# INSTRUCTOR'S SOLUTIONS MANUAL

**Richard N. Aufmann**
*Palomar College*

**Vernon C. Barker**
*Palomar College*

**Richard D. Nation**
*Palomar College*

**Christine S. Verity**

# COLLEGE ALGEBRA AND TRIGONOMETRY

## SIXTH EDITION

**Aufmann/Barker/Nation**

**HOUGHTON MIFFLIN COMPANY BOSTON NEW YORK**

Editor-in-Chief/Publisher: Richard Stratton
Sponsoring Editor: Molly Taylor
Marketing Manager: Jennifer Jones
Editorial Associate: Andrew Lipsett
Marketing Associate: Mary Legere
Editorial Assistant: Anthony D'Aries



Printed in the U.S.A.

ISBN 13: 978-0-618-82508-0
ISBN 10: 0-618-82520-7

1 2 3 4 5 6 7 8 9-XX-11 10 09 08 07

# Contents


| | | |
|---|---|---|
| **Chapter P** | **Preliminary Concepts** | **1** |
| **Chapter 1** | **Equations and Inequalities** | **36** |
| **Chapter 2** | **Functions and Graphs** | **98** |
| **Chapter 3** | **Polynomial and Rational Functions** | **177** |
| **Chapter 4** | **Exponential and Logarithmic Functions** | **238** |
| **Chapter 5** | **Trigonometric Functions** | **303** |
| **Chapter 6** | **Trigonometric Identities and Equations** | **361** |
| **Chapter 7** | **Applications of Trigonometry** | **436** |
| **Chapter 8** | **Topics in Analytic Geometry** | **496** |
| **Chapter 9** | **Systems of Equations and Inequalities** | **598** |
| **Chapter 10** | **Matrices** | **691** |
| **Chapter 11** | **Sequences, Series, and Probability** | **762** |
| **Projects** | | **818** |
| **Additional College Trigonometry Solutions** | | **871** |

# Correlation Chart

# Correlation Chart

| ***College Trigonometry, 6e*** **Solutions/Projects** | ***College Algebra and Trigonometry, 6e*** **Solutions/Projects** |
|---|---|
| **Chapter 1** | |
| Section 1.1 | **See Additional *College Trigonometry* Solutions** |
| Section 1.2* | Section 2.1 |
| Section 1.3* | Section 2.2 |
| Section 1.4* | Section 2.5 |
| Section 1.5* | Section 2.6 |
| Section 1.6* | Section 4.1 |
| Section 1.7 | Section 2.7 |
| *Exploring Concepts with Technology* | Chapter 2 *Exploring Concepts with Technology* |
| *Chapter 1 Assessing Concepts*<br>*Chapter 1 Review Exercises*<br>*Chapter 1 Quantitative Reasoning*<br>*Chapter 1 Test* | **See Additional *College Trigonometry* Solutions** |
| **Chapter 2** | |
| Chapter 2 up through *Chapter 2 Test* | Chapter 5 up through *Chapter 5 Test* |
| *Cumulative Review Exercises* | **See Additional *College Trigonometry* Solutions** |
| **Chapter 3** | |
| Chapter 3 up through *Chapter 3 Test* | Chapter 6 up through *Chapter 6 Test* |
| *Cumulative Review Exercises* | **See Additional *College Trigonometry* Solutions** |
| **Chapter 4** | |
| Section 4.1 | Section 7.1 |
| Section 4.2 | Section 7.2 |
| Section 4.3 | Section 7.3 |
| *Exploring Concepts with Technology* | Chapter 7 *Exploring Concepts with Technology* |
| *Chapter 4 Review Exercises* | Exercises 1-45 *Chapter 7 Review Exercises* |
| *Chapter 4 Assessing Concepts*<br>*Chapter 4 Test*<br>*Cumulative Review Exercises* | **See Additional *College Trigonometry* Solutions** |
| **Chapter 5** | |
| Section 5.1 Exercises 1-62 | Section P.6 Exercises 1-62 |
| Section 5.1* Exercises 63-72 | **See Additional *College Trigonometry* Solutions** |
| Section 5.2 | Section 7.4 |
| Section 5.3 | Section 7.5 |
| *Exploring Concepts with Technology*<br>*Chapter 5 Assessing Concepts*<br>*Chapter 5 Review Exercises*<br>*Chapter 5 Quantitative Reasoning*<br>*Chapter 5 Test*<br>*Cumulative Review Exercises* | **See Additional *College Trigonometry* Solutions** |
| **Chapter 6** | |
| Sections 6.1-6.6 | Chapters 8.1-8.6 |
| Section 6.7, Exercises 1-18 | Section 8.7, Exercises 1-18 |
| Section 6.7, Exercises 19-40<br>*Chapter 6 Connecting Concepts*<br>*Chapter 6 Review Exercises*<br>*Chapter 6 Test*<br>*Cumulative Review Exercises* | **See Additional *College Trigonometry* Solutions** |
| **Chapter 7** | |
| Chapter 7 up through *Chapter 7 Test* | Section 4.2-4.7 up through *Chapter 4 Test* |
| *Cumulative Review Exercises* | **See Additional *College Trigonometry* Solutions** |

* See the Additional *College Trigonometry* Solutions for solutions to the *Prepare for Section* exercises.

# Solutions

# Chapter P
# Preliminary Concepts

## Section P.1

**1.** $-\frac{1}{5}$: rational, real; 0: integer, rational, real; –44: integer, rational, real; $\pi$: irrational, real; 3.14: rational, real; 5.05005000500005…: irrational, real; $\sqrt{81} = 9$: integer, rational, prime, real; 53: integer, rational, prime, real

**2.** $\frac{5}{\sqrt{7}}$: irrational, real; $\frac{5}{7}$: rational, real; 31: integer, rational, prime, real; $-2\frac{1}{2}$: rational, real; 4.235653907493: rational, real; 51: integer, rational, real; 0.888… = $0.\bar{8} = \frac{8}{9}$: rational, real

**3.** Let $x$ = 1, 2, 3, 4. Then $\{2x \mid x \text{ is a positive integer}\} = \{2, 4, 6, 8\}$

**4.** Let $x$ = 0, 1, 2, 3. (We could have used $x$ = –3, –2, –1, 0.) Then $\{|x| \mid x \text{ is an integer}\} = \{0, 1, 2, 3\}$

**5.** Let $x$ = 1, 2, 3, 4. (Recall 0 is not a natural number.) Then $\{y \mid y = 2x + 1, x \text{ is a natural number}\} = \{3, 5, 7, 9\}$

**6.** Let $x$ = 0, 1, 2, 3. (We could have used $x$ = –3, –2, –1, 0.) Then $\{y \mid y = x^2 - 1\} = \{-1, 0, 3, 8\}$

**7.** Let $x$ = 0, 1, 2, 3. (We could have used $x$ = –3, –2, –1, 0.) Then $\{z \mid z = |x|, x \text{ is an integer}\} = \{0, 1, 2, 3\}$

**8.** Let $x$ = –1, –2, –3, –4. Then $\{z \mid z = |x| - x, x \text{ is a negative integer}\} = \{2, 4, 6, 8\}$

**9.** $A \cup B = \{-3, -2, -1, 0, 1, 2, 3, 4, 6\}$

**10.** $C \cup D = \{-3, -1, 0, 1, 2, 3, 4, 5, 6\}$

**11.** $A \cap C = \{0, 1, 2, 3\}$

**12.** $C \cap D = \{1, 3\}$

**13.** $B \cap D = \varnothing$

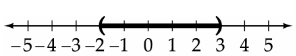
**14.** $(A \cap C) = \{0, 1, 2, 3\}$
$B \cup (A \cap C) = \{-2, 0, 1, 2, 3, 4, 6\}$

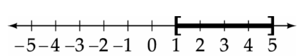
**15.** $(B \cup C) = \{-2, 0, 1, 2, 3, 4, 5, 6\}$
$D \cap (B \cup C) = \{1, 3\}$

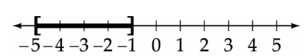
**16.** $(A \cap B) \cup (A \cap C) = \{-2, 0, 2\} \cup \{0, 1, 2, 3\} = \{-2, 0, 1, 2, 3\}$

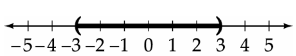
**17.** $(B \cup C) \cap (B \cup D) = \{-2, 0, 1, 2, 3, 4, 5, 6\} \cap \{-3, -2, -1, 0, 1, 2, 3, 4, 6\} = \{-2, 0, 1, 2, 3, 4, 6\}$

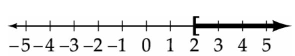
**18.** $(A \cap C) \cup (B \cap D) = \{0, 1, 2, 3\} \cup \varnothing = \{0, 1, 2, 3\}$

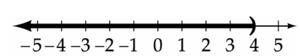
**19.** 
$\{x \mid -2 < x < 3\}$

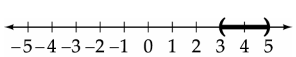
**20.** 
$\{x \mid 1 \le x \le 5\}$

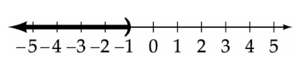
**21.** 
$\{x \mid -5 \le x \le -1\}$

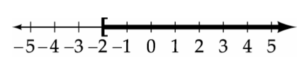
**22.** 
$\{x \mid -3 < x < 3\}$

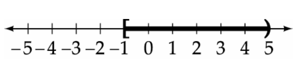
**23.** 
$\{x \mid x \ge 2\}$

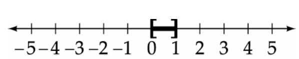
**24.** 
$\{x \mid x < 4\}$

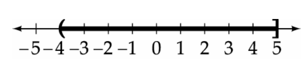
**25.** 
$(3, 5)$

**26.** 
$(-\infty, -1)$

**27.** 
$[-2, \infty)$

**28.** 
$[-1, 5)$

**29.** 
$[0, 1]$

**30.** 
$(-4, 5]$

**31.** –5

**32.** $-(4)^2 = -16$

**33.** 3(4) = 12

**34.** $|-3| - |-7| = 3 - 7 = -4$

**35.** $\pi^2 + 10$

**36.** $10 - \pi^2$

**37.** $|x-4|+|x+5| = 4-x+x+5 = 9$

**38.** $|x+6|+|x-2| = x+6+x-2 = 2x+4$

**39.** $|2x|-|x-1| = 2x-(1-x)$
$= 2x-1+x$
$= 3x-1$

**40.** $|x+1|+|x-3| = (x+1)+(x-3) = 2x-2$

**41.** $|x-3|$

**42.** $|a--2| = |a+2|$

**43.** $|x--2| = 4$
$|x+2| = 4$

**44.** $|z-5| = 1$

**45.** $|m-n|$

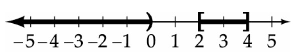
**46.** $|p-8|$

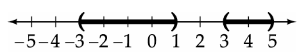
**47.** $|a-4| < 5$

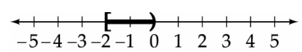
**48.** $|z-5| > 7$

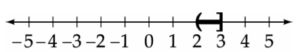
**49.** $|x+2| > 4$

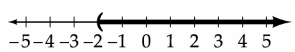
**50.** $|y+3| > 6$

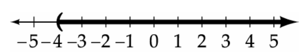
**51.** 

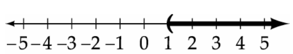
**52.** 

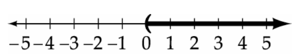
**53.** 

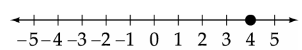
**54.** 

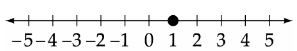
**55.** 

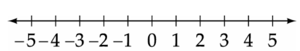
**56.** 

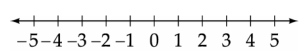
**57.** 

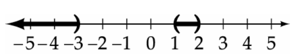
**58.** 

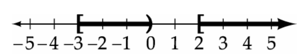
**59.** 

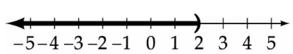
**60.** 

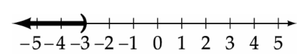
**61.** 

**62.** 

**63.** 

**64.** 

**65.** 

**66.** 

**67.** $-(-2)^3 = -(-8) = 8$

**68.** $-(-2)^2 = -(4) = -4$

**69.** $2(3)(-2)(-1) = 12$

**70.** $-3(3)(-1) = 9$

**71.** $-2(3)^2(-2)^2 = -2(9)(4) = -72$

**72.** $2(-2)^3(-1)^2 = 2(-8)(1) = -16$

**73.** $3(-2)-(-1)[3-(-2)]^2 = 3(-2)-(-1)[3+2]^2 = (3)(-2)-(-1)[5]^2 = (3)(-2)-(-1)(25) = -6+25 = 19$

**74.** $[-1-2(-2)]^2 - 3(-1)^3 = (-1+4)^2 - 3(-1)^3 = (3)^2 - 3(-1) = 9+3 = 12$

**75.** $\dfrac{3^2+(-2)^2}{3+(-2)} = \dfrac{9+4}{1} = \dfrac{13}{1} = 13$

**76.** $\dfrac{2(3)(-2)^2(-1)^4}{[-2-(-1)]^4} = \dfrac{2(3)(4)(1)}{(-2+1)^4} = \dfrac{24}{(-1)^4} = 24$

**77.** $\dfrac{3(-2)}{3} - \dfrac{2(-1)}{-2} = \dfrac{-6}{3} - \dfrac{-2}{-2} = -2-1 = -3$

**78.** $(3+1)^2(3-1)^2 = 4^2 \cdot 2^2 = 16 \cdot 4 = 64$

**79.** $(ab^2)c = a(b^2c)$
Associative property of multiplication

**80.** $2x-3y = -3y+2x$
Commutative property of addition

**81.** $4(2a-b) = 8a-4b$
Distributive property

**82.** $6+(7+a) = 6+(a+7)$
Commutative property of addition

**83.** $(3x)y = y(3x)$
Commutative property of multiplication

**84.** $4ab+0 = 4ab$
Identity property of addition

**85.** $1 \cdot (4x) = 4x$
Identity property of multiplication

**86.** $7(a + b) = 7(b + a)$
Commutative property of addition

**87.** $x^2 + 1 = x^2 + 1$
Reflexive property of equality

**88.** If $a + b = 2$, then $2 = a + b$
Symmetric property of equality

**89.** If $2x + 1 = y$ and $3x - 2 = y$, then $2x + 1 = 3x - 2$
Transitive property of equality

**90.** $4x + 2y = 7$ and $x = 3$, then $4(3) = 2y = 7$
Substitution property of equality

**91.** $4 \cdot \frac{1}{4} = 1$
Inverse property of multiplication

**92.** $ab + (-ab) = 0$
Inverse property of addition

**93.** $3(2x)$
$6x$

**94.** $-2(4y)$
$-8y$

**95.** $3(2 + x)$
$3x + 6$

**96.** $-2(4 + y)$
$-2y - 8$

**97.** $\frac{2}{3}a + \frac{5}{6}a$
$\frac{4}{6}a + \frac{5}{6}a$
$\frac{9}{6}a$
$\frac{3}{2}a$

**98.** $\frac{3}{4}x - \frac{1}{2}x$
$\frac{3}{4}x - \frac{2}{4}x$
$\frac{1}{4}x$

**99.** $2 + 3(2x - 5)$
$2 + 6x - 15$
$6x - 13$

**100.** $4 + 2(2a - 3)$
$4 + 4a - 6$
$4a - 2$

**101.** $5 - 3(4x - 2y)$
$5 - 12x + 6y$
$-12x + 6y + 5$

**102.** $7 - 2(5n - 8m)$
$7 - 10n + 16m$
$16m - 10n + 7$

**103.** $3(2a - 4b) - 4(a - 3b)$
$6a - 12b - 4a + 12b$
$6a - 4a - 12b + 12b$
$2a$

**104.** $5(4r - 7t) - 2(10r + 3t)$
$20r - 35t - 20r - 6t$
$20r - 20r - 35t - 6t$
$-41t$

**105.** $5a - 2[3 - 2(4a + 3)]$
$5a - 2(3 - 8a - 6)$
$5a - 2(-8a - 3)$
$5a + 16a + 6$
$21a + 6$

**106.** $6 + 3[2x - 4(3x - 2)]$
$6 + 3(2x - 12x + 8)$
$6 + 3(-10x + 8)$
$6 - 30x + 24$
$30x + 30$

**107.** $\text{Area} = \frac{1}{2}bh = \frac{1}{2}(3 \text{ in})(4 \text{ in}) = 6 \text{ in}^2$

**108.** $V = lwh = (40 \text{ ft})(30 \text{ ft})(12 \text{ ft}) = 14{,}400 \text{ ft}^3$

**109.** $\text{Profit} = = -0.5x^2 + 120x - 2000$
$= -0.5(110)^2 + 120(110) - 2000$
$= -0.5(110)^2 + 120(110) - 2000$
$= -0.5(12100) + 120(110) - 2000$
$= -6050 + 13200 - 2000$
$= 5150$
The profit for selling 110 bicycles is \$5150.

**110.** $\text{Circulation} = \sqrt{n^2 - n + 1}$
$= \sqrt{12^2 - 12 + 1}$
$\sqrt{144 - 12 + 1}$
$= \sqrt{133} \approx 11.5$
The circulation of the magazine after 12 months is approximately 11.5 thousand or 11,500 subscriptions.

**111.** $\text{Heart rate} = 65 + \frac{53}{4t + 1}$
$= 65 + \frac{53}{4(10) + 1}$
$= 65 + \frac{53}{41}$
$\approx 66$
Heart rate is about 66 beats per minute.

**112.** $\text{BMI} = \frac{705w}{h^2}$
$= \frac{705(160)}{(70)^2} = \frac{112800}{4900} \approx 23$
The body mass index (BMI) of a person who weighs 160 pounds and is 5 feet 10 inches (70 inches) tall is about 23.

**113.** $\text{Height} = -16t^2 + 80t + 4$
$= -16(2)^2 + 80(2) + 4$
$= -16(4) + 80(2) + 4$
$= -64 + 160 + 4$
$= 100$
After 2 seconds, the ball will have a height of 100 feet.

**114.** $\text{Concentration} = \frac{50t}{t+1}$
$= \frac{50(24)}{24+1}$
$= \frac{1200}{25} = 48$
After 24 minutes, the concentration will be 48 grams per liter.

## Connecting Concepts

**115.** For any set $A$, $A \cup A = A$.

**116.** For any set $A$, $A \cap A = A$.

**117.** For any set $A$, $A \cap \varnothing = \varnothing$.

**118.** For any set $A$, $A \cup \varnothing = A$.

**119.** If $A$ and $B$ are two sets and a $A \cup B = A$, then all elements of $B$ are contained in $A$. So $B$ is a subset of $A$.

**120.** If $A$ and $B$ are two sets and a $A \cap B = B$, then all elements of $B$ are contained in $A$. So $B$ is a subset of $A$.

**121.** No.
$(8 \div 4) \div 2 = 2 \div 2 = 1$
$8 \div (4 \div 2) = 8 \div 2 = 4$

**122.** No.
$5 - 3 = 2$
$3 - 5 = -2$

**123.** All but the multiplicative inverse property

**124.** All

**125.** $$\left|\frac{x+7}{|x|+|x-1|}\right| = \frac{|x+7|}{\left||x|+|x-1|\right|} = \frac{x+7}{|x-(x-1)|} = \frac{x+7}{|1|} = x+7$$

**126.** $$\left|\frac{x+3}{\left|x-\frac{1}{2}\right|+\left|x+\frac{1}{2}\right|}\right| = \frac{|x+3|}{\left|\left|x-\frac{1}{2}\right|+\left|x+\frac{1}{2}\right|\right|} = \frac{x+3}{\left|-\left(x-\frac{1}{2}\right)+\left(x+\frac{1}{2}\right)\right|} = \frac{x+3}{\left|\frac{1}{2}+\frac{1}{2}\right|} = x+3$$

**127.** $|x-2| < |x-6|$

**128.** $|x-a| < |x-b|$

**129.** $|x-3| > |x+7|$

**130.** $|x| > |x-5|$

**131.** $2 < |x-4| < 7$

**132.** $b < |x-a| < c$

## Prepare for Section P.2

**PS1.** $2^2 \cdot 2^3 = 4 \cdot 8 = 32$
Alternate method: $2^2 \cdot 2^3 = 2^{2+3} = 2^5 = 32$

**PS2.** $\frac{4^3}{4^5} = 4^{3-5} = 4^{-2} = \frac{1}{4^2} = \frac{1}{16}$
Alternate method: $\frac{4^3}{4^5} = \frac{1}{4^{5-3}} = \frac{1}{4^2} = \frac{1}{16}$

**PS3.** $(2^3)^2 = 8^2 = 64$
Alternate method: $(2^3)^2 = 2^{3(2)} = 2^6 = 64$

**PS4.** $3.14(10^5) = 3.14(100,000) = 314,000$
Alternate method: To multiply by $10^5$, move the decimal point 5 places to the right.
Thus, $3.14(10^5) = 3\underset{\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow}{14000}. = 314,000$

**PS5.** False
$3^4 \cdot 3^2 = 3^6$, not $9^6$.

**PS6.** False
$(3+4)^2 = 7^2 = 49$ but $3^2 + 4^2 = 9 + 16 = 25$.

## Section P.2

**1.** $-5^3 = -(5^3) = -125$

**2.** $(-5)^3 = -125$

**3.** $\left(\frac{2}{3}\right)^0 = 1$

**4.** $-6^0 = -(6^0) = -1$

**5.** $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$

**6.** $3^{-4} = \frac{1}{3^4} = \frac{1}{81}$

**7.** $\frac{1}{2^{-5}} = 2^5 = 32$

**8.** $\frac{1}{3^{-3}} = 3^3 = 27$

**9.** $\frac{2^{-3}}{6^{-3}} = \left(\frac{2}{6}\right)^{-3} = \left(\frac{1}{3}\right)^{-3} = \left(\frac{3}{1}\right)^3 = 3^3 = 27$

**10.** $\frac{4^{-2}}{2^{-3}} = \frac{2^3}{4^2} = \frac{8}{16} = \frac{1}{2}$

**11.** $-2x^0 = -2$

**12.** $\frac{x^0}{4} = \frac{1}{4}$

**13.** $2x^{-4} = 2\left(x^{-4}\right) = \frac{2}{x^4}$

**14.** $3y^{-2} = 3\left(y^{-2}\right) = \frac{3}{y^2}$

**15.** $\frac{5}{z^{-6}} = 5z^6$

**16.** $\frac{8}{x^{-5}} = 8z^5$

**17.** $(x^3y^2)(xy^5) = x^{3+1}y^{2+5} = x^4y^7$

**18.** $(uv^6)(u^2v) = u^{1+2}v^{6+1} = u^3v^7$

**19.** $(-2ab^4)(-3a^2b^5) = (-2)(-3)a^{1+2}b^{4+5} = 6a^3b^9$

**20.** $(9xy^2)(-2x^2y^5) = (9)(-2)x^{1+2}y^{2+5} = -18x^3y^7$

**21.** $\frac{16a^7}{2a} = \frac{16}{2}a^{7-1} = 8a^6$

**22.** $\frac{24z^8}{-3z^3} = \frac{24}{-3}z^{8-3} = -8z^5$

**23.** $\frac{6a^4}{8a^8} = \frac{6}{8}a^{4-8} = \frac{3}{4}a^{-4} = \frac{3}{4a^4}$

**24.** $\frac{12x^3}{16x^4} = \frac{12}{16}x^{3-4} = \frac{3}{4}x^{-1} = \frac{3}{4x}$

**25.** $\frac{12x^3y^4}{18x^5y^2} = \frac{12}{18}x^{3-5}y^{4-2} = \frac{2}{3}x^{-2}y^2 = \frac{2y^2}{3x^2}$

**26.** $\frac{5v^4w^{-3}}{10v^8} = \frac{5}{10}v^{4-8}w^{-3} = \frac{1}{2}v^{-4}w^{-3} = \frac{1}{2v^4w^3}$

**27.** $\frac{36a^{-2}b^3}{3ab^4} = \frac{36}{3}a^{-2-1}b^{3-4} = 12a^{-3}b^{-1} = \frac{12}{a^3b}$

**28.** $\frac{-48ab^{10}}{-32a^4b^3} = \frac{-48}{-32}a^{1-4}b^{10-3} = \frac{3}{2}a^{-3}b^7 = \frac{3b^7}{2a^3}$

**29.**
$$\begin{aligned}(-2m^3n^2)(-3mn^2)^2 &= (-2m^3n^2)(9m^2n^4)\\ &= (-2)(9)m^{3+2}n^{2+4}\\ &= -18m^5n^6\end{aligned}$$

**30.**
$$\begin{aligned}(2a^3b^2)^3(-4a^4b^2) &= (8a^9b^6)(-4a^4b^2)\\ &= (8)(-4)a^{9+4}b^{6+2}\\ &= -32a^{13}b^8\end{aligned}$$

**31.**
$$\begin{aligned}(x^{-2}y)^2(xy)^{-2} &= (x^{-4}y^2)(x^{-2}y^{-2})\\ &= x^{-4-2}y^{2-2}\\ &= x^{-6}y^0\\ &= \frac{1}{x^6}\end{aligned}$$

**32.**
$$\begin{aligned}(x^{-1}y^2)^{-3}(x^2y^{-4})^{-3} &= (x^3y^{-6})(x^{-6}y^{12})\\ &= x^{3-6}y^{-6+12}\\ &= x^{-3}y^6\\ &= \frac{y^6}{x^3}\end{aligned}$$

**33.** $$\left(\frac{3a^2b^3}{6a^4b^4}\right)^2 = \frac{9a^4b^6}{36a^8b^8} = \frac{9}{36}a^{4-8}b^{6-8} = \frac{1}{4}a^{-4}b^{-2} = \frac{1}{4a^4b^2}$$

**34.** $$\left(\frac{2ab^2c^3}{5ab^2}\right)^3 = \frac{8a^3b^6c^9}{125a^3b^6} = \frac{8}{125}a^{3-3}b^{6-6}c^9 = \frac{8}{125}a^0b^0c^9 = \frac{8c^9}{125}$$

**35.** $$\frac{(-4x^2y^3)^2}{(2xy^2)^3} = \frac{16x^4y^6}{8x^3y^6} = 2x^{4-3}y^{6-6} = 2x$$

**36.** $$\frac{(-3a^2b^3)^2}{(-2ab^4)^3} = \frac{9a^4b^6}{-8a^3b^{12}} = -\frac{9}{8}a^{4-3}b^{6-12} = -\frac{9}{8}ab^{-6} = -\frac{9a}{8b^6}$$

**37.** $$\left(\frac{a^{-2}b}{a^3b^{-4}}\right)^2 = \frac{a^{-4}b^2}{a^6b^{-8}} = a^{-4-6}b^{2-(-8)} = a^{-4-6}b^{2+8} = a^{-10}b^{10} = \frac{b^{10}}{a^{10}}$$

**38.** $$\left(\frac{x^{-3}y^{-4}}{x^{-2}y}\right)^{-2} = \frac{x^6y^8}{x^4y^{-2}} = x^{6-4}y^{8-(-2)} = x^{6-4}y^{8+2} = x^2y^{10}$$

**39.** $2,011,000,000,000 = 2.011\times10^{12}$

**40.** $49,100,000,000 = 4.91\times10^{10}$

**41.** $0.000000000562 = 5.62\times10^{-10}$

**42.** $0.000000402 = 4.02\times10^{-7}$

**43.** $3.14\times10^7 = 31,400,000$

**44.** $4.03\times10^9 = 4,030,000,000$

**45.** $-2.3\times10^{-6} = -0.0000023$

**46.** $6.14\times10^{-8} = 0.0000000614$

**47.** $$(3\times10^{12})(9\times10^{-5}) = (3)(9)\times10^{12-5} = 27\times10^7 = 2.7\times10^8$$

**48.** $$(8.9\times10^{-5})(3.4\times10^{-6}) = (8.9)(3.4)\times10^{-5-6} = 30.26\times10^{-11} = 3.026\times10^{-10}$$

**49.** $$\frac{9\times10^{-3}}{6\times10^8} = \frac{9}{6}\times10^{-3-8} = 1.5\times10^{-11}$$

**50.** $$\frac{2.5\times10^8}{5\times10^{10}} = \frac{2.5}{5}\times10^{8-10} = 0.5\times10^{-2} = 5\times10^{-3}$$

**51.** $$\frac{(3.2\times10^{-11})(2.7\times10^{18})}{1.2\times10^{-5}} = \frac{(3.2)(2.7)}{1.2}\times10^{-11+18-(-5)} = 7.2\times10^{-11+18+5} = 7.2\times10^{12}$$

**52.** $$\frac{(6.9\times10^{27})(8.2\times10^{-13})}{4.1\times10^{15}} = \frac{(6.9)(8.2)}{4.1}\times10^{27-13-15} = 13.8\times10^{-1} = 1.38\times10^0$$

**53.** $\dfrac{(4.0\times10^{-9})(8.4\times10^{5})}{(3.0\times10^{-6})(1.4\times10^{18})} = \dfrac{(4.0)(8.4)}{(3.0)(1.4)}\times10^{-9+5+6-18}$
$= 8\times10^{-16}$

**54.** $\dfrac{(7.2\times10^{8})(3.9\times10^{-7})}{(2.6\times10^{-10})(1.8\times10^{-8})} = \dfrac{(7.2)(3.9)}{(2.6)(1.8)}\times10^{8-7-(-10)-(-8)}$
$= 6\times10^{8-7+10+8}$
$= 6\times10^{19}$

**55.** $4^{3/2} = \sqrt{4}^{3} = 2^3 = 8$

**56.** $-16^{3/2} = -\sqrt{16}^{3} = -4^3 = -64$

**57.** $-64^{2/3} = -\sqrt[3]{64}^{2} = -4^2 = -16$

**58.** $125^{4/3} = \sqrt[3]{125}^{4} = 5^4 = 625$

**59.** $9^{-3/2} = \dfrac{1}{9^{3/2}} = \dfrac{1}{\left(\sqrt{9}\right)^3} = \dfrac{1}{3^3} = \dfrac{1}{27}$

**60.** $32^{-3/5} = \dfrac{1}{32^{3/5}} = \dfrac{1}{\left(\sqrt[5]{32}\right)^3} = \dfrac{1}{2^3} = \dfrac{1}{8}$

**61.** $\left(\dfrac{4}{9}\right)^{1/2} = \sqrt{\dfrac{4}{9}} = \dfrac{\sqrt{4}}{\sqrt{9}} = \dfrac{2}{3}$

**62.** $\left(\dfrac{16}{25}\right)^{3/2} = \left(\sqrt{\dfrac{16}{25}}\right)^3 = \left(\dfrac{4}{5}\right)^3 = \dfrac{4^3}{5^3} = \dfrac{64}{125}$

**63.** $\left(\dfrac{1}{8}\right)^{-4/3} = 8^{4/3} = \sqrt[3]{8}^{4} = 2^4 = 16$

**64.** $\left(\dfrac{8}{27}\right)^{-2/3} = \left(\dfrac{27}{8}\right)^{2/3} = \left(\sqrt[3]{\dfrac{27}{8}}\right)^2 = \left(\dfrac{\sqrt[3]{27}}{\sqrt[3]{8}}\right)^2 = \left(\dfrac{3}{2}\right)^2 = \dfrac{9}{4}$

**65.** $(4a^{2/3}b^{1/2})(2a^{1/3}b^{3/2}) = (4)(2)a^{2/3+1/3}b^{1/2+3/2} = 8a^{3/3}b^{4/2} = 8ab^2$

**66.** $(6a^{3/5}b^{1/4})(-3a^{1/5}b^{3/4}) = (6)(-3)a^{3/5+1/5}b^{1/4+3/4} = -18a^{4/5}b$

**67.** $(-3x^{2/3})(4x^{1/4}) = (-3)(4)x^{2/3+1/4} = -12x^{8/12+3/12} = -12x^{11/12}$

**68.** $(-5x^{1/3})(-4x^{1/2}) = (-5)(-4)x^{1/3+1/2} = 20x^{2/6+3/6} = 20x^{5/6}$

**69.** $(81x^8y^{12})^{1/4} = 81^{1/4}x^{8/4}y^{12/4} = \sqrt[4]{81}x^2y^3 = 3x^2y^3$

**70.** $(27x^3y^6)^{2/3} = 27^{2/3}x^{6/3}y^{12/3} = \sqrt[3]{27}^{2}x^2y^4 = 3^2x^2y^4 = 9x^2y^4$

**71.** $\dfrac{16z^{3/5}}{12z^{1/5}} = \dfrac{16z^{3/5-1/5}}{12} = \dfrac{4z^{2/5}}{3}$

**72.** $\dfrac{6a^{2/3}}{9a^{1/3}} = \dfrac{6a^{2/3-1/3}}{9} = \dfrac{2a^{1/3}}{3} = \dfrac{2a^{1/3}}{3}$

**73.** $(2x^{2/3}y^{1/2})(3x^{1/6}y^{1/3}) = (2)(3)x^{2/3+1/6}y^{1/2+1/3} = 6x^{5/6}y^{5/6}$

**74.** $\dfrac{x^{1/3}y^{5/6}}{x^{2/3}y^{1/6}} = x^{1/3-2/3}y^{5/6-1/6} = x^{-1/3}y^{4/6} = \dfrac{y^{2/3}}{x^{1/3}}$

**75.** $\dfrac{9a^{3/4}b}{3a^{2/3}b^2} = \dfrac{9a^{3/4-2/3}b^{1-2}}{3} = 3a^{9/12-8/12}b^{-1} = \dfrac{3a^{1/12}}{b}$

**76.** $\dfrac{12x^{1/6}y^{1/4}}{16x^{3/4}y^{1/2}} = \dfrac{12x^{1/6-3/4}y^{1/4-1/2}}{16} = \dfrac{3x^{2/12-9/12}y^{1/4-2/4}}{4} = \dfrac{3x^{-7/12}y^{-1/4}}{4} = \dfrac{3}{4x^{7/12}y^{1/4}}$

**77.** $\sqrt{45} = \sqrt{3^2\cdot5} = 3\sqrt{5}$

**78.** $\sqrt{75} = \sqrt{5^2\cdot3} = 5\sqrt{3}$

**79.** $\sqrt[3]{24} = \sqrt[3]{2^3\cdot3} = 2\sqrt[3]{3}$

**80.** $\sqrt[3]{135} = \sqrt[3]{3^3\cdot5} = 3\sqrt[3]{5}$

**81.** $\sqrt[3]{-135} = \sqrt[3]{(-3)^3\cdot5}$
$= -3\sqrt[3]{5}$

**82.** $\sqrt[3]{-250} = \sqrt[3]{(-5)^3\cdot2}$
$= -5\sqrt[3]{2}$

**83.** $\sqrt{24x^2y^3} = \sqrt{2^2x^2y^2}\cdot\sqrt{6y} = 2|xy|\sqrt{6y}$

**84.** $\sqrt{18x^2y^5} = \sqrt{3^2x^2y^4}\cdot\sqrt{2y} = 3|x|y^2\sqrt{2y}$

**85.** $\sqrt[3]{16a^3y^7} = \sqrt[3]{2^3a^3y^6}\cdot\sqrt[3]{2y} = 2ay^2\sqrt[3]{2y}$

**86.** $\sqrt[3]{54m^2n^7} = \sqrt[3]{3^3n^6}\cdot\sqrt[3]{2m^2n} = 3n^2\sqrt[3]{2m^2n}$

**87.** $2\sqrt{32} - 3\sqrt{98} = 2\sqrt{16}\cdot\sqrt{2} - 3\sqrt{49}\cdot\sqrt{2} = 2(4)\sqrt{2} - 3(7)\sqrt{2} = 8\sqrt{2} - 21\sqrt{2} = -13\sqrt{2}$

**88.** $5\sqrt[3]{32} + 2\sqrt[3]{108} = 5\sqrt[3]{8\cdot4} + 2\sqrt[3]{27\cdot4} = 5\sqrt[3]{8}\cdot\sqrt[3]{4} + 2\sqrt[3]{27}\cdot\sqrt[3]{4} = 5\sqrt[3]{2^3}\cdot\sqrt[3]{4} + 2\sqrt[3]{3^3}\cdot\sqrt[3]{4} = 5(2)\sqrt[3]{4} + 2(3)\sqrt[3]{4}$
$= 10\sqrt[3]{4} + 6\sqrt[3]{4} = 16\sqrt[3]{4}$

**89.** $-8\sqrt[4]{48} + 2\sqrt[4]{243} = -8\sqrt[4]{16\cdot3} + 2\sqrt[4]{81\cdot3} = -8\sqrt[4]{16}\cdot\sqrt[4]{3} + 2\sqrt[4]{81}\cdot\sqrt[4]{3} = -8\sqrt[4]{2^4}\cdot\sqrt[4]{3} + 2\sqrt[4]{3^4}\cdot\sqrt[4]{3}$
$= -8(2)\sqrt[4]{3} + 2(3)\sqrt[4]{3} = -16\sqrt[4]{3} + 6\sqrt[4]{3} = -10\sqrt[4]{3}$

**90.** $2\sqrt[3]{40} - 3\sqrt[3]{135} = 2\sqrt[3]{8\cdot5} - 3\sqrt[3]{27\cdot5} = 2\sqrt[3]{8}\cdot\sqrt[3]{5} - 3\sqrt[3]{27}\cdot\sqrt[3]{5} = 2\sqrt[3]{2^3}\cdot\sqrt[3]{5} - 3\sqrt[3]{3^3}\cdot\sqrt[3]{5} = 2\sqrt[3]{2^3}\cdot\sqrt[3]{5} - 3\sqrt[3]{3^3}\cdot\sqrt[3]{5}$
$= 2(2)\sqrt[3]{5} - 3(3)\sqrt[3]{5} = 4\sqrt[3]{5} - 9\sqrt[3]{5} = -5\sqrt[3]{5}$

**91.** $4\sqrt[3]{32y^4} + 3y\sqrt[3]{108y} = 4\sqrt[3]{8y^3\cdot4y} + 3y\sqrt[3]{27\cdot4y} = 4\sqrt[3]{8y^3}\cdot\sqrt[3]{4y} + 3y\sqrt[3]{27}\cdot\sqrt[3]{4y}$
$= 4\sqrt[3]{2^3y^3}\cdot\sqrt[3]{4y} + 3y\sqrt[3]{3^3}\cdot\sqrt[3]{4y} = 4(2y)\sqrt[3]{4y} + 3y(3)\sqrt[3]{4y} = 8y\sqrt[3]{4y} + 9y\sqrt[3]{4y} = 17y\sqrt[3]{4y}$

**92.** $-3x\sqrt[3]{54x^4} + 2\sqrt[3]{16x^7} = -3x\sqrt[3]{27x^3\cdot2x} + 2\sqrt[3]{8x^6\cdot2x} = -3x\sqrt[3]{27x^3}\cdot\sqrt[3]{2x} + 2\sqrt[3]{8x^6}\cdot\sqrt[3]{2x}$
$= -3x\sqrt[3]{3^3x^3}\cdot\sqrt[3]{2x} + 2\sqrt[3]{2^3x^6}\cdot\sqrt[3]{2x} = -3x(3x)\sqrt[3]{2x} + 2(2x^2)\sqrt[3]{2x}$
$= -9x^2\sqrt[3]{2x} + 4x^2\sqrt[3]{2x} = -5x^2\sqrt[3]{2x}$

**93.** $x\sqrt[3]{8x^3y^4} - 4y\sqrt[3]{64x^6y} = x\sqrt[3]{8x^3y^3\cdot y} - 4y\sqrt[3]{64x^6\cdot y} = x\sqrt[3]{8x^3y^3}\cdot\sqrt[3]{y} - 4y\sqrt[3]{64x^6}\cdot\sqrt[3]{y}$
$= x\sqrt[3]{2^3x^3y^3}\cdot\sqrt[3]{y} - 4y\sqrt[3]{4^3x^6}\cdot\sqrt[3]{y} = x(2xy)\sqrt[3]{y} - 4y(4x^2)\sqrt[3]{y}$
$= 2x^2y\sqrt[3]{y} - 16x^2y\sqrt[3]{y} = -14x^2y\sqrt[3]{y}$

**94.** $4\sqrt{a^5b} - a^2\sqrt{ab} = 4\sqrt{a^4\cdot ab} - a^2\sqrt{ab} = 4\sqrt{a^4}\cdot\sqrt{ab} - a^2\sqrt{ab} = 4a^2\sqrt{ab} - a^2\sqrt{ab} = 3a^2\sqrt{ab}$

**95.** $(\sqrt{5}+3)(\sqrt{5}+4) = \sqrt{5}^2 + 4\sqrt{5} + 3\sqrt{5} + (3)(4) = 5 + 7\sqrt{5} + 12 = 17 + 7\sqrt{5}$

**96.** $(\sqrt{7}+2)(\sqrt{7}-5) = \sqrt{7}^2 - 5\sqrt{7} + 2\sqrt{7} + (2)(-5) = 7 - 3\sqrt{7} - 10 = -3 - 3\sqrt{7}$

**97.** $(\sqrt{2}-3)(\sqrt{2}+3) = \sqrt{2}^2 + 3\sqrt{2} - 3\sqrt{2} + (-3)(3) = 2 - 9 = -7$

**98.** $(2\sqrt{7}+3)(2\sqrt{7}-3) = (2\sqrt{7})^2 - 3(2\sqrt{7}) + 3(2\sqrt{7}) + (3)(-3) = 4(7) - 6\sqrt{7} + 6\sqrt{7} - 9 = 28 - 9 = 19$

**99.** $(3\sqrt{z}-2)(4\sqrt{z}+3) = (3)(4)\sqrt{z}^2 + 3(3\sqrt{z}) - 2(4\sqrt{z}) + (-2)(3) = 12z + 9\sqrt{z} - 8\sqrt{z} - 6 = 12z + \sqrt{z} - 6$

**100.** $(4\sqrt{a}-\sqrt{b})(3\sqrt{a}+2\sqrt{b}) = (4\sqrt{a})(3\sqrt{a}) + (4\sqrt{a})\cdot(2\sqrt{b}) - \sqrt{b}(3\sqrt{a}) - \sqrt{b}(2\sqrt{b})$
$= (4)(3)\sqrt{a}^2 + (4)(2)(\sqrt{a})(\sqrt{b}) - 3(\sqrt{a})(\sqrt{b}) - 2\sqrt{b}^2$
$= 12a + 8\sqrt{ab} - 3\sqrt{ab} - 2b = 12a + 5\sqrt{ab} - 2b$

**101.** $(\sqrt{x}+2)^2 = \sqrt{x}^2 + 2(\sqrt{x})(2) + 2^2 = x + 4\sqrt{x} + 4$

**102.** $(3\sqrt{5y}-4)^2 = 3^2\sqrt{5y}^2 + 2(3\sqrt{5y})(-4) + (-4)^2 = 9(5y) - 24\sqrt{5y} + 16 = 45y - 24\sqrt{5y} + 16$

**103.** $(\sqrt{x-3}+2)^2 = \sqrt{x-3}^2 + 2(\sqrt{x-3})(2) + 2^2 = x - 3 + 4\sqrt{x-3} + 4 = x + 4\sqrt{x-3} + 1$

**104.** $(\sqrt{2x+1}-3)^2 = \sqrt{2x+1}^2 + 2(\sqrt{2x+1})(-3) + (-3)^2 = 2x + 1 - 6\sqrt{2x+1} + 9 = 2x - 6\sqrt{2x+1} + 10$

**105.** $\frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{2}^2} = \frac{2\sqrt{2}}{2} = \frac{\cancel{2}\sqrt{2}}{\cancel{2}} = \sqrt{2}$

**106.** $\frac{3x}{\sqrt{3}} = \frac{3x}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{3x\sqrt{3}}{\sqrt{3}^2} = \frac{3x\sqrt{3}}{3} = \frac{\cancel{3}x\sqrt{3}}{\cancel{3}} = x\sqrt{3}$

**107.** $\sqrt{\frac{5}{18}} = \sqrt{\frac{5}{2\cdot 3^2}} = \sqrt{\frac{5}{2\cdot 3^2}}\cdot\sqrt{\frac{2}{2}} = \sqrt{\frac{5\cdot 2}{2^2\cdot 3^2}} = \frac{\sqrt{5\cdot 2}}{\sqrt{2^2\cdot 3^2}} = \frac{\sqrt{10}}{2\cdot 3} = \frac{\sqrt{10}}{6}$

**108.** $\sqrt{\frac{7}{40}} = \sqrt{\frac{7}{2^3\cdot 5}} = \sqrt{\frac{7}{2^3\cdot 5}}\cdot\sqrt{\frac{2\cdot 5}{2\cdot 5}} = \sqrt{\frac{7\cdot 2\cdot 5}{2^4\cdot 5^2}} = \frac{\sqrt{7\cdot 2\cdot 5}}{\sqrt{2^4\cdot 5^2}} = \frac{\sqrt{7\cdot 2\cdot 5}}{2^2\cdot 5} = \frac{\sqrt{70}}{20}$

**109.** $\frac{3}{\sqrt[3]{2}} = \frac{3}{\sqrt[3]{2}}\cdot\frac{\sqrt[3]{2^2}}{\sqrt[3]{2^2}} = \frac{3\sqrt[3]{2^2}}{\sqrt[3]{2^3}} = \frac{3\sqrt[3]{4}}{2}$

**110.** $\frac{2}{\sqrt[3]{4}} = \frac{2}{\sqrt[3]{2^2}} = \frac{2}{\sqrt[3]{2^2}}\cdot\frac{\sqrt[3]{2}}{\sqrt[3]{2}} = \frac{2\sqrt[3]{2}}{\sqrt[3]{2^3}} = \frac{2\sqrt[3]{2}}{2} = \frac{\cancel{2}\sqrt[3]{2}}{\cancel{2}} = \sqrt[3]{2}$

**111.** $\frac{4}{\sqrt[3]{8x^2}} = \frac{4}{\sqrt[3]{2^3x^2}} = \frac{4}{2\sqrt[3]{x^2}} = \frac{\overset{2}{\cancel{4}}}{\cancel{2}\sqrt[3]{x^2}} = \frac{2}{\sqrt[3]{x^2}} = \frac{2}{\sqrt[3]{x^2}}\cdot\frac{\sqrt[3]{x}}{\sqrt[3]{x}} = \frac{2\sqrt[3]{x}}{\sqrt[3]{x^3}} = \frac{2\sqrt[3]{x}}{x}$

**112.** $\frac{2}{\sqrt[4]{4y}} = \frac{2}{\sqrt[4]{2^2y}} = \frac{2}{\sqrt[4]{2^2y}}\cdot\frac{\sqrt[4]{2^2y^3}}{\sqrt[4]{2^2y^3}} = \frac{2\sqrt[4]{2^2y^3}}{\sqrt[4]{2^4y^4}} = \frac{2\sqrt[4]{4y^3}}{2y} = \frac{\cancel{2}\sqrt[4]{4y^3}}{\cancel{2}y} = \frac{\sqrt[4]{4y^3}}{y}$

**113.** $\frac{3}{\sqrt{3}+4} = \frac{3}{\sqrt{3}+4}\cdot\frac{\sqrt{3}-4}{\sqrt{3}-4} = \frac{3(\sqrt{3}-4)}{(\sqrt{3}+4)(\sqrt{3}-4)} = \frac{3(\sqrt{3}-4)}{\sqrt{3}^2-4^2} = \frac{3(\sqrt{3}-4)}{3-16} = \frac{3\sqrt{3}-12}{-13} = -\frac{3\sqrt{3}-12}{13}$

**114.** $\frac{2}{\sqrt{5}-2} = \frac{2}{\sqrt{5}-2}\cdot\frac{\sqrt{5}+2}{\sqrt{5}+2} = \frac{2(\sqrt{5}+2)}{(\sqrt{5}-2)(\sqrt{5}+2)} = \frac{2(\sqrt{5}+2)}{\sqrt{5}^2-2^2} = \frac{2(\sqrt{5}+2)}{5-4} = \frac{2\sqrt{5}+4}{1} = 2\sqrt{5}+4$

**115.** $\frac{6}{2\sqrt{5}+2} = \frac{6}{2(\sqrt{5}+1)} = \frac{\overset{3}{\cancel{6}}}{\cancel{2}(\sqrt{5}+1)} = \frac{3}{\sqrt{5}+1} = \frac{3}{\sqrt{5}+1}\cdot\frac{\sqrt{5}-1}{\sqrt{5}-1} = \frac{3(\sqrt{5}-1)}{(\sqrt{5}+1)(\sqrt{5}-1)} = \frac{3(\sqrt{5}-1)}{\sqrt{5}^2-1}$

$= \frac{3(\sqrt{5}-1)}{5-1} = \frac{3\sqrt{5}-3}{4}$

**116.** $\frac{-7}{3\sqrt{2}-5} = \frac{-7}{3\sqrt{2}-5}\cdot\frac{3\sqrt{2}+5}{3\sqrt{2}+5} = \frac{-7(3\sqrt{2}+5)}{(3\sqrt{2}-5)(3\sqrt{2}+5)} = \frac{-7(3\sqrt{2}+5)}{(3\sqrt{2})^2-5^2} = \frac{-7(3\sqrt{2}+5)}{(9)(2)-25} = \frac{-7(3\sqrt{2}+5)}{18-25}$

$= \frac{-7(3\sqrt{2}+5)}{-7} = \frac{\cancel{-7}(3\sqrt{2}+5)}{\cancel{-7}} = 3\sqrt{2}+5$

**117.** $\frac{3}{\sqrt{5}+\sqrt{x}} = \frac{3}{\sqrt{5}+\sqrt{x}}\cdot\frac{\sqrt{5}-\sqrt{x}}{\sqrt{5}-\sqrt{x}} = \frac{3(\sqrt{5}-\sqrt{x})}{(\sqrt{5}+\sqrt{x})(\sqrt{5}-\sqrt{x})} = \frac{3\sqrt{5}-3\sqrt{x}}{(\sqrt{5})^2-(\sqrt{x})^2} = \frac{3\sqrt{5}-3\sqrt{x}}{5-x}$

**118.** $\frac{5}{\sqrt{y}-\sqrt{3}}=\frac{5}{\sqrt{y}-\sqrt{3}}\cdot\frac{\sqrt{y}+\sqrt{3}}{\sqrt{y}+\sqrt{3}}=\frac{5(\sqrt{y}+\sqrt{3})}{(\sqrt{y}-\sqrt{3})(\sqrt{y}+\sqrt{3})}=\frac{5\sqrt{y}+5\sqrt{3}}{(\sqrt{y})^2-(\sqrt{3})^2}=\frac{5\sqrt{y}+5\sqrt{3}}{y-3}$

**119.** $\frac{\$8.1\times10^{12}}{2.98\times10^{8}\text{ people}}=\frac{\$8.1}{2.98}\times10^{12-8}$ per person $\approx \$2.72\times10^{4}$ per person

**120.** $4.7\times10^{21}$ bacteria $\cdot\frac{670\text{ femtograms}}{1\text{ bacteria}}\cdot\frac{1\times10^{-15}\text{ grams}}{1\text{ femtogram}}$

$4.7\times10^{21}\cdot\frac{670}{1}\cdot\frac{1\times10^{-15}}{1}=(4.7)(6.70)(1)\times10^{21+2-15}$

$=3.149\times10^{9}$

They weigh $3.149\times10^{9}$ grams.

**121.** $\frac{1\text{ seed}}{3.2\times10^{-8}\text{ ounce}}\cdot\frac{1\text{ ounce}}{\text{package}}$

$\frac{1}{3.2\times10^{-8}}\cdot1=\frac{1}{3.2}\times10^{8}$

$=0.3125\times10^{8}$

$=3.13\times10^{7}$ seeds per package

**122.** $\frac{800\text{ nm}}{1}\cdot\frac{1\times10^{-9}\text{ m}}{1\text{ nm}}=\left(8\times10^{2}\right)\left(1\times10^{-9}\right)=8\times10^{-7}$ m

frequency $=\frac{1}{\text{wavelength}}$

$=\frac{1}{8\times10^{-7}}=1.25\times10^{8}$ cycles per second

**123.** Red shift $=\frac{\lambda_r-\lambda_s}{\lambda_s}$

$=\frac{5.13\times10^{-7}-5.06\times10^{-7}}{5.06\times10^{-7}}$

$=\frac{(5.13-5.06)\times10^{-7}}{5.06\times10^{-7}}$

$=\frac{0.07}{5.06}\cdot\frac{10^{-7}}{10^{-7}}$

$=1.38\times10^{-2}$

**124.** $5.2\text{ AU}=5.2(9.3\times10^{7}\text{ miles})$

$=(5.2)(9.3)\times10^{7}$ miles

$=48.36\times10^{7}$ miles

$=4.84\times10^{8}$ miles

**125.** $\frac{1\text{ sec}}{3\times10^{8}\text{ m}}\cdot1.5\times10^{11}\text{ m}\cdot\frac{1\text{ min}}{60\text{ sec}}$

$\frac{1}{3\times10^{8}}\cdot1.5\times10^{11}\cdot\frac{1}{60}=\frac{(1)(1.5)(1)\times10^{11}}{3(60)\times10^{8}}$

$=\frac{1.5}{180}\times10^{11-8}$

$\approx0.008\times10^{3}$

$\approx8$ minutes

**126.** $\frac{1\text{ gram}}{6.023\times10^{23}\text{ atoms}}\cdot1\text{ atom}$

$\frac{1}{6.023\times10^{23}}\cdot1=\frac{1}{6.023}\times10^{-23}$

$\approx0.1660302175\times10^{-23}$

$\approx1.66\times10^{-24}$

One hydrogen atom weighs approximately $1.66\times10^{-24}$ gram.

**127.** Evaluate $R$ when $x = 20$.

$R=1250x(2^{-0.007x})=1250(20)(2^{-0.007(20)})=25{,}000(2^{-0.14})\approx25{,}000(0.907519)\approx22{,}688$

When the company sells 20 thousand phones, the revenue is \$22,688.

**128.** **a.** Evaluate $A$ when $t = 4$. $A = 2(10^{-0.0078t}) = 2(10^{-0.0078(4)}) = 2(10^{-0.0312}) \approx 2(.930679) \approx 1.86$

Four hours after taking 2 milligrams of digoxin, about 1.86 milligrams remain in the patient's blood.

**b.** Determine the sum of the amounts of medication remaining at 6:00 PM from each of the two doses. For the 1:00 PM dose, use $t = 5$; for the 5:00 PM dose, use $t = 1$.

$$A = 2(10^{-0.0078(5)}) + 2(10^{-0.0078(1)}) = 2(10^{-0.039}) + 2(10^{-0.0078}) \approx 2(.9141) + 2(0.9822)$$
$$\approx 1.828 + 1.964 \approx 3.79$$

At 6:00 PM, the amount of digoxin remaining in the patient's blood is 3.79 milligrams.

**129.** Evaluate $P$ when $n = 50$. $P = 6.5(2^{0.016354n}) = 6.5(2^{0.016354(45)}) = 6.5(2^{0.73593}) \approx 6.5(1.665) \approx 10.8$

In 2050, the world's population will be approximately 10.8 billion.

**130.** One hour is 60 minutes. Evaluate $P$ when $t = 60$. $P = 90 - 3t^{2/3} = 90 - 3(60)^{2/3} \approx 90 - 3(15.3261) \approx 44.02$

Thus, the percent of the students who remembered the number after 1 hour was 44.02%.

**131.** **a.** Evaluate $P$ when $d = 10$. $P = 10^{2-d/40} = 10^{2-10/40} = 10^{2-0.25} = 10^{1.75} \approx 56$

The amount of light that will pass to a depth of 10 feet below the ocean's surface is about 56%.

**b.** Evaluate $P$ when $d = 25$. $P = 10^{2-d/40} = 10^{2-25/40} = 10^{2-0.625} = 10^{1.375} \approx 24$

The amount of light that will pass to a depth of 25 feet below the ocean's surface is about 24%.

## Connecting Concepts

**132.** If $2^x = y$, then $2^{x-4} = 2^x \cdot 2^{-4} = \dfrac{2^x}{2^4} = \dfrac{2^x}{2^4} = \dfrac{y}{2^4}$

**133.** No, if $a$ and $b$ are nonzero numbers and $a < b$, then the statement $a^{-1} < b^{-1}$ is not a true statement.

Let $a = 2$ and $b = 3$. Then $a < b$, but $a^{-1} = 2^{-1} = \dfrac{1}{2}$ and $b^{-1} = 3^{-1} = \dfrac{1}{3}$. $\dfrac{1}{2} > \dfrac{1}{3}$ so $a^{-1} > b^{-1}$.

**134.** $4^{50} \cdot 5^{100} = 4^{50} \cdot 5^{50} \cdot 5^{50} = (4 \cdot 5 \cdot 5)^{50} = 100^{50} = (10^2)^{50} = 10^{100}$

$10^{100}$ is 1 followed by 100 zeros, thus has 101 digits.

**135.**
$$a^{2/5}a^p = a^2$$
$$\frac{2}{5} + p = 2$$
$$p = 2 - \frac{2}{5}$$
$$p = \frac{8}{5}$$

**136.**
$$b^{-3/4}b^{2p} = b^3$$
$$-\frac{3}{4} + 2p = 3$$
$$2p = 3 + \frac{3}{4}$$
$$p = \left(\frac{15}{4}\right)\frac{1}{2}$$
$$p = \frac{15}{8}$$

**137.**
$$\frac{x^{-3/4}}{x^{3p}} = x^4$$
$$-\frac{3}{4} - 3p = 4$$
$$-3p = \frac{19}{4}$$
$$p = -\frac{19}{12}$$

**138.**
$$\left(x^4x^{2p}\right)^{1/2} = x$$
$$(4 + 2p)\frac{1}{2} = 1$$
$$2 + p = 1$$
$$p = -1$$

**139.** $\dfrac{\sqrt{4+h}-2}{h}=\dfrac{\sqrt{4+h}-2}{h}\cdot\dfrac{\sqrt{4+h}+2}{\sqrt{4+h}+2}=\dfrac{(\sqrt{4+h}-2)(\sqrt{4+h}+2)}{h(\sqrt{4+h}+2)}=\dfrac{(\sqrt{4+h})^2-2^2}{h(\sqrt{4+h}+2)}=\dfrac{4+h-4}{h(\sqrt{4+h}+2)}$

$=\dfrac{h}{h(\sqrt{4+h}+2)}=\dfrac{\cancel{h}}{\cancel{h}(\sqrt{4+h}+2)}=\dfrac{1}{\sqrt{4+h}+2}$

**140.** $\dfrac{\sqrt{9+h}-3}{h}=\dfrac{\sqrt{9+h}-3}{h}\cdot\dfrac{\sqrt{9+h}+3}{\sqrt{9+h}+3}=\dfrac{(\sqrt{9+h}-3)(\sqrt{9+h}+3)}{h(\sqrt{9+h}+3)}=\dfrac{(\sqrt{9+h})^2-3^2}{h(\sqrt{9+h}+3)}=\dfrac{9+h-9}{h(\sqrt{9+h}+3)}$

$=\dfrac{h}{h(\sqrt{9+h}+3)}=\dfrac{\cancel{h}}{\cancel{h}(\sqrt{9+h}+3)}=\dfrac{1}{\sqrt{9+h}+3}$

**141.** $\dfrac{\sqrt{n^2+1}-n}{1}=\dfrac{\sqrt{n^2+1}-n}{1}\cdot\dfrac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}=\dfrac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{\sqrt{n^2+1}+n}=\dfrac{(\sqrt{n^2+1})^2-n^2}{\sqrt{n^2+1}+n}=\dfrac{n^2+1-n^2}{\sqrt{n^2+1}+n}=\dfrac{1}{\sqrt{n^2+1}+n}$

**142.** $\dfrac{\sqrt{n^2+n}-n}{1}=\dfrac{\sqrt{n^2+n}-n}{1}\cdot\dfrac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n}=\dfrac{(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n)}{\sqrt{n^2+n}+n}=\dfrac{(\sqrt{n^2+n})^2-n^2}{\sqrt{n^2+n}+n}=\dfrac{n^2+n-n^2}{\sqrt{n^2+n}+n}=\dfrac{n}{\sqrt{n^2+n}+n}$

**143.** $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}=\sqrt{2}^{(\sqrt{2})(\sqrt{2})}=\sqrt{2}^{(\sqrt{2})^2}=\sqrt{2}^2=2$

## Prepare for Section P.3

**PS1.** $-3(2a-4b)$

$-6a+12b$

**PS2.** $5-2(2x-7)$

$5-4x+14$

$-4x+19$

**PS3.** $2x^2+3x-5+x^2-6x-1$

$2x^2+x^2+3x-6x-5-1$

$(2+1)x^2+(3-6)x-(5+1)$

$3x^2-3x-6$

**PS4.** $4x^2-6x-1-5x^2+x$

$4x^2-5x^2-6x+x-1$

$(4-5)x^2+(-6+1)x-1$

$-x^2-5x-1$

**PS5.** $4-3x-2x^2\overset{?}{=}-2x^2-4x+4$

$-2x^2-3x+4\overset{?}{=}-2x^2-4x+4$

False.

**PS6.** $\dfrac{12+15}{4}=\dfrac{27}{4}$

$=6\dfrac{3}{4}\neq 18$

False.

## Section P.3

**1.** D **2.** E **3.** H **4.** F **5.** G

**6.** I **7.** B **8.** C **9.** J **10.** A

**11.**
**a.** $x^2+2x-7$
**b.** 2
**c.** 1, 2, –7
**d.** 1
**e.** $x^2$, $2x$, $-7$

**12.**
**a.** $-12x^4-3x^2-11$
**b.** 4
**c.** –12, – 3, – 11
**d.** –12
**e.** $-12x^4$, $-3x^2$, $-11$

**13.**
**a.** $x^3-1$
**b.** 3
**c.** 1, –1
**d.** 1
**e.** $x^3$, $-1$

**14.** **a.** $4x^2 - 2x + 7$
**b.** 2
**c.** 4, −2, 7
**d.** 4
**e.** $4x^2, -2x, 7$

**15.** **a.** $2x^4 + 3x^3 + 4x^2 + 5$
**b.** 4
**c.** 2, 3, 4, 5
**d.** 2
**e.** $2x^4, 3x^3, 4x^2, 5$

**16.** **a.** $-5x^3 + 3x^2 + 7x - 1$
**b.** 3
**c.** −5, 3, 7, −1
**d.** −5
**e.** $-5x^3, 3x^2, 7x, -1$

**17.** 3

**18.** 3

**19.** 5

**20.** 6

**21.** 2

**22.** 4

**23.** $(3x^2 + 4x + 5) + (2x^2 + 7x - 2) = 5x^2 + 11x + 3$

**24.** $(5y^2 - 7y + 3) + (2y^2 + 8y + 1) = 7y^2 + y + 4$

**25.** $(4w^3 - 2w + 7) + (5w^3 + 8w^2 - 1) = 9w^3 + 8w^2 - 2w + 6$

**26.** $(5x^4 - 3x^2 + 9) + (3x^3 - 2x^2 - 7x + 3) = 5x^4 + 3x^3 - 5x^2 - 7x + 12$

**27.** $(r^2 - 2r - 5) - (3r^2 - 5r + 7) = r^2 - 2r - 5 - 3r^2 + 5r - 7 = -2r^2 + 3r - 12$

**28.** $(7s^2 - 4s + 11) - (-2s^2 + 11s - 9) = 7s^2 - 4s + 11 + 2s^2 - 11s + 9 = 9s^2 - 15s + 20$

**29.** $(u^3 - 3u^2 - 4u + 8) - (u^3 - 2u + 4) = u^3 - 3u^2 - 4u + 8 - u^3 + 2u - 4 = -3u^2 - 2u + 4$

**30.** $(5v^4 - 3v^2 + 9) - (6v^4 + 11v^2 - 10) = 5v^4 - 3v^2 + 9 - 6v^4 - 11v^2 + 10 = -v^4 - 14v^2 + 19$

**31.** $(2x^2 + 7x - 8)(4x - 5) = 8x^3 - 10x^2 + 28x^2 - 35x - 32x + 40 = 8x^3 + 18x^2 - 67x + 40$

**32.** $(3x^2 - 8x - 5)(5x - 7) = 15x^3 - 21x^2 - 40x^2 + 56x - 25x + 35 = 15x^3 - 61x^2 + 31x + 35$

**33.**

$$\begin{array}{r} 3x^2 - 2x + 5 \\ 2x^2 - 5x + 2 \\ \hline +\ 6x^2 - 4x + 10 \\ -15x^3 + 10x^2 - 25x \phantom{+10} \\ +\ 6x^4 - 4x^3 + 10x^2 \phantom{-25x+10} \\ \hline 6x^4 - 19x^3 + 26x^2 - 29x + 10 \end{array}$$

**34.**

$$\begin{array}{r} 2y^3 - 3y + 4 \\ 2y^2 - 5y + 7 \\ \hline +\ 14y^3 \qquad - 21y + 28 \\ -10y^4 \qquad + 15y^2 - 20y \phantom{+28} \\ 4y^5 \qquad - 6y^3 + 8y^2 \phantom{-20y+28} \\ \hline 4y^5 - 10y^4 + 8y^3 + 23y^2 - 41y + 28 \end{array}$$

**35.** $(2x + 4)(5x + 1) = 10x^2 + 2x + 20x + 4 = 10x^2 + 22x + 4$

**36.** $(5x - 3)(2x + 7) = 10x^2 + 35x - 6x - 21 = 10x^2 + 29x - 21$

**37.** $(y + 2)(y + 1) = y^2 + y + 2y + 2 = y^2 + 3y + 2$

**38.** $(y + 5)(y + 3) = y^2 + 3y + 5y + 15 = y^2 + 8y + 15$

**39.** $(4z - 3)(z - 4) = 4z^2 - 16z - 3z + 12 = 4z^2 - 19z + 12$

**40.** $(5z - 6)(z - 1) = 5z^2 - 5z - 6z + 6 = 5z^2 - 11z + 6$

**41.** $(a + 6)(a - 3) = a^2 - 3a + 6a - 18 = a^2 + 3a - 18$

**42.** $(a - 10)(a + 4) = a^2 + 4a - 10a - 40 = a^2 - 6a - 40$

**43.** $(5x - 11y)(2x - 7y) = 10x^2 - 35xy - 22xy + 77y^2$
$= 10x^2 - 57xy + 77y^2$

**44.** $(3a - 5b)(4a - 7b) = 12a^2 - 21ab - 20ab + 35b^2$
$= 12a^2 - 41ab + 35b^2$

**45.** $(9x + 5y)(2x + 5y) = 18x^2 + 45xy + 10xy + 25y^2$
$= 18x^2 + 55xy + 25y^2$

**46.** $(3x - 7z)(5x - 7z) = 15x^2 - 21xz - 35xz + 49z^2$
$= 15x^2 - 56xz + 49z^2$

**47.** $(3p + 5q)(2p - 7q) = 6p^2 - 21pq + 10pq - 35q^2$
$= 6p^2 - 11pq - 35q^2$

**48.** $(2r - 11s)(5r + 8s) = 10r^2 + 16rs - 55rs - 88s^2$
$= 10r^2 - 39rs - 88s^2$

**49.** $(4d - 1)^2 - (2d - 3)^2 = (16d^2 - 8d + 1) - (4d^2 - 12d + 9) = 16d^2 - 8d + 1 - 4d^2 + 12d - 9 = 12d^2 + 4d - 8$

**50.** $(5c - 8)^2 - (2c - 5)^2 = (25c^2 - 80c + 64) - (4c^2 - 20c + 25) = 25c^2 - 80c + 64 - 4c^2 + 20c - 25 = 21c^2 - 60c + 39$

**51.**
$$\begin{array}{r} r^2 - rs + s^2 \\ \underline{\qquad\qquad r + s} \\ +r^2s - rs^2 + s^3 \\ \underline{r^3 - r^2s + rs^2 \qquad} \\ r^3 \qquad\qquad + s^3 \end{array}$$

**52.**
$$\begin{array}{r} r^2 + rs + s^2 \\ \underline{\qquad\qquad r - s} \\ -r^2s - rs^2 - s^3 \\ \underline{r^3 + r^2s + rs^2 \qquad} \\ r^3 \qquad\qquad - s^3 \end{array}$$

**53.** $(3c-2)(4c+1)(5c-2) = (12c^2 - 5c - 2)(5c - 2)$
$$\begin{array}{r} 12c^2 - 5c - 2 \\ \underline{\qquad\qquad 5c - 2} \\ -24c^2 + 10c + 4 \\ \underline{60c^3 - 25c^2 - 10c \qquad} \\ 60c^3 - 49c^2 \qquad + 4 \end{array}$$

**54.** $(4d-5)(2d-1)(3d-4) = (8d^2 - 14d + 5)(3d - 4)$
$$\begin{array}{r} 8d^2 - 14d + 5 \\ \underline{\qquad\qquad 3d - 4} \\ -32d^2 + 56d - 20 \\ \underline{24d^3 - 42d^2 + 15d \qquad} \\ 24d^3 - 74d^2 + 71d - 20 \end{array}$$

**55.** $(3x+5)(3x-5) = 9x^2 - 25$

**56.** $(4x^2 - 3y)(4x^2 + 3y) = 16x^4 - 9y^2$

**57.** $(3x^2 - y)^2 = 9x^4 - 6x^2y + y^2$

**58.** $(6x + 7y)^2 = 36x^2 + 84xy + 49y^2$

**59.** $(4w + z)^2 = 16w^2 + 8wz + z^2$

**60.** $(3x - 5y^2)^2 = 9x^2 - 30xy^2 + 25y^4$

**61.** $[(x+5)+y][(x+5)-y] = (x+5)^2 - y^2$
$= x^2 + 10x + 25 - y^2$

**62.** $[(x-2y)+7][(x-2y)-7] = (x-2y)^2 - 49$
$= x^2 - 4xy + 4y^2 - 49$

**63.** $x^2 + 7x - 1 = 3^2 + 7(3) - 1 = 9 + 21 - 1 = 29$

**64.** $x^2 - 8x + 2 = 4^2 - 8(4) + 2 = 16 - 32 + 2 = -14$

**65.** $-x^2 + 5x - 3 = -(-2)^2 + 5(-2) - 3 = -4 - 10 - 3 = -17$

**66.** $-x^2 - 5x + 4 = -(-5)^2 - 5(-5) + 4 = -25 + 25 + 4 = 4$

**67.** $3x^3 - 2x^2 - x + 3 = 3(-1)^3 - 2(-1)^2 - (-1) + 3 = 3(-1) - 2(1) + 1 + 3 = -3 - 2 + 1 + 3 = -1$

**68.** $5x^3 - x^2 + 5x - 3 = 5(-1)^3 - (-1)^2 + 5(-1) - 3 = 5(-1) - (1) - 5 - 3 = -5 - 1 - 5 - 3 = -14$

**69.** $1 - x^5 = 1 - (-2)^5 = 1 - (-32) = 1 + 32 = 33$

**70.** $1 - x^3 - x^5 = 1 - 2^3 - 2^5 = 1 - 8 - 32 = -39$

**71.** Substitute the given value of $v$ into $0.016v^2$.
Then simplify

**a.** $0.016\,v^2$
$0.016(10)^2 = 1.6$
The air resistance is 1.6 pounds.

**b.** $0.016\,v^2$
$0.016(15)^2 = 3.6$
The air resistance is 3.6 pounds.

**72.** Substitute the given value of $v$ into $0.015v^2 + v + 10$.
Then simplify.

**a.** $0.015v^2 + v + 10$
$0.015(30)^2 + 30 + 10 = 53.5$
The safe distance is 53.5 feet.

**b.** $0.015v^2 + v + 10$
$0.015(55)^2 + 55 + 10 = 110.375$
The safe distance is 110.375 feet.

**73.** Substitute the given value of $h$ and $r$ into $\pi r^2 h$.
Then simplify

**a.** $\pi r^2 h$
$\pi(3)^2(8) = 72\pi$
The volume is $72\pi$ in$^3$ .

**b.** $\pi r^2 h$
$\pi(5)^2(12) = 300\pi$
The volume $300\pi$ cm$^3$.

**74.** Substitute the given value of $v$ into $-0.02v^2 + 1.5v + 2$.
Then simplify.

**a.** $0.02v^2 + 1.5v + 2$
$-0.02\,(45)^2 + 1.5(45) + 2 = 29$
The fuel efficiency is 29 miles per gallon.

**b.** $0.02v^2 + 1.5v + 2$
$-0.02\,(60)^2 + 1.5(60) + 2 = 20$
The fuel efficiency is 20 miles per gallon.

**75.** Substitute the given value of $v$ into $0.005x^2 - 0.32x + 12$.

**a.** $0.005x^2 - 0.32x + 12$
$0.005(20)^2 - 0.32(20) + 12 = 7.6$
The reaction time is 7.6 hundredths of a second or 0.076 seconds.

**b.** $0.005x^2 - 0.32x + 12$
$0.005(50)^2 - 0.32(50) + 12 = 8.5$
The reaction time is 8.5 hundredths of a second or 0.085 seconds.

**76.** $\frac{1}{6}n^3 - \frac{1}{2}n^2 + \frac{1}{3}n = \frac{1}{6}(21)^3 - \frac{1}{2}(21)^2 + \frac{1}{3}(21) = 1330$ committees

**77.** $\frac{1}{2}n^2 - \frac{1}{2}n = \frac{1}{2}(150)^2 - \frac{1}{2}(150) = 11{,}175$ chess matches

**78.** **a.** $4.3 \times 10^{-6}(1000)^2 - 2.1 \times 10^{-4}(1000) = 4.09$ sec
**b.** $4.3 \times 10^{-6}(5000)^2 - 2.1 \times 10^{-4}(5000) = 106.45$ sec
**c.** $4.3 \times 10^{-6}(10{,}000)^2 - 2.1 \times 10^{-4}(10{,}000) = 427.9$ sec

**79.** **a.** $1.9 \times 10^{-6}(4000)^2 - 3.9 \times 10^{-3}(4000) = 14.8$ sec
**b.** $1.9 \times 10^{-6}(8000)^2 - 3.9 \times 10^{-3}(8000) = 90.4$ sec

**80.** **a.**
$$\text{velocity} = 6r^2 - 10r^3 = -10r^3 + 6r^2 \text{ m/s}$$

**b.** Evaluate $-10r^3 + 6r^2$ when $r = 0.35$.
$$\begin{aligned}\text{velocity} &= -10(0.35)^3 + 6(0.35)^2\\ &= -10(0.042875) + 6(0.1225)\\ &= -0.42875 + 0.735\\ &= 0.30625\end{aligned}$$
The velocity of the air in a cough when the radius of the trachea is 0.35 centimeters is 0.31 m/s.

**81.** Evaluate $-16t^2 + 4.7881t + 6$ when $t = 0.5$
$$\begin{aligned}\text{height} &= -16t^2 + 4.7881t + 6\\ &= -16(0.5)^2 + 4.7881(0.5)t + 6\\ &= 4.39\end{aligned}$$
Yes. The ball is approximately 4.4 feet high when it crosses home plate.

**82.** **a.** Evaluate $0.0002t^3 - 0.0114t^2 + 0.0158t + 104$ when $t = 0$
$$\begin{aligned}\text{Temp} &= 0.0002t^3 - 0.0114t^2 + 0.0158t + 104\\ &= 0.0002(0)^3 - 0.0114(0)^2 + 0.0158(0) + 104\\ &= 104\end{aligned}$$
The patient's temperature was 104°F before taking the medication.

**b.** Evaluate $0.0002t^3 - 0.0114t^2 + 0.0158t + 104$ when $t = 25$
$$\begin{aligned}\text{Temp} &= 0.0002t^3 - 0.0114t^2 + 0.0158t + 104\\ &= 0.0002(25)^3 - 0.0114(25)^2 + 0.0158(25) + 104\\ &= 100.395\end{aligned}$$
The patient's temperature was 100.4°F 25 minutes after taking the medication.

