Krishna's

TEXT BOOK on Trigonometry

(*For B.A. and B.Sc. Ist year students of All Colleges affiliated to universities in Uttar Pradesh*)

As per U.P. UNIFIED Syllabus

(*w.e.f.* 2011-2012)

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(Kanpur Edition)

KRISHNA Prakashan Media (P) Ltd.
KRISHNA HOUSE, 11, Shivaji Road, Meerut-250 001 (U.P.), India

Jai Shri Radhey Shyam

Dedicated

to

Lord

Krishna

Authors & Publishers

Preface

This book on **TRIGONOMETRY** has been specially written according to the latest **Unified Syllabus** to meet the requirements of the **B.A. and B.Sc. Part-I Students** of all Universities in Uttar Pradesh.

The subject matter has been discussed in such a simple way that the students will find no difficulty to understand it. The proofs of various theorems and examples have been given with minute details. Each chapter of this book contains complete theory and a fairly large number of solved examples. Sufficient problems have also been selected from various university examination papers. At the end of each chapter an exercise containing objective questions has been given.

We have tried our best to keep the book free from misprints. The authors shall be grateful to the readers who point out errors and omissions which, inspite of all care, might have been there.

The authors, in general, hope that the present book will be warmly received by the students and teachers. We shall **indeed** be very thankful to our colleagues for their recommending this book to their students.

The authors wish to express their thanks to **Mr. S.K. Rastogi**, *Managing Director,* **Mr. Sugam Rastogi**, *Executive Director,* **Mrs. Kanupriya Rastogi** *Director* and **entire team** of **KRISHNA Prakashan Media (P) Ltd.**, **Meerut** for bringing out this book in the present nice form.

The authors will feel amply rewarded if the book serves the purpose for which it is meant. Suggestions for the improvement of the book are always welcome.

Preface to the Revised Edition

The authors feel great pleasure in presenting the thoroughly revised edition of the book **TRIGONOMETRY** and wish to record thanks to the teachers and students for their warm reception to the previous edition.

The present edition has been specially designed, made up-to-date and well organised in a systematic order according to the latest syllabus.

The authors have always endeavoured to keep the text update in the best interests of the students community- a gesture which the authors hope would be appreciated by the students and teachers alike.

Suggestions for the improvement of the book will be thankfully received.

June, 2014 ***—Authors***

Syllabus

Algebra & Trigonometry

U.P. UNIFIED (*w.e.f.* 2011-12)

B.A./B.Sc. Paper-I — M.M. : 33 / 65

Algebra

Unit-1: Sequence and its convergence (basic idea), Convergence of infinite series, Comparison test, ratio test, root test, Raabe's test, Logarithmic ratio test, Cauchy's condensation test, DeMorgan and Bertrand test and higher logarithmic ratio test. Alternating series, Leibnitz test, Absolute and conditional convergence, Congruence modulo m relation, Equivalence relations and partitions.

Unit-2: Definition of a group with examples and simple properties, Permutation groups, Subgroups, Centre and normalizer, Cyclic groups, Coset decomposition, Lagrange's theorem and its consequences.

Unit-3: Homomorphism and Isomorphism. Cayley's theorem, Normal subgroups, Quotient group, Fundamental theorem of homomorphism, Conjugacy irelation, Class equation, Direct product.

Unit-4: Introduction to rings, subrings, integral domains and fields, Characteristic of a ring, Homomorphism of rings, Ideals, Quotient rings.

Trigonometry

Unit-5: Complex functions and separation into real and imaginary parts, Exponential, Direct and inverse trigonometric and hyperbolic functions, Logarithmic functions, Gregory's series, Summation of series.

Brief Contents

SECTION

B

TRIGONOMETRY

Chapters

Chapter

1

Complex Numbers

1.1 Inverse Circular Functions

Definitions: The equation $\sin\theta = x$ means that θ is the angle whose sine is x. To express θ explicitly in terms of x, a convenient notation '$\sin^{-1} x$' (*read as sine inverse x*) is introduced. Thus $\theta = \sin^{-1} x$ means that θ is the angle whose sine is x. Similarly '$\cos^{-1} x$' expresses an angle whose cosine is x, $\tan^{-1} x$ denotes an angle whose tangent is x, and so on.

The quantities $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ etc., are called **Inverse Circular Functions.** Sometimes $\sin^{-1} x$ is written as '*arc sine x*' with similar notations for other inverse functions.

Note: $\sin^{-1} x$ should not be confused with $(\sin x)^{-1}$ as

$$(\sin x)^{-1} = 1/\sin x.$$

1.2 General and Principal Values of Inverse Circular Functions

Consider the equation $\theta = \sin^{-1} x$, where $-1 \le x \le 1$. In the set of real numbers, there are infinite values of θ which satisfy this equation. All these values of θ taken together form what we call the *general value* of $\sin^{-1} x$. Among the values of θ satisfying the equation $\theta = \sin^{-1} x$, the value which is numerically the smallest one is called the *principal value* of $\sin^{-1} x$. If θ is the principal value of $\sin^{-1} x$, obviously we must have $-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi$. If x is 0, the principal value of $\sin^{-1} x$ is 0; if x is negative, the principal value of $\sin^{-1} x$ lies between $-\frac{1}{2}\pi$ and 0; and if x is positive, the principal value of $\sin^{-1} x$ lies between 0 and $\frac{1}{2}\pi$. Thus the principal value of $\sin^{-1}(1/\sqrt{2})$ is $\frac{1}{4}\pi$, while the principal value of $\sin^{-1}(-1/\sqrt{2})$ is $-\frac{1}{4}\pi$.

If θ is the principal value of $\sin^{-1} x$, then all the angles given by $n\pi + (-1)^n \theta$, where n is any integer, have their sines equal to x. We call $n\pi + (-1)^n \theta$ as the general value of $\sin^{-1} x$ and denote it by $\text{Sin}^{-1} x$. Generally if the first letter of an inverse circular function is small, we consider the principal value, while if the first letter is capital, it means the general value.

For example, $\sin^{-1}\frac{1}{2} = \frac{1}{6}\pi$, while $\text{Sin}^{-1}\frac{1}{2} = n\pi + (-1)^n \frac{1}{6}\pi$,

where n is any integer.

Thus the inverse sine of x is a many-valued function. Its principal value is denoted by $\sin^{-1} x$ and its general value is denoted by $\text{Sin}^{-1} x$. Also we have

$$\text{Sin}^{-1} x = n\pi + (-1)^n \sin^{-1} x,$$

where n is any integer, positive or negative or zero.

Similarly we may define the concepts of the principal and general values of the other inverse circular functions. The relations between the general and principal values of various inverse circular functions are as follows :

$$\textbf{Sin}^{-1} \boldsymbol{x} = \boldsymbol{n\pi + (-1)^n} \textbf{ sin}^{-1} \boldsymbol{x}, \quad \textbf{Cosec}^{-1} \boldsymbol{x} = \boldsymbol{n\pi + (-1)^n} \textbf{ cosec}^{-1} \boldsymbol{x}$$

$$\textbf{Cos}^{-1} \boldsymbol{x} = \boldsymbol{2n\pi \pm} \textbf{ cos}^{-1} \boldsymbol{x}, \quad \textbf{Sec}^{-1} \boldsymbol{x} = \boldsymbol{2n\pi \pm} \textbf{ sec}^{-1} \boldsymbol{x},$$

$$\textbf{Tan}^{-1} \boldsymbol{x} = \boldsymbol{n\pi +} \textbf{ tan}^{-1} \boldsymbol{x}, \quad \textbf{Cot}^{-1} \boldsymbol{x} = \boldsymbol{n\pi +} \textbf{ cot}^{-1} \boldsymbol{x},$$

where n is any integer, positive or negative or zero.

The principal value of any inverse circular function is the smallest numerical value of that inverse circular function. It may be positive or negative. In case there are two values, one positive and the other negative, which are numerically equal and smallest, the principal value is taken as the positive one. For example, the principal value of $\cos^{-1}\frac{1}{2}$ is $\frac{1}{3}\pi$, as out of the two numerically smallest values $\frac{1}{3}\pi$ and $-\frac{1}{3}\pi$ of $\cos^{-1}\frac{1}{2}$, the value $\frac{1}{3}\pi$ is positive.

It is to be noted that by the value of an inverse circular function we usually mean its principal value unless the contrary is stated. If x is positive, the principal values of $\sin^{-1}x$, $\text{cosec}^{-1}x$, $\cos^{-1}x$, $\sec^{-1}x$, $\tan^{-1}x$ and $\cot^{-1}x$ all lie between 0 and $\frac{1}{2}\pi$. But if x is negative, the principal values of $\sin^{-1}x$, $\text{cosec}^{-1}x$, $\tan^{-1}x$ and $\cot^{-1}x$ lie between $-\frac{1}{2}\pi$ and 0, while those of $\cos^{-1}x$ and $\sec^{-1}x$ lie between $\frac{1}{2}\pi$ and π. Thus the principal values of $\sin^{-1}x$ and $\tan^{-1}x$ (and therefore of $\text{cosec}^{-1}x$ and $\cot^{-1}x$) lie between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, while those of $\cos^{-1}x$ and $\sec^{-1}x$ lie between 0 and π.

1.3 Relations between Inverse Functions

(a) Principle of reciprocity: We have

$$\sin^{-1}x = \text{cosec}^{-1}(1/x);\ \cos^{-1}x = \sec^{-1}(1/x);\ \text{and}\ \tan^{-1}x = \cot^{-1}(1/x).$$

(b) From the definition of an inverse circular function, it is clear that

$$\theta = \sin^{-1}(\sin\theta)\ \text{and}\ x = \sin(\sin^{-1}x).$$

Similarly $\theta = \cos^{-1}(\cos\theta)$ and $x = \cos(\cos^{-1}x)$,

$$\theta = \tan^{-1}(\tan\theta)\ \text{and}\ x = \tan(\tan^{-1}x),\ \text{etc.}$$

$$\sin^{-1}x = \cos^{-1}\sqrt{(1-x^2)} = \tan^{-1}\{x/\sqrt{(1-x^2)}\}$$

$$\cos^{-1}x = \sin^{-1}\sqrt{(1-x^2)} = \tan^{-1}\frac{\sqrt{(1-x^2)}}{x}\ \text{etc.,}$$

and $\tan^{-1}x = \sin^{-1}\{x/\sqrt{(1+x^2)}\} = \cos^{-1}\{1/\sqrt{(1+x^2)}\}$ etc.

(c) We have,

(i) $\sin^{-1}(-x) = -\sin^{-1}x$; **(ii)** $\cos^{-1}(-x) = \pi - \cos^{-1}x$;

and **(iii)** $\tan^{-1}(-x) = -\tan^{-1}x$.

1.4 Some Important Results about Inverse Functions

(a) Complementary inverse functions: We have

(i) $\sin^{-1}x + \cos^{-1}x = \pi/2$; (ii) $\tan^{-1}x + \cot^{-1}x = \pi/2$;

(iii) $\sec^{-1}x + \text{cosec}^{-1}x = \pi/2$.

(b) Important formulae: We have,

(i) $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \{(x + y) / (1 - xy)\}$;

(ii) $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \{(x - y) / (1 + xy)\}$;

(iii) $2 \tan^{-1} x = \tan^{-1} \{2x / (1 - x^2)\}$;

(iv) $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1} \dfrac{x + y + z - xyz}{1 - yz - zx - xy}$.

Remark: The formula

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \{(x + y) / (1 - xy)\}$$

does not mean that the sum of the principal values of $\tan^{-1} x$ and $\tan^{-1} y$ will necessarily be equal to the principal value of $\tan^{-1} \{(x + y) / (1 - xy)\}$.
This sum may be equal to the principal value of $\tan^{-1} \{(x + y) / (1 - xy)\}$ or it may be equal to some other value of $\tan^{-1} \{(x + y) / (1 - xy)\}$.

(c) Some more formulae: We have

(i) $\cot^{-1} x + \cot^{-1} y = \cot^{-1} \dfrac{xy - 1}{x + y}$

(ii) $\cot^{-1} x - \cot^{-1} y = \cot^{-1} \dfrac{xy + 1}{y - x}$.

1.5 Complex Numbers

The equation $x^2 = -1$ has no solution in the set of real numbers because the square of every real number is either positive or zero. Therefore we feel the necessity to extend the system of real numbers. We all know that this defect is remedied by introducing complex numbers.

Complex numbers: Definition: A number of the form $x + iy$, where $i = \sqrt{(-1)}$ and x, y are both real numbers, is called a *complex number*. A complex number is also defined as an ordered pair (x, y) of real numbers. A complex number $x + iy$ or (x, y) is usually denoted by the symbol z. If we write $z = x + iy$ or (x, y) then x is called the *real part* and *y the imaginary part* of the complex number z and these are denoted by $R(z)$ and $I(z)$ respectively. Thus in the complex number $z = \sqrt{3} + 5i$, we have $R(z) =$ the real part of $z = \sqrt{3}$, and $I(z) =$ the imaginary part of $z = 5$.

A complex number is said to be purely real if its imaginary part is zero, and purely imaginary if its real part is zero.

The complex number $a + 0i$ is simply written as a.

We shall denote the set of all complex numbers by **C**.

Equality of two complex numbers:

Definition: *Two complex numbers*

$$z_1 = x_1 + iy_1 \quad or \quad (x_1, y_1) \quad and \quad z_2 = x_2 + iy_2 \quad or \quad (x_2, y_2)$$

are said to be equal if $x_1 = x_2$ *and* $y_1 = y_2$. *Thus two complex numbers are equal if and only if the real part of one is equal to the real part of the other and the imaginary part of one is equal to the imaginary part of the other.*

1.6 Addition of Complex Numbers

If $z_1 = x_1 + iy_1$ or (x_1, y_1) and $z_2 = x_2 + iy_2$ or (x_2, y_2) are any two complex numbers, then the sum of z_1 and z_2 written as $z_1 + z_2$ is defined by

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

or $$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

Thus $$(3 + 5i) + (7 - 8i) = (3 + 7) + (5 - 8)\, i = 10 - 3i.$$

Properties of the Addition of Complex Numbers:

The addition of complex numbers is commutative, associative, admits of identity element and every complex number possesses additive inverse.

Commutativity of addition in C: *We have* $z_1 + z_2 = z_2 + z_1$ *where* z_1 *and* z_2 *are any complex numbers.*

Associativity of addition in C: *We have* $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$, *for all complex numbers* z_1, z_2 *and* z_3.

The complex number (0, 0) or $0 + i0$ is the additive identity, since for every complex number (x, y), we have

$$(x, y) + (0, 0) = (x + 0, y + 0) = (x, y) = (0, 0) + (x, y).$$

The complex number (0, 0) is called the **zero complex number** and is simply written as 0.

A complex number $x + iy$ is said to be a non-zero complex number if at least one of x and y is not zero.

The complex number $(-x, -y)$ is the **additive inverse** of the complex number (x, y) since

$$(x, y) + (-x, -y) = (x - x, y - y) = (0, 0) = \text{the additive identity}$$

and also $(-x, -y) + (x, y) = (0, 0)$.

The complex number $(-x, -y)$ is called the **negative** of the complex number (x, y) and we denote $(-x, -y)$ by $-(x, y)$.

Thus if $z = (x, y)$, then $-z = -(x, y) = (-x, -y)$.

Cancellation law for addition in C: *If* z_1, z_2, z_3 *are any complex numbers, then*

$$z_1 + z_3 = z_2 + z_3 \Rightarrow z_1 = z_2.$$

1.7 Multiplication of Complex Numbers

If $z_1 = x_1 + iy_1$ or (x_1, y_1) and $z_2 = x_2 + iy_2$ or (x_2, y_2) are any two complex numbers, then the product of z_1 and z_2 written as $z_1 z_2$ is defined by

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

or $$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2).$$

Thus $$(3 + 5i)(7 + 6i) = (3 \times 7 - 5 \times 6) + (3 \times 6 + 5 \times 7) i = -9 + 53i,$$

or using the notation of ordered pairs, we have

$$(3, 5)(7, 6) = (3 \times 7 - 5 \times 6, 3 \times 6 + 5 \times 7) = (-9, 53).$$

Properties of the Multiplication of Complex Numbers

The multiplication of complex numbers is commutative, associative, admits of identity element and every non-zero complex number possesses multiplicative inverse.

Commutativity of multiplication in C : *We have* $z_1 z_2 = z_2 z_1$, *for all complex numbers* z_1 *and* z_2.

Associativity of multiplication in C : *We have* $(z_1 z_2) z_3 = z_1 (z_2 z_3)$, *for all complex numbers* z_1, z_2 *and* z_3.

The complex number $(1, 0)$ or $1 + i0$ or simply 1 is the **multiplicative identity** since for every complex number (x, y), we have

$$(x, y)(1, 0) = (x . 1 - y . 0, \ x . 0 + y . 1) = (x, y) = (1, 0)(x, y).$$

Multiplicative inverse: The complex number (x, y) is called the multiplicative inverse of the complex number (a, b) if $(x, y)(a, b) = (1, 0)$ or simply 1.

We have $(x, y)(a, b) = (1, 0)$

$\Rightarrow$ $$(xa - yb, xb + ya) = (1, 0)$$

$\Rightarrow$ $$xa - yb = 1 \quad \text{and} \quad xb + ya = 0.$$

The equations $xa - yb = 1$ and $xb + ya = 0$ give

$$x = \frac{a}{a^2 + b^2}, \quad y = -\frac{b}{a^2 + b^2},$$

provided $a^2 + b^2 \neq 0$ which implies that a and b are not both zero *i.e.,* (a, b) is a non-zero complex number.

Thus every non-zero complex number possesses multiplicative inverse and the multiplicative inverse of the complex number $(a, b) \neq (0, 0)$ is the complex number

$$\left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right).$$

If z is a non-zero complex number, the multiplicative inverse of z is denoted by $1 / z$ or z^{-1}.

Cancellation law for multiplication in C : *If* z_1, z_2, z_3 *are complex numbers and* $z_3 \neq 0$, *then* $z_1 z_3 = z_2 z_3 \Rightarrow z_1 = z_2$.

Multiplication distributes addition in C : *We have*
$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$, *for all complex numbers* z_1, z_2 *and* z_3.

1.8 Difference of Two Complex Numbers

If z_1 and z_2 are two complex numbers, we define $z_1 - z_2 = z_1 + (-z_2)$.
Thus if $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, then

$$z_1 - z_2 = z_1 + (-z_2) = (x_1, y_1) + (-x_2, -y_2) = (x_1 - x_2, y_1 - y_2).$$

1.9 Division in C

Definition: *A complex number (a, b) is said to be divisible by a complex number (c, d) if there exists a complex number (x, y) such that $(x, y)(c, d) = (a, b)$.*

We have $(x, y)(c, d) = (a, b)$

$\Rightarrow \quad (xc - yd, xd + yc) = (a, b)$

$\Rightarrow \quad xc - yd = a$ and $xd + yc = b$.

The equations $xc - yd = a$ and $xd + yc = b$ give

$$x = \frac{ac + bd}{c^2 + d^2}, \quad y = \frac{bc - ad}{c^2 + d^2}$$

provided $c^2 + d^2 \neq 0$ which implies that c and d are not both zero.

Thus division, except by $(0, 0)$, is always possible in the set of complex numbers. If z_1 and z_2 are two complex numbers such that $z_2 \neq 0$ then the quotient of the complex numbers z_1 and z_2 is defined by the relation

$$\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2} = z_1 (z_2)^{-1}.$$

1.10 The Symbol *i* and its Powers

It is customary to denote the complex number $(0, 1)$ by the symbol i. With this notation

$$i^2 = (0, 1)(0, 1) = (0 \,.\, 0 - 1 \,.\, 1, 0 \,.\, 1 + 1 \,.\, 0) = (-1, 0).$$

But we have agreed to write the complex number $(a, 0)$ simply as a. Therefore we have $i^2 = -1$. Then

$$i^3 = -i, i^4 = 1, i^5 = i, i^6 = -1, \text{and so on.}$$

Using the symbol i, we may write the complex number (x, y) in our usual form $x + iy$. For, we have

$$\begin{aligned} x + iy &= (x, 0) + (0, 1)(y, 0) = (x, 0) + (0 \,.\, y - 1 \,.\, 0, 0 \,.\, 0 + 1 \,.\, y) \\ &= (x, 0) + (0, y) = (x + 0, 0 + y) = (x, y). \end{aligned}$$

1.11 Conjugate of a Complex Number

If $z = x + iy$ is any complex number, then the complex number $x - iy$ is called the conjugate of the complex number z and is written as $\bar{z}$. Thus if $z = 3 + 4i$, then $\bar{z} = 3 - 4i$.

If $z = x + iy$ is any complex number, then we define $|z| = \sqrt{(x^2 + y^2)}$. Obviously $|z| = |\bar{z}|$.

The following results are obvious and should be remembered.

(i) Two complex numbers are equal if and only if their conjugates are equal *i.e.,* $z_1 = z_2$ if and only if $\overline{z_1} = \overline{z_2}$.

(ii) $\overline{(\bar{z})} = z$.

(iii) We have $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$, $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

and $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$, provided $z_2 \neq 0$.

(iv) If $z = x + iy$, then

$$z + \bar{z} = (x + iy) + (x - iy) = 2x = 2\,R(z).$$

(v) A complex number $z = x + iy$ is purely imaginary if and only if $z + \bar{z} = 0$.

(vi) If $z = x + iy$, then $z - \bar{z} = x + iy - (x - iy) = 2iy = 2i\,I\,(z)$.

(vii) A complex number z is purely real if and only if $z - \bar{z} = 0$.

(viii) If $z = x + iy$, then $z\,\bar{z} = (x + iy)(x - iy) = x^2 + y^2$

$$= [\sqrt{(x^2 + y^2)}]^2 = |z|^2.$$

Thus the product of two conjugate complex numbers is a purely real number which is always ≥ 0 *i.e.*, which is never negative.

1.12 Modulus of a Complex Number

Definition: *If $z = (x, y)$ or $x + iy$ be any complex number, then the non-negative real number $\sqrt{(x^2 + y^2)}$ is called the modulus or absolute value of the complex number z and is denoted by $|z|$ or mod z.*

Thus $|3 + 4i| = \sqrt{(3^2 + 4^2)} = 5$, $|8 - 6i| = \sqrt{\{8^2 + (-6)^2\}} = 10$,

$$|\cos\alpha + i\sin\alpha| = \sqrt{(\cos^2\alpha + \sin^2\alpha)} = 1,$$

and so on. Remember that the modulus of a complex number is equal to the positive square root of the sum of the squares of the real and imaginary parts of that complex number.

The following results about the modulus of a complex number should be remembered :

(i) If z is any complex number, then $|z| = |\bar{z}|$. For, if $z = x + iy$, then

$$|z| = \sqrt{(x^2 + y^2)}.$$

Also $|\bar{z}| = |x - iy| = \sqrt{\{x^2 + (-y)^2\}} = \sqrt{(x^2 + y^2)} = |z|$.

(ii) If $z = x + iy$ be any complex number, then

$$|z| = 0 \Leftrightarrow \sqrt{(x^2 + y^2)} = 0 \Leftrightarrow x^2 + y^2 = 0 \Leftrightarrow x = 0, y = 0$$

$\Leftrightarrow \quad z = 0 + i0 \quad \Leftrightarrow \quad z = 0.$

(iii) If z is any complex number, then $z\,\bar{z} = |z|^2$. For, if $z = x + iy$, then

$$z\,\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = \{\sqrt{(x^2 + y^2)}\}^2 = |z|^2.$$

(iv) If z_1, z_2 are any two complex numbers, then

$$|z_1 z_2| = |z_1||z_2|$$

i.e., the modulus of a product is equal to the product of the moduli.

(v) If z_1, z_2 are any two complex numbers and $z_2 \neq 0$, then

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

i.e., the modulus of a quotient is equal to the quotient of the moduli.

(vi) If $z = x + iy$ is any complex number, then

$$R(z) = x \leq \sqrt{(x^2 + y^2)}. \text{ Thus } R(z) \leq |z|.$$

Similarly $I(z) \leq |z|$.

1.13 Some Important Results about Complex Numbers

(i) The separation of the complex number $\dfrac{a + ib}{c + id}$ into real and imaginary parts *i.e.*, to put it in the form $A + iB$, where A and B are real numbers.

We have $\dfrac{a + ib}{c + id} = \dfrac{(a + ib)(c - id)}{(c + id)(c - id)}$,

multiplying the Nr and the Dr by the conjugate of the Dr

$$= \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + i\,\frac{bc - ad}{c^2 + d^2}$$

$$= A + iB, \quad \text{where} \quad A = \frac{ac + bd}{c^2 + d^2} \text{ and } B = \frac{bc - ad}{c^2 + d^2}.$$

Remember: To put the complex number $(a + ib)/(c + id)$ in the form $A + iB$, multiply its numerator and denominator by the conjugate of the denominator.

(ii) If $z_1 = \cos\alpha + i\sin\alpha$ and $z_2 = \cos\beta + i\sin\beta$, then

$$\begin{aligned} z_1 z_2 &= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \\ &= \cos(\alpha + \beta) + i\sin(\alpha + \beta). \end{aligned}$$

1.14 Modulus-Argument Form or Polar Standard Form or Trigonometric Form of a Complex Number

Every non-zero complex number $x + iy$ can always be put in the form $r(\cos\theta + i\sin\theta)$, where r and θ are both real numbers.

Let $x + iy = r(\cos\theta + i\sin\theta) = r\cos\theta + ir\sin\theta$.

Then equating real and imaginary parts on both sides, we get

$$x = r\cos\theta, \quad \ldots(1)$$

and $$y = r\sin\theta. \quad \ldots(2)$$

Squaring and adding (1) and (2), we have

$$x^2 + y^2 = r^2$$

or $r = +\sqrt{(x^2 + y^2)}$, taking the positive sign before the radical sign

or $r = |z|$.

Thus r is known and is equal to the modulus of the complex number z.

Substituting this value of r in (1) and (2), we have

$$\cos\theta = \frac{x}{\sqrt{(x^2 + y^2)}} \text{ and } \sin\theta = \frac{y}{\sqrt{(x^2 + y^2)}}. \quad \ldots(3)$$

Whatever be the values of x and y, if they are not both zero, there is one and only one value of θ lying between $-\pi$ and π which satisfies the equations (3) simultaneously. Thus θ is also determined. If n is any integer, then

$$\cos(2n\pi + \theta) = \cos\theta \text{ and } \sin(2n\pi + \theta) = \sin\theta.$$

So there are an infinite number of values of θ satisfying the equations (3). Any value of θ satisfying the equations (3) is called an **argument** or **amplitude** of the complex number z and we write

$$\theta = \arg z \quad \text{or} \quad \text{amp } z.$$

Thus every non-zero complex number $x + iy$ can always be put uniquely in the form $r(\cos\theta + i\sin\theta)$, where r is positive and $-\pi < \theta \leq \pi$. This trigonometrical form of a complex number is also called its polar standard form or modulus-amplitude form.

We have seen that argument of a complex number is not unique. Thus $\arg z$ is a many-valued function. The value of argument which satisfies the inequality $-\pi < \theta \leq \pi$ is called the **principal value** of the argument and it is unique. If θ is the principal value of $\arg z$, then $2n\pi + \theta$, where n is any integer, is the general value of $\arg z$ and is represented by Arg z *i.e.*, by writing the first letter of the word arg as capital. Thus

$$\text{Arg } z = 2n\pi + \arg z,$$

where Arg z stands for the general value and $\arg z$ for the principal value of the argument of z.

Usually by argument we understand its principal value unless stated otherwise.

The expression $\cos\theta + i\sin\theta$ is sometimes written in short as cis θ. In this notation $r(\cos\theta + i\sin\theta)$ is written as r cis θ.

Remark 1: From the equations (1) and (2) to determine r and θ, we also have

$$\tan\theta = y/x \ \ i.e., \ \ \theta = \tan^{-1}(y/x).$$

But it should be noted that the value of $\tan^{-1}(y/x)$ lying between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ is not always the principal value of the argument. For example, if

$$z = -1 + \sqrt{3}i = r(\cos\theta + i\sin\theta),$$

then $\quad r\cos\theta = -1,\ r\sin\theta = \sqrt{3}.$

$\therefore \quad r^2 = (-1)^2 + (\sqrt{3})^2 = 4,$ giving $r = 2.$

For $r = 2$, we have $\cos\theta = -\frac{1}{2}, \sin\theta = \frac{1}{2}\sqrt{3}.$

The principal value of the argument θ lying between $-\pi$ and π and found from these equations is $2\pi/3$. But if we write $\tan\theta = -\sqrt{3}$ *i.e.*, $\theta = \tan^{-1}(-\sqrt{3})$, then the value of $\tan^{-1}(-\sqrt{3})$ lying between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ is $-\frac{1}{3}\pi$ which is thus not the principal value of $\arg z$. So to find the principal value of $\arg z$, we should not simply make use of the equation $\theta = \tan^{-1}(y/x)$, but we must see that the value of θ found from this equation also satisfies the equations (3).

Remark 2: The following rule for locating the quadrant in which the principal value of $\arg z$, where $z = x + iy$, lies should be committed to memory.

If x and y are both +ive, the principal value of $\arg z$ lies between 0 and $\frac{1}{2}\pi$; if x and y are both –ive, the principal value of $\arg z$ lies between $-\pi$ and $-\frac{1}{2}\pi$; if x is +ive and y –ive, it lies between $-\frac{1}{2}\pi$ and 0; and if x is –ive and y + ive, it lies between $\frac{1}{2}\pi$ and π.

Remark 3: If a complex number z is given in any of the forms $r(\cos\theta - i\sin\theta)$ or $r(\sin\theta + i\cos\theta)$ or $r(\sin\theta - i\cos\theta)$, then we cannot have $\theta = \arg z$. We can have $\theta = \arg z$ only if z is put exclusively in the form $r(\cos\theta + i\sin\theta)$, where r is positive.

Remark 4: Every complex number has a unique modulus and every non-zero complex number has an infinite number of arguments any two of them differing by an integral multiple of 2π. Two complex numbers are equal if and only if their moduli are equal and their arguments differ by integral multiples of 2π.

Some particular cases of complex numbers expressed in polar standard form. The following particular cases should be remembered:

(i) $1 = 1 + 0i = \cos 0 + i\sin 0$,

(ii) $-1 = -1 + 0i = \cos\pi + i\sin\pi$,

(iii) $i = 0 + 1i = \cos\frac{1}{2}\pi + i\sin\frac{1}{2}\pi$,

and (iv) $-i = 0 - 1i = \cos(-\frac{1}{2}\pi) + i\sin(-\frac{1}{2}\pi)$.

1.15 De Moivre's Theorem

Whatever be the value of n, positive or negative, integral or fractional, the value, or one of the values, of $(\cos\theta + i\sin\theta)^n$ *is* $(\cos n\theta + i\sin n\theta)$. **(Rohilkhand 2007)**

Corollary: *For all values of n, integral or fractional, positive or negative, the value or one of the values of* $(\cos\theta - i\sin\theta)^n$ *is* $\cos n\theta - i\sin n\theta$.

Note: The students should note the following very carefully :

(i) $(\sin\theta + i\cos\theta)^n \neq \sin n\theta + i\cos n\theta$.

But $(\sin\theta + i\cos\theta)^n = [\cos(\frac{1}{2}\pi - \theta) + i\sin(\frac{1}{2}\pi - \theta)]^n$

$$= \cos n(\frac{1}{2}\pi - \theta) + i\sin n(\frac{1}{2}\pi - \theta).$$

(ii) $(\cos\theta + i\sin\phi) \neq \cos n\theta + i\sin n\phi$

i.e., De Moivre's theorem is applied only when the real and imaginary parts are cosine and sine of the same angle.

(iii) Some authors use the notation cis θ to denote $\cos\theta + i\sin\theta$.

In this notation, De Moivre's theorem would be written as

$$(\text{cis}\,\theta)^n = \text{cis}\,n\theta.$$

(iv) To separate the complex number $(a + ib)^n$ into real and imaginary parts, we put

$$a = r\cos\theta \quad \text{and} \quad b = r\sin\theta$$

so that $r = \sqrt{(a^2 + b^2)}$ and $\theta = \tan^{-1}(b/a)$.

Then $(a + ib)^n = (r\cos\theta + i\,r\sin\theta)^n = r^n(\cos\theta + i\sin\theta)^n$

$$= r^n(\cos n\theta + i\sin n\theta)$$

$$= A + iB, \quad \text{where} \quad A = r^n\cos n\theta \quad \text{and} \quad B = r^n\sin n\theta.$$

Results to be remembered:

(i) $(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)\ldots$

$$= \cos(\alpha + \beta + \gamma + \ldots) + i\sin(\alpha + \beta + \gamma + \ldots)$$

i.e., the angles are added,

(ii) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, **(Rohilkhand 2006)**

(iii) $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$,

(iv) $(\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$,

(v) $(\cos\theta - i\sin\theta)^{-n} = \cos(-n\theta) - i\sin(-n\theta) = \cos n\theta + i\sin n\theta$,

(vi) $\dfrac{1}{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta$,

and (vii) $\dfrac{1}{\cos\theta - i\sin\theta} = \cos\theta + i\sin\theta$.

Illustrative Examples

Example 1: *Find real numbers A and B, if* $A + iB = \dfrac{3-2i}{7+4i}\cdot$

Solution: We have

$$\frac{3-2i}{7+4i} = \frac{(3-2i)(7-4i)}{(7+4i)(7-4i)},$$ multiplying the Nr. and Dr. by the conjugate complex of the Dr.

$$= \frac{21-12i-14i+8i^2}{49-16i^2} = \frac{(21-8)-26i}{49+16} \qquad [\because\ i^2 = -1]$$

$$= \frac{13-26i}{65} = \frac{13}{65} - \frac{26}{65}i = \frac{1}{5} - \frac{2}{5}i.$$

$\therefore \quad A + iB = (1/5) - (2/5)\,i.$

Equating real and imaginary parts, we get $A = 1/5, B = -2/5.$

Example 2: *Find the modulus and principal argument of* $1 + i$.

Solution: Let $1 + i = r(\cos\theta + i\sin\theta)$.

Equating real and imaginary parts, we have

$$1 = r\cos\theta, \qquad \ldots(1)$$

and $$1 = r\sin\theta. \qquad \ldots(2)$$

Squaring and adding (1) and (2), we have

$$r^2 = 1 + 1 = 2.$$

$\therefore$ $$r = +\sqrt{2}.$$

Substituting the value of r in (1) and (2), we have

$$\cos\theta = 1/\sqrt{2} \text{ and } \sin\theta = 1/\sqrt{2}.$$

$\therefore$ $$\theta = \pi/4.$$

Hence $$1 + i = \sqrt{2}\,[\cos(\pi/4) + i\sin(\pi/4)].$$

$\therefore$ modulus of $1 + i = \sqrt{2}$ and principal argument $= \pi/4$.

Example 3: *Express* $-1 - i$ *in the form* $r(\cos\theta + i\sin\theta)$.

Solution: Let $-1 - i = r(\cos\theta + i\sin\theta)$.

Equating real and imaginary parts, we have

$$-1 = r\cos\theta, \qquad \ldots(1)$$

and $$-1 = r\sin\theta. \qquad \ldots(2)$$

Squaring and adding (1) and (2), we have

$$r^2 = 1 + 1 = 2, \text{ or } r = \sqrt{2}.$$

Substituting the value of r in (1) and (2), we have

$$\cos\theta = -1/\sqrt{2} \quad \text{and} \quad \sin\theta = -1/\sqrt{2}.$$

These give $\theta = -3\pi/4$, choosing the value of θ which lies between $-\pi$ and π.

Hence $\quad -1 - i = \sqrt{2}\,[\cos(-3\pi/4) + i\sin(-3\pi/4)]$

or $\quad -1 - i = \sqrt{2}\,[\cos(3\pi/4) - i\sin(3\pi/4)]$.

Example 4: *Express* $1 + \sqrt{(-3)}$ *in the modulus-amplitude form.*

Solution: We have $1 + \sqrt{(-3)} = 1 + i\sqrt{3}$.

Let $\quad 1 + i\sqrt{3} = r(\cos\theta + i\sin\theta)$.

Equating real and imaginary parts, we have

$$1 = r\cos\theta, \quad \ldots(1)$$

and $\quad \sqrt{3} = r\sin\theta. \quad \ldots(2)$

Squaring and adding (1) and (2), we have

$$r^2 = 1 + 3 = 4, \quad \text{or} \quad r = 2.$$

Substituting the value of r in (1) and (2), we have

$$\cos\theta = 1/2 \quad \text{and} \quad \sin\theta = \sqrt{3}/2.$$

$\therefore \quad \theta = \pi/3$.

Hence $\quad 1 + \sqrt{(-3)} = 2\,[\cos(\pi/3) + i\sin(\pi/3)]$.

Example 5: *Express* $-1 - \sqrt{(-3)}$ *in the polar form.*

Solution: Here $-1 - \sqrt{(-3)} = -1 - \sqrt{\{3(-1)\}} = -1 - i\sqrt{3}$.

Let $\quad -1 - i\sqrt{3} = r(\cos\theta + i\sin\theta)$.

Equating real and imaginary parts, we have

$$-1 = r\cos\theta, \quad \ldots(1)$$

and $\quad -\sqrt{3} = r\sin\theta. \quad \ldots(2)$

Squaring and adding (1) and (2), we have

$$r^2 = 1 + 3 = 4 \text{ so that } r = 2.$$

Dividing (2) by (1), we have $\tan\theta = \sqrt{3}$. This gives $\theta = -2\pi/3$, choosing the value of θ lying between $-\pi$ and π for which both $\cos\theta$ and $\sin\theta$ are negative.

$\therefore \quad -1 - \sqrt{(-3)} = 2\,[\cos(-2\pi/3) + i\sin(-2\pi/3)]$

or $\quad -1 - \sqrt{(-3)} = 2\,[\cos(2\pi/3) - i\sin(2\pi/3)]$.

□

Chapter

2

Exponential, Trigonometric and Hyperbolic Functions of a Complex Variable (Separation into Real and Imaginary Parts)

2.1 The Exponential Function of a Complex Variable

We know that if x is any real number, then

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots \text{ad. inf.}$$

We shall take motivation from this expansion of e^x as a power series in x, to define the exponential function e^z of a complex variable $z = x + iy$, where x and y are real. Thus we define

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\ldots \text{ ad. inf.} \qquad \ldots(1)$$

It can be shown by D' Alembert's ratio test, that the power series (1) is absolutely convergent for all values of the complex variable z. If we denote the nth term of the series (1) by u_n, we have

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \frac{z^n}{n!} \cdot \frac{(n-1)!}{z^{n-1}} \right| = \lim_{n \to \infty} \frac{|z|}{n} = 0,$$

which is < 1. Therefore the series (1) is absolutely convergent and hence convergent for all values of the complex variable z. Thus the sum of the series (1) exists for all z and this justifies our definition of e^z given in (1). Hence for the complex variable z, we define e^z as the sum of the power series (1).

The other ways to denote e^z are $E(z)$ and $\exp(z)$. Either of these symbols is read as 'exponential z'.

Putting $z = 0$ on both sides of (1), we see that $e^0 = 1$.

Remark: It should be noted that when z is complex, e^z is only a symbol used to represent the sum of the series on the R.H.S. of (1). It does not mean that

$$e^z = \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots + \infty \right)^z .$$

2.2 Index Law for the Exponential Functions

Theorem : *To prove that* $e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}$

i.e., $E(z_1) \cdot E(z_2) = E(z_1 + z_2)$.

Proof: If z_1 and z_2 be two complex numbers, then by the definition of $E(z)$, we have

$$E(z_1) = 1 + \frac{z_1}{1!} + \frac{z_1^2}{2!} + \frac{z_1^3}{3!} + \ldots\ldots + \frac{z_1^n}{n!} + \ldots\ldots \infty \quad \ldots(1)$$

and $$E(z_2) = 1 + \frac{z_2}{1!} + \frac{z_2^2}{2!} + \frac{z_2^3}{3!} + \ldots\ldots + \frac{z_2^n}{n!} + \ldots\ldots \infty \quad \ldots(2)$$

Since the series (1) and (2) are absolutely convergent, therefore by Cauchy's theorem on multiplication of absolutely convergent series, we have

$$E(z_1) \cdot E(z_2) = \left(1 + \frac{z_1}{1!} + \frac{z_1^2}{2!} + \ldots + \frac{z_1^n}{n!} + \ldots\right)\left(1 + \frac{z_2}{1!} + \frac{z_2^2}{2!} + \ldots + \frac{z_2^n}{n!} + \ldots\right)$$

$$= 1 + \frac{1}{1!}(z_1 + z_2) + \left(\frac{z_1^2}{2!} + \frac{z_1 z_2}{1!} + \frac{z_2^2}{2!}\right) + \ldots$$

$$+ \left(\frac{z_1^n}{n!} + \frac{z_1^{n-1}}{(n-1)!} \cdot \frac{z_2}{1!} + \frac{z_1^{n-2}}{(n-2)!} \cdot \frac{z_2^2}{2!} + \ldots + \frac{z_2^n}{n!}\right) + \ldots$$

$$= 1 + \frac{1}{1!}(z_1 + z_2) + \frac{1}{2!}(z_1^2 + 2 z_1 z_2 + z_2^2) + \ldots$$

$$+ \frac{1}{n!}\left[z_1^n + n z_1^{n-1} z_2 + \frac{n(n-1)}{2!} z_1^{n-2} z_2^2 + \ldots + z_2^n \right] + \ldots$$

$$= 1 + \frac{(z_1 + z_2)}{1!} + \frac{(z_1 + z_2)^2}{2!} + \ldots + \frac{(z_1 + z_2)^n}{n!} + \ldots$$

$$= E(z_1 + z_2), \text{ by def. of } E(z).$$

The above result may also be written as

$$\exp(z_1) \cdot \exp(z_2) = \exp(z_1 + z_2).$$

Deductions: (i) If $z_1, z_2, \ldots, z_n$ be n complex numbers, then from the above law, we see by induction that

$$E(z_1) \cdot E(z_2) \ldots\ldots E(z_n) = E(z_1 + z_2 + \ldots + z_n).$$

If we put $z_1 = z_2 = \ldots = z_n = z$ (say), we have

$$[E(z)]^n = E(nz), \; n \text{ being a positive integer.}$$

This result may also be written as

$$[e^z]^n = e^{nz} \quad \text{or} \quad [\exp(z)]^n = \exp(nz).$$

(ii) If z be any complex number, then

$$e^z \cdot e^{-z} = e^{z-z} = e^0 = 1.$$

From this result, it follows that $e^z \neq 0$ for all z.

