha - \cos 4\alpha) + \frac{1}{2}(\cos\alpha - \cos 2\alpha) + \frac{1}{2}\left(\cos\frac{\alpha}{2} - \cos\alpha\right) + \ldots$$

that is

$$\frac{1}{2}\left(\cos\frac{\alpha}{2^{n-2}} - \cos 4\alpha\right).$$

Problem 29. $\dfrac{1}{\cos\theta + \cos 3\theta} + \dfrac{1}{\cos\theta + \cos 5\theta} + \dfrac{1}{\cos\theta + \cos 7\theta} + \ldots$

Solution.

$$\frac{1}{\cos\theta + \cos 3\theta} = \frac{1}{2\cos\theta\cos 2\theta} = \frac{1}{2\sin\theta}\cdot\frac{\sin(2\theta-\theta)}{\cos\theta\cos 2\theta}$$
$$= \frac{1}{2\sin\theta}(\tan 2\theta - \tan\theta),$$
$$\frac{1}{\cos + \cos 5\theta} = \frac{1}{2\cos 2\theta\cos 3\theta} = \frac{1}{2\sin\theta}\cdot\frac{\sin(3\theta-2\theta)}{\cos 2\theta\cos 3\theta}$$
$$= \frac{1}{2\sin\theta}(\tan 3\theta - \tan 2\theta),$$
$$\frac{1}{\cos\theta + \cos 7\theta} = \frac{1}{2\cos 3\theta\cos 4\theta} = \frac{1}{2\sin\theta}\cdot\frac{\sin(4\theta-3\theta)}{\cos 3\theta\cos 4\theta}$$
$$= \frac{1}{2\sin\theta}(\tan 4\theta - \tan 3\theta).$$

Proceeding in this way, and adding the terms, we see that all cancel on the right-hand side except two, namely

$$\frac{1}{2\sin\theta}\{\tan(n+1)\theta - \tan\theta\}.$$

Problem 30. $\dfrac{\sin\theta}{\cos 2\theta + \cos\theta} + \dfrac{\sin 2\theta}{\cos 4\theta + \cos\theta} + \dfrac{\sin 3\theta}{\cos 6\theta + \cos\theta} + \ldots$

Solution.

$$\frac{\sin\theta}{\cos 2\theta + \cos\theta} = \frac{\sin\theta}{2\cos\dfrac{\theta}{2}\cos\dfrac{3\theta}{2}}$$
$$= \frac{1}{4\sin\dfrac{\theta}{2}}\left\{\frac{1}{\cos\dfrac{3\theta}{2}} - \frac{1}{\cos\dfrac{\theta}{2}}\right\};$$

$$\frac{\sin 2\theta}{\cos 4\theta + \cos\theta} = \frac{\sin 2\theta}{2\cos\frac{3\theta}{2}\cos\frac{5\theta}{2}} = \frac{1}{4\sin\frac{\theta}{2}}\left\{\frac{1}{\cos\frac{5\theta}{2}} - \frac{1}{\cos\frac{3\theta}{2}}\right\}.$$

$$\frac{\sin 3\theta}{\cos 6\theta + \cos\theta} = \frac{\sin 3\theta}{2\cos\frac{5\theta}{2}\cos\frac{7\theta}{2}} = \frac{1}{4\sin\frac{\theta}{2}}\left\{\frac{1}{\cos\frac{7\theta}{2}} - \frac{1}{\cos\frac{5\theta}{2}}\right\}.$$

Proceeding in this way, and adding the terms, we see that all cancel on the right hand except two, namely

$$\frac{1}{4\sin\frac{\theta}{2}}\left\{\frac{1}{\cos\frac{(2n+1)\theta}{2}} - \frac{1}{\cos\frac{\theta}{2}}\right\}.$$

Problem 31. $\dfrac{\sin\theta}{1+2\cos\theta} + \dfrac{3\sin 3\theta}{1+2\cos 3\theta} + \dfrac{3^2\sin 3^2\theta}{1+2\cos 3^2\theta} + \ldots$

Solution.

$$\frac{\sin\theta}{1+2\cos\theta} = \frac{\sin\theta}{1+2\left(1-2\sin^2\frac{\theta}{2}\right)} = \frac{\sin\theta}{3-4\sin^2\frac{\theta}{2}}$$

$$= \frac{\sin\theta\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}\left(3-4\sin^2\frac{\theta}{2}\right)} = \frac{\sin\theta\sin\frac{\theta}{2}}{\sin\frac{3\theta}{2}} = \frac{\cos\frac{\theta}{2}-\cos\frac{3\theta}{2}}{2\sin\frac{3\theta}{2}}$$

$$= \frac{2\cos\frac{\theta}{2}-2\cos\frac{3\theta}{2}}{4\sin\frac{3\theta}{2}} = \frac{\cos\frac{\theta}{2}(1+2\cos\theta)+\cos\frac{\theta}{2}-2\cos\theta\cos\frac{\theta}{2}-2\cos\frac{3\theta}{2}}{4\sin\frac{3\theta}{2}}$$

$$= \frac{\cos\frac{\theta}{2}}{4\sin\frac{\theta}{2}} + \frac{\cos\frac{\theta}{2}-2\cos\theta\cos\frac{\theta}{2}-2\cos\frac{3\theta}{2}}{4\sin\frac{3\theta}{2}}$$

$$= \frac{\cos\frac{\theta}{2}}{4\sin\frac{\theta}{2}} - \frac{3\cos\frac{3\theta}{2}}{4\sin\frac{3\theta}{2}} = \frac{1}{4}\cot\frac{\theta}{2} - \frac{3}{4}\cot\frac{3\theta}{2}.$$

Similarly
$$\frac{3\sin 3\theta}{1+2\cos 3\theta} = \frac{3}{4}\cot\frac{3\theta}{2} - \frac{9}{4}\cot\frac{9\theta}{2},$$
$$\frac{3^2\sin 3^2\theta}{1+2\cos 3^2\theta} = \frac{9}{4}\cot\frac{9\theta}{2} - \frac{27}{4}\cot\frac{27\theta}{2}.$$

Proceeding in this way, and adding the terms, we see that all cancel on the right hand except two, namely

$$\frac{1}{4}\cot\frac{\theta}{2} - \frac{3^n}{4}\cot\frac{3^n\theta}{2}.$$

Problem 32. $\cot^{-1}\left(2a^{-1}+a\right) + \cot^{-1}\left(2a^{-1}+3a\right) + \cot^{-1}\left(2a^{-1}+6a\right) + \cot^{-1}\left(2a^{-1}+10a\right) + \ldots$

Solution. $\cot^{-1}\left\{2a^{-1}+\frac{m(m+1)}{2}a\right\}=\cot^{-1}\frac{m}{2}a-\cot^{-1}\frac{m+1}{2}a.$

For if we take the cotangent of $\cot^{-1}\frac{m}{2}a-\cot^{-1}\frac{m+1}{2}a$,

we obtain
$$\frac{\frac{m}{2}a\cdot\frac{m+1}{2}a+1}{\frac{m+1}{2}a-\frac{m}{2}a},$$

that is
$$\frac{m(m+1)}{2}a+2a^{-1}.$$

Apply this transformation to every term of the proposed series; thus we obtain
$$\cot^{-1}\frac{a}{2}-\cot^{-1}\frac{2a}{2}+\cot^{-1}\frac{2a}{2}-\cot^{-1}\frac{3a}{2}+\cot^{-1}\frac{3a}{2}-\cot^{-1}\frac{4a}{2}+\dots;$$

that is
$$\cot^{-1}\frac{a}{2}-\cot^{-1}\frac{n+1}{2}a.$$

Problem 33. $\frac{1}{2}\sec\theta+\frac{1}{2^2}\sec\theta\sec 2\theta+\frac{1}{2^3}\sec\theta\sec 2\theta\sec 2^2\theta+\dots$

Solution.
$$\frac{1}{2}\sec\theta=\frac{1}{2\cos\theta}=\frac{\sin\theta}{2\sin\theta\cos\theta}=\frac{\sin\theta}{\sin 2\theta}=\frac{\sin(2\theta-\theta)}{\sin 2\theta}$$
$$=\cos\theta-\cot 2\theta\sin\theta=\sin\theta(\cot\theta-\cot 2\theta);$$

$$\frac{1}{2^2}\sec\theta\sec 2\theta=\frac{1}{2}\sec\theta\sin 2\theta(\cot 2\theta-\cot 4\theta)$$
$$=\sin\theta(\cot 2\theta-\cot 4\theta);$$
$$\frac{1}{2^3}\sec\theta\sec 2\theta\sec 4\theta=\frac{1}{2}\sec\theta\sin 2\theta(\cot 4\theta-\cot 8\theta)$$
$$=\sin\theta(\cot 4\theta-\cot 8\theta).$$

Proceeding in this way, and adding the terms, we see that all cancel on the right-hand side except two, namely
$$\sin\theta(\cot 4\theta-\cot 2^n\theta).$$

Problem 34. $\frac{1}{2}\log\tan 2\theta+\frac{1}{2^2}\log\tan 2^2\theta+\frac{1}{2^3}\log\tan 2^3\theta+\dots$

Solution. $\tan 2\theta=\frac{\sin 2\theta}{\cos 2\theta}=\frac{\sin^2 2\theta}{\sin 2\theta\cos 2\theta}=\frac{2\sin^2 2\theta}{\sin 4\theta}=\frac{4\sin^2 2\theta}{2\sin 4\theta};$

therefore
$$\frac{1}{2}\log\tan 2\theta=\log 2\sin 2\theta-\frac{1}{2}\log 2\sin 4\theta,$$
$$\frac{1}{2^2}\log\tan 2^2\theta=\frac{1}{2}\log 2\sin 4\theta-\frac{1}{2^2}\log 2\sin 8\theta,$$
$$\frac{1}{2^3}\log\tan 2^3\theta=\frac{1}{2^2}\log 2\sin 8\theta-\frac{1}{2^3}\log 2\sin 16\theta.$$

Proceeding in this way, and adding the terms, we see that all cancel on the right-hand side except two, namely
$$\log 2\sin 2\theta-\frac{1}{2^n}\log 2\sin 2^{n+1}\theta.$$

Problem 35. $\cos\frac{\theta}{2} + 2\cos\frac{\theta}{2}\cos\frac{\theta}{2^2} + 2^2\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3} + \ldots$

Solution.

$$\cos\frac{\theta}{2} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}} = \frac{\sin\theta}{2\sin\frac{\theta}{2}} = \frac{\sin\theta}{2}\frac{2\cos^2\frac{\theta}{4} - \cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$= \frac{\sin\theta}{2}\left\{\frac{2\cos^2\frac{\theta}{4}}{2\sin\frac{\theta}{4}\cos\frac{\theta}{4}} - \cos\frac{\theta}{2}\right\} = \frac{\sin\theta}{2}\left\{\cot\frac{\theta}{4} - \cot\frac{\theta}{2}\right\};$$

$$2\cos\frac{\theta}{2}\cos\frac{\theta}{2^2} = 2\cos\frac{\theta}{2}\frac{\sin\frac{\theta}{2}}{2}\left\{\cot\frac{\theta}{8} - \cot\frac{\theta}{4}\right\}$$

$$= \frac{\sin\theta}{2}\left\{\cot\frac{\theta}{8} - \cot\frac{\theta}{4}\right\};$$

$$2^2\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3} = 2\cos\frac{\theta}{2}\cdot\frac{\sin\frac{\theta}{2}}{2}\left\{\cot\frac{\theta}{16} - \cot\frac{\theta}{8}\right\}$$

$$= \frac{\sin\theta}{2}\left\{\cot\frac{\theta}{16} - \cot\frac{\theta}{8}\right\}.$$

Proceeding in this way, and adding the terms, we see that all cancel on the right-hand side except two, namely

$$\frac{\sin\theta}{2}\left\{\cot\frac{\theta}{2^{n+1}} - \cot\frac{\theta}{2}\right\}.$$

Problem 36. *An equilateral polygon is inscribed in a circle and from any point in the circumference chords are drawn to the angular points : find the sum of the squares of the chords and the sum of the fourth powers of the chords.*

Solution. Let R denote the radius of the circle, n the number of sides of the polygon. Put β for $\frac{\pi}{n}$. Let 2α denote the angular distance of a fixed point in the circumference from one of the angular points; then the angular distances from the other angular points in succession will be

$$2\alpha + 2\beta,\ 2\alpha + 4\beta,\ 2\alpha + 6\beta,\ \ldots 2\alpha + 2(n-1)\beta.$$

The lengths of the succession chords will be

$$2R\sin\alpha,\ 2R\sin(\alpha+\beta),\ 2R\sin(\alpha+2\beta),\ \ldots 2R\sin\{\alpha+(n-1)\beta\}.$$

To find the sum of the squares of the chords, we have

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta);$$

and applying this transformation to every term of the proposed series, we obtain

$$2nR^2 - 2R^2\{\cos 2\alpha + \cos(2\alpha+2\beta) + \cos(2\alpha+4\beta) + \ldots\}.$$

The sum of the series of cosines is zero, as in *Art.* 304 (page 437) and thus the result is $2nR^2$.

Next, to find the sum of the fourth powers of the chords. We have

$$\sin^4\theta = \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta;$$

and applying this transformation to every term of the proposed series, we obtain

$$6nR^4 - 8R^4\{\cos 2\alpha + \cos(2\alpha + 2\beta) + \cos(2\alpha + 4\beta) + \ldots\}$$
$$+ 2R^4\{\cos 4\alpha + \cos(4\alpha + 4\beta) + \cos(4\alpha + 8\beta) + \ldots\},$$

that is $6nR^4$.

Problem 37. *Circles are inscribed in triangles, whose bases are the sides of a regular polygon of n sides, and whose vertices lie in one of the angular points : show that the sum of the radii of the circles is* $2r\left(1 - n\sin^2\dfrac{\pi}{2n}\right)$, *where r is the radius of the circle circumscribing the polygon.*

Solution. Let A be the common vertex; let B, C, ... be the successive angular points. Put β for $\dfrac{\pi}{n}$.

Let PQ be one of the sides of the polygon, such that the arc ABP contains m of the sides; then the angle $AQP = m\beta$, the angle $PAQ = \beta$, and the angle $APQ = \pi - (m+1)\beta$.

Let $PQ = c$, and let r_m denote the radius of the circle inscribed in APQ. Then

$$r_m\left\{\cot\frac{1}{2}APQ + \cot\frac{1}{2}AQP\right\} = c,$$

therefore
$$r_m\left\{\cot\frac{\pi - (m+1)\beta}{2} + \cot\frac{m\beta}{2}\right\} = c,$$

therefore
$$r_m\left\{\tan\frac{m+1}{2}\beta + \cot\frac{m\beta}{2}\right\} = c,$$

therefore
$$r_m\cos\frac{\beta}{2} = c\cos\frac{m+1}{2}\beta\sin\frac{m\beta}{2}$$
$$= \frac{c}{2}\left\{\sin\frac{2m+1}{2}\beta - \sin\frac{\beta}{2}\right\}.$$

Now there are $n-2$ circles in all which can be drawn; so that we have to sum up the values of

$$\frac{c}{2}\sec\frac{\beta}{2}\left\{\sin\frac{2m+1}{2}\beta - \sin\frac{\beta}{2}\right\}$$

for values of m from 1 to $n-2$ inclusive. The sum then is

$$\frac{c}{2}\sec\frac{\beta}{2}\left\{\frac{\sin\left\{\dfrac{3\beta}{2} + (n-3)\dfrac{\beta}{2}\right\}\sin\dfrac{n-2}{2}\beta}{\sin\dfrac{\beta}{2}} - (n-2)\sin\frac{\beta}{2}\right\},$$

that is
$$\frac{c}{2}\sec\frac{\beta}{2}\left\{\frac{\cos\beta}{\sin\dfrac{\beta}{2}} - (n-2)\sin\frac{\beta}{2}\right\},$$

that is
$$\frac{c}{2}\sec\frac{\beta}{2}\left\{\frac{1}{\sin\dfrac{\beta}{2}} - n\sin\frac{\beta}{2}\right\}.$$

But $c = 2r\sin\beta$; thus we get

$$\frac{r\sin\beta}{\cos\frac{\beta}{2}\sin\frac{\beta}{2}} - \frac{rn\sin\beta\sin\frac{\beta}{2}}{\cos\frac{\beta}{2}}, \text{ that is } 2r - 2rn\sin^2\frac{\beta}{2},$$

that is
$$2r\left(1 - n\sin^2\frac{\pi}{2n}\right).$$

Problem 38. *Circles are inscribed in triangles whose bases are the sides of a regular polygon of n sides and whose vertices lie in one of the angular points ; r is the radius of the circle circumscribing the polygon : show that the sum of the areas of the circles is*

$$16\pi r^2\sin^2\frac{\pi}{2n}\left\{\frac{n}{4}\sin^2\frac{\pi}{2n} + \frac{n-4}{8}\right\}.$$

Solution. Use the notation of the preceding solution. The area of the m^{th} circle

$$= \frac{\pi c^2}{4}\sec^2\frac{\beta}{2}\left\{\sin\frac{2m+1}{2}\beta - \sin\frac{\beta}{2}\right\}^2$$
$$= \frac{\pi c^2}{4}\sec^2\frac{\beta}{2}\left\{\sin^2\frac{2m+1}{2}\beta - 2\sin\frac{2m+1}{2}\beta\sin\frac{\beta}{2} + \sin^2\frac{\beta}{2}\right\}$$
$$= \frac{\pi c^2}{8}\sec^2\frac{\beta}{2}\left\{1 - \cos(2m+1)\beta - 4\sin\frac{2m+1}{2}\beta\sin\frac{\beta}{2} + 2\sin^2\frac{\beta}{2}\right\}.$$

Then as before we have to sum this expression for the values of m from 1 to $n-2$ inclusive. Thus we obtain

$$\frac{\pi c^2}{8}\sec^2\frac{\beta}{2}\left\{(n-2)\left(1 + 2\sin^2\frac{\beta}{2}\right) - \frac{\cos\{3\beta + (n-3)\beta\}\sin(n-2)\beta}{\sin\beta}\right.$$
$$\left. - 4\sin\frac{\beta}{2}\frac{\sin\left\{\frac{3\beta}{2} + \frac{n-3}{2}\beta\right\}\sin\frac{n-2}{2}\beta}{\sin\frac{\beta}{2}}\right\};$$

and this
$$= \frac{\pi c^2}{8}\sec^2\frac{\beta}{2}\left\{(n-2)\left(1 + 2\sin^2\frac{\beta}{2}\right) - 2\cos\beta\right\}$$
$$= \frac{\pi c^2}{8}\sec^2\frac{\beta}{2}\left\{n - 4 + 2n\sin^2\frac{\beta}{2}\right\}$$

But $c = 2r\sin\beta$; therefore $c^2\sec^2\frac{\beta}{2} = \dfrac{4r^2\sin^2\beta}{\cos^2\frac{\beta}{2}} = 16r^2\sin^2\frac{\beta}{2}$;

so that the result $= 16\pi r^2\sin^2\dfrac{\pi}{2n}\left\{\dfrac{n}{4}\sin^2\dfrac{\pi}{2n} + \dfrac{n-4}{8}\right\}$.

Problem 39. *Show that if n be a positive integer*

$$n\sin\theta + (n-1)\sin 2\theta + (n-2)\sin 3\theta + \ldots + \sin n\theta$$

$$= \frac{n+1}{2}\cot\frac{\theta}{2} - \frac{\sin(n+1)\theta}{4\sin^2\dfrac{\theta}{2}}.$$

Solution. Let S_n denote the sum of the series; so that

$$S_n = n\sin\theta + (n-1)\sin 2\theta + (n-2)\sin 3\theta + \ldots + \sin n\theta.$$

In like manner let S_{n-1} denote the sum of the series formed by changing n into $n-1$, so that

$$S_{n-1} = (n-1)\sin\theta + (n-2)\sin 2\theta + \ldots + \sin(n-1)\theta;$$

$$\therefore\ S_n - S_{n-1} = \sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta$$

$$= \frac{\sin\dfrac{n+1}{2}\theta\sin\dfrac{n\theta}{2}}{\sin\dfrac{\theta}{2}} = \frac{1}{\sin\dfrac{\theta}{2}}\left\{\cos\frac{\theta}{2} - \cos\frac{2n+1}{2}\theta\right\}.$$

Similarly we have

$$S_{n-1} - S_{n-2} = \frac{1}{2\sin\dfrac{\theta}{2}}\left\{\cos\frac{\theta}{2} - \cos\frac{2n-1}{2}\theta\right\};$$

$$S_{n-2} - S_{n-3} = \frac{1}{2\sin\dfrac{\theta}{2}}\left\{\cos\frac{\theta}{2} - \cos\frac{2n-3}{2}\theta\right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$S_2 - S_1 = \frac{1}{2\sin\dfrac{\theta}{2}}\left\{\cos\frac{\theta}{2} - \cos\frac{5\theta}{2}\right\}.$$

$$S_1 = \frac{1}{2\sin\dfrac{\theta}{2}}\left\{\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right\}.$$

Hence by addition from this series of equations we obtain

$$S_n = \frac{1}{2\sin\dfrac{\theta}{2}}\left\{n\cos\frac{\theta}{2} - \cos\frac{3\theta}{2} - \cos\frac{5\theta}{2}\ldots - \cos\frac{2n+1}{2}\theta\right\}$$

$$= \frac{1}{2\sin\dfrac{\theta}{2}}\left\{n\cos\frac{\theta}{2} - \frac{\cos\left\{\dfrac{3\theta}{2} + (n-1)\dfrac{\theta}{2}\right\}\sin\dfrac{n\theta}{2}}{\sin\dfrac{\theta}{2}}\right\}$$

$$= \frac{n}{2}\cot\frac{\theta}{2} - \frac{\cos\dfrac{(n+2)}{2}\theta\sin\dfrac{n\theta}{2}}{2\sin^2\dfrac{\theta}{2}} = \frac{n}{2}\cot\frac{\theta}{2} - \frac{\sin(n+1)\theta - \sin\theta}{4\sin^2\dfrac{\theta}{2}}$$

$$= \frac{n+1}{2}\cot\frac{\theta}{2} - \frac{\sin(n+1)\theta}{4\sin^2\dfrac{\theta}{2}}.$$

Problem 40. *Show that if n be a positive integer*

$$(n+1)n\sin\theta + n(n-1)\sin 2\theta + (n-1)(n-2)\sin 3\theta + \ldots + 2\cdot 1\sin n\theta$$

$$= \frac{n(n+3)}{2}\cot\frac{\theta}{2} - \frac{1}{4}\operatorname{cosec}^3\frac{\theta}{2}\left\{\cos\frac{3\theta}{2} - \cos\frac{2n+3}{2}\theta\right\}.$$

Solution. Let S_n denote the required sum, and S_{n-1} the sum of the series when n is changed to $n-1$. Thus

$$S_n = (n+1)n\sin\theta + n(n-1)\sin 2\theta + \ldots + 2\cdot 1\sin n\theta,$$

$$S_{n-1} = n(n-1)\sin\theta + (n-1)(n-2)\sin 2\theta + \ldots + 2\cdot 1\sin(n-1)\theta;$$

$$\therefore\quad S_n - S_{n-1} = 2\{n\sin\theta + (n-1)\sin\theta + \ldots + \sin n\theta\};$$

that is, by *Problem* 39,

$$S_n - S_{n-1} = (n+1)\cot\frac{\theta}{2} - \frac{\sin(n+1)\theta}{2\sin^2\frac{\theta}{2}}.$$

Similarly
$$S_{n-1} - S_{n-2} = n\cot\frac{\theta}{2} - \frac{\sin n\theta}{2\sin^2\frac{\theta}{2}};$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$S_2 - S_1 = 3\cot\frac{\theta}{2} - \frac{\sin 3\theta}{2\sin^2\frac{\theta}{2}};$$

$$S_1 = 2\cot\frac{\theta}{2} - \frac{\sin 2\theta}{2\sin^2\frac{\theta}{2}}.$$

Hence by addition from this series of equations we obtain

$$S_n = \frac{n(n+3)}{2}\cot\frac{\theta}{2} - \frac{1}{2\sin^2\frac{\theta}{2}}\{\sin 2\theta + \sin 3\theta + \ldots + \sin(n+1)\theta\}$$

$$= \frac{n(n+3)}{2}\cot\frac{\theta}{2} - \frac{\sin\left\{2\theta + (n-1)\frac{\theta}{2}\right\}\sin\frac{n\theta}{2}}{2\sin^2\frac{\theta}{2}\sin\frac{\theta}{2}}.$$

$$= \frac{n(n+3)}{2}\cot\frac{\theta}{2} - \frac{\cos\frac{3\theta}{2} - \cos\frac{2n+3}{2}\theta}{4\sin^3\frac{\theta}{2}}.$$

CHAPTER XXIII
Resolution of Trigonometrical Expressions Into Factors

Problem 1. *Sum the infinite series*

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots$$

Solution. By *Art.* 274 (page 432) we have

$$\frac{\sin\theta}{\theta}=1-\frac{\theta^2}{\lfloor\!\underline{3}}+\frac{\theta^4}{\lfloor\!\underline{5}}-\frac{\theta^6}{\lfloor\!\underline{7}}+\dots;$$

and by *Art.* 320 (page 445) we have

$$\frac{\sin\theta}{\theta}=\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\dots$$

Take the logarithms of the equivalent expressions; thus

$$\log\left\{1-\frac{\theta^2}{\lfloor\!\underline{3}}+\frac{\theta^4}{\lfloor\!\underline{5}}-\frac{\theta^6}{\lfloor\!\underline{7}}+\dots\right\}$$

$$=\log\left(1-\frac{\theta^2}{\pi^2}\right)+\log\left(1-\frac{\theta^2}{2^2\pi^2}\right)+\log\left(1-\frac{\theta^2}{3^2\pi^2}\right)+\dots$$

Expand the logarithms; then both sides become series arranged according to powers of θ; and by equating the coefficients of θ^2 we obtain

$$-\frac{\theta^2}{\lfloor\!\underline{3}}=-\theta^2\left(\frac{1}{\pi^2}+\frac{1}{2^2\pi^2}+\frac{1}{3^2\pi^2}+\dots\right);$$

therefore
$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots=\frac{\pi^2}{6}.$$

Problem 2. *Sum the infinite series*

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\dots$$

Solution. Equate the coefficients of θ^4 in the two equivalent series of the preceding solution; thus since

$$\log\left\{1-\frac{\theta^2}{\lfloor\!\underline{3}}+\frac{\theta^4}{\lfloor\!\underline{5}}\dots\right\}=-\left\{\frac{\theta^2}{\lfloor\!\underline{3}}-\frac{\theta^4}{\lfloor\!\underline{5}}+\dots\right\}-\frac{1}{2}\left\{\frac{\theta^2}{\lfloor\!\underline{3}}-\frac{\theta^4}{\lfloor\!\underline{5}}+\dots\right\}^2-\dots$$

we have
$$\frac{1}{\lfloor\!\underline{5}}-\frac{1}{2}\left(\frac{1}{\lfloor\!\underline{3}}\right)^2=-\frac{1}{2\pi^4}\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\dots\right);$$

therefore
$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\dots=2\pi^4\left(\frac{1}{72}-\frac{1}{120}\right)$$

$$=\frac{\pi^4}{12}\left(\frac{1}{3}-\frac{1}{5}\right)=\frac{\pi^4}{90}.$$

Problem 3. *Sum the infinite series*

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots$$

Solution.

Let $$S = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \ldots;$$

and let $$\Sigma = \frac{1}{1^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \ldots.$$

Then
$$\begin{aligned} S &= \frac{1}{1^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \ldots \\ &\quad + \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \ldots \\ &= \frac{1}{1^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \ldots \\ &\quad + \frac{1}{2^n}\left\{\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \ldots\right\} \\ &= \Sigma + \frac{1}{2^n}S. \end{aligned}$$

Therefore $$\Sigma = \frac{2^n - 1}{2^n}S.$$

Hence Σ can be found when S is known.

If $n = 2$ we have $S = \dfrac{\pi^2}{6}$ by *Problem* 1; and then $\Sigma = \dfrac{3}{4} \cdot \dfrac{\pi^2}{6} = \dfrac{\pi^2}{8}$.

Problem 4. *Sum the infinite series*

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \ldots$$

Solution. In the preceding solution suppose $n = 4$; then we have $S = \dfrac{\pi^4}{90}$ by *Problem* 2, and therefore $\Sigma = \dfrac{15}{16} \cdot \dfrac{\pi^4}{90} = \dfrac{\pi^4}{96}$.

Problem 5. $\alpha = \dfrac{\pi}{4n}$, *show that*

$$\sin\alpha \sin 5\alpha \sin 9\alpha \ldots\ldots \sin(4n - 3)\alpha = 2^{-n+\frac{1}{2}}.$$

Solution. By *Art.* 313 (page 440) we have

$$\sin n\phi = 2^{n-1} \sin\phi \sin(2\beta + \phi) \sin(4\beta + \phi) \ldots \sin(2n\beta - 2\beta + \phi)$$

where $\beta = \dfrac{\pi}{2n}$.

Let $\alpha = \dfrac{1}{2}\beta$, and let $\phi = \alpha$; then $\sin n\phi = \sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$; thus

$$\frac{1}{\sqrt{2}} = 2^{n-1} \sin\alpha \sin 5\alpha \sin 9\alpha \ldots \sin(4n - 3)\alpha;$$

therefore $$\sin\alpha \sin 5\alpha \sin 9\alpha \ldots \sin(4n - 3)\alpha = 2^{-n+\frac{1}{2}}.$$

Problem 6. *A polygon of n sides inscribed in a circle is such that its sides subtend angles* α, 2α, 3α, ... $n\alpha$ *at the centre : show that the ratio of the area of this polygon to the area of the regular inscribed polygon of n sides is equal to that of* $\sin\dfrac{n\alpha}{2}$ *to* $n\sin\dfrac{\alpha}{2}$.

Solution. Let r be the radius of the circle. The polygon can be resolved into n triangles; and thus the area of the polygon

$$= \frac{r^2}{2}\{\sin\alpha + \sin 2\alpha + \sin 3\alpha + \ldots + \sin\alpha\}$$

$$= \frac{r^2}{2} \cdot \frac{\sin\dfrac{n+1}{2}\alpha \sin\dfrac{n\alpha}{2}}{\sin\dfrac{\alpha}{2}}.$$

But $$\alpha + 2\alpha + 3\alpha + \ldots + n\alpha = 2\pi;$$

that is $$\frac{n(n+1)\alpha}{2} = 2\pi,$$

so that $$\frac{n+1}{2}\alpha = \frac{2\pi}{n}.$$

Now the area of the regular polygon of n sides

$$= \frac{nr^2}{2}\sin\frac{2\pi}{n} = \frac{nr^2}{2}\sin\frac{n+1}{2}\alpha.$$

Hence the ratio of the former area to the latter $= \dfrac{\sin\dfrac{n\alpha}{2}}{n\sin\dfrac{\alpha}{2}}$.

Problem 7. *The product of all the straight lines that can be drawn from one of the angles of a regular polygon of n sides inscribed in a circle whose radius is a to all the other angular points is* na^{n-1}.

Solution. Let $A, B, C, \ldots$ be the angles of the polygon. From A draw straight lines to the other angles. Let AP be the m^{th} straight line, so that AP subtends at the centre of the circle the angle $m\dfrac{2\pi}{n}$. Then $AP = 2a\sin m\beta$ where $\beta = \dfrac{\pi}{n}$.

Thus the product of all the straight lines

$$= (2a)^{n-1}\sin\beta\sin 2\beta\sin 3\beta\ldots\sin(n-1)\beta$$
$$= na^{n-1};$$

for by *Art*. 318 (page 443) we have

$$n = 2^{n-1}\sin\beta\sin 2\beta\sin 3\beta\ldots\sin(n-1)\beta.$$

Problem 8. *If* $p_1, p_2, \ldots p_{2n-1}, p_{2n}$ *be the perpendiculars drawn from any point in the circumference of a circle of radius a on the sides of a regular circumscribing polygon of 2n sides, show that*

$$p_1 p_3 p_5 \ldots p_{2n-1} + p_2 p_4 \ldots p_{2n} = \frac{a^n}{2^{n-2}}.$$

Solution. Let A, B, $C, \ldots$ be the points of contact of the circle with the circumscribed polygon taken in order. Let O be the fixed point, and suppose the arc OA to subtend an angle 2ϕ at the centre of the circle. Then the angle between OA and the tangent at A is ϕ; and the length of the perpendicular from O on this tangent is $OA\sin\phi$, that is $2a\sin^2\phi$. Thus we have

$$p_1 = 2a\sin^2\phi.$$

Let $\beta = \dfrac{\pi}{2n}$, then we obtain in a similar way

$$p_2 = 2a\sin^2(\phi+\beta),$$

$$p_3 = 2a\sin^2(\phi + 2\beta),$$
$$p_4 = 2a\sin^2(\phi + 3\beta),$$

and so on.

Thus $\quad p_1p_3p_5\ldots p_{2n-1} = (2a)^n\sin^2\phi\sin^2(\phi+2\beta)\ldots\sin^2\{\phi+(2n-2)\beta\}$

$$= (2a)^n\left\{\frac{\sin n\phi}{2^{n-1}}\right\}^2, \text{ by } Art.\ 318\text{(page 443)},$$
$$= \frac{a^n}{2^{n-2}}\sin^2 n\phi.$$

In the same way we have

$$p_2p_4\ldots p_{2n} = (2a)^n\sin^2(\phi+\beta)\sin^2(\phi+3\beta)\ldots\sin^2\{\phi+(2n-1)\beta\}$$
$$= \frac{a^n}{2^{n-2}}\cos^2 n\phi.$$

Hence by addition we obtain

$$\frac{a^n}{2^{n-2}}\left(\sin^2 n\phi + \cos^2 n\phi\right), \text{ that is } \frac{a^n}{2^{n-2}}.$$

Problem 9. *A polygon is described about a circle touching it at the angular points of an inscribed polygon; the product of the perpendiculars drawn to the several sides of the inscribed polygon from any point in the circumference of the circle is equal to the product of the perpendiculars drawn from the same point to the several sides of the circumscribed polygon.*

Solution. Let $A, B, C, D, \ldots$ be the angular points of the inscribed polygon. Let O be the fixed point from which the perpendiculars are drawn. Let the arc OA subtend an angle 2α at the centre of the circle, let the arc OB subtend an angle 2β, the arc OC an angle 2γ, and so on.

Let p_1, p_2, p_3, ... denote the perpendiculars from O on the sides of the circumscribed polygon which the circle at $A, B, C\ldots$ respectively. Then

$$p_1 = OA\sin\alpha, \quad p_2 = OB\sin\beta, \quad p_3 = OC\sin\gamma, \ldots$$

Again let q_1, q_2, $q_3\ldots$ denote the perpendiculars from O on the sides of the inscribed polygon AB, BC, CD, ... respectively. Then

$$q_1 = OA\sin OAB = OA\sin(\pi - \beta) = OA\sin\beta;$$

similarly $\quad q_2 = OB\sin\gamma, \quad q_3 = OC\sin\delta, \ldots$

Thus $p_1p_2p_3\ldots$ and $q_1q_2q_3\ldots$ are equal, for each is equal to the product of the same series of lengths into the same series of sines.

Problem 10. *Show that*

$$16\cos A\cos(72^\circ - A)\cos(72^\circ + A)\cos(144^\circ - A)\cos(144^\circ + A) = \cos 5A.$$

Solution. By *Art.* 313 (page 440) we have

$$\cos 5A = 16\sin(A+18^\circ)\sin(A+54^\circ)\sin(A+90^\circ)\sin(A+126^\circ)\sin(A+162^\circ)$$
$$= 16\cos(72^\circ - A)\cos(36^\circ - A)\cos A\cos(A+36^\circ)\cos(A+72^\circ);$$

and $\quad \cos(36^\circ - A) = -\cos(144^\circ + A), \quad \cos(A+36^\circ) = -\cos(144^\circ - A),$

therefore

$$\cos 5A = 16\cos(72^\circ - A)\cos(72^\circ + A)\cos A\cos(144^\circ - A)\cos(144^\circ + A).$$

Problem 11. *From the expression for* $\sin\theta$ *in factors show that*

$$\pi = 3 \cdot \frac{36}{35} \cdot \frac{144}{143} \cdot \frac{324}{323} \cdot \frac{576}{575} \cdots\cdots$$

Solution. Put $\frac{\pi}{6}$ for θ in the expression for $\sin\theta$ in *Art.* 320 (page 445); thus

$$\frac{1}{2} = \frac{\pi}{6}\left(1 - \frac{1}{6^2}\right)\left(1 - \frac{1}{2^2 6^2}\right)\left(1 - \frac{1}{3^2 6^2}\right)\cdots;$$

therefore
$$3 = \pi\frac{35}{36} \cdot \frac{143}{144} \cdot \frac{323}{324} \cdot \frac{575}{576}\cdots;$$

therefore
$$\pi = 3 \cdot \frac{36}{35} \cdot \frac{144}{143} \cdot \frac{324}{323} \cdot \frac{576}{575}\cdots$$

Problem 12. *Show that*

$$e^z + e^{-z} = 2\left(1 + \frac{4z^2}{\pi^2}\right)\left(1 + \frac{4z^2}{3^2\pi^2}\right)\left(1 + \frac{4z^2}{5^2\pi^2}\right)\cdots$$

Solution. In the general result of *Art.* 321 (page 445) put $\frac{\pi}{2}$ for θ, thus

$$e^z + e^{-z} = 2\left(1 + \frac{4z^2}{\pi^2}\right)\left(1 + \frac{4z^2}{3^2\pi^2}\right)\left(1 + \frac{4z^2}{5^2\pi^2}\right)\left(1 + \frac{4z^2}{7^2\pi^2}\right)\cdots$$

Problem 13. *Show that*

$$e^x - e^{-x} = 2x\left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{2^2\pi^2}\right)\left(1 + \frac{x^2}{3^2\pi^2}\right)\cdots$$

Solution. It is shown in *Art.* 321 (page 445) that

$$e^z - 2\cos\theta + e^{-z} = 4\sin^2\frac{\theta}{2}\left(1 + \frac{z^2}{\theta^2}\right)\left\{1 + \frac{z^2}{(2\pi+\theta)^2}\right\}\left\{1 + \frac{z^2}{(2\pi-\theta)^2}\right\}\cdots$$

The product of the first two factors on the right-hand side

$$= 4\sin^2\frac{\theta}{2} + z^2\left(\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right)^2,$$

and this is equal to z^2 when θ vanishes.

Thus, by supposing $\theta = 0$, we obtain

$$e^z - 2 + e^{-z} = z^2\left(1 + \frac{z^2}{2^2\pi^2}\right)^2\left(1 + \frac{z^2}{4^2\pi^2}\right)^2\left(1 + \frac{z^2}{6^2\pi^2}\right)^2\cdots$$

Extract the square root and put $2x$ for z; thus

$$e^x - e^{-x} = 2x\left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{2^2\pi^2}\right)\left(1 + \frac{x^2}{3^2\pi^2}\right)\cdots$$

Problem 14. *Find the sum of the series formed by multiplying together every two of the terms of the series* $\frac{1}{1}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}, \ldots.$

Solution. Let s denote the series of which we require the sum,

then $$\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\right)^2=2s+\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots$$

Hence by *Problems* 1 and 2 we have
$$\left(\frac{\pi^2}{6}\right)^2=2s+\frac{\pi^4}{90};$$
therefore $$s=\frac{\pi^4}{2}\left(\frac{1}{36}-\frac{1}{90}\right)=\frac{\pi^4}{36}\left(\frac{1}{2}-\frac{1}{5}\right)=\frac{\pi^4}{120}.$$

Problem 15. *If n be even show that*
$$\tan\phi\tan\left(\phi+\frac{\pi}{n}\right)\tan\left(\phi+\frac{2\pi}{n}\right)\ldots\tan\left(\phi+\frac{n-1}{n}\pi\right)=(-1)^{\frac{n}{2}}.$$

Solution. By *Art.* 318 (page 443) we have
$$\sin n\phi=2^{n-1}\sin\phi\sin\left(\phi+\frac{\pi}{n}\right)\sin\left(\phi+\frac{2\pi}{n}\right)\ldots\sin\left(\phi+\frac{n-1}{n}\pi\right).$$
Change ϕ into $\phi+\dfrac{\pi}{2}$; then since n is even we have
$$\sin n\left(\phi+\frac{\pi}{2}\right)=\sin\left(n\phi+\frac{n\pi}{2}\right)=\sin n\phi\cos\frac{n\pi}{2};$$
thus
$$\sin n\phi\cos\frac{n\pi}{2}=2^{n-1}\cos\phi\cos\left(\phi+\frac{\pi}{n}\right)\cos\left(\phi+\frac{2\pi}{n}\right)\ldots\cos\left(\phi+\frac{n-1}{n}\pi\right).$$
Divide the former result by this; then we obtain
$$\sec\frac{n\pi}{2}=\tan\phi\tan\left(\phi+\frac{\pi}{n}\right)\tan\left(\phi+\frac{2\pi}{n}\right)\ldots\tan\left(\phi+\frac{n-1}{n}\pi\right).$$
And $$\sec\frac{n\pi}{2}=\frac{1}{\cos\frac{n\pi}{2}}=\frac{1}{(-1)^{\frac{n}{2}}}=(-1)^{\frac{n}{2}}.$$

Problem 16. *Show that* $\sin 5A-\cos 5A$
$$=16\cos(A-27^\circ)\cos(A+9^\circ)\sin(A+27^\circ)\sin(A-9^\circ)(\cos A-\sin A).$$

Solution. $\sin 5A-\cos 5A=\sqrt{2}\sin(5A-45^\circ)=\sqrt{2}\sin 5(A-9^\circ)$.
And by *Art.* 318 (page 443) we have $\sin 5(A-9^\circ)$
$$=2^4\sin B\sin(B+36^\circ)\sin(B+72^\circ)\sin(B+108^\circ)\sin(B+144^\circ),$$
where $B=A-9^\circ$,
$$=2^4\sin(A-9^\circ)\sin(A+27^\circ)\cos(27^\circ-A)\cos(A+9^\circ)\sin(A+135^\circ)$$
$$=2^4\sin(A-9^\circ)\sin(A+27^\circ)\cos(27^\circ-A)\cos(A+9^\circ)(\cos A-\sin A)\frac{1}{\sqrt{2}}.$$
Therefore $\sin 5A-\cos 5A$
$$=2^4\sin(A-9^\circ)\sin(A+27^\circ)\cos(A-27^\circ)\cos(A+9^\circ)(\cos A-\sin A).$$

Problem 17. *Show that*
$$\frac{\pi}{2}=\frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\ \ldots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\ \ldots}$$

Solution. By *Art.* 320 (page 445) we have

$$\frac{\sin\theta}{\theta} = \left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right)\dots$$

Put $\frac{\pi}{2}$ for θ : thus

$$\frac{2}{\pi} = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{4^2}\right)\left(1 - \frac{1}{6^2}\right)\dots;$$

therefore $$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\dots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\dots}.$$

Problem 18. *By aid of the formula* $\cos\theta = \dfrac{\sin 2\theta}{2\sin\theta}$ *deduce the expression for* $\cos\theta$ *obtained in Art.* 320 *(page 445) from that for* $\sin\theta$.

Solution. We have

$$\sin 2\theta = 2\theta\left(1 - \frac{4\theta^2}{\pi^2}\right)\left(1 - \frac{4\theta^2}{2^2\pi^2}\right)\left(1 - \frac{4\theta^2}{3^2\pi^2}\right)\left(1 - \frac{4\theta^2}{4^2\pi^2}\right)\dots$$

$$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\pi^2}{3^2\pi^2}\right)\left(1 - \frac{\theta^2}{4^2\pi^2}\right).$$

Divide the first by the second : thus

$$\frac{\sin 2\theta}{\sin\theta} = 2\left(1 - \frac{4\theta^2}{\pi^2}\right)\left(1 - \frac{4\theta^2}{3^2\pi^2}\right)\left(1 - \frac{4\theta^2}{5^2\pi^2}\right)\dots;$$

therefore $$\cos\theta = \left(1 - \frac{4\theta^2}{\pi^2}\right)\left(1 - \frac{4\theta^2}{3^2\pi^2}\right)\left(1 - \frac{4\theta^2}{5^2\pi^2}\right)\dots$$

Problem 19. *Show that*

$$\sqrt{2} = \frac{4\cdot 36\cdot 100\cdot 196\cdot 324\dots}{3\cdot 35\cdot 99\cdot 195\cdot 323\dots}.$$

Solution. In the formula for $\cos\theta$ in *Art.* 320 (page 445) put $\frac{\pi}{4}$ for θ; thus

$$\frac{1}{\sqrt{2}} = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{2^2\cdot 3^2}\right)\left(1 - \frac{1}{2^2\cdot 5^2}\right)\left(1 - \frac{1}{2^2\cdot 7^2}\right)\dots$$

$$= \frac{3\cdot 35\cdot 99\cdot 195\dots}{4\cdot 36\cdot 100\cdot 196\dots};$$

therefore $$\sqrt{2} = \frac{4\cdot 36\cdot 100\cdot 196\dots}{3\cdot 35\cdot 99\cdot 195\dots}.$$

Problem 20. *Show that*

$$\frac{\sqrt{3}}{2} = \frac{8\cdot 80\cdot 224\cdot 440\dots}{9\cdot 81\cdot 225\cdot 441\dots}.$$

Solution. In the formula for $\cos\theta$ in *Art.* 320 (page 445) put $\frac{\pi}{6}$ for θ; thus

$$\frac{\sqrt{3}}{2} = \left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{3^2\cdot 3^2}\right)\left(1 - \frac{1}{3^2\cdot 5^2}\right)\left(1 - \frac{1}{3^2\cdot 7^2}\right)\dots$$

$$= \frac{8\cdot 80\cdot 224\cdot 440\dots}{9\cdot 81\cdot 225\cdot 441\dots}.$$

Problem 21. *Show that* $\cos x + \tan\frac{y}{2}\sin x =$
$$\left(1+\frac{2x}{\pi-y}\right)\left(1-\frac{2x}{\pi+y}\right)\left(1+\frac{2x}{3\pi-y}\right)\left(1-\frac{2x}{3\pi+y}\right)\left(1+\frac{2x}{5\pi-y}\right)\dots$$

Solution. $\cos x + \tan\frac{y}{2}\sin x = \dfrac{\cos x\cos\frac{y}{2}+\sin x\sin\frac{y}{2}}{\cos\frac{y}{2}} = \dfrac{\cos\left(x-\frac{y}{2}\right)}{\cos\frac{y}{2}}.$

Now by *Art.* 320 (page 445)
$$\cos\left(x-\frac{y}{2}\right) = \left\{1-\frac{(2x-y)^2}{\pi^2}\right\}\left\{1-\frac{(2x-y)^2}{3^2\pi^2}\right\}\left\{1-\frac{(2x-y)^2}{5^2\pi^2}\right\}\dots$$
$$\cos\frac{y}{2} = \left(1-\frac{y^2}{\pi^2}\right)\left(1-\frac{y^2}{3^2\pi^2}\right)\left(1-\frac{y^2}{5^2\pi^2}\right)\dots$$
Divide the former by the latter. Then
$$\frac{1-\frac{(2x-y)^2}{\pi^2}}{1-\frac{y^2}{\pi^2}} = \frac{\pi^2-(2x-y)^2}{\pi^2-y^2} = \frac{\pi^2-y^2-4x^2+4xy}{\pi^2-y^2}$$
$$= 1-\frac{4x^2}{\pi^2-y^2}+\frac{4xy}{\pi^2-y^2}$$
$$= \left(1+\frac{2x}{\pi-y}\right)\left(1-\frac{2x}{\pi+y}\right).$$

Similarly $$\frac{1-\frac{(2x-y)^2}{3^2\pi^2}}{1-\frac{y^2}{3^2\pi^2}} = \left(1+\frac{2x}{3\pi-y}\right)\left(1-\frac{2x}{3\pi+y}\right).$$

And so on. Thus the required result is obtained.

Problem 22. *Show that* $\cos x - \cot\frac{y}{2}\sin x =$
$$\left(1-\frac{2x}{y}\right)\left(1+\frac{2x}{2\pi-y}\right)\left(1-\frac{2x}{2\pi+y}\right)\left(1+\frac{2x}{4\pi-y}\right)\left(1-\frac{2x}{4\pi+y}\right)\dots$$

Solution. $\cos x - \cot\frac{y}{2}\sin x = \dfrac{\cos x\sin\frac{y}{2}-\sin x\cos\frac{y}{2}}{\sin\frac{y}{2}} = \dfrac{\sin\left(\frac{y}{2}-x\right)}{\sin\frac{y}{2}}.$

Now by *Art.* 320 (page445)
$$\sin\left(\frac{y}{2}-x\right) = \left(\frac{y}{2}-x\right)\left\{1-\frac{(y-2x)^2}{4\cdot\pi^2}\right\}\left\{1-\frac{(y-2x)^2}{4\cdot2^2\pi^2}\right\}\left\{1-\frac{(y-2x)^2}{4\cdot3^2\pi^2}\right\}\dots$$
$$\sin\frac{y}{2} = \frac{y}{2}\left(1-\frac{y^2}{4\cdot\pi^2}\right)\left(1-\frac{y^2}{4\cdot2^2\pi^2}\right)\left(1-\frac{y^2}{4\cdot3^2\pi^2}\right)\dots$$
Divide the former by the latter. Then
$$\frac{\frac{y}{2}-x}{\frac{y}{2}} = 1-\frac{2x}{y}.$$

$$\frac{1-\dfrac{(y-2x)^2}{4\cdot\pi^2}}{1-\dfrac{y^2}{4\cdot\pi^2}} = \frac{4\pi^2-(y-2x)^2}{4\pi^2-y^2} = \frac{4\pi^2-y^2-4x^2+4xy}{4\pi^2-y^2}$$

$$= 1-\frac{4x^2}{4\pi^2-y^2}+\frac{4xy}{4\pi^2-y^2} = \left(1+\frac{2x}{2\pi-y}\right)\left(1-\frac{2x}{2\pi+y}\right).$$

Similarly $$\frac{1-\dfrac{(y-2x)^2}{4\cdot 2^2\pi^2}}{1-\dfrac{y^2}{4\cdot 2^2\pi^2}} = \left(1+\frac{2x}{4\pi-y}\right)\left(1-\frac{2x}{4\pi+y}\right).$$

And so on. Thus the required result is obtained.

Problem 23.

Show that

$$\frac{\cos x-\cos y}{1-\cos y} = \left(1-\frac{x^2}{y^2}\right)\left\{1-\frac{x^2}{(2\pi-y)^2}\right\}\left\{1-\frac{x^2}{(2\pi+y)^2}\right\}$$
$$\left\{1-\frac{x^2}{(4\pi-y)^2}\right\}\left\{1-\frac{x^2}{(4\pi+y)^2}\right\}\ldots.$$

Solution. $$\frac{\cos x-\cos y}{1-\cos y} = \frac{2\sin\frac{1}{2}(y-x)\sin\frac{1}{2}(y+x)}{2\sin^2\frac{y}{2}}$$

Now by *Art.* 320 (page 445)

$$\sin\frac{1}{2}(y-x) = \frac{1}{2}(y-x)\left\{1-\frac{(y-x)^2}{4\pi^2}\right\}\left\{1-\frac{(y-x)^2}{4\cdot 2^2\pi^2}\right\}\ldots$$

$$\sin\frac{1}{2}(y+x) = \frac{1}{2}(y+x)\left\{1-\frac{(y+x)^2}{4\pi^2}\right\}\left\{1-\frac{(y+x)^2}{4\cdot 2^2\pi^2}\right\}\ldots$$

$$\sin\frac{1}{2}y = \frac{1}{2}y\left(1-\frac{y^2}{4\pi^2}\right)\left(1-\frac{y^2}{4\cdot 2^2\pi^2}\right)\ldots$$

Divide the first by the third, and divide the second by the third, and multiply the results together.

Then $\dfrac{\frac{1}{2}(y-x)}{\frac{1}{2}y} = 1-\dfrac{x}{y}$; $\dfrac{\frac{1}{2}(x+y)}{\frac{1}{2}y} = 1+\dfrac{x}{y}$; $\left(1-\dfrac{x}{y}\right)\left(1+\dfrac{x}{y}\right) = 1-\dfrac{x^2}{y^2}$.

And as in the solution of *Problem* 22,

$$\frac{1-\dfrac{(y-x)^2}{4\pi^2}}{1-\dfrac{y^2}{4\pi^2}} = \left(1+\frac{x}{2\pi-y}\right)\left(1-\frac{x}{2\pi+y}\right);$$

$$\frac{1-\dfrac{(y+x)^2}{4\pi^2}}{1-\dfrac{y^2}{4\pi^2}} = \left(1-\frac{x}{2\pi-y}\right)\left(1+\frac{x}{2\pi+y}\right);$$

$$\left(1+\frac{x}{2\pi-y}\right)\left(1-\frac{x}{2\pi+y}\right)\left(1-\frac{x}{2\pi-y}\right)\left(1+\frac{x}{2\pi+y}\right)$$

$$= \left\{1 - \frac{x^2}{(2\pi - y)^2}\right\}\left\{1 - \frac{x^2}{(2\pi + y^2)}\right\}.$$

Similarly

$$\frac{1 - \dfrac{(y-x)^2}{4\cdot 2^2\pi^2}}{1 - \dfrac{y^2}{4\cdot 2^2\pi^2}} \cdot \frac{1 - \dfrac{(y+x)^2}{4\cdot 2^2\pi^2}}{1 - \dfrac{y^2}{4\cdot 2^2\pi^2}} = \left\{1 - \frac{x^2}{(4\pi - y)^2}\right\}\left\{1 - \frac{x^2}{(4\pi + y)^2}\right\}.$$

And so on. Thus the required result is obtained.

Problem 24. *Show that* $\dfrac{\cos x + \cos y}{1 + \cos y} =$

$$\left\{1 - \frac{x^2}{(\pi - y)^2}\right\}\left\{1 - \frac{x^2}{(\pi + y)^2}\right\}\left\{1 - \frac{x^2}{(3\pi - y)^2}\right\}\left\{1 - \frac{x^2}{(3\pi + y)^2}\right\}\ldots.$$

Solution. $$\frac{\cos x + \cos y}{1 + \cos y} = \frac{2\cos\frac{1}{2}(y-x)\cos\frac{1}{2}(y+x)}{2\cos^2\frac{y}{2}}$$

Now by *Art.* 320 (page 445)

$$\cos\frac{1}{2}(y-x) = \left\{1 - \frac{(y-x)^2}{\pi^2}\right\}\left\{1 - \frac{(y-x)^2}{3^2\pi^2}\right\}\left\{1 - \frac{(y-x)^2}{5^2\pi^2}\right\}\ldots$$

$$\cos\frac{1}{2}(y+x) = \left\{1 - \frac{(y+x)^2}{\pi^2}\right\}\left\{1 - \frac{(y+x)^2}{3^2\pi^2}\right\}\left\{1 - \frac{(y+x)^2}{5^2\pi^2}\right\}\ldots$$

$$\cos\frac{y}{2} = \left(1 - \frac{y^2}{\pi^2}\right)\left(1 - \frac{y^2}{3^2\pi^2}\right)\left(1 - \frac{y^2}{5^2\pi^2}\right)\ldots$$

Divide the first by the third, and divide the second by the third, and multiply the two results together; the reductions will be similar to those in the preceding solution.

Thus $$\frac{1 - \dfrac{(y-x)^2}{\pi^2}}{1 - \dfrac{y^2}{\pi^2}} \cdot \frac{1 - \dfrac{(y+x)^2}{\pi^2}}{1 - \dfrac{y^2}{\pi^2}} = \left\{1 - \frac{x^2}{(\pi - y)^2}\right\}\left\{1 - \frac{x^2}{(\pi + y)^2}\right\};$$

$$\frac{1 - \dfrac{(y-x)^2}{3^2\pi^2}}{1 - \dfrac{y^2}{3^2\pi^2}} \cdot \frac{1 - \dfrac{(y+x)^2}{3^2\pi^2}}{1 - \dfrac{y^2}{3^2\pi^2}} = \left\{1 - \frac{x^2}{(3\pi - y)^2}\right\}\left\{1 - \frac{x^2}{(3\pi + y^2)}\right\}.$$

And so on. Thus the required result is obtained.

Or we may obtain the result in *Problem* 24 by changing y into $\pi - y$ in the result of *Problem* 23.

Problem 25. *Show that* $\dfrac{\sin x + \sin y}{\sin y} =$

$$\left(1 + \frac{x}{y}\right)\left(1 + \frac{x}{\pi - y}\right)\left(1 - \frac{x}{\pi + y}\right)\left(1 + \frac{x}{2\pi + y}\right)\left(1 - \frac{x}{2\pi - y}\right)\ldots$$

Solution. $$\frac{\sin x + \sin y}{\sin y} = \frac{2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)}{2\sin\frac{y}{2}\cos\frac{y}{2}}$$

Now in the course of the solution of *Problem* 23 we see that

$$\frac{\sin\frac{1}{2}(x+y)}{\sin\frac{1}{2}y} = \left(1+\frac{x}{y}\right)\left(1-\frac{x}{2\pi-y}\right)\left(1+\frac{x}{2\pi+y}\right)\left(1-\frac{x}{4\pi-y}\right)\left(1+\frac{x}{4\pi+y}\right)\ldots$$

And by changing y into $\pi - y$ we see that

$$\frac{\cos\frac{1}{2}(x-y)}{\cos\frac{1}{2}y} = \left(1+\frac{x}{\pi-y}\right)\left(1-\frac{x}{\pi+y}\right)\left(1+\frac{x}{3\pi-y}\right)\left(1-\frac{x}{3\pi+y}\right)\ldots$$

Hence by multiplication the required result is obtained.

Problem 26. *In Problem 21 by expanding both sides in powers of x and equating the coefficients of x, show that*

$$\tan\frac{y}{2} = \frac{2}{\pi-y} - \frac{2}{\pi+y} + \frac{2}{3\pi-y} - \frac{2}{3\pi+y} + \frac{2}{5\pi-y} - \frac{2}{5\pi+y} + \ldots$$

Solution. We have $\cos x + \tan\frac{y}{2}\sin x$

$$= 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \ldots + \tan\frac{y}{2}\left(x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \ldots\right);$$

thus the coefficient of x is $\tan\frac{y}{2}$.

Now conceive the factors on the right-hand side of the formula of *Problem* 21 multiplied together, and the product arranged according to powers of x. The first term will be unity; the second term will involve x, and the coefficient will be

$$\frac{2}{\pi-y} - \frac{2}{\pi+y} + \frac{2}{3\pi-y} - \frac{2}{3\pi+y} + \ldots$$

Hence by equating the coefficients we obtain the required result.

Problem 27. *Show in like manner from Problem 22 that*

$$\cot\frac{y}{2} = \frac{2}{y} - \frac{2}{2\pi-y} + \frac{2}{2\pi+y} - \frac{2}{4\pi-y} + \frac{2}{4\pi+y} - \ldots$$

Solution. Proceed as in the solution of *Problem* 36. Then on the left-hand side the coefficient of x will be $-\cot\frac{y}{2}$, and on the right-hand side

$$-\frac{2}{y} + \frac{2}{2\pi-y} - \frac{2}{2\pi+y} + \frac{2}{4\pi-y} - \frac{2}{4\pi+y} + \ldots$$

Equate the coefficients, and then change the signs of both sides; thus we obtain the required result.

Problem 28. *Show that*

$$\frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \frac{1}{10} - \ldots$$

Solution. In the formula of *Problem* 26 put $\frac{\pi}{3}$ for y; then

$$\tan\frac{y}{2} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}};$$

thus
$$\frac{1}{\sqrt{3}} = \frac{1}{\pi}\left\{\frac{6}{2} - \frac{6}{4} + \frac{6}{8} - \frac{6}{10} + \frac{6}{14} - \ldots\right\};$$

that is
$$\frac{1}{\sqrt{3}} = \frac{6}{\pi}\left\{\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{10} + \frac{1}{14} - \frac{1}{16} + \ldots\right\};$$

therefore
$$\frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \ldots$$

Problem 29. *Show that*

$$\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \frac{1}{19} - \ldots$$

Solution. In the formula of *Problem* 27 put $\frac{\pi}{3}$ for y; then

$$\cot\frac{y}{2} = \cot\frac{\pi}{6} = \sqrt{3};$$

thus
$$\sqrt{3} = \frac{6}{\pi}\left\{1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \frac{1}{19} - \ldots\right\};$$

therefore
$$\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \frac{1}{19} - \ldots$$

Problem 30. *Show that* $\frac{1}{\sin y} =$

$$\frac{1}{y} + \frac{1}{\pi - y} - \frac{1}{2\pi - y} - \frac{1}{\pi + y} + \frac{1}{2\pi + y} + \frac{1}{3\pi - y} - \frac{1}{4\pi - y} - \frac{1}{3\pi + y} + \ldots$$

Solution. Add together the results given in *Problems* 26 and 27; then

$$\tan\frac{y}{2} + \cot\frac{y}{2} = \frac{\sin\frac{y}{2}}{\cos\frac{y}{2}} + \frac{\cos\frac{y}{2}}{\sin\frac{y}{2}} = \frac{1}{\sin\frac{y}{2}\cos\frac{y}{2}} = \frac{2}{\sin y}.$$

Thus

$$\frac{2}{\sin y} = \frac{2}{y} + \frac{2}{\pi - y} - \frac{2}{2\pi - y} - \frac{2}{\pi + y} + \frac{2}{2\pi + y} + \frac{2}{3\pi - y} - \frac{2}{4\pi - y} - \frac{2}{3\pi + y} + \ldots$$

Then divide both sides by 2.

Or we may equate the coefficients of x in *Problem* 25.

CHAPTER XXIV
Miscellaneous Propositions

Problem 1. *Prove that* $\sin\theta\cos\frac{\theta}{2} = 8\sin\frac{\theta}{2}\sin^2\frac{\pi-\theta}{4}\sin^2\frac{\pi+\theta}{4}$.

Solution. $\sin\theta\cos\frac{\theta}{2} = 2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2} = \sin\frac{\theta}{2}\sin^2\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$

$= \sin\frac{\theta}{2}\sin^2\left(\frac{\pi}{4}-\frac{\theta}{4}\right)\cos^2\left(\frac{\pi}{4}-\frac{\theta}{4}\right) = 8\sin\frac{\theta}{2}\sin^2\left(\frac{\pi}{4}-\frac{\theta}{4}\right)\sin^2\left(\frac{\pi}{4}+\frac{\theta}{4}\right).$

Problem 2. *Prove that*

$$\left(\operatorname{cosec}^2\frac{\theta}{6} - \sec^2\frac{\theta}{2}\right)\tan\frac{\theta}{3} = \left(\tan^2\frac{\theta}{2}\operatorname{cosec}^2\frac{\theta}{6} - \sec^2\frac{\theta}{2}\right)\cot\frac{2\theta}{3}.$$

Solution. $\left(\operatorname{cosec}^2\frac{\theta}{6} - \sec^2\frac{\theta}{2}\right)\tan\frac{\theta}{3} = \left(\dfrac{1}{\sin^2\frac{\theta}{6}} - \dfrac{1}{\cos^2\frac{\theta}{2}}\right)\tan\frac{\theta}{3}$

$$= \frac{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{6}}{\sin^2\frac{\theta}{6}\cos^2\frac{\theta}{2}}\tan\frac{\theta}{3} = \frac{\cos\left(\frac{\theta}{2}-\frac{\theta}{6}\right)\cos\left(\frac{\theta}{2}+\frac{\theta}{6}\right)}{\sin^2\frac{\theta}{6}\cos^2\frac{\theta}{2}}\tan\frac{\theta}{3}$$

$$= \frac{\cos\frac{\theta}{3}\cos\frac{2\theta}{3}\tan\frac{\theta}{3}}{\sin^2\frac{\theta}{6}\cos^2\frac{\theta}{2}} = \frac{\sin\frac{\theta}{3}\cos\frac{2\theta}{3}}{\sin^2\frac{\theta}{6}\cos^2\frac{\theta}{2}}.$$

Again

$$\left(\tan^2\frac{\theta}{2}\operatorname{cosec}^2\frac{\theta}{6} - \sec^2\frac{\theta}{2}\right)\cot\frac{2\theta}{3}$$

$$= \left(\frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{6}} - \frac{1}{\cos^2\frac{\theta}{2}}\right)\cot\frac{2\theta}{3}$$

$$= \frac{\sin^2\frac{\theta}{2} - \sin^2\frac{\theta}{6}}{\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{6}}\cot\frac{2\theta}{3}$$

$$= \frac{\sin\left(\frac{\theta}{2}-\frac{\theta}{6}\right)\sin\left(\frac{\theta}{2}+\frac{\theta}{6}\right)}{\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}}\cot\frac{2\theta}{3}$$

$$= \frac{\sin\frac{\theta}{3}\sin\frac{2\theta}{3}\cot\frac{2\theta}{3}}{\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{6}} = \frac{\sin\frac{\theta}{3}\cos\frac{2\theta}{3}}{\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{6}}.$$

Problem 3. *Prove that*

$$\tan 3\theta - \tan 2\theta - \tan\theta = \tan 3\theta \tan 2\theta \tan\theta.$$

Solution. $\tan 3\theta - \tan 2\theta - \tan\theta$

$$= \frac{\sin 3\theta}{\cos 3\theta} - \frac{\sin 2\theta}{\cos 2\theta} - \tan\theta$$
$$= \frac{\sin 3\theta\cos 2\theta - \sin 2\theta\cos 3\theta}{\cos 3\theta\cos 2\theta} - \tan\theta$$
$$= \frac{\sin\theta}{\cos 3\theta\cos 2\theta} - \frac{\sin\theta}{\cos\theta}$$
$$= \frac{\sin\theta}{\cos\theta\cos 2\theta\cos 3\theta}\{\cos\theta - \cos 3\theta\cos 2\theta\}$$
$$= \frac{\sin\theta}{\cos\theta\cos 2\theta\cos 3\theta}\{\cos(3\theta-2\theta) - \cos 3\theta\cos 2\theta\}$$
$$= \frac{\sin\theta\sin 2\theta\sin 3\theta}{\cos\theta\cos 2\theta\cos 3\theta} = \tan\theta\tan 2\theta\tan 3\theta.$$

Problem 4. *Find x from the equation*

$$\tan^3 x + \cot^3 x = m^3 - 3m.$$

Solution. $\tan^3 x + \cot^3 x = m^3 - 3m$;

therefore $\quad (\tan x + \cot x)^3 - 3(\tan^2 x\cot x + \cot^2 x\tan x) = m^3 - 3m$;

therefore $\quad (\tan x + \cot x)^3 - 3(\tan x + \cot x) = m^3 - 3m$;

therefore $\quad (\tan x + \cot x)^3 - m^3 = 3(\tan x + \cot x) - 3m.$

Put y for $\tan x + \cot x$; thus

$$y^3 - m^3 = 3(y - m);$$

therefore
$$(y-m)(y^2 + ym + m^2) = 3(y - m).$$

Therefore either $y - m = 0$, or $y^2 + ym + m^2 = 3$.

Take $y - m = 0$; thus $\tan x + \cot x = m$,

therefore
$$\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = m;$$

therefore
$$\frac{1}{\sin x\cos x} = m;$$

therefore
$$\sin 2x = \frac{2}{m}.$$

Again, take $y^2 + ym + m^2 = 3$. By solving this quadratic in the usual way we obtain

$$y = \frac{-m \pm \sqrt{(12 - 3m^2)}}{2}$$

and thus we obtain two other values for $\sin 2x$.

Problem 5. *The circumference of a circle is divided into $2n$ equal parts at the points A, P, Q, ... Tangents are drawn at the points A, P, Q, ... and perpendiculars*

OA, OB, OC, ... are let fall on them from O the extremity of the diameter OA. Show that

$$OA^2 + OB^2 + OC^2 + \ldots = 3n(radius)^2.$$

Solution. Let T denote the m^{th} point, counting from A as the first, so that the angle $TOA = m\beta$, where $\beta = \dfrac{\pi}{2n}$. Then, since the angle OTA is a right angle, we have $OT = OA\cos TOA = 2r\cos m\beta$, where r is the radius of the circle. The angle between OT and the tangent at T is equal to the angle TAO, and is therefore $\dfrac{\pi}{2} - m\beta$. Thus the perpendicular from O on the tangent at $T = OT\sin\left(\dfrac{\pi}{2} - m\beta\right) = OT\cos m\beta = 2r\cos^2 m\beta$. The square of the perpendicular is $4r^2\cos^4 m\beta$.

Thus the sum of the squares of the perpendiculars.

$$= 4r^2\{1 + \cos^4\beta + \cos^4 2\beta + \ldots + \cos^4(2n-1)\beta\}.$$

Now
$$\cos^4\theta = \frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta.$$

Apply this transformation to every term in the series, observing that 1 may be considered as $\cos 0$.

Thus the sum of the squares of the perpendiculars

$$= 4r^2\cdot\frac{3}{8}\cdot 2n + 2r^2\{\cos 0 + \cos 2\beta + \cos 4\beta + \ldots + \cos 2(2n-1)\beta\}$$
$$+\frac{r^2}{2}\{\cos 0 + \cos 4\beta + \cos 8\beta + \ldots + \cos 4(2n-1)\beta\}.$$

Each series of cosines will vanish as in *Art.* 305 (page 437); thus we find that the sum of the squares of the perpendiculars $= 3nr^2$.

Problem 6. *ABC is a quadrant; AP, AQ, AR are three arcs in ascending order of magnitude, each being less than AB, and their sum equal to twice AB; radii CP, CQ, CR are produced to meet the tangent at A at p, q, r and a triangle is formed with Ap, Aq, Ar. Find the condition that this may be possible, and the inferior limit of Aq and the superior limit of Ap. Prove also that in all such triangles the radii of the inscribed and circumscribed circles are inversely proportional.*

Solution. Let $ACP = \theta$, $ACQ = \phi$, $ACR = \psi$; let $AC = \rho$. Then $\theta + \phi + \psi = \pi$. Now, in order that a triangle may exist with Ap, Aq, and Ar as sides we must have $Ap + Aq$ greater than Ar; thus $\tan\theta + \tan\phi$ must be greater than $\tan\psi$.

But by *Art.* 114 (page 409) we have

$$\tan\theta + \tan\phi + \tan\psi = \tan\theta\tan\phi\tan\psi.$$

Hence $\tan\theta\tan\phi\tan\psi$ must be greater than $2\tan\psi$, and therefore $\tan\theta\tan\phi$ greater than 2. Therefore *a fortiori* $\tan^2\phi$ must be greater than 2.

Thus the inferior limit of Aq is when $\tan QCA = \sqrt{2}$.

Again, since $\theta + \phi + \psi = \pi$, the superior limit of θ is the value of θ when θ, ϕ and ψ are all equal; that is, when $\theta = \dfrac{\pi}{3}$.

When the triangle is formed the radius of the inscribed circle, by *Art.* 248 (page 426) is $\dfrac{2S}{\rho(\tan\theta + \tan\phi + \tan\psi)}$; and the radius of the circumscribed circle by *Art.* 252 (page 428) is $\dfrac{\rho^3\tan\theta\tan\phi\tan\psi}{4S}$. The product of these radii is $\dfrac{\rho^2}{2}\cdot\dfrac{\tan\theta\tan\phi\tan\psi}{\tan\theta + \tan\phi + \tan\psi}$, that is $\dfrac{\rho^2}{2}$ by *Art.* 114 (page 409).

Hence the product of the radii is constant, whatever θ, ϕ and ψ may be; and therefore one radius varies inversely as the other.

Problem 7. *ABC is a right-angled triangle, C being the right angle, E is the point at which the inscribed circle touches BC, and F the point at which the circle drawn to touch AB and the sides CA, CB produced meets CA; show that if EF be joined the triangle FEC is half the triangle ABC.*

Solution. As in *Art.* 250 (page 427) we see that $CF = s$; also $CE = r$; hence

$$\text{the area of } CEF = \frac{1}{2}rs.$$

And, as in *Art.* 248 (page 426) the area of $ABC = rs$; therefore CEF is half ABC.

Problem 8. *Through the angular points of a triangle straight lines are drawn bisecting the exterior angles. If S be the area of the original triangle and S′ that of the new triangle, show that*

$$S' = \frac{1}{2}S \operatorname{cosec}\frac{A}{2}\operatorname{cosec}\frac{B}{2}\operatorname{cosec}\frac{C}{2}.$$

Solution. The straight lines bisecting the external angles are the same as those which join the centres of the escribed circles.

Thus, by Chapter XVI : *Problem* 34, we have $S' = \dfrac{abc}{2r}$;

therefore
$$\frac{S'}{S} = \frac{abc}{2rS} = \frac{sabc}{2S^2} = \frac{abc}{2(s-a)(s-b)(s-c)}$$
$$= \frac{1}{2}\operatorname{cosec}\frac{A}{2}\operatorname{cosec}\frac{B}{2}\operatorname{cosec}\frac{C}{2}.$$

Problem 9. *ABCD is a horizontal straight line. From a point immediately above D the known distances AB and BC are observed to subtend the same angle α. If AB = a and BC = b, show that the height of the observer's position above D is*

$$\frac{2ab(a+b)\tan\alpha}{(a-b)^2+(a+b)^2\tan^2\alpha}.$$

Solution. Let P denote the point above D. Then $PD \times AC$ = twice the area of the triangle $APC = AP \cdot PC \sin\alpha$; therefore

$$PD = \frac{AP\cdot PC\cdot\sin 2\alpha}{AC} = \frac{(a+b)AP\cdot PC\cdot\sin 2\alpha}{AC^2}$$
$$= \frac{(a+b)AP\cdot PC\cdot\sin 2\alpha}{AP^2+PC^2-2AP\cdot PC\cos 2\alpha}$$
$$= \frac{(a+b)ab\sin 2\alpha}{a^2+b^2-2ab\cos 2\alpha},\quad \text{for } \frac{AP}{PC}=\frac{a}{b} \text{ by } \textit{Euclid} \text{ VI. } 3,$$
$$= \frac{2(a+b)ab\sin\alpha\cos\alpha}{(a^2+b^2)(\sin^2\alpha+\cos^2\alpha)-2ab(\cos^2\alpha-\sin^2\alpha)}$$
$$= \frac{2(a+b)ab\tan\alpha}{(a-b)^2+(a+b)^2\tan^2\alpha}$$

Problem 10. *If in any arc not greater than a quadrant a point be taken, and from this point two straight lines be drawn, one to the extremity of the arc, the other perpendicular to its chord and terminated by it, prove that the sum of these two straight lines is less than the chord of the arc.*

Solution. Let AB be the arc, and C the centre of the circle; in AB take any point P, join PA, and draw PM perpendicular to AB.

Let $BCA = 2\gamma, PCA = 2\theta$, and $AC = r$.

Then $AB = 2r\sin\gamma,\ AP = 2r\sin\theta,\ PM = AP\sin PAB = AP\sin(\gamma-\theta)$.

We have then to show that $2r\sin\theta\{1+\sin*(\gamma-\theta)\}$ is less than $2r\sin\gamma$, or that $\sin(\gamma-\theta)\sin\theta$ is less than $\sin\gamma - \sin\theta$, or that

$$2\sin\frac{1}{2}(\gamma-\theta)\cos\frac{1}{2}(\gamma-\theta) \text{ is less than } 2\sin\frac{1}{2}(\gamma-\theta)\cos\frac{1}{2}(\gamma+\theta),$$

or that
$$\cos\frac{1}{2}(\gamma-\theta)\sin\theta \text{ is less than } \cos\frac{1}{2}(\gamma+\theta).$$

Now this is the case, for $\cos\frac{1}{2}(\gamma-\theta)\sin\theta$ is less than $\sin\theta$, and therefore less than $\sin\gamma$ and $\cos\frac{1}{2}(\gamma+\theta)$ is greater than $\cos\frac{1}{2}(\gamma+\gamma)$, that is greater than $\cos\gamma$; and $\cos\gamma$ is greater than $\sin\gamma$, since γ is less than $\frac{\pi}{4}$.