**83.**
$$\begin{aligned}&2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13\\ &= 2 \cdot 3 \cdot (2^2) \cdot 5 \cdot (2 \cdot 3) \cdot 7 \cdot (2^3) \cdot (3^2) \cdot (2 \cdot 5) \cdot 11 \cdot (2^2 \cdot 3) \cdot 13 \cdot (2 \cdot 7) \cdot (3 \cdot 5)\\ &= 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15\\ &= 15!\end{aligned}$$
$n = 15$

## Connecting Concepts

**84.** $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

**85.** $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

**86.** $(x - 1)^3 = x^3 - 3x^2 + 3x - 1$

**87.** $(y + 2)^3 = y^3 + 3y^2(2) + 3y(2)^2 + 2^3 = y^3 + 6y^2 + 12y + 8$

**88.** $(2x-3y)^3 = (2x)^3 + 3(2x)^2(-3y) + 3(2x)(-3y)^2 + (-3y)^3 = 8x^3 - 36x^2y + 54xy^2 - 27y^3$

**89.** $(3x+5y)^3 = (3x)^3 + 3(3x)^2 5y + 3(3x)(5y)^2 + (5y)^3 = 27x^3 + 135x^2y + 225xy^2 + 125y^3$

## Prepare for Section P.4

**PS1.** $\frac{6x^3}{2x} = 3x^{3-1} = 3x^2$

**PS2.** $(-12x^4)3x^2 = (-12)(3)x^{4+2} = -36x^6$

**PS3.** **a.** $x^6 = (x^2)^3$

**b.** $x^6 = (x^3)^2$

**PS4.** $6a^3b^4 \cdot ? = 18a^3b^7 = 6a^3b^4(3b^3)$
Thus, $? = 3b^3$

**PS5.** $-3(5a - ?) = -15a + 21 = -3(5a - 7)$
Thus, ? = 7

**PS6.** $2x(3x - ?) = 6x^2 - 2x = 2x(3x - 1)$
Thus, ? = 1

## Section P.4

**1.** $5x + 20 = 5(x + 4)$

**2.** $8x^2 + 12x - 40 = 4(2x^2 + 3x - 10)$

**3.** $-15x^2 - 12x = -3x(5x + 4)$

**4.** $-6y^2 - 54y = -6y(y + 9)$

**5.** $10x^2y + 6xy - 14xy^2 = 2xy(5x + 3 - 7y)$

**6.** $6a^3b^2 - 12a^2b + 72ab^3 = 6ab(a^2b - 2a + 12b^2)$

**7.** $(x-3)(a+b) + (x-3)(a+2b) = (x-3)(a+b+a+2b) = (x-3)(2a+3b)$

**8.** $(x-4)(2a-b) + (x+4)(2a-b) = (2a-b)(x-4+x+4) = (2a-b)(2x)$

**9.** $x^2 + 7x + 12 = (x+3)(x+4)$

**10.** $x^2 + 9x + 20 = (x+4)(x+5)$

**11.** $a^2 - 10a - 24 = (a-12)(a+2)$

**12.** $b^2 + 12b - 28 = (b+14)(b-2)$

**13.** $6x^2 + 25x + 4 = (6x+1)(x+4)$

**14.** $8a^2 - 26a + 15 = (4a-3)(2a-5)$

**15.** $51x^2 - 5x - 4 = (17x+4)(3x-1)$

**16.** $57y^2 + y - 6 = (19y-6)(3y+1)$

**17.** $6x^2 + xy - 40y^2 = (3x+8y)(2x-5y)$

**18.** $8x^2 + 10xy - 25y^2 = (4x-5y)(2x+5y)$

**19.** $x^4 + 6x^2 + 5 = (x^2+5)(x^2+1)$

**20.** $x^4 + 11x^2 + 18 = (x^2+9)(x^2+2)$

**21.** $6x^4 + 23x^2 + 15 = (6x^2+5)(x^2+3)$

**22.** $9x^4 + 10x^2 + 1 = (9x^2+1)(x^2+1)$

**23.** $b^2 - 4ac = 26^2 - 4(8)(15) = 196 = 14^2$
The trinomial is factorable over the integers

**24.** $b^2 - 4ac = 8^2 - 4(16)(-35) = 2304 = 48^2$
The trinomial is factorable over the integers.

**25.** $b^2 - 4ac = (-5)^2 - 4(4)(6) = -71$
The trinomial is not factorable over the integers.

**26.** $b^2 - 4ac = 8^2 - 4(6)(-3) = 136$
The trinomial is not factorable over the integers.

**27.** $b^2 - 4ac = (-14)^2 - 4(6)(5) = 76$
The trinomial is not factorable over the integers.

**28.** $b^2 - 4ac = (-4)^2 - 4(10)(-5) = 216$
The trinomial is not factorable over the integers.

**29.** $x^4 - x^2 - 6 = (x^2 - 3)(x^2 + 2)$

**30.** $x^4 + 3x^2 + 2 = (x^2 + 1)(x^2 + 2)$

**31.** $x^2y^2 - 2xy - 8 = (xy - 4)(xy + 2)$

**32.** $2x^2y^2 + xy - 1 = (2xy - 1)(xy + 1)$

**33.** $3x^4 + 11x^2 - 4 = (3x^2 - 1)(x^2 + 4)$

**34.** $2x^4 + 3x^2 - 9 = (2x^2 - 3)(x^2 + 3)$

**35.** $3x^6 + 2x^3 - 8 = (3x^3 - 4)(x^3 + 2)$

**36.** $8x^6 - 10x^3 - 3 = (4x^3 + 1)(2x^3 - 3)$

**37.** $x^2 - 9 = (x - 3)(x + 3)$

**38.** $x^2 - 64 = (x - 8)(x + 8)$

**39.** $4a^2 - 49 = (2a - 7)(2a + 7)$

**40.** $81b^2 - 16c^2 = (9b - 4c)(9b + 4c)$

**41.** $1 - 100x^2 = (1 - 10x)(1 + 10x)$

**42.** $1 - 121y^2 = (1 - 11y)(1 + 11y)$

**43.** $x^4 - 9 = (x^2 - 3)(x^2 + 3)$

**44.** $y^4 - 196 = (y^2 - 14)(y^2 + 14)$

**45.** $(x + 5)^2 - 4 = (x + 5 - 2)(x + 5 + 2) = (x + 3)(x + 7)$

**46.** $(x - 3)^2 - 16 = (x - 3 - 4)(x - 3 + 4) = (x - 7)(x + 1)$

**47.** $x^2 + 10x + 25 = (x + 5)^2$

**48.** $y^2 + 6y + 9 = (y + 3)^2$

**49.** $a^2 - 14a + 49 -= (a - 7)^2$

**50.** $b^2 - 24b + 144 = (b - 12)^2$

**51.** $4x^2 + 12x + 9 = (2x + 3)^2$

**52.** $25y^2 + 40y + 16 = (5y + 4)^2$

**53.** $z^4 + 4z^2w^2 + 4w^4 = (z^2 + 2w^2)^2$

**54.** $9x^4 - 30x^2y^2 + 25y^4 = (3x^2 - 5y^2)^2$

**55.** $x^3 - 8 = (x - 2)(x^2 + 2x + 4)$

**56.** $b^3 + 64 = (b + 4)(b^2 - 4b + 16)$

**57.** $8x^3 - 27y^3 = (2x - 3y)(4x^2 + 6xy + 9y^2)$

**58.** $64u^3 - 27v^3 = (4u - 3v)(16u^2 + 12uv + 9v^2)$

**59.** $8 - x^6 = (2 - x^2)(4 + 2x^2 + x^4)$

**60.** $1 + y^{12} = (1 + y^4)(1 - y^4 + y^8)$

**61.** $(x - 2)^3 - 1 = [(x - 2) - 1]\,[(x - 2)^2 + (x - 2) + 1] = (x - 3)(x^2 - 4x + 4 + x - 2 + 1) = (x - 3)(x^2 - 3x + 3)$

**62.** $(y + 3)^3 + 8 = ((y + 3) + 2)((y + 3)^2 - 2(y + 3) + 4) = (y + 5)(y^2 + 6y + 9 - 2y - 6 + 4) = (y + 5)(y^2 + 4y + 7)$

**63.** $3x^3 + x^2 + 6x + 2 = x^2(3x + 1) + 2(3x + 1) = (3x + 1)(x^2 + 2)$

**64.** $18w^3 + 15w^2 + 12w + 10 = 3w^2(6w + 5) + 2(6w + 5) = (6w + 5)(3w^2 + 2)$

**65.** $ax^2 - ax + bx - b = ax(x - 1) + b(x - 1) = (x - 1)(ax + b)$

**66.** $a^2y^2 - ay^3 + ac - cy = ay^2(a - y) + c(a - y) = (a - y)(ay^2 + c)$

**67.** $6w^3 + 4w^2 - 15w - 10 = 2w^2(3w + 2) - 5(3w + 2) = (3w + 2)(2w^2 - 5)$

**68.** $10z^3 - 15z^2 - 4z + 6 = 5z^2(2z - 3) - 2(2z - 3) = (2z - 3)(5z^2 - 2)$

**69.** $18x^2 - 2 = 2(9x^2 - 1) = 2(3x - 1)(3x + 1)$

**70.** $4bx^3 + 32b = 4b(x^3 + 8) = 4b(x + 2)(x^2 - 2x + 4)$

**71.** $16x^4 - 1 = (4x^2 - 1)(4x^2 + 1) = (2x - 1)(2x + 1)(4x^2 + 1)$

**72.** $81y^4 - 16 = (9y^2 - 4)(9y^2 + 4) = (3y - 2)(3y + 2)(9y^2 + 4)$

**73.** $12ax^2 - 23axy + 10ay^2 = a(12x^2 - 23xy + 10y^2) = a(3x - 2y)(4x - 5y)$

**74.** $6ax^2 - 19axy - 20ay^2 = a(6x^2 - 19xy - 20y^2) = a(6x + 5y)(x - 4y)$

**75.** $3bx^3 + 4bx^2 - 36x - 4b = bx^2(3x + 4) - b(3x + 4) = (3x + 4)(bx^2 - b) = b(3x + 4)(x^2 - 1) = b(3x + 4)(x - 1)(x + 1)$

**76.** $2x^6 - 2 = 2(x^6 - 1) = 2(x^3 - 1)(x^3 + 1) = 2(x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1)$

**77.** $72bx^2 + 24bxy + 2by^2 = 2b(36x^2 + 12xy + y^2) = 2b(6x + y)^2$

**78.** $64y^3 - 16y^2z + yz^2 = y(64y^2 - 16yz + z^2) = y(8y - z)^2$

**79.** $(w - 5)^3 + 8 = [(w - 5) + 2][(w - 5)^2 - 2(w - 5) + 4] = (w - 3)(w^2 - 10w + 25 - 2w + 10 + 4) = (w-3)(w^2 - 12w + 39)$

**80.** $5xy + 20y - 15x - 60 = 5(xy + 4y - 3x - 12) = 5[y(x + 4) - 3(x + 4)] = 5(x + 4)(y - 3)$

**81.** $x^2 + 6xy + 9y^2 - 1 = (x + 3y)^2 - 1 = (x + 3y - 1)(x + 3y + 1)$

**82.** $4y^2 - 4yz + z^2 - 9 = (2y - z)^2 - 9 = (2y - z - 3)(2y - z + 3)$

**83.** $8x^2 + 3x - 4$ is not factorable over the integers.

**84.** $16x^2 + 81$ is not factorable over the integers.

**85.** $5x(2x - 5)^2 - (2x - 5)^3 = (2x - 5)^2[5x - (2x - 5)] = (2x - 5)^2(5x - 2x + 5) = (2x - 5)^2(3x + 5)$

**86.** $6x(3x + 1)^3 - (3x + 1)^4 = (3x + 1)^3[6x - (3x + 1)] = (3x + 1)^3(6x - 3x - 1) = (3x + 1)^3(3x - 1)$

**87.** $4x^2 + 2x - y - y^2 = 4x^2 - y^2 + 2x - y = (2x - y)(2x + y) + (2x - y) = (2x - y)(2x + y + 1)$

**88.** $a^2 + a + b - b^2 = a^2 - b^2 + a + b = (a - b)(a + b) + (a + b) = (a + b)(a - b + 1)$

## Connecting Concepts

**89.** $x^2 + kx + 16 = (x + 4)^2 = x^2 + 8x + 16$, thus $k = 8$

**90.** $36x^2 + kx + 100y^2 = (6x + 10y)^2 = 36x^2 + 120x + 100y^2$, thus $k = 120$

**91.** $x^2 + 16x + k = (x + \sqrt{k})^2 = x^2 + 2x\sqrt{k} + k \Rightarrow 16x = 2x\sqrt{k} \Rightarrow 8 = \sqrt{k} \Rightarrow k = 64$

**92.** $x^2 - 14xy + ky^2 = (x - \sqrt{k}\,y)^2 = x^2 - 2xy\sqrt{k} + ky^2 \Rightarrow -14xy = -2xy\sqrt{k} \Rightarrow 7 = \sqrt{k} \Rightarrow k = 49$

**93.** $x^{4n} - 1 = (x^{2n} - 1)(x^{2n} + 1) = (x^n - 1)(x^n + 1)(x^{2n} + 1)$

**94.** $x^{4n} - 2x^{2n} + 1 = (x^{2n} - 1)(x^{2n} - 1) = (x^n - 1)(x^n + 1)(x^n - 1)(x^n + 1) = (x^n - 1)^2(x^n + 1)^2$

**95.** $A = \pi R^2 - \pi r^2 = \pi(R^2 - r^2) = \pi(R-r)(R+r)$

**96.** $A = \pi r^2 + (2r)^2 = \pi r^2 + 4r^2 = r^2(\pi + 4)$

**97.** $A = (2r)^2 - \pi r^2 = r^2(4 - \pi)$

**98.** $A = x^2 - y^2 = (x-y)(x+y)$

## Prepare for Section P.5

**PS1.** $$1 + \frac{1}{2 - \frac{1}{3}} = 1 + \frac{1}{2 - \frac{1}{3}} \cdot \left(\frac{3}{3}\right) = 1 + \frac{1 \cdot 3}{2 \cdot 3 - \frac{1}{3} \cdot 3} = 1 + \frac{3}{6-1} = 1 + \frac{3}{5} = 1\frac{3}{5} \text{ or } \frac{8}{5}$$

**PS2.** $$\left(\frac{w}{x}\right)^{-1}\left(\frac{y}{z}\right)^{-1} = \left(\frac{x}{w}\right)\left(\frac{z}{y}\right) = \frac{xz}{wy}$$

**PS3.** $x^2 + 2x - 3 = (x+3)(x-1)$
$x^2 + 7x + 12 = (x+4)(x+3)$
The common factor is $x + 3$.

**PS4.** $(2x-3)(3x+2) - (2x-3)(x+2)$
$= (2x-3)[(3x+2) - (x+2)]$
$= (2x-3)(2x)$
$= 2x(2x-3)$

**PS5.** $x^2 - 5x - 6 = (x-6)(x+1)$

**PS6.** $x^3 - 64 = (x-4)(x^2 + 4x + 16)$

## Section P.5

**1.** $$\frac{x^2 - x - 20}{3x - 15} = \frac{(x+4)(x-5)}{3(x-5)} = \frac{x+4}{3}$$

**2.** $$\frac{2x^2 - 5x - 12}{2x^2 + 5x + 3} = \frac{(2x+3)(x-4)}{(2x+3)(x+1)} = \frac{x-4}{x+1}$$

**3.** $$\frac{x^3 - 9x}{x^3 + x^2 - 6x} = \frac{x(x^2 - 9)}{x(x^2 + x - 6)} = \frac{x(x-3)(x+3)}{x(x+3)(x-2)} = \frac{x-3}{x-2}$$

**4.** $$\frac{x^3 + 125}{2x^3 - 50x} = \frac{(x+5)(x^2 - 5x + 25)}{2x(x^2 - 25)} = \frac{(x+5)(x^2 - 5x + 25)}{2x(x-5)(x+5)} = \frac{x^2 - 5x + 25}{2x(x-5)}$$

**5.** $$\frac{a^3 + 8}{a^2 - 4} = \frac{(a+2)(a^2 - 2a + 4)}{(a-2)(a+2)} = \frac{a^2 - 2a + 4}{a-2}$$

**6.** $$\frac{y^3 - 27}{-y^2 + 11y - 24} = \frac{(y-3)(y^2 + 3y + 9)}{-(y^2 - 11y + 24)} = \frac{(y-3)(y^2 + 3y + 9)}{-(y-8)(y-3)} = -\frac{y^2 + 3y + 9}{y-8}$$

**7.** $$\frac{x^2 + 3x - 40}{-(x^2 - 3x - 10)} = \frac{(x-5)(x+8)}{-(x-5)(x+2)} = -\frac{x+8}{x+2}$$

**8.** $$\frac{2x^3 - 6x^2 + 5x - 15}{9 - x^2} = \frac{2x^2(x-3) + 5(x-3)}{-(x^2 - 9)} = \frac{(x-3)(2x^2 + 5)}{-(x-3)(x+3)} = -\frac{2x^2 + 5}{x+3}$$

**9.** $$\frac{4y^3 - 8y^2 + 7y - 14}{-y^2 - 5y + 14} = \frac{4y^2(y-2) + 7(y-2)}{-(y^2 + 5y - 14)} = \frac{(y-2)(4y^2 + 7)}{-(y+7)(y-2)} = -\frac{4y^2 + 7}{y+7}$$

**10.** $\dfrac{x^3-x^2+x}{x^3+1}=\dfrac{x(x^2-x+1)}{(x+1)(x^2-x+1)}=\dfrac{x}{x+1}$

**11.** $\left(-\dfrac{4a}{3b^2}\right)\left(\dfrac{6b}{a^4}\right)=-\dfrac{24ab}{3a^4b^2}=-\dfrac{8}{a^3b}$

**12.** $\left(\dfrac{12x^2y}{5z^4}\right)\left(-\dfrac{25x^2z^3}{15y^2}\right)=-\dfrac{12\cdot 25x^4yz^3}{5\cdot 15y^2z^4}=-\dfrac{4x^4}{yz}$

**13.** $\left(\dfrac{6p^2}{5q^2}\right)^{-1}\left(\dfrac{2p}{3q^2}\right)^2=\dfrac{5q^2}{6p^2}\cdot\dfrac{4p^2}{9q^4}=\dfrac{10}{27q^2}$

**14.** $\left(\dfrac{4r^2s}{3t^3}\right)^{-1}\left(\dfrac{6rs^3}{5t^2}\right)=\dfrac{3t^3}{4r^2s}\cdot\dfrac{6rs^3}{5t^2}=\dfrac{9s^2t}{10r}$

**15.** $\dfrac{x^2+x}{2x+3}\cdot\dfrac{3x^2+19x+28}{x^2+5x+4}=\dfrac{x(x+1)}{2x+3}\cdot\dfrac{(3x+7)(x+4)}{(x+4)(x+1)}=\dfrac{x(3x+7)}{2x+3}$

**16.** $\dfrac{x^2-16}{x^2+7x+12}\cdot\dfrac{x^2-4x-21}{x^2-4x}=\dfrac{(x-4)(x+4)}{(x+3)(x+4)}\cdot\dfrac{(x+3)(x-7)}{x(x-4)}=\dfrac{x-7}{x}$

**17.** $\dfrac{3x-15}{2x^2-50}\cdot\dfrac{2x^2+16x+30}{6x+9}=\dfrac{3(x-15)}{2(x^2-25)}\cdot\dfrac{2(x^2+8x+15)}{3(2x+3)}=\dfrac{3(x-15)}{2(x-5)(x+5)}\cdot\dfrac{2(x+3)(x+5)}{3(2x+3)}=\dfrac{x+3}{2x+3}$

**18.** $\dfrac{y^3-8}{y^2+y-6}\cdot\dfrac{y^2+3y}{y^3+2y^2+4y}=\dfrac{(y-2)(y^2+2y+4)}{(y-2)(y+3)}\cdot\dfrac{y(y+3)}{y(y^2+2y+4)}=1$

**19.** $\dfrac{12y^2+28y+15}{6y^2+35y+25}\div\dfrac{2y^2-y-3}{3y^2+11y-20}=\dfrac{(6y+5)(2y+3)}{(6y+5)(y+5)}\cdot\dfrac{(3y-4)(y+5)}{(2y-3)(y+1)}=\dfrac{(2y+3)(3y-4)}{(2y-3)(y+1)}$

**20.** $\dfrac{z^2-81}{z^2-16}\div\dfrac{z^2-z-20}{z^2+5z-36}=\dfrac{(z-9)(z+9)}{(z-4)(z+4)}\cdot\dfrac{(z+9)(z-4)}{(z-5)(z+4)}=\dfrac{(z-9)(z+9)(z+9)}{(z+4)(z-5)(z+4)}=\dfrac{(z-9)(z+9)^2}{(z+4)^2(z-5)}$

**21.** $\dfrac{a^2+9}{a^2-64}\div\dfrac{a^3-3a^2+9a-27}{a^2+5a-24}=\dfrac{a^2+9}{(a-8)(a+8)}\cdot\dfrac{(a-3)(a+8)}{a^2(a-3)+9(a-3)}=\dfrac{a^2+9}{(a-8)(a+8)}\cdot\dfrac{(a-3)(a+8)}{(a-3)(a^2+9)}=\dfrac{1}{a-8}$

**22.** $\dfrac{6x^2+13xy+6y^2}{4x^2-9y^2}\div\dfrac{3x^2-xy-2y^2}{2x^2+xy-3y^2}=\dfrac{(3x+2y)(2x+3y)}{(2x-3y)(2x+3y)}\cdot\dfrac{(2x+3y)(x-y)}{(3x+2y)(x-y)}=\dfrac{2x+3y}{2x-3y}$

**23.** $\dfrac{p+5}{r}+\dfrac{2p-7}{r}=\dfrac{p+5+2p-7}{r}=\dfrac{3p-2}{r}$

**24.** $\dfrac{2s+5t}{4t}+\dfrac{-2s+3t}{4t}=\dfrac{2s+5t-2s+3t}{4t}=\dfrac{8t}{4t}=2$

**25.** $\dfrac{x}{x-5}+\dfrac{7x}{x+3}=\dfrac{x(x+3)+7x(x-5)}{(x-5)(x+3)}=\dfrac{x^2+3x+7x^2-35x}{(x-5)(x+3)}=\dfrac{8x^2-32x}{(x-5)(x+3)}=\dfrac{8x(x-4)}{(x-5)(x+3)}$

**26.** $\dfrac{2x}{3x+1}+\dfrac{5x}{x-7}=\dfrac{2x(x-7)+5x(3x+1)}{(3x+1)(x-7)}=\dfrac{2x^2-14x+15x^2+5x}{(3x+1)(x-7)}=\dfrac{17x^2-9x}{(3x+1)(x-7)}=\dfrac{x(17x-9)}{(3x+1)(x-7)}$

**27.** $\dfrac{5y-7}{y+4}-\dfrac{2y-3}{y+4}=\dfrac{(5y-7)-(2y-3)}{y+4}=\dfrac{5y-7-2y+3}{y+4}=\dfrac{3y-4}{y+4}$

**28.** $\frac{6x-5}{x-3}-\frac{3x-8}{x-3}=\frac{(6x-5)-(3x-8)}{x-3}=\frac{6x-5-3x+8}{x-3}=\frac{3x+3}{x-3}=\frac{3(x+1)}{x-3}$

**29.** $\frac{4z}{2z-3}+\frac{5z}{z-5}=\frac{4z(z-5)+5z(2z-3)}{(2z-3)(z-5)}=\frac{4z^2-20z+10z^2-15z}{(2z-3)(z-5)}=\frac{14z^2-35z}{(2z-3)(z-5)}=\frac{7z(2z-5)}{(2z-3)(z-5)}$

**30.** $\frac{3y-1}{3y+1}-\frac{2y-5}{y-3}=\frac{(3y-1)(y-3)-(2y-5)(3y+1)}{(3y+1)(y-3)}=\frac{(3y^2-10y+3)-(6y^2-13y-5)}{(3y+1)(y-3)}$

$=\frac{3y^2-10y+3-6y^2+13y+5}{(3y+1)(y-3)}=\frac{-3y^2+3y+8}{(3y+1)(y-3)}$

**31.** $\frac{x}{x^2-9}-\frac{3x-1}{x^2+7x+12}=\frac{x}{(x-3)(x+3)}-\frac{3x-1}{(x+3)(x+4)}=\frac{x(x+4)-(3x-1)(x-3)}{(x-3)(x+3)(x+4)}$

$=\frac{(x^2+4x)-(3x^2-10x+3)}{(x-3)(x+3)(x+4)}=\frac{x^2+4x-3x^2+10x-3}{(x-3)(x+3)(x+4)}=\frac{-2x^2+14x-3}{(x-3)(x+3)(x+4)}$

**32.** $\frac{m-n}{m^2-mn-6n^2}+\frac{3m-5n}{m^2+mn-2n^2}=\frac{m-n}{(m+2n)(m-3n)}+\frac{3m-5n}{(m-n)(m+2n)}=\frac{(m-n)(m-n)+(3m-5n)(m-3n)}{(m+2n)(m-3n)(m-n)}$

$=\frac{m^2-2mn+n^2+3m^2-14mn+15n^2}{(m+2n)(m-3n)(m-n)}=\frac{4m^2-16mn+16n^2}{(m+2n)(m-3n)(m-n)}$

$=\frac{4(m^2-4mn+4n^2)}{(m+2n)(m-3n)(m-n)}=\frac{4(m-2n)^2}{(m+2n)(m-3n)(m-n)}$

**33.** $\frac{1}{x}+\frac{2}{3x-1}\cdot\frac{3x^2+11x-4}{x-5}=\frac{1}{x}+\frac{2}{3x-1}\cdot\frac{(3x-1)(x+4)}{(x-5)}=\frac{1}{x}+\frac{2(x+4)}{x-5}=\frac{1(x-5)+x[2(x+4)]}{x(x-5)}$

$=\frac{x-5+2x^2+8x}{x(x-5)}=\frac{2x^2+9x-5}{x(x-5)}=\frac{(2x-1)(x+5)}{x(x-5)}$

**34.** $\frac{2}{y}-\frac{3}{y+1}\cdot\frac{y^2-1}{y+4}=\frac{2}{y}-\frac{3}{y+1}\cdot\frac{(y-1)(y+1)}{y+4}=\frac{2}{y}-\frac{3(y-1)}{y+4}=\frac{2(y+4)-3(y-1)y}{y(y+4)}=\frac{2(y+4)-y[3(y-1)]}{y(y+4)}$

$=\frac{2y+8-3y^2+3y}{y(y+4)}=\frac{-3y^2+5y+8}{y(y+4)}=-\frac{3y^2-5y-8}{y(y+4)}=-\frac{(3y-8)(y+1)}{y(y+4)}$

**35.** $\frac{q+1}{q-3}-\frac{2q}{q-3}\div\frac{q+5}{q-3}=\frac{q+1}{q-3}-\frac{2q}{q-3}\cdot\frac{q-3}{q+5}=\frac{q+1}{q-3}-\frac{2q}{q+5}=\frac{(q+1)(q+5)-2q(q-3)}{(q-3)(q+5)}$

$=\frac{q^2+6q+5-2q^2+6q}{(q-3)(q+5)}=\frac{-q^2+12q+5}{(q-3)(q+5)}$

**36.** $\frac{p}{p+5}+\frac{p}{p-4}\div\frac{p+2}{p^2-p-12}=\frac{p}{p+5}+\frac{p}{p-4}\cdot\frac{(p+3)(p-4)}{p+2}=\frac{p}{p+5}+\frac{p(p+3)}{p+2}=\frac{p(p+2)+p(p+3)(p+5)}{(p+5)(p+2)}$

$=\frac{p^2+2p+p(p^2+8p+15)}{(p+5)(p+2)}=\frac{p^2+2p+p^3+8p^2+15p}{(p+5)(p+2)}$

$=\frac{p^3+9p^2+17p}{(p+5)(p+2)}=\frac{p(p^2+9p+17)}{(p+5)(p+2)}$

**37.** $$\frac{1}{x^2+7x+12}+\frac{1}{x^2-9}+\frac{1}{x^2-16}=\frac{1}{(x+3)(x+4)}+\frac{1}{(x-3)(x+3)}+\frac{1}{(x-4)(x+4)}$$
$$=\frac{1(x-3)(x-4)+1(x-4)(x+4)+1(x-3)(x+3)}{(x+3)(x+4)(x-3)(x-4)}$$
$$=\frac{x^2-7x+12+x^2-16+x^2-9}{(x+3)(x+4)(x-3)(x-4)}=\frac{3x^2-7x-13}{(x+3)(x+4)(x-3)(x-4)}$$

**38.** $$\frac{2}{a^2-3a+2}+\frac{3}{a^2-1}-\frac{5}{a^2+3a-10}=\frac{2}{(a-1)(a-2)}+\frac{3}{(a-1)(a+1)}-\frac{5}{(a-2)(a+5)}$$
$$=\frac{2(a+1)(a+5)+3(a-2)(a+5)-5(a-1)(a+1)}{(a-1)(a-2)(a+1)(a+5)}$$
$$=\frac{2(a^2+6a+5)+3(a^2+3a-10)-5(a^2-1)}{(a-1)(a-2)(a+1)(a+5)}$$
$$=\frac{2a^2+12a+10+3a^2+9a-30-5a^2+5}{(a-1)(a-2)(a+1)(a+5)}$$
$$=\frac{21a-15}{(a-1)(a-2)(a+1)(a+5)}=\frac{3(7a-5)}{(a-1)(a-2)(a+1)(a+5)}$$

**39.** $$\left(1+\frac{2}{x}\right)\left(3-\frac{1}{x}\right)=\left(\frac{x}{x}+\frac{2}{x}\right)\left(\frac{3x}{x}-\frac{1}{x}\right)=\left(\frac{x+2}{x}\right)\left(\frac{3x-1}{x}\right)=\frac{(x+2)(3x-1)}{x^2}$$

**40.** $$\left(4-\frac{1}{z}\right)\left(4+\frac{2}{z}\right)=\left(\frac{4z}{z}-\frac{1}{z}\right)\left(\frac{4z}{z}+\frac{2}{z}\right)=\left(\frac{4z-1}{z}\right)\left(\frac{4z+2}{z}\right)=\frac{(4z-1)(4z+2)}{z^2}=\frac{(4z-1)2(2z+1)}{z^2}=\frac{2(4z-1)(2z+1)}{z^2}$$

**41.** $$\frac{4+\frac{1}{x}}{1-\frac{1}{x}}=\frac{\left(4+\frac{1}{x}\right)x}{\left(1-\frac{1}{x}\right)x}=\frac{4x+1}{x-1}$$

**42.** $$\frac{3-\frac{2}{a}}{5+\frac{3}{a}}=\frac{\left(3-\frac{2}{a}\right)a}{\left(5+\frac{3}{a}\right)a}=\frac{3a-2}{5a+3}$$

**43.** $$\frac{\frac{x}{y}-2}{y-x}=\frac{\left(\frac{x}{y}-2\right)y}{(y-x)y}=\frac{x-2y}{y(y-x)}$$

**44.** $$\frac{3+\frac{2}{x-3}}{4+\frac{1}{2+\frac{1}{x}}}=\frac{3+\frac{2}{x-3}}{4+\frac{1}{\frac{2x}{x}+\frac{1}{x}}}=\frac{3+\frac{2}{x-3}}{4+\frac{1}{\frac{2x+1}{x}}}=\frac{3+\frac{2}{x-3}}{4+1\div\frac{2x+1}{x}}=\frac{3+\frac{2}{x-3}}{4+1\cdot\frac{x}{2x+1}}=\frac{3+\frac{2}{x-3}}{4+\frac{x}{2x+1}}=\frac{\frac{3(x-3)}{x-3}+\frac{2}{x-3}}{\frac{4(2x+1)}{2x+1}+\frac{x}{2x+1}}$$
$$=\frac{\frac{3(x-3)+2}{x-3}}{\frac{4(2x+1)+x}{2x+1}}=\frac{\frac{3x-9+2}{x-3}}{\frac{8x+4+x}{2x+1}}=\frac{\frac{3x-7}{x-3}}{\frac{9x+4}{2x+1}}=\frac{3x-7}{x-3}\div\frac{9x+4}{2x+1}=\frac{3x-7}{x-3}\cdot\frac{2x+1}{9x+4}=\frac{(3x-7)(2x+1)}{(x-3)(9x+4)}$$

**45.** $$\frac{5-\frac{1}{x+2}}{1+\frac{3}{1+\frac{3}{x}}}=\frac{5-\frac{1}{x+2}}{1+\frac{3}{\frac{1(x)}{x}+\frac{3}{x}}}=\frac{5-\frac{1}{x+2}}{1+\frac{3}{\frac{x+3}{x}}}=\frac{5-\frac{1}{x+2}}{1+3\div\frac{x+3}{3}}=\frac{5-\frac{1}{x+2}}{1+3\cdot\frac{x}{x+3}}=\frac{5-\frac{1}{x+2}}{1+\frac{3x}{x+3}}=\frac{\frac{5(x+2)}{x+2}-\frac{1}{x+2}}{\frac{1(x+3)}{x+3}+\frac{3x}{x+3}}$$
$$=\frac{\frac{5(x+2)-1}{x+2}}{\frac{1(x+3)+3x}{x+3}}=\frac{\frac{5x+10-1}{x+2}}{\frac{x+3+3x}{x+3}}=\frac{\frac{5x+9}{x+2}}{\frac{4x+3}{x+3}}=\frac{5x+9}{x+2}\div\frac{4x+3}{x+3}=\frac{5x+9}{x+2}\cdot\frac{x+3}{4x+3}=\frac{(5x+9)(x+3)}{(x+2)(4x+3)}$$

**46.** $$\frac{\frac{1}{(x+h)^2}-1}{h}=\frac{\frac{1}{(x+h)^2}-\frac{1(x+h)^2}{(x+h)^2}}{h}=\frac{\frac{1-1(x+h)^2}{(x+h)^2}}{h}=\frac{\frac{1-1(x^2+2xh+h^2)}{(x+h)^2}}{h}=\frac{\frac{1-x^2-2xh-h^2}{(x+h)^2}}{h}$$
$$=\frac{1-x^2-2xh-h^2}{(x+h)^2}\div\frac{h}{1}=\frac{1-x^2-2xh-h^2}{(x+h)^2}\cdot\frac{1}{h}=\frac{1-x^2-2xh-h^2}{h(x+h)^2}$$

**47.** $$\frac{1+\frac{1}{b-2}}{1-\frac{1}{b+3}}=\frac{\left(1+\frac{1}{b-2}\right)}{\left(1-\frac{1}{b+3}\right)}\cdot\frac{(b-2)(b+3)}{(b-2)(b+3)}=\frac{1(b-2)(b+3)+1(b+3)}{1(b-2)(b+3)-1(b-2)}=\frac{b^2+b-6+b+3}{b^2+b-6-b+2}=\frac{b^2+2b-3}{b^2-4}=\frac{(b+3)(b-1)}{(b-2)(b+2)}$$

**48.** $$r-\frac{r}{r+\frac{1}{3}}=r-\frac{r}{\frac{3r}{3}+\frac{1}{3}}=r-\frac{r}{\frac{3r+1}{3}}=r-\left(r\div\frac{3r+1}{3}\right)=r-\left(r\cdot\frac{3}{3r+1}\right)=r-\frac{3r}{3r+1}$$
$$=\frac{r(3r+1)}{3r+1}-\frac{3r}{3r+1}=\frac{r(3r+1)-3r}{3r+1}=\frac{3r^2+r-3r}{3r+1}=\frac{3r^2-2r}{3r+1}=\frac{r(3r-2)}{3r+1}$$

**49.** $$\frac{1-\frac{1}{x^2}}{1+\frac{1}{x}}=\frac{\left(1-\frac{1}{x^2}\right)}{\left(1+\frac{1}{x}\right)}\cdot\frac{x^2}{x^2}=\frac{x^2-1}{x^2+x}=\frac{(x-1)(x+1)}{x(x+1)}=\frac{x-1}{x}$$

**50.** $$\frac{1}{\frac{1}{a}+\frac{1}{b}}=\frac{1}{\frac{1b}{ab}+\frac{1a}{ab}}=\frac{1}{\frac{b+a}{ab}}=1\div\frac{b+a}{ab}=1\cdot\frac{ab}{b+a}=\frac{ab}{b+a}$$

**51.** $$2-\frac{m}{1-\frac{1-m}{-m}}=2-\frac{m}{1+\frac{1-m}{m}}=2-\frac{m}{\frac{1(m)}{m}+\frac{1-m}{m}}=2-\frac{m}{\frac{m+1-m}{m}}=2-\frac{m}{\frac{1}{m}}=2-\frac{(m)}{\left(\frac{1}{m}\right)}\cdot\frac{m}{m}=2-m^2$$

**52.** $$\frac{\left(\frac{x+h+1}{x+h}-\frac{x}{x+1}\right)}{(h)}\cdot\frac{(x+h)(x+1}{(x+h)(x+1)}=\frac{(x+h+1)(x+1)-x(x+h)}{h(x+h)(x+1)}=\frac{x^2+x+xh+h+x+1-x^2-xh}{h(x+h)(x+1)}=\frac{2x+h+1}{h(x+h)(x+1)}$$

**53.** $$\frac{\left(\frac{1}{x}-\frac{x-4}{x+1}\right)}{\frac{x}{x+1}}\cdot\frac{x(x+1)}{x(x+1)}=\frac{x+1-x(x-4)}{x(x)}=\frac{x+1-x^2+4x}{x^2}=\frac{-x^2+5x+1}{x^2}$$

**54.** $$\frac{\left(\frac{2}{y}-\frac{3y-2}{y-1}\right)}{\left(\frac{y}{y-1}\right)}\cdot\frac{y(y-1)}{y(y-1)}=\frac{2(y-1)-y(3y-2)}{y(y)}=\frac{2y-2-3y^2+2y}{y^2}=\frac{-3y^2+4y-2}{y^2}$$

**55.** $$\frac{\left(\frac{1}{x+3}-\frac{2}{x-1}\right)}{\left(\frac{x}{x-1}+\frac{3}{x+3}\right)}\cdot\frac{(x+3)(x-1)}{(x+3)(x-1)}=\frac{1(x-1)-2(x+3)}{x(x+3)+3(x-1)}=\frac{x-1-2x-6}{x^2+3x+3x-3}=\frac{-x-7}{x^2+6x-3}$$

**56.** $$\frac{\frac{x+2}{x^2-1}+\frac{1}{x+1}}{\frac{x}{2x^2-x-1}+\frac{1}{x-1}}=\frac{\frac{x+2}{(x-1)(x+1)}+\frac{1}{x+1}}{\frac{x}{(2x+1)(x-1)}+\frac{1}{x-1}}\cdot\frac{(x-1)(x+1)(2x+1)}{(x-1)(x+1)(2x+1)}=\frac{(x+2)(2x+1)+1(x-1)(2x+1)}{x(x+1)+1(x+1)(2x+1)}$$
$$=\frac{2x^2+5x+2+2x^2-x-1}{x^2+x+2x^2+3x+1}=\frac{4x^2+4x+1}{3x^2+4x+1}=\frac{(2x+1)^2}{(3x+1)(x+1)}$$

**57.** $$\frac{\frac{x^2+3x-10}{x^2+x-6}}{\frac{x^2-x-30}{2x^2-15x+18}} = \frac{\frac{(x-2)(x+5)}{(x-2)(x+3)}}{\frac{(x+5)(x-6)}{(2x-3)(x-6)}} = \frac{\frac{x+5}{x+3}}{\frac{x+5}{2x-3}} = \frac{x+5}{x+3} \div \frac{x+5}{2x-3} = \frac{x+5}{x+3}\cdot\frac{2x-3}{x+5} = \frac{2x-3}{x+3}$$

**58.** $$\frac{\frac{2y^2+11y+15}{y^2-4y-21}}{\frac{6y^2+11y-10}{3y^2-23y+14}} = \frac{\frac{(2y+5)(y+3)}{(y+3)(y-7)}}{\frac{(3y-2)(2y+5)}{(3y-2)(y-7)}} = \frac{\frac{2y+5}{y-7}}{\frac{2y+5}{y-7}} = \frac{2y+5}{y-7} \div \frac{2y+5}{y-7} = \frac{2y+5}{y-7}\cdot\frac{y-7}{2y+5} = 1$$

**59.** $$\frac{a^{-1}+b^{-1}}{a-b} = \frac{\frac{1}{a}+\frac{1}{b}}{a-b} = \frac{\frac{1b}{ab}+\frac{1a}{ab}}{a-b} = \frac{\frac{b+a}{ab}}{a-b} = \frac{b+a}{ab} \div (a-b) = \frac{b+a}{ab}\cdot\frac{1}{a-b} = \frac{a+b}{ab(a-b)}$$

**60.** $$\frac{e^{-2}-f^{-1}}{ef} = \frac{\frac{1}{e^2}-\frac{1}{f}}{ef} = \frac{\frac{1f}{e^2f}-\frac{1e^2}{e^2f}}{ef} = \frac{\frac{f-e^2}{e^2f}}{ef} = \frac{f-e^2}{e^2f} \div ef = \frac{f-e^2}{e^2f}\cdot\frac{1}{ef} = \frac{f-e^2}{e^3f^2}$$

**61.** $$\frac{a^{-1}b-ab^{-1}}{a^2+b^2} = \frac{\frac{b}{a}-\frac{a}{b}}{a^2+b^2} = \frac{\frac{(b)b}{(b)a}-\frac{a(a)}{b(a)}}{a^2+b^2} = \frac{\frac{b^2}{ab}-\frac{a^2}{ab}}{a^2+b^2} = \frac{\frac{b^2-a^2}{ab}}{a^2+b^2} = \frac{b^2-a^2}{ab} \div a^2+b^2$$
$$= \frac{b^2-a^2}{ab}\cdot\frac{1}{a^2+b^2} = \frac{b^2-a^2}{ab(a^2+b^2)} = \frac{(b-a)(b+a)}{ab(a^2+b^2)}$$

**62.** $$\left(a+b^{-2}\right)^{-1} = \left(a+\frac{1}{b^2}\right)^{-1} = \left(\frac{ab^2}{b^2}+\frac{1}{b^2}\right)^{-1} = \left(\frac{ab^2+1}{b^2}\right)^{-1} = \frac{b^2}{ab^2+1}$$

**63.** **a.** $$\frac{2}{\frac{1}{180}+\frac{1}{110}} = \frac{2}{\frac{110+180}{180(110)}} = 2 \div \frac{290}{180(110)}$$
$$= 2\cdot\frac{(180)(110)}{290} \approx 136.55 \text{ mph (to the nearest hundredth)}$$

**b.** $$\frac{2}{\frac{1}{v_1}+\frac{1}{v_2}} = \frac{2}{\frac{v_2+v_1}{v_1v_2}} = \frac{2v_1v_2}{v_2+v_1} = \frac{2v_1v_2}{v_1+v_2}$$

**64.** **a.** $$\frac{v_1+v_2}{1+\frac{v_1v_2}{c^2}} = \frac{1.2\times10^8+2.4\times10^8}{1+\frac{\left(1.2\times10^8\right)\left(2.4\times10^8\right)}{\left(6.7\times10^8\right)^2}} \approx 3.4\times10^8$$

**b.** $$\frac{v_1+v_2}{1+\frac{v_1v_2}{c^2}} = \frac{\left(v_1+v_2\right)}{\left(1+\frac{v_1v_2}{c^2}\right)}\cdot\frac{c^2}{c^2} = \frac{c^2\left(v_1+v_2\right)}{c^2+v_1v_2}$$

**65.** $$\frac{1}{x}+\frac{1}{x+1} = \frac{x+1+x}{x(x+1)} = \frac{2x+1}{x(x+1)}$$

**66.** $$\frac{1}{x}-\frac{1}{x+2} = \frac{x+2-x}{x(x+2)} = \frac{2}{x(x+2)}$$

**67.** $$\frac{1}{x-2}+\frac{1}{x}+\frac{1}{x+2} = \frac{x(x+2)+(x-2)(x+2)+x(x-2)}{x(x-2)(x+2)} = \frac{x^2+2x+x^2-4+x^2-2x}{x(x-2)(x+2)} = \frac{3x^2-4}{x(x-2)(x+2)}$$

**68.** $$\frac{1}{(x-2)^2}+\frac{1}{x^2}+\frac{1}{(x+2)^2}=\frac{x^2(x+2)^2+(x-2)^2(x+2)^2+x^2(x-2)^2}{x^2(x-2)^2(x+2)^2}$$
$$=\frac{x^2(x^2+4x+4)+(x^2-4x+4)(x^2+4x+4)+x^2(x^2-4x+4)}{x^2(x-2)^2(x+2)^2}$$
$$=\frac{x^4+4x^3+4x^2+x^4-8x^2+16+x^4-4x^3+4x^2}{x^2(x-2)^2(x+2)^2}=\frac{3x^4+16}{x^2(x-2)^2(x+2)^2}$$

## Connecting Concepts

**69.** $$\frac{(x+5)-x(x+5)^{-1}}{x+5}\cdot\frac{x+5}{x+5}=\frac{(x+5)^2-x}{(x+5)^2}=\frac{x^2+10x+25-x}{(x+5)^2}=\frac{x^2+9x+25}{(x+5)^2}$$

**70.** $$\frac{(y+2)+y^2(y+2)^{-1}}{y+2}\cdot\frac{y+2}{y+2}=\frac{(y+2)^2+y^2}{(y+2)^2}=\frac{y^2+4y+4+y^2}{(y+2)^2}=\frac{2y^2+4y+4}{(y+2)^2}=\frac{2(y^2+2y+2)}{(y+2)^2}$$

**71.** $$\frac{x^{-1}-4y}{(x^{-1}-2y)(x^{-1}+2y)}=\frac{\frac{1}{x}-4y}{\left(\frac{1}{x}-2y\right)\left(\frac{1}{x}+2y\right)}=\frac{\frac{1-4xy}{x}}{\left(\frac{1-2xy}{x}\right)\left(\frac{1+2xy}{x}\right)}\cdot\frac{x^2}{x^2}=\frac{x(1-4xy)}{(1-2xy)(1+2xy)}$$

**72.** $$\frac{x+y}{x-y}\cdot\frac{x^{-1}-y^{-1}}{x^{-1}+y^{-1}}=\frac{x+y}{x-y}\cdot\frac{\frac{1}{x}-\frac{1}{y}}{\frac{1}{x}+\frac{1}{y}}\cdot\frac{xy}{xy}=\frac{x+y}{x-y}\cdot\frac{y-x}{y+x}=-1$$

**73.** $$R\left[\frac{1-\frac{1}{(1+i)^n}}{i}\right]=R\left[\frac{\frac{(1+i)^n-1}{(1+i)^n}}{i}\right]=R\left[\frac{(1+i)^n-1}{i(1+i)^n}\right]$$

**74.** $$\frac{1}{\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}}=\frac{1}{\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)}\cdot\frac{R_1R_2R_3}{R_1R_2R_3}=\frac{R_1R_2R_3}{R_2R_3+R_1R_3+R_1R_2}$$

## Prepare for Section P.6

**PS1.** $$(2-3x)(4-5x)=8-10x-12x+15x^2$$
$$=15x^2-22x+8$$

**PS2.** $$(2-5x)^2=2^2+2(2)(-5x)+(-5x)^2$$
$$=4-20x+25x^2$$
$$=25x^2-20x+4$$

**PS3.** $$\sqrt{96}=\sqrt{16\cdot 6}=4\sqrt{6}$$

**PS4.** $$\left(2+3\sqrt{5}\right)\left(3-4\sqrt{5}\right)=6-8\sqrt{5}+9\sqrt{5}-12\left(\sqrt{5}\right)^2=6+\sqrt{5}-60=-54+\sqrt{5}$$

**PS5.** $$\frac{5+\sqrt{2}}{3-\sqrt{2}}=\frac{5+\sqrt{2}}{3-\sqrt{2}}\cdot\frac{3+\sqrt{2}}{3+\sqrt{2}}=\frac{15+8\sqrt{2}+2}{9-2}=\frac{17+8\sqrt{2}}{7}$$

**PS6.** **a.** $81 - x^2$ is a difference of perfect squares with integer coefficients, which does factor over the integers.
$81 - x^2 = (9 - x)(9 + x)$

**b.** $9 + z^2$ is a sum of perfect squares with integer coefficients. If there is no common factor, sums of perfect squares with integer coefficients are not factorable over the integers.

## Section P.6

**1.** $\sqrt{-81} = i\sqrt{81} = 9i$

**2.** $\sqrt{-64} = i\sqrt{64} = 8i$

**3.** $\sqrt{-98} = i\sqrt{98} = 7i\sqrt{2}$

**4.** $\sqrt{-27} = i\sqrt{27} = 3i\sqrt{3}$

**5.** $\sqrt{16} + \sqrt{-81} = 4 + i\sqrt{81}$
$= 4 + 9i$

**6.** $\sqrt{25} + \sqrt{-9} = 5 + i\sqrt{9}$
$= 5 + 3i$

**7.** $5 + \sqrt{-49} = 5 + i\sqrt{49} = 5 + 7i$

**8.** $6 - \sqrt{-1} = 6 - i\sqrt{1} = 6 - i$

**9.** $8 - \sqrt{-18} = 8 - i\sqrt{18} = 8 - 3i\sqrt{2}$

**10.** $11 + \sqrt{-48} = 11 + i\sqrt{48} = 11 + 4i\sqrt{3}$

**11.** $(5 + 2i) + (6 - 7i) = 5 + 2i + 6 - 7i$
$= (5 + 6) + (2i - 7i)$
$= 11 - 5i$

**12.** $(4 - 8i) + (5 + 3i) = 4 - 8i + 5 + 3i$
$= (4 + 5) + (-8i + 3i)$
$= 9 - 5i$

**13.** $(-2 - 4i) - (5 - 8i) = -2 - 4i - 5 + 8i$
$= (-2 - 5) + (-4i + 8i)$
$= -7 + 4i$

**14.** $(3 - 5i) - (8 - 2i) = 3 - 5i - 8 + 2i$
$= (3 - 8) + (-5i + 2i)$
$= -5 - 3i$

**15.** $(1 - 3i) + (7 - 2i) = 1 - 3i + 7 - 2i$
$= (1 + 7) + (-3i - 2i)$
$= 8 - 5i$

**16.** $(2 - 6i) + (4 - 7i) = 2 - 6i + 4 - 7i$
$= (2 + 4) + (-6i - 7i)$
$= 6 - 13i$

**17.** $(-3 - 5i) - (7 - 5i) = -3 - 5i - 7 + 5i$
$= (-3 - 7) + (-5i + 5i)$
$= -10$

**18.** $(5 - 3i) - (2 + 9i) = 5 - 3i - 2 - 9i$
$= (5 - 2) + (-3i - 9i)$
$= 3 - 12i$

**19.** $8i - (2 - 8i) = 8i - 2 + 8i$
$= -2 + (8i + 8i)$
$= -2 + 16i$

**20.** $3 - (4 - 5i) = 3 - 4 + 5i$
$= (3 - 4) + 5i$
$= -1 + 5i$

**21.** $5i \cdot 8i = 40i^2$
$= 40(-1)$
$= -40$

**22.** $(-3i)(2i) = -6i^2$
$= -6(-1)$
$= 6$

**23.** $\sqrt{-50} \cdot \sqrt{-2} = i\sqrt{50} \cdot i\sqrt{2} = 5i\sqrt{2} \cdot i\sqrt{2}$
$= 5i^2(\sqrt{2})^2 = 5(-1)(2)$
$= -10$

**24.** $\sqrt{-12} \cdot \sqrt{-27} = i\sqrt{12} \cdot i\sqrt{27} = 2i\sqrt{3} \cdot 3i\sqrt{3}$
$= 6i^2(\sqrt{3})^2 = 6(-1)(3)$
$= -18$

**25.** $3(2 + 5i) - 2(3 - 2i) = 6 + 15i - 6 + 4i$
$= (6 - 6) + (15i + 4i)$
$= 19i$

**26.** $3i(2 + 5i) + 2i(3 - 4i) = 6i + 15i^2 + 6i - 8i^2$
$= 6i + 15(-1) + 6i - 8(-1)$
$= 6i - 15 + 6i + 8$
$= (-15 + 8) + (6i + 6i)$
$= -7 + 12i$

**27.** $(4 + 2i)(3 - 4i) = 4(3 - 4i) + (2i)(3 - 4i)$
$= 12 - 16i + 6i - 8i^2$
$= 12 - 16i + 6i - 8(-1)$
$= 12 - 16i + 6i + 8$
$= (12 + 8) + (-16i + 6i)$
$= 20 - 10i$

**28.** $(6+5i)(2-5i) = 6(2-5i)+5i(2-5i)$
$= 12-30i+10i-25i^2$
$= 12-30i+10i-25(-1)$
$= 12-30i+10i+25$
$= (12+25)+(-30i+10i)$
$= 37-20i$

**29.** $(-3-4i)(2+7i) = -3(2+7i)-4i(2+7i)$
$= -6-21i-8i-28i^2$
$= -6-21i-8i-28(-1)$
$= -6-21i-8i+28$
$= (-6+28)+(-21i-8i)$
$= 22-29i$

**30.** $(-5-i)(2+3i) = -5(2+3i)-i(2+3i)$
$= -10-15i-2i-3i^2$
$= -10-15i-2i-3(-1)$
$= -10-15i-2i+3$
$= (-10+3)+(-15i-2i)$
$= -7-17i$

**31.** $(4-5i)(4+5i) = 4(4+5i)-5i(4+5i)$
$= 16+20i-20i-25i^2$
$= 16+20i-20i-25(-1)$
$= 16+20i-20i+25$
$= (16+25)+(20i-20i)$
$= 41$

**32.** $(3+7i)(3-7i) = 3(3-7i)+7i(3-7i)$
$= 9-21i+21i-49i^2$
$= 9-21i+21i-49(-1)$
$= 9-21i+21i+49$
$= (9+49)+(-21i+21i)$
$= 58$

**33.** $(3+\sqrt{-4})(2-\sqrt{-9}) = (3+i\sqrt{4})(2-i\sqrt{9})$
$= (3+2i)(2-3i)$
$= 3(2-3i)+2i(2-3i)$
$= 6-9i+4i-6i^2$
$= 6-9i+4i-6(-1)$
$= 6-9i+4i+6$
$= (6+6)+(-9i+4i)$
$= 12-5i$

**34.** $(5+2\sqrt{-16})(1-\sqrt{-25}) = (5+2i\sqrt{16})(1-i\sqrt{25}) = [5+2i(4)][1-i(5)] = (5+8i)(1-5i) = 5(1-5i)+8i(1-5i)$
$= 5-25i+8i-40i^2 = 5-25i+8i-40(-1) = 5-25i+8i+40 = (5+40)+(-25i+8i)$
$= 45-17i$

**35.** $(3+2\sqrt{-18})(2+2\sqrt{-50}) = (3+2i\sqrt{18})(2+2i\sqrt{50}) = [3+2i(3\sqrt{2})][2+2i(5\sqrt{2})] = (3+6i\sqrt{2})(2+10i\sqrt{2})$
$= 3(2+10i\sqrt{2})+6i\sqrt{2}(2+10i\sqrt{2}) = 6+30i\sqrt{2}+12i\sqrt{2}+60i^2(\sqrt{2})^2$
$= 6+30i\sqrt{2}+12i\sqrt{2}+60(-1)(2) = 6+30i\sqrt{2}+12i\sqrt{2}-120$
$= (6-120)+(30i\sqrt{2}+12i\sqrt{2}) = -114+42i\sqrt{2}$

**36.** $(5-3\sqrt{-48})(2-4\sqrt{-27}) = (5-3i\sqrt{48})(2-4i\sqrt{27}) = [5-3i(4\sqrt{3})][2-4i(3\sqrt{3})] = (5-12i\sqrt{3})(2-12i\sqrt{3})$
$= 5(2-12i\sqrt{3})-12i\sqrt{3}(2-12i\sqrt{3}) = 10-60i\sqrt{3}-24i\sqrt{3}+144i^2(\sqrt{3})^2$
$= 10-60i\sqrt{3}-24i\sqrt{3}+144(-1)(3) = 10-60i\sqrt{3}-24i\sqrt{3}-432$
$= (10-432)+(-60i\sqrt{3}-24i\sqrt{3}) = -422-84i\sqrt{3}$

**37.** $\dfrac{6}{i} = \dfrac{6}{i}\cdot\dfrac{i}{i} = \dfrac{6i}{i^2} = \dfrac{6i}{-1} = -6i$

**38.** $\dfrac{-8}{2i} = \dfrac{-\overset{4}{\cancel{8}}}{\cancel{2}i} = \dfrac{-4}{i} = \dfrac{-4}{i}\cdot\dfrac{i}{i} = \dfrac{-4i}{i^2} = \dfrac{-4i}{-1} = 4i$

**39.** $\dfrac{6+3i}{i} = \dfrac{6+3i}{i}\cdot\dfrac{i}{i} = \dfrac{6i+3i^2}{i^2} = \dfrac{6i+3(-1)}{-1} = \dfrac{6i-3}{-1} = 3-6i$

**40.** $\dfrac{4-8i}{4i} = \dfrac{4(1-2i)}{4i} = \dfrac{\cancel{4}(1-2i)}{\cancel{4}i} = \dfrac{1-2i}{i} = \dfrac{1-2i}{i}\cdot\dfrac{i}{i} = \dfrac{i-2i^2}{i^2} = \dfrac{i-2(-1)}{-1} = \dfrac{i+2}{-1} = -2-i$

**41.** $\dfrac{1}{7+2i} = \dfrac{1}{7+2i}\cdot\dfrac{7-2i}{7-2i} = \dfrac{1(7-2i)}{(7+2i)(7-2i)} = \dfrac{7-2i}{49-4i^2} = \dfrac{7-2i}{49-4(-1)} = \dfrac{7-2i}{49+4} = \dfrac{7-2i}{53} = \dfrac{7}{53}-\dfrac{2}{53}i$

**42.** $\frac{5}{3+4i} = \frac{5}{3+4i} \cdot \frac{3-4i}{3-4i} = \frac{5(3-4i)}{(3+4i)(3-4i)} = \frac{15-20i}{9-16i^2} = \frac{15-20i}{9-16(-1)} = \frac{15-20i}{9+16} = \frac{15-20i}{25} = \frac{15}{25} - \frac{20}{25}i = \frac{3}{5} - \frac{4}{5}i$

**43.** $\frac{2i}{1+i} = \frac{2i}{1+i} \cdot \frac{1-i}{1-i} = \frac{2i(1-i)}{(1+i)(1-i)} = \frac{2i-2i^2}{1-i^2} = \frac{2i-2(-1)}{1-(-1)} = \frac{2i+2}{1+1} = \frac{2+2i}{2} = \frac{2}{2} + \frac{2}{2}i = 1+i$

**44.** $\frac{5i}{2-3i} = \frac{5i}{2-3i} \cdot \frac{2+3i}{2+3i} = \frac{5i(2+3i)}{(2-3i)(2+3i)} = \frac{10i+15i^2}{4-9i^2} = \frac{10i+15(-1)}{4-9(-1)} = \frac{10i-15}{4+9} = \frac{-15+10i}{13} = -\frac{15}{13} + \frac{10}{13}i$

**45.** $\frac{5-i}{4+5i} = \frac{5-i}{4+5i} \cdot \frac{4-5i}{4-5i} = \frac{(5-i)(4-5i)}{(4+5i)(4-5i)} = \frac{5(4-5i)-i(4-5i)}{4(4-5i)+5i(4-5i)} = \frac{20-25i-4i+5i^2}{16-20i+20i-25i^2}$

$= \frac{20-25i-4i+5(-1)}{16-25(-1)} = \frac{20-25i-4i-5}{16+25} = \frac{(20-5)+(-25i-4i)}{16+25} = \frac{15-29i}{41} = \frac{15}{41} - \frac{29}{41}i$

**46.** $\frac{4+i}{3+5i} = \frac{4+i}{3+5i} \cdot \frac{3-5i}{3-5i} = \frac{(4+i)(3-5i)}{(3+5i)(3-5i)} = \frac{4(3-5i)+i(3-5i)}{9-25i^2} = \frac{12-20i+3i-5i^2}{9-25i^2} = \frac{12-20i+3i-5(-1)}{9-25(-1)}$

$= \frac{12-20i+3i-5(-1)}{9-25(-1)} = \frac{12-20i+3i+5}{9+25} = \frac{(12+5)+(-20i+3i)}{34} = \frac{17-17i}{34} = \frac{17}{34} - \frac{17}{34}i = \frac{1}{2} - \frac{1}{2}i$

**47.** $\frac{3+2i}{3-2i} = \frac{3+2i}{3-2i} \cdot \frac{3+2i}{3+2i} = \frac{(3+2i)^2}{(3-2i)(3+2i)} = \frac{3^2+2(3)(2i)+(2i)^2}{3^2-(2i)^2} = \frac{9+12i+4i^2}{9-4i^2} = \frac{9+12i+4(-1)}{9-4(-1)}$

$= \frac{9+12i-4}{9+4} = \frac{5+12i}{13} = \frac{5}{13} + \frac{12}{13}i$

**48.** $\frac{8-i}{2+3i} = \frac{8-i}{2+3i} \cdot \frac{2-3i}{2-3i} = \frac{(8-i)(2-3i)}{(2+3i)(2-3i)} = \frac{8(2-3i)-i(2-3i)}{2^2-(3i)^2} = \frac{16-24i-2i+3i^2}{4-9i^2} = \frac{16-24i-2i+3(-1)}{4-9(-1)}$

$= \frac{16-24i-2i-3}{4+9} = \frac{13-26i}{13} = \frac{13}{13} - \frac{26i}{13} = 1-2i$

**49.** $\frac{-7+26i}{4+3i} = \frac{-7+26i}{4+3i} \cdot \frac{4-3i}{4-3i} = \frac{(-7+26i)(4-3i)}{(4+3i)(4-3i)} = \frac{-7(4-3i)+26i(4-3i)}{4^2-(3i)^2} = \frac{-28+21i+104i-78i^2}{16-9i^2}$

$= \frac{-28+21i+104i-78(-1)}{16-9(-1)} = \frac{-28+21i+104i+78}{16+9} = \frac{50+125i}{25} = \frac{50}{25} + \frac{125}{25}i = 2+5i$

**50.** $\frac{-4-39i}{5-2i} = \frac{-4-39i}{5-2i} \cdot \frac{5+2i}{5+2i} = \frac{(-4-39i)(5+2i)}{(5-2i)(5+2i)} = \frac{-4(5+2i)-39i(5+2i)}{5^2-(2i)^2} = \frac{-20-8i-195i-78i^2}{25-4i^2} = \frac{5+2i}{5+2i}$

$= \frac{-20-8i-195i-78(-1)}{25-4(-1)} = \frac{-20-8i-195i+78}{25+4} = \frac{58-203i}{29} = \frac{58}{29} - \frac{203i}{29} = 2-7i$

**51.** $(3-5i)^2 = 3^3 + 2(3)(-5i) + (-5i)^2$
$= 9-30i+25i^2$
$= 9-30i+25(-1)$
$= 9-30i-25$
$= -16-30i$

**52.** $(2+4i)^2 = 2^2 + 2(2)(4i) + (4i)^2$
$= 4+16i+16i^2$
$= 4+16i+16(-1)$
$= 4+16i-16$
$= -12+16i$

**53.** $(1+2i)^3 = (1+2i)(1+2i)^2$
$= (1+2i)[1^2 + 2(1)(2i) + (2i)^2]$
$= (1+2i)[1+4i+4i^2]$
$= (1+2i)[1+4i+4(-1)]$
$= (1+2i)[1+4i-4]$
$= (1+2i)(-3+4i)$
$= 1(-3+4i) + 2i(-3+4i)$
$= -3+4i-6i+8i^2$
$= -3+4i-6i-8$
$= -11-2i$

**54.** $(2-i)^3 = (2-i)(2-i)^2$
$= (2-i)[2^2 + 2(2)(-i) + (-i)^2]$
$= (2-i)[4-4i+i^2]$
$= (2-i)[4-4i-1]$
$= (2-i)(3-4i)$
$= 2(3-4i) - i(3-4i)$
$= 6-8i-3i+4i^2$
$= 6-8i-3i+4(-1)$
$= 6-8i-3i-4$
$= 2-11i$

**55.** Use the Powers of $i$ Theorem.
The remainder of $15 \div 4$ is 3.
$i^{15} = i^3 = -i$

**56.** Use the Powers of $i$ Theorem.
The remainder of $66 \div 4$ is 2.
$i^{66} = i^2 = -1$

**57.** Use the Powers of $i$ Theorem.
The remainder of $40 \div 4$ is 0.
$-i^{40} = -(i^0) = -1$

**58.** Use the Powers of $i$ Theorem.
The remainder of $51 \div 4$ is 3.
$-i^{51} = -(i^3) = -(-i) = i$

**59.** Use the Powers of $i$ Theorem.
The remainder of $25 \div 4$ is 1.
$\frac{1}{i^{25}} = \frac{1}{i} = \frac{1}{i}\cdot\frac{i}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$

**60.** Use the Powers of $i$ Theorem.
The remainder of $83 \div 4$ is 3.
$\frac{1}{i^{83}} = \frac{1}{i^3} = \frac{1}{i^3}\cdot\frac{i}{i} = \frac{i}{i^4} = \frac{i}{1} = i$

**61.** Use the Powers of $i$ Theorem.
The remainder of $34 \div 4$ is 2.
$i^{-34} = \frac{1}{i^{34}} = \frac{1}{i^2} = \frac{1}{-1} = -1$

**62.** Use the Powers of $i$ Theorem.
The remainder of $52 \div 4$ is 0.
$i^{-52} = \frac{1}{i^{52}} = \frac{1}{i^0} = \frac{1}{1} = 1$

**63.** Use $a = 3$, $b = -3$, $c = 3$.

$$\frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{-(-3)+\sqrt{(-3)^2-4(3)(3)}}{2(3)} = \frac{3+\sqrt{9-36}}{6} = \frac{3+\sqrt{-27}}{6} = \frac{3+i\sqrt{27}}{6} = \frac{3+3i\sqrt{3}}{6} = \frac{3}{6}+\frac{3\sqrt{3}}{6}i = \frac{1}{2}+\frac{\sqrt{3}}{2}i$$

**64.** Use $a = 2$, $b = 4$, $c = 4$.

$$\frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{-(4)+\sqrt{(4)^2-4(2)(4)}}{2(2)} = \frac{-4+\sqrt{16-32}}{4} = \frac{-4+\sqrt{-16}}{4} = \frac{-4+i\sqrt{16}}{4} = \frac{-4+4i}{4} = \frac{-4}{4}+\frac{4i}{4} = -1+i$$

**65.** Use $a = 2$ $b = 6$, $c = 6$.

$$\frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{-(6)+\sqrt{(6)^2-4(2)(6)}}{2(2)} = \frac{-6+\sqrt{36-48}}{4} = \frac{-6+\sqrt{-12}}{4} = \frac{-6+i\sqrt{12}}{4} = \frac{-6+2i\sqrt{3}}{4} = \frac{-6}{4}+\frac{2i\sqrt{3}}{4} = -\frac{3}{2}+\frac{\sqrt{3}}{2}i$$

**66.** Use $a$ =2, $b = 1$, $c = 3$.

$$\frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{-(1)+\sqrt{(1)^2-4(2)(3)}}{2(2)} = \frac{-1+\sqrt{1-24}}{4} = \frac{-1+\sqrt{-23}}{4} = \frac{-1+i\sqrt{23}}{4} = -\frac{1}{4}+\frac{\sqrt{23}}{4}i$$

**67.** Use $a = 4$, $b = -4$, $c = 2$.

$$\frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{-(-4)+\sqrt{(-4)^2-4(4)(2)}}{2(4)}$$
$$= \frac{4+\sqrt{16-32}}{8} = \frac{4+\sqrt{-16}}{8}$$
$$= \frac{4+i\sqrt{16}}{8} = \frac{4+4i}{8}$$
$$= \frac{4}{8}+\frac{4i}{8} = \frac{1}{2}+\frac{1}{2}i$$

**68.** Use $a = 3$, $b = -2$, $c = 4$.

$$\frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{-(-2)+\sqrt{(-2)^2-4(3)(4)}}{2(3)}$$
$$= \frac{2+\sqrt{4-48}}{6} = \frac{2+\sqrt{-44}}{6}$$
$$= \frac{2+i\sqrt{44}}{6} = \frac{2+2i\sqrt{11}}{6}$$
$$= \frac{2}{6}+\frac{2i\sqrt{11}}{6} = \frac{1}{3}+\frac{\sqrt{11}}{3}i$$

## Connecting Concepts

**69.** $x^2+16 = x^2+4^2 = (x+4i)(x-4i)$

**70.** $x^2+9 = x^2+3^2 = (x+3i)(x-3i)$

**71.** $z^2+25 = z^2+5^2 = (z+5i)(z-5i)$

**72.** $z^2+64 = z^2+8^2 = (z+8i)(z-8i)$

**73.** $4x^2+81 = (2x)^2+9^2 = (2x+9i)(2x-9i)$

**74.** $9x^2+1 = (3x)^2+1^2 = (3x+i)(3x-i)$

**75.** If $x = 1+2i$, then $x^2-2x+5 = (1+2i)^2-2(1+2i)+5 = 1+4i+4i^2-2-4i+5 = 1+4i+4(-1)-2-4i+5$
$$= 1+4i-4-2-4i+5 = (1-4-2+5)+(4i-4i) = 0$$

**76.** If $x = 1-2i$, then $x^2-2x+5 = (1-2i)^2-2(1-2i)+5 = 1-4i+4i^2-2+4i+5 = 1-4i+4(-1)-2+4i+5$
$$= 1-4i-4-2+4i+5 = (1-4-2+5)+(-4i+4i) = 0$$

**77.** Verify that $(-1+i\sqrt{3})^3 = 8$.

$$(-1+i\sqrt{3})^3 = (-1+i\sqrt{3})(-1+i\sqrt{3})^2 = (-1+i\sqrt{3})[(-1)^2+2(-1)(i\sqrt{3})+(i\sqrt{3})^2]$$
$$= (-1+i\sqrt{3})[1-2i\sqrt{3}+3i^2] = (-1+i\sqrt{3})[1-2i\sqrt{3}+3(-1)] = (-1+i\sqrt{3})[1-2i\sqrt{3}-3]$$
$$= (-1+i\sqrt{3})(-2-2i\sqrt{3}) = -1(-2-2i\sqrt{3})+i\sqrt{3}(-2-2i\sqrt{3}) = 2+2i\sqrt{3}-2i\sqrt{3}-2i^2(\sqrt{3})^2$$
$$= 2+2i\sqrt{3}-2i\sqrt{3}-2(-1)(3) = 2+2i\sqrt{3}-2i\sqrt{3}+6 = (2+6)+(2i\sqrt{3}-2i\sqrt{3})$$
$$= 8$$

Verify that $(-1-i\sqrt{3})^3 = 8$.

$$(-1-i\sqrt{3})^3 = (-1-i\sqrt{3})(-1-i\sqrt{3})^2 = (-1-i\sqrt{3})[(-1)^2+2(-1)(-i\sqrt{3})+(-i\sqrt{3})^2]$$
$$= (-1-i\sqrt{3})[1+2i\sqrt{3}+3i^2] = (-1-i\sqrt{3})[1+2i\sqrt{3}+3(-1)] = (-1-i\sqrt{3})[1+2i\sqrt{3}-3]$$
$$= (-1-i\sqrt{3})(-2+2i\sqrt{3}) = -1(-2+2i\sqrt{3})-i\sqrt{3}(-2+2i\sqrt{3}) = 2-2i\sqrt{3}+2i\sqrt{3}-2i^2(\sqrt{3})^2$$
$$= 2-2i\sqrt{3}+2i\sqrt{3}-2(-1)(3) = 2-2i\sqrt{3}+2i\sqrt{3}+6 = (2+6)+(-2i\sqrt{3}+2i\sqrt{3})$$
$$= 8$$

**78.** Verify that $\left[\frac{\sqrt{2}}{2}(1+i)\right]^2 = i$.

$$\left[\frac{\sqrt{2}}{2}(1+i)\right]^2 = \frac{\sqrt{2}^2}{2^2}(1+i)^2 = \frac{2}{4}(1+2i+i^2) = \frac{1}{2}[1+2i+(-1)] = \frac{1}{2}(1+2i-1) = \frac{1}{2}(2i) = i$$

**79.** $i+i^2+i^3+i^4+\ldots+i^{28} = 7(i+i^2+i^3+i^4) = 7(i+(-1)+(-i)+1) = 7(0) = 0$

**80.** $i+i^2+i^3+i^4+\ldots+i^{100} = 25(i+i^2+i^3+i^4) = 25(i+(-1)+(-i)+1) = 25(0) = 0$

## Exploring Concepts with Technology

**Can You Trust Your Calculator?**

**1.**

| Iteration | $p + 3p(1 - p)$ | $4p - 3p^2$ |
|---|---|---|
| 1 | 1.25 | 1.25 |
| 3 | 0.95703125 | 0.95703125 |
| 5 | 0.8198110957 | 0.8198110957 |
| 10 | 0.3846309658 | 0.3846309658 |
| 15 | 0.5610061236 | 0.5610061236 |
| 19 | 1.218765181 | 1.218765181 |
| 20 | 0.4188950251 | 0.4188950245 |

## Assessing Concepts

**1.** True

**2.** False

**3.** Any real number $a$ between 0 and 1

**4.** Any negative integer $n$

**5.** $-3x^2 + 6x - 7$

**6.** b

**7.** c

**8.** a

**9.** e

**10.** c

**11.** b

**12.** e

## Chapter Review

**1.** Integer, rational number, real number, prime number [P.1]

**2.** Irrational number, real number [P.1]

**3.** Rational number, real number [P.1]

**4.** Rational number, real number [P.1]

**5.** $A \cup B = \{1, 2, 3, 5, 7, 11\}$ [P.1]

**6.** $A \cap B = \{5\}$ [P.1]

**7.** Distributive property [P.1]

**8.** Commutative property of addition [P.1]

**9.** Associative property of multiplication [P.1]

**10.** Closure property of addition [P.1]

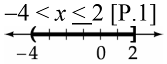
**11.** Identity property of addition [P.1]

**12.** Identity property of multiplication [P.1]

**13.** Symmetric property of equality [P.1]

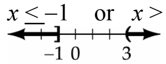
**14.** Transitive property of equality [P.1]

**15.** $-4 < x \le 2$ [P.1]



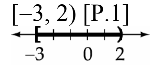

$(-4, 2]$

**16.** $x \le -1$ or $x > 3$ [P.1]



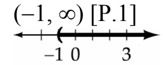

$(-\infty, -1] \cup (3, \infty)$

**17.** $[-3, 2)$ [P.1]




$-3 \le x < 2$

**18.** $(-1, \infty)$ [P.1]




$x > -1$

**19.** $|7| = 7$ [P.1]

**20.** $|2 - \pi| = -(2 - \pi) = \pi - 2$, because $\pi > 2$ [P.1]

**21.** $|4 - \pi| = 4 - \pi$, [P.1] because $4 > \pi$

**22.** $|-11| = 11$ [P.1]

**23.** $|-3 - 14| = 17$ [P.1]

**24.** $\left|\sqrt{5} - (-\sqrt{2})\right| = \sqrt{5} + \sqrt{2}$ [P.1]

**25.** $-5^2 + (-11) = -25 - 11 = -36$ [P.1]

**26.** $\dfrac{\left(2^2 \cdot 3^{-2}\right)^2}{3^{-1}2^3} = \dfrac{2^4 3^{-4}}{3^{-1}2^3} = 2^{4-3}3^{-4-(-1)} = 2^{4-3}3^{-4+1} = 2^1 3^{-3} = \dfrac{2}{3^3} = \dfrac{2}{27}$ [P.1]

**27.** $(3x^2y)(2x^3y)^2 = 3x^2y \cdot 4x^6y^2 = 12x^8y^3$ [P.2]

**28.** $\left(\dfrac{2a^2b^3c^{-2}}{3ab^{-1}}\right)^2 = \left(\dfrac{2ab^4}{3c^2}\right)^2 = \dfrac{4a^2b^8}{9c^4}$ [P.2]

**29.** $25^{1/2} = \sqrt{25} = 5$ [P.2]

**30.** $-27^{2/3} = -\left(\sqrt[3]{27}\right)^2 = -(3)^2 = -9$ [P.2]

**31.** $x^{2/3} \cdot x^{3/4} = x^{2/3\,+\,3/4} = x^{8/12\,+\,9/12} = x^{17/12}$ [P.2]

**32.** $\left(\dfrac{8x^{5/4}}{x^{1/2}}\right)^{2/3} = \left(8x^{5/4\,-\,1/2}\right)^{2/3} = \left(8x^{5/4\,-\,2/4}\right)^{2/3} = \left(8x^{3/4}\right)^{2/3} = 8^{2/3}x^{(3/4)(2/3)} = (2^3)^{2/3}x^{(3/4)(2/3)} = 2^2x^{1/2} = 4x^{1/2}$ [P.2]

**33.** $\left(\dfrac{x^2y}{x^{1/2}y^{-3}}\right)^{1/2} = \left(x^{2\,-\,1/2}y^{1\,-\,(-3)}\right)^{1/2} = \left(x^{4/2\,-\,1/2}y^{1\,+\,3}\right)^{1/2} = \left(x^{3/2}y^4\right)^{1/2} = x^{(3/2)(1/2)}y^{4(1/2)} = x^{3/4}y^2$ [P.2]

**34.** $\left(x^{1/2} - y^{1/2}\right)\left(x^{1/2} + y^{1/2}\right) = x - y$ [P.2]

**35.** $\sqrt{48a^2b^7} = \sqrt{16a^2b^6 \cdot 3b} = 4ab^3\sqrt{3b}$ [P.2]

**36.** $\sqrt{12a^3b} = \sqrt{4a^2 \cdot 3ab} = 2a\sqrt{3ab}$ [P.2]

**37.** $\sqrt{72x^2y} = \sqrt{36x^2 \cdot 2y} = 6x\sqrt{2y}$ [P.2]

**38.** $\sqrt{18x^3y^5} = \sqrt{9x^2y^4 \cdot 2xy} = 3xy^2\sqrt{2xy}$ [P.2]

**39.** $\sqrt{\dfrac{54xy^3}{10x}} = \sqrt{\dfrac{27y^3}{5}} = \dfrac{\sqrt{9y^2 \cdot 3y}}{\sqrt{5}} = \dfrac{3y\sqrt{3y}}{\sqrt{5}} \cdot \dfrac{\sqrt{5}}{\sqrt{5}} = \dfrac{3y\sqrt{15y}}{5}$ [P.2]

**40.** $-\sqrt{\dfrac{24xyz^3}{15z^6}} = -\sqrt{\dfrac{8xy}{5z^3}} = -\dfrac{2\sqrt{2xy}}{z\sqrt{5z}} \cdot \dfrac{\sqrt{5z}}{\sqrt{5z}} = -\dfrac{2\sqrt{10xyz}}{5z^2}$ [P.2]

**41.** $\dfrac{7x}{\sqrt[3]{2x^2}} = \dfrac{7x}{\sqrt[3]{2x^2}} \cdot \dfrac{\sqrt[3]{2^2x}}{\sqrt[3]{2^2x}} = \dfrac{7x\sqrt[3]{4x}}{2x} = \dfrac{7\sqrt[3]{4x}}{2}$ [P.2]

**42.** $\dfrac{5y}{\sqrt[3]{9y}} = \dfrac{5y}{\sqrt[3]{3^2y}} \cdot \dfrac{\sqrt[3]{3y^2}}{\sqrt[3]{3y^2}} = \dfrac{5y\sqrt[3]{3y^2}}{3y} = \dfrac{5\sqrt[3]{3y^2}}{3}$ [P.2]

**43.** $\sqrt[3]{-135x^2y^7} = \sqrt[3]{-27y^6 \cdot 5x^2y} = -3y^2\sqrt[3]{5x^2y}$ [P.2]

**44.** $\sqrt[3]{-250xy^6} = \sqrt[3]{-125y^6 \cdot 2x} = -5y^2\sqrt[3]{2x}$ [P.2]

**45.** $620{,}000 = 6.2 \times 10^5$ [P.2]

**46.** $0.0000017 = 1.7 \times 10^{-6}$ [P.2]

**47.** $3.5 \times 10^4 = 35{,}000$ [P.2]

**48.** $4.31 \times 10^{-7} = 0.000000431$ [P.2]

**49.** $(2a^2 + 3a - 7) + (-3a^2 - 5a + 6) = [2a^2 + (-3a^2)] + [3a + (-5a)] + [(-7) + 6] = -a^2 - 2a - 1$ [P.3]

**50.** $(5b^2 - 11) - (3b^2 - 8b - 3) = 5b^2 - 11 - 3b^2 + 8b + 3$ [P.3]

$= 2b^2 + 8b - 8$

**51.** [P.3]

$$\begin{array}{r} 2x^2 + 3x - 5 \\ 3x^2 - 2x + 4 \\ \hline +8x^2 + 12x - 20 \\ -4x^3 - 6x^2 + 10x \phantom{{}-20} \\ 6x^4 + 9x^3 - 15x^2 \phantom{{}+10x-20} \\ \hline 6x^4 + 5x^3 - 13x^2 + 22x - 20 \end{array}$$

**52.** $(3y-5)^3 = (3y-5)^2(3y-5) = (9y^2-30y+25)(3y-5) = 27y^3-45y^2-90y^2+150y+75y-125 = 27y^3-135y^2+225y-125$ [P.3]

**53.** $3x^2+30x+75=3(x^2+10x+25)=3(x+5)^2$ [P.4]

**54.** $25x^2-30xy+9y^2=(5x-3y)^2$ [P.4]

**55.** $20a^2-4b^2=4(5a^2-b^2)$ [P.4]

**56.** $16a^3+250=2(8a^3+125)=2(2a+5)(4a^2-10a+25)$ [P.4]

**57.** $\dfrac{6x^2-19x+10}{2x^2+3x-20}=\dfrac{(3x-2)(2x-5)}{(2x-5)(x+4)}=\dfrac{3x-2}{x+4}$ [P.5]

**58.** $\dfrac{4x^3-25x}{8x^4+125x}=\dfrac{x(4x^2-25)}{x(8x^3+125)}=\dfrac{x(2x-5)(2x+5)}{x(2x+5)(4x^2-10x+25)}=\dfrac{2x-5}{4x^2-10x+25}$ [P.5]

**59.** $\dfrac{10x^2+13x-3}{6x^2-13x-5}\cdot\dfrac{6x^2+5x+1}{10x^2+3x-1}=\dfrac{(2x+3)(5x-1)}{(2x-5)(3x+1)}\cdot\dfrac{(2x+1)(3x+1)}{(2x+1)(5x-1)}=\dfrac{2x+3}{2x-5}$ [P.5]

**60.** $\dfrac{15x^2+11x-12}{25x^2-9}\div\dfrac{3x^2+13x+12}{10x^2+11x+3}=\dfrac{15x^2+11x-12}{25x^2-9}\cdot\dfrac{10x^2+11x+3}{3x^2+13x+12}=\dfrac{(5x-3)(3x+4)}{(5x-3)(5x+3)}\cdot\dfrac{(5x+3)(2x+1)}{(3x+4)(x+3)}=\dfrac{2x+1}{x+3}$ [P.5]

**61.** $\dfrac{x}{x^2-9}+\dfrac{2x}{x^2+x-12}=\dfrac{x}{(x-3)(x+3)}+\dfrac{2x}{(x+4)(x-3)}=\dfrac{x(x+4)+2x(x+3)}{(x-3)(x+3)(x+4)}=\dfrac{x^2+4x+2x^2+6x}{(x-3)(x+3)(x+4)}$ [P.5]

$=\dfrac{3x^2+10x}{(x-3)(x+3)(x+4)}=\dfrac{x(3x+10)}{(x-3)(x+3)(x+4)}$

**62.** $\dfrac{3x}{x^2+7x+12}-\dfrac{x}{2x^2+5x-3}=\dfrac{3x}{(x+3)(x+4)}-\dfrac{x}{(2x-1)(x+3)}=\dfrac{3x(2x-1)-x(x+4)}{(x+3)(x+4)(2x-1)}$ [P.5]

$=\dfrac{6x^2-3x-x^2-4x}{(x+3)(x+4)(2x-1)}=\dfrac{5x^2-7x}{(x+3)(x+4)(2x-1)}=\dfrac{x(5x-7)}{(x+3)(x+4)(2x-1)}$

**63.** $\dfrac{2+\frac{1}{x-5}}{3-\frac{2}{x-5}}=\dfrac{\left(2+\frac{1}{x-5}\right)}{\left(3-\frac{2}{x-5}\right)}\cdot\dfrac{x-5}{x-5}=\dfrac{2(x-5)+1}{3(x-5)-2}=\dfrac{2x-10+1}{3x-15-2}=\dfrac{2x-9}{3x-17}$ [P.5]

**64.** $\dfrac{1}{2+\frac{3}{1+\frac{4}{x}}}=\dfrac{1}{2+\frac{3}{\frac{x}{x}+\frac{4}{x}}}=\dfrac{1}{2+\frac{3}{\frac{x+4}{x}}}=\dfrac{1}{2+\left(3\div\frac{x+4}{x}\right)}=\dfrac{1}{2+\left(3\cdot\frac{x}{x+4}\right)}=\dfrac{1}{2+\frac{3x}{x+4}}\cdot\dfrac{x+4}{x+4}=\dfrac{x+4}{2(x+4)+3x}=\dfrac{x+4}{2x+8+3x}=\dfrac{x+4}{5x+8}$ [P.5]

**65.** $5+\sqrt{-64}=5+8i$ [P.6]

**66.** $2-\sqrt{-18}=2-i\sqrt{18}$ [P.6]

$=2-i\sqrt{9\cdot 2}$

$=2-3i\sqrt{2}$

**67.** $(2-3i)+(4+2i)=2-3i+4+2i$ [P.6]

$=(2+4)+(-3i+2i)$

$=6-i$

**68.** $(4+7i)-(6-3i)=4+7i-6+3i$ [P.6]

$=(4-6)+(7i+3i)$

$=-2+10i$

**69.** $2i(3-4i)=6i-8i^2$ [P.6]

$=6i-8(-1)$

$=6i+8$

$=8+6i$

**70.** $(4-3i)(2+7i)=4(2+7i)-3i(2+7i)$ [P.6]

$=8+28i-6i-21i^2$

$=8+22i-21(-1)$

$=8+22i+21$

$=29+22i$

**71.** $(3+i)^2 = 3^2 + 2(3)(i) + i^2$ [P.6]
$= 9 + 6i + (-1)$
$= 8 + 6i$

**72.** Use the Powers of $i$ Theorem. [P.6]
The remainder of $345 \div 4$ is 1
$i^{345} = i^1 = i$

**73.** $\frac{4-6i}{2i} = \frac{2(2-3i)}{2i} = \frac{\cancel{2}(2-3i)}{\cancel{2}i} = \frac{2-3i}{i} = \frac{2-3i}{i} \cdot \frac{i}{i} = \frac{2i-3i^2}{i^2} = \frac{2i-3(-1)}{-1} = \frac{2i+3}{-1} = -3-2i$ [P.6]

**74.** $\frac{2-5i}{3+4i} = \frac{2-5i}{3+4i} \cdot \frac{3-4i}{3-4i} = \frac{(2-5i)(3-4i)}{(3+4i)(3-4i)} = \frac{2(3-4i)-5i(3-4i)}{(3)^2-(4i)^2} = \frac{6-8i-15i+20i^2}{9-16i^2} = \frac{6-8i-15i+20(-1)}{9-16(-1)}$ [P.6]
$= \frac{6-8i-15i-20}{9+16} = \frac{-14-23i}{25} = -\frac{14}{25} - \frac{23}{25}i$

## Quantitative Reasoning

**QR1.** Evaluate $z = \frac{\lambda_0 - \lambda_s}{\lambda_s}$ when $\lambda_0 = 390.5 \times 10^{-9}$ and $\lambda_s = 375.4 \times 10^{-9}$

$z = \frac{390.5 \times 10^{-9} - 375.4 \times 10^{-9}}{375.4 \times 10^{-9}} = \frac{15.1 \times 10^{-9}}{375.4 \times 10^{-9}} \approx 0.040$

**QR2.** Evaluate $z = \frac{\lambda_0 - \lambda_s}{\lambda_s}$ when $\lambda_0 = 412.3 \times 10^{-9}$ and $\lambda_s = 401.5 \times 10^{-9}$

$z = \frac{412.3 \times 10^{-9} - 401.5 \times 10^{-9}}{401.5 \times 10^{-9}} = \frac{10.8 \times 10^{-9}}{401.5 \times 10^{-9}} \approx 0.027$

**QR3.** Evaluate $v = c\left[\frac{(z+1)^2 - 1}{(z+1)^2 + 1}\right]$ when $c = 3 \times 10^5$ and $z = 0.032$.

$v = 3 \times 10^5\left[\frac{(0.032+1)^2 - 1}{(0.032+1)^2 + 1}\right] = 3 \times 10^5\left[\frac{0.065024}{2.065024}\right] \approx 9446$

The relative speed is 9446 kilometers per second.

**QR4.** Evaluate $v = c\left[\frac{(z+1)^2 - 1}{(z+1)^2 + 1}\right]$ when $c = 3 \times 10^5$ and $z = 0.041$.

$v = 3 \times 10^5\left[\frac{(0.041+1)^2 - 1}{(0.041+1)^2 + 1}\right] = 3 \times 10^5\left[\frac{0.083681}{2.083681}\right] \approx 12{,}048$

The relative speed is 12,048 kilometers per second.

## Chapter Test

**1.** Distributive property [P.1]

**2.** $A \cup B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ [P.1]

**3.** $|-12-(-5)| = |-12+5| = |-7| = 7$ [P.1]

**4.** $a\left(-2x^0y^{-2}\right)^2\left(-3x^2y^{-1}\right)^{-2} = \left(4y^{-4}\right)\left(3^{-2}x^{-4}y^2\right) = \frac{4}{9x^4y^2}$ [P.2]

**5.** $\frac{\left(2a^{-1}bc^{-2}\right)^2}{\left(3^{-1}b\right)\left(2^{-1}ac^{-2}\right)^3} = \frac{2^2a^{-2}b^2c^{-4}}{\left(3^{-1}b\right)\left(2^{-3}a^3c^{-6}\right)} = \frac{2^2 \cdot 2^3 \cdot 3^1 \cdot b^2c^6}{ba^3a^2c^4} = \frac{2^5 \cdot 3 \cdot bc^2}{a^5} = \frac{96bc^2}{a^5}$ [P.2]

**6.** $0.00137 = 1.37 \times 10^{-3}$ [P.2]

**7.** $\frac{x^{1/3}y^{-3/4}}{x^{-1/2}y^{3/2}} = x^{1/3-(-1/2)}y^{-3/4-3/2} = x^{1/3+1/2}y^{-3/4-3/2} = x^{2/6+3/6}y^{-3/4-6/4} = x^{5/6}y^{-9/4} = \frac{x^{5/6}}{y^{9/4}}$ [P.2]

**8.** $3x\sqrt[3]{81xy^4} - 2y\sqrt[3]{3x^4y} = 3x\sqrt[3]{27y^3 \cdot 3xy} - 2y\sqrt[3]{x^3 \cdot 3xy} = 3x \cdot 3y\sqrt[3]{3xy} - 2y \cdot x\sqrt[3]{3xy} = 9xy\sqrt[3]{3xy} - 2xy\sqrt[3]{3xy} = 7xy\sqrt[3]{3xy}$ [P.2]

**9.** $\frac{x}{\sqrt[4]{2x^3}} = \frac{x}{\sqrt[4]{2x^3}} \cdot \frac{\sqrt[4]{2^3x}}{\sqrt[4]{2^3x}} = \frac{x\sqrt[4]{2^3x}}{\sqrt[4]{2^4x^4}} = \frac{x\sqrt[4]{8x}}{2x} = \frac{\sqrt[4]{8x}}{2}$ [P.2]

**10.** $\frac{3}{\sqrt{x}+2} = \frac{3}{\sqrt{x}+2} \cdot \frac{\sqrt{x}-2}{\sqrt{x}-2} = \frac{3\sqrt{x}-6}{x-4}$ [P.2]

**11.** $(x-2y)(x^2-2x+y)=x^3-2x^2+xy-2x^2y+4xy-2y^2=x^3-2x^2+5xy-2x^2y-2y^2$ [P.3]

**12.** If $y=-3, 3y^3-2y^2-y+2=3(-3)^3-2(-3)^2-(-3)+2=3(-27)-2(9)+3+2=-81-18+3+2=-94$ [P.4]

**13.** $7x^2+34x-5=(7x-1)(x+5)$ [P.4]

**14.** $3ax-12bx-2a+8b=(3ax-12bx)-(2a-8b)=3x(a-4b)-2(a-4b)=(a-4b)(3x-2)$ [P.4]

**15.** $16x^4-2xy^3=2x(8x^3-y^3)=2x(2x-y)(4x^2+2xy+y^2)$ [P.4]

**16.** $\frac{x^2-2x-15}{25-x^2}=\frac{(x-5)(x+3)}{(5-x)(5+x)}=\left(\frac{x-5}{5-x}\right)\left(\frac{x+3}{x+5}\right)=-1\cdot\left(\frac{x+3}{x+5}\right)=-\frac{x+3}{x+5}$ [P.5]

**17.** $\frac{x}{x^2+x-6}-\frac{2}{x^2-5x+6}=\frac{x}{(x-2)(x+3)}-\frac{2}{(x-2)(x-3)}=\frac{x(x-3)-2(x+3)}{(x-2)(x+3)(x-3)}$ [P.5]

$=\frac{x^2-3x-2x-6}{(x-2)(x+3)(x-3)}=\frac{x^2-5x-6}{(x-2)(x+3)(x-3)}=\frac{(x-6)(x+1)}{(x-2)(x+3)(x-3)}$

**18.** $\frac{2x^2+3x-2}{x^2-3x}\div\frac{2x^2-7x+3}{x^3-3x^2}=\frac{2x^2+3x-2}{x^2-3x}\cdot\frac{x^3-3x^2}{2x^2-7x+3}=\frac{(2x-1)(x+2)}{x(x-3)}\cdot\frac{x^2(x-3)}{(2x-1)(x-3)}=\frac{x(x+2)}{x-3}$ [P.5]

**19.** $\frac{3}{a+b}\cdot\frac{a^2-b^2}{2a-b}-\frac{5}{a}=\frac{3}{a+b}\cdot\frac{(a-b)(a+b)}{2a-b}-\frac{5}{a}=\frac{3(a-b)}{2a-b}-\frac{5}{a}=\frac{3a(a-b)-5(2a-b)}{a(2a-b)}=\frac{3a^2-3ab-10a+5b}{a(2a-b)}$ [P.5]

**20.** $x-\frac{x}{x+\frac{1}{2}}=x-\frac{x}{\frac{2x}{2}+\frac{1}{2}}=x-\frac{x}{\frac{2x+1}{2}}=x-x\div\frac{2x+1}{2}=x-x\cdot\frac{2}{2x+1}=x-\frac{2x}{2x+1}$ [P.5]

$=\frac{x(2x+1)}{2x+1}-\frac{2x}{2x+1}=\frac{2x^2+x}{2x+1}-\frac{2x}{2x+1}=\frac{2x^2+x-2x}{2x+1}=\frac{2x^2-x}{2x+1}=\frac{x(2x-1)}{2x+1}$

**21.** $7+\sqrt{-20}=7+2i\sqrt{5}$ [P.6]

**22.** $(4-3i)-(2-5i)=4-3i-2+5i$ [P.6]
$=(4-2)+(-3i+5i)$
$=2+2i$

**23.** $(2+5i)(1-4i)=2(1-4i)+5i(1-4i)$ [P.6]
$=2-8i+5i-20i^2$
$=2-8i+5i-20(-1)$
$=2-8i+5i+20$
$=(2+20)+(-8i+5i)$
$=22-3i$

**24.** $\frac{3+4i}{5-i}=\frac{3+4i}{5-i}\cdot\frac{5+i}{5+i}=\frac{(3+4i)(5+i)}{(5-i)(5+i)}=\frac{3(5+i)+4i(5+i)}{5^2-i^2}=\frac{15+3i+20i+4i^2}{25-i^2}=\frac{15+3i+20i+4(-1)}{25-(-1)}$ [P.6]

$=\frac{15+3i+20i-4}{25+1}=\frac{(15-4)+(3i+20i)}{26}=\frac{11+23i}{26}=\frac{11}{26}+\frac{23}{26}i$

**25.** Use the Powers of $i$ Theorem. [P.6]

The remainder of $97\div4$ is 1.