For if there were such a value of z, then since e^{-z} would exist for this value of z, we would have $0 = 1$, which is absurd.

2.3 Trigonometrical Functions or Circular Functions of a Complex Variable

We know that if x is any real number, then

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots \text{ ad. inf.}$$

and

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots \text{ ad. inf.}$$

We shall take motivation from these expansions of $\cos x$ and $\sin x$ as power series in x, to define $\cos z$ and $\sin z$, when z is a complex number.

So if $z = x + iy$ is a complex variable, we define

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots + (-1)^n \frac{z^{2n}}{(2n)!} + \ldots \text{ad. inf.}$$

and

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots + (-1)^n \frac{z^{2n+1}}{(2n+1)} + \ldots \text{ ad. inf.}$$

Our definitions are sensible because both the series used to define $\cos z$ and $\sin z$ are absolutely convergent for all z.

Replacing z by $-z$ in the above definitions, we find that

$$\cos(-z) = \cos z, \text{ and } \sin(-z) = -\sin z.$$

The other circular functions for a complex variable are defined in the same way as those for a real variable. Thus we define

$$\tan z = \frac{\sin z}{\cos z}, \cot z = \frac{\cos z}{\sin z}, \sec z = \frac{1}{\cos z} \text{ and } \operatorname{cosec} z = \frac{1}{\sin z}.$$

Remark: For a complex variable $z = x + iy$, we have defined e^z, $\cos z$, $\sin z$ etc. in such a way that if we take

$$z = x + i0 = x \quad (i.e., \text{real}),$$

then these definitions give results which are in sound agreement with those for a real variable.

2.4 Euler's Theorem

Theorem: *If* θ *be real or complex, then prove that* $e^{i\theta} = \cos\theta + i\sin\theta$.

Proof: If z is any complex number, then by definition of e^z, we have

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^2}{3!} + \dots$$

Replacing z by $i\theta$, we have

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)$$

$= \cos\theta + i\sin\theta$, by definitions of $\cos\theta$ and $\sin\theta$.

Thus $\quad e^{i\theta} = \cos\theta + i\sin\theta.$ ...(1)

This result is known as *Euler's theorem.*

Replacing θ by $-\theta$ in (1), we get

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$

or $\quad e^{-i\theta} = \cos\theta - i\sin\theta.$ ...(2)

From (1) and (2) on addition and subtraction, we get

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta,$$

and $\quad e^{i\theta} - e^{-i\theta} = 2i\sin\theta.$

$\therefore \quad \boldsymbol{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}}$ and $\boldsymbol{\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$ ...(3)

These results are known as *Euler's exponential values* of $\cos\theta$ and $\sin\theta$.

From these exponential values of $\cos\theta$ and $\sin\theta$, we see that

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}, \quad \cot\theta = i\frac{e^{i\theta} + e^{-i\theta}}{e^{i\theta} - e^{-i\theta}}, \text{etc.}$$

Putting $\theta = 0$ in the formulae (3), we get

$$\cos 0 = \frac{1}{2}(e^0 + e^0) = \frac{1}{2}(1 + 1) = \frac{1}{2} \cdot 2 = 1,$$

and $\sin 0 = 0.$

2.5 Periodicity of Functions

(Rohilkhand 2005; Kanpur 05)

(a) *Prove that e^z is a periodic function, where z is a complex quantity.*

Proof: If $z = x + iy$, then

$$e^z = e^{x + iy} = e^x \cdot e^{iy}$$

$$= e^x (\cos y + i \sin y), \text{ by Euler's theorem}$$

$$= e^x [\cos (2n\pi + y) + i \sin (2n\pi + y)], \quad \text{where } n \text{ is any integer}$$

$$= e^x \cdot e^{i(2n\pi + y)} = e^{x + iy + 2n\pi i} = e^{z + 2n\pi i}.$$

Hence e^z is a periodic function of period $2\pi i$.

(b) *Prove that sin z, cos z, tan z etc. are periodic functions, where z is a complex quantity.*

(Kanpur 2005)

Proof: We have

$$\cos (z + 2n\pi) = \cos z \cos 2n\pi - \sin z \sin 2n\pi$$

$$= \cos z, \ n \text{ being any integer}$$

$$\sin (z + 2n\pi) = \sin z \cos 2n\pi + \cos z \sin 2n\pi$$

$$= \sin z, \ n \text{ being any integer}$$

and

$$\tan (z + n\pi) = \frac{\sin (z + n\pi)}{\cos (z + n\pi)}$$

$$= \frac{\sin z \cos n\pi + \cos z \sin n\pi}{\cos z \cos n\pi - \sin z \sin n\pi}$$

$$= \tan z, n \text{ being any integer.}$$

From these we conclude that $\cos z$ and $\sin z$ are periodic functions of period 2π and $\tan z$ is a periodic function of period π.

2.6 De Moivre's Theorem for Complex Argument

We have already mentioned De Moivre's theorem for real values of θ. Now we extend that theorem for complex values of θ. We have

$$(e^{i\theta})^n = e^{in\theta}, \text{ whether } \theta \text{ be real or complex}$$

$$\therefore \quad (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta, \text{ by Euler's theorem.}$$

This shows that De Moivre's Theorem is true whether θ be real or complex.

2.7 Standard Trigonometrical Results for Complex Arguments

Since our trigonometric functions of a complex variable are generalizations of trigonometric functions of a real variable, therefore most of our results for trigonometric functions of a real variable also hold good for trigonometric functions of a complex variable.

To prove that for all values of x, y, real or complex, the following are true :

(i) $\cos^2 x + \sin^2 x = 1,$ (ii) $\cos(-x) = \cos x,$

(iii) $\cos 2x = \cos^2 x - \sin^2 x,$ (iv) $\sin 3x = 3\sin x - 4\sin^3 x,$

(v) $\cos 3x = 4\cos^3 x - 3\cos x,$

(vi) $\sin x + \sin y = 2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y),$

(vii) $\cos x - \cos y = 2\sin\frac{1}{2}(x+y)\sin\frac{1}{2}(y-x)$

(viii) $\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y,$

(ix) $\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y.$

Proof: (i) To prove that $\mathbf{\cos^2 x + \sin^2 x = 1}$.

$$\text{L.H.S.} = \cos^2 x + \sin^2 x$$

$$= \left\{\frac{e^{ix} + e^{-ix}}{2}\right\}^2 + \left\{\frac{e^{ix} - e^{-ix}}{2i}\right\}^2 \qquad \text{[Refer article 2.4]}$$

$$= \frac{1}{4}[(e^{ix} + e^{-ix})^2 - (e^{ix} - e^{-ix})^2] \qquad [\because i^2 = -1]$$

$$= \frac{1}{4}[(e^{2ix} + e^{-2ix} + 2e^{ix}.e^{-ix}) - (e^{2ix} + e^{-2ix} - 2e^{ix}.e^{-ix})]$$

$$= \frac{1}{4}[4e^{ix}.e^{-ix}] = e^{ix-ix} = e^0 = 1 = \text{R.H.S.}$$

Aliter: We have $\cos^2 x + \sin^2 x = (\cos x + i\sin x)(\cos x - i\sin x)$

$$= e^{ix}e^{-ix} = e^0 = 1.$$

(ii) To prove that $\mathbf{\cos(-x) = \cos x}$.

$$\text{L.H.S.} = \cos(-x) = \frac{e^{i(-x)} + e^{-i(-x)}}{2} \qquad \text{[By article 2.4]}$$

$$= \frac{e^{-ix} + e^{ix}}{2} = \frac{e^{ix} + e^{-ix}}{2} = \cos x = \text{R.H.S.}$$

(iii) To prove that $\mathbf{\cos 2x = \cos^2 x - \sin^2 x}$.

$$\text{R.H.S.} = \cos^2 x - \sin^2 x = \left\{\frac{e^{ix} + e^{-ix}}{2}\right\}^2 - \left\{\frac{e^{ix} - e^{-ix}}{2i}\right\}^2,$$

by Euler's exponential values of $\cos x$ and $\sin x$

$$= \frac{1}{4}[(e^{ix} + e^{-ix})^2 + (e^{ix} - e^{-ix})^2] \qquad [\because i^2 = -1]$$

$$= \frac{1}{4}[(e^{2ix} + e^{-2ix} + 2e^{ix}.e^{-ix}) + (e^{2ix} + e^{-2ix} - 2e^{ix}.e^{-ix})]$$

$$= \frac{1}{4}[2(e^{2ix} + e^{-2ix})] = \frac{1}{2}(e^{2ix} + e^{-2ix})$$

$$= \frac{1}{2}[e^{i(2x)} + e^{-i(2x)}] = \cos 2x = \text{R.H.S.}$$

(iv) To prove that $\sin 3x = 3\sin x - 4\sin^3 x$.

$$\text{L.H.S.} = \sin 3x = \frac{e^{i(3x)} - e^{-i(3x)}}{2i} = \frac{e^{3ix} - e^{-3ix}}{2i}$$

$$= \frac{(e^{ix})^3 - (e^{-ix})^3}{2i} = \frac{(e^{ix} - e^{-ix})^3 + 3e^{ix}.e^{-ix}(e^{ix} - e^{-ix})}{2i}$$

(Note)

$$[\because a^3 - b^3 = (a-b)^3 + 3ab(a-b)]$$

$$= \frac{1}{2i}[(2i\sin x)^3 + 3.e^0.(2i\sin x)] \qquad \left[\because \sin x = \frac{e^{ix} - e^{-ix}}{2i}\right]$$

$$= \frac{1}{2i}[8i^3\sin^3 x + 6i\sin x] = \frac{1}{2i}[2i(-4\sin^3 x + 3\sin x)]$$

$$= -4\sin^3 x + 3\sin x = 3\sin x - 4\sin^3 x = \text{R.H.S.}$$

(v) To prove that $\cos 3x = 4\cos^3 x - 3\cos x$.

$$\text{L.H.S.} = \cos 3x = \frac{1}{2}[e^{i(3x)} + e^{-i(3x)}] = \frac{1}{2}[(e^{ix})^3 + (e^{-ix})^3]$$

$$= \frac{1}{2}[(e^{ix} + e^{-ix})^3 - 3e^{ix}.e^{-ix}(e^{ix} + e^{-ix})]$$

$$[\because a^3 + b^3 = (a+b)^3 - 3ab(a+b)]$$

$$= \frac{1}{2}[(2\cos x)^3 - 3.e^0.(2\cos x)] \qquad \left[\because \cos x = \frac{1}{2}(e^{ix} + e^{-ix})\right]$$

$$= \frac{1}{2}[8\cos^3 x - 6\cos x] = 4\cos^3 x - 3\cos x = \text{R.H.S.}$$

(vi) To prove that $\sin x + \sin y = 2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$.

$$\text{L.H.S.} = \sin x + \sin y$$

$$= \frac{e^{ix} - e^{-ix}}{2i} + \frac{e^{iy} - e^{-iy}}{2i} = \frac{1}{2i}[(e^{ix} - e^{-ix}) + (e^{iy} - e^{-iy})]$$

$$= (1/2i)[\{e^{i[(x+y)/2+(x-y)/2]} - e^{-i[(x+y)/2+(x-y)/2]}\}$$

$$+ \{e^{i[(x+y)/2-(x-y)/2]} - e^{-i[(x+y)/2-(x-y)/2]}\}] \textbf{ (Note)}$$

$$= (1/2i)\,[\{e^{i[(x+y)/2+(x-y)/2]} + e^{i[(x+y)/2-(x-y)/2]}$$
$$- \{e^{-i[(x+y)/2+(x-y)/2]} + e^{-i[(x+y)/2-(x-y)/2]}\}] \quad \textbf{(Note)}$$
$$= (1/2i)\,[e^{i(x+y)/2}\,\{e^{i(x-y)/2} + e^{-i(x-y)/2}\}$$
$$- e^{-i(x+y)/2}\,\{e^{-i(x-y)/2} + e^{i(x-y)/2}\}]$$
$$= \frac{1}{2i}\,[e^{i(x-y)/2} + e^{-i(x-y)/2}]\,[e^{i(x+y)/2} - e^{-i(x+y)/2}]$$
$$= \frac{1}{2i}\left[2\cos\left\{\frac{x-y}{2}\right\}\right]\cdot\left[2i\sin\left\{\frac{x+y}{2}\right\}\right]$$
$$= 2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y) = \text{R.H.S.}$$

(vii) To prove that $\cos x - \cos y = 2\sin\frac{1}{2}(x+y)\,.\,\sin\frac{1}{2}(y-x)$.

$$\text{R.H.S.} = 2\sin\frac{1}{2}(x+y)\,.\,\sin\frac{1}{2}(y-x)$$
$$= 2\left[\frac{e^{i(x+y)/2} - e^{-i(x+y)/2}}{2i}\right]\left[\frac{e^{i(y-x)/2} - e^{-i(y-x)/2}}{2i}\right]$$
$$= -\frac{1}{2}\,[e^{i(x+y+y-x)/2} - e^{i(x+y-y+x)/2} - e^{-i(x+y-y+x)/2}$$
$$+ e^{-i(x+y+y-x)/2}]$$

(simply by multiplication)

$$= -\frac{1}{2}\,[e^{iy} - e^{ix} - e^{-ix} + e^{-iy}] = \frac{1}{2}(e^{ix} + e^{-ix}) - \frac{1}{2}(e^{iy} + e^{-iy})$$
$$= \cos x - \cos y = \text{L.H.S.}$$

(viii) To prove that $\sin(x+y) = \sin x\cos y + \cos x\sin y$.

$$\text{R.H.S.} = \frac{e^{ix} - e^{-ix}}{2i} \times \frac{e^{iy} + e^{-iy}}{2} + \frac{e^{ix} + e^{-ix}}{2} \times \frac{e^{iy} - e^{-iy}}{2i}$$
$$= (1/4i)\,[\{e^{i(x+y)} + e^{i(x-y)} - e^{-i(x-y)} - e^{-i(x+y)}\}$$
$$+ \{e^{i(x+y)} - e^{i(x-y)} + e^{-i(x-y)} - e^{-i(x+y)}\}]$$
$$= \frac{1}{4i}\,[2e^{i(x+y)} - 2e^{-i(x+y)}] = \frac{e^{i(x+y)} - e^{-i(x+y)}}{2i}$$
$$= \sin(x+y) = \text{L.H.S.}$$

Similarly, $\sin(x-y) = \sin x\cos y - \cos x\sin y$.

$\therefore \quad \sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$.

(ix) To prove that $\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$.

We have $\cos x\cos y + \sin x\sin y$

$$= \frac{e^{ix} + e^{-ix}}{2} \cdot \frac{e^{iy} + e^{-iy}}{2} + \frac{e^{ix} - e^{-ix}}{2i} \cdot \frac{e^{iy} - e^{-iy}}{2i}$$

$$= \frac{1}{4} [e^{i(x+y)} + e^{i(x-y)} + e^{-i(x-y)} + e^{-i(x+y)}]$$

$$- \frac{1}{4} [e^{i(x+y)} - e^{i(x-y)} - e^{-i(x-y)} + e^{-i(x+y)}] \qquad [\because \ i^2 = -1]$$

$$= \frac{1}{4} [2e^{i(x-y)} + 2e^{-i(x-y)}] = \frac{1}{2} [e^{i(x-y)} + e^{-i(x-y)}] = \cos (x - y).$$

Similarly we can prove that

$$\cos x \cos y - \sin x \sin y = \cos (x + y).$$

$$\therefore \qquad \cos (x \pm y) = \cos x \cos y \mp \sin x \sin y.$$

Remark : If u and v are any complex numbers, we can easily prove that

$$2 \sin u \cos v = \sin (u + v) + \sin (u - v), \ 2 \cos u \sin v = \sin (u + v) - \sin (u - v)$$

$$2 \cos u \cos v = \cos (u + v) + \cos (u - v), 2 \sin u \sin v = \cos (u - v) - \cos (u + v).$$

Illustrative Examples

Example 1: *Show that* $\exp (\pm i \pi / 2) = \pm i.$ **(Avadh 2014)**

Solution: Since $\exp (\pm i\theta) = \cos \theta \pm i \sin \theta$, we have

$$\exp (\pm i \pi / 2) = \cos (\pi / 2) \pm i \sin (\pi / 2) = 0 \pm i.1 = \pm i.$$

Example 2: *Prove that*

$$\{\sin (\alpha - \theta) + e^{-\alpha i} \sin \theta\}^n = \sin^{n-1} \alpha \{\sin (\alpha - n\theta) + e^{-\alpha i} \sin n\theta\}.$$

Solution: L.H.S. $= \{\sin \alpha \cos \theta - \cos \alpha \sin \theta + (\cos \alpha - i \sin \alpha) \sin \theta\}^n$

$$= \{\sin \alpha \cos \theta - i \sin \alpha \sin \theta\}^n = \sin^n \alpha (\cos \theta - i \sin \theta)^n$$

$= \sin^n \alpha (\cos n\theta - i \sin n\theta)$, by De Moivre's theorem

and R.H.S. $= \sin^{n-1} \alpha \{\sin (\alpha - n\theta) + e^{-\alpha i} \sin n\theta\}$

$$= \sin^{n-1} \alpha \{\sin \alpha \cos n\theta - \cos \alpha \sin n\theta + (\cos \alpha - i \sin \alpha) \sin n\theta\}$$

$$= \sin^{n-1} \alpha \{\sin \alpha \cos n\theta - i \sin \alpha \sin n\theta\}$$

$$= \sin^{n-1} \alpha . \sin \alpha . (\cos n\theta - i \sin n\theta) = \sin^n \alpha (\cos n\theta - i \sin n\theta).$$

$\therefore$ L.H.S. = R.H.S.

2.8 Hyperbolic Functions

Definition: For all values of x, real or complex, the quantity $\dfrac{e^x + e^{-x}}{2}$ is called the **hyperbolic cosine** of x and is written as **cosh *x*.** Similarly the quantity $\dfrac{e^x - e^{-x}}{2}$ is called the **hyperbolic sine** of x and is written as **sinh x** (read as shine x). Thus we define

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

The hyperbolic tangent, cotangent, secant and cosecant are obtained from hyperbolic cosine and sine just as circular tangent, cotangent, secant and cosecant are obtained from circular cosine and sine. Thus

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{(e^x - e^{-x})/2}{(e^x + e^{-x})/2} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{(e^x + e^{-x})/2}{(e^x - e^{-x})/2} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\text{sech } x = \frac{1}{\cosh x} = \frac{1}{(e^x + e^{-x})/2} = \frac{2}{e^x + e^{-x}}$$

and $$\text{cosech } x = \frac{1}{\sinh x} = \frac{1}{(e^x - e^{-x})/2} = \frac{2}{e^x - e^{-x}}.$$

2.9 Relations between Hyperbolic and Circular Functions

(Bundelkhand 2006)

The hyperbolic functions can be expressed in terms of circular functions as follows:
We have

$$\sin(ix) = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = \frac{e^{i^2x} - e^{-i^2x}}{2i} = \frac{e^{-x} - e^x}{2i}$$

$$= i \cdot \frac{e^{-x} - e^x}{2i^2} = i \cdot \frac{e^x - e^{-x}}{2}, \qquad [\because \; i^2 = -1]$$

$$= i \sinh x.$$

From this relation $\sin(ix) = i \sinh x$,

we have $$\sinh x = \frac{1}{i} \sin(ix) = \frac{i}{i^2} \sin(ix) = -i \sin(ix).$$

Similarly $$\cos(ix) = \frac{e^{i(ix)} + e^{-i(ix)}}{2} = \frac{e^{i^2x} + e^{-i^2x}}{2}$$

$$= \frac{e^{-x} + e^x}{2} = \frac{e^x + e^{-x}}{2} = \cosh x.$$ **(Lucknow 2007, 08)**

Now $$\tan(ix) = \frac{\sin(ix)}{\cos(ix)} = \frac{i \sinh x}{\cosh x} = i \tanh x,$$

$$\cot(ix) = \frac{1}{\tan(ix)} = \frac{1}{i \tanh x} = \frac{-i}{\tanh x} = -i \coth x,$$

$$\textbf{sec } (ix) = \frac{1}{\cos(ix)} = \frac{1}{\cosh x} = \textbf{sech } x,$$

and $$\textbf{cosec } (ix) = \frac{1}{\sin(ix)} = \frac{1}{i \sinh x} = \frac{-i}{\sinh x} = -\, i \textbf{ cosech } x.$$

Similarly we can prove that

$\sinh(ix) = i \sin x$; **(Rohilkhand 2005)**

$\cosh(ix) = \cos x$; $\tanh(ix) = i \tan x$ and $\coth(ix) = -\, i \cot x$, etc.

2.10 Properties of Hyperbolic Functions

Formulae involving hyperbolic functions can be easily deduced with the help of the above relations. A list of some such formulae is given below.

(a) $\sinh 0 = 0, \cosh 0 = 1, \tanh 0 = 0$

(b) $\cosh^2 x - \sinh^2 x = 1$ **(Bundelkhand 2008)**

(c) $\sinh 2x = 2 \sinh x \cosh x$

(d) $\cosh 2x = \cosh^2 x + \sinh^2 x = 1 + 2 \sinh^2 x = 2 \cosh^2 x - 1$

(Lucknow 2007, 08)

(e) $\text{sech}^2 x = 1 - \tanh^2 x$

(f) $\tanh 2x = 2 \tanh x / (1 + \tanh^2 x)$

(g) *(i)* $\sinh(x + y) = \sin x \cosh y + \cosh x \sinh y$

(ii) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$ **(Rohilkhand 2006)**

(h) $e^x = \cosh x + \sinh x$ and $e^{-x} = \cosh x - \sinh x$

(i) $\sinh 3x = 3 \sinh x + 4 \sinh^3 x$

(j) $\cosh 3x = 4 \cosh^3 x - 3 \cosh x$

(k) $\tanh 3x = \dfrac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x}$.

Proof: (a) sinh 0 = 0, cosh 0 = 1, tanh 0 = 0.

By definition,

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \text{ and } \cosh x = \frac{1}{2}(e^x + e^{-x}).$$

$$\therefore \quad \sinh 0 = \frac{1}{2}(e^0 - e^{-0}) = \frac{1}{2}(1 - 1) = 0 \qquad [\because e^0 = e^{-0} = 1]$$

and $$\cosh 0 = \frac{1}{2}(e^0 + e^{-0}) = \frac{1}{2}(1 + 1) = \frac{1}{2} . 2 = 1.$$

Also $$\tanh 0 = \frac{\sinh 0}{\cosh 0} = \frac{0}{1} = 0.$$

(b) $\cosh^2 x - \sinh^2 x = 1$ **(Kashi 2014)**

We have $\cosh^2 x - \sinh^2 x = \left(\dfrac{e^x + e^{-x}}{2}\right)^2 - \left(\dfrac{e^x - e^{-x}}{2}\right)^2$

$$= \frac{1}{4}(e^{2x} + 2 + e^{-2x}) - \frac{1}{4}(e^{2x} - 2 + e^{-2x}) = \frac{1}{2} + \frac{1}{2} = 1.$$

(c) $\sinh 2x = 2 \sinh x \cosh x.$

The L.H.S. $= \sinh 2x = (1/i) \sin (2ix)$ **(Note)**

$= (1/i)\, 2 \sin (ix) \cos (ix) = (1/i)\, 2 \,.\, (i \sinh x) \,.\, (\cosh x)$

$= 2 \sinh x \cosh x =$ R.H.S.

(d) $\cosh 2x = \cosh^2 x + \sinh^2 x = 1 + 2 \sinh^2 x = 2 \cosh^2 x - 1.$

We have $\cosh 2x = \cos (2ix) = \cos 2\,(ix)$ **(Note)**

$= \cos^2 (ix) - \sin^2 (ix) = \cosh^2 x - i^2 \sinh^2 x$

$= \cosh^2 x + \sinh^2 x$ ...(1)

$= 1 + \sinh^2 x + \sinh^2 x = 1 + 2 \sinh^2 x.$ ...(2)

Also $\cosh 2x = \cosh^2 x + \sinh^2 x = \cosh^2 x + (\cosh^2 x - 1)$

$= 2 \cosh^2 x - 1.$ ...(3)

(e) $\text{sech}^2 x = 1 - \tanh^2 x.$

We have $\cosh^2 x - \sinh^2 x = 1$, by property (b).

Dividing by $\cosh^2 x$, we get $1 - \tanh^2 x = \text{sech}^2 x.$

Similarly, $\coth^2 x - 1 = \text{cosech}^2 x.$

(f) $\tanh 2x = 2 \tanh x/(1 + \tanh^2 x).$

We have $i \tanh 2x = \tan (2ix).$

$\therefore$ $\tanh 2x = (1/i) \tan (2ix) = (1/i) \tan 2(ix)$

$$= \frac{1}{i} \cdot \frac{2 \tan ix}{(1 - \tan^2 ix)} = \frac{1}{i} \cdot \frac{2i \tanh x}{(1 - i^2 \tanh^2 x)}$$ **(Note)**

$= 2 \tanh x / (1 + \tanh^2 x).$

Similarly, we find

$$\sinh 2x = \frac{2 \tanh x}{1 - \tanh^2 x} \quad \text{and} \quad \cosh 2x = \frac{1 + \tanh^2 x}{1 - \tanh^2 x}.$$

(g) (i) $\sinh (x + y) = \sinh x \cosh y + \cosh x \sinh y.$

$$\text{R.H.S.} = \frac{1}{2}(e^x - e^{-x}) . \frac{1}{2}(e^y + e^{-y}) + \frac{1}{2}(e^x + e^{-x}) . \frac{1}{2}(e^y - e^{-y})$$

$$= \frac{1}{4}[e^{x+y} + e^{x-y} - e^{-(x-y)} - e^{-(x+y)} + e^{x+y} - e^{x-y} + e^{-(x-y)} - e^{-(x+y)}]$$

$$= \frac{1}{4}\,[2e^{(x+y)} - 2e^{-(x+y)}] = \frac{1}{2}\,[e^{(x+y)} - e^{-(x+y)}]$$

$$= \sinh (x + y) = \text{L.H.S.}$$

(ii) cosh (*x* +*y*) = cosh *x* cosh *y* + sinh *x* sinh *y*. **(Rohilkhand 2006)**

$$\text{L.H.S.} = \cosh (x + y) = \cos i\,(x + y) = \cos (ix + iy)$$

$$= \cos (ix) \cos (iy) - \sin (ix) \sin (iy)$$

$$= \cosh x \cosh y - i \sinh x \,.\, i \sinh y$$

$$= \cosh x \cosh y - i^2 \sinh x \sinh y$$

$$= \cosh x \cosh y + \sinh x \sinh y = \text{R.H.S.}$$

(h) $e^x = \cosh x + \sinh x$ and $e^{-x} = \cosh x - \sinh x$.

We have $\cosh x = \frac{1}{2}(e^x + e^{-x})$; $\sinh x = \frac{1}{2}(e^x - e^{-x})$.

Adding, we get $\cosh x + \sinh x = \frac{1}{2}(e^x + e^{-x} + e^x - e^{-x}) = \frac{1}{2}\,.\,2e^x = e^x$

and subtracting, we get

$$\cosh x - \sinh x = \frac{1}{2}(e^x + e^{-x} - e^x + e^{-x}) = \frac{1}{2}\,.\,2e^{-x} = e^{-x}.$$

(i) $\sinh 3x = 3 \sinh x + 4 \sinh^3 x$.

$$\text{L.H.S.} = \sinh 3x = (1/i) \sin (3ix) = (1/i) \sin 3\,(ix)$$

$$= (1/i)\,[3 \sin (ix) - 4 \sin^3 (ix)]$$

$$= (1/i)\,[3\,(i \sinh x) - 4\,(i \sinh x)^3] \qquad [\because \sin (ix) = i \sinh x\,]$$

$$= (1/i)\,[3i \sinh x + 4i \sinh^3 x] \qquad [\because i^2 = -1\,;\, i^3 = -i]$$

$$= 3 \sinh x + 4 \sinh^3 x = \text{R.H.S.}$$

(j) $\cosh 3x = 4 \cosh^3 x - 3 \cosh x$.

$$\text{L.H.S.} = \cosh 3x = \cos (3ix) = \cos 3\,(ix) = 4 \cos^3 (ix) - 3 \cos (ix)$$

$$= 4 \cosh^3 x - 3 \cosh x. \qquad [\because \cos (ix) = \cosh x]$$

(k) $\tanh 3x = (3 \tanh x + \tan^3 x)/(1 + 3 \tanh^2 x)$. **(Kanpur 2008)**

$$\text{L.H.S.} = \tanh 3x = (1/i) \tan (3ix) = \left(\frac{1}{i}\right)\left[\frac{3 \tan (ix) - \tan^3 (ix)}{1 - 3 \tan^2 (ix)}\right]$$

$$= \frac{1}{i}\left[\frac{3i \tanh x - i^3 \tanh^3 x}{1 - 3i^2 \tanh^2 x}\right] \qquad [\because \tan (ix) = i \tanh x]$$

$$= \frac{1}{i}\left[\frac{3i \tanh x + i \tanh^3 x}{1 + 3 \tanh^2 x}\right] = \frac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x} = \text{R.H.S.}$$

2.11 Expansions in Series for sinh x and cosh x

We have $\sinh x = \frac{1}{2}[e^x - e^{-x}]$

$$= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right)\right]$$

$$= \frac{1}{2}\left[2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \ldots\right] = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \text{ ad. inf.}$$

Thus $\quad \boldsymbol{\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \text{ ad. inf.}}$

Also $\quad \cosh x = \frac{1}{2}[e^x + e^{-x}]$

$$= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right)\right]$$

$$= \frac{1}{2}\left[2x + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \ldots\right] = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \text{ ad. inf.}$$

Thus $\quad \boldsymbol{\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \text{ ad. inf.}}$

Remark: The students should note that to change a formula for the circular functions into a formula for the hyperbolic functions, we must replace cos x by cosh x and sin x by i sinh x *i.e.*, we must write $\cosh^2 x$ for $\cos^2 x$ and $-\sinh^2 x$ for $\sin^2 x$ etc.

2.12 Periods of Hyperbolic Functions

By Euler's theorem, we know that

$$e^{2n\pi i} = \cos 2n\pi + i \sin 2n\pi = 1 + i\,.\,0 = 1, \; n \text{ being any integer}$$

and $\quad e^{-2n\pi i} = \cos 2n\pi - i \sin 2n\pi = 1 - i\,.\,0 = 1.$

$\therefore \quad \sinh(x + 2n\pi i) = \frac{1}{2}[e^{x+2n\pi i} - e^{-(x+2n\pi i)}]$, by definition

$$= \frac{1}{2}[e^x e^{2n\pi i} - e^{-x} e^{-2n\pi i}] = \frac{1}{2}[e^x - e^{-x}]$$

[$\because e^{\pm 2n\pi i} = 1$, n being any integer]

$= \sinh x.$

Also, $\quad \cosh(x + 2n\pi i) = \frac{1}{2}[e^{x+2n\pi i} + e^{-(x+2n\pi i)}]$ [By definition]

$$= \frac{1}{2}[e^x \,.\, e^{2n\pi i} + e^{-x} \,.\, e^{-2n\pi i}]$$

$$= \frac{1}{2}[e^x + e^{-x}] \qquad [\because \; e^{\pm 2n\pi i} = 1]$$

$$= \cosh x.$$

It follows that sinh x and cosh x are periodic functions and both have a period $2\pi i$.

(Lucknow 2009)

Hence hyperbolic functions differ from circular functions in having no real period. Their periods are imaginary.

Again, $\tanh(x + n\pi i) = \dfrac{\sinh(x + n\pi i)}{\cosh(x + n\pi i)}$

$$= \frac{\frac{1}{2}[e^{(x + n\pi i)} - e^{-(x + n\pi i)}]}{\frac{1}{2}[e^{(x + n\pi i)} + e^{-(x + n\pi i)}]} \qquad \text{[By def.]}$$

$$= \frac{e^{n\pi i}[e^x - e^{-x} \, . \, e^{-2n\pi i}]}{e^{n\pi i}[e^x + e^{-x} \, . \, e^{-2n\pi i}]} \qquad \textbf{(Note)}$$

$$= \frac{[e^x - e^{-x}]}{[e^x + e^{-x}]} \qquad [\because \; e^{-2n\pi i} = 1]$$

$$= \frac{\sinh x}{\cosh x} = \tanh x.$$

Thus $\tanh(x + n\pi i) = \tanh x$,

i.e., tanh x is a periodic function with period πi.

Therefore, sinh x and cosh x are periodic functions of period $2\pi i$ and tanh x is a periodic function of period πi.

Remark: Note that the hyperbolic functions have imaginary periods while the circular functions have real periods.

Illustrative Examples

Example 3: *Show that* $\sinh(x+y)\cosh(x-y) = \dfrac{1}{2}(\sinh 2x + \sinh 2y)$.

(Meerut 2008, 12, 12B)

Solution: The L.H.S. $= \sinh(x + y) \, . \cosh(x - y)$

$$= \frac{1}{2}[e^{(x+y)} - e^{-(x+y)}] \, . \, \frac{1}{2}[e^{(x-y)} + e^{-(x-y)}]$$

$$= \frac{1}{4}[e^{2x} - e^{-2y} + e^{2y} - e^{-2x}]$$

$$= \frac{1}{2} \, . \left[\frac{1}{2}(e^{2x} - e^{-2x}) + \frac{1}{2}(e^{2y} - e^{-2y})\right]$$

$$= \frac{1}{2}[\sinh 2x + \sinh 2y] = \textbf{R.H.S.}$$

Example 4: *Show that*

$$\cos(\alpha + i\beta) + i \sin(\alpha + i\beta) = e^{-\beta}(\cos\alpha + i\sin\alpha).$$ **(Meerut 2012B)**

Solution: The L.H.S. $= \cos\alpha\cos i\beta - \sin\alpha\sin i\beta + i\sin\alpha\cos i\beta + i\cos\alpha\sin i\beta$

$$= \cos\alpha(\cos i\beta + i\sin i\beta) + i\sin\alpha(\cos i\beta + i\sin i\beta)$$
$$= (\cos i\beta + i\sin i\beta)(\cos\alpha + i\sin\alpha)$$
$$= (\cosh\beta - \sinh\beta)(\cos\alpha + i\sin\alpha) \quad [\because \cos i\beta = \cosh\beta, \sin i\beta = i\sinh\beta]$$
$$= e^{-\beta}(\cos\alpha + i\sin\alpha) = \text{R.H.S.}$$

Note: Similarly we can prove that

$$\cos(\alpha - i\beta) - i\sin(\alpha - i\beta) = e^{-\beta}(\cos\alpha - i\sin\alpha).$$

Example 5: *If* $\cosh\alpha = \sec\theta$, *prove that* $\tanh^2\frac{1}{2}\alpha = \tan^2\frac{1}{2}\theta$.

(Avadh 2010; Meerut 13, 13B)

Solution: We have $\cosh\alpha = \sec\theta$.

$$\therefore \quad \frac{\cosh\alpha}{1} = \frac{1}{\cos\theta}.$$

Applying componendo and dividendo, we get

$$\frac{\cosh\alpha - 1}{\cosh\alpha + 1} = \frac{1 - \cos\theta}{1 + \cos\theta} \quad \text{or} \quad \frac{2\sinh^2\frac{1}{2}\alpha}{2\cosh^2\frac{1}{2}\alpha} = \frac{2\sin^2\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta}$$

or
$$\tanh^2\frac{1}{2}\alpha = \tan^2\frac{1}{2}\theta.$$

Note: The componendo and dividendo is that if

$$\frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a-b}{a+b} = \frac{c-d}{c+d}.$$

Example 6: *If* $\tan\theta = \tanh x \cot y$ *and* $\tan\phi = \tanh x \tan y$,

show that $\dfrac{\sin 2\theta}{\sin 2\phi} = \dfrac{\cosh 2x + \cos 2y}{\cosh 2x - \cos 2y}$.

Solution: L.H.S. $= \dfrac{\sin 2\theta}{\sin 2\phi} = \dfrac{2\tan\theta/(1+\tan^2\theta)}{2\tan\phi/(1+\tan^2\phi)}$

$$= \frac{\tan\theta}{1+\tan^2\theta} \times \frac{1+\tan^2\phi}{\tan\phi} = \frac{\tan\theta}{\tan\phi}\cdot\frac{1+\tan^2\phi}{1+\tan^2\theta}$$

$$= \frac{\tanh x\cot y}{\tanh x\tan y} \times \frac{1+\tanh^2 x\tan^2 y}{1+\tanh^2 x\cot^2 y},$$

putting the given values of $\tan\theta$ and $\tan\phi$

$$= \frac{\cos^2 y}{\sin^2 y}\cdot\frac{\cosh^2 x\cos^2 y + \sinh^2 x\sin^2 y}{\cosh^2 x\cos^2 y}$$
$$\times \frac{\cosh^2 x\sin^2 y}{\cosh^2 x\sin^2 y + \sinh^2 x\cos^2 y}$$

$$= \frac{\cosh^2 x \cos^2 y + \sinh^2 x \sin^2 y}{\cosh^2 x \sin^2 y + \sinh^2 x \cos^2 y}$$

$$= \frac{(2\cosh^2 x)(2\cos^2 y) + (2\sinh^2 x)(2\sin^2 y)}{(2\cosh^2 x)(2\sin^2 y) + (2\sinh^2 x)(2\cos^2 y)} \qquad \textbf{(Note)}$$

$$= \frac{(1+\cosh 2x)(\cos 2y + 1) + (\cosh 2x - 1)(1 - \cos 2y)}{(1+\cosh 2x)(1 - \cos 2y) + (\cosh 2x - 1)(1 + \cos 2y)}$$

$$[\because 2\cosh^2 x = 1 + \cosh 2x \,;\, 2\sinh^2 x = \cosh 2x - 1]$$

$$= \frac{2(\cosh 2x + \cos 2y)}{2(\cosh 2x - \cos 2y)} = \frac{\cosh 2x + \cos 2y}{\cosh 2x - \cos 2y} = \text{R.H.S.}$$

Comprehensive Exercise 1

1. Prove that $\sin(\alpha + n\theta) - e^{i\alpha} \sin n\theta = e^{-in\theta} \sin \alpha$.
2. Prove that $\{\sin(\alpha + \theta) - e^{\alpha i} \sin \theta\}^n = \sin^n \alpha \, e^{-n\theta i}$.
3. Show that $\dfrac{1 + \tanh x}{1 - \tanh x} = \cosh 2x + \sinh 2x$. **(Gorakhpur 2005)**
4. Show that $\tanh(x + y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$.
5. Show that
   (i) $\sinh x + \sinh y = 2 \sinh \frac{1}{2}(x + y) \cosh \frac{1}{2}(x - y)$
   (ii) $\cosh x - \cosh y = 2 \sinh \frac{1}{2}(x + y) \sinh \frac{1}{2}(x - y)$.
6. If $\alpha = \log \tan [\pi/4 + \theta/2]$, prove that
   (i) $\tanh \alpha/2 = \tan \theta/2$; (ii) $\cos \theta \cosh \alpha = 1$;
   (iii) $\sinh \alpha = \tan \theta$; **(Bundelkhand 2006)** (iv) $\tanh \alpha = \sin \theta$;
   (v) If $\alpha = \theta + a_3\theta^3 + a_5\theta^5 + \ldots$, show that $\theta = \alpha - a_3\alpha^3 + a_5\alpha^5 - \ldots$
   (Kanpur 2008)
7. If α, β be the imaginary cube roots of unity, prove that
$$\alpha e^{\alpha x} + \beta e^{\beta x} = -e^{-x/2}\left[\sqrt{3} \sin \frac{\sqrt{3}}{2} x + \cos \frac{\sqrt{3}}{2} x\right].$$

2.13 Separation into Real and Imaginary Parts

By separation or resolving into real and imaginary parts, we mean to put a complex quantity in the form $x + iy$ where x and y are real quantities.

When we are given a fraction to separate into real and imaginary parts whose numerator and denominator are both complex functions, we multiply the numerator and denominator by the conjugate of the denominator and thus make the denominator real.

The conjugate complex is obtained by putting $-i$ for i, *i.e.*, $x + iy$ and $x - iy$ are conjugate complex quantities.