Problem 11. *Suppose α the angle of elevation of a cloud, β the angle of depression of the image of the cloud seen by reflection from a lake, h the height of the observer's eye above the lake, then the height of the cloud is*

$$\frac{h\sin(\beta+\alpha)}{\sin(\beta-\alpha)}.$$

Solution. Let A be the position of the observer's eye, C the cloud, B the image of the cloud formed by the lake. Draw the horizontal straight line AH. Then $HAC = \alpha$, and $HAB = \beta$.

The straight lines CB and AB are equally inclined to the surface of the water by the Laws of Optics, and thus the angle between CB and AH is equal to β.

Now
$$\frac{CB}{AB} = \frac{\sin CAB}{\sin ACB} = \frac{\sin(\beta+\alpha)}{\sin(\beta-\alpha)};$$

therefore
$$CB = \frac{AB\sin(\beta+\alpha)}{\sin(\beta-\alpha)}.$$

$$\text{The height of the cloud } = CB\sin\beta = \frac{\sin(\beta+\alpha)}{\sin(\beta-\alpha)}AB\sin\beta$$
$$= h\frac{\sin(\beta+\alpha)}{\sin(\beta-\alpha)}.$$

Problem 12. *At noon a person standing on a cliff h feet above the level of the sea, observes the altitude of a cloud in the plane of the meridian to be α and the angle of depression of its shadow on the surface of the water to be β; the sun being behind the observer when he is looking at the cloud. Show that, if γ be the sun's altitude at the time of observation, the height of the cloud above the surface of the water will be*

$$\frac{h\sin\gamma\sin(\alpha+\beta)}{\sin\beta\sin(\gamma+\alpha)}.$$

Solution. Let A be the position of the observer's eye, C the cloud, B the shadow of the cloud on the sea. Then the position of the sun is on BC produced through C. Draw from A a horizontal straight line meeting BC at H. Then $HAC = \alpha$, and $HAB = \beta$. On account of the enormous distance of the sun, the straight line HC may be considered as parallel to the straight line drawn from A to the sun; so that $CHA = \gamma$.

Now
$$\frac{BC}{BA} = \frac{\sin BAC}{\sin BCA} = \frac{\sin(\alpha+\beta)}{\sin(\pi-\alpha-\gamma)} = \frac{\sin(\alpha+\beta)}{\sin(\alpha+\gamma)}.$$

The height of the cloud above the surface of the sea
$$= BC\sin\gamma = \frac{\sin\gamma}{\sin\beta}BC\sin\beta = \frac{\sin\gamma\sin(\alpha+\beta)}{\sin\beta\sin(\alpha+\gamma)}BA\sin\beta$$
$$= h\frac{\sin\gamma}{\sin\beta}\cdot\frac{\sin(\alpha+\beta)}{\sin(\alpha+\gamma)}.$$

Problem 13. *Show that the formula of Art. 280 (page 433) may be verified by induction.*

Solution. Assume that
$$2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{1\cdot 2}\cos(n-4)\theta + \dots$$
$$+ \frac{n(n-1)\dots(n-r+1)}{\underline{|r}}\cos(n-2r)\theta + \dots$$

Multiply both sides by $2\cos\theta$. Thus
$$2^n\cos^{n+1}\theta = 2\cos n\theta\cos\theta + 2n\cos(n-2)\theta\cos\theta$$
$$+ 2\frac{n(n-1)}{1\cdot 2}\cos(n-4)\theta\cos\theta + \dots$$

Now use the formula $2\cos A\cos B = \cos(A+B) + \cos(A-B)$ for the terms on the right-hand side. Thus
$$2^n\cos^{n+1}\theta = \cos(n+1)\theta + \cos(n-1)\theta$$
$$+ n\{\cos(n-1)\theta + \cos(n-3)\theta\}$$
$$+ \frac{n(n-1)}{1\cdot 2}\{\cos(n-3)\theta + \cos(n-5)\theta\}$$
$$+ \frac{n(n-1)\dots(n-r+1)}{\underline{|r}}\{\cos(n-2r+1)\theta + \cos(n-2r-1)\theta\}$$
$$+ \dots$$

Then re-arrange the terms on the right-hand side, and we obtain a series like that with which we started except that we have $n+1$ instead of n.

For instance, the term involving $\cos(n-3)\theta$ is
$$\left\{n + \frac{n(n-1)}{1\cdot 2}\right\}\cos(n-3)\theta;$$
that is
$$\frac{(n+1)n}{1\cdot 2}\cos(n+1-4)\theta.$$

And generally the term involving $\cos(n+1-2r)\theta$ is
$$\left\{\frac{n(n-1)\dots(n-r+2)}{\underline{|r-1}} + \frac{n(n-1)\dots(n-r+1)}{\underline{|r}}\right\}\cos(n+1-2r)\theta;$$
that is
$$\frac{n(n-1)\dots(n-r+2)}{\underline{|r-1}}\left(1 + \frac{n-r+1}{r}\right)\cos(n+1-2r)\theta;$$

that is $$\frac{(n+1)n\ldots(n+1-r+1)}{\lfloor r}\cos(n+1-2r)\theta.$$

This shows that if the formula holds for an assigned value of n it holds also when n is changed into n is changed into $n+1$. Moreover the formula evidently holds when $n=1$.

We have not paid special attention to the *last term* in the expansion, but it is easy to do this if required.

Problem 14. *Show that the formula of Arts. 282 (page 434) and 283 (page 434) may be obtained from that of Art. 280 (page 433) by changing θ into $\frac{\pi}{2}-\theta$.*

Solution. Take the formula of *Art.* 280 (page 433), and suppose n *even*; thus

$$2^{n-1}\cos^n\theta=\cos n\theta+n\cos(n-2)\theta+\frac{n(n-1)}{1\cdot 2}\cos(n-4)\theta+\ldots$$

$$+\frac{n(n-1)(n-r+1)}{\lfloor r}\cos(n-2r)\theta\ldots+\frac{n(n-1)\ldots\left(\frac{1}{2}n+1\right)}{2\lfloor\frac{1}{2}n}$$

Change θ into $\frac{\pi}{2}-\theta$; thus

$\cos^n\theta$ becomes $\cos^n\left(\frac{\pi}{2}-\theta\right)$, that is $\sin^n\theta$;

$\cos n\theta$ becomes $\cos n\left(\frac{\pi}{2}-\theta\right)$, that is $\cos n\frac{\pi}{2}\cos n\theta$, that is $(-1)^{\frac{n}{2}}\cos n\theta$;

$\cos(n-2)\theta$ becomes $\cos(n-2)\left(\frac{\pi}{2}-\theta\right)$, that is $(-1)^{\frac{n-2}{2}}\cos(n-2)\theta$;

and so on.

Thus $$2^{n-1}\sin^n\theta=(-1)^{\frac{n}{2}}\cos n\theta+n(-1)^{\frac{n-2}{2}}\cos(n-2)\theta$$
$$+\frac{n(n-1)}{1\cdot 2}(-1)^{\frac{n-4}{2}}\cos(n-4)\theta+\ldots$$

Multiply both sides by $(-1)^{\frac{n}{2}}$; thus we obtain the formula of *Art.* 282 (page 434)
.

Next suppose n *odd*. Then in the same manner we deduce the formula of *Art.* 283 (page 434) from that of *Art.* 280 (page 433); we observe now that

$$\cos n\left(\frac{\pi}{2}-\theta\right)=\sin n\frac{\pi}{2}\sin n\theta=(-1)^{\frac{n-1}{2}}\sin n\theta,$$
$$\cos(n-2)\left(\frac{\pi}{2}-\theta\right)=(-1)^{\frac{n-3}{2}}\sin(n-2)\theta,$$

and so on.

Problem 15. *Express $\cos 6\left(\tan^{-1}x\right)$ in terms of x.*

Solution. Let $\tan^{-1}x=\theta$, so that $\tan\theta=x$; then

$$\cos\theta=\frac{1}{\sqrt{(1+x^2)}}\text{ and }\sin\theta=\frac{x}{\sqrt{(1+x^2)}}.$$

And by *Art.* 270 (page 432)

$$\cos 6\theta=\cos^6\theta-15\cos^4\theta\sin^2\theta+15\cos^2\theta\sin^4\theta-\sin^6\theta$$
$$=\frac{1}{(1+x^2)^3}\left\{1-15x^2+15x^4-x^6\right\}.$$

Problem 16. *If a quadrilateral can be inscribed in a circle and can also have a circle described about it, the area of the quadrilateral is equal to the square root of the product of the four sides.*

Solution. Since the quadrilateral can be inscribed in a circle, the area by *Art.* 254 (page 429) is $\sqrt{\{(s-a)(s-b)(s-c)(s-d)\}}$. But when a quadrilateral can be described about a circle it may be shown by geometry that the sum of two opposite sides is equal to the sum of the other two. Thus in the present case $a+c=b+d$.

Now $$s=\frac{1}{2}(a+b+c+d)=a+c=b+d;$$

therefore $$s-a=c,\ s-b=d,\ s-c=a,\ s-d=b;$$

therefore $$\sqrt{\{(s-a)(s-b)(s-c)(s-d)\}}=\sqrt{abcd}.$$

Thus the area $=\sqrt{abcd}$.

Problem 17. *The sides of a quadrilateral figure are a, b, c, d; and the sum of two opposite angles is θ. If S denote the area of the figure, and s half the sum of the sides, show that*

$$S^2=(s-a)(s-b)(s-c)(s-d)-abcd\cos^2\frac{\theta}{2}.$$

Solution. Let a, b, c, d denote the sides taken in succession; let B denote the angle between the first two, and D the angle between the last two. Thus $B+D=\theta$.

Then dividing the quadrilateral into two triangles, as in *Art.* 254 (page 429), we have

$$S=\frac{1}{2}ab\sin B+\frac{1}{2}cd\sin D \tag{23}$$

And from two values which can be obtained for the square of the diagonal opposite B and D we have

$$a^2+b^2-2ab\cos B=c^2+d^2-2cd\cos D,$$

therefore $$\frac{a^2+b^2-c^2-d^2}{4}=\frac{1}{2}ab\cos B-\frac{1}{2}cd\cos D \tag{24}$$

Square and add (23) and (24): thus

$$S^2+\left(\frac{a^2+b^2-c^2-d^2}{4}\right)^2=\frac{1}{4}(a^2b^2+c^2d^2)-\frac{1}{2}abcd\cos(B+D)$$

$$=\frac{1}{4}(a^2b^2+c^2d^2)-\frac{1}{2}abcd\cos\theta=\frac{1}{4}(a^2b^2+c^2d^2)-\frac{1}{2}abcd\left(2\cos^2\frac{\theta}{2}-1\right);$$

therefore $$S^2=\frac{1}{4}(ab+cd)^2-\left(\frac{a^2+b^2-c^2-d^2}{4}\right)^2-abcd\cos^2\frac{\theta}{2}.$$

Now we know that if $\theta=\pi$ the expression for S^2 must reduce to

$$(s-a)(s-b)(s-c)(s-d).$$

Hence we are sure that $\frac{1}{4}(ab+cd)^2-\left(\frac{a^2+b^2-c^2-d^2}{4}\right)^2$ must take the form just given; and this is easily verified. For this expression

$$=\left\{\frac{1}{2}(ab+cd)+\frac{a^2+b^2-c^2-d^2}{4}\right\}\left\{\frac{1}{2}(ab+cd)-\frac{a^2+b^2-c^2-d^2}{4}\right\}$$

$$=\frac{1}{16}\{(a+b)^2-(c-d)^2\}\{(c+d)^2-(a-b)^2\}$$

$$= \frac{1}{16}(a+b+c-d)(a+b-c+d)(c+d+a-b)(c+d-a+b).$$

Thus $$S^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\frac{\theta}{2}.$$

Problem 18. *Show that*

$$\cos^n\theta\cos n\theta = 1 - \frac{n(n+1)}{\underline{|2}}\tan^2\theta + \frac{n(n+1)(n+2)(n+3)}{\underline{|4}}\tan^4\theta - \dots$$

Solution. Let t stand for $\tan\theta$. By *Art.* 270 (page 432) we have

$$\cos n\theta = \cos^n\theta\left\{1 - \frac{n(n-1)}{\underline{|2}}t^2 + \frac{n(n-1)(n-2)(n-3)}{\underline{|4}}t^4 - \dots\right\}.$$

Put ι for $\sqrt{-1}$; then we may write the formula thus

$$2\cos n\theta = \cos^n\theta\{(1+\iota t)^n + (1-\iota t)^n\}.$$

Therefore $$2\cos^n\theta\cos n\theta = \cos^{2n}\theta\{(1+\iota t)^n + (1-\iota t)^n\}$$

$$= \frac{1}{(1+t^2)^n}\{(1+\iota t)^n + (1-\iota t)^n\} = \frac{1}{(1+\iota t)^n(1-\iota t)^n}\{(1+\iota t)^n + (1-\iota t)^n\}$$

$$= (1-\iota t)^{-n} + (1+\iota t)^{-n}.$$

Expand the two terms on the right-hand side by the Binomial Theorem, and the required result is obtained.

Problem 19. *Show that*

$$\cos^n\theta\sin n\theta = n\tan\theta - \frac{n(n+1)(n+2)}{\underline{|3}}\tan^3\theta + \dots$$

Solution. Proceed as in the solution of *Problem* 18. Thus we obtain

$$2\sin n\theta = \frac{\cos^n\theta}{\iota}\{(1+\iota t)^n - (1-\iota t)^n\};$$

therefore $$2\cos^n\theta\sin n\theta = \frac{\cos^n\theta}{\iota}\{(1+\iota t)^n - (1-\iota t)^n\}$$

$$= \frac{1}{\iota(1+t^2)^n}\{(1+\iota t)^n - (1-\iota t)^n\} = \frac{1}{\iota(1+\iota t)^n(1-\iota t)^n}\{(1+\iota t)^n - (1-\iota t)^n\}$$

$$= \frac{1}{\iota}\{(1-\iota t)^{-n} - (1+\iota t)^{-n}\}.$$

Expand the two terms on the right-hand side by the Binomial Theorem, and the required result is obtained.

Problem 20. *If θ is a positive angle less than $\frac{\pi}{2}$ show that $\frac{\theta}{\sin\theta}$ continually increases with θ.*

Solution. Let θ have any value between 0 and $\frac{\pi}{2}$; let h be a small positive quantity. We have then to show that $\frac{\theta+h}{\sin(\theta+h)}$ is greater than $\frac{\theta}{\sin\theta}$, that is we must show that $\frac{\theta+h}{\sin(\theta+h)} - \frac{\theta}{\sin\theta}$ is positive.

Now the sign of the last expression is the same as the sign of

$$(\theta + h)\sin\theta - \theta\sin(\theta + h),$$

and is therefore the same as the sign of

$$\theta\sin\theta(1 - \cos h) + h\sin\theta - \theta\cos\theta\sin h,$$

or as the sign of

$$\theta\sin\theta(1 - \cos h) + \sin\theta\sin h\left(\frac{h}{\sin h} - \frac{\theta}{\tan\theta}\right).$$

Now $1 - \cos h$ is positive; and $\frac{h}{\sin h}$ is greater than unity while $\frac{\theta}{\tan\theta}$ is less than unity, by *Art.* 118 (page 411); thus the expression is positive.

Problem 21. *If θ is a positive angle less than $\frac{\pi}{2}$ show that $\frac{\theta}{\tan\theta}$ continually decreases as θ increases.*

Solution. Let θ have any value between 0 and $\frac{\pi}{2}$; let h be a small positive quantity. We have then to show that $\frac{\theta}{\tan\theta}$ is greater than $\frac{\theta + h}{\tan(\theta + h)}$, that is we must show that $\frac{\theta}{\tan\theta} - \frac{\theta + h}{\tan(\theta + h)}$ is positive.

Now the sign of the last expression is the same as the sign of

$$\theta\cos\theta\sin(\theta + h) - (\theta + h)\cos(\theta + h)\sin\theta,$$

that is of

$$\theta\sin h - h\cos(\theta + h)\sin\theta,$$

that is of

$$\frac{\theta}{\sin\theta} - \frac{h}{\sin h}\cos(\theta + h).$$

But as we may suppose h less than θ, we know by *Problem* 20 that $\frac{\theta}{\sin\theta}$ is greater than $\frac{h}{\sin h}$, and therefore $\frac{\theta}{\sin\theta}$ is greater than $\frac{h}{\sin h}\cos(\theta + h)$.

Problem 22. *In the diagram of Art. 332 (page 448) if PO be joined, show that it bisects DF, and is bisected by DF.*

Solution. Let PO intersect FD at K; then by similar triangles, since $PA = 2PF$, we have $OA = 2FK$; but $DF = OA$; therefore $DF = 2FK$; therefore DF is bisected at K.

Also since PA is bisected at F, it follows by similar triangles that PO is bisected at K.

Problem 23. *Show also that PO divides DA into parts which are in the ratio of 1 to 2.*

Solution. Let PO intersect AD at L. Then the triangles ALP and DLO are similar; therefore $\frac{LD}{OD} = \frac{LA}{PA}$; therefore $\frac{LD}{LA} = \frac{OD}{PA} = \frac{FA}{PA} = \frac{1}{2}$.

Problem 24. *Show that the following four points connected with any triangle are in a straight line : the centre of the circumscribing circle, the centre of the nine* *points circle, the point of intersection of the perpendiculars from the angles on the opposite sides, and the point of intersection of the straight lines drawn from the angles to the middle points of the opposite sides.*

Solution. The point K in the solution of *Problem* 22 being the middle point of DF is the centre of the nine points circle. Thus P, K, and O are in one straight line. Also this straight line cuts AD at a point L, such that $\frac{LD}{LA} = \frac{1}{2}$; and OP is divided at L so that $\frac{OL}{LP} = \frac{1}{2}$.

In like manner the straight line from B to the middle point of AC cuts OP at the same point as AD does; and so also does the straight line from O to the middle point of AB. Hence the point L is the intersection of the three straight lines from the angles of ABC to the middle points of the opposite sides.

Problem 25. *Show that the length of the perpendicular from the centre of the nine points circle on BC is* $\frac{1}{2}R\cos(C - B)$.

Solution. The centre of the nine points circle is the middle point of OP, hence the perpendicular from it on $BC = \frac{1}{2}(OD + PG)$. Now $OD = R\cos A$, and

$$PG = BP\sin PBG = BP\cos C = \frac{BG\cos C}{\cos PBG} = \frac{BG\cos C}{\sin C} = \frac{c\cos B\cos C}{\sin C}$$
$$= 2R\cos B\cos C.$$

Hence the perpendicular required

$$= \frac{1}{2}(R\cos A + 2R\cos B\cos C) = \frac{R}{2}\{2\cos B\cos C - \cos(B + C)\}$$
$$= \frac{R}{2}(\cos B\cos C - \sin C\sin B) = \frac{R}{2}\cos(C - B).$$

Or thus : the required perpendicular

$$= \frac{R}{2}\sin HDG = \frac{R}{2}\cos OAG = \frac{R}{2}\cos(BAG - OAB)$$
$$= \frac{R}{2}\cos\{90^\circ - B - (90^\circ - C)\} = \frac{R}{2}\cos(C - B).$$

Problem 26. *Show that the length of the perpendicular from the centre of the nine points circle on AG in the diagram of Art. 332 (page 448) is* $\frac{1}{2}R\sin(C - B)$.

Solution. The perpendicular from the centre of the nine points circle on $4G$

$$= \frac{1}{2}DG = \frac{1}{2}(CD - CG) = \frac{1}{2}\left(\frac{a}{2} - b\cos C\right) = \frac{1}{2}(R\sin A - 2R\sin B\cos C)$$
$$= \frac{R}{2}\{\sin(B + C) - 2\sin B\cos C\} = \frac{R}{2}\sin(C - B).$$

Or thus : the required perpendicular

$$= \frac{R}{2}\cos HDG = \frac{R}{2}\sin OAG = \frac{R}{2}\sin(C - B).$$

Problem 27. *In the diagram of Art. 332 (page 448) show that*

$$OP^2 = R^2(1 - 8\cos A\cos B\cos C).$$

Solution. We have $AP = 2AF = 2OD = 2R\cos A$; and as shown in the solution of *Problem* 25, the angle $OAP = C - B$.

Then, from the triangle OAP,

$$\begin{aligned} OP^2 &= OA^2 + PA^2 - 2OA \cdot PA\cos(C-B) \\ &= R^2\{1 + 4\cos^2 A - 4\cos A\cos(C-B)\} \\ &= R^2 + 4R^2\cos A\{\cos A - \cos(C-B)\} \\ &= R^2 + 4R^2\cos A\{-\cos(B+C) - \cos(C-B)\} \\ &= R^2 - 8R^2\cos A\cos B\cos C. \end{aligned}$$

Problem 28. *Show that the distance of the centre of the nine points circle from the angular point A is* $\frac{R}{2}\sqrt{1 + 8\cos A\sin B\sin C}$.

Solution. Donate the centre of the nine points circle by K; then K is the middle point of OP.

Now
$$OA^2 = OK^2 + KA^2 - 2OK \cdot KA\cos OKA,$$
$$PA^2 = PK^2 + KA^2 - 2PK \cdot KA\cos PKA;$$
therefore, by addition,
$$OA^2 + PA^2 = 2OK^2 + 2KA^2;$$
thus
$$2KA^2 = R^2 + 4R^2\cos^2 A - 2OK^2 = R^2 + 4R^2\cos^2 A - \frac{1}{2}PO^2.$$

Therefore, by the aid of *Problem* 27,

$$\begin{aligned} 4KA^2 &= 2R^2 + 8R^2\cos^2 A - R^2(1 - 8\cos A\cos B\cos C) \\ &= R^2 + 8R^2\cos A(\cos A + \cos B\cos C) \\ &= R^2 + 8R^2\cos A\{-\cos(B+C) + \cos B\cos C\} \\ &= R^2 + 8R^2\cos A\sin B\sin C; \end{aligned}$$

therefore
$$KA = \frac{R}{2}\sqrt{(1 + 8\cos A\sin B\sin C)}.$$

Problem 29. *The centre of the nine points circle cannot coincide with the centre of the circumscribed circle unless the triangle is equilateral.*

Solution. Take the diagram of *Art.* 332 (page 448). The centre of the nine points circle is on a straight line which bisects DG at right angles; and so it cannot be at O unless D and G coincide, that is unless the perpendicular AG bisects BC, that is unless $B = C$. Similarly we see, by considering the side AC instead of BC, that it will be necessary to have $A = C$. Thus the triangle must be equilateral.

Problem 30. *The centre of the nine points circle cannot coincide with the centre of the inscribed circle unless the triangle is equilateral.*

Solution. Take the diagram of *Art.* 332 (page 448). The centre of inscribed circle is on the straight line AE, which bisects the angle A; and the centre of the nine points circle is on DF : hence when the two coincide it must be at the point H. Thus, by *Problem* 25;, we must have H at the middle point of OP. Then from the similar triangle OHE and AHP we find that OE must be equal to AP, that is to twice AF. Thus $R = 2R\cos A$; therefore $A = 60°$. Similarly we see by considering the side AC instead of BC, that $B = 60°$. Hence the triangle must be equilateral.

Or we may use *Problem* 25; thus we must have

$$r = \frac{R}{2}\cos(B - C) = \frac{R}{2}\cos(C - A) = \frac{R}{2}\cos(A - B);$$

so that
$$\cos(B - C) = \cos(C - A) = \cos(A - B) :$$
these lead to $A = B = C$.

CHAPTER XXV
Miscellaneous Examples

Problem 1. *If an angle of 3° be represented by .15, find how many degrees are contained in the unit of that measure. Find also what number will represent a right angle in the same measure.*

Solution. Let x denote the number of degrees in the unit. Then $3 : x :: .15 : 1$. Hence $x = \dfrac{3}{.15} = 20$. The measure of a right angle will be $\dfrac{90}{20}$, that is $4\frac{1}{2}$.

Problem 2. *The difference of two angles is 1°; the circular measure of their sum is 1. Find the circular measure of each angle.*

Solution. Let x denote the circular measure of the larger angle, y that of the smaller angle. Then, since the circular measure of 1° is $\dfrac{\pi}{180}$, we have $x - y = \dfrac{\pi}{180}$, $x + y = 1$. Hence $x = \dfrac{1}{2}\left(1 + \dfrac{\pi}{180}\right)$, $y = \dfrac{1}{2}\left(1 - \dfrac{\pi}{180}\right)$.

Problem 3. *Find* $\tan x$ *from the equation* $\tan x + ab \cot x = a + b$.

Solution. Here $\tan x + \dfrac{ab}{\tan x} = a + b$; therefore $\tan^2 x - (a + b)\tan x + ab = 0$. By solving this quadratic equation we obtain $\tan x = a$, or $\tan x = b$.

Problem 4. *If* $\sin 3\theta = \sin\theta\cos 2\theta$ *then* $\theta = \dfrac{n\pi}{2}$ *where n is zero or an integer.*

Solution. Here $\sin(2\theta + \theta) = \sin\theta\cos 2\theta$, that is

$$\sin 2\theta\cos\theta + \cos 2\theta\sin\theta = \sin\theta\cos 2\theta;$$

therefore
$$\sin 2\theta\cos\theta = 0, \text{ that is } 2\sin\theta\cos^2\theta = 0.$$

If $\cos\theta = 0$ we have θ an odd multiple of $\dfrac{\pi}{2}$; and if $\sin\theta = 0$ we have θ an even multiple of $\dfrac{\pi}{2}$: hence all the solutions are comprised in $\theta = n\dfrac{\pi}{2}$, where n is zero or an integer.

Problem 5. *If an angle be divided into two equal and also into two unequal parts, the product of the sines of the unequal parts together with the square of the sine of the angle between the dividing straight lines is equal to the square of the sine of half the angle.*

Solution. Let $2A$ denote the whole angle, and $A + x$ one of the two unequal parts; then $A - x$ denotes the other. Hence we have to show that

$$\sin(A + x)\sin(A - x) + \sin^2 x = \sin^2 A;$$

and this is obvious by *Art.* 83 (page 404).

Problem 6. *Show that*

$$\frac{(\sec\theta\sec\phi+\tan\theta\tan\phi)^2-(\tan\theta\sec\phi+\sec\theta\tan\phi)^2}{2\left(1+\tan^2\theta\tan^2\phi\right)-\sec^2\theta\sec^2\phi}$$
$$=\frac{\sec 2\theta\sec 2\phi}{\sec^2\theta\sec^2\phi}.$$

Solution. $(\sec\theta\sec\phi+\tan\theta\tan\phi)^2-(\tan\theta\sec\phi+\sec\theta\tan\phi)^2$

$$=sec^2\theta\sec^2\phi+\tan^2\theta\tan^2\phi-\tan^2\theta\sec^2\phi-\sec^2\theta\tan^2\phi$$
$$=\sec^2\phi(\sec^2\theta-\tan^2\theta)-\tan^2\phi(\sec^2\theta-\tan^2\theta)$$
$$=\sec^2\phi-\tan^2\phi=1.$$

$$2(1+\tan^2\theta\tan^2\phi)-\sec^2\theta\sec^2\phi=\frac{2(\cos^2\theta\cos^2\phi+\sin^2\phi\sin^2\theta)-1}{\cos^2\theta\cos^2\phi}$$
$$=\frac{(1+\cos 2\theta)\cos^2\phi+(1-\cos 2\theta)\sin^2\phi-1}{\cos^2\theta\cos^2\phi}$$
$$=\frac{\cos 2\theta(\cos^2\phi-\sin^2\phi)}{\cos^2\theta\cos^2\phi}=\frac{\cos 2\theta\cos 2\phi}{\cos^2\theta\cos^2\phi}.$$

And $$1\div\frac{\cos 2\theta\cos 2\phi}{\cos^2\theta\cos^2\phi}=\frac{\cos^2\theta\cos^2\phi}{\cos 2\theta\cos 2\phi}=\frac{\sec 2\theta\sec 2\phi}{\sec^2\theta\sec^2\phi}.$$

Problem 7. *If $A+B+C=360°$, show that*

$$1-\cos^2 A-\cos^2 B-\cos^2 C+2\cos A\cos B\cos C=0.$$

Solution. Since $A+B+C=360°$, we have $\cos C=\cos(A+B)$.

Thus $$1-\cos^2 A-\cos^2 B-\cos^2 C+2\cos A\cos B\cos C$$
$$=1-\cos^2 A-\cos^2 B+\cos C(2\cos A\cos B-\cos C)$$
$$=1-\cos^2 A-\cos^2 B+\cos(A+B)(\cos A\cos B+\sin A\sin B)$$
$$=1-\cos^2 A-\cos^2 B+(\cos A\cos B-\sin A\sin B)(\cos A\cos B+\sin A\sin B)$$
$$=1-\cos^2 A-\cos^2 B+\cos^2 A\cos^2 B-\sin^2 A\sin^2 B$$
$$=1-\cos^2 A-\cos^2 B+\cos^2 A\cos^2 B-(1-\cos^2 A)(1-\cos^2 B)$$
$$=0.$$

Problem 8. *If $\sin A=\frac{3}{5}$, $\sin B=\frac{12}{13}$, and $\sin C=\frac{7}{25}$, where A, B, and C are positive angles less than 90°, find* $\sin(A+B+C)$.

Solution.

$$\sin A=\frac{3}{5};\ \text{therefore}\ \cos A=\frac{4}{5}.$$
$$\sin B=\frac{12}{13};\ \text{therefore}\ \cos B=\frac{5}{13}.$$
$$\sin C=\frac{7}{25};\ \text{therefore}\ \cos C=\frac{24}{25}.$$

Hence we obtain $\sin(A+B)=\frac{63}{65}$, $\cos(A+B)=-\frac{16}{65}$;

then $$\sin(A+B+C)=\frac{63\times 24-7\times 16}{25\times 65}=\frac{1400}{25\times 65}=\frac{56}{65}.$$

Problem 9. *If* $x = r\sin\frac{1}{2}(\theta-\alpha)$ *and* $y = r\sin\frac{1}{2}(\theta+\alpha)$, *show that*

$$x^2 - 2xy\cos\alpha + y^2 = r^2\sin^2\alpha.$$

Solution. $x = r\left(\sin\frac{\theta}{2}\cos\frac{\alpha}{2} - \cos\frac{\theta}{2}\sin\frac{\alpha}{2}\right)$, $y = r\left(\sin\frac{\theta}{2}\cos\frac{\alpha}{2} + \cos\frac{\theta}{2}\sin\frac{\alpha}{2}\right)$.

From these we obtain $\sin\frac{\theta}{2} = \dfrac{x+y}{2r\cos\frac{\alpha}{2}}$, $\quad \cos\frac{\theta}{2} = \dfrac{y-x}{2r\sin\frac{\alpha}{2}}$

Square and add; thus $1 = \dfrac{1}{4r^2}\left\{\dfrac{(x+y)^2}{\cos^2\frac{\alpha}{2}} + \dfrac{(y-x)^2}{\sin^2\frac{\alpha}{2}}\right\}$;

therefore $\quad 4r^2\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2} = (x+y)^2\sin^2\frac{\alpha}{2} + (y-x)^2\cos^2\frac{\alpha}{2}$;

that is $\quad r^2\sin^2\alpha = x^2 + y^2 - 2xy\left(\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\right) = x^2 + y^2 - 2xy\cos\alpha.$

Problem 10. *Eliminate* θ *from the equations*

$$(a-b)\sin(\theta+\phi) = (a+b)\sin(\theta-\phi), \quad a\tan\frac{\theta}{2} - b\tan\frac{\phi}{2} = c.$$

Solution. Here $\dfrac{\sin(\theta+\phi)}{\sin(\theta-\phi)} = \dfrac{a+b}{a-b}$, therefore $\dfrac{\sin(\theta+\phi)+\sin(\theta-\phi)}{\sin(\theta+\phi)-\sin(\theta-\phi)} = \dfrac{a}{b}$,

that is $\quad \dfrac{\sin\theta\cos\phi}{\cos\theta\sin\phi} = \dfrac{a}{b}$; so that $a\tan\phi = b\tan\theta$.

Hence
$$\frac{a\tan\frac{\phi}{2}}{1-\tan^2\frac{\phi}{2}} = \frac{b\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}};$$

$$\therefore\ a\tan\frac{\phi}{2} - b\tan\frac{\theta}{2} = \tan\frac{\theta}{2}\tan\frac{\phi}{2}\left(a\tan\frac{\theta}{2} - b\tan\frac{\phi}{2}\right)$$
$$= c\tan\frac{\theta}{2}\tan\frac{\phi}{2};$$

$$\therefore\ a\tan\frac{\phi}{2} = \tan\frac{\theta}{2}\left(b + c\tan\frac{\phi}{2}\right).$$

Substitute for $\tan\frac{\theta}{2}$ from the second of the given equations, and we obtain

$$a^2\tan\frac{\phi}{2} = \left(b + c\tan\frac{\phi}{2}\right)\left(c + b\tan\frac{\phi}{2}\right).$$

Problem 11. *The number of dgrees in one of the acute angles of a right-angled triangle is three-tenths of the number of grades in the other : determine the angles in degrees.*

Solution. Let x denote the number of degrees in one angle; then $90-x$ denotes the number of degrees in the other angle, and consequently $\frac{10}{9}(90-x)$ the number of grades. Hence $x = \frac{3}{10}\times\frac{10}{9}(90-x) = \frac{1}{3}(90-x)$. Therefore $4x = 90$, and $x = 22\frac{1}{2}$.

Problem 12. *Show that if the circular measure of an angle is* $\frac{n\pi}{20}$, *where n is any integer, the angle can be expressed by an integer both in degrees and in grades.*

Solution. Let the circular measure of an angle be $\frac{n\pi}{20}$; then the number of degrees in it is $\frac{n\pi}{20} \cdot \frac{180}{\pi}$, that is $9n$; and the number of grades is $\frac{n\pi}{20} \cdot \frac{200}{\pi}$, that is $10n$.

Problem 13. *If* $\sin(\alpha+\beta)\cos\gamma = \sin(\alpha+\gamma)\cos\beta$, *show that* $\beta-\gamma$ *is a multiple of* π, *or* α *an odd multiple of* $\frac{\pi}{2}$.

Solution. Here $(\sin\alpha\cos\beta + \cos\alpha\sin\beta)\cos\gamma = (\sin\alpha\cos\gamma + \cos\alpha\sin\gamma)\cos\beta$;
therefore
$$\cos\alpha(\sin\beta\cos\gamma - \sin\gamma\cos\beta) = 0,$$
that is
$$\cos\alpha\sin(\beta-\gamma) = 0.$$

Either then $\cos\alpha = 0$, so that α is an odd multiple of $\frac{\pi}{2}$; or $\sin(\beta-\gamma) = 0$, so that $\beta-\gamma$ is a multiple of π.

Problem 14. *Show that*
$$\sin 4A\tan^4 A + 4\tan^3 A + 2\sin 4A\tan^2 A - 4\tan A + \sin 4A = 0.$$

Solution.
$$\sin 4A(\tan^4 A + 2\tan^2 A + 1) = \sin 4A(\tan^2 A + 1)^2$$
$$= \frac{\sin 4A}{\cos^4 A} = \frac{2\sin 2A\cos 2A}{\cos^4 A} = \frac{4\sin A\cos 2A}{\cos^3 A}$$
$$= \frac{4\sin A(\cos^2 A - \sin^2 A)}{\cos^3 A}$$
And
$$4\tan^3 A - 4\tan A = 4\tan A(\tan^2 A - 1)$$
$$= \frac{4\sin A(\sin^2 A - \cos^2 A)}{\cos^3 A}.$$
Therefore
$$\sin A(\tan^4 A + 2\tan^2 A + 1) + 4\tan^3 A - 4\tan A = 0.$$

Problem 15. *Show that* $\sin^2 24° - \sin^2 6° = \frac{\sqrt{5}-1}{8}$.

Solution.

By *Art.* 83 (page 404), $\sin^2 24° - \sin^2 6° = \sin(24° + 6°)\sin(24° - 6°)$
$$= \sin 30°\sin 18°.$$
Also
$$\sin 30° = \frac{1}{2}, \text{ and } \sin 18° = \frac{\sqrt{5}-1}{4}.$$

Problem 16. *If* $A + B + C = 360°$, *show that*
$$2(\cos A\sin B\sin C + \cos B\sin C\sin A + \cos C\sin A\sin B)$$
$$+ \sin^2 A + \sin^2 B + \sin^2 C = 0.$$

Solution. The given expression is
$$\sin A(\sin A + \cos B\sin C + \cos C\sin B)$$
$$+ \sin B(\sin B + \cos C\sin A + \cos A\sin C)$$

$$+\sin C(\sin C+\cos A\sin B+\cos B\sin A),$$

that is $\quad \sin A\{\sin A+\sin(B+C)\}+\sin B\{\sin B+\sin(C+A)\}$
$$+\sin C\{\sin C+\sin(A+B)\}.$$

Now since $A+B+C=360°$, we have

$\sin(B+C)=-\sin A,\quad \sin(C+A)=-\sin B,\quad \sin(A+B)=-\sin C:$
thus the whole expression vanishes.

Problem 17. *If α and β are the two values of θ in the equation*
$$\frac{\cos\theta\cos\gamma}{a}+\frac{\sin\theta\sin\gamma}{b}=\frac{1}{c},$$
Show that
$$(b^2+c^2-a^2)\cos\alpha\cos\beta+(a^2+c^2-b^2)\sin\alpha\sin\beta=a^2+b^2-c^2.$$

Solution. We have
$$\frac{\cos\alpha\cos\gamma}{a}+\frac{\sin\alpha\sin\gamma}{b}=\frac{1}{c},\text{ and }\frac{\cos\beta\cos\gamma}{a}+\frac{\sin\beta\sin\gamma}{b}=\frac{1}{c}.$$
From these equations we find $\cos\gamma$ and $\sin\gamma$. We get
$$\cos\gamma=\frac{a(\sin\beta-\sin\alpha)}{c(\cos\alpha\sin\beta-\cos\beta\sin\alpha)}=\frac{2a\sin\frac{1}{2}(\beta-\alpha)\cos\frac{1}{2}(\beta+\alpha)}{c\sin(\beta-\alpha)}$$
$$=\frac{a\cos\frac{1}{2}(\beta+\alpha)}{c\cos\frac{1}{2}(\beta-\alpha)};$$
$$\sin\gamma=\frac{b(\cos\alpha-\cos\beta)}{c(\cos\alpha\sin\beta-\cos\beta\sin\alpha)}=\frac{2b\sin\frac{1}{2}(\beta+\alpha)\sin\frac{1}{2}(\beta-\alpha)}{c\sin(\beta-\alpha)}$$
$$=\frac{b\sin\frac{1}{2}(\beta+\alpha)}{c\cos\frac{1}{2}(\beta-\alpha)}$$
Square, and add; thus
$$1=\frac{a^2\cos^2\frac{1}{2}(\beta+\alpha)+b^2\sin^2\frac{1}{2}(\beta+\alpha)}{c^2\cos^2\frac{1}{2}(\beta-\alpha)}$$
$$\therefore\ c^2\{1+\cos(\beta-\alpha)\}=a^2\{1+\cos(\beta+\alpha)\}+b^2\{1-\cos(\beta+\alpha)\};$$
$$\therefore\ (b^2+c^2-a^2)\cos\alpha\cos\beta+(a^2+c^2-b^2)\sin\alpha\sin\beta=a^2+b^2-c^2.$$

Problem 18. *If $\sin A=\frac{1}{3}$, and $\sin B=\frac{1}{2}$, where A and B are positive angle less than 90°, find $\sin 2(A+B)$.*

Solution. $\qquad \sin A=\frac{1}{3}$; therefore $\cos 2A=1-\frac{2}{9}=\frac{7}{9}$,

and
$$\sin 2A=\sqrt{\left(1-\frac{49}{81}\right)}=\frac{\sqrt{32}}{9}.$$
$$\sin B=\frac{1}{2};\text{ therefore }\cos 2B=1-\frac{1}{2}=\frac{1}{2},$$

and $$\sin 2B = \sqrt{\left(1-\frac{1}{4}\right)} = \frac{\sqrt{3}}{2}.$$

Hence $$\sin(2A+2B) = \frac{\sqrt{32}+7\sqrt{3}}{18} = \frac{4\sqrt{2}+7\sqrt{3}}{18}.$$

Problem 19. *Solve the equation* $\cos 4x + \cos 2x + \cos x = 0$.

Solution. $$\cos 4x + \cos 2x + \cos x = 0;$$

therefore $$2\cos 3x \cos x + \cos x = 0;$$

therefore either $\cos x = 0$ or $2\cos 3x + 1 = 0$.

If $\cos x = 0$, then $x = (2n+1)\frac{\pi}{2}$.

If $\cos 3x = -\frac{1}{2}$ then $3x = 2n\pi \pm \frac{2\pi}{3}$.

Problem 20. *If* $A+B+C+D = 360°$. *show that*

$$\cos A + \cos B + \cos C + \cos D = 4\cos\frac{A+B}{2}\cos\frac{B+C}{2}\cos\frac{C+A}{2}.$$

Solution.

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2};$$

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$= -2\cos\frac{A+B}{2}\cos\frac{C-D}{2},$$ by *Art.* 48 (page 397).

Hence, by addition,

$$\cos A + \cos B + \cos C + \cos D = 2\cos\frac{A+B}{2}\left\{\cos\frac{A-B}{2} - \cos\frac{C-D}{2}\right\}$$

$$= 4\cos\frac{A+B}{2}\sin\frac{A+C-B-D}{4}\sin\frac{C+B-A-D}{4}.$$

Also

$$\sin\frac{A+C-B-D}{4} = \sin\frac{2A+2C-360°}{4}$$

$$= \sin\left(\frac{A+C}{2} - 90°\right) = -\cos\frac{A+C}{2};$$

and in like manner $\sin\frac{C+B-A-D}{4} = -\cos\frac{B+C}{2}$.

Thus we obtain finally $4\cos\frac{A+B}{2}\cos\frac{B+C}{2}\cos\frac{C+A}{2}$.

Problem 21. *With two units of angular measurement differing by 10° the measures of an angle are as 3 is to 2: determine those units.*

Solution. Suppose that the smaller unit contains x degrees, and therefore the larger unit $x+10$ degrees. Let n denote the number of degrees in the angle measured; then $\frac{n}{x}$ is to $\frac{n}{x+10}$ as 3 is to 2. Therefore $\frac{2}{x} = \frac{3}{x+10}$; whence $x = 20$.

Problem 22. *If* $\sin x + \sin^2 x = 1$, *find* $\sin x$; *and show that*
$$\cos^2 x + \cos^4 x = 1.$$

Solution. $\sin^2 x + \sin x = 1$. Solving this quadratic in the ordinary way we obtain $\sin x = \dfrac{-1 \pm \sqrt{5}}{2}$; the upper sign must be taken, as the lower would make $\sin x$ numerically greater than unity.

Thus $$\sin^2 x = \frac{6 - 2\sqrt{5}}{4};$$

therefore $$\cos^2 x = 1 - \frac{6 - 2\sqrt{5}}{4} = \frac{-2 + 2\sqrt{5}}{4} = \frac{-1 + \sqrt{5}}{2};$$

therefore $$\cos^4 x = \frac{6 - 2\sqrt{5}}{4} = \frac{3 - \sqrt{5}}{2};$$

therefore $$\cos^2 x + \cos^4 x = 1.$$

Or thus : $$\sin x = 1 - \sin^2 x = \cos^2 x; \text{ square;}$$

therefore $$\sin^2 x = \cos^4 x, \text{ that is } 1 - \cos^2 x = \cos^4 x;$$

therefore $$1 = \cos^2 x + \cos^4 x.$$

Problem 23. *Solve the equation* $\tan^2 x + \cot^2 x = 2$.

Solution. Here $$\tan^2 x + \frac{1}{\tan^2 x} = 2;$$

therefore $$\tan^4 x - 2\tan^2 x + 1 = 0;$$

hence $$\tan^2 x = 1; \text{ therefore } \tan x = \pm 1;$$

therefore $$x = n\pi \pm \frac{\pi}{4}.$$

Problem 24. *If* $a \sin\theta + b\cos\theta = c = a \operatorname{cosec}\theta + b\sec\theta$, *show that*
$$\sin 2\theta = \frac{2ab}{c^2 - a^2 - b^2}.$$

Solution. We have $a\sin\theta + b\cos\theta = c$, $\quad \dfrac{a\cos\theta + b\sin\theta}{\sin\theta\cos\theta} = c$;

hence $$(a\sin\theta + b\cos\theta)(a\cos\theta + b\sin\theta) = c^2 \sin\theta\cos\theta;$$

therefore $$(a^2 + b^2)\sin\theta\cos\theta + ab = c^2\sin\theta\cos\theta;$$

therefore $$\sin 2\theta(c^2 - a^2 - b^2) = 2ab.$$

Problem 25. *Simplify* $\cos^2(A + B) + \cos^2(A - B) - \cos 2A\cos 2B$.

Solution.
$$\cos^2(A+B) + \cos^2(A-B) = \frac{1 + \cos(2A + 2B)}{2} + \frac{1 + \cos(2A - 2B)}{2}$$
$$= 1 + \cos 2A\cos 2B;$$

therefore $$\cos^2(A+B) + \cos^2(A-B) - \cos 2A\cos 2B = 1.$$

Problem 26. *If* $2\tan A = 3\tan B$, *then*

$$\tan(A-B) = \frac{\tan B}{2+3\tan^2 B} = \frac{\sin 2B}{5-\cos 2B}.$$

Solution.

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B} = \frac{\frac{3}{2}\tan B - \tan B}{1+\frac{3}{2}\tan^2 B}$$

$$= \frac{\tan B}{2+3\tan^2 B} = \frac{\sin B\cos B}{2\cos^2 B + 3\sin^2 B}$$

$$= \frac{\sin 2B}{2(1+\cos 2B)+3(1-\cos 2B)}$$

$$= \frac{\sin 2B}{5-\cos 2B}.$$

Problem 27. *Solve the equation*

$$\sin\frac{n+1}{2}\theta + \sin\frac{n-1}{2}\theta = \sin\theta.$$

Solution.

$$\sin\frac{n+1}{2}\theta + \sin\frac{n-1}{2}\theta = \sin\theta,$$

therefore

$$2\sin\frac{n\theta}{2}\cos\frac{\theta}{2} = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}.$$

Thus either $\cos\frac{\theta}{2} = 0$, or $\sin\frac{n\theta}{2} = \sin\frac{\theta}{2}$.

From the former we have $\frac{\theta}{2} = (2m+1)\frac{\pi}{2}$. All the solutions of the latter are comprised in $\frac{n\theta}{2} = m\pi + (-1)^m\frac{\theta}{2}$, where m is zero or an integer.

Problem 28. *If*

$$\tan(2\alpha - 3\beta) = \cot(3\alpha - 2\beta), \text{ and } \tan(2\alpha+3\beta) = \cot(3\alpha+2\beta),$$

show that both α *and* β *are multiples of* $\frac{\pi}{10}$.

Solution.

Here

$$\tan(2\alpha-3\beta) = \tan\left(\frac{\pi}{2} - 3\alpha + 2\beta\right),$$

and

$$\tan(2\alpha+3\beta) = \tan\left(\frac{\pi}{2} - 3\alpha - 2\beta\right).$$

Hence all possible solutions are comprised in

$$2\alpha - 3\beta = m\pi + \frac{\pi}{2} - 3\alpha + 2\beta, \text{ and } 2\alpha+3\beta = n\pi + \frac{\pi}{2} - 3\alpha - 2\beta,$$

where m and n are zero or integers.

From these we obtain $\alpha = (m+n+1)\frac{\pi}{10}$, $\quad \beta = (n-m)\frac{\pi}{10}$,

so that α and β are multiples of $\frac{\pi}{10}$.

Problem 29. *Solve the equation*

$$\tan(\alpha+x)\tan(\alpha-x)=\frac{1-2\cos 2\alpha}{1+2\cos 2\alpha}.$$

Solution. Here $$\frac{\sin(\alpha+x)\sin(\alpha-x)}{\cos(\alpha+x)\cos(\alpha-x)}=\frac{1-2\cos 2\alpha}{1+2\cos 2\alpha};$$

therefore, by *Art.* 83 (page 404), $$\frac{\sin^2\alpha-\sin^2 x}{\cos^2\alpha-\sin^2 x}=\frac{1-2\cos 2\alpha}{1+2\cos 2\alpha};$$

therefore $$4\cos 2\alpha\sin^2 x=\sin^2\alpha(1+2\cos 2\alpha)-\cos^2\alpha(1-2\cos 2\alpha)$$
$$=-\cos 2\alpha+2\cos 2\alpha=\cos 2\alpha;$$

therefore $\sin^2 x=\frac{1}{4}$; therefore $\sin x=\pm\frac{1}{2}$; therefore $x=n\pi\pm\frac{\pi}{6}$.

Problem 30. *If $A+B+C+D=360°$, show that*

$$\sin A+\sin B+\sin C+\sin D=4\sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+A}{2}.$$

Solution.

$$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2};$$
$$\sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$$
$$=2\sin\frac{A+B}{2}\cos\frac{C-D}{2},\ \text{by } Art.\ 48 \text{ (page 397)}.$$

Hence, by addition,

$$\sin A+\sin B+\sin C+\sin D=2\sin\frac{A+B}{2}\left\{\cos\frac{A-B}{2}+\cos\frac{C-D}{2}\right\}$$
$$=4\sin\frac{A+B}{2}\cos\frac{A+C-B-D}{4}\cos\frac{A+D-B-C}{4}.$$

Then, as in *Problem* 20, we can show that

$$\cos\frac{A+C-B-D}{4}=\sin\frac{A+C}{2},\ \text{and}\ \cos\frac{A+D-B-C}{4}=\sin\frac{B+C}{2}.$$

Problem 31. *One angle of a quadrilateral contains 60 degrees, another contains 50 grades, the circular measure of another is $\frac{3\pi}{4}$. Express all the four angles in degrees.*

Solution. The first angle contains 60 degrees; the second angle contains $\frac{9}{10}\times 50$ degrees, that is 45 degrees; the third angle contains $\frac{3\pi}{4}\times\frac{180}{\pi}$ degrees, that is 135 degrees. Therefore the fourth angle must contain $360-60-45-135$ degrees, that is 120 degrees.

Problem 32. *If $\cos A=\frac{40}{41}$ and $\cos B=\frac{60}{61}$, where A and B are angles less than a right angle, show that $\sin^2\frac{A-B}{2}=\frac{1}{41\times 61}$.*

Solution. $\sin A = \sqrt{1 - \left(\frac{40}{41}\right)^2} = \frac{\sqrt{(41-40)(41+40)}}{41} = \frac{\sqrt{81}}{41} = \frac{9}{41};$

$\sin B = \sqrt{1 - \left(\frac{60}{61}\right)^2} = \frac{\sqrt{(61-60)(61+60)}}{61} = \frac{\sqrt{121}}{61} = \frac{11}{61}.$

$\cos(A-B) = \cos A \cos B + \sin A \sin B = \frac{40 \times 60 + 9 \times 11}{41 \times 61} = \frac{2499}{2501};$

thus $$1 - 2\sin^2 \frac{1}{2}(A-B) = \frac{2499}{2501};$$

therefore $$2\sin^2 \frac{1}{2}(A-B) = \frac{2}{41 \times 61};$$

therefore $$\sin^2 \frac{1}{2}(A-B) = \frac{1}{41 \times 61}.$$

Problem 33. *Solve the equation* $\sin 3\theta = 8\sin^3\theta$.

Solution. Here $$3\sin\theta - 4\sin^3\theta = 8\sin^3\theta;$$
therefore $$3\sin\theta = 12\sin^3\theta;$$

therefore either $$\sin\theta = 0, \text{ or } \sin^2\theta = \frac{1}{4}:$$

the former gives $\theta = n\pi$; the latter gives $\theta = n\pi \pm \frac{\pi}{6}$.

Problem 34. *Eliminate* θ *and* ϕ *from the equations*
$$a^2\cos^2\theta - b^2\cos^2\phi = c^2, \quad a\cos\theta + b\cos\phi = r, \quad a\tan\theta = b\tan\phi.$$

Solution. Divide the first by the second; thus we get
$$a\cos\theta - b\cos\phi = \frac{c^2}{r};$$

therefore $$\cos\theta = \frac{1}{2a}\left(r + \frac{c^2}{r}\right), \quad \cos\phi = \frac{1}{2b}\left(r - \frac{c^2}{r}\right).$$

Now $\frac{a^2\sin^2\theta}{\cos^2\theta} = \frac{b^2\sin^2\phi}{\cos\phi}$; therefore $a^2(sec^2\theta - 1) = b^2(\sec^2\phi - 1);$

therefore $$a^2\left\{\frac{4r^2a^2}{(r^2+c^2)^2 - 1}\right\} = b^2\left\{\frac{4r^2b^2}{(r^2-c^2)^2} - 1\right\}.$$

Problem 35. *If* $\tan A$, $\tan B$, $\tan C$ *are in Arithmetical Progression, and* $\tan A$, $\tan B$, $\tan D$ *in Harmonical Progression, then*
$$\frac{\tan C}{\tan D} = 1 - \frac{8\sin^2(A-B)}{\sin 2A \sin 2B}.$$

Solution. Here $\tan A + \tan C = 2\tan B$, and $\frac{1}{\tan A} + \frac{1}{\tan D} = \frac{2}{\tan B};$

therefore $\frac{\tan C}{\tan D} = (2\tan B - \tan A)\left(\frac{2}{\tan B} - \frac{1}{\tan A}\right) = 5 - 2\left(\frac{\tan B}{\tan A} + \frac{\tan A}{\tan B}\right)$

$$= 5 - 2\left(\frac{\sin B\cos A}{\cos B\sin A} + \frac{\sin A\cos B}{\cos A\sin B}\right) = 5 - 2\frac{\sin^2 B\cos^2 A + \sin^2 A\cos^2 B}{\sin A\cos A\sin B\cos B}$$

$$= 1 - \frac{2(\sin A \cos B - \cos A \sin B)^2}{\sin A \cos A \sin B \cos B} = 1 - \frac{8 \sin^2(A - B)}{\sin 2A \sin 2B}.$$

Problem 36. *Find* $\cos x$ *from the equation* $\cos x \sin^2 x = \sin \alpha \cos^2 \alpha$, *having given as one solution* $\cos x = \sin \alpha$.

Solution. Here $\cos x(1 - \cos^2 x) = \sin \alpha(1 - \sin^2 \alpha)$; therefore

$$\cos x - \sin \alpha = \cos^3 x - \sin^3 \alpha = (\cos x - \sin \alpha)(\cos^2 x + \cos x \sin \alpha + \sin^2 \alpha);$$

therefore either $\cos x - \sin \alpha = 0$, or $1 = \cos^2 x + \cos x \sin \alpha + \sin^2 \alpha$.

The latter gives $\cos^2 x + \cos x \sin \alpha = \cos^2 \alpha$; by solving this quadratic equation we obtain

$$\cos x = \frac{-\sin \alpha \pm \sqrt{\sin^2 \alpha + 4 \cos^2 \alpha}}{2};$$

it will be found that only one of these values is numerically less than unity, namely, the numerically less of the two.

Problem 37. *Show that*

$$(2\cos\theta - 1)(2\cos 2\theta - 1)\left(2\cos 2^2\theta - 1\right)\dots\left(2\cos 2^{n-1}\theta - 1\right) = \frac{2\cos 2^n\theta + 1}{2\cos\theta + 1}.$$

Solution.

We have
$$2\cos\theta - 1 = \frac{4\cos^2\theta - 1}{2\cos\theta + 1} = \frac{2\cos 2\theta + 1}{2\cos\theta + 1},$$

$$2\cos 2\theta - 1 = \frac{4\cos^2 2\theta - 1}{2\cos 2\theta + 1} = \frac{2\cos 4\theta + 1}{2\cos 2\theta + 1},$$

and so on, which we use down to

$$2\cos 2^{n-1}\theta - 1 = \frac{4\cos^2 2^{n-1}\theta - 1}{2\cos 2^{n-1}\theta + 1} = \frac{2\cos 2^n\theta + 1}{2\cos 2^{n-1}\theta + 1}.$$

Multiply these expressions together, then by cancelling we obtain the required result.

Problem 38. *If* $\tan(\pi \cot\theta) = \cot(\pi \tan\theta)$, *show that*

$$\tan\theta = \frac{2n + 1}{4} \pm \frac{\sqrt{4n^2 + 4n - 15}}{4},$$

where n *is a positive or negative integer.*

Solution. Here
$$\tan(\pi\cot\theta) = \tan\left(\frac{\pi}{2} - \pi\tan\theta\right);$$

hence all possible solutions are comprised in the formula

$$\pi\cot\theta = n\pi + \frac{\pi}{2} - \pi\tan\theta;$$

thus
$$\tan^2\theta - \left(n + \frac{1}{2}\right)\tan\theta + 1 = 0;$$

by solving this quadratic equation we obtain the value of $\tan\theta$.

Problem 39. *Express in factors*
$$\cos^2 A + \cos^2 B + \cos^2 C - 2 \cos A \cos B \cos C - 1.$$

Solution. $\cos^2 A + \cos^2 B + \cos^2 C - 2\cos A \cos B \cos C - 1$

$$\begin{aligned}
&= (\cos A - \cos B \cos C)^2 + \cos^2 B + \cos^2 C - 1 - \cos^2 B \cos^2 C \\
&= (\cos A - \cos B \cos C)^2 - (1 - \cos^2 B)(1 - \cos^2 C) \\
&= (\cos A - \cos B \cos C)^2 - \sin^2 B \sin^2 C \\
&= (\cos A - \cos B \cos C + \sin B \sin C)(\cos A - \cos B \cos C - \sin B \sin C) \\
&= \{\cos A - \cos(B + C)\}\{\cos A - \cos(B - C)\} \\
&= 4 \sin \frac{A+B+C}{2} \sin \frac{B+C-A}{2} \sin \frac{A+B-C}{2} \sin \frac{B-C-A}{2}. \\
&= -4 \sin \frac{A+B+C}{2} \sin \frac{B+C-A}{2} \sin \frac{A+C-B}{2} \sin \frac{A+B-C}{2}.
\end{aligned}$$

Problem 40. *If* $A + B + C + D = 360°$, *show that*
$$\sin A - \sin B + \sin C - \sin D = 4 \cos \frac{A+B}{2} \cos \frac{B+C}{2} \sin \frac{C+A}{2}.$$

Solution.
$$\sin A - \sin B = 2 \sin \frac{A-B}{2} \cos \frac{A+B}{2};$$
$$\sin C - \sin D = 2 \sin \frac{C-D}{2} \cos \frac{C+D}{2}$$
$$= -2 \sin \frac{C-D}{2} \cos \frac{A+B}{2},$$ by *Art.* 48 (page 397).

Hence, by addition,
$$\sin A - \sin B + \sin C - \sin D = 2 \cos \frac{A+B}{2} \left\{ \sin \frac{A-B}{2} - \sin \frac{C-D}{2} \right\}$$
$$= 4 \cos \frac{A+B}{2} \sin \frac{A+D-B-C}{4} \cos \frac{A+C-B-D}{4}.$$
Then, as in *Problem* 20, we can show that
$\sin \frac{A+D-B-C}{4} = \cos \frac{B+C}{2}$, and $\cos \frac{A+C-B-D}{4} = \sin \frac{A+C}{2}$.

Problem 41. *Express in each system of angular measurement the angle described by the minute hand of a watch in 25 minutes.*

Solution. The angle described is $\frac{25}{60}$ of four right angles; the number of degrees $= \frac{5}{12} \times 360 = 150$; the number of grades $= \frac{5}{12} \times 400 = 166\frac{2}{3}$; the circular measure $= \frac{5}{12} \times 2\pi = \frac{5\pi}{6}$.

Problem 42. *Show that*
$$(\cos \theta + \sin \theta)(\cos 2\theta + \sin 2\theta) = \cos \theta + \cos\left(3\theta - \frac{\pi}{2}\right).$$

Solution. $\cos\theta + \sin\theta = \sqrt{2}\left(\dfrac{\cos\theta}{\sqrt{2}} + \dfrac{\sin\theta}{\sqrt{2}}\right) = \sqrt{2}\cdot\cos\left(\theta - \dfrac{\pi}{4}\right)$;

similarly $\qquad \cos 2\theta + \sin 2\theta = \sqrt{2}\cos\left(2\theta - \dfrac{\pi}{4}\right)$:

the product $= 2\cos\left(\theta - \dfrac{\pi}{4}\right)\cos\left(2\theta - \dfrac{\pi}{4}\right) = \cos\theta + \cos\left(3\theta - \dfrac{\pi}{2}\right)$.

Problem 43. *Show that*

$$\operatorname{cosec} A \operatorname{cosec} 2A + \operatorname{cosec} 2A \operatorname{cosec} 3A = \operatorname{cosec} A\,(\cot A - \cot 3A)\,.$$

Solution. $\operatorname{cosec} 2A(\operatorname{cosec} A + \operatorname{cosec} 3A) = \dfrac{1}{\sin 2A}\cdot\dfrac{\sin A + \sin 3A}{\sin A \sin 3A}$

$$= \frac{1}{\sin 2A}\cdot\frac{2\sin 2A\cos A}{\sin A\sin 3A} = \frac{2\cos A}{\sin A\sin 3A};$$

and $\qquad \operatorname{cosec} A(\cot A - \cot 3A) = \dfrac{1}{\sin A}\left(\dfrac{\cos A}{\sin A} - \dfrac{\cos 3A}{\sin 3A}\right)$

$$= \frac{1}{\sin A}\cdot\frac{\sin 3A\cos A - \cos 3A\sin A}{\sin A\sin 3A} = \frac{1}{\sin A}\cdot\frac{\sin(3A - A)}{\sin A\sin 3A}$$

$$= \frac{\sin 2A}{\sin^2 A\sin 3A} = \frac{2\sin A\cos A}{\sin^2 A\sin 3A} = \frac{2\cos A}{\sin A\sin 3A}:$$

thus the proposed expressions are equal.

Problem 44. *Show that*

$$\sec^2\frac{1}{2}A\sec A\frac{\cot^2\frac{1}{2}A - \cot^2\frac{3}{2}A}{1 + \cot^2\frac{3}{2}A} = 8.$$

Solution. $\sec^2\dfrac{1}{2}A\sec A\dfrac{\cot^2\frac{1}{2}A - \cot^2\frac{3}{2}A}{1 + \cot^2\frac{3}{2}A}$

$$= \frac{1}{\cos^2\frac{1}{2}A}\,\frac{1}{\cos A}\,\frac{\cos^2\frac{1}{2}A\sin^2\frac{3}{2}A - \cos^2\frac{3}{2}A\sin^2\frac{1}{2}A}{\sin^2\frac{1}{2}A\left(\cos^2\frac{3A}{2} + \sin^2\frac{3A}{2}\right)}$$

$$= \frac{\left(\cos\frac{1}{2}A\sin\frac{3}{2}A - \cos\frac{3}{2}A\sin\frac{1}{2}A\right)\left(\cos\frac{1}{2}A\sin\frac{3}{2}A + \cos\frac{3}{2}A\sin\frac{1}{2}A\right)}{\sin^2\frac{1}{2}A\cos^2\frac{1}{2}A\cos A}$$

$$= \frac{\sin\left(\frac{3A}{2} - \frac{A}{2}\right)\sin\left(\frac{3A}{2} + \frac{A}{2}\right)}{\sin^2\frac{1}{2}A\cos^2\frac{1}{2}A\cos A} = \frac{\sin A\sin 2A}{\sin^2\frac{1}{2}A\cos^2\frac{1}{2}A\cos A}$$

$$= \frac{2\sin^2 A}{\sin^2\frac{1}{2}A\cos^2\frac{1}{2}A} = \frac{2\left(2\sin\frac{1}{2}A\cos\frac{1}{2}A\right)^2}{\sin^2\frac{1}{2}A\cos^2\frac{1}{2}A} = 8.$$

Problem 45. *Show that*

$$\{\sec A + \operatorname{cosec} A(1+\sec A)\}\left\{1-\tan^2\frac{1}{2}A\right\}\left\{1-\tan^2\frac{1}{4}A\right\}$$
$$= \left(\sec\frac{1}{2}A + \operatorname{cosec}\frac{1}{2}A\right)\sec^2\frac{1}{4}A.$$

Solution. $\sec A + \operatorname{cosec}(1+\sec A) = \dfrac{1}{\cos A} + \dfrac{1}{\sin A}\left(1+\dfrac{1}{\cos A}\right)$

$$= \frac{1+\cos A+\sin A}{\cos A\sin A} = \frac{2\cos^2\frac{1}{2}A + 2\sin\frac{1}{2}A\cos\frac{1}{2}A}{2\sin\frac{1}{2}A\cos\frac{1}{2}A\cos A}$$

$$= \frac{\cos\frac{1}{2}A+\sin\frac{1}{2}A}{\sin\frac{1}{2}A\cos A} = \frac{\cos\frac{1}{2}A}{\cos A}\left(\sec\frac{1}{2}A + \operatorname{cosec}\frac{1}{2}A\right);$$

$$1-\tan^2\frac{1}{2}A = \frac{\cos^2\frac{1}{2}A - \sin^2\frac{1}{2}A}{\cos^2\frac{1}{2}A} = \frac{\cos A}{\cos^2\frac{1}{2}A};$$

$$1-\tan^2\frac{1}{4}A = \frac{\cos^2\frac{1}{4}A - \sin^2\frac{1}{4}A}{\cos^2\frac{1}{4}A} = \frac{\cos\frac{1}{2}A}{\cos^2\frac{1}{4}A}.$$

Hence by multiplication we obtain the required result.

Problem 46. *If*

$$(a-b)\sec\theta = \sqrt{a^4+\frac{a^2b^2}{a^2-1}},\ \text{and}\ (a+b)\sec\phi = \sqrt{a^4+\frac{a^2b^2}{a^2-1}},$$

then
$$\tan\frac{1}{2}(\theta-\phi) = \frac{b}{a\sqrt{a^2-1}}.$$

Solution. Put c^2 for $a^4 + \dfrac{a^2b^2}{a^2-1}$: thus

$$\cos\theta = \frac{a-b}{c},\ \text{and}\ \cos\phi = \frac{a+b}{c}:$$

from these we obtain

$$\sin\theta = \frac{a(a^2-1)+b}{c\sqrt{(a^2-1)}},\ \text{and}\ \sin\phi = \frac{a(a^2-1)-b}{c\sqrt{(a^2-1)}}.$$

Next we obtain $\qquad \cos(\theta-\phi) = \dfrac{a^4-a^2-b^2}{a^4-a^2+b^2};$

and thus $\qquad \tan^2\dfrac{1}{2}(\theta-\phi) = \dfrac{1-\cos(\theta-\phi)}{1+\cos(\theta-\phi)} = \dfrac{b^2}{a^4-a^2}.$

Problem 47. *Eliminate θ and ϕ from the equations*

$$\frac{a}{b} = \frac{\sin(\phi-\theta)}{\sin(\phi+\theta)},\quad \frac{c}{x} = \cos(\phi-\theta),\quad \frac{b}{x} = \frac{\sin\theta}{\sin\phi}.$$

Solution.

$$\frac{a}{b} = \frac{\sin(\phi - \theta)}{\sin(\phi + \theta)} \tag{25}$$

$$\frac{c}{x} = \cos(\phi - \theta) \tag{26}$$

$$\frac{b}{x} = \frac{\sin\theta}{\sin\phi} \tag{27}$$

From (25) we get $\dfrac{b+a}{b-a} = \dfrac{\sin\phi\cos\theta}{\sin\theta\cos\phi} = \dfrac{x\cos\theta}{b\cos\phi}$, *by* (27).

By (26) we have
$$\frac{c}{x} = \cos\phi\cos\theta + \sin\phi\sin\theta$$
$$= \frac{b}{x}\cdot\frac{b+a}{b-a}\cos^2\phi + \frac{b}{x}\sin^2\phi;$$

hence we get $\cos^2\phi = \dfrac{(c-b)(b-a)}{2ab}$;

therefore $\cos^2\theta = \dfrac{(c-b)(a+b)^2}{2ab(b-a)}\,\dfrac{b^2}{x^2}$.

But from (27) we have $\dfrac{b^2}{x^2}(1-\cos^2\phi) = 1 - \cos^2\theta$,

so that $\dfrac{b^2}{x^2} - 1 = \dfrac{b^2}{x^2}\left\{\dfrac{(c-b)(b-a)}{2ab} - \dfrac{(c-b)(a+b)^2}{2ab(b-a)}\right\}$,

or $x^2 - b^2 = \dfrac{b^2(c-b)4ab}{2ab(b-a)} = \dfrac{2b^2(c-b)}{b-a}$;

therefore $x^2(b-a) = b^2(2c-a-b)$.

Problem 48. *Find* $\cos\dfrac{x}{2}$ *from the equation*

$$\cos x\cos\left(\frac{\pi}{4} - \frac{\beta}{2}\right) = \sin\beta\cos\frac{x}{2}.$$

Solution. Put $2\cos^2\dfrac{x}{2} - 1$ for $\cos x$; thus we obtain the quadratic equation

$$2\cos^2\frac{x}{2}\cos\left(\frac{\pi}{4} - \frac{\beta}{2}\right) - \sin\beta\cos\frac{x}{2} = \cos\left(\frac{\pi}{4} - \frac{\beta}{2}\right).$$

By solving this we have

$$\cos\frac{x}{2} = \frac{\sin\beta \pm \sqrt{\left\{\sin^2\beta + 8\cos^2\left(\frac{\pi}{4} - \frac{\beta}{2}\right)\right\}}}{4\cos\left(\frac{\pi}{4} - \frac{\beta}{2}\right)}$$

and
$$\sin^2\beta + 8\cos^2\left(\frac{\pi}{4} - \frac{\beta}{2}\right) = \sin^2\beta + 4\left\{1 + \cos\left(\frac{\pi}{2} - \beta\right)\right\}$$
$$= \sin^2\beta + 4 + 4\sin\beta.$$

Hence
$$\cos\frac{x}{2} = \frac{\sin\beta \pm (2 + \sin\beta)}{4\cos\left(\frac{\pi}{4} - \frac{\beta}{2}\right)}$$

Take the upper sign; then $\cos\dfrac{x}{2} = \dfrac{1+\sin\beta}{2\cos\left(\dfrac{\pi}{4}-\dfrac{\beta}{2}\right)}$

$$= \frac{1+\cos\left(\dfrac{\pi}{2}-\beta\right)}{2\cos\left(\dfrac{\pi}{4}-\dfrac{\beta}{2}\right)} = \frac{2\cos^2\left(\dfrac{\pi}{4}-\dfrac{\beta}{2}\right)}{2\cos\left(\dfrac{\pi}{4}-\dfrac{\beta}{2}\right)} = \cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right).$$

Take the lower sign; then $\cos x = -\dfrac{1}{2\cos\left(\dfrac{\pi}{4}-\dfrac{\beta}{2}\right)}$.

Problem 49. *If*

$$\cos(\theta+3\phi) = \sin(2\theta+2\phi), \text{ and } \sin(\phi+3\theta) = \cos(2\theta+2\phi),$$

show that $\qquad \theta = (3m-5n)\dfrac{\pi}{8}+\dfrac{\pi}{16}$ and $\phi = (3n-5m)\dfrac{\pi}{8}+\dfrac{\pi}{16}$,

or else $\qquad \phi-\theta = 2m\pi - \dfrac{\pi}{2}$; where m and n are integers.

Solution. Write the first equation in the form

$$\cos\left(\frac{\pi}{2}-2\theta-2\phi\right) = \cos(\theta+3\phi);$$

hence all possible solutions are comprised in

$$\frac{\pi}{2}-2\theta-2\phi = 2m\pi \pm (\theta+3\phi).$$

If we take the upper sign we have

$$3\theta+5\phi = \frac{\pi}{2}-2m\pi \tag{28}$$

If we take the lower sign we have

$$\phi-\theta = 2m\pi-\frac{\pi}{2} \tag{29}$$

Again, write the second equation in the form

$$\cos\left(\frac{\pi}{2}-\phi-3\theta\right) = \cos(2\theta+2\phi);$$

hence all possible solutions are comprised in

$$\frac{\pi}{2}-\phi-3\theta = 2n\pi \pm (2\theta+2\phi).$$

If we take upper sign we have

$$5\theta+3\phi = \frac{\pi}{2}-2n\pi \tag{30}$$

If we take the lower sign we have

$$\phi-\theta = 2n\pi-\frac{\pi}{2};$$

this agrees with (29).

Thus either (29) holds, or both (28) and (30) hold. From (28) and (30) we obtain

$$16\theta = (3m-5n)2\pi+\pi, \quad 16\phi = (3n-5m)2\pi+\pi.$$

Problem 50. *Show that*

$$(1+\sec 2\theta)(1+\sec 4\theta)(1+\sec 8\theta)\ldots(1+\sec 2^n\theta) = \frac{\tan 2^n\theta}{\tan\theta}.$$

Solution. We have $1 + \sec 2\theta = \dfrac{1+\cos 2\theta}{\cos 2\theta} = \dfrac{\sin 2\theta}{\cos 2\theta} \cdot \dfrac{1+\cos 2\theta}{\sin 2\theta}$

$$= \tan 2\theta \cdot \cot\theta = \frac{\tan 2\theta}{\tan\theta}.$$

Similarly $$1 + \sec 4\theta = \frac{\tan 4\theta}{\tan 2\theta},$$

and so on, which we use down to

$$1 + \sec 2^n\theta = \frac{\tan 2^n\theta}{\tan 2^{n-1}\theta}.$$

Multiply these expressions together, then by cancelling we obtain the required result.

Problem 51. *The circular measure of a certain angle is equal to the ratio of the number of degrees in it to the number of grades. Find the magnitude of the angle in degrees.*

Solution. Here the circular measure of an angle is given equal to $\frac{9}{10}$; hence the number of degrees in it is $\frac{9}{10} \cdot \frac{180}{\pi}$, that is $\frac{162}{\pi}$.

Problem 52. *Show that*

$$\{\sin(A-B) + \sin(A+3B)\}\sec 2B = (\cos 2B - \cos 2A)\operatorname{cosec}(A-B).$$

Solution.

$$\sin(A-B) + \sin(A+3B) = 2\sin(A+B)\cos 2B;$$

therefore $$\{\sin(A-B) + \sin(A+3B)\}\sec 2B = 2\sin(A+B).$$

$$\cos 2B - \cos 2A = 2\sin(A-B)\sin(A+B);$$

therefore $$(\cos 2B - \cos 2A)\operatorname{cosec}(A-B) = 2\sin(A+B):$$

thus the proposed expressions are equal.

Problem 53. *If* $\dfrac{\tan\theta}{\tan\alpha} = \dfrac{1+\cos^2\theta}{1+\sin^2\theta}$, *show that* $\sin(3\theta+\alpha) = 7\sin(\theta-\alpha)$.

Solution. Here $$\frac{\sin\theta}{\cos\theta}(1+\sin^2\theta) = \frac{\sin\alpha}{\cos\alpha}(1+\cos^2\theta);$$

therefore $$\cos\alpha(\sin\theta + \sin^3\theta) = \sin\alpha(\cos\theta + \cos^3\theta);$$

therefore $$\cos\alpha\sin\theta + \cos\alpha\frac{3\sin\theta - \sin 3\theta}{4} = \sin\alpha\cos\theta + \sin\alpha\frac{3\cos\theta + \cos 3\theta}{4};$$

therefore $$7(\sin\theta\cos\alpha - \cos\theta\sin\alpha) = \sin 3\theta\cos\alpha + \cos 3\theta\sin\alpha;$$

that is $$7\sin(\theta-\alpha) = \sin(3\theta+\alpha).$$

Problem 54. *Solve the equation*

$$\cos 3\theta + \cos 5\theta + \sqrt{2}(\cos\theta + \sin\theta)\cos\theta = 0.$$

Solution. Here $$2\cos 4\theta\cos\theta + \sqrt{2}(\cos\theta + \sin\theta)\cos\theta = 0;$$

therefore either $$\cos\theta = 0 \text{ or } \cos 4\theta = -\frac{1}{\sqrt{2}}(\cos\theta + \sin\theta).$$

Take the former; then $\theta = (2n+1)\frac{\pi}{2}$.