$i^{97}=i^1=i$

# Chapter 1
# Equations and Inequalities

## Section 1.1

**1.** $$\begin{aligned} 2x+10&=40\\ 2x&=40-10\\ 2x&=30\\ x&=15 \end{aligned}$$

**2.** $$\begin{aligned} -3y+20&=2\\ -3y&=2-20\\ -3y&=-18\\ y&=6 \end{aligned}$$

**3.** $$\begin{aligned} 5x+2&=2x-10\\ 5x-2x&=-10-2\\ 3x&=-12\\ x&=-4 \end{aligned}$$

**4.** $$\begin{aligned} 4x-11&=7x+20\\ 4x-7x&=20+11\\ -3x&=31\\ x&=-\frac{31}{3} \end{aligned}$$

**5.** $$\begin{aligned} 2(x-3)-5&=4(x-5)\\ 2x-6-5&=4x-20\\ 2x-11&=4x-20\\ 2x-4x&=-20+11\\ -2x&=-9\\ x&=\frac{9}{2} \end{aligned}$$

**6.** $$\begin{aligned} 5(x-4)-7&=-2(x-3)\\ 5x-20-7&=-2x+6\\ 5x-27&=-2x+6\\ 5x+2x&=6+27\\ 7x&=33\\ x&=\frac{33}{7} \end{aligned}$$

**7.** $$\begin{aligned} 4(2r-17)+5(3r-8)&=0\\ 8r-68+15r-40&=0\\ 23r-108&=0\\ 23r&=108\\ r&=\frac{108}{23} \end{aligned}$$

**8.** $$\begin{aligned} 6(5s-11)-12(2s+5)&=0\\ 30s-66-24s-60&=0\\ 6s-126&=0\\ 6s&=126\\ s&=\frac{126}{6}\\ s&=21 \end{aligned}$$

**9.** $$\begin{aligned} \frac{3}{4}x+\frac{1}{2}&=\frac{2}{3}\\ 12\cdot\left(\frac{3}{4}x+\frac{1}{2}\right)&=12\cdot\left(\frac{2}{3}\right)\\ 9x+6&=8\\ 9x&=8-6\\ 9x&=2\\ x&=\frac{2}{9} \end{aligned}$$

**10.** $$\begin{aligned} \frac{x}{4}-5&=\frac{1}{2}\\ 4\cdot\left(\frac{x}{4}-5\right)&=4\left(\frac{1}{2}\right)\\ x-20&=2\\ x&=2+20\\ x&=22 \end{aligned}$$

**11.** $$\begin{aligned} \frac{2}{3}x-5&=\frac{1}{2}x-3\\ 6\cdot\left(\frac{2}{3}x-5\right)&=6\cdot\left(\frac{1}{2}x-3\right)\\ 4x-30&=3x-18\\ 4x-3x&=-18+30\\ x&=12 \end{aligned}$$

**12.** $$\begin{aligned} \frac{1}{2}x+7-\frac{1}{4}x&=\frac{19}{2}\\ 4\cdot\left(\frac{1}{2}x+7-\frac{1}{4}x\right)&=4\cdot\left(\frac{19}{2}\right)\\ 2x+28-x&=38\\ x+28&=38\\ x&=38-28\\ x&=10 \end{aligned}$$

**13.** $$\begin{aligned} 0.2x+0.4&=3.6\\ 0.2x&=3.6-0.4\\ 0.2x&=3.2\\ x&=16 \end{aligned}$$

**14.** $$\begin{aligned} 0.04x-0.2&=0.07\\ 0.04x&=0.07+0.2\\ 0.04x&=0.27\\ x&=6.75 \end{aligned}$$

**15.** $$\begin{aligned} x+0.08(60)&=0.20(60+x)\\ x+4.8&=12+0.20x\\ x-0.20x&=12-4.8\\ 0.80x&=7.2\\ x&=9 \end{aligned}$$

**16.** $$\begin{aligned} 6(t+1.5)&=12t\\ 6t+9&=12t\\ 6t-12t&=-9\\ -6t&=-9\\ t&=\frac{3}{2} \end{aligned}$$

**17.** $$\begin{aligned} 3(x+5)(x-1)&=(3x+4)(x-2)\\ 3\left(x^2+4x-5\right)&=3x^2-2x-8\\ 3x^2+12x-15&=3x^2-2x-8\\ 12x+2x&=-8+15\\ 14x&=7\\ x&=\frac{1}{2} \end{aligned}$$

**18.** $$\begin{aligned} 5(x+4)(x-4)&=(x-3)(5x+4)\\ 5\left(x^2-16\right)&=5x^2-11x-12\\ 5x^2-80&=5x^2-11x-12\\ 11x&=-12+80\\ 11x&=68\\ x&=\frac{68}{11} \end{aligned}$$

**19.** $$\begin{aligned} 5[x-(4x-5)]&=3-2x\\ 5(x-4x+5)&=3-2x\\ 5(-3x+5)&=3-2x\\ -15x+25&=3-2x\\ -15x+2x&=3-25\\ -13x&=-22\\ x&=\frac{22}{13} \end{aligned}$$

**20.**
$$\begin{aligned} 6[3y-2(y-1)]-2+7y&=0\\ 6(3y-2y+2)-2+7y&=0\\ 18y-12y+12-2+7y&=0\\ 13y+10&=0\\ 13y&=-10\\ y&=-\frac{10}{13} \end{aligned}$$

**21.**
$$\begin{aligned} \frac{40-3x}{5}&=\frac{6x+7}{8}\\ 40\cdot\left(\frac{40-3x}{5}\right)&=40\cdot\left(\frac{6x+7}{8}\right)\\ 8(40-3x)&=5(6x+7)\\ 320-24x&=30x+35\\ -24x-30x&=35-320\\ -54x&=-285\\ x&=\frac{95}{18} \end{aligned}$$

**22.**
$$\begin{aligned} \frac{12+x}{-4}&=\frac{5x-7}{3}+2\\ 12\cdot\left(\frac{12+x}{-4}\right)&=12\cdot\left(\frac{5x-7}{3}+2\right)\\ -3(12+x)&=4(5x-7)+24\\ -36-3x&=20x-28+24\\ -36-3x&=20x-4\\ -3x-20x&=-4+36\\ -23x&=32\\ x&=-\frac{32}{23} \end{aligned}$$

**23.**
$$\begin{aligned} -3(x-5)&=-3x+15\\ -3x+15&=-3x+15 \end{aligned}$$
Identity

**24.**
$$\begin{aligned} 2x+\frac{1}{3}&=\frac{6x+1}{3}\\ 3\cdot\left(2x+\frac{1}{3}\right)&=3\cdot\left(\frac{6x+1}{3}\right)\\ 6x+1&=6x+1 \end{aligned}$$
Identity

**25.**
$$\begin{aligned} 2x+7&=3(x-1)\\ 2x+7&=3x-3\\ 2x-3x&=-3-7\\ -x&=-10\\ x&=10 \end{aligned}$$
Conditional equation

**26.**
$$\begin{aligned} 4[2x-5(x-3)]&=6\\ 4[2x-5x+15]&=6\\ 4[-3x+15]&=6\\ -12x+60&=6\\ -12x&=-54\\ x&=\frac{9}{2} \end{aligned}$$
Conditional equation

**27.**
$$\begin{aligned} \frac{4x+8}{4}&=x+8\\ 4x+8&=4(x+8)\\ 4x+8&=4x+32\\ 8&=32 \end{aligned}$$
Contradiction

**28.**
$$\begin{aligned} 3[x-(4x-1)]&=-3(2x-5)\\ 3[x-4x+1]&=-6x+15\\ 3[-3x+1]&=-6x+15\\ -9x+3&=-6x+15\\ -3x&=12\\ x&=-4 \end{aligned}$$
Conditional equation

**29.**
$$\begin{aligned} 3[x-2(x-5)]-1&=-3x+29\\ 3[x-2x+10]-1&=-3x+29\\ 3[-x+10]-1&=-3x+29\\ -3x+30-1&=-3x+29\\ -3x+29&=-3x+29 \end{aligned}$$
Identity

**30.**
$$\begin{aligned} 4[3(x-5)+7]&=12x-32\\ 4[3x-15+7]&=12x-32\\ 4[3x-8]&=12x-32\\ 12x-32&=12x-32 \end{aligned}$$
Identity

**31.**
$$\begin{aligned} 2x-8&=-x+9\\ 3x&=17\\ x&=\frac{17}{3} \end{aligned}$$
Conditional equation

**32.**
$$\begin{aligned} |3(x-4)+7|&=|3x-5|\\ |3x-12+7|&=|3x-5|\\ |3x-5|&=|3x-5| \end{aligned}$$
Identity

**33.** $|x|=4$

$x=4$ or $x=-4$

**34.** $|x|=7$

$x=7$ or $x=-7$

**35.** $|x-5|=2$

$x-5=2$ or $x-5=-2$

$x=7$ $\qquad$ $x=3$

**36.** $|x-8|=3$

$x-8=3$ or $x-8=-3$

$x=11$ $\qquad$ $x=5$

**37.** $|2x-5|=11$

$2x-5=11$ or $2x-5=-11$

$2x=16$ $\qquad$ $2x=-6$

$x=8$ $\qquad$ $x=-3$

**38.** $|2x-3|=21$

$2x-3=21$ or $2x-3=-21$

$2x=24$ $\qquad$ $2x=-18$

$x=12$ $\qquad$ $x=-9$

**39.** $|2x+6|=10$

$2x+6=10$ or $2x+6=-10$

$2x=4$ $\qquad$ $2x=-16$

$x=2$ $\qquad$ $x=-8$

**40.** $|2x+14|=60$

$2x+14=60$ or $2x+14=-60$

$2x=46$ $\qquad$ $2x=-74$

$x=23$ $\qquad$ $x=-37$

**41.** $\left|\frac{x-4}{2}\right|=8$

$\frac{x-4}{2}=8$ or $\frac{x-4}{2}=-8$

$x-4=8(2)$ $\quad$ $x-4=-8(2)$

$x-4=16$ $\quad$ $x-4=-16$

$x=20$ $\quad$ $x=-12$

**42.** $\left|\frac{x+3}{4}\right|=6$

$\frac{x+3}{4}=6$ or $\frac{x+3}{4}=-6$

$x+3=6(4)$ $\quad$ $x+3=-6(4)$

$x+3=24$ $\quad$ $x+3=-24$

$x=21$ $\quad$ $x=-27$

**43.** $|2x+5|=-8$

$|2x+5|\geq 0$

$-8\geq 0$

Contradiction. There is no solution.

**44.** $|4x-1|=-17$

$|4x-1|\geq 0$

$-17\geq 0$

Contradiction. There is no solution.

**45.** $2|x+3|+4=34$

$2|x+3|=30$

$|x+3|=15$

$x+3=15$ or $x+3=-15$

$x=12$ $\quad$ $x=-18$

**46.** $3|x-5|-16=2$

$3|x-5|=18$

$|x-5|=6$

$x-5=6$ or $x-5=-6$

$x=11$ $\quad$ $x=-1$

**47.** $|2x-a|=b,\quad \mathrm{b}>0$

$2x-a=b$ or $2x-a=-b$

$2x=a+b$ $\quad$ $2x=a-b$

$x=\frac{a+b}{2}$ $\quad$ $x=\frac{a-b}{2}$

**48.** $3|x-d|=c,\quad \mathrm{c}>0$

$|x-d|=\frac{c}{3}$

$x-d=\frac{c}{3}$ or $x-\mathrm{d}=-\frac{c}{3}$

$x=d+\frac{c}{3}$ $\quad$ $x=d-\frac{c}{3}$

**49.** $1.6x+1.87=\text{Revenue}$

$1.6x+1.87=10$

$1.6x=8.13$

$x=\frac{8.13}{1.6}\approx 5$

2000 + 5 = 2005

The revenue first exceeded \$10 billion in 2005.

**50.** $93.8x+542.8=\text{Number of megawatts (MW)}$

$93.8x+542.8=1200$

$93.8x=657.2$

$x=\frac{657.2}{93.8}\approx 7$

2000 + 7 = 2007

The energy will exceeded 1200 MW in 2007.

**51.** $d=|210-50t|$

$60=210-50t$ or $-60=210-50t$

$-150=-50t$ $\quad$ $-270=-50t$

$t=3$ $\quad$ $t=5.4$

5.4 hours = 5 hours 24 minutes

Ruben will be exactly 60 miles from Barstow after 3 hours and after 5 hours and 24 minutes.

**52.** $m=-\frac{1}{2}|s-55|+25$

$22=-\frac{1}{2}|s-55|+25$

$-3=-\frac{1}{2}|s-55|$

$6=|s-55|$

$6=s-55$ or $-6=s-55$

$61=s$ $\quad$ $49=s$

Kate can drive at 49 mph or 61 mph to obtain gas mileage of exactly 22 miles per gallon.

**53.** $45x+550=\text{Cost}$

$45x+550=3800$

$45x=3250$

$x\approx 72$

Rounded to the nearest yard, 72 sq yards can be carpeted for \$3800.

**54.** $1.75x+8.00=\text{Retail price}$

$1.75x+8.00=156.75$

$1.75x=148.75$

$x=85$

The wholesale price of the coat is \$85.00.

**55.**

$$100-\frac{42{,}000}{500{,}000}t = \text{Percent remaining}$$
$$100-\frac{42{,}000}{500{,}000}t = 25$$
$$100-\frac{21}{250}t = 25$$
$$-\frac{21}{250}t = -75$$
$$t \approx 892.857142857 \text{ seconds}$$
$$892.857142857 \text{ sec}\cdot\left(\frac{1 \text{ min}}{60 \text{ sec}}\right) \approx 15 \text{ min}$$

**56.**

$$2650-475t = \text{miles remaining}$$
$$2650-475t = 1000$$
$$-475t = -1650$$
$$t \approx 3.5 \text{ to the nearest tenth}$$

3.5 hours

**57.**

$$\begin{aligned} \max &= 0.85(220-a) & \min &= 0.65(220-a)\\ &= 0.85(220-25) & &= 0.65(220-25)\\ &= 0.85(195) & &= 0.65(195)\\ &= 165.75 & &= 126.75 \end{aligned}$$

The maximum exercise heart rate for a person who is 25 years of age is 166 beats per minute (to the nearest beat).

The minimum exercise heart rate for a person who is 25 years of age is 127 beats per minute (to the nearest beat).

**58.**

$$\max = 0.85(220-a)$$
$$153 = 0.85(220-a)$$
$$153 = 187-0.85a$$
$$-34 = -0.85a$$
$$40 = a$$

A person should have a maximum exercise heart rate of 153 beats per minute at age 40.

**59.**

$$ax+b=c$$
$$ax=c-b$$
$$x=\frac{c-b}{a},\quad a\neq 0$$

**60.**

$$ax+b=cx+d$$
$$ax-cx=d-b$$
$$x(a-c)=d-b$$
$$x=\frac{d-b}{a-c},\quad a-c\neq 0$$

**61.**

$$|x+4| = x+4$$

$$\begin{aligned} &\text{if } x+4\geq 0 & &\text{if } x+4<0\\ &x\geq -4 & &x<-4\\ &x+4=x+4 & &x+4=-(x+4)\\ &\text{an identity} & &x+4=-x-4\\ & & &2x=-8\\ & & &x=-4 \end{aligned}$$

The case $x + 4 < 0$ has no solution since there is no real number $x$ such that $x < -4$ and $x = -4$.

$\{x \mid x\geq -4\}$

**62.**

$$|x-1| = x-1$$

$$\begin{aligned} &\text{if } x-1\geq 0 & &\text{if } x-1<0\\ &x\geq 1 & &x<1\\ &x-1=x-1 & &x-1=-(x-1)\\ &\text{an identity} & &x-1=-x+1\\ & & &2x=2\\ & & &x=1 \end{aligned}$$

The case $x-1 < 0$ has no solution since there is no real number $x$ such that $x < 1$ and $x = 1$.

$\{x \mid x\geq 1\}$

**63.**

$$|x+7| = -(x+7)$$

$$\begin{aligned} &\text{if } x+7\geq 0 & &\text{if } x+7<0\\ &x\geq -7 & &x<-7\\ &x+7=-(x+7) & &x+7=x+7\\ &x+7=-x-7 & &\text{an identity}\\ &2x=-14 & &\\ &x=-7 & & \end{aligned}$$

$\{x \mid x\leq -7\}$

**64.**

$$|x-3| = -(x-3)$$

$$\begin{aligned} &\text{if } x-3\geq 0 & &\text{if } x-3<0\\ &x\geq 3 & &x<3\\ &x-3=-(x-3) & &x-3=x-3\\ &x-3=-x+3 & &\text{an identity}\\ &2x=6 & &\\ &x=3 & & \end{aligned}$$

$\{x \mid x\leq 3\}$

**65.** $|2x+7| = 2x+7$

if $2x+7 \ge 0$

$$x \ge -\frac{7}{2}$$

$$2x+7 = 2x+7$$

an identity

if $2x+7 < 0$

$$x < -\frac{7}{2}$$

$$2x+7 = -(2x+7)$$

$$2x+7 = -2x-7$$

$$4x = -14$$

$$x = -\frac{7}{2}$$

The case $2x + 7 < 0$ has no solution since there is no real number $x$ such that $x < -\frac{7}{2}$ and $x = -\frac{7}{2}$.

$$\left\{x \mid x \ge -\frac{7}{2}\right\}$$

**66.** $|3x-11| = -3x+11$

if $3x-11 \ge 0$

$$3x \ge 11$$

$$x \ge \frac{11}{3}$$

$$3x-11 = -3x+11$$

$$6x = 22$$

$$x = \frac{11}{3}$$

if $3x-11 < 0$

$$3x < 11$$

$$x < \frac{11}{3}$$

$$3x-11 = -(-3x+11)$$

an identity

$$\left\{x \mid x \le \frac{11}{3}\right\}$$

## Prepare for Section 1.2

**PS1.** $32 - x$

$$32 - 8\frac{1}{2} = 23\frac{1}{2}$$

**PS2.** $\frac{1}{2}bh$

$$\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5} = \frac{4}{15}$$

**PS3.** $2l + 2w = 2(l + w)$
Distributive property

**PS4.** $\left(\frac{1}{2}b\right)h = \frac{1}{2}bh$
Associative property of multiplication

**PS5.** $\frac{2}{5}x + \frac{1}{3}x = \frac{6}{15}x + \frac{5}{15}x = \frac{11}{15}x$

**PS6.** $\dfrac{1}{\frac{1}{a}+\frac{1}{b}} = \dfrac{1}{\frac{1}{a}\cdot\frac{b}{b}+\frac{1}{b}\cdot\frac{a}{a}} = \dfrac{1}{\frac{b}{ab}+\frac{a}{ab}} = \dfrac{1}{\frac{b+a}{ab}} = \dfrac{ab}{a+b}$

## Section 1.2

**1.**
$$V = \frac{1}{3}\pi r^2 h$$
$$3V = \pi r^2 h$$
$$\frac{3V}{\pi r^2} = h$$

**2.**
$$P = S - Sdt$$
$$Sdt = S - P$$
$$t = \frac{S-P}{Sd}$$

**3.**
$$I = \Pr t$$
$$\frac{I}{Pr} = t$$

**4.**
$$A = P + \Pr t$$
$$A = P(1+rt)$$
$$\frac{A}{1+rt} = P$$

**5.**
$$F = \frac{Gm_1m_2}{d^2}$$
$$Fd^2 = Gm_1m_2$$
$$\frac{Fd^2}{Gm_2} = m_1$$

**6.**
$$A = \frac{1}{2}h(b_1+b_2)$$
$$2A = hb_1 + hb_2$$
$$2A - hb_2 = hb_1$$
$$\frac{2A-hb_2}{h} = b_1$$

**7.**
$$a_n = a_1 + (n-1)d$$
$$a_n - a_1 = (n-1)d$$
$$\frac{a_n - a_1}{n-1} = d$$

**8.**
$$y - y_1 = m(x - x_1)$$
$$y - y_1 = mx - mx_1$$
$$y - y_1 + mx_1 = mx$$
$$\frac{y - y_1 + mx_1}{m} = x$$

**9.**
$$S = \frac{a_1}{1-r}$$
$$S(1-r) = a_1$$
$$S - Sr = a_1$$
$$S - a_1 = Sr$$
$$\frac{S-a_1}{S} = r$$

**10.**
$$\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$$
$$P_1V_1T_2 = P_2V_2T_1$$
$$\frac{P_1V_1T_2}{P_2T_1} = V_2$$

**11.** $\text{qb rating} = \frac{100}{6}[0.05(61.3-30)+0.25(6.39-3)+0.2(3.29)+(2.375-0.25(4.78))]$
$\approx 70.8$

**12.** $\text{qb rating} = \frac{100}{6}[0.05(61.3-30)+0.25(6.64-3)+0.2(3.36)+(2.375-0.25(2.3))]$
$\approx 82.5$

**13.** $\text{SMOG} = \sqrt{42}+3$
$= 9.5$

**14.** $\text{SMOG} = \sqrt{105}+3$
$= 13.2$

**15.** $\text{GFI} = 0.4(14.8+15.1)$
$\approx 12.0$

**16.** $\text{GFI} = 0.4(18.8+14.2)$
$= 13.2$

**17.** Let $x$ = the number

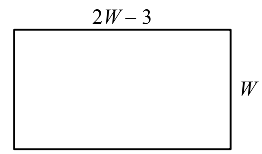
$$\frac{1}{5}x+\frac{1}{4}x=\frac{1}{2}x-5$$
$$20\left(\frac{1}{5}x+\frac{1}{4}x\right)=20\left(\frac{1}{2}x-5\right)$$
$$4x+5x=10x-100$$
$$9x=10x-100$$
$$100=x$$

**18.** The original fraction is $\frac{x-4}{x}$.

$$\frac{x-4+14}{x-10}=5$$
$$\frac{x+10}{x-10}=5$$
$$x+10=5(x-10)$$
$$x+10=5x-50$$
$$-4x=-60$$
$$x=15$$

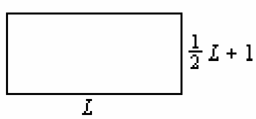
The original fraction is $\frac{15-4}{15}=\frac{11}{15}$.

**19.**




$$P=2L+2W$$
$$174=2(2W-3)+2W$$
$$174=4W-6+2W$$
$$180=6W$$
$$W=30 \text{ ft}$$
$$L=2W-3=2(30)-3=60-3=57 \text{ ft}$$

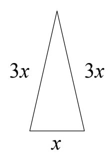
**20.**




$$P=2L+2W$$
$$110=2L+2\left[\tfrac{1}{2}L+1\right]$$
$$110=2L+L+2$$
$$108=3L$$

$$L=36 \text{ m}$$
$$W=\tfrac{1}{2}L+1=\tfrac{1}{2}(36)+1$$
$$=18+1=19 \text{ m}$$

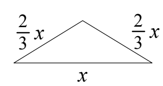
**21.**




$$3x+3x+x=84$$
$$7x=84$$
$$x=12$$
$$3x=3(12)=36$$

The shortest side is 12 cm. The longer sides are each 36 cm.

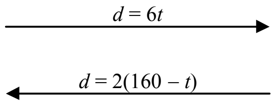
**22.**




$$\frac{2}{3}x+\frac{2}{3}x+x=161$$
$$2x+2x+3x=483$$
$$7x=483$$
$$x=69$$
$$\frac{2}{3}x=\frac{2}{3}(69)=46$$

The longest side is 69 miles. The two shorter sides are each 46 miles.

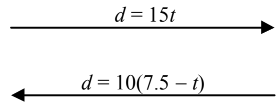
**23.**




Let $t$ = the time to run to the end of the track.
Let $160 - t$ = the time in seconds to jog back.

$$6t = 2(160 - t)$$
$$6t = 320 - 2t$$
$$8t = 320$$
$$t = 40$$
$$d = 6(40) = 240 \text{ meters}$$

**24.**



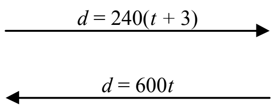

Let $t$ = the time (in hours) to travel to the island.

Let $7.5 - t$ = the time (in hours) to return.
$d = 15t$, and $d = 10(7.5 - t)$
Thus, $15t = 10(7.5 - t)$

$$15t = 75 - 10t$$
$$25t = 75$$
$$t = 3 \text{ hours}$$

$$d = 15(3) = 45 \text{ nautical miles}$$

**25.**




Let $t$ = the time (in hours) of the second plane.

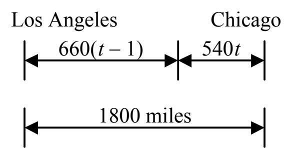
Let $t + 3$ = the time (in hours) of the first plane.

$$d = 240(t + 3)$$
$$d = 600t$$

$$240(t + 3) = 600t$$
$$240t + 720 = 600t$$
$$720 = 360t$$
$$2 = t$$
$$t = 2 \text{ hours}$$

**26.** Let $t$ = time (in hours) of the first plane.

Let $t - 1$ = time (in hours) of the second plane.




$$660(t - 1) + 540t = 1800$$
$$660t - 660 + 540t = 1800$$
$$1200t = 2460$$
$$t = 2.05 \text{ hours}$$

Thus, the planes pass each other 2.05 hours after the first plane leaves Chicago. The distance the first plane is from Chicago is 540(2.05) = 1107 miles.

**27.** Let $x + y + z$ = the distance to Jon's house, where $x$ = the distance uphill, $y$ = the distance down hill, and $z$ = the distance on level ground. Note that $x$, the distance uphill on the way to Jon's house equals the distance down hill on the way home, and that $z$, the distance down hill on the way to Jon's house equals the distance uphill on the way home. Also note that $y$ is the distance on level ground on the way to Jon's house and on the way home.

$$rt = d \Rightarrow t = \frac{d}{r}$$

| | | rate | time | distance |
|---|---|---|---|---|
| To Jon's house | Up | 4 | $\frac{x}{4}$ | $x$ |
| | Level | 6 | $\frac{y}{6}$ | $y$ |
| | Down | 12 | $\frac{z}{12}$ | $z$ |
| Back home | Up | 12 | $\frac{x}{12}$ | $x$ |
| | Level | 6 | $\frac{y}{6}$ | $y$ |
| | Down | 4 | $\frac{z}{4}$ | $z$ |

$$\frac{x}{4} + \frac{y}{6} + \frac{z}{12} + \frac{x}{12} + \frac{y}{6} + \frac{z}{4} = 1$$
$$12\left(\frac{x}{4} + \frac{y}{6} + \frac{z}{12} + \frac{x}{12} + \frac{y}{6} + \frac{z}{4}\right) = 12(1)$$
$$3x + 2y + z + x + 2y + 3z = 12$$
$$4x + 4y + 4z = 12$$
$$4(x + y + z) = 12$$
$$x + y + z = 3$$

The distance to Jon's house is 3 miles.

**28.** 30 seconds = 0.5 minutes = $\frac{0.5}{60}$ hour = $\frac{1}{120}$ hour.

500 meters = $a \ \frac{1}{2}$ km.

| | rate | time |
|---|---|---|
| Faster car | $x$ | $\frac{1}{120}$ |
| Slower car | 80 | $\frac{1}{120}$ |

$$x\left(\frac{1}{120}\right) - 80\left(\frac{1}{120}\right) = \frac{1}{2}$$
$$120\left[x\left(\frac{1}{120}\right) - 80\left(\frac{1}{120}\right)\right] = 120\left(\frac{1}{2}\right)$$
$$x - 80 = 60$$
$$x = 140$$

140 km/h

**29.** Let $x$ = the score on the next test.

$$\frac{80 + 82 + 94 + 71 + x}{5} = 85$$
$$\frac{327 + x}{5} = 85$$
$$327 + x = 425$$
$$x = 98$$

A score of 98 will produce an average of 85.

**30.** Let $x$ = the score on the final examination.

$$\frac{90 + 74 + 82 + 90 + 2x}{6} = 85$$
$$\frac{336 + 2x}{6} = 85$$
$$336 + 2x = 510$$
$$2x = 174$$
$$x = 87$$

A score of 87 on the final examination score will produce an average of 85.

**31.** Let $x$ = the number of sunglasses.
Profit = Revenue – Cost

$$17{,}884 = 29.99x - 8.95x$$
$$17{,}884 = 21.04x$$
$$x = 850$$

The manufacturer must sell 850 sunglasses to make a profit of \$17,884.

**32.** Let $x$ = the number of glasses of orange juice.
Profit = Revenue – Cost

$$2337 = 0.75x - 0.18x$$
$$2337 = 0.57x$$
$$4100 = x$$

The owner must sell 4100 glasses of orange juice to make a profit of \$2337.

**33.** Let $x$ = cost last year.

$$x - 0.20x = 750$$
$$0.80x = 750$$
$$x = 937.50$$

The cost of a computer last year was $937.50.

**34.** Let $x$ = cost last year.

$$x + 0.04x = 26$$
$$1.04x = 26$$
$$x = 25$$

The cost of the subscription last year was $25.

**35.** Let $x$ = amount invested at 8%.
$(14{,}000 - x)$ = amount invested at 6.5%.

$$0.08x + 0.065(14{,}000 - x) = 1024$$
$$0.08x + 910 - 0.065x = 1024$$
$$0.015x = 114$$
$$x = 7600$$
$$14{,}000 - x = 6400$$

$7600 was invested at 8%.
$6400 was invested at 6.5%.

**36.**

| | |
|---|---|
| 5% | $x$ |
| 7% | $7500 - x$ |

$$0.05x + 0.07(7500 - x) = 405$$
$$0.05x + 525 - 0.07x = 405$$
$$-0.02x = -120$$
$$x = 6000$$
$$7500 - x = 1500$$

$6000 was invested at 5%.
$1500 was invested at 7%.

**37.**

| | |
|---|---|
| 5.5% | 2500 |
| 8% | $x$ |
| 7% | $2500 + x$ |

$$0.055(2500) + 0.08x = 0.07(2500 + x)$$
$$137.5 + 0.08x = 175 + 0.07x$$
$$0.01x = 37.5$$
$$x = 3700$$

$3750 additional investment.

**38.**

| | |
|---|---|
| 0.068 | 4600 |
| 0.09 | $x$ |
| 0.08 | $4600 + x$ |

$$0.068(4600) + 0.09x = 0.08(4600 + x)$$
$$312.8 + 0.09x = 368 + 0.08x$$
$$0.01x = 55.2$$
$$x = 5520$$

$5520 additional investment

**39.**

| | |
|---|---|
| 1.00 | $x$ |
| 0.45 | $200 - x$ |
| 0.50 | 200 |

$$x + 0.45(200 - x) = 0.50(200)$$
$$x + 90 - 0.45x = 100$$
$$0.55x = 10$$
$$x = 18\frac{2}{11} \text{ g pure silver}$$

**40.**

| | |
|---|---|
| 0.40 | $x$ |
| 0.24 | 4 |
| 0.30 | $4 + x$ |

$$0.40x + 0.24(4) = 0.30(4 + x)$$
$$0.40x + 0.96 = 1.2 + 0.30x$$
$$0.10x = 0.24$$
$$x = 2.4$$

2.4 liters of 40% sulfuric acid

**41.**

| | |
|---|---|
| 0 | $x$ |
| 0.12 | 160 |
| 0.20 | $160 - x$ |

$$0.12(160) - 0 = 0.20(160 - x)$$
$$19.2 = 32 - 0.20x$$
$$0.20x = 12.8$$
$$x = 64$$

64 liters of water

**42.**

| | |
|---|---|
| 0.25 | 6 |
| 0.25 | $x$ |
| 1.00 | $x$ |
| 0.33 | 6 |

$$0.25(6) - 0.25x + x = 0.33(6)$$
$$1.5 + 0.75x = 1.98$$
$$0.75x = 0.48$$
$$x = 0.64$$

0.64 liter of water should be replaced.

**43.**

| | |
|---|---|
| 14 | $x$ |
| 25 | $3000 - x$ |

$$\begin{aligned} 14x + 25(3000 - x) &= 61{,}800 \\ 14x + 75{,}000 - 25x &= 61{,}800 \\ -11x &= -13{,}200 \\ x &= 1200 \\ 3000 - x &= 1800 \end{aligned}$$

1200 tickets at \$14 each
1800 tickets at \$25 each

**44.**

| | |
|---|---|
| 0.05 | $x$ |
| 0.10 | $2x$ |
| 0.25 | $255 - 3x$ |

$$\begin{aligned} 0.05x + 0.10(2x) + 0.25(255 - 3x) &= 41.25 \\ 0.05x + 0.20x + 63.75 - 0.75x &= 41.25 \\ -0.50x &= -22.50 \\ x &= 45 \end{aligned}$$

$$\begin{aligned} x &= 45 \text{ nickels} \\ 2x &= 90 \text{ dimes} \\ 255 - 3x &= 120 \text{ quarters} \end{aligned}$$

**45.**

| | |
|---|---|
| \$12 | $X$ |
| \$9 | $20 - x$ |
| \$10 | 20 |

$$\begin{aligned} 12x + 9(20 - x) &= 10(20) \\ 12x + 180 - 9x &= 200 \\ 3x &= 20 \\ x &= 6\frac{2}{3} \end{aligned}$$

$6\frac{2}{3}$ lb of \$12 coffee

$20 - 6\frac{2}{3} = 13\frac{1}{3}$ lb of \$9 coffee

**46.**

| | |
|---|---|
| 8 | $x$ (gold) |
| 5 | $42 - x$ (silver) |

$$\begin{aligned} 8x + 5(42 - x) &= 246 \\ 8x + 210 - 5x &= 246 \\ 3x &= 36 \\ x &= 12 \end{aligned}$$

12 gold coins
42 – 12 = 30 silver coins

**47.**

| | |
|---|---|
| 1.00 | $x$ |
| $\frac{14}{24}$ or $\frac{7}{12}$ | 15 |
| $\frac{18}{24}$ or $\frac{3}{4}$ | $15 + x$ |

$$\begin{aligned} x + \frac{7}{12}(15) &= \frac{3}{4}(15 + x) \\ 12 \cdot \left[x + \frac{7}{12}(15)\right] &= 12 \cdot \left[\frac{3}{4}(15 + x)\right] \\ 12x + 7(15) &= 9(15 + x) \\ 12x + 105 &= 135 + 9x \\ 3x &= 30 \\ x &= 10 \end{aligned}$$

10 g of pure gold

**48.**

| | |
|---|---|
| 1.00 | 4 |
| $\frac{14}{24}$ or $\frac{7}{12}$ | $x$ |
| $\frac{18}{24}$ or $\frac{3}{4}$ | $4 + x$ |

$$\begin{aligned} 4 + \frac{7}{12}x &= \frac{3}{4}(4 + x) \\ 12 \cdot \left[4 + \frac{7}{12}x\right] &= 12 \cdot \left[\frac{3}{4}(4 + x)\right] \\ 48 + 7x &= 9(4 + x) \\ 48 + 7x &= 36 + 9x \\ -2x &= -12 \\ x &= 6 \end{aligned}$$

6 oz of 14 karat gold

**49.** Let $t$ = the time it takes both electricians working together to wire the house.

The first electrician does $\frac{1}{14}$ of the job every hour.

The second electrician does $\frac{1}{18}$ of the job every hour.

$$\begin{aligned} \frac{1}{14}t+\frac{1}{18}t&=1 \\ 126\left[\frac{1}{14}t+\frac{1}{18}t\right]&=126\cdot 1 \\ 9t+7t&=126 \\ 16t&=126 \\ t&=7\frac{7}{8}\text{ hours} \end{aligned}$$

**50.** Let $t$ = the time it takes them working together to print the report.

Printer A does $\frac{1}{3}$ of the job every hour.

Printer B does $\frac{1}{4}$ of the job every hour.

$$\begin{aligned} \frac{1}{3}t+\frac{1}{4}t&=1 \\ 12\left[\frac{1}{3}t+\frac{1}{4}t\right]&=12\cdot 1 \\ 4t+3t&=12 \\ 7t&=12 \\ t&=1\frac{5}{7}\text{ hours} \end{aligned}$$

**51.** Let $t$ = the time it takes both painters working together to paint the kitchen.

The painter can paint $\frac{1}{10}$ of the kitchen every hour.

The apprentice can paint $\frac{1}{15}$ of the kitchen every hour.

$$\begin{aligned} \frac{1}{10}t+\frac{1}{15}t&=1 \\ 30\left[\frac{1}{10}t+\frac{1}{15}t\right]&=30\cdot 1 \\ 3t+2t&=30 \\ 5t&=30 \\ t&=6\text{ hours} \end{aligned}$$

**52.** Let $t$ = the time it takes to deposit enough snow to open the beginning trail.

The snow making machine does $\frac{1}{16}$ of the job every hour.

The natural snow fall does $\frac{1}{24}$ of the job every hour.

$$\begin{aligned} \frac{1}{16}t+\frac{1}{24}t&=1 \\ 48\left[\frac{1}{16}t+\frac{1}{24}t\right]&=48\cdot 1 \\ 3t+2t&=48 \\ 5t&=48 \\ t&=9\frac{3}{5}\text{ hours} \end{aligned}$$

**53.** Let $t$ = the time it takes the older machine to finish the job.

The new machine does $\frac{1}{12}$ of the job every hour.

The old machine does $\frac{1}{16}$ of the job every hour.

The new machine works for 4 hours: $4\left(\frac{1}{12}\right)=\frac{1}{3}$.

The old machine completes the job.

$$\begin{aligned} \frac{1}{3}+\frac{1}{16}t&=1 \\ \frac{1}{16}t&=\frac{2}{3} \\ t&=10\frac{2}{3}\text{ hours} \end{aligned}$$

**54.** Let $t$ = the time it takes the apprentice to finish the job.

The mason does $\frac{1}{12}$ of the job every hour.

The apprentice does $\frac{1}{16}$ of the job every hour.

The two people work together for 4 hours: $4\left(\frac{1}{12}+\frac{1}{16}\right)=\frac{7}{12}$

$$\begin{aligned} \frac{7}{12}+\frac{1}{16}t&=1 \\ \frac{1}{16}t&=\frac{5}{12} \\ t&=6\frac{2}{3}\text{ hours} \end{aligned}$$

**55.** Let $x$ = price of book
$10.10 - x$ = price of bookmark.

$$x = 10 + (10.10 - x)$$
$$2x = 20.10$$
$$x = 10.05$$
$$10.10 - 10.05 = 0.05$$

The price of the book is \$10.05.
The price of the bookmark is \$0.05.

**56.** Let $x$ = cost of yacht

$\frac{x}{3}$ = cost with 3 partners

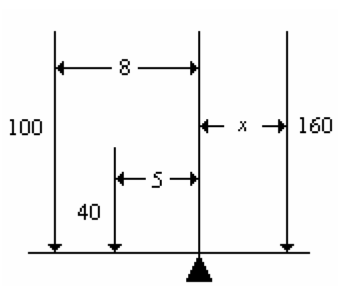
$\frac{x}{4}$ = cost with 4 partners

$$\frac{x}{3} - \frac{x}{4} = 4000$$
$$12\left[\frac{x}{3} - \frac{x}{4}\right] = 12(4000)$$
$$4x - 3x = 48{,}000$$
$$x = 48{,}000$$

Cost = \$48,000

## Connecting Concepts

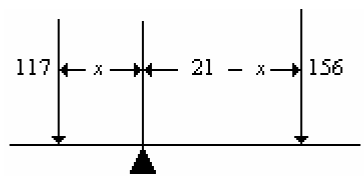
**57.**




$$8(100) + 40(5) = 160x$$
$$800 + 200 = 160x$$
$$1000 = 160x$$
$$6.25 = x$$

6.25 ft

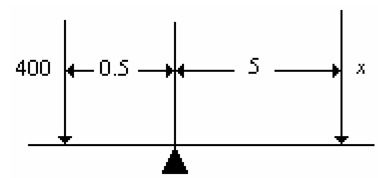
**58.**




$$117x = 156(21 - x)$$
$$117x = 3276 - 156x$$
$$273x = 3276$$
$$x = 12$$

12 ft from the 117-lb force

**59.**




$$400(0.5) = 5x$$
$$200 = 5x$$
$$40 = x$$

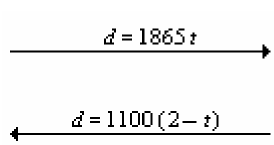
A 40-lb force is needed to lift 400 lbs.

**60.**

$$180(5)+4x=1440(1)$$
$$900+4x=1440$$
$$4x=540$$
$$x=135$$

The second worker needs to apply a force of 135 pounds.

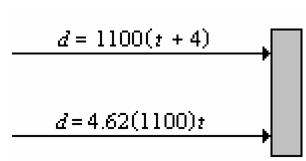
**61.**




$$1865t = 1100(2 - t)$$
$$1865t = 2200 - 1100t$$
$$2965t = 2200$$
$$t \approx 0.741989$$
$$d = 1865t$$
$$d \approx 1383.81$$

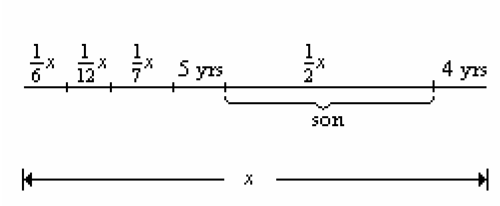
The distance to the target is 1384 feet (to the nearest foot).

**62.**




$$(t + 4)(1100) = 4.62(1100)t$$
$$1100t + 4400 = 5082t$$
$$4400 = 3982t$$
$$1.104972376 \approx t$$
$$d = 5082t$$
$$d \approx 5615 \text{ ft}$$

**63.**




$$\frac{1}{6}x+\frac{1}{12}x+\frac{1}{7}x+5+\frac{1}{2}x+4=x$$
$$84\left[\frac{1}{6}x+\frac{1}{12}x+\frac{1}{7}x+5+\frac{1}{2}x+4\right]=84x$$
$$14x+7x+12x+420+42x+336=84x$$
$$75x+756=84x$$
$$756=9x$$
$$84=x$$

Diophantus was 84 years old when he died.

**64.**
$$F=\frac{9}{5}C+32,\ \text{Let } C=F, \text{ then}$$
$$F=\frac{9}{5}F+32$$
$$5F=9F+160$$
$$-4F=160$$
$$F=-40^{\circ}$$

## Prepare for Section 1.3

**PS1.** $x^2-x-42=(x+6)(x-7)$

**PS2.** $6x^2-x-15=(2x+3)(3x-5)$

**PS3.** $3+\sqrt{-16}=3+4i$

**PS4.** $\dfrac{-(-2)-\sqrt{(-2)^2-4(-3)(5)}}{2(-3)}=\dfrac{2-\sqrt{64}}{-6}=1$

**PS5.** $\dfrac{-(-3)+\sqrt{(-3)^2-4(2)(1)}}{2(2)}=\dfrac{3+\sqrt{1}}{4}=1$

**PS6.** $(3-i)^2-6(3-i)+10=9-6i+i^2-18+6i+10$
$=0$

## Section 1.3

**1.**
$$x^2-2x-15=0$$
$$(x+3)(x-5)=0$$
$$x+3=0 \quad \text{or} \quad x-5=0$$
$$x=-3 \qquad\qquad x=5$$

**2.**
$$x^2+3x-10=0$$
$$(x-2)(x+5)=0$$
$$x-2=0 \quad \text{or} \quad x+5=0$$
$$x=2 \qquad\qquad x=-5$$

**3.**
$$2x^2-x=1$$
$$2x^2-x-1=0$$
$$(2x+1)(x-1)=0$$
$$2x+1=0 \quad \text{or} \quad x-1=0$$
$$2x=-1 \qquad\qquad x=1$$
$$x=-\frac{1}{2}$$

**4.**
$$2x^2+5x=3$$
$$2x^2+5x-3=0$$
$$(2x-1)(x+3)=0$$
$$2x-1=0 \quad \text{or} \quad x+3=0$$
$$2x=1 \qquad\qquad x=-3$$
$$x=\frac{1}{2}$$

**5.**
$$8x^2+189x-72=0$$
$$(8x-3)(x+24)=0$$
$$8x-3=0 \quad \text{or} \quad x+2=0$$
$$8x=3 \qquad\qquad x=-24$$
$$x=\frac{3}{8}$$

**6.**
$$12x^2-41x+24=0$$
$$(4x-3)(3x-8)=0$$
$$4x-3=0 \quad \text{or} \quad 3x-8=0$$
$$4x=3 \qquad\qquad 3x=8$$
$$x=\frac{3}{4} \qquad\qquad x=\frac{8}{3}$$

**7.**
$$\begin{aligned} 3x^2 - 7x &= 0 \\ x(3x-7) &= 0 \\ x = 0 \quad &\text{or} \quad 3x - 7 = 0 \\ &\quad 3x = 7 \\ &\quad x = \frac{7}{3} \end{aligned}$$

**8.**
$$\begin{aligned} 5x^2 &= -8x \\ 5x^2 + 8x &= 0 \\ x(5x+8) &= 0 \\ x = 0 \quad &\text{or} \quad 5x + 8 = 0 \\ &\quad 5x = -8 \\ &\quad x = -\frac{8}{5} \end{aligned}$$

**9.**
$$\begin{aligned} (x-5)^2 - 9 &= 0 \\ [(x-5)-3]\,[(x-5)+3] &= 0 \\ (x-8)(x-2) &= 0 \\ x - 8 = 0 \quad \text{or} &\quad x - 2 = 0 \\ x = 8 \qquad &\quad x = 2 \end{aligned}$$

**10.**
$$\begin{aligned} (3x+4)^2 - 16 &= 0 \\ [(3x+4)-4][(3x+4+4] &= 0 \\ (3x)(3x+8) &= 0 \\ 3x = 0 \quad &\text{or} \quad 3x + 8 = 0 \\ x = 0 \quad &\quad 3x = -8 \\ &\quad x = -\frac{8}{3} \end{aligned}$$

**11.**
$$\begin{aligned} x^2 &= 81 \\ x &= \pm\sqrt{81} \\ x &= \pm 9 \end{aligned}$$

**12.**
$$\begin{aligned} x^2 &= 225 \\ x &= \pm\sqrt{225} \\ x &= \pm 15 \end{aligned}$$

**13.**
$$\begin{aligned} 2x^2 &= 48 \\ x^2 &= 24 \\ x &= \pm\sqrt{24} \\ x &= \pm 2\sqrt{6} \end{aligned}$$

**14.**
$$\begin{aligned} 3x^2 &= 144 \\ x^2 &= 48 \\ x &= \pm\sqrt{48} \\ x &= \pm 4\sqrt{3} \end{aligned}$$

**15.**
$$\begin{aligned} 3x^2 + 12 &= 0 \\ 3x^2 &= -12 \\ x^2 &= -4 \\ x &= \pm\sqrt{-4} \\ x &= \pm 2i \end{aligned}$$

**16.**
$$\begin{aligned} 4x^2 + 20 &= 0 \\ 4x^2 &= -20 \\ x^2 &= -5 \\ x &= \pm\sqrt{-5} \\ x &= \pm i\sqrt{5} \end{aligned}$$

**17.**
$$\begin{aligned} (x-5)^2 &= 36 \\ x - 5 &= \pm\sqrt{36} \\ x - 5 &= \pm 6 \\ x &= 5 \pm 6 \\ x = 5 + 6 \quad &\text{or} \quad x = 5 - 6 \\ x = 11 \quad &\quad x = -1 \end{aligned}$$

**18.**
$$\begin{aligned} (x+4)^2 &= 121 \\ x + 4 &= \pm\sqrt{121} \\ x + 4 &= \pm 11 \\ x &= -4 \pm 11 \\ x = -4 + 11 \quad &\text{or} \quad x = -4 - 11 \\ x = 7 \quad &\quad x = -15 \end{aligned}$$

**19.**
$$\begin{aligned} (x-3)^2 + 16 &= 0 \\ (x-3)^2 &= -16 \\ x - 3 &= \pm\sqrt{-16} \\ x - 3 &= \pm 4i \\ x &= 3 \pm 4\mathrm{i} \end{aligned}$$

**20.**
$$\begin{aligned} (x+2)^2 + 28 &= 0 \\ (x+2)^2 &= -28 \\ x + 2 &= \pm\sqrt{-28} \\ x + 2 &= \pm 2i\sqrt{7} \\ x &= -2 \pm 2i\sqrt{7} \end{aligned}$$

**21.**
$$\begin{aligned} x^2 + 6x + 1 &= 0 \\ x^2 + 6x + 9 &= -1 + 9 \\ (x+3)^2 &= 8 \\ x + 3 &= \pm\sqrt{8} \\ x &= -3 \pm 2\sqrt{2} \\ x = -3 + 2\sqrt{2} \quad &\text{or} \quad x = -3 - 2\sqrt{2} \end{aligned}$$

**22.**
$$\begin{aligned} x^2 + 8x - 10 &= 0 \\ x^2 + 8x + 16 &= 10 + 16 \\ (x+4)^2 &= 26 \\ x + 4 &= \pm\sqrt{26} \\ x &= -4 \pm \sqrt{26} \\ x = -4 + \sqrt{26} \quad &\text{or} \quad x = -4 - \sqrt{26} \end{aligned}$$

**23.**
$$\begin{aligned} x^2 - 2x - 15 &= 0 \\ x^2 - 2x + 1 &= 15 + 1 \\ (x-1)^2 &= 16 \\ x - 1 &= \pm\sqrt{16} \\ x &= 1 \pm 4 \\ x = 1 + 4 \quad &\text{or} \quad x = 1 - 4 \\ x = 5 \quad &\quad x = -3 \end{aligned}$$

**24.**
$$x^2+2x-8=0$$
$$x^2+2x+1=8+1$$
$$(x+1)^2=9$$
$$x+1=\pm\sqrt{9}$$
$$x=-1\pm 3$$
$$x=-1+3 \quad \text{or} \quad x=-1-3$$
$$x=2 \qquad\qquad x=-4$$

**25.**
$$x^2+4x+5=0$$
$$x^2+4x+4=-5+4$$
$$(x+2)^2=-1$$
$$x+2=\pm\sqrt{-1}$$
$$x+2=\pm i$$
$$x=-2\pm i$$
$$x=-2-i \quad \text{or} \quad x=-2+i$$

**26.**
$$x^2-6x+10=0$$
$$x^2-6x+9=-10+9$$
$$(x-3)^2=-1$$
$$x-3=\pm\sqrt{-1}$$
$$x=3\pm i$$
$$x=3+i \quad \text{or} \quad x=3-i$$

**27.**
$$x^2+3x-1=0$$
$$x^2+3x+\frac{9}{4}=1+\frac{9}{4}$$
$$\left(x+\frac{3}{2}\right)^2=\frac{13}{4}$$
$$x+\frac{3}{2}=\pm\sqrt{\frac{13}{4}}$$
$$x=-\frac{3}{2}\pm\frac{\sqrt{13}}{2}$$
$$x=\frac{-3+\sqrt{13}}{2}, \quad \text{or} \quad x=\frac{-3-\sqrt{13}}{2}$$

**28.**
$$x^2+7x-2=0$$
$$x^2+7x+\frac{49}{4}=2+\frac{49}{4}$$
$$\left(x+\frac{7}{2}\right)^2=\frac{57}{4}$$
$$x+\frac{7}{2}=\pm\sqrt{\frac{57}{4}}$$
$$x=-\frac{7}{2}\pm\frac{\sqrt{57}}{2}$$
$$x=\frac{-7+\sqrt{57}}{2}, \quad \text{or} \quad x=\frac{-7-\sqrt{57}}{2}$$

**29.**
$$2x^2+4x-1=0$$
$$2x^2+4x=1$$
$$x^2+2x=\frac{1}{2}$$
$$x^2+2x+1=\frac{1}{2}+1$$
$$(x+1)^2=\frac{3}{2}$$
$$x+1=\pm\sqrt{\frac{3}{2}}$$
$$x=-1\pm\sqrt{\frac{3}{2}}=-1\pm\sqrt{\frac{3}{2}\cdot\frac{2}{2}}$$
$$x=-1\pm\frac{\sqrt{6}}{\sqrt{4}}=-1\pm\frac{\sqrt{6}}{2}=-\frac{2}{2}\pm\frac{\sqrt{6}}{2}$$
$$x=\frac{-2+\sqrt{6}}{2}, \quad \text{or} \quad x=\frac{-2-\sqrt{6}}{2}$$

**30.**
$$2x^2+10x-3=0$$
$$2x^2+10x=3$$
$$x^2+5x=\frac{3}{2}$$
$$x^2+5x+\frac{25}{4}=\frac{3}{2}+\frac{25}{4}$$
$$\left(x+\frac{5}{2}\right)^2=\frac{31}{4}$$
$$x+\frac{5}{2}=\pm\sqrt{\frac{31}{4}}$$
$$x=-\frac{5}{2}\pm\frac{\sqrt{31}}{2}$$
$$x=\frac{-5+\sqrt{31}}{2} \quad \text{or} \quad x=\frac{-5-\sqrt{31}}{2}$$

**31.**

$$3x^2 - 8x + 1 = 0$$
$$3x^2 - 8x = -1$$
$$x^2 - \frac{8}{3}x = -\frac{1}{3}$$
$$x^2 - \frac{8}{3}x + \frac{16}{9} = -\frac{1}{3} + \frac{16}{9}$$
$$\left(x - \frac{4}{3}\right)^2 = \frac{13}{9}$$
$$x - \frac{4}{3} = \pm\sqrt{\frac{13}{9}}$$
$$x = \frac{4}{3} \pm \frac{\sqrt{13}}{3}$$
$$x = \frac{4+\sqrt{13}}{3} \quad \text{or} \quad x = \frac{4-\sqrt{13}}{3}$$

**32.**

$$4x^2 - 4x + 15 = 0$$
$$4x^2 - 4x = -15$$
$$x^2 - x = -\frac{15}{4}$$
$$x^2 - x + \frac{1}{4} = -\frac{15}{4} + \frac{1}{4}$$
$$\left(x - \frac{1}{2}\right)^2 = -\frac{14}{4}$$
$$x - \frac{1}{2} = \pm\sqrt{-\frac{14}{4}}$$
$$x = \frac{1}{2} \pm \frac{i\sqrt{14}}{2}$$
$$x = \frac{1}{2} + \frac{\sqrt{14}}{2}i \quad \text{or} \quad x = \frac{1}{2} - \frac{\sqrt{14}}{2}i$$

**33.**

$$x^2 - 2x - 15 = 0,\ a = 1,\ b = -2,\ c = -15$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-15)}}{2(1)}$$
$$x = \frac{2 \pm \sqrt{4+60}}{2} = \frac{2 \pm \sqrt{64}}{2} = \frac{2 \pm 8}{2}$$
$$x = \frac{2+8}{2} = \frac{10}{2} = 5 \quad \text{or} \quad x = \frac{2-8}{2} = \frac{-6}{2} = -3$$
$x = 5$ or $x = -3$

**34.**

$$x^2 - 5x - 24 = 0$$
$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-24)}}{2(1)}$$
$$x = \frac{5 \pm \sqrt{25+96}}{2} = \frac{5 \pm \sqrt{121}}{2} = \frac{5 \pm 11}{2}$$
$$x = \frac{5+11}{2} = \frac{16}{2} = 8 \quad \text{or} \quad x = \frac{5-11}{2} = \frac{-6}{2} = -3$$
$x = 8$ or $x = -3$

**35.**

$$x^2 + x - 1 = 0$$
$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)}$$
$$x = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$
$$x = \frac{-1+\sqrt{5}}{2} \quad \text{or} \quad x = \frac{-1-\sqrt{5}}{2}$$

**36.**

$$x^2 + x + 1 = 0$$
$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)}$$
$$x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}$$
$$x = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \quad \text{or} \quad x = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

**37.**

$$2x^2 + 4x + 1 = 0$$
$$x = \frac{-4 \pm \sqrt{4^2 - 4(2)(1)}}{2(2)}$$
$$x = \frac{-4 \pm \sqrt{16-8}}{4} = \frac{-4 \pm \sqrt{8}}{4}$$
$$x = \frac{-4 \pm 2\sqrt{2}}{4} = \frac{-2 \pm \sqrt{2}}{2}$$
$$x = \frac{-2+\sqrt{2}}{2}, \quad \text{or} \quad x = \frac{-2-\sqrt{2}}{2}$$

**38.**

$$2x^2 + 4x - 1 = 0$$
$$x = \frac{-4 \pm \sqrt{4^2 - 4(2)(-1)}}{2(2)}$$
$$x = \frac{-4 \pm \sqrt{16+8}}{4} = \frac{-4 \pm \sqrt{24}}{4}$$
$$x = \frac{-4 \pm 2\sqrt{6}}{4} = \frac{-2 \pm \sqrt{6}}{2}$$
$$x = \frac{-2+\sqrt{6}}{2}, \quad \text{or} \quad x = \frac{-2-\sqrt{6}}{2}$$

**39.** $3x^2 - 5x + 3 = 0$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(3)}}{2(3)}$$

$$x = \frac{5 \pm \sqrt{25-36}}{6} = \frac{5 \pm \sqrt{-11}}{6}$$

$$x = \frac{5 \pm i\sqrt{11}}{6}$$

$$x = \frac{5}{6} + \frac{\sqrt{11}}{6}i, \text{ or } x = \frac{5}{6} - \frac{\sqrt{11}}{6}i$$

**40.** $3x^2 - 5x + 4 = 0$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(4)}}{2(3)}$$

$$x = \frac{5 \pm \sqrt{25-48}}{6} = \frac{5 \pm \sqrt{-23}}{6}$$

$$x = \frac{5 \pm i\sqrt{23}}{6}$$

$$x = \frac{5}{6} + \frac{\sqrt{23}}{6}i, \text{ or } x = \frac{5}{6} - \frac{\sqrt{23}}{6}i$$

**41.** $\frac{1}{2}x^2 + \frac{3}{4}x - 1 = 0$

$$4\left(\frac{1}{2}x^2 + \frac{3}{4}x - 1\right) = 4(0)$$

$$2x^2 + 3x - 4 = 0$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(2)(-4)}}{2(2)}$$

$$x = \frac{-3 \pm \sqrt{9+32}}{4}$$

$$x = \frac{-3 \pm \sqrt{41}}{4}$$

$$x = \frac{-3 + \sqrt{41}}{4}, \text{ or } x = \frac{-3 - \sqrt{41}}{4}$$

**42.** $\frac{2}{3}x^2 - 5x + \frac{1}{2} = 0$

$$6\left(\frac{2}{3}x^2 - 5x + \frac{1}{2}\right) = 6(0)$$

$$4x^2 - 30x + 3 = 0$$

$$x = \frac{-(-30) \pm \sqrt{(-30)^2 - 4(4)(3)}}{2(4)}$$

$$x = \frac{30 \pm \sqrt{900-48}}{8}$$

$$x = \frac{30 \pm \sqrt{852}}{8}$$

$$x = \frac{30 \pm 2\sqrt{213}}{8}$$

$$x = \frac{15 \pm \sqrt{213}}{4}$$

**43.** $24x^2 - 22x - 35 = 0$

$$x = \frac{-(-22) \pm \sqrt{(-22)^2 - 4(24)(-35)}}{2(24)}$$

$$x = \frac{22 \pm \sqrt{484+3360}}{48}$$

$$x = \frac{22 \pm \sqrt{3844}}{48}$$

$$= \frac{22 \pm 62}{48}$$

$$x = -\frac{5}{6}, \text{ or } x = \frac{7}{4}$$

**44.** $72x^2 + 13x - 15 = 0$

$$x = \frac{-13 \pm \sqrt{(13)^2 - 4(72)(-15)}}{2(72)}$$

$$x = \frac{-13 \pm \sqrt{169+4320}}{144}$$

$$x = \frac{-13 \pm \sqrt{4489}}{144}$$

$$= \frac{-13 \pm 67}{144}$$

$$x = -\frac{5}{9}, \text{ or } x = \frac{3}{8}$$

**45.** $0.5x^2+0.6x-0.8=0$

$$x=\frac{-0.6\pm\sqrt{(0.6)^2-4(0.5)(-0.8)}}{2(0.5)}$$
$$x=\frac{-0.6\pm\sqrt{0.36+0.6}}{1}$$
$$x=-0.6\pm\sqrt{1.96}$$
$$=-0.6\pm1.4$$
$$x=-2, \text{ or } x=\frac{4}{5}$$

**46.** $1.2x^2+0.4x-0.5=0$

$$x=\frac{-0.4\pm\sqrt{(0.4)^2-4(1.2)(-0.5)}}{2(1.2)}$$
$$x=\frac{-0.4\pm\sqrt{0.16+2.4}}{2.4}$$
$$x=\frac{-0.4\pm\sqrt{2.56}}{2.4}$$
$$=\frac{-0.4\pm1.6}{2.4}$$
$$x=-\frac{5}{6}, \text{ or } x=\frac{1}{2}$$

**47.** $2x^2-5x-7=0$

$b^2-4ac=(-5)^2-4(2)(-7)$
$=25+56=81$

Two real solutions

**48.** $x^2+3x-11=0$

$b^2-4ac=(3)^2-4(1)(-11)$
$=9+44=53$

Two real solutions

**49.** $3x^2-2x+10=0$

$b^2-4ac=(-2)^2-4(3)(10)$
$=4-120=-116$

No real solutions

**50.** $x^2+3x+3=0$

$b^2-4ac=(3)^2-4(1)(3)$
$=9-12=-3$

No real solutions

**51.** $x^2-20x+100=0$

$b^2-4ac=(-20)^2-4(1)(100)$
$=400-400=0$

One real solution

**52.** $4x^2+12x+9=0$

$b^2-4ac=(12)^2-4(4)(9)$
$=144-144=0$

One real solution

**53.** $24x^2+10x-21=0$

$b^2-4ac=(10)^2-4(24)(-21)$
$=100+2016=2116$

Two real solutions

**54.** $32x^2-44x+15=0$

$b^2-4ac=(-44)^2-4(32)(15)$
$=1936-1920=16$

Two real solutions

**55.** $12x^2+15x+7=0$

$b^2-4ac=(15)^2-4(12)(7)$
$=225-336=-111$

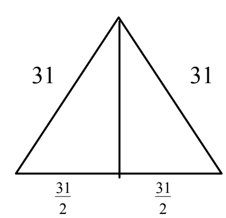
No real solutions

**56.** $8x^2-5x+3=0$

$b^2-4ac=(-5)^2-4(8)(3)$
$=25-96=-71$

No real solutions

**57.**




$$a^2+\left(\frac{31}{2}\right)^2=31^2$$
$$a^2=31^2-\left(\frac{31}{2}\right)^2$$
$$d=\sqrt{31^2-\left(\frac{31}{2}\right)^2}$$
$$d\approx 26.8 \text{ in.}$$

**58.**
$$a^2+b^2=c^2$$
$$(90)^2+(90)^2=c^2$$
$$16,200=c^2$$
$$127.3=c$$
The distance is 127.3 ft.

**59.**
$$\frac{a}{b}=\frac{4}{3}$$
$$a=\frac{4b}{3}$$
$$a^2+b^2=c^2$$
$$\left(\frac{4b}{3}\right)^2+b^2=54^2$$
$$\frac{16b^2}{9}+b^2=2916$$
$$\frac{25b^2}{9}=2916$$
$$b^2=1049.76$$
$$b=32.4$$
$$a=\frac{4(32.4)}{3}=43.2$$
The TV is 32.4 in. high and 43.2 in. wide.

**60.**
$$250,000=40,000+20x+0.0001x^2$$
$$0=0.0001x^2+20x-210,000$$
$$a=0.0001,\ b=20,\ c=-210,000$$
$$x=\frac{-20\pm\sqrt{20^2-4(0.0001)(-210,000)}}{2(0.0001)}$$
$$=\frac{-20\pm\sqrt{484}}{0.0002}=\frac{-20+22}{0.0002}=10,000 \text{ books}$$
(Reject negative $x$-value; $x$ must be positive.)

**61.**
$$19,000=38t^2+291t+15,208$$
$$0=38t^2+291t-3792$$
$$a=38,\ b=291,\ c=-3292$$
$$t=\frac{-291\pm\sqrt{291^2-4(38)(-3292)}}{2(38)}$$
$$=\frac{-291\pm\sqrt{585065}}{76}\approx\frac{-291\pm 764.8954}{76}$$
$t\approx 6.23$ or $t\approx -13.89$ (not $0\le t\le 14$)
6.23 years after 1990 is 1996.

**62.**
$$R=xp$$
$$16,500=x(26-0.01x)$$
$$16,500=26x-0.01x^2$$
$$0=-0.01x^2+26x-16,500$$
$$a=-0.01,\ b=26,\ c=-16,500$$
$$x=\frac{-26\pm\sqrt{26^2-4(-0.01)(-16,500)}}{2(-0.01)}$$
$$=\frac{-26\pm\sqrt{16}}{-0.02}=\frac{-26\pm 4}{-0.02}$$
$$x=\frac{-26+4}{-0.02}=\frac{-22}{-0.02}=1100 \quad\text{or}\quad x=\frac{-26-4}{-0.02}=\frac{-30}{-0.02}=1500$$
1100 or 1500 items must be sold.

**63.**
$$518,000=-0.01x^2+168x-120,000$$
$$0=-0.01x^2+168x-638,000$$
$$a=-0.01,\ b=168,\ c=-638,000$$
$$x=\frac{-168\pm\sqrt{168^2-4(-0.01)(-638,000)}}{2(-0.01)}$$
$$=\frac{-168\pm\sqrt{2704}}{-0.02}=\frac{-168\pm 52}{-0.02}$$
$$x=\frac{-168+52}{-0.02}=\frac{-116}{-0.02}=5800 \quad\text{or}\quad x=\frac{-168-52}{-0.02}=\frac{-220}{-0.02}=11000$$
5800 or 11,000 racquets must be sold.

**64.** **a.** Evaluate $A = 0.72(1.28)h^2$ with $h = 7$.

$$A = 0.72(1.28)(7)^2$$
$$= 0.72(1.28)(49)$$
$$\approx 45.2 \text{ square inches}$$

**b.** Solve $A = 0.72(1.28)h^2$ for $h$ with $A = 92$.

$$92 = 0.72(1.28)h^2$$
$$92 = 0.9216h^2$$
$$99.82639 \approx h^2$$
$$h \approx \sqrt{99.82639}$$
$$\approx 10.0 \text{ inches}$$

**65.** Let $w$ = width of region

Then $\frac{132-3w}{2}$ = length.

$$\text{Area} = \text{length(width)}$$
$$576 = \frac{132-3w}{2} \cdot w$$
$$1152 = 132w - 3w^2$$
$$3w^2 - 132w + 1152 = 0$$
$$3(w^2 - 44w + 384) = 0$$
$$w^2 - 44w + 384 = 0$$
$$(w-32)(w-12) = 0$$

$w - 32 = 0$ or $w - 12 = 0$

$w = 32$ $\quad$ $w = 12$

$\frac{132-3w}{2} = \frac{132-3(32)}{2} = 18$ $\qquad$ $\frac{132-3w}{2} = \frac{132-3(12)}{2} = 48$

The region is either 32 feet wide and 18 feet long, or 12 feet wide and 48 feet long.

**66.** Let $x$ = side of cardboard.
Then $(x - 6)$ = length of box.

$$\text{Volume} = \text{(length)(width)(height)}$$
$$126.75 = (x-6)(x-6)(3)$$
$$126.75 = (x^2 - 12x + 36)(3)$$
$$126.75 = 3x^2 - 36x + 108$$
$$0 = 3x^2 - 36x - 18.75$$
$$a = 3,\ b = -36,\ c = -18.75$$
$$x = \frac{-(-36) \pm \sqrt{(-36)^2 - 4(3)(-18.75)}}{2(3)}$$
$$= \frac{36 \pm \sqrt{1521}}{6} = \frac{36 \pm 39}{6}$$

$x = \frac{36+39}{6} = \frac{75}{6} = 12.5$ or $x = \frac{36-39}{6} = \frac{-3}{6} = -0.5$ (no)

Since length cannot be a negative number, each side of the cardboard is 12.5 inches.

**67.** Solve $D = -45x^2 + 190x + 200$ for $x$ with $D = 250$.

$$250 = -45x^2 + 190x + 200$$
$$0 = -45x^2 + 190x - 50$$
$$a = -45,\ b = 190,\ c = -50$$
$$x = \frac{-190 \pm \sqrt{190^2 - 4(-45)(-50)}}{2(-45)}$$
$$= \frac{-190 \pm \sqrt{27100}}{-90} \approx \frac{-190 \pm 164.2}{-90}$$

$x \approx \frac{-190+164.2}{-90} \approx \frac{-25.8}{-90} \approx 0.3$ mile or $x \approx \frac{-190-164.2}{-90} \approx \frac{-354.2}{-90} \approx 3.9$ miles

**68.** Solve $N = -5t^2 + 80t - 280$ for $t$ with $N = 35$.

$$35 = -5t^2 + 80t - 280$$
$$0 = -5t^2 + 80t - 315$$
$$0 = -5(t^2 - 16t + 63)$$
$$0 = -5(t-7)(t-9)$$

$t - 7 = 0$ or $t - 9 = 0$

$t = 7$ A.M. $\qquad$ $t = 9$ A.M.

**69.** Solve $h = -16t^2 + 25.3t + 20$ for $t$ where $h = 17$.

$$17 = -16t^2 + 25.3t + 20$$
$$0 = -16t^2 + 25.3t + 3$$
$$t = \frac{-25.3 \pm \sqrt{(25.3)^2 - 4(-16)(3)}}{2(-16)}$$
$$= \frac{-25.3 \pm \sqrt{832.09}}{-32}$$

$t = 1.7$ or $t = -0.11$

He was in the air for 1.7 s.

**70.** Original candy bar

$lwh = V$

$5 \cdot 2 \cdot 0.5 = 5$ cubic inches

80%(5) = 0.80(5) = 4 cubic inches

Let $w$ represent the width.

Let $w + 2.5$ represent the length.

$$lwh = V$$
$$(w+2.5)(w)(0.5) = 4$$
$$(w+2.5)w = 8$$
$$w^2 + 2.5w - 8 = 0$$
$$w = \frac{-2.5 \pm \sqrt{(2.5)^2 - 4(1)(-8)}}{2(1)}$$
$$= \frac{-2.5 \pm \sqrt{38.25}}{2}$$

$w \approx 4.5$ in or $w \approx 1.8$ in

The dimensions are 1.8 in by 4.5 in by 0.5 in.

**71.** Solve $h = -16t^2 + 220t$ for $t$ where $h = 350$.

$$350 = -16t^2 + 220t$$
$$0 = -16t^2 + 220t - 350$$
$$a = -16,\ b = 220,\ c = -350$$
$$t = \frac{-220 \pm \sqrt{220^2 - 4(-16)(-350)}}{2(-16)}$$
$$= \frac{-220 \pm \sqrt{26000}}{-32} \approx \frac{-220 \pm 161.245}{-32}$$

$t \approx \frac{-220 + 161.245}{-32}$ or $t \approx \frac{-220 - 161.245}{-32}$

$\approx \frac{-58.755}{-32}$ $\approx \frac{-381.245}{-32}$

$\approx 1.8$ seconds $\approx 11.9$ seconds

**72.** When the ball hits the ground, the height is 0.

Solve $h = -16t^2 + 52t + 4.5$ for $t$ where $h = 0$.

$$0 = -16t^2 + 52t + 4.5$$
$$a = -16,\ b = 52,\ c = 4.5$$
$$t = \frac{-52 \pm \sqrt{52^2 - 4(-16)(4.5)}}{2(-16)}$$
$$t = \frac{-52 \pm \sqrt{2992}}{-32} \approx \frac{-52 \pm 54.699}{-32}$$
$$t \approx \frac{-52 - 54.699}{-32}$$
$$\approx \frac{-106.699}{-32}$$

$\approx 3.3$ seconds

**73.** Solve $s = 103.9t$ for $t$ where $s = 360$ to find the time it takes the ball to reach the fence.

$$360 = 103.9t$$

$t \approx 3.465$ seconds

Next, evaluate $h = -16t^2 + 50t + 4.5$ where $t = 3.465$ to determine if the ball is at least 10 feet in the air when it reaches the fence.

$$h = -16(3.465)^2 + 50(3.465) + 4.5$$
$$h \approx -14.3$$

No, the ball will not clear the fence.

**74.** Solve $s = -16t^2 + 26.6t$ for $t$ where $s = 0$.

$$0 = -16t^2 + 26.6t$$
$$0 = t(-16t + 26.6)$$

$t = 0$ or $-16t + 26.6 = 0$

$t \approx 1.7$ seconds

**75.** Solve $h = \frac{1}{2}n(n-1)$ for $n$ where $h = 36$.

$$36 = \frac{1}{2}n(n-1)$$
$$72 = n(n-1) = n^2 - n$$
$$0 = n^2 - n - 72 = (n-9)(n+8)$$

$n - 9 = 0$ or $n + 8 = 0$

$n = 9$ people $n = -8$ (no)

**76.** Solve $D = 1.525x^2 - 21.35x + 72.225$ for $x$ where $D = 100$.

$$100 = 1.525x^2 - 21.35x + 72.225$$
$$0 = 1.525x^2 - 21.35x - 27.775$$
$$a = 1.525, b = -21.35, c = -27.775$$
$$x = \frac{-(-21.35) \pm \sqrt{(-21.35)^2 - 4(1.525)(-27.775)}}{2(1.525)}$$
$$= \frac{21.35 \pm \sqrt{625.25}}{3.05} \approx \frac{21.35 \pm 25.005}{3.05}$$
$$x \approx \frac{21.35 + 25.005}{3.05} \quad \text{or} \quad x \approx \frac{21.35 - 25.005}{3.05}$$
$$\approx 15.198 \qquad \approx -1.198 \text{ (not in the future)}$$

The year that the NARA will first exceed 100 petabytes is 15 years, after 2000, which is in 2015.

**77.** Solve $R = 37.5x^2 - 225t + 342.5$ for $t$ where $R = 1000$.

$$1000 = 37.5x^2 - 225t + 342.5$$
$$0 = 37.5x^2 - 225t - 657.5$$
$$a = 37.5, b = -225, c = -657.5$$
$$t = \frac{-225 \pm \sqrt{(-225)^2 - 4(37.5)(-657.5)}}{2(37.5)}$$
$$= \frac{225 \pm \sqrt{149{,}250}}{75} \approx \frac{225 \pm 386.33}{75}$$
$$t \approx \frac{225 + 386.33}{75} \quad \text{or} \quad t \approx \frac{225 - 386.33}{75}$$
$$\approx 8.2 \qquad \approx -2.1 \text{ (not in the future)}$$

Revenue for cellular phone software will first reach \$1 billion 8.2 years from 2000, which is in 2008.

**78.** Solve $N = -0.015v^2 + 3v$ for $v$ where $N = 100$.

$$100 = -0.015v^2 + 3v$$
$$0 = -0.015v^2 + 3v - 100$$
$$a = -0.015, b = 3, c = -100$$
$$v = \frac{-3 \pm \sqrt{3^2 - 4(-0.015)(-100)}}{2(-0.015)}$$
$$= \frac{-3 \pm \sqrt{9 - 6}}{-0.03}$$
$$= \frac{-3 \pm \sqrt{3}}{-0.03}$$
$$v = \frac{-3 + \sqrt{3}}{-0.03} \qquad v = \frac{-3 - \sqrt{3}}{-0.03}$$
$$\approx 42 \text{ miles per hour} \qquad \approx 158 \text{ (not } 0 \le v \le 90)$$

**79.** **a.** If $t = 0$ represents the year 1995, then the year 2006 is represented by $t = 11$.

Evaluate $A = 0.05t^2 + 2.25t + 14$ for $t = 11$.

$$A = 0.05(11)^2 + 2.25(11) + 14$$
$$= 0.05(121) + 2.25(11) + 14$$
$$= 6.05 + 24.75 + 14$$
$$= 44.8 \text{ million pounds}$$

**b.** Solve $A = 0.05t^2 + 2.25t + 14$ for $t$ where $A = 50$.

$$50 = 0.05t^2 + 2.25t + 14$$
$$0 = 0.05t^2 + 2.25t - 36$$
$$a = 0.05, b = 2.25, c = -36$$
$$t = \frac{-2.25 \pm \sqrt{2.25^2 - 4(0.05)(-36)}}{2(0.05)}$$
$$= \frac{-2.25 \pm \sqrt{5.0625 + 7.2}}{0.1}$$
$$= \frac{-2.25 \pm \sqrt{12.2625}}{0.1} \approx \frac{-2.25 \pm 3.50}{0.1}$$
$$t \approx \frac{-2.25 + 3.50}{0.1} \quad \text{or} \quad t \approx \frac{-2.25 - 3.50}{0.1}$$
$$\approx 12.5 \qquad \approx -57.5 \text{ (not } 0 \le t \le 15)$$

12.5 years from 1995 will be in 2007.

## Connecting Concepts

**80.** **a.** $b^2 - 4ac$

$(-b)^2 - 4(1)(-4) = b^2 + 16$

The discriminant is $b^2 + 16$, which is positive for any value of $b$.

**b.** $b^2 - 4ac$

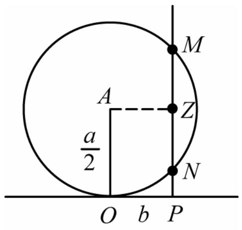
$(-6)^2 - 4(1)(k) = 36 - 4k$

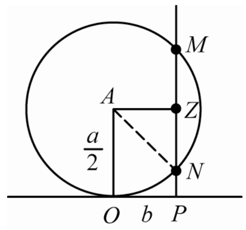
$36 - 4k > 0$

$-4k > -36$

$k < 9$

**81.** A line extending from $A$ to $MN$ is equal in length to $b$.
Draw a line from $A$ to $Z$ (a point on $MP$) that is parallel to line $OP$ and perpendicular to $MP$. Draw a line from $A$ to $N$.




Since $AN$ is also a radius of the circle, then its length is $\frac{a}{2}$. AZ is equal in length to $OP$, so $OP = AZ = b$.

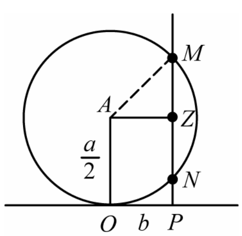
We have a right triangle $AZN$, where angle $AZN$ is the 90° angle. Also, ZN = ZP - NP, where $ZP = \frac{a}{2}$. Using the Pythagorean Theorem,

$$(AZ)^2 + (NZ)^2 = (AN)^2$$
$$b^2 + (ZP - NP)^2 = \left(\frac{a}{2}\right)^2$$
$$b^2 + \left(\frac{a}{2} - NP\right)^2 = \frac{a^2}{4}$$
$$b^2 + \frac{a^2}{4} - aNP + (NP)^2 = \frac{a^2}{4}$$
$$(NP)^2 - aNP + b^2 = 0$$

or $x^2 - ax + b^2 = 0$, where $x = NP$

Thus $NP$ is a solution to $x^2 - ab + b^2 = 0$.

Draw line $AM$. Since $AM$ is also a radius of the circle, its length is $\frac{a}{2}$.




We have a right triangle $AZM$, where angle $AZM$ is the 90° angle. Also, $MZ = MP - ZP$, where $ZP = \frac{a}{2}$. Using the Pythagorean Theorem,

$$(AZ)^2 + (MZ)^2 = (AM)^2$$
$$b^2 + (MP - ZP)^2 = \left(\frac{a}{2}\right)^2$$
$$b^2 + \left(MP - \frac{a}{2}\right)^2 = \frac{a^2}{4}$$
$$b^2 + (MP)^2 - aMP + \frac{a^2}{4} = \frac{a^2}{4}$$
$$(MP)^2 - aMP + b^2 = 0$$

or $x^2 - ax + b^2 = 0$, where $x = MP$.

Thus $MP$ is a solution to $x^2 - ab + b^2 = 0$.

**82.** $-7+3=-4;\ -\dfrac{b}{a}=-\dfrac{4}{1}=-4$

$(-7)(3)=-21;\ \dfrac{c}{a}=\dfrac{-21}{1}=-21$

Yes, –7 and 3 are roots of $x^2+4x-21=0$.

**83.** $\dfrac{-5}{2}+6=\dfrac{7}{2};\ -\dfrac{b}{a}=-\dfrac{(-7)}{2}=\dfrac{7}{2}$

$\left(\dfrac{-5}{2}\right)(6)=-15;\ \dfrac{c}{a}=\dfrac{(-30)}{2}=-15$

Yes, $\dfrac{-5}{2}$ and 6 are roots of $2x^2-7x-30=0$.

**84.** $\dfrac{2+\sqrt{5}}{3}+\dfrac{2-\sqrt{5}}{3}=\dfrac{4}{3};\ -\dfrac{b}{a}=-\dfrac{(-12)}{9}=\dfrac{4}{3}$

$\left(\dfrac{2+\sqrt{5}}{3}\right)\left(\dfrac{2-\sqrt{5}}{3}\right)=\dfrac{4-5}{9}-\dfrac{1}{9};\ \dfrac{c}{a}=\dfrac{-1}{9}-\dfrac{1}{9}$

Yes, $\dfrac{2+\sqrt{5}}{3}$ and $\dfrac{2-\sqrt{5}}{3}$ are roots of $9x^2-12x-1=0$.

**85.** $(1+i)+(1-i)=2;\ -\dfrac{b}{a}=-\dfrac{(-2)}{1}=2$

$(1+i)(1-i)=1-i^2=1+1=2;\ \dfrac{c}{a}=\dfrac{(2)}{1}=2$

Yes, $1+i$ and $1-i$ are roots of $x^2-2x+2=0$.

**86.** $(2+3i)+(2-3i)=4;\ -\dfrac{b}{a}=-\dfrac{(-4)}{1}=4$

$(2+3i)(2-3i)=4-9i^2=4+9=13;\ \dfrac{c}{a}=\dfrac{12}{1}=12$

No, $2+3i$ and $2-3i$ are not roots of $x^2-4x+12=0$.

## Prepare for Section 1.4

**PS1.** $x^3-16x$

$x(x^2-16)$

$x(x+4)(x-4)$

**PS2.** $x^4-36x^2$

$x^2(x^2-36)$

$x^2(x+6)(x-6)$

**PS3.** $8^{2/3}=(\sqrt[3]{8})^2$

$=2^2$

$=4$

**PS4.** $16^{3/2}=(\sqrt{16})^3$

$=4^3$

$=64$

**PS5.** $(1+\sqrt{x-5})^2$

$1^2+2\sqrt{x-5}+(\sqrt{x-5})^2$

$1+2\sqrt{x-5}+x-5$

$x+2\sqrt{x-5}-4$

**PS6.** $(2-\sqrt{x+3})^2$

$2^2-2(2)\sqrt{x+3}+(\sqrt{x+3})^2$

$4-4\sqrt{x+3}+x+3$

$x-4\sqrt{x+3}+7$

## Section 1.4

**1.** $x^3-25x=0$

$x(x^2-25)=0$

$x(x-5)(x+5)=0$

$x=0,\ x=5,\ \text{or}\ x=-5$

**2.** $x^3-x=0$

$x(x^2-1)=0$

$x(x-1)(x+1)=0$

$x=0,\ x=1,\ \text{or}\ x=-1$

**3.** $x^3-2x^2-x+2=0$

$x^2(x-2)-(x-2)=0$

$(x-2)(x^2-1)=0$

$(x-2)(x-1)(x+1)=0$

$x=2,\ x=1,\ \text{or}\ x=-1$

**4.** $x^3-4x^2-2x+8=0$

$x^2(x-4)-2(x-4)=0$

$(x-4)(x^2-2)=0$

$x-4=0 \quad \text{or} \quad x^2-2=0$

$x=4 \qquad x^2=2$

$x=\pm\sqrt{2}$

**5.** $2x^5-18x^3=0$

$2x^3(x^2-9)=0$

$2x^3(x-3)(x+3)=0$

$x=0,\ x=3,\ \text{or}\ x=-3$

**6.** $x^4-36x^2=0$

$x^2(x^2-36)=0$

$x^2(x-6)(x+6)=0$

$x=0,\ x=6,\ \text{or}\ x=-6$

**7.**
$$x^4 - 3x^3 - 40x^2 = 0$$
$$x^2(x^2 - 3x - 40) = 0$$
$$x^2(x+5)(x-8) = 0$$
$$x = 0,\ x = -5,\ \text{or}\ x = 8$$

**8.**
$$x^4 + 3x^3 - 8x - 24 = 0$$
$$x^3(x+3) - 8(x+3) = 0$$
$$(x+3)(x^3 - 8) = 0$$
$$(x+3)(x-2)(x^2 + 2x + 4) = 0$$
$$x = -3,\ x = 2,\ \text{or}\ x = \frac{-2 \pm \sqrt{2^2 - 4(1)(4)}}{2}$$
$$x = \frac{-2 \pm \sqrt{4-16}}{2}$$
$$x = \frac{-2 \pm \sqrt{-12}}{2}$$
$$x = \frac{-2 \pm 2i\sqrt{3}}{2}$$
$$x = -1 \pm i\sqrt{3}$$
$$x = -1 + i\sqrt{3}\ \text{or}\ x = -1 - i\sqrt{3}$$
Thus the solutions are $-3, 2, -1 + i\sqrt{3}, -1 - i\sqrt{3}$.