Illustrative Examples

Example 7: *Resolve into real and imaginary parts :*

(i) $\sin(\alpha \pm i\beta)$ **(Bundelkhand 2004)** (ii) $\cos(\alpha \pm i\beta)$ **(Rohilkhand 2007)**

(iii) $\tan(\alpha + i\beta)$ **(Bundelkhand 2006; Meerut 12)**

Solution: **(i)** We have

$$\sin(\alpha \pm i\beta) = \sin\alpha\cos(i\beta) \pm \cos\alpha\sin(i\beta)$$

$$= \sin\alpha\cosh\beta \pm i\cos\alpha\sinh\beta.$$

$$[\because \sin(i\beta) = i\sinh\beta \text{ and } \cos(i\beta) = \cosh\beta]$$

(ii)
$$\cos(\alpha \pm i\beta) = \cos\alpha\cos(i\beta) \mp \sin\alpha\sin(i\beta)$$

$$= \cos\alpha\cosh\beta \mp \sin\alpha(i\sinh\beta)$$

$$= \cos\alpha\cosh\beta \mp i\sin\alpha\sinh\beta.$$

Remark: From the expressions for $\sin(\alpha + i\beta)$ and $\sin(\alpha - i\beta)$ into real and imaginary parts, we observe that $\sin(\alpha - i\beta)$ is the conjugate complex of $\sin(\alpha + i\beta)$. Similarly $\cos(\alpha + i\beta)$ and $\cos(\alpha - i\beta)$ are conjugate complex numbers.

(iii) We have $\tan(\alpha + i\beta) = \dfrac{\sin(\alpha + i\beta)}{\cos(\alpha + i\beta)}$.

In this case, the denominator is a complex quantity so we have to multiply the numerator and denominator by the conjugate of the denominator. The conjugate complex of $\cos(\alpha + i\beta)$ is $\cos(\alpha - i\beta)$. Therefore multiplying the Nr. and Dr. by $\cos(\alpha - i\beta)$, we have

$$\tan(\alpha + i\beta) = \frac{\sin(\alpha + i\beta)}{\cos(\alpha + i\beta)} \times \frac{2\cos(\alpha - i\beta)}{2\cos(\alpha - i\beta)} \qquad \textbf{(Note)}$$

$$= \frac{\sin[(\alpha + i\beta) + (\alpha - i\beta)] + \sin[(\alpha + i\beta) - (\alpha - i\beta)]}{\cos[(\alpha + i\beta) + (\alpha - i\beta)] + \cos[(\alpha + i\beta) - (\alpha - i\beta)]}$$

$$= \frac{\sin(2\alpha) + \sin(2i\beta)}{\cos(2\alpha) + \cos(2i\beta)} = \frac{\sin 2\alpha + i\sinh 2\beta}{\cos 2\alpha + \cosh 2\beta}$$

$$= \left(\frac{\sin 2\alpha}{\cos 2\alpha + \cosh 2\beta}\right) + i\left(\frac{\sinh 2\beta}{\cos 2\alpha + \cosh 2\beta}\right).$$

Remark: As proved in part (iii), we have

$$\tan(\alpha + i\beta) = \left(\frac{\sin 2\alpha}{\cos 2\alpha + \cosh 2\beta}\right) + i\left(\frac{\sinh 2\beta}{\cos 2\alpha + \cosh 2\beta}\right).$$

Replacing β by – β on both sides, we have

$$\tan(\alpha - i\beta) = \left(\frac{\sin 2\alpha}{\cos 2\alpha + \cosh 2\beta}\right) - i\left(\frac{\sinh 2\beta}{\cos 2\alpha + \cosh 2\beta}\right).$$

[∵ $\cosh(-x) = \cosh x$, and $\sinh(-x) = -\sinh x$]

From these expressions for $\tan(\alpha + i\beta)$ and $\tan(\alpha - i\beta)$ into real and imaginary parts, we observe that $\tan(\alpha - i\beta)$ is the conjugate complex of $\tan(\alpha + i\beta)$. Thus $\tan(\alpha + i\beta)$ and $\tan(\alpha - i\beta)$ are conjugate complex numbers.

Example 8: *Separate* $\dfrac{\cos(x+iy)}{(x+iy)+1}$ *into real and imaginary parts.*

Solution: We have

$$\frac{\cos(x+iy)}{(x+iy)+1} = \frac{\cos(x+iy)}{(x+1)+iy} = \frac{\cos(x+iy)\{(x+1)-iy\}}{\{(x+1)+iy\}\{(x+1)-iy)\}},$$

multiplying the Nr. and the Dr. by the conjugate complex of the Dr.

$$= \frac{[\cos x \cos(iy) - \sin x \sin(iy)][(x+1)-iy]}{(x+1)^2 - i^2 y^2}$$

$$= \frac{(\cos x \cosh y - i \sin x \sinh y)[(x+1)-iy)]}{(x+1)^2 + y^2}$$

$$= \frac{[(x+1)\cos x \cosh y - y \sin x \sinh y] - i[(x+1)\sin x \sinh y + y \cos x \cosh y]}{(x+1)^2 + y^2}$$

$$= a - ib$$

where $a = [(x+1)\cos x \cosh y - y \sin x \sinh y] / \{(x+1)^2 + y^2\}$

and $b = [(x+1)\sin x \sinh y + y \cos x \cosh y] / \{(x+1)^2 + y^2\}$.

Example 9: *Resolve* $e^{\sin(x+iy)}$ *into real and imaginary parts.*

Solution: We have $e^{\sin(x+iy)} = e^{\sin x \cos(iy) + \cos x \sin(iy)}$

$$= e^{\sin x \cosh y + i \cos x \sinh y} = e^{\sin x \cosh y} \cdot e^{i \cos x \sinh y}$$

$$= e^{\sin x \cosh y}[\cos(\cos x \sinh y) + i \sin(\cos x \sinh y)],$$

[∵ $e^{i\theta} = \cos\theta + i\sin\theta$]

which is of the form $a + ib$.

Example 10: *Resolve* $\sin^2(x+iy)$ *into real and imaginary parts.* **(Meerut 2009)**

Solution: We have $\sin^2(x+iy) = \dfrac{1}{2}[2\sin^2(x+iy)]$

$$= \frac{1}{2}[1 - \cos 2(x + iy)] \qquad [\because 2\sin^2\theta = 1 - \cos 2\theta]$$

$$= \frac{1}{2}[1 - \cos(2x + 2iy)] = \frac{1}{2}[1 - \{\cos 2x \cos(2iy) - \sin 2x \sin(2iy)\}]$$

$$= \frac{1}{2}[1 - \cos 2x \cosh 2y + i \sin 2x \sinh 2y]$$

$$= \frac{1}{2}(1 - \cos 2x \cosh 2y) + i\frac{1}{2}(\sin 2x \sinh 2y).$$

Comprehensive Exercise 2

1. Resolve into real and imaginary parts :

(i) $\cot(\alpha + i\beta)$ (ii) $\sec(\alpha + i\beta)$

(iii) $\operatorname{cosec}(\alpha - i\beta)$

2. Resolve into real and imaginary parts :

(i) $\sinh(\alpha + i\beta)$ **(Bundelkhand 2009)** (ii) $\cosh(\alpha + i\beta)$

(iii) $\tanh(\alpha + i\beta)$ (iv) $\coth(\alpha + i\beta)$.

3. Resolve $e^{\cosh(x \pm iy)}$ into real and imaginary parts.

4. Resolve $e^{\sinh(x+iy)}$ into real and imaginary parts. **(Kumaun 2008)**

5. Split into real and imaginary parts $e^{i\theta}/(1 - ke^{i\phi})$. **(Meerut 2010B)**

6. If $E\left(\dfrac{x - a + iy}{x + a + iy}\right) = P + iQ$, find P and Q.

Answers 2

1. (i) $\left(\dfrac{\sin 2\alpha}{\cosh 2\beta - \cos 2\alpha}\right) - i\left(\dfrac{\sinh 2\beta}{\cosh 2\beta - \cos 2\alpha}\right)$

(ii) $\left(\dfrac{2\cos\alpha\cosh\beta}{\cos 2\alpha + \cosh 2\beta}\right) + i\left(\dfrac{2\sin\alpha\,.\,\sinh\beta}{\cos 2\alpha + \cosh 2\beta}\right)$

(iii) $\left(\dfrac{2\sin\alpha\,.\,\cosh\beta}{\cosh 2\beta - \cos 2\alpha}\right) + i\left(\dfrac{2\cos\alpha\,.\,\sinh\beta}{\cosh 2\beta - \cos 2\alpha}\right)$

2. (i) $(\sinh\alpha\cos\beta) + i(\cosh\alpha\sin\beta)$ $\sinh\alpha\cos\beta - i\cosh\alpha\sin\beta$

(ii) $(\cosh\alpha\cos\beta) + i(\sinh\alpha\sin\beta)$ $\cosh\alpha\cos\beta - i\sinh\alpha\sin\beta$

(iii) $\dfrac{\sinh 2\alpha}{\cosh 2\alpha + \cos 2\beta} + i\dfrac{\sin 2\beta}{\cosh 2\alpha + \cos 2\beta}$

(iv) $\left(\dfrac{\sinh 2\alpha}{\cosh 2\alpha - \cos 2\beta}\right) - i\left(\dfrac{\sin 2\beta}{\cosh 2\alpha - \cos 2\beta}\right)$.

3. $e^{\cosh x \cos y}$ [cos (sinh x sin y) $\pm$ i sin (sinh x sin y),

4. $e^{\sinh x \cos y}$ [cos (cosh x sin y) + i sin (cosh x sin y)]

5. $\left(\dfrac{\cos\theta - k\cos(\theta-\phi)}{1-2k\cos\phi+k^2}\right) + i\left(\dfrac{\sin\theta - k\sin(\theta-\phi)}{1-2k\cos\phi+k^2}\right)$

6. $P = e^p \cos q$ and $Q = e^p \sin q$,

where $p = \dfrac{x^2 + y^2 - a^2}{(x+a)^2 + y^2}$ and $q = \dfrac{2ay}{(x+a)^2 + y^2}$.

Problems Involving Trigonometric and Hyperbolic Funtions:

The problems involving trigonometric and hyperbolic functions can be solved by using the relations given in article 2.9.

Example 11: *If tan* $(\theta + i\phi)$ = *tan* α + *i sec* α, *then prove that*

$$e^{2\phi} = \pm \cot \frac{1}{2}\alpha \quad \text{and} \quad 2\theta = n\pi + \frac{1}{2}\pi + \alpha.$$

(Meerut 2010; Bundelkhand 10; Purvanchal 14; Kashi 14)

Solution: We have $\tan(\theta + i\phi) = \tan\alpha + i\sec\alpha$,

so that $\tan(\theta - i\phi) = \tan\alpha - i\sec\alpha$.

Now $\tan 2\theta = \tan[(\theta + i\phi) + (\theta - i\phi)] = \dfrac{\tan(\theta+i\phi) + \tan(\theta - i\phi)}{1 - \tan(\theta+i\phi)\tan(\theta-i\phi)}$

$$= \frac{(\tan\alpha + i\sec\alpha) + (\tan\alpha - i\sec\alpha)}{1 - (\tan\alpha + i\sec\alpha)(\tan\alpha - i\sec\alpha)} = \frac{2\tan\alpha}{1 - (\tan^2\alpha + \sec^2\alpha)}$$

$$= \frac{2\tan\alpha}{-2\tan^2\alpha} = -\frac{1}{\tan\alpha} = -\cot\alpha.$$

Thus $\tan 2\theta = -\cot\alpha = \tan(\frac{1}{2}\pi + \alpha)$.

$\therefore$ $2\theta = \frac{1}{2}\pi + \alpha$ (principal value)

or $2\theta = n\pi + \frac{1}{2}\pi + \alpha$ (general value)

Again, $\tan 2i\phi = \tan[(\theta+i\phi) - (\theta - i\phi)] = \dfrac{\tan(\theta+i\phi) - \tan(\theta-i\phi)}{1 + \tan(\theta+i\phi)\tan(\theta-i\phi)}$

$$= \frac{(\tan\alpha + i\sec\alpha) - (\tan\alpha - i\sec\alpha)}{1 + (\tan^2\alpha + \sec^2\alpha)} = \frac{2i\sec\alpha}{2\sec^2\alpha} = i\cos\alpha.$$

$\therefore$ $i\tanh 2\phi = i\cos\alpha$, or $\tanh 2\phi = \cos\alpha$

or $\dfrac{e^{2\phi} - e^{-2\phi}}{e^{2\phi} + e^{-2\phi}} = \dfrac{\cos\alpha}{1}$.

Applying componendo and dividendo, we have

or $$\frac{2e^{2\phi}}{2e^{-2\phi}} = \frac{1+\cos\alpha}{1-\cos\alpha}, \text{ or } e^{4\phi} = \frac{2\cos^2 \frac{1}{2}\alpha}{2\sin^2 \frac{1}{2}\alpha} = \cot^2 \frac{1}{2}\alpha.$$

$\therefore$ $$e^{2\phi} = \pm \cot \frac{1}{2}\alpha.$$

Example 12: *If* $\sin(\theta + i\phi) = \cos\alpha + i\sin\alpha$, *then prove that*

$$\cos^2\theta = \sinh^2\phi = \pm\sin\alpha.$$

(Meerut 2009B)

Solution: We have $\sin(\theta + i\phi) = \cos\alpha + i\sin\alpha$

or $$\sin\theta\cosh\phi + i\cos\theta\sinh\phi = \cos\alpha + i\sin\alpha.$$

Equating real and imaginary parts, we have

$$\sin\theta\cosh\phi = \cos\alpha \quad \ldots(1)$$

$$\cos\theta\sinh\phi = \sin\alpha. \quad \ldots(2)$$

First we shall eliminate ϕ between (1) and (2).

We have $\cosh^2\phi - \sinh^2\phi = 1$.

$\therefore$ $$\frac{\cos^2\alpha}{\sin^2\theta} - \frac{\sin^2\alpha}{\cos^2\theta} = 1 \quad \text{or} \quad \frac{\cos^2\alpha\cos^2\theta - \sin^2\alpha\sin^2\theta}{\sin^2\theta\cos^2\theta} = 1$$

or $$\cos^2\alpha\cos^2\theta - \sin^2\alpha\sin^2\theta = \sin^2\theta\cos^2\theta$$

or $$\cos^2\theta(1-\sin^2\alpha) - \sin^2\alpha(1-\cos^2\theta) = \cos^2\theta(1-\cos^2\theta)$$

[changing all the terms to $\cos\theta$ and $\sin\alpha$]

or $$\cos^2\theta - \cos^2\theta\sin^2\alpha - \sin^2\alpha + \sin^2\alpha\cos^2\theta = \cos^2\theta - \cos^4\theta$$

or $$\cos^4\theta = \sin^2\alpha.$$

$\therefore$ $$\cos^2\theta = \pm\sin\alpha.$$

Now we shall eliminate θ between (1) and (2) .

We have $\sin^2\theta + \cos^2\theta = 1$.

Therefore from (1) and (2), we have

$$\frac{\cos^2\alpha}{\cosh^2\phi} + \frac{\sin^2\alpha}{\sinh^2\phi} = 1$$

or $$\cos^2\alpha\sinh^2\phi + \cosh^2\phi\sin^2\alpha = \cosh^2\phi\sinh^2\phi$$

or $$\sinh^2\phi(1-\sin^2\alpha) + \sin^2\alpha(1+\sinh^2\phi) = (1+\sinh^2\phi)\sinh^2\phi$$

[changing all the terms to $\sinh\phi$ and $\sin\alpha$]

or $$\sinh^2\phi - \sinh^2\phi\sin^2\alpha + \sin^2\alpha + \sinh^2\phi\sin^2\alpha = \sinh^2\phi + \sinh^4\phi$$

or $$\sinh^4\phi = \sin^2\alpha.$$

$\therefore$ $$\sinh^2\phi = \pm\sin\alpha.$$

Thus $$\cos^2\theta = \sinh^2\phi = \pm\sin\alpha.$$

Example 13: *If* $\sin(\theta + i\phi) = \rho(\cos\alpha + i\sin\alpha)$, *prove that*

$$\rho^2 = \frac{1}{2}[\cosh 2\phi - \cos 2\theta] \text{ and } \tan\alpha = \tanh\phi\cot\theta.$$

(Garhwal 2001)

Solution: We have

$$\sin(\theta + i\phi) = \rho\cos\alpha + i\rho\sin\alpha$$

or $\sin\theta\cosh\phi + i\cos\theta\sinh\phi = \rho\cos\alpha + i\rho\sin\alpha.$

$\therefore$ $\sin\theta\cosh\phi = \rho\cos\alpha$ ...(1)

and $\cos\theta\sinh\phi = \rho\sin\alpha.$ ...(2)

Squaring (1) and (2) and adding, we get

$$\rho^2 = \sin^2\theta\cosh^2\phi + \cos^2\theta\sinh^2\phi.$$

$\therefore$ $2\rho^2 = 2\sin^2\theta\cosh^2\phi + 2\cos^2\theta\sinh^2\phi$

$= (1 - \cos 2\theta)\cosh^2\phi + (1 + \cos 2\theta)\sinh^2\phi,$

changing to double angles

$= (\cosh^2\phi + \sinh^2\phi) - \cos 2\theta(\cosh^2\phi - \sinh^2\phi)$

$= \cosh 2\phi - \cos 2\theta.$ $[\because \cosh^2\phi + \sinh^2\phi = \cosh 2\phi]$

$\therefore$ $\rho^2 = \frac{1}{2}(\cosh 2\phi - \cos 2\theta).$

Again, dividing (2) by (1), we get

$$\frac{\cos\theta\sinh\phi}{\sin\theta\cosh\phi} = \frac{\sin\alpha}{\cos\alpha} \text{ or } \tan\alpha = \tanh\phi\cot\theta.$$

Comprehensive Exercise 3

1. If $\sin(\alpha + i\beta) = x + iy$, prove that

   (i) $x^2\operatorname{cosec}^2\alpha - y^2\sec^2\alpha = 1,$ **(Bundelkhand 2005, 14; Avadh 11)**

   (ii) $x^2\operatorname{sech}^2\beta + y^2\operatorname{cosech}^2\beta = 1.$

   (Kanpur 2005, 12; Avadh 11; Bundelkhand 14)

2. If $\cos^{-1}(x + iy) = A + iB$, prove that

   (i) $x^2\sec^2 B + y^2\operatorname{cosech}^2 B = 1$

   (ii) $x^2\sec^2 A - y^2\operatorname{cosec}^2 A = 1.$ **(Garhwal 2000)**

3. If $\tan(\theta + i\phi) = \sin(x + iy)$, then prove that

   $\coth y\sinh 2\phi = \cot x\sin 2\theta.$ **(Purvanchal 2011; Agra 14)**

4. If $\tanh(\alpha + i\beta) = \sin(x + iy)$, prove that $\sinh 2\alpha\operatorname{cosec} 2\beta = \tan x\coth y.$

5. If $\tan(\alpha + i\beta) = x + iy$, prove that

   (i) $x^2 + y^2 + 2x\cot 2\alpha = 1;$ **(Kanpur 2010)**

   (ii) $x^2 + y^2 - 2y\coth 2\beta + 1 = 0;$ **(Kanpur 2010)**

   (iii) $x\coth 2\alpha + y\coth 2\beta = 1.$

6. If $\tan(\theta + i\phi) = \cos\alpha + i\sin\alpha$, prove that

$$\theta = \frac{n\pi}{2} + \frac{\pi}{4} \text{ and } \phi = \frac{1}{2}\log\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) \text{ or } e^{2\phi} = \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right).$$

(Garhwal 2002; Agra 07; Gorakhpur 07; Meerut 08; Purvanchal 10)

7. If $\tan(\alpha + i\beta) = i$, where α and β are real, then prove that α is indeterminate and β is infinite. **(Bundelkhand 2007, 14; Purvanchal 14)**

8. If $\sin(\theta + i\phi) = \tan\alpha + i\sec\alpha$, then prove that $\cos 2\theta \cosh 2\phi = 3$.
(Garhwal 2001; Avadh 07, 09; Purvanchal 08; Agra 09, 10; Kashi 13)

9. If $\cos(x + iy) = \cos\alpha + i\sin\alpha$, prove that
(i) $\cosh 2y + \cos 2x = 2$ (ii) $\sin^4 x = \sin^2\alpha$
(iii) $\sinh^4 y = \sin^2\alpha$.

10. If $\cos(\theta + i\phi) = r(\cos\alpha + i\sin\alpha)$, then prove that $\phi = \frac{1}{2}\log\frac{\sin(\theta - \alpha)}{\sin(\theta + \alpha)}$.

(Bundelkhand 2003; Avadh 05; Kanpur 06; Kashi 12; Rohilkhand 09)

11. If $\cosh u = \sec\theta$, show that

$$u = \log\tan\left(\frac{1}{4}\pi + \frac{1}{2}\theta\right)$$ **(Meerut 2013)**

and $$\tanh^2 \frac{1}{2}u = \tan^2 \frac{1}{2}\theta.$$ **(Meerut 2011; Avadh 10)**

12. If $A + iB = C\tan(x + iy)$, then prove that $\tan 2x = \dfrac{2CA}{C^2 - A^2 - B^2}$.

13. If $u + iv = \cot(x + iy)$, show that $v = -\dfrac{\sinh 2y}{\cosh 2y - \cos 2x}$.

14. If $\cos(u + iv) = x + iy$, prove that
$(1 + x)^2 + y^2 = (\cosh v + \cos u)^2$ and $(1 - x)^2 + y^2 = (\cosh v - \cos u)^2$,
where x, y, u, v are all real.

15. If $\cosh(x + iy) = p + iq$, where x, y, p and q are real, show that

$$\frac{p^2}{\cos^2 y} - \frac{q^2}{\sin^2 y} = 1.$$

16. If $\sin(x + iy)\sin(\theta + i\phi) = 1$, show that
(i) $\tanh^2 y \cosh^2\phi = \cos^2\theta$; **(Kanpur 2010)**
(ii) $\tanh^2\phi \cosh^2 y = \cos^2 x$. **(Kanpur 2010)**

17. If $\cos(\theta + i\phi)\cos(\alpha + i\beta) = 1$, prove that
$\tanh^2\phi \cosh^2\beta = \sin^2\alpha$ and $\tanh^2\beta \cosh^2\phi = \sin^2\theta$. **(Kanpur 2009)**

18. If $x = 2\cos\alpha\cosh\beta$ and $y = \sin\alpha\sinh\beta$, prove that

(i) $\sec(\alpha + i\beta) + \sec(\alpha - i\beta) = \dfrac{4x}{x^2 + y^2}$.

(ii) $\sec(\alpha + i\beta) - \sec(\alpha - i\beta) = \dfrac{4iy}{x^2 + y^2}$.

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. If $\sin(x + iy) = p + iq$, where p and q are real, then

(a) $q = \sin x \cos y$ (b) $q = \cos x \sin y$

(c) $q = \sin x \cosh y$ (d) $q = \cos x \sinh y$.

2. If θ is real, then

(a) $\cos(i\theta) = i\cosh\theta$ (b) $\sin(i\theta) = i\sinh\theta$

(c) $\tan(i\theta) = \tanh\theta$ (d) $\cot(i\theta) = i\coth\theta$.

Fill in the Blanks

Fill in the blanks "……" so that the following statements are complete and correct.

1. The Euler's theorem states that $e^{i\theta}$ = ……
2. The exponential value of $\cos\theta$ is …… .
3. The exponential value of $\sin\theta$ is …… .
4. If $e^{x+iy} = a + ib$, where a and b are real, then b = ……
5. If $\cos(x + iy) = a + ib$, where a and b are real, then a = ……
6. If $\cos(x + iy) = a + ib$, where a and b are real, then b = ……
7. The complex conjugate of $\cos(x - iy)$ is …… .
8. If $e^{\sin(x+iy)} = a + ib$, where a and b are real, then b = ……

True or False

Write 'T' for true and 'F' for false statement.

1. $\cot(i\theta) = -i\coth\theta$.
2. If $\sin(x - iy) = a + ib$, where a and b are real, then $b = \cos x \sinh y$.
3. If $\sin(\alpha + i\beta) = x + iy$, then $\dfrac{x^2}{\sin^2\alpha} + \dfrac{y^2}{\cos^2\alpha} = 1$.
4. $\cos(\alpha + i\beta) = \cos\alpha\cosh\beta + i\sin\alpha\sinh\beta$.

5. $\tan(\alpha + i\beta) = \left(\dfrac{\sin 2\alpha}{\cos 2\alpha + \cosh 2\beta}\right) + i\left(\dfrac{\sinh 2\beta}{\cos 2\alpha + \cosh 2\beta}\right).$

Answers

Multiple Choice Questions

1. (d) 2. (b)

Fill in the Blanks

1. $\cos\theta + i\sin\theta$ 2. $\dfrac{e^{i\theta} + e^{-i\theta}}{2}$ 3. $\dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

4. $e^x \sin y$ 5. $\cos x \cosh y$ 6. $-\sin x \sinh y$

7. $\cos(x + iy)$ 8. $e^{\sin x \cosh y} \sin(\cos x \sinh y)$

True or False

1. *T* 2. *F* 3. *F* 4. *F* 5. *T*

□

Chapter

3

Logarithms of Complex Numbers

3.1 Logarithms in the Set of Real Numbers

We know that if a and x are two real numbers such that $e^x = a$, then x is called the logarithm of a to the base e and we write it as

$$x = \log_e a.$$

Since for all real numbers x, we have $e^x > 0$, therefore, in the case of a real variable, $\log_e a$ exists if and only if a is positive. Also if a is a positive real number, $\log_e a$ is a unique real number.

3.2 Logarithms of Complex Numbers

(Gorakhpur 2006)

Now we shall extend our definition of logarithmic function to the set of complex numbers.

Definition: Let z and w be two complex numbers such that $e^z = w$. Then z is called a *logarithm of* w to the base e and we write it as $z = \log_e w$ or simply as $z = \log w$, the base e remaining understood.

Thus by definition, $z = \log w$ if and only if $w = e^z$.

Since $e^z \neq 0$ for any complex number z, therefore $\log w$ does not exist if $w = 0$.

Logarithm of a complex number is a many-valued function:

Let $\log_e w = z$. Then $e^z = w$.

If n is any integer, we have

$$e^{2n\pi i} = \cos 2\,n\pi + i\sin 2n\pi = 1 + i0 = 1.$$ **(Bundelkhand 2005)**

$$\therefore \qquad e^z = e^z \,.\, 1 = e^z\, e^{2n\pi i} = e^{z + 2n\pi i},$$

which means that if $\log w = z$, then we also have

$$\log w = z + 2n\pi i.$$

Thus logarithm of a complex number is a many-valued function.

3.3 Principal and General Values of Logarithm of a Non-Zero Complex Number

(Gorakhpur 2006)

Let $z = x + iy$ be a non-zero complex number. Suppose

$$\log_e z = \alpha + i\beta,$$

where α and β are real. Then

$$z = e^{\alpha + i\beta} = e^{\alpha}\, e^{i\beta} = e^{\alpha}\,(\cos\beta + i\sin\beta).$$

Since e^{α} is a positive real number, therefore

$$z = e^{\alpha}\,(\cos\beta + i\sin\beta)$$

is a representation of z in a modulus-argument form. We have

$$e^{\alpha} = |z| = \sqrt{(x^2 + y^2)} \;\text{ and }\; \beta = \arg z = \tan^{-1}(y/x).$$

The equation $e^{\alpha} = |z|$ has a unique solution $\alpha = \log |z|$, the real logarithm of the positive number $|z|$.

$$\therefore \qquad \log_e z = \log |z| + i\arg z$$

$$i.e., \qquad \log_e (x + iy) = \log \sqrt{(x^2 + y^2)} + i\tan^{-1}(y/x).$$

Now $\arg z$ is a many-valued function and consequently $\log z$ is a many-valued function. If β_0 is the principal value of $\arg z$ *i.e.*, the value of $\arg z$ lying between $-\pi$ and π, then $\log |z| + i\beta_0$ is called *the principal value* of $\log z$. Again if β_0 is the principal value of $\arg z$, then $2n\pi + \beta_0$ is the general value of $\arg z$ and

$$\log |z| + i\beta_0 + 2n\pi i$$

is called *the general value of* $\log z$. *Thus every non-zero complex number has infinitely many logarithms which differ from one another by an integral multiple of* $2\pi i$.

It is usual to denote the general value of $\log_e z$ by $\text{Log}_e\, z$, using the first letter L as the capital letter, and the principal value by $\log_e z$, using the first letter l as the small letter. Thus we have

$$\text{Log}_e\ z = \log_e z + 2n\pi i, \text{ where } n \text{ is any integer.}$$

If in the general value, we put $n = 0$, we get the principal value.

Remember :

$$\log_e (x + iy) = \log \sqrt{(x^2 + y^2)} + i \tan^{-1} (y / x),$$

where $\tan^{-1} (y / x)$ represents the principal value of arg $(x + iy)$

and $$\text{Log}_e (x + iy) = \log \sqrt{(x^2 + y^2)} + i \tan^{-1} (y / x) + 2n\pi i,$$

where n is any integer.

3.4 Properties of the Logarithmic Function

If u and v are two non-zero complex numbers, then

$$Log\ (uv) = Log\ u + Log\ v \quad \ldots(1)$$

and $$Log\ (u / v) = Log\ u - Log\ v. \quad \ldots(2)$$

Proof: Let Log $u = z_1$ and Log $v = z_2$.

Then $e^{z_1} = u$ and $e^{z_2} = v$. $\therefore$ $e^{z_1} e^{z_2} = uv$, or $e^{z_1 + z_2} = uv$.

By the definition of logarithm, we have

Log $uv = z_1 + z_2 =$ Log u + Log v, which proves the result (1).

The result (2) can be proved similarly.

The equality (1) does not mean that the principal value of log uv is equal to the sum of the principal values of log u and log v. Note that the sum of the principal arguments of u and v need not be equal to the principal argument of uv. The equality (1) implies that each value of log u + log v is equal to some value of log uv and each value of log uv is equal to some value of

$$\log u + \log v.$$

A similar remark holds for the equality (2) also.

Remark: If z be a non-zero complex number and n be a positive integer, the equality $\log z^n = n \log z$ may not be true even for the general values.

As an illustration, we have

$$\text{Log } i^3 = \text{Log } (-i) = (2n\pi - \frac{1}{2}\pi)\, i, \quad \text{while} \quad 3 \text{ Log } i = 3\, (2m\pi + \frac{1}{2}\pi)\, i.$$

Now it is not correct to say that

$$3\, (2m\pi + \frac{1}{2}\pi) = 2n\pi - \frac{1}{2}\pi.$$

For, the left hand side may be written as $2\,(3m + 1)\,\pi - \frac{1}{2}\pi$,

which shows that the solution set on the left is only a subset of the solution set on the right.

Thus $$\text{Log } i^3 \neq 3 \text{ Log } i.$$

3.5 Working Rule to Evaluate Log $(x+iy)$ *i.e.*, to Express Log $(x+iy)$ in the Form $A+iB$

(Meerut 2010)

First we put $x + iy$ in the modulus-argument form.

So let $\qquad x + iy = r(\cos\theta + i\sin\theta)$.

Then $x = r\cos\theta$, $y = r\sin\theta$. $\qquad \therefore\ r = \sqrt{(x^2 + y^2)}$ and $\theta = \tan^{-1}(y/x)$.

Now Log $(x + iy) = \text{Log}\,(r\cos\theta + ir\sin\theta) = \text{Log}\, r(\cos\theta + i\sin\theta)$

$$= \log\,[r\{\cos(2n\pi+\theta) + i\sin(2n\pi+\theta)\}] = \log r\, e^{i(2n\pi+\theta)}$$

$$= \log r + \log e^{i(2n\pi+\theta)} = \log r + i(2n\pi+\theta)$$

$$= \log\sqrt{(x^2+y^2)} + i\{2n\pi + \tan^{-1}(y/x)\}$$

$$= \frac{1}{2}\log(x^2+y^2) + i\{2n\pi + \tan^{-1}(y/x)\} = A + iB,$$

where $\qquad A = \dfrac{1}{2}\log(x^2+y^2),\ B = 2n\pi + \tan^{-1}(y/x)$.

If we put $n = 0$, we obtain the principal value of Log $(x + iy)$ written as $\log(x+iy)$.

Thus $\qquad \log(x+iy) = \dfrac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$.

(Rohilkhand 2006, 08)

Putting $-y$ for y on both sides, we get

$$\log(x-iy) = \frac{1}{2}\log(x^2+y^2) - i\tan^{-1}(y/x).$$

3.6 Logarithm of a Positive Real Number in the Set of Complex Numbers

Let x be a positive real number.

Let $\qquad x = x + i0 = r(\cos\theta + i\sin\theta)$. $\qquad$ Then $x = r\cos\theta$, $0 = r\sin\theta$.

$\therefore \qquad r = x$ and $\theta = 0$.

$\therefore \qquad \text{Log}\ x = \log x + i0 + 2n\pi i = \log x + 2n\pi i.$

Obviously only $n = 0$ gives a real value of Log x.

Thus in the set of complex numbers a positive real number has an infinite number of logarithms out of which only one is real.

3.7 Logarithm of a Negative Real Number

Let x be a positive real number so that $-x$ is a negative real number. We have to find Log $(-x)$.

Let $\quad -x = -x + i0 = r(\cos\theta + i\sin\theta)$. Then $\quad -x = r\cos\theta, 0 = r\sin\theta$.

$\therefore \quad r = x$, and $\theta = \pi$.

$$\therefore \quad \text{Log}(-x) = \text{Log}\, x(\cos\pi + i\sin\pi)$$
$$= \log x\{\cos(2n\pi + \pi) + i\sin(2n\pi + \pi)\}$$
$$= \log xe^{(2n\pi + \pi)i} = \log x + \log e^{(2n\pi + \pi)i}$$
$$= \log x + (2n + 1)\pi i, \text{ which is never real.}$$

The principal value of $\log(-x)$ is $\log x + i\pi$.

Hence the principal value of the logarithm of a negative real number is the logarithm of the corresponding positive number added with πi.

Illustrative Examples

Example 1: *Find the principal and general value of log* $(-1 + i)$.

Solution: Let $-1 + i = r(\cos\theta + i\sin\theta)$, so that $r\cos\theta = -1$ and $r\sin\theta = 1$.

Squaring and adding, we have $r^2 = 2$, *i.e.*, $r = \sqrt{2}$.

Now $\quad \cos\theta = -1/\sqrt{2}$ and $\sin\theta = 1/\sqrt{2}$, so that $\quad \theta = \frac{3}{4}\pi$.

$$\therefore \quad -1 + i = \sqrt{2}\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right) = \sqrt{2}\, e^{(3\pi/4)i}.$$

$\therefore$ the general value is

$$\text{Log}(-1 + i) = \log\{\sqrt{2}\, e^{(3\pi/4)i}\, e^{2n\pi i}\}$$
$$= \log\sqrt{2} + \frac{3}{4}\pi i + 2n\pi i = \frac{1}{2}\log 2 + \left(2n + \frac{3}{4}\right)\pi i.$$

Putting $n = 0$, the principal value is given by

$$\log(-1 + i) = \frac{1}{2}\log 2 + \frac{3}{4}\pi i.$$

Example 2: *Find the general value of log* (-3).

(Bundelkhand 2006; Rohilkhand 06, 08)

Solution: Let $-3 = -3 + i0 = r(\cos\theta + i\sin\theta)$,

so that $\quad -3 = r\cos\theta$ and $0 = r\sin\theta$.

These give $r^2 = 9$ *i.e.*, $r = 3$. Putting $r = 3$, we get

$$\cos\theta = -1 \text{ and } \sin\theta = 0, \text{ giving } \theta = \pi.$$

$$\therefore \quad -3 = 3(\cos\pi + i\sin\pi) = 3\,.\,e^{i\pi}.$$

$$\therefore \quad \text{Log}(-3) = \log\{3\,.\,e^{i\pi}\,.\,e^{2n\pi i}\} \qquad [\because e^{2n\pi i} = 1]$$
$$= \log 3 + \log e^{(2n\pi + \pi)i} = \log 3 + (2n + 1)\pi i.$$

The principal value of Log (–3) *i.e.*, log (–3) is obtained by putting $n = 0$ in the above result.

Thus $\log(-3) = \log 3 + i\pi$.

Example 3: *Find the general value of log (–i).* **(Agra 2007)**

Solution: We have $-i = (\cos \frac{1}{2}\pi - i \sin \frac{1}{2}\pi) = e^{-i\pi/2}$

so that $\log(-i) = \log e^{-i\pi/2} = -i\pi/2$, giving the principal value.

$\therefore$ $\text{Log}(-i) = \log(-i) + 2n\pi i = -(i\pi/2) + 2n\pi i = \frac{1}{2}(4n-1)\pi i.$

Example 4: *Express* $\log(1+i)^{(1-i)}$ *in the form* $A + iB$.

Solution: We have

$$\log(1+i)^{(1-i)} = (1-i)\log(1+i)$$

$$= (1-i)[\frac{1}{2}\log(1^2 + 1^2) + i\tan^{-1} 1]$$

$$= (1-i)[\frac{1}{2}\log 2 + i\frac{1}{4}\pi]$$

$$= (\frac{1}{2}\log 2 + \frac{1}{4}\pi) + i(\frac{1}{4}\pi - \frac{1}{2}\log 2),$$

which is of the form $A + iB$.

Example 5: *Prove that* $\log\left(\frac{1}{1-e^{i\alpha}}\right) = \log\left(\frac{1}{2}\operatorname{cosec}\frac{\alpha}{2}\right) + i\left(\frac{\pi}{2} - \frac{\alpha}{2}\right).$

(Meerut 2009B; Kashi 14)

Solution: We have

$$\log\frac{1}{1-e^{i\alpha}} = \log\frac{1}{1-\cos\alpha - i\sin\alpha} \qquad [\because e^{i\alpha} = \cos\alpha + i\sin\alpha]$$

$$= \log\frac{1}{2\sin^2\frac{1}{2}\alpha - 2i\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha}$$

$$= \log\frac{1}{2\sin\frac{1}{2}\alpha(\sin\frac{1}{2}\alpha - i\cos\frac{1}{2}\alpha)}$$

$$= \log\frac{1}{2\sin\frac{1}{2}\alpha\{\cos(\frac{1}{2}\pi - \frac{1}{2}\alpha) - i\sin(\frac{1}{2}\pi - \frac{1}{2}\alpha)\}}$$

$$= \log\frac{1}{2\sin\frac{1}{2}\alpha\, e^{-i(\pi/2-\alpha/2)}} \qquad [\because e^{-i\theta} = \cos\theta - i\sin\theta]$$

$$= \log[(\frac{1}{2}\operatorname{cosec}\frac{1}{2}\alpha) \,.\, e^{i(\pi/2-\alpha/2)}]$$

$$= \log\left(\frac{1}{2}\operatorname{cosec}\frac{1}{2}\alpha\right) + \log e^{i(\pi/2-\alpha/2)}$$

$$= \log\left(\frac{1}{2}\operatorname{cosec}\frac{1}{2}\alpha\right) + i\left(\frac{1}{2}\pi - \frac{1}{2}\alpha\right).$$

Example 6: *Prove that* $\log \tan\left(\frac{1}{4}\pi + i\frac{1}{2}\alpha\right) = i\tan^{-1}(\sinh\alpha)$.

(Agra 2005, 06; Meerut 07, 08, 11, 13B; Bundelkhand 10; Avadh 09, 11; Kanpur 11; Kashi 13)

Solution: L.H.S. $= \log\tan\left(\frac{1}{4}\pi + i\frac{1}{2}\alpha\right) = \log\left\{\frac{\sin\left(\frac{1}{4}\pi + i\frac{1}{2}\alpha\right)}{\cos\left(\frac{1}{4}\pi + i\frac{1}{2}\alpha\right)}\right\}$

$$= \log\left\{\frac{2\sin\left(\frac{1}{4}\pi + \frac{1}{2}\alpha i\right)\cos\left(\frac{1}{4}\pi - \frac{1}{2}\alpha i\right)}{2\cos\left(\frac{1}{4}\pi + \frac{1}{2}\alpha i\right)\cos\left(\frac{1}{4}\pi - \frac{1}{2}\alpha i\right)}\right\},$$ multiplying the Nr.

and the Dr. by the conjugate complex of the Dr.

$$= \log\left[\frac{\sin\left(\frac{1}{4}\pi + \frac{1}{4}\pi\right) + \sin\left(\frac{1}{2}\alpha i + \frac{1}{2}\alpha i\right)}{\cos\left(\frac{1}{4}\pi + \frac{1}{4}\pi\right) + \cos\left(\frac{1}{2}\alpha i + \frac{1}{2}\alpha i\right)}\right]$$

$$= \log\left[\frac{\sin\frac{1}{2}\pi + \sin(\alpha i)}{\cos\frac{1}{2}\pi + \cos(\alpha i)}\right] \qquad = \log\left[\frac{1 + i\sinh\alpha}{\cosh\alpha}\right]$$

[$\because \sin(i\alpha) = i\sinh\alpha, \cos(i\alpha) = \cosh\alpha$]

$$= \log\left[\frac{1}{\cosh\alpha} + i\frac{\sinh\alpha}{\cosh\alpha}\right]$$

$$= \frac{1}{2}\log\left(\frac{1}{\cosh^2\alpha} + \frac{\sinh^2\alpha}{\cosh^2\alpha}\right) + i\tan^{-1}\left(\frac{\sinh\alpha/\cosh\alpha}{1/\cosh\alpha}\right)$$

[By article 3.5, here $x = 1/\cosh\alpha$, $y = \sinh\alpha/\cosh\alpha$]

$$= \frac{1}{2}\log\left(\frac{1+\sinh^2\alpha}{\cosh^2\alpha}\right) + i\tan^{-1}(\sinh\alpha)$$

$$= \frac{1}{2}\log\left(\frac{\cosh^2\alpha}{\cosh^2\alpha}\right) + i\tan^{-1}(\sinh\alpha)$$

$$= \frac{1}{2}\log 1 + i\tan^{-1}(\sinh\alpha) = i\tan^{-1}(\sinh\alpha) = \text{R.H.S.}$$

Example 7: *Separate Log sin* $(x + iy)$ *into real and imaginary parts.*

Solution: We have

$$\text{Log}\sin(x+iy) = \text{Log}(\sin x\cos iy + \cos x\sin iy)$$

$= \text{Log}(\sin x \cosh y + i \cos x \sinh y)$

$= \frac{1}{2} \log (\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y)$

$+ i \tan^{-1} \left(\frac{\cos x \sinh y}{\sin x \cosh y} \right) + 2 n\pi i$ [By article 3.5]

$= \frac{1}{2} \log \{\frac{1}{2}(1 - \cos 2x) \cosh^2 y + \frac{1}{2}(1 + \cos 2x) \sinh^2 y\}$

$+ i \tan^{-1} (\cot x \tanh y) + 2n\pi i$

$= \frac{1}{2} \log \{\frac{1}{2}(\cosh^2 y + \sinh^2 y) - \frac{1}{2} \cos 2x (\cosh^2 y - \sinh^2 y)\}$

$+ i \tan^{-1} (\cot x \tanh y) + 2n\pi i$

$= \frac{1}{2} \log (\frac{1}{2} \cosh 2y - \frac{1}{2} \cos 2x) + i [2n\pi + \tan^{-1} (\cot x \tanh y)]$

$= \frac{1}{2} \log [\frac{1}{2}(\cosh 2y - \cos 2x)] + i [2n\pi + \tan^{-1} (\cot x \tanh y)],$

which is of the form $P + iQ$.