Take the latter; thus $\cos 4\theta = \cos\left(\frac{3\pi}{4}+\theta\right)$;

thereofore $$4\theta = 2n\pi \pm \left(\frac{3\pi}{4}+\theta\right).$$

Problem 55. *Eliminate ϕ from the equations*
$$n\sin\theta - m\cos\theta = 2m\sin\phi, \quad n\sin 2\theta - m\cos 2\phi = n.$$

Solution. We have $$\sin\phi = \frac{n\sin\theta - m\cos\theta}{2m};$$
and $$n\sin 2\theta - m(1-2\sin^2\phi) = n;$$
therefore $$n\sin 2\theta + 2m\left(\frac{n\sin\theta - m\cos\theta}{2m}\right)^2 = m+n;$$
therefore $$2mn\sin 2\theta + (n\sin\theta - m\cos\theta)^2 = 2m(m+n);$$
therefore $$(n\sin\theta + m\cos\theta)^2 = 2m(m+n).$$

Problem 56. *Solve the equation*
$$8\sin\left(\theta-\frac{\pi}{3}\right)\cos^3\theta + 8\cos\left(\theta-\frac{\pi}{3}\right)\sin^3\theta - 6\sin\left(2\theta-\frac{\pi}{3}\right) = \sqrt{3}.$$

Solution. Substitute $\frac{3\cos\theta+\cos 3\theta}{4}$ for $\cos^3\theta$ and $\frac{3\sin\theta-\sin 3\theta}{4}$ for $\sin^3\theta$ thus the equation becomes
$$2\sin\left(\theta-\frac{\pi}{3}\right)\{3\cos\theta+\cos 3\theta\} + 2\cos\left(\theta-\frac{\pi}{3}\right)\{3\sin\theta-\sin 3\theta\}$$
$$-6\sin\left(2\theta-\frac{\pi}{3}\right) = \sqrt{3},$$
that is $$2\sin\left(\theta-\frac{\pi}{3}\right)\cos 3\theta - 2\cos\left(\theta-\frac{\pi}{3}\right)\sin 3\theta = \sqrt{3},$$
that is $$-\sin\left(3\theta-\theta+\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2};$$
thus $$\sin\left(2\theta+\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2};$$
therefore $$2\theta+\frac{\pi}{3} = n\pi + (-1)^n\frac{4\pi}{3}.$$

Problem 57. *Among all values of θ between 0 and π find that which makes $\sin\theta\cos(\beta-\theta)$ greatest; β being a given angle between 0 and $\frac{\pi}{2}$.*

Solution. $$\sin\theta\cos(\beta-\theta) = \frac{1}{2}\{\sin\beta + \sin(2\theta-\beta)\}.$$

The greatest value of $\sin(2\theta-\beta)$ is obviously when $2\theta-\beta = \frac{\pi}{2}$, so that $\theta = \frac{\beta}{2}+\frac{\pi}{4}$.

Problem 58. *Show that*

$$\cos 55^\circ + \cos 65^\circ + \cos 175^\circ = 0,$$
$$\cos 55^\circ \cos 65^\circ + \cos 65^\circ \cos 175^\circ + \cos 55^\circ \cos 175^\circ = -\frac{3}{4},$$
$$\cos 55^\circ \cos 65^\circ \cos 175^\circ = -\frac{1+\sqrt{3}}{8\sqrt{2}}.$$

Solution. $\cos 55^\circ + \cos 65^\circ = 2\cos 60^\circ \cos 5^\circ = \cos 5^\circ = -\cos 175^\circ$;

therefore $$\cos 55^\circ + \cos 65^\circ + \cos 175^\circ = 0.$$

$$\cos 55^\circ \cos 65^\circ = \frac{1}{2}(\cos 10^\circ + \cos 120^\circ),$$
$$\cos 65^\circ \cos 175^\circ = \frac{1}{2}(\cos 110^\circ + \cos 240^\circ),$$
$$\cos 55^\circ \cos 175^\circ = \frac{1}{2}(\cos 120^\circ + \cos 230^\circ);$$

hence by addition we obtain $-\frac{3}{4} + \frac{1}{2}(\cos 10^\circ + \cos 110^\circ + \cos 230^\circ)$,

that is $$-\frac{3}{4} + \frac{1}{2}(\cos 10^\circ + \cos 110^\circ - \cos 50^\circ),$$

that is $$-\frac{3}{4} + \frac{1}{2}(2\cos 60^\circ \cos 50^\circ - \cos 50^\circ), \text{ that is } -\frac{3}{4}.$$

$$\begin{aligned}\cos 55^\circ \cos 65^\circ \cos 175^\circ &= \frac{1}{2}(\cos 10^\circ + \cos 120^\circ)\cos 175^\circ \\ &= \frac{1}{4}(\cos 165^\circ + \cos 185^\circ) + \frac{1}{4}(\cos 55^\circ + \cos 295^\circ) \\ &= \frac{1}{4}(-\cos 15^\circ + \cos 175^\circ + \cos 55^\circ + \cos 65^\circ) \\ &= -\frac{1}{4}\cos 15^\circ, \text{ by what has been already shown,} \\ &= -\frac{1}{4}\cdot\frac{\sqrt{3}+1}{2\sqrt{2}} = -\frac{\sqrt{3}+1}{8\sqrt{2}}.\end{aligned}$$

Problem 59. *If* $x\cos(\alpha+\beta) + \cos(\alpha-\beta)$

$$= x\cos(\beta+\gamma) + \cos(\beta-\gamma) = x\cos(\gamma+\alpha) + \cos(\gamma-\alpha)$$

then $$\frac{\tan\alpha}{\tan\frac{1}{2}(\beta+\gamma)} = \frac{\tan\beta}{\tan\frac{1}{2}(\gamma+\alpha)} = \frac{\tan\gamma}{\tan\frac{1}{2}\alpha+\beta)}.$$

Solution. From $x\cos(\alpha+\beta) + \cos(\alpha-\beta) = x\cos(\beta+\gamma) + \cos(\beta-\gamma)$

we obtain $$x = \frac{\cos(\beta-\gamma) - \cos(\alpha-\beta)}{\cos(\alpha+\beta) - \cos(\beta+\gamma)} = -\frac{\sin\left(\frac{\alpha+\gamma}{2} - \beta\right)}{\sin\left(\frac{\alpha+\gamma}{2} + \beta\right)}.$$

Two other expressions for the value of x may be obtained; and thus we have

$$\frac{\sin\left(\frac{\alpha+\gamma}{2} - \beta\right)}{\sin\left(\frac{\alpha+\gamma}{2} + \beta\right)} = \frac{\sin\left(\frac{\beta+\alpha}{2} - \gamma\right)}{\sin\left(\frac{\beta+\alpha}{2} + \gamma\right)} = \frac{\sin\left(\frac{\gamma+\beta}{2} - \alpha\right)}{\sin\left(\frac{\gamma+\beta}{2} + \alpha\right)} = -x;$$

hence $$\frac{\sin\frac{\alpha+\gamma}{2}\cos\beta}{\cos\frac{\alpha+\gamma}{2}\sin\beta} = \frac{1-x}{1+x}, \text{ that is } \frac{\tan\frac{\alpha+\gamma}{2}}{\tan\beta} = \frac{1-x}{1+x}.$$

Similarly $\dfrac{\tan\frac{\beta+\alpha}{2}}{\tan\gamma}$ and $\dfrac{\tan\frac{\gamma+\beta}{2}}{\tan\alpha}$ are also equal to $\dfrac{1-x}{1+x}$.

Problem 60. *If $A+B+C+D=360°$, show that*
$$\cos A - \cos B + \cos C - \cos D = 4\sin\frac{A+B}{2}\sin\frac{B+C}{2}\cos\frac{C+A}{2}.$$

Solution.
$$\cos A - \cos B = 2\sin\frac{A+B}{2}\sin\frac{B-A}{2};$$
$$\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$$
$$= 2\sin\frac{A+B}{2}\sin\frac{D-C}{2}, \text{ by } \textit{Art.}\ 48 \text{ (page 397)}.$$

Hence by addition,
$$\cos A - \cos B + \cos C - \cos D = 2\sin\frac{A+B}{2}\left\{\sin\frac{B-A}{2} + \sin\frac{D-C}{2}\right\}$$
$$= 4\sin\frac{A+B}{2}\sin\frac{B+D-A-C}{4}\cos\frac{B+C-A-D}{4}.$$

Then, as in *Problem* 20, we can show that
$$\sin\frac{B+D-A-C}{4} = \cos\frac{A+C}{2}, \text{ and } \cos\frac{B+C-A-D}{4} = \sin\frac{B+C}{2}.$$

Problem 61. *The number of degrees in an angle of one regular polygon is to the number of grades in an angle of another as 3 is to 5. Find the number of sides in each polygon, showing that there are only three solutions.*

Solution. Let x denote the number of sides in the first regular polygon, and y the number of sides in the second. All the angles of the first polygon are equal to $2x-4$ right angles; therefore each angle is equal to $\dfrac{2x-4}{x}$ right angles, and therefore contains $\dfrac{2x-4}{x}90$ degrees. In the same way each angle of the second polygon contains $\dfrac{2y-4}{y}100$ grades. Then, by supposition, we have $\dfrac{2x-4}{x}90 : \dfrac{2y-4}{y}100 :: 3:5$;

therefore $$5\frac{2x-4}{x}90 = 3\frac{2y-4}{y}100;$$

therefore $$\frac{3(x-2)}{x} = \frac{2(y-2)}{y};$$

therefore $3y(x-2)=2x(y-2)$; therefore $y(6-x)=4x$. This formula shows that x cannot be greater than 5; for if $x=6$ we should have $y\times 0 = 24$, which is absurd; and if x is greater than 6 we should have a negative value for y, which is also absurd. And x cannot be less than 3. Thus the only possible solutions are $x=3$, $x=4$, and $x=5$; which give respectively $y=4$, $y=8$, and $y=20$.

Problem 62. *Solve the equation* $\sec^2 \frac{x}{2} + \operatorname{cosec}^2 \frac{x}{2} = 16 \cot x.$

Solution. Here $$\frac{1}{\cos^2 \frac{x}{2}} + \frac{1}{\sin^2 \frac{x}{2}} = \frac{16 \cos x}{\sin x};$$

therefore $$\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} = \frac{16 \cos x \cos^2 \frac{x}{2} \sin^2 \frac{x}{2}}{\sin x} = 8 \cos x \cos \frac{x}{2} \sin \frac{x}{2};$$

therefore $$1 = 4 \cos x \sin x = 2 \sin 2x;$$

therefore $$\sin 2x = \frac{1}{2}; \text{ therefore } 2x = n\pi + (-1)^n \frac{\pi}{6}.$$

Problem 63. *Eliminate* θ *from the equations*
$$m \sin 2\theta = n \sin \theta, \quad p \cos 2\theta = q \cos \theta.$$

Solution. Here $$m \sin 2\theta = n \sin \theta, \quad p \cos 2\theta = q \cos \theta;$$
from the first equation $2m \sin \theta \cos \theta = n \sin \theta$; therefore $\cos \theta = \dfrac{n}{2m}$.

Substitute in the second equation, that is in $p(2 \cos^2 \theta - 1) = q \cos \theta$;

thus $$p\left\{2\left(\frac{n}{2m}\right)^2 - 1\right\} = \frac{qn}{2m}; \text{ therefore } p(n^2 - 2m^2) = qmn.$$

Problem 64. *Find* θ *from the equation*
$$\cos \theta - \sin \theta = \cos \alpha - \sin \alpha.$$

Solution. We may write the equation thus :
$$\frac{1}{\sqrt{2}} \cos \theta - \frac{1}{\sqrt{2}} \sin \theta = \frac{1}{\sqrt{2}} \cos \alpha - \frac{1}{\sqrt{2}} \sin \alpha;$$

therefore $$\cos\left(\theta + \frac{\pi}{4}\right) = \cos\left(\alpha + \frac{\pi}{4}\right);$$

therefore $$\theta + \frac{\pi}{4} = 2n\pi \pm \left(\alpha + \frac{\pi}{4}\right).$$

Problem 65. *Show that if* $\sin(A + B + C + D) = 0$, *then*
$$\sin(A + C) \sin(A + D) = \sin(B + C) \sin(B + D).$$

Solution. $\sin(A + C) \sin(A + D) - \sin(B + C) \sin(B + D)$
$$= \frac{1}{2}\{\cos(C - D) - \cos(2A + C + D)\} - \frac{1}{2}\{\cos(C - D) - \cos(2B + C + D)\}$$
$$= \frac{1}{2}\{\cos(2B + C + D) - \cos(2A + C + D)\}$$
$$= \sin(A - B) \sin(A + B + C + D).$$

Thus, if $\sin(A + B + C + D)$ vanishes, the difference between the two proposed expressions vanishes; and therefore the two expressions are equal.

Problem 66. *Show that all the values of* θ *which satisfy the equations*
$$\sin \theta + \sin \phi = p, \quad \cos \theta + \cos \phi = q,$$

are contained in the expression $n\pi - \alpha + (-1)^n\beta$, *where* α *and* β *are angles determined by the equations*

$$\tan\alpha = \frac{q}{p}, \quad \sin\beta = \frac{1}{2}\sqrt{p^2+q^2}.$$

Solution. We have $\sin\phi = p - \sin\theta$, $\quad \cos\phi = q - \cos\theta$; square and add,

thus $$1 = p^2 + q^2 - 2p\sin\theta - 2q\cos\theta + 1;$$

therefore $$2p\sin\theta + 2q\cos\theta = p^2 + q^2.$$

Now assume that $\tan\alpha = \frac{q}{p}$, so that

$$\sin\alpha = \frac{q}{\sqrt{(p^2+q^2)}} \text{ and } \cos\alpha = \frac{p}{\sqrt{(p^2+q^2)}};$$

thus $$2\sqrt{(p^2+q^2)}\{\sin\theta\cos\alpha + \cos\theta\sin\alpha\} = p^2 + p^2,$$

therefore $$\sin(\theta+\alpha) = \frac{\sqrt{(p^2+q^2)}}{2} = \sin\beta \text{ say };$$

therefore $$\theta + \alpha = n\pi + (-1)^n\beta.$$

Problem 67. *Show that*

$$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15} = \left(\frac{1}{2}\right)^7.$$

Solution.

$$\cos\frac{\pi}{15}\cos\frac{4\pi}{15} = \frac{1}{2}\left(\cos\frac{\pi}{3} + \cos\frac{\pi}{5}\right) = \frac{1}{2}\left(\frac{1}{2} + \frac{\sqrt{5}+1}{4}\right) = \frac{3+\sqrt{5}}{8};$$

$$\cos\frac{2\pi}{15}\cos\frac{7\pi}{15} = \frac{1}{2}\left(\cos\frac{\pi}{3} + \cos\frac{3\pi}{5}\right) = \frac{1}{2}\left(\frac{1}{2} - \frac{\sqrt{5}-1}{4}\right) = \frac{3-\sqrt{5}}{8};$$

$$\cos\frac{3\pi}{15}\cos\frac{6\pi}{15} = \frac{1}{2}\left(\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}\right) = \frac{1}{2}\left(\frac{\sqrt{5}+1}{4} - \frac{\sqrt{5}-1}{4}\right) = \frac{1}{4};$$

$$\cos\frac{5\pi}{15} = \frac{1}{2};$$

therefore $$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}$$

$$= \frac{3+\sqrt{5}}{8}\cdot\frac{3-\sqrt{5}}{8}\cdot\frac{1}{8} = \frac{4}{8^3} = \frac{1}{2^7}.$$

Problem 68. *Show that whatever* θ *may be,*

$$a\sin^2\theta + b\sin\theta\cos\theta + c\cos^2\theta$$

lies in value between

$$\frac{1}{2}(a+c) + \frac{1}{2}\sqrt{b^2+(a-c)^2} \text{ and } \frac{1}{2}(a+c) - \frac{1}{2}\sqrt{b^2+(a-c)^2}.$$

Solution. $a\sin^2\theta + b\sin\theta\cos\theta + c\cos^2\theta$

$$= \frac{1}{2}\{a(1-\cos 2\theta) + b\sin 2\theta + c(1+\cos 2\theta)\}$$

$$= \frac{1}{2}\{a + c + b\sin 2\theta - (a-c)\cos 2\theta\}.$$

Now let α be an angle such that $\tan\alpha = \dfrac{a-c}{b}$, so that $\cos\alpha = \dfrac{b}{\sqrt{b^2+(a-c)^2}}$, and $\sin\alpha = \dfrac{a-c}{\sqrt{b^2+(a-c)^2}}$. Then the above expression

$$= \frac{1}{2}(a+c) + \frac{1}{2}\sqrt{b^2+(a-c)^2}\{\sin 2\theta\cos\alpha - \cos 2\theta \sin\alpha\}$$
$$= \frac{1}{2}(a+c) + \frac{1}{2}\sqrt{b^2+(a-c)^2}\sin(2\theta-\alpha).$$

Then as $\sin(2\theta-\alpha)$ must lie between -1 and $+1$, we obtain the required result.

Problem 69. *Show that*

$$\cos\alpha + \cos\left(\frac{2\pi}{3}+\alpha\right) + \cos\left(\frac{2\pi}{3}-\alpha\right) = 0,$$
$$\cos\alpha\cos\left(\frac{2\pi}{3}+\alpha\right) + \cos\alpha\cos\left(\frac{2\pi}{3}-\alpha\right) + \cos\left(\frac{2\pi}{3}+\alpha\right)\cos\left(\frac{2\pi}{3}-\alpha\right) = -\frac{3}{4},$$
$$\cos\alpha\cos\left(\frac{2\pi}{3}+\alpha\right)\cos\left(\frac{2\pi}{3}-\alpha\right) = \frac{\cos 3\alpha}{4}.$$

Solution.

$$\cos\left(\frac{2\pi}{3}+\alpha\right) + \cos\left(\frac{2\pi}{3}-\alpha\right) = 2\cos\frac{2\pi}{3}\cos\alpha = -\cos\alpha;$$

therefore $$\cos\alpha + \cos\left(\frac{2\pi}{3}+\alpha\right) + \cos\left(\frac{2\pi}{3}-\alpha\right) = 0.$$

$$\cos\alpha\cos\left(\frac{2\pi}{3}+\alpha\right) = \frac{1}{2}\left\{\cos\frac{2\pi}{3} + \cos\left(\frac{2\pi}{3}+2\alpha\right)\right\}$$
$$\cos\alpha\cos\left(\frac{2\pi}{3}-\alpha\right) = \frac{1}{2}\left\{\cos\frac{2\pi}{3} + \cos\left(\frac{2\pi}{3}-2\alpha\right)\right\}$$
$$\cos\left(\frac{2\pi}{3}+\alpha\right)\cos\left(\frac{2\pi}{3}-\alpha\right) = \frac{1}{2}\left\{\cos\frac{4\pi}{3} + \cos 2\alpha\right\}.$$

Now $\cos 2\alpha + \cos\left(\dfrac{2\pi}{3}+2\alpha\right) + \cos\left(\dfrac{2\pi}{3}-2\alpha\right)$ is zero, in the manner already shown; and $\cos\dfrac{2\pi}{3}$ and $\cos\dfrac{4\pi}{3}$ are each $-\dfrac{1}{2}$: thus the sum is $-\dfrac{3}{4}$.

$$\cos\alpha\cos\left(\frac{2\pi}{3}+\alpha\right)\cos\left(\frac{2\pi}{3}-\alpha\right)$$
$$= \cos\alpha\left(\cos^2\alpha - \sin^2\frac{2\pi}{3}\right), \text{ by } \textit{Art. } 83 \text{ (page 404)},$$
$$= \cos\alpha\left(\cos^2\alpha - \frac{3}{4}\right) = \frac{1}{4}\cos\alpha(4\cos^2\alpha - 3) = \frac{\cos 3\alpha}{4}.$$

Problem 70. *Find an expression for the product*

$$\left(\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}\right)\left(\cos\frac{\alpha}{2^2}+\cos\frac{\beta}{2^2}\right)\dots\left(\cos\frac{\alpha}{2^n}+\cos\frac{\beta}{2^n}\right).$$

Solution.

$$\cos\frac{\alpha}{2^n}+\cos\frac{\beta}{2^n}=\frac{\cos^2\dfrac{\alpha}{2^n}-\cos^2\dfrac{\beta}{2^n}}{\cos\dfrac{\alpha}{2^n}-\cos\dfrac{\beta}{2^n}}=\frac{1}{2}\frac{\cos\dfrac{\alpha}{2^{n-1}}-\cos\dfrac{\beta}{2^{n-1}}}{\cos\dfrac{\alpha}{2^n}-\cos\dfrac{\beta}{2^n}};$$

similarly $$\cos\frac{\alpha}{2^{n-1}}+\cos\frac{\beta}{2^{n-1}}=\frac{1}{2}\frac{\cos\dfrac{\alpha}{2^{n-2}}-\cos\dfrac{\beta}{2^{n-2}}}{\cos\dfrac{\alpha}{2^{n-1}}-\cos\dfrac{\beta}{2^{n-1}}};$$

and we use a series of these transformation down to

$$\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}=\frac{1}{2}\frac{\cos\alpha-\cos\beta}{\cos\dfrac{1}{2}\alpha-\cos\dfrac{1}{2}\beta}.$$

Then by multiplication we obtain for the product

$$\frac{1}{2^n}\frac{\cos\alpha-\cos\beta}{\cos\dfrac{\alpha}{2^n}-\cos\dfrac{\beta}{2^n}}.$$

Problem 71. *An angle is the excess of $a°b'$ above $p^g q'$. Find the ratio of this angle to a right angle.*

Solution. The angle $a°b'$ is $\dfrac{60a+b}{60\times 90}$ of a right angle; the angle $p^g q'$ is $\dfrac{100p+q}{100\times 100}$ of a right angle.

Hence the excess of the former above the latter is $\left\{\dfrac{60a+b}{60\times 90}-\dfrac{100p+q}{100\times 100}\right\}$ of a right angle.

Problem 72. *Solve the equation $2\sin^2 x+\sin^2 2x=2$.*

Solution.

Here $$1-\cos 2x+\sin^2 2x=2;$$

therefore $$1-\cos 2x+1-\cos^2 2x=2;$$

therefore $$\cos 2x(1+\cos 2x)=0.$$

Therefore either $$\cos 2x=0,\ \text{or}\ 1+\cos 2x=0.$$

If $\cos 2x=0$, we have $2x=2n\pi\pm\dfrac{\pi}{2}$, which may be written more simply as $2x=(2m+1)\dfrac{\pi}{2}$.

If $1+\cos 2x=0$ we have $\cos 2x=-1$, and therefore $2x=2n\pi\pm\pi$ which may be written more simply as $2x=(2m+1)\pi$.

Problem 73. *Show that*

$$\tan A+2\tan 2A+4\tan 4A+8\cot 8A=\cot A.$$

Solution.

$$\tan A-\cot A=\frac{\sin A}{\cos A}-\frac{\cos A}{\sin A}=\frac{\sin^2 A-\cos^2 A}{\sin A\cos A}=\frac{2(\sin^2 A-\cos^2 A)}{2\sin A\cos A}$$
$$=-\frac{2\cos 2A}{\sin 2A}=-2\cot 2A.$$

Similarly $\qquad 2\tan 2A - 2\cot 2A = -4\cot 4A,$

and $\qquad 4\tan 4A - 4\cot 4A = -8\cot 8A.$

Therefore by addition and cancelling

$$\tan A - \cot A + 2\tan 2A + 4\tan 4A = -8\cot 8A;$$

therefore
$$\tan A + 2\tan 2A + 4\tan 4A + 8\cot 8A = \cot A.$$

Problem 74. *Solve the equation* $\cos 2x - \cos 4x = \sin x$.

Solution. Here $2\sin 3x \sin x = \sin x$; therefore either $\sin x = 0$ or $2\sin 3x = 1$. If $\sin x = 0$, then $x = n\pi$. If $\sin 3x = \frac{1}{2}$, then $3x = n\pi + (-1)^n \frac{\pi}{6}$.

Problem 75. *If the sum of the angles* A, B, C, D *be four right angles, and their tangents in geometrical progression, show that the ratio* $= -1$; *or else that* $\tan A \tan D = \tan B \tan C = 1$.

Solution. Let r denote the common ratio of the Geometrical Progression, so that $\tan B = r\tan A, \tan C = r\tan B, \tan D = r\tan C$;

therefore
$$\tan A \tan D = \tan B \tan C.$$

Now since $A + D = 360° - B - C$, we have $\tan(A + D) = -\tan(B + C)$;

therefore
$$\frac{\tan A + \tan D}{1 - \tan A \tan D} = -\frac{\tan B + \tan C}{1 - \tan B \tan C}.$$

Thus we must have either $\tan A \tan D = \tan B \tan C = 1$,

or else
$$\tan A + \tan D = -(\tan B + \tan C).$$

The latter gives
$$(1 + r^3)\tan A = -(r + r^2)\tan A,$$

so that
$$1 + r^3 + r + r^2 = 0, \text{ that is } (1 + r)(1 + r^2) = 0.$$

the only possible solution is $1 + r = 0$, so that $r = -1$.

Problem 76. *The angles* A, B, C *of a triangle are in Arithmetical Progression; and* cosec $2A$, cosec $2B$, cosec $2C$ *are in Arithmetical Progression. Show that the cosine of the common difference of the angles is* $\sqrt{\frac{3}{8}}$.

Solution. Let A, B, C denote the angles; then $A + B + C = 180°$; and since the angles are in Arithmetical Progression $A + C = 2B$; thus $3B = 180°$; therefore $B = 60°$.

Again we have $\frac{1}{\sin 2A} + \frac{1}{\sin 2C} = \frac{2}{\sin 2B} = \frac{4}{\sqrt{3}}$. Let x denote the common difference of the angles; so that $A = 60° - x$, and $C = 60° + x$. Then

$$\frac{\sin 2A + \sin 2C}{\sin 2A \sin 2C} = \frac{4}{\sqrt{3}}; \text{ therefore } \frac{2\sin(A + C)\cos(A - C)}{\sin(120° - 2x)\sin(120° + 2x)} = \frac{4}{\sqrt{3}};$$

therefore $\frac{\sqrt{3}\cos 2x}{\sin^2 120° - \sin^2 2x} = \frac{4}{\sqrt{3}}$; therefore

$$\cos 2x = \frac{4}{3}(\sin^2 120° - \sin^2 2x) = \frac{4}{3}\left(\frac{3}{4} - 1 + \cos^2 2x\right) = -\frac{1}{3} + \frac{4}{3}\cos^2 2x.$$

By solving this quadratic we obtain $\cos 2x = 1$, or $-\frac{1}{4}$. The latter must be taken : then $\cos^2 x = \frac{1}{2}\left(1 - \frac{1}{4}\right) = \frac{3}{8}$.

Problem 77. *Show that* $\cos A + \cos 2A + \cos 3A = \dfrac{\cos 2A \sin \frac{3A}{2}}{\sin \frac{A}{2}}$.

Solution.

$$\cos A + \cos 2A + \cos 3A = 2\cos 2A \cos A + \cos 2A$$
$$= \cos 2A(2\cos A + 1) = \cos 2A\left(2 - 4\sin^2 \frac{1}{2}A + 1\right)$$
$$= \cos 2A\left(3 - 4\sin^2 \frac{1}{2}A\right) = \frac{\cos 2A}{\sin \frac{1}{2}A}\left(3\sin \frac{1}{2}A - 4\sin^3 \frac{1}{2}A\right)$$
$$= \frac{\cos 2A}{\sin \frac{1}{2}A} \sin \frac{3A}{2}.$$

Problem 78. *Show that*

$$\left(x - 2\cos\frac{2\pi}{7}\right)\left(x - 2\cos\frac{4\pi}{7}\right)\left(x - 2\cos\frac{6\pi}{7}\right) = x^3 + x^2 - 2x - 1.$$

Solution. Multiply the given expression out. The coefficient of x^2 is

$$-2\left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\right);$$

by *Problem* 77 this $= -\dfrac{2\cos\frac{4\pi}{7}\sin\frac{3\pi}{7}}{\sin\frac{\pi}{7}} = -\dfrac{1}{\sin\frac{\pi}{7}}\left(\sin\frac{7\pi}{7} - \sin\frac{\pi}{7}\right) = 1.$

The coefficient of x is

$$4\left(\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} + \cos\frac{2\pi}{7}\cos\frac{6\pi}{7} + \cos\frac{4\pi}{7}\cos\frac{6\pi}{7}\right);$$

this $= 2\left(\cos\frac{2\pi}{7} + \cos\frac{6\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{10\pi}{7}\right)$

$$= 2\left(\cos\frac{2\pi}{7} + \cos\frac{6\pi}{7} + \cos\frac{4\pi}{7}\right) + 2\left(\cos\frac{8\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{10\pi}{7}\right)$$
$$= 2\left(\cos\frac{2\pi}{7} + \cos\frac{6\pi}{7} + \cos\frac{4\pi}{7}\right) + 2\left(\cos\frac{8\pi}{7} + \cos\frac{16\pi}{7} + \cos\frac{24\pi}{7}\right).$$

The former expression $= -1$, as we have already shown. And by *Problem* 77 the latter expression

$$= \frac{2\cos\frac{16\pi}{7}\sin\frac{12\pi}{7}}{\sin\frac{4\pi}{7}} = \frac{1}{\sin\frac{4\pi}{7}}\left(\sin 4\pi - \sin\frac{4\pi}{7}\right) = -1.$$

Hence the entire coefficient is -2.

The term independent of x is $-8\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}$; this

$$= -4\cos\frac{6\pi}{7}\left(\cos\frac{2\pi}{7}+\cos\frac{6\pi}{7}\right)$$

$$= -2\left(\cos\frac{8\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{12\pi}{7}+1\right)$$

$$= -\frac{2\cos\frac{8\pi}{7}\sin\frac{6\pi}{7}}{\sin\frac{2\pi}{7}} - 2 = -\frac{1}{\sin\frac{2\pi}{7}}\left(\sin 2\pi - \sin\frac{2\pi}{7}\right) - 2$$

$$= 1 - 2 = -1.$$

Problem 79. *If $A + B + C = 180°$, show that*

$$\sin^3 A + \sin^3 B + \sin^3 C = 3\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} + \cos\frac{3A}{2}\cos\frac{3B}{2}\cos\frac{3C}{2}.$$

Solution. $\sin^3 A + \sin^3 B + \sin^3 C$

$$= \frac{1}{4}(3\sin A + 3\sin B + 3\sin C - \sin 3A - \sin 3B - \sin 3C).$$

Then by *Problem* 32 of *Chap.* VIII we have

$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},$$

and

$$\sin 3A + \sin 3B + \sin 3C = -4\cos\frac{3A}{2}\cos\frac{3B}{2}\cos\frac{3C}{2}.$$

Problem 80. *Investigate the conditions which must hold in order that the equation $\sin^2 x + 2b\sin x + c = 0$ may give two admissible values for $\sin x$, when b is positive.*

Solution. By solving the quadratic equation we obtain

$$\sin x = -b \pm \sqrt{(b^2 - c)}.$$

Hence $b^2 - c$ must not be negative, and $b + \sqrt{(b^2 - c)}$ must not be greater than unity, in order that both values may be admissible.

Problem 81. *The number of the sides of one regular polygon is to the number of the sides of another as m is to n; and the number of degrees in an angle of the first is to the number of grades in an angle of the second as p is to q. Determine the number of sides in each polygon.*

Solution. Let mx denote the number of sides in the first regular polygon, and nx the number of sides in the second. Then, proceeding as in *Problem* 61, we find that the number of degrees in an angle of the first polygon is $\frac{2mx-4}{nx}90$, and the number of grades in an angle of the second polygon is $\frac{2nd-4}{nx}100$. Therefore $\frac{2mx-4}{mx}90 : \frac{2nx-4}{nx}100 :: p : q$.

Therefore $9q\frac{mx-2}{m} = 10p\frac{nx-2}{n}$; therefore $x = \frac{2(9qn-10pm)}{mn(9q-10p)}$.

Hence mx and nx are known.

Problem 82. *Solve the equation* $\cos x + \cos 7x = \cos 4x$.

Solution. Here $2\cos 3x \cos 4x = \cos 4x$. Therefore either $\cos 4x = 0$ or $2\cos 3x = 1$.
If $\cos 4x = 0$ then $4x = (2n+1)\frac{\pi}{2}$.
If $\cos 3x = \frac{1}{2}$ then $3x = 2n\pi \pm \frac{\pi}{3}$.

Problem 83. *Eliminate* α *from the equations*

$$x\tan(\alpha-\beta) = y\tan(\alpha+\beta), \quad (x-y)\cos 2\alpha + (x+y)\cos 2\beta = z.$$

Solution. From the first equation we have

$$x\sin(\alpha-\beta)\cos(\alpha+\beta) = y\sin(\alpha+\beta)\cos(\alpha-\beta),$$

therefore
$$x(\sin 2\alpha - \sin 2\beta) = y(\sin 2\alpha + \sin 2\beta),$$

therefore
$$(x-y)\sin 2\alpha = (x+y)\sin 2\beta.$$

Thus
$$\sin 2\alpha = \frac{(x+y)\sin 2\beta}{x-y},$$

and
$$\cos 2\alpha = \frac{z-(x+y)\cos 2\beta}{x-y}.$$

Square and add; thus

$$1 = \frac{(x+y)^2\sin^2 2\beta}{(x-y)^2} + \frac{\{z-(x+y)\cos 2\beta\}^2}{(x-y)^2}.$$

Therefore
$$\begin{aligned}(x-y)^2 =&(x+y)^2\sin^2 2\beta + \{z-(x+y)\cos 2\beta\}^2\\ =&(x+y)^2 + z^2 - 2z(x+y)\cos 2\beta;\end{aligned}$$

therefore
$$z^2 + 4xy = 2z(x+y)\cos 2\beta.$$

Problem 84. *If* x *and* y *vary so that their sum is constant, find between what limits* $\sin x \sin y$ *ranges, and its greatest value.*

Solution. Let A denote the sum of x and y. Suppose $x = \frac{A}{2} + z$, then $y = \frac{A}{2} - z$; and $\sin x \sin y = \sin\left(\frac{A}{2} - z\right)\sin\left(\frac{A}{2} + z\right) = \sin^2\frac{A}{2} - \sin^2 z$. Now $\sin^2 z$ ranges between the values 0 and 1; hence $\sin x \sin y$ ranges between the values $\sin^2\frac{A}{2}$ and $-\cos^2\frac{A}{2}$: the former is always the greatest value *algebraically*.

Problem 85. *If* $A + B + C = 180°$, *show that*

$$\sin\left(A+\frac{B}{2}\right) + \sin\left(B+\frac{C}{2}\right) + \sin\left(C+\frac{A}{2}\right) + 1$$
$$= 4\cos\frac{A-B}{4}\cos\frac{B-C}{4}\cos\frac{C-A}{4}.$$

Solution. We have $\sin\left(A+\frac{B}{2}\right) = \sin\left(\frac{A-C}{2} + \frac{A+B+C}{2}\right) = \cos\frac{A-C}{2}$;

similarly $\quad \sin\left(B+\frac{C}{2}\right) = \cos\frac{B-A}{2}, \quad \sin\left(C+\frac{A}{2}\right) = \cos\frac{C-B}{2}.$

Then $$\cos\frac{A-C}{2}+\cos\frac{B-A}{2}=2\cos\frac{B-C}{4}\cos\frac{2A-B-C}{4},$$

and $$1+\cos\frac{C-B}{2}=2\cos^2\frac{B-C}{4};$$

$$\text{therefore}\ \cos\frac{A-C}{2}+\cos\frac{B-A}{2}+\cos\frac{C-B}{2}+1$$
$$=2\cos\frac{B-C}{4}\left\{\cos\frac{B-C}{4}+\cos\frac{2A-B-C}{4}\right\}$$
$$=4\cos\frac{B-C}{4}\cos\frac{A-C}{4}\cos\frac{A-B}{4}$$
$$=4\cos\frac{A-B}{4}\cos\frac{B-C}{4}\cos\frac{C-A}{4}.$$

Problem 86. *If* $\cos^2 A+\cos^2 B+\cos^2 C=1,\quad \cos^2\alpha+\cos^2\beta+\cos^2\gamma=1,$

and $$\cos A\cos\alpha+\cos B\cos\beta+\cos C\cos\gamma=0;$$

Show that
$$\frac{\sin\alpha\sin 2\alpha}{\cos A}+\frac{\sin\beta\sin 2\beta}{\cos B}+\frac{\sin\gamma\sin 2\gamma}{\cos C}+\frac{2\cos\alpha\cos\beta\cos\gamma}{\cos A\cos B\cos C}=0.$$

Solution. Bring the proposed expression to a common denominator; then the numerator

$$=2\cos\alpha(1-\cos^2\alpha)\cos B\cos C+2\cos\beta(1-\cos^2\beta)\cos C\cos A$$
$$+2\cos\gamma(1-\cos^2\gamma)\cos A\cos B+2\cos\alpha\cos\beta\cos\gamma$$
$$=2\cos\alpha(\cos^2\beta+\cos^2\gamma)\cos B\cos C+2\cos\beta(\cos^2\gamma+\cos^2\alpha)\cos C\cos A$$
$$+2\cos\gamma(\cos^2\alpha+\cos^2\beta)\cos A\cos B+2\cos\alpha\cos\beta\cos\gamma$$
$$=2\cos\alpha\cos\beta(\cos\alpha\cos A+\cos B\cos\beta)\cos C$$
$$+2\cos\beta\cos\gamma(\cos\beta\cos B+\cos\gamma\cos C)\cos A$$
$$+2\cos\gamma\cos\alpha(\cos\gamma\cos C+\cos\alpha\cos A)\cos B+2\cos\alpha\cos\beta\cos\gamma$$
$$=-2\cos\alpha\cos\beta\cos\gamma\cos^2 C-2\cos\alpha\cos\beta\cos\gamma\cos^2 A$$
$$-2\cos\alpha\cos\beta\cos\gamma\cos^2 B+2\cos\alpha\cos\beta\cos\gamma$$
$$=2\cos\alpha\cos\beta\cos\gamma(1-\cos^2 C-\cos^2 A-\cos^2 B)=0.$$

Problem 87. *Show that* $\sin\frac{\pi}{24}=\frac{1}{4}(1+\sqrt{2}-\sqrt{3})\sqrt{2-\sqrt{2}}.$

Solution. $\sin^2 7\frac{1}{2}^\circ=\frac{1}{2}(1-\cos 15^\circ)=\frac{1}{2}\{1-\cos(45^\circ-30^\circ)\}=\frac{1}{2}\left\{1-\frac{\sqrt{3}+1}{2\sqrt{2}}\right\}$

$$=\frac{2\sqrt{2}-\sqrt{3}-1}{4\sqrt{2}}=\frac{8-2\sqrt{6}-2\sqrt{2}}{16}.$$

Now it will be found that
$$8-2\sqrt{6}-2\sqrt{2}=(2-\sqrt{2})(6+2\sqrt{2}-2\sqrt{3}-2\sqrt{6})=(2-\sqrt{2})(1+\sqrt{2}-\sqrt{3})^2;$$

therefore $$\sin 7\frac{1}{2}^\circ=\frac{1+\sqrt{2}-\sqrt{3}}{4}\sqrt{2-\sqrt{2}}.$$

Problem 88. *If* $\tan\phi = \dfrac{1+2c^2}{1-c^2}\tan\theta$,

and $$\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right) = \frac{1+c}{1-c}\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right),$$

then either $\sin\theta = \dfrac{2}{c}$ *or else* $\cos\theta = 0$.

Solution. The second equation gives $\dfrac{1+\tan\frac{\phi}{2}}{1-\tan\frac{\phi}{2}} = \dfrac{1+c}{1-c}\dfrac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}$

therefore $$\tan\frac{\phi}{2} = \frac{c+\tan\frac{\theta}{2}}{1+c\tan\frac{\theta}{2}}.$$

The first equation gives $\dfrac{2\tan\frac{\phi}{2}}{1-\tan^2\frac{\phi}{2}} = \dfrac{1+2c^2}{1-c^2}\dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$;

therefore $$\frac{\left(c+\tan\frac{\theta}{2}\right)\left(1+c\tan\frac{\theta}{2}\right)}{\left(1+c\tan\frac{\theta}{2}\right)^2-\left(c+\tan\frac{\theta}{2}\right)^2} = \frac{1+2c^2}{1-c^2}\frac{\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}.$$

therefore $$\frac{c+(1+c^2)\tan\frac{\theta}{2}+c\tan^2\frac{\theta}{2}}{(1-c^2)\left(1-\tan^2\frac{\theta}{2}\right)} = \frac{(1+2c^2)\tan\frac{\theta}{2}}{(1-c^2)\left(1-\tan^2\frac{\theta}{2}\right)};$$

therefore either $1-\tan^2\dfrac{\theta}{2} = 0$, or $c\tan^2\dfrac{\theta}{2} - c^2\tan\dfrac{\theta}{2} + c = 0$.

The former gives $\cos\theta = 0$; the latter gives $1 - c\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 0$, therefore $2 - c\sin\theta = 0$.

Problem 89. *Find* $\cos x$ *from the equation* $\cos 2x + b\cos x + c = 0$. *Investigate the conditions which must hold in order that there may be at least one admissible value of* $\cos x$, *supposing* b *positive.*

Solution. Put $2\cos^2 x - 1$ for $\cos 2x$; then $2\cos^2 x + b\cos x + c - 1 = 0$. By solving this quadratic equation we obtain
$$\cos x = \frac{-b \pm \sqrt{b^2 - 8(c-1)}}{4}.$$

Hence $b^2 + 8 - 8c$ must not be negative, and $\sqrt{(b^2+8-8c)} - b$ must not be numerically greater than 4.

Problem 90. *If* γ *is not greater than* $\dfrac{\pi}{4}$ *show that* $\sin\theta\,\{1+\sin(\gamma-\theta)\}$ *continually increases as* θ *increases from* 0 *to* γ.

Solution. Suppose θ_2 greater than θ_1, and each between 0 and γ. Now

$$\sin\theta\{1+\sin(\gamma-\theta)\} = \sin\theta + \frac{1}{2}\sin\gamma\sin 2\theta - \cos\gamma\sin^2\theta.$$

Put θ_1 and θ_2 in succession for θ, and subtract the second value of the expression from the first. Thus we get

$$(\sin\theta_2 - \sin\theta_1)\left\{1+\frac{1}{2}\sin\gamma\frac{\sin 2\theta_2 - \sin 2\theta_1}{\sin\theta_2-\sin\theta_1} - (\sin\theta_2+\sin\theta_1)\cos\gamma\right\}.$$

Now $(\sin\theta_2+\sin\theta_1)\cos\gamma$ is less than $2\sin\gamma\cos\gamma$, that is less than $\sin 2\gamma$, and therefore less than 1. Hence the preceding expression is necessarily positive, and this is what was to be proved.

Problem 91. *Show that there are eleven, and only eleven, pairs of regular polygons which are such that the number of degrees in an angle of one of them is equal to the number of grades in an angle of the other; and that there are only four pairs when these angles are expressed as integers.*

Solution. Let x be the number of sides in one regular polygon, and y the number of sides in another. Then, as in *Problem* 61, the number of degrees in an angle of the first polygon is $\frac{2x-4}{x}90$, and the number of grades in an angle of the second polygon is $\frac{2y-4}{y}100$. Hence we must have $\frac{2x-4}{x}90 = \frac{2y-4}{y}100$; therefore $9y(x-2) = 10x(y-2)$; therefore

$$x(20-y) = 18y.$$

We must then try in succession values of y from 3 to 19 inclusive, and ascertain in how many cases we obtain an integral value of x. The admissible values will be found to be these :

y	5	8	10	11	12	14	15	16	17	18	19
x	6	12	18	22	27	42	54	72	102	162	342.

The cases in which the angles are expressed by integers are when

$$y = 5,\ 8,\ 10 \text{ or } 16.$$

Problem 92. *If* $\sec\alpha\sec\beta+\tan\alpha\tan\beta = \tan\gamma$, *show that* $\cos 2\gamma$ *cannot be positive.*

Solution. We have

$$\tan\gamma = \frac{1+\sin\alpha\sin\beta}{\cos\alpha\cos\beta},$$

and

$$\cos 2\gamma = \frac{1-\tan^2\gamma}{1+\tan^2\gamma} = \frac{\cos^2\alpha\cos^2\beta - (1+\sin\alpha\sin\beta)^2}{\cos^2\alpha\cos^2\beta + (1+\sin\alpha\sin\beta)^2}.$$

The numerator

$$= (\cos\alpha\cos\beta + 1 + \sin\alpha\sin\beta)(\cos\alpha\cos\beta - 1 - \sin\alpha\sin\beta)$$
$$= -\{1+\cos(\alpha-\beta)\}\{1-\cos(\alpha+\beta)\};$$

and this cannot be positive, for $1+\cos(\alpha-\beta)$ and $1-\cos(\alpha+\beta)$ cannot be negative.

Problem 93. *Find the general value of an angle such that its cosine is to its tangent as 3 is to 2.*

Solution. Denote the angle by θ; then $\dfrac{\cos\theta}{\tan\theta} = \dfrac{3}{2}$; therefore $\cos^2\theta = \dfrac{3}{2}\sin\theta$, therefore $1 - \sin^2\theta = \dfrac{3}{2}\sin\theta$. By solving this quadratic equation we get $\sin\theta = \dfrac{1}{2}$, or -2; the former is the only admissible value, and hence

$$\theta = n\pi + (-1)^n \frac{\pi}{6}.$$

Problem 94. *If x and y vary so that their sum is constant, find between what limits $\sin x + \sin y$ ranges, and its greatest value.*

Solution. Let A denote the sum of x and y. Then

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2} = 2\sin\frac{A}{2}\cos\frac{x-y}{2};$$

and as $\cos\dfrac{x-y}{2}$ ranges between -1 and $+1$ the value of $\sin x + \sin y$ ranges between $-2\sin\dfrac{A}{2}$ and $2\sin\dfrac{A}{2}$: and the positive value out of these two is algebraically and arithmetically the greatest value of $\sin x + \sin y$.

Problem 95. *Solve the equation $\cos\theta - \sin\theta = \sqrt{2}$.*

Solution. Here $\dfrac{\cos\theta}{\sqrt{2}} - \dfrac{\sin\theta}{\sqrt{2}} = 1$; therefore $\cos\left(\theta + \dfrac{\pi}{4}\right) = 1$; therefore

$$\theta + \frac{\pi}{4} = 2n\pi.$$

Problem 96. *Express in four factors*

$$\sin^2 A + \sin^2 B + \sin^2 C - 2\sin A\sin B\sin C - 1.$$

Solution.

$$\begin{aligned}
&\sin^2 A + \sin^2 B + \sin^2 C - 2\sin A\sin B\sin C - 1\\
&= (\sin A - \sin B\sin C)^2 + \sin^2 B + \sin^2 C - 1 - \sin^2 B\sin^2 C\\
&= (\sin A - \sin B\sin C)^2 - (1-\sin^2 B)(1-\sin^2 C).\\
&= (\sin A - \sin B\sin C)^2 - \cos^2 B\cos^2 C\\
&= (\sin A - \sin B\sin C - \cos B\cos C)(\sin A - \sin B\sin C + \cos B\cos C)\\
&= \{\sin A - \cos(B-C)\}\{\sin A + \cos(B+C)\}\\
&= \left\{\cos\left(\frac{\pi}{2} - A\right) - \cos(B-C)\right\}\left\{\cos\left(\frac{\pi}{2} - A\right) + \cos(B+C)\right\}\\
&= \text{the product of } 4\sin\left(\frac{B-C-A}{2} + \frac{\pi}{4}\right)\sin\left(\frac{B-C+A}{2} - \frac{\pi}{4}\right)\\
&\qquad \text{into } \cos\left(\frac{B+C-A}{2} + \frac{\pi}{4}\right)\cos\left(\frac{B+C+A}{2} - \frac{\pi}{4}\right).
\end{aligned}$$

Instead of $\sin\left(\dfrac{B-C-A}{2} + \dfrac{\pi}{4}\right)\sin\left(\dfrac{B-C+A}{2} - \dfrac{\pi}{4}\right)$ we may put

$$-\cos\left(\frac{A+C-B}{2} + \frac{\pi}{4}\right)\cos\left(\frac{A+B-C}{2} + \frac{\pi}{4}\right).$$

Thus the expression becomes the product of

$$-4\cos\left(\frac{A+B+C}{2}-\frac{\pi}{4}\right)\cos\left(\frac{B+C-A}{2}+\frac{\pi}{4}\right)$$

$$\text{into } \cos\left(\frac{A+C-B}{2}+\frac{\pi}{4}\right)\cos\left(\frac{A+B-C}{2}+\frac{\pi}{4}\right).$$

Problem 97. *Show that* $\cos\frac{\pi}{24}=\frac{1}{4}(-1+\sqrt{2}+\sqrt{3})\sqrt{2+\sqrt{2}}.$

Solution. $\cos^2 7\frac{1}{2}^\circ=\frac{1}{2}(1+\cos 15^\circ)=\frac{1}{2}\{1+\cos(45^\circ-30^\circ)\}=\frac{1}{2}\left\{1+\frac{\sqrt{3}+1}{2\sqrt{2}}\right\}$

$$=\frac{2\sqrt{2}+\sqrt{3}+1}{4\sqrt{2}}=\frac{8+2\sqrt{6}+2\sqrt{2}}{16}.$$

Now it will be found that

$$8+2\sqrt{6}+2\sqrt{2}=(2+\sqrt{2})(6-2\sqrt{2}-2\sqrt{3}+2\sqrt{6})=(2+\sqrt{2})(-1+\sqrt{2}+\sqrt{3})^2;$$

therefore $$\cos 7\frac{1}{2}^\circ=\frac{-1+\sqrt{2}+\sqrt{3}}{4}\sqrt{2+\sqrt{2}}.$$

Problem 98. *Eliminate* θ *between*

$$a\sin\left(\theta+\frac{\pi}{4}\right)+b\sin\left(\theta-\frac{\pi}{4}\right)=\frac{c}{\sqrt{2}},$$

and $$a\cos\left(\theta-\frac{\pi}{4}\right)+b\cos\left(\theta+\frac{\pi}{4}\right)=c\sin\left(2\theta+\frac{\pi}{4}\right).$$

Solution.

By addition $$2a(\sin\theta+\cos\theta)=c(1+\sin 2\theta+\cos 2\theta)$$
$$=2c\cos\theta(\sin\theta+\cos\theta);$$

therefore $$a=c\cos\theta \tag{31}$$

Again, by subtraction, $$2b(\sin\theta-\cos\theta)=c(1-\sin 2\theta-\cos 2\theta)$$
$$=2c\sin\theta(\sin\theta-\cos\theta);$$

therefore $$b=c\sin\theta \tag{32}$$

Square and add (31) and (32); thus $a^2+b^2=c^2$. This assumes that $\tan\theta$ is neither equal to 1 nor to -1.

Problem 99. *If* A, B, C *be any quantities, and* α, β, γ *angles such that*

$$A\cot\alpha+B\cot\beta+C\cot\gamma=(A+B+C)\cot\alpha\cot\beta\cot\gamma,$$

and

$$(B+C)\cot\beta\cot\gamma+(C+A)\cot\gamma\cot\alpha+(A+B)\cot\alpha\cot\beta=0;$$

Show that $$A\sin 2\alpha+B\sin 2\beta+C\sin 2\gamma=0.$$

Solution. We have

$$A\cot\alpha(1-\cot\beta\cot\gamma)+B\cot\beta(1-\cot\alpha\cot\gamma)+C\cot\gamma(1-\cot\beta\cot\alpha)=0,$$

and

$$A\cot\alpha(\cot\beta+\cot\gamma)+B\cot\beta(\cot\gamma+\cot\alpha)+C\cot\gamma(\cot\alpha+\cot\beta)=0.$$

These may be written

$$A\cos\alpha\cos(\beta+\gamma)+B\cos\beta\cos(\gamma+\alpha)+C\cos\gamma\cos(\alpha+\beta)=0,$$
$$A\cos\alpha\sin(\beta+\gamma)+B\cos\beta\sin(\gamma+\alpha)+C\cos\gamma\sin(\alpha+\beta)=0,$$

Hence by Algebra, *Art.* 385 (page 449), we have

$$A=k\cos\beta\cos\gamma\{\cos(\gamma+\alpha)\sin(\alpha+\beta)-\cos(\alpha+\beta)\sin(\gamma+\alpha)\},$$
$$B=k\cos\gamma\cos\alpha\{\cos(\alpha+\beta)\sin(\beta+\gamma)-\cos(\beta+\gamma)\sin(\alpha+\beta)\},$$
$$C=k\cos\alpha\cos\beta\{\cos(\beta+\gamma)\sin(\gamma+\alpha)-\cos(\gamma+\alpha)\sin(\beta+\gamma)\}.$$

where k is some constant.

Thus
$$A=k\cos\beta\cos\gamma\sin(\beta-\gamma),$$
$$B=k\cos\gamma\cos\alpha\sin(\gamma-\alpha),$$
$$C=k\cos\alpha\cos\beta\sin(\alpha-\beta).$$

Therefore
$$A\sin 2\alpha+B\sin 2\beta+C\sin 2\gamma$$
$$=2k\cos\alpha\cos\beta\cos\gamma\{\sin\alpha\sin(\beta-\gamma)+\sin\beta\sin(\gamma-\alpha)+\sin\gamma\sin(\alpha-\beta)\}.$$

The term within the brackets will be seen to vanish, since

$$\sin\alpha\sin(\beta-\gamma)=\frac{1}{2}\{\cos(\gamma+\alpha-\beta)-\cos(\alpha+\beta-\gamma)\},$$
$$\sin\beta\sin(\gamma-\alpha)=\frac{1}{2}\{\cos(\alpha+\beta-\gamma)-\cos(\beta+\gamma-\alpha)\},$$

and
$$\sin\gamma\sin(\alpha-\beta)=\frac{1}{2}\{\cos(\beta+\gamma-\alpha)-\cos(\gamma+\alpha-\beta)\}.$$

Or we might proceed thus : let π stand for $\alpha+\beta+\gamma$; then the two given relations may be written

$$A\cos\alpha\cos(\sigma-\alpha)+B\cos\beta\cos(\sigma-\beta)+C\cos\gamma\cos(\sigma-\gamma)=0,$$
$$A\cos\alpha\sin(\sigma-\alpha)+B\cos\beta\sin(\sigma-\beta)+C\cos\gamma\sin(\sigma-\gamma)=0;$$

therefore
$$(A\cos^2\alpha+B\cos^2\beta+C\cos^2\gamma)\cos\sigma$$
$$=-(A\sin\alpha\cos\alpha+B\sin\beta\cos\beta+C\sin\gamma\cos\gamma)\sin\sigma \qquad (33)$$

And
$$(A\cos^2\alpha+B\cos^2\beta+C\cos^2\gamma)\sin\sigma$$
$$=(A\sin\alpha\cos\alpha+B\sin\beta\cos\beta+C\sin\gamma\cos\gamma)\cos\sigma \qquad (34)$$

Multiply (33) by $\sin\sigma$ and (34) by $\cos\sigma$ and subtract : thus

$$A\sin\alpha\cos\alpha+B\sin\beta\cos\beta+C\sin\gamma\cos\gamma=0,$$

which is the required result.

Again, multiply (33) by $\cos\sigma$ and (34) by $\sin\sigma$ and add; then we obtain the additional result $A\cos^2\alpha+B\cos^2\beta+C\cos^2\gamma=0$.

Problem 100.

If
$$\cos\theta=\cos\alpha\cos\beta-\sin\alpha\sin\beta\sqrt{1-c^2\sin^2\theta},$$
and
$$\cos\phi=\cos\alpha\cos\beta+\sin\alpha\sin\beta\sqrt{1-c^2\sin^2\phi};$$

Show that

$$\cos\theta+\cos\phi=\frac{2\cos\alpha\cos\beta}{1-c^2\sin^2\alpha\sin^2\beta},$$

and

$$1+\cos\theta\cos\phi=\frac{\cos^2\alpha+\cos^2\beta}{1-c^2\sin^2\alpha\sin^2\beta}.$$

Hence find $\sin\theta\sin\phi$ *and* $\tan\frac{\theta}{2}\tan\frac{\phi}{2}$.

Solution. From the first equation

$$(\cos\theta - \cos\alpha\cos\beta)^2 = \sin^2\alpha\sin^2\beta(1 - c^2\sin^2\theta);$$

substitute $1 - \cos^2\theta$ for $\sin^2\theta$; thus

$$\cos^2\theta(1 - c^2\sin^2\alpha\sin^2\beta) - 2\cos\theta\cos\alpha\cos\beta$$
$$+ \cos^2\alpha\cos^2\beta - (1 - c^2)\sin^2\alpha\sin^2\beta = 0.$$

The second equation leads to the same quadratic for finding $\cos\phi$. Hence we infer that $\cos\theta$ is one root of the quadratic and $\cos\phi$ the other. Hence by the theory of quadratic equations, *Algebra*, *Chapter* XXII,

$$\cos\theta + \cos\phi = \frac{2\cos\alpha\cos\beta}{1 - c^2\sin^2\alpha\sin^2\beta},$$

$$\cos\theta\cos\phi = \frac{\cos^2\alpha\cos^2\beta - (1 - c^2)\sin^2\alpha\sin^2\beta}{1 - c^2\sin^2\alpha\sin^2\beta};$$

therefore $$1 + \cos\theta\cos\phi = \frac{1 - \sin^2\alpha\sin^2\beta + \cos^2\alpha\cos^2\beta}{1 - c^2\sin^2\alpha\sin^2\beta}$$
$$= \frac{\cos^2\alpha + \cos^2\beta}{1 - c^2\sin^2\alpha\sin^2\beta}.$$

Then $$\sin^2\theta\sin^2\phi = (1 - \cos^2\theta)(1 - \cos^2\phi)$$
$$= (1 + \cos\theta\cos\phi)^2 - (\cos\theta + \cos\phi)^2$$
$$= \frac{(\cos^2\alpha - \cos^2\beta)^2}{(1 - c^2\sin^2\alpha\sin^2\beta)^2};$$

therefore $$\sin\theta\sin\phi = \pm\frac{\cos^2\alpha - \cos^2\beta}{1 - c^2\sin^2\alpha\sin^2\beta}.$$

And $$\tan\frac{\theta}{2}\tan\frac{\phi}{2} = \frac{1 - \cos\theta}{\sin\theta}\cdot\frac{1 - \cos\phi}{\sin\phi}$$
$$= \frac{1 - (\cos\theta + \cos\phi) + \cos\theta\cos\phi}{\sin\theta\sin\phi}$$
$$= \frac{(\cos\alpha - \cos\beta)^2}{\pm(\cos^2\alpha - \cos^2\beta)} = \pm\frac{\cos\alpha - \cos\beta}{\cos\alpha + \cos\beta}.$$

Problem 101. *Show that*

$$\cos 11A + 3\cos 9A + 3\cos 7A + \cos 5A = 16\cos^3 A\cos\left(4A + \frac{\pi}{4}\right)\cos\left(4A - \frac{\pi}{4}\right).$$

Solution.

$$\cos 11A + \cos 5A = 2\cos 8A\cos 3A,$$
$$3\cos 9A + 3\cos 7A = 6\cos 8A\cos A;$$

hence by addition we find that the proposed expression

$$= 2\cos 8A(\cos 3A + 3\cos A) = 8\cos 8A\cos^3 A$$
$$= 8\cos^3 A(2\cos^2 4A - 1) = 16\cos^3 A\left(\cos^2 4A - \frac{1}{2}\right)$$
$$= 16\cos^3 A\left(\cos^2 4A - \sin^2\frac{\pi}{4}\right) = 16\cos^3 A\cos\left(4A + \frac{\pi}{4}\right)\cos\left(4A - \frac{\pi}{4}\right);$$

see *Art.* 83 (page 404).

Problem 102. *Find approximately the distance at which a coin an inch in diameter must be held from the eye looking towards the moon so as just to conceal it, the apparent angular diameter of the moon being half a degree.*

Solution. Let the distance be denoted by x inches; then we must have $\frac{\frac{1}{2}}{x}$ = the tangent of a quarter of a degree. As the tangent of a small angle is approximately equal to its circular measure we have approximately $\frac{1}{2x} = \frac{1}{4} \cdot \frac{\pi}{180}$; therefore $x = \frac{2 \times 180}{\pi} = 114.6$ nearly.

Problem 103. *Solve the equation* $\cos^3 \theta \sin 3\theta + \sin^3 \theta \cos 3\theta = \frac{3}{4}$.

Solution. Let *Problem* 27 of Chapter VI. this becomes $\sin 4\theta = 1$; therefore

$$4\theta = (4n+1)\frac{\pi}{2}.$$

Problem 104. *Eliminate* θ *and* ϕ *from*

$$c \sin\theta = a\sin(\theta+\phi), \quad a\sin\phi = b\sin\theta,$$
$$\cos\theta - \cos\phi = 2m.$$

Solution. Here $c\sin\theta = a\sin\theta\cos\phi + a\cos\theta\sin\phi$;
substitute for $\sin\phi$ and $\cos\phi$; thus

$$c\sin\theta = a(\cos\theta - 2m)\sin\theta + a\cos\theta \times \frac{b}{a}\sin\theta;$$

therefore
$$c = a(\cos\theta - 2m) + b\cos\theta;$$

therefore
$$\cos\theta = \frac{c+2am}{a+b}.$$

Therefore
$$\cos\phi = \frac{c+2am}{a+b} - 2m = \frac{c-2bm}{a+b}.$$

But $a^2\sin^2\phi = b^2\sin^2\theta$; therefore

$$a^2 - b^2 = a^2\cos^2\phi - b^2\cos^2\theta = \frac{a^2(c-2bm)^2 - b^2(c+2am)^2}{(a+b)^2}$$
$$= \frac{(a-b)c^2 - 4abcm}{a+b}.$$

Problem 105. *In any triangle*

$$a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0.$$

Solution. We have, by *Art.* 252 (page 428)

$$a = 2R\sin A, \quad b = 2R\sin B, \quad c = 2R\sin C;$$

hence the proposed expression

$$= 2R\{\sin A\sin(B-C) + \sin B\sin(C-A) + \sin C\sin(A-B)\}$$
$$= 2R\{\sin(B+C)\sin(B-C) + \sin(C+A)\sin(C-A) + \sin(A+B)\sin(A-B)\}$$
$$= 2R\{\sin^2 B - \sin^2 C + \sin^2 C - \sin^2 A + \sin^2 A - \sin^2 B\}$$
$$= 0.$$

Problem 106. *Show that*

$$\cos^2 18^\circ \sin^2 36^\circ - \cos 36^\circ \sin 18^\circ = \frac{1}{16}.$$

Solution. By *Art*. 108 (page 409) the proposed expression

$$= \frac{10+2\sqrt{5}}{16} \cdot \frac{10-2\sqrt{5}}{16} - \frac{\sqrt{5}+1}{4} \cdot \frac{\sqrt{5}-1}{4} = \frac{5}{16} - \frac{1}{4} = \frac{1}{16}.$$

Problem 107. *The perpendiculars from the angles of a triangle on the opposite sides meet at O; and $OA = x$, $OB = y$, $OC = z$; show that*

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{xyz}.$$

Solution. From the triangle OAB we have

$$\frac{OA}{AB} = \frac{\sin OBA}{\sin AOB} = \frac{\cos A}{\sin C};$$

therefore
$$x = \frac{c\cos A}{\sin C} = \frac{a\cos A}{\sin A}.$$

Similarly
$$y = \frac{b\cos B}{\sin B}, \text{ and } z = \frac{c\cos C}{\sin C}.$$

Hence we have only to show that

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C;$$

and this is known to be true by *Art*. 114 (page 409).

Problem 108. *The top of a pole placed against a wall at an angle A with the horizontal plane just touches the coping; and when its foot is moved a yards further from the wall and its angle of inclination is B, it rests on the sill of a window. Show that the perpendicular distance between the coping and the sill is*

$$a \cot \frac{A+B}{2}.$$

Solution. Let l denote the length of the pole. The distance of the coping from the ground is $l \sin A$, and the distance of the sill from the ground is $l \sin B$; hence the distance from the coping to the sill $= l(\sin A - \sin B)$.

The distance of the foot of the ladder from the wall is $l \cos A$ at first, and $l \cos B$ afterwards; therefore $a = l(\cos B - \cos A)$.

Substitute for l in the former expression, and we obtain

$$\frac{a(\sin A - \sin B)}{\cos B - \cos A}, \text{ that is } \frac{a\cos\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A+B)}, \text{ that is } a\cot\frac{1}{2}(A+B).$$

Problem 109. *AB is the diameter of a circle, C its centre; a straight line AP is drawn dividing the semicircle into two equal parts; θ is the circular measure of the complement of the angle PCB. Show that* $\cos\theta = \theta$.

Solution. Let r denote the radius. Then the area of the sector $PCB = \frac{r^2}{2}\left(\frac{\pi}{2} - \theta\right)$, by *Art*. 258 (page 431);

and the area of the triangle $ACP = \frac{r^2}{2}\sin\left(\frac{\pi}{2}+\theta\right)$, by *Art.* 247 (page 425).

The sum of these two areas by supposition is equal to half the area of the semicircle; thus

$$\frac{r^2}{2}\left(\frac{\pi}{2}-\theta\right)+\frac{r^2}{2}\cos\theta=\frac{\pi r^2}{4};$$

then by simplifying we obtain $\cos\theta=\theta$.

Problem 110. *The sides of a regular pentagon and of a regular decagon inscribed in a circle of radius R are a and a′, and the radii of the circles inscribed in them are r and r′ respectively. Show that*

$$a^2-a'^2=R^2,\ \text{and}\ \frac{a}{r}+\frac{a'}{r'}=\frac{2R}{r'}.$$

Solution. By *Art.* 255 (page 430) we have $a=2R\sin 36°$, and $a'=2R\sin 18°$;

therefore $a^2-a'^2=4R^2\left\{\frac{10-2\sqrt{5}}{16}-\frac{(\sqrt{5}-1)^2}{16}\right\}=\frac{4R^2\times 4}{16}=R^2.$

Also $\frac{a}{r}=2\tan 36°$, and $\frac{a'}{r'}=2\tan 18°$;

therefore
$$\frac{a}{r}+\frac{a'}{r'}=2\left(\frac{\sin 36°}{\cos 36°}+\frac{\sin 18°}{\cos 18°}\right)=\frac{2\sin(36°+18°)}{\cos 36°\cos 18°}.$$
$$=\frac{2\sin 54°}{\cos 36°\cos 18°}=\frac{2}{\cos 18°}=\frac{2R}{r'}.$$

Problem 111. *If $A+B+C=(2n+1)\pi$, show that*

$$\cos^4\frac{A}{2}+\cos^4\frac{B}{2}+\cos^4\frac{C}{2}-2\left(\cos^2\frac{A}{2}\cos^2\frac{B}{2}+\cos^2\frac{B}{2}\cos^2\frac{C}{2}+\cos^2\frac{C}{2}\cos^2\frac{A}{2}\right)$$
$$+4\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}=0.$$

Solution.

$$\cos^2\frac{A}{2}=\frac{1}{2}(1+\cos A).$$
$$\cos^4\frac{A}{2}=\frac{1}{4}(1+\cos A)^2=\frac{1}{4}(1+2\cos A+\cos^2 A)$$
$$=\frac{1}{4}\left\{1+2\cos A+\frac{1}{2}(1+\cos 2A)\right\}=\frac{3}{8}+\frac{1}{2}\cos A+\frac{1}{8}\cos 2A.$$

In this way the proposed expression becomes

$$\frac{9}{8}+\frac{1}{2}(\cos A+\cos B+\cos C)+\frac{1}{8}(\cos 2A+\cos 2B+\cos 2C)$$
$$-\frac{1}{2}(1+\cos A)(1+\cos B)-\frac{1}{2}(1+\cos B)(1+\cos C)-\frac{1}{2}(1+\cos C)(1+\cos A)$$
$$+\frac{1}{2}(1+\cos A)(1+\cos B)(1+\cos C)$$
$$=\frac{1}{8}+\frac{1}{8}(\cos 2A+\cos 2B+\cos 2C)+\frac{1}{2}\cos A\cos B\cos C$$
$$=\frac{1}{8}-\frac{1}{8}=0;\ \text{see Example VIII. 18.}$$

Problem 112. *If* $\tan^2\theta$ *is less than unity, show that*

$$\tan^2\theta - \frac{1}{2}\tan^4\theta + \frac{1}{3}\tan^6\theta\ldots\ldots = \sin^2\theta + \frac{1}{2}\sin^4\theta + \frac{1}{3}\sin^6\theta + \ldots\ldots$$

Solution. We have $1+\tan^2\theta = \dfrac{1}{\cos^2\theta} = \dfrac{1}{1-\sin^2\theta}$; take the logarithms of both sides; thus

$$\log\left(1+\tan^2\theta\right) = \log\frac{1}{1-\sin^2\theta} = -\log\left(1-\sin^2\theta\right);$$

therefore, by *Art.* 146 (page 415),

$$\tan^2\theta - \frac{1}{2}\tan^4\theta + \frac{1}{3}\tan^6\theta - \ldots = \sin^2\theta + \frac{1}{2}\sin^4\theta + \frac{1}{3}\sin^6\theta + \ldots$$

The series are convergent, since $\tan^2\theta$ is supposed to be less than unity.

Problem 113. *Solve the equation* $\cos\theta - \cos 2\theta = \sin 3\theta$.

Solution.

Here $$2\sin\frac{3\theta}{2}\sin\frac{\theta}{2} = 2\sin\frac{3\theta}{2}\cos\frac{3\theta}{2};$$

therefore either $$\sin\frac{3\theta}{2} = 0, \text{ or } \sin\frac{\theta}{2} = \cos\frac{3\theta}{2}.$$

If $\sin\dfrac{3\theta}{2} = 0$, then $\dfrac{3\theta}{2} = n\pi$.

If $\sin\dfrac{\theta}{2} = \cos\dfrac{3\theta}{2}$, or $\cos\left(\dfrac{\pi}{2} - \dfrac{\theta}{2}\right) = \cos\dfrac{3\theta}{2}$, then $\dfrac{\pi}{2} - \dfrac{\theta}{2} = 2n\pi \pm \dfrac{3\theta}{2}$.

Problem 114. *In any triangle*

$$\frac{a^2\sin(B-C)}{\sin A} + \frac{b^2\sin(C-A)}{\sin B} + \frac{c^2\sin(A-B)}{\sin C} = 0.$$

Solution. We have

$$a = 2R\sin A,\ b = 2R\sin B,\ c = 2R\sin C;$$

hence the proposed expression

$$= 4R^2\{\sin A\sin(B-C) + \sin B\sin(C-A) + \sin C\sin(A-B)\},$$

and this is zero, as in the solution of *Problem* 105.

Problem 115. *Solve the equation* $\cos 3\theta + \sin 3\theta = \cos\theta + \sin\theta$.

Solution. We have $\dfrac{\cos 3\theta}{\sqrt{2}} + \dfrac{\sin 3\theta}{\sqrt{2}} = \dfrac{\cos\theta}{\sqrt{2}} + \dfrac{\sin\theta}{\sqrt{2}}$;

therefore $$\cos\left(3\theta - \frac{\pi}{4}\right) = \cos\left(\theta - \frac{\pi}{4}\right);$$

therefore $$3\theta - \frac{\pi}{4} = 2n\pi \pm \left(\theta - \frac{\pi}{4}\right).$$

Problem 116. *If* $\tan^2 x = \tan(\alpha + x)\tan(\alpha - x)$ *then* $\sin 2x = \sqrt{2}\cdot\sin\alpha$.

Solution. We have $\tan^2 x = \dfrac{\sin(\alpha + x)\sin(\alpha - x)}{\cos(\alpha + x)\cos(\alpha - x)} = \dfrac{\sin^2\alpha - \sin^2 x}{\cos^2 x - \sin^2\alpha}$,

therefore $$\sin^2 x(\cos^2 x - \sin^2\alpha) = \cos^2 x(\sin^2\alpha - \sin^2 x),$$

therefore $$2\sin^2 x\cos^2 x = \sin^2\alpha(\sin^2 x + \cos^2 x) = \sin^2\alpha,$$

therefore $$4\sin^2 x\cos^2 x = 2\sin^2\alpha,$$

therefore $$2\sin x\cos x = \sqrt{2}.\sin\alpha,$$

therefore $$\sin 2x = \sqrt{2}.\sin\alpha,$$

Problem 117. *If*

$$\tan^2 A\tan A' = \tan^2 B\tan B' = \tan^2 C\tan C' = \tan A\tan B\tan C,$$

and $$\operatorname{cosec} 2A + \operatorname{cosec} 2B + \operatorname{cosec} 2C = 0,$$

show that $$\tan(A - A') = \tan(B - B') = \tan(C - C').$$

Solution. Put x for $\tan A$, y for $\tan B$, z for $\tan C$, x' for $\tan A'$, y' for $\tan B'$, and z' for $\tan C'$, for the sake of abbreviation. Then we have given that

$$x^2x' = y^2y' = z^2z' = xyz \tag{35}$$

and $$\frac{1+x^2}{2x} + \frac{1+y^2}{2y} + \frac{1+z^2}{2z} = 0 \tag{36}$$

Now $$\tan(A - A') = \frac{x - x'}{1 + xx'} = \frac{x - \dfrac{yz}{x}}{1 + yz}, \text{ by (35)}, = \frac{x^2 - yz}{x(1 + yz)}.$$

But from (36) we have

$$x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} = 0,$$

therefore $$xyz(x + y + z) + xy + yz + zx = 0,$$

$$\begin{aligned}\therefore\ x^2 - yz &= x^2 + xyz(x + y + z) + xy + zx \\ &= x(x + y + z) + xyz(x + y + z) = x(1 + yz)(x + y + z);\end{aligned}$$

therefore $$\frac{x^2 - yz}{x(1 + yz)} = x + y + z$$

Thus $$\tan(A - A') = \tan A + \tan B + \tan C.$$

Similarly $\tan(B - B')$ and $\tan(C - C')$ may be shown to be equal to the same expression.

Problem 118. *If*

$$\cos 60^\circ = \sin 36^\circ\cos A, \quad \cos 36^\circ = \sin 60^\circ\cos B,$$

and $$\cos C = \cos A\cos B,$$

then one value of $A + B + C$ is 90°.

Solution.

Here $$\cos A = \frac{\cos 60^\circ}{\sin 36^\circ},$$

therefore $$\sin A = \frac{\sqrt{\sin^2 36^\circ - \cos^2 60^\circ}}{\sin 36^\circ},$$

and $$\sin^2 36^\circ - \cos^2 60^\circ = \frac{10 - 2\sqrt{5}}{16} - \frac{1}{4} = \frac{6 - 2\sqrt{5}}{16} = \left(\frac{\sqrt{5} - 1}{4}\right)^2;$$

therefore $$\sin A = \frac{\sqrt{5}-1}{4\sin 36^\circ}.$$

Hence $$\tan A = \frac{\sqrt{5}-1}{4\cos 60^\circ} = \frac{\sqrt{5}-1}{2}.$$

Again, $$\cos B = \frac{\cos 36^\circ}{\sin 60^\circ};$$

$$\therefore\ \sin B = \frac{\sqrt{\sin^2 60^\circ - \cos^2 36^\circ}}{\sin 60^\circ} = \frac{\sqrt{\sin^2 36^\circ - \cos^2 60^\circ}}{\sin 60^\circ} = \frac{\sqrt{5}-1}{4\sin 60^\circ}.$$

Hence

$$\tan B = \frac{\sqrt{5}-1}{4\cos 36^\circ} = \frac{\sqrt{5}-1}{\sqrt{5}+1} = \frac{(\sqrt{5}-1)^2}{(\sqrt{5}+1)(\sqrt{5}-1)} = \frac{6-2\sqrt{5}}{4} = \frac{3-\sqrt{5}}{2}.$$

Therefore $$\tan A + \tan B = \frac{\sqrt{5}-1}{2} + \frac{3-\sqrt{5}}{2} = 1;$$

therefore $\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B} = 1$, therefore $\sin(A+B) = \cos A\cos B = \cos C$; therefore $A + B = 90^\circ - C$ is one solution.