**9.**
$$x^4 - 16x^2 = 0$$
$$x^2(x^2 - 16) = 0$$
$$x^2(x-4)(x+4) = 0$$
$$x = 0,\ x = 4,\ \text{or}\ x = -4$$

**10.**
$$x^4 - 16 = 0$$
$$(x^2 - 4)(x^2 + 4) = 0$$
$$(x-2)(x+2)(x^2 + 4) = 0$$
$$x = 2,\ x = -2,\ \text{or}\ x^2 = -4$$
$$x = \pm\sqrt{-4}$$
$$x = 2i,\ x = -2i$$
Thus the solutions are 2, –2, $2i$, $-2i$.

**11.**
$$x^3 - 8 = 0$$
$$(x-2)(x^2 + 2x + 4) = 0$$
$$x = 2,\ \text{or}\ x^2 + 2x + 4 = 0$$
$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(4)}}{2}$$
$$x = \frac{-2 \pm \sqrt{-12}}{2} = \frac{-2 \pm 2i\sqrt{3}}{2} = -1 \pm i\sqrt{3}$$
$$x = -1 + i\sqrt{3}\ \text{or}\ x = -1 - i\sqrt{3}$$
Thus the solutions are $2, -1 + i\sqrt{3}, -1 - i\sqrt{3}$

**12.**
$$x^3 + 8 = 0$$
$$(x+2)(x^2 - 2x + 4) = 0$$
$$x = -2\ \text{or}\ x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)}$$
$$x = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm 2i\sqrt{3}}{2} = 1 \pm i\sqrt{3}$$
$$x = -2,\ x = 1 + i\sqrt{3},\ \text{or}\ x = 1 - i\sqrt{3}$$

**13.**
$$\frac{3}{x+2} = \frac{5}{2x-7}$$
$$3(2x-7) = 5(x+2)$$
$$6x - 21 = 5x + 10$$
$$6x - 5x = 10 + 21$$
$$x = 31$$

**14.**
$$\frac{4}{y+2} = \frac{7}{y-4}$$
$$4(y-4) = 7(y+2)$$
$$4y - 16 = 7y + 14$$
$$4y - 7y = 14 + 16$$
$$-3y = 30$$
$$y = -10$$

**15.**
$$\frac{30}{10+x} = \frac{20}{10-x}$$
$$30(10-x) = 20(10+x)$$
$$300 - 30x = 200 + 20x$$
$$-30x - 20x = 200 - 300$$
$$-50x = -100$$
$$x = 2$$

**16.** $\dfrac{6}{8+x}=\dfrac{4}{8-x}$

$$\begin{aligned}6(8-x)&=4(8+x)\\48-6x&=32+4x\\-6x-4x&=32-48\\-10x&=-16\\x&=\frac{8}{5}\end{aligned}$$

**17.** $\dfrac{3x}{x+4}=2-\dfrac{12}{x+4}$

$$\begin{aligned}(x+4)\cdot\left(\frac{3x}{x+4}\right)&=(x+4)\cdot\left(2-\frac{12}{x+4}\right)\\3x&=2(x+4)-12\\3x&=2x+8-12\\3x&=2x-4\\3x-2x&=-4\\x&=-4\end{aligned}$$

No solution because each side is undefined when $x=-4$.

**18.** $\dfrac{8}{2m+1}-\dfrac{1}{m-2}=\dfrac{5}{2m+1}$

$$\begin{aligned}(2m+1)(m-2)\left(\frac{8}{2m+1}-\frac{1}{m-2}\right)&=(2m+1)(m-2)\left(\frac{5}{2m+1}\right)\\8(m-2)-1(2m+1)&=5(m-2)\\8m-16-2m-1&=5m-10\\6m-17&=5m-10\\6m-5m&=-10+17\\m&=7\end{aligned}$$

**19.** $2+\dfrac{9}{r-3}=\dfrac{3r}{r-3}$

$$\begin{aligned}(r-3)\cdot\left(2+\frac{9}{r-3}\right)&=(r-3)\cdot\left(\frac{3r}{r-3}\right)\\2(r-3)+9&=3r\\2r-6+9&=3r\\2r+3&=3r\\2r-3r&=-3\\-r&=-3\\r&=3\end{aligned}$$

No solution because each side is undefined when $r=3$.

**20.** $\dfrac{t}{t-4}+3=\dfrac{4}{t-4}$

$$\begin{aligned}(t-4)\cdot\left(\frac{t}{t-4}+3\right)&=(t-4)\cdot\left(\frac{4}{t-4}\right)\\t+3(t-4)&=4\\t+3t-12&=4\\4t-12&=4\\4t&=4+12\\4t&=16\\t&=4\end{aligned}$$

No solution because each side is undefined when $t=4$.

**21.** $\dfrac{5}{x-3}-\dfrac{3}{x-2}=\dfrac{4}{x-3}$

$$\begin{aligned}(x-3)(x-2)\cdot\left(\frac{5}{x-3}-\frac{3}{x-2}\right)&=(x-3)(x-2)\cdot\left(\frac{4}{x-3}\right)\\5(x-2)-3(x-3)&=4(x-2)\\5x-10-3x+9&=4x-8\\2x-1&=4x-8\\2x-4x&=-8+1\\-2x&=-7\\x&=\frac{7}{2}\end{aligned}$$

**22.** $\dfrac{4}{x-1}+\dfrac{7}{x+7}=\dfrac{5}{x-1}$

$$\begin{aligned}(x-1)(x+7)\cdot\left(\frac{4}{x-1}+\frac{7}{x+7}\right)&=(x-1)(x+7)\cdot\left(\frac{5}{x-1}\right)\\4(x+7)+7(x-1)&=5(x+7)\\4x+28+7x-7&=5x+35\\11x+21&=5x+35\\11x-5x&=35-21\\6x&=14\\x&=\frac{7}{3}\end{aligned}$$

**23.**
$$\frac{x}{x-3}=\frac{x+4}{x+2}$$
$$x(x+2)=(x+4)(x-3)$$
$$x^2+2x=x^2+x-12$$
$$2x-x=-12$$
$$x=-12$$

**24.**
$$\frac{x}{x-5}=\frac{x+7}{x+1}$$
$$x(x+1)=(x+7)(x-5)$$
$$x^2+x=x^2+2x-35$$
$$x-2x=-35$$
$$-x=-35$$
$$x=35$$

**25.**
$$\frac{x+3}{x+5}=\frac{x-3}{x-4}$$
$$(x+3)(x-4)=(x-3)(x+5)$$
$$x^2-x-12=x^2+2x-15$$
$$-x-2x=-15+12$$
$$-3x=-3$$
$$x=1$$

**26.**
$$\frac{x-6}{x+4}=\frac{x-1}{x+2}$$
$$(x-6)(x+2)=(x-1)(x+4)$$
$$x^2-4x-12=x^2+3x-4$$
$$-4x-3x=-4+12$$
$$-7x=8$$
$$x=-\frac{8}{7}$$

**27.**
$$\sqrt{x-4}-6=0$$
$$\sqrt{x-4}=6$$
$$x-4=36$$
$$x=40$$

Check $\sqrt{40-4}-6=0$
$$\sqrt{36}-6=0$$
$$6-6=0$$
$$0=0$$

The solution is 40.

**28.**
$$\sqrt{10-x}=4$$
$$10-x=16$$
$$-x=6$$
$$x=-6$$

Check $\sqrt{10-(-6)}=4$
$$\sqrt{16}=4$$
$$4=4$$

The solution is –6.

**29.**
$$x=3+\sqrt{3-x}$$
$$x-3=\sqrt{3-x}$$
$$(x-3)^2=(\sqrt{3-x})^2$$
$$x^2-6x+9=3-x$$
$$x^2-5x+6=0$$
$$(x-3)(x-2)=0$$
$$x=3 \text{ or } x=2$$

| Check | | |
|---|---|---|
| | $3=3+\sqrt{3-3}$ | $2=3+\sqrt{3-2}$ |
| | $3=3+0$ | $2=3+1$ |
| | $3=3$ | $2=4$ (No) |

The solution is 3.

**30.**
$$x=\sqrt{5-x}+5$$
$$(x-5)^2=(\sqrt{5-x})^2$$
$$x^2-10x+25=5-x$$
$$x^2-9x+20=0$$
$$(x-5)(x-4)=0$$
$$x=5 \text{ or } x=4$$

| Check | | |
|---|---|---|
| | $5=\sqrt{5-5}+5$ | $4=\sqrt{5-4}+5$ |
| | $5=0+5$ | $4=1+5$ |
| | $5=5$ | $4=6$ (No) |

The solution is 5.

**31.** $\sqrt{3x-5}-\sqrt{x+2}=1$

$$(\sqrt{3x-5})^2=(1+\sqrt{x+2})^2$$
$$3x-5=1+2\sqrt{x+2}+x+2$$
$$2x-8=2\sqrt{x+2}$$
$$(x-4)^2=(\sqrt{x+2})^2$$
$$x^2-8x+16=x+2$$
$$x^2-9x+14=0$$
$$(x-7)(x-2)=0$$

$x=7$, or $x=2$

Check $\sqrt{3(7)-5}-\sqrt{7+2}=1$
$$\sqrt{16}-\sqrt{9}=1$$
$$4-3=1$$
$$1=1$$

$$\sqrt{3(2)-5}-\sqrt{2+2}=1$$
$$\sqrt{1}-\sqrt{4}=1$$
$$1-2=1$$
$$-1=1 \text{ (No)}$$

The solution is 7.

**32.** $\sqrt{6-x}+\sqrt{5x+6}=6$

$$(\sqrt{5x+6})^2=(6-\sqrt{6-x})^2$$
$$5x+6=36-12\sqrt{6-x}+6-x$$
$$6x-36=-12\sqrt{6-x}$$
$$(x-6)^2=(-2\sqrt{6-x})^2$$
$$x^2-12x+36=4(6-x)$$
$$x^2-12x+36=24-4x$$
$$x^2-8x+12=0$$
$$(x-6)(x-2)=0$$

$x=6$, or $x=2$

Check $\sqrt{6-6}+\sqrt{30+6}=6$
$$0+6=6$$
$$6=6$$

$$\sqrt{6-2}+\sqrt{10+6}=6$$
$$\sqrt{4}+\sqrt{16}=6$$
$$2+4=6$$
$$6=6$$

Both 6 and 2 check as solutions.

**33.** $\sqrt{2x+11}-\sqrt{2x-5}=2$

$$(\sqrt{2x+11})^2=(2+\sqrt{2x-5})^2$$
$$2x+11=4+4\sqrt{2x-5}+2x-5$$
$$12=4\sqrt{2x-5}$$
$$(3)^2=(\sqrt{2x-5})^2$$
$$9=2x-5$$
$$14=2x$$
$$7=x$$

Check $\sqrt{2(7)+11}-\sqrt{2(7)-5}=2$
$$\sqrt{25}-\sqrt{9}=2$$
$$5-3=2$$
$$2=2$$

7 checks as the solution.

**34.** $(\sqrt{x+7}-2)^2=(\sqrt{x-9})^2$

$$x+7-4\sqrt{x+7}+4=x-9$$
$$-4\sqrt{x+7}=-20$$
$$(\sqrt{x+7})^2=(5)^2$$
$$x+7=25$$
$$x=18$$

Check $\sqrt{18+7}-2=\sqrt{18-9}$
$$\sqrt{25}-2=\sqrt{9}$$
$$5-2=3$$
$$3=3$$

18 checks as a solution.

**35.** $\sqrt{x+7}+\sqrt{x-5}=6$

$$(\sqrt{x+7})^2=(6-\sqrt{x-5})^2$$
$$x+7=36-12\sqrt{x-5}+x-5$$
$$12\sqrt{x-5}=24$$
$$(\sqrt{x-5})^2=(2)^2$$
$$x-5=4$$
$$x=9$$

Check $\sqrt{9+7}+\sqrt{9-5}=6$

$$\sqrt{16}+\sqrt{4}=6$$
$$4+2=6$$
$$6=6$$

9 checks as a solution.

**36.** $x=\sqrt{12x-35}$

$$x^2=(\sqrt{12x-35})^2$$
$$x^2=12x-35$$
$$x^2-12x+35=0$$
$$(x-5)(x-7)=0$$
$$x=5,\text{ or } x=7$$

Check $5=\sqrt{12(5)-35}$ $\quad 7=\sqrt{12(7)-35}$

$$5=\sqrt{60-35} \qquad 7=\sqrt{84-35}$$
$$5=\sqrt{25} \qquad 7=\sqrt{49}$$
$$5=5 \qquad 7=7$$

5 and 7 check as solutions

**37.** $2x=\sqrt{4x+15}$

$$(2x)^2=(\sqrt{4x+15})^2$$
$$4x^2=4x+15$$
$$4x^2-4x-15=0$$
$$(2x+3)(2x-5)=0$$
$$x=-\frac{3}{2},\text{ or } x=\frac{5}{2}$$

Check $2\left[-\frac{3}{2}\right]=\sqrt{4\left[-\frac{3}{2}\right]+15}$ $\quad 2\left[\frac{5}{2}\right]=\sqrt{4\left[\frac{5}{2}\right]+15}$

$$-3=\sqrt{-6+15} \qquad 5=\sqrt{10+15}$$
$$-3=\sqrt{9} \qquad 5=\sqrt{25}$$
$$-3=3 \quad \text{(No)} \qquad 5=5$$

The solution is $\frac{5}{2}$.

**38.** $\sqrt[3]{7x-3}=\sqrt[3]{2x+7}$

$$(\sqrt[3]{7x-3})^3=(\sqrt[3]{2x+7})^3$$
$$7x-3=2x+7$$
$$5x=10$$
$$x=2$$

Check $\sqrt[3]{7(2)-3}=\sqrt[3]{2(2)+7}$

$$\sqrt[3]{11}=\sqrt[3]{11}$$

The solution is 2.

**39.** $\sqrt[3]{2x^2+5x-3}=\sqrt[3]{x^2+3}$

$$\left[\sqrt[3]{2x^2+5x-3}\right]^3=\left[\sqrt[3]{x^2+3}\right]^3$$
$$2x^2+5x-3=x^2+3$$
$$x^2+5x-6=0$$
$$(x-1)(x+6)=0$$
$$x=1,\text{ or } x=-6$$

Check $\sqrt[3]{2(1)^2+5-3}=\sqrt[3]{(1)^2+3}$

$$\sqrt[3]{4}=\sqrt[3]{4}$$

$$\sqrt[3]{2(-6)^2+5(-6)-3}=\sqrt[3]{(-6)^2+3}$$
$$\sqrt[3]{39}=\sqrt[3]{39}$$

The solutions are 1 and –6.

**40.** $\sqrt[4]{x^2+20}=\sqrt[4]{9x}$

$$\left[\sqrt[4]{x^2+20}\right]^4=(\sqrt[4]{9x})^4$$
$$x^2+20=9x$$
$$x^2-9x+20=0$$
$$(x-4)(x-5)=0$$
$$x=4,\text{ or } x=5$$

Check $\sqrt[4]{4^2+20}=\sqrt[4]{9(4)}$

$$\sqrt[4]{36}=\sqrt[4]{36}$$

$$\sqrt[4]{5^2+20}=\sqrt[4]{9(5)}$$
$$\sqrt[4]{45}=\sqrt[4]{45}$$

The solutions are 4 and 5.

**41.**
$$x^{1/3} = 2$$
$$\left(x^{1/3}\right)^3 = (2)^3$$
$$x = 8$$

**42.**
$$x^{1/2} = 5$$
$$\left(x^{1/2}\right)^2 = (5)^2$$
$$x = 25$$

**43.**
$$x^{2/5} = 9$$
$$\left(x^{2/5}\right)^{5/2} = (9)^{5/2}$$
$$|x| = 243$$
$$x = -243,\ 243$$

**44.**
$$x^{4/3} = 81$$
$$\left(x^{4/3}\right)^{3/4} = (81)^{3/4}$$
$$|x| = 27$$
$$x = -27,\ 27$$

**45.**
$$x^{3/2} = 27$$
$$\left(x^{3/2}\right)^{2/3} = (27)^{2/3}$$
$$x = 9$$

**46.**
$$x^{3/4} = 125$$
$$\left(x^{3/4}\right)^{4/3} = (125)^{4/3}$$
$$x = 625$$

**47.**
$$2x^{2/3} - 16 = 59$$
$$2x^{2/3} = 75$$
$$x^{2/3} = 25$$
$$\left(x^{2/3}\right)^{3/2} = (25)^{3/2}$$
$$|x| = 125$$
$$x = -125,\ 125$$

**48.**
$$4x^{4/5} - 27 = 37$$
$$4x^{4/5} = 64$$
$$x^{4/5} = 16$$
$$\left(x^{4/5}\right)^{5/4} = (16)^{5/4}$$
$$|x| = 32$$
$$x = -32,\ 32$$

**49.**
$$2x^{3/2} - 31 = 23$$
$$2x^{3/2} = 54$$
$$x^{3/2} = 27$$
$$\left(x^{3/2}\right)^{2/3} = (27)^{2/3}$$
$$x = 9$$

**50.**
$$3x^{3/5} + 25 = 49$$
$$3x^{3/5} = 24$$
$$x^{3/5} = 8$$
$$\left(x^{3/5}\right)^{5/3} = (8)^{5/3}$$
$$x = 32$$

**51.**
$$4x^{3/4} - 31 = 77$$
$$4x^{3/4} = 108$$
$$x^{3/4} = 27$$
$$\left(x^{3/4}\right)^{4/3} = (27)^{4/3}$$
$$x = 81$$

**52.**
$$4x^{4/5} - 54 = 270$$
$$4x^{4/5} = 324$$
$$x^{4/5} = 81$$
$$\left(x^{4/5}\right)^{5/4} = (81)^{5/4}$$
$$|x| = 243$$
$$x = -243,\ 243$$

**53.**
$$x^4 - 9x^2 + 14 = 0$$
Let $u = x^2$.
$$u^2 - 9u + 14 = 0$$
$$(u-7)(u-2) = 0$$
$$u = 7 \quad \text{or} \quad u = 2$$
$$x^2 = 7 \qquad x^2 = 2$$
$$x = \pm\sqrt{7} \qquad x = \pm\sqrt{2}$$
The solutions are $\sqrt{7},\ -\sqrt{7},\ \sqrt{2},\ -\sqrt{2}$.

**54.**
$$x^4 - 10x^2 + 9 = 0$$
Let $u = x^2$.
$$u^2 - 10u + 9 = 0$$
$$(u-9)(u-1) = 0$$
$$u = 9 \quad \text{or} \quad u = 1$$
$$x^2 = 9 \qquad x^2 = 1$$
$$x = \pm 3 \qquad x = \pm 1$$
The solutions are $3,\ -3,\ 1,\ -1$.

**55.** $2x^4 - 11x^2 + 12 = 0$

Let $u = x^2$.

$2u^2 - 11u + 12 = 0$

$(2u - 3)(u - 4) = 0$

$u = \frac{3}{2}$ or $u = 4$

$x^2 = \frac{3}{2}$ $x^2 = 4$

$x = \pm\sqrt{\frac{3}{2}} = \pm\frac{\sqrt{6}}{2}$ $x = \pm 2$

The solutions are $\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{2}, 2, -2$.

**56.** $6x^4 - 7x^2 + 2 = 0$

Let $u = x^2$.

$6u^2 - 7u + 2 = 0$

$(2u - 1)(3u - 2) = 0$

$u = \frac{1}{2}$ or $u = \frac{2}{3}$

$x^2 = \frac{1}{2}$ $x^2 = \frac{2}{3}$

$x = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$ $x = \pm\sqrt{\frac{2}{3}} = \pm\frac{\sqrt{6}}{3}$

The solutions are $\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{3}$.

**57.** $x^6 + x^3 - 6 = 0$

Let $u = x^3$.

$u^2 + u - 6 = 0$

$(u - 2)(u + 3) = 0$

$u = 2$ or $u = -3$

$x^3 = 2$ $x^3 = -3$

$x = \sqrt[3]{2}$ $x = \sqrt[3]{-3} = -\sqrt[3]{3}$

The solutions are $\sqrt[3]{2}$ and $-\sqrt[3]{3}$.

**58.** $6x^6 + x^3 - 15 = 0$

Let $u = x^3$.

$6u^2 + u - 15 = 0$

$(2u - 3)(3u + 5) = 0$

$u = \frac{3}{2}$ or $u = -\frac{5}{3}$

$x^3 = \frac{3}{2}$ $x^3 = -\frac{5}{3}$

$x = \sqrt[3]{\frac{3}{2}} = \sqrt[3]{\frac{3 \cdot 4}{2 \cdot 4}}$ $x = -\sqrt[3]{\frac{5}{3}} = -\sqrt[3]{\frac{5 \cdot 9}{3 \cdot 9}}$

$x = \frac{\sqrt[3]{12}}{2}$ $x = -\frac{\sqrt[3]{45}}{3}$

The solutions are $\frac{\sqrt[3]{12}}{2}$ and $-\frac{\sqrt[3]{45}}{3}$.

**59.** $x^{1/2} - 3x^{1/4} + 2 = 0$

Let $u = x^{1/4}$.

$u^2 - 3u + 2 = 0$

$(u - 1)(u - 2) = 0$

$u = 1$ or $u = 2$

$x^{1/4} = 1$ $x^{1/4} = 2$

$x = 1$ $x = 16$

The solutions are 1 and 16.

**60.** $2x^{1/2} - 5x^{1/4} - 3 = 0$

Let $u = x^{1/4}$.

$2u^2 - 5u - 3 = 0$

$(2u + 1)(u - 3) = 0$

$u = -\frac{1}{2}$ or $u = 3$

$x^{1/4} = -\frac{1}{2}$ $x^{1/4} = 3$

$x = 81$

$\left(x^{1/4} \neq -\frac{1}{2} \quad \text{since } x^{1/4} \geq 0\right)$

The solution is 81.

**61.** $3x^{2/3} - 11x^{1/3} - 4 = 0$

Let $u = x^{1/3}$.

$3u^2 - 11u - 4 = 0$

$(3u+1)(u-4) = 0$

$u = -\frac{1}{3}$ or $u = 4$

$x^{1/3} = -\frac{1}{3}$ $\quad x^{1/3} = 4$

$x = -\frac{1}{27}$ $\quad x = 64$

The solutions are $-\frac{1}{27}$ and 64.

**62.** $6x^{2/3} - 7x^{1/3} - 20 = 0$

Let $u = x^{1/3}$.

$6u^2 - 7u - 20 = 0$

$(3u+4)(2u-5) = 0$

$u = -\frac{4}{3}$ or $u = \frac{5}{2}$

$x^{1/3} = -\frac{4}{3}$ $\quad x^{1/3} = \frac{5}{2}$

$x = -\frac{64}{27}$ $\quad x = \frac{125}{8}$

The solutions are $-\frac{64}{27}$ and $\frac{125}{8}$.

**63.** $9x^4 = 30x^2 - 25$

Let $u = x^2$.

$9u^2 - 30u + 25 = 0$

$(3u-5)(3u-5) = 0$

$u = \frac{5}{3}$

$x^2 = \frac{5}{3}$

$x = \pm\sqrt{\frac{5}{3}} = \pm\frac{\sqrt{15}}{3}$

The solutions are $\frac{\sqrt{15}}{3}$ and $-\frac{\sqrt{15}}{3}$.

**64.** $4x^4 - 28x^2 = -49$

Let $u = x^2$.

$4u^2 - 28u + 49 = 0$

$(2u-7)(2u-7) = 0$

$u = \frac{7}{2}$

$x^2 = \frac{7}{2}$

$x = \pm\sqrt{\frac{7}{2}} = \pm\frac{\sqrt{14}}{2}$

The solutions are $\frac{\sqrt{14}}{2}$ and $-\frac{\sqrt{14}}{2}$.

**65.** $x^{2/5} - 1 = 0$

$x^{2/5} = 1$

$(x^{2/5})^{5/2} = \pm(1)^{5/2}$

$x = \pm 1$

The solutions are 1 and $-1$.

**66.** $2x^{2/5} - x^{1/5} = 6$

Let $u = x^{1/5}$.

$2u^2 - u - 6 = 0$

$(2u+3)(u-2) = 0$

$u = -\frac{3}{2}$ or $u = 2$

$x^{1/5} = -\frac{3}{2}$ $\quad x^{1/5} = 2$

$x = \left[-\frac{3}{2}\right]^5$ $\quad x = 32$

$x = -\frac{243}{32}$

The solutions are $-\frac{243}{32}$ and 32.

**67.** $9x - 52\sqrt{x} + 64 = 0$

Let $\sqrt{x} = u$.

$9u^2 - 52u + 64 = 0$

$(9u - 16)(u - 4) = 0$

$u = \frac{16}{9}$ or $u = 4$

$\sqrt{x} = \frac{16}{9}$ $\quad$ $\sqrt{x} = 4$

$x = \frac{256}{81}$ $\quad$ $x = 16$

The solutions are $\frac{256}{81}$ and 16.

**68.** $8x - 38\sqrt{x} + 9 = 0$

Let $u = \sqrt{x}$.

$8u^2 - 38u + 9 = 0$

$(4u - 1)(2u - 9) = 0$

$u = \frac{1}{4}$ or $u = \frac{9}{2}$

$\sqrt{x} = \frac{1}{4}$ $\quad$ $\sqrt{x} = \frac{9}{2}$

$x = \frac{1}{16}$ $\quad$ $x = \frac{81}{4}$

The solutions are $\frac{1}{16}$ and $\frac{81}{4}$.

**69.** Let $x$ = the number of hours the assistant would take to build the fence working alone.

The worker does $\frac{1}{8}$ of the job per hour; the assistant does $\frac{1}{x}$ of the job per hour.

| | | |
|---|---|---|
| worker | $\frac{1}{8}$ | 5 |
| assistant | $\frac{1}{x}$ | 5 |

$$\left(\tfrac{1}{8}\right)(5) + \left(\tfrac{1}{x}\right)(5) = 1$$

$$\frac{5}{8} + \frac{5}{x} = 1$$

$$8x\left(\frac{5}{8} + \frac{5}{x}\right) = 1(8x)$$

$$5x + 40 = 8x$$

$$40 = 3x$$

$$\frac{40}{3} = x$$

$$x = 13\tfrac{1}{3} \text{ hours}$$

**70.** Let $x$ = the additional number of hours for the assistant to finish the job. In two hours, $\frac{1}{3}$ of the job is done, $\frac{2}{3}$ of the job is left to do.

$$\frac{1}{14}x = \frac{2}{3}$$

$$x = \frac{2}{3} \cdot 14$$

$$x = \frac{28}{3} = 9\frac{1}{3} \text{ hours}$$

**71.** Let $x$ = number of rounds the golfer needs to play.

$$88 = \frac{4(92) + 86x}{4 + x}$$

$$88(4 + x) = 368 + 86x$$

$$352 + 88x = 368 + 86x$$

$$2x = 16$$

$$x = 8 \text{ rounds}$$

**72.** Let $A$ = the age of the child.

$$\frac{1}{2} = \frac{A}{A + 12}$$

$$A + 12 = 2A$$

$$12 = A$$

The child is 12 years old.

**73.** $\text{SMOG} = \sqrt{w} + 3$

$6 = \sqrt{w} + 3$

$3 = \sqrt{w}$

$9 = w$

9 words with three or more syllables.

**74.** $\text{SMOG} = \sqrt{w} + 3$

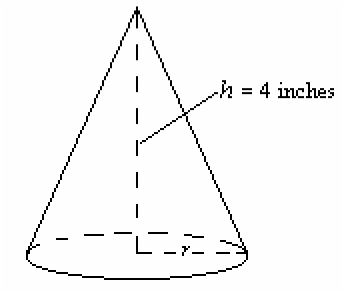
$4 = \sqrt{w} + 3$

$1 = \sqrt{w}$

$1 = w$

1 word with three or more syllables.

**75.**




$$L = \pi r\sqrt{r^2 + h^2}$$
$$15\pi = \pi r\sqrt{r^2 + 4^2}$$
$$15 = r\sqrt{r^2 + 16}$$
$$225 = r^2(r^2 + 16)$$
$$0 = r^4 + 16r^2 - 225$$

Let $u = r^2$.

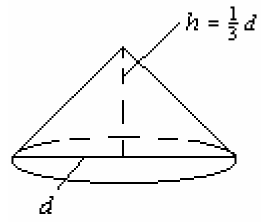
$$u^2 + 16u - 225 = 0$$
$$(u+25)(u-9) = 0$$

$u = 9$ or $u = -25$ (No)

$$r^2 = 9$$
$$r = 3$$

The radius is 3 in.

**76.**

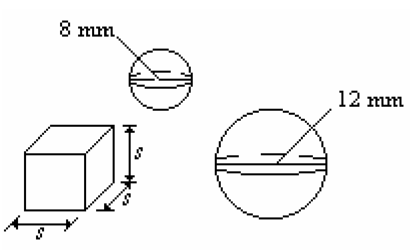



$$h = \frac{1}{3}d = \frac{1}{3}(2r) = \frac{2}{3}r$$
$$V = \frac{1}{3}\pi r^2 h$$
$$192 = \frac{1}{3}\pi r^2\left[\frac{2}{3}r\right]$$
$$192 = \frac{2}{9}\pi r^2$$
$$\frac{192(9)}{2\pi} = r^3$$
$$275 \approx r^3$$
$$6.50 \approx r$$

$r \approx 6.50$ in.

diameter $d = 2r \approx 13.0$ in.

**77.**




$d_1 = 8$ mm, $d_2 = 12$ mm

$V_c = s^3$ $\quad V_s = \frac{4}{3}\pi r^3$

$$s^3 = \frac{4}{3}\pi(4)^3 + \frac{4}{3}\pi(6)^3 = \frac{4}{3}\pi(64 + 216) \approx 1172.86$$

$s \approx 10.5$ mm

The side is approximately 10.5 mm.

**78.**

$$T = 2\pi\sqrt{\frac{L}{32}}$$
$$4 = 2\pi\sqrt{\frac{L}{32}}$$
$$\frac{2}{\pi} = \sqrt{\frac{L}{32}}$$
$$\frac{4}{\pi^2} = \frac{L}{32}$$
$$L = \frac{32(4)}{\pi^2}$$
$$L \approx 12.969$$

The length is 13.0 ft (to the nearest tenth).

**79.**

$$d = 1.5\sqrt{h}$$
$$14 = 1.5\sqrt{h}$$
$$\frac{2}{3}(14) = \sqrt{h}$$
$$\frac{28}{3} = \sqrt{h}$$
$$\frac{784}{9} = h$$

The height is approximately 87 ft.

## Connecting Concepts

**80.** **a.** $s=\frac{1}{2}(a+b+c)=\frac{1}{2}(5+6+7)=9$

$r=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$

$r=\sqrt{\frac{(9-5)(9-6)(9-7)}{9}}=\sqrt{\frac{(4)(3)(2)}{9}}$

$r=\sqrt{\frac{24}{9}}\approx 1.63$

The radius is approximately 1.63 inches.

**b.** $s=\frac{1}{2}(a+a+a)=\frac{3}{2}a$

$r=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$

$2=\sqrt{\frac{\left[\frac{3}{2}a-a\right]\left[\frac{3}{2}a-a\right]\left[\frac{3}{2}a-a\right]}{\frac{3}{2}a}}$

$2=\sqrt{\frac{\frac{a}{2}\cdot\frac{a}{2}\cdot\frac{a}{2}}{\frac{3}{2}a}}=\sqrt{\frac{a^3}{12a}}=\sqrt{\frac{a^2}{12}}$

$2=\frac{a}{2\sqrt{3}} \qquad a=4\sqrt{3}$

Each side is $4\sqrt{3}$ in.

**81.** **a.** $s=\frac{1}{2}(a+b+c)=\frac{1}{2}(10+7+15)=16$

$r=\frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$

$r=\frac{(10)(7)(15)}{4\sqrt{16(16-10)(16-7)(16-15)}}$

$r=\frac{10(7)(15)}{4\sqrt{16(6)(9)(1)}}=\frac{10(7))(15)}{4\cdot 4\cdot\sqrt{6}\cdot 3}\approx 8.93$

The radius is approximately 8.93 in.

**b.** $a=\text{side}, a=b=c, r=5$

$s=\frac{1}{2}(a+b+c)=\frac{1}{2}(a+a+a)=\frac{3a}{2}$

$r=\frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}=\frac{a\cdot a\cdot a}{4\sqrt{s(s-a)(s-a)(s-a)}}$

$5=\frac{a\cdot a\cdot a}{4\sqrt{\frac{3}{2}a\left[\frac{3}{2}a-a\right]\left[\frac{3}{2}a-a\right]\left[\frac{3}{2}a-a\right]}}$

$5=\frac{a^3}{4\sqrt{\frac{3}{2}a\left[\frac{a}{2}\right]\left[\frac{a}{2}\right]\left[\frac{a}{2}\right]}}=\frac{a^3}{4\frac{\sqrt{3a^4}}{16}}=\frac{a^3}{\frac{4a^2\sqrt{3}}{4}}$

$a=5\sqrt{3}$

Each side is $5\sqrt{3}$ in.

**82.** $T=\frac{\sqrt{s}}{4}+\frac{s}{1100}$

$T=\frac{\sqrt{7100}}{4}+\frac{7100}{1100}$

$T\approx 27.5$ seconds

**83.** $T=\frac{\sqrt{s}}{4}+\frac{s}{1100}$

$4400T=1100\sqrt{s}+4s$

Let $\sqrt{s}=u$.

$0=4u^2+1100u-4400T$

$u=\frac{-1100\pm\sqrt{(1100)^2-4(4)(-4400)T}}{8}$

$u=\frac{-1100\pm\sqrt{1,210,000+70,400T}}{8}$

$\sqrt{s}=\frac{-1100+\sqrt{1,210,000+70,400T}}{8}$

$s=\left(\frac{-1100+\sqrt{1,210,000+70,400T}}{8}\right)^2$

$s=\left(\frac{-275+5\sqrt{3025+176T}}{2}\right)^2$

**84.** $T = 3$

$$s = \left[\frac{-275 + 5\sqrt{3025 + 176(3)}}{2}\right]^2$$

$s \approx 11.5176^2 \approx 132.65$

The distance is approximately 133 ft.

## Prepare for Section 1.5

**PS1.** $\{x \mid x > 5\}$

**PS2.** $3(-3)^2 - 2(-3) + 5 = 38$

**PS3.** $\dfrac{7+3}{7-2} = 2$

**PS4.** $10x^2 + 9x - 9 = (3x+5)(5x-3)$

**PS5.** $\dfrac{x-3}{2x-7},\ 2x-7 \neq 0$

It is undefined for $x = \dfrac{7}{2}$.

**PS6.** $2x^2 - 11x + 15 = 0$

$(2x-5)(x-3) = 0$

$2x - 5 = 0 \quad x - 3 = 0$

$x = \dfrac{5}{2} \quad x = 3$

## Section 1.5

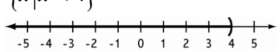
**1.** $2x + 3 < 11$

$2x < 11 - 3$

$2x < 8$

$x < 4$

$\{x | x < 4\}$

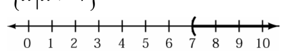


**2.** $3x - 5 > 16$

$3x > 21$

$x > 7$

$\{x | x > 7\}$

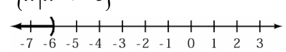


**3.** $x + 4 > 3x + 16$

$-2x > 12$

$x < -6$

$\{x | x < -6\}$

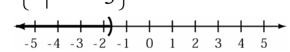


**4.** $5x + 6 < 2x + 1$

$3x < -5$

$x < -\dfrac{5}{3}$

$\left\{x \middle| x < -\dfrac{5}{3}\right\}$



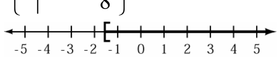
**5.** $-3(x+2) \le 5x + 7$

$-3x - 6 \le 5x + 7$

$-8x \le 13$

$x \ge -\dfrac{13}{8}$

$\left\{x \middle| x \ge -\dfrac{13}{8}\right\}$



**6.** $-4(x-5) \ge 2x + 15$

$-4x + 20 \ge 2x + 15$

$-6x \ge -5$

$x \le \dfrac{5}{6}$

$\left\{x \middle| x \le \dfrac{5}{6}\right\}$

**7.** $-4(3x-5) > 2(x-4)$

$-12x + 20 > 2x - 8$

$-14x > -28$

$x < 2$

$\{x | x < 2\}$

**8.** $3(x+7) \le 5(2x-8)$

$3x + 21 \le 10x - 40$

$-7x \le -61$

$x \ge \dfrac{61}{7}$

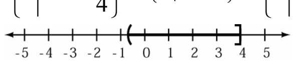
$\left\{x \middle| x \ge \dfrac{61}{7}\right\}$

**9.** $4x + 1 > -2$ and $4x + 1 \le 17$

$4x > -3$ and $4x \le 16$

$x > -\dfrac{3}{4}$ and $x \le 4$

$\left\{x \middle| x > -\dfrac{3}{4}\right\} \cap \{x | x \le 4\} = \left\{x \middle| -\dfrac{3}{4} < x \le 4\right\}$

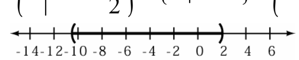


**10.** $2x + 5 > -16$ and $2x + 5 < 9$

$2x > -21$ and $2x < 4$

$x > -\dfrac{21}{2}$ and $x < 2$

$\left\{x \middle| x > -\dfrac{21}{2}\right\} \cap \{x \mid x < 2\} = \left\{x \middle| -\dfrac{21}{2} < x < 2\right\}$

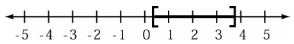


**11.** $10 \ge 3x - 1 \ge 0$

$11 \ge \quad 3x \quad \ge 1$

$\frac{11}{3} \ge \quad x \quad \ge \frac{1}{3}$

$\left\{ x \middle| \frac{1}{3} \le x \le \frac{11}{3} \right\}$

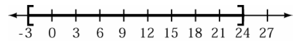



**12.** $0 \le 2x + 6 \le 54$

$-6 \le \quad 2x \quad \le 48$

$-3 \le \quad x \quad \le 24$

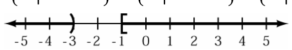
$\{x | -3 \le x \le 24\}$




**13.** $x + 2 < -1 \quad \text{or} \quad x + 3 \ge 2$

$x < -3 \quad \text{or} \quad x \ge -1$

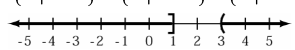
$\{x | x < -3\} \cup \{x | x \ge -1\} = \{x | x < -3 \quad \text{or} \quad x \ge -1\}$




**14.** $x + 1 > 4 \quad \text{or} \quad x + 2 \le 3$

$x > 3 \quad \text{or} \quad x \le 1$

$\{x | x \le 1\} \cup \{x | x > 3\} = \{x | x \le 1 \quad \text{or} \quad x > 3\}$

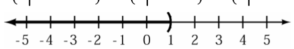



**15.** $-4x + 5 > 9 \quad \text{or} \quad 4x + 1 < 5$

$-4x > 4 \quad \text{or} \quad 4x < 4$

$x < -1 \quad \text{or} \quad x < 1$

$\{x | x < -1\} \cup \{x | x < 1\} = \{x | x < 1\}$

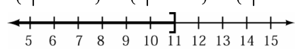



**16.** $2x - 7 \le 15 \quad \text{or} \quad 3x - 1 \le 5$

$2x \le 22 \quad \text{or} \quad 3x \le 6$

$x \le 11 \quad \text{or} \quad x \le 2$

$\{x | x \le 11\} \cup \{x | x \le 2\} = \{x | x \le 11\}$




**17.** $|2x - 1| > 4$

$2x - 1 < -4 \quad \text{or} \quad 2x - 1 > 4$

$2x < -3 \quad \text{or} \quad 2x > 5$

$x < -\frac{3}{2} \quad \text{or} \quad x > \frac{5}{2}$

$\left(-\infty, -\frac{3}{2}\right) \cup \left(\frac{5}{2}, \infty\right)$

**18.** $|2x - 9| < 7$

$-7 < 2x - 9 < 7$

$2 < \quad 2x \quad < 16$

$1 < \quad x \quad < 8$

$(1, 8)$

**19.** $|x + 3| \ge 5$

$x + 3 \le -5 \quad \text{or} \quad x + 3 \ge 5$

$x \le -8 \qquad x \ge 2$

$(-\infty, -8] \cup [2, \infty)$

**20.** $|x - 10| \ge 2$

$x - 10 \le -2 \quad \text{or} \quad x - 10 \ge 2$

$x \le 8 \qquad x \ge 12$

$(-\infty, -8] \cup [12, \infty)$

**21.** $|3x - 10| \le 14$

$-14 \le 3x - 10 \le 14$

$-4 \le \quad 3x \quad \le 24$

$-\frac{4}{3} \le \quad x \quad \le 8$

$\left[-\frac{4}{3}, 8\right]$

**22.** $|2x - 5| \ge 1$

$2x - 5 \le -1 \quad \text{or} \quad 2x - 5 \ge 1$

$2x \le 4 \qquad 2x \ge 6$

$x \le 2 \qquad x \ge 3$

$(-\infty, 2] \cup [3, \infty)$

**23.** $|4 - 5x| \ge 24$

$4 - 5x \le -24 \quad \text{or} \quad 4 - 5x \ge 24$

$-5x \le -28 \qquad -5x \ge 20$

$x \ge \frac{28}{5} \qquad x \le -4$

$(-\infty, \ -4] \cup \left[\frac{28}{5}, \ \infty\right)$

**24.** $|3 - 2x| \le 5$

$-5 \le 3 - 2x \le 5$

$-8 \le \ -2x \ \le 2$

$4 \ge \quad x \quad \ge -1$

$[-1, 4]$

**25.** $|x-5| \geq 0$

(Note: The absolute value of *any* real number is greater than or equal to 0.)

$(-\infty, \infty)$

**26.** $|x-7| \geq 0$

$(-\infty, \infty)$

**27.** $|x-4| \leq 0$

(Note: No absolute value is less than 0.)

$x-4=0$

$x=4$

$\{4\}$

**28.** $|2x+7| \leq 0$

$2x+7=0$

$2x=-7$

$x=-\dfrac{7}{2}$

$\left\{-\dfrac{7}{2}\right\}$

**29.** $x^2+7x>0$

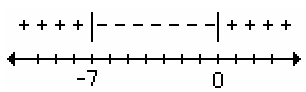
$x(x+7)>0$

The product $x(x+7)$ is positive.

$x=0$ is a critical value.

$x+7=0 \Rightarrow x=-7$ is a critical value.

$x(x+7)$




$(-\infty, -7) \cup (0, \infty)$

**30.** $x^2-5x \leq 0$

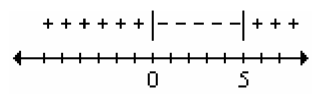
$x(x-5) \leq 0$

The product $x(x-5)$ is negative or zero.

$x=0$ is a critical value.

$x-5=0 \Rightarrow x=5$ is a critical value.

$x(x-5)$




$[0, 5]$

**31.** $x^2-16 \leq 0$

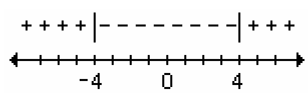
$(x-4)(x+4) \leq 0$

The product $(x-4)(x+4)$ is negative or zero.

$x-4=0 \Rightarrow x=4$ is a critical value.

$x+4=0 \Rightarrow x=-4$ is a critical value.

$(x-4)(x+4)$




$[-4, 4]$

**32.** $x^2-49>0$

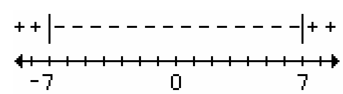
$(x-7)(x+7)>0$

The product $(x-7)(x+7)$ is positive.

$x=7$ is a critical value.

$x+7=0 \Rightarrow x=-7$ is a critical value.

$(x-7)(x+7)$




$(-\infty, -7) \cup (7, \infty)$

**33.** $x^2+7x+10<0$

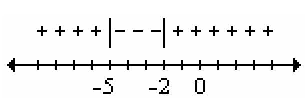
$(x+5)(x+2)<0$

The product $(x+5)(x+2)$ is negative.

$x+5=0 \Rightarrow x=-5$ is a critical value.

$x+2=0 \Rightarrow x=-2$ is a critical value.

$(x+5)(x+2)$




$(-5, -2)$

**34.** $x^2+5x+6<0$

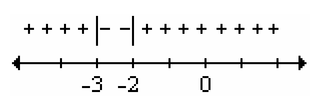
$(x+3)(x+2)<0$

The product $(x+3)(x+2)$ is negative.

$x+3=0 \Rightarrow x=-3$ is a critical value.

$x+2=0 \Rightarrow x=-2$ is a critical value.

$(x+3)(x+2)$




$(-3, -2)$

**35.** $x^2 - 3x \geq 28$

$x^2 - 3x - 28 \geq 0$

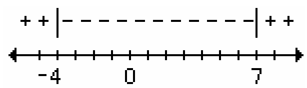
$(x-7)(x+4) \geq 0$

The product $(x-7)(x+4)$ is positive or zero.

$x - 7 = 0 \Rightarrow x = 7$ is a critical value.

$x + 4 = 0 \Rightarrow x = -4$ is a critical value.

$(x-7)(x+4)$




$(-\infty, -4] \cup [7, \infty)$

**36.** $x^2 < -x + 30$

$x^2 + x - 30 < 0$

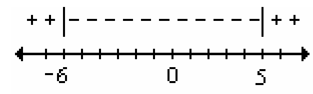
$(x+6)(x-5) < 0$

The product $(x+6)(x-5)$ is negative.

$x + 6 = 0 \Rightarrow x = -6$ is a critical alue.

$x - 5 = 0 \Rightarrow x = 5$ is a critical value.

$(x+6)(x-5)$




$(-6, 5)$

**37.** $\dfrac{x+4}{x-1} < 0$

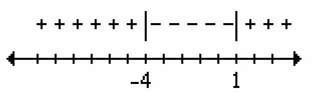
The quotient $\dfrac{x+4}{x-1}$ is negative.

$x + 4 = 0 \Rightarrow x = -4$

$x - 1 = 0 \Rightarrow x = 1$

The critical values are –4 and 1.

$\dfrac{x+4}{x-1}$




$(-4, 1)$

**38.** $\dfrac{x-2}{x+3} > 0$

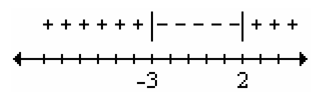
The quotient $\dfrac{x-2}{x+3}$ is positive.

$x - 2 = 0 \Rightarrow x = 2$

$x + 3 = 0 \Rightarrow x = -3$

The critical values are 2 and –3.

$\dfrac{x-2}{x+3}$




$(-\infty, -3) \cup (2, \infty)$

**39.**

$$\frac{x-5}{x+8} \geq 3$$

$$\frac{x-5}{x+8} - 3 \geq 0$$

$$\frac{x-5-3(x+8)}{x+8} \geq 0$$

$$\frac{x-5-3x-24}{x+8} \geq 0$$

$$\frac{-2x-29}{x+8} \geq 0$$

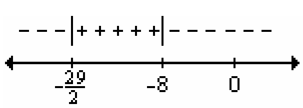
The quotient $\dfrac{-2x-29}{x+8}$ is positive or zero.

$-2x - 29 = 0 \Rightarrow x = -\dfrac{29}{8}$

$x + 8 = 0 \Rightarrow x = -8$

The critical values are $-\dfrac{29}{2}$ and $-8$.

$\dfrac{-2x-29}{x+8}$




The denominator cannot equal zero $\Rightarrow x \neq -8$.

$\left[-\dfrac{29}{2}, -8\right)$

**40.**

$$\frac{x-4}{x+6} \leq 1$$

$$\frac{x-4}{x+6} - 1 \leq 0$$

$$\frac{x-4-1(x+6)}{x+6} \leq 0$$

$$\frac{x-4-x-6}{x+6} \leq 0$$

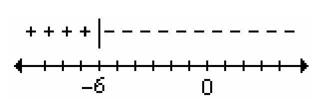
$$\frac{-10}{x+6} \leq 0$$

The quotient $\dfrac{-10}{x+6}$ is negative or zero.

$x + 6 = 0 \Rightarrow x = -6$

The critical value is –6.

$\dfrac{-10}{x+6}$




The denominator cannot equal zero $\Rightarrow x \neq -6$.

$(-6, \infty)$

**41.**

$$\frac{x}{2x+7} \geq 4$$
$$\frac{x}{2x+7} - 4 \geq 0$$
$$\frac{x - 4(2x+7)}{2x+7} \geq 0$$
$$\frac{x - 8x - 28}{2x+7} \geq 0$$
$$\frac{-7x-28}{2x+7} \geq 0$$

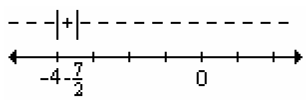
The quotient $\frac{-7x-28}{2x+7}$ is positive or zero.

$-7x - 28 = 0 \Rightarrow x = -4$

$2x + 7 = 0 \Rightarrow x = -\frac{7}{2}$

The critical values are $-4$ and $-\frac{7}{2}$.

$\frac{-7x-28}{2x+7}$




The denominator cannot equal zero $\Rightarrow x \neq -\frac{7.}{2}$

$\left[-4, -\frac{7}{2}\right)$

**42.**

$$\frac{x}{3x-5} \leq -5$$
$$\frac{x}{3x-5} + 5 \leq 0$$
$$\frac{x + 5(3x-5)}{3x-5} \leq 0$$
$$\frac{x + 15x - 25}{3x-5} \leq 0$$
$$\frac{16x-25}{3x-5} \leq 0$$

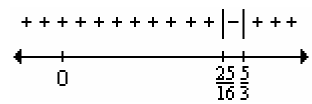
The quotient $\frac{16x-25}{3x-5}$ is negative or zero.

$16x - 25 = 0 \Rightarrow x = \frac{25}{16}$

$3x - 5 = 0 \Rightarrow x = \frac{5}{3}$

The critical values are $\frac{25}{16}$ and $\frac{5}{3}$.

$\frac{16x-25}{3x-5}$




The denominator cannot equal zero $\Rightarrow x \neq \frac{5}{3}$.

$\left[\frac{25}{16}, \frac{5}{3}\right)$

**43.**

$$\frac{(x+1)(x-4)}{x-2} < 0$$

The quotient $\frac{(x+1)(x-4)}{x-2}$ is negative.

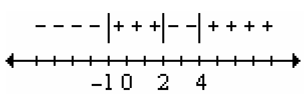
$x + 1 = 0 \Rightarrow x = -1$

$x - 4 = 0 \Rightarrow x = 4$

$x - 2 = 0 \Rightarrow x = 2$

The critical values are –1, 4, and 2.

$\frac{(x+1)(x-4)}{x-2}$




$(-\infty, -1) \cup (2, 4)$

**44.**

$$\frac{x(x-4)}{x+5} > 0$$

The quotient $\frac{x(x-4)}{x+5} > 0$ is positive.

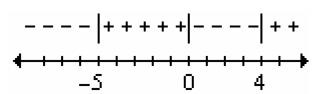
$x = 0$

$x - 4 = 0 \Rightarrow x = 4$

$x + 5 = 0 \Rightarrow x = -5$

The critical values are 0, 4, and –5.

$\frac{x(x-4)}{x+5}$




$(-5, 0) \cup (4, \infty)$

**45.** $\dfrac{x+2}{x-5} \le 2$

$\dfrac{x+2}{x-5} - 2 \le 0$

$\dfrac{x+2-2(x-5)}{x-5} \le 0$

$\dfrac{x+2-2x+10}{x-5} \le 0$

$\dfrac{-x+12}{x-5} \le 0$

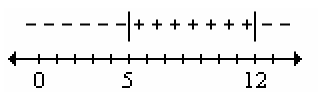
The quotient $\dfrac{-x+12}{x-5}$ is negative or zero.

$-x+12 = 0 \Rightarrow x = 12$

$x-5 = 0 \Rightarrow x = 5$

The critical values are 12 and 5.

$\dfrac{-x+12}{x-5}$




The denominator cannot equal zero $\Rightarrow x \ne 5$.

$(-\infty, 5) \cup [12, \infty)$

**46.** $\dfrac{3x+1}{x-2} \ge 4$

$\dfrac{3x+1}{x-2} - 4 \ge 0$

$\dfrac{3x+1-4(x-2)}{x-2} \ge 0$

$\dfrac{3x+1-4x+8}{x-2} \ge 0$

$\dfrac{-x+9}{x-2} \ge 0$

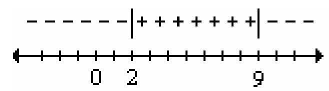
The quotient $\dfrac{-x+9}{x-2}$ is positive or zero.

$-x+9 = 0 \Rightarrow x = 9$

$x-2 = 0 \Rightarrow x = 2$

The critical values are 9 and 2.

$\dfrac{-x+9}{x-2}$




The denominator cannot equal zero $\Rightarrow x \ne 2$.

$(2, 9]$

**47.** $\dfrac{6x^2-11x-10}{x} > 0$

$\dfrac{(3x+2)(2x-5)}{x} > 0$

The quotient $\dfrac{(3x+2)(2x-5)}{x}$ is positive.

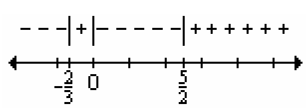
$3x+2 = 0 \Rightarrow x = -\dfrac{2}{3}$

$2x-5 = 0 \Rightarrow x = \dfrac{5}{2}$

$x = 0$

The critical values are $-\dfrac{2}{3}, \dfrac{5}{2}$, and 0.

$\dfrac{(3x+2)(2x-5)}{x}$




$\left(-\dfrac{2}{3}, 0\right) \cup \left(\dfrac{5}{2}, \infty\right)$

**48.** $\dfrac{3x^2-2x-8}{x-1} \ge 0$

$\dfrac{(3x+4)(x-2)}{x-1} \ge 0$

The quotient $\dfrac{(3x+4)(x-2)}{x-1}$ is positive or zero.

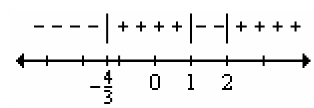
$3x+4 = 0 \Rightarrow x = -\dfrac{4}{3}$

$x-2 = 0 \Rightarrow x = 2$

$x-1 \Rightarrow x = 1$

The critical values are $-\dfrac{4}{3}$, 2, and 1.

$\dfrac{(3x+4)(x-2)}{x-1}$




The denominator cannot equal zero $\Rightarrow x \ne 1$.

$\left[-\dfrac{4}{3}, 1\right) \cup [2, \infty)$

**49.** $\dfrac{x^2-6x+9}{x-5} \le 0$

$\dfrac{(x-3)(x-3)}{x-5} \le 0$

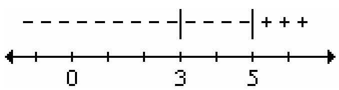
The quotient $\dfrac{(x-3)(x-3)}{x-5}$ is negative or zero.

$x-3=0 \Rightarrow x=3$

$x-5=0 \Rightarrow x=5$

The critical values are 3 and 5.

$\dfrac{(x-3)(x-3)}{x-5}$




The denominator cannot equal zero $\Rightarrow x \ne 5$.

$(-\infty, 5)$

**50.** $\dfrac{x^2+10x+25}{x+1} \ge 0$

$\dfrac{(x+5)(x+5)}{x+1} \ge 0$

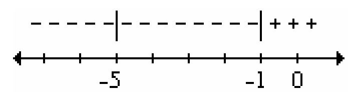
The quotient $\dfrac{(x+5)(x+5)}{x+1}$ is positive or zero.

$x+5=0 \Rightarrow x=-5$

$x+1=0 \Rightarrow x=-1$

The critical values are −5 and −1.

$\dfrac{(x+5)(x+5)}{x+1}$




The denominator cannot equal zero $\Rightarrow x \ne -1$.

$\{-5\} \cup (-1, \infty)$

**51.** Plan A: $5 + 0.01x$
Plan B: $1 + 0.08x$
$5 + 0.01x < 1 + 0.08x$
$4 < 0.07x$
$57.1 < x$
Plan A is less expensive if you use more than 57 checks.

**52.** Company A: $19 + 0.12m$
Company B: $12 + 0.21m$
$19 + 0.12m < 12 + 0.21m$
$7 < 0.09m$
$77.7 < m$
Company A is less expensive if you drive at least 78 miles.

**53.** Let $h$ = the height of the package.

$$\begin{aligned} \text{length} + \text{girth} &\le 130 \\ \text{length} + 2(\text{width}) + 2(\text{height}) &\le 130 \\ 34 + 2(22) + 2h &\le 130 \\ 34 + 44 + 2h &\le 130 \\ 78 + 2h &\le 130 \\ 2h &\le 52 \\ h &\le 26 \end{aligned}$$

The height must be more than 0 but less than or equal to 26 inches.

**54.**
$$\begin{aligned} -0.05x + 1.73 &< 1.25 \\ -0.05x &< -0.48 \\ x &> 9.6 \end{aligned}$$

9.6 years after 2000 is in the year 2009.

**55.**
$$\begin{aligned} 17.1895x + 95.2065 &> 600 \\ 17.1895x &> 504.7935 \\ x &> 29.366 \end{aligned}$$

29 months after September 2004 is in January 2007.

**56.** Plan A: $15 + 1.49x$
Plan B: $1.99x$
$1.99x < 15 + 1.49x$
$0.50x < 15$
$x < 30$
Plan B is less expensive if fewer than 30 videos are rented.

**57.**
$$\begin{aligned} 68 \le \quad F \quad &\le 104 \\ 68 \le \frac{9}{5}C + 32 &\le 104 \\ 36 \le \quad \frac{9}{5}C \quad &\le 72 \\ \frac{5}{9}(36) \le \frac{5}{9}\left(\frac{9}{5}C\right) &\le \frac{5}{9}(72) \\ 20° \le \quad C \quad &\le 40° \end{aligned}$$

**58.**
$$\begin{aligned} -2.33 < \frac{1.63-\mu}{1.79} &< 2.33 \\ -4.1707 < 1.63 - \mu &< 4.1707 \\ -167.2 < \quad -\mu \quad &< -158.8 \\ 167.2 > \quad \mu \quad &> 158.8 \\ 158.8 < \quad \mu \quad &< 167.2 \text{ lb} \end{aligned}$$

**59.**
$$\begin{aligned} -2.575 < \frac{190-\mu}{2.45} &< 2.575 \\ -6.30875 < 1.90 - \mu &< 6.30875 \\ -196.30875 < \quad -\mu \quad &< -183.69125 \\ 196.30875 > \quad \mu \quad &> 183.69125 \\ 183.7 < \quad \mu \quad &< 196.3 \text{ lb} \end{aligned}$$

**60.**
$$\begin{aligned} 63 < x + (x+2) + (x+4) &< 81 \\ 63 < \quad 3x + 6 \quad &< 81 \\ 57 < \quad 3x \quad &< 75 \\ 19 < \quad x \quad &< 25 \end{aligned}$$

$x$ must be odd, thus $x = 21$ or $x = 23$.
Therefore, the numbers are $\{21, 23, 25\}$ or $\{23, 25, 27\}$.

**61.** Solve $|h-(2.47f+54.10)| \le 3.72$ for $h$ where $f = 32.24$.

$$|h-(2.47f+54.10)| \le 3.72$$
$$|h-[2.47(32.24)+54.10]| \le 3.72$$
$$|h-(79.6328+54.10)| \le 3.72$$
$$|h-133.7328| \le 3.72$$

$h-133.7328 \le 3.72$ or $h-133.7328 \ge -3.72$

$h \le 137.4528$ $\qquad$ $h \ge 130.0128$

The height, to the nearest 0.1 cm, is from 130.0 cm to 137.5 cm.

**62.** To determine potential stature, solve $|h-(3.32r+85.43)| \le 4.57$ for $h$ where $r = 26.36$.

$$|h-(3.32r+85.43)| \le 4.57$$
$$|h-[3.32(26.36)+85.43]| \le 4.57$$
$$|h-(87.5152+85.43)| \le 4.57$$
$$|h-172.9452| \le 4.57$$

$h-172.9452 \le 4.57$ or $h-172.9452 \ge -4.57$

$h \le 177.5152$ $\qquad$ $h \ge 168.3752$

The potential stature, to the nearest 0.1 cm, is from 168.4 to 177.5 cm.

**63.** $R = 420x - 2x^2$

$420x - 2x^2 > 0$

$2x(210 - x) > 0$

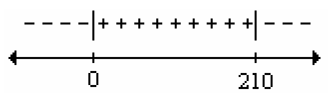
The product is positive.

$2x = 0 \Rightarrow x = 0$

$210 - x = 0 \Rightarrow x = 210$

Critical values are 0 and 210.

$2x(210-x)$




($0, $210)

**64.** $R = 312x - 3x^2$

$312x - 3x^2 \ge 5925$

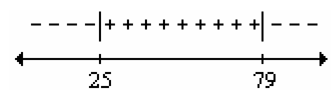
$-3x^2 + 312x - 5925 \ge 0$

$3(-x^2 + 104x - 1975) \ge 0$

$3(-x+25)(x-79) \ge 0$

Critical values are 25 and 79.

$3(-x+25)(x-79)$




[$25, $79]

**65.** $\dfrac{14.25x + 350{,}000}{x} < 50$

$14.25x + 350{,}000 < 50x$

$-35.75x < -350{,}000$

$x > 9790.2$

At least 9791 books must be published.

**66.** $\overline{C} = \dfrac{0.00014x^2 + 12x + 400{,}000}{x} < 30$

$0.00014x^2 + 12x + 400{,}000 < 30x$

$0.00014x^2 - 18x + 400{,}000 < 0$

Solve $0.00014x^2 - 18x + 400{,}000 = 0$ to find the critical values.

$$x = \frac{-(-18) \pm \sqrt{(-18)^2 - 4(0.00014)(400{,}000)}}{2(0.00014)} = \frac{18 \pm \sqrt{324-224}}{0.00028} = \frac{18 \pm \sqrt{100}}{0.00028} = \frac{18 \pm 10}{0.00028}$$

$x = \dfrac{18+10}{0.00028}$ or $x = \dfrac{18-10}{0.00028}$

$= \dfrac{28}{0.00028}$ $\qquad$ $= \dfrac{8}{0.00028}$

$= 100{,}000$ $\qquad$ $\approx 28{,}571.4$

The critical values are 100,000 and $\approx 28{,}571.4$.

Since $x$ is a non-negative integer, the intervals are $(0,\ 26{,}571.4)$, $(28{,}571.4,\ 100{,}000)$, and $(100{,}000,\ \infty)$.

Test 1: $\dfrac{0.00014(1)^2 + 12(1) + 400{,}000}{1} < 30 \Rightarrow 400{,}012.00014 < 30$, which is false.

Test 50,000: $\dfrac{0.00014(50{,}000)^2 + 12(50{,}000) + 400{,}000}{50{,}000} < 30 \Rightarrow 27 < 30$, which is true.

Test 150,000: $\dfrac{0.00014(150000)^2 + 12(150000) + 400{,}000}{150000} < 30 \Rightarrow 35.\overline{6} < 30$, which is false

The company should manufacture from 28,572 to 99,999 pairs of running shoes.

## Connecting Concepts

**67.** $28 - 0.15 \le C \le 28 + 0.15$

$27.85 \le 2\pi r \le 28.15$

$\frac{27.85}{2\pi} \le r \le \frac{28.15}{2\pi}$

$4.432 \le r \le 4.480$

The radius of the cylinder must be between 4.432 inches and 4.480 inches.

**68.** $750 - 15 \le V \le 750 + 15$

$735 \le \pi r^2 h \le 765$

$\frac{735}{\pi r^2} \le h \le \frac{765}{\pi r^2}$

$\frac{735}{4\pi} \le h \le \frac{765}{4\pi}$

$58.5 \le h \le 60.9$

The height of the beaker must be between 58.5 cm and 60.9 cm.

**69.** $\frac{(x-3)^2}{(x-6)^2} > 0$

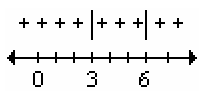
The quotient is positive.

$x - 3 = 0 \Rightarrow x = 3$

$x - 6 = 0 \Rightarrow x = 6$

Critical values are 3 and 6.

$\frac{(x-3)^2}{(x-6)^2}$




$(-\infty, 3) \cup (3, 6) \cup (6, \infty)$

**70.** $\frac{(x-1)^2}{(x-4)^4} \ge 0$

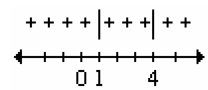
The quotient is positive or zero.

$x - 1 = 0 \Rightarrow x = 1$

$x - 4 = 0 \Rightarrow x = 4$

Critical values are 1 and 4.

$\frac{(x-1)^2}{(x-4)^4}$




$(-\infty, 4) \cup (4, \infty)$

**71.** $\frac{(x-4)^2}{(x+3)^3} \ge 0$

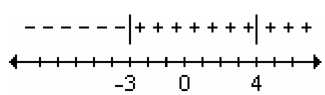
The quotient is positive or zero.

$x - 4 = 0 \Rightarrow x = 4$

$x + 3 = 0 \Rightarrow x = -3$

Critical values are 4 and –3.

$\frac{(x-4)^2}{(x+3)^3}$




Denominator not 0 $\Rightarrow x \ne -3$.

$(-3, \infty)$

**72.** $\frac{2x-7}{(x-1)^2(x+2)^2} \ge 0$

The quotient is positive or zero.

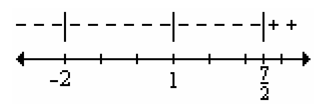
$2x - 7 = 0 \Rightarrow x = \frac{7}{2}$

$x - 1 = 0 \Rightarrow x = 1$

$x + 2 = 0 \Rightarrow x = -2$

Critical values are $\frac{7}{2}$, 1, –2.

$\frac{2x-7}{(x-1)^2(x+2)^2}$




$\left[\frac{7}{2}, \infty\right)$

**73.** $1 < |x| < 5$

if $x \ge 0$ $\quad 1 < x < 5$

if $x < 0$ $\quad 1 < -x < 5$

$-1 > x > -5$

$(-5, -1) \cup (1, 5)$

**74.** $2 < |x| < 3$

if $x \ge 0$ $\quad 2 < x < 3$

if $x < 0$ $\quad 2 < -x < 3$

$-2 > x > -3$

$(-3, -2) \cup (2, 3)$

**75.** $3 \le |x| < 7$

if $x \ge 0$ $\quad 3 \le x < 7$

if $x < 0$ $\quad 3 \le -x < 7$

$-3 \ge x > -7$

$(-7, -3] \cup [3, 7)$

**76.** $0<|x|\le 3$

if $x\ge 0$ $\quad 0< \; x\le 3$

if $x<0$ $\quad 0<-x\le 3$

$\quad 0> \; x\ge -3$

$[-3,0)\cup(0,3]$

**77.** $0<|x-\alpha|<\delta, \quad \delta>0$

if $x-a\ge 0$ $\quad 0< \; x-a \; <\delta$

$\quad \alpha< \; x \; <\delta+\alpha$

if $x-a<0$ $\quad 0<-(x-a)<\delta$

$\quad 0> \; x-a \; >-\delta$

$\quad a> \; x \; >a-\delta$

$(a-\delta,a)\cup(a,a+\delta)$

**78.** $0<|x-5|<2$

if $x-5\ge 0$ $\quad 0< \; x-5 \; <2$

$\quad 5< \; x \; <7$

if $x-5<0$ $\quad 0<-(x-5)<2$

$\quad 0> \; x-5 \; >-2$

$\quad 5> \; x \; >3$

$(3,5)\cup(5,7)$

**79.** $s=-16t^2+v_0t+s_0, \quad s>48, v_0=64, s_0=0$

$-16t^2+64t>48$

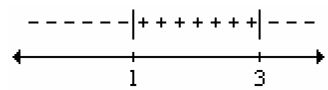
$-16t^2+64t-48>0$

$-16(t^2-4t+3)>0$

$-16(t-1)(t-3)>0$

The product is positive.
The critical values are 1 and 3.

$(t-1)(t-3)$




1 second $< t <$ 3 seconds
The ball is higher than 48 ft between 1 and 3 seconds.

**80.** $s=-16t^2+v_0t+s_0, \quad s>96, \; t>0, \; v_0=80, \; s_0=32$

$-16t^2+80t+32>96$

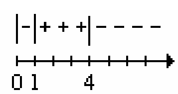
$-16t^2+80t-64>0$

$-16(t^2-5t+4)>0$

$-16(t-1)(t-4)>0$

The product is positive.
The critical values are 1 and 4.

$(t-1)(t-4)$




1 second $< t <$ 4 seconds
The ball is higher than 96 ft between 1 and 4 seconds.

## Prepare for Section 1.6

**PS1.** $1820=k(28)$

$65=k$

**PS2.** $20=\dfrac{k}{1.5^2}$

$45=k$

**PS3.** $k\dfrac{3}{5^2}$

$(225)\dfrac{3}{5^2}=27$

**PS4.** $k\dfrac{4.5\cdot 32}{8^2}$

$(12.5)\dfrac{4.5\cdot 32}{8^2}=28.125$

**PS5.** The area becomes 4 times as large.

**PS6.** No. The volume becomes 9 times as large.

## Section 1.6

**1.** $d = kt$

**2.** $r = ks^2$

**3.** $y = \frac{k}{x}$

**4.** $p = \frac{k}{q}$

**5.** $m = knp$

**6.** $t = krs^3$

**7.** $V = klwh$

**8.** $u = \frac{kv}{w^2}$

**9.** $A = ks^2$

**10.** $A = khr^2$

**11.** $F = \frac{km_1m_2}{d^2}$

**12.** $T = ktra^2$

**13.**
$$y = kx$$
$$64 = k \cdot 48$$
$$\frac{64}{48} = k$$
$$\frac{4}{3} = k$$

**14.**
$$m = kn$$
$$92 = k \cdot 23$$
$$\frac{92}{23} = k$$
$$4 = k$$

**15.**
$$r = kt^2$$
$$144 = k \cdot 108^2$$
$$\frac{144}{108^2} = k$$
$$\frac{2^4 \cdot 3^2}{2^4 \cdot 3^6} = k$$
$$\frac{1}{81} = k$$

**16.**
$$C = kr$$
$$94.2 = k \cdot 15$$
$$\frac{94.2}{15} = k$$
$$6.28 = k$$

**17.**
$$T = krs^2$$
$$210 = k \cdot 30 \cdot 5^2$$
$$\frac{210}{30 \cdot 5^2} = k$$
$$\frac{7}{25} = k$$
$$0.28 = k$$

**18.**
$$u = \frac{kv}{\sqrt{w}}$$
$$0.04 = \frac{k \cdot 8}{\sqrt{0.04}}$$
$$\frac{0.04\sqrt{0.04}}{8} = k$$
$$\frac{(0.04))0.2)}{8} = k$$
$$0.001 = k$$

**19.**
$$V = klwh$$
$$240 = k \cdot 8 \cdot 6 \cdot 5$$
$$\frac{240}{8 \cdot 6 \cdot 5} = k$$
$$1 = k$$

**20.**
$$t = \frac{kr^3}{\sqrt{s}}$$
$$10 = \frac{k \cdot 5^3}{\sqrt{0.09}}$$
$$\frac{10\sqrt{0.09}}{5^3} = k$$
$$\frac{2(0.3)}{8} = k$$
$$\frac{.06}{25} = k$$
$$0.024 = k$$

**21.**
$$V = kT$$
$$0.85 = k \cdot 270$$
$$\frac{0.85}{270} = k$$
$$\frac{0.17}{54} = k$$
Thus $V = \frac{0.17}{54}T = \frac{0.17}{54} \cdot 324 = (0.17)6 = 1.02$ liters

**22.**
$$d = k \cdot w$$
$$6 = k \cdot 80$$
$$6 = k \cdot 80$$
$$\frac{6}{80} = k$$
$$\frac{3}{40} = k$$
Therefore $d = \frac{3}{40} \cdot 100 = 7.5$ inches

**23.**
$$s = k \cdot q$$
$$34 = k \cdot 51$$
$$\frac{2}{3} = k$$
$$p = \frac{2}{3} \cdot 93$$
$p = 62$ semester hours

**24.**
$$p = kd$$
$$187.5 = k \cdot 3$$
$$62.5 = k$$
$$p = 62.5 \cdot 7$$
$p = 437.5$ lb/ft$^2$

**25.**
$$j = k \cdot d^3$$
$$6 = k \cdot (4)^3$$
$$\frac{3}{32} = k$$
$$p = \frac{3}{32} \cdot (5)^3$$
$p \approx 11.7$ fl oz

**26.**
$$r = kv^2$$
$$140 = k \cdot 60^2$$
$$\frac{140}{60^2} = k$$
$$\frac{7}{180} = k$$
Thus $r = \frac{7}{180} \cdot 65^2$
$r \approx 164.3$ ft.

**27.**
$$T = k\sqrt{l}$$
$$1.8 = k\sqrt{3}$$
$$\frac{1.8}{\sqrt{3}} = k$$
$$1.03923 \approx k$$

**a.**
$$T = \frac{1.8}{\sqrt{3}}\sqrt{10}$$
$$= \frac{1.8\sqrt{30}}{3}$$
$$= 0.6\sqrt{30}$$
$$\approx 3.3 \text{ seconds}$$

**b.**
$$T = k\sqrt{l}$$
$$\frac{T}{k} = \sqrt{l}$$
$$\frac{2}{1.03923} = \sqrt{l}$$
$$\frac{4}{1.03923^2} = \sqrt{l}$$
$$3.7 \text{ ft} \approx l$$

**28.**
$$A = kd^2$$
$$64 = k \cdot 20^2$$
$$6 = k \cdot 80$$
$$\frac{64}{400} = k$$
$$\frac{4}{25} = k$$
$$100 = \frac{4}{25} \cdot d^2$$
$$625 = d^2$$
$$d = 25 \text{ ft}$$

**29.**
$$r = \frac{k}{t}$$
$$30 = \frac{k}{64}$$
$$1920 = k$$
$$r = \frac{1920}{48}$$
$$r = 40 \text{ revolutions per minute}$$

**30.**
$$f = \frac{k}{l}$$
$$144 = \frac{k}{20}$$
$$2880 = k$$
$$f = \frac{2880}{18}$$
$$f = 160 \text{ vibrations per second}$$

**31.**
$$l = \frac{k}{d^2}$$
$$28 = \frac{k}{8^2}$$
$$28 \cdot 64 = k$$
$$1792 = k$$
$$l = \frac{1792}{4^2} = \frac{1792}{16}$$
$$l = 112 \text{ decibels}$$

**32.**
$$l = \frac{k}{d^2}$$
$$50 = \frac{k}{10^2}$$
$$5000 = k$$
$$\text{Thus } I = \frac{5000}{d^2}$$
$$I = \frac{5000}{15^2} = \frac{5000}{225}$$
$$I \approx 22.2 \text{ footcandles}$$

**33.** **a.**
$$V = kr^2h$$
$$V_1 = k(3r)^2h$$
$$= 9(kr^2h)$$
$$= 9V$$

Thus the new volume is 9 times the original volume.

**b.**
$$V_2 = kr^2(3h)$$
$$= 3(kr^2h)$$
$$= 3V$$

Thus the new volume is 3 times the original volume.

**c.**
$$V_3 = k(3r)^2(3h)$$
$$= k9r^2 \cdot 3 \cdot h$$
$$= 27(kr^2h)$$
$$= 27V$$

Thus the new volume is 27 times the original volume.

**34.**
$$L = kwd^2$$
$$200 = k \cdot 2 \cdot 6^2$$
$$\frac{200}{2 \cdot 6^2} = k$$
$$\frac{25}{9} = k$$
$$\text{Thus L} = \frac{25}{9} \cdot 4 \cdot 4^2$$
$$= \frac{1600}{9}$$
$$\approx 178 \text{ lb}$$

**35.**
$$V = \frac{knT}{P}$$
$$V_1 = \frac{k(3n)T}{\left(\frac{1}{2}p\right)}$$
$$= 6\left(\frac{knT}{p}\right)$$
$$= 6V$$
Thus the new volume is 6 times larger than the original volume.

**36.**
$$L = \frac{k \cdot d^4}{h^2}$$
$$6 = \frac{k \cdot 2^4}{10^2}$$
$$\frac{6 \cdot 10^2}{2^4} = k$$
$$\frac{600}{16} = 37.5 = k$$
$$\text{Thus L} = \frac{37.5 \cdot 3^4}{14^2}$$
$$\approx 15.5 \text{ tons}$$

**37.** For Randy Johnson,
$$\text{ERA} = \frac{kr}{i}$$
$$2.32 = \frac{k(67)}{(260)}$$
$$9.00 = k$$
For Tom Glavine,
$$\text{ERA} = \frac{9(74)}{(224.2)}$$
$$= 2.97$$

**38.**
$$L = \frac{kbd^2}{l}$$
$$800 = \frac{k \cdot 4 \cdot 8^2}{12}$$
$$\frac{800 \cdot 12}{4 \cdot 8^2} = k$$
$$37.5 = k$$
$$\text{Thus L} = \frac{37.5(3.5)(6)^2}{16}$$
$$\approx 295 \text{ pounds}$$

**39.**
$$F = \frac{kws^2}{r}$$
$$2800 = \frac{k \cdot 1800 \cdot 45^2}{425}$$
$$\frac{2800 \cdot 425}{1800 \cdot 45^2} = k$$
$$\frac{14 \cdot 425}{9 \cdot 45^2} = k$$
$$0.3264746 \approx k$$
$$\text{Thus F} = \frac{(0.3264746) \cdot 1800 \cdot 55^2}{450}$$
$$\approx 3950 \text{ pounds}$$

## Connecting Concepts

**40.** $S = kwd^3$

when $d = 10,\ w = \sqrt{18^2 - 10^2} \approx 15 \Rightarrow S \approx k(15)(10)^3 = 15{,}000k$

$d = 12,\ w = \sqrt{18^2 - 12^2} \approx 13.4 \Rightarrow S \approx k(13.4)(12)^3 = 23{,}155k$

$d = 14,\ w = \sqrt{18^2 - 14^2} \approx 11.3 \Rightarrow S \approx k(11.3)(14)^3 = 31{,}007k$

$d = 16,\ w = \sqrt{18^2 - 16^2} \approx 8.2 \Rightarrow S \approx k(8.2)(16)^3 = 33{,}587k$

The strongest beam occurs when $d$ = 16 inches.

**41.**
$$T = kd^{3/2}$$
$$365 = k \cdot 93^{3/2}$$
$$\frac{365}{93^{3/2}} = k$$
$$\text{Thus} \quad 686 = \frac{365}{93^{3/2}} \cdot d^{3/2}$$
$$\frac{686 \cdot 93^{3/2}}{365} = d^{3/2}$$
$$\left(\frac{686 \cdot 93^{3/2}}{365}\right)^{2/3} = d$$
$$93\left(\frac{686}{365}\right)^{2/3} = d$$
$$142 \text{ million miles} \approx d$$

## Assessing Concepts

**1.** No. The solution set of $x = 3$ is {3} but the solution set of $x^2 = 9$ is {–3, 3}.

**2.** $a < 0$

**3.** $-a < -b$

**4.** Equal to 0

**5.** False.
$(\sqrt{x}+3)^2 = x + 6\sqrt{x} + 9 \neq x + 9$

**6.** c

**7.** b

**8.** a

**9.** g

**10.** f

## Chapter Review

**1.** $x - 2(5x-3) = -3(-x+4)$ [1.1]
$$\begin{aligned} x - 10x + 6 &= 3x - 12 \\ -9x + 6 &= 3x - 12 \\ -12x &= -18 \\ x &= \frac{3}{2} \end{aligned}$$

**2.** $3x - 5(2x-7) = -4(5-2x)$ [1.1]
$$\begin{aligned} 3x - 10x + 35 &= -20 + 8x \\ -7x + 35 &= -20 + 8x \\ -15x &= -55 \\ x &= \frac{11}{3} \end{aligned}$$

**3.** $\frac{4x}{3} - \frac{4x-1}{6} = \frac{1}{2}$ [1.1]
$$\begin{aligned} 6\left(\frac{4x}{3} - \frac{4x-1}{6}\right) &= 6\left(\frac{1}{2}\right) \\ 2(4x) - (4x-1) &= 3 \\ 8x - 4x + 1 &= 3 \\ 4x + 1 &= 3 \\ 4x &= 2 \\ x &= \frac{1}{2} \end{aligned}$$

**4.** $\frac{3x}{4} - \frac{2x-1}{8} = \frac{3}{2}$ [1.1]
$$\begin{aligned} 8\left(\frac{3x}{4} - \frac{2x-1}{8}\right) &= 8\left(\frac{3}{2}\right) \\ 2(3x) - (2x-1) &= 4(3) \\ 6x - 2x + 1 &= 12 \\ 4x + 1 &= 12 \\ 4x &= 11 \\ x &= \frac{11}{4} \end{aligned}$$

**5.**
$$\begin{aligned} \frac{x}{x+2} + \frac{1}{4} &= 5 \\ 4(x+2)\left(\frac{x}{x+2} + \frac{1}{4}\right) &= 5(4)(x+2) \\ 4x + x + 2 &= 20(x+2) \\ 5x + 2 &= 20x + 40 \\ -15x &= 38 \\ x &= -\frac{38}{15} \end{aligned}$$
[1.5]

**6.**
$$\begin{aligned} \frac{y-1}{y+1} - 1 &= \frac{2}{y} \\ y(y+1)\left(\frac{y-1}{y+1} - 1\right) &= y(y+1)\left(\frac{2}{y}\right) \\ y(y-1) - y(y+1) &= 2(y+1) \\ y^2 - y - y^2 - y &= 2y + 2 \\ -4y &= 2 \\ y &= -\frac{1}{2} \end{aligned}$$
[1.4]

**7.** $x^2 - 5x + 6 = 0$ [1.3]
$$\begin{aligned} &(x-2)(x-3) = 0 \\ &x - 2 = 0 \quad \text{or} \quad x - 3 = 0 \\ &x = 2 \qquad\qquad x = 3 \end{aligned}$$

**8.** $6x^2 + x - 12 = 0$ [1.3]
$$\begin{aligned} &(3x-4)(2x+3) = 0 \\ &3x - 4 = 0 \quad \text{or} \quad 2x + 3 = 0 \\ &3x = 4 \qquad\qquad 2x = -3 \\ &x = \frac{4}{3} \qquad\qquad x = -\frac{3}{2} \end{aligned}$$

**9.** $3x^2 - x - 1 = 0$ [1.3]
$$\begin{aligned} x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(-1)}}{2(3)} \\ x &= \frac{1 \pm \sqrt{13}}{6} \\ x &= \frac{1+\sqrt{13}}{6} \quad \text{or} \quad x = \frac{1-\sqrt{13}}{6} \end{aligned}$$

**10.** $x^2 - x + 1 = 0$ [1.3]

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

$$x = \frac{1}{2} + \frac{\sqrt{3}}{2}i \quad \text{or} \quad x = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

**11.** $3x^3 - 5x^2 = 0$ [1.4]

$$x^2(3x - 5) = 0$$

$$x^2 = 0 \Rightarrow x = 0$$

$$3x - 5 = 0 \Rightarrow x = \frac{5}{3}$$

$$x = 0 \quad \text{or} \quad x = \frac{5}{3}$$

**12.** $2x^3 - 8x = 0$ [1.4]

$$2x(x^2 - 4) = 0$$

$$2x(x-2)(x+2) = 0$$

$$x = 0, \ x = 2, \quad \text{or} \quad x = -2$$

**13.** $6x^4 - 23x^2 + 20 = 0$ [1.4]

Let $u = x^2$.

$$6u^2 - 23u + 20 = 0$$

$$(3u - 4)(2u - 5) = 0$$

$$a = \frac{4}{3} \quad \text{or} \quad u = \frac{5}{2}$$

$$x^2 = \frac{4}{3} \qquad x^2 = \frac{5}{2}$$

$$x = \pm\sqrt{\frac{4}{3}} \qquad x = \pm\sqrt{\frac{5}{2}}$$

$$x = \pm\frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) \qquad x = \pm\frac{\sqrt{5}}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)$$

$$x = \pm\frac{2\sqrt{3}}{3} \qquad x = \pm\frac{\sqrt{10}}{2}$$

**14.** $3x + 16\sqrt{x} - 12 = 0$ [1.4]

Let $u = \sqrt{x}$.

$$3u^2 + 16u - 12 = 0$$

$$(3u - 2)(u + 6) = 0$$

$$u = \frac{2}{3} \quad \text{or} \quad u = -6$$

$$\sqrt{x} = \frac{2}{3} \qquad \sqrt{x} = -6$$

No solution.

$$x = \frac{4}{9}$$

Thus, $x = \frac{4}{9}$.

**15.** $\sqrt{x^2 - 15} = \sqrt{-2x}$ [1.4]

$$\left[\sqrt{x^2 - 15}\right]^2 = \left[\sqrt{-2x}\right]^2$$

$$x^2 - 15 = -2x$$

$$x^2 + 2x - 15 = 0$$

$$(x + 5)(x - 3) = 0$$

$$x = -5 \quad \text{or} \quad x = 3$$

Check $\sqrt{(-5)^2 - 15} = \sqrt{-2(-5)}$

$$\sqrt{10} = \sqrt{10}$$

$$\sqrt{3^2 - 15} = \sqrt{-2(3)}$$

$$\sqrt{-6} = \sqrt{-6}$$

The solutions are –5 and 3.

**16.** $\sqrt{x^2 - 24} = \sqrt{2x}$ [1.4]

$$\left[\sqrt{x^2 - 24}\right]^2 = \left[\sqrt{2x}\right]^2$$

$$x^2 - 24 = 2x$$

$$x^2 - 2x - 24 = 0$$

$$(x - 6)(x + 4) = 0$$

$$x = 6 \quad \text{or} \quad x = -4$$

Check $\sqrt{(6)^2 - 24} = \sqrt{2(6)}$

$$\sqrt{36 - 24} = \sqrt{12}$$

$$\sqrt{12} = \sqrt{12}$$

$$\sqrt{(-4)^2 - 24} = \sqrt{2(-4)}$$

$$\sqrt{16 - 24} = \sqrt{-8}$$

$$\sqrt{-8} = \sqrt{-8}$$

The solutions are 6 and –4.