To find out the principal value, we put $n = 0$ in the above result. So we have

$$\log \sin (x + iy) = \frac{1}{2} \log [\frac{1}{2}(\cosh 2y - \cos 2x)] + i \tan^{-1} (\cot x \tanh y).$$

Example 8: *If* $\log \sin (x + iy) = \alpha + i\beta$, *show that*

(i) $\alpha = \frac{1}{2} \log \frac{1}{2} (\cosh 2y - \cos 2x)$, **(Meerut 2013B)**

(ii) $2 \cos 2x = e^{2y} + e^{-2y} - 4e^{2\alpha}$, (iii) $\beta = \tan^{-1} (\cot x \tanh y)$,

(iv) $\cos (x - \beta) = e^{2y} \cos (x + \beta)$.

Solution: $\log \sin (x + iy) = \alpha + i\beta, \quad \Rightarrow \sin (x + iy) = e^{\alpha + i\beta} = e^{\alpha} e^{i\beta}$

or $\quad \sin x \cos iy + \cos x \sin iy = e^{\alpha} (\cos \beta + i \sin \beta)$

or $\quad \sin x \cosh y + i \cos x \sinh y = e^{\alpha} \cos \beta + i e^{\alpha} \sin \beta.$

Equating real and imaginary parts, we have

$\sin x \cosh y = e^{\alpha} \cos \beta$ ...(1)

and $\quad \cos x \sinh y = e^{\alpha} \sin \beta.$ ...(2)

Squaring and adding (1) and (2), we get

$$\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y = e^{2\alpha} (\cos^2 \beta + \sin^2 \beta)$$

or $\quad \frac{1}{2}(1 - \cos 2x) \cosh^2 y + \frac{1}{2}(1 + \cos 2x) \sinh^2 y = e^{2\alpha}$

or $\quad \frac{1}{2}(\cosh^2 y + \sinh^2 y) - \frac{1}{2} \cos 2x (\cosh^2 y - \sinh^2 y) = e^{2\alpha}$

or $\quad \frac{1}{2} \cosh 2y - \frac{1}{2} \cos 2x = e^{2\alpha}$ or $\frac{1}{2}(\cosh 2y - \cos 2x) = e^{2\alpha}$. ...(A)

$\therefore \qquad 2\alpha = \log \frac{1}{2}(\cosh 2y - \cos 2x),$

or $\qquad \alpha = \frac{1}{2} \log \frac{1}{2}(\cosh 2y - \cos 2x)$, which proves the result (i).

From the result (A), we have

$$\cosh 2y - \cos 2x = 2e^{2\alpha} \quad \text{or} \quad \cosh 2y = \cos 2x + 2e^{2\alpha}$$

or $\qquad 2 \cosh 2y = 2 \cos 2x + 4e^{2\alpha}$

or $\qquad 2 \cos 2x = 2 \cosh 2y - 4e^{2\alpha} = e^{2y} + e^{-2y} - 4e^{2\alpha},$

which proves the result (ii).

Dividing (2) by (1), we get

$$\tan \beta = \frac{\cos x \sinh y}{\sin x \cosh y} = \cot x \tanh y. \qquad \text{...(B)}$$

$\therefore \qquad \beta = \tan^{-1}(\cot x \tanh y)$, which proves the result (iii).

From the result (B), we have

$$\tanh y = \frac{\tan \beta}{\cot x} \quad \text{or} \quad \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{\sin \beta \sin x}{\cos \beta \cos x} \cdot$$

Applying componendo and dividendo, we have

$$\frac{(e^y + e^{-y}) + (e^y - e^{-y})}{(e^y + e^{-y}) - (e^y - e^{-y})} = \frac{\cos \beta \cos x + \sin \beta \sin x}{\cos \beta \cos x - \sin \beta \sin x}$$

or $\qquad \dfrac{2e^y}{2e^{-y}} = \dfrac{\cos (x - \beta)}{\cos (x + \beta)} \quad \text{or} \quad e^{2y} = \dfrac{\cos (x - \beta)}{\cos (x + \beta)}$

or $\qquad \cos (x - \beta) = e^{2y} \cos (x + \beta)$, which proves the result (iv).

Comprehensive Exercise 1

1. (i) Prove that $\text{Log}(1 + i) = \frac{1}{2} \log 2 + i\,(2n\pi + \frac{1}{4}\pi)$. **(Meerut 2012; Avadh 14)**

(ii) Prove that $\text{Log}(-5) = \log 5 + (2n\pi + \pi)\, i$.

(iii) Find the general value of $\log i$. **(Kanpur 2005)**

2. (i) Find the general value of $\log \sqrt{i}$. **(Rohilkhand 2007)**

(ii) Show that $\log (1 + e^{i\theta}) = \log (2 \cos \frac{1}{2} \theta) + \frac{1}{2} i\theta$, if $-\pi < \theta < \pi$.

3. (i) Prove that $\log \left(\dfrac{a + ib}{a - ib} \right) = 2i \tan^{-1} \left(\dfrac{b}{a} \right)$. **(Avadh 2009; Purvanchal 11)**

(ii) Show that $i \log \dfrac{x - i}{x + i} = \pi - 2 \tan^{-1} x$. **(Kanpur 2008; Avadh 08, 10; Purvanchal 11; Kashi 13)**

4. (i) Show that $\tan\left(i \log \dfrac{a - ib}{a + ib}\right) = \dfrac{2ab}{a^2 - b^2}$.

(Meerut 2004, 10, 10B; Avadh 05; Purvanchal 07, 09)

(ii) Prove that $\log(1 + i \tan \alpha) = \log \sec \alpha + \alpha i$.

5. Prove that $\sin(\log i^i) = -1$.

6. If $u = \log \tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)$, prove that

(i) $\tanh \dfrac{u}{2} = \tan \dfrac{\theta}{2}$ **(Kanpur 2007; Meerut 10, 12B; Kashi 11; Agra 14)**

(ii) $\theta = -i \log \tan\left(\dfrac{\pi}{4} + \dfrac{iu}{2}\right)$.

(Gorakhpur 2005; Purvanchal 08; Avadh 13)

7. Prove that $\log(1 + \cos 2\theta + i \sin 2\theta) = \log(2 \cos \theta) + i\theta$, if $-\pi < \theta \le \pi$.

8. Prove that $\log\left[\dfrac{\sin(x + iy)}{\sin(x - iy)}\right] = 2i \tan^{-1}(\cot x \tanh y)$. **(Meerut 2012, 13)**

9. Prove that

$$\log \cos(x + iy) = \frac{1}{2} \log \frac{1}{2}(\cosh 2y + \cos 2x) - i \tan^{-1}(\tan x \tanh y).$$

(Kumaun 2008)

10. Prove that $\log\left\{\dfrac{\cos(x - iy)}{\cos(x + iy)}\right\} = 2\, i \tan^{-1}(\tan x \tanh y)$.

11. Show that one of the values of $\log \dfrac{(1 + i)(1 + i\sqrt{3})}{\sqrt{3} + i}$ is $\dfrac{1}{2} \log 2 + i \dfrac{5}{12} \pi$.

12. If $\tan \log(x + iy) = a + ib$, where $a^2 + b^2 \ne 1$, prove that

$$\tan\{\log(x^2 + y^2)\} = \frac{2a}{1 - a^2 - b^2}.$$

(Meerut 2005B; Avadh 13)

13. If $\log \log(x + iy) = p + iq$, prove that

$$y = x \tan[\tan q \log(x^2 + y^2)^{1/2}].$$ **(Purvanchal 2006; Avadh 10)**

14. If $\log \log \log(\alpha + i\beta) = p + iq$, prove that

(i) $e^{e^p \cos q} . \cos(e^p \sin q) = \dfrac{1}{2} \log(\alpha^2 + \beta^2)$,

(Kanpur 2011)

(ii) $e^{e^p \sin q} . \sin(e^p \sin q) = \tan^{-1}(\beta / \alpha)$.

15. If $(a_1 + ib_1)(a_2 + ib_2) \ldots\ldots (a_n + ib_n) = A + iB$, prove that

$$\tan^{-1}\left(\frac{b_1}{a_1}\right) + \tan^{-1}\left(\frac{b_2}{a_2}\right) + \ldots + \tan^{-1}\left(\frac{b_n}{a_n}\right) = \tan^{-1} \frac{B}{A},$$

(Avadh 2009)

and $(a_1^2 + b_1^2)(a_2^2 + b_2^2) \ldots (a_n^2 + b_n^2) = A^2 + B^2$.

Answers 1

1. (iii) $\frac{1}{2}(4n+1)\pi i$ **2.** (i) $\frac{1}{4}(8n+1)\pi i$

3.8 The General Exponential Function a^z

If a and z are any two complex numbers, the general exponential function a^z is defined as

$$a^z = e^{z \operatorname{Log} a} = \exp(z \operatorname{Log} a).$$

Since Log a is a many-valued function, a^z is also a many-valued function. Thus

$$a^z = e^{z \operatorname{Log} a} = e^{z(\log a + 2n\pi i)} = \exp[z(\log a + 2n\pi i)].$$

This gives the *general value of* a^z. The *principal value* of a^z is obtained by putting $n = 0$.

Hence the principal value of $a^z = e^{z \log a} = \exp(z \log a)$.

3.9 To Separate $(\alpha + i\beta)^{p+iq}$ into Real and Imaginary Parts

By the definition of the general exponential function, we have

$$(\alpha + i\beta)^{p+iq} = e^{(p+iq)\operatorname{Log}(\alpha+i\beta)} = \exp[(p+iq)\operatorname{Log}(\alpha+i\beta)]$$

$$= \exp[(p+iq)\{\tfrac{1}{2}\log(\alpha^2+\beta^2) + i\tan^{-1}(\beta/\alpha) + 2n\pi i\}]$$

[By article 3.5]

$$= \exp[\{\tfrac{1}{2}p\log(\alpha^2+\beta^2) - q\tan^{-1}(\beta/\alpha) - 2nq\pi\}$$

$$+ i\{\tfrac{1}{2}q\log(\alpha^2+\beta^2) + p\tan^{-1}(\beta/\alpha) + 2np\pi\}]$$

$$= \exp(A + iB),$$

where $A = \frac{1}{2}p\log(\alpha^2+\beta^2) - q\{\tan^{-1}(\beta/\alpha) + 2n\pi\}$

and $B = \frac{1}{2}q\log(\alpha^2+\beta^2) + p\{\tan^{-1}(\beta/\alpha) + 2n\pi\}$

$$= e^{A+iB} = e^A e^{iB} = e^A(\cos B + i\sin B).$$

Therefore the real part is $e^A \cos B$ and the imaginary part is $e^A \sin B$.

The principal value is obtained by putting $n = 0$.

Note: In the following examples it is sometimes found convenient to write e^x as exp (x).

Illustrative Examples

Example 9: *Prove that* $i^i = e^{-(4n+1)\pi/2}$. **(Meerut 2004B; Gorakhpur 06; Agra 08; Purvanchal 10; Rohilkhand 09; Bundelkhand 14)**

Solution: We have $i^i = e^{i \text{ Log } i}$, by def. of a^z

$$= \exp(i \text{ Log } i) = \exp[i(\log i + 2n\pi i)]. \quad \ldots(1)$$

Now $\log i = \log(\cos \frac{1}{2}\pi + i \sin \frac{1}{2}\pi)$ $\quad [\because i = \cos \frac{1}{2}\pi + i \sin \frac{1}{2}\pi]$

$$= \log e^{i\pi/2} = i\pi/2.$$

Substituting for log i in (1), we have

$$i^i = \exp[i(i\frac{1}{2}\pi + 2n\pi i)] = \exp[i^2(2n + \frac{1}{2})\pi]$$

$$= \exp[-(4n+1)\pi/2] = e^{-(4n+1)\pi/2}. \quad \ldots(2)$$

Note 1: Putting $n = 0$ in (2), we get the principal value of $i^i = e^{-\pi/2}$.

Note 2: Putting $n = 0, 1, 2, \ldots$ in (2), the various values of i^i are

$$e^{-\pi/2}, e^{-5\pi/2}, e^{-9\pi/2}, e^{-13\pi/2}, \ldots$$

which is a geometrical progression with common ratio $e^{-2\pi}$.

Example 10: *If* $i^{i^{i \ldots ad\ inf.}} = A + iB$, *principal values only being considered, prove that*

(i) $\tan \frac{1}{2}\pi A = B/A$, *and* (ii) $A^2 + B^2 = e^{-\pi B}$. **(Bundelkhand 2007; Kanpur 09; Purvanchal 14)**

Solution: (i) We have $i^{i^{i \ldots ad\ inf.}} = A + iB$.

$\therefore$ $\quad i^{A+iB} = A + iB$

or $\quad \exp[(A+iB)\log i] = A + iB$, taking the principal value of i^{A+iB}

or $\quad \exp[(A+iB)\log(\cos \frac{1}{2}\pi + i\sin\frac{1}{2}\pi)] = A + iB$

or $\quad \exp[(A+iB)\log e^{i\pi/2}] = A + iB$

or $\quad \exp[(A+iB)\, i\frac{1}{2}\pi] = A + iB$

or $\quad \exp[-\frac{1}{2}\pi B + i\frac{1}{2}\pi A] = A + iB$ or $e^{-B\pi/2} e^{\pi A i/2} = A + iB$

or $\quad e^{-B\pi/2}(\cos\frac{1}{2}\pi A + i\sin\frac{1}{2}\pi A) = A + iB.$

Equating real and imaginary parts on both sides, we have

$$e^{-B\pi/2}\cos\frac{1}{2}\pi A = A, \quad \ldots(1)$$

and

$$e^{-B\pi/2}\sin\frac{1}{2}\pi A = B. \quad \ldots(2)$$

Dividing (2) by (1), we have $\tan \frac{1}{2} \pi A = B / A$.

Squaring and adding (1) and (2), we get

(ii) $$(e^{-B\pi/2})^2 \ (\cos^2 \frac{1}{2} \pi A + \sin^2 \frac{1}{2} \pi A) = A^2 + B^2$$

or $$e^{-B\pi} = A^2 + B^2.$$

Example 11: *If* $p^{\alpha + i\beta} = (x + iy)^{m + in}$ *and the principal values are considered, prove that*

(i) $\alpha = \frac{1}{2} m \log_p (x^2 + y^2) - n \tan^{-1} (y / x) \log_p e$,

(Meerut 2006B; Kanpur 08)

(ii) $\log_p (x^2 + y^2) = \dfrac{2(\alpha m + \beta n)}{m^2 + n^2}$.

(Kanpur 2008)

Solution: We have $p^{\alpha + i\beta} = (x + iy)^{m + in}$.

Taking logarithm of both sides, we have

$$(\alpha + i\beta) \log_e p = (m + in) \log_e (x + iy)$$

$$= (m + in) [\frac{1}{2} \log_e (x^2 + y^2) + i \tan^{-1} (y / x)],$$ refer article 3.5

$$= [\frac{1}{2} m \log_e (x^2 + y^2) - n \tan^{-1} (y / x)]$$

$$+ i [\frac{1}{2} n \log_e (x^2 + y^2) + m \tan^{-1} (y / x)].$$

Equating real and imaginary parts on both sides, we get

$$\alpha \log_e p = \frac{1}{2} m \log_e (x^2 + y^2) - n \tan^{-1} (y / x) \qquad \text{...(1)}$$

and $$\beta \log_e p = \frac{1}{2} m \log_e (x^2 + y^2) + m \tan^{-1} (y / x). \qquad \text{...(2)}$$

(i) From (1), we have

$$\alpha \log_e p = \frac{1}{2} m \log_e (x^2 + y^2) - n \tan^{-1} (y / x)$$

or $$(\alpha \log_e p) . (\log_p e) = [\frac{1}{2} m \log_e (x^2 + y^2) - n \tan^{-1} (y / x)]. \log_e e,$$

multiplying both sides by $\log_p e$

or $$\alpha = \frac{1}{2} m \log_p (x^2 + y^2) - n \tan^{-1} (y / x) . \log_p e.$$

$$[\because \log_b a \times \log_a b = 1; \log_a m \times \log_b a = \log_b m]$$

$\therefore$ $$\alpha = \frac{1}{2} m \log_p (x^2 + y^2) - n \tan^{-1} (y / x) \log_p e.$$ **Proved.**

(ii) Multiplying (1) by m and (2) by n and adding, we get

$$(\alpha m+\beta n)\log_e p=\frac{1}{2}(m^2+n^2)\log_e(x^2+y^2)$$

or $$(\alpha m+\beta n)\log_e p\times\log_p e=\frac{1}{2}(m^2+n^2)\log_e(x^2+y^2)\times\log_p e,$$

multiplying both sides by $\log_p e$

or $$(\alpha m+\beta n)\,.\,1=\frac{1}{2}(m^2+n^2)\log_p(x^2+y^2)$$

or $$\log_p(x^2+y^2)=\frac{2(\alpha m+\beta n)}{(m^2+n^2)}\cdot$$

Example 12: *If* $(a+ib)^p=m^{x+iy}$*, then prove that* $\dfrac{y}{x}=\dfrac{2\tan^{-1}(b/a)}{\log(a^2+b^2)}$, *where only principal values are considered .* **(Kanpur 2007)**

Solution: Given that $(a+ib)^p=m^{(x+iy)}$.

Taking logarithm of both sides, we have

$$p\log(a+ib)=(x+iy)\log m$$

or $$p\left[\frac{1}{2}\log(a^2+b^2)+i\tan^{-1}(b/a)\right]=x\log m+iy\log m.$$

Equating real and imaginary parts, we get

$$\frac{1}{2}p\log(a^2+b^2)=x\log m \qquad \ldots(1)$$

and $$p\tan^{-1}(b/a)=y\log m. \qquad \ldots(2)$$

Dividing (2) by (1), we get

$$\frac{y}{x}=\frac{p\tan^{-1}(b/a)}{(p/2)\log(a^2+b^2)}=\frac{2\tan^{-1}(b/a)}{\log(a^2+b^2)}\cdot$$

Example 13: *Prove that if* $(1+i\tan\alpha)^{1+i\tan\beta}$ *can have real values, one of them is* $(\sec\alpha)^{\sec^2\beta}$. **(Kanpur 2008, 10; Rohilkhand 05)**

Solution: We have $(1+i\tan\alpha)^{1+i\tan\beta}$

$=\exp[(1+i\tan\beta)\log(1+i\tan\alpha)]$, taking the principal value

$=\exp[(1+i\tan\beta)\{\log\sqrt{(1+\tan^2\alpha)}+i\tan^{-1}(\tan\alpha)\}]$

$=\exp[(1+i\tan\beta)(\log\sec\alpha+i\alpha)]$

$=\exp[\log\sec\alpha-\alpha\tan\beta+i(\alpha+\tan\beta\log\sec\alpha)]$

$=\exp[\log\sec\alpha-\alpha\tan\beta]\,.\exp[i(\alpha+\tan\beta\log\sec\alpha)]$

$=\exp[\log\sec\alpha-\alpha\tan\beta]\,.[\cos\{\alpha+\tan\beta\log\sec\alpha\}$
$+i\sin\{\alpha+\tan\beta\log\sec\alpha\}].$

Now if this value is real, the imaginary part is equal to zero.

$\therefore$ $\sin\{\alpha + \tan\beta \log\sec\alpha\} = 0$ or $\alpha + \tan\beta \log\sec\alpha = 0$

or $\alpha = -\tan\beta \log\sec\alpha.$ ...(1)

Also then $(1 + i\tan\alpha)^{1 + i\tan\beta}$

$$= \exp[\log\sec\alpha - \alpha\tan\beta] \,.\, (\cos 0 + i\sin 0)$$

$$= \exp[\log\sec\alpha + \tan\beta \,.\, \tan\beta \log\sec\alpha],$$

substituting for α from (1)

$$= \exp[(1 + \tan^2\beta)\log\sec\alpha] = \exp[\sec^2\beta \log\sec\alpha]$$

$$= \exp[\log(\sec\alpha)^{\sec^2\beta}] = (\sec\alpha)^{\sec^2\beta}.$$

This is one of the values since $(1 + i\tan\alpha)^{1 + i\tan\beta}$ can have infinite number of values.

Example 14: *Find the general value of* $\log_4(-2)$. **(Meerut 2009)**

Solution: We have $\mathrm{Log}_4(-2) = \dfrac{\mathrm{Log}_e(-2)}{\mathrm{Log}_e 4}$

$$= \frac{\log_e(-2) + 2n\pi i}{\log_e 4 + 2m\pi i}, \text{ where } m \text{ and } n \text{ are any integers}$$

$$= \frac{\log_e\{2(\cos\pi + i\sin\pi)\} + 2n\pi i}{\log_e 4 + 2m\pi i} = \frac{\log_e(2e^{i\pi}) + 2n\pi i}{\log_e 4 + 2m\pi i}$$

$$= \frac{\log_e 2 + i\pi + 2n\pi i}{\log_e 4 + 2m\pi i} = \frac{\log_e 2 + i(2n\pi + \pi)}{2\log_e 2 + 2m\pi i}$$

$$= \frac{[\log_e 2 + i(2n\pi + \pi)][2\log_e 2 - 2m\pi i]}{(2\log_e 2 + 2m\pi i)(2\log_e 2 - 2m\pi i)}$$

$$= \frac{2(\log_e 2)^2 + 2m\pi(2n\pi + \pi) + 2i\{(2n+1)\pi\log_e 2 - m\pi\log_e 2\}}{4(\log_e 2)^2 + 4m^2\pi^2}$$

$$= \frac{(\log_e 2)^2 + (2n+1)m\pi^2}{2(\log_e 2)^2 + 2m^2\pi^2} + i\frac{(2n+1-m)\pi\log_e 2}{2(\log_e 2)^2 + 2m^2\pi^2},$$

which is required general value of $\log_4(-2)$.

Example 15: *If* $\sin\log(i^i) = a + ib$, *find a and b. Hence find* $\cos(\log i^i)$.

Solution: First, we have

$$i^i = \exp(i\log i) = \exp[i\log(\cos\tfrac{1}{2}\pi + i\sin\tfrac{1}{2}\pi)]$$

$$= \exp[i\log e^{i\pi/2}] = \exp[i(i\pi/2)] = \exp(-\pi/2) = e^{-\pi/2}.$$

$\therefore$ $\log i^i = \log e^{-\pi/2} = -\pi/2.$

$\therefore$ $\sin(\log i^i) = \sin(-\pi/2) = -1.$

Hence $\sin(\log i^i) = a + ib \;\Rightarrow\; -1 = a + ib.$

Equating real and imaginary parts, we get $a = -1, b = 0.$

$\therefore$ $\sin(\log i^i) = -1.$

Now $\cos(\log i^i) = \sqrt{\{1 - \sin^2(\log i^i)\}} = \sqrt{\{1 - 1\}} = 0.$

Comprehensive Exercise 2

1. If $i^{\alpha + i\beta} = \alpha + i\beta$, show that $\alpha^2 + \beta^2 = e^{-(4n+1)\pi\beta}$.

(Agra 2005; Kanpur 09; Bundelkhand 11)

2. Prove that $i^a = \cos\left[\left(2m + \frac{1}{2}\right)\pi a\right] + i\sin\left[\left(2m + \frac{1}{2}\right)\pi a\right].$

3. If $i^{i^{i}} = \cos\theta + i\sin\theta$, prove that

$$\theta = \left(2m + \frac{1}{2}\right)\pi \exp\left[-\left(2n + \frac{1}{2}\right)\pi\right].$$

(Kanpur 2008)

4. If $(i^i)^i = \cos\theta - i\sin\theta$, show that $\theta = \frac{1}{2}\pi(4n + 1).$

5. Show that the sum of the moduli of the values of $(1 + i)^{1+i}$, which are less than unity is $(1/\sqrt{2}) \cdot e^{3\pi/4} \cdot \operatorname{cosech} \pi.$

6. Show that the principal value of $(a + ib)^{p+iq} / (a - ib)^{p-iq}$ is

$$\cos 2(p\alpha + q\log r) + i\sin 2(p\alpha + q\log r),$$

where $r = \sqrt{(a^2 + b^2)}$ and $\alpha = \tan^{-1}(b/a).$

7. Prove that the real part of the principal value of

$$(i)^{\log(1+i)} \text{ is } \exp(-\pi^2/8) \times \cos\left(\frac{1}{4}\pi\log 2\right).$$

(Purvanchal 2008)

8. Prove that $(x + ix\tan y)^{\log(x\sec y) - iy}$ is real, when only principal values are considered.

9. Prove that the principal value of $(a + ib)^{c+id}$ is wholly real or wholly imaginary according as $\frac{1}{2}d\log(a^2 + b^2) + c\tan^{-1}(b/a)$ is an even or an odd multiple of $\frac{1}{2}\pi$. **(Gorakhpur 2005; Avadh 07; Rohilkhand 10)**

In case it is wholly real, prove that $(a+ib)^{c+id} = (a^2+b^2)^{(c^2+d^2)/2c}$.

10. Prove that the general value of $(1+i\tan\alpha)^{-i}$ is **(Purvanchal 2006)**

$$\exp(\alpha+2m\pi).\,[\cos(\log\cos\alpha)+i\sin(\log\cos\alpha)].$$

11. Prove that $\text{Log}_i\, i = \dfrac{4m+1}{4n+1}$, where m and n are integers.

(Rohilkhand 2005; Kanpur 06)

12. Prove that

$$\log\left(\frac{a+ib-x}{a+ib+x}\right) = \frac{1}{2}\log\left\{\frac{(a-x)^2+b^2}{(a+x)^2+b^2}\right\} + i\tan^{-1}\left[\frac{b}{a-x} - \tan^{-1}\frac{b}{a+x}\right].$$

13. If $\dfrac{(1+i)^{p+iq}}{(1-i)^{p-iq}} = \alpha + i\beta$, prove that one value of $\tan^{-1}(\beta/\alpha)$ is $(\pi p/2) + q\log 2$.

14. If $[\cos(\theta - i\phi)]^{x+iy} = A + iB$ and principal values are taken into consideration, then

$$\tan^{-1}(B/A) = \frac{1}{2}y\log(\cosh^2\phi - \sin^2\theta) + x\tan^{-1}(\tan\theta\tanh\phi).$$

15. If $\left\{\dfrac{a+x+iy}{a-x-iy}\right\}^{\lambda+\mu i} = X + iY$, prove that one of the values of

$$\tan^{-1}(Y/X) \text{ is } \lambda\tan^{-1}\left\{\frac{2ay}{a^2-x^2-y^2}\right\} + \frac{\mu}{2}\log\left\{\frac{(a+x)^2+y^2}{(a-x)^2+y^2}\right\}.$$

16. If $i^{\alpha+i\beta} = e^x(\cos y + i\sin y)$, then prove that

$$x = -\frac{1}{2}(4n+1)\pi\beta \quad\text{and}\quad y = \frac{1}{2}(4n+1)\pi\alpha.$$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The general value of $\log(-i)$ is

(a) $\left(\dfrac{2n+1}{2}\right)\pi i$ (b) $\left(\dfrac{4n-1}{2}\right)\pi i$ (c) $\left(\dfrac{2n-1}{2}\right)\pi i$ (d) $\left(\dfrac{3n-1}{2}\right)\pi i$

2. If $\log (x - iy) = A + iB$, where A and B are real, then

(a) $A = \frac{1}{2} \log (x^2 - y^2)$ (b) $A = \log (x^2 + y^2)$

(c) $A = \frac{1}{2} \log (x^2 + y^2)$ (d) $A = \log (x^2 - y^2)$

Fill in the Blanks

Fill in the blanks "……" so that the following statements are complete and correct.

1. If only the principal value is considered, then

$$\log_e (x + iy) = \frac{1}{2} \log_e (x^2 + y^{2)} + \ldots\ldots$$

2. If the general value is taken into account, then

$$\text{Log}_e (x + iy) = \frac{1}{2} \log (x^2 + y^2) + i \tan^{-1} \frac{y}{x} + \ldots\ldots$$

3. The principal value of $\log (-3)$ is …… .
4. The general value of $\text{Log} (-5)$ is …… .
5. $\log (1 + i \tan \alpha) = \log \sec \alpha + \ldots\ldots$

True or False

Write 'T' for true and 'F' for false statement.

1. Logarithm of a complex number is a many-valued function.
2. $i \log \frac{x - i}{x + i} = \pi - 2 \tan^{-1} x$.
3. The general value of $\log \sqrt{i}$ is $\frac{1}{2} (8n + 1) \pi i$.

Answers

Multiple Choice Questions

1. (b) 2. (c)

Fill in the Blanks

1. $i \tan^{-1} \frac{y}{x}$
2. $2n\pi i$, where n is any integer
3. $\log 3 + i\pi$
4. $\log 5 + (2n + 1) \pi i$, where n is any integer
5. $i\alpha$

True or False

1. *T* 2. *T* 3. *F*

□

Chapter

4

Inverse Circular and Hyperbolic Functions of Complex Numbers

4.1 Inverse Circular Functions of Complex Numbers

If $\cos(x + iy) = u + iv$, then $x + iy$ is called an *inverse cosine* of $u + iv$ and is denoted by $\cos^{-1}(u + iv)$. Thus

if $\qquad \cos(x + iy) = u + iv$, then $\cos^{-1}(u + iv) = x + iy$.

Also, if $\cos(x + iy) = u + iv$, then

$$u + iv = \cos\{2n\pi \pm (x + iy)\}, \text{ where } n \text{ is any integer.}$$

Therefore by the above definition the *general value* of inverse cosine of $u + iv$ is

$$2n\pi \pm (x + iy),$$

and is denoted by $\text{Cos}^{-1}(u + iv)$ *i.e.*, by writing the first letter '*C*' as capital.

It follows that the inverse cosine of $u + iv$ is a many-valued function. Its *principal value* is that value of $2n\pi \pm (x + iy)$ in which the real part lies between 0 and π. It is denoted by $\cos^{-1}(u + iv)$ *i.e.*, by writing the first letter '*c*' as small. Thus

$$\text{Cos}^{-1}(u + iv) = 2n\pi \pm (x + iy) = 2n\pi \pm \cos^{-1}(u + iv).$$

In a like manner if $\sin(x+iy) = u+iv$, then $x+iy$ is called an inverse sine of $u+iv$ and is denoted by $\sin^{-1}(u+iv)$. Thus

if $\qquad \sin(x+iy) = u+iv$, then $\sin^{-1}(u+iv) = x+iy$.

Also, if $\sin(x+iy) = u+iv$, then

$$u+iv = \sin\{n\pi + (-1)^n (x+iy)\},$$ where n is any integer.

Therefore the general value of inverse sine of $u+iv$ is

$$n\pi + (-1)^n (x+iy),$$

and is denoted by $\text{Sin}^{-1}(u+iv)$, the first letter 'S' being written capital.

Therefore the inverse sine of $u+iv$ is a many-valued function. Its principal value is that value of $n\pi + (-1)^n (x+iy)$ for which the real part lies between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. It is denoted by $\sin^{-1}(u+iv)$, the first letter 's' being written small. Thus

$$\text{Sin}^{-1}(u+iv) = n\pi + (-1)^n \sin^{-1}(u+iv).$$

The other inverse circular functions may be defined similarly. For example, if

$$\tan(x+iy) = u+iv,$$

then $\qquad \text{Tan}^{-1}(u+iv) = n\pi + (x+iy) = n\pi + \tan^{-1}(x+iy)$,

the principal value being that for which the real part lies between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$.

The relations connecting the general and principal values of the remaining inverse circular functions may be written as follows :

$$\text{Sec}^{-1}(u+iv) = 2n\pi \pm \sec^{-1}(u+iv),$$

$$\text{Cosec}^{-1}(u+iv) = n\pi + (-1)^n \text{cosec}^{-1}(u+iv),$$

and $\qquad \text{Cot}^{-1}(u+iv) = n\pi + \cot^{-1}(u+iv)$.

It should be noted that the principal value for the case of sin, cosec, tan and cot is that value for which the real part lies between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, while for the case of cos and sec it is that value for which its real part lies between 0 and π.

4.2 Inverse Hyperbolic Functions

Let z and w be two complex numbers. If $\sinh w = z$, then w is called the inverse hyperbolic sine of z and is written as

$$w = \sinh^{-1} z.$$

The other inverse hyperbolic functions $\cosh^{-1} z$, $\tanh^{-1} z$, $\text{cosech}^{-1} z$, $\text{sec h}^{-1} z$ and $\coth^{-1} z$ are defined similarly.

The values of inverse hyperbolic functions can also be expressed in terms of the logarithmic functions as shown below.

(i) *To prove that* $\sinh^{-1} z = \log [z + \sqrt{(z^2 + 1)}]$.

(Meerut 2004; Agra 05; Kumaun 08; Kashi 12)

Proof: Let $\sinh^{-1} z = w$. Then

$$\sinh w = z. \qquad ...(1)$$

$$\therefore \quad \cosh w = \sqrt{(1 + \sinh^2 w)} = \sqrt{(1 + z^2)}. \qquad ...(2)$$

Adding (1) and (2), we get

$$\sinh w + \cosh w = z + \sqrt{(1 + z^2)}$$

i.e., $\frac{1}{2}(e^w - e^{-w}) + \frac{1}{2}(e^w + e^{-w}) = z + \sqrt{(1 + z^2)}$ *i.e.*, $e^w = z + \sqrt{(1 + z^2)}$.

$$\therefore \quad w = \log_e [z + \sqrt{(1 + z^2)}].$$

Hence $\mathbf{\sinh^{-1} z = \log [z + \sqrt{(z^2 + 1)}]}$.

(ii) *To prove that* $\cosh^{-1} z = \log [z + \sqrt{(z^2 - 1)}]$.

Proof: Let $\cosh^{-1} z = w$. Then

$$\cosh w = z. \qquad ...(1)$$

$$\therefore \quad \sinh w = \sqrt{(\cosh^2 w - 1)} = \sqrt{(z^2 - 1)}. \qquad ...(2)$$

Adding (1) and (2), we get

$$\cosh w + \sinh w = z + \sqrt{(z^2 - 1)} \quad \text{or} \quad e^w = z + \sqrt{(z^2 - 1)}.$$

$$\therefore \quad w = \log [z + \sqrt{(z^2 - 1)}].$$

Hence $\mathbf{\cosh^{-1} z = \log [z + \sqrt{(z^2 - 1)}]}$.

(iii) *To prove that* $\tanh^{-1} z = \frac{1}{2} \log \frac{1+z}{1-z}$. **(Bundelkhand 2005)**

Proof: Let $\tanh^{-1} z = w$. Then $\tanh w = z$

or $$\frac{e^w - e^{-w}}{e^w + e^{-w}} = \frac{z}{1}.$$

Applying componendo and dividendo, we have

$$\frac{(e^w + e^{-w}) + (e^w - e^{-w})}{(e^w + e^{-w}) - (e^w - e^{-w})} = \frac{1+z}{1-z}$$

or $$\frac{2e^w}{2e^{-w}} = \frac{1+z}{1-z} \quad \text{or} \quad e^{2w} = \frac{1+z}{1-z}.$$

$$\therefore \quad 2w = \log \frac{1+z}{1-z} \quad \text{or} \quad w = \frac{1}{2} \log \frac{1+z}{1-z}.$$

Hence $\quad \tanh^{-1} z = \frac{1}{2} \log \frac{z+1}{z-1}$.

(iv) Similarly, we can prove that

$$\coth^{-1} z = \frac{1}{2} \log \frac{z+1}{z-1}.$$ **(Meerut 2007B)**

4.3 Relations between Inverse Hyperbolic Functions and Inverse Circular Functions

(i) *To prove that* $\sinh^{-1} x = -i \sin^{-1}(ix)$.

Proof: Let $\sinh^{-1} x = y$. Then $x = \sinh y$.

$\therefore \quad ix = i \sinh y = \sin(iy)$.

$\therefore \quad iy = \sin^{-1}(ix) \quad$ or $\quad y = (1/i) \sin^{-1}(ix) = -i \sin^{-1}(ix)$.

Hence $\quad \mathbf{\sinh^{-1} x = -i \sin^{-1}(ix)}$.

(ii) *To prove that* $\cosh^{-1} x = -i \cos^{-1}(x)$.

Proof: Let $\cosh^{-1} x = y$. Then $x = \cosh y = \cos(iy)$.

$\therefore \quad iy = \cos^{-1} x \quad$ or $\quad y = (1/i) \cos^{-1} x = -i \cos^{-1} x$.

Hence $\quad \mathbf{\cosh^{-1} x = -i \cos^{-1} x}$.

(iii) *To prove that* $\tanh^{-1} x = -i \tan^{-1}(ix)$.

Proof: Let $\tanh^{-1} x = y$. Then $x = \tanh y$.

$\therefore \quad ix = i \tanh y = \tan(iy)$.

$\therefore \quad iy = \tan^{-1}(ix) \quad$ or $\quad y = (1/i) \tan^{-1}(ix) = -i \tan^{-1}(ix)$.

Hence $\quad \mathbf{\tanh^{-1} x = -i \tan^{-1}(ix)}$.

Illustrative Examples

Example 1: *Express* $\cos^{-1}(x + iy)$ *in the form* $A + iB$. **(Kanpur 2009; Avadh 08)**

Solution: Let $\cos^{-1}(x + iy) = A + iB$.

Then $\quad \cos(A + iB) = x + iy$

or $\quad \cos A \cos(iB) - \sin A \sin(iB) = x + iy$

or $\quad \cos A \cosh B - \sin A \sinh B = x + iy$.

Equating real and imaginary parts on both sides, we get

$$\cos A \cosh B = x, \qquad \ldots(1)$$

and $$\sin A \sinh B = -y. \qquad \ldots(2)$$

Let us first eliminate B between (1) and (2). From (1) and (2), we have

$$\cosh B = x/\cos A \quad \text{and} \quad \sinh B = -y/\sin A.$$

$$\therefore \quad \frac{x^2}{\cos^2 A} - \frac{y^2}{\sin^2 A} = \cosh^2 B - \sinh^2 B = 1$$

or $$x^2 \sin^2 A - y^2 \cos^2 A = \cos^2 A \sin^2 A$$

or $$x^2 \sin^2 A - y^2 (1 - \sin^2 A) = (1 - \sin^2 A) \sin^2 A$$

or $$x^2 \sin^2 A - y^2 + y^2 \sin^2 A = \sin^2 A - \sin^4 A$$

or $$\sin^4 A + (x^2 + y^2 - 1) \sin^2 A - y^2 = 0,$$

which is a quadratic in $\sin^2 A$.

$$\therefore \quad \sin^2 A = \frac{-(x^2 + y^2 - 1) \pm \sqrt{[(x^2 + y^2 - 1)^2 + 4y^2]}}{2}.$$

Since $\sin^2 A$ must be positive, therefore neglecting the –ive sign, we have

$$\sin^2 A = \frac{\sqrt{\{(x^2 + y^2 - 1)^2 + 4y^2\}} - (x^2 + y^2 - 1)}{2}.$$

$$\therefore \quad \sin A = \pm \left[\frac{\sqrt{\{(x^2 + y^2 - 1)^2 + 4y^2\}} - (x^2 + y^2 - 1)}{2}\right]^{1/2}$$

or $$A = \pm \sin^{-1} \left[\frac{\sqrt{\{(x^2 + y^2 - 1)^2 + 4y^2\}} - (x^2 + y^2 - 1)}{2}\right]^{1/2} \qquad \ldots(3)$$

Now let us eliminate A between (1) and (2). From (1) and (2), we have

$$\cos A = x/\cosh B \quad \text{and} \quad \sin A = -y/\sinh B.$$

$$\therefore \quad \frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = \cos^2 A + \sin^2 A = 1$$

or $$x^2 \sinh^2 B + y^2 \cosh^2 B = \cosh^2 B \sinh^2 B$$

or $$x^2 \sinh^2 B + y^2 (1 + \sinh^2 B) = (1 + \sinh^2 B) \sinh^2 B$$

or $$x^2 \sinh^2 B + y^2 + y^2 \sinh^2 B = \sinh^2 B + \sinh^4 B$$

or $$\sinh^4 B + (1 - x^2 - y^2) \sinh^2 B - y^2 = 0,$$

which is a quadratic in $\sinh^2 B$.