Problem 119. *Two rows of houses of equal height stand at an angle to each other. On a sunshiny day the distance of the corner of the shadow from the corner where the rows meet is observed twice and found to be a and b respectively. Show that if h be the height of the houses and α the difference between the altitudes of the sun on the two occasions*

$$h^2 + h(b-a)\cot\alpha + ab = 0.$$

Solution. Let θ be the sun's altitude at the first observation, and $\theta+\alpha$ that at the second observation; then

$$h = a\tan\theta,\ \text{and}\ h = b\tan(\theta+\alpha).$$

Thus $$h = \frac{b(\tan\theta + \tan\alpha)}{1-\tan\theta\tan\alpha} = \frac{\dfrac{hb}{a} + b\tan\alpha}{1 - \dfrac{h}{a}\tan\alpha},$$

therefore $$h\left(1 - \frac{h}{a}\tan\alpha\right) = \frac{hb}{a} + b\tan\alpha,$$

therefore $$h^2\tan\alpha + h(b-a) + ab\tan\alpha = 0,$$

therefore $$h^2\tan\alpha + h(b-a)\cot\alpha + ab = 0.$$

Problem 120. *P is any point in the arc of a semicircle APB; two circles are described touching the semicircle and also touching AP, BP at their middle points. Show that the rectangle contained by the radii of these circles is one eighth of the square described on the radius of the circle which is inscribed in the triangle APB.*

Solution.

Let $$AP = b,\ BP = a,\ AB = c.$$

The diameter of the circle which touches the semicircle and also touches AP at its middle point is $\dfrac{c}{2} - \dfrac{c}{2}\sin PAB$, that is $\dfrac{c}{2} - \dfrac{c}{2}\dfrac{a}{c}$, that is $\dfrac{c-a}{2}$; therefore the radius of this circle is $\dfrac{c-a}{4}$.

Similarly the radius of the circle which touches the semicircle and also touches BP at its middle point is $\dfrac{c-b}{4}$. We have then to show that

$$\frac{(c-a)(c-b)}{16} = \frac{r^2}{8}, \text{ that is } \frac{(c-a)(c-b)}{2} = r^2.$$

But $r = \dfrac{S}{s} = \dfrac{ab}{a+b+c} = \dfrac{ab(a+b-c)}{(a+b)^2-c^2} = \dfrac{a+b-c}{2}$, since $c^2 = a^2+b^2$; therefore

$$r^2 = \frac{(a+b-c)^2}{4} = \frac{2c^2+2ab-2c(a+b)}{4} = \frac{(c-a)(c-b)}{2}.$$

Problem 121. *Show that*

$$\sin\alpha\sin(\beta-\gamma)\cos(\beta+\gamma-\alpha) + \sin\beta\sin(\gamma-\alpha)\cos(\gamma+\alpha-\beta) \\ + \sin\gamma\sin(\alpha-\beta)\cos(\alpha+\beta-\gamma) = 0.$$

Solution.

$$\begin{aligned} &\sin\alpha\sin(\beta-\gamma)\cos(\beta+\gamma-\alpha) \\ &= \frac{1}{2}\{\cos(\alpha-\beta+\gamma) - \cos(\alpha+\beta-\gamma)\}\cos(\beta+\gamma-\alpha) \\ &= \frac{1}{4}\{\cos 2\gamma + \cos 2(\alpha-\beta) - \cos 2\beta - \cos 2(\alpha-\gamma)\}. \end{aligned}$$

The other two terms may be transformed in a similar manner, and then it will be obvious that the sum is zero.

Problem 122. *Show that*

$$\log\cot\theta = \cos 2\theta + \frac{1}{3}(\cos 2\theta)^3 + \frac{1}{5}(\cos 2\theta)^5 + \ldots\ldots$$

Solution.

$$\begin{aligned} \cot\theta &= \frac{\cos\theta}{\sin\theta} = \frac{2\cos^2\theta}{2\sin\theta\cos\theta} = \frac{1+\cos 2\theta}{\sin 2\theta} \\ &= \frac{1+\cos 2\theta}{\sqrt{(1-\cos^2 2\theta)}} = \sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}. \end{aligned}$$

Hence, taking logarithms we have

$$\log\cot\theta = \frac{1}{2}\log\frac{1+\cos 2\theta}{1-\cos 2\theta} = \cos 2\theta + \frac{1}{3}(\cos 2\theta)^3 + \frac{1}{5}(\cos 2\theta)^5 + \ldots$$

Problem 123. *If $A+B+C = (2n+1)\pi$, show that*

$$\begin{aligned} &\cot A + \cot B + \cot C - 2(\cot 2A + \cot 2B + \cot 2C) \\ &= \left(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right)(\sec A - 1)(\sec B - 1)(\sec C - 1). \end{aligned}$$

Solution. We have shown in the solution of *Example* VIII. 15 that

$$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2};$$

hence the expression on the right hand-side

$$= \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\cdot\frac{8\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}}{\cos A\cos B\cos C}$$

$$= \frac{8\sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}}{\cos A\cos B\cos C} = \tan A\tan B\tan C.$$

Again $\qquad \cot A - 2\cot 2A = \cot A - \dfrac{2(\cos^2 A - \sin^2 A)}{2\cos A\sin A} = \tan A;$

similarly $\quad \cot B - 2\cot 2B = \tan B,$ and $\cot C - 2\cot 2C = \tan C;$

thus the expression on the left-hand side

$$= \tan A + \tan B + \tan C = \tan A\tan B\tan C, \text{ by } Art.\ 114 \text{ (page 409)}.$$

Thus the two expressions are equivalent.

Problem 124. *Show that*

$$\sin A\sin B\sin(A-B) + \sin B\sin C\sin(B-C) + \sin C\sin A\sin(C-A)$$
$$= \frac{1}{4}\left\{\sin(2A-2B) + \sin(2B-2C) + \sin(2C-2A)\right\}.$$

Solution. $\sin A\sin B\sin(A-B) = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\}\sin(A-B)$

$$= \frac{1}{4}\sin(2A-2B) - \frac{1}{4}(\sin 2A - \sin 2B).$$

Transform the second and third terms in the same way; then by addition we obtain the required result.

Problem 125. *In any triangle*

$$\frac{1}{a}\cos^2\frac{A}{2} + \frac{1}{b}\cos^2\frac{B}{2} + \frac{1}{c}\cos^2\frac{C}{2} = \frac{(a+b+c)^2}{4abc}.$$

Solution. $\dfrac{1}{a}\cos^2\dfrac{A}{2} + \dfrac{1}{b}\cos^2\dfrac{B}{2} + \dfrac{1}{c}\cos^2\dfrac{C}{2} = \dfrac{s(s-a) + s(s-b) + s(s-c)}{abc}$

$$= \frac{3s^2 - s(a+b+c)}{abc} = \frac{3s^2 - 2s^2}{abc} = \frac{s^2}{abc} = \frac{(a+b+c)^2}{4abc}.$$

Problem 126. *Find x from the equation*

$$\sec\left(\frac{\pi}{4} + x\right) + \sec\left(\frac{\pi}{4} - x\right) = 2\sqrt{2}.$$

Solution. Here $\dfrac{1}{\cos\left(\frac{\pi}{4} + x\right)} + \dfrac{1}{\cos\left(\frac{\pi}{4} - x\right)} = 2\sqrt{2},$

therefore $\qquad \dfrac{\sqrt{2}}{\cos x - \sin x} + \dfrac{\sqrt{2}}{\cos x + \sin x} = 2\sqrt{2},$

therefore $\qquad \cos x = \cos^2 x - \sin^2 x = \cos 2x,$

therefore $\qquad 2x = 2n\pi \pm x.$

Problem 127. *Show that*

$$\frac{\sin(\theta-\alpha)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)} + \frac{\sin(\theta-\beta)}{\sin(\beta-\alpha)\sin(\beta-\gamma)} + \frac{\sin(\theta-\gamma)}{\sin(\gamma-\alpha)\sin(\gamma-\beta)} = 0.$$

Solution. Express the fractions with the common denominator

$$\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha):$$

then the numerator becomes

$$-\{\sin(\beta-\gamma)\sin(\theta-\alpha)+\sin(\gamma-\alpha)\sin(\theta-\beta)+\sin(\alpha-\beta)\sin(\theta-\gamma)\}.$$

Now $\sin(\beta-\gamma)\sin(\theta-\alpha)=\frac{1}{2}\cos(\theta-\alpha-\beta+\gamma)-\frac{1}{2}\cos(\theta-\alpha+\beta-\gamma),$

$$\sin(\gamma-\alpha)\sin(\theta-\beta)=\frac{1}{2}\cos(\theta-\beta+\alpha-\gamma)-\frac{1}{2}\cos(\theta-\beta+\gamma-\alpha),$$

$$\sin(\alpha-\beta)\sin(\theta-\gamma)=\frac{1}{2}\cos(\theta-\gamma+\beta-\alpha)-\frac{1}{2}\cos(\theta-\gamma+\alpha-\beta);$$

thus the sum of the expressions is zero.

Problem 128. *From each of two stations on a horizontal plane, at a distance c from each other, a pillar on a distant hill, in the vertical plane passing through the stations, is seen under the same visual angle, and the angles of elevation of the top of the pillar at the stations are α and β. Show that the length of the pillar is equal to*

$$\frac{c\cos(\beta+\alpha)}{\sin(\beta-\alpha)}.$$

Solution. Let x denote the length of the pillar, h the height of the foot of the pillar above the horizontal plane, b the horizontal distance of the pillar from the first station. Let θ be the angle subtended by the pillar. Then

$$\frac{h}{b}=\tan(\alpha-\theta),\quad \frac{h}{b+c}=\tan(\beta-\theta),\quad \frac{h+x}{b}=\tan\alpha,\quad \frac{h+x}{b+c}=\tan\beta.$$

And from the fact that a circle would pass through the two stations and the top and the foot of the pillar we have $\alpha+\beta-\theta=\frac{\pi}{2}$. Thus

$$\frac{h}{b}=\cot\beta,\ \frac{h+x}{b}=\tan\alpha;\quad \text{therefore}$$

$$\frac{x}{b}=\tan\alpha-\cot\beta=-\frac{\cos(\alpha+\beta)}{\cos\alpha\sin\beta}.$$

Similarly $$\frac{x}{b+c}=\tan\beta-\cot\alpha=-\frac{\cos(\alpha+\beta)}{\cos\beta\sin\alpha}.$$

Therefore $$\frac{c}{x}=\frac{\cos\alpha\sin\beta-\cos\beta\sin\alpha}{\cos(\alpha+\beta)}=\frac{\sin(\beta-\alpha)}{\cos(\beta+\alpha)};$$

therefore $$x=\frac{c\cos(\beta+\alpha)}{\sin(\beta-\alpha)}.$$

Problem 129. *If in a right-angled triangle twice the distance between the centres of the inscribed and circumscribed circles is a mean proportional between the hypotenuse and the excess of the sum of the sides over the hypotenuse, shew that the radius of the inscribed circle is equal to one sixth of the hypotenuse.*

Solution. In every *right-angled* triangle $r=\frac{1}{2}(a+b-c)$; see the solution of *Problem* 120. In the present case $2\sqrt{R^2-2Rr}=\sqrt{c(a+b-c)}$; and $R=\frac{c}{2}$: thus $4\left(\frac{c^2}{4}-cr\right)=c(a+b-c)$; therefore $c-4r=a+b-c$; therefore $4r=2c-a-b$;

therefore $2(a+b-c)=2c-a-b$; therefore $a+b=\frac{4c}{3}$; and therefore $r=\frac{c}{6}$.

Problem 130. *In any triangle show that*

$$\cos^2\frac{1}{2}A+\cos^2\frac{1}{2}B+\cos^2\frac{1}{2}C=2+\frac{r}{2R}.$$

Solution. $\cos^2\frac{1}{2}A+\cos^2\frac{1}{2}B+\cos^2\frac{1}{2}C=\frac{1}{2}(3+\cos A+\cos B+\cos C)$

$$=\frac{1}{2}\left(4+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right),$$ by *Art.* 114 (page 409).

$$=2+\frac{2S^2}{sabc}=2+\frac{2rS}{abc}=2+\frac{r}{2R}.$$

Problem 131. *If* $\frac{\sin(x+A)}{\sin(x+B)}=\sqrt{\frac{\sin 2A}{\sin 2B}}$, *then* $\tan^2 x=\tan A\tan B$.

Solution.

$$\frac{\sin(x+A)}{\sqrt{\sin 2A}}=\frac{\sin(x+B)}{\sqrt{\sin 2B}};$$

therefore $$\frac{\sin x\cos A+\cos x\sin A}{\sqrt{\sin 2A}}=\frac{\sin x\cos B+\cos x\sin B}{\sqrt{\sin 2B}},$$

therefore $$\frac{\cos A(\tan x+\tan A)}{\sqrt{\sin 2A}}=\frac{\cos B(\tan x+\tan B)}{\sin 2B},$$

therefore $$\frac{\tan x+\tan A}{\sqrt{\tan A}}=\frac{\tan x+\tan B}{\sqrt{\tan B}},$$

$$\therefore\ \tan x(\sqrt{\tan B}-\sqrt{\tan A})=\sqrt{\tan A\tan B}(\sqrt{\tan B}-\sqrt{\tan A}),$$

therefore $$\tan x=\sqrt{\tan A\tan B}.$$

Problem 132. *If* $A+B+C=(2n+1)\pi$, *show that*

$$\sin^2 2A+\sin^2 2B+\sin^2 2C+2\cos 2A\cos 2B\cos 2C=2.$$

Solution. $\sin^2 2A+\cos 2A\cos 2B\cos 2C=1-\cos^2 2A+\cos 2A\cos 2B\cos 2C$

$$=1+\cos 2A\{\cos 2B\cos 2C-\cos 2A\}$$
$$=1+\cos 2A\{\cos 2B\cos 2C-\cos(2B+2C)\}$$
$$=1+\cos 2A\sin 2B\sin 2C.$$

Similarly

$$\sin^2 2B+\cos 2A\cos 2B\cos 2C=1+\cos 2B\sin 2A\sin 2C.$$

Hence the proposed expression

$$=2+\sin 2C\{\sin 2C+\sin 2B\cos 2A+\sin 2A\cos 2B\}$$
$$=2+\sin 2C\{-\sin(2A+2B)+\sin(2A+2B)\}$$
$$=2.$$

Problem 133. *Sum the infinite series*

$$(1+2)\log 2 + \frac{1+2^2}{\lfloor 2}(\log 2)^2 + \frac{1+2^3}{\lfloor 3}(\log 2)^3 + \ldots.$$

Solution. The series may be separated into two, namely

$$\log 2 + \frac{1}{\lfloor 2}(\log 2)^2 + \frac{1}{\lfloor 3}(\log 2)^3 + \ldots$$

and

$$2\log 2 + \frac{1}{\lfloor 2}(2\log 2)^2 + \frac{1}{\lfloor 3}(2\log 2)^3 + \ldots$$

and is therefore equal to $e^{\log 2} - 1 + e^{2\log 2} - 1$, that is to $e^{\log 2} - 1 + e^{\log 4} - 1$, that is to $2 - 1 + 4 - 1$, that is to 4.

Problem 134. *Show that*

$$\sin(A-B)\cos(C-B)\cos(A-C) + \sin(B-C)\cos(A-C)\cos(B-A)$$
$$+ \sin(C-A)\cos(B-A)\cos(C-B)$$
$$= \frac{1}{4}\{\sin(2B-2A) + \sin(2C-2B) + \sin(2A-2C)\}.$$

Solution.

$$\sin(A-B)\cos(C-B)\cos(A-C)$$
$$= \frac{1}{2}\sin(A-B)\{\cos(A+B-2C) + \cos(A-B)\}$$
$$= \frac{1}{4}\sin 2(A-C) + \frac{1}{4}\sin 2(C-B) + \frac{1}{4}\sin 2(A-B).$$

Transform the second and third terms in like manner, then by addition we obtain the required result.

Problem 135. *In any triangle* $\dfrac{\sin(A-B)}{\sin(A+B)} = \dfrac{a^2-b^2}{c^2}$.

Solution. $\dfrac{\sin(A-B)}{\sin(A+B)} = \dfrac{\sin(A+B)\sin(A-B)}{\sin^2(A+B)} = \dfrac{\sin^2 A - \sin^2 B}{\sin^2 C}$, by *Art.* 83 (page 404),

$$= \left(\frac{\sin A}{\sin C}\right)^2 - \left(\frac{\sin B}{\sin C}\right)^2 = \left(\frac{a}{c}\right)^2 - \left(\frac{b}{c}\right)^2 = \frac{a^2-b^2}{c^2}.$$

Problem 136. *The diagonals of a quadrilateral figure are in length h and k respectively, and inclined at an angle A. Show that the area of the greatest rectangle which can be drawn with its four sides passing through the four corners of this quadrilateral is*

$$\frac{1}{2}hk(1+\sin A).$$

Solution. Suppose the diagonal h of the quadrilateral to make an angle θ with the sides of the rectangle which pass through its extremities; then each of the other sides is equal to $h\sin\theta$. It will be seen from a diagram that the diagonal k of the quadrilateral will make an angle $\dfrac{3\pi}{2} - (\theta + A)$ with the sides of the rectangle which pass through its extremities; then each of the other sides is equal to $k\sin\left(\frac{3\pi}{2}-\theta-A\right)$, that is to $-k\cos(\theta+A)$. Hence the area of the rectangle $= -hk\sin\theta\cos(\theta+A) = \frac{hk}{2}\{-\sin(2\theta+A)+\sin A\}$. The greatest value of this is when $2\theta+A=\frac{3\pi}{2}$, and is $\frac{hk}{2}(1+\sin A)$.

Problem 137. *A person standing at the door of a house observes that he can just see the top of a church spire over the intervening wall at an angle α; he then ascends to the root, whence he is just able to get a view of the entire building, and he observes that the elevation of the spire top is β. Having given the height of the house, that of the observer being neglected, determine the heights of the spire and the wall.*

Solution. Let h denote the height of the house, x the height of the wall, y the height of the church. Then $x\cot\alpha$ is the distance of the wall from the house, and $y\cot\alpha$ is the distance of the church from the house. By similar triangles $\frac{h}{x\cot\alpha} = \frac{y}{y\cot\alpha - x\cot\alpha}$; therefore $h(y-x)=xy$.

Also $\frac{y-h}{y\cot\alpha}=\tan\beta$; therefore $y=\frac{h}{1-\cot\alpha\tan\beta}=\frac{h\tan\alpha}{\tan\alpha-\tan\beta}$.

Then
$$x=\frac{hy}{h+y}=\frac{h\tan\alpha}{2\tan\alpha-\tan\beta}.$$

Problem 138. *In any triangle show that*
$$a\cos^2\frac{1}{2}A+b\cos^2\frac{1}{2}B+c\cos^2\frac{1}{2}C=s+\frac{S}{R}.$$

Solution. $a\cos^2\frac{1}{2}A+b\cos^2\frac{1}{2}B+c\cos^2\frac{1}{2}C$.
$$\begin{aligned}
&=\frac{1}{2}(a+b+c+a\cos A+b\cos B+c\cos C)\\
&=s+\frac{1}{2}R(\sin 2A+\sin 2B+\sin 2C)\\
&=s+2R\sin A\sin B\sin C,\ \text{by } Art.\ 114 \text{ (page 409)},\\
&=s+2R\frac{8S^3}{a^2b^2c^2}=s+\frac{4S^2}{abc}=s+\frac{S}{R}.
\end{aligned}$$

Problem 139. *A, B, C are three points on a plane inclined to the horizon, C being the lowest; it is found that CA is inclined to the horizon at an angle α, and CB at an angle β; and the angle ACB is γ. If θ be the inclination of the plane to the horizon, show that*
$$\sin^2\theta\sin^2\gamma=\sin^2\alpha+\sin^2\beta-2\sin\alpha\sin\beta\cos\gamma.$$

Solution. Through C draw a plane parallel to the horizon; from A draw AP perpendicular to the intersection of this plane with that which contains A, B and C; from B draw BQ perpendicular to the same intersection. Let $ACP=\phi$, and

$BCQ = \psi$; so that $\phi + \psi + \gamma = \pi$. Therefore

$$\cos\gamma = \sin\phi\sin\psi - \cos\phi\cos\psi.$$

Now $AP = AC\sin\phi$; thus the perpendicular from A on the plane drawn through C parallel to the horizon $= AP\sin\theta = AC\sin\theta\sin\phi$; but this perpendicular also $= AC\sin\alpha$; therefore

$$\sin\alpha = \sin\theta\sin\phi.$$

Similarly $$\sin\beta = \sin\theta\sin\psi.$$

Hence $$\cos\gamma = \frac{\sin\alpha\sin\beta}{\sin^2\theta} - \frac{\sqrt{(\sin^2\theta - \sin^2\alpha)(\sin^2\theta - \sin^2\beta)}}{\sin^2\theta},$$

therefore $$(\cos\gamma\sin^2\theta - \sin\alpha\sin\beta)^2 = (\sin^2\theta - \sin^2\alpha)(\sin^2\theta - \sin^2\beta),$$

therefore $$\cos^2\gamma\sin^2\theta - 2\cos\gamma\sin\alpha\sin\beta = \sin^2\theta - \sin^2\alpha - \sin^2\beta,$$

therefore $$\sin^2\theta\sin^2\gamma = \sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta\cos\gamma.$$

Problem 140. *If a', b', c' are the sides of the triangle formed by joining the points of contact of the inscribed circle with the sides of a triangle, show that*

$$\frac{a'b'c'}{abc} = \frac{r^2}{2R^2}.$$

Solution. With the diagram of *Art.* 248 (page 426) we see that

$$a' = 2r\sin FOA = 2r\cos\frac{1}{2}A;$$

and we have similar values for b' and c'

Thus $$a'b'c' = 8r^3\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C = 8r^3\frac{sS}{abc};$$

therefore $$\frac{a'b'c'}{abc} = \frac{8r^2S^2}{a^2b^2c^2} = \frac{8r^2}{(4R)^2} = \frac{r^2}{2R^2}.$$

Problem 141. *Find* $\tan\theta$ *from the equation* $\tan 3\theta + \tan 2\theta + \tan\theta = 0$.

Solution. Here $$\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} + \frac{2\tan\theta}{1 - \tan^2\theta} + \tan\theta = 0.$$

Therefore either $\tan\theta = 0$ or $\dfrac{3 - \tan^2\theta}{1 - 3\tan^2\theta} + \dfrac{2}{1 - \tan^2\theta} + 1 = 0.$

The latter gives

$$(3 - \tan^2\theta)(1 - \tan^2\theta) + 2(1 - 3\tan^2\theta) + (1 - \tan^2\theta)(1 - 3\tan^2\theta) = 0;$$

therefore $$4\tan^4\theta - 14\tan^2\theta + 6 = 0.$$

By solving this quadratic we obtain $\tan^2\theta = 3$ or $\dfrac{1}{2}$.

Problem 142. *Show that*

$$\left(\cos\frac{\pi}{8}\right)^8 + \left(\cos\frac{3\pi}{8}\right)^8 + \left(\cos\frac{5\pi}{8}\right)^8 + \left(\cos\frac{7\pi}{8}\right)^8 = \frac{17}{16}.$$

Solution. We may obtain the result by taking the values of the four cosines and raising them to the eighth power. Or we may proceed thus :

$$\cos^8\frac{\pi}{8}+\cos^8\frac{3\pi}{8}+\cos^8\frac{5\pi}{8}+\cos^8\frac{7\pi}{8}$$
$$=2\left(\cos^8\frac{\pi}{8}+\cos^8\frac{3\pi}{8}\right)=2\left(\cos^8\frac{\pi}{8}+\sin^8\frac{\pi}{8}\right)$$
$$=\frac{1}{32}\left(\cos\pi+28\cos\frac{\pi}{2}+35\right),\ \text{by Example IX. 13,}$$
$$=\frac{34}{32}=\frac{17}{16}.$$

Problem 143. *If* $a\cos\phi=b\cos\theta$ *then* $\cot\frac{1}{2}(\phi+\theta)\cot\frac{1}{2}(\phi-\theta)=\frac{a+b}{a-b}$.

Solution. Here $\frac{b}{a}=\frac{\cos\phi}{\cos\theta}$; therefore $\frac{a+b}{a-b}=\frac{\cos\theta+\cos\phi}{\cos\theta-\cos\phi}$

$$=\frac{2\cos\frac{1}{2}(\phi+\theta)\cos\frac{1}{2}(\phi-\theta)}{2\sin\frac{1}{2}(\phi+\theta)\sin\frac{1}{2}(\phi-\theta)}=\cot\frac{1}{2}(\phi+\theta)\cot\frac{1}{2}(\phi-\theta).$$

Problem 144. *If* A, B, C *be the angles of a triangle, show that* $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ *cannot be greater than* $\frac{3\sqrt{3}}{8}$.

Solution. $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)\sin\left(\frac{\pi}{2}-\frac{B}{2}\right)\sin\left(\frac{\pi}{2}-\frac{C}{2}\right)$;
and the sum of $\frac{\pi}{2}-\frac{A}{2}$, $\frac{\pi}{2}-\frac{B}{2}$, and $\frac{\pi}{2}-\frac{C}{2}$ is a fixed quantity, namely π.

Hence proceeding as in Chapter XIII : *Problem* 40, we see that the proposed product is greatest when

$$\frac{\pi}{2}-\frac{A}{2}=\frac{\pi}{2}-\frac{B}{2}=\frac{\pi}{2}-\frac{C}{2},$$

that is when
$$\frac{A}{2}=\frac{B}{2}=\frac{C}{2}=\frac{\pi}{6};$$

and then the product
$$-\left(\frac{\sqrt{3}}{2}\right)^3=\frac{3\sqrt{3}}{8}.$$

Problem 145. *In any triangle*

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{a+b+c}{b+c-a}\cot\frac{A}{2}.$$

Solution. We have $\cot\frac{B}{2}=\sqrt{\frac{s(s-b)}{(s-a)(s-c)}}=\frac{s-b}{s-a}\cot\frac{A}{2}$;

similarly
$$\cot\frac{C}{2}=\frac{s-c}{s-a}\cot\frac{A}{2}.$$

Hence
$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\left(1+\frac{s-b}{s-a}+\frac{s-c}{s-a}\right)\cot\frac{A}{2}$$

$$= \frac{3s-a-b-c}{s-a}\cot\frac{A}{2} = \frac{s}{s-a}\cot\frac{A}{2}$$
$$= \frac{a+b+c}{b+c-a}\cot\frac{A}{2}.$$

Problem 146. *The diagonals of a four-sided figure are h and k, and the area is C. Show that the area of the circumscribing square is*
$$\frac{h^2k^2-4C^2}{h^2+k^2-4C}.$$

Solution. Let A denote the angle between the diagonals; then $C = \frac{1}{2}hk\sin A$; and by the solution of *Problem* 136 the area of the circumscribed rectangle is $-hk\sin\theta\cos(A+\theta)$. And since the rectangle is to be a square, we have by the solution of *Problem* 136
$$h\sin\theta = -k\cos(A+\theta);$$
therefore
$$h = -k(\cos A\cot\theta - \sin A);$$
therefore
$$\cot\theta = \frac{k\sin A - h}{k\cos A};$$
therefore
$$\sin^2\theta = \frac{k^2\cos^2 A}{(k\sin A-h)^2+k^2\cos^2 A} = \frac{k^2-k^2\sin^2 A}{k^2-2kh\sin A+h^2}$$
$$= \frac{k^2-\dfrac{4C^2}{h^2}}{h^2+k^2-4C}.$$
And the area of the circumscribing square $= h^2\sin^2\theta = \dfrac{h^2k^2-4C^2}{h^2+k^2-4C}$.

Problem 147. *Show that x, y, z can be so taken that for all values of θ the following expression shall have a given constant value,*
$$x\sin(\theta-\beta)\sin(\theta-\gamma)+y\sin(\theta-\gamma)\sin(\theta-\alpha)+z\sin(\theta-\alpha)\sin(\theta-\beta).$$

Solution. We may put the proposed expression in the form
$$L\sin^2\theta + M\sin\theta\cos\theta + N\cos^2\theta,$$
where L, M, N involve the angles α, β, γ and also x, y, z : moreover x, y, z occur only in the first power. Now if we put $M = 0$ and $L = N =$ the given constant, the expression is equal to the given constant whatever θ may be. So we have only to determine x, y and z from the three simple equations $M = 0$, and $L = N =$ the given constant.

As soon as we have thus shown that such values of x, y, z as we require must exist, we can determine the values more simply. For let C denote the given constant; put α for θ, then
$$x\sin(\alpha-\beta)\sin(\alpha-\gamma) = C.$$
This finds x. Similarly, by putting β for θ we find y, and by putting γ for θ we find z.

Problem 148. *If from the extremities of a side of a regular pentagon inscribed in a circle straight lines be drawn to the middle of the arc subtended by the adjacent* *side, their difference is equal to the radius of the circle, their product is equal to the square on the radius, and the sum of their squares is equal to three times the square on the radius.*

Solution. Let AB denote the side of the regular pentagon, P the middle point of the arc subtended by the side adjacent to AB at B. Then the angle APB is the angle subtended at the circumference of the circle by the side of a regular pentagon inscribed in the circle, so that the angle $= \frac{\pi}{5}$. Similarly the angle $PAB = \frac{\pi}{10}$; and therefore the angle $ABP = \frac{7\pi}{10}$.

Let r denote the radius of the circle, so that

$$Ab = 2r\sin\frac{\pi}{5}, \quad PB = 2r\sin\frac{\pi}{10}, \text{ and } PA = 2r\sin\frac{7\pi}{10} = 2r\sin\frac{3\pi}{10}.$$

Hence
$$PA - PB = 2r\left\{\frac{\sqrt{5}+1}{4} - \frac{\sqrt{5}-1}{4}\right\} = r,$$

$$PA \cdot PB = \frac{4r^2(\sqrt{5}+1)(\sqrt{5}-1)}{16} = r^2,$$

$$PA^2 + PB^2 = 4r^2\left\{\left(\frac{\sqrt{5}+1}{4}\right)^2 + \left(\frac{\sqrt{5}-1}{4}\right)^2\right\} = 3r^2.$$

Problem 149. *If a flag-staff at the top of a tower of height a subtend a small angle θ at the eye of an observer when at the distance b from the tower, show that the length of the flag-staff is $\frac{a^2+b^2}{b}\theta$ nearly.*

Solution. Suppose the tower to subtend an angle ϕ at the eye of the observer; let x be the length of the flag-staff : then

$$\frac{a}{b} = \tan\phi, \quad \frac{a+x}{b} = \tan(\phi+\theta) = \frac{\tan\phi + \tan\theta}{1-\tan\phi\tan\theta} = \frac{a+b\tan\theta}{b-a\tan\theta};$$

therefore
$$\frac{x}{b} = \frac{a+b\tan\theta}{b-a\tan\theta} - \frac{a}{b} = \frac{(b^2+a^2)\tan\theta}{b(b-a\tan\theta)};$$

then if θ be very small we may put θ for $\tan\theta$, and neglect $a\tan\theta$ in comparison with b, so that $x = \frac{b^2+a^2}{b}\theta$ nearly.

Problem 150. *In any triangle*

$$(s-a)^2\sin A + (s-b)^2\sin B + (s-c)^2\sin C$$
$$= 4r(2R-r)\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Solution. We have by *Art.* 249 (page 426)

$$(s-a)^2\sin A + (s-b)^2\sin B + (s-c)^2\sin C$$
$$= r\left\{(s-a)\sin A\cot\frac{A}{2} + (s-b)\sin B\cot\frac{B}{2} + (s-c)\sin C\cot\frac{C}{2}\right\}$$
$$= 2r\left\{(s-a)\cos^2\frac{A}{2} + (s-b)\cos^2\frac{B}{2} + (s-c)\cos^2\frac{C}{2}\right\};$$

and by *Problems* 130 and 138, this

$$= 2r\left\{\left(2+\frac{r}{2R}\right)s-\left(s+\frac{S}{R}\right)\right\} = 2r\left(s+\frac{S}{2R}-\frac{S}{R}\right)$$

$$= 2r\left(s-\frac{S}{2R}\right) = 2r\left(\frac{S}{r}-\frac{S}{2R}\right) = \frac{S(2R-r)}{R}.$$

And
$$\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{Ss}{abc} = \frac{s}{4R};$$

so that
$$4r(2R-r)\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = 4r(2R-r)\frac{s}{4R} = \frac{S(2R-r)}{R}.$$

Thus the proposed expressions are equal.

Problem 151*. Show that*

$$2\sin 7A\cos A + 16\sin A\cos^3 A = \sin 6A + 4\sin 2A(1+2\cos^3 2A).$$

Solution.
$$2\sin 7A\cos A = \sin 8A + \sin 6A;$$

Therefore $2\sin 7A\cos A + 16\sin A\cos^3 A = \sin 6A + \sin 8A + 16\sin A\cos^3 A$

$$= \sin 6A + 2\sin 4A\cos 4A + 8\sin 2A\cos^2 A$$

$$= \sin 6A + 4\sin 2A\cos 2A\cos 4A + 8\sin 2A\cos^2 A$$

$$= \sin 6A + 4\sin 2A\left(2\cos^2 A + \cos 2A\cos 4A\right)$$

$$= \sin 6A + 4\sin 2A\left\{1+\cos 2A(1+\cos 4A)\right\}$$

$$= \sin 6A + 4\sin 2A\left(1+2\cos^3 2A\right).$$

Problem 152*. Find the logarithm of 32 to the base $\sqrt[3]{4}$, and the logarithm of 81 $\sqrt[3]{3}$ to the base $\sqrt[3]{9}$.*

Solution. Let x denote the logarithm of 32 to the base $\sqrt[3]{4}$; then $32 = (\sqrt[3]{1})^x$, that is $2^5 = 4^{\frac{x}{3}} = 2^{\frac{2x}{3}}$; therefore $5 = \frac{2x}{3}$; therefore $x = \frac{15}{2}$.

Let x denote the logarithm of $81\sqrt[3]{3}$ to the base $\sqrt[3]{9}$; then $81\sqrt[3]{3} = (\sqrt[3]{9})^x$, that is $3^{4+\frac{1}{3}} = 9^{\frac{x}{3}} = 3^{\frac{2x}{3}}$; therefore $\frac{2x}{3} = 4\frac{1}{3} = \frac{13}{3}$; therefore $x = \frac{13}{2}$.

Problem 153*. If* $\tan(A+B) = 3\tan A$, *show that*

$$\sin(2A+2B) + \sin 2A = 2\sin 2B.$$

Solution. Here $\dfrac{\sin(A+B)}{\cos(A+B)} = \dfrac{3\sin A}{\cos A}$;

therefore $\sin(A+B)\cos A - \cos(A+B)\sin A = 2\sin A\cos(A+B);$

therefore $\sin(A+B-A) = 2\sin A\cos(A+B);$

that is $\sin B = 2\sin A\cos(A+B) = \sin(2A+B) - \sin B;$

therefore $2\sin B = \sin(2A+B);$

therefore $2\sin B\cos B = \sin(2A+B)\cos B;$

therefore $\sin 2B = \frac{1}{2}\{\sin(2A+2B) + \sin 2A\}$

therefore $2\sin 2B = \sin(2A+2B) + \sin 2A.$

Problem 154. *In any triangle*

$$\frac{1}{a}\sin^2\frac{1}{2}A+\frac{1}{b}\sin^2\frac{1}{2}B+\frac{1}{c}\sin^2\frac{1}{2}C=\frac{2ab+2bc+2ca-a^2-b^2-c^2}{4abc}.$$

Solution. $\frac{1}{a}\sin^2\frac{A}{2}+\frac{1}{b}\sin^2\frac{B}{2}+\frac{1}{c}\sin^2\frac{C}{2}$

$$=\frac{1}{abc}\{(s-b)(s-c)+(s-a)(s-c)+(s-a)(s-b)\}$$
$$=\frac{1}{abc}\{3s^2-2s(a+b+c)+ab+bc+ca\}$$
$$=\frac{1}{abc}\{ab+bc+ca-s^2\}$$
$$=\frac{1}{4abc}\{4ab+4bc+4ca-(a+b+c)^2\}$$
$$=\frac{1}{4abc}\{2ab+2bc+2ca-a^2-b^2-c^2\}$$

Problem 155. *The sides of a quadrilateral figure are divided in order in the ratio of m to n, and a new quadrilateral is formed by joining the points of division. Show that the area of this quadrilateral is to the area of the original quadrilateral as m^2+n^2 is to $(m+n)^2$.*

Solution. Let $ABCD$ denote the quadrilateral figure. Let P, Q, R, S be taken in AB, BC, CD, DA respectively, such that

$$\frac{AP}{PB}=\frac{BQ}{QC}=\frac{CR}{RD}=\frac{DS}{SA}=\frac{m}{n}.$$

Then
$$\frac{PB}{AB}=\frac{n}{m+n},\quad \frac{BQ}{BC}=\frac{m}{m+n};$$

and the area of the triangle $PBQ=\frac{1}{2}BP\cdot BQ\sin B=\frac{mn}{2(m+n)^2}AB\cdot BC\sin B$

$$=\frac{mn}{(m+n)^2}\text{ area of the triangle }ABC.$$

Similarly the area of the triangle $RDS=\frac{mn}{(m+n)^2}$ area of the triangle ADC.

Therefore the area of the triangle PBQ and $RDS=\frac{mnH}{(m+n)^2}$, where H denotes the area of the quadrilateral figure $ABCD$.

In the same way we show that the area of the triangles QCR and $SAP=\frac{mnH}{(m+n)^2}$.

Thus the area of the four triangles PBQ, QCR, RDS, and $SAP=\frac{2mnH}{(m+n)^2}$.

Therefore the area of the quadrilateral figure $PQRS$

$$=H\left\{1-\frac{2mn}{(m+n)^2}\right\}=\frac{H(m^2+n^2)}{(m+n)^2}.$$

Problem 156. *Show that $\cos\theta=-\frac{1}{2}$ is a solution of the equation*

$$\cos\theta+\cos 3\theta=\frac{1}{2};$$

and find the other values of $\cos\theta$.

Solution. $\cos\theta+\cos 3\theta=\frac{1}{2}$; therefore $\cos\theta+4\cos^3\theta-3\cos\theta=\frac{1}{2}$; therefore $4\cos^3\theta-2\cos\theta-\frac{1}{2}=0$; therefore $4\left(\cos^3\theta+\frac{1}{8}\right)-2\left(\cos\theta+\frac{1}{2}\right)=0$;

therefore $$2\left(\cos\theta+\frac{1}{2}\right)\left(\cos^2\theta-\frac{1}{2}\cos\theta+\frac{1}{4}\right)-\left(\cos\theta+\frac{1}{2}\right)=0;$$

thereofore $$\left(\cos\theta+\frac{1}{2}\right)\left(2\cos^2\theta-\cos\theta-\frac{1}{2}\right)=0.$$

Thus either $\cos\theta+\frac{1}{2}=0$ or $2\cos^2\theta-\cos\theta-\frac{1}{2}=0$;
the former gives $\cos\theta=-\frac{1}{2}$; the latter gives $\cos\theta=\frac{1\pm\sqrt{5}}{4}$.

Problem 157. *Show that*

$$\cos\beta\cos\gamma\sin(\gamma-\beta)+\cos\gamma\cos\alpha\sin(\alpha-\gamma)+\cos\alpha\cos\beta\sin(\beta-\alpha)$$
$$=\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha).$$

Solution. $\cos\beta\cos\gamma\sin(\gamma-\beta)=\frac{1}{2}\{\cos(\beta-\gamma)+\cos(\beta+\gamma)\}\sin(\gamma-\beta)$
$$=\frac{1}{4}\{\sin(2\gamma-2\beta)+\sin 2\gamma-\sin 2\beta\}.$$

Transform the other two terms in the same way; and thus we obtain finally as the sum

$$\frac{1}{4}\{\sin(2\gamma-2\beta)+\sin(2\alpha-2\gamma)+\sin(2\beta-2\alpha)\}.$$

Again, $\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)$
$$=\frac{1}{2}\{\cos(\alpha+\gamma-2\beta)-\cos(\alpha-\gamma)\}\sin(\gamma-\alpha)$$
$$=\frac{1}{4}\{\sin(2\gamma-2\beta)+\sin(2\beta-2\alpha)+\sin(2\alpha-2\gamma)\}.$$

Thus the proposed expressions are equal.

Or thus : from Chapter VIII : *Problem* 12 we see that

$$\sin\beta\sin\gamma\sin(\gamma-\beta)+\sin\gamma\sin\alpha\sin(\alpha-\gamma)+\sin\alpha\sin\beta\sin(\beta-\alpha)$$
$$=\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha).$$

In this formula change α, β, γ into $\frac{\pi}{2}+\alpha$, $\frac{\pi}{2}+\beta$, $\frac{\pi}{2}+\gamma$ respectively; and thus we obtain the required result.

Problem 158. *If* $A+B+C=180°$, *show that*

$$\sin A\sin(A-B)\sin(A-C)+\sin B\sin(B-C)\sin(B-A)$$
$$+\sin C\sin(C-A)\sin(C-B)$$
$$=\sin A\sin B\sin C\{1-8\cos A\cos B\cos C\}.$$

Solution. $\sin A\sin(A-B)\sin(A-C)=\frac{1}{2}\sin A\{\cos(C-B)-\cos(2A-B-C)\}$
$$=\frac{1}{4}\{\sin(A+C-B)+\sin(A+B-C)-\sin(3A-B-C)-\sin(B+C-A)\}$$

$= \frac{1}{4}\{\sin 2B + \sin 2C + \sin 4A - \sin 2A\}.$

In this way we see that the expression on the left-hand side in the proposed formula

$$= \frac{1}{4}\{\sin 2A + \sin 2B + \sin 2C + \sin 4A + \sin 4B + \sin 4C\}.$$

Then by Chapter VIII : *Problem* 33 we have

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C,$$

$$\begin{aligned}\sin 4A + \sin 4B + \sin 4C &= -4 \sin 2A \sin 2B \sin 2C \\ &= -32 \sin A \sin B \sin C \cos A \cos B \cos C.\end{aligned}$$

Thus we obtain the required result.

Problem 159. *A person with his eye on the level of the ground close to a pole, observed that he saw the top of a distant windw over an intervening wall at an elevation α. He then climbed up the pole c feet, when he saw the whole window, and the elevations of the tops of the window and wall were then β and γ. Show that the height of the lowest part of the window above the ground was*

$$\frac{c(\tan\alpha - \tan\beta + \tan\gamma)}{\tan\alpha - \tan\beta}.$$

Solution. Let A denote the bottom of the pole, B the point on the pole to which the man climbs, F the top of the window, E the bottom. Let AF and BE intersect at D, which is therefore the top of the wall. Draw DC perpendicular to the ground, and produce FE to meet the ground at H. Draw from B a horizontal straight line meeting FH at G.

Then from the triangle BAF we get $BF = \dfrac{c\cos\alpha}{\sin(\alpha-\beta)}$;

$$BG = BF\cos\beta = \frac{c\cos\alpha\cos\beta}{\sin(\alpha-\beta)} = \frac{c}{\tan\alpha - \tan\beta},$$

$$EG = BG\tan\gamma = \frac{c\tan\gamma}{\tan\alpha - \tan\beta},$$

$$EH = c + EG = \frac{c(\tan\alpha - \tan\beta + \tan\gamma)}{\tan\alpha - \tan\beta}.$$

Problem 160. *ABC is a triangle; straight lines bisecting the angles A and B meet the opposite sides at D and E respectively. Show that the area of the triangle CED is*

$$\frac{S\sin\dfrac{A}{2}\sin\dfrac{B}{2}}{\cos\dfrac{C-A}{2}\cos\dfrac{C-B}{2}}.$$

Solution. From the triangle CEB we have $\dfrac{CE}{a} = \dfrac{\sin\frac{1}{2}B}{\sin\left(C + \frac{1}{2}B\right)}$;

and from the triangle CDA we have $\dfrac{CD}{b} = \dfrac{\sin\frac{1}{2}A}{\sin\left(C + \frac{1}{2}A\right)}$.

Thus the area of the triangle $CED = \frac{1}{2}CE \cdot CD \sin C$

$$= \frac{ab\sin C \sin\frac{1}{2}A \sin\frac{1}{2}B}{2\sin\left(C + \frac{1}{2}B\right)\sin\left(C + \frac{1}{2}A\right)} = \frac{S\sin\frac{1}{2}A\sin\frac{1}{2}B}{\cos\frac{C-A}{2}\cos\frac{C-B}{2}}.$$

Problem 161.

If $$p = 2\cos A - 5\cos^3 A + 4\cos^5 A,$$

and $$q = 2\sin A - 5\sin^3 A + 4\sin^5 A;$$

Show that $$p\cos 3A + q\sin 3A = \cos 2A,$$

and $$p\sin 3A - q\cos 3A = \frac{1}{2}\sin 2A.$$

Solution. We have $p = 2\cos A + \cos^3 A(-5 + 4\cos^2 A)$

$$= 2\cos A + \frac{1}{4}(\cos 3A + 3\cos A)(-5 + 4\cos^2 A)$$
$$= 2\cos A + \frac{1}{4}(\cos 3A + 3\cos A)(-3 + 2\cos 2A)$$
$$= 2\cos A - \frac{3}{4}\cos 3A - \frac{9}{4}\cos A + \frac{1}{2}\cos 3A\cos 2A + \frac{3}{2}\cos A\cos 2A$$
$$= -\frac{3}{4}\cos 3A - \frac{1}{4}\cos A + \frac{1}{4}(\cos 5A + \cos A) + \frac{3}{4}(\cos 3A + \cos A)$$
$$= \frac{1}{4}(\cos 5A + 3\cos A).$$

In the same way we find that

$$q = \frac{1}{4}(\sin 5A + 3\sin A).$$

Therefore

$$p\cos 3A + q\sin 3A = \frac{1}{4}(\cos 5A + 3\cos A)\cos 3A + \frac{1}{4}(\sin 5A + 3\sin A)\sin 3A$$
$$= \frac{1}{4}(\cos 5A\cos 3A + \sin 5A\sin 3A) + \frac{3}{4}(\cos 3A\cos A + \sin 3A\sin A)$$
$$= \frac{1}{4}\cos(5A - 3A) + \frac{3}{4}\cos(3A - A) = \cos 2A.$$

And

$$p\sin 3A - q\cos 3A = \frac{1}{4}(\cos 5A + 3\cos A)\sin 3A - \frac{1}{4}(\sin 5A + 3\sin A)\cos 3A$$
$$= \frac{1}{4}(\cos 5A\sin 3A - \sin 5A\cos 3A) + \frac{3}{4}(\sin 3A\cos A - \cos 3A\sin A)$$
$$= -\frac{1}{4}\sin(5A - 3A) + \frac{3}{4}\sin(3A - A) = \frac{1}{2}\sin 2A.$$

Problem 162. *Find the limit of* $\left(\cos\frac{\alpha}{n}\right)^{\cot^2\frac{\beta}{n}}$ *when n is infinite.*

Solution. Let $u = \left(\cos\frac{\alpha}{n}\right)^{\cot^2\frac{\beta}{n}}$; therefore

$$\log u = \cot^2\frac{\beta}{n}\log\cos\frac{\alpha}{n} = \cos^2\frac{\beta}{n} \times \operatorname{cosec}^2\frac{\beta}{n}\log\cos\frac{\alpha}{n}.$$

Now as in the solution of Chapter XII : *Problem* 33 we can show that

$$\operatorname{cosec}^2\frac{\beta}{n}\log\cos\frac{\alpha}{n} = -\frac{\alpha^2}{2\beta^2} \text{ when } n \text{ is infinite.}$$

And $\cos^2\frac{\beta}{n} = 1$ when n is infinite.

Thus $\log u = -\frac{\alpha^2}{2\beta^2}$; and therefore $u = e^{-\frac{\alpha^2}{2\beta^2}}$.

Problem 163. *Sum the infinite series*

$$1 + \frac{1+2}{\lfloor\underline{2}} + \frac{1+2+2^2}{\lfloor\underline{3}} + \frac{1+2+2^2+2^3}{\lfloor\underline{4}} + \ldots$$

Solution. If n be a positive integer, we have $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$,
Hence the infinite series

$$= 2 - 1 + \frac{2^2-1}{\lfloor\underline{2}} + \frac{2^3-1}{\lfloor\underline{3}} + \frac{2^4-1}{\lfloor\underline{4}} + \ldots$$

$$= 2 + \frac{2^2}{\lfloor\underline{2}} + \frac{2^3}{\lfloor\underline{3}} + \frac{2^4}{\lfloor\underline{4}} + \ldots - \left\{1 + \frac{1}{\lfloor\underline{2}} + \frac{1}{\lfloor\underline{3}} + \frac{1}{\lfloor\underline{4}} + \ldots\right\}$$

$$= e^2 - 1 - \{e - 1\} = e^2 - e.$$

Problem 164. *Find* $\cos(x-y)$ *and* $\cos(x+y)$ *from the equations*

$$\sec\alpha\cos(x+y) = 1 + \tan x\tan y, \quad \sec\beta\cos(x-y) = 1 - \tan x\tan y.$$

Solution. Here $\qquad \sec\alpha\cos(x+y) = \frac{\cos(x-y)}{\cos x\cos y},$

and $\qquad \sec\beta\cos(x-y) = \frac{\cos(x+y)}{\cos x\cos y};$

therefore, by division, $\qquad \frac{\cos\beta}{\cos\alpha}\cdot\frac{\cos(x+y)}{\cos x - y} = \frac{\cos(x-y)}{\cos(x+y)},$

so that
$$\frac{\cos(x-y)}{\cos(x+y)} = \sqrt{\frac{\cos\beta}{\cos\alpha}} \tag{37}$$

And
$$\cos(x-y) + \cos(x+y) = 2\cos x\cos y = 2\cos\alpha\frac{\cos(x-y)}{\cos(x+y)}$$
$$= 2\sqrt{\cos\alpha\cos\beta} \tag{38}$$

From (37) and (38) we have

$$\cos(x+y)\left\{\sqrt{\frac{\cos\beta}{\cos\alpha}} + 1\right\} = 2\sqrt{\cos\alpha\cos\beta};$$

therefore
$$\cos(x+y) = \frac{2\cos\alpha\sqrt{\cos\beta}}{\sqrt{\cos\alpha} + \sqrt{\cos\beta}}.$$

Then by (37) we have $\cos(x-y) = \dfrac{2\cos\beta\sqrt{\cos\alpha}}{\sqrt{\cos\alpha}+\sqrt{\cos\beta}}$.

Problem 165. *In any triangle*

$$\frac{a\cos\frac{1}{2}(B-C)}{bc\cos\frac{1}{2}(B+C)} + \frac{b\cos\frac{1}{2}(C-A)}{ca\cos\frac{1}{2}(C+A)} + \frac{c\cos\frac{1}{2}(A-B)}{ab\cos\frac{1}{2}(A+B)} = \frac{2(ab+bc+ca)}{abc}.$$

Solution. It may be shown as in the solution of *Example* XX. 4 that

$$\frac{a\cos\frac{1}{2}(B-C)}{bc\cos\frac{1}{2}(B+C)} = \frac{b+c}{bc} = \frac{a(b+c)}{abc};$$

similarly $\dfrac{b\cos\frac{1}{2}(C-A)}{ca\cos\frac{1}{2}(C+A)} = \dfrac{b(c+a)}{abc}$, and $\dfrac{c\cos\frac{1}{2}(A-B)}{ab\cos\frac{1}{2}(A+B)} = \dfrac{c(a+b)}{abc}$.

Hence by addition we obtain the required result.

Problem 166. *O is any point in the interior of a quadrilateral ABCD; OP, OQ, OR, OS are perpendiculars on the sides AB, BC, CD, DA respectively. Show that the area of PQRS is*

$$\frac{1}{2}ABCD - \frac{1}{8}\left(OA^2\sin 2A + OB^2\sin 2B + OC^2\sin 2C + OD^2\sin 2D\right).$$

Solution. $AP = OA\cos OAP,\ AS = OA\cos OAS;$

therefore the area of the triangle $APS = \frac{1}{2}OA^2\cos OAP\cos OAS\sin A$.

In the same way the area of the triangle OPS

$$= \frac{1}{2}OA^2\sin OAP\sin OAS\sin POS = \frac{1}{2}OA^2\sin OAP\sin OAS\sin(180^\circ - A)$$
$$= \frac{1}{2}OA^2\sin OAP\sin OAS\sin A.$$

Hence triangle APS - triangle OPS

$$= \frac{1}{2}OA^2\sin A\{\cos OAP\cos OAS - \sin OAP\sin OAS\}$$
$$= \frac{1}{2}OA^2\sin A\cos(OAP+OAS) = \frac{1}{2}OA^2\sin A\cos A = \frac{1}{4}OA^2\sin 2A.$$

In the same way we obtain

$$\text{triangle } BQP - \text{ triangle } OQP = \frac{1}{4}OB^2\sin 2B,$$
$$\text{triangle } CRQ - \text{ triangle } ORQ = \frac{1}{4}OC^2\sin 2C,$$

and $$\text{triangle } DSR - \text{ triangle } OSR = \frac{1}{4}OD^2\sin 2D,$$

Hence by addition we have

triangle $APS+$ triangle $BQP+$ triangle $CRQ+$ triangle $DSR-$ quadrilateral $PQRS$

$$= \frac{1}{4}\left\{OA^2\sin 2A + OB^2\sin 2B + OC^2\sin 2C + OD^2\sin 2D\right\}.$$

But the sum of the four triangles and the quadrilateral

$$= \text{ the quadrilateral } ABCD.$$

Hence by subtraction we have
twice the quadrilateral $PQRS =$ the quadrilateral $ABCD$
$-\frac{1}{4}\{OA^2 \sin 2A + OB^2 \sin 2B + OC^2 \sin 2C + OD^2 \sin 2D\}.$

Problem 167. *In any triangle*

$$a \sin \frac{B-C}{2} \operatorname{cosec} \frac{A}{2} + b \sin \frac{C-A}{2} \operatorname{cosec} \frac{B}{2} + c \sin \frac{A-B}{2} \operatorname{cosec} \frac{C}{2} = 0.$$

Solution. We have $a = 2R \sin A,\ b = 2R \sin B,\ c = 2R \sin C$;
thus the proposed expression

$$= 4R\left\{\sin \frac{B-C}{2} \cos \frac{A}{2} + \sin \frac{C-A}{2} \cos \frac{B}{2} + \sin \frac{A-B}{2} \cos \frac{C}{2}\right\}$$
$$= 4R\left\{\sin \frac{B-C}{2} \sin \frac{B+C}{2} + \sin \frac{C-A}{2} \sin \frac{C+A}{2} + \sin \frac{A-B}{2} \sin \frac{A+B}{2}\right\}$$
$$= 4R\left\{\sin^2 \frac{B}{2} - \sin^2 \frac{C}{2} + \sin^2 \frac{C}{2} - \sin^2 \frac{A}{2} + \sin^2 \frac{A}{2} - \sin^2 \frac{B}{2}\right\}$$
$$= 0.$$

Problem 168. *Show by the aid of Trigonometry that if $x + y + z = xyz$, then*

$$\frac{3x - x^3}{1 - 3x^2} + \frac{3y - y^3}{1 - 3y^2} + \frac{3z - z^3}{1 - 3z^2} = \frac{(3x - x^3)(3y - y^3)(3z - z^3)}{(1 - 3x^2)(1 - 3y^2)(1 - 3z^2)}.$$

Solution. Assume $x = \tan A,\ y = \tan B,\ z = \tan c$; then since $x+y+z = xyz$ it will follow in the manner of *Art.* 114 (page 409) that $\tan(A + B + C)$ is zero; therefore $A + B + C = n\pi$ where n is zero or some integer. Therefore $3A + 3B + 3C = 3n\pi$; and therefore in the manner of *Art.* 114 (page 409) we have

$$\tan 3A + \tan 3B + \tan 3C = \tan 3A \tan 3B \tan 3C.$$

But $$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A} = \frac{3x - x^3}{1 - 3x^2};$$

similarly $$\tan 3B = \frac{3y - y^2}{1 - 3y^2}, \quad \tan 3C = \frac{3z - z^3}{1 - 3z^2}:$$

thus the required result follows.

Problem 169. *If l, m, n be the perpendiculars from the centre of the circumscribed circle on the sides of a triangle, show that*

$$4\left(\frac{a}{l} + \frac{b}{m} + \frac{c}{n}\right) = \frac{abc}{lmn}.$$

Solution. We have $l = R \cos A,\ m = R \cos B,\ n = R \cos C$; thus we have to show that

$$\frac{4a}{R \cos A} + \frac{4b}{R \cos B} + \frac{4c}{R \cos C} = \frac{abc}{R^2 \cos A \cos B \cos C}.$$

Now $a = 2R\sin A$, $b = 2R\sin B$, $c = 2R\sin C$; thus the proposed identity becomes

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C;$$

and this is true by *Art.* 114 (page 409) .

Problem 170. *If A, B and C are the angles of a triangle, show that:* $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2}$ *cannot be less than* $\frac{3}{4}$.

Solution. $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin\frac{C}{2} = \frac{3}{2} - \frac{1}{2}(\cos A + \cos B + \cos C)$

$$= 1 - 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}, \text{ by } Art. \text{ 114 (page 409)}.$$

Now we have seen in Chapter XIII : *Problem* 40 that $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ cannot be greater than $\frac{1}{8}$; hence $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin\frac{C}{2}$ cannot be less than $1 - \frac{1}{4}$, that is than $\frac{3}{4}$.

Problem 171. *Find the limit when* θ *is indefinitely diminished of*

$$\frac{\sin a\theta}{\sin b\theta} \textit{ and of } \frac{\text{vers } a\theta}{\text{vers } b\theta}.$$

Solution. $\frac{\sin a\theta}{\sin b\theta} = \frac{a}{b}\frac{\sin a\theta}{a\theta}\cdot\frac{b\theta}{\sin b\theta}$; and when θ is indefinitely diminished the limit of $\frac{\sin a\theta}{a\theta}$ is unity, and so also is the limit of $\frac{b\theta}{\sin b\theta}$: thus the limit of $\frac{\sin a\theta}{\sin b\theta}$ is $\frac{a}{b}$.

Also
$$\frac{\text{vers } a\theta}{\text{vers } b\theta} = \frac{1-\cos a\theta}{1-\cos b\theta} = \frac{\sin^2\frac{a\theta}{2}}{\sin^2\frac{b\theta}{2}} = \left\{\frac{\sin\frac{a\theta}{2}}{\sin\frac{b\theta}{2}}\right\}^2.$$

Now the limit of $\frac{\sin\frac{a\theta}{2}}{\sin\frac{b\theta}{2}}$ is $\frac{a}{b}$ in the manner just shown; therefore the limit of $\frac{\text{vers } a\theta}{\text{vers } b\theta}$ is $\frac{a^2}{b^2}$.

Problem 172. *Show that*

$$\log\sqrt{2} = \frac{1}{4}\left\{\frac{1}{1\cdot 1} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 5} + \ldots\right\}.$$

Solution.

$$\frac{1}{4}\left\{\frac{1}{1\cdot 1} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{4\cdot 7} + \ldots\right\}$$
$$= \frac{1}{2}\left\{\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \frac{1}{7\cdot 8} + \ldots\right\}$$

$$=\frac{1}{2}\log 2, \text{ by } Art.\ 146(\text{ page } 415), = \log\sqrt{2}.$$

Problem 173. *If $A+B+C=180°$, show that*

$$\tan\frac{A}{2}+\cos\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2}=\tan\frac{B}{2}+\cos\frac{B}{2}\sec\frac{C}{2}\sec\frac{A}{2}$$
$$=\tan\frac{C}{2}+\cos\frac{C}{2}\sec\frac{A}{2}\sec\frac{B}{2}.$$

Solution.

$$\tan\frac{A}{2}+\cos\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}+\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}$$
$$=\frac{\cos^2\frac{A}{2}+\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$

The numerator of this fraction $=1-\sin^2\frac{A}{2}+\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

$$=1+\sin\frac{A}{2}\left\{\cos\frac{B}{2}\cos\frac{C}{2}-\cos\frac{B+C}{2}\right\}=1+\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Thus the fraction $=\sec\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2}+\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$.

Similarly the other two proposed expressions may be reduced to this symmetrical form; and thus the three expressions are equal.

Problem 174. *If* $\sin(\pi\cot\theta)=\cos(\pi\tan\theta)$, *show that either* cosec 2θ *or* $\cot 2\theta$ *is of the form* $m+\frac{1}{4}$, *where m is an integer positive or negative.*

Solution.

$$\sin(\pi\cot\theta)=\cos(\pi\tan\theta);$$

therefore
$$\cos(\pi\tan\theta)=\cos\left(\frac{\pi}{2}-\pi\cot\phi\right);$$

therefore all the solutions are comprised in

$$\pi\tan\theta=2n\pi\pm\left(\frac{\pi}{2}-\pi\cot\theta\right),$$

where n is zero, or some integer, positive or negative.

Take the upper sign; thus $2n+\frac{1}{2}=\tan\theta+\cot\theta=\frac{1}{\sin\theta\cos\theta}$, so that $n+\frac{1}{4}=\frac{1}{\sin 2\theta}$.

Take the lower sign; thus $\frac{1}{2}-2n=\cot\theta-\tan\theta=2\cot 2\theta$, so that $\cot 2\theta=\frac{1}{4}-n$.

Thus either cosec 2θ or $\cot 2\theta$ takes the prescribed form.

Problem 175. *In any triangle*

$$\left(\frac{\sin A+\sin B+\sin C}{a+b+c}\right)^2=\frac{a\cos A+b\cos B+c\cos C}{2abc}.$$

Solution. We have $a = 2R\sin A,\ b = 2R\sin B,\ c = 2R\sin C$.

Thus the left-hand member $= \left(\dfrac{1}{2R}\right)^2$. And,

the right-hand member $= \dfrac{\sin 2A + \sin 2B + \sin 2C}{16R^2 \sin A \sin B \sin C} = \dfrac{1}{4R^2}$, by *Art*. 114 (page 409).

Problem 176. *A circle of radius r is inscribed in a sector of a circle of radius a, and 2c is the chord of the sector. Show that*

$$\frac{1}{r} = \frac{1}{a} + \frac{1}{c}.$$

Solution. Let θ be the angle of the sector; then we see from a diagram that

$$\frac{r}{a-r} = \sin\frac{\theta}{2}.$$

But
$$2c = 2a\sin\frac{\theta}{2}.$$

$$\therefore\ \frac{r}{a-r} = \frac{c}{a};$$

$$\therefore\ \frac{a-r}{r} = \frac{a}{c};$$

$$\therefore\ \frac{1}{r} = \frac{1}{a} + \frac{1}{c}.$$

Problem 177. *If* $\tan x + \tan 2x + \tan 3x + \tan 4x = 0$, *show that either* $5x = n\pi$, *or* $2x = (2m+1)\pi$, *or* $8\cos 2x = 1 \pm \sqrt{17}$.

Solution.
$$\frac{\sin x}{\cos x} + \frac{\sin 4x}{\cos 4x} + \frac{\sin 2x}{\cos 2x} + \frac{\sin 3x}{\cos 3x} = 0;$$

therefore
$$\frac{\sin x\cos 4x + \cos x\sin 4x}{\cos x\cos 4x} + \frac{\sin 2x\cos 3x + \cos 2x\sin 3x}{\cos 2x\cos 3x} = 0;$$

therefore
$$\frac{\sin 5x}{\cos x\cos 4x} + \frac{\sin 5x}{\cos 2x\cos 3x} = 0;$$

therefore either $\quad \sin 5x = 0$ or $\dfrac{1}{\cos x\cos 4x} + \dfrac{1}{\cos 2x\cos 3x} = 0;$

If we take the former, then $5x = n\pi$.

If we take the latter, then $\cos 2x\cos 3x + \cos x\cos 4x = 0;$

therefore
$$\cos 2x(4\cos^3 x - 3\cos x) + \cos x\cos 4x = 0;$$

therefore either $\quad \cos x = 0\ $ or $\ \cos 2x(4\cos^2 x - 3) + \cos 4x = 0.$

If we take the former, then $x = (2m+1)\dfrac{\pi}{2}$.

If we take the latter, then $\cos 2x(2 + 2\cos 2x - 3) + 2\cos^2 2x - 1 = 0;$

therefore
$$4\cos^2 2x - \cos 2x - 1 = 0;$$

and by solving this quadratic we obtain $\cos 2x = \dfrac{1 \pm \sqrt{17}}{8}$.

Problem 178. *Show that*

$$\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)} + \frac{\sin(\theta-\gamma)\sin(\theta-\alpha)}{\sin(\beta-\gamma)\sin(\beta-\alpha)} + \frac{\sin(\theta-\alpha)\sin(\theta-\beta)}{\sin(\gamma-\alpha)\sin(\gamma-\beta)} = 1.$$

Solution. We may proceed as in the solution of *Problem* 147, and seek for the values of x, y, and z, which make

$$x\sin(\theta-\beta)\sin(\theta-\gamma)+y\sin(\theta-\gamma)\sin(\theta-\alpha)+z\sin(\theta-\alpha)\sin(\theta-\beta)$$

always equal to 1. Then we shall find that $x=\dfrac{1}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}$, and so on.

Or we may verify the formula by direct work. For reduce the three fractions to the common denominator $\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)$. Then the numerator will become $L\sin^2\theta+M\sin\theta\cos\theta+N\cos^2\theta$, where

$$L=\cos\beta\cos\gamma\sin(\gamma-\beta)+\cos\gamma\cos\alpha\sin(\alpha-\gamma)+\cos\alpha\cos\beta\sin(\beta-\alpha),$$
$$M=-\sin(\gamma+\beta)\sin(\gamma-\beta)-\sin(\alpha+\gamma)\sin(\alpha-\gamma)-\sin(\beta+\alpha)\sin(\beta-\alpha),$$
$$N=\sin\beta\sin\gamma\sin(\gamma-\beta)+\sin\gamma\sin\alpha\sin(\alpha-\gamma)+\sin\alpha\sin\beta\sin(\beta-\alpha).$$

It is obvious by *Art*. 83 (page 404) that $M=0$; and we have seen in the solution of *Problem* 157 that L and N are each equal to the common denominator; so that $L\sin^2\theta+N\cos^2\theta$ is also equal to this denominator, and the expression is equal to unity.

Problem 179. *ABC is a triangle; straight lines are drawn bisecting the angles A, B, C and meeting the opposite sides at D, E, F respectively. Show that the area of the triangle DEF*

$$=\frac{2S\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}}{\cos\dfrac{B-C}{2}\cos\dfrac{C-A}{2}\cos\dfrac{A-B}{2}}.$$

Solution. By *Euclid* VI. 2 we find that $BD=\dfrac{ac}{b+c}$, and $CD=\dfrac{ab}{b+c}$.
Similar expressions hold for the segments of the other sides of ABC.
Therefore the area of the triangle DCE

$$=\frac{1}{2}\frac{ab}{b+c}\cdot\frac{ab}{a+c}\sin C=\frac{Sab}{(a+c)(b+c)}.$$

Similar expressions hold for the areas of EFA and FDB.
Therefore the area of DEF

$$=S\left\{1-\frac{ab}{(a+c)(b+c)}-\frac{bc}{(b+a)(c+a)}-\frac{ca}{(c+b)(a+b)}\right\}$$
$$=\frac{S}{(a+b)(b+c)(c+a)}\{(a+b)(b+c)(c+a)-ab(a+b)-bc(b+c)-ca(c+a)\}$$
$$=\frac{2abcS}{(a+b)(b+c)(c+a)}=2S\cdot\frac{a}{b+c}\cdot\frac{b}{c+a}\cdot\frac{c}{a+b}\cdot.$$

Now $$\frac{a}{b+c}=\frac{\sin A}{\sin B+\sin C}=\frac{\sin\dfrac{A}{2}\cos\dfrac{A}{2}}{\sin\dfrac{B+C}{2}\cos\dfrac{B-C}{2}}=\frac{\sin\dfrac{A}{2}}{\cos\dfrac{B-C}{2}}.$$

Similarly $$\frac{b}{c+a}=\frac{\sin\dfrac{B}{2}}{\cos\dfrac{C-A}{2}},\text{ and }\frac{c}{a+b}=\frac{\sin\dfrac{C}{2}}{\cos\dfrac{A-B}{2}}.$$

Thus the required result is obtained.

Problem 180. *From the top of a mountain the angles of depression of two stations in the plane at its foot are observed to be α and β, and the difference of their bearings is observed to be γ. Show that if c be the distance between the two stations, the height of the mountain will be*

$$\frac{c\sin\alpha\sin\beta}{\sin(\alpha+\beta)\cos\phi}, \text{ where } \sin^2\phi = \frac{\sin 2\alpha \sin 2\beta}{\sin^2(\alpha+\beta)}\cos^2\frac{\gamma}{2}.$$

Solution. Let x denote the height of the mountain; then the distances of the two stations from the point in the horizontal plane which is vertically under the top of the mountain are $x\cot\alpha$ and $x\cot\beta$ respectively.

Thus $\quad c^2 = x^2\cot^2\alpha + x^2\cot^2\beta - 2x^2\cot\alpha\cot\beta\cos\gamma$; (*Art.* 215 : page 419)

$$\therefore\ x^2 = \frac{c^2}{\cot^2\alpha + \cot^2\beta \cdot 2\cot\alpha\cot\beta\cos\gamma}$$

$$= \frac{c^2\sin^2\alpha\sin^2\beta}{\sin^2\beta\cos^2\alpha + \sin^2\alpha\cos^2\beta - 2\sin\alpha\cos\alpha\sin\beta\cos\beta\cos\gamma}$$

The denominator of this fraction may be put in the form

$$(\sin\beta\cos\alpha + \cos\beta\sin\alpha)^2 - \sin 2\alpha\sin 2\beta\cos^2\frac{\gamma}{2},$$

so that with the specified value of ϕ it becomes $\sin^2(\alpha+\beta)\cos^2\phi$;

and therefore $$x = \frac{c\sin\alpha\sin\beta}{\sin(\alpha+\beta)\cos\phi}.$$

Problem 181. *Find approximately the angle subtended by a target two feet wide at the distance of 450 yards.*

Solution. Let θ denote the angle; then $\tan\dfrac{\theta}{2} = \dfrac{1}{3\times 450}$; therefore approximately $\dfrac{\theta}{2} = \dfrac{1}{1350}$; therefore $\theta = \dfrac{1}{675}$. Hence the number of degrees in the angle is $\dfrac{180}{\pi}\times\dfrac{1}{675}$, and the number of minutes is $\dfrac{180}{\pi}\times\dfrac{60}{675}$, that is $\dfrac{4}{45}\times\dfrac{180}{\pi}$, that is $\dfrac{4}{45}\times 57.29\ldots$ that is about 5.

Problem 182. *Sum the following infinite series :*

$$\frac{1}{\underline{|2}} + \frac{4}{\underline{|3}} + \frac{9}{\underline{|4}} + \frac{16}{\underline{|5}} + \ldots$$

Solution. The general term of the series is $\dfrac{n^2}{\underline{|n+1}}$; for we obtain all the terms by putting successively $1, 2, 3, \ldots$ for n in this expression.

Now $$\frac{n^2}{\underline{|n+1}} = \frac{n(n+1)-(n+1)+1}{\underline{|n+1}} = \frac{1}{\underline{|n-1}} - \frac{1}{\underline{|n}} + \frac{1}{\underline{|n+1}}.$$

If then we split up each term into three in this manner, beginning with the second term, we obtain

$$\frac{1}{\underline{|2}} + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \ldots$$
$$-\left\{\frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \frac{1}{\underline{|5}} + \ldots\right\}$$

$$+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\frac{1}{\underline{|5}}+\ldots;$$

that is $\frac{1}{2}+e-1-(e-2)+e-2-\frac{1}{2}$, that is $e-1$.

Problem 183. *If $2\sec\theta=\sec(\theta+2\alpha)+\sec(\theta-2\alpha)$, show that*

$$\cos^2\theta=2\cos^2\alpha.$$

Solution.

Here $$\frac{2}{\cos\theta}=\frac{1}{\cos(\theta+2\alpha)}+\frac{1}{\cos(\theta-2\alpha)}=\frac{2\cos\theta\cos 2\alpha}{\cos(\theta+2\alpha)\ cos(\theta-2\alpha)}$$
$$=\frac{2\cos\theta\cos 2\alpha}{\cos^2\theta-\sin^2 2\alpha};$$

therefore $$\cos^2\theta-\sin^2 2\alpha=\cos^2\theta\cos 2\alpha,$$

therefore $$\cos^2\theta(1-\cos 2\alpha)=\sin^2 2\alpha=4\sin^2\alpha\cos^2\alpha;$$

therefore $$\cos^2\theta=2\cos^2\alpha.$$

Problem 184. *Solve the equations*

$$2(\sin 2\theta+\sin 2\phi)=1=2\sin(\theta+\phi).$$

Solution. Here $4\sin(\theta+\phi)\cos(\theta-\phi)=1$, and $2\sin(\theta+\phi)=1$;

therefore $$\sin(\theta+\phi)=\frac{1}{2},\ \text{and}\ \cos(\theta-\phi)=\frac{1}{2};$$

therefore $$\theta+\phi=n\pi+(-1)^n\frac{\pi}{6},\ \text{and}\ \theta-\phi=2m\pi\pm\frac{\pi}{3}.$$

Problem 185. *In any triangle*

$$(\sin A+\sin B+\sin C)\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)=4+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Solution. $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$, by Chapter VIII : *Problem* 16.

And $$\tan\frac{A}{2}+\tan\frac{B}{2}=\frac{\sin\frac{A+B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}=\frac{\cos\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}};$$

therefore $$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}=\frac{\cos\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}$$
$$=\frac{\cos^2\frac{C}{2}+\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}};$$

the numerator $$=1-\sin^2\frac{C}{2}+\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}$$
$$=1+\sin\frac{C}{2}\left\{\cos\frac{A}{2}\cos\frac{B}{2}-\cos\frac{A+B}{2}\right\}=1+\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2};$$

and thus the fraction $= \dfrac{1+\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$

Hence by multiplication we obtain the required result.

Problem 186. *A, B, C, D, E, ... are the angular points of a polygon inscribed in a circle; from the centre of the circle perpendiculars are drawn to the sides, and a second polygon is formed by joining the feet of the perpendiculars. Show that if the area of the first polygon is double that of the second,*

$$\sin 2A + \sin 2B + \sin 2C + \sin 2D + \sin 2E + \ldots = 0.$$

Solution. Proceed as in the solution of *Problem* 166. Then we obtain the following expression for the excess of the sum of all triangles at the corners above the second polygon

$$\frac{r^2}{4}\left\{\sin 2A + \sin 2B + \sin 2C + \sin 2D + \ldots\right\},$$

where r is the radius of the circle.

Hence this vanishes if $\sin 2A + \sin 2B + \ldots = 0$, and then the sum of the triangles at the corners is equal to the second polygon, and therefore the first polygon is double the second.