**17.**

$$\sqrt{3x+4}+\sqrt{x-3}=5$$
$$\sqrt{3x+4}=5-\sqrt{x-3}$$
$$\left[\sqrt{3x+4}\right]^2=\left[5-\sqrt{x-3}\right]^2$$
$$3x+4=25-10\sqrt{x-3}+x-3$$
$$2x-18=-10\sqrt{x-3}$$
$$x-9=-5\sqrt{x-3}$$
$$(x-9)^2=\left[-5\sqrt{x-3}\right]^2$$
$$x^2-18x+81=25(x-3)$$
$$x^2-18x+81=25x-75$$
$$x^2-43x+156=0$$
$$(x-4)(x-39)=0$$
$$x=4 \quad \text{or} \quad x=39$$

Check $\sqrt{3(4)+4}+\sqrt{4-3}=5$

$$\sqrt{16}+\sqrt{1}=5$$
$$4+1=5$$
$$5=5$$
$$\sqrt{3(39)+4}+\sqrt{39-3}=5$$
$$\sqrt{121}+\sqrt{36}=5$$
$$11+6=5$$
$$17=5 \quad \text{(No)}$$

The solution is 4. [1.4]

**18.**

$$\sqrt{2x+2}-\sqrt{x+2}=\sqrt{x-6}$$
$$\left[\sqrt{2x+2}-\sqrt{x+2}\right]^2=\left[\sqrt{x-6}\right]^2$$
$$2x+2-2\sqrt{(2x+2)(x+2)}+x+2=x-6$$
$$-2\sqrt{(2x+2)(x+2)}=-2(x+5)$$
$$\sqrt{(2x+2)(x+2)}=x+5$$
$$\left[\sqrt{(2x+2)(x+2)}\right]^2=[x+5]^2$$
$$(2x+2)(x+2)=x^2+10x+25$$
$$2x^2+4x+2x+4=x^2+10x+25$$
$$x^2-4x-21=0$$
$$(x-7)(x+3)=0$$
$$x=7 \quad \text{or} \quad x=-3$$

Check $\sqrt{2(7)+2}-\sqrt{7+2}=\sqrt{7-6}$

$$\sqrt{16}-\sqrt{9}=\sqrt{1}$$
$$4-3=1$$
$$1=1$$
$$\sqrt{2(-3)+2}-\sqrt{-3+2}=\sqrt{-3-6}$$
$$\sqrt{-4}-\sqrt{-1}=\sqrt{-9}$$
$$2i-i=3i$$
$$i=3i \quad \text{(No)}$$

The solution is 7. [1.4]

**19.** [1.4]

$$\sqrt{4-3x}-\sqrt{5-x}=\sqrt{5+x}$$
$$\left[\sqrt{4-3x}-\sqrt{5-x}\right]^2=\left[\sqrt{5+x}\right]^2$$
$$-2\sqrt{(4-3x)(5-x)}=5x-4$$
$$\left[-2\sqrt{(4-3x)(5-x)}\right]^2=[5x-4]^2$$
$$4(4-3x)(5-x)=25x^2-40x+16$$
$$4(20-19x+3x^2)=25x^2-40x+16$$
$$0=13x^2+36x-64$$
$$0=(13x-16)(x+4)$$
$$x=\frac{16}{13} \quad \text{or} \quad x=-4$$

Check $\sqrt{4-3\left(\frac{16}{13}\right)}-\sqrt{5-\frac{16}{13}}=\sqrt{5+\frac{16}{13}}$

$$\sqrt{\frac{52}{13}-\left(\frac{48}{13}\right)}-\sqrt{\frac{65}{13}-\frac{16}{13}}=\sqrt{\frac{65}{13}+\frac{16}{13}}$$
$$\sqrt{\frac{4}{13}}-\sqrt{\frac{49}{13}}=\sqrt{\frac{81}{13}}$$
$$\frac{2}{\sqrt{13}}-\frac{7}{\sqrt{13}}=\frac{9}{\sqrt{13}} \quad \text{(No)}$$
$$\sqrt{4-3(-4)}-\sqrt{5-(-4)}=\sqrt{5-4}$$
$$\sqrt{16}-\sqrt{9}=\sqrt{1}$$
$$4-3=1$$
$$1=1$$

The solution is –4.

**20.**

$$\sqrt{3x+9}-\sqrt{2x+4}=\sqrt{x+1}$$
$$\left[\sqrt{3x+9}-\sqrt{2x+4}\right]^2=\left[\sqrt{x+1}\right]^2$$
$$3x+9-2\sqrt{(3x+9)(2x+4)}+2x+4=x+1$$
$$-2\sqrt{(3x+9)(2x+4)}=-4x-12$$
$$\left[\sqrt{(3x+9)(2x+4)}\right]^2=\left[2x+6\right]^2$$
$$(3x+9)(2x+4)=4x^2+24x+36$$
$$6x^2+30x+36=4x^2+24x+36$$
$$2x^2+6x=0$$
$$2x(x+3)=0$$
$x=0$ or $x=-3$ [1.4]

Check $\sqrt{3(0)+9}-\sqrt{2(0)+4}=\sqrt{0+1}$
$$\sqrt{9}-\sqrt{4}=\sqrt{1}$$
$$3-2=1$$
$$1=1$$
$$\sqrt{3(-3)+9)}-\sqrt{2(-3)+4}=\sqrt{-3+1}$$
$$\sqrt{0}-\sqrt{-2}=\sqrt{-2}$$
$$0-\sqrt{-2}=\sqrt{-2}$$
$-\sqrt{-2}=\sqrt{-2}$ (No)

The solution is 0.

**21.** $\dfrac{1}{(y+3)^2}=1$ [1.4]
$$1=(y+3)^2$$
$$1=y^2+6y+9$$
$$0=y^2+6y+8$$
$$0=(y+2)(y+4)$$
$y=-2$ or $y=-4$

**22.** $\dfrac{1}{(2s-5)^2}=4$ [1.4]
$$1=4(4s^2-20s+25)$$
$$1=16s^2-80s+100$$
$$0=16s^2-80s+99$$
$$0=(4s-11)(4s-9)$$
$s=\dfrac{11}{4}$ or $s=\dfrac{9}{4}$

**23.** $|x-3|=2$ [1.1]
$x-3=2$ or $x-3=-2$
$x=5$ $\quad x=1$

**24.** $|x+5|=4$ [1.1]
$x+5=4$ or $x+5=-4$
$x=-1$ $\quad x=-9$

**25.** $|2x+1|=5$ [1.1]
$2x+1=5$ or $2x+1=-5$
$2x=4$ $\quad 2x=-6$
$x=2$ $\quad x=-3$

**26.** $|3x-7|=8$ [1.1]
$3x-7=8$ or $3x-7=-8$
$3x=15$ $\quad 3x=-1$
$x=5$ $\quad x=-\dfrac{1}{3}$

**27.** $(x+2)^{1/2}+x(x+2)^{3/2}=0$ [1.4]
$$(x+2)^{1/2}\left[1+x(x+2)\right]=0$$
$$(x+2)^{1/2}\left[1+x^2+2x\right]=0$$
$$(x+2)^{1/2}(x^2+2x+1)=0$$
$$(x+2)^{1/2}(x+1)^2=0$$
$(x+2)^{1/2}=0$ or $(x+1)^2=0$
$x+2=0$ $\quad x+1=0$
$x=-2$ $\quad x=-1$

**28.** $x^2(3x-4)^{1/4}+(3x-4)^{5/4}=0$ [1.4]
$$(3x-4)^{1/4}(x^2+3x-4)=0$$
$(3x-4)^{1/4}=0$ or $x^2+3x-4=0$
$3x-4=0$ $\quad (x+4)(x-1)=0$
$3x=4$ $\quad x+4=0$ or $x-1=0$
$x=\dfrac{4}{3}$ $\quad x=-4$ $\quad x=1$

**29.** $-3x+4\ge-2$ [1.5]
$$-3x\ge-2-4$$
$$-3x\ge-6$$
$$x\le2$$
$(-\infty, 2]$

**30.** $-2x+7\le5x+1$ [1.5]
$$-7x\le-6$$
$$x\ge\frac{6}{7}$$
$\left[\dfrac{6}{7}, \infty\right)$

**31.** $x^2+3x-10\le 0$ [1.5]

$(x+5)(x-2)\le 0$

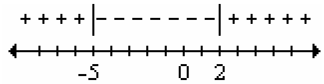
The product is negative or zero.

$x+5=0\Rightarrow x=-5$

$x-2=0\Rightarrow x=2$

Critical values are –5 and 2.

$(x+5)(x-2)$




$[-5, 2]$

**32.** $x^2-2x-3>0$ [1.5]

$(x+1)(x-3)>0$

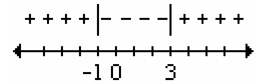
The product is positive.

$x+1=0\Rightarrow x=-1$

$x-3=0\Rightarrow x=3$

Critical values are –1 and 3.

$(x+1)(x-3)$




$(-\infty,\ -1)\cup(3,\ \infty)$

**33.** $61\le \frac{9}{5}C+32\le 95$ [1.5]

$29\le \quad \frac{9}{5}C \quad \le 63$

$\frac{145}{9}\le \quad C \quad \le 35$

$\left[\frac{145}{9},\ 35\right]$

**34.** $30<\frac{5}{9}(F-32)<65$ [1.5]

$54< \ F-32 \quad <117$

$86< \quad F \quad <149$

(86, 149)

**35.** $x^3-7x^2+12x\le 0$ [1.5]

$x(x^2-7x+12)\le 0$

$x(x-3)(x-4)\le 0.$

The product is negative or zero.

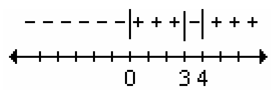
$x=0$

$x-3=0\Rightarrow x=3$

$x-4=0\Rightarrow x=4$

The critical values are 0, 3, and 4.

$x(x-3)(x-4)$




$(-\infty, 0]\cup[3, 4]$

**36.** $x^3+4x^2-21x>0$ [1.5]

$x(x^2+4x-21)>0$

$x(x+7)(x-3)>0.$

The product is positive.

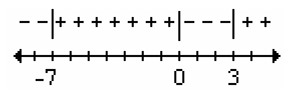
$x=0$

$x+7=0\Rightarrow x=-7$

$x-3=0\Rightarrow x=3$

The critical values are $0,\ -7$, and 3.

$x(x+7)(x-3)$




$(-7,\ 0)\cup(3,\ \infty)$

**37.** $\frac{x+3}{x-4}>0$ [1.5]

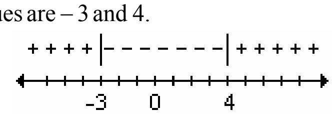
The quotient is positive.

$x+3=0\Rightarrow x=-3$

$x-4=0\Rightarrow x=4$

The critical values are $-3$ and 4.

$\frac{x+3}{x-4}$




$(-\infty,\ -3)\cup(4,\ \infty)$

**38.** $\frac{x(x-5)}{x+7}\le 0$ [1.5]

The quotient is negative or zero.

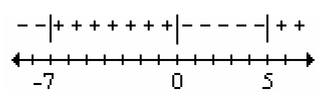
$x=0$

$x-5=0\Rightarrow x=5$

$x+7=0\Rightarrow x=-7$

The critical values are 0, 5 and $-7$.

$\frac{x(x-5)}{x+7}$




Denominator $\ne 0\Rightarrow x\ne -7$.

$(-\infty, -7)\cup[0, 5]$

**39.**

$$\frac{2x}{3-x} \le 10 \quad [1.5]$$
$$\frac{2x}{3-x} - 10 \le 0$$
$$\frac{2x-10(3-x)}{3-x} \le 0$$
$$\frac{2x-30+10x}{3-x} \le 0$$
$$\frac{12x-30}{3-x} \le 0$$

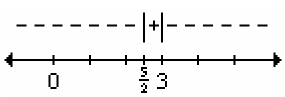
The quotient is negative or zero.

$12x-30=0 \Rightarrow x=\frac{5}{2}$

$3-x=0 \Rightarrow x=3$

The critical values are $\frac{5}{2}$ and 3.

$\frac{12x-30}{3-x}$




Denominator $\ne 0 \Rightarrow x \ne 3$.

$\left(-\infty, \frac{5}{2}\right] \cup (3, \infty)$

**40.**

$$\frac{x}{5-x} \ge 1 \quad [1.5]$$
$$\frac{x}{5-x} - 1 \ge 0$$
$$\frac{x-(5-x)}{5-x} \ge 0$$
$$\frac{x-5+x}{5-x} \ge 0$$
$$\frac{2x-5}{5-x} \ge 0$$

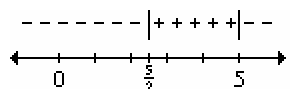
The quotient is positive or zero.

$2x-5=0 \Rightarrow x=\frac{5}{2}$

$5-x=0 \Rightarrow x=5$

The critical values are $\frac{5}{2}$ and 5.

$\frac{2x-5}{5-x}$




Denominator $\ne 0 \Rightarrow x \ne 5$.

$\left[\frac{5}{2}, 5\right)$

**41.**

$|3x-4| < 2 \quad [1.5]$

$-2 < 3x-4 < 2$

$2 < 3x < 6$

$\frac{2}{3} < x < 2$

$\left(\frac{2}{3}, 2\right)$

**42.**

$|2x-3| \ge 1 \quad [1.5]$

$2x-3 \ge 1$ or $2x-3 \le -1$

$2x \ge 4 \qquad 2x \le 2$

$x \ge 2 \qquad x \le 1$

$(-\infty, 1] \cup [2, \infty)$

**43.**

$0 < |x-2| < 1 \quad [1.5]$

If $x-2 \ge 0$, then $2 < x < 3$.

If $x-2 < 0$, then $0 < x-2 < -1$

$2 > x > 1$.

$(1, 2) \cup (2, 3)$

**44.**

$0 < |x-a| < b \quad [1.5]$

If $x-a \ge 0$, then $a < x < a+b$.

If $x-a < 0$, then $0 < x-a < -b$

$a > x > a-b$.

$[a-b, a) \cup (a, a+b]$

**45.**

$$V = \pi r^2 h \quad [1.2]$$
$$\frac{V}{\pi r^2} = h$$

**46.**

$$P = \frac{A}{1+rt} \quad [1.2]$$
$$P(1+rt) = A$$
$$P + Prt = A$$
$$Prt = A - P$$
$$t = \frac{A-P}{Pr}$$

**47.**

$$A = \frac{h}{2}(b_1 + b_2) \quad [1.2]$$
$$2A = h(b_1 + b_2)$$
$$2A = hb_1 + hb_2$$
$$2A - hb_2 = hb_1$$
$$\frac{2A - hb_2}{h} = b_1$$

**48.**

$$P = 2(l+w) \quad [1.2]$$
$$P = 2l + 2w$$
$$P - 2l = 2w$$
$$\frac{P-2l}{2} = w$$

**49.**

$$e = mc^2 \quad [1.2]$$
$$\frac{e}{c^2} = m$$

**50.**

$$F = G\frac{m_1 m_2}{s^2} \quad [1.2]$$
$$Fs^2 = Gm_1 m_2$$
$$\frac{Fs^2}{Gm_2} = m_1$$

**51.** Let $x$ = the number [1.2]

$\frac{1}{2}x - \frac{1}{4}x = 4 + \frac{1}{5}x$

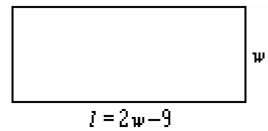
$20\left(\frac{1}{2}x - \frac{1}{4}x\right) = 20\left(4 + \frac{1}{5}x\right)$

$10x - 5x = 80 + 4x$

$5x = 80 + 4x$

$x = 80$

**52.**




$P = 54$

$54 = 2l + 2w$

$54 = 2(2w - 9) + 2w$

$54 = 4w - 18 + 2w$

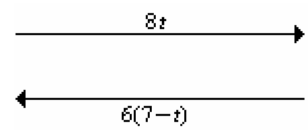
$72 = 6w$

$12 = w$

$2w - 9 = 2(12) - 9 = 24 - 9 = 15$

width = 12 ft, length = 15 ft [1.2]

**53.**




$d = rt$

$d = 8t \qquad d = 6(7 - t)$

$6(7 - t) = 8t$

$42 - 6t = 8t$

$42 = 14t$

$3 = t$

$d = 8(3) = 24$ nautical miles [1.2]

**54.** Let $x$ = cost last year

Cost = last year + raise

Let $x$ = the number.

$21 = x + 0.05x$

$21 = 1.05x$

$\frac{21}{1.05} = x$

$20 = x$

The cost last year was \$20.00 . [1.2]

**55.**

| | |
|---|---|
| 4% | $x$ |
| 6% | $5500 - x$ |

$0.04x + 0.06(5500 - x) = 295$

$0.04x + 330 - 0.06x = 295$

$-0.02x = -35$

$x = 1750$

$5500 - 1750 = 3750$

\$1750 in the 4% account

\$3750 in the 6% account [1.2]

**56.** Let $x$ = price of battery

$x + 20$ = price of calculator

$x + x + 20 = 21$

$2x + 20 = 21$

$2x = 1$

$x = 0.50$

$x + 20 = 20.50$

Price of calculator is \$20.50.

Price of battery is \$0.50. [1.2]

**57.** Let $x$ = monthly maintenance cost per owner

$18x = 24(x - 12)$

$18x = 24x - 288$

$-6x = -288$

$x = 48$

$18x = 864$

The total monthly maintenance cost is \$864. [1.2]

**58.** $P = 40$

$A = 96$

$40 = 2l + 2w$

$20 = l + w$

$l = 20 - w$

$96 = lw$

$96 = (20 - w)w$

$96 = 20w - w^2$

$w^2 - 20w + 96 = 0$

$(w - 12)(w - 8) = 0$

$w = 12 \quad \text{or} \quad w = 8$

$l = 20 - 12 \quad \text{or} \quad l = 20 - 8$

$l = 8 \qquad l = 12$

Length = 8 in. and width = 12 in.,
or length = 12 in. and width = 8 in. [1.2]

**59.**

| | Time | Part completed In 1 hour |
|---|---|---|
| Mason | $x-9$ | $\frac{1}{x-9}$ |
| Apprentice | $x$ | $\frac{1}{x}$ |

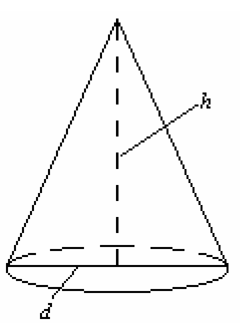
$$6\left(\frac{1}{x}+\frac{1}{x-9}\right)=1$$
$$6x(x-9)\left(\frac{1}{x}+\frac{1}{x-9}\right)=1x(x-9)$$
$$6(x-9)+6x=x^2-9x$$
$$6x-54+6x=x^2-9x$$
$$0=x^2-21x+54$$
$$0=(x-18)(x-3)$$
$$x=18 \quad \text{or} \quad x=3$$
(Note: $x=3 \Rightarrow$ mason's time $=-6$ hours. Thus $x\neq 3$.)

Apprentice takes 18 hours to build the wall. [1.4]

**60.** Let $x=$ number of adult tickets
$4526-x=$ number of student tickets
$$8x+2(4526-x)=33{,}196$$
$$8x+9052-2x=33{,}196$$
$$6x=24{,}144$$
$$x=4024$$
$$4526-x=502$$

4024 adult tickets, 502 student tickets [1.2]

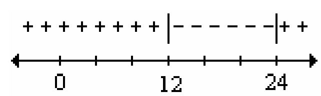
**61.**




$$h=\frac{1}{4}d$$
$$V=144$$
$$d=2r \quad \Rightarrow \quad r=\frac{d}{2}$$
$$V=\frac{1}{3}\pi r^2 h$$
$$144=\frac{1}{3}\pi\left(\frac{d}{2}\right)^2\left(\frac{1}{4}d\right)$$
$$144=\frac{\pi d^3}{48}$$
$$\frac{144(48)}{\pi}=d^3$$
$$d\approx 13\text{ ft}$$ [1.2]

**62.** $R=72x-2x^2,\ R>576$
$$72x-2x^2>576$$
$$0>2x^2-72x+576$$
$$2x^2-72x+576<0$$
$$x^2-36x+288<0$$
$$(x-24)(x-12)<0$$
The product is negative.
$x-24=0\Rightarrow x=24$
$x-12=0\Rightarrow x=12$
Critical values are 24 and 12.

$(x-24)(x-12)$




(12, 24)

The revenue is greater than \$576
when the price is between \$12 and \$24. [1.5]

**63.**
$$-1.96<\frac{x-50}{5}<1.96$$
$$-9.8<1.63-x<9.8$$
$$40.2< -x <59.8$$
$$41< x <59, \text{ where } x \text{ is an integer}$$

**64.**
$$-1.645<\frac{63.8-\mu}{0.45}<1.645$$
$$-0.74025<63.8-\mu<0.74025$$
$$-64.54025< -\mu <-63.05975$$
$$64.5> \mu >63.1$$
$$63.1< \mu <64.5\text{ lb}$$

**65.**

$$-1.96 < \frac{39-\mu}{0.53} < 1.96$$
$$-1.0388 < 39-\mu < 1.0388$$
$$-40.0388 < -\mu < -37.9612$$
$$40.0 > \mu > 38.0$$
$$38.0 < \mu < 40.0 \text{ lb}$$

**66.** Let $x$ = the score on the fifth test.

$$68 \le \frac{82+72+64+95+x}{5} \le 79$$
$$68 \le \frac{313+x}{5} \le 79$$
$$340 \le 313+x \le 395$$
$$27 \le x \le 82$$

The student needs to earn a score in the interval [27, 82] to receive a C grade for the course. [1.5]

**67.** Let $C$ = the circumference, $r$ = the radius, and $d$ = the diameter.

$$C = 2\pi r = \pi d$$
$$29.5 \le C \le 30.0$$
$$29.5 \le \pi d \le 30.0$$
$$\frac{29.5}{\pi} \le d \le \frac{30.0}{\pi}$$
$$9.39 \le d \le 9.55$$

The diameter of the basketball is from 9.39 to 9.55 inches. [1.5]

**68.**

$$300 = -45x^2 + 190x + 200$$
$$45x^2 - 190x + 100 = 0$$
$$9x^2 - 38x + 20 = 0$$
$$x = \frac{-(-38) \pm \sqrt{(-38)^2 - 4(9)(20)}}{2(9)}$$
$$= \frac{38 \pm \sqrt{724}}{18}$$
$$x \approx 0.6 \text{ or } x \approx 3.6$$

More than 0.6 mi but less than 3.6 mi from the city center. [1.5]

**69.**

$$A = \frac{km}{r^2}$$
$$9.8 = \frac{k(5.98\times 10^{26})}{(6,370,000)^2}$$
$$9.8(6,370,000)^2 = k(5.98\times 10^{26})$$
$$\frac{9.8(6,370,000)^2}{5.98\times 10^{26}} = k$$
$$k \approx 6.6497\times 10^{-13}$$
$$A = \frac{6.6497\times 10^{-13} m}{r^2}$$
$$A = \frac{(6.6497\times 10^{-13})(7.46\times 10^{24})}{(1,740,000)^2}$$
$$A \approx 1.64 \text{ meters/sec}^2$$

[1.6]

**70.**

$$L = \frac{kd^4}{h^2}$$
$$4 = \frac{k(1.5)^4}{8^2}$$
$$4(8^2) = k(1.5)^4$$
$$\frac{4(8^2)}{(1.5)^4} = k$$
$$k \approx 50.5679$$
$$L = \frac{50.5679 d^4}{h^2}$$
$$L = \frac{50.5679(4)^4}{12^2}$$
$$L \approx 89.9 \text{ tons}$$

[1.6]

## Quantitative Reasoning

**QR1.**

$$\frac{x}{1-x} = \frac{1}{x}$$
$$x^2 = 1-x$$
$$x^2 + x - 1 = 0$$
$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \frac{-1\pm\sqrt{1+4}}{2} = \frac{-1\pm\sqrt{5}}{2}$$

**QR2.**

$$EG = ED$$
$$EG = 1+x$$
$$EG = \sqrt{EF^2 + FG^2} = \sqrt{1^2+2^2} = \sqrt{5}$$
$$\sqrt{5} = 1+x$$
$$x = \sqrt{5} - 1$$
$$\frac{AD}{AB} = \frac{2+x}{2} = \frac{2+\sqrt{5}-1}{2} = \frac{1+\sqrt{5}}{2} = \phi$$

**QR3.** Answers will vary.

**QR4.** Answers will vary.

**QR5.** Answers will vary.

## Chapter Test

**1.** $3(2x-5)+1=-2(x-5)$ [1.1]

$$6x-15+1=-2x+10$$
$$6x-14=-2x+10$$
$$8x=24$$
$$x=3$$

**2.** $|x-3|=8$ [1.1]

$x-3=8$ or $x-3=-8$

$x=11$ $\quad$ $x=-5$

**3.** $6x^2-13x-8=(3x-8)(2x+1)=0$ [1.3]

$3x-8=0$ or $2x+1=0$

$x=\frac{8}{3}$ $\quad$ $x=-\frac{1}{2}$

**4.** $2x^2-8x+1=0 \Rightarrow x^2-4x \quad =-\frac{1}{2}$ [1.3]

$$x^2-4x+4=-\frac{1}{2}+4 \Rightarrow (x-2)^2=\frac{7}{2}$$
$$x-2=\pm\sqrt{\frac{7}{2}}=\pm\frac{\sqrt{7}}{\sqrt{2}}=\pm\frac{\sqrt{7\cdot 2}}{\sqrt{2\cdot 2}}=\pm\frac{\sqrt{14}}{2}$$
$$x=2\pm\frac{\sqrt{14}}{2}=\frac{4}{2}\pm\frac{\sqrt{14}}{2}=\frac{4\pm\sqrt{14}}{2}$$

**5.** $3x^2-5x-1=0$ [1.3]

$a=3,\ b=-5,\ c=-1$

$$x=\frac{-(-5)\pm\sqrt{(-5)^2-4(3)(-1)}}{2(3)}$$
$$=\frac{5\pm\sqrt{25+12}}{6}=\frac{5\pm\sqrt{37}}{6}$$

**6.** $2x^2+3x+1=0$ [1.3]

$a=2,\ b=3,\ c=1$

$b^2-4ac=(3)^2-4(2)(1)=9-8=1$

The discriminant, 1, is a positive number. Therefore, there are two real solutions.

**7.**

$$ax-c=c(x-d)$$
$$ax-c=cx-cd$$
$$ax-cx=c-cd$$
$$x(a-c)=c-cd$$
$$x=\frac{c-cd}{a-c},\ a\neq c \quad [1.2]$$

**8.**

$$\sqrt{x-2}-1=\sqrt{3-x}$$
$$\left(\sqrt{x-2}-1\right)^2=\left(\sqrt{3-x}\right)^2$$
$$x-2-2\sqrt{x-2}+1=3-x$$
$$2x-4=2\sqrt{x-2}$$
$$x-2=\sqrt{x-2}$$
$$(x-2)^2=\left(\sqrt{x-2}\right)^2$$
$$x^2-4x+4=x-2$$
$$x^2-5x+6=0$$
$$(x-3)(x-2)=0$$

$x-3=0 \Rightarrow x=3$

$x-2=0 \Rightarrow x=2$

Check $\sqrt{2-2}-1=\sqrt{3-2}$

$-1=1$ (No)

$\sqrt{3-2}-1=\sqrt{3-3}$

$1-1=0$

$0=0$

The solution is 3. [1.4]

**9.** $3x^{2/3}+10x^{1/3}-8=0$ [1.4]

Let $u=x^{1/3}$

$3u^2+10u-8=0$

$(3u-2)(u+4)=0$

$u=\frac{2}{3}$ or $u=-4$

$x^{1/3}=\frac{2}{3}$ $\quad$ $x^{1/3}=-4$

$\left(x^{1/3}\right)^3=\left(\frac{2}{3}\right)^3$ $\quad$ $\left(x^{1/3}\right)^3=(-4)^3$

$x=\frac{8}{27}$ $\quad$ $x=-64$

**10.** $\frac{3}{x+2}-\frac{3}{4}=\frac{5}{x+2}$ [1.4]

$$4(x+2)\left(\frac{3}{x+2}-\frac{3}{4}\right)=4(x+2)\left(\frac{5}{x+2}\right)$$

$$4(3)-3(x+2)=4(5)$$

$$12-3x-6=20$$

$$-3x=14$$

$$x=-\frac{14}{3}$$

**11.** **a.** $2x-5\le 11$ or $-3x+2>14$

$2x\le 16$ $\quad$ $-3x>12$

$x\le 8$ $\quad$ $x<-4$

$\{x\mid x\le 8\}\cup\{x\mid x<-4\}=\{x\mid x\le 8\}$ [1.5]

**b.** $2x-1<9$ and $-3x+1\le 7$

$2x<10$ $\quad$ $-3x\le 6$

$x<5$ $\quad$ $x\ge -2$

$\{x\mid x<5\}\cap\{x\mid x\ge -2\}=[-2,\ 5)$ [1.5]

**12.** $\frac{x^2+x-12}{x+1}\ge 0$

$\frac{(x+4)(x-3)}{x+1}\ge 0$

The quotient is positive or zero.

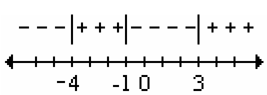
$x+4=0\Rightarrow x=-4$

$x-3=0\Rightarrow x=3$

$x+1=0\Rightarrow x=-1$

Critical values are –4, 3, and –1.

$\frac{(x+4)(x-3)}{x+1}$




Denominator $\ne 0\Rightarrow x\ne -1$.

$[-4,-1)\cup[3,\infty)$ [1.5]

**13.** $\left|x-11\frac{5}{32}\right|\le\frac{9}{32}$

$$-\frac{9}{32}\le x-11\frac{5}{32}\le\frac{9}{32}$$

$$10\frac{7}{8}\le x\le 11\frac{7}{16}$$

The range is from $10\frac{7}{8}$ in. to $11\frac{7}{16}$ in. [1.5]

**14.** Let $x$ = the rate of the current.

Rate with current $=5+x$.

Rate against current $=5-x$.

$d=rt$

$21=(5+x)t$ $\quad$ $9=(5-x)t$

$\frac{21}{5+x}=t$ $\quad$ $\frac{9}{5-x}=t$

$$\frac{21}{5+x}=\frac{9}{5-x}$$

$$21(5-x)=9(5+x)$$

$$105-21x=45+9x$$

$$60=30x$$

$$2=x$$

The current is 2 mph. [1.2]

**15.**

| | | |
|---|---|---|
| $x$ | 0.20 | Remove $x$ amount of 20% |
| $x$ | 1.00 | Add $x$ amount of 100% |

$$6(0.20)-x(0.20)+x(1.00)=6(0.50)$$

$$1.2+0.8x=3$$

$$0.8x=1.8$$

$x=2.25$ liters [1.2]

**16.** Let $x$ = number of hours the assistant needs to cover the parking lot.

$$6\left[\frac{1}{10}+\frac{1}{x}\right]=1$$
$$10x(6)\left[\frac{1}{10}+\frac{1}{x}\right]=10x(1)$$
$$6x+60=10x$$
$$-4x=-60$$
$$x=15$$

The assistant takes 15 hours to cover the parking lot. [1.4]

**17.** $10+0.18x>18+0.10x$

$$0.08x>8$$
$$x>100$$

If you drive more than 100 miles, then company A is less expensive. [1.5]

**18.** $0.5=-0.0002348x^2+0.0375x$

$$0.0002348x^2-0.0375x+0.5=0$$
$$x=\frac{-(-0.0375)\pm\sqrt{(-0.0375)^2-4(0.0002348)(0.5)}}{2(0.0002348)}$$
$$=\frac{0.0375\pm\sqrt{0.00094}}{0.00047}$$
$$=\frac{0.0375\pm0.0306}{0.0004696}$$

$x\approx145.0$ or $x\approx14.7$

More than 14.7 ft but less than 145.0 ft from a side line. [1.5]

**19.** $200=\dfrac{4500x}{2x^2+25}$

$$200(2x^2+25)=4500x$$
$$400x^2-4500x+5000=0$$
$$4x^2-45x+50=0$$
$$(x-10)(4x-5)=0$$

$x-10=0 \quad 4x-5=0$

$x=10 \quad x=1.25$

More than 1.25 mi but less than 10 mi from the city center. [1.5]

**20.** $v=\dfrac{k}{\sqrt{d}}$

$$4=\frac{k}{\sqrt{3000}}$$
$$k=4\sqrt{3000}=40\sqrt{30}$$
$$v=\frac{40\sqrt{30}}{\sqrt{2500}}=\frac{40\sqrt{30}}{50}$$
$$v=\frac{4\sqrt{30}}{5}\approx 4.4 \text{ miles/second} \quad [1.6]$$

## Cumulative Review

**1.** $4+3(-5)=4-15=-11$ [P.1]

**2.** $0.00017=1.7\times10^{-4}$ [P.2]

**3.** $(3x-5)^2-(x+4)(x-4)=(9x^2-30x+25)-(x^2-16)$ [P.3]

$$=9x^2-30x+25-x^2+16$$
$$=8x^2-30x+41$$

**4.** $8x^2+19x-15=(8x-5)(x+3)$ [P.4]

**5.** $\dfrac{7x-3}{x-4}-5=\dfrac{7x-3-5x+20}{x-4}=\dfrac{2x+17}{x-4}$ [P.5]

**6.** $a^{2/3}\cdot a^{1/4}=a^{2/3+1/4}=a^{11/12}$ [P.2]

**7.** $(2+5i)(2-5i)=4-25i^2=4+25=29$ [P.6]

**8.** $2(3x-4)+5=17$ [1.1]

$$2(3x-4)=12$$
$$6x=20$$
$$x=\frac{10}{3}$$

**9.** $2x^2 - 4x = 3$ [1.3]

$2x^2 - 4x - 3 = 0$

$x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)} = \dfrac{4 \pm 2\sqrt{10}}{4} = \dfrac{2 \pm \sqrt{10}}{2}$

**10.** $|2x - 6| = 4$ [1.1]

$2x - 6 = 4$ $\quad$ $2x - 6 = -4$

$2x = 10$ or $2x = 2$

$x = 5$ $\quad$ $x = 1$

**11.** $x = 3 + \sqrt{9 - x}$ [1.4]

$x - 3 = \sqrt{9 - x}$

$(x - 3)^2 = \left(\sqrt{9 - x}\right)^2$

$x^2 - 6x + 9 = 9 - x$

$x^2 - 5x = x(x - 5) = 0$

$x = 0$ or $x = 5$

Check 0:

$0 = 3 + \sqrt{9 - 0}$

$0 = 3 + \sqrt{9}$

$0 = 3 + 3$

$0 = 6$

No

Check 5:

$5 = 3 + \sqrt{9 - 5}$

$5 = 3 + \sqrt{4}$

$5 = 3 + 2$

$5 = 5$

The solution is 5.

**12.** $x^3 - 36x = x(x^2 - 36) = x(x + 6)(x - 6) = 0$

The solutions are 0, –6, 6. [1.4]

**13.** $2x^4 - 11x^2 + 15 = 0$ Let $u = x^2$.

$2u^2 - 11u + 15 = (2u - 5)(u - 3) = 0$

$2u - 5 = 0$ or $u - 3 = 0$

$u = x^2 = \dfrac{5}{2}$ $\quad$ $u = 3$

$x^2 = 3$

$x = \pm\sqrt{\dfrac{5}{2}} = \pm\dfrac{\sqrt{10}}{2}$ $\quad$ $x = \pm\sqrt{3}$

The solutions are $-\dfrac{\sqrt{10}}{2},\ \dfrac{\sqrt{10}}{2}, -\sqrt{3},\ \sqrt{3}$. [1.4]

**14.** $3x - 1 > 2$ or $-3x + 5 \ge 8$

$3x > 3$ $\quad$ $-3x \ge 3$

$x > 1$ $\quad$ $x \le -1$

The solution is $\{x \mid x \le -1 \text{ or } x > 1\}$. [1.5]

**15.** $|x - 6| \ge 2 \Rightarrow x - 6 \ge 2$ or $x - 6 \le -2$

$x \ge 8$ $\quad$ $x \le 4$

The solution is $(-\infty,\ 4] \cup [8,\ \infty)$. [1.5]

**16.** $\dfrac{x-2}{2x-3} \ge 4 \Rightarrow \dfrac{x-2}{2x-3} - 4 \ge 0 \Rightarrow \dfrac{x-2}{2x-3} - \dfrac{4(2x-3)}{2x-3} \ge 0 \Rightarrow \dfrac{x - 2 - 8x + 12}{2x-3} \ge 0 \Rightarrow \dfrac{-7x+10}{2x-3} \ge 0$

Solve $-7x + 10 = 0$ and $2x - 3 - 0$ to find the critical values.

$-7x + 10 = 0$ $\quad$ $2x - 3 = 0$

$x = \dfrac{10}{7}$ $\quad$ $x = \dfrac{3}{2}$

The critical values are $\dfrac{10}{7}$ and $\dfrac{3}{2}$. The intervals are $\left(-\infty,\ \dfrac{10}{7}\right)$, $\left(\dfrac{10}{7},\ \dfrac{3}{2}\right)$ and $\left(\dfrac{3}{2},\ \infty\right)$

Test 0, in the interval $\left(-\infty,\ \frac{10}{7}\right)$: $\dfrac{0-2}{2(0)-3} \ge 4 \Rightarrow \dfrac{-2}{-3} \ge 4 \Rightarrow \dfrac{2}{3} \ge 4$, which is false.

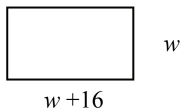
Test 1.45, in the interval $\left(\dfrac{10}{7},\ \dfrac{3}{2}\right)$: $\dfrac{1.45-2}{2(1.45)-3} \ge 4 \Rightarrow \dfrac{-0.55}{-0.1} \ge 4 \Rightarrow 5.5 \ge 4$, which is true.

Test 2, in the interval $\left(\dfrac{3}{2},\ \infty\right)$: $\dfrac{2-2}{2(2)-3} \ge 4 \Rightarrow \dfrac{0}{1} \ge 4 \Rightarrow 0 \ge 4$, which is false.

The denominator cannot equal zero $\Rightarrow x \ne \dfrac{3}{2}$.

The solution is $\left\{x \,\middle|\, \dfrac{10}{7} \le x < \dfrac{3}{2}\right\}$. [1.5]

**17.**




$$\begin{aligned} \text{Perimeter} &= 2(\text{Length}) + 2(\text{Width}) \\ 200 &= 2(w+16) + 2w \\ 200 &= 2w + 32 + 2w \\ 168 &= 4w \\ 42 &= w \\ w &= 42 \\ w + 16 &= 58 \end{aligned}$$

The width is 42 feet; the length is 58 feet. [1.2]

**18.**

$$\begin{aligned} P &= R - C \\ &= 200x - 0.004x^2 - (65x + 320{,}000) \\ &= -0.004x^2 + 135x - 320{,}000 \end{aligned}$$

Profits must be greater than or equal to 600,000.

$$-0.004x^2 + 135x - 320{,}000 \ge 600{,}000$$

$$-0.004x^2 + 135x - 920{,}000 \ge 0$$

$$\begin{aligned} x &= \frac{-135 \pm \sqrt{(135)^2 - 4(-0.004)(-920{,}000)}}{2(-0.004)} \\ &= \frac{-135 \pm \sqrt{3505}}{-0.008} \\ &= 9475 \text{ or } 24{,}275 \end{aligned}$$

9475 to 24,275 printers should be manufactured. [1.5]

**19.** Let $x$ = the score on the fourth test.

$$80 \le \frac{86 + 72 + 94 + x}{4} < 90 \quad \text{and} \quad 0 \le x \le 100$$

$$80 \le \frac{252 + x}{4} < 90$$

$$320 \le 252 + x < 360$$

$$68 \le x < 108$$

$$[68,\ 108) \cap [0,\ 100] = [68,\ 100]$$

The fourth test score must be from 68 to 100. [1.5]

**20.**

$$\begin{aligned} \frac{600p}{100 - p} &\ge 100 \\ 600p &\ge 100(100 - p) \\ 600p &\ge 10{,}000 - 100p \\ 700p &\ge 10{,}000 \\ p &\ge 14.3 \end{aligned}$$

and

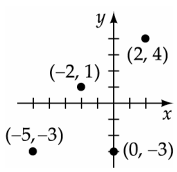
$$\begin{aligned} \frac{600p}{100 - p} &\le 180 \\ 600p &\le 180(100 - p) \\ 600p &\le 18{,}000 - 180p \\ 780p &\le 18{,}000 \\ p &\le 23.1 \end{aligned}$$

They can expect to ticket from 14.3% to 23.1% of the speeders. [1.5]

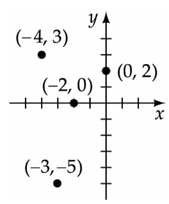
# Chapter 2
# Functions and Graphs

## Section 2.1

**1.**




**2.**




**3.** **a.** \$31,500

**b.** Increase from 2004 to 2005
$33.5-32.9=0.6$
Increase from 2005 to 2006
$33.50+0.6=34.1$
The per capita income for 2006 would be \$34,100.

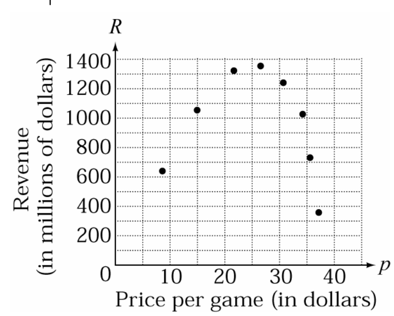
**c.** Percent increase from 2004 to 2005
$\frac{33.5-32.9}{32.9}=1.824\%$
Percent increase from 2005 to 2006
$33.50(1.01824)=34.111$
The per capita income for 2006 would be \$34,111.

**4.** **a.** When the cost of a game is \$22, 50 million games can be sold.

**b.** The projected numbers of sales decreases as the price of this game increases.

**c.**

| $p$ | $R=p\cdot N$ |
|---|---|
| 8 | $8\cdot 80=640$ |
| 15 | $15\cdot 70=1050$ |
| 22 | $22\cdot 60=1320$ |
| 27 | $27\cdot 50=1350$ |
| 31 | $31\cdot 40=1240$ |
| 34 | $34\cdot 30=1020$ |
| 36 | $36\cdot 20=720$ |
| 37 | $37\cdot 10=370$ |




**d.** The revenue increases to a certain point and then decreases as the price of the game increases.

**5.**
$$\begin{aligned} d &= \sqrt{(-8-6)^2+(11-4)^2} \\ &= \sqrt{(-14)^2+(7)^2} \\ &= \sqrt{196+49} \\ &= \sqrt{245} \\ &= 7\sqrt{5} \end{aligned}$$

**6.**
$$\begin{aligned} d &= \sqrt{(-10-(-5))^2+(14-8)^2} \\ &= \sqrt{(-5)^2+(6)^2} \\ &= \sqrt{25+36} \\ &= \sqrt{61} \end{aligned}$$

**7.**
$$\begin{aligned} d &= \sqrt{(-10-(-4))^2+(15-(-20))^2} \\ &= \sqrt{(-6)^2+(35)^2} \\ &= \sqrt{36+1225} \\ &= \sqrt{1261} \end{aligned}$$

**8.**
$$\begin{aligned} d &= \sqrt{(36-40)^2+(20-32)^2} \\ &= \sqrt{(-4)^2+(-12)^2} \\ &= \sqrt{16+144} \\ &= \sqrt{160} \\ &= 4\sqrt{10} \end{aligned}$$

**9.**
$$\begin{aligned} d &= \sqrt{(0-5)^2+(0-(-8))^2} \\ &= \sqrt{(-5)^2+(8)^2} \\ &= \sqrt{25+64} \\ &= \sqrt{89} \end{aligned}$$

**10.**
$$\begin{aligned} d &= \sqrt{(5-0)^2+(13-0)^2} \\ &= \sqrt{5^2+13^2} \\ &= \sqrt{25+169} \\ &= \sqrt{194} \end{aligned}$$

**11.**
$$\begin{aligned} d &= \sqrt{(\sqrt{12}-\sqrt{3})^2+(\sqrt{27}-\sqrt{8})^2} \\ &= \sqrt{(2\sqrt{3}-\sqrt{3})^2+(3\sqrt{3}-2\sqrt{2})^2} \\ &= \sqrt{(\sqrt{3})^2+(3\sqrt{3}-2\sqrt{2})^2} \\ &= \sqrt{3+(27-12\sqrt{6}+8)} \\ &= \sqrt{3+27-12\sqrt{6}+8} \\ &= \sqrt{38-12\sqrt{6}} \end{aligned}$$

**12.**
$$\begin{aligned} d &= \sqrt{(6-\sqrt{125})^2+(2\sqrt{5}-\sqrt{20})^2} \\ &= \sqrt{(6-5\sqrt{5})^2+(2\sqrt{5}-2\sqrt{5})^2} \\ &= \sqrt{(6-5\sqrt{5})^2+0^2} \\ &= \sqrt{(6-5\sqrt{5})^2} = \left|6-5\sqrt{5}\right| = 5\sqrt{5}-6 \end{aligned}$$

Note: for another form of the solution,

$$\begin{aligned} d &= \sqrt{(6-5\sqrt{5})^2} \\ &= \sqrt{36-60\sqrt{5}+125} = \sqrt{161-60\sqrt{5}} \end{aligned}$$

**13.**
$$\begin{aligned} d &= \sqrt{(-a-a)^2+(-b-b)^2} \\ &= \sqrt{(-2a)^2+(-2b)^2} \\ &= \sqrt{4a^2+4b^2} \\ &= \sqrt{4(a^2+b^2)} \\ &= 2\sqrt{a^2+b^2} \end{aligned}$$

**14.**
$$\begin{aligned} d &= \sqrt{(a-(a-b))^2+(a+b-b)^2} \\ &= \sqrt{(a-a+b)^2+(a)^2} \\ &= \sqrt{b^2+a^2} \\ &= \sqrt{a^2+b^2} \end{aligned}$$

**15.**
$$\begin{aligned} d &= \sqrt{(-2x-x)^2+(3x-4x)^2} \text{ with } x<0 \\ &= \sqrt{(-3x)^2+(-x)^2} \\ &= \sqrt{9x^2+x^2} \\ &= \sqrt{10x^2} \\ &= -x\sqrt{10} \quad (\text{Note: since } x<0, \sqrt{x^2}=-x) \end{aligned}$$

**16.**
$$\begin{aligned} d &= \sqrt{(-2x-x)^2+(3x-4x)^2} \text{ with } x>0 \\ &= \sqrt{(-3x)^2+(-x)^2} \\ &= \sqrt{9x^2+x^2} \\ &= \sqrt{10x^2} \\ &= x\sqrt{10} \quad (\text{since } x>0, \sqrt{x^2}=x) \end{aligned}$$

**17.**
$$\begin{aligned} \sqrt{(4-x)^2+(6-0)^2} &= 10 \\ \left(\sqrt{(4-x)^2+(6-0)^2}\right)^2 &= 10^2 \\ 16-8x+x^2+36 &= 100 \\ x^2-8x-48 &= 0 \\ (x-12)(x+4) &= 0 \end{aligned}$$

$x=12$ or $x=-4$

The points are (12, 0), ( – 4, 0).

**18.**
$$\begin{aligned} \sqrt{(5-0)^2+(y-(-3)^2} &= 12 \\ \left(\sqrt{(5)^2+(y+3)^2}\right)^2 &= 12^2 \\ 25+y^2+6y+9 &= 144 \\ y^2+6y-110 &= 0 \end{aligned}$$

$$y=\frac{-6\pm\sqrt{6^2-4(1)(-110)}}{2(1)}$$

$$y=\frac{-6\pm\sqrt{36+440}}{2}$$

$$y=\frac{-6\pm\sqrt{476}}{2}$$

$$y=\frac{-6\pm2\sqrt{119}}{2}$$

$$y=-3\pm\sqrt{119}$$

The points are $(0,\ -3+\sqrt{119}),\ (0,\ -3-\sqrt{119})$.

**19.** $M = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$
$= \left(\dfrac{1+5}{2}, \dfrac{-1+5}{2}\right)$
$= \left(\dfrac{6}{2}, \dfrac{4}{2}\right)$
$= (3, 2)$

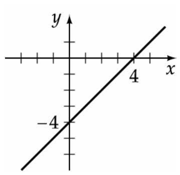
**20.** $M = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$
$= \left(\dfrac{-5+6}{2}, \dfrac{-2+10}{2}\right)$
$= \left(\dfrac{1}{2}, \dfrac{8}{2}\right)$
$= \left(\dfrac{1}{2}, 4\right)$

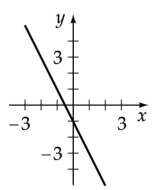
**21.** $M = \left(\dfrac{6+6}{2}, \dfrac{-3+11}{2}\right)$
$= \left(\dfrac{12}{2}, \dfrac{8}{2}\right)$
$= (6, 4)$

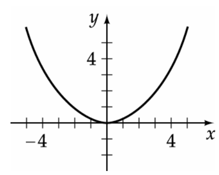
**22.** $M = \left(\dfrac{4+(-10)}{2}, \dfrac{7+7}{2}\right)$
$= \left(\dfrac{-6}{2}, \dfrac{14}{2}\right)$
$= (-3, 7)$

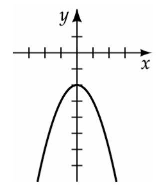
**23.** $M = \left(\dfrac{1.75+(-3.5)}{2}, \dfrac{2.25+5.57}{2}\right)$
$= \left(-\dfrac{1.75}{2}, \dfrac{7.82}{2}\right)$
$= (-0.875, 3.91)$

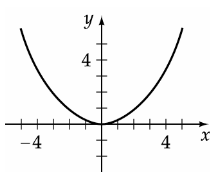
**24.** $\left(\dfrac{-8.2+(-2.4)}{2}, \dfrac{10.1+(-5.7)}{2}\right)$
$= \left(-\dfrac{10.6}{2}, \dfrac{4.4}{2}\right)$
$= (-5.3, 2.2)$

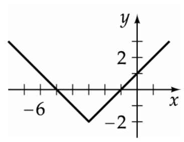
**25.**


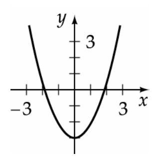

**26.**


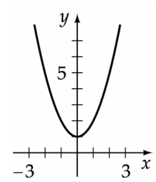

**27.**


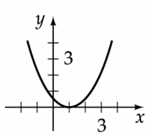

**28.**


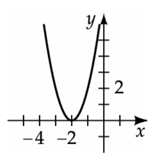

**29.**


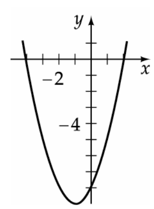

**30.**


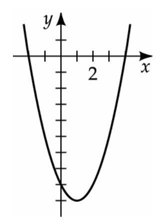

**31.**


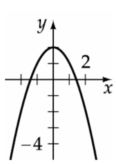

**32.**


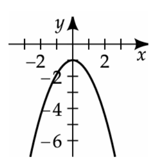

**33.**



**34.**


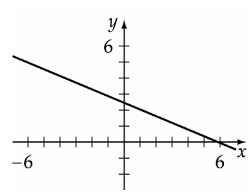

**35.**



**36.**


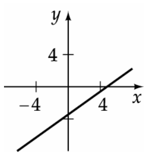

**37.**



**38.**


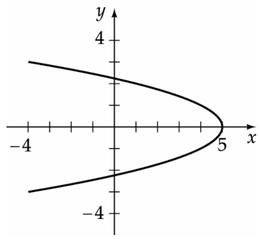

**39.** Intercepts: $\left(0, \dfrac{12}{5}\right), (6, 0)$



**40.** Intercepts: $\left(0, -\frac{15}{4}\right), (5, 0)$

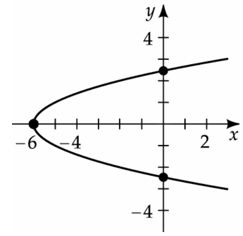


**41.** $\left(0, \sqrt{5}\right), \left(0, -\sqrt{5}\right), (5, 0)$



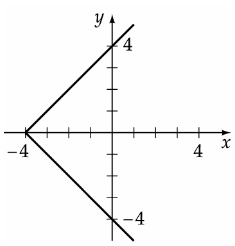
$x = -y^2 + 5$

**42.** $\left(0, \sqrt{6}\right), \left(0, -\sqrt{6}\right), (-6, 0)$



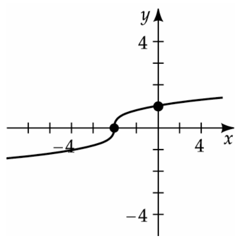
$x = y^2 - 6$

**43.** (0, 4), (0, –4), (–4, 0)



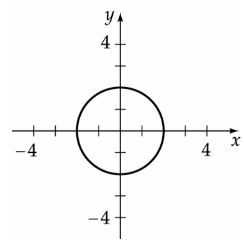
$x = |y| - 4$

**44.** $\left(0, \sqrt[3]{2}\right), (-2, 0)$



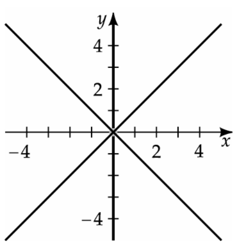
$x = y^3 - 2$

**45.** $(0, \pm 2), (\pm 2, 0)$



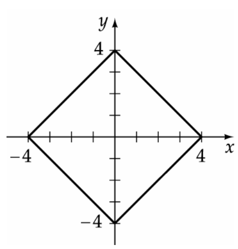
$x^2 + y^2 = 4$

**46.** (0, 0)



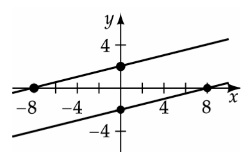
$x^2 = y^2$

**47.** $(0, \pm 4), (\pm 4, 0)$



$|x| + |y| = 4$

**48.** $(0, \pm 2), (\pm 8, 0)$



$|x - 4y| = 8$

**49.** center (0, 0), radius 6

**50.** center (0, 0), radius 7

**51.** center (1, 3), radius 7

**52.** center (2, 4), radius 5

**53.** center (–2, –5), radius 5

**54.** center (–3, –5), radius 11

**55.** center (8, 0), radius $\frac{1}{2}$

**56.** center (0, 12), radius 1

**57.** $(x-4)^2 + (y-1)^2 = 2^2$

**58.** $(x-5)^2 + (y+3)^2 = 4^2$

**59.** $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{4}\right)^2 = \left(\sqrt{5}\right)^2$

**60.** $(x-0)^2 + \left(y - \frac{2}{3}\right)^2 = \left(\sqrt{11}\right)^2$

**61.**
$$(x-0)^2 + (y-0)^2 = r^2$$
$$(-3-0)^2 + (4-0)^2 = r^2$$
$$(-3)^2 + 4^2 = r^2$$
$$9 + 16 = r^2$$
$$25 = 5^2 = r^2$$
$$(x-0)^2 + (y-0)^2 = 5^2$$

**62.**
$$(x-0)^2 + (y-0)^2 = r^2$$
$$(5-0)^2 + (12-0)^2 = r^2$$
$$5^2 + 12^2 = r^2$$
$$25 + 144 = r^2$$
$$169 = 13^2 = r^2$$
$$(x-0)^2 + (y-0)^2 = 13^2$$

**63.**
$$(x+2)^2+(y-5)^2=r^2$$
$$(x-1)^2+(y-3)^2=r^2$$
$$(4-1)^2+(-1-3)^2=r^2$$
$$3^2+(-4)^2=r^2$$
$$9+16=r^2$$
$$25=5^2=r^2$$
$$(x-1)^2+(y-3)^2=5^2$$

**64.**
$$(1+2)^2+(7-5)^2=r^2$$
$$3^2+2^2=r^2$$
$$9+4=r^2$$
$$13=\left(\sqrt{13}\right)^2=r^2$$
$$(x+2)^2+(y-5)^2=\left(\sqrt{13}\right)^2$$

**65.**
$$x^2-6x \quad +y^2=-5$$
$$x^2-6x+9+y^2=-5+9$$
$$(x-3)^2+y^2=2^2$$
center (3, 0), radius 2

**66.**
$$x^2-6x \quad +y^2-4y \quad =-12$$
$$x^2-6x+9+y^2-4y+4=-12+9+4$$
$$(x-3)^2+(y-2)^2=1^2$$
center (3, 2), radius 1

**67.**
$$x^2-14x \quad +y^2+8y \quad =-56$$
$$x^2-14x+49+y^2+8y+16=-56+49+16$$
$$(x-7)^2+(y+4)^2=3^2$$
center (7, –4), radius 3

**68.**
$$x^2-10x \quad +y^2+2y \quad =-25$$
$$x^2-10x+25+y^2+2y+1=-25+25+1$$
$$(x-5)^2+(y+1)^2=1^2$$
center (5, –1), radius 1

**69.**
$$4x^2+4x \quad +4y^2=63$$
$$x^2+\ x \quad +\ y^2=\frac{63}{4}$$
$$x^2+\ x+\ \frac{1}{4}\ +\ y^2=\frac{63}{4}+\frac{1}{4}$$
$$\left(x+\frac{1}{2}\right)^2+y^2=16$$
$$\left(x+\frac{1}{2}\right)^2+(y-0)^2=4^2$$
center $\left(-\frac{1}{2},0\right)$, radius 4

**70.**
$$9x^2+9y^2-6y \quad =17$$
$$x^2+y^2-\frac{2}{3}y \quad =\frac{17}{9}$$
$$x^2+y^2-\frac{2}{3}y\ +\frac{1}{9}=\frac{17}{9}+\frac{1}{9}$$
$$x^2+\left(y-\frac{1}{3}\right)^2=2$$
$$(x-0)^2+\left(y-\frac{1}{3}\right)^2=\left(\sqrt{2}\right)^2$$
center $\left(0,\frac{1}{3}\right)$, radius $\sqrt{2}$

**71.**
$$x^2-x \quad +y^2+\frac{3}{2}y \quad =\frac{15}{4}$$
$$x^2-x+\frac{1}{4}\ +y^2+\frac{3}{2}y\ +\frac{9}{4}\ =\frac{15}{4}+\frac{1}{4}+\frac{9}{4}$$
$$\left(x-\frac{1}{2}\right)^2+\left(y+\frac{3}{2}\right)^2=\left(\frac{5}{2}\right)^2$$
center $\left(\frac{1}{2},\ -\frac{3}{2}\right)$, radius $\frac{5}{2}$

**72.**
$$x^2+3x \quad +y^2-5y \quad =-\frac{25}{4}$$
$$x^2+3x+\frac{9}{4}\ +y^2-5y\ +\frac{25}{4}\ =-\frac{25}{4}+\frac{9}{4}+\frac{25}{4}$$
$$\left(x+\frac{3}{2}\right)^2+\left(y-\frac{5}{2}\right)^2=\left(\frac{3}{2}\right)^2$$
center $\left(-\frac{3}{2},\ \frac{5}{2}\right)$, radius $\frac{3}{2}$

**73.** $d=\sqrt{(-4-2)^2+(11-3)^2}$

$=\sqrt{36+64}$

$=\sqrt{100}$

$=10$

Since the diameter is 10, the radius is 5.
The center is the midpoint of the line segment from (2,3) to (-4,11).

$\left(\frac{2+(-4)}{2},\frac{3+11}{2}\right)=(-1,7)$ center

$(x+1)^2+(y-7)^2=5^2$

**74.** $d=\sqrt{(-3-7)^2+(5-(-2))^2}=\sqrt{100+49}=\sqrt{149}$

Since the diameter is $\sqrt{149}$, the radius is $\frac{\sqrt{149}}{2}$.

Center is $\left(\frac{7+(-3)}{2},\frac{(-2)+5}{2}\right)=\left(2,\frac{3}{2}\right)$

$$(x-2)^2+\left(y-\frac{3}{2}\right)^2=\left(\frac{\sqrt{149}}{2}\right)^2$$

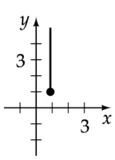
**75.** Since it is tangent to the $x$-axis, its radius is 11.

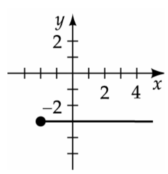
$(x-7)^2+(y-11)^2=11^2$

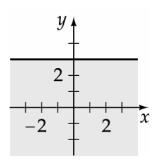
**76.** Since it is tangent to the $y$-axis, its radius is 2.

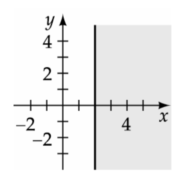
$(x+2)^2+(y-3)^2=2^2$

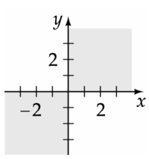
## Connecting Concepts

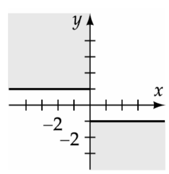
**77.**


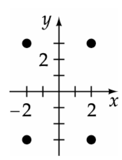

**78.**


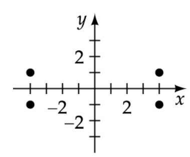

**79.**


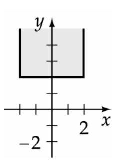

**80.**


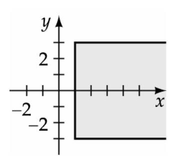

**81.**



**82.**



**83.**



**84.**



**85.**



**86.**



**87.** $\left(\frac{x+5}{2},\frac{y+1}{2}\right)=(9,3)$

therefore $\frac{x+5}{2}=9$ and $\frac{y+1}{2}=3$

$x+5=18$ $\quad y+1=6$

$x=13$ $\quad y=5$

Thus (13, 5) is the other endpoint.

**88.** $\left(\frac{x+4}{2},\frac{y+(-6)}{2}\right)=(-2,11)$

therefore $\frac{x+4}{2}=-2$ and $\frac{y+(-6)}{2}=11$

$x+4=-4$ $\quad y-6=22$

$x=-8$ $\quad y=28$

Thus (−8, 28) is the other endpoint.

**89.** $\left(\frac{x+(-3)}{2},\frac{y+(-8)}{2}\right)=(2,-7)$

therefore $\frac{x-3}{2}=2$ and $\frac{y-8}{2}=-7$

$x-3=4$ $\quad y-8=-14$

$x=7$ $\quad y=-6$

Thus (7, −6) is the other endpoint.

**90.** $\left(\frac{x+5}{2},\frac{y+(-4)}{2}\right)=(0,0)$

therefore $\frac{x+5}{2}=0$ and $\frac{y-4}{2}=0$

$x+5=0$ $\quad y-4=0$

$x=-5$ $\quad y=4$

Thus (−5, 4) is the other endpoint.

**91.**
$$\sqrt{(3-x)^2+(4-y)^2}=5$$
$$(3-x)^2+(4-y)^2=5^2$$
$$9-6x+x^2+16-18y+y^2=25$$
$$x^2-6x+y^2-8y=0$$

**92.**
$$\sqrt{(-5-x)^2+(12-y)^2}=13$$
$$(-5-x)^2+(12-y)^2=13^2$$
$$25+10x+x^2+144-24y+y^2=169$$
$$x^2+10x+y^2-24y=0$$

**93.**
$$\sqrt{(4-x)^2+(0-y)^2}+\sqrt{(-4-x)^2+(0-y)^2}=10$$
$$(4-x)^2+(0-y)^2=100-20\sqrt{(-4-x)^2+(0-y)^2}+(-4-x)^2+(-y)^2$$
$$16-8x+x^2+y^2=100-20\sqrt{(-4-x)^2+(-y)^2}+16+8x+x^2+y^2$$
$$-16x-100=-20\sqrt{(-4-x)^2+(-y)^2}$$
$$4x+25=5\sqrt{(-4-x)^2+(-y)^2}$$
$$16x^2+200x+625=25\left[(-4-x)^2+(-y)^2\right]$$
$$16x^2+200x+625=25\left[16+8x+x^2+y^2\right]$$
$$16x^2+200x+625=400+200x+25x^2+25y^2$$

Simplifying yields $9x^2+25y^2=225$.

**94.**
$$\left|\sqrt{(0-x)^2+(4-y)^2}-\sqrt{(0-x)^2+(-4-y)^2}\right|=6$$
$$\left(\sqrt{x^2+(4-y)^2}-\sqrt{x^2+(4+y)^2}\right)^2=6^2$$
$$x^2+(4-y)^2-2\sqrt{x^2+(4-y)^2}\sqrt{x^2+(4+y^2}+x^2+(4+y)^2=36$$
$$x^2+16-8y+y^2-2\sqrt{x^2+(4-y)^2}\sqrt{x^2+(4+y^2}+x^2+16+8y+y)^2=36$$
$$2x^2+2y^2-4=2\sqrt{x^2+(4-y)^2}\sqrt{x^2+(4+y)^2}$$
$$(x^2+y^2-2)^2=\left(\sqrt{x^2+(4-y)^2}\sqrt{x^2+(4+y)^2}\right)^2$$
$$x^4+x^2y^2-2x^2+x^2y^2+y^4-2y^2-2x^2-2y^2+4=(x^2+(4-y)^2)(x^2+(4+y)^2)$$
$$x^4+x^2y^2-4x^2-4y^2+y^4+4=(x^2+16-8y+y^2)(x^2+16+8y+y^2)$$
$$x^4+2x^2y^2-4x^2-4y^2+y^4+4=x^4+16x+8x^2y+x^2y^2+16x^2+256+128y+16y^2$$
$$-8x^2y-128y-64y^2-8y^3+x^2y^2+16y^2+8y^3+y^4$$
$$-36x^2+28y^2=252 \quad \text{or} \quad -9x^2+7y^2=63.$$

**95.** The center is (-3,3). The radius is 3.
$$(x+3)^2+(y-3)^2=3^2$$

**96.** The center is $\left(-\frac{\sqrt{5}}{2},-\frac{\sqrt{5}}{2}\right)$. The radius is $\frac{\sqrt{5}}{2}$.
$$\left(x+\frac{\sqrt{5}}{2}\right)^2+\left(y+\frac{\sqrt{5}}{2}\right)^2=\left(\frac{\sqrt{5}}{2}\right)^2$$

## Prepare for Section 2.2

**PS1.** $x^2+3x-4$
$$(-3)^2+3(-3)-4=9-9-4=-4$$

**PS2.** $D=\{-3,-2,-1,0,2\}$
$R=\{1,2,4,5\}$

**PS3.** $d=\sqrt{(3-(-4))^2+(-2-1)^2}=\sqrt{49+9}=\sqrt{58}$

**PS4.** $2x-6\geq 0$
$2x\geq 6$
$x\geq 3$

**PS5.** $x^2-x-6=0$
$(x+2)(x-3)=0$
$x+2=0 \quad x-3=0$
$x=-2 \quad x=3$
$-2, 3$

**PS6.** $a=3x+4,\quad a=6x-5$
$3x+4=6x-5$
$9=3x$
$3=x$
$a=3(3)+4=13$

## Section 2.2

**1.** Given $f(x)=3x-1$,

**a.** $f(2)=3(2)-1 = 6-1 = 5$

**b.** $f(-1)=3(-1)-1=-3-1=-4$

**c.** $f(0)=3(0)-1=0-1=-1$

**d.** $f\left(\frac{2}{3}\right)=3\left(\frac{2}{3}\right)-1=2-1=1$

**e.** $f(k)=3(k)-1=3k-1$

**f.** $f(k+2)=3(k+2)-1=3k+6-1=3k+5$

**2.** Given $g(x)=2x^2+3$,

**a.** $g(3)=2(3)^2+3=18+3=21$

**b.** $g(-1)=2(-1)^2+3=2+3=5$

**c.** $g(0)=2(0)^2+3=0+3=3$

**d.** $g\left(\frac{1}{2}\right)=2\left(\frac{1}{2}\right)^2+3=\frac{1}{2}+3=\frac{7}{2}$

**e.** $g(c)=2(c)^2+3=2c^2+3$

**f.** $g(c+5)=2(c+5)^2+3=2c^2+20c+50+3=2c^2+20c+53$

**3.** Given $A(w)=\sqrt{w^2+5}$,

**a.** $A(0)=\sqrt{(0)^2+5}=\sqrt{5}$

**b.** $A(2)=\sqrt{(2)^2+5}=\sqrt{9}=3$

**c.** $A(-2)=\sqrt{(-2)^2+5}=\sqrt{9}=3$

**d.** $A(4)=\sqrt{4^2+5}=\sqrt{21}$

**e.** $A(r+1)=\sqrt{(r+1)^2+5}=\sqrt{r^2+2r+1+5}=\sqrt{r^2+2r+6}$

**f.** $A(-c)=\sqrt{(-c)^2+5}=\sqrt{c^2+5}$

**4.** Given $J(t) = 3t^2 - t$,

**a.** $J(-4) = 3(-4)^2 - (-4)$
$= 48 + 4$
$= 52$

**b.** $J(0) = 3(0)^2 - (0)$
$= 0 - 0$
$= 0$

**c.** $J\left(\frac{1}{3}\right) = 3\left(\frac{1}{3}\right)^2 - \frac{1}{3}$
$= \frac{1}{3} - \frac{1}{3}$
$= 0$

**d.** $J(-c) = 3(-c)^2 - (-c)$
$= 3c^2 + c$

**e.** $J(x+1) = 3(x+1)^2 - (x+1)$
$= 3x^2 + 6x + 3 - x - 1$
$= 3x^2 + 5x + 2$

**f.** $J(x+h) = 3(x+h)^2 - (x+h)$
$= 3x^2 + 6xh + 3h^2 - x - h$

**5.** Given $f(x) = \frac{1}{|x|}$,

**a.** $f(2) = \frac{1}{|2|} = \frac{1}{2}$

**b.** $f(-2) = \frac{1}{|-2|} = \frac{1}{2}$

**c.** $f\left(-\frac{3}{5}\right) = \frac{1}{\left|-\frac{3}{5}\right|}$
$= \frac{1}{3/5}$
$= 1 \div \frac{3}{5} = 1 \cdot \frac{5}{3} = \frac{5}{3}$

**d.** $f(2) + f(-2) = \frac{1}{2} + \frac{1}{2} = 1$

**e.** $f(c^2 + 4) = \frac{1}{|c^2 + 4|} = \frac{1}{c^2 + 4}$

**f.** $f(2 + h) = \frac{1}{|2 + h|}$

**6.** Given $T(x) = 5$,

**a.** $T(-3) = 5$

**b.** $T(0) = 5$

**c.** $T\left(\frac{2}{7}\right) = 5$

**d.** $T(3) + T(1) = 5 + 5 = 10$

**e.** $T(x + h) = 5$

**f.** $T(3k + 5) = 5$

**7.** Given $s(x) = \frac{x}{|x|}$,

**a.** $s(4) = \frac{4}{|4|} = \frac{4}{4} = 1$

**b.** $s(5) = \frac{5}{|5|} = \frac{5}{5} = 1$

**c.** $s(-2) = \frac{-2}{|-2|} = \frac{-2}{2} = -1$

**d.** $s(-3) = \frac{-3}{|-3|} = \frac{-3}{3} = -1$

**e.** Since $t > 0, |t| = t$.
$s(t) = \frac{t}{|t|} = \frac{t}{t} = 1$

**f.** Since $t < 0, |t| = -t$.
$s(t) = \frac{t}{|t|} = \frac{t}{-t} = -1$

**8.** Given $r(x) = \frac{x}{x+4}$,

**a.** $r(0) = \frac{0}{0+4} = \frac{0}{4} = 0$

**b.** $r(-1) = \frac{-1}{-1+4} = \frac{-1}{3} = -\frac{1}{3}$

**c.** $r(-3) = \frac{-3}{-3+4} = \frac{-3}{1} = -3$

**d.** $r\left(\frac{1}{2}\right) = \frac{\frac{1}{2}}{\frac{1}{2} + 4} = \frac{\left(\frac{1}{2}\right)}{\left(\frac{9}{2}\right)}$
$= \frac{1}{2} \div \frac{9}{2} = \frac{1}{2} \cdot \frac{2}{9} = \frac{1}{9}$

**e.** $r(0.1) = \frac{0.1}{0.1 + 4} = \frac{0.1}{4.1} = \frac{1}{41}$

**f.** $r(10{,}000) = \frac{10.000}{10{,}000 + 4}$
$= \frac{10{,}000}{10{,}004} = \frac{2500}{2501}$

**9.** **a.** Since $x = -4 < 2$, use $P(x) = 3x + 1$.
$P(-4) = 3(-4) + 1 = -12 + 1 = -11$

**b.** Since $x = \sqrt{5} \ge 2$, use $P(x) = -x^2 + 11$.
$P(\sqrt{5}) = -(\sqrt{5})^2 + 11 = -5 + 11 = 6$

**c.** Since $x = c < 2$, use $P(x) = 3x + 1$.
$P(c) = 3c + 1$

**d.** Since $k \ge 1$, then $x = k + 1 \ge 2$,
so use $P(x) = -x^2 + 11$.
$$P(k+1) = -(k+1)^2 + 11 = -(k^2 + 2k + 1) + 11$$
$$= -k^2 - 2k - 1 + 11$$
$$= -k^2 - 2k + 10$$

**10.** **a.** Since $t = -4$ and $0 \le t \le 5$, use $Q(t) = 4$.
$Q(0) = 4$

**b.** Since $t = e$ and $6 < e < 7$, then $5 < t \le 8$,
so use $Q(t) = -t + 9$.
$Q(e) = -e + 9$

**c.** Since $t = n$ and $1 < n < 2$, then $0 \le t \le 5$,
so use $Q(t) = 4$
$Q(0) = 4$

**d.** Since $t = m^2 + 7$ and $1 < m \le 2$,
then $1^2 < m^2 \le 2^2$
$1^2 + 7 < m^2 + 7 \le 2^2 + 7$
$1 + 7 < m^2 + 7 \le 4 + 7$
$8 < m^2 + 7 \le 11$
thus $8 < t \le 11$,
so use $Q(t) = \sqrt{t - 7}$
$$Q(m^2 + 7) = \sqrt{(m^2 + 7) - 7}$$
$$= \sqrt{m^2} = |m| = m \text{ since } m > 0$$

**11.** $2x + 3y = 7$
$3y = -2x + 7$
$y = -\frac{2}{3}x + \frac{7}{3}$, $y$ is a function of $x$.

**12.** $5x + y = 8$
$y = -5x + 8$, $y$ is a function of $x$.

**13.** $-x + y^2 = 2$
$y^2 = x + 2$
$y = \pm\sqrt{x + 2}$, $y$ is a not function of $x$.

**14.** $x^2 - 2y = 2$
$-2y = -x^2 + 2$
$y = \frac{1}{2}x^2 - 1$, $y$ is a function of $x$.

**15.** $y = 4 \pm \sqrt{x}$, $y$ is not a function of $x$ since for each $x > 0$ there are two values of $x$.

**16.** $x^2 + y^2 = 9$
$y^2 = 9 - x^2$
$y = \pm\sqrt{9 - x^2}$, $y$ is a not function of $x$.

**17.** $y = \sqrt[3]{x}$, $y$ is a function of $x$.

**18.** $y = |x| + 5$, $y$ is a function of $x$.

**19.** $y^2 = x^2$
$y = \pm\sqrt{x^2}$, $y$ is a not function of $x$.

**20.** $y^3 = x^3$
$y = \sqrt[3]{x^3}$
$= x$, $y$ is a function of $x$.

**21.** Function; each $x$ is paired with exactly one $y$.

**22.** Not a function; 5 is paired with 10 and 8.

**23.** Function; each $x$ is paired with exactly one $y$.

**24.** Function; each $x$ is paired with exactly one $y$.

**25.** Function; each $x$ is paired with exactly one $y$.

**26.** Function; each $x$ is paired with exactly one $y$.

**27.** $f(x) = 3x - 4$ Domain is the set of all real numbers.

**28.** $f(x) = -2x + 1$ Domain is the set of all real numbers.

**29.** $f(x) = x^2 + 2$ Domain is the set of all real numbers.

**30.** $f(x) = 3x^2 + 1$ Domain is the set of all real numbers.

**31.** $f(x)=\dfrac{4}{x+2}$ Domain is $\{x \mid x\neq -2\}$.

**32.** $f(x)=\dfrac{6}{x-5}$ Domain is $\{x \mid x\neq 5\}$.

**33.** $f(x)=\sqrt{7+x}$ Domain is $\{x \mid x\geq -7\}$.

**34.** $f(x)=\sqrt{4-x}$ Domain is $\{x \mid x\leq 4\}$.

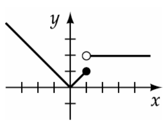
**35.** $f(x)=\sqrt{4-x^2}$ Domain is $\{x \mid -2\leq x\leq 2\}$.

**36.** $f(x)=\sqrt{12-x^2}$ Domain is $\{x \mid -2\sqrt{3}\leq x\leq 2\sqrt{3}\}$

**37.** $f(x)=\dfrac{1}{\sqrt{x+4}}$ Domain is $\{x \mid x>-4\}$.

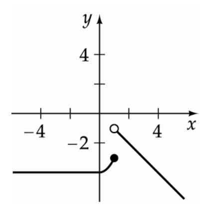
**38.** $f(x)=\dfrac{1}{\sqrt{5-x}}$ Domain is $\{x \mid x<5\}$.

**39.**



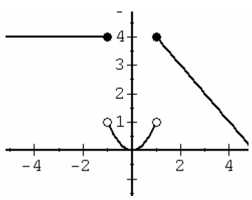

Domain: the set of all real numbers

**40.**



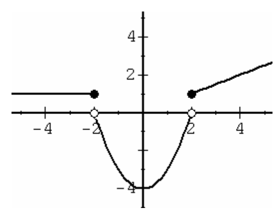

Domain: the set of all real numbers

**41.**



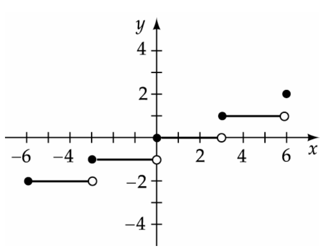

Domain: the set of all real numbers

**42.**



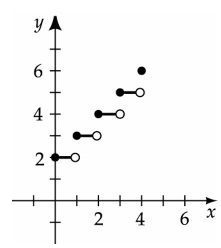

Domain: the set of all real numbers

**43.**



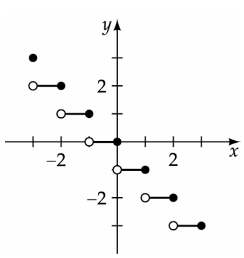

Domain: $\{x \mid -6\leq x\leq 6\}$

**44.**



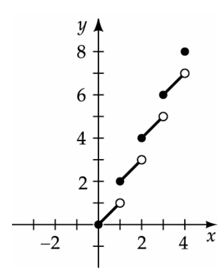

Domain: $\{x \mid 0\leq x\leq 4\}$

**45.**




Domain: $\{x \mid -3\leq x\leq 3\}$

**46.**




Domain: $\{x \mid 0\leq x\leq 4\}$

**47.** $\dfrac{\text{int}\left[10^2(2.3458)+0.5\right]}{10^2}=\dfrac{\text{int}[235.08]}{100}=\dfrac{235}{100}=2.35$

**48.** $\dfrac{\text{int}\left[10(34.567)+0.5\right]}{10}=\dfrac{\text{int}[346.17]}{10}=\dfrac{346}{10}=34.6$

**49.** $\dfrac{\text{int}\left[10^3(34.05622)+0.5\right]}{10^3}=\dfrac{\text{int}[34{,}056.72]}{1000}=\dfrac{34{,}056}{1000}=34.056$

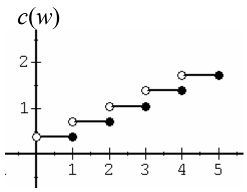
**50.** $\dfrac{\text{int}\left[10^0(109.83)+0.5\right]}{10^0}=\dfrac{\text{int}[110.33]}{1}=\dfrac{110}{100}=110$

**51.** $\dfrac{\text{int}\left[10^4(0.08951)+0.5\right]}{10^4}=\dfrac{\text{int}[895.6]}{10{,}000}=\dfrac{895}{10{,}000}=0.0895$

**52.** $\dfrac{\text{int}\left[10^3(2.98245)+0.5\right]}{10^3}=\dfrac{\text{int}[2982.95]}{1000}=\dfrac{2982}{1000}=2.982$

**53.** **a.**
$$\begin{aligned}C(2.8)&=0.39-0.34\text{int}(1-2.8)\\&=0.39-0.34\text{int}(-1.8)\\&=0.39-0.34(-2)\\&=0.39+0.68\\&=\$1.07\end{aligned}$$

**b.**




**54.** **a.** Domain: $[0, \infty)$

**b.**
$$\begin{aligned}T(31{,}250)&=0.25(31{,}250-30{,}650)+4220\\&=0.25(600)+4220\\&=150+4220\\&=\$4370\end{aligned}$$

**c.**
$$\begin{aligned}T(78{,}900)&=0.28(78{,}900-74{,}200)+15{,}107.50\\&=0.28(4700)+15{,}107.50\\&=1316+15{,}107.50\\&=\$16{,}423.50\end{aligned}$$

**55.** **a.** Yes; every vertical line intersects the graph in one point.
**b.** Yes; every vertical line intersects the graph in one point.
**c.** No; some vertical lines intersect the graph at more than one point.
**d.** Yes; every vertical line intersects the graph in at most one point.

**56.** **a.** Yes; every vertical line intersects the graph in at most one point.
**b.** No; some vertical lines intersect the graph at more than one point.
**c.** No; a vertical line intersects the graph at more than one point.
**d.** Yes; every vertical line intersects the graph in one point.

**57.** Decreasing on $(-\infty, 0]$; increasing on $[0, \infty)$

**58.** Decreasing on $(-\infty, \infty)$

**59.** Increasing on $(-\infty, \infty)$

**60.** Increasing on $(-\infty, 2]$; decreasing on $[2, \infty)$

**61.** Decreasing on $(-\infty, -3]$; increasing on $[-3, 0]$; decreasing on $[0, 3]$; increasing on $[3, \infty)$

**62.** Increasing on $(-\infty, \infty)$

**63.** Constant on $(-\infty, 0]$; increasing on $[0, \infty)$

**64.** Constant on $(-\infty, \infty)$

**65.** Decreasing on $(-\infty, 0]$; constant on [0, 1]; increasing on $[1, \infty)$

**66.** Constant on $(-\infty, 0]$; decreasing on $[0, 3]$; constant on $[3, \infty)$

**67.** *g* and *F* are one-to-one since every horizontal line intersects the graph at one point.
*f*, *V*, and *p* are not one-to-one since some horizontal lines intersect the graph at more than one point.

**68.** *s* is one-to-one since every horizontal line intersects the graph at one point.
*t*, *m*, *r* and *k* are not one-to-one since some horizontal lines intersect the graph at more than one point.

**69.** **a.**
$$2l + 2w = 50$$
$$2w = 50 - 2l$$
$$w = 25 - l$$

**b.**
$$A = lw$$
$$A = l(25 - l)$$
$$A = 25l - l^2$$

**70.** **a.**
$$\frac{4}{l} = \frac{12}{d + l}$$
$$4(d + l) = 12l$$
$$4d + 4l = 12l$$
$$4d = 8l$$
$$\frac{1}{2}d = l$$
$$l(d) = \frac{1}{2}d$$

**b.** Domain: $[0, \infty)$

**c.** $l(8) = \frac{1}{2}(8) = 4$ ft

**71.** $v(t) = 80{,}000 - 6500t, \quad 0 \le t \le 10$

**72.** $v(t) = 44{,}000 - 4200t, \quad 0 \le t \le 8$

**73.** **a.**
$$C(x) = 5(400) + 22.80x$$
$$= 2000 + 22.80x$$

**b.** $R(x) = 37.00x$

**c.**
$$P(x) = 37.00x - C(x)$$
$$= 37.00 - [2000 + 22.80x]$$
$$= 37.00x - 2000 - 22.80x$$
$$= 14.20x - 2000$$

Note $x$ is a natural number.

**74.** **a.**
$$V = lwh$$
$$V = (30 - 2x)(30 - 2x)(x)$$
$$V = (900 - 120x + 4x^2)(x)$$
$$V = 900x - 120x^2 + 4x^3$$

**b.** $V = lwh \Rightarrow$ the domain of $V$ is dependent on the domains of $l$, $w$, and $h$. Length, width and height must be positive values $\Rightarrow$ $30 - 2x > 0$ and $x > 0$.
$$-2x > -30$$
$$x < 15$$
Thus, the domain of $V$ is $\{x \mid 0 < x < 15\}$.