$\therefore$ $$\sinh^2 B = \frac{-(1 - x^2 - y^2) \pm \sqrt{\{(1 - x^2 - y^2)^2 + 4y^2\}}}{2}.$$

But $\sinh^2 B$ is positive, so neglecting the –ive sign, we have

$$\sinh^2 B = \frac{\sqrt{\{(1 - x^2 - y^2)^2 + 4y^2\}} - (1 - x^2 - y^2)}{2}.$$

$\therefore$ $$\sinh B = \pm \left[\frac{\sqrt{\{(1 - x^2 - y^2)^2 + 4y^2\}} - (1 - x^2 - y^2)}{2}\right]^{1/2}$$

or $$B = \pm \sinh^{-1} \left[\frac{\sqrt{\{(1 - x^2 - y^2)^2 + 4y^2\}} - (1 - x^2 - y^2)}{2}\right]^{1/2}. \quad \ldots(4)$$

Remark: The general value of $\cos^{-1}(x + iy)$ *i.e.*, $\text{Cos}^{-1}(x + iy)$ is given by

$$\text{Cos}^{-1}(x + iy) = 2n\pi \pm \cos^{-1}(x + iy) = 2n\pi \pm (A + iB),$$

where A and B are as found in (3) and (4).

Example 2: *Explain the meaning of* $\sin^{-1} x$*, when x is real and greater than* 1*, and find the value of* $\text{Sin}^{-1} 2$.

Solution: We know that if α is a real number, then $\sin \alpha$ lies between -1 and 1. Therefore if x is a real number greater than 1, then $\sin^{-1} x$ is not a real number, but is a complex number.

We have $\text{Sin}^{-1} 2 = n\pi + (-1)^n \sin^{-1} 2$.

Let $\sin^{-1} 2 = u + iv$, where u and v are real.

Then $\sin(u + iv) = 2$

or $\sin u \cos iv + \cos u \sin iv = 2$

or $\sin u \cosh v + i \cos u \sinh v = 2$.

Equating real and imaginary parts, we get

$$\sin u \cosh v = 2, \quad \ldots(1)$$

and $$\cos u \sinh v = 0. \quad \ldots(2)$$

From (2), $\cos u \sinh v = 0$. But $\sinh v = 0$ gives $v = 0$ which makes $\sin^{-1} 2$ real. So we have $\sinh v \neq 0$.

$\therefore$ $\cos u = 0$, or $u = \pm \frac{1}{2}\pi$.

Putting $u = \pm \frac{1}{2}\pi$ in (1), we get

$$\cosh v = \pm 2.$$

$\therefore \qquad v = \cosh^{-1} (\pm 2) = \log (\sqrt{3} \pm 2).$

$$[\because \cosh^{-1} x = \log \{x + \sqrt{(x^2 - 1)}\}]$$

$\therefore \qquad \sin^{-1} 2 = u + iv = \pm \frac{1}{2} \pi + i \log (\sqrt{3} \pm 2).$

$\therefore \qquad \text{Sin}^{-1} 2 = n\pi + (-1)^n [\pm \frac{1}{2} \pi + i \log (\sqrt{3} \pm 2)].$

Example 3: *Express* $\tan^{-1} (x + iy)$ *as the sum of real and imaginary parts.*

(Rohilkhand 2007; Agra 08; Bundelkhand 08; Meerut 10B; Purvanchal 10; Avadh 12)

Solution: Let $\tan^{-1} (x + iy) = A + iB$.

Then $\qquad \tan (A + iB) = x + iy,$

and $\qquad \tan (A - iB) = x - iy,$ by equating complex conjugates.

Now $\qquad \tan 2A = \tan [(A + iB) + (A - iB)]$

$$= \frac{\tan (A + iB) + \tan (A - iB)}{1 - \tan (A + iB) . \tan (A - iB)} = \frac{(x + iy) + (x - iy)}{1 - (x + iy)(x - iy)}$$

$$= \frac{2x}{1 - (x^2 + y^2)} = \frac{2x}{1 - x^2 - y^2}.$$

$\therefore \qquad 2A = \tan^{-1} \frac{2x}{1 - x^2 - y^2} \quad \text{or} \quad A = \frac{1}{2} \tan^{-1} \frac{2x}{1 - x^2 - y^2}.$

Again $\qquad \tan (2iB) = \tan [(A + iB) - (A - iB)]$

$$= \frac{\tan (A + iB) - \tan (A - iB)}{1 + \tan (A + iB) \tan (A - iB)} = \frac{(x + iy) - (x - iy)}{1 + (x + iy)(x - iy)}$$

$$= \frac{2iy}{1 + x^2 + y^2}.$$

$\therefore \qquad i \tanh 2B = \frac{2iy}{1 + x^2 + y^2} \qquad [\because \tan (i\theta) = i \tanh \theta]$

or $\qquad \tanh 2B = \frac{2y}{1 + x^2 + y^2} \quad \text{or} \quad 2B = \tanh^{-1} \frac{2y}{1 + x^2 + y^2}$

or $\qquad B = \frac{1}{2} \tanh^{-1} \frac{2y}{1 + x^2 + y^2}.$

Hence $\qquad \tan^{-1} (x + iy) = A + iB$

$$= \frac{1}{2} \tan^{-1} \frac{2x}{1 - x^2 - y^2} + i \frac{1}{2} \tanh^{-1} \frac{2y}{1 + x^2 + y^2}.$$

Note: If we are to find the general value $\text{Tan}^{-1} (x + iy)$, then

$$\text{Tan}^{-1} (x + iy) = n\pi + \tan^{-1} (x + iy)$$

$$= n\pi + \frac{1}{2} \tan^{-1} \frac{2x}{1 - x^2 - y^2} + i \frac{1}{2} \tanh^{-1} \frac{2y}{1 + x^2 + y^2}.$$

Example 4: *Prove that* $\sinh^{-1}(\cot x) = \log(\cot x + \operatorname{cosec} x)$. **(Kanpur 2014)**

Solution: Let $\sinh^{-1}(\cot x) = y$.

Then $\quad \sinh y = \cot x.$ ...(1)

$\therefore \quad \cosh y = \sqrt{(1 + \sinh^2 y)}$

$\quad = \sqrt{(1 + \cot^2 x)} = \operatorname{cosec} x.$ ...(2)

Adding (1) and (2), we have

$\sinh y + \cosh y = \cot x + \operatorname{cosec} x$

or $\quad e^y = \cot x + \operatorname{cosec} x$

or $\quad y = \log(\cot x + \operatorname{cosec} x)$.

$\therefore \quad \sinh^{-1}(\cot x) = \log(\cot x + \operatorname{cosec} x)$.

Example 5: *Express* $\tanh^{-1}(x + iy)$ *into the form* $\alpha + i\beta$ *and hence deduce the value of* $\tanh^{-1}(iy)$. **(Avadh 2014)**

Solution: Let $\tanh^{-1}(x + iy) = \alpha + i\beta$

so that $\quad x + iy = \tanh(\alpha + i\beta) = (1/i)\tan[i(\alpha + i\beta)]$. $\quad [\because \tan i\theta = i \tanh \theta]$

$\therefore \quad i(x + iy) = \tan(i\alpha - \beta) \quad$ or $\quad ix - y = \tan\{-(\beta - i\alpha)\}$

or $\quad -(y - ix) = -\tan(\beta - i\alpha) \quad [\because \tan(-z) = -\tan z]$

or $\quad \tan(\beta - i\alpha) = y - ix.$ ...(1)

Equating the complex conjugates of both sides of (1), we have

$\tan(\beta + i\alpha) = y + ix.$ ...(2)

Now $\quad \tan 2\beta = \tan[(\beta - i\alpha) + (\beta + i\alpha)]$

$$= \frac{\tan(\beta - i\alpha) + \tan(\beta + i\alpha)}{1 - \tan(\beta - i\alpha)\tan(\beta + i\alpha)} = \frac{(y - ix) + (y + ix)}{1 - (y - ix)(y + ix)},$$

from (1) and (2)

$$= \frac{2y}{1 - (y^2 + x^2)} = \frac{2y}{1 - x^2 - y^2}.$$

$$\therefore \quad 2\beta = \tan^{-1}\frac{2y}{1 - x^2 - y^2} \quad \text{or} \quad \beta = \frac{1}{2}\tan^{-1}\left(\frac{2y}{1 - x^2 - y^2}\right) \quad ...(3)$$

Again, $\quad \tan(2i\alpha) = \tan[(\beta + i\alpha) - (\beta - i\alpha)]$

$$= \frac{\tan(\beta + i\alpha) - \tan(\beta - i\alpha)}{1 + \tan(\beta + i\alpha)\tan(\beta - i\alpha)} = \frac{(y + ix) - (y - ix)}{1 + (y + ix)(y - ix)}$$

$$= \frac{2ix}{1 + y^2 + x^2}.$$

$$\therefore \quad i\tanh 2\alpha = \frac{2ix}{1 + x^2 + y^2} \quad \text{or} \quad \tanh 2\alpha = \frac{2x}{1 + x^2 + y^2}$$

or $$\alpha = \frac{1}{2} \tanh^{-1} \left(\frac{2x}{1 + x^2 + y^2} \right). \quad ...(4)$$

$\therefore$ $$\tanh^{-1} (x + iy) = \alpha + i\beta$$

$$= \frac{1}{2} \tanh^{-1} \left(\frac{2x}{1 + x^2 + y^2} \right) + i \frac{1}{2} \tan^{-1} \frac{2y}{(1 - x^2 - y^2)}. \quad ...(5)$$

[From (3) and (4)]

To deduce the value of $\tanh^{-1} (iy)$, put $x = 0$ on both sides of (5).

Then $$\tanh^{-1} (iy) = \frac{1}{2} \tanh^{-1} 0 + i \frac{1}{2} \tan^{-1} \frac{2y}{1 - y^2}$$

$$= 0 + i \frac{1}{2} . 2 \tan^{-1} y \qquad \left[\because 2 \tan^{-1} y = \tan^{-1} \frac{2y}{1 - y^2} \right]$$

$$= i \tan^{-1} y.$$

$\therefore$ $$\tanh^{-1} (iy) = i \tan^{-1} y.$$

Example 6: *If $\sin^{-1} (\theta + i\phi) = \alpha + i\beta$, then prove that $\sin^2 \alpha$ and $\cosh^2 \beta$ are the roots of the equation $x^2 - x (1 + \theta^2 + \phi^2) + \theta^2 = 0$.*

Solution: We have $\sin^{-1} (\theta + i\phi) = \alpha + i\beta$ so that $\theta + i\phi = \sin (\alpha + i\beta)$

or $$\theta + i\phi = \sin \alpha \cosh \beta + i \cos \alpha \sinh \beta.$$

Equating real and imaginary parts, we have

$$\sin \alpha \cosh \beta = \theta, \quad ...(1)$$

and $$\cos \alpha \sinh \beta = \phi. \quad ...(2)$$

Now $1 + \theta^2 + \phi^2 = 1 + \sin^2 \alpha \cosh^2 \beta + \cos^2 \alpha \sinh^2 \beta$, from (1) and (2)

$$= 1 + \sin^2 \alpha \cosh^2 \beta + (1 - \sin^2 \alpha) (\cosh^2 \beta - 1)$$

$$= 1 + \sin^2 \alpha \cosh^2 \beta + \cosh^2 \beta - 1 - \sin^2 \alpha \cosh^2 \beta + \sin^2 \alpha$$

$$= \sin^2 \alpha + \cosh^2 \beta.$$

Thus $\sin^2 \alpha + \cosh^2 \beta = 1 + \theta^2 + \phi^2$.

Also from (1), $\sin^2 \alpha \cosh^2 \beta = \theta^2$.

$\therefore$ $\sin^2 \alpha$ and $\cosh^2 \beta$ are the roots of the equation

$$x^2 - (\sin^2 \alpha + \cosh^2 \beta) x + \sin^2 \alpha \cosh^2 \beta = 0$$

or $$x^2 - (1 + \theta^2 + \phi^2) x + \theta^2 = 0.$$

Comprehensive Exercise 1

1. Show that

$$\sin^{-1}(\cos\theta + i\sin\theta) = \cos^{-1}\sqrt{(\sin\theta)} + i\log[\sqrt{(\sin\theta)} + \sqrt{(1+\sin\theta)}],$$

where θ is a positive acute angle.

(Purvanchal 2006; Rohilkhand 07; Agra 09; Avadh 10)

2. (i) Express $\sin^{-1}(x + iy)$ in the form $A + iB$. **(Purvanchal 2007)**

(ii) Express $\cosh^{-1}(x + iy)$ in the form $\alpha + i\beta$.

3. (i) Show that

$$\cos^{-1}(\cos\theta + i\sin\theta) = \sin^{-1}\sqrt{(\sin\theta)} + i\log\{\sqrt{(1+\sin\theta)} - \sqrt{(\sin\theta)}\},$$

where θ is a positive acute angle.

(ii) Separate $\cos^{-1}(\cos\theta + i\sin\theta)$ into real and imaginary parts.

4. If $\cos^{-1}(\alpha + i\beta) = \theta + i\phi$, then prove that

(i) $\alpha^2\,\text{sech}^2\,\phi + \beta^2\,\text{cosech}^2\,\phi = 1$, (ii) $\alpha^2\sec^2\theta - \beta^2\,\text{cosec}^2\,\theta = 1$.

5. If $\sin^{-1}(x + iy) = \tan^{-1}(u + iv)$, show that

$$[(x-1)^2 + y^2][(x+1)^2 + y^2] = \frac{(x^2+y^2)^2}{(u^2+v^2)^2}.$$

6. Prove that $\text{Sin}^{-1}(\text{cosec}\,\theta) = \{2n + (-1)^n\}\frac{1}{2}\pi + i(-1)^n\log\cot\frac{1}{2}\theta.$

(Gorakhpur 2007)

7. Show that $\text{Sin}^{-1}(ix) = n\pi + i(-1)^n\log\{x + \sqrt{(1+x^2)}\}$.

8. (i) Prove that

$$\text{Tan}^{-1}(\cos\theta + i\sin\theta) = n\pi + \frac{1}{4}\pi - \frac{1}{2}i\log\tan\left(\frac{1}{4}\pi - \frac{1}{2}\theta\right).$$

(Gorakhpur 2005; Avadh 09; Rohilkhand 14)

(ii) Prove that $\text{Tan}^{-1}(\cos\theta + i\sin\theta) = n\pi + \frac{1}{4}\pi + \frac{1}{2}i\log\tan\left(\frac{1}{4}\pi + \frac{1}{2}\theta\right)$.

(Bundelkhand 2009)

9. If $x > y$, then show that $\tan^{-1}\left(\frac{x+iy}{x-iy}\right) = \frac{\pi}{4} + \frac{i}{2}\log\frac{x+y}{x-y}$.

(Meerut 2011, 12, 12B)

10. Prove that $\tan^{-1}\left[i\,\frac{x-a}{x+a}\right] = -\frac{1}{2}i\log\left(\frac{a}{x}\right)$.

(Meerut 2004B; Purvanchal 09)

11. If $\cosh x = \sec\theta$, then prove that $x = \log(\sec\theta \pm \tan\theta)$.

12. (i) Show that $\sinh^{-1} x = \tanh^{-1} \dfrac{x}{\sqrt{(1+x^2)}}$.

(ii) Prove that $\tanh^{-1} x = \sinh^{-1} \{x / \sqrt{(1-x^2)}\}$. **(Kashi 2012)**

(iii) Prove that $\coth^{-1} (2/x) = \sinh^{-1} \{x / \sqrt{(4-x^2)}\}$.

13. Prove that

$$\text{Tan}^{-1}\left(\frac{\tan 2\theta + \tanh 2\phi}{\tan 2\theta - \tanh 2\phi}\right) + \text{Tan}^{-1}\left(\frac{\tan \theta - \tanh \phi}{\tan \theta + \tanh \phi}\right) = \text{Tan}^{-1}(\cot \theta \coth \phi).$$

14. If $\cos^{-1} (u + iv) = \alpha + i\beta$, prove that $cos^2 \alpha$ and $\cosh^2 \beta$ are the roots of the equation

$$x^2 - (1 + u^2 + v^2)\, x + u^2 = 0.$$ **(Agra 2007; Kanpur 09)**

15. If $\cosh^{-1} (x + iy) + \cosh^{-1} (x - iy) = \cosh^{-1} a$,

show that $2 (a-1)\, x^2 + 2 (a+1)\, y^2 = a^2 - 1$. **(Meerut 2006; Kanpur 10)**

Answers 1

2. (i) $$\frac{1}{2}\pi \pm \sin^{-1}\left[\frac{\sqrt{\{(x^2+y^2-1)^2+4y^2\}}-(x^2+y^2-1)}{2}\right]^{1/2}$$

$$\pm\, i \sinh^{-1}\left[\frac{\sqrt{\{(1-x^2-y^2)^2+4y^2\}}-(1-x^2-y^2)}{2}\right]^{1/2}$$

(ii) $$\alpha = \pm \sinh^{-1}\left[\frac{\sqrt{\{(1-x^2-y^2)^2+4y^2\}}-(1-x^2-y^2)}{2}\right]^{1/2}$$

$$\beta = \pm \sin^{-1}\left[\frac{\sqrt{\{(x^2+y^2-1)^2+4y^2\}}-(x^2+y^2-1)}{2}\right]^{1/2}$$

Objective Type Questions

Fill In The Blanks

Fill in the blanks "……" so that the following statements are complete and correct.

1. If w and z are any complex numbers then we define $w = \sinh^{-1} z$ if …… .

2. $\sinh^{-1} z = \log [z + \text{……}]$.

3. $\cosh^{-1} z = \log [z + \text{……}]$.

4. $\tanh^{-1} z = \ldots\ldots \log \dfrac{1+z}{1-z}$.

5. If w and z are any complex numbers then we define $w = \tanh^{-1} z$ if ....

True or False

Write 'T' for true and 'F' for false statement.

1. $\sinh^{-1} x = i \sin^{-1} (ix)$.
2. $\coth^{-1} z = \dfrac{1}{2} \log \dfrac{z+1}{z-1}$.
3. $\cosh^{-1} z = \log [z + \sqrt{(1 - z^2)}]$.
4. $\tanh^{-1} x = - i \tan^{-1} (ix)$.
5. $\cosh^{-1} x = i \cos^{-1} x$.

Answers

Fill in the Blanks

1. $\sinh w = z$ 2. $\sqrt{(z^2 + 1)}$ 3. $\sqrt{(z^2 - 1)}$
4. $\dfrac{1}{2}$ 5. $\tanh w = z$

True or False

1. *F* 2. *T* 3. *F* 4. *T* 5. *F*

□

Chapter

5

Gregory's Series

5.1 Gregory's Series

To prove that, if θ *lies within the closed interval* $[-\pi/4, \pi/4]$, *i.e., if* $-\pi/4 \le \theta \le \pi/4$, *then*

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \frac{1}{7}\tan^7\theta + \ldots\ldots \textit{ad. inf}.$$

(Kanpur 2005, 08; Meerut 10B; Avadh 06; Bundelkhand 14; Agra 14)

Proof: We have

$$(1 + i\tan\theta) = \left(1 + i\frac{\sin\theta}{\cos\theta}\right) = \frac{1}{\cos\theta}(\cos\theta + i\sin\theta) = \sec\theta \,.\, e^{i\theta}.$$

Taking logarithm of both sides, we have

$$\log(1 + i\tan\theta) = \log\sec\theta + \log e^{i\theta},$$

(considering only principal values)

or $$\log(1 + i\tan\theta) = \log\sec\theta + i\theta.$$

Now since θ lies between $-\pi/4$ and $\pi/4$, $\tan\theta$ lies between -1 and 1, *i.e.*, $\tan\theta$ is numerically not greater than unity.

Therefore,

$$\log \sec\theta + i\theta = \log(1 + i\tan\theta)$$

$$= i\tan\theta - \frac{i^2\tan^2\theta}{2} + \frac{i^3\tan^3\theta}{3} - \frac{i^4\tan^4\theta}{4} + \ldots\ldots, \qquad \ldots(1)$$

(expanding the R.H.S. by logarithmic series which is justified because $|i\tan\theta| = |\tan\theta| \le 1$)

$$= i\tan\theta + \frac{1}{2}\tan^2\theta - \frac{1}{3}i\tan^3\theta - \frac{1}{4}\tan^4\theta + \frac{1}{5}i\tan^5\theta + \ldots$$

Equating the imaginary parts on both sides, we have

$$\boldsymbol{\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \ldots\ldots} \qquad \ldots(2)$$

The general term on the R.H.S. is $\dfrac{(-1)^{n-1}}{2n-1}\tan^{2n-1}\theta$.

This expansion (2) is known as Gregory's series after the name of James Gregory (1638 – 1675).

Another Form of Gregory's Series.

In the series (2) if we put $\tan\theta = x$ so that $\theta = \tan^{-1} x$, then we have another form of the Gregory's series as

$$\boldsymbol{\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots \text{ ad. inf.}}$$

where $-1 \le x \le 1$ i.e., $|x| \le 1$.

Corollary: Equating real parts from both sides of (1), we have

$$\log\sec\theta = \frac{1}{2}\tan^2\theta - \frac{1}{4}\tan^4\theta + \frac{1}{6}\tan^6\theta - \ldots\ldots$$

(Bundelkhand 2008)

5.2 General Theorem on Gregory's Series

If θ *lies between* $n\pi - \frac{1}{4}\pi$ *and* $n\pi + \frac{1}{4}\pi$, *both limits being inclusive, then*

$$\theta - n\pi = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \ldots\ldots \textit{ ad. inf.}$$

Proof: Let $\theta - n\pi = \phi$ *i.e.,* $\theta = n\pi + \phi$.

Then ϕ lies between $-\dfrac{\pi}{4}$ and $\dfrac{\pi}{4}\cdot$

Now $\quad 1 + i\tan\theta = 1 + i\tan(n\pi + \phi) = 1 + i\tan\phi$

$$= \sec\phi\,(\cos\phi + i\sin\phi) = \sec\phi \,.\, e^{i\phi}.$$

Taking logarithm of both sides, we get

$$\log(1 + i\tan\theta) = \log\sec\phi + \log e^{i\phi}$$

or $\quad \log \sec \phi + i\phi = \log (1 + i \tan \theta)$,

the expansion of which is valid because θ lies between $n\pi - \frac{1}{4}\pi$ and $n\pi + \frac{1}{4}\pi$ implies that tan θ is not numerically greater than 1.

$$\therefore \quad \log \sec \phi + i\phi = i \tan \theta - \frac{1}{2} i^2 \tan^2 \theta + \frac{1}{3} i^3 \tan^3 \theta - \frac{1}{4} i^4 \tan^4 \theta + \ldots\ldots \infty,$$

$$\left[\because \log (1 + z) = z - \frac{1}{2} z^2 + \frac{1}{3} z^3 - \ldots \infty, \text{ if } |z| \le 1\right]$$

$$= i \tan \theta + \frac{1}{2} \tan^2 \theta - \frac{1}{3} i \tan^3 \theta - \frac{1}{4} \tan^4 \theta + \ldots\ldots \infty.$$

Equating imaginary parts from both sides, we have

$$\phi = \tan \theta - \frac{1}{3} \tan^3 \theta + \frac{1}{5} \tan^5 \theta - \ldots\ldots \text{ad. inf.}$$

$$\therefore \quad \boldsymbol{\theta - n\pi = \tan \theta - \frac{1}{3} \tan^3 \theta + \frac{1}{5} \tan^5 \theta - \ldots\ldots \text{ ad. inf.}}$$

5.3 Value of π

The main use of Gregory's series is to find the value of π to various decimal places. Some mathematicians have designed different expressions based on Gregory's series for finding the value of π. Some important cases are given below :

(a) Gregory's series:

In the Gregory's series

$$\tan^{-1} x = x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \frac{1}{7} x^7 + \ldots\ldots \infty,$$

if we put $x = 1$, then we have

$$\tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\ldots \infty \quad \text{or} \quad \frac{1}{4}\pi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\ldots \infty.$$

From this the value of π can be calculated. But this series does not converge rapidly and so a large number of terms will have to be taken in order to evaluate π correct to a certain decimal place. So several other series have been designed for this purpose.

(b) Euler's series: We have

$$\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \tan^{-1} \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}} = \tan^{-1} \frac{5/6}{5/6} = \tan^{-1} 1 = \frac{\pi}{4}.$$

Thus $\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$

$$= \left(\frac{1}{2} - \frac{1}{3}\cdot\frac{1}{2^3} + \frac{1}{5}\cdot\frac{1}{2^5} - \ldots\ldots\right) + \left(\frac{1}{3} - \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} - \ldots\ldots\right),$$

expanding both $\tan^{-1}\frac{1}{2}$ and $\tan^{-1}\frac{1}{3}$

by Gregory's series because $\frac{1}{2} < 1$ and $\frac{1}{3} < 1$

$$= \left(\frac{1}{2} + \frac{1}{3}\right) - \frac{1}{3}\left(\frac{1}{2^3} + \frac{1}{3^3}\right) + \frac{1}{5}\left(\frac{1}{2^5} + \frac{1}{3^5}\right) - \ldots\ldots$$

From this the value of π can be calculated. This series is more rapidly convergent than the preceding one.

(c) Machin's series: We have

$$4\tan^{-1}\frac{1}{5} = 2\,.\,2\tan^{-1}\frac{1}{5} = 2\tan^{-1}\frac{2/5}{1-(1/25)} = 2\tan^{-1}\frac{5}{12}$$

$$= \tan^{-1}\left\{\frac{2\,.\,(5/12)}{1-(5/12)^2}\right\} = \tan^{-1}\frac{120}{119}.$$

Now $4\tan^{-1}\frac{1}{5} - \frac{1}{4}\pi = \tan^{-1}\frac{120}{119} - \tan^{-1}1$ $\left[\because \frac{1}{4}\pi = \tan^{-1}1\right]$

$$= \tan^{-1}\frac{\frac{120}{119} - 1}{1 + \frac{120}{119}\times 1} = \tan^{-1}\frac{1}{239}.$$

Therefore, $\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$

or $\frac{\pi}{4} = 4\left[\frac{1}{5} - \frac{1}{3}\cdot\frac{1}{5^3} + \frac{1}{5}\cdot\frac{1}{5^5} - \ldots\ldots\right] - \left[\frac{1}{239} - \frac{1}{3}\cdot\frac{1}{(239)^3} + \frac{1}{5}\cdot\frac{1}{(239)^5} - \ldots\ldots\right],$

expanding both the terms on the R.H.S. by Gregory's series because $\frac{1}{5} < 1$ and $\frac{1}{239} < 1$.

The above is Machin's series and it is more rapidly convergent than Euler's series.

(d) Rutherford's Series: We have

$$\tan^{-1}\frac{1}{70} - \tan^{-1}\frac{1}{99} = \tan^{-1}\frac{\frac{1}{70} - \frac{1}{99}}{1 + \frac{1}{70}\cdot\frac{1}{99}} = \tan^{-1}\frac{99-70}{70\,.\,99+1}$$

$$= \tan^{-1}\frac{29}{6931} = \tan^{-1}\frac{1}{239} = 4\tan^{-1}\frac{1}{5} - \frac{\pi}{4},$$

as shown in Machin's series.

$\therefore \quad \dfrac{\pi}{4} = 4 \tan^{-1} \dfrac{1}{5} - \tan^{-1} \dfrac{1}{70} + \tan^{-1} \dfrac{1}{99}$

i.e., $\quad \dfrac{\pi}{4} = 4 \left\{ \dfrac{1}{5} - \dfrac{1}{3} \cdot \dfrac{1}{5^3} + \dfrac{1}{5} \cdot \dfrac{1}{5^5} - \dots \right\} - \left\{ \dfrac{1}{70} - \dfrac{1}{3} \cdot \dfrac{1}{(70)^3} + \dfrac{1}{5} \cdot \dfrac{1}{(70)^5} - \dots \right\}$

$$+ \left\{ \frac{1}{99} - \frac{1}{3} \cdot \frac{1}{(99)^3} + \frac{1}{5} \cdot \frac{1}{(99)^5} - \dots\dots \right\}.$$

It is Rutherford's series. This is more convenient for expansion than Machin's series and converges equally rapidly.

Illustrative Examples

Example 1: *Assuming that* $\theta - n\pi = \tan\theta - \dfrac{1}{3}\tan^3\theta + \dfrac{1}{5}\tan^5\theta - \dots,$ *when* θ *lies between* $(n\pi - \dfrac{1}{4}\pi)$ *and* $(n\pi + \dfrac{1}{4}\pi)$, *write down the value of n when* θ *lies between*

(i) $\dfrac{7\pi}{4}$ *and* $\dfrac{9\pi}{4}$ *(ii)* $\dfrac{-3\pi}{4}$ *and* $\dfrac{-5\pi}{4}$ *(iii)* $\dfrac{11\pi}{4}$ *and* $\dfrac{13\pi}{4}$

(iv) $\dfrac{-15\pi}{4}$ *and* $\dfrac{-17\pi}{4}$ *(v)* $\dfrac{19\pi}{4}$ *and* $\dfrac{21\pi}{4}$.

Solution: **(i)** θ lies between $(7\pi/4)$ and $(9\pi/4)$

i.e., between $(2\pi - \dfrac{1}{4}\pi)$ and $(2\pi + \dfrac{1}{4}\pi)$. $\quad \therefore \quad n = 2.$

(ii) θ lies between $-(3\pi/4)$ and $-(5\pi/4)$

i.e., between $(-\pi + \dfrac{1}{4}\pi)$ and $(-\pi - \dfrac{1}{4}\pi)$. $\quad \therefore \quad n = -1.$

(iii) θ lies between $(11\pi/4)$ and $(13\pi/4)$

i.e., between $(3\pi - \dfrac{1}{4}\pi)$ and $(3\pi + \dfrac{1}{4}\pi)$. $\quad \therefore \quad n = 3.$

(iv) θ lies between $-(15\pi/4)$ and $-(17\pi/4)$

i.e., between $(-4\pi + \dfrac{1}{4}\pi)$ and $(-4\pi - \dfrac{1}{4}\pi)$. $\quad \therefore \quad n = 4.$

(v) θ lies between $(19\pi/4)$ and $(21\pi/4)$

i.e., between $(5\pi - \dfrac{1}{4}\pi)$ and $(5\pi + \dfrac{1}{4}\pi)$. $\quad \therefore \quad n = 5.$

Example 2: *Sum the series* $\dfrac{1}{2^3} - \dfrac{1}{3 \cdot 2^7} + \dfrac{1}{5 \cdot 2^{11}} - \dots\dots$ *ad. inf.*

(Meerut 2013B; Kanpur 14)

Solution: We have $\dfrac{1}{2^3} - \dfrac{1}{3 \cdot 2^7} + \dfrac{1}{5 \cdot 2^{11}} + \dots\dots$

$$= \frac{1}{2}\left[\frac{1}{2^2} - \frac{1}{3 \cdot 2^6} + \frac{1}{5 \cdot 2^{10}} - \dots\dots\right], \qquad \text{(taking } \tfrac{1}{2} \text{ common)}$$

$$= \frac{1}{2}\left[\frac{1}{(2^2)} - \frac{1}{3\,(2^2)^3} + \dots\dots\right] = \frac{1}{2}\tan^{-1}\left(\frac{1}{2^2}\right),$$

by Gregory's series because $1/2^2 < 1$.

[Note that by Gregory's series
$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots \infty$, if $|x| \le 1$]

$$= \frac{1}{2}\tan^{-1}\frac{1}{4}.$$

Example 3: *Sum to infinity the series*

(i) $1 - \frac{1}{3\,.\,4^2} + \frac{1}{5\,.\,4^4} - \dots$ *ad. inf.*; **(Meerut 2013)**

(ii) $1 - \frac{1}{3^2} + \frac{1}{5\,.\,3^2} - \frac{1}{7\,.\,3^3} + \dots$ *ad. inf.*

Solution: (i) The given series $1 - \frac{1}{3\,.\,4^2} + \frac{1}{5\,.\,4^4} - \dots$ ad. inf.

$$= 4\left[\frac{1}{4} - \frac{1}{3\,.\,4^3} + \frac{1}{5\,.\,4^5} - \dots \text{ ad. inf.}\right], \text{ (taking 4 common)}$$

$= 4\tan^{-1}\frac{1}{4}$, by Gregory's series because $1/4 < 1$.

(ii) The given series

$$1 - \frac{1}{3^2} + \frac{1}{5\,.\,3^2} - \frac{1}{7\,.\,3^3} + \dots = 1 - \frac{1}{3\,.\,3} + \frac{1}{5\,.\,3^2} - \frac{1}{7\,.\,3^3} + \dots$$

$$= \sqrt{3}\left[\frac{1}{\sqrt{3}} - \frac{1}{3\,.\,(\sqrt{3})^3} + \frac{1}{5\,.\,(\sqrt{3})^5} - \frac{1}{7\,.\,(\sqrt{3})^7} + \dots\right]$$

$= \sqrt{3}\,\{\tan^{-1}(1/\sqrt{3})\}$, by Gregory's series because $1/\sqrt{3} < 1$

$= \sqrt{3}\,.\,(\pi/6) = (\pi\sqrt{3})/6$.

Note: This question can also be written as

Prove that $\pi = 2\sqrt{3}\left\{1 - \frac{1}{3^2} + \frac{1}{5\,.\,3^2} - \frac{1}{7\,.\,3^3} + \dots\right\}$.

(Agra 2005; Rohilkhand 08; Kashi 12)

Example 4: *Prove that*

$$\frac{\pi}{4} = \frac{17}{21} - \frac{713}{81 \times 343} + \dots\dots + \frac{(-1)^{n+1}}{2n-1}\left\{\frac{2}{3}\,9^{1-n} + 7^{1-2n}\right\} + \dots$$

(Kanpur 2008, 10; Agra 08)

Solution: The nth term of the given series on the R.H.S. is

$$T_n = \frac{(-1)^{n+1}}{2n-1}\left\{\frac{2}{3}\cdot(3^2)^{1-n} + 7^{1-2n}\right\} = \frac{(-1)^{n+1}}{2n-1}\left\{\frac{2}{3}\cdot 3^{2-2n} + 7^{1-2n}\right\}$$

$$= \frac{(-1)^{n+1}}{(2n-1)} \left[\frac{2}{3^{2n-1}} + \frac{1}{7^{2n-1}} \right].$$

Putting $n = 1, 2, 3, \ldots\ldots$etc., we have

$$\text{first term} = T_1 = \left(\frac{2}{3} + \frac{1}{7} \right);$$

$$\text{second term} = T_2 = -\frac{1}{3}\left(\frac{2}{3^3} + \frac{1}{7^3} \right) = -\frac{2}{3.3^3} - \frac{1}{3.7^3};$$

$$\text{third term} = T_3 = \frac{1}{5}\left(\frac{2}{3^5} + \frac{1}{7^5} \right) = \frac{2}{5} \cdot \frac{1}{3^5} + \frac{1}{5.7^5}; \text{ and so on.}$$

$\therefore$ the sum of the series on the R.H.S.

$$= T_1 + T_2 + T_3 + \ldots \text{ ad. inf.}$$

$$= \left(\frac{2}{3} + \frac{1}{7} \right) - \left(\frac{2}{3.3^3} + \frac{1}{3.7^3} \right) + \left(\frac{2}{5.3^5} + \frac{1}{5.7^5} \right) - \ldots$$

$$= 2\left(\frac{1}{3} - \frac{1}{3.3^3} + \frac{1}{5.3^5} - \ldots \right) + \left(\frac{1}{7} - \frac{1}{3.7^3} + \frac{1}{5.7^5} - \ldots \right)$$

$$= 2 \tan^{-1} (1/3) + \tan^{-1} (1/7), \quad \text{(by Gregory's series)}$$

$$= \tan^{-1} \left\{ \frac{\frac{1}{3} + \frac{1}{3}}{1 - \frac{1}{3} \cdot \frac{1}{3}} \right\} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{3}{4} + \tan^{-1} \frac{1}{7}$$

$$= \tan^{-1} \left\{ \frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{4} \cdot \frac{1}{7}} \right\} = \tan^{-1} 1 = \frac{\pi}{4}.$$

Example 5: *Express* $\tan^{-1} (\cos\theta + i \sin\theta)$ *in the form* $A + iB$ *and deduce that*

(i) $\cos\theta - \frac{1}{3} \cos 3\theta + \frac{1}{5} \cos 5\theta - \ldots = \pm \frac{1}{4} \pi$; *and* **(Kanpur 2014)**

(ii) $\sin\theta - \frac{1}{3} \sin 3\theta + \frac{1}{5} \sin 5\theta - \ldots = \frac{1}{2} \log \{\pm \tan (\frac{1}{4} \pi + \frac{1}{2} \theta)\}.$

Solution: Let $\tan^{-1} (\cos\theta + i \sin\theta) = A + iB$

so that $\tan^{-1} (\cos\theta - i \sin\theta) = A - iB$, (complex conjugates).

$$\therefore \quad 2A = (A + iB) + (A - iB)$$

$$= \tan^{-1} (\cos\theta + i \sin\theta) + \tan^{-1} (\cos\theta - i \sin\theta)$$

$$= \tan^{-1} \left\{ \frac{(\cos\theta + i \sin\theta) + (\cos\theta - i \sin\theta)}{1 - (\cos\theta + i \sin\theta)(\cos\theta - i \sin\theta)} \right\}$$

$$= \tan^{-1} \left\{ \frac{2 \cos\theta}{1 - (\cos^2\theta + \sin^2\theta)} \right\} = \tan^{-1} \left(\frac{2 \cos\theta}{1 - 1} \right)$$

$$= \tan^{-1} \left(\frac{2 \cos\theta}{0} \right) = \tan^{-1} (\pm \infty) = \pm \frac{1}{2} \pi.$$

[$\because$ θ lies between $-\pi$ and π]

Therefore, $A = \pm \frac{1}{4}\pi$. ...(1)

Again, $2iB = (A + iB) - (A - iB)$

$= \tan^{-1}(\cos\theta + i\sin\theta) - \tan^{-1}(\cos\theta - i\sin\theta)$

$= \tan^{-1}\left\{\frac{(\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)}{1 + (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}\right\}$

$= \tan^{-1}\left\{\frac{2i\sin\theta}{1 + (\cos^2\theta + \sin^2\theta)}\right\} = \tan^{-1}\left\{\frac{2i\sin\theta}{1+1}\right\} = \tan^{-1}(i\sin\theta)$

or $\tan(2iB) = i\sin\theta$ or $\tanh 2B = \sin\theta$

or $\frac{e^{2B} - e^{-2B}}{e^{2B} + e^{-2B}} = \frac{\sin\theta}{1}$ or $\frac{2e^{2B}}{2e^{-2B}} = \frac{1 + \sin\theta}{1 - \sin\theta}$

[By componendo and dividendo]

i.e., $e^{4B} = \frac{1 + \sin\theta}{1 - \sin\theta} = \frac{\{\cos(\theta/2) + \sin(\theta/2)\}^2}{\{\cos(\theta/2) - \sin(\theta/2)\}^2}$

or $e^{2B} = \pm\frac{\{\cos(\theta/2) + \sin(\theta/2)\}}{\{\cos(\theta/2) - \sin(\theta/2)\}} = \pm\frac{\{1 + \tan(\theta/2)\}}{\{1 - \tan(\theta/2)\}} = \pm\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$.

$\therefore$ $B = \frac{1}{2}\log\left\{\pm\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right\}$ ...(2)

Now $\tan^{-1}(\cos\theta + i\sin\theta) = A + iB$

or $\tan^{-1}(e^{i\theta}) = A + iB$ or $e^{i\theta} - \frac{1}{3}e^{3i\theta} + \frac{1}{5}e^{5i\theta} - \ldots = A + iB$,

using Gregory's series

or $(\cos\theta + i\sin\theta) - \frac{1}{3}(\cos 3\theta + i\sin 3\theta) + \frac{1}{5}(\cos 5\theta + i\sin 5\theta) - \ldots = A + iB$

or $(\cos\theta - \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta - \ldots) + i(\sin\theta - \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta - \ldots)$

$= A + iB = \pm\frac{1}{4}\pi + \frac{1}{2}i\log\{\pm\tan(\frac{1}{4}\pi + \frac{1}{2}\theta)\}$, by (1) and (2).

Equating real and imaginary parts, we have

$$\cos\theta - \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta - \ldots = \pm\frac{1}{4}\pi$$

and $$\sin\theta - \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta - \ldots = \frac{1}{2}\log\{\pm\tan(\frac{1}{4}\pi + \frac{1}{2}\theta)\}.$$

Comprehensive Exercise 1

1. (i) Prove that $\frac{\pi}{4} = \left[\frac{2}{3} + \frac{1}{7}\right] - \frac{1}{3}\left[\frac{2}{3^3} + \frac{1}{7^3}\right] + \frac{1}{5}\left[\frac{2}{3^5} + \frac{1}{7^5}\right] - \ldots$

(Kanpur 2005, 08, 10; Agra 08; Bundelkhand 09; Purvanchal 10; Meerut 12; Kashi 13)

(ii) Prove that $\dfrac{\pi}{8} = \dfrac{1}{1.3} + \dfrac{1}{5.7} + \dfrac{1}{9.11} + \ldots$ad. inf.