Problem 187. *In any triangle*

$$a\sin\frac{B-C}{2}\sec\frac{A}{2} + b\sin\frac{C-A}{2}\sec\frac{B}{2} + c\sin\frac{A-B}{2}\sec\frac{C}{2} = 0.$$

Solution. We have $a = 2R\sin A,\ b = 2R\sin B,\ c = 2R\sin C$;
hence the proposed expression

$$= 4R\left\{\sin\frac{B-C}{2}\sin\frac{A}{2} + \sin\frac{C-A}{2}\sin\frac{B}{2} + \sin\frac{A-B}{2}\sin\frac{C}{2}\right\}$$

$$= 2R\left\{\cos\frac{A+C-B}{2} - \cos\frac{A+B-C}{2} + \cos\frac{B+A-C}{2} - \cos\frac{B+C-A}{2}\right.$$
$$\left. + \cos\frac{B+C-A}{2} - \cos\frac{A+C-B}{2}\right\} = 0.$$

Problem 188. *In any triangle*

$$\frac{a^2\sin(B-C)}{\sin B+\sin C} + \frac{b^2\sin(C-A)}{\sin C+\sin A} + \frac{c^2\sin(A-B)}{\sin A+\sin B} = 0.$$

Solution. We have $a = 2R\sin A,\ b = 2R\sin B,\ c = 2R\sin C$;
hence the proposed expression

$$= 4R^2\left\{\frac{\sin^2 A\sin(B-C)}{\sin B+\sin C} + \frac{\sin^2 B\sin(C-A)}{\sin C+\sin A} + \frac{\sin^2 C\sin(A-B)}{\sin A+\sin B}\right\}.$$

Now $\quad \dfrac{\sin^2 A\sin(B-C)}{\sin B+\sin C} = \dfrac{\sin A\sin(B+C)\sin(B-C)}{\sin B+\sin C}$

$$= \frac{\sin A(\sin^2 B - \sin^2 C)}{\sin B+\sin C} = \sin A(\sin B - \sin C).$$

In this way the proposed expression
$= 4R^2\{\sin A(\sin B - \sin C) + \sin B(\sin C - \sin A) + \sin C(\sin A - \sin B)\} = 0.$

Problem 189. *If a be the side of a regular polygon inscribed in a circle of radius r, and b the side of another regular polygon of twice the number of sides inscribed in the same circle, show that*

$$b = \sqrt{r\left(r + \frac{a}{2}\right)} - \sqrt{r\left(r - \frac{a}{2}\right)}.$$

Solution. If n be the number of sides in the first polygon we have

$$a = 2r\sin\frac{\pi}{n},\ b = 2r\sin\frac{\pi}{2n}.$$

By *Art.* 100 (page 407), since $\frac{\pi}{n}$ lies between 0 and $\frac{\pi}{2}$, we have

$$2\sin\frac{\pi}{2n} = \sqrt{\left(1 + \sin\frac{\pi}{n}\right)} - \sqrt{\left(1 - \sin\frac{\pi}{n}\right)};$$

therefore $$\frac{b}{r} = \sqrt{\left(1 + \frac{a}{2r}\right)} - \sqrt{\left(1 - \frac{a}{2r}\right)}.$$

Multiply by r, and we obtain the required result.

Problem 190. *If $A + B + C = 180°$, show that*

$$\left(1 - \sin\frac{B}{2}\right)\left(1 - \sin\frac{C}{2}\right)\cos\frac{A}{2} + \left(1 - \sin\frac{C}{2}\right)\left(1 - \sin\frac{A}{2}\right)\cos\frac{B}{2} + \left(1 - \sin\frac{A}{2}\right)\left(1 - \sin\frac{B}{2}\right)\cos\frac{C}{2} = \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Solution. $\left(1 - \sin\frac{B}{2}\right)\left(1 - \sin\frac{C}{2}\right)\cos\frac{A}{2}$

$$= \cos\frac{A}{2} - \left(\sin\frac{B}{2} + \sin\frac{C}{2}\right)\cos\frac{A}{2} + \sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}.$$

Develop each of the other two terms in the same way; the aggregate
$= \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} - \sin\frac{A+B}{2} - \sin\frac{B+C}{2} - \sin\frac{C+A}{2} + \sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2} + \sin\frac{C}{2}\sin\frac{A}{2}\cos\frac{B}{2} + \sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}.$
But $\cos\frac{A}{2} = \sin\frac{B+C}{2}, \cos\frac{B}{2} = \sin\frac{C+A}{2}, \cos\frac{C}{2} = \sin\frac{A+B}{2};$
thus the expression

$$= \sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2} + \sin\frac{C}{2}\sin\frac{A}{2}\cos\frac{B}{2} + \sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$$
$$= \sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2} + \sin\frac{A}{2}\left(\sin\frac{C}{2}\cos\frac{B}{2} + \sin\frac{B}{2}\cos\frac{C}{2}\right)$$
$$= \sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2} + \sin\frac{A}{2}\sin\frac{B+C}{2}$$

$$= \cos\frac{A}{2}\left\{\sin\frac{A}{2} + \sin\frac{B}{2}\sin\frac{C}{2}\right\} = \cos\frac{A}{2}\left\{\cos\frac{B+C}{2} + \sin\frac{B}{2}\sin\frac{C}{2}\right\}$$
$$= \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Or instead of the last four lines we may use *Art*. 113 (page 409), observing that here
$$\cos\left(\frac{A}{2}+\frac{B}{2}+\frac{C}{2}\right) = 0.$$

Problem 191. *A person walks in a straight line towards a very distant object, and observes that at three points A, B, C, the elevations of the objects are α, 2α, 3α respectively. Show that $AB = 3\cdot BC$ nearly.*

Solution. Let D denote the top of the object.

From the triangle ABD we have $\dfrac{AB}{BD} = 1$, for the angles BAD and BDA are equal. From the triangle BDC we have $\dfrac{BD}{BC} = \dfrac{\sin 3\alpha}{\sin\alpha}$.

Therefore $$\frac{AB}{BC} = \frac{\sin 3\alpha}{\sin\alpha} = 3 - 4\sin^2\alpha.$$

Since the object is very distant α is very small; therefore $AB = 3BC$ nearly.

Problem 192. *If x is less than unity so also is*
$$\frac{1}{x} + \frac{1}{\log(1-x)}.$$

Solution. $$\frac{1}{x} + \frac{1}{\log_e(1-x)} = \frac{1}{x} - \frac{1}{x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots} = \frac{\frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \dots}{x^2 + \frac{1}{2}x^3 + \frac{1}{3}x^4 + \dots}.$$

Here every term in the numerator is less than the corresponding term in the denominator, and thus the fraction is less than unity.

Problem 193. *Having given*
$$\frac{6\sin B}{\cos(A+B)} = \frac{3\sin 2B}{\cos(A+2B)} = \frac{2\sin 3B}{\cos(A+3B)},$$
show that it is impossible that any value can be assigned to B other than $n\pi$.

Solution. Here $\dfrac{6\sin B}{\cos(A+B)} = \dfrac{6\sin B\cos B}{\cos(A+2B)}$;

thus either $\sin B = 0$ or $\cos(A+2B) = \cos(A+B)\cos B$;

the latter gives $\cos(A+B)\cos B - \sin(A+B)\sin B = \cos(A+B)\cos B$,

so that either $\sin B = 0$ or $\sin(A+B) = 0$.

Suppose that $\sin(A+B) = 0$; then since
$$\frac{3\sin 2B}{\cos(A+B+B)} = \frac{2\sin 3B}{\cos(A+B+2B)},$$
we have $$\frac{3\sin 2B}{\cos(A+B)\cos B} = \frac{2\sin 3B}{\cos(A+B)\cos 2B},$$

so that
$$\frac{3\sin B}{\cos(A+B)} = \frac{\sin 3B}{\cos(A+B)\cos 2B};$$
therefore
$$3\sin B\cos 2B = \sin 3B,$$
therefore
$$3\sin B(1-2\sin^2 B) = 3\sin B - 4\sin^3 B,$$
therefore
$$6\sin^3 B = 4\sin^3 B,$$
therefore
$$\sin B = 0.$$

Problem 194. *If* $\frac{\sin\theta}{x} = \frac{\cos\theta}{y}$, *and* $\frac{\sin^2\theta}{y^2} + \frac{\cos^2\theta}{x^2} = \frac{6}{x^2+y^2}$, *find a general expression for the value of* θ.

Solution. Here $\tan\theta = \frac{x}{y}$; therefore $\sin^2\theta = \frac{x^2}{x^2+y^2}$, and $\cos^2\theta = \frac{y^2}{x^2+y^2}$. Substitute in the second given equation; thus
$$\left(\frac{x^2}{y^2} + \frac{y^2}{x^2}\right)\frac{1}{x^2+y^2} = \frac{6}{x^2+y^2}; \text{ therefore } \frac{x^2}{y^2} + \frac{y^2}{x^2} = 6.$$

From this quadratic in $\frac{x^2}{y^2}$ we find $\frac{x^2}{y^2} = 3 \pm 2\sqrt{2} = (\sqrt{2}\pm 1)^2$; therefore $\tan\theta = \pm(\sqrt{2}+1)$ or $\pm(\sqrt{2}-1)$. The former gives $\theta = n\pi \pm \frac{3\pi}{8}$; and the latter gives $\theta = n\pi \pm \frac{\pi}{8}$.

Problem 195. *If* A, B C *are the angles of a triangle, and* $\sin A$, $\sin B$, $\sin C$ *are in Harmonical Progression, then* $1-\cos A$, $1-\cos B$, $1-\cos C$ *are in Harmonical Progression.*

Solution. Since the sines of the angles are in Harmonical Progression, so are the sides of the opposite angles. Thus a, b, c are in Harmonical Progression, and we have to show that $\frac{(s-b)(s-c)}{bc}$, $\frac{(s-c)(s-a)}{ca}$, $\frac{(s-a)(s-b)}{ab}$ are so also. Multiply each term by $\frac{abc}{(s-a)(s-b)(s-c)}$; thus we see it is sufficient to show that $\frac{a}{s-a}$, $\frac{b}{s-b}$, $\frac{c}{s-c}$; are in Harmonical Progression, or that $\frac{s-a}{a}$, $\frac{s-b}{b}$, $\frac{s-c}{c}$ are in Arithmetical Progression, or that $\frac{s}{a}$, $\frac{s}{b}$, $\frac{s}{c}$ are in Arithmetical Progression; and this is the case since a, b, c are in Harmonical Progression.

Problem 196. *If* R *be the radius of a circle described about a regular pentagon whose side is* a, *show that* $\frac{R}{a} = \frac{17}{20}$ *nearly.*

Solution. We have $a = 2R\sin\frac{\pi}{5}$;
$$\therefore\ \frac{R}{a} = \frac{1}{2\sin\frac{\pi}{5}} = \frac{2}{\sqrt{10-2\sqrt{5}}} = \frac{2\sqrt{10+2\sqrt{5}}}{\sqrt{80}} = \frac{2\sqrt{200+40\sqrt{5}}}{\sqrt{80\times 20}}$$

$$= \frac{\sqrt{200+40\sqrt{5}}}{20} = \frac{\sqrt{289.44\ldots}}{20} = \frac{17}{20} \text{ nearly.}$$

Problem 197. *In a triangle* $\dfrac{\sin A}{\sin B} = \dfrac{m}{n}$, *and* $\dfrac{\cos A}{\cos B} = \dfrac{p}{q}$. *Show that*

$$\cos C = \frac{mp - nq}{np - mq}.$$

Solution. Here $\dfrac{\sin(B+C)}{\sin B} = \dfrac{m}{n}, \; \dfrac{\cos(B+C)}{\cos B} = -\dfrac{p}{q};$

therefore $\cos C + \cot B \sin C = \dfrac{m}{n}, \; \tan B \sin C - \cos C = \dfrac{p}{q};$

therefore $\sin^2 C = \left(\dfrac{m}{n} - \cos C\right)\left(\dfrac{p}{q} + \cos C\right);$

therefore $1 = \dfrac{mp}{nq} + \cos C\left(\dfrac{m}{n} - \dfrac{p}{q}\right);$

therefore $\cos C = \dfrac{mp - nq}{np - mq}.$

Problem 198. *If O be the centre of the circle inscribed in a triangle ABC*, *and* D, E, F *the points of contact with BC*, *CA*, *AB respectively , show that*

$$OA \cdot OB \cdot OC(AF + BD + CE) = 4R \cdot AF \cdot BD \cdot CE.$$

Solution. We have $OA = \dfrac{r}{\sin\dfrac{A}{2}}, \; OB = \dfrac{r}{\sin\dfrac{B}{2}}, \; OC = \dfrac{r}{\sin\dfrac{C}{2}};$

$$AF = r\cot\frac{A}{2}, \; BD = r\cot\frac{B}{2}, \; CE = r\cot\frac{C}{2}.$$

Hence we have to show that

$$\frac{r^4}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\left\{\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right\} = 4Rr^3\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$$

By *Chapter* VIII : *Problem* 15 the left-hand member

$$= \frac{r^4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}};$$

thus we have to show that

$$4R\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C = r.$$

The left-hand member

$$= 4R\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}} = \frac{4RS^2}{sabc} = \frac{S}{s} = r.$$

Problem 199. *In the semi-circle ABC, whose centre is O, and radius ρ, the straight line OB is drawn at an angle 2α to OC. Circles are inscribed in the* *triangles OAB, OCB. Show that the distance between their centres is*

$$\frac{\rho\sqrt{2-\sin 2\alpha}}{\sqrt{(1+\sin\alpha)(1+\cos\alpha)}}.$$

Solution. The radius of the circle inscribed in the triangle OBC

$$=\frac{\text{area of the triangle}}{\text{semi-perimeter}}=\frac{\frac{1}{2}\rho^2\sin 2\alpha}{\rho(1+\sin\alpha)}=\frac{\rho\sin 2\alpha}{2(1+\sin\alpha)}.$$

Let P denote the position of the centre; then

$$OP=\frac{\rho\sin 2\alpha}{2(1+\sin\alpha)}\times\frac{1}{\sin\alpha}=\frac{\rho\cos\alpha}{1+\sin\alpha}.$$

Again, let Q denote the position of the centre of the circle inscribed in the triangle OAB; then, as 2α is now to be changed to $\pi-2\alpha$, we have

$$OQ=\frac{\rho\cos\left(\frac{\pi}{2}-\alpha\right)}{1+\sin\left(\frac{\pi}{2}-\alpha\right)}=\frac{\rho\sin\alpha}{1+\cos\alpha}.$$

And since POQ is a right angle, $PQ^2=OP^2+OQ^2$

$$=\rho^2\left\{\frac{\cos^2\alpha}{(1+\sin\alpha)^2}+\frac{\sin^2\alpha}{(1+\cos\alpha)^2}\right\}$$

$$=\rho^2\left\{\frac{1-\sin\alpha}{1+\sin\alpha}+\frac{1-\cos\alpha}{1+\cos\alpha}\right\}=\frac{\rho^2(2-\sin 2\alpha)}{(1+\sin\alpha)(1+\cos\alpha)};$$

therefore $$PQ=\frac{\rho\sqrt{2-\sin 2\alpha}}{\sqrt{(1+\sin\alpha)(1+\cos\alpha)}}.$$

Problem 200. *A man walking along a straight road observes the directions with respect to the road of two objects when the angle which they subtend is greatest, and then measures the distance from the point of observation to the point whence they appear in the same straight line. Find the distance between them.*

Solution. Let A and B be the two objects. Suppose a circle to pass through A and B, and to *touch* the straight line at P; then P is the point at which the greatest angle is subtended: see *Appendix to Euclid*, page 308. Produce AB to meet the straight line at Q. Let the angle $BPQ=\alpha$, and let β be the angle between AP and the straight line. Then also $PAB=\alpha$, and $PBA=\beta$, by *Euclid* III. 32. Let $PQ=c$.

Then $$\frac{BP}{PQ}=\frac{\sin(\beta-\alpha)}{\sin\beta},\quad \frac{AB}{BP}=\frac{\sin(\beta+\alpha)}{\sin\alpha};$$

therefore $$\frac{AB}{c}=\frac{\sin(\beta+\alpha)\sin(\beta-\alpha)}{\sin\alpha\sin\beta}.$$

Problem 201. *Show that $r_1+r_2+r_3-r=4R$.*

Solution. We have

$$r_1+r_2+r_3-r=S\left\{\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}\right\}$$

$$= S\left\{\frac{2s-a-b}{(s-a)(s-b)}+\frac{c}{s(s-c)}\right\}$$
$$= cS\left\{\frac{1}{(s-a)(s-b)}+\frac{1}{s(s-c)}\right\}$$
$$= \frac{cS}{S^2}\{s(s-c)+(s-a)(s-b)\} = \frac{c}{S}\{2s^2-s(a+b+c)+ab\}$$
$$= \frac{abc}{S} = 4R.$$

Problem 202. *Show that* $\sin^{-1}\dfrac{1}{\sqrt{5}}+\cot^{-1}3=\dfrac{\pi}{4}$.

Solution. $\sin^{-1}\dfrac{1}{\sqrt{5}}=\tan^{-1}\dfrac{1}{2},\quad \cot^{-1}3=\tan^{-1}\dfrac{1}{3};$

$$\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}}=\tan^{-1}1=\frac{\pi}{4}.$$

Problem 203. *ABC is a triangle; a second triangle is formed by the external bisectors of the angles of ABC; then a third triangle is in like manner formed from the second, and so on. Determine the angles of the* n^{th} *triangle.*

Solution. The angle of the second triangle which is opposite to the angle C of the first triangle will be found to be $\dfrac{\pi}{2}-\dfrac{C}{2}$; Similarly the corresponding angle of the third triangle will be $\dfrac{\pi}{2}-\dfrac{1}{2}\left(\dfrac{\pi}{2}-\dfrac{C}{2}\right)$, that is $\dfrac{\pi}{2}-\dfrac{\pi}{4}+\dfrac{C}{4}$. Proceeding in this way we find that the corresponding angle of the n^{th} triangle is

$$\pi\left\{\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\ldots-\frac{(-1)^{n-1}}{2^{n-1}}\right\}+\frac{(-1)^{n-1}C}{2^{n-1}},$$

that is
$$\frac{\pi}{2}\,\frac{1-\left(-\frac{1}{2}\right)^{n-1}}{1+\frac{1}{2}}+\frac{(-1)^{n-1}C}{2^{n-1}};$$

that is
$$\frac{\pi}{3}\left\{1-\frac{(-1)^{n-1}}{2^{n-1}}\right\}+\frac{(-1)^{n-1}C}{2^{n-1}}.$$

Similar expressions hold for the other angles.

Problem 204. *Find in terms of a the value of* $\cos 4\left(\tan^{-1} a\right)$.

Solution. Suppose $\theta=\tan^{-1}a$; then we require $\cos 4\theta$.

Now
$$\tan\theta=a,\quad \cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{1-a^2}{1+a^2},$$
$$\cos 4\theta = 2\cos^2 2\theta-1=\frac{2(1-a^2)^2}{(1+a^2)^2}-1=\frac{1-6a^2+a^4}{(1+a^2)^2}.$$

Problem 205. *Find the general term in the expansion of* $e^{ax}\cos(bx+c)$ *in powers of* x.

Solution. We have $e^{ax}\cos(bx+c) = e^{ax}(\cos bx\cos c - \sin bx \sin c)$;
then by *Art.* 300 (page 436) the required general term is

$$\frac{\left(a^2+b^2\right)^{\frac{n}{2}}}{\underline{\lfloor n}}(\cos n\theta\cos c - \sin n\theta \sin c),$$

that is
$$\frac{\left(a^2+b^2\right)^{\frac{n}{2}}}{\underline{\lfloor n}}\cos(n\theta + c),$$

where θ is such that $\tan\theta = \dfrac{b}{a}$.

Problem 206. *A circle touches the sides* AB *and* AC *of a triangle produced, and touches the side* BC *at* D. *Show that*

$$a\left(s^2 - AD^2\right) = 4s(s-b)(s-c).$$

Solution.

We have
$$\begin{aligned} AD^2 &= AB^2 + BD^2 - 2AB\cdot BD\cos B \\ &\quad \text{by } Art.\ 250 \text{ (page 427) :} \\ &= c^2 + (s-c)^2 - 2c(s-c)\cos B, \end{aligned}$$

therefore
$$\begin{aligned} s^2 - AD^2 &= s^2 - c^2 - (s-c)^2 + 2c(s-c)\cos B \\ &= (s-c)\{s + c - (s-c) + 2c\cos B\} \\ &= 2c(s-c)(1+\cos B) = 4c(s-c)\cos^2\frac{B}{2}; \end{aligned}$$

therefore $a(s^2 - AD^2) = 4ac(s-c)\cos^2\dfrac{B}{2} = 4s(s-b)(s-c)$.

Problem 207. *If* $\cos^{-1}\left(\alpha + \beta\sqrt{-1}\right) = \theta + \phi\sqrt{-1}$, *where the letters denote real quantities, show that*

$$\frac{\alpha^2}{\cos^2\theta} - \frac{\beta^2}{\sin^2\theta} = 1, \text{ and } \frac{\alpha^2}{\left(e^{\phi}+e^{-\phi}\right)^2} + \frac{\beta^2}{\left(e^{\phi}-e^{-\phi}\right)^2} = \frac{1}{4}.$$

Solution. We have

$$\begin{aligned} \alpha + \beta\sqrt{-1} = \cos\left(\theta + \phi\sqrt{-1}\right) &= \cos\theta\cos\phi\sqrt{-1} - \sin\theta\sin\phi\sqrt{-1} \\ &= \cos\theta\frac{e^{-\phi}+e^{\phi}}{2} - \sin\theta\frac{e^{-\phi}-e^{\phi}}{2\sqrt{-1}} \\ &= \cos\theta\frac{e^{\phi}+e^{-\phi}}{2} - \sin\theta\frac{e^{\phi}-e^{-\phi}}{2}\sqrt{-1}. \end{aligned}$$

Hence by equating the possible and the impossible parts we have

$$\alpha = \cos\theta\frac{e^{\phi}+e^{-\phi}}{2},\ \beta = -\sin\theta\frac{e^{\phi}-e^{-\phi}}{2}.$$

Therefore
$$\frac{\alpha^2}{\cos^2\theta} - \frac{\beta^2}{\sin^2\theta} = \left(\frac{e^{\phi}+e^{-\phi}}{2}\right)^2 - \left(\frac{e^{\phi}-e^{-\phi}}{2}\right)^2 = 1;$$

and
$$\frac{\alpha^2}{(e^{\phi}+e^{-\phi})^2} + \frac{\beta^2}{(e^{\phi}-e^{-\phi})^2} = \frac{\cos^2\theta + \sin^2\theta}{4} = \frac{1}{4}.$$

Problem 208. *Show that* $\log\sec\theta$

$$= 2\left\{\sin^2\theta - \frac{1}{2}\sin^2 2\theta + \frac{1}{3}\sin^2 3\theta - \frac{1}{4}\sin^2 4\theta + \dots\right\}.$$

Solution.

$$\begin{aligned}\log\sec\theta &= \frac{1}{2}\log\frac{1}{\cos^2\theta} = \frac{1}{2}\log\frac{2}{1+\cos 2\theta}\\ &= \frac{1}{2}\log\frac{4}{2+e^{2\iota\theta}+e^{-2\iota\theta}} = \frac{1}{2}\log\frac{4}{(1+e^{2\iota\theta})(1+e^{-2\iota\theta})}\\ &= \frac{1}{2}\left\{2\log 2 - \log\left(1+e^{2\iota\theta}\right) - \log\left(1+e^{-2\iota\theta}\right)\right\};\end{aligned}$$

$$\begin{aligned}\therefore\ 2\log\sec\theta &= 2\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots\right)\\ &\quad - \left(e^{2\iota\theta} - \frac{1}{2}e^{4\iota\theta} + \frac{1}{3}e^{6\iota\theta} - \frac{1}{4}e^{8\iota\theta} + \dots\right)\\ &\quad - \left(e^{-2\iota\theta} - \frac{1}{2}e^{-4\iota\theta} + \frac{1}{3}e^{-6\iota\theta} - \frac{1}{4}e^{-8\iota\theta} + \dots\right).\end{aligned}$$

Now

$$2 - e^{2\iota\theta} - e^{-2\iota\theta} = -(e^{\iota\theta} - e^{-\iota\theta})^2 = 4\sin^2\theta;$$

$$\frac{1}{2}\left(-2 + e^{4\iota\theta} + e^{-4\iota\theta}\right) = \frac{1}{2}\left(e^{2\iota\theta} - e^{-2\iota\theta}\right)^2 = -\frac{4}{2}\sin^2 2\theta,$$

$$\frac{1}{3}(2 - e^{6\iota\theta} - e^{-6\iota\theta}) = -\frac{1}{3}(e^{3\iota\theta} - e^{-3\iota\theta})^2 = \frac{4}{3}\sin^2 3\theta,$$

and so on; thus

$$2\log\sec\theta = 4\left\{\sin^2\theta - \frac{1}{2}\sin^2 2\theta + \frac{1}{3}\sin^2 3\theta - \dots\right\};$$

therefore

$$\log\sec\theta = 2\left\{\sin^2\theta - \frac{1}{2}\sin^2 2\theta + \frac{1}{3}\sin^2 3\theta - \dots\right\}.$$

Problem 209. *Show that the sum of n terms of the series*

$$\sec\alpha\sec(\alpha+\beta) + \sec(\alpha+\beta)\sec(\alpha+2\beta) + \sec(\alpha+2\beta)\sec(\alpha+3\beta) + \dots$$
$$= \operatorname{cosec}\beta\left\{\tan(\alpha+n\beta) - \tan\alpha\right\}.$$

Solution.

$$\sec\alpha\sec(\alpha+\beta) = \frac{1}{\sin\beta}\left\{\tan(\alpha+\beta) - \tan\alpha\right\},$$

$$\sec(\alpha+\beta)\sec(\alpha+2\beta) = \frac{1}{\sin\beta}\left\{\tan(\alpha+2\beta) - \tan(\alpha+\beta)\right\},$$

and so on.

Then by addition we obtain the required result.

Problem 210. *In a regular hexagon, one of whose sides is equal to a, a circle is inscribed; and in this circle another regular hexagon, and so on until there are in all n hexagons. Show that the sum of the areas of the hexagons is*

$$6\sqrt{3}\left\{1 - \left(\frac{3}{4}\right)^n\right\}a^2.$$

Solution. The regular hexagon may be divided into six equilateral triangles; and thus the area of the first hexagon $= \dfrac{6a^2\sqrt{3}}{4}$.

By *Art.* 255 (page 430) the radius of the first circle $= \dfrac{a}{2}\cot 30^0 = \dfrac{a\sqrt{3}}{2}$; and the side of the second hexagon is equal to this, so that the area of the second hexagon $= \dfrac{6a^2\sqrt{3}}{4}\left(\dfrac{\sqrt{3}}{2}\right)^2$. In this way we see that the areas of the hexagons form a geometrical progression of which the ratio is $\dfrac{3}{4}$; and the sum of the areas $=$

$$\frac{6a^2\sqrt{3}}{4}\frac{1-\left(\frac{3}{4}\right)^n}{1-\frac{3}{4}} = 6a^2\sqrt{3}\left\{1-\left(\frac{3}{4}\right)^n\right\}.$$

Problem 211. *Adapt the expression $a\cos A + b\cos B + c\cos C$ to logarithmic computation, the letters denoting the sides and the angles of a triangle.*

Solution. We have $a = 2R\sin A$, $b = 2R\sin B$, $c = 2R\sin C$; thus the proposed expression

$$\begin{aligned} &= 2R(\sin A\cos A + \sin B\cos B + \sin C\cos C)\\ &= R(\sin 2A + \sin 2B + \sin 2C)\\ &= 4R\sin A\sin B\sin C, \text{ by } Art.\ 114 \text{ (page 409)},\\ &= 2a\sin B\sin C. \end{aligned}$$

The expression is now adapted to logarithms.

Problem 212. *A, B, C are telegraph posts at equal intervals by the side of a rail-road; t and t′ are the tangents of the angles which AB and BC subtend at any point P; T is the tangent of the angle which the road makes with PB. Show that*

$$\frac{2}{T} = \frac{1}{t'} - \frac{1}{t}.$$

Solution. Let θ denote the angle APB, ϕ the angle BPC, and ψ the angle ABP.

We have $$\frac{AB}{PB} = \frac{\sin\theta}{\sin(\psi+\theta)},\quad \frac{BC}{PB} = \frac{\sin\phi}{\sin(\psi-\phi)};$$

but AB is supposed equal to BC, and thus

$$\frac{\sin(\psi+\theta)}{\sin\theta} = \frac{\sin(\psi-\phi)}{\sin\phi};$$

therefore $$\sin\psi\cot\theta + \cos\psi = \sin\psi\cot\phi - \cos\psi;$$

therefore $$2\cot\psi = \cot\phi - \cot\theta;$$

therefore $$\frac{2}{T} = \frac{1}{t'} - \frac{1}{t}.$$

Problem 213. *Show that*

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - 2\tan^{-1}\frac{1}{408} + \tan^{-1}\frac{1}{1393}.$$

Solution. It is shown in *Art.* 296 (page 435) that $\dfrac{\pi}{4} = 4\tan^{-1}\dfrac{1}{5} - \tan^{-1}\dfrac{1}{239}$;

hence we have only to show that

$$\tan^{-1}\frac{1}{239} = 2\tan^{-1}\frac{1}{408} - \tan^{-1}\frac{1}{1393},$$

or that

$$\tan^{-1}\frac{1}{239} + \tan^{-1}\frac{1}{1393} = 2\tan^{-1}\frac{1}{408}.$$

Now

$$\tan^{-1}\frac{1}{239} + \tan^{-1}\frac{1}{1393} = \tan^{-1}\frac{\frac{1}{239}+\frac{1}{1393}}{1-\frac{1}{239\times 1393}}$$

$$= \tan^{-1}\frac{1393+239}{239\times 1393 - 1} = \tan^{-1}\frac{1632}{332926} = \tan^{-1}\frac{816}{166463};$$

and

$$2\tan^{-1}\frac{1}{408} = \tan^{-1}\frac{\frac{2}{408}}{1-\left(\frac{1}{408}\right)^2} = \tan^{-1}\frac{2\times 408}{166463}.$$

Thus the required result is established.

Problem 214. *If P denote the point of intersection of the perpendiculars from the angles of a triangle on the opposite sides, show that* $PA^2 = 4R^2 - a^2$.

Solution. By the diagram of *Art.* 332 (page 448) we see that

$$\frac{PA}{AB} = \frac{\sin PBA}{\sin APB} = \frac{\sin\left(\frac{\pi}{2} - A\right)}{\sin(A+B)} = \frac{\cos A}{\sin C};$$

therefore

$$PA = \frac{c\cos A}{\sin C} = \frac{a\cos A}{\sin A};$$

therefore

$$PA^2 = \frac{a^2\left(1-\sin^2 A\right)}{\sin^2 A} = \frac{a^2}{\sin^2 A} - a^2 = 4R^2 - a^2.$$

Problem 215. *If* $\alpha = 15°$, *find the value of*

$$\frac{\left(\cos\alpha + \sqrt{-1}\sin\alpha\right)\left(\cos 2\alpha + \sqrt{-1}\sin 2\alpha\right)}{\cos 3\alpha - \sqrt{-1}\sin 3\alpha}.$$

Solution. $\dfrac{(\cos\alpha + \sqrt{-1}\sin\alpha)(\cos 2\alpha + \sqrt{-1}\sin 2\alpha)}{\cos 3\alpha - \sqrt{-1}\sin 3\alpha} = \dfrac{\cos 3\alpha + \sqrt{-1}\sin 3\alpha}{\cos 3\alpha - \sqrt{-1}\sin 3\alpha};$
multiply both numerator and denominator by $\cos 3\alpha + \sqrt{-1}\sin 3\alpha$; thus we obtain unity in the denominator, and $\cos 6\alpha + \sqrt{-1}\sin 6\alpha$ in the numerator: and this numerator $= \sqrt{-1}$ since $\alpha = 15°$.

Problem 216. *ABC is a triangle; a new triangle is formed by the external bisectors of the angles. Show that the sides of the new triangle are* $4R\cos\frac{A}{2}$, $4R\cos\frac{B}{2}$, *and* $4R\cos\frac{C}{2}$ *respectively.*

Solution. The new triangle will have for its angular points the centres of the escribed circles of the original triangle. Now from *Art.* 250 (page 427) we have

$$OC = CE\sec OCE = (s-b)\operatorname{cosec}\frac{C}{2};$$

and in the same manner the distance from C of the centre of the circle which touches BC and BA produced $= (s-a)\operatorname{cosec}\frac{C}{2}$. Hence the sum of these two $= (2s-b-a)\operatorname{cosec}\frac{C}{2} = c\operatorname{cosec}\frac{C}{2} = 2R\sin C\operatorname{cosec}\frac{C}{2} = 4R\cos\frac{C}{2}$.

This is the length of the side of the second triangle which passes through the point C; similar expressions hold for the other two sides.

Problem 217. *Show that the sum of the squares on the sides of the triangle formed as in the preceding Problem* $= 8R(4R+r)$.

Solution. By the preceding Example the sum of the squares

$$\begin{aligned} &= 16R^2\left\{\cos^2\frac{1}{2}A + \cos^2\frac{1}{2}B + \cos^2\frac{1}{2}C\right\} \\ &= 8R^2\left\{3 + \cos A + \cos B + \cos C\right\} \\ &= 8R^2\left\{4 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right\}, \text{ by } Art.\ 114 (\text{page } 409) \\ &= 32R^2 + \frac{32R^2S^2}{sabc} = 32R^2 + 8Rr. \end{aligned}$$

Problem 218. *Reduce to its simplest form*

$$\frac{\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10}{\cos 5\theta + 5\cos 3\theta + 10\cos\theta}.$$

Solution. The numerator can be expressed in powers of θ; and it will be found to reduce to $2^5\cos^6\theta$; in like manner the denominator will be found to reduce to $2^4\cos^5\theta$: see *Art.* 280 (page 433). Hence the expression reduces to $2\cos\theta$.

Problem 219. *If θ be a positive angle less than $\frac{\pi}{2}$, show that $\sqrt{\cos\theta}$ is less than* $\cos\frac{\theta}{\sqrt{2}}$.

Solution. $\cos\theta$ is less than $1 - \frac{\theta^2}{2} + \frac{\theta^4}{16}$, that is less than $\left(1-\frac{\theta^2}{4}\right)^2$, therefore $\sqrt{\cos\theta}$ is less than $1-\frac{\theta^2}{4}$; this holds if θ lies between 0 and $\frac{\pi}{2}$: see *Art.* 328 (page 446). Again, $\cos\frac{\theta}{\sqrt{2}}$ is greater than $1 - \frac{1}{2}\left(\frac{\theta}{\sqrt{2}}\right)^2$, that is greater than $1-\frac{\theta^2}{4}$; this holds as long as $\cos\frac{\theta}{\sqrt{2}}$ and $1-\frac{\theta^2}{4}$ remain both positive, and this certainly holds if θ lies between 0 and $\frac{\pi}{2}$. Hence $\cos\theta$ is less than $\cos\frac{\theta}{\sqrt{2}}$.

Problem 220. *If any point be taken within a regular polygon of an even number of sides from which perpendiculars are drawn to the sides taken in order, then the sum of one set of alternate perpendiculars is equal to the sum of the other set.*

Solution. Suppose the polygon has n sides. Let O be the centre of the circle inscribed in the polygon, and S the assumed point. Let $OS = c$; and suppose OS to be inclined at an angle α to the first perpendicular which is drawn; put β for $\frac{2\pi}{n}$, and r for the radius of the inscribed circle. Then the length of the first perpendicular will be $r + c\cos\alpha$, that of the second $r + c\cos(\alpha+\beta)$, that of the third $r + c\cos(\alpha + 2\beta)$, and so on. Hence the sum of one set of perpendiculars

$$= \frac{nr}{2} + c\left\{\cos\alpha + \cos(\alpha+2\beta) + \cos(\alpha+4\beta) + \dots \text{ to } \frac{n}{2} \text{ terms}\right\}.$$

By *Art.* 304 (page 437) the sum of the series of cosines contains the factor $\sin\frac{n}{2}\beta$, that is $\sin\pi$, that is zero.

Hence the sum of the set of perpendiculars $= \frac{nr}{2}$.

Similarly the sum of the other set of perpendiculars has the same value.

Problem 221. *Show that* $\sqrt{rr_1r_2r_3} = S$.

Solution.
$$r = \frac{S}{s}, \quad r_1 = \frac{S}{s-a}, \quad r_2 = \frac{S}{s-b}, \quad \frac{S}{s-c};$$

therefore
$$rr_1r_2r_3 = \frac{S^4}{s(s-a)(s-b)(s-c)} = \frac{S^4}{S^2} = S^2;$$

therefore
$$\sqrt{rr_1r_2r_3} = S.$$

Problem 222. *Show that* $\frac{\pi}{4} = 5\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{3}{79}$.

Solution. We have $2\tan^{-1}\frac{1}{7} = \tan^{-1}\dfrac{\frac{2}{7}}{1-\frac{1}{49}} = \tan^{-1}\frac{7}{24}$;

thus
$$4\tan^{-1}\frac{1}{7} = 2\tan^{-1}\frac{7}{24} = \frac{2\times\frac{7}{24}}{1-\left(\frac{7}{24}\right)^2} = \tan^{-1}\frac{2\times7\times24}{(24+7)(24-7)}$$
$$= \tan^{-1}\frac{336}{527}.$$

Then
$$5\tan^{-1}\frac{1}{7} = 4\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{336}{527} + \tan^{-1}\frac{1}{7}$$
$$= \tan^{-1}\frac{\frac{336}{527}+\frac{1}{7}}{1-\frac{336}{7\times527}} = \tan^{-1}\frac{2879}{3353}.$$

Again
$$2\tan^{-1}\frac{3}{79} = \tan^{-1}\frac{2\times\frac{3}{79}}{1-\left(\frac{3}{79}\right)^2} = \tan^{-1}\frac{2\times3\times79}{(79+3)(79-3)}$$
$$= \tan^{-1}\frac{237}{3116}.$$

Finally $$\tan\left(\frac{\pi}{4}-\tan^{-1}\frac{2879}{3353}\right)=\frac{1-\frac{2879}{3353}}{1+\frac{2879}{3353}}=\frac{474}{6232}=\frac{237}{3116};$$

so that $$\frac{\pi}{4}-\tan^{-1}\frac{2879}{3353}=\tan^{-1}\frac{237}{3116}.$$

Problem 223. *Assuming the expression for* $\tan n\theta$ *in terms of* $\tan\theta$, *show that if* n *be an odd integer the following two series are numerically equal,*

$$1-\frac{n(n-1)}{\lfloor 2}+\frac{n(n-1)(n-2)(n-3)}{\lfloor 4}-\ldots,$$

$$n-\frac{n(n-1)(n-2)}{\lfloor 3}+\frac{n(n-1)(n-2)(n-3)(n-4)}{\lfloor 5}-\ldots;$$

and if n *be an even integer one of the two series is zero.*

Solution. In the expression for $\tan n\theta$ put $\frac{\pi}{4}$ for θ; then $\theta = 1$.

If n is an odd number we have $\tan n\theta = (-1)^{\frac{n-1}{2}}$, so that the numerator of the expression is numerically equal to the denominator.

If n is an even number, $\tan\theta$ is either zero or infinite; so that in the former case the numerator of the expression must vanish must vanish, and in the latter case the denominator must vanish.

Problem 224. *Show that* $\sin^4\theta\cos^5\theta$

$$=\frac{1}{256}(\cos 9\theta+\cos 7\theta)-\frac{1}{64}(\cos 5\theta+\cos 3\theta)+\frac{3}{128}\cos\theta.$$

Solution. We have $\sin^4\theta\cos^5\theta=(1-\cos^2\theta)^2\cos^5\theta=\cos^9\theta-2\cos^7\theta+\cos^5\theta$.

Now $$\cos^9\theta=\frac{1}{2^8}\{\cos 9\theta+9\cos 7\theta+36\cos 5\theta+84\cos 3\theta+126\cos\theta\},$$

$$\cos^7\theta=\frac{1}{2^6}\{\cos 7\theta+7\cos 5\theta+21\cos 3\theta+35\cos\theta\},$$

$$\cos^5\theta=\frac{1}{2^4}\{\cos 5\theta+5\cos 3\theta+10\cos\theta\}.$$

Hence

$$\cos^9\theta-2\cos^7\theta+\cos^5\theta=\frac{1}{256}(\cos 9\theta+\cos 7\theta)-\frac{1}{64}(\cos 5\theta+\cos 3\theta)+\frac{3}{128}\cos\theta.$$

Or we may proceed thus :

$$\sin^4\theta\cos^5\theta=\sin^4\theta\cos^4\theta\sin\theta=\frac{1}{16}(\sin 2\theta)^4\cos\theta$$

$$=\frac{1}{16}\left\{\frac{1}{8}\cos 8\theta-\frac{1}{2}\cos 4\theta+\frac{3}{8}\right\}\cos\theta$$

$$=\frac{1}{256}(\cos 9\theta+\cos 7\theta)-\frac{1}{64}(\cos 5\theta+\cos 3\theta)+\frac{3}{128}\cos\theta.$$

Problem 225. *Show that* $\sin^3 x$

$$=\frac{3}{4}\left\{\frac{3^2-1}{\lfloor 3}x^3-\frac{3^4-1}{\lfloor 5}x^5+\ldots+(-1)^{n-1}\frac{3^{2n}-1}{\lfloor 2n+1}x^{2n+1}+\ldots\right\}.$$

Solution. We have

$$\begin{aligned}\sin^3 x &= \frac{1}{4}(3\sin x - \sin 3x)\\ &= \frac{3}{4}\left\{x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \ldots + \frac{(-1)^n x^{2n+1}}{\underline{|2n+1}} + \ldots\right\}\\ &\quad - \frac{1}{4}\left\{3x - \frac{(3x)^3}{\underline{|3}} + \frac{(3x)^5}{\underline{|5}} - \frac{(3x)^7}{\underline{|7}} + \ldots + \frac{(-1)^n (3x)^{2n+1}}{\underline{|2n+1}} + \ldots\right\};\end{aligned}$$

then by arranging according to powers of x we obtain the required result.

Problem 226. *If* $\phi = \dfrac{\pi}{13}$, *show that*

$$\cos\phi + \cos 3\phi + \cos 9\phi = \frac{1+\sqrt{13}}{4},$$

and

$$\cos 5\phi + \cos 7\phi + \cos 11\phi = \frac{1-\sqrt{13}}{4}.$$

Solution. Put s for $\cos\phi + \cos 3\phi + \cos 9\phi$, and t for $\cos 5\phi + \cos 7\phi + \cos 11\phi$.

Then $$s + t = \cos\phi + \cos 3\phi + \cos 5\phi + \cos 7\phi + \cos 9\phi + \cos 11\phi$$

$$= \frac{\cos(\phi + 5\phi)\sin 6\phi}{\sin\phi} (Art.\ 304)\ (\text{page } 437) = \frac{\sin 12\phi}{2\sin\phi} = \frac{\sin\phi}{2\sin\phi} = \frac{1}{2}.$$

And $$\begin{aligned}st &= (\cos\phi + \cos 3\phi + \cos 9\phi)(\cos 5\phi + \cos 7\phi + \cos 11\phi)\\ &= \cos\phi(\cos 5\phi + \cos 7\phi + \cos 11\phi) + \ldots\end{aligned}$$

Resolve each product into the sum of two cosines by *Art.* 84 (page 405); thus we get

$$\begin{aligned}2st &= \cos 6\phi + \cos 4\phi + \cos 8\phi + \cos 6\phi + \cos 12\phi + \cos 10\phi\\ &\quad + \cos 8\phi + \cos 2\phi + \cos 10\phi + \cos 4\phi + \cos 14\phi + \cos 8\phi\\ &\quad + \cos 14\phi + \cos 4\phi + \cos 16\phi + \cos 2\phi + \cos 20\phi + \cos 2\phi\\ &= 3\cos 2\phi + 3\cos 4\phi + 2\cos 6\phi + 3\cos 8\phi + 2\cos 10\phi\\ &\quad + \cos 12\phi + 2\cos 14\phi + \cos 16\phi + \cos 20\phi.\end{aligned}$$

Now since $\phi = \dfrac{\pi}{13}$ we have $\cos 14\phi = \cos 12\phi$, $\cos 20\phi = -\cos 7\phi = \cos 6\phi$, $\cos 16\phi = -\cos 3\phi = \cos 10\phi$. Thus

$$\begin{aligned}2st &= 3\{\cos 2\phi + \cos 4\phi + \cos 6\phi + \cos 8\phi + \cos 10\phi + \cos 12\phi\}\\ &= \frac{3\cos(2\phi + 5\phi)\sin 6\phi}{\sin\phi} = -\frac{3\cos 6\phi\sin 6\phi}{\sin\phi} = -\frac{3\sin 12\phi}{2\sin\phi} = -\frac{3}{2};\end{aligned}$$

therefore $$st = -\frac{3}{4}.$$

Then, since $s + t = \dfrac{1}{2}$, and $st = -\dfrac{3}{4}$, we find by Algebra that

$$s = \frac{1 \pm \sqrt{13}}{4} \text{ and } t = \frac{1 \mp \sqrt{13}}{4};$$

and it is obvious that the upper sign must be taken, because s is positive for $\cos\phi$ and $\cos 3\phi$, which are positive, are both numerically greater than $\cos 9\phi$, which is negative.

Problem 227. *A point is taken inside a regular polygon and perpendiculars are* *drawn from it to the sides of the polygon; a new polygon is formed by joining the successive feet of the perpendiculars : find the sum of the squares on the sides of the new polygon.*

Solution. Suppose the polygon has n sides. Let O be the centre of the circle inscribed in the polygon, and S the assumed point. Let $OS = c$; and suppose OS to be inclined at an angle α to the first perpendicular which is drawn; put β for $\frac{2\pi}{n}$, and r for the radius of the inscribed circle. Then the length of the first perpendicular will be $r + c\cos\alpha$, that of the second $r + c\cos(\alpha+\beta)$, that of the third $r + c\cos(\alpha + 2\beta)$, and so on.

Then for the squares on the sides of the new polygon we obtain the expressions

$$\{r + c\cos\alpha\}^2 + \{r + c\cos(\alpha+\beta)\}^2 - 2\{r + c\cos\alpha\}\{r + c\cos(\alpha+\beta)\}\cos\beta,$$

$$\{r + c\cos(\alpha+\beta)\}^2 + \{r + c\cos(\alpha+2\beta)\}^2 - 2\{r + c\cos(\alpha+\beta)\}\{r + c\cos(\alpha+2\beta)\}\cos\beta,$$

and so on.

Thus for the squares on the m^{th} side of the new polygon we shall obtain

$$2r^2(1-\cos\beta) + 2rc\{\cos(\alpha+m\beta-\beta) + \cos(\alpha+m\beta)\}(1-\cos\beta)$$
$$+c^2\{\cos^2(\alpha+m\beta-\beta) + \cos^2(\alpha+m\beta) - 2\cos\beta\cos(\alpha+m\beta-\beta)\cos(\alpha+m\beta)\};$$

that is

$$2r^2(1-\cos\beta) + 2rc\{\cos(\alpha+m\beta-\beta) + \cos(\alpha+m\beta)\}(1-\cos\beta)$$
$$+\frac{c^2}{2}\{1 + \cos(2\alpha+2m\beta-2\beta) + 1 + \cos(2\alpha+2m\beta)$$
$$-2\cos\beta[\cos\beta + \cos(2\alpha+2m\beta-\beta)]\}.$$

We have to obtain the sum formed from this expression by giving to m all integral values from 1 to n, both inclusive; the result, by *Art.* 305 (page 437),

$$= 2nr^2(1-\cos\beta) + \frac{c^2}{2}\{2n - 2n\cos^2\beta\}$$
$$= 4nr^2\sin^2\frac{\beta}{2} + nc^2\sin^2\beta.$$

Problem 228. *Show that if* $\beta = \frac{\pi}{20}$ *then* $\cos 5\theta + \sin 5\theta$

$$= -2^4\sin(\theta - 3\beta)\sin(\theta+\beta)\cos(\theta+3\beta)\cos(\theta-\beta)(\cos\theta + \sin\theta).$$

Solution. We have $\cos 5\theta + \sin 5\theta = \sqrt{2}\cos\left(5\theta - \frac{\pi}{4}\right)$

$$= \sqrt{2}\cos 5\left(\theta - \frac{\pi}{20}\right) = \sqrt{2}\cos 5(\theta-\beta).$$

And by *Art.* 318 (page 443) we have $\cos 5(\theta-\beta)\sin\frac{5\pi}{2}$

$$= 2^4\cos(\theta-\beta)\cos(\theta-\beta+2\alpha)\cos(\theta-\beta+4\alpha)\cos(\theta-\beta+6\alpha)\cos(\theta-\beta+8\alpha),$$

where $\alpha = \frac{\pi}{10} = 2\beta$.

Thus $\cos 5(\theta-\beta)$

$$= 2^4\cos(\theta-\beta)\cos(\theta+3\beta)\cos(\theta+7\beta)\cos(\theta+11\beta)\cos(\theta+15\beta).$$

Also
$$\cos(\theta+7\beta) = \sin(3\beta-\theta) = -\sin(\theta-3\beta),$$
$$\cos(\theta+11\beta) = \cos\left(\theta+\beta+\frac{\pi}{2}\right) = -\sin(\theta+\beta),$$

$$\cos(\theta + 15\beta) = \cos\left(\theta + \frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}(\cos\theta + \sin\theta).$$

Hence $\sqrt{2}\cos 5(\theta - \beta)$

$$= -2^4 \cos(\theta - \beta)\cos(\theta + 3\beta)\sin(\theta - 3\beta)\sin(\theta + \beta)(\sin\theta + \cos\theta);$$

hence also $\cos 5\theta + \sin 5\theta$ is equal to the last expression, which had to be shown.

Problem 229. *Given* $\sin\theta\left\{1 + \tan^2\alpha\tan^2\beta\right\}^{\frac{1}{2}} + \cos\theta\left\{1 - \tan^2\alpha\tan^2\beta\right\}^{\frac{1}{2}} = \tan\alpha + \tan\beta,$

show how to determine θ *by a formula suitable to logarithmic computation.*

Solution. We have

$$\sin\theta\sqrt{(\cos^2\alpha\cos^2\beta + \sin^2\alpha\sin^2\beta)} + \cos\theta\sqrt{(\cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta)} = \sin(\alpha + \beta).$$

Assume $\qquad r\cos\phi = \sqrt{(\cos^2\alpha\cos^2\beta + \sin^2\alpha\sin^2\beta)},$

and $\qquad r\sin\phi = \sqrt{(\cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta)};$

so that $\qquad r^2 = 2\cos^2\alpha\cos^2\beta \qquad (39)$

and $\qquad \tan^2\phi = \dfrac{\cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta}{\cos^2\alpha\cos^2\beta + \sin^2\alpha\sin^2\beta} \qquad (40)$

Thus $\qquad r\sin(\theta + \phi) = \sin(\alpha + \beta) \qquad (41)$

Now it is obvious that r may be found from (39) by logarithms. Also ϕ may be determined by logarithms; for we have from (40)

$$\frac{1 - \tan^2\phi}{1 + \tan^2\phi} = \frac{\sin^2\alpha\sin^2\beta}{\cos^2\alpha\cos^2\beta},$$

that is $\qquad \cos 2\phi = \tan^2\alpha\tan^2\beta,$

which is adapted to logarithms.

Thus θ can be found from (41) by logarithms.

Problem 230. *If* A, B, C *are angles of a triangle, show that* $\sin A + \sin B + \sin C$ *is never less than* $\sin 2A + \sin 2B + \sin 2C$.

Solution. If A, B and C are angles of a triangle, we have by *Art.* 114, (page 409) and Chapter VIII : *Problem* 16,

$$\begin{aligned} &\sin A + \sin B + \sin C - (\sin 2A + \sin 2B + \sin 2C) \\ &= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} - 4\sin A\sin B\sin C \\ &= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\left\{1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right\}; \end{aligned}$$

and by Chapter XIII : *Problem* 40 this expression can never be negative.

Problem 231. *A regular polygon of n sides is inscribed in a circle, and from any point in the circumference chords are drawn to the angular points; if these chords be denoted by* c_1, c_2, $\ldots c_n$, *beginning with the chord drawn to the nearest angular* *point, and taking the rest in order, show that the quantity*

$$c_1 c_2 + c_2 c_3 + \ldots + c_{n-1} c_n - c_n c_1$$

is independent of the position of the point from which the chords are drawn.

Solution. Let A, B, $C, \ldots M$, N denote the angular points of the polygon taken in order; and let $\alpha = \dfrac{\pi}{n}$. Suppose P the point in the circumstance from which chords are drawn, so that P is between N and A.

Then $\quad \dfrac{1}{2}c_1c_2 \sin\alpha =$ the area of the triangle PAB,

$\dfrac{1}{2}c_2c_3 \sin\alpha =$ the area of the triangle PBC,

....

$\dfrac{1}{2}c_{n-1}c_n \sin\alpha =$ the area of the triangle PMN.

Therefore $\quad \dfrac{1}{2}(c_1c_2 + c_2c_3 + \ldots + c_{n-1}c_n) \sin\alpha.$

$=$ the area of the triangles $PAB, PBC, \ldots PMN$.

Also $\quad \dfrac{1}{2}c_nc_1 \sin\alpha =$ the area of the triangle PNA.

Thus $\quad \dfrac{1}{2}(c_1c_2 + c_2c_3 + \ldots + c_{n-1}c_n - c_nc_1) \sin\alpha$

$=$ the area of the regular polygon;

so that $\quad c_1c_2 + c_2c_3 + \ldots + c_{n-1}c_n - c_nc_1$

$= \dfrac{2}{\sin\alpha} \times$ the area of the regular polygon.

This result is the same for all positions of P on the circumstance of the circle.

Problem 232. *Two circular sectors have a common chord and equal areas, and their angles are as* 2 *is to* 1 : *show that one of the sectors must be a semicircle and the other a quadrant.*

Solution. Let θ be the angle of one sector, and 2θ the angle of the other. Let a and b be the corresponding radii. Then, since the areas are equal, $a^2\dfrac{\theta}{2} = b^2\dfrac{2\theta}{2}$; and since there is a common chord, $2a\sin\dfrac{\theta}{2} = 2b\sin\dfrac{2\theta}{2}$.

Thus $\quad a\sin\dfrac{\theta}{2} = b\sin\theta = 2b\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$; therefore $\cos\dfrac{\theta}{2} = \dfrac{a}{2b}$;

therefore $\quad \cos^2\dfrac{\theta}{2} = \dfrac{a^2}{4b^2} = \dfrac{2b^2}{4b^2} = \dfrac{1}{2}$; therefore $\dfrac{\theta}{2} = \dfrac{\pi}{4}$.

Therefore $\quad \theta = \dfrac{\pi}{2}$ and $2\theta = \pi$.

Problem 233. *If* $\phi = \tan^{-1}\dfrac{x\sqrt{3}}{2k - x}$, *and* $\theta = \tan^{-1}\dfrac{2x - k}{k\sqrt{3}}$, *show that one value of* $\phi - \theta$ *is* $\dfrac{\pi}{6}$.

Solution. We have $\tan(\phi-\theta) = \dfrac{\tan\phi - \tan\theta}{1+\tan\phi\tan\theta} = \dfrac{\dfrac{x\sqrt{3}}{2k-x} - \dfrac{2x-k}{k\sqrt{3}}}{1+\dfrac{x(2x-k)\sqrt{3}}{(2k-x)k\sqrt{3}}}$

$$= \frac{1}{\sqrt{3}}\cdot\frac{3kx-(2k-x)(2x-k)}{(2k-x)k+x(2x-k)} = \frac{1}{\sqrt{3}}.\frac{2x^2-2kx+2k^2}{2x^2-2kx+2k^2} = \frac{1}{\sqrt{3}}.$$

Therefore one value of $\phi-\theta$ is $\dfrac{\pi}{6}$.

Problem 234. *If* $\dfrac{\sin\theta}{\theta} = \dfrac{863}{864}$, *show that* θ *contains very nearly* $5°$.

Solution. We have here $\dfrac{\sin\theta}{\theta}$ very nearly equal to unity; so we may infer that θ is small : hence $\sin\theta = \theta - \dfrac{\theta^3}{6}$ nearly. Therefore $1-\dfrac{\theta^2}{6} = \dfrac{863}{864}$ nearly; therefore $\dfrac{\theta^2}{6} = \dfrac{1}{864}$ nearly; therefore $\theta^2 = \dfrac{1}{144}$ nearly; therefore $\theta = \dfrac{1}{12}$ nearly. Hence the number of degrees in the angle is nearly $\dfrac{1}{12}\cdot\dfrac{180}{\pi}$, so that is about 5.

Problem 235. *ABC is a triangle, and DEF is another triangle formed by joining the centres of the escribed circles of ABC. Show that the circle described round ABC is the nine points circle of DEF.*

Solution. Let ABC be any triangle; let D,E,F be the centers of the escrowed circles opposite to A,B,C respectively.

Then AD bisected the angle of the triangle at A, and EF bisects the *exterior* angle at A. Therefore AD is perpendicular to EF.

Similarly EB is perpendicular to FD, and FC is perpendicular to DE.

Therefore by *Art.* 332 (page 448) the circle described round ABC is the nine points circle of DEF.

Problem 236. *From the expansion of* $\sin^{2n+1}\theta$ *in terms of the sines of the multiples of* θ, *show that zero is the sum of* $n+1$ *terms of the following series :*

$$1-(2n-1)+\frac{2n(2n-3)}{\underline{|2}} - \frac{2n(2n-1)(2n-5)}{\underline{|3}} + \ldots.$$

Solution. As in *Art.* 283 (page 434) we have

$$2^{2n}(-1)^n\sin^{2n+1}\theta$$
$$= \sin(2n+1)\theta - (2n+1)\sin(2n-1)\theta + \frac{(2n+1)2n}{\underline{|2}}\sin(2n-3)\theta - \ldots.$$

Now suppose each side were to be expanded in powers of θ; on the left-hand side we should have $2^{2n}(-1)^n\left\{\theta - \dfrac{\theta^3}{\underline{|3}} + \ldots\right\}^{2n+1}$, by *Art.* 274 (page 432).

On the right-hand side each sine gives rise to a series. Since the lowest power of θ on the left-hand side is θ^{2n+1} it follows that the whole coefficient of every lower power of θ on the right-hand side must be zero. The whole coefficient of θ is

$$2n+1-(2n+1)(2n-1)+\frac{(2n+1)2n}{\lfloor 2}(2n-3)-\dots \text{ to } n+1 \text{ terms};$$

hence this is zero; and dividing by $2n+1$ we obtain the required result.

Similarly, supposing n to be greater than unity, we can obtain another result by equating to zero the whole coefficient of θ^3 on the right-hand side. And so on.

Problem 237. *If* $\cos\left(\theta+\phi\sqrt{-1}\right)=\cos\alpha+\sqrt{-1}\sin\alpha$, *where the letters denote real quantities, show that* $\sin^2\theta=\pm\sin\alpha$.

Solution. We have

$\cos\alpha+\sqrt{-1}\sin\alpha=\cos\left(\theta+\phi\sqrt{-1}\right)=\cos\theta\cos\phi\sqrt{-1}-\sin\theta\sin\phi\sqrt{-1}.$

$$=\cos\theta\frac{e^{-\phi}+e^{\phi}}{2}-\sin\theta\frac{e^{-\phi}-e^{\phi}}{2\sqrt{-1}}=\cos\theta\frac{e^{-\phi}+e^{\phi}}{2}+\sin\theta\frac{e^{-\phi}-e^{\phi}}{2}\sqrt{-1}.$$

Hence, by equating the possible and the impossible parts, we have

$$\cos\theta\frac{e^{-\phi}+e^{\phi}}{2}=\cos\alpha,\ \sin\theta\frac{e^{-\phi}-e^{\phi}}{2}=\sin\alpha;$$

so that
$$\frac{e^{-\phi}+e^{\phi}}{2}=\frac{\cos\alpha}{\cos\theta},\ \frac{e^{-\phi}-e^{\phi}}{2}=\frac{\sin\alpha}{\sin\theta}.$$

Square and subtract; thus

$$1=\frac{\cos^2\alpha}{\cos^2\theta}-\frac{\sin^2\alpha}{\sin^2\theta};$$

therefore
$$\sin^2\theta\cos^2\alpha-\cos^2\theta\sin^2\alpha=\sin^2\theta\cos^2\theta;$$

therefore
$$\sin^2\theta(1-\cos^2\theta)=\sin^2\alpha;$$

therefore
$$\sin^4\theta=\sin^2\alpha;$$

therefore
$$\sin^2\theta=\pm\sin\alpha.$$

Problem 238. *Show that*

$$\frac{\sin x-\sin 3x+\sin 5x-\dots \text{ to } n \text{ terms}}{\cos x-\cos 3x+\cos 5x-\dots \text{ to } n \text{ terms}}=\tan\left(nx+\frac{n-1}{2}\pi\right).$$

Solution. On the left-hand side the numerator

$$=\sin x+\sin(3x+\pi)+\sin(5x+2\pi)+\dots \text{ to } n \text{ terms},$$

$$=\frac{\sin\left\{x+\frac{n-1}{2}(2x+\pi)\right\}\sin\frac{n}{2}(2x+\pi)}{\sin\frac{1}{2}(2x+\pi)};$$

in like manner the denominator

$$=\frac{\cos\left\{x+\frac{n-1}{2}(2x+\pi)\right\}\sin\frac{n}{2}(2x+\pi)}{\sin\frac{1}{2}(2x+\pi)}$$

Divide the former by the latter and we obtain

$$\tan\left\{x+\frac{n-1}{2}(2x+\pi)\right\},\ \text{that is}\ \tan\left(nx+\frac{n-1}{2}\pi\right).$$

Problem 239. *ABCP and DEFQ are two concentric circles, ABC and DEF being any two equilateral triangles inscribed in them. If P and Q are any two points on the circumferences of the circles, show that*

$$QA^2 + QB^2 + QC^2 = PD^2 + PE^2 + PF^2.$$

Solution. Let O denote the centre of the circles, r the radius of the circle $ABCP$, and R the radius of the circle $DEFQ$.

Suppose the angle QOA is equal to θ, then the angle QOB will be $\theta + \frac{2\pi}{3}$, and the angle QOC will be $\theta + \frac{4\pi}{3}$; or at least the angles may be so denoted by suitably choosing A, B, and C. Then

$$QA^2 = QO^2 + OA^2 - 2QO \cdot OA \cos\theta$$
$$= R^2 + r^2 - 2Rr\cos\theta,$$

similarly
$$QB^2 = R^2 + r^2 - 2Rr\cos\left(\theta + \frac{2\pi}{3}\right),$$

and
$$QC^2 = R^2 + r^2 - 2Rr\cos\left(\theta + \frac{4\pi}{3}\right).$$

Hence by addition, and *Art.* 305 (page 437), we have

$$QA^2 + QB^2 + QC^2 = 3(R^2 + r^2).$$

In the same way we find that

$$PD^2 + PE^2 + PF^2 = 3(R^2 + r^2).$$

Problem 240. *If* $\tan\theta = \dfrac{a\sin cx}{1 - a\cos cx}$ *and* $r^2 = 1 - 2a\cos cx + a^2$, *show that*

$$1 - \frac{n}{1}a\cos cx + \frac{n(n-1)}{\underline{|2}}a^2\cos 2cx - \ldots + (-1)^n a^n \cos ncx = r^n \cos n\theta.$$

Solution. Put for each cosine its exponential value; then the proposed series

$$= \frac{1}{2}(1 - ae^{\iota cx})^n + \frac{1}{2}(1 - ae^{-\iota cx})^n$$
$$= \frac{1}{2}(1 - a\cos cx - \iota a\sin cx)^n + \frac{1}{2}(1 - a\cos cx + \iota a\sin cx)^n.$$

Now assume $\quad 1 - a\cos cx = r\cos\theta$ and $a\sin cx = r\sin\theta$;

then the sum
$$= \frac{1}{2}(r\cos\theta - \iota r\sin\theta)^n + \frac{1}{2}(r\cos\theta + \iota r\sin\theta)^n$$
$$= \frac{r^n}{2}(\cos n\theta - \iota\sin n\theta) + \frac{r^n}{2}(\cos n\theta + \iota\sin n\theta)$$
$$= r^n\cos n\theta.$$

Problem 241. *Eliminate* θ *between*

$$\sin^2\theta - p\sin\theta + 1 = 0 \text{ and } \cos^2\theta - q\cos\theta + 1 = 0.$$

Solution. By addition $3 - p\sin\theta - q\cos\theta = 0$.

By subtraction
$$\cos^2\theta - \sin^2\theta = -p\sin\theta + q\cos\theta.$$

Therefore
$$3(\cos^2\theta - \sin^2\theta) = q^2\cos^2\theta - p^2\sin^2\theta;$$

therefore
$$3(2\cos^2\theta - 1) = q^2\cos^2\theta - p^2 + p^2\cos^2\theta;$$

therefore $$\cos^2\theta = \frac{p^2-3}{p^2+q^2-6};$$

therefore $$\sin^2\theta = \frac{q^2-3}{p^2+q^2-6}.$$

Substitute in the equation $3 = p\sin\theta + q\cos\theta$; thus

$$3\sqrt{(p^2+q^2-6)} = p\sqrt{(q^2-3)} + q\sqrt{(p^2-3)}.$$

This is the result of the elimination; the radicals are not necessarily positive. By squaring, transposing, and squaring again, we obtain finally

$$\left\{p^2q^2 - 6(p^2+q^2) + 27\right\}^2 = p^2q^2(p^2-3)(q^2-3).$$

Problem 242. *Show that the area of a triangle*

$$= \frac{(a+b+c)^2}{4\left(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right)}.$$

Solution. $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}$

$$= \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} + \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \frac{\sqrt{s}}{\sqrt{(s-a)(s-b)(s-c)}}\{s-a+s-b+s-c\} = \frac{s^2}{S}.$$

Hence the proposed expression $= s^2 \div \frac{s^2}{S} = S$.

Problem 243. *If* $\tan^{-1} ax + \frac{1}{2}\sec^{-1} bx = \frac{\pi}{4}$, *then one solution is*

$$x^2 = \frac{1}{2ab - a^2}.$$

Solution. Here $$2\tan^{-1} ax + \sec^{-1} bx = \frac{\pi}{2};$$

therefore $$\sin^{-1}\frac{2ax}{1+a^2x^2} = \frac{\pi}{2} - \cos^{-1}\frac{1}{bx} = \sin^{-1}\frac{1}{bx};$$

therefore $$\frac{2ax}{1+a^2x^2} = \frac{1}{bx};\ \text{therefore } 2abx^2 = 1 + a^2x^2;$$

therefore $$x^2 = \frac{1}{2ab - a^2}.$$

Problem 244. *If O is the centre of the circle described round a triangle, and P the point of intersection of the perpendiculars from the angles on the opposite sides, show that*

$$OP^2 = 2R^2\left(\frac{3}{2} + \cos 2A + \cos 2B + \cos 2C\right).$$

Solution. With the diagram of *Art.* 332 (page 448) we have $OA = R$, also the angle

$OAB = \dfrac{\pi}{2} - C$, and the angle $BAP = \dfrac{\pi}{2} - B$; so that the angle $OAP = C - B$.

Hence $$OP^2 = R^2 + AP^2 - 2R \cdot AP \cos(B - C).$$

Now, as in *Problem* 214, we have $AP = \dfrac{a \cos A}{\sin A} = 2R \cos A$;

so that
$$\begin{aligned} OP^2 &= R^2 + 4R^2 \cos^2 A - 4R^2 \cos A \cos(B - C) \\ &= R^2 + 2R^2(1 + \cos 2A) + 4R^2 \cos(B + C) \cos(B - C) \\ &= 3R^2 + 2R^2 \cos 2A + 2R^2(cos2B + \cos 2C) \\ &= 3R^2 + 2R^2(\cos 2A + \cos 2B + \cos 2C). \end{aligned}$$

Problem 245. *If α, β, γ are the lengths of the straight lines joining the centres of the escribed circles of a triangle with the centre of the inscribed circle, and x, y, z the lengths of the straight lines joining the centres of the escribed circles, show that*
$$\frac{\beta z + \gamma y}{x} = \frac{\gamma x + \alpha z}{y} = \frac{\alpha y + \beta x}{z}.$$

Solution. The values of x, y, z are given in *Problem* 216; and the values of α, β, γ in *Example* XVI. 31. Hence
$$\begin{aligned} \frac{\beta z + \gamma y}{x} &= \frac{4R \cos \frac{1}{2}C b \sec \frac{1}{2}B + 4R \cos \frac{1}{2}B c \sec \frac{1}{2}C}{4R \cos \frac{1}{2}A} \\ &= \frac{4R\left(\cos \frac{1}{2}C \sin \frac{1}{2}B + \cos \frac{1}{2}B \sin \frac{1}{2}C\right)}{\cos \frac{1}{2}A} = \frac{4R \sin \frac{1}{2}(B + C)}{\cos \frac{1}{2}A} \\ &= 4R. \end{aligned}$$

Similarly the other expressions are also equal to $4R$.

Problem 246. *If $\pi - \theta$ denote the angle opposite to the side b of a triangle, and θ be very small, show that approximately*
$$c = (b - a)\left\{1 + \frac{a}{b}\frac{\theta^2}{\underline{|2}} - \left(\frac{a}{b} - \frac{3a^2}{b^2} - \frac{3a^3}{b^3}\right)\frac{\theta^4}{\underline{|4}}\right\}.$$

Solution.

We have $c = b \cos A + a \cos B = b \cos A - a \cos(\pi - B)$
$$= b\sqrt{(1 - \sin^2 A)} - a \cos \theta = b\sqrt{\left(1 - \frac{a^2}{b^2}\sin^2 \theta\right)} - a \cos \theta.$$

We wish to expand this in powers of θ, as far as terms involving θ^4.

Now $$\sqrt{\left(1 - \frac{a^2}{b^2}\sin^2 \theta\right)} = 1 - \frac{a^2}{2b^2}\sin^2 \theta - \frac{a^4}{8b^4}\sin^4 \theta - \dots$$

Put for $\sin \theta$ its value $\theta - \dfrac{\theta^3}{6} + \dots$; thus we obtain
$$1 - \frac{a^2}{2b^2}\left(\theta - \frac{\theta^3}{6} + \dots\right)^2 - \frac{a^4}{8b^4}\left(\theta - \frac{\theta^3}{6} + \dots\right)^4,$$

that is $$1 - \frac{a^2}{2b^2}\left(\theta^2 - \frac{\theta^4}{3}\right) - \frac{a^4}{8b^4}\theta^4 + \dots$$

And $$\cos\theta = 1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \dots$$

Hence approximately

$$c = b - \frac{a^2}{2b}\left(\theta^2 - \frac{\theta^4}{3}\right) - \frac{a^4}{8b^3}\theta^4 - \alpha\left(1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4}\right)$$
$$= b - a + \left(a - \frac{a^2}{b}\right)\frac{\theta^2}{2} + \left(\frac{a^2}{6b} - \frac{a^4}{8b^3} - \frac{a}{24}\right)\theta^4$$
$$= b - a + \frac{(b-a)a\theta^2}{2b} + \frac{\theta^4}{\lfloor 4}\left(\frac{4a^2}{b} - \frac{3a^4}{b^3} - a\right);$$

and $$\frac{4a^2}{b} - \frac{3a^4}{b^3} - a = \frac{a^2}{b} - a + 3\left(\frac{a^2}{b} - \frac{a^4}{b^3}\right)$$
$$= \frac{a(a-b)}{b} + \frac{3a^2(b^2-a^2)}{b^3} = (a-b)\left\{\frac{a}{b} - \frac{3a^2}{b^3}(a+b)\right\}.$$

Thus we obtain the required result.

Problem 247. *Express $\sin^5\theta\cos^6\theta$ in terms of sines of multiples of θ.*

Solution. $\sin^5\theta\cos^6\theta = \cos\theta(\sin\theta\cos\theta)^5 = \frac{\cos\theta}{2^5}(\sin 2\theta)^5$

$$= \frac{\cos\theta}{2^5} \times \frac{1}{2^4}\{\sin 10\theta - 5\sin 6\theta + 10\sin 2\theta\}$$
$$= \frac{1}{2^{10}}\{\sin 11\theta + \sin 9\theta - 5(\sin 7\theta + \sin 5\theta) + 10(\sin 3\theta + \sin\theta)\}.$$

Problem 248. *If $\tan\left(\theta + \phi\sqrt{-1}\right) = \cos\alpha + \sqrt{-1}\sin\alpha$, where the letters denote real quantities, show that $\theta = n\pi \pm \frac{\pi}{4}$ where n is an integer.*

Solution. We have $\cos\alpha + \sqrt{-1}\sin\alpha = \dfrac{\sin\left(\theta + \sqrt{-1}\phi\right)}{\cos\left(\theta + \sqrt{-1}\phi\right)}$

$$= \frac{\sin\theta\cos\sqrt{-1}\phi + \cos\theta\sin\sqrt{-1}\phi}{\cos\theta\cos\sqrt{-1}\phi - \sin\theta\sin\sqrt{-1}\phi} = \frac{\sin\theta(e^{\phi} + e^{-\phi}) - \sqrt{-1}\cos\theta(e^{-\phi} - e^{\phi})}{\cos\theta(e^{\phi} + e^{-\phi}) + \sqrt{-1}\sin\theta(e^{-\phi} - e^{\phi})}$$
$$= \frac{\sin\theta + \sqrt{-1}k\cos\theta}{\cos\theta - \sqrt{-1}k\sin\theta}, \text{ where } k = \frac{e^{\phi} - e^{-\phi}}{e^{\phi} + e^{-\phi}}.$$

Hence $$\sin\theta + \sqrt{-1}k\cos\theta = (\cos\alpha + \sqrt{-1}\sin\alpha)(\cos\theta - \sqrt{-1}k\sin\theta)$$
$$= \cos\alpha\cos\theta + k\sin\alpha\sin\theta + \sqrt{-1}(\sin\alpha\cos\theta - k\sin\theta\cos\alpha);$$

therefore $$\sin\theta = \cos\alpha\cos\theta + k\sin\alpha\sin\theta,$$

and $$k\cos\theta = \sin\alpha\cos\theta - k\sin\theta\cos\alpha$$

therefore $$\frac{\sin\theta - \cos\alpha\cos\theta}{\sin\alpha\sin\theta} = \frac{\sin\alpha\cos\theta}{\cos\theta + \sin\theta\cos\alpha}.$$

Multiply up; thus we get $\cos\alpha(\sin^2\theta - \cos^2\theta) = 0$;

therefore $$\tan^2\theta = 1, \text{ and therefore } \theta = n\pi \pm \frac{\pi}{4}.$$

Problem 249. *Show that*

$$\frac{1}{\pi} = \frac{1}{4}\tan\frac{\pi}{4} + \frac{1}{8}\tan\frac{\pi}{8} + \frac{1}{16}\tan\frac{\pi}{16} + \ldots$$

Solution. By *Art.* 309 (page 438) we have

$$\frac{1}{x} - 2\cot 2x = \tan x + \frac{1}{2}\tan\frac{x}{2} + \frac{1}{2^2}\tan\frac{x}{2^2} + \ldots,$$

and, since $2\cot 2x + \tan x = \cot x$, we have

$$\frac{1}{x} - \cot x = \frac{1}{2}\tan\frac{x}{2} + \frac{1}{2^2}\tan\frac{x}{2^2} + \ldots;$$

then put $\frac{\pi}{2}$ for x, and divide by 2; thus $\frac{1}{\pi} = \frac{1}{4}\tan\frac{\pi}{4} + \frac{1}{8}\tan\frac{\pi}{8} + \ldots.$

Problem 250. *Show that the coefficient of x^n in the product*

$$(1+x)\left(1+\frac{x}{2^2}\right)\left(1+\frac{x}{3^2}\right)\ldots \textit{ ad infinitum is } \frac{\pi^{2n}}{\underline{|2n+1}}.$$

Solution. Put $-\frac{\theta^2}{\pi^2}$ for x; then we require the coefficient of $\left(-\frac{\theta^2}{\pi^2}\right)^n$, that is of $\frac{(-1)^n\theta^{2n}}{\pi^{2n}}$ in the development of

$$\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots.$$

Thus we require the coefficient of $\frac{(-1)^n\theta^{2n+1}}{\pi^{2n}}$ in the development of

$$\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots.$$

that is in $\sin\theta$. See *Art.* 320 (page 445).

But the general term in the expansion of $\sin\theta$ is $\frac{(-1)^n\theta^{2n+1}}{\underline{|2n+1}}$.

Hence $\frac{1}{\pi^{2n}}\times$ the required coefficient $= \frac{1}{\underline{|2n+1}}$; so that the required coefficient is $\frac{\pi^{2n}}{\underline{|2n+1}}$.