**75.**
$$\frac{15}{3} = \frac{15 - h}{r}$$
$$5 = \frac{15 - h}{r}$$
$$5r = 15 - h$$
$$h = 15 - 5r$$
$$h(r) = 15 - 5r$$

**76.** **a.**
$$\frac{r}{h} = \frac{2}{4}$$
$$r = \frac{2}{4}h$$
$$r = \frac{1}{2}h$$

**b.**
$$V = \frac{1}{3}\pi r^2 h$$
$$V = \frac{1}{3}\pi \left(\frac{1}{2}h\right)^2 h = \frac{1}{3}\pi \left(\frac{1}{4}h^2\right)h$$
$$V = \frac{1}{12}\pi h^3$$

**77.**
$$d = \sqrt{(3t)^2 + (50)^2}$$
$$d = \sqrt{9t^2 + 2500} \text{ meters}, \ 0 \le t \le 60$$

**78.**
$$t = \frac{d}{r}$$
$$t = \frac{\sqrt{1 + x^2}}{2} + \frac{3 - x}{8} \text{ hours}$$

**79.** $d = \sqrt{(45-8t)^2 + (6t)^2}$ miles
where $t$ is the number of hours after 12:00 noon

**80.** **a.**
$$A = xy$$
$$A(x) = x\left(-\frac{1}{2}x + 4\right)$$
$$A(x) = -\frac{1}{2}x^2 + 4x$$

**b.**

| $x$ | 1 | 2 | 4 | 6 | 7 |
|---|---|---|---|---|---|
| Area | 3.5 | 6 | 8 | 6 | 3.5 |

**c.** Domain: [0, 8].

**81.** **a.**

Circle
$$C = 2\pi r$$
$$x = 2\pi r$$
$$r = \frac{x}{2\pi}$$
$$\text{Area} = \pi r^2 = \pi\left(\frac{x}{2\pi}\right)^2 = \frac{x^2}{4\pi}$$

Square
$$C = 4s$$
$$20 - x = 4s$$
$$s = 5 - \frac{x}{4}$$
$$\text{Area} = s^2 = \left(5 - \frac{x}{4}\right)^2 = 25 - \frac{5}{2}x + \frac{x^2}{16}$$

$$\text{Total Area} = \frac{x^2}{4\pi} + 25 - \frac{5}{2}x + \frac{x^2}{16} = \left(\frac{1}{4\pi} + \frac{1}{16}\right)x^2 - \frac{5}{2}x + 25$$

**b.**

| $x$ | 0 | 4 | 8 | 12 | 16 | 20 |
|---|---|---|---|---|---|---|
| Total Area | 25 | 17.27 | 14.09 | 15.46 | 21.37 | 31.83 |

**c.** Domain: [0, 20].

**82.** **a.**
$$m_{PB} = \frac{0-2}{x-2} = \frac{-2}{x-2}$$
$$m_{AB} = \frac{0-y}{x-0} = \frac{-y}{x}$$
$$m_{PB} = m_{AB}$$
$$\frac{-2}{x-2} = \frac{-y}{x}$$
$$\frac{2x}{x-2} = y$$
$$\text{Area} = \frac{1}{2}bh = \frac{1}{2}xy = \frac{1}{2}x\frac{2x}{x-2} = \frac{x^2}{x-2}$$

**b.** Domain: $(2, \infty)$

**83.** **a.**

Left side triangle
$$c^2 = 20^2 + (40-x)^2$$
$$c = \sqrt{400 + (40-x)^2}$$

Right side triangle
$$c^2 = 30^2 + x^2$$
$$c = \sqrt{900 + x^2}$$

$$\text{Total length} = \sqrt{900 + x^2} + \sqrt{400 + (40-x)^2}$$

**b.**

| $x$ | 0 | 10 | 20 | 30 | 40 |
|---|---|---|---|---|---|
| Total Length | 74.72 | 67.68 | 64.34 | 64.79 | 70 |

**c.** Domain: [0, 40].

**84.**

| $p$ | 40 | 50 | 60 | 75 | 90 |
|---|---|---|---|---|---|
| $f(p)$ | 4900 | 4300 | 3800 | 3200 | 2800 |

answers accurate to nearest 100 feet

**85.**

| $x$ | 5 | 10 | 12.5 | 15 | 20 |
|---|---|---|---|---|---|
| $Y(x)$ | 275 | 375 | 385 | 390 | 394 |

answers accurate to nearest apple

**86.**

| $x$ | 100 | 200 | 500 | 750 | 1000 |
|---|---|---|---|---|---|
| $C(x)$ | 57,121 | 59,927 | 65,692 | 69,348 | 72,507 |

answers accurate to nearest dollar

**87.** $f(c) = c^2 - c - 5 = 1$

$$c^2 - c - 6 = 0$$
$$(c-3)(c+2) = 0$$
$$c - 3 = 0 \quad \text{or} \quad c + 2 = 0$$
$$c = 3 \qquad\qquad c = -2$$

**88.** $g(c) = -2c^2 + 4c - 1 = -4$

$$= -2c^2 + 4c + 3 = 0$$
$$c = \frac{-4 \pm \sqrt{4^2 - 4(-2)(3)}}{2(-2)}$$
$$c = \frac{-4 \pm \sqrt{16+24}}{-4}$$
$$c = \frac{-4 \pm \sqrt{40}}{-4}$$
$$c = \frac{-4 \pm 2\sqrt{10}}{-4}$$
$$c = \frac{2 \pm \sqrt{10}}{2}$$

**89.** 1 is not in the range of $f(x)$, since

$1 = \dfrac{x-1}{x+1}$ only if $x + 1 = x - 1$ or $1 = -1$.

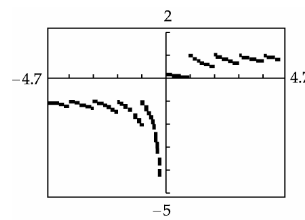
**90.** 0 is not in the range of $g(x)$, since

$0 = \dfrac{1}{x-3}$ only if $(x-3)(0) = 1$ or $0 = 1$.

**91.** Set the graphing utility to "dot" mode.

```
Y1=int X/abs X
Y2=
Y3=
Y4=
Y5=
Y6=
Y7=
Y8=
```

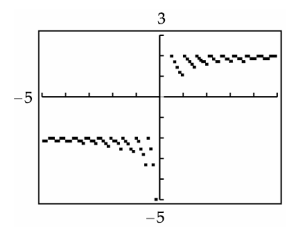
```
WINDOW FORMAT
  Xmin=-4.7
  Xmax=4.7
  Xscl=1
  Ymin=-5
  Ymax=2
  Yscl=1
```




**92.** Set the graphing utility to "dot" mode.

```
Y1=int 2X/abs X
Y2=
Y3=
Y4=
Y5=
Y6=
Y7=
Y8=
```

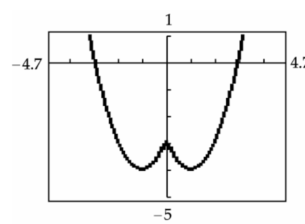
```
WINDOW FORMAT
  Xmin=-5
  Xmax=5
  Xscl=1
  Ymin=-5
  Ymax=3
  Yscl=1
```




**93.**

```
Y1=X²-2abs X-3
Y2=
Y3=
Y4=
Y5=
Y6=
Y7=
Y8=
```

```
WINDOW FORMAT
  Xmin=-4.7
  Xmax=4.7
  Xscl=1
  Ymin=-5
  Ymax=1
  Yscl=1
```

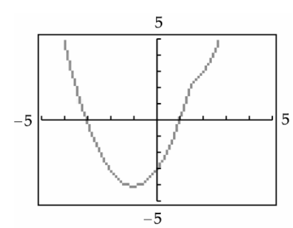



**94.**

```
Y1=X²-abs (2X-3)

Y2=
Y3=
Y4=
Y5=
Y6=
Y7=
```

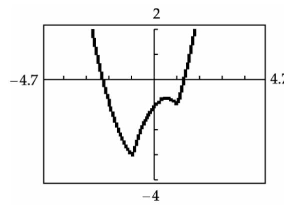
```
WINDOW FORMAT
  Xmin=-5
  Xmax=5
  Xscl=1
  Ymin=-5
  Ymax=5
  Yscl=1
```



**95.**

```
Y1=abs (X²-1)-
abs (X-2)
Y2=
Y3=
Y4=
Y5=
Y6=
Y7=
```

```
WINDOW FORMAT
  Xmin=-4.7
  Xmax=4.7
  Xscl=1
  Ymin=-4.7
  Ymax=4.7
  Yscl=1
```

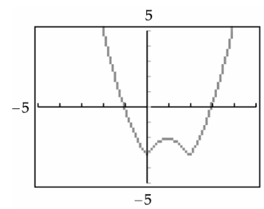



**96.**

```
Y1=abs (X²-2X)-3

Y2=
Y3=
Y4=
Y5=
Y6=
Y7=
```

```
WINDOW FORMAT
  Xmin=-4.7
  Xmax=4.7
  Xscl=1
  Ymin=-4.7
  Ymax=4.7
  Yscl=1
```




## Connecting Concepts

**97.** $f(x)\Big|_2^3 = (9-3)-(4-2) = 6-2 = 4$

**98.** $f(x)\Big|_4^7 = (-21+2)-(-12+2)$

$= -19-(-10) = -19+10 = -9$

**99.** $f(x)\Big|_0^2 = (16-12-2)-0 = 2$

**100.** $f(x)\Big|_0^8 = 0-\sqrt{8} = -\sqrt{8} = -2\sqrt{2}$

**101.**
**a.** $f(1,7) = 3(1)+5(7)-2 = 3+35-2 = 36$
**b.** $f(0,3) = 3(0)+5(3)-2 = 13$
**c.** $f(-2,4) = 3(-2)+5(4)-2 = 12$
**d.** $f(4,4) = 3(4)+5(4)-2 = 30$
**e.** $f(k,2k) = 3(k)+5(2k)-2 = 13k-2$
**f.** $f(k+2,k-3) = 3(k+2)+5(k-3)-2 = 3k+6+5k-15-2 = 8k-11$

**102.**
**a.** $g(3,-4) = 2(3)^2-|-4|+3 = 18-4+3 = 17$
**b.** $g(-1,2) = 2(-1)^2-|2|+3 = 2-2+3 = 3$
**c.** $g(0,-5) = 2(0)^2-|-5|+3 = -2$
**d.** $g\left(\frac{1}{2},-\frac{1}{4}\right) = 2\left(\frac{1}{2}\right)^2-\left|-\frac{1}{4}\right|+3 = \frac{1}{2}-\frac{1}{4}+3 = \frac{13}{4}$
**e.** $g(c,3c) = 2(c)^2-|3c|+3 = 2c^2-3c+3 \quad (|3c| = 3c \text{ since } c>0)$
**f.** $g(c+5,c-2) = 2(c+5)^2-|c-2|+3 \quad (\text{Since } c<0,\ c-2<0)$

$= 2c^2+20c+50-(-(c-2))+3 \quad (\text{Thus } |c-2| = -c+2)$

$= 2c^2+20c+50+c-2+3 = 2c^2+21c+51$

**103.** $s = \dfrac{5+8+11}{2} = 12$

$A(5,8,11) = \sqrt{12(12-5)(12-8)(12-11)}$

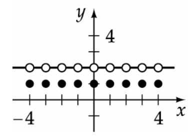
$= \sqrt{12(7)(4)(1)} = \sqrt{336} = 4\sqrt{21}$

**104.** $C(18,11) = 15(18)+14(11) = 270+154 = \$424$

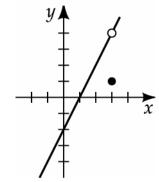
**105.**
$$a^2+3a-3=a$$
$$a^2+2a-3=0$$
$$(a-1)(a+3)=0$$
$$a=1 \quad \text{or} \quad a=-3$$

**106.**
$$\frac{a}{a+5}=a$$
$$a=a(a+5)$$
$$a=a^2+5a$$
$$0=a^2+4a$$
$$0=a(a+4)$$
$$a=0 \quad \text{or} \quad a=-4$$

**107.**




**108.**




## Prepare for Section 2.3

**PS1.** $d=5-(-2)=7$

**PS2.** The product of any number and its negative reciprocal is –1.
$$-7\cdot\frac{1}{7}=-1$$

**PS3.** $\frac{-4-4}{2-(-3)}=\frac{-8}{5}$

**PS4.**
$$y-3=-2(x-3)$$
$$y-3=-2x+6$$
$$y=-2x+9$$

**PS5.**
$$3x-5y=15$$
$$-5y=-3x+15$$
$$y=\frac{3}{5}x-3$$

**PS6.**
$$y=3x-2(5-x)$$
$$0=3x-2(5-x)$$
$$0=3x-10+2x$$
$$10=5x$$
$$2=x$$

## Section 2.3

**1.** $m=\frac{y_2-y_1}{x_2-x_1}=\frac{7-4}{1-3}=\frac{3}{-2}=-\frac{3}{2}$

**2.** $m=\frac{1-4}{5-(-2)}=\frac{-3}{7}=-\frac{3}{7}$

**3.** $m=\frac{2-0}{0-4}=-\frac{1}{2}$

**4.** $m=\frac{4-4}{2-(-3)}=\frac{0}{5}=0$

**5.** $m=\frac{4-0}{0-0}=\frac{4}{0}$ undefined

**6.** $m=\frac{0-0}{3-0}=\frac{0}{3}=0$

**7.** $m=\frac{-2-4}{-4-(-3)}=\frac{-6}{-1}=6$

**8.** $m=\frac{4-(-1)}{-3-(-5)}=\frac{5}{2}$

**9.** $m=\frac{\frac{7}{2}-\frac{1}{2}}{\frac{7}{3}-(-4)}=\frac{\frac{6}{2}}{\frac{19}{3}}=3\cdot\frac{3}{19}=\frac{9}{19}$

**10.** $m=\frac{2-4}{\frac{7}{4}-\frac{1}{2}}=\frac{-2}{\frac{5}{4}}=-\frac{8}{5}$

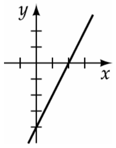
**11.** $m=\frac{f(3+h)-f(3)}{3+h-3}=\frac{f(3+h)-f(3)}{h}$

**12.** $m=\frac{f(-2+h)-f(-2+h)}{-2+h-(-2)}=\frac{0}{h}=0$

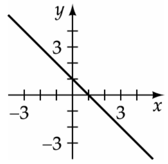
**13.** $m=\frac{f(h)-f(0)}{h-0}=\frac{f(h)-f(0)}{h}$

**14.** $m=\frac{f(a+h)-f(a)}{a+h-a}=\frac{f(a+h)-f(a)}{h}$

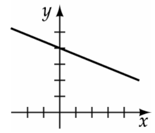
**15.** $m = 2$
$y$-intercept $(0, -4)$



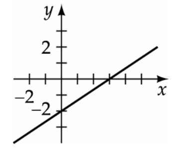

**16.** $m = -1$
$y$-intercept $(0, 1)$



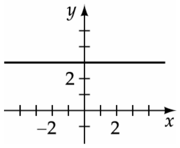

**17.** $m = -\frac{1}{3}$
$y$-intercept $(0, 4)$



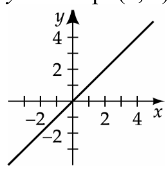

**18.** $m = \frac{2}{3}$
$y$-intercept $(0, -2)$



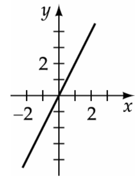

**19.** $m = 0$
$y$-intercept $(0, 3)$



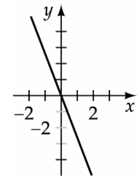

**20.** $m = 1$
$y$-intercept $(0, 0)$



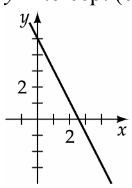

**21.** $m = 2$
$y$-intercept $(0, 0)$



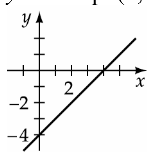

**22.** $m = -3$
$y$-intercept $(0, 0)$



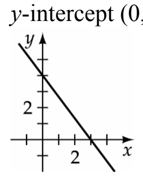

**23.** $m = -2$
$y$-intercept $(0, 5)$



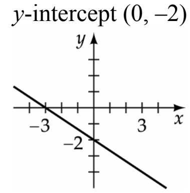

**24.** $m = 1$
$y$-intercept $(0, -4)$




**25.** $m = -\frac{3}{4}$
$y$-intercept $(0, 4)$




**26.** $m = -\frac{2}{3}$
$y$-intercept $(0, -2)$




**27.** Use $y = mx + b$ with $m = 1,\ b = 3.$
$y = x + 3$

**28.** Use $y = mx + b$ with $m = -2,\ b = 5.$
$y = -2x + 5$

**29.** Use $y = mx + b$ with $m = \frac{3}{4},\ b = \frac{1}{2}.$

$$y = \frac{3}{4}x + \frac{1}{2}$$

**30.** Use $y = mx + b$ with $m = -\frac{2}{3},\ b = \frac{3}{4}.$

$$y = -\frac{2}{3}x + \frac{3}{4}$$

**31.** Use $y = mx + b$ with $m = 0,\ b = 4.$
$y = 4$

**32.** Use $y = mx + b$ with $m = \frac{1}{2},\ b = -1.$

$$y = \frac{1}{2}x - 1$$

**33.**
$$y - 2 = -4(x - (-3))$$
$$y - 2 = -4x - 12$$
$$y = -4x - 10$$

**34.**
$$y + 1 = -3(x + 5)$$
$$y = -3x - 15 - 1$$
$$y = -3x - 16$$

**35.**
$$m = \frac{4-1}{-1-3}$$
$$= \frac{3}{-4} = -\frac{3}{4}$$
$$y - 1 = -\frac{3}{4}(x - 3)$$
$$y = -\frac{3}{4}x + \frac{9}{4} + \frac{4}{4}$$
$$y = -\frac{3}{4}x + \frac{13}{4}$$

**36.**
$$m = \frac{-8-(-6)}{2-5}$$
$$= \frac{-2}{-3} = \frac{2}{3}$$
$$y - (-6) = \frac{2}{3}(x - 5)$$
$$y + 6 = \frac{2}{3}x - \frac{10}{3}$$
$$y = \frac{2}{3}x - \frac{10}{3} - 6$$
$$y = \frac{2}{3}x - \frac{28}{3}$$

**37.** $m=\dfrac{-1-11}{2-7}$

$=\dfrac{-12}{-5}=\dfrac{12}{5}$

$y-11=\dfrac{12}{5}(x-7)$

$y-11=\dfrac{12}{5}x-\dfrac{84}{5}$

$y=\dfrac{12}{5}x-\dfrac{84}{5}+\dfrac{55}{5}$

$=\dfrac{12}{5}x-\dfrac{29}{5}$

**38.** $m=\dfrac{-4-6}{-3-(-5)}$

$=\dfrac{-10}{2}=-5$

$y-6=-5(x+5)$

$y-6=-5x-25$

$y=-5x-25+6$

$y=-5x-19$

**39.** $f(x)=2x+3=-1$

$2x=-4$

$x=-2$

**40.** $f(x)=4-3x=7$

$-3x=3$

$x=-1$

**41.** $f(x)=1-4x=3$

$-4x=2$

$x=-\dfrac{1}{2}$

**42.** $f(x)=\dfrac{2x}{3}+2=4$

$\dfrac{2x}{3}=2$

$x=2\left(\dfrac{3}{2}\right)$

$x=3$

**43.** $f(x)=3-\dfrac{x}{2}=5$

$-\dfrac{x}{2}=2$

$x=2(-2)$

$x=-4$

**44.** $f(x)=4x-3=-2$

$4x=1$

$x=\dfrac{1}{4}$

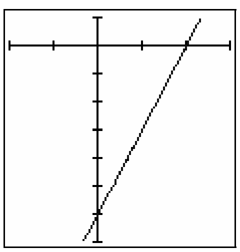
**45.** $f(x)=3x-12$

$3x-12=0$

$3x=12$

$x=4$

The $x$-intercept of the graph of $f(x)$ is (4,0).



Xmin = – 4, Xmax = 6, Xscl=2,
Ymin = –12.2, Ymax = 2, Yscl = 2

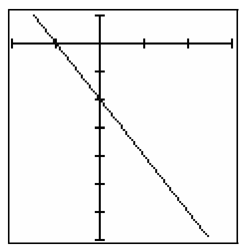
**46.** $f(x)=-2x-4$

$-2x-4=0$

$-2x=4$

$x=-2$

The $x$-intercept of the graph of $f(x)$ *(x)* is (–2,0).



Xmin = – 4, Xmax = 6, Xscl=2,
Ymin = –12.2, Ymax = 2, Yscl = 2

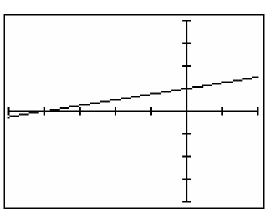
**47.** $f(x)=\dfrac{1}{4}x+5$

$\dfrac{1}{4}x+5=0$

$\dfrac{1}{4}x=-5$

$x=-20$

The $x$-intercept of the graph of $f(x)$ is (–20,0).



Xmin = –30, Xmax = 30, Xscl = 10,
Ymin = –10, Ymax = 10, Yscl = 1

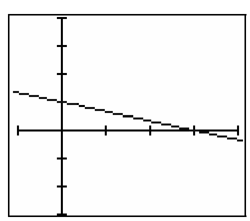
**48.** $f(x)=-\dfrac{1}{3}x+2$

$-\dfrac{1}{3}x+2=0$

$-\dfrac{1}{3}x=-2$

$x=6$

The $x$-intercept of the graph of $f(x)$ is (6,0).



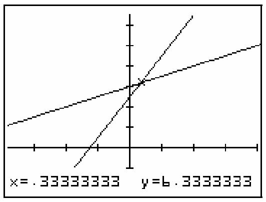
Xmin = –2, Xmax = 8, Xscl = 2,
Ymin = –6, Ymax = 8, Yscl = 2

**49.** Algebraic method:

$$f_1(x)=f_2(x)$$
$$4x+5=x+6$$
$$3x=1$$
$$x=\frac{1}{3}$$

Graphical method: Graph $y=4x+5$ and $y=x+6$

They intersect at $x=\frac{1}{3}$, $y=6\frac{1}{3}$.



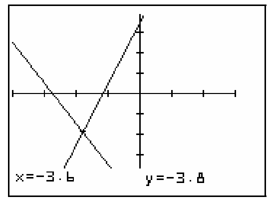

Xmin = –7.8, Xmax = 7.8, Xscl = 2,
Ymin = –2 ,Ymax = 10, Yscl = 2

**50.** Algebraic method:

$$f_1(x)=f_2(x)$$
$$-2x-11=3x+7$$
$$-5x=18$$
$$x=-\frac{18}{5}$$

Graphical method: Graph $y=-2x-11$ and $y=3x+7$.

They intersect at $x=-3.6$, $y=-3.8$.



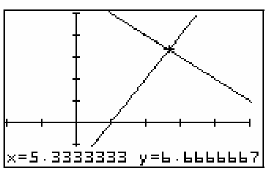

Xmin = –6, Xmax = 6, Xscl = 2,
Ymin = –7, Ymax =7.8, Yscl = 2

**51.** Algebraic method:

$$f_1(x)=f_2(x)$$
$$2x-4=-x+12$$
$$3x=16$$
$$x=\frac{16}{3}$$

Graphical method: Graph $y=2x-4$ and $y=-x+12$

They intersect at $x=5\frac{1}{3}$, $y=6\frac{2}{3}$.



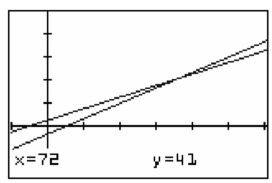

Xmin = – 4, Xmax = 10, Xscl = 2,
Ymin = –2, Ymax = 10, Yscl = 2

**52.** Algebraic method:

$$f_1(x)=f_2(x)$$
$$\frac{1}{2}x+5=\frac{2}{3}x-7$$
$$6\left(\frac{1}{2}x+5\right)=6\left(\frac{2}{3}x-7\right)$$
$$3x+30=4x-42$$
$$72=x$$

Graphical method: Graph $y=\frac{1}{2}x+5$ and $y=\frac{2}{3}x-7$

They intersect at $x=72$, $y=41$.




Xmin = – 20, Xmax = 120, Xscl = 20,
Ymin = –20, Ymax = 100, Yscl = 20

**53.** $m=\frac{1505-1482}{28-20}=2.875$

The value of the slope indicates that the speed of sound in water increases 2.875 m for a one-degree increase in temperature.

**54.** $m=\frac{4-1}{100-25}=0.04$

The value of the slope indicates that the file is being downloaded at 0.04 megabytes per second.

**55.** **a.** $m=\frac{29-13}{20-9}\approx 1.45$

$H(c)-13=1.45(c-9)$

$H(c)=1.45c$

**b.** $H(18)=1.45(18)\approx 26$ mpg

**56.** **a.** $m=\frac{799.1-675.7}{2005-2000}=24.68$

$C(t)-675.7=24.68(t-2000)$

$C(t)=24.68t-48,684.3$

**b.** $900=24.68t-48,684.3$

$49584.3=24.68t$

$2009.1\approx t$

The debt will exceed \$900 billion in 2009.

**57.** **a.** $m=\dfrac{63,000-38,000}{2010-2000}=2500$

$N(t)-63,000=2500(t-2010)$

$N(t)=2500t-4,962,000$

**b.** $60,000=2500t-4,962,000$

$5,022,000=2500t$

$2008.8=t$

The number of jobs will exceed 60,000 in 2008.

**58.** **a.** $m=\dfrac{2200-2150}{15-20}=-10$

$T(t)-2200=-10(t-15)$

$T(t)=-10t+2350$

**b.** The value of the slope means that the temperature is decreasing at a rate of 10 degrees per minute.

**c.** $T(180)=-10(180)+2350=550^\circ\text{F}$

After 3 hours, the temperature will be 550°F.

**59.** **a.** $m=\dfrac{240-180}{18-16}=30$

$B(d)-180=30(d-16)$

$B(d)=30d-300$

**b.** The value of the slope means that a 1-inch increase in the diameter of a log 32 ft long results in an increase of 30 board-feet of lumber that can be obtained from the log.

**c.** $B(19)=30(19)-300=270$ board feet

**60.** **a.** $m=\dfrac{1640-800}{60-40}=42$

$E(T)-800=42(T-40)$

$E(T)=42T-880$

**b.** The value of the slope means that an additional 42 acre-feet of water evaporate for a one degree increase in temperature.

**c.** $E(75)=42(75)-880=2270$ acre-feet

**61.** Line A represents Michelle
Line B represents Amanda
Line C represents the distance between Michelle and Amanda.

**62.** **a.** $m_{AB}=\dfrac{1-9}{8-6}=-4^\circ\text{F}$

**b.** $m_{AB}=\dfrac{1-9}{8-6}=-4^\circ\text{F}$

$m_{DE}=\dfrac{-4-5}{5-6}=9^\circ\text{F}$

The temperature changed most rapidly between points D and E.

**c.** The temperature remained constant (zero slope) between points C and D.

**63.** **a.** Find the slope of the line.

$m=\dfrac{180-110}{108-70}=\dfrac{70}{38}\approx 1.842$

Use the point-slope formula to find the equation.

$y-y_1=m(x-x_1)$

$y-110=1.842(x-70)$

$y-110=1.842x-128.94$

$y=1.842x-18.94$

**b.** $y=1.842(90)-18.94$

$y=165.78-18.94$

$y=146.84\approx 147$

**64.** **a.** Find the slope of the line.

$m=\dfrac{11.2-76.5}{75-0}=\dfrac{-65.3}{75}\approx -0.87$

Use the point-slope formula to find the equation.

$y-y_1=m(x-x_1)$

$y-76.5=-0.87(x-0)$

$y-76.5=-0.87x$

$y=-0.87x+76.5$

**b.** $y=-0.87(25)+76.5$

$y=-21.75+76.5$

$y=54.75\approx 55$ years

**65.** $P(x)=92.50x-(52x+1782)$

$P(x)=92.50x-52x-1782$

$P(x)=40.50x-1782$

$40.50x-1782=0$

$40.50x=1782$

$x=\dfrac{1782}{40.50}$

$x=44,$ the break-even point

**66.** $P(x)=124x-(78.5x+5005)$

$P(x)=124x-78.5x-5005$

$P(x)=45.5x-5005$

$45.5x-5005=0$

$45.5x=5005$

$x=\dfrac{5005}{45.5}$

$x=110,$ the break-even point

**67.** $P(x) = 259x - (180x + 10,270)$
$P(x) = 259x - 180x - 10,270$
$P(x) = 79x - 10,270$

$79x - 10,270 = 0$
$79x = 10.270$
$x = \frac{10.270}{79}$
$x = 130,$ the break-even point

**68.** $P(x) = 14,220x - (8010x + 1,602,180)$
$P(x) = 14,220x - 8010x - 1,602,180$
$P(x) = 6210x - 1,602,180$

$6210x - 1,602,180 = 0$
$6210x = 1,602,180$
$x = \frac{1,602,180}{6210}$
$x = 258,$ the break-even point

**69.** **a.** $C(0) = 8(0) + 275 = 0 + 275 = \$275$
**b.** $C(1) = 8(1) + 275 = 8 + 275 = \$283$
**c.** $C(10) = 8(10) + 275 = 80 + 275 = \$355$
**d.** The marginal cost is the slope of $C(x) = 8x + 275,$ which is \$8 per unit.

**70.** **a.** $R(0) = 210(0) = \$0$
**b.** $R(1) = 210(1) = \$210$
**c.** $R(10) = 210(10) = \$2100$
**d.** The marginal revenue is the slope of $R(x) = 210x,$ which is \$210 per unit.

**71.** **a.** $C(t) = 19{,}500.00 + 6.75t$
**b.** $R(t) = 55.00t$
**c.** $P(t) = R(t) - C(t)$
$P(t) = 55.00t - (19,500.00 + 6.75t)$
$P(t) = 55.00t - 19,500.00 - 6.75t$
$P(t) = 48.25t - 19,500.00$
**d.** $48.25t = 19,500.00$
$t = \frac{19,500.00}{48.25}$
$t = 404.1451$ days $\approx$ 405 days

**72.** $m = \frac{117{,}500 - 98{,}000}{35{,}000 - 32{,}000} = \frac{19{,}500}{2000} = 6.5$
**a.** $P(s) - 98,000 = 6.5(s - 32,000)$
$P(s) = 6.5s - 208,000 + 98,000$
$P(s) = 6.5s - 110,000$
**b.** $P(50,000) = 6.5(50,000) - 110,000$
$= 325,000 - 110,000$
$= \$215,000$
**c.** Let $6.5s - 110{,}000 = 0.$ Then
$6.5s = 110,000$
$s = \frac{110,000}{6.5} \approx 16,924$ subscribers

**73.** The graph of $3x + y = -24$ has $m = -\frac{3}{4}$.

$y - 3 = -\frac{3}{4}(x - 1)$
$y = -\frac{3}{4}x + \frac{3}{4} + 3$
$y = -\frac{3}{4}x + \frac{15}{4}$

**74.** The graph of $x + y = 10$ has $m = -1$.
$y + 1 = (-1)(x - 2)$
$y = -x + 2 - 1$
$y = -x + 1$

**75.** The graph of $y = -\frac{1}{2}x + 6$ has $m = -\frac{1}{2}$.

$y - 10 = -\frac{1}{2}(x - 4)$
$y = -\frac{1}{2}x + 2 + 10$
$y = -\frac{1}{2}x + 12$

**76.** The graph of $y = \frac{5}{2}x + 5$ has $m = \frac{5}{2}$.

$y + 2 = \frac{5}{2}(x - 10)$
$y = \frac{5}{2}x - 25 - 2$
$y = \frac{5}{2}x - 27$

**77.** The graph of $y=\frac{3}{2}x+6$ has $m=\frac{3}{2}$.

Thus we use a slope of $-\frac{2}{3}$.

$$y-7=-\frac{2}{3}(x+9)$$
$$y=-\frac{2}{3}x-6+7$$
$$y=-\frac{2}{3}x+1$$

**78.** The graph of $y=-\frac{2}{3}x+3$ has $m=-\frac{2}{3}$.

Thus we use a slope of $\frac{3}{2}$.

$$y+6=\frac{3}{2}(x-4)$$
$$y=\frac{3}{2}x-6-6$$
$$y=\frac{3}{2}x-12$$

**79.** The graph of $y=\frac{4}{7}x-9$ has $m=\frac{4}{7}$.

$$y-3=\frac{4}{7}(x-4)$$
$$y=\frac{4}{7}x-\frac{16}{7}+3$$
$$y=\frac{4}{7}x+\frac{5}{7}$$

**80.** The graph of $y=\frac{9}{7}x+2$ has $m=\frac{9}{7}$.

Thus we use a slope of $-\frac{7}{9}$.

$$y+3=-\frac{7}{9}(x-1)$$
$$y=-\frac{7}{9}x+\frac{7}{9}-3$$
$$y=-\frac{7}{9}x-\frac{20}{9}$$

**81.** The graph of $y=\frac{1}{2}x+9$ has $m=\frac{1}{2}$.

Thus we use a slope of –2.

$$y+1=-2(x+3)$$
$$y=-2x-6-1$$
$$y=-2x-7$$

**82.** The graph of $y=-\frac{5}{4}x+7$ has $m=-\frac{5}{4}$.

$$y+5=-\frac{5}{4}(x+2)$$
$$y=-\frac{5}{4}x-\frac{5}{2}-5$$
$$y=-\frac{5}{4}x-\frac{15}{2}$$

**83.** The graph of $x+y=4$ has $m=-1$.

Thus we use a slope of 1.

$$y-2=1(x-1)$$
$$y=x-1+2$$
$$y=x+1$$

**84.** The graph of $2x-y=7$ has $m=2$.

Thus we use a slope of $-\frac{1}{2}$.

$$y-4=-\frac{1}{2}(x+3)$$
$$y=-\frac{1}{2}x-\frac{3}{2}+\frac{8}{2}$$
$$y=-\frac{1}{2}x+\frac{5}{2}$$

**85.** The equation of the line through (0,0) and $P(3,4)$ has slope $\frac{4}{3}$.

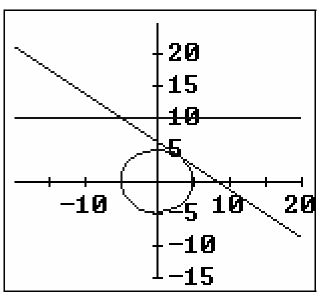
The path of the rock is on the line through $P(3,4)$ with slope $-\frac{3}{4}$, so $y-4=-\frac{3}{4}(x-3)$.

$$y-4=-\frac{3}{4}x+\frac{9}{4}$$
$$y=-\frac{3}{4}x+\frac{9}{4}+4$$
$$y=-\frac{3}{4}x+\frac{25}{4}$$

The point where the rock hits the wall at $y = 10$ is the point of intersection of $y=-\frac{3}{4}x+\frac{25}{4}$ and $y = 10$.

$$-\frac{3}{4}x+\frac{25}{4}=10$$
$$-3x+25=40$$
$$-3x=15$$
$$x=-5 \text{ feet}$$

Therefore the rock hits the wall at $(-5,10)$.
The $x$-coordinate is –5.




**86.** The equation of the line through (0,0) and $P(\sqrt{15},1)$ has slope $\frac{1}{\sqrt{15}}$.

The path of the rock is on the line through $P(\sqrt{15}, 1)$ with slope $-\sqrt{15}$ so $y-1=-\sqrt{15}(x-\sqrt{15})$

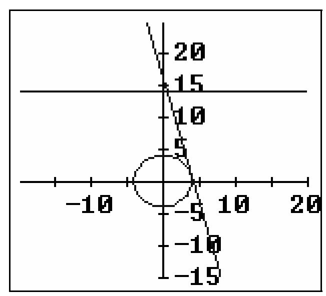
$$y-1=-\sqrt{15}x+15$$
$$y=-\sqrt{15}x+15+1$$
$$y=-\sqrt{15}x+16$$

The point of impact with the wall at $y = 14$ is the point of intersection of $y=-\sqrt{15}x+16$ and $y = 14$ intersect.

$$-\sqrt{15}x+16=14$$
$$-\sqrt{15}x=-2$$
$$x=\frac{2}{\sqrt{15}}\approx 0.52 \text{ feet}$$

Therefore, the rock hits the wall at $\left(\frac{2}{\sqrt{15}}, 14\right)$.

The $x$-coordinate is $\frac{2}{\sqrt{15}}$ or approximately 0.52.




**87.** **a.** $h=1$ so $Q(2+h,[2+h]^2+1)=Q(3,3^2+1)=Q(3,10)$

$$m=\frac{10-5}{3-2}=\frac{5}{1}=5$$

**b.** $h=0.1$ so $Q(2+h,[2+h]^2+1)=Q(2.1,2.1^2+1)=Q(2.1,5.41)$

$$m=\frac{5.41-5}{2.1-2}=\frac{0.41}{0.1}=4.1$$

**c.** $h=0.01$ so $Q(2+h,[2+h]^2+1)=Q(2.01,2.01^2+1)=Q(2.01,5.0401)$

$$m=\frac{5.0401-5}{2.01-2}=\frac{0.0401}{0.01}=4.01$$

**d.** As $h$ approaches 0, the slope of $PQ$ seems to be approaching 4.

**e.** $x_1=2, y_1=5, x_2=2+h,\ y_2=[2+h]^2+1$

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{[2+h]^2+1-5}{(2+h)-2}=\frac{(4+4h+h^2)+1-5}{h}=\frac{4h+h^2}{h}=4+h$$

**88.** **a.** $h = 1$ so $Q(-1+h, 3[-1+h]^2) = Q(0, 0)$

$$m = \frac{0-3}{0-(-1)} = \frac{-3}{1} = -3$$

**b.** $h = 0.1$ so $Q(-1+h, 3[-1+h]^2) = Q(-0.9, 3(-0.9)^2) = Q(-0.9, 2.43)$

$$m = \frac{2.43-3}{-0.9-(-1)} = \frac{-0.57}{0.1} = -5.7$$

**c.** $h = 0.01$ so $Q(-1+h, 3[-1+h]^2) = Q(-0.99, 3(-0.99)^2) = Q(-0.99, 2.9403)$

$$m = \frac{2.9403-3}{-0.99-(-1)} = \frac{-0.0597}{0.01} = -5.97$$

**d.** As $h$ approaches 0, the slope of $PQ$ seems to be approaching $-6$.

**e.** $x_1 = -1,\ y_1 = 3,\ x_2 = -1+h,\ y_2 = 3[-1+h]^2$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3[-1+h]^2 - 3}{(-1+h)-(-1)} = \frac{3(1-2h+h^2)-3}{h} = \frac{3-6h+3h^2-3}{h} = \frac{-6h+3h^2}{h} = -6+3h$$

**89.** $$m = \frac{(x+h)^2 - x^2}{x+h-x} = \frac{x^2+2xh+h^2-x^2}{h} = \frac{2xh+h^2}{h} = \frac{h(2x+h)}{h} = 2x+h$$

**90.** $$m = \frac{4(x+h)^2 - 4x^2}{x+h-x} = \frac{4(x^2+2xh+h^2)-4x^2}{h} = \frac{4x^2+8xh+4h^2-4x^2}{h} = \frac{8xh+4h^2}{h} = \frac{h(8x+4h)}{h} = 8x+4h$$

## Connecting Concepts

**91.** Substitute $\frac{y_2 - y_1}{x_2 - x_1}$ for $m$ in the point-slope form $y - y_1 = m(x - x_1)$ to yield $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, the two-point form.

**92.**
$$y - 0 = \frac{b-0}{0-a}(x-a)$$
$$y = \frac{b}{-a}(x-a)$$
$$y = -\frac{bx}{a} + b$$

$\frac{bx}{a} + y = b$ Then divide by $b$ to produce $\frac{x}{a} + \frac{y}{b} = 1.$

**93.**
$$y - 1 = \frac{3-1}{4-5}(x-5)$$
$$y - 1 = \frac{2}{-1}(x-5)$$
$$y = -2(x-5)$$
$$y - 1 = -2x + 10$$
$$y = -2x + 10 + 1$$
$$y = -2x + 11$$

**94.**
$$y - 7 = \frac{6-7}{-1-2}(x-2)$$
$$y - 7 = \frac{-1}{-3}(x-2)$$
$$y - 7 = \frac{1}{3}(x-2)$$
$$y - 7 = \frac{1}{3}x - \frac{2}{3}$$
$$y = \frac{1}{3}x - \frac{2}{3} + \frac{21}{3}$$
$$y = \frac{1}{3}x + \frac{19}{3}$$

**95.** Use $\frac{x}{a} + \frac{y}{b} = 1$ with $a = 3$ and $b = 5$.
$$\frac{x}{3} + \frac{y}{5} = 1$$
$$15\left(\frac{x}{3} + \frac{y}{5}\right) = 15(1)$$
$$5x + 3y = 15$$

**96.** Use $\frac{x}{a}+\frac{y}{b}=1$ with $a=-2$ and $b=7$.

$$\frac{x}{-2}+\frac{y}{7}=1$$
$$14\left(\frac{x}{-2}+\frac{y}{7}\right)=14(1)$$
$$-7x+2y=14$$

**97.** Use $\frac{x}{a}+\frac{y}{b}=1$ with $b=3a$.

$\frac{x}{a}+\frac{y}{3a}=1$ Since (5, 2) is on the line,

$$\frac{5}{a}+\frac{2}{3a}=1$$
$$3a\left(\frac{5}{a}+\frac{2}{3a}\right)=3a(1)$$
$$15+2=3a$$
$$17=3a$$
$$\frac{17}{3}=a$$

Thus $\frac{x}{\left(\frac{17}{3}\right)}+\frac{y}{3\left(\frac{17}{3}\right)}=1$

$$\frac{3x}{17}+\frac{y}{17}=1$$
$$3x+y=17$$

**98.** $\frac{x}{-b}+\frac{y}{2b}=1$ Since $(-3,10)$ is on the line,

$$\frac{-3}{-b}+\frac{10}{2b}=1$$
$$2b\left(\frac{3}{b}+\frac{10}{2b}\right)=2b(1)$$
$$6+10=2b$$
$$16=2b$$
$$8=b$$
$$\frac{x}{-8}+\frac{y}{16}=1$$
$$-2x+y=16$$

**99.**
$$\frac{3(1+h)^3-3}{1+h-1}=\frac{3(1+3h+3h^2+h^3)-3}{h}$$
$$=\frac{3+9h+9h^2+3h^3-3}{h}$$
$$=\frac{9h+9h^2+3h^3}{h}$$
$$=\frac{h(9+9h+3h^2)}{h}$$
$$=9+9h+3h^2$$

**100.** The slope of the radius from (0, 0) to $(x, y)$ is 0.5, so $\frac{y-0}{x-0}=\frac{y}{x}=0.5$ thus $y=0.5x$.

Substitute $y=0.5x$ into $x^2+y^2=25$.

$$x^2+(0.5x)^2=25$$
$$x^2+0.25x^2=25$$
$$1.25x^2=25$$
$$x^2=20$$
$$x=\pm\sqrt{20}=\pm2\sqrt{5}$$

If $x=2\sqrt{5}$, then $y=0.5(2\sqrt{5})=\sqrt{5}$. If $x=-2\sqrt{5}$, then $y=0.5(-2\sqrt{5})=-\sqrt{5}$.

The points are $(2\sqrt{5}, \sqrt{5})$ and $(-2\sqrt{5}, -\sqrt{5})$.

**101.** The slope of the line through (3, 9) and $(x, y)$ is $\frac{15}{2}$, so $\frac{y-9}{x-3}=\frac{15}{2}$.

Therefore

$$2(y-9)=15(x-3)$$
$$2y-18=15x-45$$
$$2y-15x+27=0 \quad \text{Substitute } y=x^2 \text{ into this equation.}$$
$$2x^2-15x+27=0$$
$$(2x-9)(x-3)=0$$
$$x=\frac{9}{2} \quad \text{or} \quad x=3$$

If $x=\frac{9}{2},\ y=x^2=\left(\frac{9}{2}\right)^2=\frac{81}{4}\Rightarrow\left(\frac{9}{2},\frac{81}{4}\right)$.

If $x=3,\ y=x^2=(3)^2=9\Rightarrow(3,9)$, but this is the point itself.

The point $\left(\frac{9}{2},\frac{81}{4}\right)$ is on the graph of $y=x^2$, and the slope of the line containing (3, 9) and $\left(\frac{9}{2},\frac{81}{4}\right)$ is $\frac{15}{2}$.

**102.** The slope of the line through (3, 2) and $(x, y)$ is $\frac{3}{8}$, so $\frac{y-2}{x-3}=\frac{3}{8}$.

Therefore

$$8(y-2)=3(x-3).$$
$$8y-16=3x-9$$
$$8y=3x+7 \quad \text{Substitute } y=\sqrt{x+1} \text{ into this equation.}$$
$$8\sqrt{x+1}=3x+7$$
$$\left(8\sqrt{x+1}\right)^2=(3x+7)^2$$
$$64(x+1)=9x^2+42x+49$$
$$64x+64=9x^2+42x+49$$
$$0=9x^2-22x-15$$
$$0=(9x+5)(x-3)$$
$$x=-\frac{5}{9} \quad \text{or} \quad x=3$$

If $x=-\frac{5}{9},\ y=\sqrt{x+1}=\sqrt{-\frac{5}{9}+\frac{9}{9}}=\sqrt{\frac{4}{9}}=\frac{2}{3}\Rightarrow\left(-\frac{5}{9},\ \frac{2}{3}\right)$.

If $x=3,\ y=\sqrt{x+1}=\sqrt{3+1}=\sqrt{4}=2\Rightarrow(3,\ 2)$, but this is the point itself.

The point $\left(-\frac{5}{9},\ \frac{2}{3}\right)$ is on the graph of $y=\sqrt{x+1}$, and the slope of the line containing (3, 2) and $\left(-\frac{5}{9},\ \frac{2}{3}\right)$ is $\frac{3}{8}$.

## Prepare for Section 2.4

**PS1.** $3x^2+10x-8=(3x-2)(x+4)$

**PS2.** $x^2-8x=x^2-8x+16=(x-4)^2$

**PS3.**
$$f(-3)=2(-3)^2-5(-3)-7$$
$$=18+15-7$$
$$=26$$

**PS4.**
$$2x^2-x-1=0$$
$$(2x+1)(x-1)=0$$
$$2x+1=0 \qquad x-1=0$$
$$x=-\frac{1}{2} \qquad x=1$$

**PS5.** $x^2 + 3x - 2 = 0$

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(-2)}}{2(1)}$$

$$= \frac{-3 \pm \sqrt{17}}{2}$$

**PS6.** $53 = -16t^2 + 64t + 5$

$16t^2 - 64t + 48 = 0$

$t^2 - 4t + 3 = 0$

$(t-1)(t-3) = 0$

$t = 1, 3$

## Section 2.4

**1.** d **2.** f **3.** b **4.** h

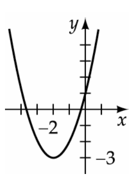
**5.** g **6.** e **7.** c **8.** a

**9.** $f(x) = (x^2 + 4x) + 1$

$= (x^2 + 4x + 4) + 1 - 4$

$= (x+2)^2 - 3$ standard form,

vertex $(-2, -3)$, axis of symmetry $x = -2$

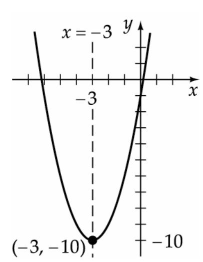



**10.** $f(x) = (x^2 + 6x) - 1$

$= (x^2 + 6x + 9) - 1 - 9$

$= (x+3)^2 - 10$ standard form,

vertex $(-3, -10)$, axis of symmetry $x = -3$

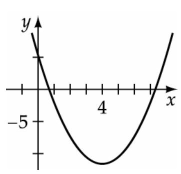



**11.** $f(x) = (x^2 - 8x) + 5$

$= (x^2 - 8x + 16) + 5 - 16$

$= (x-4)^2 - 11$ standard form,

vertex $(4, -11)$, axis of symmetry $x = 4$

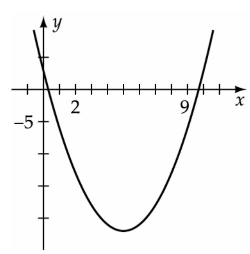



**12.** $f(x) = (x^2 - 10x) + 3$

$= (x^2 - 10x + 25) + 3 - 25$

$= (x-5)^2 - 22$ standard form,

vertex $(5, -22)$, axis of symmetry $x = 5$



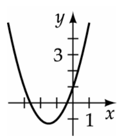
**13.** $f(x)=(x^2+3x)+1$

$=\left(x^2+3x+\frac{9}{4}\right)+1-\frac{9}{4}$

$=\left(x+\frac{3}{2}\right)^2+\frac{4}{4}-\frac{9}{4}$

$=\left(x+\frac{3}{2}\right)^2-\frac{5}{4}$ standard form,

vertex $\left(-\frac{3}{2},-\frac{5}{4}\right)$, axis of symmetry $x=-\frac{3}{2}$



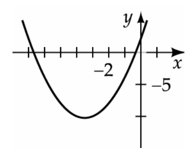

**14.** $f(x)=(x^2+7x)+2$

$=\left(x^2+7x+\frac{49}{4}\right)+2-\frac{49}{4}$

$=\left(x+\frac{7}{2}\right)^2+\frac{8}{4}-\frac{49}{4}$

$=\left(x+\frac{7}{2}\right)^2-\frac{41}{4}$ standard form,

vertex $\left(-\frac{7}{2},-\frac{41}{4}\right)$, axis of symmetry $x=-\frac{7}{2}$



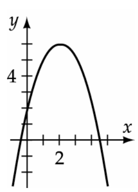

**15.** $f(x)=-x^2+4x+2$

$=-(x^2-4x)+2$

$=-(x^2-4x+4)+2+4$

$=-(x-2)^2+6$ standard form,

vertex (2, 6), axis of symmetry $x = 2$



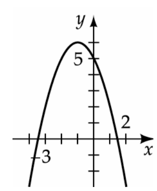

**16.** $f(x)=-x^2-2x+5$

$=-(x^2+2x)+5$

$=-(x^2+2x+1)+5+1$

$=-(x+1)^2+6$ standard form,

vertex (–1, 6), axis of symmetry $x = -1$




**17.** $f(x)=-3x^2+3x+7$

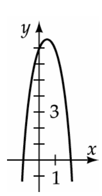
$=-3(x^2-1x)+7$

$=-3\left(x^2-1x+\frac{1}{4}\right)+7+\frac{3}{4}$

$=-3\left(x-\frac{1}{2}\right)^2+\frac{28}{4}+\frac{3}{4}$

$=-3\left(x-\frac{1}{2}\right)^2+\frac{31}{4}$ standard form,

vertex $\left(\frac{1}{2},\frac{31}{4}\right)$, axis of symmetry $x=\frac{1}{2}$



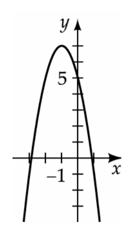

**18.** $f(x)=-2x^2-4x+5$

$=-2(x^2+2x)+5$

$=-2(x^2+2x+1)+5+2$

$=-2(x+1)^2+7$ standard form,

vertex (–1, 7), axis of symmetry $x = -1$



**19.** $x = \frac{-b}{2a} = \frac{10}{2(1)} = 5$

$y = f(5) = (5)^2 - 10(5)$
$= 25 - 50 = -25$
vertex $(5, -25)$

$f(x) = (x-5)^2 - 25$

**20.** $x = \frac{-b}{2a} = \frac{6}{2(1)} = 3$

$y = f(3) = (3)^2 - 6(3)$
$= 9 - 18 = -9$
vertex $(3, -9)$

$f(x) = (x-3)^2 - 9$

**21.** $x = \frac{-b}{2a} = \frac{0}{2(1)} = 0$

$y = f(0) = (0)^2 - 10 = -10$
vertex $(0, -10)$

$f(x) = x^2 - 10$

**22.** $x = \frac{-b}{2a} = \frac{0}{2(1)} = 0$

$y = f(0) = (0)^2 - 4 = -4$
vertex $(0, -4)$

$f(x) = x^2 - 4$

**23.** $x = \frac{-b}{2a} = \frac{-6}{2(-1)} = \frac{-6}{-2} = 3$

$y = f(3) = -(3)^2 + 6(3) + 1$
$-9 + 18 + 1$
$= 10$
vertex $(3, 10)$

$f(x) = -(x-3)^2 + 10$

**24.** $x = \frac{-b}{2a} = \frac{-4}{2(-1)} = \frac{-4}{-2} = 2$

$y = f(2) = -(2)^2 + 4(2) + 1$
$= -4 + 8 + 1$
$= 5$
vertex $(2, 5)$

$f(x) = -(x-2)^2 + 5$

**25.** $x = \frac{-b}{2a} = \frac{3}{2(2)} = \frac{3}{4}$

$y = f\left(\frac{3}{4}\right) = 2\left(\frac{3}{4}\right)^2 - 3\left(\frac{3}{4}\right) + 7$

$= 2\left(\frac{9}{16}\right) - \frac{9}{4} + 7$

$= \frac{9}{8} - \frac{9}{4} + 7$

$= \frac{9}{8} - \frac{18}{8} + \frac{56}{8}$

$= \frac{47}{8}$

vertex $\left(\frac{3}{4}, \frac{47}{8}\right)$

$f(x) = 2\left(x - \frac{3}{4}\right)^2 + \frac{47}{8}$

**26.** $x = \frac{-b}{2a} = \frac{10}{2(3)} = \frac{10}{6} = \frac{5}{3}$

$y = f\left(\frac{5}{3}\right) = 3\left(\frac{5}{3}\right)^2 - 10\left(\frac{5}{3}\right) + 2$

$= 3\left(\frac{25}{9}\right) - \frac{50}{3} + 2$

$= \frac{25}{3} - \frac{50}{3} + 2$

$= \frac{25}{3} - \frac{50}{3} + \frac{6}{3}$

$= -\frac{19}{3}$

vertex $\left(\frac{5}{3}, -\frac{19}{3}\right)$

$f(x) = 3\left(x - \frac{5}{3}\right)^2 - \frac{19}{3}$

**27.** $x = \frac{-b}{2a} = \frac{-1}{2(-4)} = \frac{1}{8}$

$y = f\left(\frac{1}{8}\right) = -4\left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right) + 1$

$= -4\left(\frac{1}{64}\right) + \frac{1}{8} + 1$

$= -\frac{1}{16} + \frac{1}{8} + 1$

$= -\frac{1}{16} + \frac{2}{16} + \frac{16}{16}$

$= \frac{17}{16}$

vertex $\left(\frac{1}{8}, \frac{17}{16}\right)$

$f(x) = -4\left(x - \frac{1}{8}\right)^2 + \frac{17}{16}$

**28.** $x = \frac{-b}{2a} = \frac{6}{2(-5)} = \frac{6}{-10} = -\frac{3}{5}$

$y = f\left(-\frac{3}{5}\right) = -5\left(-\frac{3}{5}\right)^2 - 6\left(-\frac{3}{5}\right) + 3$

$= -5\left(\frac{9}{25}\right) + \frac{18}{5} + 3$

$= -\frac{9}{5} + \frac{18}{5} + 3$

$= -\frac{9}{5} + \frac{18}{5} + \frac{15}{5}$

$= \frac{24}{5}$

vertex $\left(-\frac{3}{5}, \frac{24}{5}\right)$

$f(x) = -5\left(x + \frac{3}{5}\right)^2 + \frac{24}{5}$

**29.** $f(x) = x^2 - 2x - 1$

$= (x^2 - 2x) - 1$

$= (x^2 - 2x + 1) - 1 - 1$

$= (x-1)^2 - 2$

vertex (1, –2)
The $y$-value of the vertex is –2.
The parabola opens up since $a = 1 > 0$.
Thus the range is $\{y \mid y \geq -2\}$.

$f(x) = 2 = x^2 - 2x - 1$

$0 = x^2 - 2x - 3$

$0 = (x-3)(x+1)$

$x - 3 = 0$ or $x + 1 = 0$

$x = 3$ $\quad x = -1$

**30.** $f(x) = -x^2 - 6x - 2$

$= -(x^2 + 6x) - 2$

$= -(x^2 + 6x + 9) - 2 + 9$

$= -(x+3)^2 + 7$

vertex (–3, 7)
The $y$-value of the vertex is 7.
The parabola opens down since $a = -1 < 0$.
Thus the range is $\{y \mid y \leq 7\}$.

$f(x) = 3 = -x^2 - 6x - 2$

$x^2 + 6x + 5 = 0$

$(x+5)(x+1) = 0$

$x + 5 = 0$ or $x + 1 = 0$

$x = -5$ $\quad x = -1$

**31.** $f(x)=-2x^2+5x-1$

$$=-2\left(x^2-\frac{5}{2}x\right)-1$$

$$=-2\left(x^2-\frac{5}{2}x+\frac{25}{16}\right)-1+2\left(\frac{25}{16}\right)$$

$$=-2\left(x-\frac{5}{4}\right)^2-\frac{8}{8}+\frac{25}{8}$$

$$=-2\left(x-\frac{5}{4}\right)^2+\frac{17}{8}$$

vertex $\left(\frac{5}{4}, \frac{17}{8}\right)$

The $y$-value of the vertex is $\frac{17}{8}$.

The parabola opens down since $a=-2<0$.

Thus the range is $\left\{y \middle| y \le \frac{17}{8}\right\}$.

$$f(x)=2=-2x^2+5x-1$$
$$2x^2-5x+3=0$$
$$(2x-3)(x-1)=0$$
$$2x-3=0 \quad \text{or} \quad x-1=0$$
$$x=\frac{3}{2} \qquad x=1$$

**32.** $f(x)=2x^2+6x-5$

$$=2(x^2+3x)-5$$

$$=2\left(x^2+3x+\frac{9}{4}\right)-5-2\left(\frac{9}{4}\right)$$

$$=2\left(x+\frac{3}{2}\right)^2-5-\frac{9}{2}$$

$$=2\left(x+\frac{3}{2}\right)^2-\frac{10}{2}-\frac{9}{2}$$

$$=2\left(x+\frac{3}{2}\right)^2-\frac{19}{2}$$

vertex $\left(-\frac{3}{2}, -\frac{19}{2}\right)$

The $y$-value of the vertex is $-\frac{19}{2}$.

The parabola opens up since $a=2>0$.

Thus the range is $\left\{y \middle| y \ge -\frac{19}{2}\right\}$.

$$f(x)=15=2x^2+6x-5$$
$$0=2x^2+6x-20$$
$$0=2(x^2+3x-10)$$
$$0=2(x-2)(x+5)$$
$$x-2=0 \quad \text{or} \quad x+5=0$$
$$x=2 \qquad x=-5$$

**33.** $f(x)=x^2+3x+6$

$$=(x^2+3x)+6$$

$$=\left(x^2+3x+\frac{9}{4}\right)+6-\frac{9}{4}$$

$$=\left(x+\frac{3}{2}\right)^2+6-\frac{9}{4}$$

$$=\left(x+\frac{3}{2}\right)^2+\frac{24}{4}-\frac{9}{4}$$

$$=\left(x+\frac{3}{2}\right)^2+\frac{15}{4}$$

vertex $\left(-\frac{3}{2}, \frac{15}{4}\right)$

The $y$-value of the vertex is $\frac{15}{4}$.

The parabola opens up since $a=1>0$.

Thus the range is $\left\{y \middle| y \ge \frac{15}{4}\right\}$.

No, $3 \notin \left\{y \middle| y \ge \frac{15}{4}\right\}$.

**34.** $f(x)=-2x^2-x+1$

$$=-2\left(x^2+\frac{1}{2}x\right)+1$$

$$=-2\left(x^2+\frac{1}{2}x+\frac{1}{16}\right)+1+2\left(\frac{1}{16}\right)$$

$$=-2\left(x+\frac{1}{4}\right)^2+\frac{8}{8}+\frac{1}{8}$$

$$=-2\left(x+\frac{1}{4}\right)^2+\frac{9}{8}$$

vertex $\left(-\frac{1}{4}, \frac{9}{8}\right)$

The $y$-value of the vertex is $\frac{9}{8}$.

The parabola opens down since $a=-2<0$.

Thus the range is $\left\{y \middle| y \le \frac{9}{8}\right\}$.

Yes, $-2 \in \left\{y \middle| y \le \frac{9}{8}\right\}$.

**35.** $f(x)=x^2+8x$

$=(x^2+8x+16)-16$

$=(x+4)^2-16$

minimum value of –16 when $x = -4$

**36.** $f(x)=-x^2-6x$

$=-(x^2+6x)$

$=-(x^2+6x+9)+9$

$=-(x+3)^2+9$

maximum value of 9 when $x = -3$

**37.** $f(x)=-x^2+6x+2$

$=-(x^2-6x)+2$

$=-(x^2-6x+9)+2+9$

$=-(x-3)^2+11$

maximum value of 11 when $x = 3$

**38.** $f(x)=-x^2+10x-3$

$=-(x^2-10x)-3$

$=-(x^2-10x+25)-3+25$

$=-(x-5)^2+22$

maximum value of 22 when $x = 5$

**39.** $f(x)=2x^2+3x+1$

$=2\left(x^2+\frac{3}{2}x\right)+1$

$=2\left(x^2+\frac{3}{2}x+\frac{9}{16}\right)+1-2\left(\frac{9}{16}\right)$

$=2\left(x+\frac{3}{4}\right)^2+\frac{8}{8}-\frac{9}{8}$

$=2\left(x+\frac{3}{4}\right)^2-\frac{1}{8}$

minimum value of $-\frac{1}{8}$ when $x = -\frac{3}{4}$

**40.** $f(x)=3x^2+x-1$

$=3\left(x^2+\frac{1}{3}x\right)-1$

$=3\left(x^2+\frac{1}{3}x+\frac{1}{36}\right)-1-3\left(\frac{1}{36}\right)$

$=3\left(x+\frac{1}{6}\right)^2-\frac{12}{12}-\frac{1}{12}$

$=3\left(x+\frac{1}{6}\right)^2-\frac{13}{12}$

minimum value of $-\frac{13}{12}$ when $x = -\frac{1}{6}$

**41.** $f(x)=5x^2-11$

$=5(x^2)-11$

$=5(x-0)^2-11$

minimum value of –11 when $x = 0$

**42.** $f(x)=3x^2-41$

$=3(x^2)-41$

$=3(x-0)^2-41$

minimum value of –41 when $x = 0$

**43.** $f(x)=-\frac{1}{2}x^2+6x+17$

$=-\frac{1}{2}(x^2-12x)+17$

$=-\frac{1}{2}(x^2-12x+36)+17+18$

$=-\frac{1}{2}(x-6)^2+35$

maximum value of 35 when $x = 6$

**44.** $f(x)=-\frac{3}{4}x^2-\frac{2}{5}x+7$

$=-\frac{3}{4}\left(x^2+\frac{8}{15}x\right)+7$

$=-\frac{3}{4}\left(x^2+\frac{8}{15}x+\frac{16}{225}\right)+7+\frac{4}{75}$

$=-\frac{3}{4}\left(x+\frac{4}{15}\right)^2+\frac{529}{75}$

maximum value of $\frac{529}{75}=7\frac{4}{75}$ when $x = -\frac{4}{15}$

**45.** $h(x) = -\frac{3}{64}x^2 + 27 = -\frac{3}{64}(x-0)^2 + 27$

**a.** The maximum height of the arch is 27 feet.

**b.**
$$\begin{aligned} h(10) &= -\frac{3}{64}(10)^2 + 27 \\ &= -\frac{3}{64}(100) + 27 \\ &= -\frac{75}{16} + 27 \\ &= -\frac{75}{16} + \frac{432}{16} \\ &= \frac{357}{16} = 22\frac{5}{16} \quad \text{feet} \end{aligned}$$

**c.**
$$\begin{aligned} h(x) = 8 &= -\frac{3}{64}x^2 + 27 \\ 8 - 27 &= -\frac{3}{64}x^2 \\ -19 &= -\frac{3}{64}x^2 \\ 64(-19) &= -3x^2 \\ \frac{64(-19)}{-3} &= x^2 \\ \sqrt{\frac{64(-19)}{-3}} &= x \\ 8\sqrt{\frac{19}{3}} &= x \\ \frac{8\sqrt{19}\sqrt{3}}{3} &= x \\ \frac{8\sqrt{57}}{3} &= x \\ 20.1 &\approx x \end{aligned}$$

$h(x) = 8$ when $x \approx 20.1$ feet

**46.** $l + w = 240$

**a.** $w = 240 - l$

**b.**
$$A = l(240 - l)$$
$$A = 240l - l^2$$

**c.**
$$\begin{aligned} A &= -l^2 + 240l \\ A &= -(l^2 - 240l) \\ A &= -(l^2 - 240l + 120^2) + 120^2 \\ A &= -(l - 120)^2 + 120^2 \end{aligned}$$

Thus $l = 120$ and $w = 120$ produce the greatest area.

**47.** **a.**
$$\begin{aligned} 3w + 2l &= 600 \\ 3w &= 600 - 2l \\ w &= \frac{600 - 2l}{3} \end{aligned}$$

**b.**
$$\begin{aligned} A &= w \cdot l \\ A &= \left(\frac{600 - 2l}{3}\right) l \\ &= 200l - \frac{2}{3}l^2 \end{aligned}$$

**c.**
$$A = -\frac{2}{3}(l^2 - 300l)$$
$$A = -\frac{2}{3}(l^2 - 300l + 150^2) + 15{,}000$$

In standard form,

$$A = -\frac{2}{3}(l - 150)^2 + 15{,}000$$

The maximum area of 15,000 $\text{ft}^2$ is produced when $l = 150$ ft and the width $w = \frac{600 - 2(150)}{3} = 100$ ft .

**48.**
$$\begin{aligned} 4w + 2l &= 1200 \\ 2l &= 1200 - 4w \\ l &= \frac{1200 - 4w}{2} \\ l &= 600 - 2w \\ A &= w(600 - 2w) \\ A &= 600w - 2w^2 \\ A &= -2w^2 + 600w \\ A &= -2(w^2 - 300w) \\ A &= -2(w^2 - 300w + 150^2) + 2 \cdot 150^2 \\ A &= -2(w - 150)^2 + 45{,}000 \end{aligned}$$

Thus when $w = 150$, the length $l = \frac{1200 - 4(150)}{2} = 300$.

Thus the dimensions that yield the greatest enclosed area are $w = 150$ ft and $l = 300$ ft.

**49.** **a.** $T(t) = -0.7t^2 + 9.4t + 59.3$

$$= -0.7\left(t^2 - \frac{9.4}{0.7}t\right) + 59.3$$

$$= -0.7\left(t^2 - \frac{94}{7}t\right) + 59.3$$

$$= -0.7\left(t^2 - \frac{94}{7}t + \left[\frac{47}{7}\right]^2\right) + 59.3 + 0.7\left[\frac{47}{7}\right]^2$$

$$\approx -0.7\left(t - \frac{47}{7}\right)^2 + 90.857$$

$$\approx -0.7\left(t - 6\frac{5}{7}\right)^2 + 91$$

The temperature is a maximum when

$t = \frac{47}{7} = 6\frac{5}{7}$ hours after 6:00 A.M.

Note $\frac{5}{7}$(60 minutes) $\approx$ 43 minutes.

Thus the temperature is a maximum at 12:43 P.M.

**b.** The maximum temperature is approximately 91°F.

**50.** **a.** $N(t) = -0.6t^2 + 32.1t - 350$

$$= -0.6\left(t^2 - \frac{32.1}{0.6}t\right) - 350$$

$$= -0.6(t^2 - 53.5t) - 350$$

$$= -0.6\,[t^2 - 53.5t + (26.75)^2] - 350 + 0.6(26.75)^2$$

$$= -0.6(t - 26.75)^2 + 79.3375$$

$$\approx -0.6(t - 27)^2 + 79$$

The maximum number of larvae will survive at 27°C.

**b.** The maximum number of larvae that will survive is 79.

**c.** $N(t) = 0 = -0.6t^2 + 32.1t - 350$

$$t = \frac{-32.1 \pm \sqrt{(-32.1)^2 - 4(-0.6)(-350)}}{2(-0.6)}$$

$$t = \frac{-32.1 \pm \sqrt{1030.41 - 840}}{-1.2}$$

$$t = \frac{-32.1 \pm \sqrt{191.41}}{-1.2} \approx \frac{-32.1 \pm 13.8}{-1.2}$$

$$t = \frac{-32.1 + 13.8}{-1.2} = 15.25 \approx 15 \quad \text{or} \quad t = \frac{-32.1 - 13.8}{-1.2} = 38.25 \approx 38$$

Thus the $x$-intercepts to the nearest whole number for $N(t)$ are (15, 0) and (38, 0).

**d.** When the temperature is less than 15°C or greater than 38°C, none of the larvae survive.

**51.** $t = -\frac{b}{2a} = -\frac{82.86}{2(-279.67)} = 0.14814$

$E(0.14814) = -279.67(0.14814)^2 + 82.86(0.14814) \approx 6.1$

The maximum energy is 6.1 joules.

**52.** $h(t) = -9.8t^2 + 100t$

$h(t) = -9.8(t^2 - 10.2t)$

$h(t) = -9.8(t - 5.1)^2 + 254.9$

The maximum height is 255 m.

**53.** $h(x) = -0.002x^2 - 0.03x + 8$

$h(39) = -0.002(39)^2 - 0.03(39) + 8 = 3.788 > 3$

Solve for $x$ using quadratic formula.

$-0.002x^2 - 0.03x + 8 = 0$

$x^2 + 15x - 4000 = 0$

$$x = \frac{-15 \pm \sqrt{(15)^2 - 4(1)(-4000)}}{2(1)}$$

$$= \frac{-15 \pm \sqrt{16{,}225}}{2}$$, use positive value of $x$

$x \approx 56.2$

Yes, the conditions are satisfied.

**54.** $h(x) = -0.0009x^2 + 6$

$h(60.5) = -0.0009(60.5)^2 + 6 \approx 2.7$

Since 2.7 is less than 5.4 and greater than 2.5, yes, the pitch is a strike.

**55.** **a.** $E(v) = -0.018v^2 + 1.476v + 3.4$

$$= -0.018\left(v^2 - \frac{1.476}{0.018}v\right) + 3.4$$
$$= -0.018\,(v^2 - 82v) + 3.4$$
$$= -0.018\left(v^2 - 82v + 41^2\right) + 3.4 + 0.018(41)^2$$
$$= -0.018\,(v - 41)^2 + 33.658$$

The maximum fuel efficiency is obtained at a speed of 41 mph.

**b.** The maximum fuel efficiency for this car, to the nearest mile per gallon, is 34 mpg.

**56.** $h(x) = -0.0002348x^2 + 0.0375x$

$$= -0.0002348\left(x^2 - \frac{0.0375}{0.0002348}x\right)$$
$$= -0.0002348\left(x^2 - \frac{0.0375}{0.0002348}x + \left[\frac{1}{2}\cdot\frac{0.0375}{0.0002348}\right]^2\right) + 0.0002348\left[\frac{1}{2}\cdot\frac{0.0375}{0.0002348}\right]^2$$
$$\approx -0.0002348\left(x^2 - \frac{0.0375}{0.0002348}x + \left[\frac{1}{2}\cdot\frac{0.0375}{0.0002348}\right]^2\right) + 1.5$$

The maximum height of the field, to the nearest tenth of a foot, is 1.5 feet.

**57.** Let $y = 0$, then $0 = x^2 + 6x$

$$0 = x(x + 6)$$
$$x = 0 \quad \text{or} \quad x + 6 = 0$$
$$x = -6$$

The $x$-intercepts are (0, 0) and (–6, 0).

Let $x = 0$, then $f(x) = 0^2 + 6(0) = 0$

The $y$-intercept is (0, 0).

**58.** Let $y = 0$, then $0 = -x^2 + 4x$

$$0 = x(-x + 4)$$
$$x = 0 \quad \text{or} \quad -x + 4 = 0$$
$$x = 4$$

The $x$-intercepts are (0, 0) and (4, 0).

Let $x = 0$, then $f(x) = -0^2 + 4(0) = 0$

The $y$-intercept is (0, 0).

**59.** Let $y = 0$, then $0 = -3x^2 + 5x - 6$

$$x = \frac{-5 \pm \sqrt{5^2 - 4(-3)(-6)}}{2(-3)}$$

Since the discriminant $5^2 - 4(-3)(-6) = -47$ is negative, there are no $x$-intercepts.

Let $x = 0$, then $f(x) = -3(0)^2 + 5(0) - 6 = -6$

The $y$-intercept is (0, –6).

**60.** Let $y = 0$, then $0 = 2x^2 + 3x + 4$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(2)(4)}}{2(2)}$$

Since the discriminant $3^2 - 4(2)(4) = -23$ is negative, there are no $x$-intercepts.

Let $x = 0$, then $f(x) = 2(0)^2 + 3(0) + 4 = 4$

The $y$-intercept is (0, 4).

**61.** $-\frac{b}{2a} = -\frac{296}{2(-0.2)} = 740$

$R(740) = 296(740) - 0.2(740)^2 = 109{,}520$

Thus, 740 units yield a maximum revenue of $109,520.

**62.** $-\frac{b}{2a} = -\frac{810}{2(-0.6)} = 675$

$R(675) = 810(675) - 0.6(675)^2 = 273{,}375$

Thus, 675 units yield a maximum revenue of $273,375.

**63.** $-\frac{b}{2a} = -\frac{1.7}{2(-0.01)} = 85$

$P(85) = -0.01(85)^2 + 1.7(85) - 48 = 24.25$

Thus, 85 units yield a maximum profit of $24.25.

**64.** $-\frac{b}{2a} = -\frac{1.68}{2\left(-\frac{1}{14{,}000}\right)} = 11{,}760$

$P(11{,}760) = -\frac{(11{,}760)^2}{14{,}000} + 1.68(11{,}760) - 4000 = 5878.40$

Thus, 11,760 units yield a maximum profit of $5878.40.

**65.** $P(x) = R(x) - C(x)$
$= x(102.50 - 0.1x) - (52.50x + 1840)$
$= -0.1x^2 + 50x - 1840$

The break-even points occur when $R(x) = C(x)$ or $P(x) = 0$.

Thus, $0 = -0.1x^2 + 50x - 1840$

$$x = \frac{-50 \pm \sqrt{50^2 - 4(-0.1)(-1840)}}{2(-0.1)} = \frac{-50 \pm \sqrt{1764}}{-0.2} = \frac{-50 \pm 42}{-0.2}$$

$x = 40$ or $x = 460$

The break-even points occur when $x = 40$ or $x = 460$.

**66.** $P(x) = R(x) - C(x)$
$= x(210 - 0.25x) - (78x + 6399)$
$= -0.25x^2 + 132x - 6399$

$$-\frac{b}{2a} = -\frac{132}{2(-0.25)} = 264$$

$P(264) = -0.25(264)^2 + 132(264) - 6399$
$= \$11,025$, the maximum profit

The break-even points occur when $P(x) = 0$.

Thus, $0 = -0.25x^2 + 132x - 6399$

$$x = \frac{-132 \pm \sqrt{132^2 - 4(-0.25)(-6399)}}{2(-0.25)} = \frac{-132 \pm \sqrt{11025}}{-0.5}$$

$$= \frac{-132 \pm 105}{-0.5} \Rightarrow x = 54 \quad \text{or} \quad x = 474$$

The break-even points occur when $x = 54$ or $x = 474$.

**67.** Let $x$ = the number of people that take the tour.

**a.** $R(x) = x(15.00 + 0.25(60 - x))$
$= x(15.00 + 15 - 0.25x)$
$= -0.25x^2 + 30.00x$

**b.** $P(x) = R(x) - C(x)$
$= (-0.25x^2 + 30.00x) - (180 + 2.50x)$
$= -0.25x^2 + 27.50x - 180$

**c.** $$-\frac{b}{2a} = -\frac{27.50}{2(-0.25)} = 55$$

$P(55) = -0.25(55)^2 + 27.50(55) - 180$
$= \$576.25$

**d.** The maximum profit occurs when $x = 55$.

**68.** Let $x$ = the number of parcels.

**a.** $R(x) = xp = x(22 - 0.01x) = -0.01x^2 + 22x$

**b.** $P(x) = R(x) - C(x)$
$= (-0.01x^2 + 22x) - (2025 + 7x)$
$= -0.01x^2 + 15x - 2025$

**c.** $$-\frac{b}{2a} = -\frac{15}{2(-0.01)} = 750$$

$P(750) = -0.01(750)^2 + 15(750) - 2025$
$= \$3600$

**d.** $p(750) = 22 - 0.01(750) = \$14.50$

**e.** The break-even points occur when $R(x) = C(x)$.

$-0.01x^2 + 22x = 2025 + 7x$

$-0.01x^2 + 15x - 2025 = 0$

$$x = \frac{-(15) \pm \sqrt{15^2 - 4(-0.01)(-2025)}}{2(-0.01)}$$

$x = 150$ or $x = 1350$ are the break-even points. Thus the minimum number of parcels the air freight company must ship to break even is 150.

**69.** $h(t) = -16t^2 + 128t$

**a.** $$-\frac{b}{2a} = -\frac{128}{2(-16)} = 4 \text{ seconds}$$

**b.** $h(4) = -16(4)^2 + 128(4) = 256$ feet

**c.** $0 = -16t^2 + 128t$
$0 = -16t(t - 8)$
$-16t = 0$ or $t - 8 = 0$
$t = 0$ $\quad$ $t = 8$

The projectile hits the ground at $t = 8$ seconds.

**70.** $h(t) = -16t^2 + 64t + 80$

**a.** $-\frac{b}{2a} = -\frac{64}{2(-16)} = 2$

$h(2) = -16(2)^2 + 64(2) + 80$
$= 144$ feet

**b.** $-\frac{b}{2a} = -\frac{64}{2(-16)}$
$= 2$ seconds

**c.** $0 = -16t^2 + 64t + 80$
$0 = -16(t^2 - 4t - 5)$
$0 = -16(t-5)(t+1)$
$t - 5 = 0$ or $t + 1 = 0$
$t = 5$ $t = -1$ No

The projectile has height 0 feet at $t = 5$ seconds.

**71.** $y(x) = -0.014x^2 + 1.19x + 5$

$-\frac{b}{2a} = -\frac{1.19}{2(-0.014)}$
$= 42.5$

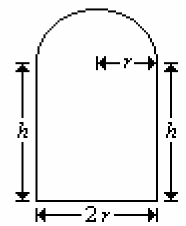
$y(42.5) = -0.014(42.5)^2 + 1.19(42.5) + 5$
$= 30.2875$
$\approx 30$ feet

**72.** $h(t) = -204.8t^2 + 256t$

$-\frac{b}{2a} = -\frac{256}{2(-204.8)}$
$= 0.625$

$h(0.625) = -204.5(0.625)^2 + 256(0.625)$
$= 80.1171875$
$\approx 80$ inches

**73.**




The perimeter is $48 = \pi r + h + 2r + h$.
Solve for *h*.

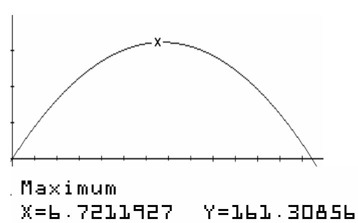
$$48 - \pi r - 2r = 2h$$

$$\frac{1}{2}(48 - \pi r - 2r) = h$$

Area = semicircle + rectangle

$$\begin{aligned} A &= \frac{1}{2}\pi r^2 + 2rh \\ &= \frac{1}{2}\pi r^2 + 2r\left(\frac{1}{2}\right)(48 - \pi r - 2r) \\ &= \frac{1}{2}\pi r^2 + r(48 - \pi r - 2r) \\ &= \frac{1}{2}\pi r^2 + 48r - \pi r^2 - 2r^2 \\ &= \left(\frac{1}{2}\pi - \pi - 2\right)r^2 + 48r \\ &= \left(-\frac{1}{2}\pi - 2\right)r^2 + 48r \end{aligned}$$

Graph the function *A* to find that its maximum occurs when $r \approx 6.72$ feet.




Xmin = 0, Xmax = 14, Xscl = 1
Ymin = –50, Ymax = 200, Yscl = 50

$$\begin{aligned} h &= \frac{1}{2}(48 - \pi r - 2r) \\ &\approx \frac{1}{2}(48 - \pi(6.72) - 2(6.72)) \\ &\approx 6.72 \text{ feet} \end{aligned}$$

Hence the optimal window has its semicircular radius equal to its height.

Note: Using calculus it can be shown that the exact value of $r = h = \dfrac{48}{\pi + 4}$.

**74.** $y = a(x-h)^2 + k$

$y = a(x-0)^2 + 6$

$y = ax^2 + 6$

$$500 = a(2100)^2 + 6$$

$$494 = a(2100)^2$$

$$\frac{494}{2100^2} = a$$

$$0.000112018 \approx a$$

$y = 0.000112018x^2 + 6$

## Connecting Concepts

**75.** $f(x)=x^2-(a+b)x+ab$

**a.** $x$-intercepts occur when $y=0$.

$$0=x^2-(a+b)x+ab$$
$$0=(x-a)(x-b)$$
$$x-a=0 \quad \text{or} \quad x-b=0$$
$$x=a \qquad x=b$$

Thus the x-intercepts are ($a$, 0) and ($b$, 0).

**b.** $-\dfrac{b}{2a}=\dfrac{(a+b)}{2(1)}=\dfrac{a+b}{2}$ which is the $x$-coordinate of the midpoint of the segment joining ($a$, 0) and ($b$, 0).

**76.** $f(x)=ax^2+bx+c$

**a.** $a<0$, $b$ and $c$ any real numbers

**b.** $a>0$, $b$ and $c$ any real numbers

**c.** $b^2-4ac>0$

**77.** Let $f(x)=ax^2+bx+c.$ We know

$f(2)=a(2)^2+b(2)+c=1 \qquad (1)$

$f(0)=a(0)^2+b(0)+c=4$

This implies $c=4$ and from Equation (1) we have

$4a+2b+4=1$ or $4a+2b=-2 \qquad (2)$

The $x$-value of the vertex is 2, and by the vertex formula we have $2=\frac{-b}{2a}$, which implies $b=-4a$.

Substituting $-4a$ for $b$ in Equation (2) gives us

$$4a+2(-4a)=-3$$
$$4a-8a=-3$$
$$-4a=-3$$
$$a=\frac{3}{4}$$

Substituting $\frac{3}{4}$ for $a$ in Equation (2) gives us

$$4\left(\frac{3}{4}\right)+2b=-3$$
$$3+2b=-3$$
$$2b=-6$$
$$b=-3$$

Thus the desired quadratic function is

$$f(x)=\frac{3}{4}x^2-3x+4.$$

**78.** Let $f(x)=ax^2+bx+c.$ We know $a<0$.

$f(-3)=a(-3)^2+b(-3)+c=2 \qquad (1)$

$f(0)=0+0+c=-5$, which implies $c=-5$.

Now the vertex is $(-3, 2)$, so $-3=\dfrac{-b}{2a}$ or $6a=b$ or

$a=\dfrac{b}{6}$. Thus, substituting in Equation (1) gives us

$$9(a)-3b-5=2$$
$$9\left(\frac{b}{6}\right)-3b-5=2$$
$$\frac{3}{2}b-3b-5=2$$
$$-\frac{3}{2}b=7$$
$$b=\frac{7\cdot 2}{-3}=-\frac{14}{3}$$
$$a=\frac{b}{6}$$
$$a=\frac{-\frac{14}{3}}{6}$$
$$a=-\frac{14}{18}=-\frac{7}{9}$$

Hence $f(x)=-\dfrac{7}{9}x^2-\dfrac{14}{3}x-5.$

**79.** $P=32=2x+2w$

$16=x+w$

**a.** $w=16-x$

**b.** Area $A=xw$

$$A=x(16-x)$$
$$A=16x-x^2$$

**80.** $A=16x-x^2$ attains its maximum when

$$x=-\frac{b}{2a}=-\frac{16}{2(-1)}=8.$$

Now $x=8$ implies

$$w=16-x$$
$$w=16-8$$
$$w=8$$

Thus the rectangle with perimeter 32 inches that has the largest area is the square with each side of length 8 inches.

**81.** The discriminant is $b^2-4(1)(-1)=b^2+4$, which is always positive. Thus the equation has two real zeros for all values of $b$.

**82.** The discriminant is $b^2 - 4(-1)(1) = b^2 + 4$, which is always positive. Thus the equation has two real zeros for all values of $b$.

**83.** Increasing the constant $c$ increases the height of each point on the graph by $c$ units.

**84.** Decreasing the coefficient $a$ shrinks the graph of the parabola toward the $x$-axis.

**85.** Let $x$ = one number. Then $8 - x$ = the other number. $P = x(8-x) = 8x - x^2$, vertex at $x = \frac{-b}{2a} = \frac{-8}{-2} = 4$.

Thus, $x = 4$ and $8 - x = 4$. The numbers are 4 and 4 .

**86.** Let $x$ = one number. Let $x + 12$ = the other number. $P = x(x+12) = x^2 + 12x$, vertex at $x = \frac{-b}{2a} = \frac{-12}{2} = -6$.

Thus, $x = -6$ and $x + 12 = -6 + 12 = 6$. The numbers are $-6$ and 6.