(Meerut 2004B, 07, 09, 13B; Agra 07, 08; Kanpur 12)

(iii) Show that

$$\frac{\pi}{12} = \left\{1 - \frac{1}{3^{1/2}}\right\} - \frac{1}{2}\left\{1 - \frac{1}{3^{3/2}}\right\} + \frac{1}{5}\left\{1 - \frac{1}{3^{5/2}}\right\} - \ldots \infty.$$

(Meerut 2012B, 13)

(iv) Show that

$$\frac{\pi}{4} = \left[\frac{1}{2} + \frac{1}{5} + \frac{1}{8}\right] - \frac{1}{3}\left[\frac{1}{2^3} + \frac{1}{5^3} + \frac{1}{8^3}\right] + \frac{1}{5}\left[\frac{1}{2^5} + \frac{1}{5^5} + \frac{1}{8^5}\right] - \ldots\ldots$$

2. If $x < (\sqrt{2} - 1)$, prove that $2\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \ldots \text{ ad. inf.}\right)$

$$= \left(\frac{2x}{1 - x^2}\right) - \frac{1}{3}\left(\frac{2x}{1 - x^2}\right)^3 + \frac{1}{5}\left(\frac{2x}{1 - x^2}\right)^5 - \ldots$$

(Kashi 2014)

3. If $x > 0$, prove that $\tan^{-1} x = \dfrac{\pi}{4} + \dfrac{x - 1}{x + 1} - \dfrac{1}{3}\left\{\dfrac{x - 1}{x + 1}\right\}^3 + \ldots$

(Garhwal 2000)

4. When θ lies between 0 and $\frac{1}{2}\pi$, prove that

$$\tan^{-1}\left(\frac{1 - \cos\theta}{1 + \cos\theta}\right) = \tan^2\frac{\theta}{2} - \frac{1}{3}\tan^6\frac{\theta}{2} + \frac{1}{5}\tan^{10}\frac{\theta}{2} - \ldots\ldots \infty.$$

(Bundelkhand 2010)

5. When both θ and $\tan^{-1}(\sec\theta)$ lie between 0 and $\frac{1}{2}\pi$, prove that

$$\tan^{-1}(\sec\theta) = \frac{1}{4}\pi + \tan^2\frac{1}{2}\theta - \frac{1}{3}\tan^6\frac{1}{2}\theta + \frac{1}{5}\tan^{10}\frac{1}{2}\theta - \ldots$$

(Kanpur 2006)

6. Prove that

$$\tan^{-1}\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) = n\pi + \frac{1}{4}\pi + \tan\theta - (1/3)\tan^3\theta + (1/5)\tan^5\theta - \ldots$$

(Meerut 2006; Kanpur 14)

7. (i) If x lies between $-\pi/4$ and $\pi/4$, show that

$$\tan x - \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x - \ldots \text{ ad. inf.} = \tanh x + \frac{1}{3}\tanh^3 x + \frac{1}{5}\tanh^5 x + \ldots \text{ ad. inf.}$$

(Kumaun 2000)

(ii) If ϕ lies between $(\pi/4)$ and $(3\pi/4)$, show that

$$\phi = \frac{1}{2}\pi - \cot\phi + \frac{1}{3}\cot^3\phi - \frac{1}{5}\cot^5\phi + \ldots\ldots \infty.$$

(Kanpur 2005)

8. If $\tan x < 1$, show that

$$\tan^2 x - \frac{1}{2}\tan^4 x + \frac{1}{3}\tan^6 x - \ldots = \sin^2 x + \frac{1}{2}\sin^4 x + \frac{1}{3}\sin^6 x + \ldots$$

(Meerut 2011)

9. Prove that if x, y, z are cube roots of unity,

$$\frac{\tan^{-1} x}{x} + \frac{\tan^{-1} y}{y} + \frac{\tan^{-1} z}{z} = 3\left[1 - \frac{1}{7} + \frac{1}{13} - \frac{1}{19} + \frac{1}{25} - \ldots\right].$$

(Kanpur 2009)

Objective Type Questions

Fill in the Blanks

Fill in the blanks "……" so that the following statements are complete and correct.

1. If $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$, then by Gregory's series, we have $\theta = \ldots\ldots$.

2. If $|x| \le 1$, then by Gregory's series, we have $\tan^{-1} x = \ldots\ldots$.

3. The sum of the infinite series

$$\frac{1}{2^3} - \frac{1}{3 \cdot 2^7} + \frac{1}{5 \cdot 2^{11}} - \ldots\ldots \infty \text{ is } \ldots\ldots .$$

True or False

Write 'T' for true and 'F' for false statement.

1. $1 - \frac{1}{3 \cdot 4^2} + \frac{1}{5 \cdot 4^4} - \ldots\ldots \infty = 4 \tan^{-1} \frac{1}{4}$

2. If $|x| \le 1$, then $\tan^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\ldots \infty.$

Answers

Fill in the Blanks

1. $\tan \theta - \frac{\tan^3 \theta}{3} + \frac{\tan^5 \theta}{5} - \frac{\tan^7 \theta}{7} + \ldots\ldots \infty$

2. $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots \infty$

3. $\frac{1}{2} \tan^{-1} \frac{1}{4}$

True or False

1. *T* **2.** *F*

□

Chapter

6

Summation of Trigonometrical Series

6.1 Introduction

In this chapter we shall give some important methods for summing up trigonometric series which may be finite or infinite. There are two main methods for summation, known as the $C + iS$ *method* and the *difference method*. First we shall discuss the $C + iS$ method.

6.2 $C + iS$ Method for Summing Up Trigonometric Series

Consider the series

$$C = a_0 \cos\alpha + a_1 \cos(\alpha + \beta) + a_2 \cos(\alpha + 2\beta) + \ldots \qquad \ldots(1)$$

and

$$S = a_0 \sin\alpha + a_1 \sin(\alpha + \beta) + a_2 \sin(\alpha + 2\beta) + \ldots \qquad \ldots(2)$$

These series may be finite or infinite. The coefficients a_0, a_1, a_2 etc. and α, β may be any numbers real or complex.

The series (1) and (2) are companion series because each is summed up with the help of the other.

In the series (1) we have terms which contain cosines of numbers which increase in arithmetic progression. It is called *cosine series* and its sum is denoted by C. The series (2) is obtained from the series (1) by simply replacing cos by sin in various terms where we have cosines of numbers which increase in arithmetic progression, the coefficients a_0, a_1, a_2 etc. are kept as they are. The series (2) is called sine series and its sum is denoted by S.

In case we are given one of these two series and are required to find its sum, we first write the other series called its *companion series* or its *auxiliary series.*

After writing both C and S, we find $C + iS$ and $C - iS$ by making use of the Euler's theorem

$$e^{iz} = \cos z + i \sin z, \text{ and } e^{-iz} = \cos z - i \sin z.$$

Thus we get

$$C + iS = a_0 e^{i\alpha} + a_1 e^{i(\alpha+\beta)} + a_2 \; e^{i(\alpha+2\beta)} + \ldots \qquad \ldots(3)$$

and $$C - iS = a_0 e^{-i\alpha} + a_1 e^{-i(\alpha+\beta)} + a_2 \; e^{-i(\alpha+2\beta)} + \ldots \qquad \ldots(4)$$

The series in (3) and (4) can generally be summed up easily. Depending upon the coefficients a_0, a_1, a_2 etc., these series are generally in one of the following forms :

(i) Series in geometric progression, or arithmetico-geometric series.

(ii) Binomial series.

(iii) Exponential series or its sub-case sine or cosine series.

(iv) Logarithmic series or its sub-case Gregory's series.

Having found the sums of the series (3) and (4), we use

$$C = \frac{1}{2}[(C + iS) + (C - iS)] \quad \text{and} \quad S = (1/2i)[(C + iS) - (C - iS)]$$

to find the values of C and S respectively.

In the above discussion the coefficients a_0, a_1, a_2 etc. and α, β are real or complex. However, if these are real, we need not find $C - iS$. To get C and S we simply find the sum of the series (3) *i.e.*, we find $C + iS$. After separating $C + iS$ into real and imaginary parts, we get the values of C and S by equating the real and imaginary parts.

Remark: We are always to find $C + iS$ and never $S + iC$ whether we first write C or S.

Now we shall illustrate $C + iS$ method by taking examples based upon series of different types.

6.3 Series Based on Geometric Progression or Arithmetico-Geometric Series

We know that a geometric progression (G.P.) is of the form

$$a, ar, ar^2, ar^3, \ldots, ar^n, \ldots \infty.$$

Its common ratio is r and nth term is ar^{n-1}.

Sum of n terms in G.P.

i.e., $$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} = a\left(\frac{1-r^n}{1-r}\right) \quad \text{or} \quad = a\left(\frac{r^n - 1}{r-1}\right).$$

Sum of an infinite series in geometrical progression

i.e., $$a + ar + ar^2 + ar^3 + \ldots + ar^n + \ldots \text{ad. inf.} = a/(1-r), \text{ provided } |r| < 1.$$

An arithmetico-geometric progression is of the form

$$a, (a+d)\,r, (a+2d)\,r^2, (a+3d)\,r^3, \ldots.$$

Illustrative Examples

Example 1: *Sum the series*

$$\sin\alpha + c\sin(\alpha+\beta) + c^2\sin(\alpha+2\beta) + \ldots \text{ upto } n \text{ terms.}$$

Deduce the sum to infinity if $|c| < 1$ *i.e.,* $-1 < c < 1$.

Solution: Let the given series be denoted by S. Then

$$S = \sin\alpha + c\sin(\alpha+\beta) + c^2\sin(\alpha+2\beta) + \ldots \text{to } n \text{ terms.}$$

So the auxiliary series is given by

$$C = \cos\alpha + c\cos(\alpha+\beta) + c^2\cos(\alpha+2\beta) + \ldots \text{to } n \text{ terms.}$$

Now multiplying the first series by i and adding to the second, we get

$$C + iS = (\cos\alpha + i\sin\alpha) + c\{\cos(\alpha+\beta) + i\sin(\alpha+\beta)\}$$
$$+ c^2\{\cos(\alpha+2\beta) + i\sin(\alpha+2\beta)\} + \ldots \text{ to } n \text{ terms}$$
$$= e^{i\alpha} + ce^{i(\alpha+\beta)} + c^2 e^{i(\alpha+2\beta)} + \ldots \quad \text{to } n \text{ terms.} \quad \ldots(1)$$

The series on the R.H.S. is a geometrical progression having n terms. The first term is $e^{i\alpha}$ and the common ratio is $ce^{i\beta}$.

Therefore summing up the G.P., we have

$$C + iS = \frac{e^{i\alpha}\{1-(ce^{i\beta})^n\}}{1-ce^{i\beta}} = \frac{e^{i\alpha} - c^n e^{i(\alpha+n\beta)}}{(1-ce^{i\beta})}$$

$$= \frac{[e^{i\alpha} - c^n e^{i(\alpha+n\beta)}]}{(1-ce^{i\beta})} \times \frac{(1-ce^{-i\beta})}{(1-ce^{-i\beta})}, \quad \text{multiplying the Nr.}$$

and the Dr. by the conjugate complex of the Dr.

$$= \frac{e^{i\alpha} - c^n e^{i(\alpha+n\beta)} - ce^{i(\alpha-\beta)} + c^{n+1} e^{i[\alpha+(n-1)\beta]}}{1 - c\,(e^{i\beta} + e^{-i\beta}) + c^2 \,.\, e^{i\beta} \,.\, e^{-i\beta}}.$$

Now the numerator of the R.H.S.

$$= [(\cos\alpha + i\sin\alpha) - c^n\{\cos(\alpha+n\beta) + i\sin(\alpha+n\beta)\}$$
$$- c\{\cos(\alpha-\beta) + i\sin(\alpha-\beta)\}$$
$$+ c^{n+1}\{\cos(\alpha + \overline{n-1}\,\beta) + i\sin(\alpha + \overline{n-1}\,\beta)\}]$$

$$= [\cos\alpha - c\cos(\alpha-\beta) - c^n\cos(\alpha+n\beta) + c^{n+1}\cos\{\alpha+(n-1)\beta\}]$$
$$+ i[\sin\alpha - c\sin(\alpha-\beta) - c^n\sin(\alpha+n\beta) + c^{n+1}\sin\{\alpha+(n-1)\beta\}],$$
(putting into the form $A + iB$)

and the denominator of the R.H.S.

$$= 1 - 2c\cdot\frac{1}{2}(e^{i\beta}+e^{-i\beta}) + c^2 e^{i\beta}\cdot e^{-i\beta}$$
$$= 1 - 2c\cos\beta + c^2, \text{ which is real.}$$

Now we are required to find the sum of the sine series (S) which is the imaginary part of $C + iS$. Hence equating imaginary parts from both sides, we have the required sum

$$S = \frac{\sin\alpha - c\sin(\alpha-\beta) - c^n\sin(\alpha+n\beta) + c^{n+1}\sin\{\alpha+(n-1)\beta\}}{1-2c\cos\beta+c^2}. \quad \ldots(2)$$

To find sum upto infinity, c must be numerically less than 1, and then $c^n \to 0$ as $n \to \infty$.

$\therefore$ from (2), as $c^n \to 0$, the required sum

$$S_\infty = \frac{\sin\alpha - c\sin(\alpha-\beta) - 0\,.\sin(\alpha+n\beta) + 0.\sin\{\alpha+(n-1)\beta\}}{1-2c\cos\beta+c^2}$$
$$= \frac{\sin\alpha - c\sin(\alpha-\beta)}{1-2c\cos\beta+c^2}.$$

Example 2: *Sum the series*

$$1 + c\cosh\theta + c^2\cosh 2\theta + c^3\cosh 3\theta + \ldots \text{ to } n \text{ terms.}$$

Solution: The given series may be written as

$$1 + c\left(\frac{e^{\theta}+e^{-\theta}}{2}\right) + c^2\left(\frac{e^{2\theta}+e^{-2\theta}}{2}\right) + c^3\left(\frac{e^{3\theta}+e^{-3\theta}}{2}\right) + \ldots \text{ to } n \text{ terms}$$

(Note)

$$= \frac{1}{2}[2 + c(e^{\theta}+e^{-\theta}) + c^2(e^{2\theta}+e^{-2\theta)} + c^3(e^{3\theta}+e^{-3\theta}) + \ldots \text{ to } n \text{ terms}]$$
$$= \frac{1}{2}[(1 + ce^{\theta} + c^2e^{2\theta} + c^3e^{3\theta} + \ldots \text{ to } n \text{ terms})$$
$$+ (1 + ce^{-\theta} + c^2e^{-2\theta} + c^3e^{-3\theta} + \ldots \text{ to } n \text{ terms})]$$
$$= \frac{1}{2}\left[\left(\frac{1-c^n e^{n\theta}}{1-ce^{\theta}}\right) + \left(\frac{1-c^n e^{-n\theta}}{1-ce^{-\theta}}\right)\right],$$

summing up both the geometric series upto n terms

$$= \frac{1}{2}\left[\frac{(1-c^ne^{n\theta})(1-ce^{-\theta}) + (1-c^ne^{-n\theta})(1-ce^{\theta})}{(1-ce^{\theta})(1-ce^{-\theta})}\right]$$

$$= \frac{1}{2}\left\{\frac{1 - c^{n} e^{n\theta} - ce^{-\theta} + c^{n+1} e^{(n-1)\theta} + 1 - c^{n} e^{-n\theta} - ce^{\theta} + c^{n+1} e^{-(n-1)\theta}}{1 - ce^{\theta} - ce^{-\theta} + c^{2} e^{\theta} . e^{-\theta}}\right\}$$

$$= \frac{1}{2}\left[\frac{2 - c\,(e^{\theta} + e^{-\theta}) - c^{n}\,(e^{n\theta} + e^{-n\theta}) + c^{n+1}\,\{e^{(n-1)\theta} + e^{-(n-1)\theta}\}}{1 - c\,(e^{\theta} + e^{-\theta}) + c^{2}}\right]$$

$$= \frac{1}{2}\left[\frac{2 - c\,(2\cosh\theta) - c^{n}\,(2\cosh n\theta) + c^{n+1}\,\{2\cosh (n-1)\,\theta\}}{1 - c\,(2\cosh\theta) + c^{2}}\right]$$

$$= \frac{1 - c\cosh\theta - c^{n}\cosh n\theta + c^{n+1}\cosh (n-1)\,\theta}{1 - 2c\cosh\theta + c^{2}}.$$

Example 3: *Sum the series* $1 - 2\cos\alpha + 3\cos 2\alpha - 4\cos 3\alpha + \ldots$ *to n terms.*

Solution: Let

$$C = 1 - 2\cos\alpha + 3\cos 2\alpha - 4\cos 3\alpha + \ldots + (-1)^{n-1}\, n\cos\{(n-1)\,\alpha\},$$

and $$S = -\,2\sin\alpha + 3\sin 2\alpha - 4\sin 3\alpha + \ldots + (-1)^{n-1}\, n\sin\{(n-1)\,\alpha\}.$$

$\therefore$ $$C + iS = 1 - 2\,(\cos\alpha + i\sin\alpha) + 3\,(\cos 2\alpha + i\sin 2\alpha) - 4\,(\cos 3\alpha + i\sin 3\alpha) + \ldots\ldots + (-1)^{n-1}\, n\,[\cos\{(n-1)\,\alpha\} + i\sin\{(n-1)\,\alpha\}]$$

$$= 1 - 2e^{i\alpha} + 3e^{2i\alpha} - 4e^{3i\alpha} + \ldots + (-1)^{n-1}\, ne^{i(n-1)\alpha}$$

or $$C + iS = 1 - 2x + 3x^{2} - 4x^{3} + \ldots + (-1)^{n-1}\, nx^{n-1}, \qquad \ldots(1)$$

where $x = e^{i\alpha}$

The right hand side of (1) is an arithmetico-geometrical progression. To sum it up multiply both sides of (1) by $-x$ which is the common ratio of the geometric factors of the terms.

So multiplying both sides of (1) by $-x$, we have

$$(C + iS)\,(-x) = -\,x + 2x^{2} - 3x^{3} + 4x^{4} - \ldots + (-1)^{n}\, nx^{n}. \qquad \ldots(2)$$

Subtracting (2) from (1), we have

$$(C + iS)\,(1 + x) = (1 - x + x^{2} - x^{3} + x^{4} - \ldots \text{to } n \text{ terms}) - (-1)^{n}\, nx^{n}.$$

There are $(n + 1)$ terms in this right hand side series and the first n terms form a geometric progression of common ratio $-x$.

$\therefore$ $$(C + iS)\,(1 + x) = \frac{1 - (-x)^{n}}{1 - (-x)} - (-1)^{n}\, nx^{n}.$$

Dividing both sides by $(1 + x)$ and putting $x = e^{i\alpha}$, we have

$$(C + iS) = \frac{1 - (-1)^{n}\, e^{ni\alpha}}{(1 + e^{i\alpha})^{2}} - \frac{(-1)^{n}\, ne^{ni\alpha}}{(1 + e^{i\alpha})}$$

$$= \frac{1 + (-1)^{n-1} \ e^{ni\alpha}}{(1 + e^{i\alpha})^2} + \frac{(-1)^{n-1} \ ne^{ni\alpha}}{(1 + e^{i\alpha})}, \quad [\text{Putting } (-1)^n = (-1) \cdot (-1)^{n-1}]$$

$$= \frac{[1 + (-1)^{n-1} \ e^{ni\alpha}] + (1 + e^{i\alpha}) [(-1)^{n-1} \ ne^{ni\alpha}]}{(1 + e^{i\alpha})^2}$$

$$= \frac{1 + (-1)^{n-1} \ [(n+1) \ e^{in\alpha} + ne^{i(n+1)\alpha}]}{e^{i\alpha} \ [e^{-i\alpha/2} + e^{i\alpha/2}]^2} \qquad \textbf{(Note)}$$

$$= \frac{e^{-i\alpha} + (-1)^{n-1} \ [(n+1) \ e^{i(n-1)\alpha} + ne^{in\alpha}]}{(2 \cos \frac{1}{2} \alpha)^2},$$

dividing Nr. and Dr. by $e^{i\alpha}$

$$= \frac{(\cos \alpha - i \sin \alpha) + (-1)^{n-1} \ \{(n+1) \cos (n-1) \alpha + i (n+1) \sin (n-1) \alpha + n \cos n\alpha + in \sin n\alpha\}}{4 \cos^2 \frac{1}{2} \alpha}$$

$$= \frac{[\cos \alpha + (-1)^{n-1} \ \{(n+1) \cos (n-1) \alpha + n \cos n\alpha\}] + i [(-1)^{n-1} \ \{(n+1) \sin (n-1) \alpha + n \sin n\alpha\} - \sin \alpha]}{2 (1 + \cos \alpha)}.$$

Now equating real parts on both sides, we have

$$C = \frac{\cos \alpha + (-1)^{n-1} \ \{(n+1) \cos (n-1) \alpha + n \cos n\alpha\}}{2 (1 + \cos \alpha)}.$$

Comprehensive Exercise 1

Sum the following series. In the case of infinite series it may be assumed that $-1 < c < 1$.

1. $\cos \alpha + c \cos (\alpha + \beta) + c^2 \cos (\alpha + 2\beta) + \ldots$ to n terms.

   Deduce the sum to infinity if $|c| < 1$ *i.e.,* $-1 < c < 1$.

2. (i) $\cos \alpha + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + \ldots$ to n terms.

   (Purvanchal 2006; Bundelkhand 09; Agra 14)

   (ii) $\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + \ldots$ to n terms. **(Agra 2014)**

3. $1 + c \cos \alpha + c^2 \cos 2\alpha + c^3 \cos 3\alpha + \ldots$ to n terms. **(Purvanchal 2008)**

4. (i) $1 + c \cos \alpha + c^2 \cos 2\alpha + \ldots$ ad. inf.

   (ii) $c \sin \alpha + c^2 \sin 2\alpha + c^3 \sin 3\alpha + \ldots$ ad. inf. **(Purvanchal 2008)**

5. $\sin \alpha + \frac{1}{2} \sin 2\alpha + (1/2^2) \sin 3\alpha + \ldots$ ad. inf. **(Kumaun 2000; Kanpur 07)**

6. (i) $\cos\alpha\sin\alpha + \cos 2\alpha\sin^2\alpha + \cos 3\alpha\sin^3\alpha + \ldots$ ad. inf. **(Meerut 2012)**

(ii) $\sin\alpha\sin\alpha + \sin 2\alpha\sin^2\alpha + \sin 3\alpha\sin^3\alpha + \ldots$ ad. inf., where $\alpha \neq \pm\frac{1}{2}\pi$.

(Meerut 2012B)

7. $\cos\alpha\cos\alpha + \cos^2\alpha\cos 2\alpha + \cos^3\alpha\cos 3\alpha + \ldots$ ad. inf.

(Bundelkhand 2005, 08)

8. $1 + \frac{\cos\theta}{\cos\theta} + \frac{\cos 2\theta}{\cos^2\theta} + \frac{\cos 3\theta}{\cos^3\theta} + \ldots$ to n terms.

Or $1 + \sec\theta\cos\theta + \sec^2\theta\cos 2\theta + \sec^3\theta\cos 3\theta + \ldots$ to n terms.

9. $1 + \cos\theta\cos\theta + \cos^2\theta\cos 2\theta + \cos^3\theta\cos 3\theta + \ldots$ ad. inf.

10. (i) $\frac{\sin\theta}{\pi} - \frac{\sin 2\theta}{\pi^2} + \frac{\sin 3\theta}{\pi^3} - \ldots$ ad. inf.

(ii) $\frac{\sin\theta}{\tan\phi} - \frac{\sin 2\theta}{\tan^2\phi} + \frac{\sin 3\theta}{\tan^3\phi} - \ldots$ ad. inf., $(\tan\phi > 1)$.

11. $c\sinh\theta + c^2\sinh 2\theta + c^3\sinh 3\theta + \ldots$ ad. inf. **(Kumaun 2008)**

12. $2\sin\alpha - 3\sin 2\alpha + 4\sin 3\alpha - \ldots$ to n terms.

13. (i) $3\cos\theta + 5\cos 2\theta + 7\cos 3\theta + \ldots$ to n terms **(Meerut 2009; Kanpur 11)**

(ii) $3\sin\theta + 5\sin 2\theta + 7\sin 3\theta + \ldots$ to n terms.

14. (i) $\sin\alpha + n\sin(\alpha+\beta) + \frac{n(n-1)}{1.2}\sin(\alpha + 2\beta) + \ldots$ *to* $(n+1)$ terms.

(Kumaun 2008)

(ii) $n\sin\alpha + \frac{n(n-1)}{1.2}\sin 2\alpha + \frac{n(n-1)(n-2)}{1.2.3}\sin 3\alpha + \ldots$ to n terms.

Answers 1

1. $$C = \frac{\cos\alpha - c\cos(\alpha-\beta) - c^n\cos(\alpha + n\beta) + c^{n+1}\cos\{\alpha + (n-1)\beta\}}{1 - 2c\cos\beta + c^2}$$

Sum to infinity $= \frac{\cos\alpha - c\cos(\alpha-\beta)}{1 - 2c\cos\beta + c^2}$

2. (i) $C = \cos\{\alpha + (n-1)\beta/2\}\sin(n\beta/2)\operatorname{cosec}(\beta/2)$

(ii) $S = \sin\{\alpha + (n-1)\beta/2)\sin(n\beta/2)\operatorname{cosec}(\beta/2)$

3. $$C = \frac{1 - c\cos\alpha - c^n\cos n\alpha + c^{n+1}\cos(n-1)\alpha}{1 - 2c\cos\alpha + c^2}$$

4. (i) $\frac{1 - c\cos\alpha}{1 - 2c\cos\alpha + c^2}$ (ii) $\frac{c\sin\alpha}{1 - 2c\cos\alpha + c^2}$

5. $(4 \sin \alpha)/(5 - 4 \cos \alpha)$

6. (i) $\dfrac{\cos \alpha \sin \alpha - \sin^2 \alpha}{1 - \sin 2\alpha + \sin^2 \alpha}$ (ii) $\dfrac{\sin^2 \alpha}{1 - \sin 2\alpha + \sin^2 \alpha}$

7. 0 **8.** $\dfrac{\sec^n \theta \sin n\theta}{\tan \theta}$ **9.** 1

10. (i) $\dfrac{\pi \sin \theta}{\pi^2 + 2\pi \cos \theta + 1}$ (ii) $\dfrac{\tan \phi \,.\, \sin \theta}{\tan^2 \phi + 2 \tan \phi \cos \theta + 1}$

11. $\dfrac{c \sinh \theta}{1 - 2c \cosh \theta + c^2}$

12. $\dfrac{\sin \alpha + (-1)^{n+1} [(n+2) \sin n\alpha + (n+1) \sin (n+1) \alpha]}{2 (1 + \cos \alpha)}$

13. (i) $\dfrac{\cos \theta + (2n+3) \cos n\theta - (2n+1) \cos (n+1) \theta - 3}{2 (1 - \cos \theta)}$

(ii) $\dfrac{\sin \theta + (2n+3) \sin n\theta - (2n+1) \sin (n+1) \theta}{2 (1 - \cos \theta)}$

14. (i) $(2 \cos \frac{1}{2} \beta)^n \sin (\alpha + \frac{1}{2} n\beta)$ (ii) $\left(2 \cos \frac{1}{2} \alpha\right)^n \sin \frac{1}{2} n\alpha$

6.4 Series Based on Binomial Expansions

Remember the following formulae :

(i) When n is a positive integer and x, a are any complex numbers, we have

$$(x + a)^n = x^n + nx^{n-1}.a + \frac{n(n-1)}{2!} x^{n-2} a^2 + \ldots \text{to } (n+1) \text{ terms},$$

$$(x - a)^n = x^n - nx^{n-1}.a + \frac{n(n-1)}{2!} x^{n-2} a^2 - \ldots \text{to } (n+1) \text{ terms},$$

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1.2} x^2 + \frac{n(n-1)(n-2)}{1.2.3} x^3 + \ldots \text{ to } (n+1) \text{ terms},$$

and
$$(1 - x)^n = 1 - nx + \frac{n(n-1)}{1.2} x^2 - \frac{n(n-1)(n-2)}{1.2.3} x^3 + \ldots \text{ to } (n+1) \text{ terms}.$$

(ii) When n is any rational index and x is a complex number such that $|x| < 1$, we have

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1.2} x^2 + \frac{n(n-1)(n-2)}{1.2.3} x^3 + \ldots \infty.$$

When $|x| = 1$, this result is still true if either

$n > 0$ or if $-1 < n < 0$ and $x \neq -1$.

Also remember that with suitable restrictions on the values of x as mentioned above, we have

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{1.2} x^2 - \frac{n(n+1)(n+2)}{1.2.3} x^3 + \ldots \infty$$

$$(1-x)^{n} = 1 - nx + \frac{n(n-1)}{1.2} x^2 - \frac{n(n-1)(n-2)}{1.2.3} x^3 + \ldots \infty,$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{1.2} x^2 + \frac{n(n+1)(n+2)}{1.2.3} x^3 + \ldots \infty,$$

$$(1+x)^{1/2} = 1 + \frac{1}{2} x - \frac{1}{2.4} x^2 + \frac{1.3}{2.4.6} x^3 - \ldots \infty,$$

$$(1-x)^{-1/2} = 1 + \frac{1}{2} x + \frac{1.3}{2.4} x^2 + \frac{1.3.5}{2.4.6} x^3 + \ldots \infty,$$

$$(1+x)^{1/3} = 1 + \frac{1}{3} x - \frac{1.2}{3.6} x^2 + \frac{1.2.5}{3.6.9} x^3 - \ldots \infty,$$

and $$(1-x)^{-1/3} = 1 + \frac{1}{3} x + \frac{1.4}{3.6} x^2 + \frac{1.4.7}{3.6.9} x^3 + \ldots \infty.$$

Illustrative Examples

Example 4: *Sum the series* $1 + \frac{1}{2} \cos 2\theta - \frac{1}{2.4} \cos 4\theta + \frac{1.3}{2.4.6} \cos 6\theta - \ldots$ *ad. inf.*,

where $-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi$. **(Garhwal 2001; Rohilkhand 06; Meerut 09)**

Solution: Let $C = 1 + \frac{1}{2} \cos 2\theta - \frac{1}{2.4} \cos 4\theta + \frac{1}{2.4.6} \cos 6\theta - \ldots$ ad. inf.

The companion sine series is

$$S = \frac{1}{2} \sin 2\theta - \frac{1}{2.4} \sin 4\theta + \frac{1.3}{2.4.6} \sin 6\theta - \ldots \text{ ad. inf.}$$

$$\therefore \quad C + iS = 1 + \frac{1}{2}(\cos 2\theta + i \sin 2\theta) - \frac{1}{2.4}(\cos 4\theta + i \sin 4\theta)$$

$$+ \frac{1.3}{2.4.6}(\cos 6\theta + i \sin 6\theta) - \ldots \text{ ad. inf.}$$

$$= 1 + \frac{1}{2} e^{2i\theta} - \frac{1}{2.4} e^{4i\theta} + \frac{1.3}{2.4.6} e^{6i\theta} - \ldots \text{ ad. inf.}$$

$$= [1 + e^{2i\theta}]^{1/2}, \quad \text{summing the binomial series}$$

$$= (1 + \cos 2\theta + i \sin 2\theta)^{1/2} = (2\cos^2\theta + i \,.\, 2 \sin\theta \cos\theta)^{1/2}$$

$$= (2\cos\theta)^{1/2} (\cos\theta + i \sin\theta)^{1/2}$$

$$= (2\cos\theta)^{1/2} (\cos \tfrac{1}{2}\theta + i \sin \tfrac{1}{2}\theta).$$

Hence C, the sum of the given series

$$= \text{real part of } C + iS = (2\cos\theta)^{1/2} . \cos\frac{1}{2}\theta = [2\cos\theta . \cos^2\frac{1}{2}\theta]^{1/2}$$

$$= [\cos\theta . 2\cos^2\frac{1}{2}\theta]^{1/2} = \sqrt{\{(\cos\theta\,(1+\cos\theta)\}}.$$

Example 5: *Find the sum of the following series :*

$$n\sin\alpha + \frac{n(n+1)}{1.2}\sin 2\alpha + \frac{n(n+1)(n+2)}{1.2.3}\sin 3\alpha + \ldots \text{ ad. inf}.$$

(Bundelkhand 2010)

Solution: Let

$$S = n\sin\alpha + \frac{n(n+1)}{1.2}\sin 2\alpha + \frac{n(n+1)(n+2)}{1.2.3}\sin 3\alpha + \ldots \text{ ad. inf.}$$

The companion cosine series is

$$C = 1 + n\cos\alpha + \frac{n(n+1)}{1.2}\cos 2\alpha + \frac{n(n+1)(n+2)}{1.2.3}\cos 3\alpha + \ldots \text{ ad. inf.} \quad \textbf{(Note)}$$

$$\therefore \quad C + iS = 1 + n(\cos\alpha + i\sin\alpha) + \frac{n(n+1)}{1.2}(\cos 2\alpha + i\sin 2\alpha) + \frac{n(n+1)(n+2)}{1.2.3}(\cos 3\alpha + i\sin 3\alpha) + \ldots \text{ ad. inf.}$$

$$= 1 + ne^{i\alpha} + \frac{n(n+1)}{1.2}e^{2i\alpha} + \frac{n(n+1)(n+2)}{1.2.3}e^{3i\alpha} + \ldots \text{ ad. inf.}$$

$$= (1 - e^{i\alpha})^{-n}, \text{ by binomial theorem}$$

$$= [1 - (\cos\alpha + i\sin\alpha)]^{-n} = [(1-\cos\alpha) - i\sin\alpha]^{-n}$$

$$= [2\sin^2\frac{1}{2}\alpha - i . 2\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha]^{-n}$$

$$= (2\sin\frac{1}{2}\alpha)^{-n}[\sin\frac{1}{2}\alpha - i\cos\frac{1}{2}\alpha]^{-n}$$

$$= (2\sin\frac{1}{2}\alpha)^{-n}[\cos(\frac{1}{2}\pi - \frac{1}{2}\alpha) - i\sin(\frac{1}{2}\pi - \frac{1}{2}\alpha)]^{-n}$$

$$= (2\sin\frac{1}{2}\alpha)^{-n}[\cos\{-n(\frac{1}{2}\pi - \frac{1}{2}\alpha)\} - i\sin\{-n(\frac{1}{2}\pi - \frac{1}{2}\alpha)\}],$$

by De Moivre's theorem

$$= (2\sin\frac{1}{2}\alpha)^{-n}[\cos\{n(\frac{1}{2}\pi - \frac{1}{2}\alpha)\} + i\sin\{n(\frac{1}{2}\pi - \frac{1}{2}\alpha)\}].$$

Hence equating imaginary parts on both sides, we have the required sum

$$S = (2\sin\frac{1}{2}\alpha)^{-n}\sin\{n(\frac{1}{2}\pi - \frac{1}{2}\alpha)\}.$$

Comprehensive Exercise 2

Sum the following series:

1. $\cos^n \alpha - n \cos^{n-1} \alpha \cos \alpha + \dfrac{n(n-1)}{1.2} \cos^{n-2} \alpha \cos 2\alpha - \ldots$ to $(n+1)$ terms .
(Meerut 2006B; Agra 07)

2. $\sin \alpha + \dfrac{1}{2} \sin 3\alpha + \dfrac{1.3}{2.4} \sin 5\alpha + \ldots$ to ∞.

3. $\dfrac{1}{2} \sin \theta + \dfrac{1.3}{2.4} \sin 2\theta + \dfrac{1.3.5}{2.4.6} \sin 3\theta + \ldots$ ad. inf. **(Rohilkhand 2006)**

4. (i) $1 - \dfrac{1}{2} \cos \theta + \dfrac{1.3}{2.4} \cos 2\theta - \dfrac{1.3.5}{2.4.6} \cos 3\theta + \ldots$ ad. inf. $(-\pi < \theta < \pi)$.
(Meerut 2005; Purvanchal 09)

   (ii) $1 + \dfrac{1}{2} \cos \alpha + \dfrac{1.3}{2.4} \cos 2\alpha + \dfrac{1.3.5}{2.4.6} \cos 3\alpha + \ldots$ ad. inf.
(Rohilkhand 2006)

5. $1 + \dfrac{1}{3} y \cos \alpha + \dfrac{1.4}{3.6} y^2 \cos 2\alpha + \dfrac{1.4.7}{3.6.9} y^3 \cos 3\alpha + \ldots$ to ∞, if $y < 1$.

6. $1 + n \cos \alpha + \dfrac{n(n+1)}{1.2} \cos 2\alpha + \dfrac{n(n+1)(n+2)}{1.2.3} \cos 3\alpha + \ldots$ ad. inf.

7. $\sinh \alpha + n \sinh 2\alpha + \dfrac{n(n-1)}{2!} \sinh 3\alpha + \ldots$ to $(n+1)$ terms,

   where n is a positive integer.

8. $\dfrac{1}{c^n} \left\{ 1 + n \dfrac{a}{c} \cos B + \dfrac{n(n+1)}{1.2} \cdot \dfrac{a^2}{c^2} \cos 2B + \ldots \text{ad. inf.} \right\} = \dfrac{\cos nA}{b^n}$,

   where a, b, c are the sides of the triangle ABC and $a < c$.
(Meerut 2004; Kanpur 10)

   Also show that

$$\frac{1}{c^n} \left[\frac{na}{c} \sin B + \frac{n(n+1)}{1.2} \cdot \frac{a^2}{c^2} \sin 2B + \ldots \right] = \frac{\sin nA}{b^n}.$$
(Kanpur 2010)

Answers 2

1. $(-1)^{n/2} \sin^n \alpha$ when n is even; 0, when n is odd

2. $(2 \sin \alpha)^{-1/2} \sin \left(\frac{1}{4} \pi + \frac{1}{2} \alpha\right)$ **3.** $\left(2 \sin \frac{1}{2} \theta\right)^{-1/2} \sin \left(\frac{1}{4} \pi - \frac{1}{4} \theta\right)$

4. (i) $\left(\cos \frac{1}{4} \theta\right) / \left(2 \cos \frac{1}{2} \theta\right)^{1/2}$ (ii) $\dfrac{\cos \frac{1}{4} \alpha}{\left(-2 \cos \frac{1}{2} \alpha\right)^{1/2}}$

5. $r^{-1/3} \cos \frac{1}{3} \phi$,

where $r = (1 - 2y \cos \alpha + y^2)^{1/2}$ and $\phi = \tan^{-1} \left\{ \frac{y \sin \alpha}{(1 - y \cos \alpha)} \right\}$

6. $2^n \cosh^n (\alpha/2) . \sinh \{\frac{1}{2} (n+2) \alpha\}$

6.5 Series Based on Exponential Series

We know that if x is any complex number, then

(i) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ ad. inf.

(ii) $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$ ad. inf.

(iii) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ ad. inf.

(iv) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$ ad. inf.

(v) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$ ad. inf.

(vi) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$ ad. inf.