Problem 251. *Show how to eliminate θ between*

$$\sin^2\theta - p\sin\theta + m = 0, \text{ and } \cos^2\theta - q\cos\theta + n = 0.$$

Solution. Proceed as in *Problem* 241. We have

$$1+m+n = p\sin\theta + q\cos\theta,\ \cos^2\theta - \sin^2\theta + n - m = q\cos\theta - p\sin\theta;$$

$$\therefore (1+m+n)(2\cos^2\theta - 1 + n - m) = q^2\cos^2\theta - p^2\sin^2\theta;$$

$$\therefore \cos^2\theta = \frac{p^2+n^2-(1+m)^2}{p^2+q^2-2(1+m+n)},$$

$$\text{and } \sin^2\theta = \frac{q^2+m^2-(1+n)^2}{p^2+q^2-2(1+m+n)}.$$

Substitute in $1+m+n = p\sin\theta + q\cos\theta$, and the elimination will be effected.

Problem 252. *The internal bisectors of the angles of a triangle are produced to meet the circumference of the circumscribing circle : show that the area of the triangle formed by joining the three points thus obtained* $= \frac{Rs}{2}$.

Solution. Let D, E, F be the points at which the bisectors of the angles A, B, C respectively meet the circumference. Then the angle $DAC = \frac{1}{2}A$, and the angle $CAE =$ the angle $CBE = \frac{1}{2}B$; therefore $DAE = \frac{1}{2}(A+B)$; and therefore DE subtends at the centre of the circle an angle equal to $A+B$: thus $DE = 2R\sin\frac{1}{2}(A+B) = 2R\cos\frac{1}{2}C$. Similarly $EF = 2R\cos\frac{1}{2}A$; and the angle $DEF = \frac{1}{2}(A+C)$; thus the area of the triangle DEF

$$= \frac{1}{2}\cdot 4R^2\cos\frac{1}{2}A\cos\frac{1}{2}C\sin\frac{1}{2}(A+C) = 2R^2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$$

$$= \frac{2R^2sS}{abc} = \frac{Rs}{2}.$$

Problem 253. *If* $\sin^{-1}\frac{x}{a} + \sin^{-1}\frac{y}{b} = \sin^{-1}\frac{c^2}{ab}$, *then*

$$b^2x^2 + 2xy\left(a^2b^2 - c^4\right)^{\frac{1}{2}} + a^2y^2 = c^4.$$

Solution. Here $$\sin^{-1}\frac{x}{a} + \sin^{-1}\frac{y}{b} = \sin^{-1}\frac{c^2}{ab}.$$

Take the cosines of both sides; thus

$$\sqrt{\left(1-\frac{x^2}{a^2}\right)}\sqrt{\left(1-\frac{y^2}{b^2}\right)} - \frac{xy}{ab} = \sqrt{\left(1-\frac{c^4}{a^2b^2}\right)};$$

$$\therefore \sqrt{\left(1-\frac{x^2}{a^2}\right)}\sqrt{\left(1-\frac{y^2}{b^2}\right)} = \frac{xy}{ab} + \sqrt{\left(1-\frac{c^4}{a^2b^2}\right)};$$

square both sides; thus

$$1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{x^2y^2}{a^2b^2} = \frac{x^2y^2}{a^2b^2} + 2\frac{xy}{ab}\sqrt{\left(1-\frac{c^4}{a^2b^2}\right)} + 1 - \frac{c^4}{a^2b^2};$$

$$\therefore b^2x^2 + a^2y^2 + 2xy\sqrt{(a^2b^2 - c^4)} = c^4.$$

Problem 254. *From a point P each of two straight lines CA and CB, which are at right angles, subtends an angle γ. If $CA = a$, and $CB = b$, show that*

$$CP = \frac{ab\cos 2\gamma}{\sin\gamma\sqrt{a^2+b^2-2ab\sin 2\gamma}}.$$

Solution. Let $PCA = \theta$; then $PCB = \frac{\pi}{2} - \theta$;

$$\frac{PC}{a} = \frac{\sin(\theta+\gamma)}{\sin\gamma}, \quad \frac{PC}{b} = \frac{\sin\left(\frac{\pi}{2}-\theta+\gamma\right)}{\sin\gamma};$$

thus $$a\sin(\theta+\gamma) = b\sin\left(\frac{\pi}{2}-\theta+\gamma\right) = b\cos(\theta-\gamma);$$

$$\therefore a(\sin\theta\cos\gamma + \cos\theta\sin\gamma) = b(\cos\theta\cos\gamma + \sin\theta\sin\gamma);$$

$$\therefore \tan\theta = \frac{b\cos\gamma - a\sin\gamma}{a\cos\gamma - b\sin\gamma}.$$

Hence
$$\sin\theta = \frac{b\cos\gamma - a\sin\gamma}{\sqrt{\{(a\cos\gamma - b\sin\gamma)^2 + (b\cos\gamma - a\sin\gamma)^2\}}}$$
$$= \frac{b\cos\gamma - a\sin\gamma}{\sqrt{(a^2+b^2-2ab\sin 2\gamma)}};$$

and
$$\cos\theta = \frac{a\cos\gamma - b\sin\gamma}{\sqrt{(a^2+b^2-2ab\sin 2\gamma)}}.$$

Then
$$PC = \frac{a(\sin\theta\cos\gamma + \cos\theta\sin\gamma)}{\sin\gamma} = \frac{ab\cos 2\gamma}{\sin\gamma\sqrt{(a^2+b^2-2ab\sin 2\gamma)}}.$$

Problem 255. *Show that the roots of the equation $x^4 - x^3 + x^2 - x + 1 = 0$ are $\cos 36^\circ \pm \sqrt{-1}\sin 36^\circ$ and $\cos 108^\circ \pm \sqrt{-1}\sin 108^\circ$.*

Solution. $x^4 - x^3 + x^2 - x + 1 = \dfrac{x^5+1}{x+1}$. Hence we must find the roots of $x^5 + 1 = 0$, and omit the root -1.

Now if $x^5 = -1$ we may put $x^5 = \cos n\pi \pm \sqrt{-1}\sin n\pi$, where n is any odd integer. Hence $x = (\cos n\pi \pm \sqrt{-1}\sin n\pi)^{\frac{1}{5}} = \cos\dfrac{n\pi}{5} \pm \sqrt{-1}\sin\dfrac{n\pi}{5}$.

Put in succession 1 and 3 for n; thus we obtain the assigned values. If we put 5 for n we obtain the root -1, which we had to omit.

Problem 256. *If an angle of a triangle be 30°, one of the adjacent sides 1 foot, and the opposite side 250 feet, find approximately the number of minutes in the other acute angle.*

Solution. Let θ denote the angle opposite to the side 1; then

$$\frac{\sin\theta}{\sin\frac{\pi}{6}} = \frac{1}{250};\ \text{therefore}\ \sin\theta = \frac{1}{500}.$$

As θ is very small we may put θ for $\sin\theta$; thus $\theta = \dfrac{1}{500}$ approximately. Therefore the number of degrees in the angle $= \dfrac{1}{500}\times\dfrac{180}{\pi}$; and therefore the number of minutes $= \dfrac{60}{500}\times\dfrac{180}{\pi} = \dfrac{3}{25}\times\dfrac{180}{\pi} = \dfrac{3}{25}\times 57.3 = 7$ nearly.

Problem 257. *Show that the area of the triangle formed by joining the points of contact of the inscribed circle, or an escribed circle, of a triangle is $\dfrac{\rho S}{2R}$, where ρ is the radius of the circle.*

Solution. First take the inscribed circle : see *Art.* 248 (page 426).

$$FE = 2r\sin FOA = 2r\cos\frac{A}{2};\ \text{similarly}\ FD = 2r\cos\frac{B}{2}.$$

The angle $EFA = \frac{1}{2}(\pi - A)$; the angle $DFB = \frac{1}{2}(\pi - B)$; therefore the angle $EFD = \frac{1}{2}(A + B)$.

Hence the area of the triangle DFE

$$= \frac{1}{2} \cdot 4r^2 \cos\frac{A}{2} \cos\frac{B}{2} \sin\frac{A+B}{2} = 2r^2 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2} = \frac{2r^2 Ss}{abc} = \frac{rS}{2R}.$$

Now take one of the escribed circles, as for instance that opposite to the angle A : see *Art.* 250 (page 427).

$$DF = 2r_1 \sin DOB = 2r_1 \sin\frac{B}{2}; \text{ similarly } DE = 2r_1 \sin\frac{C}{2}.$$

The angle FDE = the angle FDO + the angle EDO

$$= \frac{1}{2}(\pi - B) + \frac{1}{2}(\pi - C) = \pi - \frac{1}{2}(B + C).$$

Hence the area of the triangle DFE

$$= \frac{1}{2} \cdot 4r_1^2 \sin\frac{B}{2} \sin\frac{C}{2} \sin\frac{B+C}{2}$$

$$= 2r_1^2 \sin\frac{B}{2} \sin\frac{C}{2} \cos\frac{A}{2} = \frac{2r_1^2(s-a)S}{abc} = \frac{2r_1 S^2}{abc} = \frac{r_1 S}{2R}.$$

Problem 258. *If* $\tan\left(\theta + \phi\sqrt{-1}\right) = \cos\alpha + \sqrt{-1}\sin\alpha$, *where the letters denote real quantities, show that* $e^{2\phi} = \pm\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$.

Solution. Proceed as in *Problem* 248; thus we obtain $\sin^2\theta = \cos^2\theta$.
If we take $\sin\theta = \cos\theta$ we get $1 = \cos\alpha + k\sin\alpha$; thus

$$k = \frac{1 - \cos\alpha}{\sin\alpha} = \tan\frac{\alpha}{2}, \text{ that is } \frac{e^{\phi} - e^{-\phi}}{e^{\phi} + e^{-\phi}} = \tan\frac{\alpha}{2};$$

$$\therefore e^{2\phi} = \frac{1 + \tan\frac{\alpha}{2}}{1 - \tan\frac{\alpha}{2}} = \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right).$$

If we take $\sin\theta = -\cos\theta$ we get $1 = -\cos\alpha + k\sin\alpha$; thus

$$k = \frac{1 + \cos\alpha}{\sin\alpha} = \cot\frac{\alpha}{2}, \text{ that is } \frac{e^{\phi} - e^{-\phi}}{e^{\phi} + e^{-\phi}} = \cot\frac{\alpha}{2};$$

$$\therefore e^{2\phi} = \frac{1 + \cot\frac{\alpha}{2}}{1 - \cot\frac{\alpha}{2}} = \frac{\tan\frac{\alpha}{2} + 1}{\tan\frac{\alpha}{2} - 1} = -\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right).$$

Problem 259. If $s = 1 + z\cos\theta + \frac{z^2}{\underline{|2}}\cos 2\theta + \frac{z^3}{\underline{|3}}\cos 3\theta + \dots,$

and $$\sigma = z\sin\theta + \frac{z^2}{\underline{|2}}\sin 2\theta + \frac{z^3}{\underline{|3}}\sin 3\theta + \dots,$$

then $$z\sin\theta = \tan^{-1}\frac{\sigma}{s}, \text{ and } z\cos\theta = \frac{1}{2}\log\left(s^2 + \sigma^2\right).$$

Find the values of s *and* σ *when* $\theta = \frac{\pi}{2}$.

Solution. Put the exponential values for the cosines in the series demoted by s : thus

$$s = 1 + \frac{1}{2}z(e^{\iota\theta} + e^{-\iota\theta}) + \frac{z^2}{2\underline{|2}}\left(e^{2\iota\theta} + e^{-2\iota\theta}\right) + \frac{z^3}{2\underline{|3}}\left(e^{3\iota\theta} + e^{-3\iota\theta}\right) + \dots\dots$$

$$= \frac{1}{2}(e^x + e^y),$$

where $\qquad x = ze^{\iota\theta} = (\cos\theta + \iota\sin\theta),$

and $\qquad y = ze^{-\iota\theta} = z(\cos\theta - \iota\sin\theta).$

Thus $\qquad s = \frac{1}{2}e^{z\cos\theta}(e^{\iota z\sin\theta} + e^{-\iota z\sin\theta}) = e^{z\cos\theta}\cos(z\sin\theta).$

Similarly we find that $\sigma = \frac{1}{2\iota}(e^x - e^y) = e^{z\cos\theta}\sin(z\sin\theta).$

Therefore $\qquad \frac{\sigma}{s} = \frac{\sin(z\sin\theta)}{\cos(z\sin\theta)} = \tan(z\sin\theta);$

so that $\qquad z\sin\theta = \tan^{-1}\frac{\sigma}{s}.$

And $\qquad s^2 + \sigma^2 = e^{2z\cos\theta}\left\{\cos^2(z\sin\theta) + \sin^2(z\sin\theta)\right\} = e^{2z\cos\theta};$

so that $\qquad z\cos\theta = \frac{1}{2}\log\left(s^2+\sigma^2\right).$

If $\theta = \frac{\pi}{2}$, we have $\frac{\sigma}{s} = \tan z$ and $s^2 + \sigma^2 = 1$, so that $\sigma = \sin z$ and $s = \cos z$.

Problem 260. *Through a given point straight lines are drawn parallel to the sides of a regular polygon; and from another given point perpendiculars are drawn to the straight lines. Find the sum of the squares on the perpendiculars.*

Solution. Let c be the distance of the two given points, n the number of sides in the polygon; and put $\beta = \frac{2\pi}{n}$. Let α be the angle which the distance between the two given points makes with the first straight line which is drawn. Then the numerical values of the successive perpendiculars are

$$c\sin\alpha,\ c\sin(\alpha+\beta),\ c\sin(\alpha+2\beta),\ \ldots$$

Hence the sum of the squares on the perpendiculars

$$= c^2\left\{\sin^2\alpha + \sin^2(\alpha+\beta) + \sin^2(\alpha+2\beta) + \ldots \text{ to } n \text{ terms}\right\}$$

$$= \frac{c^2}{2}\{1 - \cos 2a + 1 - \cos 2(\alpha+\beta) + 1 - \cos 2(\alpha+2\beta) + \ldots\}$$

$$= \frac{nc^2}{2}\cdot$$ See *Art.* 305 (page 437).

Problem 261. *Show how to eliminate θ between*

$$a\tan\theta + b\sec\theta = h, \text{ and } a\cot\theta + b\cos\theta = k.$$

Solution. We have

$$a\sin\theta + b = h\cos\theta, \text{ and } \cos\theta(a + b\sin\theta) = k\sin\theta.$$

Find $\cos\theta$ from the first equation, and substitute it in the second; thus we get

$$\sin^2\theta + \frac{a^2+b^2-hk}{ab}\sin\theta + 1 = 0.$$

Again, find $\sin\theta$ from the first equation, and substitute it in the second; thus we get

$$\cos^2\theta + \frac{a^2-b^2-hk}{bh}\cos\theta + \frac{k}{h} = 0.$$

Then we may employ the process on *Problem* 251.

Problem 262. *Perpendiculars are drawn from the angles of an acute-angled triangle on the opposite sides, and the feet of the perpendiculars joined. Show that the perimeter of the triangle thus formed* $= \dfrac{2S}{R}$.

Solution. By *Example* XVI. 50 we know that the sides of the new triangle are respectively $a\cos A$, $b\cos B$, and $c\cos C$. Thus the perimeter

$$= a\cos A + b\cos B + c\cos C = 4R\sin A\sin B\sin C, \text{ by Example XVI. 22},$$

$$= \frac{4R\cdot 8S^3}{(abc)^2} = \frac{2S}{R}.$$

Problem 263. *The internal bisectors of the angles of a triangle are produced to meet the circumscribing circle. Show that the area of the triangle formed by joining the points of intersection is never less than the area of the original triangle.*

Solution. As in *Problem* 252 we show that the area of the triangle thus formed is $2R^2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$; denote this by Σ.

Also
$$S = \frac{1}{2}ab\sin C = 2R^2\sin A\sin B\sin C.$$

Hence
$$\frac{\Sigma}{S} = \frac{\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}{\sin A\sin B\sin C} = \frac{1}{8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}.$$

Now, as in *Example* XIII. 40 we see that $8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ cannot be greater than unity; and therefore S cannot be greater than Σ.

Problem 264. *Show that* $4R = \dfrac{(r_1+r_2)(r_2+r_3)(r_3+r_1)}{r_1r_2+r_2r_3+r_3r_1}$.

Solution. $r_1 = \dfrac{S}{s-a}$, $r_2 = \dfrac{S}{s-b}$, $r_3 = \dfrac{S}{s-c}$.

Hence

$$(r_1+r_2)(r_2+r_3)(r_3+r_1) = S^3\left(\frac{1}{s-a}+\frac{1}{s-b}\right)\left(\frac{1}{s-b}+\frac{1}{s-c}\right)\left(\frac{1}{s-c}+\frac{1}{s-a}\right)$$

$$= \frac{S^3abc}{(s-a)^2(s-b)^2(s-c)^2} = \frac{s^2S^3abc}{S^4} = \frac{s^2abc}{S}.$$

And

$$r_1r_2+r_2r_3+r_3r_1 = S^2\left\{\frac{1}{(s-a)(s-b)}+\frac{1}{(s-b)(s-c)}+\frac{1}{(s-c)(s-a)}\right\}$$

$$= \frac{S^2(3s-a-b-c)}{(s-a)(s-b)(s-c)} = \frac{S^2s^2}{S^2} = s^2.$$

Divide the first result by the second; and thus we get $\dfrac{abc}{S}$.

Problem 265. *The shadows of two vertical walls which are inclined to each other at an angle γ, and are a and a' feet in height, are observed when the Sun is due South to be b and b' feet in breadth. Show that if α be the Sun's altitude above the horizon, and β the inclination of the first wall to the meridian,*

$$\cot^2\alpha = \left(\frac{b^2}{a^2} + \frac{b'^2}{a'^2}\right)\operatorname{cosec}^2\gamma + \frac{2bb'}{aa'}\cot\gamma\operatorname{cosec}\gamma,$$

$$\cot\beta = \frac{ab'}{a'b}\operatorname{cosec}\gamma + \cot\gamma.$$

Solution. The wall a feet high casts a shadow which extends $a\cot\alpha$ feet from the wall *measured in the direction of the meridian*; hence $a\cot\alpha\sin\beta$ is the breadth of the shadow *measured in the direction at right angles to the wall.*

Thus $b = a\cot\alpha\sin\beta$. Similarly $b' = a'\cot\alpha\sin(\gamma-\beta)$.

From these two equations we have to find α and β.

We get $\dfrac{a\sin\beta}{b} = \dfrac{a'\sin(\gamma-\beta)}{b'}$; so that

$$\frac{a}{b} = \frac{a'}{b'}(\sin\gamma\cot\beta - \cos\gamma);$$

$$\therefore \cot\beta = \cot\gamma + \frac{ab'}{a'b}\operatorname{cosec}\gamma.$$

Then
$$\cot^2\alpha = \frac{b^2}{a^2\sin^2\beta} = \frac{b^2}{a^2}(1+\cot^2\beta) = \frac{b^2}{a^2}\left\{1+\left(\cot\gamma + \frac{ab'}{a'b}\operatorname{cosec}\gamma\right)^2\right\}$$
$$= \frac{b^2}{a^2}\left\{1+\cot^2\gamma + \frac{a^2b'^2}{a'^2b^2}\mathit{cosec}^2\gamma + \frac{2ab'}{a'b}\cot\gamma\operatorname{cosec}\gamma\right\}$$
$$= \left(\frac{b^2}{a^2} + \frac{b'^2}{a'^2}\right)\operatorname{cosec}^2\gamma + \frac{2bb'}{aa'}\cot\gamma\operatorname{cosec}\gamma.$$

Problem 266. *Show that $\left(a+b\sqrt{-1}\right)^{\alpha+\beta\sqrt{-1}}$ will be wholly real if*

$$\frac{\beta}{2}\log\left(a^2+b^2\right) + \alpha\tan^{-1}\frac{b}{a}$$

is zero or an even multiple of $\dfrac{\pi}{2}$.

Solution. Assume $a = r\cos\theta$, and $b = r\sin\theta$; so that $r^2 = a^2+b^2$, and $\tan\theta = \dfrac{b}{a}$.

Also assume $\alpha = \rho\cos\phi$, $\beta = \rho\sin\phi$; so that $\rho^2 = \alpha^2+\beta^2$, and $\tan\phi = \dfrac{\beta}{\alpha}$.

Then the proposed expression

$$= (r\cos\theta + \iota r\sin\theta)^{\rho\cos\phi+\iota\rho\sin\phi} = \left(re^{\iota\theta}\right)^{\rho\cos\phi+\iota\rho\sin\phi}.$$

Denote this by u; then

$$\log u = (\rho\cos\phi + \iota\rho\sin\phi)\log\left(re^{\iota\theta}\right)$$
$$= (\rho\cos\phi + \iota\rho\sin\phi)(\iota\theta + \log r)$$
$$= \rho(\cos\phi\log r - \theta\sin\phi) + \rho\iota(\sin\phi\log r + \theta\cos\phi)$$
$$= \sigma + \iota\tau \text{ say;}$$

therefore
$$u = e^{\sigma+\iota\tau} = e^{\sigma}e^{\iota\tau} = e^{\sigma}(\cos\tau + \iota\sin\tau).$$

To make this wholly real the term involving ι must vanish, therefore $\sin\tau$ must vanish; therefore τ must be zero or a multiple of π; therefore $\rho(\sin\phi\log r + \theta\cos\phi)$

must be zero or an even multiple of $\frac{\pi}{2}$; but $\rho\sin\phi = \beta$ and $\rho\cos\phi = \alpha$; so that $\frac{\beta}{2}\log\left(a^2+b^2\right) + \alpha\tan^{-1}\frac{b}{a}$ must be zero or an even multiple of $\frac{\pi}{2}$.

Problem 267. *Apply the exponential values of the sine and cosine to show that*

$$\log\frac{a^2}{a^2\cos^2\theta + b^2\sin^2\theta} = 4\left\{c\sin^2\theta - \frac{1}{2}c^2\sin^2 2\theta + \frac{1}{3}c^3\sin^2 3\theta - \dots\right\}$$

when $c = \frac{a-b}{a+b}$.

Solution.

$$\frac{a^2}{a^2\cos^2\theta + b^2\sin^2\theta} = \frac{2a^2}{a^2(1+\cos 2\theta) + b^2(1-\cos 2\theta)}$$
$$= \frac{2a^2}{a^2+b^2+(a^2-b^2)\cos 2\theta}$$
$$= \frac{4a^2}{2(a^2+b^2)+(a^2-b^2)(e^{2\theta\iota}+e^{-2\theta\iota})} = \frac{4a^2}{(a+b)^2(1+ce^{2\theta\iota})(1+ce^{-2\theta\iota})}$$
$$= \frac{(1+c)^2}{(1+ce^{2\theta\iota})(1+ce^{-2\theta\iota})}.$$

Therefore

$$\log\frac{a^2}{a^2\cos^2\theta + b^2\sin^2\theta} = 2\log(1+c) - \log\left(1+ce^{2\theta\iota}\right) - \log\left(1+ce^{-2\theta\iota}\right)$$
$$= 2\left\{c - \frac{c^2}{2} + \frac{c^3}{3} - \frac{c^4}{4} + \dots\right\}$$
$$- \left\{ce^{2\theta\iota} - \frac{c^2}{2}e^{4\theta\iota} + \frac{c^3}{3}e^{6\theta\iota} - \frac{c^4}{4}e^{8\theta\iota} + \dots\right\}$$
$$- \left\{ce^{-2\theta\iota} - \frac{c^2}{2}e^{-4\theta\iota} + \frac{c^3}{3}e^{-6\theta\iota} - \frac{c^4}{4}e^{-8\theta\iota} + \dots\right\}.$$

The term which involves c is $-c(e^{\theta\iota} - e^{-\theta\iota})^2$, that is $4c\sin^2\theta$.

The term which involves c^2 is $\frac{c^2}{2}(e^{2\theta\iota} - e^{-2\theta\iota})^2$, that is $-\frac{4c^2}{2}\sin^2 2\theta$.

The term which involves c^3 is $-\frac{c^3}{3}(e^{3\theta\iota} - e^{-3\theta\iota})^2$, that is $\frac{4c^3}{3}\sin^2 3\theta$.

And so on.

Thus we obtain the required result.

Problem 268. *A triangle is formed by joining the centre of the inscribed circle of a triangle with the centres of the escribed circles which are opposite to the angles A and B. Show that the area of this triangle is* $\frac{abc}{2s}\cot\frac{C}{2}$.

Solution. Let O denote the centre of the inscribed circle, D and E the centers of the escribed circles. Then D, C, E are on a straight line which is at right angles to OC. The area of the triangle ODE

$$= \frac{1}{2}OC\cdot DE = \frac{1}{2}r\operatorname{cosec}\frac{C}{2}(r_1+r_2)\sec\frac{C}{2}$$
$$= \frac{r(r_1+r_2)}{\sin C} = \frac{S}{s\sin C}\left(\frac{S}{s-a} + \frac{S}{s-b}\right) = \frac{S^2 c}{s(s-a)(s-b)\sin C}$$

$$= \frac{(s-c)c}{\sin C} = \frac{abc\cos^2\frac{1}{2}C}{s\sin C} = \frac{abc}{2s}\cot\frac{C}{2}.$$

Problem 269. *If O be the centre of the circle inscribed in a triangle ABC, and r_a, r_b, r_c the radii of the circles inscribed in the triangles OBC, OCA, OAB, show that*

$$\frac{a}{r_a} + \frac{b}{r_b} + \frac{c}{r_c} = 2\left(\cot\frac{A}{4} + \cot\frac{B}{4} + \cot\frac{C}{4}\right).$$

Solution. The angle $OBC = \frac{1}{2}B$, and the angle $OCB = \frac{1}{2}C$. Hence, as in *Art.* 249 (page 426), we have

$$r_a\left(\cot\frac{B}{4} + \cot\frac{C}{4}\right) = a;$$

$$\therefore \frac{a}{r_a} = \cot\frac{B}{4} + \cot\frac{C}{4}.$$

Similarly $\qquad \frac{b}{r_b} = \cot\frac{C}{4} + \cot\frac{A}{4}, \quad$ and $\frac{c}{r_c} = \cot\frac{A}{4} + \cot\frac{B}{4}.$

Hence by addition we get the required result.

Problem 270. *Sum the series*

$$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{8} + \ldots + \tan^{-1}\frac{1}{2n^2}.$$

Solution. We easily see that $\tan^{-1}\frac{1}{2r^2} = \tan^{-1}(2r+1) - \tan^{-1}(2r-1)$.

Resolve each of the given terms into two by this formula. Then by addition we find that the sum $= \tan^{-1}(2n+1) - \tan^{-1}1 = \tan^{-1}\frac{n}{n+1}$.

Problem 271. *If a series of triangles be described of the same area, show that the sum of the cotangents of the angles varies as the sum of the squares on the sides.*

Solution. $\qquad \cot A = \frac{\cos A}{\sin A} = \frac{b^2+c^2-a^2}{2bc\sin A} = \frac{b^2+c^2-a^2}{4S};$

similar expressions hold for $\cot B$ and $\cot C$. Thus

$$\cot A + \cot B + \cot C = \frac{a^2+b^2+c^2}{4S}.$$

Hence if S be given the sum of the cotangents of the angles varies as the sum of the squares of the sides.

Problem 272. *Let I denote the centre of the circle inscribed in a triangle, O the centre of the circumscribed circle, D, E, F the centres of the escribed circles : then show that*

$$OI^2 + OD^2 + OE^2 + OF^2 = 12R^2.$$

Solution. By *Art.* 253 (page 428) we have

$$OI^2 = R^2 - 2Rr;$$

and $$OD^2 + OE^2 + OF^2 = 3R^2 + 2R(r_1 + r_2 + r_3).$$

Thus by addition we obtain

$$\begin{aligned} OI^2 + OD^2 + OE^2 + OF^2 &= 4R^2 + 2R(r_1 + r_2 + r_3 - r) \\ &= 4R^2 + 8R^2, \text{ by } \textit{Problem } 201, = 12R^2. \end{aligned}$$

Problem 273. *Show that* $2\tan^{-1}\left(\sqrt{\dfrac{a-b}{a+b}}\tan\dfrac{x}{2}\right) = \cos^{-1}\dfrac{b + a\cos x}{a + b\cos x}$.

Solution. Let $\theta = \tan^{-1}\left(\sqrt{\dfrac{a-b}{a+b}}\tan\dfrac{x}{2}\right)$, then $\tan\theta = \sqrt{\dfrac{a-b}{a+b}}\tan\dfrac{x}{2}$,

$$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{a+b-(a-b)\tan^2\dfrac{x}{2}}{a+b+(a-b)\tan^2\dfrac{x}{2}}$$

$$= \frac{b + a\left(cos^2\dfrac{x}{2} - \sin^2\dfrac{x}{2}\right)}{a + b\left(\cos^2\dfrac{x}{2} - \sin^2\dfrac{x}{2}\right)} = \frac{b + a\cos x}{a + b\cos x}.$$

Problem 274. *A chord is drawn cutting two concentric circles whose radii are as 1 to n, so that the intercepted portions subtend angles 2α and 2β at the centre. Show that the chord is divided at either point of intersection with the inner circle in the ratio of*

$$n^2 - 2n\cos(\alpha - \beta) + 1 \textit{ to } n^2 - 1.$$

Solution. Let O denote the centre of the circles. Let $ABCD$ be a straight line cutting the outer circumference at A and D, and the inner circumference at B and C. Let $OB = r$, and $OA = nr$. Let the angle $AOD = 2\alpha$, and the angle $BOC = 2\beta$; so that the angle $AOB = \alpha - \beta$.

Then $$AB^2 = n^2r^2 + r^2 - 2nr^2\cos(\alpha - \beta).$$

Now $$\frac{AB}{BD} = \frac{AB^2}{AB\cdot BD} = \frac{AB^2}{AB\cdot AC}.$$

But $AB\cdot AC =$ the square on the straight line drawn from A to touch the inner circumference $= (n^2-1)r^2$.

Therefore $$\frac{AB}{BD} = \frac{n^2 - 2n\cos(\alpha-\beta) + 1}{n^2 - 1}.$$

Problem 275. *Show that* $\left(a + b\sqrt{-1}\right)^{\alpha + \beta\sqrt{-1}}$ *will be wholly imaginary if*

$$\frac{\beta}{2}\log\left(a^2 + b^2\right) + \alpha\tan^{-1}\frac{b}{a}$$

is an odd multiple of $\dfrac{\pi}{2}$.

Solution. Proceed as in the solution of *Problem* 266. That the expression may be wholly imaginary we must have $\cos\tau = 0$, and therefore τ must be an odd multiple of $\dfrac{\pi}{2}$, therefore $\rho(\sin\phi\log r + \theta\cos\phi)$ must be an odd multiple of $\dfrac{\pi}{2}$; but $\rho\sin\phi = \beta$, and $\rho\cos\phi = \alpha$, so that $\dfrac{\beta}{2}\log\left(a^2+b^2\right)+\alpha\tan^{-1}\dfrac{b}{a}$ must be an odd multiple of $\dfrac{\pi}{2}$.

Problem 276. *Show that the area of the triangle formed by joining the centres of the escribed circles of a triangle is*

$$\frac{abc}{2}\left\{\left(\frac{1}{a}+\frac{1}{b}\right)\tan\frac{C}{2}+\left(\frac{1}{b}+\frac{1}{c}\right)\tan\frac{A}{2}+\left(\frac{1}{c}+\frac{1}{a}\right)\tan\frac{B}{2}\right\}.$$

Solution.

$$\frac{1}{a}\left(\tan\frac{C}{2}+\tan\frac{B}{2}\right)=\frac{1}{a}\left(\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}+\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}\right)=\frac{\cos\frac{A}{2}}{a\cos\frac{C}{2}\cos\frac{B}{2}}$$

$$=\frac{1}{r}\tan\frac{C}{2}\tan\frac{B}{2},\ \text{by } Art.\ 249\ (\text{page } 426).$$

In this way we find that the proposed expression

$$=\frac{abc}{2r}\left\{\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}\right\}$$

$$=\frac{abc}{2r},\ \text{by } Example\ \text{VIII. } 15;$$

and this is the area of the triangle by *Example* XVI. 34.

Or thus. Let I denote the centre of the inscribed circle, O the centre of the escribed circle opposite to A; then the area of the quadrilateral $IBOC = \dfrac{a}{2}(r+r_1) = \dfrac{a}{2}(s-a+s)\tan\dfrac{A}{2} = \dfrac{a}{2}(b+c)\tan\dfrac{A}{2}$: see *Arts.* 249 (page 426) and 250 (page 427).

In this way we obtain for the whole required area the given expression.

Problem 277. *Sum the following series to n terms :*

$$\log(1+2\cos\theta)+\log(1+2\cos 3\theta)+\log\left(1+2\cos 3^2\theta\right)+\dots$$

Solution.

We have

$$1+2\cos\theta=\frac{\sin\frac{3\theta}{2}}{\sin\frac{\theta}{2}},$$

$$1+2\cos 3\theta=\frac{\sin\frac{3^2\theta}{2}}{\sin\frac{3\theta}{2}},$$

and so on. Then, as the sum of the logarithms of any set of quantities is equal to the logarithm of the product of those quantities, we see that the required sum is the logarithm of

$$\frac{\sin\frac{3\theta}{2}}{\sin\frac{\theta}{2}}\cdot\frac{\sin\frac{3^2\theta}{2}}{\sin\frac{3\theta}{2}}\cdots\cdots\frac{\sin\frac{3^n\theta}{2}}{\sin\frac{3^{n-1}\theta}{2}},\ \text{that is the logarithm of } \frac{\sin\frac{3^n\theta}{2}}{\sin\frac{\theta}{2}}.$$

Problem 278. *A regular polygon of n sides is placed with one of its sides in contact with a fixed straight line, and is turned about one extremity of this side until the next side is in contact with the straight line, and so on for a complete revolution. Show that the length of the path described by any one of the angular points of the polygon is* $\frac{4\pi R}{n}\cot\frac{\pi}{2n}$, *where R is the radius of the circle circumscribing the polygon.*

Solution. Put β for $\frac{\pi}{n}$. The path consists of a set of arcs of circles, each of which corresponds to the angle 2β, and the radii of which are the respective distances of any assumed point from all the other angular points. The radii thus are $2R\sin\beta, 2R\sin 2\beta, 2R\sin 3\beta, \ldots$

Hence the required sum

$$= 2R\{\sin\beta + \sin 2\beta + \sin 3\beta + \ldots + \sin n\beta\}2\beta.$$

The term $\sin n\beta$ is zero, and may be omitted if we please.

By *Art.* 303 (page 436) this expression

$$= 4R\beta\frac{\sin\left(\beta + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{1}{2}\beta} = 4R\beta\frac{\sin\frac{n+1}{2n}\pi}{\sin\frac{\pi}{2n}}$$

$$= 4R\beta\cot\frac{\pi}{2n} = \frac{4R\pi}{n}\cot\frac{\pi}{2n}.$$

Problem 279. *Show that the sum of the areas of the sectors of the circles which correspond to the path in the preceding Problem* $= 2\pi R^2$.

Solution. The sum of the areas of all the sectors will be

$$4R^2\left\{\sin^2\beta + \sin^2 2\beta + \sin^2 3\beta + \ldots + \sin^2 n\beta\right\}\beta$$

$$= 2R^2\beta\{1 - \cos 2\beta + 1 - \cos 4\beta + \ldots\}$$

$$= 2R^2\beta\left\{n - \frac{\cos(2\beta + \overline{n-1}\beta)\sin n\beta}{\sin\beta}\right\} = 2R^2 n\beta = 2R^2\pi.$$

If we wish to have the whole area of the figure bounded by the straight line and by the arcs between two points where they cross the straight line, we must add to the above a set of triangles which make up the whole polygon, that is $\frac{n}{2}R^2\sin\frac{2\pi}{n}$.

Problem 280. *Sum the following series to 2n terms :*

$$\frac{\sin 2\theta}{\sin\theta\sin 3\theta} - \frac{\sin 4\theta}{\sin 3\theta\sin 5\theta} + \frac{\sin 6\theta}{\sin 5\theta\sin 7\theta} - \ldots$$

Solution. We have

$$\frac{\sin 2r\theta}{\sin(2r-1)\theta\sin(2r+1)\theta} = \frac{1}{2\cos\theta}\left\{\frac{1}{\sin(2r-1)\theta} + \frac{1}{\sin(2r+1)\theta}\right\}.$$

If we resolve each term of the proposed series into two by the aid of this formula we find that the sum of $2n$ terms $= \frac{1}{2\cos\theta}\left\{\frac{1}{\sin\theta} - \frac{1}{\sin(4n+1)\theta}\right\}$.

Problem 281. *In an acute-angled triangle let P denote the point of intersection of the perpendiculars from the angles on the opposite sides; and let $AP = a,\ BP = \beta,\ CP = \gamma$: then*

$$S = \frac{1}{4}(a\alpha + b\beta + c\gamma),$$
$$2abc = a^2\alpha \operatorname{cosec} A + b^2\beta \operatorname{cosec} B + c^2\gamma \operatorname{cosec} C.$$

Solution. As in *Problem* 214 we have

$$\alpha = \frac{a\cos A}{\sin A}, \quad \beta = \frac{b\cos B}{\sin B}, \quad \gamma = \frac{c\cos C}{\sin C}.$$

Hence $\frac{1}{4}(a\alpha + b\beta + c\gamma) = \frac{1}{4}\left(\frac{a^2\cos A}{\sin A} + \frac{b^2\ cosB}{\sin B} + \frac{c^2\cos C}{\sin C}\right)$

$= R^2(\sin A\cos A + \sin B\cos B + \sin C\cos C)$

$= \frac{R^2}{2}(\sin 2A + \sin 2B + \sin 2C) = 2R^2\sin A\sin B\sin C,$ by *Art.* 114 (page 409),

$= \frac{1}{2}ab\sin C,$ by *Art.* 252 (page 428), $= S.$

Also, $a^2\alpha \operatorname{cosec} A + b^2\beta \operatorname{cosec} B + c^2\gamma \operatorname{cosec} C = \frac{a^3\cos A}{\sin^2 A} + \frac{b^3\cos B}{\sin^2 B} + \frac{c^3\cos C}{\sin^2 C}$

$= 8R^3(\sin A\cos A + \sin B\cos B + \sin C\cos C)$

$= 4R^3(\sin 2A + \sin 2B + \sin 2C) = 16R^3\sin A\sin B\sin C$

$= 8RS,$ by the former part of the Problem, $= 2abc.$

Problem 282. *Let R denote the radius of the circle circumscribing a triangle; and let r', r'', r''' denote the radii of the greatest circles which touch the former circle and the sides of the triangle, being outside the triangle : then show that*

$$64Rr'r''r''' = \left(\frac{abc}{a+b+c}\right)^2.$$

Solution. We obtain immediately from a diagram

$$2r' = R(1-\cos A), \quad 2r'' = R(1-\cos B), \quad 2r''' = R(1-\cos C);$$

hence $$8r'r''r''' = 8R^3\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2} = \frac{8R^3S^4}{a^2b^2c^2s^2}.$$

$$\therefore 64Rr'r''r''' = \frac{64R^4S^4}{a^2b^2c^2s^2} = \frac{a^2b^2c^2}{4s^2} = \left(\frac{abc}{a+b+c}\right)^2.$$

Problem 283. *Show that one value of*

$$\sin^{-1}\frac{\sqrt{x^2-c^2}}{\sqrt{a^2-c^2}} - \sin^{-1}\frac{c\sqrt{a^2-x^2}}{x\sqrt{a^2-c^2}} \text{ is } \sin^{-1}\frac{x^2-ac}{x(a-c)}.$$

Solution. Let $\theta = \sin^{-1}\frac{\sqrt{(x^2-c^2)}}{\sqrt{(a^2-c)}}$ and $\phi = \sin^{-1}\frac{c\sqrt{(a^2-x^2)}}{x\sqrt{(a^2-c^2)}}$;

then $$\cos\theta = \frac{\sqrt{(a^2-x^2)}}{\sqrt{(a^2-c^2)}} \text{ and } \cos\phi = \frac{\alpha\sqrt{(x^2-c^2)}}{x\sqrt{(a^2-c^2)}};$$

$$\therefore \sin(\theta-\phi) = \frac{\sqrt{(x^2-c^2)}}{\sqrt{(a^2-c^2)}}\cdot\frac{a\sqrt{(x^2-c^2)}}{x\sqrt{(a^2-c^2)}} - \frac{\sqrt{(a^2-x^2)}}{\sqrt{(a^2-c^2)}}\cdot\frac{c\sqrt{(a^2-x^2)}}{x\sqrt{(a^2-c^2)}}$$

$$= \frac{a(x^2-c^2)}{x(a^2-c^2)} - \frac{c(a^2-x^2)}{x(a^2-c^2)} = \frac{x^2(a+c)-ac(a+c)}{x(a^2-c^2)} = \frac{x^2-ac}{x(a-c)}.$$

Problem 284. *If the lengths of the three straight lines drawn from the angles of a triangle to bisect the opposite sides be denoted by h, k, l, show that*

$$4(h^2+k^2+l^2) = 3(a^2+b^2+c^2),$$
$$16(h^2k^2+k^2l^2+l^2h^2) = 9(a^2b^2+b^2c^2+c^2a^2),$$
$$16(h^4+k^4+l^4) = 9(a^4+b^4+c^4).$$

Solution. Suppose D the middle point of BC. Then

$$AB^2 = AD^2 + BD^2 - 2AD\cdot BD\cos ADB,$$
$$AC^2 = AD^2 + CD^2 - 2AD\cdot CD\cos ADC;$$

therefore by addition $b^2+c^2 = 2h^2 + \dfrac{a^2}{2}$; so that $h^2 = \dfrac{1}{2}(b^2+c^2) - \dfrac{a^2}{4}$; similarly $k^2 = \dfrac{1}{2}(c^2+a^2) - \dfrac{b^2}{4}$, and $l^2 = \dfrac{1}{2}(a^2+b^2) - \dfrac{c^2}{4}$.

Therefore by addition $4(h^2+k^2+l^2) = 3(a^2+b^2+c^2)$.

Also $(4h^2)^2 + (4k^2)^2 + (4l^2)^2$

$$= (2b^2+2c^2-a^2)^2 + (2c^2+2a^2-b^2)^2 + (2a^2+2b^2-c^2)^2$$
$$= 9(a^4+b^4+c^4), \text{ by development.}$$

Again, from what has been already shewn,

$$16(h^2+k^2+l^2)^2 = 9(a^2+b^2+c^2),$$

and

$$16(h^4+k^4+l^4) = 9(a^4+b^4+c^4);$$

subtract and divide by 2; thus

$$16(h^2k^2+k^2l^2+l^2h^2) = 9(a^2b^2+b^2c^2+c^2a^2).$$

Problem 285. *The area of any triangle is to the area of the triangle whose sides are respectively equal to the straight lines joining its angular points with the middle points of the opposite sides, as 4 is to 3.*

Solution. The area of the triangle which can be formed with the straight lines h, k, l, by *Arts.* 247 (page 425) and 218 (page 420),

$$= \frac{1}{4}\sqrt{(2h^2k^2+2k^2l^2+2l^2h^2-h^4-k^4-l^4)}$$
$$= \frac{1}{4}\sqrt{\frac{9}{16}(2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4)}$$
$$= \frac{3}{16}\sqrt{(2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4)} = \frac{3}{4}S.$$

Problem 286. *Show that one value of*

$$\left\{a\left(\cos\theta+\sqrt{-1}\sin\theta\right) - b\left(\cos\overline{\theta-\alpha}+\sqrt{-1}\sin\overline{\theta-\alpha}\right)\right\}^{\frac{1}{n}}$$

is

$$k^{\frac{1}{n}}\left\{\cos\frac{\theta+\beta}{n} + \sqrt{-1}\sin\frac{\theta+\beta}{n}\right\}$$

when

$$k^2 = a^2+b^2-2ab\cos\alpha, \text{ and } \tan\beta = \frac{b\sin\alpha}{a-b\cos\alpha}.$$

Solution. $a\cos\theta - b\cos(\theta-\alpha) = \cos\theta(a - b\cos\alpha) - b\sin\alpha\sin\theta$;
assume $a - b\cos\alpha = k\cos\beta$, and $b\sin\alpha = k\sin\beta$;
then $a\cos\theta - b\cos(\theta-\alpha) = k(\cos\theta\cos\beta - \sin\theta\sin\beta) = k\cos(\theta+\beta)$.
Similarly

$$a\sin\theta - b\sin(\theta-\alpha) = \sin\theta(a - b\cos\alpha) + b\sin\alpha\cos\theta = k\sin(\beta+\theta).$$

Thus the proposed expression

$$= \left\{k\cos(\theta+\beta) + k\sqrt{-1}\sin(\theta+\beta)\right\}^{\frac{1}{n}}$$
$$= k^{\frac{1}{n}}\left\{\cos\frac{\theta+\beta}{n} + \sqrt{-1}\sin\frac{\theta+\beta}{n}\right\}.$$

Problem 287. *Any point is taken within a triangle ABC; its distances from A, B, C are h, k, l respectively, and its perpendicular distances from BC, CA, AB are α, β, γ respectively. Show that*

$$h^2\alpha\sin A + k^2\beta\sin B + l^2\gamma\sin C = a\beta\gamma + b\gamma\alpha + c\alpha\beta.$$

Solution. Denote the point by O; and let OD, OE, OF be the perpendiculars on BC, CA, AB respectively. Then OA is the diameter of the circle which would go round $OEAF$; so that $OA = \dfrac{EF}{\sin A}$, by *Art.* 252 (page 428). Therefore

$$OA^2\cdot\alpha\sin A = \alpha\cdot OA\cdot EF = \alpha(OE\cdot FA + OF\cdot AE),\ \text{by Euclid VI. } D,$$
$$= \alpha\cdot\beta\cdot AF + \alpha\cdot\gamma\cdot AE.$$

Transform the other two terms in like manner; thus we obtain

$$\alpha\beta(AF+BF) + \beta\gamma(BD+CD) + \gamma\alpha(AE+EC) = \alpha\beta c + \beta\gamma\alpha + \gamma\alpha b.$$

Problem 288. *If D, E, F be the feet of the perpendiculars drawn from the point in the preceding Problem on the sides of the triangle, show that*

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = 4R\times \text{ area of } DEF.$$

Solution. We have $\alpha\beta c = \dfrac{c}{\sin C}\alpha\beta\sin C = 2R\times 2$ area of OED.
Transform the other two terms similarly; thus we obtain

$$4R(\text{ area of } OED + \text{ area of } ODF + \text{ area of } OFE).$$

Problem 289. *Sum the following series to n terms :*

$$\frac{\sin\theta}{\cos^2\theta} + \frac{\sin 3\theta}{\cos^2 2\theta\cos^2\theta} + \frac{\sin 5\theta}{\cos^2 3\theta\cos^2 2\theta} + \ldots$$

Solution. We have

$$\frac{1}{\cos^2 B} - \frac{1}{\cos^2 A} = \frac{\cos^2 A - \cos^2 B}{\cos^2 A\cos^2 B} = \frac{\sin^2 B - \sin^2 A}{\cos^2 B\cos^2 A} = \frac{\sin(B-A)\sin(B+A)}{\cos^2 B\cos^2 A};$$

so that
$$\frac{\sin(B+A)}{\cos^2 B\cos^2 A} = \frac{1}{\sin(B-A)}\left\{\frac{1}{\cos^2 B} - \frac{1}{\cos^2 A}\right\}$$

Apply this transformation to every term of the proposed series; then we find that the sum

$$= \frac{1}{\sin\theta}\left\{\frac{1}{\cos^2 n\theta} - \frac{1}{\cos^2 0}\right\} = \frac{1}{\sin\theta}\left\{\frac{1}{\cos^2 n\theta} - 1\right\} = \operatorname{cosec}\theta\tan^2 n\theta.$$

Problem 290. *Find the general value of θ which satisfies the equation*

$$\left(\cos\theta + \sqrt{-1}\sin\theta\right)\left(\cos 2\theta + \sqrt{-1}\sin 2\theta\right)\dots\left(\cos n\theta + \sqrt{-1}\sin n\theta\right) = 1.$$

Solution. By De Moivre's Theorem the equation becomes

$$\cos(\theta + 2\theta + \dots + n\theta) + \sqrt{-1}\sin(\theta + 2\theta + \dots + n\theta) = 1,$$

that is $$\cos\frac{n(n+1)}{2}\theta + \sqrt{-1}\sin\frac{n(n+1)}{2}\theta = 1.$$

Hence we must have $\cos\frac{n(n+1)}{2}\theta = 1$, and $\sin\frac{n(n+1)}{2}\theta = 0$; so that $\frac{n(n+1)}{2}\theta = 2m\pi$, where m is zero or any integer.

Problem 291. *D, E, F are the centres of the escribed circles of a triangle opposite to A, B, C respectively : if r', r'', r''' denote the radii of the circles described round DBC, ECA, FAB, show that*

$$r'r''r''' = 2R^2r.$$

Solution. We have $r' = \frac{BC}{2\sin BDC} = \frac{a}{2\sin\frac{B+C}{2}} = \frac{a}{2\cos\frac{A}{2}}$; and similar expressions hold for r'' and r'''. Thus

$$r'r''r''' = \frac{abc}{8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{a^2b^2c^2}{8sS} = \frac{16R^2S^2}{8sS} = \frac{2R^2S}{s} = 2R^2r.$$

Problem 292. *If two sides a, b, and the included angle C of a triangle are given, and a small error γ exist in C, the corresponding error in the radius of the circumscribed circle is $\frac{ab\gamma}{2c}\cot A\cot B$.*

Solution. We have $$R = \frac{a}{2\sin A}, \text{ so that } R\sin A = \frac{a}{2} \tag{42}$$

Suppose that in consequence of the error γ in C there is an error α in A, and an error ρ in R. Thus

$$(R+\rho)\sin(A+\alpha) = \frac{a}{2};$$

therefore approximately by *Art.* 181 (page 416)

$$(R+\rho)(\sin A + \alpha\cos A) = \frac{a}{2} \tag{43}$$

From (42) and (43) by subtraction, neglecting the product $\alpha\rho$,

$$\alpha R\cos A + \rho\sin A = 0;$$

so that $$\alpha = -\frac{\rho}{R}\tan A.$$

Similarly if β be the error in B arising from the error γ in C, we have

$$\beta = -\frac{\rho}{R}\tan B.$$

But $\alpha + \beta + \gamma = 0$, since the sum of the three angles of a triangle is equal to a fixed quantity, namely two right angles.

Thus $$\gamma - \frac{\rho}{R}(\tan A + \tan B) = 0;$$

$$\therefore \rho = \frac{R\gamma}{\tan A + \tan B} = \frac{R\gamma \cos A \cos B}{\sin(A+B)} = \frac{c\gamma \cos A \cos B}{2\sin^2 C}.$$

And since $\sin C = \dfrac{c \sin A}{a}$ and $= \dfrac{c \sin B}{b}$; we have $\rho = \dfrac{ab\gamma \cot A \cot B}{2c}$.

Problem 293. *A quadrilateral is formed by connecting two points in the produced sides of a right-angled isosceles triangle, equidistant from the vertex, by a straight line whose length is n times that of the base. Show that the angle between the diagonals of the quadrilateral is* $2\tan^{-1}\dfrac{n-1}{n+1}$.

Solution. Let C denote the right angle, CA and CB the equal sides; produce CA to D and CB to E; then since DE is n times AB, it follows that CD and CE are each n times CA or CB. Let AE and BD intersect at O. Then the angle $DOA =$ the sum of the angles OBA and OAB, and these are equal; so that the angle $DOA =$ twice the angle OAB. But the angle $OAB =$ the angle $EAC-$ the angle BAC, so that

$$\tan OAB = \tan(EAC - BAC) = \frac{\tan EAC - \tan BAC}{1 + \tan EAC \tan BAC} = \frac{n-1}{1+n}.$$

Problem 294. *In the equation $\theta = \cos\theta$, show that there must be a solution, and only one; and that the value of θ is less than* $\dfrac{\pi}{4}$.

Solution. As θ continually increases from 0 to $\dfrac{\pi}{2}$ the value of $\cos\theta$ continually decreases from 1 to 0; so that there must be one value of θ, and only one, in this range, which makes $\theta = \cos\theta$. Also as $\cos\theta$ is greater than θ when $\theta = 0$, and is less than θ when $\theta = \dfrac{\pi}{4}$, this value is less than $\dfrac{\pi}{4}$. As θ changes from 0 to $-\dfrac{\pi}{2}$, the cosine is always positive, and so we cannot have $\cos\theta = \theta$.

When θ is numerically greater than $\dfrac{\pi}{2}$ it is numerically greater than unity, and so cannot be equal to $\cos\theta$.

Hence there must be one, and only one, solution of the equation $\theta = \cos\theta$.

Problem 295. *If β is an approximate value of θ in the equation $\cos\theta = \theta$, and too large, show that* $\beta - \dfrac{\beta - \cos\beta}{1 + \cos\beta}$ *is a closer value, and also too large.*

Solution. Suppose β the circular measure of an angle between 0 and $\dfrac{\pi}{2}$, which is greater than the solution of $\theta = \cos\theta$, so that $\beta - \cos\beta$ is positive. Let $\beta - \alpha$ denote the solution, so that $\beta - \alpha = \cos(\beta - \alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha$; therefore $\alpha = \dfrac{\beta - \cos\beta\cos\alpha}{1 + \sin\beta\dfrac{\sin\alpha}{\alpha}}$. Now $\dfrac{\sin\alpha}{\alpha}$ is less than unity, and so is $\cos\alpha$; hence $\dfrac{\beta - \cos\beta}{1 + \sin\beta}$

is less than the true value of α, and is a positive quantity. Therefore $\beta - \dfrac{\beta - \cos\beta}{1+\sin\beta}$ is nearer than β to the solution of the equation, and is still too large.

Problem 296. *If* $\tan\left(\theta + \phi\sqrt{-1}\right) = \tan\alpha + \sqrt{-1}\sec\alpha$, *where the letters denote real quantities, show that* $e^{2\phi} = \pm\cot\dfrac{\alpha}{2}$.

Solution. As in the solution of *Problem* 248 we get

$$\tan\alpha + \sqrt{-1}\sec\alpha = \frac{\sin\theta + \sqrt{-1}k\cos\theta}{\cos\theta - \sqrt{-1}k\sin\theta};$$

$$\therefore\ (\tan\alpha + \sqrt{-1}\sec\alpha)(\cos\theta - \sqrt{-1}k\sin\theta) = \sin\theta + \sqrt{-1}k\cos\theta;$$

$$\therefore\ \sin\theta = \tan\alpha\cos\theta + k\sin\theta\sec\alpha,$$

and
$$k\cos\theta = \sec\alpha\cos\theta - k\sin\theta\tan\alpha;$$

$$\therefore\ (\sin\theta - \tan\alpha\cos\theta)(\cos\theta + \sin\theta\tan\alpha) = \sin\theta\cos\theta\sec^2\alpha;$$

$$\therefore\ \sin\theta\cos\theta(1 - \sec^2\alpha - \tan^2\alpha) = \tan\alpha(\cos^2\theta - \sin^2\theta);$$

$$\therefore\ -\tan\alpha = \frac{\cos 2\theta}{\sin 2\theta} = \cot 2\theta.$$

Hence
$$\cot 2\theta = \cot\left(\frac{\pi}{2} + \alpha\right); \quad \therefore 2\theta = n\pi + \frac{\pi}{2} + \alpha.$$

And
$$k = \frac{\sin\theta - \tan\alpha\cos\theta}{\sin\theta\sec\alpha} = \frac{\sin(\theta - \alpha)}{\sin\theta};$$

$$\therefore\ \frac{1+k}{1-k} = \frac{\sin(\theta-\alpha) + \sin\theta}{\sin\theta - \sin(\theta - \alpha)} = \tan\left(\theta - \frac{\alpha}{2}\right)\cot\frac{\alpha}{2}.$$

Now
$$\tan\left(\theta - \frac{\alpha}{2}\right) = \tan\left(\frac{n\pi}{2} + \frac{\pi}{4}\right) = \pm 1;$$

thus
$$\frac{1+k}{1-k} = \pm\cot\frac{\alpha}{2}, \text{ that is } e^{2\phi} = \pm\cot\frac{\alpha}{2}.$$

Problem 297. *A regular pentagon and a regular hexagon are inscribed in a circle of radius r, so as to have an angular point in common; and the other adjacent angular points are joined. Show that the perimeter of the figure so formed is* $\dfrac{4r\sin 18^\circ\sin 15^\circ}{\sin 3^\circ}$.

Solution. When the figure is constructed it will be found to have ten sides, five of which are respectively equal to the other five.

The sum of five sides will be found to be

$$2r\left\{\sin 30^\circ + \sin 6^\circ + \sin 24^\circ + \sin 12^\circ + \sin 18^\circ\right\};$$

and by *Art*. 303 (page 436) this $= \dfrac{2r\sin(6^\circ + 12^\circ)\sin 15^\circ}{\sin 3^\circ} = \dfrac{2r\sin 18^\circ\sin 15^\circ}{\sin 3^\circ}$

Problem 298. *Sum the following series to n terms :*

$$\cos\theta\cos^2\frac{1}{2}\theta\,\mathrm{cosec}^2\frac{3\theta}{2} + \cos 3\theta\cos^2\frac{3\theta}{2}\,\mathrm{cosec}^2\frac{3^2\theta}{2} + \cos 3^2\theta\cos^2\frac{3^2\theta}{2}\,\mathrm{cosec}^2\frac{3^3\theta}{2} + \ldots$$

Solution. The first term $= \dfrac{\cos\theta(1+\cos\theta)}{1-\cos 3\theta} = \dfrac{\cos\theta(1+\cos\theta)}{(1-\cos\theta)(1+2\cos\theta)^2}$

$$= \frac{\cos\theta(1+\cos\theta)+\frac{1}{4}-\frac{1}{4}}{(1-\cos\theta)(1+2\cos\theta)^2} = \frac{\frac{1}{4}}{1-\cos\theta} - \frac{\frac{1}{4}}{(1-\cos\theta)(1+2\cos\theta)^2}$$

$$= \frac{\frac{1}{4}}{1-\cos\theta} - \frac{\frac{1}{4}}{1-\cos 3\theta}.$$

Each term is to be resolved into two in this manner; so that the sum

$$= \frac{1}{4}\left\{\frac{1}{1-\cos\theta} - \frac{1}{1-\cos 3^n\theta}\right\}.$$

Problem 299. *A series of radii divide the circumference of a circle into 2n equal parts. Show that the product of the perpendiculars let fall from any point of the circumference on n successive radii* $= \dfrac{r^n}{2^{n-1}}\sin n\phi$, *where r is the radius, and ϕ the angle between the radius to the given point and one of the extreme radii.*

Solution. Put β for $\dfrac{\pi}{n}$. The first perpendicular $= r\sin\phi$, the second perpendicular $= r\sin(\phi+\beta)$, the third $= r\sin(\phi+2\beta)$, and so on. Hence the product

$$= r^n \sin\phi\sin(\phi+\beta)\sin(\phi+\beta)\sin(\phi+2\beta)\ldots\ldots\sin(\phi+n\beta-\beta);$$

and this by *Art.* 318 (page 443) $= \dfrac{r^n}{2^{n-1}}\sin n\phi$.

Problem 300. *There are n stones arranged at equal intervals round the circumference of a circle : compare the labour of carrying them all to the centre with that of heaping them all round one of the stones; and show that when the number of stones is indefinitely increased the ratio is that of π to 4.*

Solution. Let r denote the radius. When all the stones are taken to the centre each stone is carried over a length r, so that the labour may be denoted by nr. When all the stones are taken to the position of one stone the labour in like manner may be represented by the sum of the straight lines drawn from one corner of the polygon to all the other corners.

Let $\beta = \dfrac{\pi}{n}$: then this sum

$$= 2r\{\sin\beta + \sin 2\beta + \sin 3\beta + \ldots + \sin n\beta\}$$

$$= \frac{2r\sin\left(\beta + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} = \frac{2r\sin\frac{n+1}{2}\beta}{\sin\frac{\beta}{2}} = 2r\cot\frac{\beta}{2}.$$

Hence the required ratio $= \dfrac{nr}{2r\cot\frac{\beta}{2}} = \dfrac{n}{2}\tan\dfrac{\beta}{2} = \dfrac{n}{2}\tan\dfrac{\pi}{2n}$.

To find the value of this when n is indefinitely increased we put it in the form $\dfrac{\pi}{4}\dfrac{\tan\frac{\pi}{2n}}{\frac{\pi}{2n}}$; then by *Art.* 118 (page 411) the limit is $\dfrac{\pi}{4}$.

APPENDIX
Miscellaneous Articles and Propositions

Art. 8 *To compare the number of degrees in any angle with the number of grades in the same angle.*

Let D be the number of *degrees* contained in any angle, G the number of *grades* contained in the same angle. Then since there are 90 degrees in a right angle, $\frac{D}{90}$ expresses the ratio of the given angle to a right angle; and since there are 100 grades in a right angle, $\frac{G}{100}$ also expresses the ratio of the given angle to a right angle.

Hence
$$\frac{D}{90} = \frac{G}{100};$$
therefore
$$D = \frac{90}{100}G = \frac{9}{10}G = G - \frac{1}{10}G,$$
and
$$G = \frac{100}{90}D = \frac{10}{9}D = D + \frac{1}{9}D.$$

The formula $D = G - \frac{1}{10}G$ gives the following rule : *From the number of grades contained in any angle subtract one-tenth of that number, the remainder is the number of degrees contained in the angle.*

The formula $G = D + \frac{1}{9}D$ gives the following rule : *To the number of degrees contained in any angle add one-ninth of that number, the sum is the number of grades contained in the angle.*

Art. 9 Again, let m be the number of English minutes contained in any angle, μ the number of French minutes contained in the same angle. Then since there are 90×60 English minutes in a right angles, $\frac{m}{90 \times 60}$ expresses the ratio of the given angle to a right angle; and since there are 100×100 French minutes in a right angle, $\frac{\mu}{100 \times 100}$ also expresses the ratio of the given angle to a right angle. Hence
$$\frac{m}{90 \times 60} = \frac{\mu}{100 \times 100};$$
therefore
$$m = \frac{9 \times 6}{10 \times 10}\mu = \frac{27}{50}\mu,$$
and
$$\mu = \frac{50}{27}m.$$
Similarly, if s be the number of English seconds contained in any angle, and σ the number of French seconds contained in the same angle,
$$\frac{s}{90 \times 60 \times 60} = \frac{\sigma}{100 \times 100 \times 100};$$
therefore
$$s = \frac{81}{250}\sigma,$$
and
$$\sigma = \frac{250}{81}s.$$

Art. 22 *We will now show how to connect the circular measure of any angle with the measure of the same angle in degrees.*

Let x denote the number of degrees in any given angle, θ the circular measure of the same angle. Since there are 180 degrees in two right angles, $\frac{x}{180}$ expresses the ratio of the given angle to two right angles. And since π is the circular measure of two right angles, $\frac{\theta}{\pi}$ also expresses the ratio of the given angle to two right angles. Hence

$$\frac{x}{180} = \frac{\theta}{\pi};$$

thus

$$x = \frac{180\theta}{\pi},$$

and

$$\theta = \frac{\pi x}{180}.$$

Art. 24 *Similarly we may connect the circular measure of any angle with the measure of the same angle in grades.*

Let y denote the number of grades in any given angle, θ the circular measure of the same angle; then the ratio of the given angle to two right angles is expressed by $\frac{y}{200}$ and also by $\frac{\theta}{\pi}$.

Hence

$$\frac{y}{200} = \frac{\theta}{\pi};$$

thus

$$y = \frac{200\theta}{\pi};$$

and

$$\theta = \frac{\pi y}{200}.$$

The number of grades in the angle which is the unit of circular measure is $\frac{200}{\pi}$, that is, 63·661977...

Art. 32 *To prove that* $(\sin A)^2 + (\cos A)^2 = 1$.

In the right-angled triangle APM we have

$$PM^2 + AM^2 = AP^2;$$

therefore

$$\frac{PM^2 + AM^2}{AP^2} = 1;$$

therefore

$$\left(\frac{PM}{AP}\right)^2 + \left(\frac{AM}{AP}\right)^2 = 1;$$

that is

$$(\sin A)^2 + (\cos A)^2 = 1.$$

Art. 34 *To prove that*

$$(\sec A)^2 = 1 + (\tan A)^2, \text{ and } (\text{cosec } A)^2 = 1 + (\cot A)^2.$$

In the right-angled triangle APM we have

$$AP^2 = AM^2 + PM^2;$$

therefore

$$\frac{AP^2}{AM^2} = 1 + \frac{PM^2}{AM^2},$$

therefore

$$\left(\frac{AP}{AM}\right)^2 = 1 + \left(\frac{PM}{AM}\right)^2,$$

that is

$$(\sec A)^2 = 1 + (\tan A)^2.$$

Again, since

$$AP^2 = PM^2 + AM^2,$$

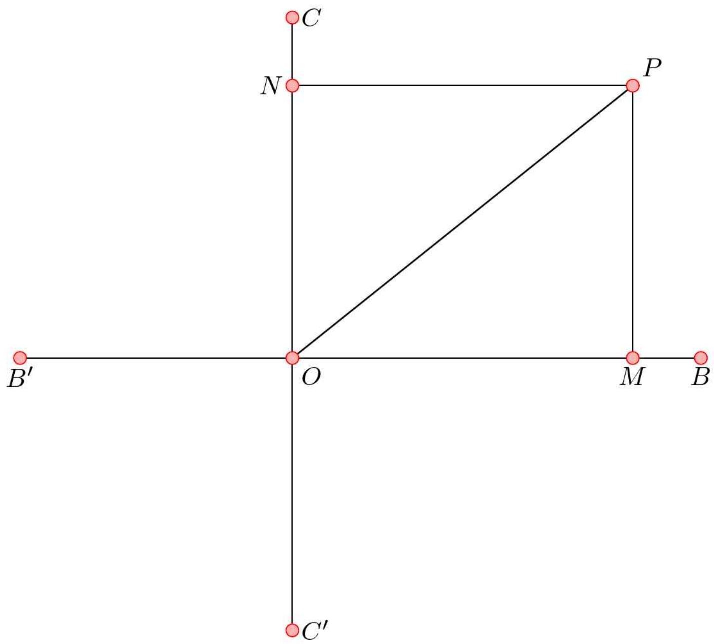
$$\left(\frac{AP}{PM}\right)^2 = 1 + \left(\frac{AM}{PM}\right)^2,$$

that is $$(\text{cosec } A)^2 = 1 + (\cot A)^2.$$

The results here obtained are usually written thus,

$$\sec^2 A = 1 + \tan^2 A, \quad \text{cosec}^2 A = 1 + \cot^2 A.$$

Art. 42 Let OB, OC be two straight lines which meet at right

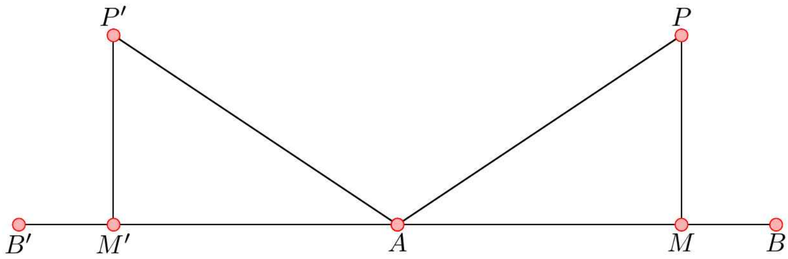


angles; produce BO to any point B' and CO to any point C'. Let P be any point in the plane containing the two straight lines. The position of P will be known if we know the distance of P from each of the straight lines BB' and CC', and also know *on which side* of each of these straight lines it is situated. Draw PM and PN perpendicular to the straight lines BB' and CC' respectively. We shall adopt the following conventions : the distance ON or PM will be denoted by a *positive* number when P is *above* the straight line BB', and by a *negative* number when P is *below* the straight line BB'; the distance OM or PN will be denoted by a *positive* number when P is to the *right* of CC', and by a *negative* number when P is to the *left* of CC'.

Art. 45 *It follows immediately from the definitions, that if two angles differ by four right angles or by any multiple of four right angles the Trigonometrical Ratios of the two angles are the same.*

Art. 48 *To compare the Trigonometrical Ratios of any angle and of its supplement.*

Let PAB be any angle, produce BA to B' and make $P'AB' = PAB$;



take $AP' = AP$, and draw PM and $P'M'$ perpendicular to BB'.

The angle $P'AB = 180^\circ - P'AB' = 180^\circ - PAB$; thus $P'AB$ is the supplement of PAB. The triangles PAM and $P'AM'$ are geometrically equal in all respects; now

$$\sin A = \frac{PM}{AP}, \quad \sin(180^\circ - A) = \frac{P'M'}{AP'};$$

and since PM and $P'M'$ are equal in magnitude and *of the same sign,* we have

$$\sin A = \sin(180^\circ - A).$$

Also $$\cos A = \frac{AM}{AP}, \quad \cos(180^\circ - A) = \frac{AM'}{AP'};$$

now AM and AM' are equal in magnitude, but since they are measured in opposite directions from A, they are of *opposite sign;* thus

$$\cos A = -\cos(180^\circ - A).$$

The other Trigonometrical Ratios of the angle A may be compared with those of the supplement either by direct use of the figure, or by employing the two results already established; thus adopting the latter method,

$$\tan(180^\circ - A) = \frac{\sin(180^\circ - A)}{\cos(180^\circ - A)} = \frac{\sin A}{-\cos A} = -\tan A,$$

$$\cot(180^\circ - A) = \frac{\cos(180^\circ - A)}{\sin(180^\circ - A)} = \frac{-\cos A}{\sin A} = -\cot A,$$

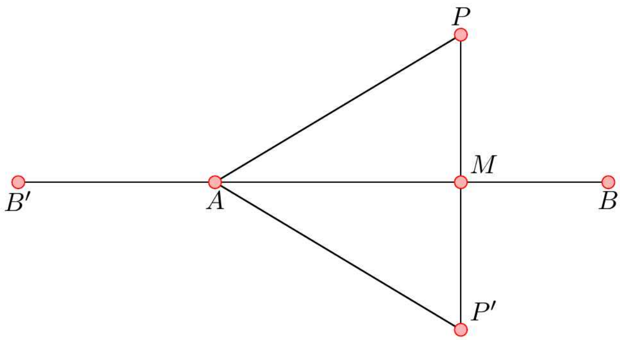
$$\sec(180^\circ - A) = \frac{1}{\cos(180^\circ - A)} = \frac{1}{-\cos A} = -\sec A,$$

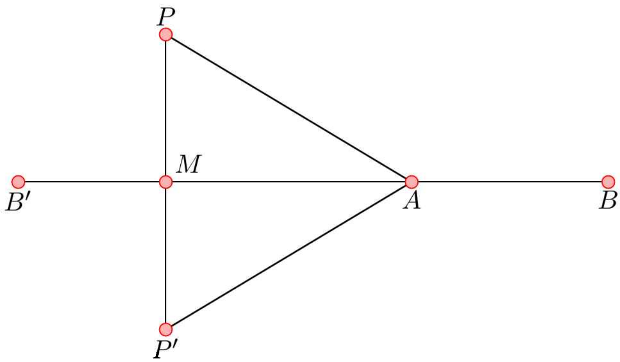
$$\operatorname{cosec}(180^\circ - A) = \frac{1}{\sin(180^\circ - A)} = \frac{1}{\sin A} = \operatorname{cosec} A,$$

$$\operatorname{vers}(180^\circ - A) = 1 - \cos(180^\circ - A) = 1 + \cos A.$$

Thus the sine and the cosecant of any angle are respectively the same as the sine and cosecant of the supplement of the angle; all the other Trigonometrical Ratios of any angle, except the versed sine, are *numerically* equal to the corresponding Ratios of the supplement of the angle, but are of *opposite sign.*

Art. 49 *To prove that* $\sin(-A) = -\sin A$ *and* $\cos(-A) = \cos A$.





Let PAB be any angle; draw PM perpendicular to BAB', and produce it to P' so that MP' may be equal in length to MP, and join AP'. Then the angles $P'AB$ and PAB which are measured in opposite directions from AB are numerically equal, and if PAB be denoted by A, then $P'AB$ will be denoted by $-A$. And

$$\sin A = \frac{PM}{AP}, \quad \sin(-A) = \frac{P'M}{AP'};$$

and $P'M$ is numerically equal to PM, but of opposite sign; thus

$$\sin(-A) = -\sin A.$$

Also
$$\cos(-A) = \frac{AM}{AP'} = \frac{AM}{AP} = \cos A.$$

Moreover,
$$\tan(-A) = \frac{\sin(-A)}{\cos(-A)} = \frac{-\sin A}{\cos A} = -\tan A;$$

$$\cot(-A) = \frac{\cos(-A)}{\sin(-A)} = \frac{\cos A}{-\sin A} = -\cot A;$$

$$\sec(-A) = \frac{1}{\cos(-A)} = \frac{1}{\cos A} = \sec A;$$

$$\operatorname{cosec}(-A) = \frac{1}{\sin(-A)} = \frac{1}{-\sin A} = -\operatorname{cosec} A;$$

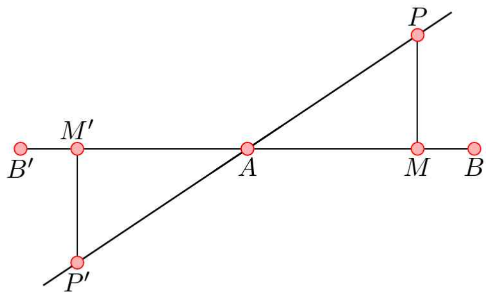
$$\operatorname{vers}(-A) = 1 - \cos(-A) = 1 - \cos A = \operatorname{vers} A.$$

All these results may if we please be obtained by direct use of the figure.

Art. 50 *To prove that*

$$\sin(180° + A) = -\sin A \text{ and } \cos(180° + A) = -\cos A.$$

Let PAB be any angle, produce PA to P' so that AP' may be equal in length to AP. Draw PM and $P'M'$ perpendicular to BAB'. Then if PAB be denoted by A, the angle $P'AB$ measured in the same direction from AB will be denoted by $180° + A$.



The triangles PAM and $P'AM'$ are geometrically equal in all respects;

and $$\sin A = \frac{PM}{AP}, \quad \sin(180° + A) = \frac{P'M'}{AP'};$$

$$\cos A = \frac{AM}{AP}, \quad \cos(180° + A) = \frac{AM'}{AP'}.$$

Now PM and $P'M'$ are numerically equal but of opposite sign; also AM and AM' are numerically equal but of opposite sign; thus

$$\sin(180° + A) = -\sin A, \quad \cos(180° + A) = -\cos A;$$

moreover $$\tan(180° + A) = \frac{\sin(180° + A)}{\cos(180° + A)} = \frac{-\sin A}{-\cos A} = \tan A,$$

$$\cot(180° + A) = \frac{\cos(180° + A)}{\sin(180° + A)} = \frac{-\cos A}{-\sin A} = \cot A;$$

similarly $$\sec(180° + A) = -\sec A, \quad \operatorname{cosec}(180° + A) = -\operatorname{cosec} A.$$

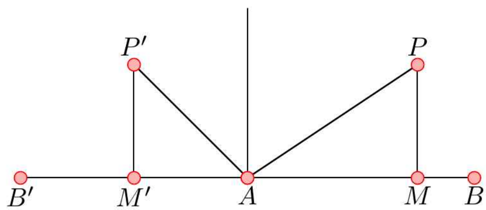
$$\operatorname{vers}(180° + A) = 1 - \cos(180° + A) = 1 + \cos A.$$

All these results may if we please be obtained by direct use of the figure.

It is obviously only another mode of expressing the two fundamental results if we write

$$\sin A = -\sin(A - 180°), \quad \cos A = -\cos(A - 180°).$$

Art. 52 *To prove that* $\sin(90° + A) = \cos A$, *and* $\cos(90° + A) = -\sin A$.