**87.** $x_1 = x,\ \ y_1 = x^3,\ \ x_2 = x+h,\ \ y_2 = (x+h)^3$

$$m = \frac{y_2 - y_1}{x_2 - x_1} - \frac{(x+h)^3 - x^3}{x+h-x} = \frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h} = \frac{3hx^2 + 3h^2x + h^3}{h} = \frac{h(3x^2 + 3hx + h^2)}{h} = 3x^2 + 3hx + h^2$$

**88.** $x_1 = x,\ \ y_1 = 4x^3 + x,\ \ x_2 = x+h,\ \ y_2 = 4(x+h)^3 + (x+h)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4(x+h)^3 + (x+h) - (4x^3 + x)}{x+h-x} = \frac{4(x^3 + 3hx^2 + 3h^2x + h^3) + x + h - 4x^3 - x}{h}$$

$$= \frac{4x^3 + 12hx^2 + 12h^2x + 4h^3 + x + h - 4x^3 - x}{h}$$

$$= \frac{12hx^2 + 12h^2x + 4h^3 + h}{h}$$

$$= \frac{h(12x^2 + 12hx + 4h^2 + 1)}{h}$$

$$= 12x^2 + 12hx + 4h^2 + 1$$

## Prepare for Section 2.5

**PS1.** $f(x) = x^2 + 4x - 6$

$-\frac{b}{2a} = -\frac{4}{2(1)} = -2$

$x = -2$

**PS2.** $f(3) = \frac{3(3)^4}{(3)^2 + 1} = \frac{243}{10} = 24.3$

$f(-3) = \frac{3(-3)^4}{(-3)^2 + 1} = \frac{243}{10} = 24.3$

$f(3) = f(-3)$

**PS3.** $f(-2) = 2(-2)^3 - 5(-2) = -16 + 10 = -6$

$-f(2) = -[2(2)^3 - 5(2)] = -[16 - 10] = -6$

$f(-2) = -f(2)$

**PS4.** $f(-2) - g(-2) = (-2)^2 - [-2 + 3] = 4 - 1 = 3$

$f(-1) - g(-1) = (-1)^2 - [-1 + 3] = 1 - 2 = -1$

$f(0) - g(0) = (0)^2 - [0 + 3] = 0 - 3 = -3$

$f(1) - g(1) = (1)^2 - [1 + 3] = 1 - 4 = -3$

$f(2) - g(2) = (2)^2 - [2 + 3] = 4 - 5 = -1$

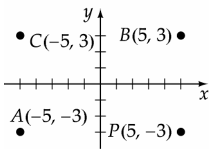
**PS5.** $\frac{-a+a}{2} = 0,\ \ \frac{b+b}{2} = b$

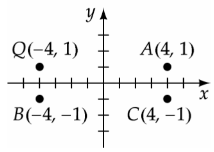
midpoint is $(0, b)$

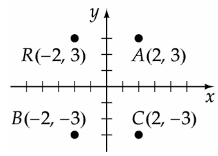
**PS6.** $\frac{-a+a}{2} = 0,\ \ \frac{-b+b}{2} = 0$

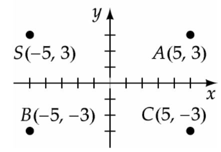
midpoint is (0, 0)

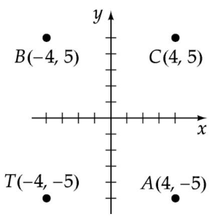
## Section 2.5

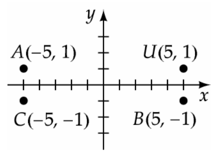
**1.** 

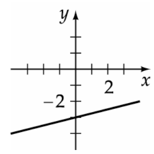

**2.** 

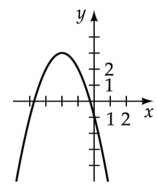

**3.** 

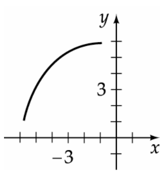

**4.** 

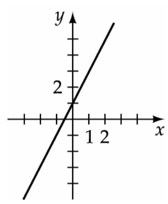

**5.** 

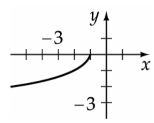

**6.** 

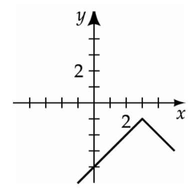

**7.** 

**8.** 

**9.** 

**10.** 

**11.** 

**12.** 

**13.** **a.** No **b.** Yes

**14.** **a.** Yes **b.** No

**15.** **a.** No **b.** No

**16.** **a.** No **b.** No

**17.** **a.** Yes **b.** Yes

**18.** **a.** Yes **b.** Yes

**19.** **a.** Yes **b.** Yes

**20.** **a.** No **b.** No

**21.** **a.** Yes **b.** Yes

**22.** Not symmetric with respect to the origin since $(-y) = (-x) + 1$ does not simplify to the original equation $y = x + 1$.

**23.** No, since $(-y) = 3(-x) - 2$ simplifies to $(-y) = -3x - 2$, which is not equivalent to the original equation $y = 3x - 2$.

**24.** Yes, since $(-y) = (-x)^3 - (-x)$ simplifies to $-y = -x^3 + x$, which is equivalent to the original equation $y = x^3 - x$.

**25.** Yes, since $(-y) = -(-x)^3$ implies $-y = x^3$ or $y = -x^3$, which is the original equation.

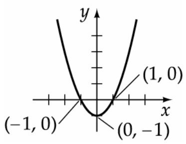
**26.** Yes, since $(-y) = \dfrac{9}{(-x)}$ is equivalent to the original equation $y = \dfrac{9}{x}$.

**27.** Yes, since $(-x)^2 + (-y)^2 = 10$ simplifies to the original equation.

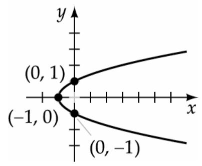
**28.** Yes, since $(-x)^2 - (-y)^2 = 4$ simplifies to the original equation.

**29.** Yes, since $-y = \dfrac{-x}{|-x|}$ simplifies to the original equation.

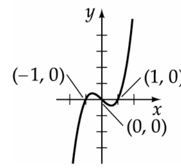
**30.** Yes, since $|-y| = |-x|$ simplifies to the original equation.

**31.** 

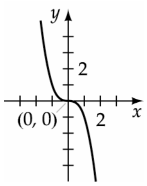

symmetric with respect to the $y$-axis

**32.** 

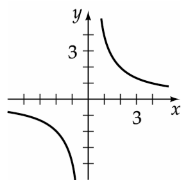

symmetric with respect to the $x$-axis

**33.** 

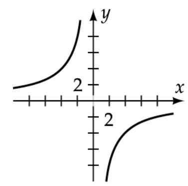

symmetric with respect to the origin

**34.** 

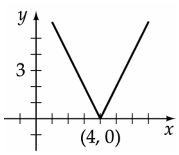

symmetric with respect to the origin

**35.** 

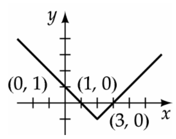

symmetric with respect to the origin

**36.** 

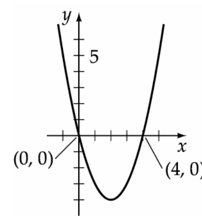

symmetric with respect to the origin

**37.** 

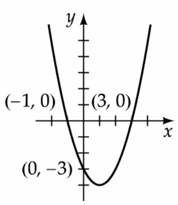

symmetric with respect to the line $x = 4$

**38.** 

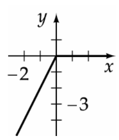

symmetric with respect to the line $x = 2$

**39.** 

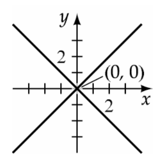

symmetric with respect to the line $x = 2$

**40.** 


symmetric with respect to the line $x = 2$

**41.** 


no symmetry

**42.** 


symmetric with respect to the $x$-axis, $y$-axis, and origin

**43.** Even since $g(-x) = (-x)^2 - 7 = x^2 - 7 = g(x)$.

**44.** Even, since $h(-x) = (-x)^2 + 1 = x^2 + 1 = h(x)$.

**45.** Odd, since
$$\begin{aligned} F(-x) &= (-x)^5 + (-x)^3 \\ &= -x^5 - x^3 \\ &= -F(x). \end{aligned}$$

**46.** Neither, since $G(-x) \neq G(x)$ and $G(-x) \neq -G(x)$.

**47.** Even

**48.** Even

**49.** Even

**50.** Neither

**51.** Even

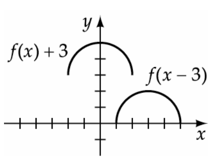
**52.** Even

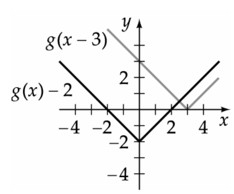
**53.** Even

**54.** Neither

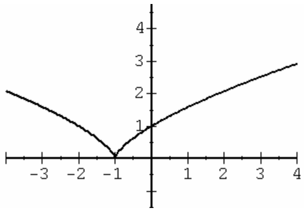
**55.** Neither

**56.** Odd

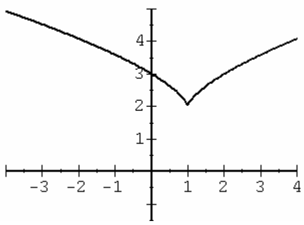
**57.** 


**58.** 

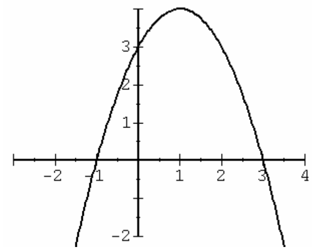
**59.** **a.** $f(x+2)$



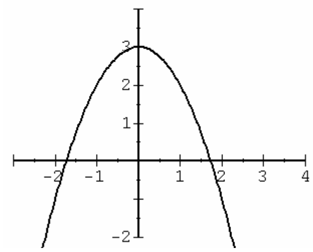
**b.** $f(x)+2$



**60.** **a.** $g(x-1)$



**b.** $g(x)-1$



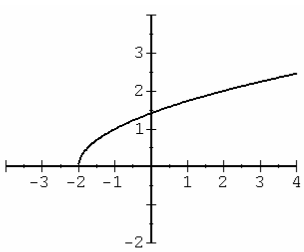
**61.** **a.** $f(x+3)$
$(-2-3, 5)=(-5, 5)$
$(0-3, -2)=(-3, -2)$
$(1-3, 0)=(-2, 0)$

**b.** $f(x)+1$
$(-2, 5+1)=(-2, 6)$
$(0, -2+1)=(0, -1)$
$(1, 0+1)=(1, 1)$

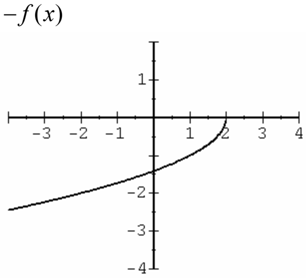
**62.** **a.** $g(x-2)$
$(-3+2, -1)=(-1, -1)$
$(1+2, -3)=(3, -3)$
$(4+2, 2)=(6, 2)$

**b.** $g(x)-2$
$(-3, -1-2)=(-3, -3)$
$(1, -3-2)=(1, -5)$
$(4, 2-2)=(4, 0)$

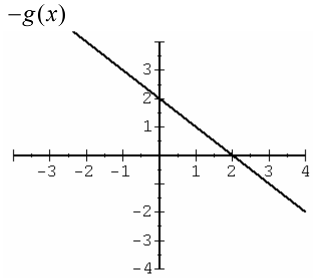
**63.** **a.** $f(-x)$



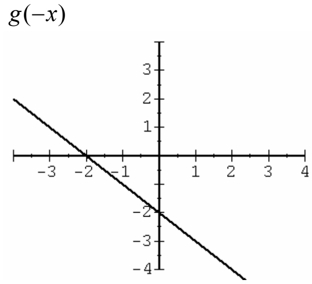
**b.** $-f(x)$



**64.** **a.** $-g(x)$



**b.** $g(-x)$



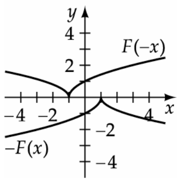
**65.** **a.** $f(-x)$
$(--1, 3)=(1, 3)$
$(-2, -4)$

**b.** $-f(x)$
$(-1, -3)$
$(2, --4)=(2, 4)$

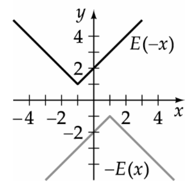
**66.** **a.** $-g(x)$
$(4, --5)=(4, 5)$
$(-3, -2)$

**b.** $g(-x)$
$(-4, -5)$
$(--3, 2)=(3, 2)$

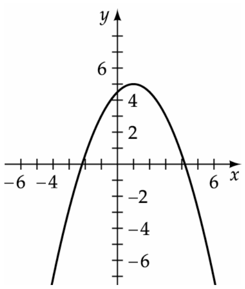
**67.**




**68.**

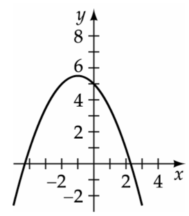



**69.**



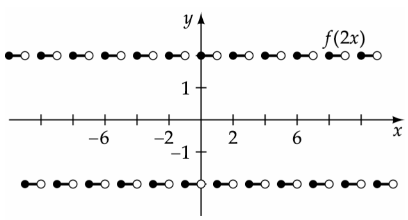
$y = -\frac{1}{2}m(x) + 3$

**70.**

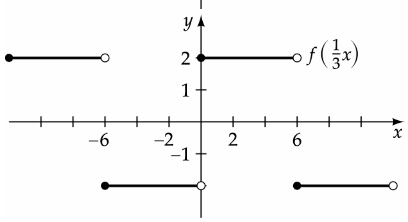


$y = -\frac{1}{2}n(x) + 1$

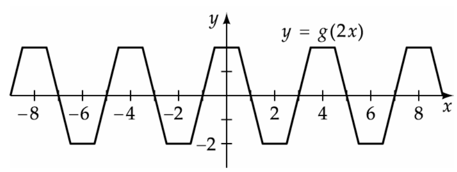
**71.** **a.**



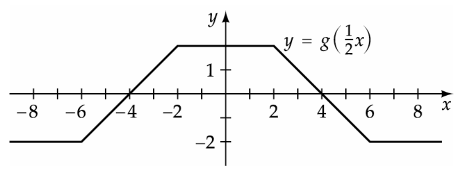

**b.**



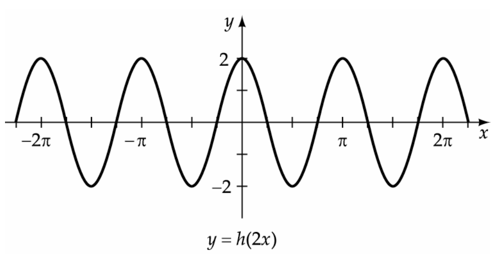

**72.** **a.**



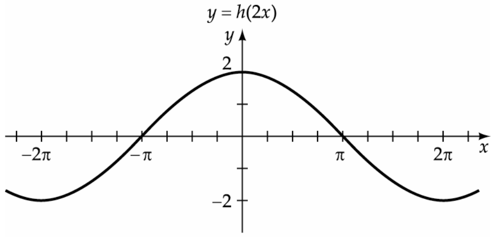

**b.**




**73.** **a.**

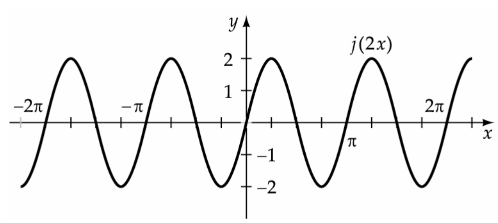



**b.**

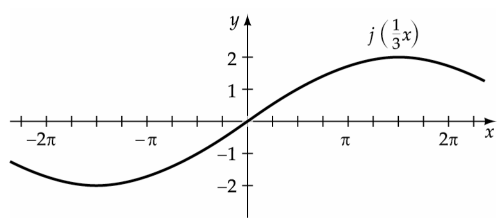


$y = h\left(\frac{1}{2}x\right)$

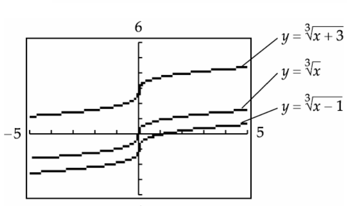
**74.** **a.**



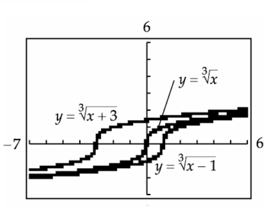

**b.**



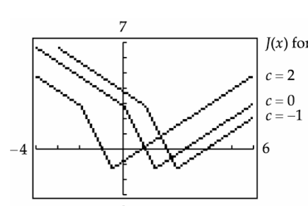

**75.**



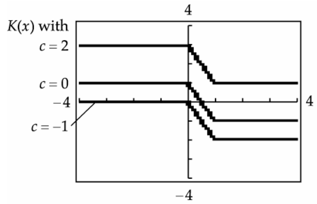

**76.**



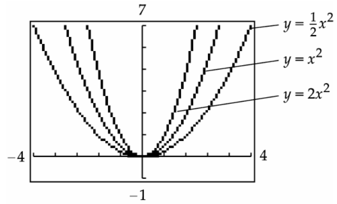

**77.**



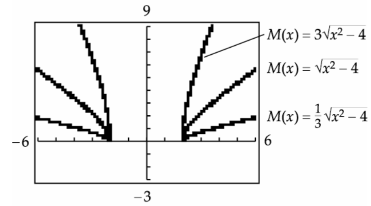
**78.**



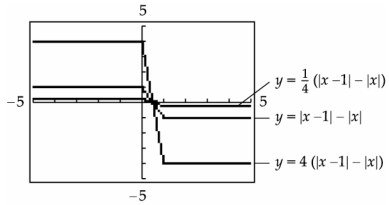

**79.**



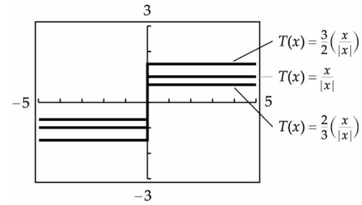

**80.**



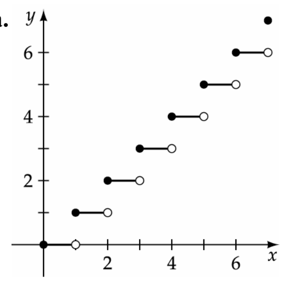

**81.**



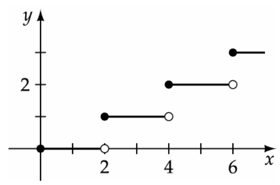

**82.**



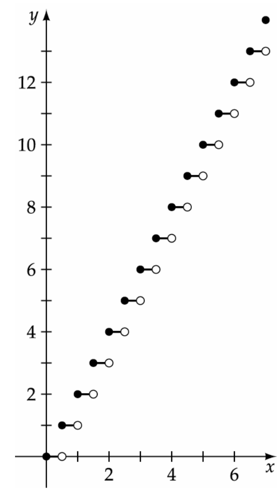

**83. a.**



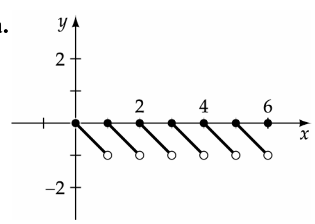
**b.**



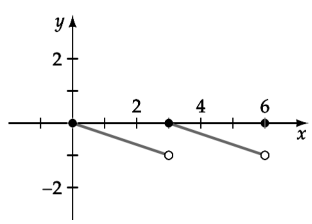
**c.**



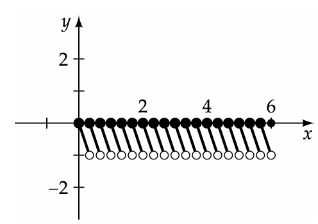
**84. a.**



**b.**



**c.**



## Connecting Concepts

**85.** **a.** $f(x) = \dfrac{2}{(x+1)^2 + 1} + 1$

**b.** $f(x) = -\dfrac{2}{(x-2)^2 + 1}$

**86.** **a.** $f(x) = (x-2)\sqrt{2 + (x-2)} - 3$

$f(x) = (x-2)\sqrt{x} - 3$

**b.** $f(x) = -\left[(x-3)\sqrt{2 + (x-3)}\right] - 2$

$= -\left[(x-3)\sqrt{x-1}\right] - 2$

$f(x) = (3-x)\sqrt{x-1} - 2$

## Prepare for Section 2.6

**PS1.** $(2x^2+3x-4)-(x^2+3x-5)=x^2+1$

**PS2.** $(3x^2-x+2)(2x-3)=6x^3-2x^2+4x-9x^2+3x-6$
$=6x^3-11x^2+7x-6$

**PS3.** $f(3a)=2(3a)^2-5(3a)+2$
$=18a^2-15a+2$

**PS4.** $f(2+h)=2(2+h)^2-5(2+h)+2$
$=2h^2+8h+8-5h-10+2$
$=2h^2+3h$

**PS5.** Domain: all real numbers except $x=1$

**PS6.** $2x-8=0$
$x=4$
Domain: $x\geq 4$ or $[4,\infty)$

## Section 2.6

**1.** $f(x)+g(x)=(x^2-2x-15)+(x+3)$
$=x^2-x-12$ Domain all real numbers
$f(x)-g(x)=(x^2-2x-15)-(x+3)$
$=x^2-3x-18$ Domain all real numbers
$f(x)g(x)=(x^2-2x-15)(x+3)$
$=x^3+x^2-21x-45$ Domain all real numbers
$f(x)/g(x)=(x^2-2x-15)/(x+3)$
$=x-5$ Domain $\{x|x\neq-3\}$

**2.** $f(x)+g(x)=(x^2-25)+(x-5)$
$=x^2+x-30$ Domain all real numbers
$f(x)-g(x)=(x^2-25)-(x-5)$
$=x^2-x-20$ Domain all real numbers
$f(x)g(x)=(x^2-25)(x-5)$
$=x^3-5x^2-25x+125$ Domain all real numbers
$f(x)/g(x)=(x^2-25)/(x-5)$
$=x+5$ Domain $\{x|x\neq 5\}$

**3.** $f(x)+g(x)=(2x+8)+(x+4)$
$=3x+12$ Domain all real numbers
$f(x)-g(x)=(2x+8)-(x+4)$
$=x+4$ Domain all real numbers
$f(x)g(x)=(2x+8)(x+4)$
$=2x^2+16x+32$ Domain all real numbers
$f(x)/g(x)=(2x+8)/(x+4)$
$=[2(x+4)]/(x+4)$
$=2$ Domain $\{x|x\neq-4\}$

**4.** $f(x)+g(x)=(5x-15)+(x-3)$
$=6x-18$ Domain all real numbers
$f(x)-g(x)=(5x-15)-(x-3)$
$=4x-12$ Domain all real numbers
$f(x)g(x)=(5x-15)(x-3)$
$=5x^2-30x+45$ Domain all real numbers
$f(x)/g(x)=(5x-15)/(x-3)$
$=[5(x-3)]/(x-3)$
$=5$ Domain $\{x|x\neq 3\}$

**5.** $f(x)+g(x)=(x^3-2x^2+7x)+x$
$=x^3-2x^2+8x$ Domain all real numbers
$f(x)-g(x)=(x^3-2x^2+7x)-x$
$=x^3-2x^2+6x$ Domain all real numbers
$f(x)g(x)=(x^3-2x^2+7x)x$
$=x^4-2x^3+7x^2$ Domain all real numbers
$f(x)/g(x)=(x^3-2x^2+7x)/x$
$=x^2-2x+7$ Domain $\{x\mid x\neq 0\}$

**6.** $f(x)+g(x)=(x^2-5x-8)+(-x)$
$=x^2-6x-8$ Domain all real numbers
$f(x)-g(x)=(x^2-5x-8)-(-x)$
$=x^2-4x-8$ Domain all real numbers
$f(x)g(x)=(x^2-5x-8)(-x)$
$=-x^3+5x^2+8x$ Domain all real numbers
$f(x)/g(x)=(x^2-5x-8x)/(-x)$
$=-x+5+\dfrac{8}{x}$ Domain $\{x|x\neq 0\}$

**7.** $f(x)+g(x)=(4x-7)+(2x^2+3x-5)$

$=2x^2+7x-12$ Domain all real numbers

$f(x)-g(x)=(4x-7)-(2x^2+3x-5)$

$=-2x^2+x-2$ Domain all real numbers

$f(x)g(x)=(4x-7)(2x^2+3x-5)$

$=6x^3-10x^2+12x^2-20x-21x+35$

$=6x^3+2x^2-41x+35$ Domain all real numbers

$f(x)/g(x)=(4x-7)/(2x^2+3x-5)$

$=\dfrac{4x-7}{2x^2+3x-5}$ Domain $\left\{x|x\neq 1, x\neq -\dfrac{5}{2}\right\}$

**8.** $f(x)+g(x)=(6x+10)+(3x^2+x-10)$

$=3x^2+7x$ Domain all real numbers

$f(x)-g(x)=(6x+10)-(3x^2+x-10)$

$=-3x^2+5x+20$ Domain all real numbers

$f(x)g(x)=(6x+10)(3x^2+x-10)$

$=18x^3+6x^2-60x+30x^2+10x-100$

$=18x^3+36x^2-50x-100$ Domain all real numbers

$f(x)/g(x)=(6x+10)/(3x^2+x-10)$

$=\dfrac{6x+10}{3x^2+x-10}$ Domain $\left\{x|x\neq -2, x\neq \dfrac{5}{3}\right\}$

**9.** $f(x)+g(x)=\sqrt{x-3}+x$ Domain $\{x \mid x\geq 3\}$

$f(x)-g(x)=\sqrt{x-3}-x$ Domain $\{x \mid x\geq 3\}$

$f(x)g(x)=x\sqrt{x-3}$ Domain $\{x \mid x\geq 3\}$

$f(x)/g(x)=\dfrac{\sqrt{x-3}}{x}+x$ Domain $\{x \mid x\geq 3\}$

**10.** $f(x)+g(x)=\sqrt{x-4}-x$ Domain $\{x \mid x\geq 4\}$

$f(x)-g(x)=\sqrt{x-4}+x$ Domain $\{x \mid x\geq 4\}$

$f(x)g(x)=-x\sqrt{x-4}$ Domain $\{x \mid x\geq 4\}$

$f(x)/g(x)=-\dfrac{\sqrt{x-4}}{x}$ Domain $\{x \mid x\geq 4\}$

**11.** $f(x)+g(x)=\sqrt{4-x^2}+2+x$ Domain $\{x \mid -2\leq x\leq 2\}$

$f(x)-g(x)=\sqrt{4-x^2}-2-x$ Domain $\{x \mid -2\leq x\leq 2\}$

$f(x)g(x)=\left(\sqrt{4-x^2}\right)(2+x)$ Domain $\{x \mid -2\leq x\leq 2\}$

$f(x)/g(x)=\dfrac{\sqrt{4-x^2}}{2+x}$ Domain $\{x \mid -2\leq x\leq 2\}$

**12.** $f(x)+g(x)=\sqrt{x^2-9}+x-3$ Domain $\{x \mid x\leq -3 \text{ or } x\geq 3\}$

$f(x)-g(x)=\sqrt{x^2-9}-x+3$ Domain $\{x \mid x\leq -3 \text{ or } x\geq 3\}$

$f(x)g(x)=\left(\sqrt{x^2-9}\right)(x-3)$ Domain $\{x \mid x\leq -3 \text{ or } x\geq 3\}$

$f(x)/g(x)=\dfrac{\sqrt{x^2-9}}{x-3}$ Domain $\{x \mid x\leq -3 \text{ or } x\geq 3\}$

**13.** $(f+g)(x) = x^2 - x - 2$

$(f+g)(5) = (5)^2 - (5) - 2$
$= 25 - 5 - 2$
$= 18$

**14.** $(f+g)(x) = x^2 - x - 2$

$(f+g)(-7) = (-7)^2 - (-7) - 2$
$= 49 + 7 - 2$
$= 54$

**15.** $(f+g)(x) = x^2 - x - 2$

$(f+g)\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) - 2$
$= \frac{1}{4} - \frac{1}{2} - 2$
$= -\frac{9}{4}$

**16.** $(f+g)(x) = x^2 - x - 2$

$(f+g)\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right) - 2$
$= \frac{4}{9} - \frac{2}{3} - 2$
$= -\frac{20}{9}$

**17.** $(f-g)(x) = x^2 - 5x + 6$

$(f-g)(-3) = (-3)^2 - 5(-3) + 6$
$= 9 + 15 + 6$
$= 30$

**18.** $(f-g)(x) = x^2 - 5x + 6$

$(f-g)(24) = (24)^2 - 5(24) + 6$
$= 576 - 120 + 6$
$= 462$

**19.** $(f-g)(x) = x^2 - 5x + 6$

$(f-g)(-1) = (-1)^2 - 5(-1) + 6$
$= 1 + 5 + 6$
$= 12$

**20.** $(f-g)(x) = x^2 - 5x + 6$

$(f-g)(0) = (0)^2 - 5(0) + 6$
$= 6$

**21.** $(fg)(x) = \left(x^2 - 3x + 2\right)(2x - 4)$
$= 2x^3 - 6x^2 + 4x - 4x^2 + 12x - 8$
$= 2x^3 - 10x^2 + 16x - 8$

$(fg)(7) = 2(7)^3 - 10(7)^2 + 16(7) - 8$
$= 686 - 490 + 112 - 8$
$= 300$

**22.** $(fg)(x) = 2x^3 - 10x^2 + 16x - 8$

$(fg)(-3) = 2(-3)^3 - 10(-3)^2 + 16(-3) - 8$
$= -54 - 90 - 48 - 8$
$= -200$

**23.** $(fg)(x) = 2x^3 - 10x^2 + 16x - 8$

$(fg)\left(\frac{2}{5}\right) = 2\left(\frac{2}{5}\right)^3 - 10\left(\frac{2}{5}\right)^2 + 16\left(\frac{2}{5}\right) - 8$
$= \frac{16}{125} - \frac{40}{25} + \frac{32}{5} - 8$
$= \frac{-384}{125} = -3.072$

**24.** $(fg)(x) = 2x^3 - 10x^2 + 16x - 8$

$(fg)(-100) = 2(-100)^3 - 10(-100)^2 + 16(-100) - 8$
$= -2{,}000{,}000 - 100{,}000 - 1600 - 8$
$= -2{,}101{,}608$

**25.** $\left(\frac{f}{g}\right)(x) = \frac{x^2 - 3x + 2}{2x - 4}$

$\left(\frac{f}{g}\right)(x) = \frac{1}{2}x - \frac{1}{2}$

$\left(\frac{f}{g}\right)(-4) = \frac{1}{2}(-4) - \frac{1}{2}$
$= -2 - \frac{1}{2}$
$= -2\frac{1}{2}$ or $-\frac{5}{2}$

**26.** $\left(\frac{f}{g}\right)(x) = \frac{1}{2}x - \frac{1}{2}$

$\left(\frac{f}{g}\right)(11) = \frac{1}{2}(11) - \frac{1}{2}$
$= \frac{11}{2} - \frac{1}{2}$
$= \frac{10}{2} = 5$

**27.** $\left(\frac{f}{g}\right)(x) = \frac{1}{2}x - \frac{1}{2}$

$$\begin{aligned}\left(\frac{f}{g}\right)\left(\frac{1}{2}\right) &= \frac{1}{2}\left(\frac{1}{2}\right) - \frac{1}{2}\\ &= \frac{1}{4} - \frac{1}{2}\\ &= -\frac{1}{4}\end{aligned}$$

**28.** $\left(\frac{f}{g}\right)(x) = \frac{1}{2}x - \frac{1}{2}$

$$\begin{aligned}\left(\frac{f}{g}\right)\left(\frac{1}{4}\right) &= \frac{1}{2}\left(\frac{1}{4}\right) - \frac{1}{2}\\ &= \frac{1}{8} - \frac{1}{2}\\ &= -\frac{3}{8}\end{aligned}$$

**29.**
$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{[2(x+h)+4]-(2x+4)}{h}\\ &= \frac{2x+2(h)+4-2x-4}{h}\\ &= \frac{2h}{h}\\ &= 2\end{aligned}$$

**30.**
$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{[4(x+h)-5]-(4x-5)}{h}\\ &= \frac{4x+4(h)-5-4x+5}{h}\\ &= \frac{4(h)}{h}\\ &= 4\end{aligned}$$

**31.**
$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{[(x+h)-6]-(x^2-6)}{h}\\ &= \frac{x^2+2x(h)+(h)^2-6-x^2+6}{h}\\ &= \frac{2x(h)+h^2}{h}\\ &= 2x+h\end{aligned}$$

**32.**
$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\left[(x+h)^2+11\right]-(x^2+11)}{h}\\ &= \frac{x^2+2xh+(h)^2+11-x^2-11}{h}\\ &= \frac{2xh+h^2}{h}\\ &= 2x+h\end{aligned}$$

**33.**
$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{2(x+h)^2+4(x+h)-3-(2x^2+4x-3)}{h}\\ &= \frac{2x^2+4xh+2h^2+4x+4h-3-2x^2-4x+3}{h}\\ &= \frac{4xh+2h^2+4h}{h}\\ &= 4x+2h+4\end{aligned}$$

**34.**
$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{2(x+h)^2-5(x+h)-(2x^2-5x+7)}{h}\\ &= \frac{2x^2+4xh+2h^2-5x-5h+7-2x^2+5x-7}{h}\\ &= \frac{4xh+2h^2-5h}{h}\\ &= 4x+2h-5\end{aligned}$$

**35.**
$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{-4(x+h)^2+6-(-4x^2+6)}{h}\\ &= \frac{-4x^2-8xh-4h^2+6+4x^2-6}{h}\\ &= \frac{-8xh-4h^2}{h}\\ &= -8x-4h\end{aligned}$$

**36.**

$$f(x) = -5x^2 - 4x$$

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{-5(x+h)^2 - 4(x+h) - (-5x^2 - 4x)}{h} \\ &= \frac{-5x^2 - 10x(h) - 5h^2 - 4x^2 - 4h + 5x^2 + 4x}{h} \\ &= \frac{-10x(h) - 5h^2 - 4h}{h} \\ &= -10x - 5h - 4\end{aligned}$$

**37.**

$$\begin{aligned}(g\circ f)(x) &= g[f(x)] \\ &= g[3x+5] \\ &= 2[3x+5] \\ &= 6x + 10 - 7 \\ &= 6x + 3\end{aligned}$$

$$\begin{aligned}(f\circ g)(x) &= f[g(x)] \\ &= f[2x-7] \\ &= 3[2x-7]+5 \\ &= 6x - 21 + 5 \\ &= 6x - 16\end{aligned}$$

**38.**

$$\begin{aligned}(g\circ f)(x) &= g[f(x)] \\ &= g[2x-7] \\ &= 3[2x-7]+2 \\ &= 6x - 21 + 2 \\ &= 6x - 19\end{aligned}$$

$$\begin{aligned}(f\circ g)(x) &= f[g(x)] \\ &= f[3x+2] \\ &= 2[3x+2]-7 \\ &= 6x + 4 - 7 \\ &= 6x - 3\end{aligned}$$

**39.**

$$\begin{aligned}(g\circ f)(x) &= g\left[x^2+4x-1\right] \\ &= \left[x^2+4x-1\right]+2 \\ &= x^2 + 4x + 1\end{aligned}$$

$$\begin{aligned}(f\circ g)(x) &= f[x+2] \\ &= [x+2]^2 + 4[x+2] - 1 \\ &= x^2 + 4x + 4 + 4x + 8 - 1 \\ &= x^2 + 8x + 11\end{aligned}$$

**40.**

$$\begin{aligned}(g\circ f)(x) &= g\left[x^2-11x\right] \\ &= 2\left[x^2-11x\right]+3 \\ &= 2x^2 - 22x + 3\end{aligned}$$

$$\begin{aligned}(f\circ g)(x) &= f[2x+3] \\ &= [2x+3]^2 - 11[2x+3] \\ &= 4x^2 + 12x + 9 - 22x - 33 \\ &= 4x^2 - 10x - 24\end{aligned}$$

**41.**

$$\begin{aligned}(g\circ f)(x) &= g[f(x)] \\ &= g\left[x^3+2x\right] \\ &= -5\left[x^3+2x\right] \\ &= -5x^3 - 10x\end{aligned}$$

$$\begin{aligned}(f\circ g)(x) &= f[g(x)] \\ &= f[-5x] \\ &= [-5x]^3 + 2[-5x] \\ &= -125x^3 - 10x\end{aligned}$$

**42.**

$$\begin{aligned}(g\circ f)(x) &= g[f(x)] \\ &= g\left[-x^3-7\right] \\ &= \left[-x^3-7\right]+1 \\ &= -x^3 - 6\end{aligned}$$

$$\begin{aligned}(f\circ g)(x) &= f[g(x)] \\ &= f[x+1] \\ &= -[x+1]^3 - 7 \\ &= -x^3 - 3x^2 - 3x - 1 - 7 \\ &= -x^3 - 3x^2 - 3x - 8\end{aligned}$$

**43.** 
$$(g\circ f)(x)=g\left[f(x)\right]=g\left[\frac{2}{x+1}\right]=3\left[\frac{2}{x+1}\right]-5=\frac{6}{x+1}-\frac{5(x+1)}{x+1}=\frac{6-5x-5}{x+1}=\frac{1-5x}{x+1}$$

$$(f\circ g)(x)=f\left[g(x)\right]=f\left[3x-5\right]=\frac{2}{\left[3x-5\right]+1}=\frac{2}{3x-4}$$

**44.** 
$$(g\circ f)(x)=g\left[f(x)\right]=g\left[\sqrt{x+4}\right]=\frac{1}{\sqrt{x+4}}=\frac{\sqrt{x+4}}{x+4}$$

$$(f\circ g)(x)=f\left[g(x)\right]=f\frac{1}{x}=\sqrt{\frac{1}{x}+4}=\sqrt{\frac{1+4x}{x}}=\frac{\sqrt{1+4x}}{\sqrt{x}}=\frac{\sqrt{x}\sqrt{1+4x}}{x}=\frac{\sqrt{x+4x^2}}{x}$$

**45.** 
$$(g\circ f)(x)=g\left[f(x)\right]=g\left[\frac{1}{x^2}\right]=\sqrt{\left[\frac{1}{x^2}\right]-1}=\sqrt{\frac{1-x^2}{x^2}}=\frac{\sqrt{1-x^2}}{|x|}$$

$$(f\circ g)(x)=f\left[g(x)\right]=f\left[\sqrt{x-1}\right]=\frac{1}{\left[\sqrt{x-1}\right]^2}=\frac{1}{x-1}$$

**46.** 
$$(g\circ f)(x)=g\left[f(x)\right]=g\left[\frac{6}{x-2}\right]=\frac{3}{5\left[\frac{6}{x-2}\right]}=\frac{3}{\left(\frac{30}{x-2}\right)}=3\cdot\frac{x-2}{30}=\frac{x-2}{10}$$

$$(f\circ g)(x)=f\left[g(x)\right]=f\left[\frac{3}{5x}\right]=\frac{6}{\left[\frac{3}{5x}\right]-2}=\frac{6}{\left(\frac{3\text{-}10\text{x}}{5\text{x}}\right)}=\frac{30x}{3-10x}$$

**47.** $(g\circ f)(x)=g\left[\dfrac{3}{|5-x|}\right]$

$=-\dfrac{2}{\left[\dfrac{3}{|5-x|}\right]}$

$=\dfrac{-2|5-x|}{3}$

$(f\circ g)(x)=f\left[-\dfrac{2}{x}\right]$

$=\dfrac{3}{\left|5-\left[-\dfrac{2}{x}\right]\right|}$

$=\dfrac{3}{\left|5+\dfrac{2}{x}\right|}$

$=\dfrac{3}{\dfrac{[5\text{x}+2]}{|x|}}$

$=\dfrac{3|x|}{|5x+2|}$

**48.** $(g\circ f)(x)=g\left[|2x+1|\right]$

$=3\left[|2x+1|\right]^2-1$

$=3(2x+1)^2-1$

$=3(4x^2+4x+1)-1$

$=12x^2+12x+3-1$

$=12x^2+12x+2$

$(f\circ g)(x)=f\left[g(x)\right]$

$=f\left[3x^2-1\right]$

$=\left|2\left[3x^2-1\right]+1\right|$

$=\left|6x^2-2+1\right|$

$=\left|6x^2-1\right|$

Use the results to work Exercises 49 to 64.

**49.** $(g\circ f)(x)=4x^2+2x-6$

$(g\circ f)(4)=4(4)^2+2(4)-6$

$=64+8-6$

$=66$

**50.** $(f\circ g)(x)=2x^2-10x+3$

$(f\circ g)(4)=2(4)^2-10(4)+3$

$=32-40+3$

$=-5$

**51.** $(f\circ g)(x)=2x^2-10x+3$

$(f\circ g)(-3)=2(-3)^2-10(-3)+3$

$=18+30+3$

$=51$

**52.** $(g\circ f)(x)=4x^2+2x-6$

$(g\circ f)(-1)=4(-1)^2+2(-1)-6$

$=4-2-6$

$=-4$

**53.** $(g\circ h)(x)=9x^4-9x^2-4$

$(g\circ h)(0)=9(0)^4-9(0)^2-4$

$=-4$

**54.** $(h\circ g)(x)=-3x^4+30x^3-75x^2+4$

$(h\circ g)(0)=-3(0)^4+30(0)^3-75(0)^2+4$

$=4$

**55.** $(f\circ f)(x)=4x+9$

$(f\circ f)(8)=4(8)+9$

$=41$

**56.** $(f\circ f)(x)=4x+9$

$(f\circ f)(-8)=4(-8)+9$

$=-23$

**57.** $(h\circ g)(x)=-3x^4+30x^3-75x^2+4$

$(h\circ g)\left(\dfrac{2}{5}\right)=-3\left(\dfrac{2}{5}\right)^4+30\left(\dfrac{2}{5}\right)^3-75\left(\dfrac{2}{5}\right)^2+4$

$=-\dfrac{48}{625}+\dfrac{240}{125}-\dfrac{300}{25}+4$

$=\dfrac{-48+1200-7500+2500}{625}$

$=-\dfrac{3848}{625}$

**58.** $(g\circ h)(x)=9x^4-9x^2-4$

$(g\circ h)\left(-\dfrac{1}{3}\right)=9\left(-\dfrac{1}{3}\right)^4-9\left(-\dfrac{1}{3}\right)^2-4$

$=\dfrac{9}{81}-\dfrac{9}{9}-4$

$=\dfrac{1}{9}-1-4$

$=-4\dfrac{8}{9}$ or $-\dfrac{44}{9}$

**59.** $(g \circ f)(x) = 4x^2 + 2x - 6$

$$\begin{aligned}(g \circ f)(\sqrt{3}) &= 4(\sqrt{3})^2 + 2(\sqrt{3}) - 6\\ &= 12 + 2\sqrt{3} - 6\\ &= 6 + 2\sqrt{3}\end{aligned}$$

**60.** $(f \circ g)(x) = 2x^2 - 10x + 3$

$$\begin{aligned}(f \circ g)(\sqrt{2}) &= 2(\sqrt{2})^2 - 10(\sqrt{2}) + 3\\ &= 4 - 10\sqrt{2} + 3\\ &= 7 - 10\sqrt{2}\end{aligned}$$

**61.** $(g \circ f)(x) = 4x^2 + 2x - 6$

$$\begin{aligned}(g \circ f)(2c) &= 4(2c)^2 + 2(2c) - 6\\ &= 16c^2 + 4c - 6\end{aligned}$$

**62.** $(f \circ g)(x) = 2x^2 - 10x + 3$

$$\begin{aligned}(f \circ g)(3k) &= 2(3k)^2 - 10(3k) + 3\\ &= 18k^2 - 30k + 3\end{aligned}$$

**63.** $(g \circ h)(x) = 9x^4 - 9x^2 - 4$

$$\begin{aligned}(g \circ h)(k+1) &= 9(k+1)^4 - 9(k+1)^2 - 4\\ &= 9(k^4 + 4k^3 + 6k^2 + 4k + 1) - 9k^2 - 18k - 9 - 4\\ &= 9k^4 + 36k^3 + 54k^2 + 36k + 9 - 9k^2 - 18k - 13\\ &= 9k^4 + 36k^3 + 45k^2 + 18k - 4\end{aligned}$$

**64.** $(h \circ g)(x) = -3x^4 + 30x^3 - 75x^2 + 4$

$$\begin{aligned}(h \circ g)(k-1) &= -3(k-1)^4 + 30(k-1)^3 - 75(k-1)^2 + 4\\ &= -3k^4 + 12k^3 - 18k^2 + 12k - 3 + 30k^3 - 90k^2 + 90k - 30 - 75k^2 + 150k - 75 + 4\\ &= -3k^4 + 42k^3 - 183k^2 + 252k - 104\end{aligned}$$

**65.** **a.** $r = 1.5t$ and $A = \pi r^2$

so $A(t) = \pi[r(t)]^2$

$= \pi(1.5t)^2$

$A(2) = 2.25\pi(2)^2$

$= 9\pi$ square feet

$\approx 28.27$ square feet

**b.** $r = 1.5t$

$h = 2r = 2(1.5t) = 3t$ and

$V = \frac{1}{3}\pi r^2 h$ so

$V(t) = \frac{1}{3}\pi(1.5t)^2[3t]$

$= 2.25\pi t^3$

Note: $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}(\pi r^2) = \frac{1}{3}hA$

$= \frac{1}{3}(3t)(2.25\pi t^2) = 2.25\pi t^3$

$V(3) = 2.25\pi(3)^3$

$= 60.75\pi$ cubic feet

$\approx 190.85$ cubic feet

**66.** **a.** $l = 3 - 0.5t$ for $0 \le t \le 6$

$= -3 + 0.5t$ for $6 \le t \le 14$

or $l = |3 - 0.5t|$

$w = 2 - 0.2t$ for $0 \le t \le 10$

$= -2 + 0.2t$ for $10 \le t \le 14$

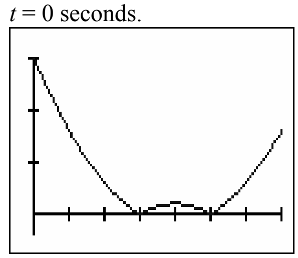
or $w = |2 - 0.2t|$

**b.** $l = lw = |3 - 0.5t|\ |2\text{-}0.2t|$

$= |(3 - 0.5t)(2 - 0.2t)|$

**c.** $A$ is increasing on $[6, 8]$ and on $[10, 14]$; and $A$ is decreasing on $[0, 6]$ and on $[8, 10]$.

**d.** The highest point on the graph of $A$ occurs when $t = 0$ seconds.



Xmin = 0, Xmax =14, Xscl = 2,
Ymin = –1, Ymax = 6, Yscl = 2

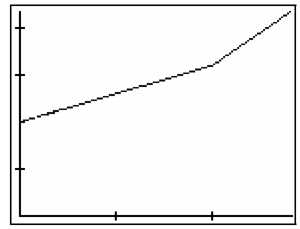
**67.** **a.** Since $d^2+4^2=s^2$,

$$d^2=s^2-16$$
$$d=\sqrt{s^2-16}$$
$$d=\sqrt{(48-t)^2-16} \quad \cdot\ s=48-t$$
$$=\sqrt{2304-96t+t^2-16}$$
$$=\sqrt{t^2-96t+2288}$$

**b.** $s(35)=48-35=13$ ft

$$d(35)=\sqrt{35^2-96(35)+2288}$$
$$=\sqrt{153}\approx 12.37 \text{ ft}$$

**68.** The perimeter is an increasing function over $0 \le t \le 14$.
The graph of $P=2(3+0.5t)+2|2-0.2t|$ is shown below.



Xmin = 0, Xmax = 14, Xscl = 5,
Ymin = −1, Ymax = 22, Yscl =5

**69.** $(Y \circ F)(x)=Y(F(x))$ converts $x$ inches to yards.
$F$ takes $x$ inches to feet, and then $Y$ takes feet to yards.

**70.** $(I \circ F)(x)=I(F(x))$ converts $x$ yards to inches.
$F$ takes $x$ yards to feet, and then $I$ takes feet to inches.

**71.** **a.** On $[0,\ 1]$, $a=0$

$\Delta t=1-0=1$

$C(a+\Delta t)=C(1)=99.8$

$C(a)=C(0)=0$

Average rate of change $=\dfrac{C(1)-C(0)}{1}=99.8-0=99.8$

This is identical to the slope of the line through
$(0,\ C(0))$ and $(1,\ C(1))$ since $m=\dfrac{C(1)-C(0)}{1-0}=C(1)-C(0)$

**b.** On $[0,\ 0.5]$, $a=0$, $\Delta t=0.5$

Average rate of change $=\dfrac{C(0.5)-C(0)}{0.5}=\dfrac{78.1-0}{0.5}=156.2$

**c.** On $[1,\ 2]$, $a=1$, $\Delta t=2-1=1$

Average rate of change $=\dfrac{C(2)-C(1)}{1}=\dfrac{50.1-99.8}{1}=-49.7$

**d.** On $[1,\ 1.5]$, $a=1$, $\Delta t=1.5-1=0.5$

Average rate of change $=\dfrac{C(1.5)-C(1)}{0.5}=\dfrac{84.4-99.8}{0.5}=\dfrac{-15.4}{0.5}=-30.8$

**e.** On $[1,\ 1.25]$, $a=1$, $\Delta t=1.25-1=0.25$

Average rate of change $=\dfrac{C(1.25)-C(1)}{0.25}=\dfrac{95.7-99.8}{0.25}=\dfrac{-4.1}{0.25}=-16.4$

**f.** On $[1,\ 1+\Delta t]$, $Con\,(1+\Delta t)=25(1+\Delta t)^3-150(1+\Delta t)^2+225(1+\Delta t)$

$$=25(1+3(\Delta t)+3(\Delta t)^3)-150(1+2(\Delta t)+(\Delta t)^2)+225(1+\Delta t)$$
$$=25+75(\Delta t)+75(\Delta t)^2)+25(\Delta t)^3-150-300(\Delta t)-150(\Delta t)^2+225+225(\Delta t)$$
$$=100-75(\Delta t)^2+25(\Delta t)^3$$
$$Con(1)=100$$

Average rate of change $=\dfrac{Con(1+\Delta t)-Con(1)}{\Delta t}=\dfrac{100-75(\Delta t)^2+25(\Delta t)^3-100}{\Delta t}$

$$=\frac{-75(\Delta t)^2+25(\Delta t)^3}{\Delta t}$$
$$=-75(\Delta t)+25(\Delta t)^2$$

As $\Delta t$ approaches 0, the average rate of change over $[1,\ 1+\Delta t]$ seems to approach 0.

**72.** **a.** On $[2,\ 3]$, $a=2$

$\Delta t = 3-2=1$

$f(a+\Delta t)=f(3)=6\cdot 3^2=54$

$f(a)=f(2)=6\cdot 2^2=24$

Average velocity $=\dfrac{f(a+\Delta t)-f(a)}{\Delta t}=\dfrac{f(3)-f(2)}{1}=54-24=30$ feet per second

This is identical to the slope of the line through $(2, f(2))$ and $(3, f(3))$ since $m=\dfrac{f(3)-f(2)}{3-2}=f(3)-f(2)$.

**b.** On $[2,\ 2.5]$, $a=2$ ,

$\Delta t = 2.5-2=0.5$

$f(a+\Delta t)=f(2.5)=6(2.5)^2=37.5$

Average velocity $=\dfrac{f(2.5)-f(2)}{0.5}=\dfrac{37.5-24}{0.5}=\dfrac{13.5}{0.5}=27$ feet per second

**c.** On $[2,\ 2.1]$, $a=2$

$\Delta t = 2.1-2=0.1$

$f(a+\Delta t)=f(2.1)=6(2.1)^2=26.46$

Average velocity $=\dfrac{f(2.1)-f(2)}{0.1}=\dfrac{26.46-24}{0.1}=\dfrac{2.46}{0.1}=24.6$ feet per second

**d.** On $[2,\ 2.01]$, $a=2$

$\Delta t = 2.01-2=0.01$

$f(a+\Delta t)=f(2.01)=6(2.01)^2=24.2406$

Average velocity $=\dfrac{f(2.01)-f(2)}{0.01}=\dfrac{24.2406-24}{0.01}=\dfrac{0.2406}{0.01}=24.06$ feet per second

**e.** On $[2,\ 2.001]$, $a=2$

$\Delta t = 2.001-2=0.001$

$f(a+\Delta t)=f(2.001)=6(2.001)^2=24.024006$

Average velocity $=\dfrac{f(2.001)-f(2)}{0.001}=\dfrac{24.024006-24}{0.001}=\dfrac{0.24006}{0.001}=24.006$ feet per second

**f.** On $[2,\ 2+\Delta t]$, $Con\ \dfrac{f(2+\Delta t)-f(2)}{\Delta t}=\dfrac{6(2+\Delta t)^2-24}{\Delta t}=\dfrac{6(4+4(\Delta t)+(\Delta t)^2-24}{\Delta t}=\dfrac{24+24(\Delta t)+6(\Delta t)^2-24}{(\Delta t)}$

$$=\frac{24\Delta t+6(\Delta t)^2}{\Delta t}=24+6(\Delta t)$$

As $\Delta t$ approaches 0, the average velocity seems to approach 24 feet per second.

## Connecting Concepts

**73.** 
$$\begin{aligned}(g\circ f)(x)&=g[f(x)]\\&=g[2x+3]\\&=5(2x+3)+12\\&=10x+15+12\\&=10x+27\end{aligned}\qquad\begin{aligned}(f\circ g)(x)&=f[g(x)]\\&=f[5x+12]\\&=2(5x+12)+3\\&=10x+24+3\\&=10x+27\end{aligned}$$

$(g\circ f)(x)=(f\circ g)(x)$

**74.**
$$\begin{aligned}(g\circ f)(x)&=g[f(x)]\\&=g[4x-2]\\&=7(4x-2)-4\\&=28x-14-4\\&=28x-18\end{aligned}$$
$$\begin{aligned}(f\circ g)(x)&=f[g(x)]\\&=f[7x-4]\\&=4(7x-4)-2\\&=28x-16-2\\&=28x-18\end{aligned}$$
$$(g\circ f)(x)=(f\circ g)(x)$$

**75.**
$$\begin{aligned}(g\circ f)(x)&=g[f(x)]\\&=g\left[\frac{6x}{x-1}\right]\\&=\frac{5\left(\frac{6x}{x-1}\right)}{\frac{6x}{x-1}-2}\\&=\frac{\frac{30x}{x-1}}{\frac{6x-2x+2}{x-1}}=\frac{\frac{30x}{x-1}}{\frac{4x+2}{x-1}}\\&=\frac{30x}{x-1}\cdot\frac{x-1}{2(2x+1)}\\&=\frac{15x}{2x+1}\end{aligned}$$
$$\begin{aligned}(f\circ g)(x)&=f[g(x)]\\&=f\left[\frac{5x}{x-2}\right]\\&=\frac{6\left(\frac{5x}{x-2}\right)}{\frac{5x}{x-2}-1}\\&=\frac{\frac{30x}{x-2}}{\frac{5x-x+2}{x-2}}=\frac{\frac{30x}{x-2}}{\frac{4x+2}{x-2}}\\&=\frac{30x}{x-2}\cdot\frac{x-2}{2(2x+1)}\\&=\frac{15x}{2x+1}\end{aligned}$$
$$(g\circ f)(x)=(f\circ g)(x)$$

**76.**
$$\begin{aligned}(g\circ f)(x)&=g[f(x)]\\&=g\left[\frac{5x}{x+3}\right]\\&=-\frac{2\left(\frac{5x}{x+3}\right)}{\frac{5x}{x+3}-4}\\&=-\frac{\frac{10x}{x+3}}{\frac{5x-4x-12}{x+3}}=-\frac{\frac{10x}{x+3}}{\frac{x-12}{x+3}}\\&=-\frac{10x}{x+3}\cdot\frac{x+3}{x-12}\\&=-\frac{10x}{x-12}\end{aligned}$$
$$\begin{aligned}(f\circ g)(x)&=f[g(x)]\\&=f\left[-\frac{2x}{x-4}\right]\\&=\frac{5\left(-\frac{2x}{x-4}\right)}{-\frac{2x}{x-4}+3}\\&=\frac{\frac{-10x}{x-4}}{\frac{-2x+3x-12}{x-4}}=\frac{\frac{-10x}{x-4}}{\frac{x-12}{x-4}}\\&=\frac{-10x}{x-4}\cdot\frac{x-4}{x-12}\\&=\frac{10x}{x-12}\end{aligned}$$
$$(g\circ f)(x)=(f\circ g)(x)$$

**77.**
$$\begin{aligned}(g\circ f)(x)&=g[f(x)]\\&=g[2x+3]\\&=\frac{[2x+3]-3}{2}\\&=\frac{2x}{2}\\&=x\end{aligned}$$
$$\begin{aligned}(f\circ g)(x)&=f[g(x)]\\&=f\left[\frac{x-3}{2}\right]\\&=2\left[\frac{x-3}{2}\right]+3\\&=x-3+3\\&=x\end{aligned}$$

**78.** $(g\circ f)(x)=g\left[f(x)\right]$
$=g\left[4x-5\right]$
$=\dfrac{\left[4x-5\right]+5}{4}$
$=\dfrac{4x}{4}$
$=x$

$(f\circ g)(x)=f\left[g(x)\right]$
$=f\left[\dfrac{x+5}{4}\right]$
$=4\left[\dfrac{x+5}{4}\right]-5$
$=x+5-5$
$=x$

**79.** $(g\circ f)(x)=g\left[f(x)\right]$
$=g\left[\dfrac{4}{x+1}\right]$
$=\dfrac{4-\left[\dfrac{4}{x+1}\right]}{\left[\dfrac{4}{x+1}\right]}$
$=\dfrac{\dfrac{4x+4-4}{x+1}}{\dfrac{4}{x+1}}$
$=\dfrac{4x}{x+1}\cdot\dfrac{x+1}{4}$
$=x$

$(f\circ g)(x)=f\left[g(x)\right]$
$=f\left[\dfrac{4-x}{x}\right]$
$=\dfrac{4}{\left[\dfrac{4-x}{x}\right]+1}$
$=\dfrac{4}{\dfrac{4-x+x}{x}}$
$=\dfrac{4}{\dfrac{4}{x}}$
$=4\cdot\dfrac{x}{4}$
$=x$

**80.** $(g\circ f)(x)=g\left[f(x)\right]$
$=g\left[\dfrac{2}{1-x}\right]$
$=\dfrac{\left[\dfrac{2}{1-x}\right]-2}{\left[\dfrac{2}{1-x}\right]}$
$=\dfrac{\dfrac{2-2+2x}{1-x}}{\dfrac{2}{1-x}}$
$=\dfrac{2x}{1-x}\cdot\dfrac{1-x}{2}$
$=x$

$(f\circ g)(x)=f\left[g(x)\right]$
$=f\left[\dfrac{x-2}{x}\right]$
$=\dfrac{2}{1-\left[\dfrac{x-2}{x}\right]}$
$=\dfrac{2}{\dfrac{x-x+2}{x}}$
$=\dfrac{2}{1}\cdot\dfrac{x}{x-x+2}$
$=\dfrac{2}{1}\cdot\dfrac{x}{2}$
$=x$

**81.** $(g\circ f)(x)=g\left[f(x)\right]$
$=g\left[x^3-1\right]$
$=\sqrt[3]{\left[x^3-1\right]+1}$
$=\sqrt[3]{x^3}$
$=x$

$(f\circ g)(x)=f\left[g(x)\right]$
$=f\left[\sqrt[3]{x+1}\right]$
$=\left[\sqrt[3]{x+1}\right]^3-1$
$=x+1-1$
$=x$

**82.** $(g \circ f)(x) = g[f(x)]$
$= g[-x^3+2]$
$= \sqrt[3]{2-[-x^3+2]}$
$= \sqrt[3]{2+x^3-2}$
$= \sqrt[3]{x^3}$
$= x$

$(f \circ g)(x) = f[g(x)]$
$= f[\sqrt[3]{2-x}]$
$= -[\sqrt[3]{2-x}]^3+2$
$= -(2-x)+2$
$= -2+x+2$
$= x$

## Prepare for Section 2.7

**PS1.** Slope: $-\frac{1}{3}$; $y$-intercept: (0, 4)

**PS2.** $3x - 4y = 12$
$y = \frac{3}{4}x - 3$
Slope: $\frac{3}{4}$; $y$-intercept: (0, –3)

**PS3.** $y = -0.45x + 2.3$

**PS4.** $y + 4 = -\frac{2}{3}(x-3)$
$y = -\frac{2}{3}x - 2$

**PS5.** $f(2) = 3(2)^2 + 4(2) - 1 = 12 + 8 - 1 = 19$

**PS6.** $|f(x_1) - y_1| + |f(x_2) - y_2| = |(2)^2 - 3 - (-1)| + |(4)^2 - 3 - 14|$
$= |4-3+1| + |16-3-14|$
$= 2 + 1$
$= 3$

## Section 2.7

**1.** The scatter diagram suggests no relationship between $x$ and $y$.

**2.** The scatter diagram suggest a nonlinear relationship between $x$ and $y$.

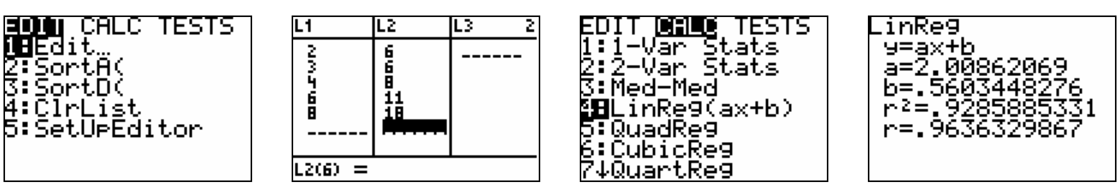
**3.** The scatter diagram suggests a linear relationship between $x$ and $y$.

**4.** The scatter diagram suggests a linear relationship between $x$ and $y$.

**5.** Figure A better approximates a graph that can be modeled by an equation than does Figure B. Thus Figure A has a coefficient of determination closer to 1.

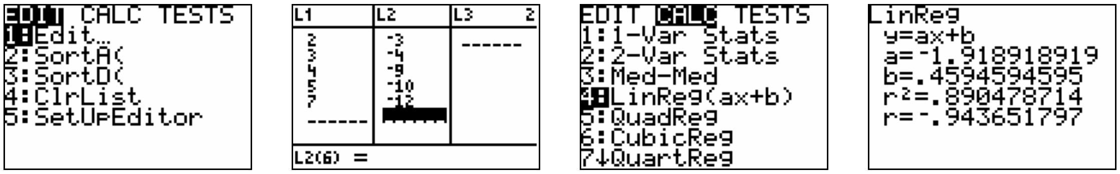
**6.** Figure A better approximates a graph that can be modeled by an equation than does Figure B. Thus Figure A has a coefficient of determination closer to 1.

**7.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.



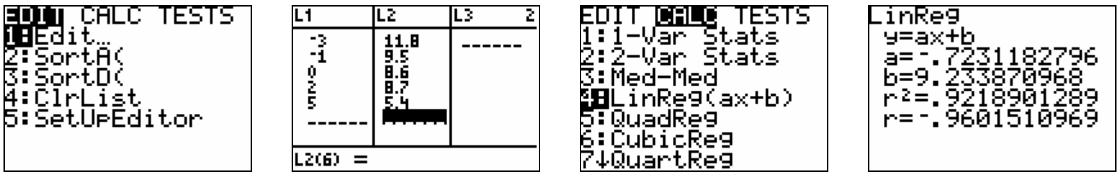

$y = 2.00862069x + 0.5603448276$

**8.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.



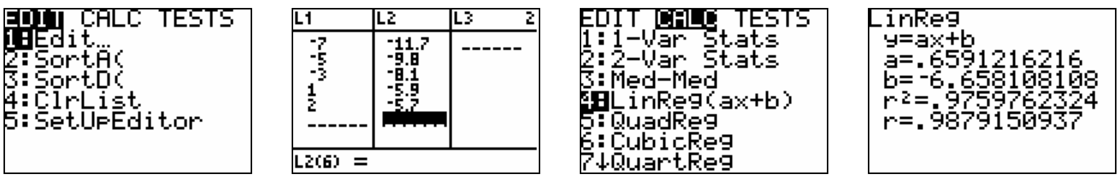

$y = -1.918918919x + 0.4594594595$

**9.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.



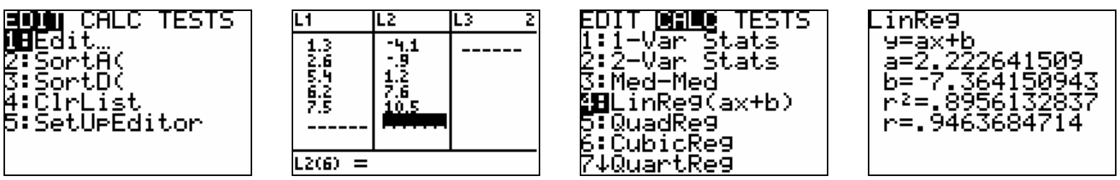

$y = -0.7231182796x + 9.233870968$

**10.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.



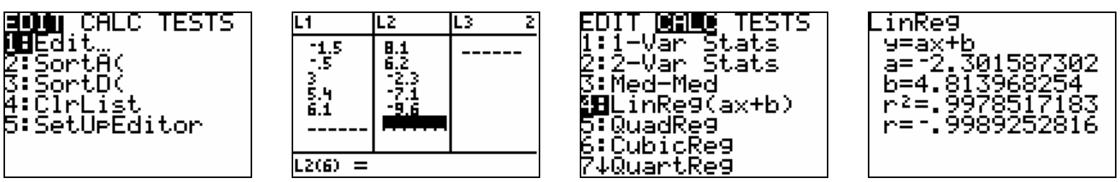

$y = 0.6591216216x - 6.658108108$

**11.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.



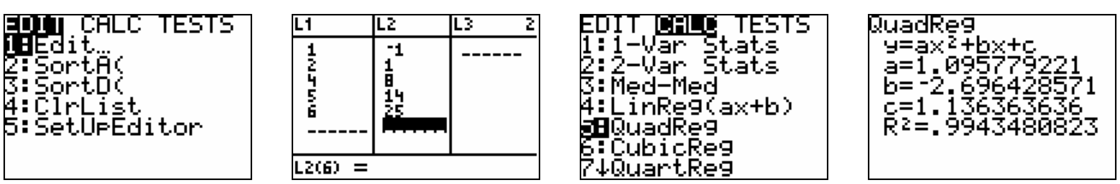

$y = 2.222641509x - 7.364150943$

**12.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.



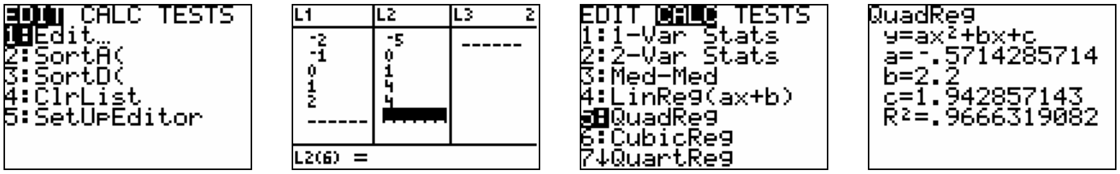

$y = -2.301587302x + 4.813968254$

**13.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.



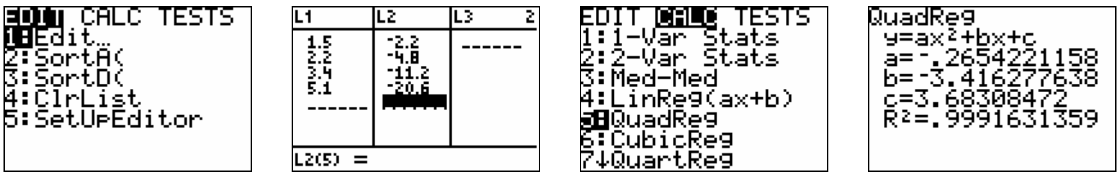

$y = 1.095779221x^2 - 2.69642857x + 1.136363636$

**14.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.



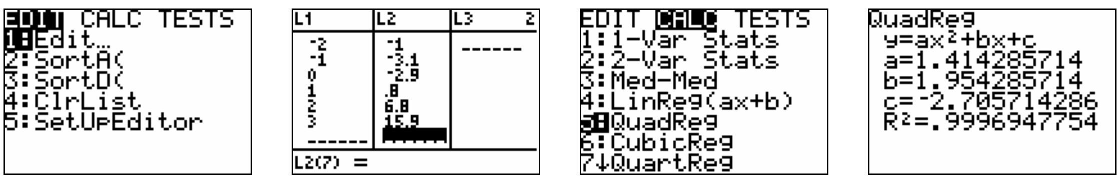

$y = -0.5714285714x^2 + 2.2x + 1.942857143$

**15.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.



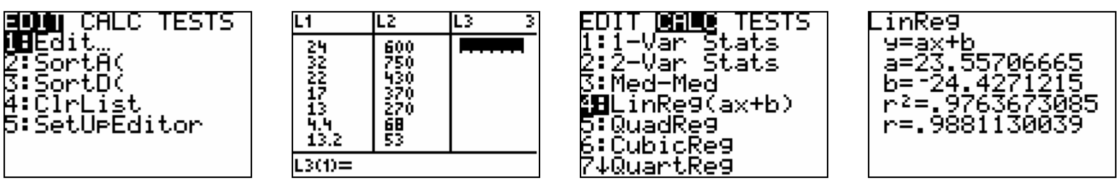

$y = -0.2987274717x^2 - 3.20998141x + 3.416463667$

**16.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.




$y = 1.414285714x^2 + 1.954285714x - 2.705714286$

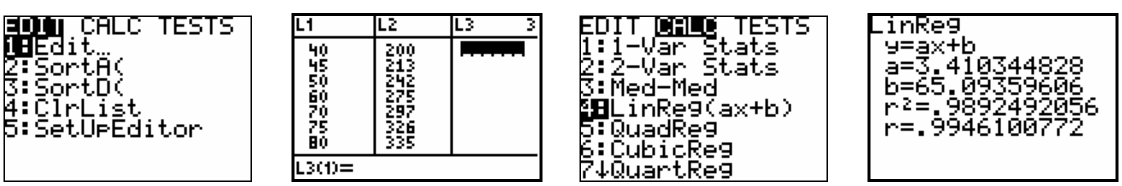
**17.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.




**a.** $y = 23.55706665x - 24.4271215$

**b.** $y = 23.55706665(54) - 24.4271215 \approx 1248$ cm

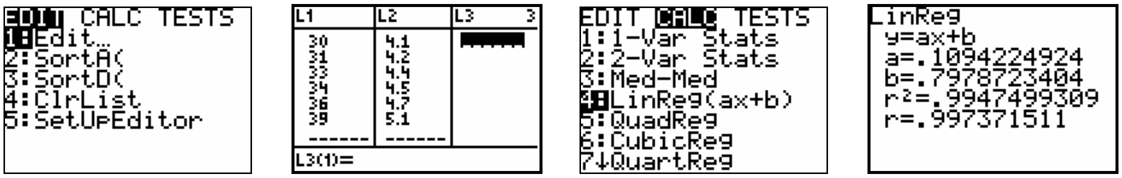
**18.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.




**a.** $y = 3.410344828x + 65.09359606$

**b.** $y = 3.410344828x + 65.09359606 \approx 263$ ft

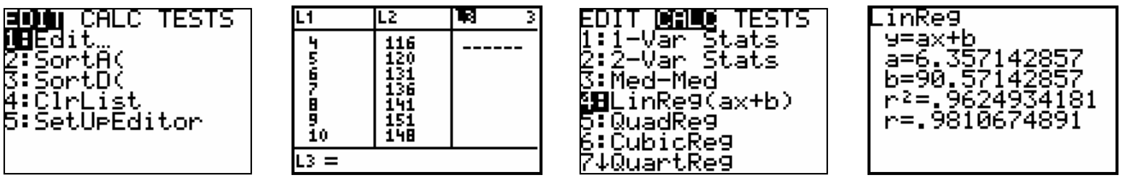
**19.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.




**a.** $y = 0.194224924x + 0.7978723404$

**b.** $y = 0.194224924(32) + 0.7978723404 \approx 4.3$ m/s

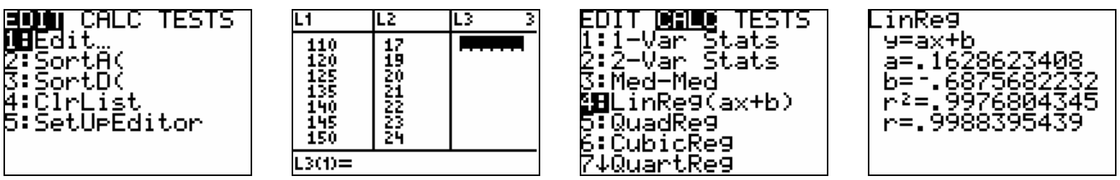
**20.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.




**a.** $y = 6.357142857x + 90.57142857$

**b.** $y = 6.357142857(7.5) + 90.57142857 = 138.25$ or 138,000 bacteria

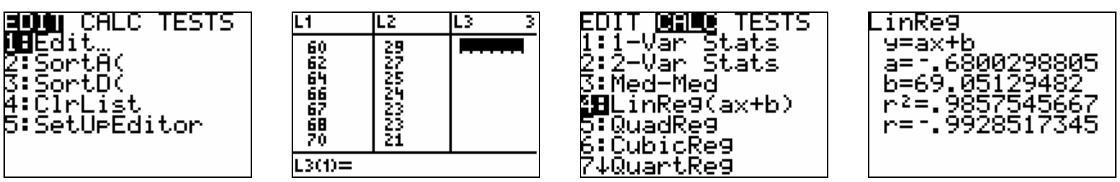
**21.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.




**a.** $y = 0.1628623408x - 0.6875682232$

**b.** $y = 0.1628623408(158) - 0.6875682232 \approx 25$

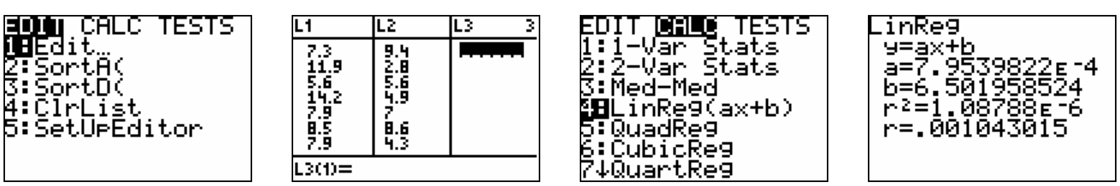
**22.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.




**a.** $y = -0.6800298805x + 69.05129482$

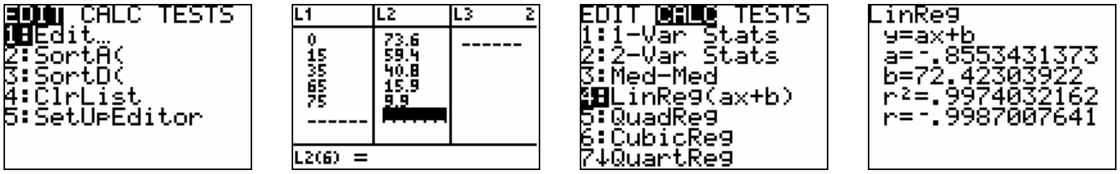
**b.** 5 feet 8 inches = 68 inches; $y = -0.6800298805(68) + 69.05129482 \approx 23$

**23.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.



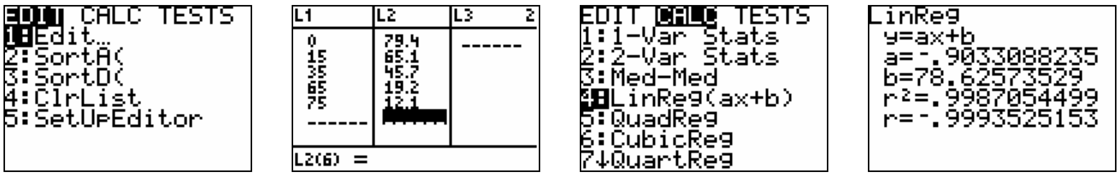

The value of $r$ is close to 0. Therefore, no, there is not a strong linear relationship between the current and the torque.

**24.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.




The value of $r$ is close to $-1$, so, yes, there is a strong correlation between a man's age and his average remaining lifetime.

**25.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.

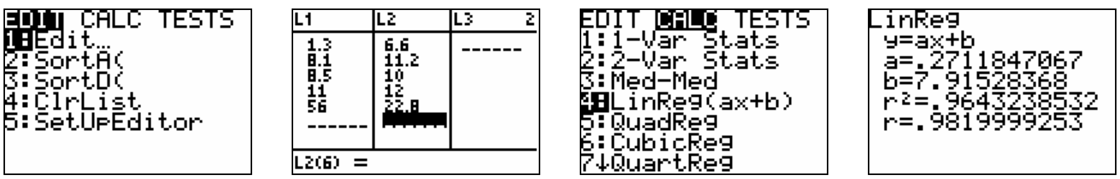



**a.** The value of $r$ is close to $-1$, so, yes, there is a strong linear correlation.

**b.** $y = -0.9033088235x + 78.62573529$

**c.** $y = -0.9033088235(25) + 78.62573529 \approx 56$ years

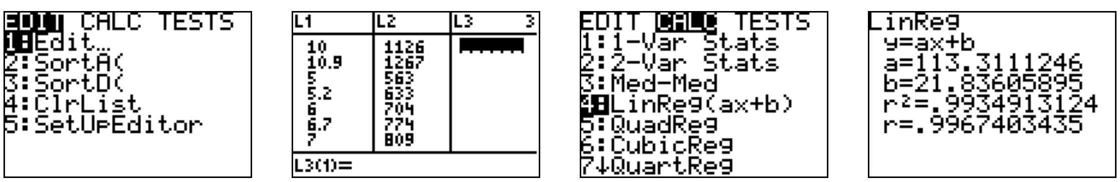
**26.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.




$y = 0.2711847067x + 7.91528368$

$y = 0.2711847067(41) + 7.91528368 \approx 19$ meters per second

**27.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.

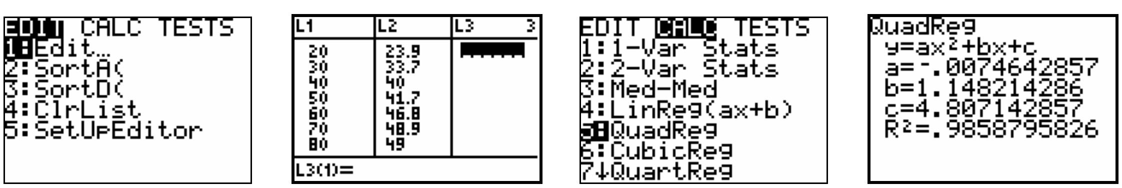



$y = 113.3111246x + 21.83605895$

**a.** Positively

**b.** $y = 113.3111246(9.5) + 21.83605895 \approx 1098$ calories

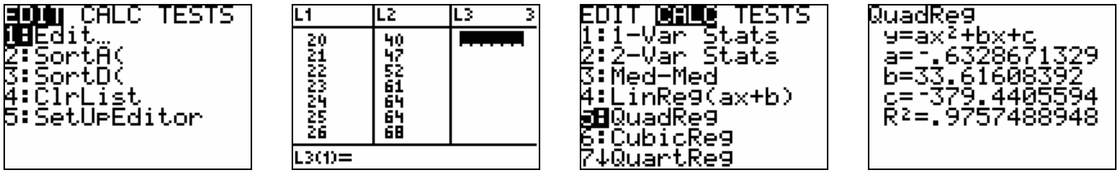
**28.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.




**a.** $y = -0.0074642857x^2 + 1.148214286x + 4.807142857$

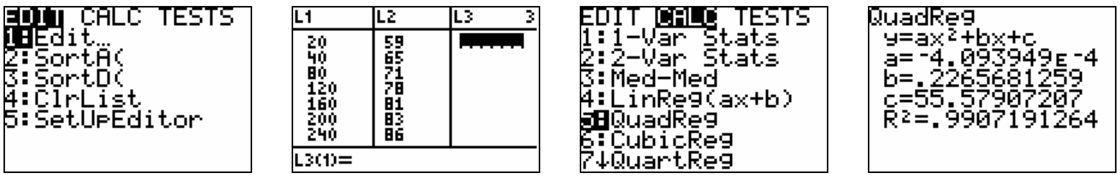
**b.** $y = -0.0074642857(65)^2 + 1.148214286(65) + 4.807142857 \approx 47.9$ ft

**29.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.




$y = -0.6328671329x^2 + 33.6160839x - 379.4405594$

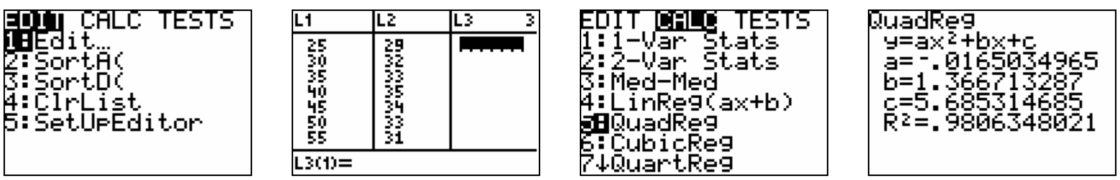
**30.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.




**a.** $y = -0.0004093949t^2 + 0.2265681259t + 55.57907207$

**b.** 1:00 P.M. is 7 hours, or 420 minutes, after 6:00 A.M.
$y = -0.0004093949(420)^2 + 0.2265681259(420) + 55.57907207 \approx 78.5°$

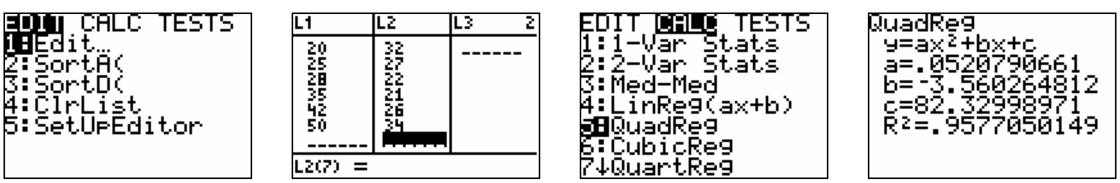
**31.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.




**a.** $y = -0.0165034965x^2 + 1.366713287x + 5.685314685$

**b.** $y = -0.0165034965(50)^2 + 1.366713287(50) + 5.685314685 \approx 32.8$ mpg

**32.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.



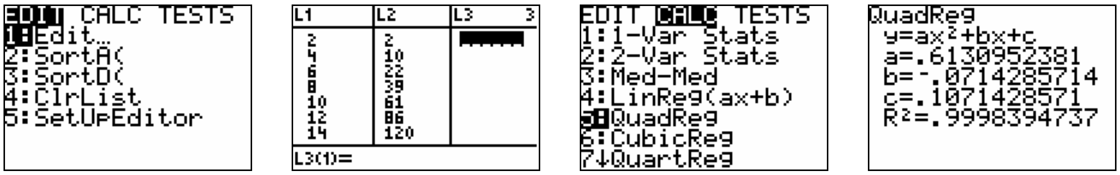

**a.** $y = 0.05208x^2 - 3.56026x + 82.32999$

**b.** $-\frac{b}{2a} = -\frac{-3.56026}{2(0.05208)} \approx 34$ kilometers per hour

**Connecting Concepts**

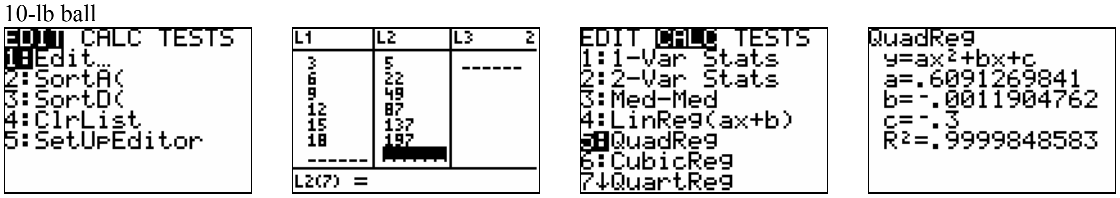
**33.** **a.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.

5-lb ball



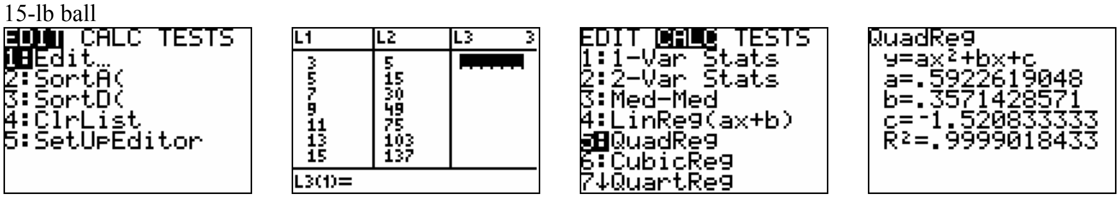

$y = 0.6130952381t^2 - 0.0714285714t + 0.1071428571$

10-lb ball




$y = 0.6091269841t^2 - 0.0011904762t - 0.3$

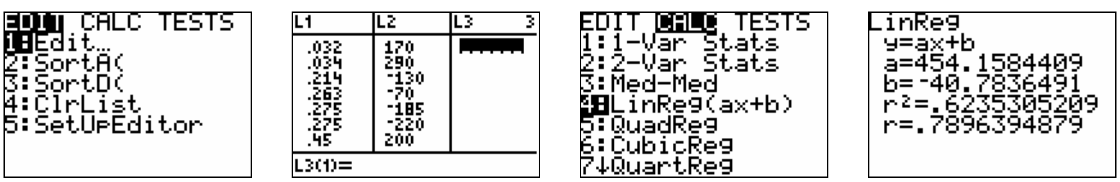
15-lb ball




$y = 0.5922619048t^2 + 0.3571428571t - 1.520833333$

**b.** All the regression equations are approximately the same. Therefore, there is one equation of motion.

**34.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here.



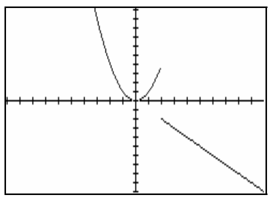

**a.** $y = 454.1584409x - 40.78364910$

**b.** $y = 454.1584409(1.5) - 40.78364910 \approx 640$ kilometers per second

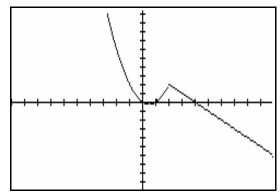
## Exploring Concepts with Technology

**Graphing Piecewise Functions with a Graphing Calculator**

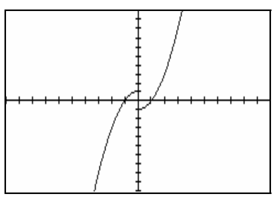
**1.** Use Dot mode. Enter the function as
`Y1=X^2*(X<2)-X*(X≥2)`
and graph this in the standard viewing window.



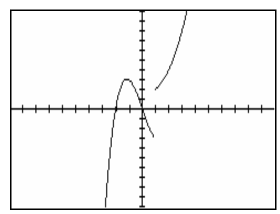
**2.** Use Dot mode. Enter the function as
`Y1=(X^2-X)*(X<2)+(-X+4)*(X≥2)`
and graph this in the standard viewing window.



**3.** Use Dot mode. Enter the function as
`Y1=(-X^2+1)*(X<0)+(X^2-1)*(X≥0)`
and graph this in the standard viewing window.



**4.** Use Dot mode. Enter the function as
`Y1=(X^3-4X)*(X<1)+(X^2-X+2)*(X≥1)`
and graph this in the standard viewing window.