Illustrative Examples

Example 6: *Sum the series*

(*i*) $\cos \alpha + c \cos (\alpha + \beta) + \frac{1}{2!} c^2 \cos (\alpha + 2\beta) + \dots$ *ad. inf.* **(Meerut 2012)**

(*ii*) $\sin \alpha + c \sin (\alpha + \beta) + \frac{1}{2!} c^2 \sin (\alpha + 2\beta) + \dots$ *ad. inf.* **(Meerut 2012B)**

Solution: Let

$$C = \cos \alpha + c \cos (\alpha + \beta) + \frac{1}{2!} c^2 \cos (\alpha + 2\beta) + \dots \text{ad. inf.}$$

and
$$S = \sin \alpha + c \sin (\alpha + \beta) + \frac{1}{2!} c^2 \sin (\alpha + 2\beta) + \dots \text{ad. inf.}$$

Multiplying the second series by i, and adding to the first, we get

$$C + iS = (\cos \alpha + i \sin \alpha) + c \{\cos (\alpha + \beta) + i \sin (\alpha + \beta)\}$$
$$+ (1/2!) c^2 \{\cos (\alpha + 2\beta) + i \sin (\alpha + 2\beta)\} + \dots \text{ad. inf.}$$
$$= e^{i\alpha} + ce^{i(\alpha + \beta)} + (1/2!) c^2 e^{i(\alpha + 2\beta)} + \dots \text{ad. inf.}$$

$$= e^{i\alpha}\ [1 + ce^{i\beta} + (1/2!)\, c^2 e^{2i\beta} + \ldots \text{ad. inf.}]$$

$$= e^{i\alpha} \left[e^{ce^{i\beta}} \right], \qquad \text{summing up the exponential series}$$

$$= e^{i\alpha}\ [e^{c(\cos\beta + i\sin\beta)}] = e^{i\alpha} \,.\, e^{c\cos\beta} \,.\, e^{ic\sin\beta}$$

$$= e^{c\cos\beta} \,.\, e^{i(\alpha + c\sin\beta)}$$

$$= e^{c\cos\beta}\ [\cos(\alpha + c\sin\beta) + i\sin(\alpha + c\sin\beta)].$$

Equating real and imaginary parts on both sides, we have

$$C = \text{the sum of the series (i)} = e^{c\cos\beta} \,.\cos(\alpha + c\sin\beta)$$

and $$S = \text{the sum of the series (ii)} = e^{c\cos\beta} \,.\sin(\alpha + c\sin\beta).$$

Example 7: *Sum the series*

$$1 + e^{\cos\alpha} \cos(\sin\alpha) + \frac{1}{2!} e^{2\cos\alpha} \cos(2\sin\alpha) + \ldots \textit{ad. inf}.$$

Solution: If $C = 1 + e^{\cos\alpha} \cos(\sin\alpha) + \frac{1}{2!} e^{2\cos\alpha} \cos(2\sin\alpha) + \ldots \text{ad. inf.}$

then $$S = e^{\cos\alpha} \sin(\sin\alpha) + \frac{1}{2!} e^{2\cos\alpha} \sin(2\sin\alpha) + \ldots \text{ad. inf.}$$

$\therefore$ $$C + iS = 1 + e^{\cos\alpha} [\cos(\sin\alpha) + i\sin(\sin\alpha)] + \frac{1}{2!} e^{2\cos\alpha} [\cos(2\sin\alpha) + i\sin(2\sin\alpha)] + \ldots \text{ad. inf.}$$

$$= 1 + \frac{1}{1!} e^{\cos\alpha} \,.\, e^{i\sin\alpha} + \frac{1}{2!} e^{2\cos\alpha} \,.\, e^{2i\sin\alpha} + \ldots \text{ad. inf.}$$

$$= 1 + \frac{1}{1!} \cdot y\, e^{i\sin\alpha} + \frac{1}{2!} y^2\, e^{2i\sin\alpha} + \ldots \infty, \text{ where } y = e^{\cos\alpha}$$

$$= e^{y\,.\,e^{i\sin\alpha}} = e^{y\{\cos(\sin\alpha) + i\sin(\sin\alpha)\}}$$

$$= e^{y\cos(\sin\alpha)} \,.\, e^{iy\sin(\sin\alpha)}$$

$$= e^{y\cos(\sin\alpha)} .\, [\cos\{y\sin(\sin\alpha)\} + i\sin\{y\sin(\sin\alpha)\}].$$

Equating real parts on both sides, we have

$$C = e^{y\cos(\sin\alpha)} .\cos\{y\sin(\sin\alpha)\}, \text{ where } y = e^{\cos\alpha}.$$

Example 8: *Sum the series*

(i) $\cos\alpha - \frac{\cos(\alpha + 2\beta)}{3!} + \frac{\cos(\alpha + 4\beta)}{5!} - \ldots \textit{ad. inf}.$ **(Meerut 2010B)**

(ii) $\sin\alpha - \frac{\sin(\alpha + 2\beta)}{3!} + \frac{\sin(\alpha + 4\beta)}{5!} - \ldots \textit{ad. inf}.$

Solution: **(i)** Let

$$C = \cos\alpha - \frac{\cos(\alpha + 2\beta)}{3!} + \frac{\cos(\alpha + 4\beta)}{5!} - \ldots \text{ad. inf.}$$

so that $$S = \sin\alpha - \frac{\sin(\alpha + 2\beta)}{3!} + \frac{\sin(\alpha + 4\beta)}{5!} - \ldots \text{ad. inf.}$$

$\therefore \quad C + iS = (\cos\alpha + i\sin\alpha) - (1/3!)\{\cos(\alpha + 2\beta) + i\sin(\alpha + 2\beta)\}$
$+ (1/5!)\{\cos(\alpha + 4\beta) + i\sin(\alpha + 4\beta)\} - \dots$ ad. inf.

$= e^{i\alpha} - (1/3!)\, e^{i(\alpha + 2\beta)} + (1/5!)\, e^{i(\alpha + 4\beta)} - \dots$ ad. inf.

$= e^{i\alpha}\,[1 - (1/3!)\, e^{2i\beta} + (1/5!)\, e^{4i\beta} + \dots$ ad. inf.$]$

$= e^{i\alpha} . e^{-i\beta}\,[e^{i\beta} - (1/3!)\, e^{3i\beta} + (1/5!)\, e^{5i\beta} - \dots$ ad. inf.$]$ **(Note)**

$= e^{i(\alpha - \beta)} . \sin(e^{i\beta}) = e^{i(\alpha - \beta)} . \sin(\cos\beta + i\sin\beta)$

$= [\cos(\alpha - \beta) + i\sin(\alpha - \beta)]\,[\sin(\cos\beta) . \cos(i\sin\beta)$
$+ \cos(\cos\beta)\sin(i\sin\beta)]$

$= [\cos(\alpha - \beta) + i\sin(\alpha - \beta)]\,[\sin(\cos\beta)\cosh(\sin\beta)$
$+ i\cos(\cos\beta)\sinh(\sin\beta)]$.

Hence the sum of the given series
= the real part of $C + iS$
$= \cos(\alpha - \beta)\sin(\cos\beta)\cosh(\sin\beta)$
$- \sin(\alpha - \beta)\cos(\cos\beta)\sinh(\sin\beta)$.

(ii) Do yourself.

Comprehensive Exercise 3

Sum the following series :

1. (i) $\dfrac{\sin\theta}{1!} - \dfrac{\sin 2\theta}{2!} + \dfrac{\sin 3\theta}{3!} - \dots$ ad. inf.

(ii) $\dfrac{\sin\theta}{1!} + \dfrac{\sin 2\theta}{2!} + \dfrac{\sin 3\theta}{3!} + \dots$ ad. inf.

2. (i) $\cos\theta + \dfrac{\cos\phi\cos(\theta + \phi)}{1!} + \dfrac{\cos^2\phi\cos(\theta + 2\phi)}{2!} + \dots$ to ∞.

(ii) $\cos\theta + \dfrac{\cos\theta}{1!}\cos 2\theta + \dfrac{\cos^2\theta}{2!}\cos 3\theta + \dots$ to ∞.

3. $\cos\theta + \dfrac{\sin\theta}{1!}\cos 2\theta + \dfrac{\sin^2\theta}{2!}\cos 3\theta + \dots$ ad. inf.

(Rohilkhand 2007; Kashi 14; Rohilkhand 14)

4. (i) $1 + \cos\theta\cos\theta + \dfrac{\cos 2\theta\cos^2\theta}{2!} + \dfrac{\cos 3\theta\cos^3\theta}{3!} + \dots$ to ∞,

(ii) $\sin\theta\cos\theta + \dfrac{\sin 2\theta\cos^2\theta}{2!} + \dfrac{\sin 3\theta\cos^3\theta}{3!} + \dots$ to ∞.

(Rohilkhand 2005; Kanpur 05)

5. (i) $1 + \dfrac{\cos\alpha}{\cos\alpha} + \dfrac{\cos 2\alpha}{2!\cos^2\alpha} + \dfrac{\cos 3\alpha}{3!\cos^3\alpha} + \dots$ ad. inf. **(Rohilkhand 2006)**

(ii) $\dfrac{\cos\alpha}{\cos\alpha} + \dfrac{\cos 2\alpha}{2!\cos^2\alpha} + \dfrac{\cos 3\alpha}{3!\cos^3\alpha} + \dots$ ad. inf.

6. $1 - \cos\alpha \cos\beta + \frac{\cos^2\alpha}{2!}\cos 2\beta - \frac{\cos^3\alpha}{3!}\cos 3\beta + \ldots$ ad. inf.

(Bundelkhand 2007; Kanpur 09; Purvanchal 10)

7. $1 + e^{\sin\alpha}\cos(\cos\alpha) + \frac{e^{2\sin\alpha}}{2!}\cos(2\cos\alpha) + \frac{e^{3\sin\alpha}}{3!}\cos(3\cos\alpha) + \ldots$ ad. inf.

8. $\sin\alpha - \frac{\sin(\alpha + 2\beta)}{2!} + \frac{\sin(\alpha + 4\beta)}{4!} - \ldots$ ad. inf.

(Agra 2005, 06)

9. $\sin\theta - \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{4!} - \ldots$ ad. inf.

(Kanpur 2014)

10. (i) $1 + \frac{c^2\cos 2\theta}{2!} + \frac{c^4\cos 4\theta}{4!} + \ldots$ ad. inf. **(Avadh 2006; Gorakhpur 07)**

(ii) $1 + \frac{c^2\sin 2\theta}{2!} + \frac{c^4\sin 4\theta}{4!} + \ldots\infty$.

11. $\sin\theta\cos\theta + \frac{\sin^3\theta\cos 3\theta}{3!} + \frac{\sin^5\theta\cos 5\theta}{5!} + \ldots$ ad. inf.

12. $\frac{5\cos\theta}{1!} + \frac{7\cos 3\theta}{3!} + \frac{9\cos 5\theta}{5!} + \ldots$ ad. inf.

(Kanpur 2010)

13. $1 + \cosh\alpha + \frac{\cosh 2\alpha}{2!} + \frac{\cosh 3\alpha}{3!} + \ldots$ ad. inf.

14. $\sinh\alpha + \frac{\sinh 2\alpha}{2!} + \frac{\sinh 3\alpha}{3!} + \ldots$ ad. inf.

15. $\cosh\theta + \frac{\sin\theta}{1!}\cosh 2\theta + \frac{\sin^2\theta}{2!}\cosh 3\theta + \ldots$ ad. inf.

Answers 3

1. (i) $e^{-\cos\theta}\sin(\sin\theta)$ (ii) $e^{\cos\theta}.\sin(\sin\theta)$
2. (i) $e^{\cos^2\phi}.\cos(\theta + \cos\phi\sin\phi)$ (ii) $e^{\cos^2\theta}.\cos(\theta + \cos\theta\sin\theta)$
3. $e^{\sin\theta\cos\theta}.\cos(\theta + \sin^2\theta)$
4. (i) $e^{\cos^2\theta}.\cos(\sin\theta\cos\theta)$ (ii) $e^{\cos^2\theta}.\sin(\sin\theta\cos\theta)$
5. (i) $e.\cos(\tan\alpha)$ (ii) $[e\cos(\tan\alpha)] - 1$
6. $e^{-\cos\alpha\cos\beta}.\cos(\cos\alpha\sin\beta)$
7. $e^{y\cos(\cos\alpha)}.\cos\{y\sin(\cos\alpha)\}$, where $y = e^{\sin\alpha}$
8. $\sin\alpha\cos(\cos\beta)\cosh(\sin\beta) - \cos\alpha\sin(\cos\beta)\sinh(\sin\beta)$
9. $S = \sin\theta\cos(\cos\frac{1}{2}\theta)\cosh(\sin\frac{1}{2}\theta) - \cos\theta\sin(\cos\frac{1}{2}\theta)\sinh(\sin\frac{1}{2}\theta)$

10. (i) $\cosh (c \cos \theta) \cos (c \sin \theta)$ (ii) $1 + \sinh (c \cos \theta) \sin (c \sin \theta)$.

11. $\sinh (\sin \theta \cos \theta) \cos (\sin^2 \theta)$.

12. $4 \sinh (\cos \theta) \cos (\sin \theta) + \cos \theta \cosh (\cos \theta) \cos (\sin \theta)$

$- \sin \theta \sinh (\cos \theta) \sin (\sin \theta)$.

13. $e^{\cosh \alpha} \cosh (\sinh \alpha)$. 14. $e^{\cosh \alpha} \sinh (\sinh \alpha)$.

14. $e^{\sin \theta \cosh \theta} . \cosh (\theta + \sin \theta \sinh \theta)$.

6.6 Series Based on Logarithmic Series and its Sub-Case Gregory's Series

Remember the following results :

If z is any complex number, then

(i) $\log (1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots$ ad. inf. provided $|z| \leq 1$ and $z \neq -1$.

(ii) $\log (1 - z) = -\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \ldots \text{ ad. inf.}\right)$ provided $|z| \leq 1$ and $z \neq 1$.

(iii) $\log (1 + z) - \log (1 - z) = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \ldots \text{ ad. inf.}\right)$,

provided $|z| \leq 1$ and $z \neq \pm 1$.

This result may also be put in the form

$$\frac{1}{2} \log \frac{1 + z}{1 - z} = z + \frac{z^3}{3} + \frac{z^5}{5} + \ldots \infty.$$

(Kumaun 2008)

(iv) $\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \ldots$ ad. inf., provided $|z| \leq 1$.

This is Gregory's series.

Also remember that if α and β are any real numbers, then

(v) $\log (\alpha + i\beta) = \frac{1}{2} \log (\alpha^2 + \beta^2) + i \tan^{-1} (\beta / \alpha)$

(vi) $\log (\alpha - i\beta) = \frac{1}{2} \log (\alpha^2 + \beta^2) - i \tan^{-1} (\beta / \alpha)$

(vii) $\log \{(\alpha + i\beta) / (\alpha - i\beta)\} = 2i \tan^{-1} (\beta / \alpha)$.

Illustrative Examples

Example 9: *Sum the series when c is positive and* < 1.

(*i*) $c \cos \alpha - \frac{1}{2} c^2 \cos 2\alpha + \frac{1}{3} c^3 \cos 3\alpha - \ldots$ *ad. inf*. **(Garhwal 2002)**

(*ii*) $c \sin \alpha - \frac{1}{2} c^2 \sin 2\alpha + \frac{1}{3} c^3 \sin 3\alpha - \ldots$ *ad. inf*.

(Garhwal 2002; Purvanchal 09)

Solution: Let $C = c\cos\alpha - \frac{1}{2}c^2\cos 2\alpha + \frac{1}{3}c^3\cos 3\alpha - \dots$ ad. inf.

and $$S = c\sin\alpha - \frac{1}{2}c^2\sin 2\alpha + \frac{1}{3}c^3\sin 3\alpha - \dots \text{ ad. inf.}$$

$\therefore$ $$C + iS = c(\cos\alpha + i\sin\alpha) - \frac{1}{2}c^2(\cos 2\alpha + i\sin 2\alpha) + \frac{1}{3}c^3(\cos 3\alpha + i\sin 3\alpha) - \dots \text{ ad. inf.}$$

$$= c\,e^{i\alpha} - \frac{1}{2}c^2 e^{2i\alpha} + \frac{1}{3}c^3 e^{3i\alpha} - \dots \text{ ad. inf.}$$

$$= \log(1 + c\,e^{i\alpha}), \quad \left[\because \log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots \infty, \text{ if } |x| < 1.\right.$$

Here $|c\,e^{i\alpha}| = |c|$ which is given to be < 1.]

$$= \log[1 + c(\cos\alpha + i\sin\alpha)] = \log[(1 + c\cos\alpha) + ic\sin\alpha]$$

$$= \frac{1}{2}\log[(1 + c\cos\alpha)^2 + c^2\sin^2\alpha] + i\tan^{-1}\left(\frac{c\sin\alpha}{1 + c\cos\alpha}\right),$$

$$\left[\because \log(\alpha + i\beta) = \frac{1}{2}\log(\alpha^2 + \beta^2) + i\tan^{-1}(\beta/\alpha)\right]$$

$$= \frac{1}{2}\log[1 + 2c\cos\alpha + c^2\cos^2\alpha + c^2\sin^2\alpha] + i\tan^{-1}\left(\frac{c\sin\alpha}{1 + c\cos\alpha}\right)$$

$$= \frac{1}{2}\log(1 + 2c\cos\alpha + c^2) + i\tan^{-1}\left(\frac{c\sin\alpha}{1 + c\cos\alpha}\right).$$

Equating real and imaginary parts, we have

$$C = \frac{1}{2}\log(1 + 2c\cos\alpha + c^2), \text{ which is the sum of the series (i)}$$

and $$S = \tan^{-1}\{(c\sin\alpha)/(1 + c\cos\alpha)\}, \text{ which is the sum of the series (ii).}$$

Example 10: *Sum the series*

$$\sin\alpha\sin\beta + \frac{1}{2}\sin 2\alpha\sin 2\beta + \frac{1}{3}\sin 3\alpha\sin 3\beta + \dots \textit{ ad. inf.}$$

Solution: We know that

$$\sin\alpha\sin\beta = \frac{1}{2}(2\sin\alpha\sin\beta) = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)],$$

$$\sin 2\alpha\sin 2\beta = \frac{1}{2}[\cos 2(\alpha - \beta) - \cos 2(\alpha + \beta)], \text{ and so on.}$$

$\therefore$ the given series may be written as

$$\frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)] + \frac{1}{2}\cdot\frac{1}{2}[\cos 2(\alpha - \beta) - \cos 2(\alpha + \beta)] + \dots \text{ ad. inf.}$$

$$= \frac{1}{2}\left[\left\{\cos(\alpha - \beta) + \frac{1}{2}\cos 2(\alpha - \beta) + \frac{1}{3}\cos 3(\alpha - \beta) + \dots \text{ ad. inf.}\right\}\right.$$

$$-\{\cos(\alpha+\beta)+\frac{1}{2}\cos 2(\alpha+\beta)+\frac{1}{3}\cos 3(\alpha+\beta)+\ldots \text{ ad. inf.}\}].$$

Thus the given series is equal to $\frac{1}{2}$ of the difference of the two series

$$\cos(\alpha-\beta)+\frac{1}{2}\cos 2(\alpha-\beta)+\frac{1}{3}\cos 3(\alpha-\beta)+\ldots \text{ ad. inf.} \qquad \ldots(1)$$

and $$\cos(\alpha+\beta)+\frac{1}{2}\cos 2(\alpha+\beta)+\frac{1}{3}\cos 3(\alpha+\beta)+\ldots \text{ ad. inf.} \qquad \ldots(2)$$

Now to find the sum of the series (1), put

$$C=\cos(\alpha-\beta)+\frac{1}{2}\cos 2(\alpha-\beta)+\frac{1}{3}\cos 3(\alpha-\beta)+\ldots \text{ ad. inf.}$$

so that $$S=\sin(\alpha-\beta)+\frac{1}{2}\sin 2(\alpha-\beta)+\frac{1}{3}\sin 3(\alpha-\beta)+\ldots \text{ ad. inf.}$$

$\therefore$ $$C+iS=\{\cos(\alpha-\beta)+i\sin(\alpha-\beta)\}+\frac{1}{2}\{\cos 2(\alpha-\beta)+i\sin 2(\alpha-\beta)\}$$
$$+\frac{1}{3}\{\cos 3(\alpha-\beta)+i\sin 3(\alpha-\beta)\}+\ldots \text{ ad. inf.}$$
$$=e^{i(\alpha-\beta)}+\frac{1}{2}e^{i2(\alpha-\beta)}+\frac{1}{3}e^{i3(\alpha-\beta)}+\ldots \text{ ad. inf.}$$
$$=-\log[1-e^{i(\alpha-\beta)}]=-\log[1-\cos(\alpha-\beta)-i\sin(\alpha-\beta)]$$
$$=-\frac{1}{2}\log[\{1-\cos(\alpha-\beta)\}^2+\sin^2(\alpha-\beta)]$$
$$+i\tan^{-1}\left\{\frac{\sin(\alpha-\beta)}{1-\cos(\alpha-\beta)}\right\}$$
$$=-\frac{1}{2}\log[2-2\cos(\alpha-\beta)]+i\tan^{-1}\left\{\frac{\sin(\alpha-\beta)}{1-\cos(\alpha-\beta)}\right\}$$
$$=-\frac{1}{2}\log\left[2\,.\,2\sin^2\frac{1}{2}(\alpha-\beta)\right]+i\tan^{-1}\left\{\frac{\sin(\alpha-\beta)}{1-\cos(\alpha-\beta)}\right\}$$
$$=\log\left\{4\sin^2\frac{1}{2}(\alpha-\beta)\right\}^{-1/2}+i\tan^{-1}\left\{\frac{\sin(\alpha-\beta)}{1-\cos(\alpha-\beta)}\right\}$$
$$=\log\left[\frac{1}{2}\operatorname{cosec}\frac{1}{2}(\alpha-\beta)\right]+i\tan^{-1}\left\{\frac{\sin(\alpha-\beta)}{1-\cos(\alpha-\beta)}\right\}.$$

Equating real parts on both sides, we get

$$C=\log[\frac{1}{2}\operatorname{cosec}\frac{1}{2}(\alpha-\beta)].$$

$\therefore$ sum of the series (1) $=\log[\frac{1}{2}\operatorname{cosec}\frac{1}{2}(\alpha-\beta)]$.

Now the series (2) differs from the series (1) in having $-\beta$ in place of β.

$\therefore$ sum of the series (2) can be written simply by replacing β by $-\beta$ in the sum of the series (1).

Hence the sum of the series (2) = $\log [\frac{1}{2} \text{cosec} \frac{1}{2} (\alpha + \beta)]$.

$\therefore$ the required sum = $\frac{1}{2}$ [sum of the series (1) – sum of the series (2)]

$$= \frac{1}{2} [\log \{\frac{1}{2} \text{cosec} \frac{1}{2} (\alpha - \beta)\} - \log \{\frac{1}{2} \text{cosec} \frac{1}{2} (\alpha + \beta)\}]$$

$$= \frac{1}{2} \log \left[\frac{\text{cosec} \frac{1}{2} (\alpha - \beta)}{\text{cosec} \frac{1}{2} (\alpha + \beta)}\right] = \frac{1}{2} \log \left[\frac{\sin \frac{1}{2} (\alpha + \beta)}{\sin \frac{1}{2} (\alpha - \beta)}\right].$$

Example 11: *Sum the series*

(*i*) $c \cos \alpha + \frac{1}{3} c^3 \cos 3\alpha + \frac{1}{5} c^5 \cos 5\alpha + \ldots$ *ad. inf*.

(*ii*) $c \sin \alpha + \frac{1}{3} c^3 \sin 3\alpha + \frac{1}{5} c^5 \sin 5\alpha + \ldots$ *ad. inf*.

Solution: Let C and S denote the sums of the two given series respectively. Then multiplying the second series by i, and adding it to the first, we get

$$C + iS = c (\cos \alpha + i \sin \alpha) + \frac{1}{3} c^3 (\cos 3\alpha + i \sin 3\alpha)$$
$$+ \frac{1}{5} c^5 (\cos 5\alpha + i \sin 5\alpha) + \ldots \text{ ad. inf.}$$

$$= ce^{i\alpha} + \frac{1}{3} c^3 e^{3i\alpha} + \frac{1}{5} c^5 e^{5i\alpha} + \ldots \text{ ad. inf.}$$

$$= \frac{1}{2} \log \left\{\frac{1 + c\, e^{i\alpha}}{1 - c\, e^{i\alpha}}\right\} \quad \left[\because \frac{1}{2} \log \frac{1 + x}{1 - x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots \infty\right]$$

$$= \frac{1}{2} \log (1 + c\, e^{i\alpha}) - \frac{1}{2} \log (1 - c\, e^{i\alpha})$$

$$= \frac{1}{2} \log \{(1 + c \cos \alpha) + i c \sin \alpha\} - \frac{1}{2} \log \{(1 - c \cos \alpha) - ic \sin \alpha\}$$

$$= \frac{1}{2} \left[\frac{1}{2} \log \{(1 + c \cos \alpha)^2 + c^2 \sin^2 \alpha\} + i \tan^{-1} \left\{\frac{c \sin \alpha}{1 + c \cos \alpha}\right\}\right]$$
$$- \frac{1}{2} \left[\frac{1}{2} \log \{(1 - c \cos \alpha)^2 + c^2 \sin^2 \alpha\} - i \tan^{-1} \left\{\frac{c \sin \alpha}{1 - c \cos \alpha}\right\}\right]$$

$$= \frac{1}{4} [\log (1 + 2c \cos \alpha + c^2) - \log (1 - 2c \cos \alpha + c^2)]$$
$$+ i \cdot \frac{1}{2} \left[\tan^{-1} \left\{\frac{c \sin \alpha}{1 + c \cos \alpha}\right\} + \tan^{-1} \left\{\frac{c \sin \alpha}{1 - c \cos \alpha}\right\}\right].$$

Equating real and imaginary parts on both sides, we get

$$C = \frac{1}{4} [\log (1 + 2c \cos \alpha + c^2) - \log (1 - 2c \cos \alpha + c^2)]$$

$$= \frac{1}{4} \log \{(1 + 2c \cos \alpha + c^2) / (1 - 2c \cos \alpha + c^2)\},$$

which is the sum of the series (i)

and $$S = \frac{1}{2} \left[\tan^{-1} \left\{ \frac{c \sin \alpha}{1 + c \cos \alpha} \right\} + \tan^{-1} \left\{ \frac{c \sin \alpha}{1 - c \cos \alpha} \right\} \right]$$

$$= \frac{1}{2} \tan^{-1} \left[\frac{\{(c \sin \alpha)/(1 + c \cos \alpha)\} + \{(c \sin \alpha)/(1 - c \cos \alpha)\}}{1 - \{(c \sin \alpha)/(1 + c \cos \alpha)\}.\{(c \sin \alpha)/(1 - c \cos \alpha)\}} \right]$$

$$= \frac{1}{2} \tan^{-1} \left[\frac{(c \sin \alpha) \{(1 - c \cos \alpha) + (1 + c \cos \alpha)\}}{(1 - c^2 \cos^2 \alpha) - c^2 \sin^2 \alpha} \right]$$

$$= \frac{1}{2} \tan^{-1} \left\{ \frac{2c \sin \alpha}{1 - c^2} \right\},$$ which is sum of the series (ii).

Comprehensive Exercise 4

Sum the following series. It may be assumed that $-1 < c < 1$:

1. (i) $\cos \theta - \frac{1}{2} \cos 2\theta + \frac{1}{3} \cos 3\theta - \dots$ ad. inf.

(ii) $\sin \theta - \frac{1}{2} \sin 2\theta + \frac{1}{3} \sin 3\theta - \dots$ ad. inf.

2. (i) $\cos \alpha \cos \alpha - \frac{1}{2} \cos^2 \alpha \cos 2\alpha + \frac{1}{3} \cos^3 \alpha \cos 3\alpha - \dots$ ad. inf.

(ii) $\cos \alpha \sin \alpha - \frac{1}{2} \cos^2 \alpha \sin 2\alpha + \frac{1}{3} \cos^3 \alpha \sin 3\alpha - \dots$ ad. inf.

(iii) $\sin \alpha \sin \alpha - \frac{1}{2} \sin^2 \alpha \sin 2\alpha + \frac{1}{3} \sin^3 \alpha \sin 3\alpha - \dots$ ad. inf.

3. (i) $c \cos \alpha + \frac{1}{2} c^2 \cos 2\alpha + \frac{1}{3} c^3 \cos 3\alpha + \dots$ ad. inf. **(Gorakhpur 2005)**

(ii) $c \sin \alpha + \frac{1}{2} c^2 \sin 2\alpha + \frac{1}{3} c^3 \sin 3\alpha + \dots$ ad. inf. **(Avadh 2008)**

(iii) $\frac{\cos \alpha}{2} + \frac{1}{2} \frac{\cos 2\alpha}{2^2} + \frac{1}{3} \frac{\cos 3\alpha}{2^3} + \dots$ ad. inf.

(iv) $\cos \alpha \sin \alpha + \frac{1}{2} \cos^2 \alpha \sin 2\alpha + \frac{1}{3} \cos^3 \alpha \sin 3\alpha + \dots$ ad. inf.

4. $\cos \frac{\pi}{3} + \frac{1}{2} \cos \frac{2\pi}{3} + \frac{1}{3} \cos \frac{3\pi}{3} + \dots$ ad. inf. **(Kanpur 2008, 09)**

5. (i) $c \cos \alpha + \frac{1}{2} c^2 \cos (\alpha + \beta) + \frac{1}{3} c^3 \cos (\alpha + 2\beta) + \dots$ ad. inf.

(ii) $c \sin \alpha + \frac{1}{2} c^2 \sin (\alpha + \beta) + \frac{1}{3} c^3 \sin (\alpha + 2\beta) + \dots$ ad. inf., **(Gorakhpur 2005; Bundelkhand 11)**

6. (i) $c\cos\alpha - \frac{1}{2}c^2\cos(\alpha+\beta) + \frac{1}{3}c^3\cos(\alpha+2\beta) - \ldots$ ad. inf.

(ii) $c\sin\alpha - \frac{1}{2}c^2\sin(\alpha+\beta) + \frac{1}{3}c^3\sin(\alpha+2\beta) - \ldots$ ad. inf.

7. $c\sin^2\theta - \frac{1}{2}c^2\sin^2 2\theta + \frac{1}{3}c^3\sin^2 3\theta - \ldots$ ad. inf.

8. $\cos\frac{1}{3}\pi + \frac{1}{3}\cos\frac{2}{3}\pi + \frac{1}{5}\cos\frac{3}{3}\pi + \frac{1}{7}\cos\frac{4}{3}\pi + \ldots$ad. inf.

9. (i) $c\cos\theta - \frac{1}{3}c^3\cos 3\theta + \frac{1}{5}c^5\cos 5\theta - \ldots$ ad. inf.

(ii) $c\sin\theta - \frac{1}{3}c^3\sin 3\theta + \frac{1}{5}c^5\sin 5\theta - \ldots$ ad. inf.

10. $\cos^2\theta - \frac{1}{3}\cos^3\theta\cos 3\theta + \frac{1}{5}\cos^5\theta\cos 5\theta - \ldots$ ad. inf.

11. $e^{\alpha}\cos\beta - \frac{1}{3}e^{3\alpha}\cos 3\beta + \frac{1}{5}e^{5\alpha}\cos 5\beta - \ldots$ ad. inf.

12. (i) $\cosh\theta - \frac{1}{2}\cosh 2\theta + \frac{1}{3}\cosh 3\theta - \ldots$ ad. inf.,

(ii) $\sinh\theta - \frac{1}{2}\sinh 2\theta + \frac{1}{3}\sinh 3\theta - \ldots$ ad. inf.

13. If $\theta - \alpha = \tan^2(\frac{1}{2}\phi)\sin 2\theta - \frac{1}{2}\tan^4(\frac{1}{2}\phi)\sin 4\theta$

$$+ \frac{1}{3}\tan^6(\frac{1}{2}\phi)\sin 6\theta - \ldots \text{ ad. inf.},$$

show that $\tan\alpha = \tan\theta\cos\phi$. **(Agra 2010)**

14. Prove that

$$\tanh x + \frac{1}{3}\tanh^3 x + \frac{1}{5}\tanh^5 x + \ldots = \tan x - \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x - \ldots,$$

where x lies between $-(\pi/4)$ and $+(\pi/4)$. **(Meerut 2010; Kanpur 14)**

15. Prove that

$$\sin\theta + \frac{1}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta + \ldots \text{ ad. inf.}$$

$$= 2\left[\sin\theta - \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta - \ldots \text{ ad. inf.}\right], \text{ where } \theta \neq (2n+1)\frac{1}{2}\pi.$$

(Kanpur 2007)

Answers 4

1. (i) $\log 2 + \log\cos\frac{1}{2}\alpha$ (ii) $\frac{1}{2}\alpha$

2. (i) $\frac{1}{2}\log(1+3\cos^2\alpha)$ (ii) $\tan^{-1}\frac{\sin\alpha\cos\alpha}{1+\cos^2\alpha}$

(iii) $\tan^{-1} \dfrac{\sin^2 \alpha}{1 + \sin \alpha \cos \alpha}$

3. (i) $-\frac{1}{2} \log (1 - 2c \cos \alpha + c^2)$ (ii) $\tan^{-1} \{(c \sin \alpha) / (1 - c \cos \alpha)\}$

(iii) $-\frac{1}{2} \log (\frac{5}{4} - \cos \alpha)$ (iv) $\frac{1}{2} \pi - \alpha$ **4.** 0

5. (i) $-\sin (\alpha - \beta) . \tan^{-1} \{(c \sin \beta) / (1 - c \cos \beta)\}$

$- \frac{1}{2} \cos (\alpha - \beta) \log (1 - 2c \cos \beta + c^2)$

(ii) $\cos (\alpha - \beta) . \tan^{-1} \{(c \sin \beta) / (1 - c \cos \beta)\}$

$- \frac{1}{2} \sin (\alpha - \beta) \log (1 - 2c \cos \beta + c^2)$

6. (i) $\frac{1}{2} \cos (\alpha - \beta) \log (1 + 2c \cos \beta + c^2)$

$- \sin (\alpha - \beta) . \tan^{-1} \{(c \sin \beta) / (1 + c \cos \beta)\}$

(ii) $\frac{1}{2} \sin (\alpha - \beta) \log (1 + 2c \cos \beta + c^2)$

$+ \cos (\alpha - \beta) . \tan^{-1} \{(c \sin \beta) / (1 + c \cos \beta)\}$.

7. $\frac{1}{2} \left[\log \dfrac{(1 + c)}{\sqrt{(1 + 2c \cos 2\theta + c^2)}} \right]$

8. $\frac{1}{8} [2 \sqrt{3} \log (2 + \sqrt{3}) - \pi]$

9. (i) $\frac{1}{2} \tan^{-1} (\dfrac{2c \cos \theta}{1 - c^2})$ (ii) $\frac{1}{4} \log \dfrac{1 + 2c \sin \theta + c^2}{1 - 2c \sin \theta + c^2}$

10. $\frac{1}{2} \tan^{-1} (2 \cot^2 \theta)$

11. $-\frac{1}{2} \tan^{-1} \{\cos \beta / \sinh \alpha\}$

12. (i) $\log (2 \cosh \frac{1}{2} \theta)$ (ii) $\frac{1}{2} \theta$

6.7 The Difference Method

Sometimes in order to sum a series it is convenient to split each term of the series as difference of two terms. The splitting is done in such a way that when all the terms of the series are added together, the component terms cancel in pairs. Ultimately we are left usually with two components, one from the first term and one from the last term. This method is not always easy to apply, since sometimes it requires a considerable amount of mathematical jugglery to split the nth term of the series as the difference of two terms. It is however learnt by practice. Use of certain trigonometric identities is sometimes helpful. If the answer is given, then the mode of splitting can often be found by putting $n = 1$ in the answer, since then it reduces to first term of the series.

Suppose the rth term u_r of the series is expressed as

$$u_r = f(r + 1) - f(r).$$

Then $u_1 = f(2) - f(1)$

$u_2 = f(3) - f(2)$

$\dots \quad \dots \quad \dots \quad \dots$

$u_{n-1} = f(n) - f(n-1)$

$u_n = f(n+1) - f(n).$

Adding these n relations, we have

$$S_n = f(n+1) - f(1),$$

where $S_n = u_1 + u_2 + \dots + u_n$

= the sum of the first n terms of the series.

If the given series is convergent and sum to infinity is required, we deduce it as follows :

$$S_\infty = \lim_{n \to \infty} S_n = \lim_{n \to \infty} [f(n+1) - f(1)].$$

Thus to find the sum upto infinity we have to first find the sum upto n terms and then proceed to the limits as $n \to \infty$.

The following trigonometric identities are often useful :

(i) $\operatorname{cosec} \alpha = \cot \frac{1}{2} \alpha - \cot \alpha$,

(ii) $\tan \alpha = \cot \alpha - 2 \cot 2\alpha$,

(iii) $\tan \alpha \sec 2\alpha = \tan 2\alpha - \tan \alpha$,

(iv) $\sin^3 \alpha = \frac{1}{4}(3 \sin \alpha - \sin 3\alpha)$.

If the n th term is of the form $\tan^{-1}(a/b)$, we put it in the form

$$\tan^{-1} x - \tan^{-1} y.$$

Illustrative Examples

Example 12: *Sum the series*

$$\sec \theta \sec 2\theta + \sec 2\theta \sec 3\theta + \sec 3\theta \sec 4\theta + \dots \text{ to } n \text{ terms.}$$

Or $$\frac{1}{\cos \theta \cos 2\theta} + \frac{1}{\cos 2\theta \cos 3\theta} + \frac{1}{\cos 3\theta \cos 4\theta} + \dots \text{ to } n \text{ terms.}$$

(Kanpur 2011; Avadh 12)

Solution: The given series is

$$\sec \theta \sec 2\theta + \sec 2\theta \sec 3\theta + \dots \text{ to } n \text{ terms}$$

$$= \frac{1}{\cos \theta \cos 2\theta} + \frac{1}{\cos 2\theta \cos 3\theta} + \frac{1}{\cos 3\theta \cos 4\theta} + \dots \text{ to } n \text{ terms.}$$

The first term of the above series is $[1/(\cos \theta \cos 2\theta)]$. We should try to split it up as a difference of two terms. If T_1, T_2, etc. represent the successive terms of the given series, then

$$T_1 = \sec\theta \sec 2\theta = \frac{1}{\cos\theta\cos 2\theta} = \frac{1}{\sin\theta}\left[\frac{\sin\theta}{\cos\theta\cos 2\theta}\right],$$

multiplying the Nr. and Dr. by $\sin\theta$

$$= \frac{1}{\sin\theta}\left[\frac{\sin(2\theta-\theta)}{\cos\theta\cos 2\theta}\right] = \frac{1}{\sin\theta}\left[\frac{\sin 2\theta\cos\theta - \cos 2\theta\sin\theta}{\cos\theta\cos 2\theta}\right]$$

$$= (1/\sin\theta)\,[\tan 2\theta - \tan\theta].$$

Similarly the second term

$$T_2 = \sec 2\theta\sec 3\theta = \frac{1}{\cos 2\theta\cos 3\theta} = \frac{1}{\sin\theta}[\tan 3\theta - \tan 2\theta].$$

The third term

$$T_3 = \sec 3\theta\sec 4\theta = \frac{1}{\cos 3\theta\cos 4\theta} = \frac{1}{\sin\theta}[\tan 4\theta - \tan 3\theta].$$

... ...

Lastly, the nth term

$$T_n = \sec n\theta\sec(n+1)\theta = \frac{1}{\cos n\theta\cos(n+1)\theta}$$

$$= (1/\sin\theta)\,[\tan(n+1)\theta - \tan n\theta].$$

Adding up the above n relations, we have the required sum

$$= T_1 + T_2 + T_3 + \ldots + T_n = (1/\sin\theta)\,[\tan(n+1)\theta - \tan\theta],$$

the other terms on the R.H.S. cancelling each other.

Aliter: By inspection the nth term of the given series is

$$T_n = \sec n\theta\sec(n+1)\theta = \frac{1}{\cos n\theta\cos(n+1)\theta}$$

$$= \frac{1}{\sin\theta}\left[\frac{\sin\theta}{\cos n\theta\cos(n+1)\theta}\right],$$ multiplying the Nr. and Dr. by $\sin\theta$

$$= \frac{1}{\sin\theta}\left[\frac{\sin\{(n+1)\theta - n\theta\}}{\cos n\theta\cos(n+1)\theta}\right]$$ **(Note)**

$$= \frac{1}{\sin\theta}\left[\frac{\sin(n+1)\theta\cos n\theta - \cos(n+1)\theta\sin n\theta}{\cos n\theta\cos(n+1)\theta}\right]$$

$$= (1/\sin\theta)\,[\tan(n+1)\theta - \tan n\theta].$$

Putting $n = 1, 2, 3, \ldots, n$, we get

$$T_1 = (1/\sin\theta)\,[\tan 2\theta - \tan\theta],$$

$$T_2 = (1/\sin\theta)\,[\tan 3\theta - \tan 2\theta],$$

... ...

$$T_{n-1} = (1/\sin\theta)\,[\tan n\theta - \tan(n-1)\theta],$$

$$T_n = (1/\sin\theta)\,[\tan(n+1)\theta - \tan n\theta].$$

Adding these we get the required sum

$$T_1 + T_2 + T_3 + \ldots + T_{n-1} + T_n$$
$$= (1/\sin\theta)\,[\tan(n+1)\,\theta - \tan\theta],$$ other terms cancelling each other.

Example 13: *Sum the series* $\operatorname{cosec}\theta + \operatorname{cosec} 2\theta + \operatorname{cosec} 2^2\theta + \ldots$ *to n terms.*

(Purvanchal 2011)

Solution: The first term

$$T_1 = \operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{\sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta\,.\,\sin\theta} \qquad \textbf{(Note)}$$

$$= \frac{\sin(\theta - \frac{1}{2}\theta)}{\sin\frac{1}{2}\theta\,.\,\sin\theta} = \frac{\sin\theta\cos\frac{1}{2}\theta - \cos\theta\sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta\sin\theta} = \cot\frac{1}{2}\theta - \cot\theta.$$

Similarly, $T_2 = \cot\theta - \cot 2\theta,$

$$T_3 = \cot 2\theta - \cot 2^2\theta,$$
$$T_4 = \cot 2^2\theta - \cot 2^3\theta,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$T_n = \cot 2^{n-2}\theta - \cot 2^{n-1}\theta.$$

Adding these, we have the required sum

$$S_n = T_1 + T_2 + \ldots + T_n = \cot\frac{1}{2}\theta - \cot 2^{n-1}\theta,$$

other terms on the R.H.S. cancelling each other.

Example 14 : *Sum the series*

$$\tan\theta + 2\tan 2\theta + 2^2\tan 2^2\theta + 2^3\tan 2^3\theta + \ldots \text{ to } n \text{ terms.}$$

Solution: Here

$$T_1 = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin^2\theta}{\sin\theta\cos\theta} = \frac{\cos^2\theta - \cos 2\theta}{\sin\theta\cos\theta} \qquad \textbf{(Note)}$$

$$[\because \cos 2\theta = \cos^2\theta - \sin^2\theta]$$

$$= \frac{\cos^2\theta}{\sin\theta\cos\theta} - \frac{\cos 2\theta}{\sin\theta\cos\theta} = \frac{\cos\theta}{\sin\theta} - \frac{\cos 2\theta}{\frac{1}{2}\sin 2\theta} = \cot\theta - 2\cot 2\theta.$$

Similarly, $T_2 = 2\tan 2\theta = 2\,[\cot 2\theta - 2\cot 2(2\theta)]$

$$= 2\,[\cot 2\theta - 2\cot 2^2\theta]$$
$$= 2\cot 2\theta - 2^2\cot 2^2\theta,$$
$$T_3 = 2^2\cot 2^2\theta - 2^3\cot 2^3\theta,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$T_n = 2^{n-1}\cot 2^{n-1}\theta - 2^n\cot 2^n\theta.$$

Adding these, we have the required sum

$$= \cot\theta - 2^n \cot 2^n \theta,$$

the other terms on the R.H.S. cancelling one another.