Let PAB be any angle; let AP' be at right angles to AP and so situated that a moveable straight line can pass from the position AP to the position AP' by revolving round A in the *positive direction* through a right angle. Then if PAB be denoted by A we can denote $P'AB$ by $90° + A$. Take $AP' = AP$ and draw PM and $P'M'$ perpendicular to BAB'. Then the angle PAM is geometrically equal to the angle $AP'M'$, and the triangles PAM and $P'AM'$ are geometrically equal in all respects. And

$$\sin(90° + A) = \frac{P'M'}{AP'}, \quad \cos A = \frac{AM}{AP};$$

now $P'M'$ is numerically equal to AM and both are of the same sign (*Art.* 42) (page 397); thus

$$\sin(90° + A) = \cos A.$$

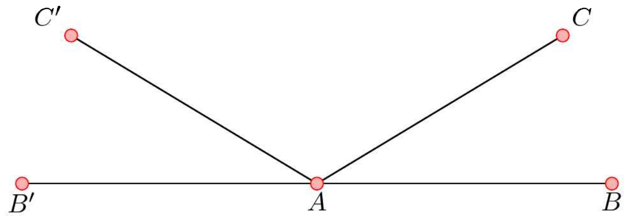
Again $$\cos(90° + A) = \frac{AM'}{AP'}, \quad \sin A = \frac{PM}{AP};$$

now AM' and PM are numerically equal but of opposite sign (*Art.* 42)(page 397); thus

$$\cos(90° + A) = -\sin A.$$

Art. 66 *To find an expression for all the angles which have a given sine.*

Let BAC be the least positive angle which has the given sine;



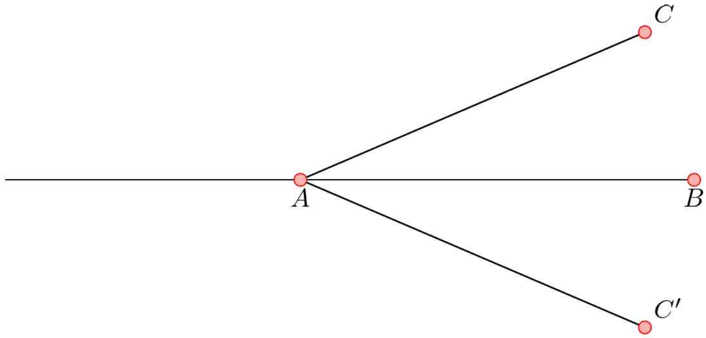
denote this angle by α. Produce BA to any point B' and make the angle $B'AC' = BAC$; then $BAC' = \pi - \alpha$.

Now it is obvious from the figure that the only *positive* angles which have the same sine as α are $\pi - \alpha$, and the angles formed by adding any multiple of four right angles to α or to $\pi - \alpha$; that is, angles included in the formulae $2n\pi + \alpha$ and $2n\pi + \pi - \alpha$, where n is zero or any positive integer. Also the only *negative* angles which have the same sine as α are $-(\pi + \alpha)$, and $-(2\pi - \alpha)$, and the angles formed by adding to these any multiple of four right angles taken negatively; that is angles included in the formulae $2n\pi - (\pi + \alpha)$, and $2n\pi - (2\pi - \alpha)$, where n is zero or any negative integer. All the angles which have been indicated will be found on trial to be included in the formula $n\pi + (-1)^n\alpha$, where n is zero, or any integer positive or negative. Also all the angles included in this formula will be found among the angles which have been indicated.

Thus the formula $n\pi + (-1)^n\alpha$ includes all the angles which have the same sine as α, and all the angles which it includes have the same sine as α.

This formula also determines all the angles which have the same cosecant as α.

Art. 67 *To find an expression for all the angles which have a given cosine.*



Let BAC be the least positive angle which has the given cosine; denote this angle by α. Make the angle $BAC' = BAC$.

Now it is obvious from the figure, that the only *positive* angles which have the same cosine as α are $2\pi - \alpha$, and the angles formed by adding any multiple of four right angles to α or to $2\pi - \alpha$; that is, angles included in the formulae $2n\pi + \alpha$ and $2n\pi + 2\pi - \alpha$, where n is zero or any positive integer. Also the only negative angles which have the same cosine as α are $-\alpha$, and $-(2\pi - \alpha)$, and the angles formed by adding to these any multiple of four right angles taken negatively; that is, angles included in the formulae $2n\pi - \alpha$ and $2n\pi - (2\pi - \alpha)$, where n is zero or any negative integer.

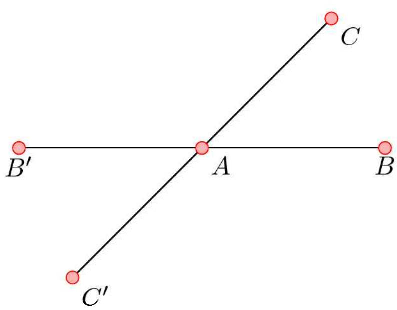
All the angles which have been indicated will be found on trial to be included in the formulae $2n\pi \pm \alpha$, where n is zero or any integer positive or negative. Also all the angles included in this formula will be found among the angles which have been indicated.

Thus the formula $2n\pi \pm \alpha$ included all the angles which have the same cosine as α, and all the angles which it includes have the same cosine as α.

This formula also determines all the angles which have the same secant or the same versed sine as α.

Art. 68 *To find an expression for all the angles which have a given tangent,*

Let BAC be the least positive angle which has the given tangent; denote this angle by α. Produce BA to any point B' and CA to any point C'.

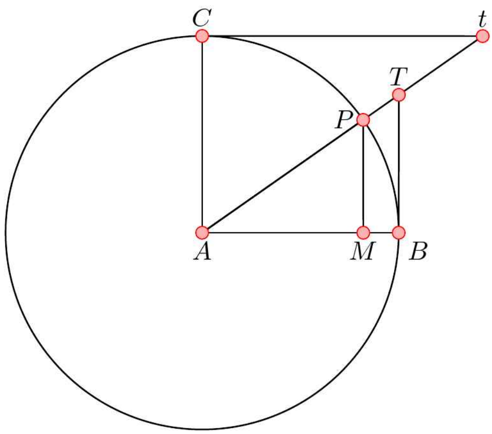


Now it is obvious from the figure that the only *positive* angles which have the same tangent as α are $\pi + \alpha$, and the angles formed by adding any multiple of four right angles to α or to $\pi + \alpha$; that is, angles included in the formulae $2n\pi + \alpha$ and $2n\pi + \pi + \alpha$, where n is zero or any positive integer. Also the only *negative* angles which have the same tangent as α are $-(\pi - \alpha)$, and $-(2\pi - \alpha)$, and the angles formed by adding to these any multiple of four right angles taken negatively; that is, angles included in the formulae $2n\pi - (\pi - \alpha)$ and $2n\pi - (2\pi - \alpha)$, where n is zero or any negative integer. All the angles which have been indicated will be found on trial to be included in the formula $n\pi + \alpha$, where n is zero, or any integer positive or negative. Also all the angles included in this formula will be found among the angles which have been indicated.

Thus the formula $n\pi + \alpha$ includes all the angles which have the same tangent as α, and all the angles which it includes have the same tangent as α.

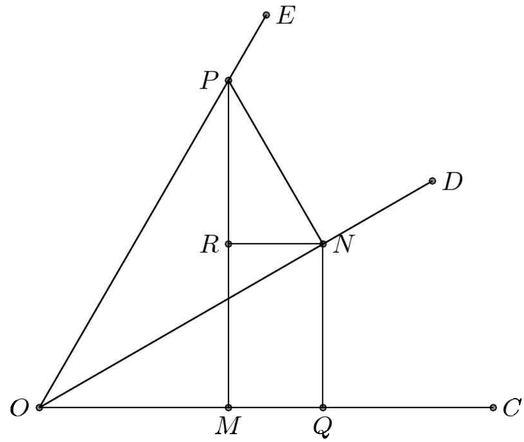
Thus the formula also determines all the angles which have the same *cotangent* as α.

Art. 71 Before leaving this part of the subject we will recur to the definitions of the Trigonometrical Ratios; we considered them as ratios formed by comparing the sides of a right-angled triangle, but formerly they were differently defined, and it is advisable to notice the old definitions in order that the student may understand allusions to them which will occur in his reading.



Let A be the centre of any circle, AB a radius, BP any arc; draw the radius AC at right angles to AB, and draw tangents to the circle at the points B and C; produce AP to meet the first tangent at T and the second tangent at t; draw PM perpendicular to AB. Then the old definitions are as follows, in which the straight *lines* of the figure are considered to be functions of the arc BP. PM is the sine of the arc BP, AM is its cosine, BT is its tangent, Ct is its cotangent, AT is its secant, At is its cosecant, BM is its versed sine; also the straight line joining B and P is the *chord* of the arc BP. Thus the terms *sine, cosine,* &c., formerly denoted certain straight *lines* and not certain *ratios.* On the old system the lengths of the *sine, cosine,* &c. depended on the radius of the circle considered, so that it became necessary to state what length was ascribed to this radius in any investigation.

Art. 76 *To express the sine and the cosine of the sum of two angles in terms of the sines and the cosines of the angles themselves.*



Let the angle COD be denoted by A, and the angle DOE by B; then the angle COE will be denoted by $A+B$. In OE take any point P, draw PM perpendicular to OC, and PN perpendicular to OD; draw NR perpendicular to PM and NQ perpendicular to OC.

Then the angle NPR is the complement of PNR, and is therefore equal to RNO, which is equal to NOC or A.

Now
$$\begin{aligned}\sin(A+B) &= \frac{PM}{OP} = \frac{RM+PR}{OP} = \frac{NQ}{OP} + \frac{PR}{OP}\\ &= \frac{NQ}{ON}\cdot\frac{ON}{OP} + \frac{PR}{PN}\cdot\frac{PN}{OP}\\ &= \sin A\cos B + \cos A\sin B.\\ \cos(A+B) &= \frac{OM}{OP} = \frac{OQ-QM}{OP} = \frac{OQ}{OP} - \frac{NR}{OP}\\ &= \frac{OQ}{ON}\cdot\frac{ON}{OP} - \frac{NR}{NP}\cdot\frac{NP}{OP}\\ &= \cos A\cos B - \sin A\sin B.\end{aligned}$$

Art. 82 In the expressions for $\sin(A+B)$ and $\cos(A+B)$ put $B=A$; thus
$$\sin 2A = 2\sin A\cos A;$$
$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1.$$

Thus
$$1+\cos 2A = 2\cos^2 A,$$
$$1-\cos 2A = 2\sin^2 A,$$

and
$$\frac{1-\cos 2A}{1+\cos 2A} = \tan^2 A.$$

Also
$$\frac{\sin 2A}{1+\cos 2A} = \tan A, \qquad \frac{\sin 2A}{1-\cos 2A} = \cot A.$$

Art. 83
$$\begin{aligned}&\sin(A+B)\sin(A-B)\\ &\quad= (\sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B)\\ &\quad= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B\\ &\quad= \sin^2 A\left(1-\sin^2 B\right) - \left(1-\sin^2 A\right)\sin^2 B\end{aligned}$$

$$= \sin^2 A - \sin^2 B.$$

This result is very important.

And $\cos(A+B)\cos(A-B)$

$$\begin{aligned}
&= (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B) \\
&= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B \\
&= \cos^2 A \left(1 - \sin^2 B\right) - \left(1 - \cos^2 A\right) \sin^2 B \\
&= \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A.
\end{aligned}$$

Art. 84 From the four fundamental formulae we have

$$\begin{aligned}
\sin(A+B) + \sin(A-B) &= 2 \sin A \cos B, \\
\sin(A+B) - \sin(A-B) &= 2 \cos A \sin B, \\
\cos(A+B) + \cos(A-B) &= 2 \cos A \cos B, \\
\cos(A-B) - \cos(A+B) &= 2 \sin A \sin B.
\end{aligned}$$

Let $A + B = C$ and $A - B = D$; therefore

$$A = \frac{1}{2}(C+D) \text{ and } B = \frac{1}{2}(C-D); \text{ thus}$$

$$\begin{aligned}
\sin C + \sin D &= 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}, \\
\sin C - \sin D &= 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}, \\
\cos C + \cos D &= 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}, \\
\cos D - \cos C &= 2 \sin \frac{C+D}{2} \sin \frac{C-D}{2}.
\end{aligned}$$

These formulae will be found to be extremely useful in mathematical investigations; they enable us to put the *sum or the difference* of two sines or two cosines in the form of a *product;* or to replace the *product* of a sine or a cosine into a sine or a cosine by half the *sum* or half the *difference* of two such Ratios.

Art. 87

$$\sin 2A = 2 \sin A \cos A = \frac{2 \sin A \cos A}{\sin^2 A + \cos^2 A}$$ (*Arts.* 82 (page 404) and 32 (page 396));

Divide both numerator and denominator of the last expression by $\cos^2 A$;

thus we get
$$\frac{\dfrac{2 \sin A}{\cos A}}{1 + \dfrac{\sin^2 A}{\cos^2 A}};$$

therefore
$$\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}.$$

Also
$$\cos 2A = \cos^2 A - \sin^2 A = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A}$$

(*Arts.* 82 (page 404) and 32 (page 396))

$$= \frac{1 - \dfrac{\sin^2 A}{\cos^2 A}}{1 + \dfrac{\sin^2 A}{\cos^2 A}} = \frac{1 - \tan^2 A}{1 + \tan^2 A}.$$

Art. 88

$$\frac{\sin A+\sin B}{\sin A-\sin B}=\frac{2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}}{2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}}(\mathit{Art.}\ 84)\ (\text{page } 405)$$

$$=\frac{\tan\dfrac{A+B}{2}}{\tan\dfrac{A-B}{2}};$$

$$\frac{\cos A+\cos B}{\cos B-\cos A}=\frac{2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}}{2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}}(\mathit{Art.}\ 84)\ (\text{page } 405)$$

$$=\cot\frac{A+B}{2}\cot\frac{A-B}{2};$$

$$\frac{\sin A+\sin B}{\cos A+\cos B}=\frac{2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}}{2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}}=\tan\frac{A+B}{2}.$$

$$\frac{\sin A-\sin B}{\cos B-\cos A}=\frac{2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}}{2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}}=\cot\frac{A+B}{2}.$$

Art. 91

$$\begin{aligned}\sin 3A&=\sin(2A+A)=\sin 2A\cos A+\cos 2A\sin A\ (\mathit{Art.}\ 76)(\text{page } 403)\\&=2\sin A\cos^2 A+(1-2\sin^2 A)\sin A\\&=2\sin A(1-\sin^2 A)+(1-2\sin^2 A)\sin A\\&=3\sin A-4\sin^3 A.\end{aligned}$$

$$\begin{aligned}\cos 3A&=\cos(2A+A)=\cos 2A\cos A-\sin 2A\sin A\ (\mathit{Art.}\ 76)(\text{page } 403)\\&=(2\cos^2 A-1)\cos A-2\cos A\sin^2 A\\&=(2\cos^2 A-1)\cos A-2\cos A(1-\cos^2 A)\\&=4\cos^3 A-3\cos A.\end{aligned}$$

Hence $\tan 3A=\dfrac{\sin 3A}{\cos 3A}=\dfrac{3\sin A-4\sin^3 A}{4\cos^3 A-3\cos A}.$

Divide both numerator and denominator by $\cos^3 A$; thus

$$\begin{aligned}\tan 3A&=\frac{\dfrac{3\tan A}{\cos^3}-4\tan^3 A}{4-\dfrac{3}{\cos^2 A}}\\&=\frac{3\tan A(1+\tan^2 A)-4\tan^3 A}{4-3(1+\tan^2 A)}(\mathit{Art.}\ 34)\ (\text{page } 396)\\&=\frac{3\tan A-\tan^3 A}{1-3\tan^2 A}.\end{aligned}$$

Art. 93 *If* $\sin A=\sin B$ *and* $\cos A=\cos B$, *then either* A *and* B *are equal, or they differ by some multiple of four right angles.*

For
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$
$$= \cos^2 A + \sin^2 A = 1$$
therefore $A - B = 0$, or a multiple of four right angles taken positively or negatively. (*Art.* 67 : page 402.)

Art. 94 *If* $\cos A = \cos B$ *and* $\sin A = -\sin B$*, then* $A + B$ *is zero, or a multiple of four right angles positive or negative.*

For the given relations may be written
$$\cos A = \cos(-B), \quad \sin A = \sin(-B). \quad (\textit{Art. } 49 : \text{page } 399.)$$
Hence by the preceding Article $A - (-B)$, that is $A + B$, is zero or some multiple of four right angles taken positively or negatively.

Art. 98 By *Art.* 82 (page 404)
$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2},$$
also
$$1 = \sin^2 \frac{A}{2} + \cos^2 \frac{A}{2},$$
thus
$$\left(\sin \frac{A}{2} + \cos \frac{A}{2}\right)^2 = 1 + \sin A,$$
and
$$\left(\sin \frac{A}{2} - \cos \frac{A}{2}\right)^2 = 1 - \sin A;$$
therefore
$$\sin \frac{A}{2} + \cos \frac{A}{2} = \sqrt{1 + \sin A} \tag{44}$$
and
$$\sin \frac{A}{2} - \cos \frac{A}{2} = \sqrt{1 - \sin A} \tag{45}$$
therefore
$$2 \sin \frac{A}{2} = \sqrt{1 + \sin A} + \sqrt{1 - \sin A},$$
and
$$2 \cos \frac{A}{2} = \sqrt{1 + \sin A} - \sqrt{1 - \sin A}.$$

Art. 100 If $\sin A$ only be given and nothing more be known respecting A, then the ambiguities of sign which occur in *Art.* 98 (page 407) cannot be removed. If however A itself be given, or if we merely know in which quadrant the angle A lies, we can determine the proper signs; for in any particular case we may proceed thus. We have
$$\sin \frac{A}{2} + \cos \frac{A}{2} = \pm\sqrt{1 + \sin A} \tag{46}$$
$$\sin \frac{A}{2} - \cos \frac{A}{2} = \pm\sqrt{1 - \sin A} \tag{47}$$
Now suppose, for example, that A lies between 0 and $90°$, then $\frac{A}{2}$ lies between 0 and $45°$; therefore $\cos \frac{A}{2}$ and $\sin \frac{A}{2}$ are both positive and $\cos \frac{A}{2}$ is greater than $\sin \frac{A}{2}$; hence the left-hand member of (46) is a *positive* quantity, and we must therefore take the *positive* sign in (46), and the left-member of (47) is a *negative* quantity, and we must therefore take the *negative* sign in (47). Therefore if A lies between 0

and 90°, we have

$$\sin\frac{A}{2}+\cos\frac{A}{2}=+\sqrt{1+\sin A},\quad \sin\frac{A}{2}-\cos\frac{A}{2}=-\sqrt{1-\sin A};$$

$$\therefore\ 2\sin\frac{A}{2}=+\sqrt{1+\sin A}-\sqrt{1-\sin A},$$

$$2\cos\frac{A}{2}=+\sqrt{1+\sin A}+\sqrt{1-\sin A}.$$

For another example, suppose that A lies between 270° and 360°, then $\frac{A}{2}$ lies between 135° and 180°; therefore $\cos\frac{A}{2}$ is negative, and $\sin\frac{A}{2}$ is positive, and $\cos\frac{A}{2}$ is numerically greater than $\sin\frac{A}{2}$; hence the left-hand member of (46) is a *negative* quantity, and we must therefore take the *negative* sign in (46), and the left-hand member of (47) is a *positive* quantity, and we must therefore take the *positive* sign in (47). Therefore if A lies between 270° and 360°, we have

$$\sin\frac{A}{2}+\cos\frac{A}{2}=-\sqrt{1+\sin A},\quad \sin\frac{A}{2}-\cos\frac{A}{2}=+\sqrt{1-\sin A};$$

$$\therefore\ 2\sin\frac{A}{2}=-\sqrt{1+\sin A}+\sqrt{1-\sin A},$$

$$2\cos\frac{A}{2}=-\sqrt{1+\sin A}-\sqrt{1-\sin A}.$$

Art. 101 *It is easy to give general formulae for determining the signs of* $\sin\frac{A}{2}+\cos\frac{A}{2}$ *and* $\sin\frac{A}{2}-\cos\frac{A}{2}$.

For $\sin\frac{A}{2}+\cos\frac{A}{2}=\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\frac{A}{2}+\frac{1}{\sqrt{2}}\cos\frac{A}{2}\right)=\sqrt{2}\sin\left(\frac{A}{2}+\frac{\pi}{4}\right)$;

Now $\sin\left(\frac{A}{2}+\frac{\pi}{4}\right)$ is *positive* if $\frac{A}{2}+\frac{\pi}{4}$ lies between $2n\pi$ and $(2n+1)\pi$, and *negative* if $\frac{A}{2}+\frac{\pi}{4}$ lies between $(2n+1)\pi$ and $(2n+2)\pi$, where n is zero or any integer positive or negative. Thus $\sin\frac{A}{2}+\cos\frac{A}{2}$ is positive if $\frac{A}{2}$ lies between $2n\pi-\frac{\pi}{4}$ and $2n\pi+\frac{3\pi}{4}$, and negative if $\frac{A}{2}$ lies between $2n\pi+\frac{3\pi}{4}$ and $2n\pi+\frac{7\pi}{4}$.

Similarly $\sin\frac{A}{2}-\cos\frac{A}{2}=\sqrt{2}\sin\left(\frac{A}{2}-\frac{\pi}{4}\right)$; and hence we can infer that $\sin\frac{A}{2}-\cos\frac{A}{2}$ is positive if $\frac{A}{2}$ lies between $2m\pi+\frac{\pi}{4}$ and $2m\pi+\frac{5\pi}{4}$, and negative if $\frac{A}{2}$ lies between $2m\pi+\frac{5\pi}{4}$ and $2m\pi+\frac{9\pi}{4}$, where m is zero or any integer positive or negative.

We will apply this to an example : required the limits between which $\frac{A}{2}$ must lie in order that

$$2\sin\frac{A}{2}=-\sqrt{1+\sin A}-\sqrt{1-\sin A}.$$

To obtain this result the *lower* sign must be taken in (46) and in (47) of *Art.* 100 (page 407); thus from (46) we infer that $\frac{A}{2}$ must lie between $2n\pi+\frac{3\pi}{4}$ and $2n\pi+\frac{7\pi}{4}$; and from (47) we infer that $\frac{A}{2}$ must lie between $2m\pi+\frac{5\pi}{4}$ and $2m\pi+\frac{9\pi}{4}$: hence, combining these results, we see that $\frac{A}{2}$ must lie between $2n\pi + \frac{5\pi}{4}$ and $2n\pi + \frac{7\pi}{4}$, where n is zero or any integer positive or negative.

Art. 108 *To find the sine and the cosine of an angle of 36°.*

$$\cos 36^\circ = 1 - 2\sin^2 18^\circ = 1 - 2\left(\frac{\sqrt{5}-1}{4}\right)^2 = 1 - \frac{6-2\sqrt{5}}{8}$$
$$= 1 - \frac{3-\sqrt{5}}{4} = \frac{1+\sqrt{5}}{4},$$
$$\sin 36^\circ = \sqrt{1-\cos^2 36^\circ} = \frac{\sqrt{10-2\sqrt{5}}}{4}.$$

Art. 113 It is easy to find expressions for the Trigonometrical Ratios of any compound angle in terms of the Ratios of the component angles. For example,

$$\sin(A+B+C) = \sin(A+B)\cos C + \cos(A+B)\sin C$$
$$= \sin A\cos B\cos C + \sin B\cos C\cos A + \sin C\cos A\cos B - \sin A\sin B\sin C.$$
$$\cos(A+B+C) = \cos(A+B)\cos C - \sin(A+B)\sin C$$
$$= \cos A\cos B\cos C - \cos A\sin B\sin C - \cos B\sin A\sin C - \cos C\sin A\sin B.$$
$$\tan(A+B+C) = \frac{\sin(A+B+C)}{\cos(A+B+C)}$$
$$= \frac{\sin A\cos B\cos C + \sin B\cos C\cos A + \sin C\cos A\cos B - \sin A\sin B\sin C}{\cos A\cos B\cos C - \cos A\sin B\sin C - \cos B\sin A\sin C - \cos C\sin A\sin B};$$

divide both numerator and denominator of the last expression by $\cos A\cos B\cos C$; thus we obtain

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan B\tan C - \tan C\tan A - \tan A\tan B}.$$

Suppose B and C each equal to A; thus we have

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

Art. 114 When three or more angles are connected by some relation, we may often find that some simple relation exists among some of their Trigonometrical Ratios, thus, for example,

if

$$A + B + C = 180^\circ,$$

then will

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C.$$

For

$$\sin 2A + \sin 2B = 2\sin(A+B)\cos(A-B) = 2\sin C\cos(A-B),$$

and

$$\sin 2C = 2\sin C\cos C = -2\sin C\cos(A+B),$$
$$\because \cos(C) = \cos(180^\circ - (A+B)) = -\cos(A+B).$$
$$\therefore \sin 2A + \sin 2B + \sin 2C = 2\sin C\{\cos(A-B) - \cos(A+B)\}$$
$$= 4\sin C\sin A\sin B.$$

Again, if $A+B+C=180°$, then will

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C.$$

For $\tan 180°=0$, therefore $\tan(A+B+C)=0$; therefore by *Art.* 113, (page 409)

$$\tan A+\tan B+\tan C-\tan A\tan B\tan C=0$$

Again, by *Art.* 113, (page 409)

$$\cot(A+B+C)=\frac{1-\tan B\tan C-\tan C\tan A-\tan A\tan B}{\tan A+\tan B+\tan C-\tan A\tan B\tan C};$$

now $\cot 90°=0$; hence we have the following result,
if $A+B+C=90°$, then will

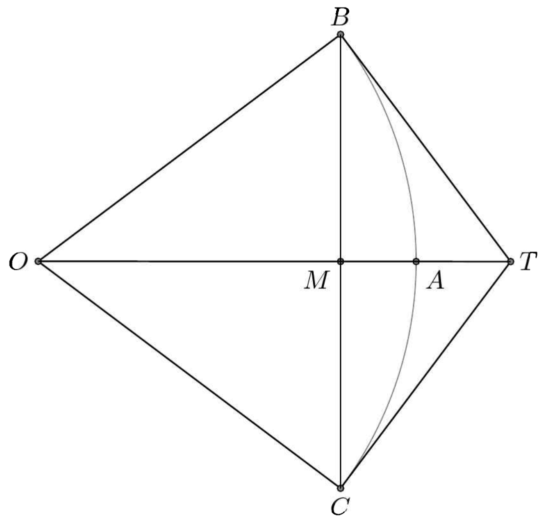
$$1=\tan B\tan C+\tan C\tan A+\tan A\tan B.$$

Art. 116. *If θ be the circular measure of a positive angle less than a right angle, θ is greater than $\sin\theta$ and less than $\tan\theta$.*

Let AOB be an angle less than a right angle and let $OB=OA$; from B draw BM perpendicular to OA and produce it to C so that $MC=MB$; draw BT at right angles to OB meeting OA produced at T, and join CT and OC. Then the triangles MOC and MOB are equal in all respects, so that the angle $TOC=$ the angle TOB; therefore the triangles TOC and TOB are equal in all respects, so that TCO is a right angle, and $TC=TB$.

With centre O and radius OB describe OB describe an arc of a circle BAC; this will touch BT at B and CT at C.

Now we assume as an axiom that the straight line BC is less that the arc BAC; thus BM the half of BC is less than BA the half of the arc BAC; therefore $\frac{BM}{OB}$ is less than $\frac{BA}{OB}$; that is, the sine of AOB is less than the circular measure of AOB.



Again, we assume as an axiom that the arc BAC is less than *the sum of the* two exterior lines BT and TC; thus BA is less than BT; therefore $\frac{BA}{OB}$ is less than $\frac{BT}{OB}$; that is, the circular measure of AOB is less than the tangent of AOB.

Hence $\sin\theta$, θ, and $\tan\theta$ are in ascending order of magnitude if θ be less than $\frac{\pi}{2}$.

Art. 118. *The limit of* $\frac{\sin\theta}{\theta}$ *when* θ *is indefinitely diminished is unity.*

For $\sin\theta$, θ, and $\tan\theta$ are in ascending order of magnitude ; divide by $\sin\theta$; therefore 1, $\frac{\theta}{\sin\theta}$, and $\frac{1}{\cos\theta}$ are in ascending order of magnitude. Thus $\frac{\theta}{\sin\theta}$ lies in value between 1 and $\frac{1}{\cos\theta}$; but when θ is zero, $\cos\theta$ is unity; hence as θ diminishes indefinitely $\frac{\theta}{\sin\theta}$ approaches the limit unity. Therefore also $\frac{\sin\theta}{\theta}$ approaches the limit unity.

Art. 119. From the preceding Article we see that the *limit of* $m\sin\frac{\alpha}{m}$ *when* m *increases indefinitely* is α.

For $m\sin\frac{\alpha}{m} = \alpha\sin\frac{\alpha}{m} \div \frac{\alpha}{m}$; and when m is indefinitely great $\sin\frac{\alpha}{m} \div \frac{\alpha}{m}$ is unity.

Similarly the *limit of* $m\tan\frac{\alpha}{m}$ *when* m *increases indefinitely* is α.

It must be carefully remembered that in the important proposition of the preceding Article, θ is the *circular measure* of the angle considered. If any other unit of angular measurement be adopted instead of the unit of circular measure, the limit under consideration will *not* be unity. For example, let us find the limit of $\frac{\sin n^\circ}{n}$ when n is indefinitely diminished. Let θ be the circular measure of an angle of n degree, then $\theta = \frac{n\pi}{180}$; thus

$$\frac{\sin n^\circ}{n} = \frac{\sin\theta}{\frac{180}{\pi}\theta} = \frac{\pi}{180}\cdot\frac{\sin\theta}{\theta}.$$

Now when n diminishes indefinitely, θ does so also, and the limit of $\frac{\sin\theta}{\theta}$ is unity; hence the limit of $\frac{\sin n^\circ}{n}$ when n is diminished indefinitely is $\frac{\pi}{180}$, which is the circular measure of an angle of *one degree.* Similarly we may prove that the limit of $\frac{\sin n'}{n}$ when n is indefinitely diminished is the circular measure of an angle of *one minute;* and so on. Thus we shall find that, whatever be the unit of angular measurement, the limit of the ratio of the sine of an angle to the angle, when the angle is indefinitely diminished, is the *circular measure of the unit.*

Art. 120. *If* θ *be the circular measure of a positive angle less than a right angle,* $\sin\theta$ *is greater than* $\theta - \frac{\theta^3}{4}$.

For $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$; and $\tan\frac{\theta}{2}$ is greater than $\frac{\theta}{2}$, therefore $\sin\frac{\theta}{2}$ is greater than $\frac{\theta}{2}\cos\frac{\theta}{2}$; therefore $\sin\theta$ is greater than $2\frac{\theta}{2}\cos^2\frac{\theta}{2}$, that is greater than $\theta\cos^2\frac{\theta}{2}$, that is greater than $\theta\left(1-\sin^2\frac{\theta}{2}\right)$. And $\sin^2\frac{\theta}{2}$ is less than $\left(\frac{\theta}{2}\right)^2$, therefore *a* *fortiori* $\sin\theta$ is greater than $\theta\left(1-\frac{\theta^2}{4}\right)$; that is, $\sin\theta$ is greater than $\theta-\frac{\theta^3}{4}$.

Art. 121. Thus we see that if θ lie between zero and a right angle $\sin\theta$ is less than θ and greater than $\theta-\frac{\theta^3}{4}$; and therefore $\sin\frac{\theta}{2}$ is less than $\frac{\theta}{2}$ and greater than $\frac{\theta}{2}-\frac{\theta^3}{32}$.

Now $\cos\theta = 1-2\sin^2\frac{\theta}{2}$. Thus $\cos\theta$ is greater than $1-2\left(\frac{\theta}{2}\right)^2$, that is greater than $1-\frac{\theta^2}{2}$. Also $\cos\theta$ is less than $1-2\left(\frac{\theta}{2}-\frac{\theta^3}{32}\right)^2$, that is less than $1-\frac{\theta^2}{2}+\frac{\theta^4}{16}-2\left(\frac{\theta^3}{32}\right)^2$; therefore *a fortiori* $\cos\theta$ is less than $1-\frac{\theta^2}{2}+\frac{\theta^4}{16}$.

Art. 129. *The limit of* $\cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}\dots\cos\frac{x}{2^n}$ *when the integer* n *is indefinitely increased is* $\frac{\sin x}{x}$.

For

$$\begin{aligned}\sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2}\\ &= 4\sin\frac{x}{4}\cos\frac{x}{4}\cos\frac{x}{2}\\ &= 8\sin\frac{x}{8}\cos\frac{x}{8}\cos\frac{x}{4}\cos\frac{x}{2}\\ &\dots\dots\dots\dots\\ &= 2^n\sin\frac{x}{2^n}\cos\frac{x}{2^n}\dots\cos\frac{x}{8}\cos\frac{x}{4}\cos\frac{x}{2}.\end{aligned}$$

$$\therefore \cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}\dots\cos\frac{x}{2^n} = \frac{\sin x}{2^n\sin\frac{x}{2^n}}.$$

Now

$$\frac{\sin x}{2^n\sin\frac{x}{2^n}} = \frac{\sin x}{x}\,\frac{\frac{x}{2^n}}{\sin\frac{x}{2^n}},$$

and the limit of this when n is indefinitely increased is $\frac{\sin x}{x}$, since by *Art.* 118 (page 411), the limit of $\frac{\frac{x}{2^n}}{\sin\frac{x}{2^n}}$ is unity. This result is sometimes cited as *Euler's Formula*.

Art. 130. *To prove that if* x *be the circular measure of a positive angle less than a right angle* $\sin x$ *is greater than* $x-\frac{x^3}{6}$.

We have, by *Art.* 91 (page 406),

$$\begin{aligned}\sin x &= 3\sin\frac{x}{3}-4\sin^3\frac{x}{3}\\ &= 3\left(3\sin\frac{x}{3^2}-4\sin^3\frac{x}{3^2}\right)-4\sin^3\frac{x}{3}\end{aligned}$$

$$= 3^2 \sin \frac{x}{3^2} - 4 \sin^3 \frac{x}{3} - 4 \times 3 \sin^3 \frac{x}{3^2}$$
$$= 3^3 \left(3 \sin \frac{x}{3^3} - 4 \sin^3 \frac{x}{3^3}\right) - 4 \sin^3 \frac{x}{3} - 4 \times 3 \sin^3 \frac{x}{3^2}$$
$$= 3^3 \sin \frac{x}{3^3} - 4 \left\{\sin^3 \frac{x}{3} + 3 \sin^3 \frac{x}{3^2} + 3^2 \sin^3 \frac{x}{3^3}\right\}.$$

Proceeding in this way we see that

$$\sin x = 3^n \sin \frac{x}{3^n} - 4 \left\{\sin^3 \frac{x}{3} + 3 \sin^3 \frac{x}{3^2} + \ldots + 3^{n-1} \sin^3 \frac{x}{3^n}\right\}.$$

Hence, by *Art.* 116 (page 410), $\sin x$ is greater than

$$3^n \sin \frac{x}{3^n} - \frac{4x^3}{3^3} \left\{1 + \frac{1}{3^2} + \frac{1}{3^4} + \ldots + \frac{1}{3^{2n-2}}\right\},$$

that is greater than $3^n \sin \frac{x}{3^n} - \frac{4x^3}{3} \frac{1 - \frac{1}{3^{2n}}}{3^2 - 1}$. Thus $\sin x$ exceeds the last expression; and the excess does not vanish however great n may be : therefore $\sin x$ exceeds the limit to which the last expression approaches when n is made infinite.

But the limit of $3^n \sin \frac{x}{3^n}$ is x by *Art.* 119 (page 411); and the limit of $1 - \frac{1}{3^{2n}}$ is 1 : thus

$$\sin x \text{ is greater than } x - \frac{x^3}{6}.$$

By proceeding as in *Art.* 121 (page 412), we may now show that

$$\cos x \text{ is less than } 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

(Le Cointe's *Trigonometry*, and *Messenger of Mathematics*, III. 101.)

Art. 135. *The logarithm of the base itself is unity.*

For $a^x = a$ when $x = 1$.

Art. 139. *To find the relation between the logarithms of the same number to different bases.*

Let

$$x = \log_a m, \ y = \log_b m;$$
$$\therefore m = a^x \text{ and } = b^y;$$
$$\therefore a^x = b^y;$$
$$\therefore a^{\frac{x}{y}} = b, \text{ and } b^{\frac{y}{x}} = a;$$
$$\therefore \frac{x}{y} = \log_a b, \text{ and } \frac{y}{x} = \log_b a.$$
$$\therefore y = x \log_b a, \text{ and } = \frac{x}{\log_a b}.$$

Hence the logarithm of a number to the base b may be found by multiplying the logarithm of the number to the base a by

$$\log_b a, \text{ or by } \frac{1}{\log_a b}.$$

We may notice that $log_b a \times \log_a b = 1$.

Art. 143. *To expand a^x in a series of ascending powers of x; that is, to expand a number in a series of ascending powers of its logarithm to a given base.*

$$a^x = \{1+(a-1)\}^x = 1 + x(a-1) + \frac{x(x-1)}{1\cdot 2}(a-1)^2$$
$$+ \frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}(a-1)^3 + \frac{x(x-1)(x-2)(x-3)}{1\cdot 2\cdot 3\cdot 4}(a-1)^4 + \dots$$
$$= 1 + x\left\{a - 1 - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \frac{1}{4}(a-1)^4 + \dots\right\}$$
$$+ \text{ terms involving } x^2, x^3, \dots.$$

This shows that a^x *can* be expanded in a series beginning with 1 and proceeding in ascending powers of x; we may therefore suppose that

$$a^x = 1 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots$$

where c_1, c_2, c_3, ... are quantities which do not depend on x, and which therefore remain unchanged however x may be changed; also

$$c_1 = a - 1 - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \frac{1}{4}(a-1)^4 + \dots$$

while c_2, c_3, ... are at present unknown; we proceed to find their values. Changing x into $x+y$ we have

$$a^{x+y} = 1 + c_1(x+y) + c_2(x+y)^2 + c_3(x+y)^3 + \dots;$$
$$\text{but } a^{x+y} = a^x a^y = a^y\left\{1 + c_1x + c_2x^2 + c_3x^3 + \dots\right\}.$$

Since the two expressions for a^{x+y} are identically equal, we may *assume* that the coefficients of x in the two expressions are equal, thus

$$c_1 + 2c_2y + 3c_3y^2 + 4c_4y^3 + \dots = c_1a^y$$
$$= c_1\left\{1 + c_1y + c_2y^2 + c_3y^3 + \dots\right\}.$$

In this identity we may *assume* the coefficients of the corresponding powers of y are equal; thus

$$2c_2 = c_1^2;\ \therefore c_2 = \frac{c_1^2}{2},$$
$$3c_3 = c_1c_2;\therefore c_3 = \frac{c_1c_2}{3} = \frac{c_1^3}{1\cdot 2\cdot 3},$$
$$4c_4 = c_1c_3;\therefore c_4 = \frac{c_1c_3}{4} = \frac{c_1^4}{1\cdot 2\cdot 3\cdot 4}.$$
$$\dots\dots\dots\dots$$
$$\text{Thus, } a^x = 1 + c_1x + \frac{c_1^2x^2}{\underline{|2}} + \frac{c_1^3x^3}{\underline{|3}} + \frac{c_1^4x^4}{\underline{|4}} + \dots$$

Since this result is true for all values of x, take x such that $c_1x = 1$, then $x = \frac{1}{c_1}$ and

$$a^{\frac{1}{c_1}} = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \dots;$$

this series is usually denoted by e; thus $a^{\frac{1}{c_1}} = e$, therefore $a = e^{c_1}$ and $c_1 = \log_e a$; hence,

$$a^x = 1 + (\log_e a)\,x + \frac{(\log_e a)^2\,x^2}{\underline{|2}} + \frac{(\log_e a)^3\,x^3}{\underline{|3}} + \dots$$

This result is called the *Exponential Theorem.*

Put e for a then $\log_e a$ becomes $\log_e e$, that is, unity, (*Art.* 135) (page 413); thus,

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \dots$$

With respect to the *assumption* which has been made twice in the course of this article, the student is referred to the chapter on *Indeterminate Coefficients* in the *Algebra.*

Art. 145. *To expand* $\log_e(1+x)$ *in a series of ascending powers of* x.

We have seen in *Art.* 143 (page 414), that $c_1 = \log_e a$; that is, by the same Article,

$$\log_e a = a - 1 - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \frac{1}{4}(a-1)^4 + \dots$$

For a put $1 + x$; hence,

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

This series may be applied to calculate $\log_e(1 + x)$ if x is a proper fraction; but unless x be very small, the terms diminish so slowly that we shall have to retain a large number of them ; if x be greater than unity, the series is altogether unsuitable. We shall therefore deduce some more convenient formulae.

Art. 146. We have

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\therefore \log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots,$$

by subtraction we obtain the value of $log_e(1+x) - \log_e(1-x)$, that is, of $\log_e \dfrac{1+x}{1-x}$;

$$\therefore \log_e \frac{1+x}{1-x} = 2\left\{x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right\}\dots$$

In this series write $\dfrac{m-n}{m+n}$ for x, and therefore $\dfrac{m}{n}$ for $\dfrac{1+x}{1-x}$, thus

$$\log_e \frac{m}{n} = 2\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \dots\right\}\dots \qquad (48)$$

Put $n = 1$, then

$$\log_e m = 2\left\{\frac{m-1}{m+1} + \frac{1}{3}\left(\frac{m-1}{m+1}\right)^3 + \frac{1}{5}\left(\frac{m-1}{m+1}\right)^5 + \dots\right\}\dots \qquad (49)$$

Again in (48) put $m = n + 1$, thus we obtain the value of $\log_e \dfrac{n+1}{n}$; therefore

$$\log_e(n+1) - \log_e n = 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \dots\right\}\dots \qquad (50)$$

Art. 148. From *Art.* 139 (page 413), we see that the logarithm of a number to the base 10 can be found by multiplying the Napierian logarithm by $\dfrac{1}{\log_e 10}$, that is, by $\dfrac{1}{2.30258509}$, or by .43429448; this multiplier is called the *modulus* of the common system.

The series in *Art.* 146 (page 415), may be so adjusted as to give common logarithms; for example, take the series (50), multiply throughout by the modulus which we shall denote by μ; thus

$$\mu \log_e(n+1) - \mu \log_e n = 2\mu \left\{ \frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \dots \right\};$$

$$\therefore \log_{10}(n+1) - \log_{10} n = 2\mu \left\{ \frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \dots \right\};$$

Similarly from *Art.* 145 (page 415), we have

$$\log_{10}(1+x) = \mu \left\{ x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \right\}.$$

Art. 150. *To find the limit of* $\left(\cos \frac{\alpha}{n}\right)^n$ *when n is increased indefinitely.*

Let $u = \left(\cos \frac{\alpha}{n}\right)^n = \left(1 - \sin^2 \frac{\alpha}{n}\right)^{\frac{n}{2}}$; then

$$\log u = \log \left(1 - \sin^2 \frac{\alpha}{n}\right)^{\frac{n}{2}} = \frac{n}{2} \log \left(1 - \sin^2 \frac{\alpha}{n}\right)$$

$$= -\frac{n}{2} \left(\sin^2 \frac{\alpha}{n} + \frac{1}{2} \sin^4 \frac{\alpha}{n} + \frac{1}{3} \sin^6 \frac{\alpha}{n} + \dots \right).$$

Now $n \sin \frac{\alpha}{n} = \alpha \frac{\sin \frac{\alpha}{n}}{\frac{\alpha}{n}} = \alpha$ when n is increased indefinitely (*Art.* 118) (page 416); therefore $n \sin^2 \frac{\alpha}{n} = \alpha \sin \frac{\alpha}{n} = 0$ ultimately; and similarly $n \sin^4 \frac{\alpha}{n}$, $n \sin^6 \frac{\alpha}{n}, \dots$ vanish ultimately. Therefore $\log u = 0$; therefore $u = 1$. Thus the required limit is unity.

Art. 181. *To prove that in general the change of the sine of an angle is approximately proportional to the change of the angle.*

We have
$$\sin(\theta + h) - \sin \theta = \sin h \cos \theta - \sin \theta (1 - \cos h)$$

$$= \sin h \cos \theta \left(1 - \tan \theta \frac{1 - \cos h}{\sin h}\right)$$

$$= \sin h \cos \theta \left(1 - \tan \theta \tan \frac{h}{2}\right).$$

Let us now suppose that h is the circular measure of a very small angle so that $\sin h = h$ approximately; thus, approximately,

$$\sin(\theta + h) - \sin \theta = h \cos \theta \left(1 - \tan \theta \tan \frac{h}{2}\right);$$

Let us also suppose that θ is not very nearly equal to $\frac{\pi}{2}$ so that $\tan \theta$ is not very large, and thus $\tan \theta \tan \frac{h}{2}$ may be neglected.

We have then, approximately,

$$\sin(\theta + h) - \sin \theta = h \cos \theta,$$

and this establishes the proposition.

Similarly
$$\sin(\theta - h) - \sin \theta = -h \cos \theta \text{ approximately.}$$

Art. 188. *To prove that in general the change of the tangent of an angle is approx-*

imately proportional to the change of the angle.

We have

$$\tan(\theta+h)-\tan\theta=\frac{\sin(\theta+h)}{\cos(\theta+h)}-\frac{\sin\theta}{\cos\theta}$$

$$=\frac{\sin(\theta+h)\cos\theta-\cos(\theta+h)\sin\theta}{\cos(\theta+h)\cos\theta}=\frac{\sin(\theta+h-\theta)}{\cos(\theta+h)\cos\theta}=\frac{\sin h}{\cos(\theta+h)\cos\theta}$$

$$=\frac{\sin h}{\cos^2\theta(\cos h-\sin h\tan\theta)}=\frac{\tan h}{\cos^2\theta(1-\tan\theta\tan h)}.$$

Let us now suppose that h is so small that we may put h for $\tan h$, and also that θ is not nearly equal to $\frac{\pi}{2}$ so that $\tan\theta\tan h$ may be neglected. We have then, approximately,

$$\tan(\theta+h)-\tan\theta=\frac{h}{\cos^2\theta}=h\sec^2\theta,$$

also by changing the sign of h

$$\tan(\theta-h)-\tan\theta=-h\sec^2\theta;$$

this establishes the proposition.

Art. 194.

We have shown that approximately

$$\sec(\theta+h)-\sec\theta=h\sin\theta\sec^2\theta;$$

change θ into $\frac{\pi}{2}-\theta'$, thus

$$\sec\left(\frac{\pi}{2}-\theta'+h\right)-\sec\left(\frac{\pi}{2}-\theta'\right)=h\sin\left(\frac{\pi}{2}-\theta'\right)\sec^2\left(\frac{\pi}{2}-\theta'\right),$$

that is
$$\operatorname{cosec}(\theta'-h)-\operatorname{cosec}\theta'=h\cos\theta'\operatorname{cosec}^2\theta',$$

and by changing the sign of h

$$\operatorname{cosec}(\theta'+h)-\operatorname{cosec}\theta'=-h\cos\theta'\operatorname{cosec}^2\theta'.$$

This may also be proved independently.

Art. 196. *To prove that in general the change of the tabular logarithmic sine of an angle is approximately proportional to the change of the angle.*

We have approximately

$$\sin(\theta+h)=\sin\theta+h\cos\theta,$$

$$\therefore\frac{\sin(\theta+h)}{\sin\theta}=1+h\cot\theta;$$

$$\therefore\log\sin(\theta+h)-\log\sin\theta=\log\frac{\sin(\theta+h)}{\sin\theta}=\log(1+h\cot\theta),\ \text{and}$$

$\log(1+h\cot\theta)=\mu h\cot\theta$ approximately (*Art.* 148) (page 415),

where μ is the *modulus*; thus approximately

$$\log\sin(\theta+h)-\log\sin\theta=\mu h\cot\theta,$$

also by changing the sign of h,

$$\log\sin(\theta-h)-\log\sin\theta=-\mu h\cot\theta.$$

If L stand for *tabular* logarithm, we have

$$L\sin(\theta+h)=10+\log\sin(\theta+h),$$

$$L\sin\theta=10+\log\sin\theta;$$

$$\therefore L\sin(\theta\pm h)-L\sin\theta=\pm\mu h\cot\theta.$$

This establishes the proposition.

Art. 208. If we have to determine an angle from its natural sine or cosine it will be advisable to employ the natural sine if the angle be less than 45°, and the natural cosine if the angle be greater than 45°. For the differences of consecutive sines vary approximately as the cosine of the angle, and the differences of consecutive cosines vary approximately as the sine of the angle; thus the differences of consecutive sines are greater or less than the differences of consecutive cosines according as the angle is less or greater than 45°. A similar remark holds for the logarithmic sine and cosine.

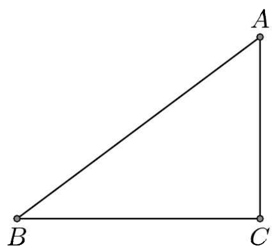
Art. 212. *In a right-angled triangle each side is equal to the product of the hypotenuse into the cosine of the adjacent angle.*

Let ABC be a triangle having a right angle at C; then

$$\frac{AC}{AB} = \cos A, \ \frac{BC}{AB} = \cos B;$$

$$\therefore b = c\cos A, \ a = c\cos B.$$

Since $\cos A = \sin B$ and $\cos B = \sin A$, we may also enunciate the proposition thus : *in a right-angled triangle each side is equal to the product of the hypotenuse into the sine of the opposite angle.*



Art. 214. *In any triangle the sides are proportional to the sines of the opposite angles.*

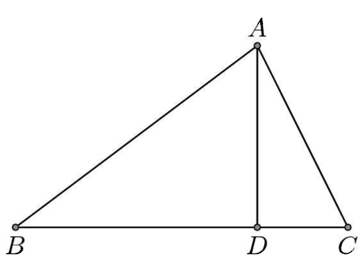
Let ABC be any triangle, and from A draw AD perpendicular to the opposite side meeting that side, or that side produced, at D.

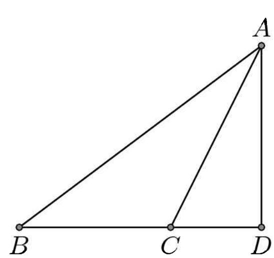
If B and C are *acute* angles we have from the left-hand figure,

$$AD = AB\sin B, \text{ and } AD = AC\sin C;$$

$$\therefore AB\sin B = AC\sin C,$$

$$\therefore \frac{c}{b} = \frac{\sin C}{\sin B}.$$





If the angle C be *obtuse* we have from the right-handed figure,

$$AD = AB\sin B, \text{ and } AD = AC\sin ACD = AC\sin(180° - C) = AC\sin C;$$

$$\therefore AB\sin B = AC\sin C,$$

$$\therefore \frac{c}{b} = \frac{\sin C}{\sin B}.$$

If the angle C be a *right angle*, we have from the figure of *Art.* 212 (page 418),

$$AC = AB \sin B,$$

$$\therefore \frac{c}{b} = \frac{1}{\sin B} = \frac{\sin C}{\sin B}.$$

Thus it is proved that in every case

$$\frac{c}{b} = \frac{\sin C}{\sin B}.$$

Similarly

$$\frac{a}{b} = \frac{\sin A}{\sin B}, \text{ and } \frac{a}{c} = \frac{\sin A}{\sin C}.$$

The results may be written symmetrically thus,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c};$$

and we shall show hereafter that each of these is equal to $\frac{1}{2R}$, where R is the radius of the circle described round the triangle.

Art. 215. *To express the cosine of an angle of a triangle in terms of the sides.*

Let ABC be a triangle, and suppose C an *acute* angle. (See the left-hand figure of the preceding Article.) Then by Euclid II. 13,

$$AB^2 = BC^2 + AC^2 - 2BC \cdot CD, \text{ and}$$

$$CD = AC \cos C;$$

$$\therefore c^2 = a^2 + b^2 - 2ab \cos C.$$

Next suppose C an *obtuse* angle. (See the right-hand figure of the preceding Article.) Then by Euclid II. 12,

$$AB^2 = BC^2 + AC^2 + 2BC \cdot CD, \text{ and}$$

$$CD = AC \cos(180° - C) = -AC \cos C,$$

$$\therefore c^2 = a^2 + b^2 - 2ab \cos C.$$

Thus in both cases we have $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$.

Moreover when C is a right angle, $a^2 + b^2 = c^2$ and $\cos C$ is zero; thus the formula just found for $\cos C$ is true in every case.

Similarly

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \ \cos B = \frac{c^2 + a^2 - b^2}{2ca}.$$

Art. 216. *In every triangle each side is equal to the sum of the product of each of the other sides into the cosine of the angle which it makes with the first side.*

From the left-hand figure in *Art.* 214 (page 418), we have

$$BC = BD + DC = AB \cos B + AC \cos C,$$

$$\therefore a = c \cos B + b \cos C.$$

From the right-hand figure in *Art.* 214 (page 418), we have

$$BC = BD - DC = AB \cos B - AC \cos(180° - C),$$

$$= AB \cos B + AC \cos C,$$

$$\therefore a = c \cos B + b \cos C.$$

Similarly we shall have $b = a \cos C + c \cos A$, and $c = b \cos A + a \cos B$.

Art. 217. *To express the sine, the cosine, and the tangent of half an angle of a* *triangle of the sides.*

We have by *Art.* 215 (page 419),

$$\cos A = \frac{b^2+c^2-a^2}{2bc},$$

$$\therefore 1-\cos A = 1-\frac{b^2+c^2-a^2}{2bc} = \frac{a^2-(b-c)^2}{2bc};$$

$$\therefore \sin^2\frac{A}{2} = \frac{(a+b-c)(a+c-b)}{4bc}.$$

Let $2s = a+b+c$, so that s is half the sum of the sides of the triangle; then

$$a+b-c = a+b+c-2c = 2(s-c),$$
$$a+c-b = a+b+c-2b = 2(s-b).$$

$$\therefore \sin^2\frac{A}{2} = \frac{(s-b)(s-c)}{bc}, \therefore \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

Also

$$1+\cos A = 1+\frac{b^2+c^2-a^2}{2bc} = \frac{(b+c)^2-a^2}{2bc};$$

$$\therefore \cos^2\frac{A}{2} = \frac{(a+b+c)(b+c-a)}{4bc} = \frac{s(s-a)}{bc}, \text{ and}$$

$$\therefore \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}.$$

From the values of $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ we deduce

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

The positive sign must be given to the radicals which occur in this Article, because $\frac{A}{2}$ is less than a right angle, and therefore its sine, cosine, and tangent are all positive.

Similar expressions hold for the sine, the cosine, and the tangents of half of each of the other angles.

Art. 218. *To express the sine of an angle of a triangle in terms of the sides.*

Since $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$, we obtain

$$\sin A = 2\sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\sqrt{\frac{s(s-a)}{bc}}$$
$$= \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

Or we may find $\sin A$ directly from the known value of $\cos A$;

$$\therefore \sin^2 A = 1-\frac{\left(b^2+c^2-a^2\right)^2}{4b^2c^2} = \frac{2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4}{4b^2c^2};$$

$$\therefore \sin A = \frac{\sqrt{2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4}}{2bc};$$

the former expression may be shown to agree with this by forming the product of the factors s, $s-a$, $s-b$, and $s-c$.

Similar expressions hold for $\sin B$ and $\sin C$.

Art. 228. *To solve a triangle having given two angles and a side.*

Suppose A and C the given angles, and b the given side;

then $$B = 180^\circ - A - C;$$

$$\frac{a}{b} = \frac{\sin A}{\sin B}, \quad \text{therefore } a = \frac{b \sin A}{\sin B},$$

therefore $\log a = \log b + \log \sin A - \log \sin B = \log b + L \sin A - L \sin B$;

similarly $\log c = \log b + L \sin C - L \sin B$.

Thus B, a, and c are determined.

If A and B are the given angles then

$$C = 180^\circ - B - A,$$

and we may proceed as before to find a and c.

Art. 229. *To solve a triangle having given two sides and the included angle.*

Suppose b and c the given sides and A the included angle.

We have $\dfrac{\sin B}{\sin C} = \dfrac{b}{c}$; therefore $\dfrac{\sin(A+C)}{\sin C} = \dfrac{b}{c}$;

therefore $\dfrac{\sin A \cos C + \cos A \sin C}{\sin C} = \dfrac{b}{c}$; that is $\sin A \cot C + \cos A = \dfrac{b}{c}$:

thus $\cot C$ is determined, and therefore C can be found; and then B.

But as this process is not adapted to logarithmic computation another is usually given :

we have $$\frac{\sin B}{\sin C} = \frac{b}{c}, \text{ therefore } \frac{\sin B - \sin C}{\sin B + \sin C} = \frac{b-c}{b+c},$$

therefore $$\frac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)} = \frac{b-c}{b+c}, (\textit{Art. } 88) \text{ (page 406)},$$

and $$\tan \frac{1}{2}(B+C) = \tan \frac{1}{2}(180^\circ - A) = \cot \frac{A}{2},$$

therefore $$\tan \frac{1}{2}(B-C) = \frac{b-c}{b+c} \cot \frac{A}{2},$$

therefore $$\log \tan \frac{1}{2}(B-C) = \log(b-c) - \log(b+c) + \log \cot \frac{A}{2},$$

therefore $$L \tan \frac{1}{2}(B-C) = \log(b-c) - \log(b+c) + L \cot \frac{A}{2};$$

this formula determines $\frac{1}{2}(B-C)$; and $\frac{1}{2}(B+C)$ is known since it is $90^\circ - \dfrac{A}{2}$; thus B and C can be immediately found.

Also $\dfrac{a}{c} = \dfrac{\sin A}{\sin C}$, from which a can be found.

We have supposed b and c unequal; if however $b = c$ then $B = C$, and all the angles will be known, so that a can be found as in *Art.* 228 (page 420).

Art. 233. *To solve a triangle having given two sides and the angle opposite to one of them.*

Let a and b be the given sides, and A the given angle;

then $$\frac{\sin B}{\sin A} = \frac{b}{a}; \quad \text{therefore } \sin B = \frac{b}{a} \sin A;$$

therefore $$L \sin B = \log b - \log a + L \sin A.$$

If $\dfrac{b \sin A}{a}$ is less than unity, two different angles may be found less than 180° which have $\dfrac{b\sin A}{a}$ for sine, one of these angles being less than a right angle, and the other greater. If a be greater than b, then A must be greater than B, and therefore B must be an *acute* angle; thus only the smaller value is admissible for B. If a be less than b, then either value may be taken for B. When B is determined, C is known since it is $180° - A - B$, and then c can be found from

$$\frac{c}{a} = \frac{\sin C}{\sin A}.$$

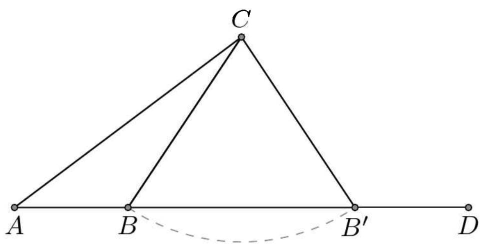
Thus if two values are admissible for B we obtain two corresponding values for C and c, so that *two* triangles can be found from the given parts.

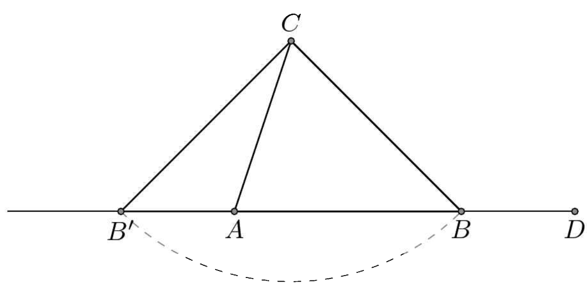
If $\dfrac{b\sin A}{a} = 1$, then B is a right angle, so that only one triangle can be found from the given parts.

If $\dfrac{b\sin A}{a}$ is greater than unity, no triangle exists with the given parts.

Thus, when two sides are given and the angle opposite the less we can *generally* find two triangles from the given parts, and this case in the solution of triangles is therefore called the *ambiguous case.* We say that two triangles can be *generally* found in order to have regard to the exceptions; for the triangle may be *right angled,* and then only one triangle can be found, or the triangle may be *impossible.*

Art. 234. The *ambiguous case* may be illustrated by figures.





Let CAD be the given angle A, and AC the given side b; suppose a circle described from C as a centre with radius equal to a. The perpendicular from C on AD is equal to $b\sin A$; therefore if a be greater than $b\sin A$, the circle will meet the straight line AD at two points, which we will denote by B and B'. If a be less than b, then B and B' are on the same side of A, as in the first figure; thus two triangles, namely ABC and $AB'C$, can be obtained, each having the given parts a, b, A. If a be greater than b, then B' and B are on opposite sides of A, as in the second figure; thus only one triangle, namely CAB, can be obtained having the given parts a, b, A; the triangle CAB' has an angle CAB' which is $180° - A$ instead of A.

If a be *equal* to $b \sin A$, the circle *touches* the straight line AD, and the two points B and B' in the first figure coincide; thus one triangle in obtained which has a right angle at B.

If a be *less* than $b \sin A$ the circle does not meet the straight line AD, and no triangle exists with the given parts a, b, A.

Art. 238. We have seen in *Chapter* XII. that the Tables of trigonometrical functions cannot always be used with advantage; this circumstance guides us in selecting the method of solution of a triangle to be adopted when more than one method is theoretically applicable, and leads us to modify the method of solution in some cases. For example, suppose we have to find A from the equation $\sin A = n$, where n is nearly equal to unity; this is an inconvenient equation for determining A, because the difference of consecutive sines is nearly insensible when the angles are nearly right angles. We have however

$$\sin\left(45^\circ - \frac{A}{2}\right) = \sqrt{\frac{1-\cos(90^\circ - A)}{2}}$$
$$= \sqrt{\frac{1-\sin A}{2}} = \sqrt{\frac{1-n}{2}};$$

and this formula is free from the objection.

Similarly if we have to find A from the equation

$$\cos A = n,$$

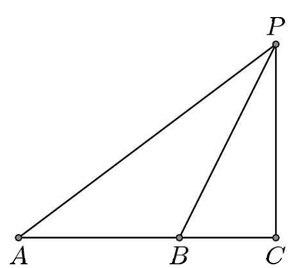
where n is nearly equal to unity, we may advantageously transform the equation thus

$$\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}} = \sqrt{\frac{1-n}{2}};$$

or thus
$$\frac{1-\cos A}{1+\cos A} = \frac{1-n}{1+n};$$

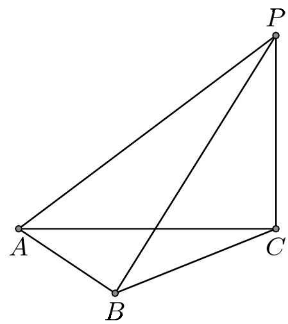
therefore
$$\tan\frac{A}{2} = \sqrt{\frac{1-n}{1+n}}.$$

Art. 240. *To find the height and the distance of an inaccessible object on a horizontal plane.*



Let P be the top of an object, and let it be required to find its height PC, and the distance of the object from a point A in the horizontal plane through C. At A observe the angle PAC; then measure any length AB directly towards the object, and at B observe the angle PBC. Then in the triangle APB the side AB is known, and the angle PAB; also the angle PBA is known, since it is the supplement of PBC;

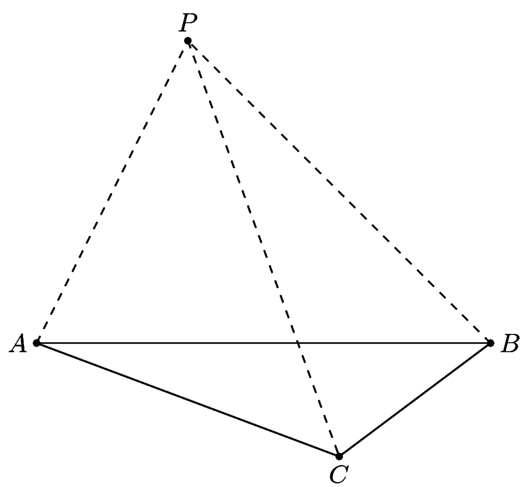
therefore AP can be found. Then $PC = AP \sin PAC$, and $AC = AP \cos PAC$; thus the height PC and the distance AC are determined.



If however it is not convenient to measure the length AB directly towards the object, we may proceed thus; measure the length AB in *any* direction from A; at A observe the angles PAC and PAB, and at B observe the angle PBA. Then in the triangle APB the side AB and the angles PAB and PBA are known; therefore AP can be found. Then, as before, $PC = AP \sin PAC$, and $AC = AP \cos PAC$.

Art. 242. *The lengths of the straight lines which join three points A, B, C, are known; at any point P in the same plane as A, B, C, the angles APC and BPC are observed: it is required to find the distance of P from each of the points A, B, C.*

Let the angle APC be denoted by α, the angle BPC by β, the angle PAC by x, and the angle PBC by y; then α and β are



known, and when x and y are found the required distances PA, PB, PC can be found; for in each of the triangles PAC and PBC two angles and a side will then be known. We will show how x and y may be found.

Since the angles of the triangles APC and BPC are together equal to four right angles, we have

$$x + y = 2\pi - \alpha - \beta - C;$$

thus the *sum* of x and y is known.

From the triangle ACP we have

$$PC = \frac{AC \sin PAC}{\sin PAC} = \frac{b \sin x}{\sin \alpha};$$

from the triangle BCP we have

$$PC = \frac{BC \sin PBC}{\sin BPC} = \frac{a \sin y}{\sin \beta};$$

therefore $$\frac{b \sin x}{\sin \alpha} = \frac{a \sin y}{\sin \beta};$$

therefore $$\frac{\sin x}{\sin y} = \frac{a \sin \alpha}{b \sin \beta}.$$

Now assume $\tan \phi = \dfrac{a \sin \alpha}{b \sin \beta}$, then the value of ϕ can be found from the Trigonometrical Tables; thus

$$\frac{\sin x}{\sin y} = \tan \phi;$$

therefore $$\frac{\sin x - \sin y}{\sin x + \sin y} = \frac{\tan \phi - 1}{\tan \phi + 1} = \tan\left(\phi - \frac{\pi}{4}\right);$$

$\therefore$ (*Art.* 88 : (page 406)) $$\frac{\tan \frac{1}{2}(x-y)}{\tan \frac{1}{2}(x+y)} = \tan\left(\phi - \frac{\pi}{4}\right);$$

from the last equation we can determine $x - y$, since $x + y$ is known; thus x and y can be found.

Art. 247. *To find expressions for the area of a triangle.*

A triangle is half a rectangle on the same base and altitude; thus if ABC be any triangle, and AD the perpendicular from A on the opposite side, we have (see the figures in *Art.* 214 : page 418)

$$\text{area of triangle} = \frac{1}{2} BC \cdot AD, \text{ and}$$
$$AD = AB \sin B,$$
$$\therefore \text{ area of triangle} = \frac{1}{2} ac \sin B \qquad (51)$$

thus the area of a triangle is half the product of two sides into the sine of the included angle.

By *Art.* 218 (page 420),

$$\sin B = \frac{2}{ac} \sqrt{\{s(s-a)(s-b)(s-c)\}};$$

substitute the value of $\sin B$ in (51) and we obtain

$$\text{area of triangle} = \sqrt{\{s(s-a)(s-b)(s-c)\}} \qquad (52)$$

this furnishes a convenient expression for the area when all the sides are known; the expression $\sqrt{\{s(s-a)(s-b)(s-c)\}}$ is often for abbreviation denoted by S.

By *Art.* 214 (page 418), $a = \dfrac{b \sin A}{\sin B}$, $c = \dfrac{b \sin C}{\sin B}$,

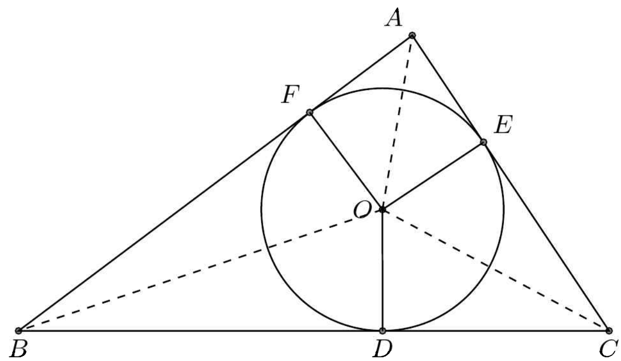
substitute these values in (51); thus we obtain

$$\text{area of triangle} = \frac{b^2 \sin A \sin C}{2 \sin B} \qquad (53)$$

thus we can find the area when a side and two angles are given, *for if two angles* are given the third angle is also known.

Art. 248. *To find the radius of the circle inscribed in a triangle.*

Let ABC be a triangle, O the centre of the circle inscribed in the triangle and touching the sides at the points D, E, F. Join OD, OE, and OF. The angles at D, E, and F are right angles by Euclid III. 18. Let r denote the radius of the circle; then



$$\text{area of triangle } BOC = \frac{1}{2} BC \cdot OD = \frac{ar}{2},$$
$$\text{area of triangle } COA = \frac{1}{2} CA \cdot OE = \frac{br}{2},$$
$$\text{area of triangle } AOB = \frac{!}{2} AB \cdot OF = \frac{cr}{2};$$

therefore, by addition,

$$(a+b+c)\frac{r}{2} = \text{ area of triangle } ABC = S, \qquad (Art.\ 247)\ (page\ 425),$$

that is $rs = S$, therefore $r = \dfrac{S}{s}$.

The radius of the inscribed circle is thus equal to the area of the triangle divided by half the sum of the sides; and hence different forms can be obtained for the radius by employing the different expressions already given for the area of the triangle.

It is easy to show by Geometry that

$$AE = AF = s - a, \quad BF = BD = s - b, \quad CD = CE = s - c.$$

Art. 249. We may also obtain the value of r in another form, which will be often useful.

By *Euclid* IV. 4, the straight lines OA, OB, OC bisect the angles A, B, C respectively. Thus

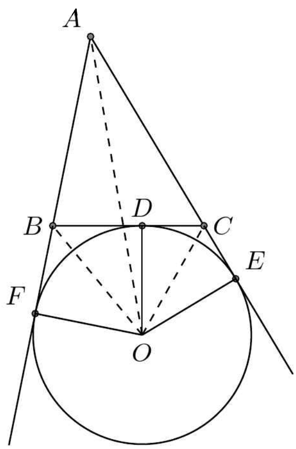
$$BD = r\cot\frac{B}{2}, \quad CD = r\cot\frac{C}{2}, \quad \therefore\ r\left(\cot\frac{B}{2} + \cot\frac{C}{2}\right) = a,$$

$$\therefore\ r\sin\frac{B+C}{2} = a\sin\frac{B}{2}\sin\frac{C}{2}, \quad \therefore\ r = \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}.$$

Or thus : $$r = AE\tan\frac{A}{2} = (s-a)\tan\frac{A}{2}.$$

Art. 250. *To find the radius of the circle which touches one side of a triangle and the other sides produced.*

Let ABC be a triangle, and let O be the centre of the circle which touches the side BC, and the other sides produced at the points D, E, F. Join OD, OE, and OF. The angles at D, E, and F are right angles by Euclid III. 18. Let r_1 denote the radius of the circle.



The quadrilateral $OBAC$ may be divided into the two triangles OAB, OAC; therefore the area of this quadrilateral is $\frac{c}{2}r_1 + \frac{b}{2}r_1$.

Again, the same quadrilateral may be divided into the triangles OBC and ABC; therefore the area of this quadrilateral is $\frac{a}{2}r_1 + S$. Thus

$$\frac{c}{2}r_1 + \frac{b}{2}r_1 = \frac{a}{2}r_1 + S;$$

$$\therefore (c+b-a)\frac{r_1}{2} = S,$$

$$\therefore r_1(s-a) = S,$$

$$\therefore r_1 = \frac{S}{s-a}.$$

It is easy to show by Geometry that

$$AF = AE = s, \quad BD = BF = s-c, \quad CD = CE = s-b.$$

The centre of the inscribed circle is also on AO, and the distance between it and O subtends a right angle at B and at C.

Similarly, if r_2 be the radius of the circle which touches CA and the other sides produced, and r_3 the radius of the circle which touches AB and the other sides produced,

$$r_2 = \frac{S}{s-b}, \quad r_3 = \frac{S}{s-c}.$$

A circle which touches one side of a triangle and the other sides produced is called an *escribed* circle.

Art. 251. We may also obtain an expression for the radius of an escribed circle similar to that in *Art.* 249 (page 426) for the radius of the inscribed circle.

For, in the figure of *Art.* 250 (page 427), the straight line OB bisects the angle which is the supplement of B, and the straight line OC bisects the angle which is the supplement of C. Thus

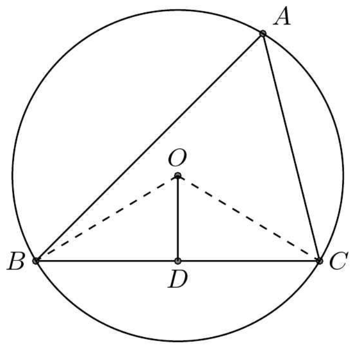
$$BD = r_1\cot\left(90^\circ - \frac{B}{2}\right), \quad CD = r_1\cot\left(90^\circ - \frac{C}{2}\right);$$

$$\therefore r_1\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right) = a;$$

$$\therefore r_1 = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\sin\frac{B+C}{2}} = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}.$$

Or thus: $$r_1 = AF\tan\frac{A}{2} = s\tan\frac{A}{2}.$$

Art. 252. *To find the radius of the circle described round a triangle.*



Let ABC be a triangle, and O the centre of the circle described round it. Draw OD perpendicular to BC, then BC is bisected at D by Euclid III. 3. Let R denote the radius of the circle.

The angle BOC is double the angle BAC, by Euclid III. 20;

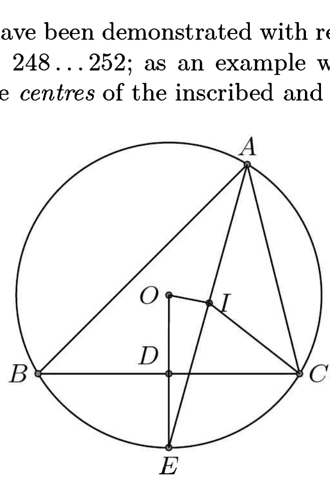
therefore $$BOD = A;$$

and $$BD = R\sin A = \frac{a}{2}; \quad \therefore R = \frac{a}{2\sin A};$$

thus R is expressed in terms of a side and the opposite angle.

By *Art.* 218 (page 420), $\sin A = \frac{2S}{bc}$, $\quad \therefore R = \frac{abc}{4S}$.

Art. 253. Many theorems have been demonstrated with respect to the circles which have been noticed in *Arts.* 248 . . . 252; as an example we will find an expression for the distance between the *centres* of the inscribed and circumscribed circles of a triangle.



Let ABC be a triangle, let O be the centre of the circumscribed circle. From O draw a perpendicular OD on BC, and produce it to meet the circumference of the circle at E. Then the arc BE is equal to the arc CE; and therefore the straight line

AE bisects the angle BAC. Thus the centre of the inscribed circle will be on AE; let the point I denote it. Join OI and IC.

The angle $EIC = \frac{1}{2}(A+C)$ by *Euclid* I. 32; and the angle $ECI = ECB+BCI = \frac{1}{2}(A+C)$: therefore the angle $EIC =$ the angle ECI : and therefore $EI = EC$.

And $$EC = 2R\sin\frac{A}{2}; \quad \text{therefore } EI = 2R\sin\frac{A}{2}.$$

Hence $$EI \times IA = 2R\sin\frac{A}{2} \times \frac{r}{\sin\frac{A}{2}} = 2Rr.$$

And $$(R - OI)(R + OI) = 2Rr, \text{ by } \textit{Euclid} \text{ III. 35};$$

therefore $$OI^2 = R^2 - 2Rr.$$

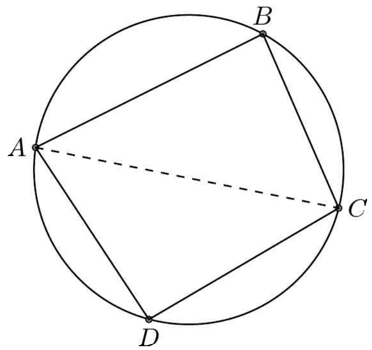
If we suppose IE produced through E to a point J such that $EJ = EI$, the point J will be the centre of the escribed circle which is opposite the angle A; and we shall have

$$OJ^2 = R^2 + 2Rr.$$

Art. 254. *To find the area of a quadrilateral which can be inscribed in a circle.*

Let $ABCD$ be the quadrilateral; let

$$AB = a, \quad BC = b, \quad CD = c, \quad DA = d.$$



The figure can be divided into the triangles ABC, ADC; its area therefore

$$= \frac{1}{2}(ab\sin B + cd\sin D) = \frac{1}{2}(ab + cd)\sin B,$$

for the angles B and D are supplemental by *Euclid* III. 22.

Now from the triangle ABC,

$$AC^2 = a^2 + b^2 - 2ab\cos B,$$

and from the triangle CDA,

$$AC^2 = c^2 + d^2 - 2cd\cos D = c^2 + d^2 + 2cd\cos B;$$

therefore $$c^2 + d^2 + 2cd\cos B = a^2 + b^2 - 2ab\cos B,$$

therefore $$\cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)};$$

$$\therefore \quad \sin^2 B = 1 - \frac{(a^2 + b^2 - c^2 - d^2)^2}{4(ab + cd)^2}$$

$$= \frac{\left\{2(ab + cd) + c^2 + d^2 - a^2 - b^2\right\}\left\{2(ab + cd) - c^2 - d^2 + a^2 + b^2\right\}}{4(ab + cd)^2}$$

$$= \frac{\left\{(c + d)^2 - (a - b)^2\right\}\left\{(a + b)^2 - (c - d)^2\right\}}{4(ab + cd)^2}$$

$$= \frac{(c+b+d-a)(a+c+d-b)(a+b+d-c)(a+b+c-d)}{4(ab+cd)^2}$$

Now let $\frac{1}{2}(a+b+c+d) = s$; thus

$$\sin^2 B = \frac{16(s-a)(s-b)(s-c)(s-d)}{4(ab+cd)^2}.$$

Hence the area of the quadrilateral

$$= \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

If we substitute the value of $\cos B$ in the expression for AC^2, we obtain

$$AC^2 = c^2 + d^2 + \frac{2cd(a^2+b^2-c^2-d^2)}{2(ab+cd)}$$

$$= c^2 + d^2 + \frac{cd(a^2+b^2-c^2-d^2)}{ab+cd}$$

$$= \frac{(ac+bd)(ad+bc)}{ab+cd}.$$

Similarly it may be shown that

$$\cos A = \frac{a^2+d^2-b^2-c^2}{2(ad+bc)},$$

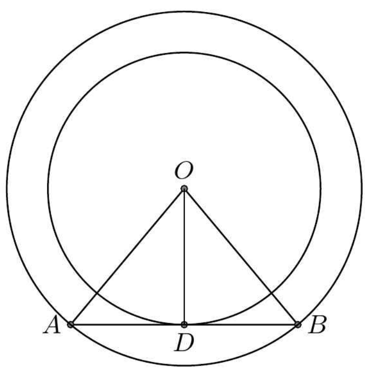
$$BD^2 = \frac{(ac+bd)(ab+cd)}{ad+bc}.$$

The radius of the circle described round the quadrilateral may be easily expressed; for this circle passes round the triangle ABC, hence by *Art.* 252 (page 428), its radius

$$= \frac{AC}{2\sin B} = \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}.$$

Art. 255. *To find the radii of the inscribed and circumscribed circles of a regular polygon, that is of a polygon which has all its sides equal and all its angles equal.*

Let AB be the side of a regular polygon of n sides; let O be the centre of the circles, OD the radius of the inscribed circle, OA the radius of the circumscribed circle.



Let $AB = a$, $OA = R$, $OD = r$.

The angle AOB is the n^{th} part of 4 right angles, that is,

$$AOB = \frac{2\pi}{n}, \quad AOD = \frac{\pi}{n}.$$

$$AD = \frac{a}{2} = R\sin\frac{\pi}{n} = r\tan\frac{\pi}{n};$$

therefore $$R = \frac{a}{2\sin\frac{\pi}{n}}, \quad r = \frac{a}{2\tan\frac{\pi}{n}}.$$

Art. 258. *To find the area of a sector of a circle.*

Let θ be the circular measure of the angle of the sector; then
$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\theta}{2\pi}, (\textit{Euclid}\ \text{VI. 33});$$
therefore the area of the sector $= \pi r^2 \times \dfrac{\theta}{2\pi} = \dfrac{r^2\theta}{2}$.

Since θ is the circular measure of the angle of the sector, the length of the arc of the sector is $r\theta$; hence the area of a sector is equal to half the product of the length of the arc into the radius.

The area of a *segment* of a circle can now be found. For a segment of a circle which is less than a semicircle is equal to the difference between a sector and a triangle; so that if θ be the circular measure of the angle of the sector the area of the *segment* is $\dfrac{r^2}{2}(\theta - \sin\theta)$. A segment of a circle which is greater than a semicircle is equal to the difference between the circle, and a segment less than a semicircle.

Art. 267. ***De Moivre's Theorem :*** *Whatever be the value of n positive or negative, integral or fractional,* $\cos n\theta + \sqrt{-1}\sin n\theta$ *is one of the values of* $\left\{\cos\theta + \sqrt{-1}\sin\theta\right\}^n$.

Multiply $\cos\alpha + \sqrt{-1}\sin\alpha$ by $\cos\beta + \sqrt{-1}\sin\beta$;
the product is
$$\cos\alpha\cos\beta - \sin\alpha\sin\beta + \sqrt{-1}\left\{\sin\alpha\cos\beta + \cos\alpha\sin\beta\right\},$$
that is, $$\cos(\alpha+\beta) + \sqrt{-1}\sin(\alpha+\beta);$$
multiply the last expression by $\cos\gamma + \sqrt{-1}\sin\gamma$; the product is
$$\cos(\alpha+\beta+\gamma) + \sqrt{-1}\sin(\alpha+\beta+\gamma).$$
By proceeding in this way we obtain the product of any number of factors of the form $\cos\alpha + \sqrt{-1}\sin\alpha$. Suppose there are n of these factors, each factor being $\cos\theta + \sqrt{-1}\sin\theta$; we then have
$$\left\{\cos\theta + \sqrt{-1}\sin\theta\right\}^n = \cos n\theta + \sqrt{-1}\sin n\theta.$$
This proves De Moivre's theorem when n is a *positive integer.*

Next, let n be a *negative integer;* suppose $n = -m$, then
$$\left\{\cos\theta + \sqrt{-1}\sin\theta\right\}^n = \left\{\cos\theta + \sqrt{-1}\sin\theta\right\}^{-m}$$
$$= \frac{1}{\left\{\cos\theta + \sqrt{-1}\sin\theta\right\}^m} = \frac{1}{\cos m\theta + \sqrt{-1}\sin m\theta};$$
multiply both numerator and denominator by
$$\cos m\theta - \sqrt{-1}\sin m\theta,$$
thus we obtain $$\frac{\cos m\theta - \sqrt{-1}\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta};$$
that is $$\cos m\theta - \sqrt{-1}\sin m\theta;$$
that is $$\cos(-m\theta) + \sqrt{-1}\sin(-m\theta),$$
or $$\cos n\theta + \sqrt{-1}\sin n\theta.$$

This proves De Moivre's theorem when n is a *negative integer.*

Thus, since when n is any integer,
$$\left\{\cos\theta + \sqrt{-1}\sin\theta\right\}^n = \cos n\theta + \sqrt{-1}\sin n\theta,$$

it follows that $\cos\theta + \sqrt{-1}\sin\theta$ is one of the values of $\left\{\cos n\theta + \sqrt{-1}\sin n\theta\right\}^{\frac{1}{n}}$, when n is any integer.

Lastly, let n be a fraction; suppose $n = \dfrac{p}{q}$, then

$$\left\{\cos\theta + \sqrt{-1}\sin\theta\right\}^n = \left\{\cos\theta + \sqrt{-1}\sin\theta\right\}^{\frac{p}{q}}$$
$$= \left\{\cos p\theta + \sqrt{-1}\sin p\theta\right\}^{\frac{1}{q}},$$

and, by what has just been shown, one of the values of the last expression is

$$\cos\frac{p\theta}{q} + \sqrt{-1}\sin\frac{p\theta}{q}.$$

Thus De Moivre's theorem is completely established.

Art. 270. *We proceed to deduce some important results from De Moivre's theorem.* In the equation

$$\cos n\theta + \sqrt{-1}\sin n\theta = \left\{\cos\theta + \sqrt{-1}\sin\theta\right\}^n,$$

suppose n a positive integer. Expand the right-hand member by the Binomial Theorem, and equate the possible and impossible parts of the two members; thus

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{1\cdot 2}\cos^{n-2}\theta\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{\underline{|4}}\cos^{n-4}\theta\sin^4\theta - \ldots$$

$$\sin n\theta = n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{\underline{|3}}\cos^{n-3}\theta\sin^3\theta + \frac{n(n-1)(n-2)(n-3)(n-4)}{\underline{|5}}\cos^{n-5}\theta\sin^5\theta - \ldots$$

Art. 274. *We shall now prove formulae for the expansion of* $\sin\alpha$ *and* $\cos\alpha$ *in series of powers of* α.

We have, when n is a positive integer,

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{1\cdot 2}\cos^{n-2}\theta\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{\underline{|4}}\cos^{n-4}\theta\sin^4\theta - \ldots.$$

Let $n\theta = \alpha$; and suppose n to increase without limit, and let θ so change that n may remain a positive integer and $n\theta$ be always equal to α; thus θ must diminish without limit. The preceding equation may be written

$$\cos\alpha = \cos^n\theta - \frac{\alpha(\alpha-\theta)}{1\cdot 2}\cos^{n-2}\theta\left(\frac{\sin\theta}{\theta}\right)^2 + \frac{\alpha(\alpha-\theta)(\alpha-2\theta)(\alpha-3\theta)}{\underline{|4}}\cos^{n-4}\theta\left(\frac{\sin\theta}{\theta}\right)^4 - \ldots.$$

Now when n increases without limit, and, therefore, θ diminishes without limit, $\dfrac{\sin\theta}{\theta}$ is equal to unity, and so is every power of $\dfrac{\sin\theta}{\theta}$ up to $\left(\dfrac{\sin\theta}{\theta}\right)^n$; also $\cos\theta$ is unity and so is every power of $\cos\theta$ up to $\cos^n\theta$ (*Art.* 150 : page 416). Hence the above formula becomes

$$\cos\alpha = 1 - \frac{\alpha^2}{1\cdot 2} + \frac{\alpha^4}{\underline{|4}} - \frac{\alpha^6}{\underline{|6}} + \ldots.$$

Also $\quad \sin n\theta = n\cos^{n-1}\theta\sin\theta - \dfrac{n(n-1)(n-2)}{\lfloor 3}\cos^{n-3}\theta\sin^3\theta + \ldots$

thus $\quad \sin\alpha = \alpha\cos^{n-1}\theta\dfrac{\sin\theta}{\theta} - \dfrac{\alpha(\alpha-\theta)(\alpha-2\theta)}{\lfloor 3}\cos^{n-3}\theta\left(\dfrac{\sin\theta}{\theta}\right)^3 + \ldots.$

Hence, by supposing n to increase without limit, we obtain

$$\sin\alpha = \alpha - \frac{\alpha^3}{\lfloor 3} + \frac{\alpha^5}{\lfloor 5} - \frac{\alpha^7}{\lfloor 7} + \ldots.$$

The results of this Article are of the greatest importance; we shall make some remarks upon them in the next three Articles.

Art. 280. *To express* $\cos^n\theta$ *in terms of cosines of multiples of* θ *when* n *is a positive integer.*

$$2^n\cos^n\theta = \left(x+\frac{1}{x}\right)^n = x^n + nx^{n-1}\cdot\frac{1}{x} + \frac{n(n-1)}{1\cdot 2}x^{n-2}\cdot\frac{1}{x^2} + \ldots$$
$$+ \frac{n(n-1)}{1\cdot 2}x^2\cdot\frac{1}{x^{n-2}} + nx\cdot\frac{1}{x^{n-1}} + \frac{1}{x^n}.$$

Now rearrange the terms on the right-hand side, putting together the first term and the last, the second and the last but one, and so on; thus we obtain

$$x^n + \frac{1}{x^n} + n\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + \frac{n(n-1)}{1\cdot 2}\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \ldots;$$

but $\quad x^n + \dfrac{1}{x^n} = 2\cos n\theta, \quad x^{n-2} + \dfrac{1}{x^{n-2}} = 2\cos(n-2)\theta,$ and so on;

therefore

$$2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{1\cdot 2}\cos(n-4)\theta + \ldots$$
$$+ \frac{n(n-1)\ldots(n-r+1)}{\lfloor r}\cos(n-2r)\theta + \ldots$$

The last term of the series on the right-hand side will take different forms according as n is even or odd. In the expansion of $\left(x+\dfrac{1}{x}\right)^n$ by the Binomial Theorem there are $n+1$ terms; thus when n is even, there will be a middle term, namely the $\left(\dfrac{n}{2}+1\right)^{th}$, which is

$$\frac{n(n-1)\ldots\left(n-\frac{1}{2}n+1\right)}{\lfloor \frac{1}{2}n}x^{\frac{n}{2}}\cdot\frac{1}{x^{\frac{n}{2}}}; \text{ that is } \frac{n(n-1)\ldots\left(\frac{1}{2}n+1\right)}{x^{\frac{n}{2}}}.$$

Hence, when n is even, the last term of $2^{n-1}\cos^n\theta$ is

$$\frac{n(n-1)\ldots\left(\frac{1}{2}n+1\right)}{2\lfloor \frac{1}{2}n}.$$

When n is odd suppose it $= 2m+1$; there are two middle terms in the expansion of $\left(x+\dfrac{1}{x}\right)^n$, namely, the $(m+1)^{th}$ and $(m+2)^{th}$: their sum is

$$\frac{n(n-1)\ldots(n-m+1)}{\lfloor m}\left(x+\frac{1}{x}\right).$$

Hence when n is odd, the last term of $2^{n-1}\cos^n\theta$ is

$$\frac{n(n-1)\dots\frac{1}{2}(n+3)}{\lfloor\frac{1}{2}(n-1)}\cos\theta.$$

Art. 281. We shall find that $\sin^n\theta$ can be expressed in terms of cosines of multiples of θ if n be an even positive integer, and in terms of sines of multiples of θ if n be an odd positive integer; this will appear in the following two Articles.

Art. 282. *To express* $\sin^n\theta$ *in terms of cosines of multiples of* θ, *when n is an even positive integer.*

$$2^n(-1)^{\frac{n}{2}}\sin^n\theta = \left(x-\frac{1}{x}\right)^n = x^n - nx^{n-1}\cdot\frac{1}{x} + \frac{n(n-1)}{1\cdot 2}x^{n-2}\cdot\frac{1}{x^2} + \dots$$
$$+\frac{n(n-1)}{1\cdot 2}x^2\cdot\left(-\frac{1}{x}\right)^{n-2} + nx\left(-\frac{1}{x}\right)^{n-1} + \left(-\frac{1}{x}\right)^n.$$

Now rearrange the terms on the right-hand side, putting together the first term and the last, the second and the last but one, and so on; thus we obtain

$$x^n + \frac{1}{x^n} - n\left(x^{n-2}+\frac{1}{x^{n-2}}\right) + \frac{n(n-1)}{1\cdot 2}\left(x^{n-4}+\frac{1}{x^{n-4}}\right) - \dots$$
$$+(-1)^{\frac{n}{2}}\frac{n(n-1)\dots\left(\frac{n}{2}+1\right)}{\lfloor\frac{n}{2}}.$$

Therefore

$$2^{n-1}(-1)^{\frac{n}{2}}\sin^n\theta = \cos n\theta - n\cos(n-2)\theta + \frac{n(n-1)}{1\cdot 2}\cos(n-4)\theta - \dots$$
$$+(-1)^{n-r}\frac{n(n-1)\dots(n-r+1)}{\lfloor r}\cos(n-2r)\theta + \dots$$
$$+(-1)^{\frac{n}{2}}\frac{n(n-1)\dots\left(\frac{n}{2}+1\right)}{2\lfloor\frac{n}{2}}.$$

Art. 283. *To express* $\sin^n\theta$ *in terms of sines of multiples of* θ *when n is an odd positive integer.*

$$2^n(-1)^{\frac{n}{2}}\sin^n\theta = \left(x-\frac{1}{x}\right)^n = x^n - nx^{n-1}\cdot\frac{1}{x} + \frac{n(n-1)}{1\cdot 2}x^{n-2}\cdot\frac{1}{x^2}$$
$$-\dots - \frac{n(n-1)}{1\cdot 2}x^2\cdot\frac{1}{x^{n-2}} + nx\cdot\frac{1}{x^{n-1}} - \frac{1}{x^n}.$$

Now rearrange the terms on the right-hand side, putting together the first term and the last, the second and the last but one, and so on; thus we obtain

$$x^n - \frac{1}{x^n} - n\left(x^{n-2}-\frac{1}{x^{n-2}}\right) + \frac{n(n-1)}{1\cdot 2}\left(x^{n-4}-\frac{1}{x^{n-4}}\right) - \dots$$

$$+(-1)^{\frac{n-1}{2}} \frac{n(n-1)\ldots\frac{(n+3)}{2}}{\left\lfloor\frac{(n-1)}{2}\right.} \left(x - \frac{1}{x}\right);$$

but
$$x^n - \frac{1}{x^n} = 2\sqrt{(-1)}\sin n\theta,$$
$$x^{n-2} - \frac{1}{x^{n-2}} = 2\sqrt{(-1)}\sin(n-2)\theta,$$

and so on; therefore

$$2^{n-1}(-1)^{\frac{n-1}{2}}\sin^n\theta = \sin n\theta - n\sin(n-2)\theta + \frac{n(n-1)}{1\cdot 2}\sin(n-4)\theta$$
$$-\frac{n(n-1)(n-2)}{\lfloor 3}\sin(n-6)\theta + (-1)^{\frac{n-1}{2}}\frac{n(n-1)\ldots\frac{(n+3)}{2}}{\frac{(n-1)}{2}}\sin\theta.$$

Art. 293. Moreover $\tan^{-1} x$ has an infinite number of values corresponding to the same value of x, so that one member of what appears as an equation admits of more values than the other; this point is left unexplained in the investigation which has been given.

The subject of series cannot be adequately treated without using the Differential Calculus. The student must therefore be referred to treatises on that subject for a satisfactory demonstration of Gregory's Series. It is there shown that so long as θ lies between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$, the result $\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \ldots$ is absolutely true. (See *Differential Calculus*, Chapter VII.)

If, however, $\theta = n\pi + \phi$, where ϕ lies between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$, then

$$\phi = \tan\phi - \frac{1}{3}\tan^3\phi + \frac{1}{5}\tan^5\phi - \ldots;$$

that is,
$$\theta - n\pi = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \ldots;$$

Art. 294. In Gregory's Series put $\theta = \frac{\pi}{4}$; then since $\tan\frac{\pi}{4} = 1$,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$

This series might be used for calculating the value of π; but it is very slowly convergent, so that a large number of terms would have to be taken to calculate π to a close approximation.

Art. 296. *Machin's Series.* We shall first show that

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}.$$

$$2\tan^{-1}\frac{1}{5} = \tan^{-1}\frac{\frac{2}{5}}{1-\frac{1}{25}} = \tan^{-1}\frac{10}{24} = \tan^{-1}\frac{5}{12},$$

$$4\tan^{-1}\frac{1}{5} = 2\tan^{-1}\frac{5}{12} = \tan^{-1}\frac{\frac{10}{12}}{1-\frac{25}{144}} = \tan^{-1}\frac{120}{119}.$$

Hence $4\tan^{-1}\dfrac{1}{5}$ is a little greater than $\dfrac{\pi}{4}$; suppose

$$4\tan^{-1}\frac{1}{5}=\frac{\pi}{4}+\tan^{-1}x,$$

then
$$\frac{120}{119}=\tan\left(\frac{\pi}{4}+\tan^{-1}x\right)=\frac{1+x}{1-x};$$

from this we find $x=\dfrac{1}{239}$;

therefore
$$\frac{\pi}{4}=4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}.$$

Therefore
$$\frac{\pi}{4}=4\left\{\frac{1}{5}-\frac{1}{3\cdot 5^3}+\frac{1}{5\cdot 5^5}-\frac{1}{7\cdot 5^7}+\ldots\right\}$$
$$-\left\{\frac{1}{239}-\frac{1}{3(239)^3}+\frac{1}{5(239)^5}-\frac{1}{7(239)^7}+\ldots\right\}.$$

Art. 300. *To find the coefficient of x^n in the expansion of $e^{ax}\cos bx$ in powers of x.*

Here
$$e^{ax}\cos bx=\frac{1}{2}e^{ax}\left(e^{bx\iota}+e^{-bx\iota}\right)=\frac{1}{2}e^{(a+b\iota)x}+\frac{1}{2}e^{(a-b\iota)x}.$$

Expand these two exponential expressions by the exponential theorem; then the coefficient of x^n is

$$\frac{1}{2\underline{|n}}\left\{(a+b\iota)^n+(a-b\iota)^n\right\}=\frac{r^n}{2\underline{|n}}\left\{\left(\frac{a}{r}+\frac{b}{r}\iota\right)^n+\left(\frac{a}{r}-\frac{b}{r}\iota\right)^n\right\}.$$

Now suppose $\dfrac{a}{r}=\cos\theta$, $\quad\dfrac{b}{r}=\sin\theta$, so that $r^2=a^2+b^2$.

Thus the coefficient of x^n becomes

$$\frac{\left(a^2+b^2\right)^{\frac{n}{2}}}{2\underline{|n}}\left\{(\cos\theta+\iota\sin\theta)^n+(\cos\theta-\iota\sin\theta)^n\right\}$$
$$=\frac{\left(a^2+b^2\right)^{\frac{n}{2}}}{2\underline{|n}}(\cos n\theta+\iota\sin n\theta+\cos n\theta-\iota\sin n\theta)$$
$$=\frac{\left(a^2+b^2\right)^{\frac{n}{2}}}{\underline{|n}}\cos n\theta.$$

Similarly the coefficient of x^n in the expansion of $e^{ax}\sin bx$ in powers of x is $\dfrac{\left(a^2+b^2\right)^{\frac{n}{2}}}{\underline{|n}}\sin n\theta.$

Art. 303. *To find the sum of the sines of a series of angles which are in arithmetical progression.*

Let the proposed series consist of the following n terms,
$$\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots+\sin\left\{\alpha+(n-1)\beta\right\}.$$

We have

$$\cos\left(\alpha-\frac{1}{2}\beta\right)-\cos\left(\alpha+\frac{1}{2}\beta\right)=2\sin\frac{1}{2}\beta\sin\alpha,$$
$$\cos\left(\alpha+\frac{1}{2}\beta\right)-\cos\left(\alpha+\frac{3}{2}\beta\right)=2\sin\frac{1}{2}\beta\sin(\alpha+\beta),$$
$$\cos\left(\alpha+\frac{3}{2}\beta\right)-\cos\left(\alpha+\frac{5}{2}\beta\right)=2\sin\frac{1}{2}\beta\sin(\alpha+2\beta),$$

. . . .

$$\cos\left(\alpha+\frac{2n-3}{2}\beta\right)-\cos\left(\alpha+\frac{2n-1}{2}\beta\right)=2\sin\frac{1}{2}\beta\sin\{\alpha+(n-1)\beta\}.$$

Let S denote the proposed series; then, by addition,

$$\cos\left(\alpha-\frac{1}{2}\beta\right)-\cos\left(\alpha+\frac{2n-1}{2}\beta\right)=2S\sin\frac{1}{2}\beta;$$

therefore

$$S=\frac{\cos\left(\alpha-\frac{1}{2}\beta\right)-\cos\left(\alpha+\frac{2n-1}{2}\beta\right)}{2\sin\frac{1}{2}\beta}$$

$$=\frac{\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{1}{2}\beta}.$$

Art. 304. *To find the sum of the cosines of a series of angles which are in arithmetical progression.*

Let the proposed series consist of the following n terms,

$$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots+\cos\{\alpha+(n-1)\beta\}.$$

We have

$$\sin\left(\alpha+\frac{1}{2}\beta\right)-\sin\left(\alpha-\frac{1}{2}\beta\right)=2\sin\frac{1}{2}\beta\cos\alpha,$$

$$\sin\left(\alpha+\frac{3}{2}\beta\right)-\sin\left(\alpha+\frac{1}{2}\beta\right)=2\sin\frac{1}{2}\beta\cos(\alpha+\beta),$$

$$\sin\left(\alpha+\frac{5}{2}\beta\right)-\sin\left(\alpha+\frac{3}{2}\beta\right)=2\sin\frac{1}{2}\beta\cos(\alpha+2\beta),$$

. . . .

$$\sin\left(\alpha+\frac{2n-1}{2}\beta\right)-\sin\left(\alpha+\frac{2n-3}{2}\beta\right)=2\sin\frac{1}{2}\beta\cos\{\alpha+(n-1)\beta\}.$$

Let S denote the proposed series; then by addition,

$$\sin\left(\alpha+\frac{2n-1}{2}\beta\right)-\sin\left(\alpha-\frac{1}{2}\beta\right)=2S\sin\frac{1}{2}\beta;$$

therefore

$$S=\frac{\sin\left(\alpha+\frac{2n-1}{2}\beta\right)-\sin\left(\alpha-\frac{1}{2}\beta\right)}{2\sin\frac{1}{2}\beta}$$

$$=\frac{\cos\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{1}{2}\beta}.$$

Art. 305. Suppose in *Arts*. 303 (page 436) and 304 (page 437) that $\beta=\frac{2\pi}{n}$; then since $\sin\dfrac{n\beta}{2} = \sin\pi = 0$, then sum of the sines or the sum of the cosines of the series of angles $\alpha,\ \alpha + \dfrac{2\pi}{n},\ \alpha + \dfrac{4\pi}{n}, \ldots \alpha + \dfrac{2(n-1)\pi}{n}$ is zero.

This is a very important result, and the student should pay great attention to it. Moreover we may give this wide extension to our result : *let m and n be positive integers, m being less than n, and $\beta = \dfrac{2\pi}{n}$, then the following sum is a number independent of angles,*

$$\sin^m \alpha + \sin^m(\alpha+\beta) + \sin^m(\alpha+2\beta) + \ldots + \sin^m\left(\alpha + \overline{n-1}\beta\right).$$

The same theorem is true when sine is changed into cosine. The theorem is established by the aid of *Arts*. 280...283 (page 433, 434, 434, 434). Suppose, for example, we take $m = 4$. We have

$$\sin^4\alpha = \frac{1}{8}\left\{\cos 4\alpha - 4\cos 2\alpha + 3\right\},$$

$$\sin^4(\alpha+\beta) = \frac{1}{8}\left\{\cos(4\alpha+4\beta) - 4\cos(2\alpha+2\beta) + 3\right\},$$

and so on.

Thus the proposed series can be replaced by other series; the sum to n terms of $\cos 4\alpha + \cos(4\alpha+4\beta) + \ldots$ is zero by *Art*. 304 (page 437); the sum to n terms of $\cos 2\alpha + \cos(2\alpha+2\beta) + \ldots$ is zero by the same Article; thus the proposed series reduces to $\dfrac{3n}{8}$.

The condition that m is less than n ensures that the denominators in the expressions for the sums of the sines and cosines do not vanish.

Art. 307. *We may now deduce the sum of the following n terms:*

$$\sin\alpha - \sin(\alpha+\beta) + \sin(\alpha+2\beta) - \ldots + (-1)^{n-1}\sin\left\{\alpha + (n-1)\beta\right\}.$$

This series may be written

$$\sin\alpha + \sin(\alpha+\beta+\pi) + \sin(\alpha+2\beta+2\pi) + \ldots + \sin\left\{\alpha+(n-1)(\beta+\pi)\right\}.$$

We have then only to change β into $\beta+\pi$ in the result of *Art*. 303 (page 436). Hence the required sum is

$$\frac{\sin\left\{\alpha + \dfrac{(n-1)(\beta+\pi)}{2}\right\}\sin\dfrac{n(\beta+\pi)}{2}}{\sin\dfrac{\beta+\pi}{2}}.$$

Similarly

$$\cos\alpha - \cos(\alpha+\beta) + \cos(\alpha+2\beta) - \ldots + (-1)^{n-1}\cos\left\{\alpha+(n-1)\beta\right\}$$

$$= \frac{\cos\left\{\alpha + \dfrac{(n-1)(\beta+\pi)}{2}\right\}\sin\dfrac{n(\beta+\pi)}{2}}{\sin\dfrac{\beta+\pi}{2}}.$$

Art. 309. *To find the sum of the following n terms :*

$$\tan x + \frac{1}{2}\tan\frac{x}{2} + \frac{1}{2^2}\tan\frac{x}{2^2} + \ldots + \frac{1}{2^{n-1}}\tan\frac{x}{2^{n-1}}.$$

We have

$$\tan x = \cot x - 2\cot 2x,$$

$$\frac{1}{2}\tan\frac{x}{2} = \frac{1}{2}\cot\frac{x}{2} - \cot x,$$

$$\frac{1}{2^2}\tan\frac{x}{2^2} = \frac{1}{2^2}\cot\frac{x}{2^2} - \frac{1}{2}\cot\frac{x}{2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{1}{2^{n-1}}\tan\frac{x}{2^{n-1}} = \frac{1}{2^{n-1}}\cot\frac{x}{2^{n-1}} - \frac{1}{2^{n-2}}\cot\frac{x}{2^{n-2}}.$$

Let S denote the proposed series; then, by addition,

$$S = \frac{1}{2^{n-1}}\cot\frac{x}{2^{n-1}} - 2\cot 2x.$$

The term $\frac{1}{2^{n-1}}\cot\frac{x}{2^{n-1}} = \frac{1}{x}\cos\beta\frac{\beta}{\sin\beta}$, where $\beta = \frac{x}{2^{n-1}}$; if we suppose n to increase indefinitely, $\cos\beta = 1$, and $\frac{\beta}{\sin\beta} = 1$.

Thus the limit of the proposed series, when n is indefinitely increased, is $\frac{1}{x} - 2\cot 2x$.

Art. 310. *To find the sum of the following n terms :*

$$\sin\alpha + c\sin(\alpha+\beta) + c^2\sin(\alpha+2\beta) + \ldots + c^{n-1}\sin\{\alpha+(n-1)\beta\}.$$

Let S denote the proposed series; substitute for the sines their exponential values : thus

$$2\iota S = e^{\alpha\iota} + ce^{(\alpha+\beta)\iota} + c^2e^{(\alpha+2\beta)\iota} + \ldots + c^{n-1}e^{(\alpha+n\beta-\beta)\iota}$$
$$- e^{-\alpha\iota} - ce^{-(\alpha+\beta)\iota} - c^2e^{-(\alpha+2\beta)\iota} - \ldots - c^{n-1}e^{-(\alpha+n\beta-\beta)\iota}$$

We have now two geometrical progressions; thus

$$2\iota S = e^{\alpha\iota}\frac{1-c^ne^{n\beta\iota}}{1-ce^{\beta\iota}} - e^{-\alpha\iota}\frac{1-c^ne^{-n\beta\iota}}{1-ce^{-\beta\iota}}$$

$$= \frac{e^{\alpha\iota} - e^{-\alpha\iota} - c\left\{e^{(\alpha-\beta)\iota} - e^{-(\alpha-\beta)\iota}\right\} - c^n\left\{e^{(\alpha+n\beta)\iota} - e^{-(\alpha+n\beta)\iota}\right\} + c^{n+1}\left\{e^{(n\beta+\alpha-\beta)\iota} - e^{-(n\beta+\alpha-\beta)\iota}\right\}}{1 - c\left(e^{\beta\iota}+e^{-\beta\iota}\right) + c^2}$$

therefore

$$S = \frac{\sin\alpha - c\sin(\alpha-\beta) - c^n\sin(\alpha+n\beta) + c^{n+1}\sin\{\alpha+(n-1)\beta\}}{1-2c\cos\beta+c^2}.$$

If c be less than unity, then when n is indefinitely increased c^n and c^{n+1} diminish without limit; hence if c be less than unity, the limit of the proposed series when n is indefinitely increased is

$$\frac{\sin\alpha - c\sin(\alpha-\beta)}{1-2c\cos\beta+c^2}.$$

Similarly we can show that

$$\cos\alpha + c\cos(\alpha+\beta) + c^2\cos(\alpha+2\beta) + \ldots + c^{n--1}\cos\{\alpha+(n-1)\beta\}$$
$$= \frac{\cos\alpha - c\cos(\alpha-\beta) - c^n\cos(\alpha+n\beta) + c^{n+1}\cos\{\alpha+(n-1)\beta\}}{1-2c\cos\beta+c^2}.$$

This result may also be obtained from the preceding by changing α into $\alpha+\frac{\pi}{2}$. If c be less than unity the limit of the proposed series, when n is indefinitely increased, is

$$\frac{\cos\alpha - c\cos(\alpha-\beta)}{1-2c\cos\beta+c^2}.$$

Art. 311. *To sum the infinite series*

$$c\sin(\alpha+\beta)+\frac{c^2}{2}\sin(\alpha+2\beta)+\frac{c^3}{\underline{|3}}\sin(\alpha+3\beta)+\ldots,$$

and
$$c\cos(\alpha+\beta)+\frac{c^2}{2}\cos(\alpha+2\beta)+\frac{c^3}{\underline{|3}}\cos(\alpha+3\beta)+\ldots$$

Denote the former series by S and the latter by C; multiply the former by ι and add it to the latter : thus

$$C+\iota S = ce^{\iota(\alpha+\beta)}+\frac{c^2}{2}e^{\iota(\alpha+2\beta)}+\frac{c^3}{\underline{|3}}e^{\iota(\alpha+3\beta)}+\ldots$$
$$= e^{\iota\alpha}\left(e^{ce^{\iota\beta}}-1\right)=e^{\iota\alpha}\left(e^{c\cos\beta+\iota c\sin\beta}-1\right)=e^{c\cos\beta}e^{\iota(\alpha+c\sin\beta)}-e^{\iota\alpha}$$
$$= e^{c\cos\beta}\left\{\cos(\alpha+c\sin\beta)+\iota\sin(\alpha+c\sin\beta)\right\}-(\cos\alpha+\iota\sin\alpha).$$

Equate the real and imaginary parts : thus

$$C=e^{c\cos\beta}\cos(\alpha+c\sin\beta)-\cos\alpha.$$
$$S=e^{c\sin\beta}\sin(\alpha+c\sin\beta)-\sin\alpha.$$

The method of this Article might be used in *Art.* 310 (page 439); or the method of that Article might be used here.

Art. 313. It is known from the treatises on the Theory of Equations that the expression x^n-1, where n is a positive integer, can be resolved into n factors, each of the form $x-a$, where a is either a real quantity or an expression of the form $\alpha+\beta\sqrt{-1}$, where α and β are real : and there is only one such set of factors. We proceed now to resolve the expression x^n-1, and some similar expressions, into component factors. The factors of the *expression* x^n-1 are found by solving the *equation* $x^n-1=0$; every root of the equation determines one factor of the expression : thus if a denote a root the corresponding factor is $x-a$.

Art. 314. *To resolve x^n-1 into factors.*

The expression $\cos\dfrac{2r\pi}{n}\pm\sqrt{(-1)}\sin\dfrac{2r\pi}{n}$, where r is any integer, is a root of the equation $x^n=1$; for the n^{th} power of this expression is by De Moivre's Theorem $\cos 2r\pi\pm\sqrt{(-1)}\sin 2r\pi$, that is 1.

First, suppose n is even. If we put $r=0$ we obtain a real root 1, and the corresponding factor is $x-1$; if we put $r=\dfrac{n}{2}$, we obtain a real root -1, and the corresponding factor is $x+1$.

If we put for r in succession the values 1, 2, 3, … $\dfrac{n}{2}-1$ we obtain $n-2$ additional roots, since each value of r gives rise to two roots. These roots are all different, for the angles are less than π and all different, and thus $\cos\dfrac{2r\pi}{n}$ cannot have two coincident values. Therefore $x^n-1=(x-1)(x+1)P$, where P is the product of $n-2$ factors obtained by ascribing to r in succession the values 1, 2, 3, … $\dfrac{n}{2}-1$ in the expression $x-\cos\dfrac{2r\pi}{n}\mp\sqrt{(-1)}\sin\dfrac{2r\pi}{n}$.

The product of the two factors $x-\cos\dfrac{2r\pi}{n}-\sqrt{(-1)}\sin\dfrac{2r\pi}{n}$, and $x-\cos\dfrac{2r\pi}{n}+\sqrt{(-1)}\sin\dfrac{2r\pi}{n}$, is the real quadratic factor $\left(x-\cos\dfrac{2r\pi}{n}\right)^2+\sin^2\dfrac{2r\pi}{n}$, that is,

$x^2 - 2x\cos\dfrac{2r\pi}{n} + 1.$

Hence when n is even

$$x^n - 1 = (x-1)(x+1)\left(x^2 - 2x\cos\frac{2\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{4\pi}{n} + 1\right)\dots$$
$$\dots\left\{x^2 - 2x\cos\frac{n-4}{n}\pi + 1\right\}\left\{x^2 - 2x\cos\frac{n-2}{n}\pi + 1\right\} \qquad (54)$$

Secondly, suppose n is odd. The only real root of $x^n = 1$ is now 1; the other $n-1$ roots are obtained by giving to r in succession the values $1, 2, 3, \dots \dfrac{n-1}{2}$ in the expression $\cos\dfrac{2r\pi}{n} \pm \sqrt{(-1)}\sin\dfrac{2r\pi}{n}$.

Hence when n is odd

$$x^n - 1 = (x-1)\left(x^2 - 2x\cos\frac{2\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{4\pi}{n} + 1\right)\dots$$
$$\dots\left\{x^2 - 2x\cos\frac{n-3}{n}\pi + 1\right\}\left\{x^2 - 2x\cos\frac{n-1}{n}\pi + 1\right\} \qquad (55)$$

Art. 315. *To resolve $x^n + 1$ into factors.*

The expression $\cos\dfrac{2r+1}{n}\pi \pm \sqrt{-1}\sin\dfrac{2r+1}{n}\pi$, where r is any integer, is a root of the equation $x^n = -1$; for the n^{th} power of this expression is $\cos(2r+1)\pi \pm \sqrt{-1}\sin(2r+1)\pi$, by De Moivre's Theorem, that is -1.

First, suppose n is even; there is no real root of the equation $x^n = -1$; the n roots are all imaginary, and are found by giving to r in succession the values $0, 1, 2, 3, \dots \dfrac{n}{2} - 1$, in the expression $\cos\dfrac{2r+1}{n}\pi \pm \sqrt{-1}\sin\dfrac{2r+1}{n}\pi$.

The product of the two factors $x - \cos\dfrac{2r+1}{n}\pi - \sqrt{-1}\sin\dfrac{2r+1}{n}\pi$, and $x - \cos\dfrac{2r+1}{n}\pi + \sqrt{-1}\sin\dfrac{2r+1}{n}\pi$, is the real quadratic factor $\left(x - \cos\dfrac{2r+1}{n}\pi\right)^2 + \sin^2\dfrac{2r+1}{n}\pi$, that is, $x^2 - 2x\cos\dfrac{2r+1}{n}\pi + 1$.

Hence when n is even

$$x^n + 1 = \left(x^2 - 2x\cos\frac{\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{3\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{5\pi}{n} + 1\right)\dots$$
$$\dots\left(x^2 - 2x\cos\frac{n-3}{n}\pi + 1\right)\left(x^2 - 2x\cos\frac{n-1}{n}\pi + 1\right) \qquad (56)$$

Secondly, suppose n is odd. The only real root of $x^n = -1$ is -1; the other $n-1$ roots are obtained by giving to r in succession the values $0, 1, 2, 3, \dots \dfrac{n-3}{2}$ in the expression $\cos\dfrac{2r+1}{n}\pi \pm \sqrt{-1}\sin\dfrac{2r+1}{n}\pi$.

Hence when n is odd

$$x^n + 1 = (x+1)\left(x^2 - 2x\cos\frac{\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{3\pi}{n} + 1\right)\dots$$
$$\dots\left(x^2 - 2x\cos\frac{n-4}{n}\pi + 1\right)\left(x^2 - 2x\cos\frac{n-2}{n}\pi + 1\right) \qquad (57)$$

Art. 316. The four formulae established in the two preceding Articles are *identically* true; we may deduce many particular results by supposing particular values assigned to x. Thus in (54) of *Art.* 314 (page 440), divide both sides by $x-1$; the quotient on the left-hand side will be $x^{n-1}+x_{n-2}+\ldots+x+1$. Now put $x=1$; thus *when n is even*

$$n=2^{\frac{n}{2}}\left(1-\cos\frac{2\pi}{n}\right)\left(1-\cos\frac{4\pi}{n}\right)\ldots\left(1-\cos\frac{n-4}{n}\pi\right)\left(1-\cos\frac{n-2}{n}\pi\right);$$

and by extracting the square root,

$$\sqrt{n}=2^{\frac{n-1}{2}}\sin\frac{2\pi}{2n}\sin\frac{4\pi}{2n}\ldots\sin\frac{n-4}{2n}\pi\sin\frac{n-2}{2n}\pi \tag{58}$$

The positive sign of the radical must be taken on the left-hand side, because the right-hand side is obviously positive.

Again in (55) of *Art.* 314 (page 440), divide both sides by $x-1$, and afterwards put $x=1$; thus *when n is odd*

$$n=2^{\frac{n-1}{2}}\left(1-\cos\frac{2\pi}{n}\right)\left(1-\cos\frac{4\pi}{n}\right)\ldots\left(1-\cos\frac{n-3}{n}\pi\right)\left(1-\cos\frac{n-1}{n}\pi\right);$$

and by extracting the square root,

$$\sqrt{n}=2^{\frac{n-1}{2}}\sin\frac{2\pi}{2n}\sin\frac{4\pi}{2n}\ldots\sin\frac{n-3}{2n}\pi\sin\frac{n-1}{2n}\pi \tag{59}$$

Again, in (56) of *Art.* 315 (page 441), put $x=1$; thus *when n is even*

$$2=2^{\frac{n}{2}}\left(1-\cos\frac{\pi}{n}\right)\left(1-\cos\frac{3\pi}{n}\right)\ldots\left(1-\cos\frac{n-3}{n}\pi\right)\left(1-\cos\frac{n-1}{n}\pi\right);$$

and by extracting the square root,

$$1=2^{\frac{n-1}{2}}\sin\frac{\pi}{2n}\sin\frac{3\pi}{2n}\ldots\sin\frac{n-3}{2n}\pi\sin\frac{n-1}{2n}\pi \tag{60}$$

Again, in (57) of *Art.* 315 (page 441), put $x=1$; thus *when n is odd*

$$2=2^{\frac{n+1}{2}}\left(1-\cos\frac{\pi}{n}\right)\left(1-\cos\frac{3\pi}{n}\right)\ldots\left(1-\cos\frac{n-4}{n}\pi\right)\left(1-\cos\frac{n-2}{n}\pi\right);$$

and by extracting the square root,

$$1=2^{\frac{n-1}{2}}\sin\frac{\pi}{2n}\sin\frac{3\pi}{2n}\ldots\sin\frac{n-4}{2n}\pi\sin\frac{n-2}{2n}\pi \tag{61}$$

Four other results may apparently be deduced from the four formulae of the two preceding Articles by putting $x=-1$; but it will be found on trial that these results do not differ really from those already deduced. Thus, for example, in (54) of *Art.* 314 (page 440) , divide both sides by $x+1$, afterwards put $x=-1$, and extract the square root; thus *when n is even*

$$\sqrt{n}=2^{\frac{n-1}{2}}\cos\frac{2\pi}{2n}\cos\frac{4\pi}{2n}\ldots\cos\frac{n-4}{2n}\pi\cos\frac{n-2}{2n}\pi;$$

this however is the same result as that in (58) of the present Article, the factors on the right-hand side being merely differently arranged; for

$$\cos\frac{2\pi}{2n}=\sin\frac{n-2}{2n}\pi,\quad \cos\frac{4\pi}{2n}=\sin\frac{n-4}{2n}\pi,\ldots$$

Art. 317. *To resolve $x^{2n}-2x^n\cos\theta+1$ into factors.*

If $\cos\theta=1$ the expression becomes $(x^n-1)^2$, and if $\cos\theta=-1$ it becomes $(x^n+1)^2$; in these cases the resolution into factors is effected by what has already been given in *Arts.* 314 (page 440) and 315 (page 441), and we will therefore suppose these cases excluded from what follows.

If we put

$$x^{2n}-2x^n\cos\theta+1=0,$$

we obtain $x^n=\cos\theta\pm\sqrt{-1}\sin\theta$; hence x is an n^{th} root of $\cos\theta+\sqrt{-1}\sin\theta$; the

n^{th} roots are found from the expression $\cos\dfrac{2r\pi+\theta}{n}\pm\sqrt{-1}\sin\dfrac{2r\pi+\theta}{n}$ by ascribing integral values to r, for it is obvious from De Moivre's Theorem that the n^{th} power of the last expression is $\cos(2r\pi+\theta)\pm\sqrt{-1}\sin(2r\pi+\theta)$, and if r be an integer this reduces to $\cos\theta\pm\sqrt{-1}\sin\theta$. If we ascribe to r in succession the values 0, 1, 2, ... $n-1$ in the expression $\cos\dfrac{2r\pi+\theta}{n}\pm\sqrt{-1}\sin\dfrac{2r\pi+\theta}{n}$ we obtain $2n$ *different* values for the expression. For if $r=p$ and $r=q$ could give the same value to the expression we should have

$$\cos\frac{2p\pi+\theta}{n}\pm\sqrt{-1}\sin\frac{2p\pi+\theta}{n}=\cos\frac{2q\pi+\theta}{n}\pm\sqrt{-1}\sin\frac{2q\pi+\theta}{n};$$

now by *Art.* 93 (page 406) we cannot have $\cos\dfrac{2p\pi+\theta}{n}=\cos\dfrac{2q\pi+\theta}{n}$ and $\sin\dfrac{2p\pi+\theta}{n}=\sin\dfrac{2q\pi+\theta}{n}$; nor can $\cos\dfrac{2p\pi+\theta}{n}=\cos\dfrac{2q\pi+\theta}{n}$ and $\sin\dfrac{2p\pi+\theta}{n}=-\sin\dfrac{2q\pi+\theta}{n}$, for that, by *Art.* 94 (page 407), would require $\dfrac{2p\pi+\theta}{n}+\dfrac{2q\pi+\theta}{n}$ to be a multiple of 2π, so that θ would be a multiple of π, and this value of θ has been expressly excluded above. Thus we obtain $2n$ *different* values of x.

The product of the two factors $x-\cos\dfrac{2r\pi+\theta}{n}-\sqrt{-1}\sin\dfrac{2r\pi+\theta}{n}$, and $x-\cos\dfrac{2r\pi+\theta}{n}+\sqrt{-1}\sin\dfrac{2r\pi+\theta}{n}$, is the real quadratic factor $\left(x-\cos\dfrac{2r\pi+\theta}{n}\right)^2+\sin^2\dfrac{2r\pi+\theta}{n}$, that is, $x^2-2x\cos\dfrac{2r\pi+\theta}{n}+1$.

Thus $x^{2n}-2x^n\cos\theta+1$

$$=\left(x^2-2x\cos\frac{\theta}{n}+1\right)\left(x^2-2x\cos\frac{2\pi+\theta}{n}+1\right)\left(x^2-2x\cos\frac{4\pi+\theta}{n}+1\right)$$
$$\ldots\left\{x^2-2x\cos\frac{(2n-4)\pi+\theta}{n}+1\right\}\left\{x^2-2x\cos\frac{(2n-2)\pi+\theta}{n}+1\right\}.$$

Art. 318. We shall now deduce some important results from the preceding general theorem. Suppose $x=1$; then

$$2(1-\cos\theta)=2^n\left(1-\cos\frac{\theta}{n}\right)\left(1-\cos\frac{2\pi+\theta}{n}\right)\left(1-\cos\frac{4\pi+\theta}{n}\right)\ldots$$
$$\ldots\left(1-\cos\frac{2n\pi-2\pi+\theta}{n}\right).$$

Let $\theta=2n\phi$ and $\dfrac{\pi}{2n}=\alpha$; extract the square root; thus

$$\sin n\phi=2^{n-1}\sin\phi\sin(2\alpha+\phi)\sin(4\alpha+\phi)\ldots\sin(2n\alpha-2\alpha+\phi).$$

We shall now prove that the *upper* sign must always be taken on the left-hand side. First, suppose ϕ to lie between 0 and 2α; then every factor on the right-hand side is positive, and so is $\sin n\phi$. Next suppose ϕ to lie between 2α and 4α; then every factor on the right-hand side is positive *except the last,* and $\sin n\phi$ is negative. Next suppose ϕ to lie between 4α and 6α, then every factor on the right-hand side is positive *except the last two,* and $\sin n\phi$ is positive. By proceeding in this way we see that for every value of ϕ between 0 and $2n\alpha$, the upper sign must be taken, so that we have *for all values of* ϕ *between* 0 *and* π

$$\sin n\phi=2^{n-1}\sin\phi\sin(2\alpha+\phi)\sin(4\alpha+\phi)\ldots\sin(2n\alpha-2\alpha+\phi).$$

We shall next show that this formula is true for *all values* of ϕ; for suppose $\phi=$

$m\pi + \psi$ where m is any integer, positive or negative, and ψ is between 0 and π; then we know that

$$\sin n\psi = 2^{n-1} \sin\psi \sin(2\alpha+\psi)\sin(4\alpha+\psi)\ldots\sin(2n\alpha-2\alpha+\psi);$$

but
$$\sin n\psi = \sin(n\phi - nm\pi) = \sin n\phi \cos nm\pi = (-1)^{nm}\sin n\phi,$$
$$\sin\psi = \sin(\phi - m\pi) = \sin\phi\cos m\pi = (-1)^m \sin\phi,$$
$$\sin(2\alpha+\psi) = \sin(2\alpha+\phi-m\pi) = \sin(2\alpha+\phi)\cos m\pi = (-1)^m\sin(2\alpha+\phi),$$

and so on.

Substitute these values of $\sin n\psi$, $\sin\psi$, $\sin(2\alpha+\psi),\ldots$ in the formula which expresses $\sin n\psi$ in factors; then divide both sides by $(-1)^{nm}$ and we obtain the required formula for $\sin n\phi$, whatever may be the value of ϕ.

In the expression for $\sin n\phi$ change ϕ into $\phi+\alpha$; then $n\phi$ is changed into $n\phi + \frac{\pi}{2}$; hence

$$\cos n\phi = 2^{n-1}\sin(\phi+\alpha)\sin(\phi+3\alpha)\sin(\phi+5\alpha)\ldots\sin(2n\alpha-\alpha+\phi).$$

In the last result put $\phi = 0$; thus

$$1 = 2^{n-1}\sin\alpha\sin3\alpha\sin5\alpha\ldots\sin(2n\alpha-\alpha),$$

where
$$\alpha = \frac{\pi}{2n}.$$

Again we have

$$\frac{\sin n\phi}{\sin\phi} = 2^{n-1}\sin(2\alpha+\phi)\sin(4\alpha+\phi)\ldots\sin(2n\alpha-2\alpha+\phi);$$

now let ϕ diminish without limit; then since the limit of $\frac{\sin n\phi}{\sin\phi}$ is n, we obtain

$$n = 2^{n-1}\sin2\alpha\sin4\alpha\sin6\alpha\ldots\sin(2n\alpha-2\alpha).$$

These two formulae are sometimes useful; the first includes (60) and (61) of *Art.* 316 (page 441), and the second included (58) and (59) of *Art.* 316 (page 441).

If we divide the expression for $\sin n\phi$ by that for $\cos n\phi$ we obtain an expression for $\tan n\phi$; *when n is odd* this takes a simple form which we may obtain more readily thus : in the expression for $\sin n\phi$, change ϕ into $\phi + \frac{\pi}{2}$; we obtain

$$\cos n\phi \sin\frac{n\pi}{2} = 2^{n-1}\cos\phi\cos(2\alpha+\phi)\ldots\cos(2n\alpha-2\alpha+\phi).$$

Divide the expression for $\sin n\phi$ by this; hence *when n is odd*

$$\tan n\phi = (-1)^{\frac{n-1}{2}}\tan\phi\tan\left(\phi+\frac{\pi}{n}\right)\ldots\tan\left(\phi+\frac{n-1}{n}\pi\right).$$

Art. 319. The expression for $\sin n\phi$ in *Art.* 318 (page 443) may be put into a different form; for

$$\sin(2n\alpha-2\alpha+\phi) = \sin(\pi-2\alpha+\phi) = \sin(2\alpha-\phi),$$
$$\sin(2n\alpha-4\alpha+\phi) = \sin(\pi-4\alpha+\phi) = \sin(4\alpha-\phi).$$

and so on.

Then by multiplying together the second factor and the last, the third and the last but one, and so on, we have

$$\sin n\phi = 2^{n-1}\sin\phi\left(\sin^2 2\alpha - \sin^2\phi\right)\left(\sin^2 4\alpha - \sin^2\phi\right)\ldots$$

It will be necessary to examine separately the cases hen n is even and when n is odd.

First suppose n is even; then the factor $\sin(n\alpha+\phi)$, that is, $\cos\phi$, will occur without any factor to multiply it : hence if n be even, we have

$$\sin n\phi = 2^{n-1}\sin\phi\cos\phi\left(\sin^2 2\alpha - \sin^2\phi\right)\left(\sin^2 4\alpha - \sin^2\phi\right)\ldots$$

$$\ldots\left\{\sin^2(n-4)\alpha - \sin 2\phi\right\}\left\{\sin^2(n-2)\alpha - \sin^2\phi\right\}.$$

Next suppose n is odd; then we have

$$\sin n\phi = 2^{n-1}\sin\phi\left(\sin^2 2\alpha - \sin^2\phi\right)\left(\sin^2 4\alpha - \sin^2\phi\right)\ldots$$
$$\ldots\left\{\sin^2(n-3)\alpha - \sin^2\phi\right\}\left\{\sin^2(n-1)\alpha - \sin^2\phi\right\}.$$

Similarly from the formula

$$\cos n\phi = 2^{n-1}\sin(\phi+\alpha)\sin(\phi+3\alpha)\sin(\phi+5\alpha)\ldots\sin(2n\alpha-\alpha+\phi)$$

We obtain if n be even

$$\cos n\phi = 2^{n-1}\left(\sin^2\alpha - \sin^2\phi\right)\left(\sin^2 3\alpha - \sin^2\phi\right)\ldots$$
$$\ldots\left\{\sin^2(n-3)\alpha - \sin^2\phi\right\}\left\{\sin^2(n-1)\alpha - \sin^2\phi\right\}.$$

and if n be odd

$$\cos n\phi = 2^{n-1}\cos\phi\left(\sin^2\alpha - \sin^2\phi\right)\left(\sin^2 3\alpha - \sin^2\phi\right)\ldots$$
$$\ldots\left\{\sin^2(n-4)\alpha - \sin^2\phi\right\}\left\{\sin^2(n-2)\alpha - \sin^2\phi\right\}.$$

Art. 320. We can now resolve $\sin\theta$ and $\cos\theta$ into their factors. Suppose $n\phi = \theta$ and that n is odd; then by the preceding Article

$$\sin\theta = 2^{n-1}\sin\frac{\theta}{n}\left(\sin^2 2\alpha - \sin^2\frac{\theta}{n}\right)\left(\sin^2 4\alpha - \sin^2\frac{\theta}{n}\right)\ldots$$

Divide both sides by $\sin\frac{\theta}{n}$, and then diminish θ indefinitely; since the limit of $\sin\theta \div \sin\frac{\theta}{n}$ is n we obtain

$$n = 2^{n-1}\sin^2 2\alpha\sin^2 4\alpha\ldots;$$

therefore by division,

$$\sin\theta = n\sin\frac{\theta}{n}\left(1 - \frac{\sin^2\frac{\theta}{n}}{\sin^2 2\alpha}\right)\left(1 - \frac{\sin^2\frac{\theta}{n}}{\sin^2 4\alpha}\right)\ldots$$

Now suppose n to increase without limit; then since $\alpha = \frac{\pi}{2n}$, the limit of $\frac{\sin\frac{\theta}{n}}{\sin 2\alpha}$ is $\frac{\theta}{\pi}$, the limit of $\frac{\sin\frac{\theta}{n}}{\sin 4\alpha}$ is $\frac{\theta}{2\pi}$, and so on.

Thus finally,

$$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right)\ldots$$

We shall obtain the same result if we begin by supposing n even.

Similarly we may show that

$$\cos\theta = \left(1 - \frac{4\theta^2}{\pi^2}\right)\left(1 - \frac{4\theta^2}{3^2\pi^2}\right)\left(1 - \frac{4\theta^2}{5^2\pi^2}\right)\ldots$$

Art. 321. In the same way as $x^{2n} - 2x^n\cos\theta + 1$ was decomposed in *Art.* 317 (page 442) we may decompose $x^{2n} - 2x^n a^n\cos\theta + a^{2n}$, and each quadratic factor of the last expression will be of the form $x^2 - 2xa\cos\frac{2r\pi+\theta}{n} + a^2$, where r is an integer; and all the factors are found by giving to r in succession the values $0, 1, 2, \ldots n-1$.

And $\cos\frac{2(n-1)\pi+\theta}{n} = \cos\frac{2\pi-\theta}{n}, \quad \cos\frac{2(n-2)\pi+\theta}{n} = \cos\frac{4\pi-\theta}{n},$

and so on; thus all the factors will be found if we take $x^2 - 2xa\cos\dfrac{2r\pi \pm \theta}{n} + a^2$, and use both signs and give to r in succession the values 0, 1, 2, ... up to $\dfrac{n-1}{2}$ if n be odd, and up to $\dfrac{n}{2}$ if n be even; in the latter case when $r = \dfrac{n}{2}$ we must take only *one* factor $x^2 - 2xa\cos\dfrac{n\pi + \theta}{n} + a^2$.

Now suppose $x = 1 + \dfrac{z}{2n}$, and $a = 1 - \dfrac{z}{2n}$; thus

$$\left(1 + \frac{z}{2n}\right)^{2n} - 2\left(1 - \frac{z^2}{4n^2}\right)^n \cos\theta + \left(1 - \frac{z}{2n}\right)^{2n}$$

is the expression to be decomposed into factors; and the general form of the factors is

$$\left(1 + \frac{z}{2n}\right)^2 - 2\left(1 - \frac{z^2}{4n^2}\right)\cos\frac{2r\pi \pm \theta}{n} + \left(1 - \frac{z}{2n}\right)^2,$$

that is, $$2\left(1 + \frac{z^2}{4n^2}\right) - 2\left(1 - \frac{z^2}{4n^2}\right)\cos\frac{2r\pi \pm \theta}{n},$$

that is, $$4\sin^2\frac{2r\pi \pm \theta}{2n}\left(1 + \frac{z^2}{4n^2}\cot^2\frac{2r\pi \pm \theta}{2n}\right).$$

Suppose n to increase indefinitely; then

$$\left(1 + \frac{z}{2n}\right)^{2n} = e^z, \quad \left(1 - \frac{z}{2n}\right)^{2n} = e^{-z},$$ (*Algebra, Art.* 552)(page 452),

also $$\frac{z^2}{4n^2}\cot^2\frac{2r\pi \pm \theta}{2n} = \frac{z^2}{(2r\pi \pm \theta)^2};$$

and by putting $z = 0$ we obtain

$$4\sin^2\frac{\theta}{2} = 4\sin^2\frac{\theta}{2n} 4\sin^2\frac{2\pi \pm \theta}{2n} 4\sin^2\frac{4\pi \pm \theta}{2n}\ldots;$$

thus finally

$$e^z - 2\cos\theta + e^{-z} = 4\sin^2\frac{\theta}{2}\left\{1 + \frac{z^2}{\theta^2}\right\}\left\{1 + \frac{z^2}{(2\pi \pm \theta)^2}\right\}\left\{1 + \frac{z^2}{(4\pi \pm \theta)^2}\right\}\ldots$$

Let ι stand for $\sqrt{-1}$; then we may put

$$e^z - 2\cos\theta + e^{-z} = 2(\cos\iota z - \cos\theta) = 4\sin\frac{\theta + \iota z}{2}\sin\frac{\theta - \iota z}{2}:$$

it will be a useful exercise to resolve $\sin\dfrac{\theta + \iota z}{2}$ and $\sin\dfrac{\theta - \iota z}{2}$ into factors by *Art.* 320 (page 445), and to show that the result agrees with that which has just been obtained.

For θ put $\pi + \phi$; thus we can obtain a formula for resolving $e^z + 2\cos\phi + e^{-z}$ into factors.

Art. 328. The propositions which are given in *Chapter* IX. admit of some extensions beyond the enunciations to which, for the sake of simplicity, we have there confined ourselves. It will be sufficient if we consider only *positive* angles.

We have shown in *Art. 116* (page 410) that $\sin\theta$ is less than θ so long as θ is less than $\dfrac{\pi}{2}$; it is obvious then that $\sin\theta$ is less than θ for *every* value of θ.

Now consider *Art. 120* (page 411). The demonstration there given depends on the fact that $\tan\dfrac{\theta}{2}$ is greater than $\dfrac{\theta}{2}$. Thus it is really shown that $\sin\theta$ is *algebraically* greater than $\theta - \dfrac{\theta^3}{4}$ as long as θ is less than π; that is, $\sin\theta - \left(\theta - \dfrac{\theta^3}{4}\right)$ is always positive. For we find, by calculation, that $\dfrac{\theta^3}{4} - \theta$ is greater than unity when $\theta = \pi$, and it increases beyond this value : thus $\dfrac{\theta^3}{4} - \theta + \sin\theta$ is always positive. And $\sin\theta$ is *arithmetically* greater than $\theta - \dfrac{\theta^3}{4}$ certainly as long as both are positive, that is certainly up to $\theta = 2$, which is beyond $\theta = \dfrac{\pi}{2}$.

Next consider *Art. 121* (page 412). From that article combined with the extension just given to *Art. 116* (page 410), it follows that cos *theta* is always *algebraically* greater than $1 - \dfrac{\theta^2}{2}$. And from *Art. 121* (page 412) combined with the extension just given to *Art. 120* (page 411), it follows that $\cos\theta$ is *algebraically* less than $\left(1 - \dfrac{\theta^2}{4}\right)^2$, certainly as long as $\dfrac{\theta}{2}$ is less than 2 : hence it follows that $\cos\theta$ is always *algebraically* less than $\left(1 - \dfrac{\theta^2}{4}\right)^2$, for this expression is greater than unity if $\dfrac{\theta}{2}$ is not less than 2. But $\cos\theta$ is not always *arithmetically* greater than $1 - \dfrac{\theta^2}{2}$, even if θ is less than $\dfrac{\pi}{2}$. On the other hand $\cos\theta$ is *arithmetically* less than $\left(1 - \dfrac{\theta^2}{4}\right)^2$, while θ lies between 0 and some value which is greater than $\dfrac{\pi}{2}$ and less than π.

Now consider *Art. 130* (page 412). In the same manner as we extended *Art. 120* (page 411) we can show that $\sin\theta$ is *algebraically* greater than $\theta - \dfrac{\theta^3}{6}$ for every value of θ, and that $\sin\theta$ is *arithmetically* greater than $\theta - \dfrac{\theta^3}{6}$, certainly up to $\theta = \sqrt{6}$. And $\cos\theta$ is *algebraically* less than $1 - \dfrac{\theta^2}{2} + \dfrac{\theta^4}{24}$, certainly as long as $\dfrac{\theta}{2}$ is less than $\sqrt{6}$: hence it will follow that $\cos\theta$ is always algebraically less than $1 - \dfrac{\theta^2}{2} + \dfrac{\theta^4}{24}$. And $\cos\theta$ is *arithmetically* less than $1 - \dfrac{\theta^2}{2} + \dfrac{\theta^4}{24}$, certainly while θ lies between 0 and $\dfrac{\pi}{2}$; for $1 - \dfrac{\theta^2}{2} + \dfrac{\theta^4}{24}$ is positive throughout this range, and until $\theta = \sqrt{6 - \sqrt{12}}$.

Art. 331. The following theorem is given for the sake of an important application : EDH is a triangle having the sides ED and DH equal.

Produce DH to any point N; and EH to a point I, such that $EI^2 = 4DH \cdot DN$.

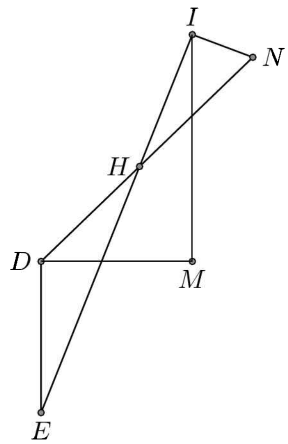
Draw DM at right angles to DE, and IM parallel to DE. Then the circle which has the centre I and the radius IM will touch the circle which has the centre N and the radius ND.

Then $DH = h,\ EI = i,\ DN = n;\ \angle DEH = \theta$.

Then $EH = 2h\cos\theta$; and from the triangle IHN

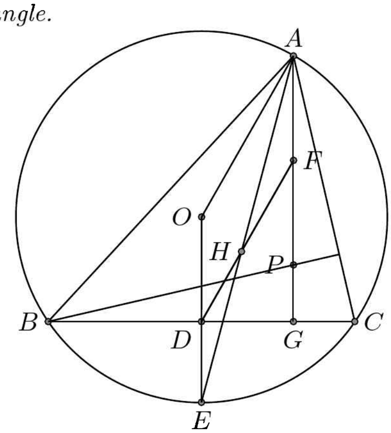
$$\begin{aligned} IN^2 &= (i - 2h\cos\theta)^2 + (n-h)^2 - 2(n-h)(i - 2h\cos\theta)\cos\theta \\ &= i^2 + (n-h)^2 - 2i(n+h)\cos\theta + 4nh\cos^2\theta \\ &= (n+h)^2 - 2i(n+h)\cos\theta + i^2\cos^2\theta \end{aligned}$$

$$= (n + h - i\cos\theta)^2.$$



Thus $IN = DN - IM$; therefore $DN = IN + IM$, which demonstrates the theorem.

Art. 332. The application of the preceding theorem which we propose to make is this : *the nine points circle of any triangle touches the inscribed circle and the escribed circles of the triangle.*



For an account of the nine points circle the student is referred to the *Appendix to Euclid*, pages 317, 318, where the following theorems are demonstrated : ABC is a triangle, and P is the intersection of the perpendiculars from A, B, C on the opposite sides; the circle which passes through the middle points of PA, PB, PC passes through the feet of the perpendiculars and through the middle points of the sides of the triangle; the diameter of the nine points circle is equal to the radius of the circumscribed circle of the triangle.

Let ABC be a triangle, O the centre of the circumscribed circle, D the middle point of BC; let AG be perpendicular to BC, let P be the intersection of the perpendiculars, F the middle point of PA. Let OD be produced to meet the circumference of the circumscribed circle at E; join OA, AE, and FD.

Since the nine points circle passes through D, F, G it follows that DF is a diameter; and therefore $DF = OA$. Also $OD = AF$, for it may shown that each

$= \dfrac{c\cos A}{2\sin C}$. Hence, since the opposite sides of $OAFD$ are equal, DF is parallel to OA. This if H be the point of intersection of EA and FD we have $ED = DH$.

Suppose that in *Art.* 331 (page 447) the letters D, E, I indicate the same points as in *Art.* 253 (page 428). Let

$$i = 2R\sin\frac{A}{2},\quad h = 2R\sin^2\frac{A}{2};\quad \text{then } n = \frac{R}{2};$$

thus N is the centre of the nine points circle of the triangle; and therefore the nine points circle touches the inscribed circle.

Again, suppose that in *Art.* 331 (page 447) the letters D, E indicate the same points as in *Art.* 253 (page 428); and let I now denote what was denoted by J in *Art.* 253 (page 428); then we see that the nine points circle touches the escribed circle which is opposite the angle A.

Since the nine points circle of the triangle ABC passes through the middle points of AB, BP, and PA, it is also the nine points circle of the triangle APB; and so it touches the inscribed and escribed circles of that triangle. A similar remark holds with regard to the triangles BPC and CPA.

[**Algebra**] **Art. 342**. Some particular cases of the equation $ax^2 + bx + c = 0$ may now be investigated. The roots of the equation are

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \frac{-b-\sqrt{b^2-4ac}}{2a};$$

we will first examine the results of supposing $a = 0$.

The numerator of the first root becomes $-b + b$, that is, 0; thus this root takes the form $\dfrac{0}{0}$. The numerator of the second root becomes $-2b$; thus this root takes the form $\dfrac{-2b}{0}$. If in the original equation we put $a = 0$, it becomes $bx + c = 0$, so that $x = -\dfrac{c}{b}$; and we may arrive at this result from the expression which takes the form $\dfrac{0}{0}$ by a suitable transformation. For multiply both numerator and denominator of $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ by $b+\sqrt{b^2-4ac}$; thus we obtain $\dfrac{-2c}{b+\sqrt{b^2-4ac}}$, and if we now put $a = 0$, we obtain $\dfrac{-2c}{2b}$, that is, $\dfrac{-c}{b}$. If the root $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$ be transformed by multiplying its numerator and denominator by $b-\sqrt{b^2-4ac}$ it becomes $\dfrac{-2c}{b-\sqrt{b^2-4ac}}$, and the smaller a is the smaller is the denominator of this fraction, and the greater the fraction itself : and equivalent result may obviously be obtained without effecting any transformation of the root. Thus we may enunciate our results as follows : in the equation $ax^2+bx+c = 0$, if a be very small compared with b and c, one root is very large and the other root is nearly equal to $-\dfrac{c}{b}$, and the smaller a is, the larger one root becomes, and the nearer the other root approaches to $-\dfrac{c}{b}$.

[**Algebra**] **Art. 385**. Suppose that we have *three* unknown quantities x, y, z connected by the *two* equations

$$ax + by + cz = 0,\quad a'x + b'y + c'z = 0;$$

these equations are not sufficient to determine the unknown quantities, but they will determine the ratios subsisting between them. For multiply the first equation by c', and the second by c, and subtract : thus

$$(ac' - a'c)x + (bc' - b'c)y = 0;$$

therefore

$$\frac{x}{bc' - b'c} = \frac{y}{ca' - c'a}.$$

Again, multiply the first equation by b', and the second by b, and subtract : thus we obtain

$$\frac{x}{bc' - b'c} = \frac{z}{ab' - a'b}.$$

Hence we may write the results in this form :

$$\frac{x}{bc' - b'c} = \frac{y}{ca' - c'a} = \frac{z}{ab' - a'b}.$$

These results are very important, and should be carefully remembered; the second denominator may be derived from the first, and the third from the second.

Denote the common value of these fractions by k; then

$$x = k(bc' - b'c), \quad y = k(ca' - c'a), \quad z = k(ab' - a'b).$$

Now suppose that we have also a third equation connecting the unknown quantities x, y, z; then by substituting in it for x, y, z; the expressions just given, we shall obtain an equation which will determine k : thus the values of x, y, z become known.

Suppose, for example, the third equation is

$$lx^2 + my^2 + nz^2 = 1,$$

then k is determined by

$$k^2\left\{l(bc' - b'c)^2 + m(ca' - c'a)^2 + n(ab' - a'b)^2\right\} = 1.$$

[**Algebra**] **Art. 526**. *Following is the third example which illustrates the use of the Binomial Theorem.*

Example (3) : Required approximate values of the roots of the quadratic equation $ax^2 + bx + c = 0$, when ac is very small compared with b^2.

The roots are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

And by the Binomial Theorem,

$$\sqrt{b^2 - 4ac} = b\left(1 - \frac{4ac}{b^2}\right)^{\frac{1}{2}}$$

$$= b\left\{1 - \frac{1}{2}\frac{4ac}{b^2} - \frac{1}{8}\left(\frac{4ac}{b^2}\right)^2 - \frac{1}{16}\left(\frac{4ac}{b^2}\right)^3 - \ldots\ldots\right\}.$$

Thus for the root with the upper sign we get

$$-\frac{c}{b} - \frac{ac^2}{b^3} - \frac{2a^2c^3}{b^5} - \ldots\ldots$$

and for the root with the lower sign we get

$$-\frac{b}{a} + \frac{c}{b} + \frac{ac^2}{b^3} + \frac{2a^2c^3}{b^5} + \ldots\ldots$$

If a be very small, while b and c are not small, the former root does not differ much from $-\dfrac{c}{b}$, and the latter root is numerically very large. See *Art.* 342 (page 449).

It is deserving of notice that the approximate value of the root in the former case coincides with what we shall obtain in the following way. Write the equation thus,

$$bx + c = -ax^2.$$

For an approximate result neglect the term ax^2 as small; thus we obtain $x = -\dfrac{c}{b}$.

Then substitute this approximate value of x in the term ax^2; thus we obtain

$$bx + c = -\frac{ac^2}{b^2},$$

that is,
$$x = -\frac{c}{b} - \frac{ac^2}{b^3}.$$

Again, substitute this new approximate value of x in the term ax^2, and preserve the terms involving a and a^2; thus we obtain

$$bx + c = -\frac{ac^2}{b^2} - \frac{2a^2c^3}{b^4},$$

that is,
$$x = -\frac{c}{b} - \frac{ac^2}{b^3} - \frac{2a^2c^3}{b^5},$$

and so on.

[**Algebra**] **Art. 550**. We will give another method of arriving at the exponential theorem. By the Binomial Theorem

$$\left(1+\frac{1}{n}\right)^{nx} = 1 + nx\frac{1}{n} + \frac{nx(nx-1)}{\lfloor 2}\frac{1}{n^2} + \frac{nx(nx-1)(nx-2)}{\lfloor 3}\frac{1}{n^3} + \frac{nx(nx-1)(nx-2)(nx-3)}{\lfloor 4}\frac{1}{n^4} + \ldots\ldots$$

that is,
$$\left(1+\frac{1}{n}\right)^{nx} = 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{\lfloor 2} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\lfloor 3} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)\left(x-\frac{3}{n}\right)}{\lfloor 4} + \ldots\ldots$$

Put $x = 1$, then $\left(1+\frac{1}{n}\right)^{n}$

$$= 1 + 1 + \frac{1-\frac{1}{n}}{\lfloor 2} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor 3} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)}{\lfloor 4} + \ldots$$

But
$$\left(1+\frac{1}{n}\right)^{nx} = \left\{\left(1+\frac{1}{n}\right)^{n}\right\}^{x};$$

hence
$$1 + x + \frac{x\left(x-\frac{1}{n}\right)}{\lfloor 2} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\lfloor 3} + \ldots\ldots = \left\{1 + 1 + \frac{1-\frac{1}{n}}{\lfloor 2} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor 3} + \ldots\ldots\right\}^{x}$$

Now this being true however large n may be, will be true when n is mae infinite; then $\frac{1}{n}$ vanishes and we obtain

$$1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \ldots\ldots = \left\{1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \ldots\ldots\right\}^{x},$$

that is, $= e^x$.

We have thus obtained the expansion of e^x in powers of x; to find the expansion of a^x suppose $a = e^c$ so that $c = \log_e a$, thus

$$a^x = e^{cx} = 1 + cx + \frac{c^2x^2}{\underline{|2}} + \frac{c^3x^3}{\underline{|3}} + \frac{c^4x^4}{\underline{|4}} + \ldots\ldots$$

[**Algebra**] **Art. 552**. We have found in *Art.* 550 (page 451), that when n increases without limit $\left(1 + \frac{1}{n}\right)^{nx}$ ultimately becomes e^x; in the same way we may show that when n increases without limit $\left(1 + \frac{r}{n}\right)^{nx}$ ultimately becomes e^{rx}.

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