Example 15: *Sum the series*

$$\sin^3 \frac{\theta}{3} + 3 \sin^3 \frac{\theta}{3^2} + 3^2 \sin^3 \frac{\theta}{3^3} + \ldots \text{ to } n \text{ terms}$$

and also sum to infinity.

Solution: We know that $\sin 3\theta = 3 \sin\theta - 4 \sin^3 \theta$ giving

$$\sin^3\theta = \frac{1}{4}(3 \sin\theta - \sin 3\theta). \qquad \ldots(1)$$

Now the first term of the given series

$$T_1 = \sin^3 \frac{1}{3}\theta = \frac{1}{4}[3 \sin \frac{1}{3}\theta - \sin 3 \,.\, \frac{1}{3}\theta]$$

[Putting $\frac{1}{3}\theta$ for θ in (1)]

$$= \frac{1}{4}[3 \sin \frac{1}{3}\theta - \sin\theta].$$

Similarly, $T_2 = 3 \sin^3 (\theta/3^2) = 3 \cdot \frac{1}{4}[3 \sin(\theta/3^2) - \sin 3.(\theta/3^2)]$

[Putting $(\theta/3^2)$ for θ in (1)]

$$= \frac{1}{4}[3^2 \sin(\theta/3^2) - 3 \sin(\theta/3)],$$

$$T_3 = 3^2 \sin^3 (\theta/3^3) = \frac{1}{4}[3^3 \sin(\theta/3^3) - 3^2 \sin(\theta/3^2)],$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$T_n = \frac{1}{4}[3^n \sin(\theta/3^n) - 3^{n-1} \sin(\theta/3^{n-1})].$$

Adding up these n relations, we have

$$T_1 + T_2 + T_3 + \ldots + T_n = \frac{1}{4}[3^n \sin(\theta/3^n) - \sin\theta].$$

Thus the required sum upto n terms

$$S_n = \frac{1}{4}[3^n \sin(\theta/3^n) - \sin\theta].$$

The sum to infinity S_∞ is given by

$$S_\infty = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{4}\left[3^n \sin\frac{\theta}{3^n} - \sin\theta\right]$$

$$= \frac{1}{4} \cdot \lim_{n\to\infty} \left[\theta \cdot \frac{\sin(\theta/3^n)}{(\theta/3^n)} - \sin\theta\right] \qquad \textbf{(Note)}$$

Let $(\theta/3^n) = h$, then as $n \to \infty$, $h \to 0$.

$$\therefore \qquad S_\infty = \frac{1}{4} \lim_{h \to 0} \left[\theta \cdot \frac{\sin h}{h} - \sin\theta\right] = \frac{1}{4}[\theta - \sin\theta].$$

$$\left[\because \lim_{h \to 0} (\sin h / h) = 1\right]$$

Example 16: *Sum the series*

$$\tan\alpha \tan(\alpha+\beta) + \tan(\alpha+\beta)\tan(\alpha+2\beta) + \tan(\alpha+2\beta)\tan(\alpha+3\beta) + \ldots \text{ to } n \text{ terms.}$$

Solution: We have

$$\tan\beta = \tan[(\alpha+\beta)-\alpha] = \frac{\tan(\alpha+\beta) - \tan\alpha}{1 + \tan(\alpha+\beta) \,.\, \tan\alpha}$$

or $\quad \tan\beta\,[1 + \tan(\alpha+\beta)\tan\alpha] = \tan(\alpha+\beta) - \tan\alpha$

or $\quad 1 + \tan(\alpha+\beta)\tan\alpha = \cot\beta\,[\tan(\alpha+\beta) - \tan\alpha]$

or $\quad \tan(\alpha+\beta)\tan\alpha = \cot\beta\,[\tan(\alpha+\beta) - \tan\alpha] - 1. \qquad \ldots(1)$

$\therefore$ first term of the given series *i.e.*,

$$T_1 = \cot\beta\,[\tan(\alpha+\beta) - \tan\alpha] - 1.$$

Similarly, $\quad T_2 = \cot\beta\,[\tan(\alpha+2\beta) - \tan(\alpha+\beta)] - 1,$

putting $(\alpha+\beta)$ for α in (1)

$$T_3 = \cot\beta\,[\tan(\alpha+3\beta) - \tan(\alpha+2\beta)] - 1,$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$T_n = \cot\beta\,[\tan(\alpha+n\beta) - \tan\{\alpha+(n-1)\beta\}] - 1.$$

Adding the above n relations, we have

the required sum $= \cot\beta\,[\tan(\alpha+n\beta) - \tan\alpha] - n,$

the other terms on the R.H.S. cancelling one another.

Example 17: *Sum the series*

$$\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} + \tan^{-1}\frac{1}{21} + \ldots \text{ to } n \text{ terms.}$$

(Meerut 2005B, 09B; Kanpur 10; Kashi 11; Avadh 13; Purvanchal 14)

Solution: The given series may be written as

$$\tan^{-1}\frac{1}{1+1\,.\,2} + \tan^{-1}\frac{1}{1+2\,.\,3} + \tan^{-1}\frac{1}{1+3\,.\,4} + \ldots + \tan^{-1}\frac{1}{1+n(n+1)}\,.$$

We have

$$T_n = \text{the } n\text{th term of the series}$$

$$= \tan^{-1}\frac{1}{1+n(n+1)} = \tan^{-1}\frac{(n+1)-n}{1+(n+1)\,n}$$

$$= \tan^{-1}(n+1) - \tan^{-1} n.$$

$$[\because \tan^{-1} x - \tan^{-1} y = \tan^{-1}\{(x-y)/(1+xy)\}]$$

Putting $n = 1, 2, 3, \ldots, n$, we have

$$T_1 = \tan^{-1} 2 - \tan^{-1} 1,$$
$$T_2 = \tan^{-1} 3 - \tan^{-1} 2,$$
$$T_3 = \tan^{-1} 4 - \tan^{-1} 3,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\ldots$$
$$T_n = \tan^{-1} (n+1) - \tan^{-1} n.$$

Adding the above n relations , we have

the required sum $S_n = T_1 + T_2 + \ldots + T_n = \tan^{-1} (n+1) - \tan^{-1} 1,$

the other terms on the R.H.S. cancelling one another.

Comprehensive Exercise 5

Sum the following series:

1. (i) $\sec \theta \sec (\theta + \phi) + \sec (\theta + \phi) \sec (\theta + 2\phi) + \sec (\theta + 2\phi) \sec (\theta + 3\phi) + \ldots$ to n terms.

(ii) $\dfrac{1}{\cos \theta + \cos 3\theta} + \dfrac{1}{\cos \theta + \cos 5\theta} + \dfrac{1}{\cos \theta + \cos 7\theta} + \ldots$ to n terms. **(Gorakhpur 2007)**

(iii) $\operatorname{cosec} \theta \operatorname{cosec} 2\theta + \operatorname{cosec} 2\theta \operatorname{cosec} 3\theta + \operatorname{cosec} 3\theta \operatorname{cosec} 4\theta + \ldots$ to n terms. **(Purvanchal 2007)**

(iv) $\dfrac{1}{\cos \theta - \cos 3\theta} + \dfrac{1}{\cos \theta - \cos 5\theta} + \dfrac{1}{\cos \theta - \cos 7\theta} + \ldots$ to n terms.

2. (i) $\dfrac{\sin \theta}{\sin 2\theta \sin 3\theta} + \dfrac{\sin \theta}{\sin 3\theta \sin 4\theta} + \dfrac{\sin \theta}{\sin 4\theta \sin 5\theta} + \ldots$ to n terms.

(ii) $\dfrac{\sin \theta}{\cos \theta + \cos 2\theta} + \dfrac{\sin 2\theta}{\cos \theta + \cos 4\theta} + \dfrac{\sin 3\theta}{\cos \theta + \cos 6\theta} + \ldots$ to n terms.

3. (i) $\cos \dfrac{\theta}{2} + 2 \cos \dfrac{\theta}{2} \cos \dfrac{\theta}{2^2} + 2^2 \cos \dfrac{\theta}{2} \cos \dfrac{\theta}{2^2} \cos \dfrac{\theta}{2^3} + \ldots$ to n terms.

(ii) $\tan \dfrac{\theta}{2} \sec \theta + \tan \dfrac{\theta}{2^2} \sec \dfrac{\theta}{2} + \tan \dfrac{\theta}{2^3} \sec \dfrac{\theta}{2^2} + \ldots$ to n terms **(Garhwal 2000)**

4. $\sin \theta \sec 3\theta + \sin 3\theta \sec 3^2 \theta + \sin 3^2 \theta \,.\, \sec 3^3 \theta + \ldots$ to n terms.

5. (i) $2 \operatorname{cosec} 2\theta \cot 2\theta + 4 \operatorname{cosec} 4\theta \cot 4\theta + 8 \operatorname{cosec} 8\theta \cot 8\theta + \ldots$ to n terms.

(ii) $\sin 2\theta \cos^2 \theta - \dfrac{1}{2} \sin 4\theta \cos^2 2\theta + \dfrac{1}{4} \sin 8\theta \cos^2 4\theta - \ldots$ to n terms.

6. $\tan \theta + \dfrac{1}{2} \tan \dfrac{\theta}{2} + \dfrac{1}{2^2} \tan \dfrac{\theta}{2^2} + \dfrac{1}{2^3} \tan \dfrac{\theta}{2^3} + \ldots$ ad. inf.

(Meerut 2004B; Avadh 07, 10)

7. $\tanh\theta + \frac{1}{2}\tanh\frac{\theta}{2} + \frac{1}{2^2}\tanh\frac{\theta}{2^2} + \ldots$ to n terms.

8. $\tan^2\theta\tan 2\theta + \frac{1}{2}\tan^2 2\theta\tan 4\theta + (1/2^2)\tan^2 4\theta\tan 8\theta + \ldots$ to n terms.
(Avadh 2008)

9. $\cot 2\alpha\cot 3\alpha + \cot 3\alpha\cot 4\alpha + \ldots + \cot(n+1)\alpha\cot(n+2)\alpha$.

10. $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{2}{9} + \tan^{-1}\frac{4}{33} + \ldots + \tan^{-1}\frac{2^{n-1}}{1+2^{2n-1}} + \ldots$ ad. inf.
(Purvanchal 2007)

11. $\tan^{-1}x + \tan^{-1}\frac{x}{1+1.2x^2} + \tan^{-1}\frac{x}{1+2.3x^2} + \ldots$ to n terms.
(Avadh 2009)

12. $\tan^{-1}\frac{4}{1+3.4} + \tan^{-1}\frac{6}{1+8.9} + \tan^{-1}\frac{8}{1+15.16} + \ldots +$ to n terms.
(Meerut 2010; 13)

13. (i) $\tan^{-1}\frac{1}{1+1+1^2} + \tan^{-1}\frac{1}{1+2+2^2} + \tan^{-1}\frac{1}{1+3+3^2} + \ldots +$ to n terms.
(Meerut 2008; 13B)

(ii) Evaluate $\sum_{n=1}^{\infty}\cot^{-1}(1+n+n^2)$.

14. $\tan^{-1}\frac{4}{7} + \tan^{-1}\frac{4}{19} + \tan^{-1}\frac{4}{39} + \ldots + \tan^{-1}\frac{4}{4n^2+3}$.

15. $\tan\alpha\tan 2\alpha + \tan 2\alpha\tan 3\alpha + \tan 3\alpha\tan 4\alpha + \ldots$ to $(n-1)$ terms.
(Meerut 2006)

16. $\tan^{-1}\frac{1}{2.1^2} + \tan^{-1}\frac{1}{2.2^2} + \tan^{-1}\frac{1}{2.3^2} + \ldots + \tan^{-1}\frac{1}{2n^2}$.

Deduce its sum to infinity. **(Rohilkhand 2008)**

17. $\cot^{-1}\left(2^2+\frac{1}{2}\right) + \cot^{-1}\left(2^3+\frac{1}{2^2}\right) + \cot^{-1}\left(2^4+\frac{1}{2^3}\right) + \ldots$ ad. inf.
(Gorakhpur 2005)

Answers 5

1. (i) $(1/\sin\phi)[\tan(\theta+n\phi) - \tan\theta]$ (ii) $\frac{1}{2}\operatorname{cosec}\theta\{\tan(n+1)\theta - \tan\theta\}$

(iii) $(1/\sin\theta)[\cot\theta - \cot(n+1)\theta]$ (iv) $\frac{1}{2}\operatorname{cosec}\theta[\cot\theta - \cot(n+1)\theta]$

2. (i) $\cot 2\theta - \cot(n+2)\theta$

(ii) $(1/4\sin\frac{1}{2}\theta)[\sec\frac{1}{2}(2n+1)\theta - \sec\frac{1}{2}\theta]$

3. (i) $\frac{\sin n\theta}{2}\left[\cot \frac{\theta}{2^{n+1}} - \cot \frac{\theta}{2}\right]$ (ii) $\tan\theta$

5. (i) $\text{cosec}^2\theta - 2^n \text{ cosec}^2 2^n\theta$ (ii) $\frac{1}{2}\sin 2\theta + (-1)^{n-1}(1/2^{n+1})\sin 2^{n+1}\theta$

6. $\frac{1}{\theta} - 2\cot 2\theta$ **7.** $[2\coth 2\theta - (1/2^{n-1})\coth(\theta/2^{n-1})]$

8. $(1/2^{n-1})\tan 2^n\theta - 2\tan\theta$ **9.** $\cot\alpha\,[\cot 2\alpha - \cot(n+2)\alpha] - n$

10. $\frac{1}{4}\pi$ **11.** $\tan^{-1} nx$

12. $\tan^{-1}(n+1)(n+2) - \tan^{-1} 2$

13. (i) $\tan^{-1}\{n/(n+2)\}$ (ii) $\frac{1}{4}\pi$

14. $\tan^{-1}\frac{4n}{2n+5}$ **15.** $\cot\alpha\tan n\alpha - n.$

16. $\tan^{-1}(2n+1) - \tan^{-1}(2n-1); \frac{1}{4}\pi$ **17.** $\cot^{-1} 2$

6.8 Angles in Arithmetical Progression

(a) *To find the sum of the sines of a series of angles, the angles being in arithmetical progression.*

Let $\alpha, \alpha+\beta, \alpha+2\beta, \ldots, \{\alpha+(n-1)\beta\}$ be n angles in arithmetical progression, the common difference of the angles being β.

Then the series of sines of above angles will be

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin\{\alpha+(n-1)\beta\}.$$

Suppose S_n denotes the sum of the above series, so that

$$S_n = \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin\{\alpha+(n-1)\beta\}. \quad \ldots(1)$$

The common difference of angles is β.

Multiplying both sides of (1) by $2\sin\frac{1}{2}\beta$, we have

$$2\sin\frac{1}{2}\beta\,.\,S_n = 2\sin\alpha\sin\frac{1}{2}\beta + 2\sin(\alpha+\beta)\,.\sin\frac{1}{2}\beta + \ldots$$
$$\ldots + 2\sin\{\alpha+(n-1)\beta\}\sin\frac{1}{2}\beta.$$

Now applying the trigonometric formula

$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$

to each term on the right hand side , we have

$$2\sin\frac{1}{2}\beta\,.\,S_n = \cos(\alpha - \frac{1}{2}\beta) - \cos(\alpha + \frac{1}{2}\beta)$$
$$+ \cos(\alpha + \frac{1}{2}\beta) - \cos(\alpha + \frac{3}{2}\beta)$$

$$+ \cos (\alpha + \frac{3}{2} \beta) - \cos (\alpha + \frac{5}{2} \beta)$$

$$+ \dots \qquad \dots \qquad \dots$$

$$+ \dots \qquad \dots \qquad \dots$$

$$+ \cos \{\alpha + (n - \frac{3}{2}) \beta\} - \cos \{\alpha + (n - \frac{1}{2}) \beta\}$$

$$= \cos (\alpha - \frac{1}{2} \beta) - \cos \{\alpha + (n - \frac{1}{2}) \beta\},$$

the other terms on the right hand side cancelling one another.

$$= 2 \sin \{\alpha + \frac{1}{2} (n - 1) \beta\} \sin \frac{1}{2} n\beta.$$

Therefore

$$S_n = \frac{\sin \left\{\alpha + \frac{1}{2} (n + 1) \beta\right\} \sin \frac{1}{2} n\beta}{\sin \frac{1}{2} \beta} \qquad \dots(2) \quad \textbf{(Remember)}$$

The sum of the sine series obtained above may be remembered in the following form :

$$\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + \dots + \sin \{\alpha + (n - 1) \beta\}$$

$$= \sin \left[\frac{\text{first angle} + \text{last angle}}{2}\right] \frac{\sin \left(n \times \frac{\text{diff.}}{2}\right)}{\sin \left(\frac{\text{diff.}}{2}\right)}$$

Particular Cases: I. If $\beta = \alpha$, then the expression denoting the sum of the sine series (1) becomes

$$S_n = \sin \alpha + \sin 2\alpha + \sin 3\alpha + \dots + \sin n\alpha$$

$$= \sin \left(\frac{n+1}{2} \alpha\right) \frac{\sin \frac{1}{2} n\alpha}{\sin \frac{1}{2} \alpha}, \text{ from (2)},$$

II. If we put $\beta = 2\pi / n$, then since

$$\sin \frac{1}{2} n\beta = \sin \pi = 0 \text{ and } \sin \frac{1}{2} \beta \neq 0,$$

we have $\sin \alpha + \sin \left(\alpha + \frac{2\pi}{n}\right) + \sin \left(\alpha + \frac{4\pi}{n}\right) + \dots$ to n terms $= 0$.

In general if $\beta = p.(2\pi / n)$, where p is any integer, even then

$$\sin \frac{n\beta}{2} = \sin \left(\frac{n 2 p\pi}{2n}\right) = \sin p\pi = 0.$$

Also if p is not divisible by n, then $\sin \frac{1}{2} \beta \neq 0$.

Therefore the sum of sines of n angles, which are in arithmetic progression, vanishes if the common difference of the angles is $p(2\pi / n)$, *where p is any integer not divisible by n.*

(b) *To find the sum of the cosines of a series of angles, the angles being in arithmetical progression.*

Let n angles in arithmetical progression be

$$\alpha, (\alpha + \beta), (\alpha + 2\beta), \ldots, \{\alpha + (n-1)\beta\}.$$

Let S_n denote the sum of the series of cosines of above angles, so that

$$S_n = \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos\{\alpha+(n-1)\beta\}. \quad \ldots(3)$$

Multiplying both sides by $2\sin\frac{1}{2}\beta$, we get

$$2\sin\frac{1}{2}\beta \, . \, S_n = 2\cos\alpha\sin\frac{1}{2}\beta + 2\cos(\alpha+\beta)\sin\frac{1}{2}\beta$$
$$+ 2\cos(\alpha+2\beta)\sin\frac{1}{2}\beta + \ldots\ldots + 2\cos\{\alpha+(n-1)\beta\}\sin\frac{1}{2}\beta.$$

Now applying the trigonometric relation

$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$

to each term on the right hand side, we have

$$2\sin\frac{1}{2}\beta \, . \, S_n = \sin(\alpha + \frac{1}{2}\beta) - \sin(\alpha - \frac{1}{2}\beta)$$
$$+ \sin(\alpha + \frac{3}{2}\beta) - \sin(\alpha + \frac{1}{2}\beta)$$
$$+ \sin(\alpha + \frac{5}{2}\beta) - \sin(\alpha + \frac{3}{2}\beta)$$
$$+ \quad \ldots \quad \ldots \quad \ldots$$
$$+ \quad \ldots \quad \ldots \quad \ldots$$
$$+ \sin\{\alpha + (n - \frac{1}{2})\beta\} - \sin\{\alpha + (n - \frac{3}{2})\beta\}$$

$$= \sin\{\alpha + (n - \frac{1}{2})\beta\} - \sin(\alpha - \frac{1}{2}\beta),$$ the other terms on the right hand side cancelling one another

$$= 2\cos\{\alpha + \frac{1}{2}(n-1)\beta\}\sin\frac{1}{2}n\beta.$$

Therefore

$$S_n = \frac{\cos\left\{\alpha + \frac{1}{2}(n-1)\beta\right\}\sin\frac{1}{2}n\beta}{\sin\frac{1}{2}\beta} \quad \ldots(4) \qquad \textbf{(Remember)}$$

Particular Cases: I. If $\beta = \alpha$, then the expression denoting the sum of the cosine series (3) becomes

$$S_n = \cos\alpha + \cos 2\alpha + \cos 3\alpha + \ldots + \cos n\alpha$$

$$= \cos\left(\frac{n+1}{2}\alpha\right)\frac{\sin\frac{1}{2}n\alpha}{\sin\frac{1}{2}\alpha}, \text{ from (4).}$$

II. If $\beta = 2\pi/n$, then since $\sin\frac{1}{2}n\beta = \sin\pi = 0$ and $\sin\frac{1}{2}\beta \neq 0$,

we have $\cos\alpha + \cos\left(\alpha + \frac{2\pi}{n}\right) + \cos\left(\alpha + \frac{4\pi}{n}\right) + \ldots$ to n terms $= 0$.

In general if $\beta = p(2\pi/n)$, where p is any integer not divisible by n, even then the sum of the above cosine series is zero.

Illustrative Examples

Example 18: *Sum the series*

(i) $\sin\alpha + \sin 3\alpha + \sin 5\alpha + \ldots +$ *to n terms* ;

(ii) $\cos\alpha + \cos 3\alpha + \cos 5\alpha + \ldots$ *to n terms.*

(iii) *Prove that*

$$\tan n\alpha = \frac{\sin\alpha + \sin 3\alpha + \sin 5\alpha + \ldots \text{ to } n \text{ terms}}{\cos\alpha + \cos 3\alpha + \cos 5\alpha + \ldots \text{ to } n \text{ terms}}.$$

Solution: **(i)** Let S_n denote the sum of the given series (i) ; then

$$S_n = \sin\alpha + \sin 3\alpha + \sin 5\alpha + \ldots \text{ to } n \text{ terms.} \qquad \ldots(1)$$

In the above series (1) the angles $\alpha, 3\alpha, 5\alpha, \ldots$ are in arithmetical progression. The common difference of angles is $(3\alpha - \alpha) = 2\alpha$. Here we shall multiply both sides of (1) by

$$2\sin\frac{1}{2}(2\alpha) \text{ i.e., by } 2\sin\alpha.$$

$\therefore$ multiplying both the sides of (1) by $2\sin\alpha$, we have

$$S_n \cdot 2\sin\alpha = 2\sin\alpha\sin\alpha + 2\sin 3\alpha\sin\alpha + 2\sin 5\alpha\sin\alpha + \ldots$$
$$+ 2\sin\{(2n-1)\alpha\}\sin\alpha. \qquad \ldots(2)$$

Now applying the formula

$$2\sin A\sin B = \cos(A - B) - \cos(A + B)$$

to each term on the R.H.S. of (2), we have

the first term $= \cos(\alpha - \alpha) - \cos(\alpha + \alpha) = \cos 0 - \cos 2\alpha$,

the second term $= \cos(3\alpha - \alpha) - \cos(3\alpha + \alpha) = \cos 2\alpha - \cos 4\alpha$,

the third term $= \cos(5\alpha - \alpha) - \cos(5\alpha + \alpha) = \cos 4\alpha - \cos 6\alpha$,

$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

the nth term $= \cos[\{(2n-1)\alpha\} - \alpha] - \cos[\{(2n-1)\alpha + \alpha]$

$= \cos(2n-2)\alpha - \cos 2n\alpha$.

Adding up the above n rows, we have the sum of the n terms on the R.H.S. of (2)

$$= \cos 0 - \cos 2n\alpha = 1 - \cos 2n\alpha,$$

the other terms cancelling one another.

Hence from (2), we have

$$S_n \,.\, 2 \sin \alpha = 1 - \cos 2n\alpha = 2 \sin^2 n\alpha.$$

$$\therefore \qquad S_n = \sin^2 n\alpha / \sin \alpha.$$

Thus the required sum of the series (i) is equal to

$$\sin^2 n\alpha / \sin \alpha.$$

(ii) Let S_n denote the sum of the given series (ii) ; then

$$S_n = \cos \alpha + \cos 3\alpha + \cos 5\alpha + \ldots + \cos (2n - 1)\, \alpha. \qquad \ldots(3)$$

The angles in the above series are in A.P. with common difference

$$(3\alpha - \alpha) = 2\alpha.$$

$\therefore$ multiplying both sides of (3) by $2 \sin \frac{1}{2} (2\alpha)$ *i.e.*, by $2 \sin \alpha$, we have

$$2 \sin \alpha \,.\, S_n = 2 \sin \alpha \cos \alpha + 2 \sin \alpha \cos 3\alpha + 2 \sin \alpha \cos 5\alpha + \ldots + 2 \sin \alpha \cos (2n - 1)\, \alpha.$$

Now using the formula

$$2 \cos A \sin B = \sin (A + B) - \sin (A - B)$$

term by term on the right hand side of (3), we have

the first term $= \sin (\alpha + \alpha) - \sin (\alpha - \alpha) = \sin 2\alpha - \sin 0,$

the second term $= \sin (3\alpha + \alpha) - \sin (3\alpha - \alpha) = \sin 4\alpha - \sin 2\alpha,$

the third term $= \sin (5\alpha + \alpha) - \sin (5\alpha - \alpha) = \sin 6\alpha - \sin 4\alpha,$

… … … … …

the nth term $= \sin \{(2n - 1)\, \alpha + \alpha\} - \sin \{(2n - 1)\, \alpha - \alpha\}$

$= \sin 2n\alpha - \sin (2n - 2)\, \alpha.$

Adding up the above n rows , the sum of the n terms on the R.H.S. of (3)

$= \sin 2n\alpha - \sin 0 = \sin 2n\alpha,$ the other terms cancelling one another.

Hence from (3), we have

$$2 \sin \alpha \,.\, S_n = \sin 2n\alpha$$

or $$S_n = (\sin 2n\alpha) / 2 \sin \alpha.$$

Thus the required sum of the series (ii) is equal to

$$(\sin 2n\alpha) / (2 \sin \alpha).$$

(iii) We have

$$\frac{\sin \alpha + \sin 3\alpha + \sin 5\alpha + \ldots \text{ to } n \text{ terms}}{\cos \alpha + \cos 3\alpha + \cos 5\alpha + \ldots \text{ to } n \text{ terms}}$$

$$= \frac{(\sin^2 n\alpha) / (\sin \alpha)}{(\sin 2n\alpha) / (2 \sin \alpha)}, \text{ from the solutions of parts (i) and (ii)}$$

(prove here)

$$= \frac{2 \sin^2 n\alpha}{\sin 2n\alpha} = \frac{2 \sin^2 n\alpha}{2 \sin n\alpha \cos n\alpha} = \frac{\sin n\alpha}{\cos n\alpha} = \tan n\alpha.$$

Remark: The above question can also be done by direct application of the formulae derived in article 6.8 as shown below.

We know that

$$\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + \ldots \text{ upto } n \text{ terms}$$

$$= \sin \{\alpha + \frac{1}{2} (n - 1) \beta\} \frac{\sin \frac{1}{2} n\beta}{\sin \frac{1}{2} \beta}.$$

Here $\beta = 3\alpha - \alpha = 2\alpha$.

$\therefore$ $$\sin \alpha + \sin 3\alpha + \sin 5\alpha + \ldots \text{ upto } n \text{ terms}$$

$$= \sin \{\alpha + \frac{1}{2} (n - 1) 2\alpha\} \frac{\sin \{\frac{1}{2} n (2\alpha)\}}{\sin \{\frac{1}{2} (2\alpha)\}}$$

$$= \sin n\alpha \frac{\sin n\alpha}{\sin \alpha} = \frac{\sin^2 n\alpha}{\sin \alpha}.$$

Similarly we may find the sum of the cosine series.

Example 19: *If β be the exterior angle of a regular polygon of n sides and α is any constant, prove that*

(i) $\sum_{r=0}^{n-1} \sin (\alpha + \beta r) = 0$, (ii) $\sum_{r=0}^{n-1} \cos (\alpha + \beta r) = 0$.

Solution: We know that the sum of all the exterior angles of a polygon is equal to 2π. Here the polygon is a regular one of n sides. Therefore the sum of the n equal exterior angles $= 2\pi$ *i.e.*, each exterior angle is equal to $(2\pi / n)$ *i.e.*, $\beta = (2\pi / n)$.

(i) By putting $r = 0, 1, 2, \ldots, (n - 1)$, we find that the series in (i) is

$$\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + \ldots \text{ to } n \text{ terms.}$$

By article 6.8 (a), we have

$$\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + \ldots \text{ to } n \text{ terms}$$

$$= \frac{\sin \frac{1}{2} n\beta}{\sin \frac{1}{2} \beta} \sin \left\{\alpha + \frac{1}{2} (n - 1) \beta\right\}$$

$$= \frac{\sin \{\frac{1}{2} n . (2\pi / n)\} \sin \{\alpha + \frac{1}{2} (n - 1) (2\pi / n)\}}{\sin \frac{1}{2} (2\pi / n)} \qquad \left[\because \beta = \frac{2\pi}{n}\right]$$

$$= \frac{\sin \pi . \sin \{\alpha + (n - 1) \pi / n\}}{\sin (\pi / n)} = 0. \qquad [\because \sin \pi = 0 \text{ and } \sin (\pi / n) \neq 0]$$

(ii) Putting $r = 0, 1, 2, \ldots, (n - 1)$, we find that the series given in (ii) is

$$\cos \alpha + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + \ldots \text{ to } n \text{ terms.}$$

By article 6.8 (b), we have

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \dots \text{ to } n \text{ terms}$$

$$= \frac{\sin \frac{1}{2} n\beta}{\sin \frac{1}{2}\beta} \cos\left\{\alpha + \frac{1}{2}(n-1)\beta\right\}$$

$$= \frac{\sin\pi \,.\, \cos\{\alpha + \frac{1}{2}(n-1)(2\pi/n)\}}{\sin(\pi/n)} \qquad \left[\because \beta = \frac{2\pi}{n}\right]$$

$$= 0. \qquad [\because \sin\pi = 0 \text{ and } \sin(\pi/n) \neq 0]$$

Example 20: *Find the sum to n terms of the series*

(i) $sin^4\ \alpha + sin^4\left(\alpha + \frac{2\pi}{n}\right) + sin^4\left(\alpha + \frac{4\pi}{n}\right) + \dots$

(ii) $cos^4\ \alpha + cos^4\left(\alpha + \frac{2\pi}{n}\right) + cos^4\left(\alpha + \frac{4\pi}{n}\right) + \dots$

Solution: **(i)** We know that $\sin^2\alpha = \frac{1}{2}[1 - \cos 2\alpha]$.

$$\therefore \quad \sin^4\alpha = [\sin^2\alpha]^2 = \frac{1}{4}[1-\cos 2\alpha]^2 = \frac{1}{4}[1 + \cos^2 2\alpha - 2\cos 2\alpha]$$

$$= \frac{1}{4}\cdot\frac{1}{2}[2 + 2\cos^2 2\alpha - 4\cos 2\alpha]$$

$$= \frac{1}{8}[2 + (1 + \cos 4\alpha) - 4\cos 2\alpha]$$

$$= \frac{1}{8}[\cos 4\alpha - 4\cos 2\alpha + 3]. \qquad \dots(1)$$

Similarly $\sin^4\left(\alpha + \frac{2\pi}{n}\right) = \frac{1}{8}\left[\cos\left(4\alpha + \frac{8\pi}{n}\right) - 4\cos\left(2\alpha + \frac{4\pi}{n}\right) + 3\right]$,

replacing α by $\alpha + (2\pi/n)$ in (1)

$$\sin^4\left(\alpha + \frac{4\pi}{n}\right) = \frac{1}{8}\left[\cos\left(4\alpha + \frac{16\pi}{n}\right) - 4\cos\left(2\alpha + \frac{8\pi}{n}\right) + 3\right],$$

and so on upto n terms.

Adding the above n relations, we have

$$\sin^4\alpha + \sin^4\left(\alpha + \frac{2\pi}{n}\right) + \sin^4\left(\alpha + \frac{4\pi}{n}\right) + \dots \text{ to } n \text{ terms}$$

$$= \frac{1}{8}\left[\left\{\cos(4\alpha) + \cos\left(4\alpha + \frac{8\pi}{n}\right) + \dots \text{ to } n \text{ terms}\right\}\right.$$

$$\left. - 4\left\{\cos 2\alpha + \cos\left(2\alpha + \frac{4\pi}{n}\right) + \dots \text{ to } n \text{ terms}\right\} + 3(1 + 1 + \dots \text{ to } n \text{ terms})\right]\cdot$$

In the first series on the R.H.S. the common difference of the angles is $8\pi / n$ and in the second series it is $4\pi / n$ and each of them is a multiple of $2\pi / n$. Therefore the sum of each of these two series vanishes.

Hence the required sum of the given series $= \frac{1}{8}[0 - 0 + 3n] = \frac{3}{8} n$.

(ii) The given series is

$$\cos^4 \alpha + \cos^4 \left(\alpha + \frac{2\pi}{n}\right) + \cos^4 \left(\alpha + \frac{4\pi}{n}\right) + \ldots \text{ to } n \text{ terms.}$$

We know that $\cos^2 \alpha = \frac{1}{2}[1 + \cos 2\alpha]$.

$$\therefore \quad \cos^4 \alpha = \frac{1}{4}[1 + \cos 2\alpha]^2 = \frac{1}{4}[1 + \cos^2 2\alpha + 2 \cos 2\alpha]$$

$$= \frac{1}{8}[2 + 2\cos^2 2\alpha + 4 \cos 2\alpha] = \frac{1}{8}[2 + (1 + \cos 4\alpha) + 4 \cos 2\alpha]$$

$$= \frac{1}{8}[3 + \cos 4\alpha + 4 \cos 2\alpha] = \frac{1}{8}[\cos 4\alpha + 4 \cos 2\alpha + 3].$$

Now proceeding exactly as in part (i) we find that the required sum of the given series

$$= \frac{1}{8}[0 + 4 \,.\, 0 + 3n] = \frac{3n}{8}.$$

Comprehensive Exercise 6

1. Find the sum of n terms of the series
   (i) $\sin \alpha - \sin(\alpha + \beta) + \sin(\alpha + 2\beta) - \ldots$
   (ii) $\cos \alpha - \cos(\alpha + \beta) + \cos(\alpha + 2\beta) - \ldots$
2. Sum the series $\sin \alpha - \sin 2\alpha + \sin 3\alpha - \sin 4\alpha + \ldots$ to n terms.
3. Sum the series $\cos \alpha - \sin 2\alpha - \cos 3\alpha + \sin 4\alpha + \ldots$ to n terms.
4. Sum the series $\cos \frac{\pi}{2n+1} + \cos \frac{3\pi}{2n+1} + \cos \frac{5\pi}{2n+1} + \ldots$ to n terms.
5. Find the sum of the series $\sin \alpha + \sin 2\alpha + \sin 3\alpha + \ldots + \sin n\alpha$
   and hence deduce the sum of the series $1 + 2 + 3 + \ldots + n$.
6. Find the sum of the series
   (i) $\sin^2 \alpha + \sin^2(\alpha + \beta) + \sin^2(\alpha + 2\beta) + \ldots$ to n terms
   (ii) $\cos^2 \alpha + \cos^2(\alpha + \beta) + \cos^2(\alpha + 2\beta) + \ldots$ to n terms.

Answers 6

1. (i) $\sin\{\alpha + \frac{1}{2}(n-1)(\pi+\beta)\} \sin\frac{1}{2}n(\pi+\beta) \sec\frac{1}{2}\beta$

(ii) $\cos\{\alpha + \frac{1}{2}(n-1)(\pi+\beta)\} \sin\frac{1}{2}n(\pi+\beta) \sec\frac{1}{2}\beta$

2. $\sin\{\alpha + \frac{1}{2}(n-1)(\pi+\alpha)\} \sin\frac{1}{2}n(\pi+\alpha) \sec\frac{1}{2}\alpha$

3. $\dfrac{\sin\{\frac{1}{2}(n+1)(\frac{1}{2}\pi+\alpha)\} \sin\{\frac{1}{2}n(\frac{1}{2}\pi+\alpha)\}}{\sin\{\frac{1}{2}.(\frac{1}{2}\pi+\alpha)\}}$.

4. $\frac{1}{2}$

5. $\frac{1}{2}n(n+1)$

6. (i) $\frac{1}{2}n - \frac{1}{2}\left[\dfrac{\cos\{2\alpha+(n-1)\beta\}\sin n\beta}{\sin\beta}\right]$

(ii) $\frac{n}{2} + \frac{1}{2}\left[\dfrac{\cos\{2\alpha+(n-1)\beta\}\sin n\beta}{\sin\beta}\right]$

Objective Type Questions

Multiple Choice Questions.

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. If $|c| < 1$, then the sum of the infinite series $ce^{i\beta} + \frac{1}{2}c^2 e^{i2\beta} + \frac{1}{3}c^3 e^{i3\beta} + \ldots\ldots \infty$ is

(a) $\log(1+ce^{i\beta})$

(b) $\log(1-ce^{i\beta})$

(c) $-\log(1-ce^{i\beta})$

(d) $-\log(1+ce^{i\beta})$.

2. If $\tan^{-1}\dfrac{1}{2n^2} = \tan^{-1}(2n+1) - \tan^{-1}y$, then the value of y is

(a) $2n-1$

(b) $n-1$

(c) $n+1$

(d) $2n$.

Fill In The Blanks

Fill in the blanks "......" so that the following statements are complete and correct.

1. In $C + iS$ method, if $C = \cos\alpha + c\cos(\alpha+\beta) + \frac{1}{2!}c^2\cos(\alpha+2\beta) + \ldots\ldots \infty$, then $S = \ldots\ldots$.
2. If $C + iS = 1 - e^{-e^{i\theta}}$, then $S = \ldots\ldots$.
3. The sum of the infinite series $e^{i\theta} + \frac{\cos\phi}{1!}e^{i(\theta+\phi)} + \frac{\cos^2\phi}{2!}e^{i(\theta+2\phi)} + \ldots\infty$ is ....
4. If $C + iS = e^{\cos\theta\, e^{i\theta}} - 1$, then $C = \ldots\ldots$.
5. If $|c| < 1$, then the sum of the infinite series $ce^{i\alpha} - \frac{1}{2}c^2e^{i2\alpha} + \frac{1}{3}c^3e^{i3\alpha} - \ldots\infty$ is ....
6. If $|z| < 1$, then the sum of the infinite series $z + \frac{z^3}{3} + \frac{z^5}{5} + \ldots\ldots\infty$ is .... .
7. If $C + iS = \tan^{-1}(c\cos\theta + ic\sin\theta)$, then $C = \ldots\ldots$.
8. $\frac{1}{\cos\theta\cos 2\theta} = \frac{1}{\sin\theta}[\tan 2\theta - \ldots\ldots]$.
9. $\operatorname{cosec}\theta = \cot\frac{\theta}{2} - \ldots\ldots$.
10. $\tan^{-1}\frac{1}{1+n(n+1)} = \tan^{-1}(n+1) - \ldots\ldots$.

True or False

Write 'T' for true and 'F' for false statement.

1. If $C + iS = e^{i\theta}[e^{\cos\phi\, e^{i\phi}}]$, then $C = e^{\cos^2\phi}.\sin(\theta + \cos\phi\sin\phi)$.
2. If $C + iS = e^{\sec\alpha\,.\,e^{i\alpha}}$, then $C = e\,.\cos(\tan\alpha)$.
3. $\sec\theta\sec(\theta+\phi) = \frac{1}{\sin\phi}[\tan(\theta+\phi) - \tan\theta]$.
4. $\tan\theta = \cot\theta - 2\cot 2\theta$.
5. $\tan^{-1}\frac{2}{(n+1)^2} = \tan^{-1}(n+2) - \tan^{-1}(n+1)$.
6. The sum of the series $\sec\theta\sec 2\theta + \sec 2\theta\sec 3\theta + \sec 3\theta\sec 4\theta + \ldots\ldots$ to n terms is
$$\frac{1}{\sin\theta}[\tan(n+1) - \tan\theta].$$

Answers

Multiple Choice Questions

1. (c) 2. (a)

Fill in the Blanks

1. $\sin\alpha + c\sin(\alpha+\beta) + \frac{1}{2!}c^2 \sin(\alpha+2\beta) + \ldots\ldots \infty$
2. $e^{-\cos\theta}\sin(\sin\theta)$
3. $e^{i\theta}\,[e^{\cos\phi\, e^{i\phi}}]$
4. $e^{\cos^2\theta}\cos(\cos\theta\sin\theta) - 1$
5. $\log(1 + ce^{i\alpha})$
6. $\frac{1}{2}\log\frac{1+z}{1-z}$
7. $\frac{1}{2}\tan^{-1}\left(\frac{2c\cos\theta}{1+c^2}\right)$
8. $\tan\theta$
9. $\cot\theta$
10. $\tan^{-1} n$

True or False

1. *F* 2. *T* 3. *T* 4. *T* 5. *F*
6. *T*

□