## Assessing Concepts

**1.** Circle

**2.** There are two values of $y$ for one value of $x$.

**3.** Let $f(x)=x^2$, then $f(3)=f(-3)$ but $3 \neq -3$.

**4.** No. If $f(a)=f(b)$, then $f$ is not a one-to-one function.

**5.** $f(0)$ is greater than 7.

**6.** All real numbers.

**7.** (–2, –5)

**8.** $f(1)$ is not defined.

**9.** (–3, 5)

**10.** (3, –10)

**11.** (3, 1)

**12.** (–4, –2)

## Chapter Review

**1.** $d=\sqrt{(7-(-3))^2+(11-2)^2}$ [2.1]
$=\sqrt{10^2+9^2}=\sqrt{100+81}=\sqrt{181}$

**2.** $d=\sqrt{(-3-5)^2+(-8-(-4))^2}$ [2.1]
$=\sqrt{(-8)^2+(-4)^2}=\sqrt{64+16}=\sqrt{80}=4\sqrt{5}$

**3.** $\left(\frac{2+(-3)}{2},\frac{8+12}{2}\right)=\left(\frac{-1}{2},\frac{20}{2}\right)=\left(-\frac{1}{2},10\right)$ [2.1]

**4.** $\left(\frac{-4+8}{2},\frac{7+(-11)}{2}\right)=\left(\frac{4}{2},\frac{-4}{2}\right)=(2,-2)$ [2.1]

**5.** center (3, –4), radius 9 [2.1]

**6.** $x^2+10x+y^2+4y=-20$ [2.1]

$x^2+10x+25+y^2+4y+4=-20+25+4$

$(x+5)^2+(y+2)^2=9$

center (–5, –2), radius 3

**7.** $(x-2)^2+(y+3)^2=5^2$ [2.1]

**8.** $(x+5)^2+(y-1)^2=r^2$ [2.1]

$(3+5)^2+(1-1)^2=r^2$

$8^2+0^2=r^2$

$8^2=r^2$

$(x+5)^2+(y-1)^2=8^2$

**9.** **a.** $f(1)=3(1)^2+4(1)-5$ [2.2]

$=3(1)+4-5$

$=3+4-5$

$=2$

**b.** $f(-3)=3(-3)^2+4(-3)-5$

$=3(9)-12-5$

$=27-12-5$

$=10$

**c.** $f(t)=3t^2+4t-5$

**d.** $f(x+h)=3(x+h)^2+4(x+h)-5$

$=3(x^2+2xh+h^2)+4x+4h-5$

$=3x^2+6xh+3h^2+4x+4h-5$

**e.** $3f(t)=3(3t^2+4t-5)$

$=9t^2+12t-15$

**f.** $f(3t)=3(3t)^2+4(3t)-5$

$=3(9t^2)+12t-5$

$=27t^2+12t-5$

**10.** **a.** $g(3)=\sqrt{64-3^2}$ [2.2]

$=\sqrt{64-9}$

$=\sqrt{55}$

**b.** $g(-5)=\sqrt{64-(-5)^2}$

$=\sqrt{64-25}$

$=\sqrt{39}$

**c.** $g(8)=\sqrt{64-(8)^2}$

$=\sqrt{64-64}$

$=\sqrt{0}$

$=0$

**d.** $g(-x)=\sqrt{64-(-x)^2}$

$=\sqrt{64-x^2}$

**e.** $2g(t)=2\sqrt{64-t^2}$

**f.** $g(2t)=\sqrt{64-(2t)^2}$

$=\sqrt{64-4t^2}$

$=\sqrt{4(16-t^2)}$

$=2\sqrt{16-t^2}$

**11.** **a.** $(f\circ g)(3) = f[g(3)]$ [2.6]
$= f[3-8]$
$= f[-5]$
$= (-5)^2 + 4(-5)$
$= 25-20$
$= 5$

**b.** $(g\circ f)(-3) = g[f(-3)]$
$= g[(-3)^2 + 4(-3)]$
$= g[9-12]$
$= g[-3]$
$= [-3]-8$
$= -11$

**c.** $(f\circ g)(x) = f[g(x)]$
$= f[x-8]$
$= (x-8)^2 + 4(x-8)$
$= x^2 -16x + 64 + 4x - 32$
$= x^2 -12x + 32$

**d.** $(g\circ f)(x) = g[f(x)]$
$= g[x^2 + 4x)]$
$= [x^2 + 4x] - 8$
$= x^2 + 4x - 8$

**12.** **a.** $(f\circ g)(-5) = f[g(-5)]$ [2.6]
$= f\left[|(-5)-1|\right]$
$= f[6]$
$= 2(6)^2 + 7$
$= 72 + 7$
$= 79$

**b.** $(g\circ f)(-5) = g[f(-5)]$
$= g[2(-5)^2 + 7]$
$= g[57]$
$= |57-1|$
$= 56$

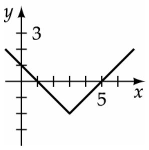
**c.** $(f\circ g)(x) = f[g(x)]$
$= f\left[|x-1|\right]$
$= 2(|x-1|)^2 + 7$
$= 2(x-1)^2 + 7$
$= 2(x^2 - 2x + 1) + 7$
$= 2x^2 - 4x + 2 + 7$
$= 2x^2 - 4x + 9$

**d.** $(g\circ f)(x) = g[f(x)]$
$= g[2x^2 + 7]$
$= |2x^2 + 7 - 1|$
$= |2x^2 + 6|$
$= 2x^2 + 6$

**13.** $\dfrac{f(x+h)-f(x)}{h} = \dfrac{4(x+h)^2-3(x+h)-1-(4x^2-3x-1)}{h}$ [2.6]
$= \dfrac{4(x^2+2xh+h^2)-3x-3h-1-4x^2+3x+1}{h}$
$= \dfrac{4x^2+8xh+4h^2-3x-3h-1-4x^2+3x+1}{h}$
$= \dfrac{8xh+4h^2-3h}{h}$
$= \dfrac{h(8x+4h-3)}{h}$
$= 8x+4h-3$

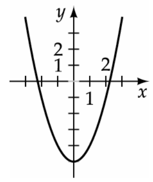
**14.** $\dfrac{g(x+h)-g(x)}{h} = \dfrac{(x+h)^3-(x+h)-(x^3-x)}{h}$ [2.6]
$= \dfrac{x^3+3x^2h+3xh^2+h^3-x-h-x^3+x}{h}$
$= \dfrac{3x^2h+3xh^2+h^3-h}{h}$
$= \dfrac{h(3x^2+3xh+h^2-1)}{h}$
$= 3x^2+3xh+h^2-1$

**15.**



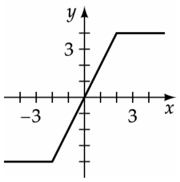

$f$ is increasing on $[3, \infty)$
$f$ is decreasing on $(-\infty, 3]$ [2.2]

**16.**



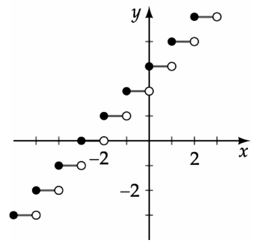

$f$ is increasing on $[0, \infty)$
$f$ is decreasing on $(-\infty, 0]$ [2.2]

**17.**



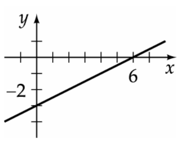

$f$ is increasing on $[-2, 2]$
$f$ is constant on $(-\infty, -2] \cup [2, \infty)$ [2.2]

**18.**



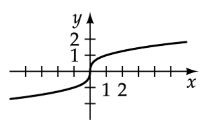

$f$ is constant on $\ldots, [-6, -5), [-5, -4), [-4, -3), [-3, -2),$
$[-2, -1), [-1, 0), [0, 1), \ldots$ [2.2]

**19.**




$f$ is increasing on $(-\infty, \infty)$ [2.2]

**20.**




$f$ is increasing on $(-\infty, \infty)$ [2.2]

**21.** Domain $\{x \mid x \text{ is a real number}\}$ [2.2]

**22.** Domain $\{x \mid x \le 6\}$ [2.2]

**23.** Domain $\{x \mid -5 \le x \le 5\}$ [2.2]

**24.** Domain $\{x \mid x \ne -3,\ x \ne 5\}$ [2.2]

**25.**
$$m = \frac{-7-3}{4-(-1)} = \frac{-10}{5} = -2 \quad [2.3]$$
$$y - 3 = -2(x+1) \qquad \text{point-slope form}$$
$$y - 3 = -2x - 2$$
$$y = -2x + 1$$

**26.**
$$m = \frac{11-0}{7-0} = \frac{11}{7} \quad [2.3]$$
$$y - 0 = \frac{11}{7}(x - 0)$$
$$y = \frac{11}{7}x$$

**27.** $3x-4y=8$ [2.3]

$$-4y=-3x+8$$
$$y=\frac{3}{4}x-2$$

Slope of parallel line is $\frac{3}{4}$.

$$y-11=\frac{3}{4}(x-2)$$
$$y-11=\frac{3}{4}x-\frac{3}{2}$$
$$y=\frac{3}{4}x-\frac{3}{2}+11$$
$$y=\frac{3}{4}x-\frac{3}{2}+\frac{22}{2}$$
$$y=\frac{3}{4}x+\frac{19}{2}$$

**28.** $2x=-5y+10$ [2.3]

$$5y=-2x+10$$
$$y=-\frac{2}{5}x+2$$

Slope of perpendicular line is $\frac{5}{2}$.

$$y-(-7)=\frac{5}{2}[x-(-3)]$$
$$y+7=\frac{5}{2}(x+3)$$
$$y+7=\frac{5}{2}x+\frac{15}{2}$$
$$y=\frac{5}{2}x+\frac{15}{2}-7$$
$$y=\frac{5}{2}x+\frac{15}{2}-\frac{14}{2}$$
$$y=\frac{5}{2}x+\frac{1}{2}$$

**29.** $f(x)=(x^2+6x)+10$ [2.4]

$$f(x)=(x^2+6x+9)+10-9$$
$$f(x)=(x+3)^2+1$$

**30.** $f(x)=(2x^2+4x)+5$ [2.4]

$$f(x)=2(x^2+2x)+5$$
$$f(x)=2(x^2+2x+1)+5-2$$
$$f(x)=2(x+1)^2+3$$

**31.** $f(x)=-x^2-8x+3$ [2.4]

$$f(x)=-(x^2+8x)+3$$
$$f(x)=-(x^2+8x+16)+3+16$$
$$f(x)=-(x+4)^2+19$$

**32.** $f(x)=(4x^2-6x)+1$ [2.4]

$$f(x)=4\left(x^2-\frac{3}{2}x\right)+1$$
$$f(x)=4\left(x^2-\frac{3}{2}x+\frac{9}{16}\right)+1-\frac{9}{4}$$
$$f(x)=4\left(x-\frac{3}{4}\right)^2+\frac{4}{4}-\frac{9}{4}$$
$$f(x)=4\left(x-\frac{3}{4}\right)^2-\frac{5}{4}$$

**33.** $f(x)=-3x^2+4x-5$ [2.4]

$$f(x)=-3\left(x^2-\frac{4}{3}x\right)-5$$
$$f(x)=-3\left(x^2-\frac{4}{3}x+\frac{4}{9}\right)-5+\frac{4}{3}$$
$$f(x)=-3\left(x-\frac{2}{3}\right)^2-\frac{15}{3}+\frac{4}{3}$$
$$f(x)=-3\left(x-\frac{2}{3}\right)^2-\frac{11}{3}$$

**34.** $f(x)=x^2-6x+9$ [2.4]

$$f(x)=(x^2-6x)+9$$
$$f(x)=(x^2-6x+9)+9-9$$
$$f(x)=(x-3)^2+0$$

**35.** $\dfrac{-b}{2a}=\dfrac{-(-6)}{2(3)}=\dfrac{6}{6}=1$ [2.4]

$$\begin{aligned} f(1) &= 3(1)^2-6(1)+11\\ &= 3(1)-6+11\\ &= 3-6+11\\ &= 8 \end{aligned}$$

Thus the vertex is (1, 8).

**36.** $\dfrac{-b}{2a}=\dfrac{0}{2(4)}=0$ [2.4]

$$\begin{aligned} f(0) &= 4(0)^2-10\\ &= 0-101\\ &= -10 \end{aligned}$$

Thus the vertex is (0, –10).

**37.** $\dfrac{-b}{2a}=\dfrac{-(60)}{2(-6)}=\dfrac{-60}{-12}=5$ [2.4]

$$\begin{aligned} f(5) &= -6(5)^2+60(5)+11\\ &= -6(25)+300+11\\ &= -150+300+11\\ &= 161 \end{aligned}$$

Thus the vertex is (5, 161).

**38.** $\dfrac{-b}{2a}=\dfrac{-(-8)}{2(-1)}=\dfrac{8}{-2}=-4$ [2.4]

$$\begin{aligned} f(-4) &= 14-8(-4)-(-4)^2\\ &= 14+32-16\\ &= 30 \end{aligned}$$

Thus the vertex is (–4, 30).

**39.** $d=\dfrac{|mx_1+b-y_1|}{\sqrt{1+m^2}},\ y=2x-3,\ (x_1,y_1)=(1,3)$ [2.3]

$$d=\frac{|2(1)+(-3)-3|}{\sqrt{1+2^2}}$$

$$d=\frac{|2-3-3|}{\sqrt{1+4}}$$

$$d=\frac{|-4|}{\sqrt{5}}$$

$$d=\frac{4}{\sqrt{5}}=\frac{4\sqrt{5}}{5}$$

**40.** **a.** Revenue = $13x$

**b.** Profit = Revenue – Cost

$P=13x-(0.5x+1050)$

$P=13x-0.5x-1050$

$P=12.5x-1050$

**c.** Break even $\Rightarrow$ Revenue = Cost

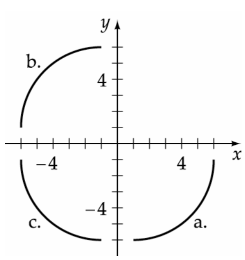
$13x=0.5x+1050$

$12.5x=1050$

$x=84$

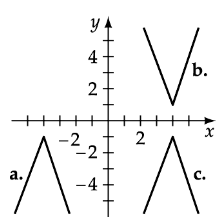
The company must ship 84 parcels. [2.5]

**41.**




[2.5]

**42.**




[2.5]

**43.** The graph of $y=x^2-7$ is symmetric with respect to the $y$-axis. [2.5]

**44.** The graph of $x=y^2+3$ is symmetric with respect to the $x$-axis. [2.5]

**45.** The graph of $y=x^3-4x$ is symmetric with respect to the origin. [2.5]

**46.** The graph of $y^2=x^2+4$ is symmetric with respect to the $x$-axis, $y$-axis, and the origin. [2.5]

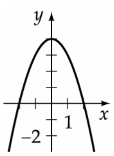
**47.** The graph of $\dfrac{x^2}{3^2}+\dfrac{y^2}{4^2}=1$ is symmetric with respect to the $x$-axis, $y$-axis, and the origin. [2.5]

**48.** The graph of $xy=8$ is symmetric with respect to the origin. [2.5]

**49.** The graph of $|y| = |x|$ is symmetric with respect to the $x$-axis, $y$-axis, and the origin. [2.5]

**50.** The graph of $|x + y| = 4$ is symmetric with respect to the origin. [2.5]

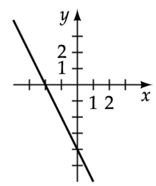
**51.**




**a.** Domain all real numbers
Range $\{y | y \le 4\}$

**b.** $g$ is an even function [2.5]

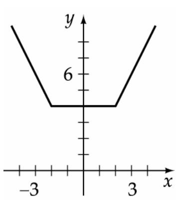
**52.**




**a.** Domain all real numbers
Range all real numbers

**b.** $g$ is neither even nor odd [2.5]

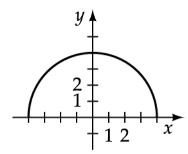
**53.**




**a.** Domain all real numbers
Range $\{y | y \ge 4\}$

**b.** $g$ is an even function [2.5]

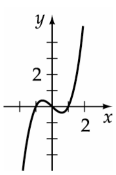
**54.**




**a.** Domain $\{x | -4 \le x \le 4\}$
Range $\{y | 0 \le y \le 4\}$

**b.** $g$ is an even function [2.5]

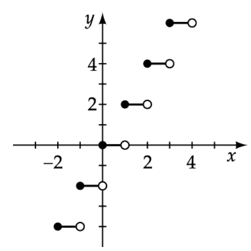
**55.**




**a.** Domain all real numbers
Range all real numbers

**b.** $g$ is an odd function [2.5]

**56.**




**a.** Domain all real numbers
Range $\{y | y \text{ is an even integer}\}$

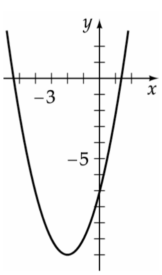
**b.** $g$ is neither even nor odd [2.5]

**57.** $F(x) = x^2 + 4x - 7$ [2.5]

$F(x) = (x^2 + 4x) - 7$

$F(x) = (x^2 + 4x + 4) - 7 - 4$

$F(x) = (x + 2)^2 - 11$

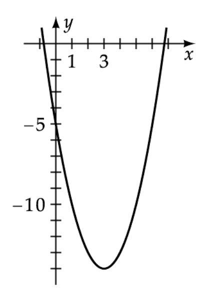



**58.** $A(x) = x^2 - 6x - 5$ [2.5]

$A(x) = (x^2 - 6x) - 5$

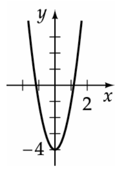
$A(x) = (x^2 - 6x + 9) - 5 - 9$

$A(x) = (x - 3)^2 - 14$



**59.** $P(x) = 3x^2 - 4$ [2.5]

$P(x) = 3(x-0)^2 - 4$

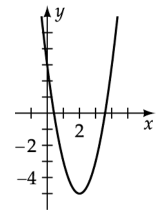



**60.** $G(x) = 2x^2 - 8x + 3$ [2.5]

$G(x) = 2(x^2 - 4x) + 3$

$G(x) = 2(x^2 - 4x + 4) + 3 - 8$

$G(x) = 2(x-2)^2 - 5$



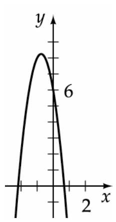

**61.** $W(x) = -4x^2 - 6x + 6$ [2.5]

$W(x) = -4\left(x^2 + \frac{3}{2}x\right) + 6$

$W(x) = -4\left(x^2 + \frac{3}{2}x + \frac{9}{16}\right) + 6 + \frac{9}{4}$

$W(x) = -4\left(x + \frac{3}{4}\right)^2 + \frac{24}{4} + \frac{9}{4}$

$W(x) = -4\left(x + \frac{3}{4}\right)^2 + \frac{33}{4}$

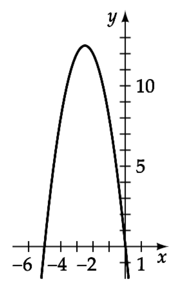



**62.** $T(x) = -2x^2 - 10x$ [2.5]

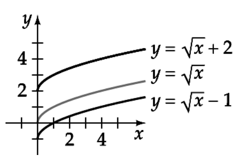
$T(x) = -2(x^2 + 5x)$

$T(x) = -2\left(x^2 + 5x + \frac{25}{4}\right) + \frac{25}{2}$

$T(x) = -2\left(x + \frac{5}{2}\right)^2 + \frac{25}{2}$

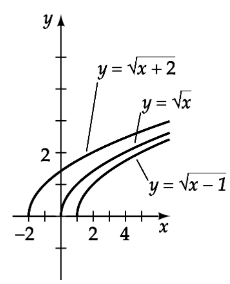



**63.**



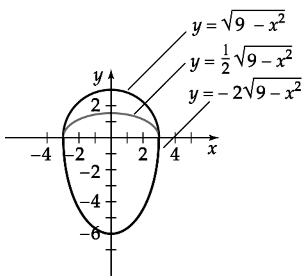

[2.5]

**64.**



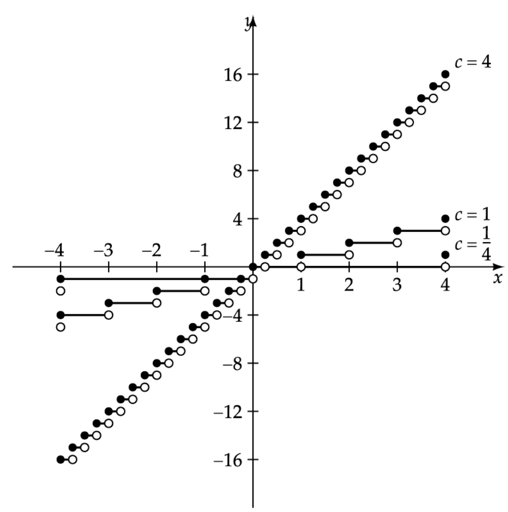

[2.5]

**65.**



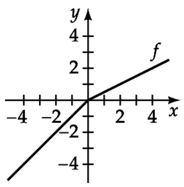

[2.5]

**66.**



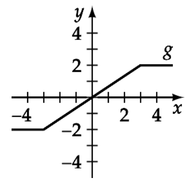

[2.2]

**67.**




[2.2]

**68.**




[2.2]

**69.** $(f+g)(x) = x^2 - 9 + x + 3$ [2.6]
$= x^2 + x - 6$
The domain is all real numbers.

$(f-g)(x) = x^2 - 9 - (x+3)$
$= x^2 - 9 - x - 3$
$= x^2 - x - 12$
The domain is all real numbers.

$(fg)(x) = (x^2 - 9)(x+3)$
$= x^3 + 3x^2 - 9x - 27$
The domain is all real numbers.

$\left(\frac{f}{g}\right)(x) = \frac{x^2 - 9}{x+3}$
$= \frac{(x-3)(x+3)}{x+3}$
$= x - 3$
The domain is $\{x \mid x \neq -3\}$.

**70.** $(f+g)(x) = x^3 + 8 + x^2 - 2x + 4$ [2.6]
$= x^3 + x^2 - 2x + 12$
The domain is all real numbers.

$(f-g)(x) = x^3 + 8 - (x^2 - 2x + 4)$
$= x^3 + 8 - x^2 + 2x - 4$
$= x^3 - x^2 + 2x + 4$
The domain is all real numbers.

$(fg)(x) = (x^3 + 8)(x^2 - 2x + 4)$
$= x^5 - 2x^4 + 4x^3 + 8x^2 - 16x + 32$
The domain is all real numbers.

$\left(\frac{f}{g}\right)(x) = \frac{x^3 + 8}{x^2 - 2x + 4}$
$= \frac{(x+2)(x^2 - 2x + 4)}{x^2 - 2x + 4}$
$= x + 2$

The domain is restricted when $x^2 - 2x + 4 = 0$.

$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)}$

$x = \frac{2 \pm \sqrt{4 - 16}}{2}$

$x = \frac{2 \pm \sqrt{-12}}{2}$ which is not a real number

Therefore the domain is all real numbers.

**71.** Let $x$ = one of the numbers and $50 - x$ = the other number. Their product is given by
$y = x(50 - x) = 50x - x^2 = -x^2 + 50x.$

Now $y$ takes on its maximum value when
$x = \frac{-b}{2a} = \frac{-50}{2(-1)} = \frac{-50}{-2} = 25.$

Thus the two numbers are 25 and $(50 - 25) = 25$. That is, both numbers are 25. [2.4]

**72.** Let $x$ = the smaller number. Let $x + 10$ equal the larger number. The sum of their squares $y$ is given by
$y = x^2 + (x+10)^2$
$= x^2 + x^2 + 20x + 100$
$= 2x^2 + 20x + 100$

Now $y$ takes on its minimum value when
$x = \frac{-b}{2a} = \frac{-20}{2(2)} = \frac{-20}{4} = -5$

Thus the numbers are $-5$ and $(-5 + 10) = 5$. [2.4]

**73.** $s(t) = 3t^2$ [2.4]

**a.** Average velocity $= \dfrac{3(4)^2 - 3(2)^2}{4-2} = \dfrac{3(16)-3(4)}{2} = \dfrac{48-12}{2} = \dfrac{36}{2} = 18$ ft/sec

**b.** Average velocity $= \dfrac{3(3)^2 - 3(2)^2}{3-2} = \dfrac{3(9)-3(4)}{1} = \dfrac{27-12}{1} = 15$ ft/sec

**c.** Average velocity $= \dfrac{3(2.5)^2 - 3(2)^2}{2.5-2} = \dfrac{3(6.25)-3(4)}{0.5} = \dfrac{18.75-12}{0.5} = \dfrac{6.75}{0.5} = 13.5$ ft/sec

**d.** Average velocity $= \dfrac{3(2.01)^2 - 3(2)^2}{2.01-2} = \dfrac{3(4.0401)-3(4)}{0.01} = \dfrac{12.1203-12}{0.01} = \dfrac{0.1203}{0.01} = 12.03$ ft/sec

**e.** It appears that the average velocity of the ball approaches 12 ft/sec.

**74.** $s(t) = 2t^2 + t$ [2.4]

**a.** Average velocity $= \dfrac{2(5)^2+5-[2(3)^2+3]}{5-3} = \dfrac{2(25)+5-[2(9)+3]}{2} = \dfrac{50+5-[18+3]}{2} = \dfrac{50+5-18-3}{2} = \dfrac{34}{2} = 17$ ft/sec

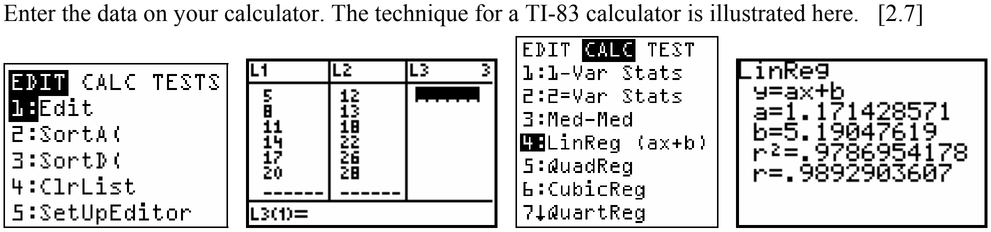
**b.** Average velocity $= \dfrac{2(4)^2+4-[2(3)^2+3]}{4-3} = \dfrac{2(16)+4-[2(9)+3]}{1} = \dfrac{32+4-[18+3]}{1} = \dfrac{32+4-18-3}{1} = \dfrac{15}{1} = 15$ ft/sec

**c.** Average velocity $= \dfrac{2(3.5)^2+3.5-[2(3)^2+3]}{3.5-3} = \dfrac{2(12.25)+3.5-[2(9)+3]}{0.5} = \dfrac{24.5+3.5-[18+3]}{0.5} = \dfrac{24.5+3.5-18-3}{0.5} = \dfrac{7}{0.5} = 14$ ft/sec

**d.** Average velocity $= = \dfrac{2(3.01)^2+3.01-[2(3)^2+3]}{3.01-3} = \dfrac{2(9.0601)+3.01-[2(9)+3]}{0.01} = \dfrac{18.1202+3.01-[18+3]}{0.01} = \dfrac{18.1202+3.01-18-3}{0.01} = \dfrac{0.1302}{2} = 13.02$ ft/sec

**e.** It appears that the average velocity of the ball approaches 13 ft/sec.

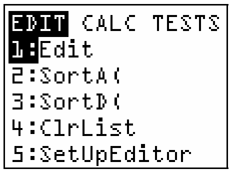
**75.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here. [2.7]

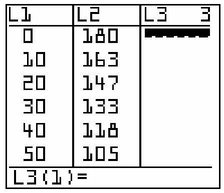


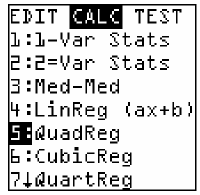

**a.** $y = 1.171428571x + 5.19047619$

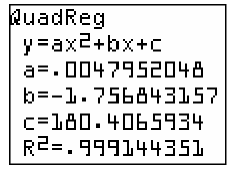
**b.** $y = 1.171428571(12) + 5.19047619 \approx 19$ m/s

**76.** **a.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here. [2.7]



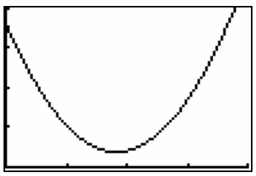




$h = 0.0047952048t^2 - 1.756843157t + 180.4065934$

**b.** Empty $\Rightarrow y = 0 \Rightarrow$ the graph intersects the $x$-axis.
Graph the equation, and notice that it never intersects the $x$-axis.



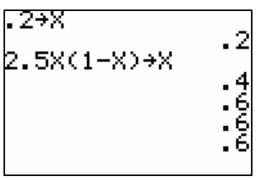
Xmin = 0, Xmax = 400, Xscl = 100
Ymin = 0, Ymax = 200, Xscl = 50

Thus, no, on the basis of this model, the can never empties.

**c.** The regression line is a model of the data and is not based on physical principles.

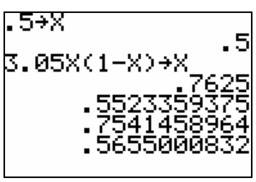
## Quantitative Reasoning

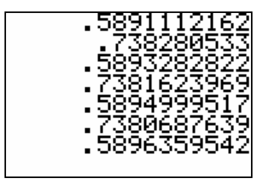
**QR1.**




The apparent attractor is 0.6.

**QR2.**




The behavior is not chaotic. 0.59 and 0.74 are the approximate values of the two attractors.

**QR3.** Four different values could be attractors: approximately 0.50, 0.58, 0.83, and 0.87.

**QR4.** There are no attractors.

**QR5.** The behavior of the function becomes more chaotic, and there are no attractors.

## Chapter Test

**1.** midpoint $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-2+4}{2}, \frac{3+(-1)}{2}\right) = \left(\frac{2}{2}, \frac{2}{2}\right) = (1, 1)$ [2.1]

length $= d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(-2-4)^2 + (3-(-1))^2} = \sqrt{(-6)^2 + 4^2} = \sqrt{36+16} = \sqrt{52} = 2\sqrt{13}$

**2.** $x = 2y^2 - 4$ [2.1]

$y = 0 \Rightarrow x = 2(0)^2 - 4 = -4$

Thus the $x$-intercept is (–4, 0).

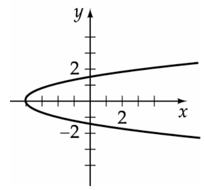
$x = 0 \Rightarrow 0 = 2y^2 - 4$

$4 = 2y^2$

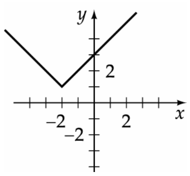
$2 = y^2$

$\pm\sqrt{2} = y$

Thus the $y$-intercepts are $(0, -\sqrt{2})$ and $(0, \sqrt{2})$.




**3.** $y = |x + 2| + 1$ [2.1]




**4.**

$$x^2 - 4x + y^2 + 2y - 4 = 0 \qquad [2.1]$$
$$(x^2 - 4x) + (y^2 + 2y) = 4$$
$$(x^2 - 4x + 4) + (y^2 + 2y + 1) = 4 + 4 + 1$$
$$(x - 2)^2 + (y + 1)^2 = 9$$

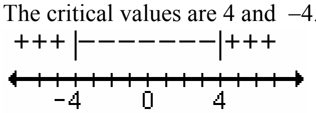
center (2, –1), radius 3

**5.**

$$x^2 - 16 \ge 0$$
$$(x - 4)(x + 4) \ge 0$$

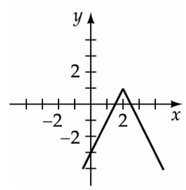
The product is positive or zero.

The critical values are 4 and –4.




The domain is $\{x | x \ge 4 \text{ or } x \le -4\}$. [2.2]

**6.**




**a.** increasing on $(-\infty, 2]$

**b.** never constant

**c.** decreasing on $[2, \infty)$ [2.2]

**7.** **a.** $R = 12x$

**b.** $P$ = revenue – cost

$P = 12x - (6.75x + 875)$

$P = 11.25x - 875$

**c.** break-even $\Rightarrow P = 0$

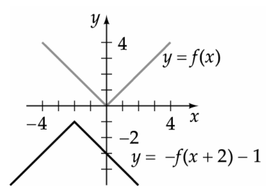
$0 = 11.25x - 875$

$875 = 11.25x$

$78 \approx x$

78 parcels must be sent to break even. [2.4]

**8.**




[2.5]

**9.** **a.** $f(x) = x^4 - x^2$ [2.5]

$f(-x) = (-x)^4 - (-x)^2 = x^4 - x^2 = f(x)$

$f(x)$ is an even function.

**b.** $f(x) = x^3 - x$

$f(-x) = (-x)^3 - (-x) = -x^3 + x$

$= -(x^3 - x) = -f(x)$

$f(x)$ is an odd function.

**c.** $f(x) = x - 1$

$f(-x) = -x - 1 \ne f(x)$ not an even function

$f(-x) = -x - 1 \ne -f(x)$ not an odd function

Neither

**10.** $3x - 2y = 4$ [2.3]

$-2y = -3x + 4$

$y = \frac{3}{2}x - 2$

Slope of perpendicular line is $-\frac{2}{3}$.

$y - y_1 = m(x - x_1)$

$y + 2 = -\frac{2}{3}(x - 4)$

$y + 2 = -\frac{2}{3}x + \frac{8}{3}$

$y = -\frac{2}{3}x + \frac{8}{3} - \frac{6}{3}$

$y = -\frac{2}{3}x + \frac{2}{3}$

**11.** $-\dfrac{b}{2a}=-\dfrac{-4}{2(1)}=2$ [2.4]

$f(2)=2^2-4(2)-8$
$=4-8-8$
$=-12$

The minimum value of the function is –12.

**12.** $(f+g)(x)=f(x)+g(x)$ [2.6]
$=(x^2-1)+(x-2)$
$=x^2+x-3$

$\left(\dfrac{f}{g}\right)(x)=\dfrac{f(x)}{g(x)}$
$=\dfrac{x^2-1}{x-2},\ x\neq 2$

**13.** $f(x)=x^2+1$ [2.6]

$$\frac{f(x+h)-f(x)}{h}=\frac{(x+h)^2+1-(x^2+1)}{h}$$
$$=\frac{x^2+2xh+h^2+1-x^2-1}{h}$$
$$=\frac{2xh+h^2}{h}=\frac{h(2x+h)}{h}$$
$$=2x+h$$

**14.** $f(x)=x^2-2x \quad g(x)=2x+5$ [2.6]

$$(f\circ g)(x)=f[g(x)]=f(2x+5)$$
$$=(2x+5)^2-2(2x+5)$$
$$=4x^2+20x+25-4x-10$$
$$=4x^2+16x+15$$

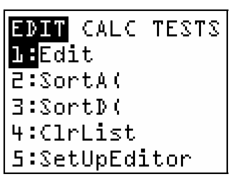
**15.** $s(t)=5t^2$ [2.6]

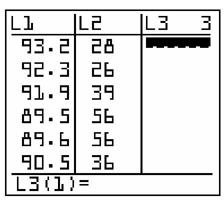
**a.** Average velocity $=\dfrac{5(3)^2-5(2)^2}{3-2}=\dfrac{5(9)-5(4)}{1}=45-20=25$ ft/sec

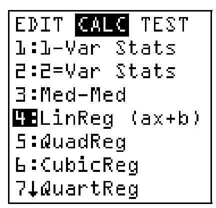
**b.** Average velocity $=\dfrac{5(2.5)^2-5(2)^2}{2.5-2}=\dfrac{5(6.25)-5(4)}{0.5}=\dfrac{31.25-20}{0.5}=22.5$ ft/sec

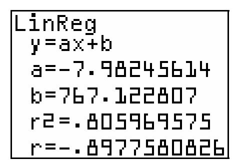
**c.** Average velocity $=\dfrac{5(2.01)^2-5(2)^2}{2.01-2}=\dfrac{5(4.0401)-5(4)}{0.01}=\dfrac{20.2005-20}{0.01}=20.05$ ft/sec

**16.** **a.** Enter the data on your calculator. The technique for a TI-83 calculator is illustrated here. [2.7]




$y=-7.98245614x+767.122807$

**b.** $y=-7.98245614(89)+767.122807$
$\approx 57$ calories

## Cumulative Review

**1.** Commutative Property of Addition [P.1]

**2.** $\dfrac{6}{\pi},\sqrt{2}$ are not rational numbers [P.1]

**3.** $3+4(2x-9)$ [P.1]
$3+8x-36$
$8x-33$

**4.** $(-4xy^2)^3(-2x^2y^4)=(-64x^3y^6)(-2x^2y^4)$ [P.2]
$=(-64)(-2)(x^{3+2}y^{6+4})$
$=128x^5y^{10}$

**5.** $\dfrac{24a^4b^3}{18a^4b^5}=\dfrac{4a^{4-4}b^{3-5}}{3}=\dfrac{4b^{-2}}{3}=\dfrac{4}{3b^2}$ [P.2]

**6.** $(2x+3)(3x-7)=6x^2-5x-21$ [P.3]

**7.** $\dfrac{x^2+6x-27}{x^2-9}=\dfrac{(x+9)(x-3)}{(x+3)(x-3)}=\dfrac{x+9}{x+3}$ [P.5]

**8.** $\frac{4}{2x-1}-\frac{2}{x-1}=\frac{4(x-1)}{(2x-1)(x-1)}-\frac{2(2x-1)}{(2x-1)(x-1)}=\frac{4x-4-4x+2}{(2x-1)(x-1)}=\frac{-2}{(2x-1)(x-1)}$ [P.5]

**9.** $6-2(2x-4)=14 \Rightarrow 6-4x+8=14 \Rightarrow -4x=0 \Rightarrow x=0$ [1.1]

**10.** $x^2-x-1=0 \Rightarrow x=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-1)}}{2(1)}=\frac{1\pm\sqrt{1+4}}{2}=\frac{1\pm\sqrt{5}}{2}$ [1.3]

**11.** $(2x-1)(x+3)=4 \Rightarrow 2x^2+5x-3=4 \Rightarrow 2x^2+5x-7=(2x+7)(x-1)=0 \Rightarrow x=-\frac{7}{2}$ or $x=1$ [1.3]

**12.** $3x+2y=15$ [1.1]

$3x=-2y+15$

$x=-\frac{2}{3}y+5$

**13.** $x^4-x^2-2=0$ [1.4]

Let $u=x^2$.

$u^2-u-2=0$

$(u-2)(u+1)=0$

$u-2=0$ or $u+1=0$

$u=2$ $\quad u=-1$

$x^2=2$ $\quad x^2=-1$

$x=\pm\sqrt{2}$ $\quad x=\pm i$

**14.** $3x-1<5x+7$ [1.5]

$-2x<8$

$x>-4$

**15.** distance $=\sqrt{[-2-2]^2+[-4-(-3)]^2}=\sqrt{(-4)^2+(-1)^2}=\sqrt{16+1}=\sqrt{17}$ [2.1]

**16.** $G(x)=2x^3-4x-7$ [2.2]

$G(-2)=2(-2)^3-4(-2)-7=2(-8)+8-7=-15$

**17.** The slope is $m=\frac{-1-(-3)}{-2-2}=\frac{-1+3}{-2-2}=\frac{2}{-4}=-\frac{1}{2}$ [2.3]

The equation is $y-(-3)=-\frac{1}{2}(x-2) \Rightarrow y=-\frac{1}{2}x-2$

**18.**

| 0 | $x$ |
|---|---|
| 0.08 | 60 |
| 0.03 | $60+x$ |

$0.08(60)+0x=0.03(60+x)$

$4.8=1.8+0.03x$

$3=0.03x$

$100=x$

100 ounces of water [1.1]

**19.** $h(x)=-0.002x^2-0.03x+8$

$h(39)=-0.002(39)^2-0.03(39)+8=3.788$ ft

Yes. [2.4]

**20.** 0.04°F/min [2.3]

# Chapter 3
# Polynomial and Rational Functions

## Section 3.1

**1.**

$$\begin{array}{r} 5x^2 - 9x + 10 - \frac{10}{x+3} \\ x+3\overline{)5x^3 + 6x^2 - 17x + 20} \\ \underline{5x^3 + 15x^2} \qquad\qquad \\ -9x^2 - 17x \qquad \\ \underline{-9x^2 - 27x} \qquad \\ 10x + 20 \\ \underline{10x + 30} \\ -10 \end{array}$$

**2.**

$$\begin{array}{r} 6x^2 - 9x + 28 - \frac{110}{x+4} \\ x+4\overline{)6x^3 + 15x^2 - 8x + 2} \\ \underline{6x^3 + 24x^2} \qquad\qquad \\ -9x^2 - 8x \qquad \\ \underline{-9x^2 - 36x} \qquad \\ 28x + 2 \\ \underline{28x + 112} \\ -110 \end{array}$$

**3.**

$$\begin{array}{r} x^3 + 2x^2 - x + 1 + \frac{1}{x-2} \\ x-2\overline{)x^4 \qquad - 5x^2 + 3x - 1} \\ \underline{x^4 - 2x^3} \qquad\qquad\qquad \\ 2x^3 - 5x^2 \qquad\qquad \\ \underline{2x^3 - 4x^2} \qquad\qquad \\ -x^2 + 3x \qquad \\ \underline{-x^2 + 2x} \qquad \\ x - 1 \\ \underline{x - 2} \\ 1 \end{array}$$

**4.**

$$\begin{array}{r} x^3 - 4x^2 - 4x - 3 - \frac{7}{x-1} \\ x-1\overline{)x^4 - 5x^3 \qquad + x - 4} \\ \underline{x^4 - x^3} \qquad\qquad\qquad \\ -4x^3 \qquad\qquad\qquad \\ \underline{-4x^3 + 4x^2} \qquad\qquad \\ -4x^2 + x \qquad \\ \underline{-4x^2 + 4x} \qquad \\ -3x - 4 \\ \underline{-3x + 3} \\ 7 \end{array}$$

**5.**

$$\begin{array}{r} x^2 + 4x + 10 + \frac{25}{x-3} \\ x-3\overline{)x^3 + x^2 - 2x - 5} \\ \underline{x^3 - 3x^2} \qquad\qquad \\ 4x^2 - 2x \qquad \\ \underline{4x^2 - 12x} \qquad \\ 10x - 5 \\ \underline{10x - 30} \\ 25 \end{array}$$

**6.**

$$\begin{array}{r} x^2 + 5x + 14 + \frac{23}{x-2} \\ x-2\overline{)x^3 + 3x^2 + 4x - 5} \\ \underline{x^3 - 2x^2} \qquad\qquad \\ 5x^2 + 4x \qquad \\ \underline{5x^2 - 10x} \qquad \\ 14x - 5 \\ \underline{14x - 34} \\ 23 \end{array}$$

**7.**

$$\begin{array}{r} x^4 + 2x^3 + 2x + 1 - \frac{8}{x-1} \\ x-1\overline{)x^5 + x^4 - 2x^3 + 2x^2 - 3x - 7} \\ \underline{x^5 - x^4} \qquad\qquad\qquad\qquad\qquad \\ 2x^4 - 2x^3 \qquad\qquad\qquad \\ \underline{2x^4 - 2x^3} \qquad\qquad\qquad \\ 0 + 2x^2 - 3x \qquad \\ \underline{2x^2 - 2x} \qquad \\ -x - 7 \\ \underline{-x - 1} \\ -8 \end{array}$$

**8.**

$$\begin{array}{r} x^4 - 6x^3 + 23x^2 - 89x + 351 - \frac{1396}{x+4} \\ x+4\overline{)x^5 - 2x^4 - x^3 + 3x^2 - 5x + 8} \\ \underline{x^5 + 4x^4} \qquad\qquad\qquad\qquad\qquad \\ -6x^4 - x^3 \qquad\qquad\qquad\qquad \\ \underline{-6x^4 - 24x^3} \qquad\qquad\qquad\qquad \\ 23x^3 + 3x^2 \qquad\qquad\qquad \\ \underline{23x^3 + 92x^2} \qquad\qquad\qquad \\ -89x^2 - 5x \qquad\qquad \\ \underline{-89x^2 - 356x} \qquad\qquad \\ 351x + 8 \\ \underline{351x + 1404} \\ -1396 \end{array}$$

**9.**

$$x^2 + 3x - 2 + \frac{-x+5}{2x^2-x+1}$$
$$2x^2-x+1\overline{)2x^4+5x^3-6x^2+4x+3}$$
$$\underline{2x^4 - x^3 + x^2}$$
$$6x^3 - 7x^2 + 4x$$
$$\underline{6x^3 - 3x^2 + 3x}$$
$$-4x^2 + x + 3$$
$$\underline{-4x^2 + 2x - 2}$$
$$-x + 5$$

**10.**

$$3x + 7 + \frac{3x-12}{x^2-2x+2}$$
$$x^2-2x+2\overline{)3x^3+x^2-5x+2}$$
$$\underline{3x^3 - 6x^2 + 6x}$$
$$7x^2 - 11x + 2$$
$$\underline{7x^2 - 14x + 14}$$
$$3x - 12$$

**11.**

$$x^3 - x^2 + 2x - 1 + \frac{-x+3}{2x^2+x-3}$$
$$2x^2+2x-3\overline{)2x^5 - x^3 + 5x^2 - 9x + 6}$$
$$\underline{2x^5 + 2x^4 - 3x^3}$$
$$-2x^4 + 2x^3 + 5x^2$$
$$\underline{-2x^4 - 2x^3 + 3x^2}$$
$$4x^3 - 2x^2 - 9x$$
$$\underline{4x^3 + 4x^2 - 6x}$$
$$-2x^2 - 3x + 6$$
$$\underline{-2x^2 - 2x + 3}$$
$$-x + 3$$

**12.**

$$x^3 + 3x^2 - 3x - 10 + \frac{2x+14}{x^2+1}$$
$$x^2+1\overline{)x^5 + 3x^4 - 2x^3 - 7x^2 - x + 4}$$
$$\underline{x^3 + x^3}$$
$$3x^4 - 3x^3 - 7x^2$$
$$\underline{3x^4 + 3x^2}$$
$$-3x^3 - 10x^2 - x$$
$$\underline{-3x^3 - 3x}$$
$$-10x^2 + 2x + 4$$
$$\underline{-10x^2 - 10}$$
$$2x + 14$$

**13.**

| | | | | |
|---|---|---|---|---|
| 2 | 4 | –5 | 6 | –7 |
| | | 8 | 6 | 24 |
| | 4 | 3 | 12 | 17 |

$4x^2 + 3x + 12 + \frac{17}{x-2}$

**14.**

| | | | | |
|---|---|---|---|---|
| 5 | 5 | 6 | –8 | 1 |
| | | 25 | 155 | 735 |
| | 5 | 31 | 147 | 736 |

$5x^2 + 31x + 147 + \frac{736}{x-5}$

**15.**

| | | | | |
|---|---|---|---|---|
| –1 | 4 | 0 | –2 | 3 |
| | | –4 | 4 | –2 |
| | 4 | –4 | 2 | 1 |

$4x^2 - 4x + 2 + \frac{1}{x+1}$

**16.**

| | | | | |
|---|---|---|---|---|
| –3 | 6 | –4 | 0 | 17 |
| | | –18 | 66 | –198 |
| | 6 | –22 | 66 | –181 |

$6x^2 - 22x + 66 - \frac{181}{x+3}$

**17.**

| | | | | | | |
|---|---|---|---|---|---|---|
| 4 | 1 | 0 | –10 | 0 | 5 | –1 |
| | | 4 | 16 | 24 | 96 | 404 |
| | 1 | 4 | 6 | 24 | 101 | 403 |

$x^4 + 4x^3 + 6x^2 + 24x + 101 + \frac{403}{x-4}$

**18.**

| | | | | | |
|---|---|---|---|---|---|
| 5 | 6 | –2 | –3 | –1 | 0 |
| | | 30 | 140 | 685 | 3420 |
| | 6 | 28 | 137 | 684 | 3420 |

$6x^3 + 28x^2 + 137x + 684 + \frac{3420}{x-5}$

**19.**

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | –1 |
| | | 1 | 1 | 1 | 1 | 1 |
| | 1 | 1 | 1 | 1 | 1 | 0 |

$x^4 + x^3 + x^2 + x + 1$

**20.**

| | | | | | |
|---|---|---|---|---|---|
| –1 | 1 | 0 | 0 | 0 | 1 |
| | | –1 | 1 | –1 | 1 |
| | 1 | –1 | 1 | –1 | 2 |

$x^3 - x^2 + x - 1 + \frac{2}{x+1}$

**21.**

| | | | | |
|---|---|---|---|---|
| $\frac{1}{2}$ | 8 | –4 | 6 | –3 |
| | | 4 | 0 | 3 |
| | 8 | 0 | 6 | 0 |

$8x^2 + 6$

**22.**

| | | | | |
|---|---|---|---|---|
| $\frac{3}{4}$ | 12 | 5 | 5 | 6 |
| | | –9 | 3 | –6 |
| | 12 | –4 | 8 | 0 |

$12x^2 - 4x + 8$

**23.**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 4 |
| | | 2 | 4 | 10 | 20 | 42 | 84 | 170 | 340 |
| | 1 | 2 | 5 | 10 | 21 | 42 | 85 | 170 | 344 |

$x^7+2x^6+5x^5+10x^4+21x^3+42x^2+85x+170+\dfrac{344}{x-2}$

**24.**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| −1 | −1 | 0 | −1 | 0 | −1 | 0 | −1 | −5 |
| | | 1 | −1 | 2 | −2 | 3 | −3 | 4 |
| | −1 | 1 | −2 | 2 | −3 | 3 | −4 | −1 |

$-x^6+x^5-2x^4+2x^3-3x^2+3x-4-\dfrac{1}{x+1}$

**25.**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| −3 | 1 | 0 | 0 | 0 | 0 | 1 | −10 |
| | | −3 | 9 | −27 | 81 | −243 | 726 |
| | 1 | −3 | 9 | −27 | 81 | −242 | 716 |

$x^5-3x^4+9x^3-27x^2+81x-242+\dfrac{716}{x+3}$

**26.**

| | | | | | | |
|---|---|---|---|---|---|---|
| 4 | 2 | −3 | 0 | −5 | 0 | −10 |
| | | 8 | 20 | 80 | 300 | 1200 |
| | 2 | 5 | 20 | 75 | 300 | 1190 |

$2x^4+5x^3+20x^2+75x+300+\dfrac{1190}{x-4}$

**27.**

| | | | | |
|---|---|---|---|---|
| 2 | 3 | 1 | 1 | −5 |
| | | 6 | 14 | 30 |
| | 3 | 7 | 15 | 25 |

$P(c)=P(2)=25$

**28.**

| | | | | |
|---|---|---|---|---|
| 3 | 2 | −1 | 3 | −1 |
| | | 6 | 15 | 54 |
| | 2 | 5 | 18 | 53 |

$P(c)=P(3)=53$

**29.**

| | | | | | |
|---|---|---|---|---|---|
| −2 | 4 | 0 | −6 | 0 | 5 |
| | | −8 | 16 | −20 | 40 |
| | 4 | −8 | 10 | −20 | 45 |

$P(c)=P(-2)=45$

**30.**

| | | | | |
|---|---|---|---|---|
| −3 | 6 | −1 | 4 | 0 |
| | | −18 | 57 | −183 |
| | 6 | −19 | 61 | −183 |

$P(c)=P(-3)=-183$

**31.**

| | | | | |
|---|---|---|---|---|
| 10 | −2 | −2 | −1 | −20 |
| | | −20 | −220 | −2210 |
| | −2 | −22 | −221 | −2230 |

$P(c)=P(10)=-2230$

**32.**

| | | | | |
|---|---|---|---|---|
| 8 | −1 | 3 | 5 | 30 |
| | | −8 | −40 | −280 |
| | −1 | −5 | −35 | −250 |

$P(c)=P(8)=-250$

**33.**

| | | | | | |
|---|---|---|---|---|---|
| 3 | −1 | 0 | 0 | 0 | 1 |
| | | −3 | −9 | −27 | −81 |
| | −1 | −3 | −9 | −27 | −80 |

$P(c)=P(3)=-80$

**34.**

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | −1 |
| | | 1 | 1 | 1 | 1 | 1 |
| | 1 | 1 | 1 | 1 | 1 | 0 |

$P(c)=P(1)=0$

**35.**

| | | | | | |
|---|---|---|---|---|---|
| 3 | 1 | −10 | 0 | 0 | 2 |
| | | 3 | −21 | −63 | −189 |
| | 1 | −7 | −21 | −63 | −187 |

$P(c)=P(3)=-187$

**36.**

| | | | | | | |
|---|---|---|---|---|---|---|
| −4 | 1 | 0 | 0 | 20 | 0 | −1 |
| | | −4 | 16 | −64 | 176 | −704 |
| | 1 | −4 | 16 | −64 | 176 | −705 |

$P(c)=P(-4)=-705$

**37.**

| | | | | |
|---|---|---|---|---|
| 2 | 1 | 2 | −5 | -6 |
| | | 2 | 8 | 6 |
| | 1 | 4 | 3 | 0 |

A remainder of 0 implies that $x-2$ is a factor of $P(x)$.

**38.**

| | | | | |
|---|---|---|---|---|
| −6 | 1 | 4 | −27 | −90 |
| | | −6 | 12 | 90 |
| | 1 | −2 | −15 | 0 |

A remainder of 0 implies that $x+6$ is a factor of $P(x)$.

**39.**

| | | | | |
|---|---|---|---|---|
| −1 | 2 | 1 | −3 | −1 |
| | | −2 | 1 | 2 |
| | 2 | −1 | −2 | 1 |

A remainder of 1 implies that $x+1$ is not a factor of $P(x)$.

**40.**

| | | | | |
|---|---|---|---|---|
| 4 | 3 | 4 | −27 | −36 |
| | | 12 | 64 | 148 |
| | 3 | 16 | 37 | 112 |

A remainder of 112 implies that $x-4$ is not a factor of $P(x)$.

**41.**

| | | | | | |
|---|---|---|---|---|---|
| −3 | 1 | 0 | −25 | 0 | 144 |
| | | −3 | 9 | 48 | −144 |
| | 1 | −3 | −16 | 48 | 0 |

A remainder of 0 implies that $x+3$ is a factor of $P(x)$.

**42.**

| | | | | | |
|---|---|---|---|---|---|
| 3 | 1 | 0 | −25 | 0 | 144 |
| | | 3 | 9 | −48 | −144 |
| | 1 | 3 | −16 | −48 | 0 |

A remainder of 0 implies that $x-3$ is a factor of $P(x)$.

**43.**

| | | | | | | |
|---|---|---|---|---|---|---|
| 5 | 1 | 2 | −22 | −50 | −75 | 0 |
| | | 5 | 35 | 65 | 75 | 0 |
| | 1 | 7 | 13 | 15 | 0 | 0 |

A remainder of 0 implies that $x-5$ is a factor of $P(x)$.

**44.**

| | | | | | |
|---|---|---|---|---|---|
| −1 | 9 | −6 | −23 | −4 | 4 |
| | | −9 | 15 | 8 | −4 |
| | 9 | −15 | −8 | 4 | 0 |

A remainder of 0 implies that $x+1$ is a factor of $P(x)$.

**45.**

| | | | | | |
|---|---|---|---|---|---|
| $\frac{1}{4}$ | 16 | −8 | 9 | 14 | 4 |
| | | 4 | −1 | 2 | 4 |
| | 16 | −4 | 8 | 16 | 8 |

A remainder of 8 implies that $x-1/4$ is not a factor of $P(x)$.

**46.**

| | | | | | |
|---|---|---|---|---|---|
| $-\frac{1}{2}$ | 10 | 9 | −4 | 9 | 6 |
| | | -5 | -2 | 3 | -6 |
| | 10 | 4 | -6 | 12 | 0 |

A remainder of 0 implies that $x+1/2$ is a factor of $P(x)$.

**47.**

| | | | | |
|---|---|---|---|---|
| 2 | 3 | −8 | −10 | 28 |
| | | 6 | −4 | −28 |
| | 3 | −2 | −14 | 0 |

**48.**

| | | | | |
|---|---|---|---|---|
| 3 | 4 | −10 | −8 | 6 |
| | | 12 | 6 | −6 |
| | 4 | 2 | −2 | 0 |

**49.**

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | −1 |
| | | 1 | 1 | 1 | 1 |
| | 1 | 1 | 1 | 1 | 0 |

**50.**

| | | | | |
|---|---|---|---|---|
| −2 | 1 | 0 | 0 | 8 |
| | | −2 | 4 | −8 |
| | 1 | −2 | 4 | 0 |

**51.**

| | | | | | |
|---|---|---|---|---|---|
| −2 | 3 | 8 | 10 | 2 | −20 |
| | | −6 | −4 | −12 | 20 |
| | 3 | 2 | 6 | −10 | 0 |

**52.**

| | | | | | |
|---|---|---|---|---|---|
| 5 | 1 | 0 | −2 | −100 | −75 |
| | | 5 | 25 | 115 | 75 |
| | 1 | 5 | 23 | 15 | 0 |

**53.**

| | | | | |
|---|---|---|---|---|
| 11 | 2 | −18 | −50 | 66 |
| | | 22 | 44 | −66 |
| | 2 | 4 | −6 | 0 |

**54.**

| | | | | | |
|---|---|---|---|---|---|
| 15 | 2 | −34 | 70 | −153 | 45 |
| | | 30 | −60 | 150 | −45 |
| | 2 | −4 | 10 | −3 | 0 |

**55.**

| | | | |
|---|---|---|---|
| $\frac{2}{3}$ | 3 | −8 | 4 |
| | | 2 | −4 |
| | 3 | −6 | 0 |

**56.**

| | | | |
|---|---|---|---|
| $-\frac{2}{5}$ | 5 | 12 | 4 |
| | | −2 | −4 |
| | 5 | 10 | 0 |

**57.**

| | | | | |
|---|---|---|---|---|
| 2 | 1 | 1 | 1 | −14 |
| | | 2 | 6 | 14 |
| | 1 | 3 | 7 | 0 |

A remainder of 0 implies that $x-2$ is a factor of $P(x)$.

$P(x)=(x-2)(x^2+3x+7)$

**58.**

| | | | | | |
|---|---|---|---|---|---|
| −1 | 1 | 5 | 3 | −5 | −4 |
| | | −1 | −4 | 1 | 4 |
| | 1 | 4 | −1 | −4 | 0 |

A remainder of 0 implies that $x+1$ is a factor of $P(x)$.

$P(x)=(x+1)(x^3+4x^2-x-4)$

**59.**

| | | | | | |
|---|---|---|---|---|---|
| 4 | 1 | −1 | −9 | −11 | −4 |
| | | 4 | 12 | 12 | 4 |
| | 1 | 3 | 3 | 1 | 0 |

A remainder of 0 implies that $x-4$ is a factor of $P(x)$.

$P(x)=(x-4)(x^3+3x^2+3x+1)$

**60.**

| | | | | | | |
|---|---|---|---|---|---|---|
| 2 | 2 | −1 | −7 | 1 | 7 | −10 |
| | | 4 | 6 | −2 | −2 | 10 |
| | 2 | 3 | −1 | −1 | 5 | 0 |

A remainder of 0 implies that $x-2$ is a factor of $P(x)$.

$P(x)=(x-2)(2x^4+3x^3-x^2-x+5)$

**61.** **a.**

| | | | | |
|---|---|---|---|---|
| 8 | 1 | −3 | 2 | 0 |
| | | 8 | 40 | 336 |
| | 1 | 5 | 42 | 336 |

336 ways

**b.**

$$\begin{aligned} P(8) &= 8^3-3(8)^2+2(8) \\ &= 512-3(64)+2(8) \\ &= 512-192+16 \\ &= 336 \text{ ways} \end{aligned}$$

They are the same.

**62.** **a.**

| | | | | | | |
|---|---|---|---|---|---|---|
| 7 | 1 | −10 | 35 | −50 | 24 | 0 |
| | | 7 | −21 | 98 | 336 | 2520 |
| | 1 | −3 | 14 | 48 | 360 | 2520 |

2520 ways

$$\begin{aligned} P(7) &= 7^5-10(7)^4+35(7)^3-50(7)^2+24(7) \\ &= 16{,}807-10(2401)+35(343)-50(49)+24(7) \\ &= 16{,}807-24{,}010+12{,}005-2{,}450+168 \\ &= 2520 \text{ ways} \end{aligned}$$

They are the same.

**63.** **a.**

| | | | |
|---|---|---|---|
| 8 | 1.5 | 0.5 | 0 |
| | | 12 | 100 |
| | 1.5 | 12.5 | 100 |

100 cards

**b.**

| | | | |
|---|---|---|---|
| 20 | 1.5 | 0.5 | 0 |
| | | 30 | 610 |
| | 1.5 | 30.5 | 610 |

610 cards

**64.** **a.**

| | | | | |
|---|---|---|---|---|
| 6 | $\frac{1}{3}$ | $\frac{1}{2}$ | $\frac{1}{6}$ | 0 |
| | | 2 | 15 | 91 |
| | $\frac{1}{3}$ | $\frac{5}{2}$ | $\frac{91}{6}$ | 91 |

91cans

**b.**

| | | | | |
|---|---|---|---|---|
| 12 | $\frac{1}{3}$ | $\frac{1}{2}$ | $\frac{1}{6}$ | 0 |
| | | 4 | 54 | 650 |
| | $\frac{1}{3}$ | $\frac{9}{2}$ | $\frac{325}{6}$ | 650 |

650 cans

**65.** **a.**

| | | | | | |
|---|---|---|---|---|---|
| 12 | 1 | –6 | 11 | –6 | 0 |
| | | 12 | 72 | 996 | 11880 |
| | 1 | 6 | 83 | 990 | 11880 |

11,880 ways

**b.**

| | | | | | |
|---|---|---|---|---|---|
| 24 | 1 | –6 | 11 | –6 | 0 |
| | | 24 | 432 | 10632 | 255024 |
| | 1 | 18 | 443 | 10626 | 255024 |

255,024 ways

**66.** **a.**

| | | | | |
|---|---|---|---|---|
| 7 | 1 | 3 | 0 | 0 |
| | | 7 | 70 | 490 |
| | 1 | 10 | 70 | 490 |

490 cubic inches

**b.**

| | | | | |
|---|---|---|---|---|
| 11 | 1 | 3 | 0 | 0 |
| | | 11 | 154 | 1694 |
| | 1 | 14 | 154 | 1694 |

1694 cubic inches

**67.** **a.**

| | | | | |
|---|---|---|---|---|
| 6 | 1 | 1 | 10 | –8 |
| | | 6 | 42 | 312 |
| | 1 | 7 | 52 | 304 |

304 cubic inches

**b.**

| | | | | |
|---|---|---|---|---|
| 9 | 1 | 1 | 10 | –8 |
| | | 9 | 90 | 900 |
| | 1 | 10 | 100 | 892 |

892 cubic inches

**68.**

| | | | | |
|---|---|---|---|---|
| –2 | 1 | 10 | 31 | 30 |
| | | –2 | –16 | –30 |
| | 1 | 8 | 15 | 0 |

| | | | |
|---|---|---|---|
| –3 | 1 | 8 | 15 |
| | | –3 | –15 |
| | 1 | 5 | 0 |

$(x^3+10x^2+31x+30)\div(x+2)=x^2+8x+15$

$(x^2+8x+15)\div(x+3)=(x+5)$ in.

**69.**

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 0 | 0 | –1 |
| | | 1 | 1 | 1 |
| | 1 | 1 | 1 | 0 |

$(x^3-1)\div(x-1)=x^2+x+1$

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | –1 |
| | | 1 | 1 | 1 | 1 | 1 |
| | 1 | 1 | 1 | 1 | 1 | 0 |

$(x^5-1)\div(x-1)=x^4+x^3+x^2+x+1$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | –1 |
| | | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |

$(x^7-1)\div(x-1)=x^6+x^5+x^4+x^3+x^2+x+1$

$(x^9-1)\div(x-1)=x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1$

**70.**

| | | | | |
|---|---|---|---|---|
| 2 | 1 | –1 | –14 | $k$ |
| | | 2 | 2 | –24 |
| | 1 | 1 | –12 | 0 |

$k=24$

**71.**

| | | | | |
|---|---|---|---|---|
| 3 | 2 | 1 | –25 | $k$ |
| | | 6 | 21 | –12 |
| | 2 | 7 | –4 | 0 |

$k=-12$

**72.**

| | | | | |
|---|---|---|---|---|
| –2 | 3 | 14 | $k$ | –6 |
| | | –6 | –16 | $-2k+32$ |
| | 3 | 8 | $k-16$ | 0 |

$-6-2k+32=0$

$-2k=-26$

$k=13$

**73.**

| | | | | | |
|---|---|---|---|---|---|
| –4 | 1 | 3 | –8 | $k$ | 16 |
| | | –4 | 4 | 16 | $-4k-64$ |
| | 1 | –1 | –4 | $k+16$ | 0 |

$16-4k-64=0$

$-4k=48$

$k=-12$

## Connecting Concepts

**74.** $P(c) = P(1) = 1^n - 1 = 0$

Thus, by the Factor Theorem, $(x - 1)$ is a factor of $P(x)$ for any positive integer $n$.

**75.** $5(1)^{48} + 6(1)^{10} - 5(1) + 7 = 5 + 6 - 5 + 7 = 13$

**76.** $18(-1)^{80} - 6(-1)^{50} + 4(-1)^{20} - 2 = 18 - 6 + 4 - 2 = 14$

**77.**

| | | | | |
|---|---|---|---|---|
| $i$ | 1 | $-3$ | 1 | $-3$ |
| | | $i$ | $-1-3i$ | 3 |
| | 1 | $-3+i$ | $-3i$ | 0 |

A remainder of 0 implies that $x - i$ is a factor of $x^3 - 3x^2 + x - 3$.

**78.**

| | | | | |
|---|---|---|---|---|
| $-2i$ | 1 | $-2$ | 1 | $-8$ |
| | | $-2i$ | $-4-4i$ | $8+6i$ |
| | 1 | $-2-2i$ | $-3+4i$ | $6i$ |

A remainder of 0 implies that $x + 2i$ is a factor of $x^4 - 2x^3 + x^2 - 8x - 12$.

## Prepare for Section 3.2

**PS1.** $P(x) = x^2 - 4x + 6$

$$-\frac{b}{2a} = -\frac{-4}{2(1)} = -\frac{-4}{2} = 2$$

$$P(2) = (2)^2 - 4(2) + 6 = 4 - 8 + 6 = 2$$

The minimum value is 2.

**PS2.** $P(x) = -2x^2 - x + 1$

$$-\frac{b}{2a} = -\frac{-1}{2(-2)} = -\frac{-1}{-4} = -\frac{1}{4}$$

$$P\left(-\frac{1}{4}\right) = -2\left(-\frac{1}{4}\right)^2 - \left(-\frac{1}{4}\right) + 1 = -2\left(\frac{1}{16}\right) - \left(-\frac{1}{4}\right) + 1 = -\frac{1}{8} + \frac{2}{8} + \frac{8}{8} = \frac{9}{8}$$

The maximum value is $\frac{9}{8}$.

**PS3.** $P(x) = x^2 + 2x + 7$ is a parabola that opens up.

The $x$-value of the vertex is

$$-\frac{b}{2a} = -\frac{2}{2(1)} = -\frac{2}{2} = -1$$

The graph decreases from the left until it reaches the vertex, and then it increases.

$P(x)$ increases on the interval $[-1, \infty)$.

**PS4.** $P(x) = -2x^2 + 4x + 5$ is a parabola that opens downward.

The $x$-value of the vertex is

$$-\frac{b}{2a} = -\frac{4}{2(-2)} = -\frac{4}{-4} = 1$$

The graph increases from the left until it reaches the vertex, and then it decreases.

$P(x)$ decreases on the interval $[1, \infty)$.

**PS5.** $x^4 - 5x^2 + 4$

$(x^2 - 4)(x^2 - 1)$

$(x + 2)(x - 2)(x + 1)(x - 1)$

**PS6.** $P(x) = 6x^2 - x - 2$

$0 = 6x^2 - x - 2$

$0 = (3x - 2)(2x + 1)$

$3x - 2 = 0$ or $2x + 1 = 0$

$x = \frac{2}{3}$ $\quad$ $x = -\frac{1}{2}$

The $x$-intercepts are $\left(\frac{2}{3}, 0\right)$ and $\left(-\frac{1}{2}, 0\right)$.

# Section 3.2

**1.** Since $a_n = 3$ is positive and $n = 4$ is even, the graph of $P$ goes up to its far left and up to its far right.

**2.** Since $a_n = -2$ is negative and $n = 3$ is odd, the graph of $P$ goes up to its far left and down to its far right.

**3.** Since $a_n = 5$ is positive and $n = 5$ is odd, the graph of $P$ goes down to its far left and up to its far right.

**4.** Since $a_n = -6$ is negative and $n = 4$ is even, the graph of $P$ goes down to its far left and down to its far right.

**5.** $P(x) = -4x^2 - 3x + 2$

Since $a_n = -4$ is negative and $n = 2$ is even, the graph of $P$ goes down to its far left and down to its far right.

**6.** $P(x) = x^4 - 16$

Since $a_n = 1$ is positive and $n = 4$ is even, the graph of $P$ goes up to its far left and up to its far right.

**7.** $P(x) = \frac{1}{2}x^3 + \frac{5}{2}x^2 - 1$

Since $a_n = \frac{1}{2}$ is positive and $n = 3$ is odd, the graph of $P$ goes down to its far left and up to its far right.

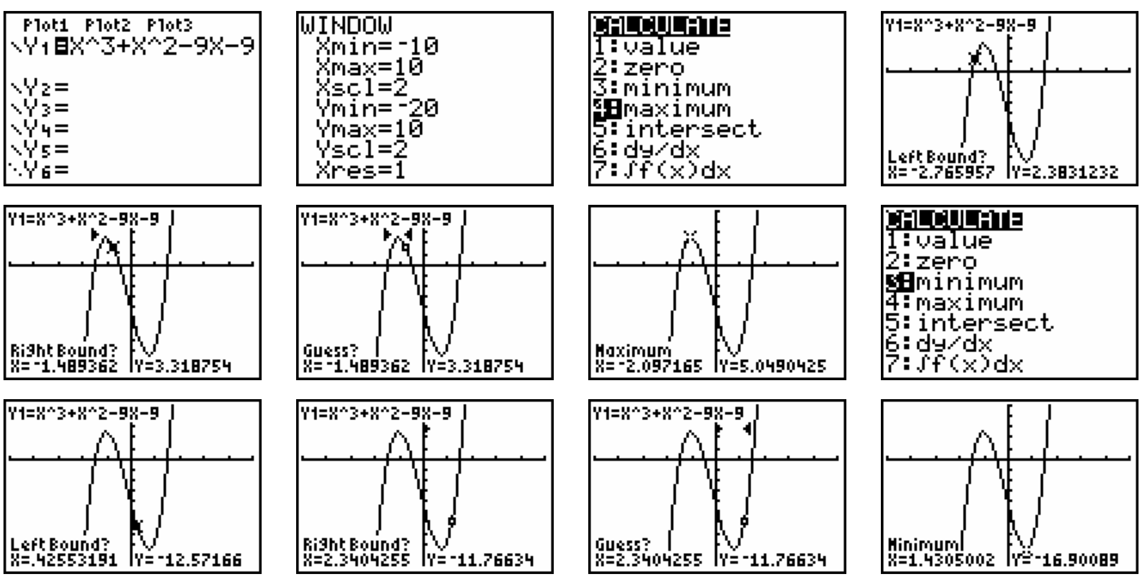
**8.** $P(x) = -\frac{1}{4}x^4 - \frac{3}{4}x^3 + \frac{1}{2}x - \frac{3}{2}$

Since $a_n = -\frac{1}{2}$ is negative and $n = 4$ is even, the graph of $P$ goes down to its far left and down to its far right.

**9.** Up to the far left and down to the far right $\Rightarrow a < 0$.

**10.** Down to the far left and down to the far right $\Rightarrow a < 0$.

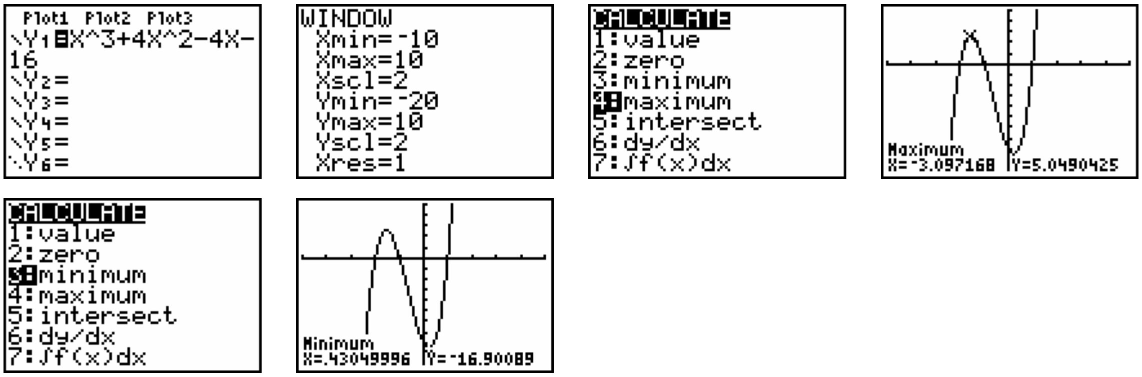
**11.**




On a TI-83 calculator, the CALC feature is located above the TRACE key.

There is a relative maximum of $y \approx 5.0$ at $x \approx -2.1$. There is a relative minimum of $y \approx -16.9$ at $x \approx 1.4$.

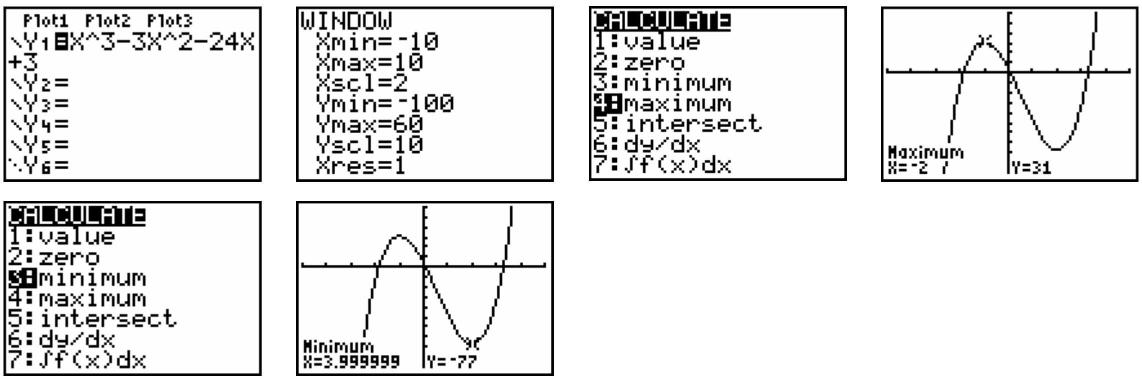
**12.**




The step-by-step technique for a TI-83 calculator is illustrated in the solution to Exercise **15.**
The CALC feature is located above the TRACE key.

There is a relative maximum of $y \approx 5.0$ at $x \approx -3.1$. There is a relative minimum of $y \approx -16.9$ at $x \approx 0.4$.

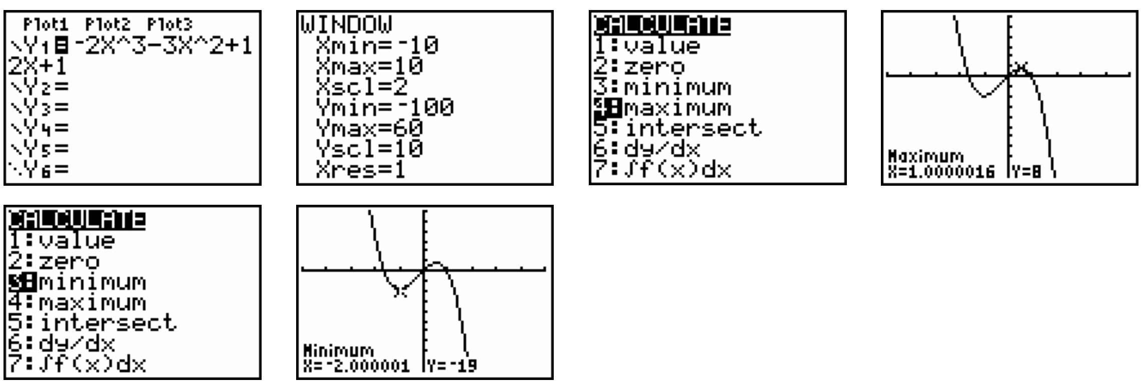
**13.**




The step-by-step technique for a TI-83 calculator is illustrated in the solution to Exercise **15.**
The CALC feature is located above the TRACE key.

There is a relative maximum of $y \approx 31.0$ at $x \approx -2.0$. There is a relative minimum of $y \approx -77.0$ at $x \approx 4$.

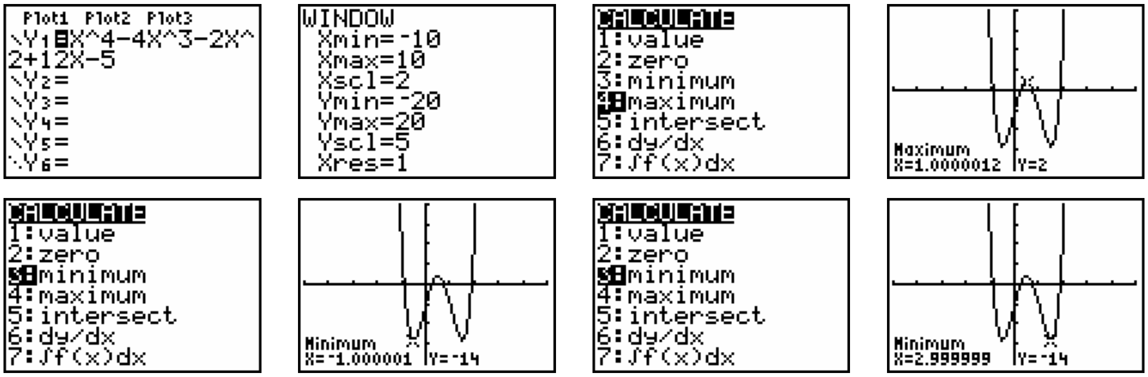
**14.**




The step-by-step technique for a TI-83 calculator is illustrated in the solution to Exercise **15.**
The CALC feature is located above the TRACE key.

There is a relative maximum of $y \approx 8.0$ at $x \approx 1.0$ There is a relative minimum of $y \approx -19.0$ at $x \approx -2.0$.

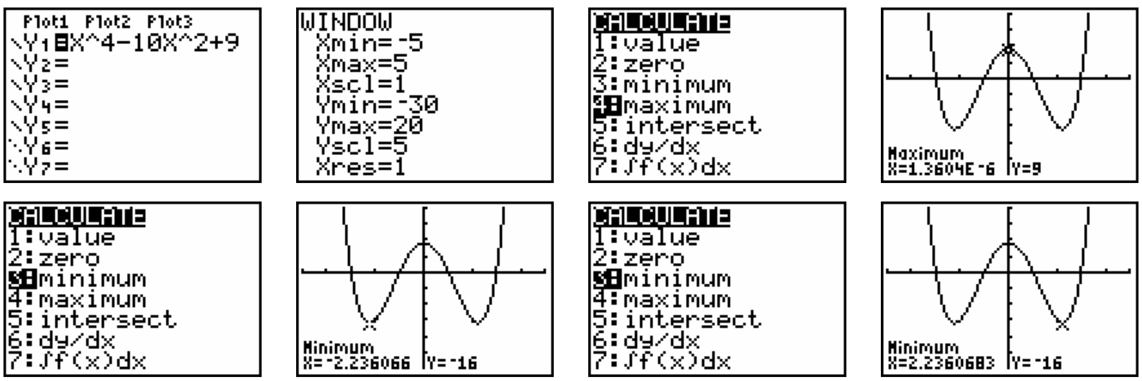
**15.**




The step-by-step technique for a TI-83 calculator is illustrated in the solution to Exercise **15.**
The CALC feature is located above the TRACE key.

There is a relative maximum of $y \approx 2.0$ at $x \approx 1.0$. There is a relative minimum of $y \approx -14.0$ at $x \approx -1.0$, and another relative minimum of $y \approx -14.0$ at $x \approx 3.0$.

**16.**




The step-by-step technique for a TI-83 calculator is illustrated in the solution to Exercise **15.**
The CALC feature is located above the TRACE key.

There is a relative maximum of $y \approx 9.0$ at $x \approx 0.0$. There is a relative minimum of $y \approx -16.0$ at $x \approx -2.2$, and another relative minimum of $y \approx -16.0$ at $x \approx 2.2$.

**17.** $P(x) = x^3 - 2x^2 - 15x$

$0 = x(x^2 - 2x - 15)$

$0 = x(x-5)(x+3)$

The zeros are 0, 5, –3.

**18.** $P(x) = x^3 - 6x^2 + 8x$

$0 = x(x^2 - 6x + 8)$

$0 = x(x-2)(x-4)$

The zeros are 0, 2, 4.

**19.** $P(x) = x^4 - 13x^2 + 36$

$0 = (x^2 - 9)(x^2 - 4)$

$0 = (x+3)(x-3)(x+2)(x-2)$

The zeros are –3, 3, –2, 2.

**20.** $P(x) = 4x^4 - 37x^2 + 9$

$0 = (4x^2 - 1)(x^2 - 9)$

$0 = (2x+1)(2x-1)(x+3)(x-3)$

The zeros are $-\frac{1}{2},\ \frac{1}{2},\ -3,\ 3.$

**21.** $P(x) = x^5 - 5x^3 + 4x$

$0 = x(x^4 - 5x^2 + 4)$

$0 = x(x^2 - 4)(x^2 - 1)$

$0 = x(x+2)(x-2)(x+1)(x-1)$

The zeros are 0, –2, 2, –1, 1.

**22.** $P(x) = x^5 - 25x^3 + 144x$

$0 = x(x^4 - 25x^2 + 144$

$0 = x(x^2 - 16)(x^2 - 9)$

$0 = x(x+4)(x-4)(x+3)(x-3)$

The zeros are 0, –4, 4, –3, 3.

**23.** $P(x) = 2x^3 + 3x^2 - 23x - 42$

| | | | | |
|---|---|---|---|---|
| 3 | 2 | 3 | −23 | −42 |
| | | 6 | 27 | 12 |
| | 2 | 9 | 4 | −30 |

| | | | | |
|---|---|---|---|---|
| 4 | 2 | 3 | −23 | −42 |
| | | 8 | 44 | 84 |
| | 2 | 11 | 21 | 42 |

$P(3)$ is negative; $P(4)$ is positive.
Therefore $P(x)$ has a zero between 3 and 4.

**24.** $P(x) = 4x^3 - x^2 - 6x + 1$

| | | | | |
|---|---|---|---|---|
| 0 | 4 | −1 | −6 | 1 |
| | | 0 | 0 | 0 |
| | 4 | −1 | −6 | 1 |

| | | | | |
|---|---|---|---|---|
| 1 | 4 | −1 | −6 | 1 |
| | | 4 | 3 | −3 |
| | 4 | 3 | −3 | −2 |

$P(0)$ is positive; $P(1)$ is negative.
Therefore $P(x)$ has a zero between 0 and 1.

**25.** $P(x) = 3x^3 + 7x^2 + 3x + 7$

| | | | | |
|---|---|---|---|---|
| −3 | 3 | 7 | 3 | 7 |
| | | −9 | 6 | −27 |
| | 3 | −2 | 9 | −20 |

| | | | | |
|---|---|---|---|---|
| −2 | 3 | 7 | 3 | 7 |
| | | −6 | −2 | −2 |
| | 3 | 1 | 1 | 5 |

$P(-3)$ is negative; $P(-2)$ is positive.
Therefore $P(x)$ has a zero between −3 and −2.

**26.** $P(x) = 2x^3 - 21x^2 - 2x + 25$

| | | | | |
|---|---|---|---|---|
| 1 | 2 | −21 | −2 | 25 |
| | | 2 | −19 | −21 |
| | 2 | −19 | −21 | 4 |

| | | | | |
|---|---|---|---|---|
| 2 | 2 | −21 | −2 | 21 |
| | | 4 | −17 | −38 |
| | 2 | −17 | −19 | −17 |

$P(1)$ is positive; $P(2)$ is negative.
Therefore $P(x)$ has a zero between 1 and 2.

**27.** $P(x) = 4x^4 + 7x^3 - 11x^2 + 7x - 15$

| | | | | | |
|---|---|---|---|---|---|
| 1 | 4 | 7 | −11 | 7 | −15 |
| | | 4 | 11 | 0 | 7 |
| | 4 | 11 | 0 | 7 | −8 |

| | | | | | |
|---|---|---|---|---|---|
| $1\frac{1}{2} = 1.5$ | 4 | 7 | −11 | 7 | −15 |
| | | 6 | 19.5 | 12.75 | 29.625 |
| | 4 | 13 | 8.5 | 19.75 | 14.625 |

$P(1)$ is negative; $P(1.5)$ is positive.
Therefore $P(x)$ has a zero between 1 and 1.5.

**28.** $P(x) = 5x^3 - 16x^2 - 20x + 64$

| | | | | |
|---|---|---|---|---|
| 3 | 5 | −16 | −20 | 64 |
| | | 15 | −3 | −69 |
| | 5 | −1 | −23 | −5 |

| | | | | |
|---|---|---|---|---|
| $3\frac{1}{2} = 3.5$ | 5 | −16 | −20 | 64 |
| | | 17.5 | 5.25 | −51.625 |
| | 5 | 1.5 | −14.75 | 12.375 |

$P(3)$ is negative; $P(3.5)$ is positive.
Therefore $P(x)$ has a zero between 3 and 3.5.

**29.** $P(x) = x^4 - x^2 - x - 4$

| | | | | | |
|---|---|---|---|---|---|
| 1.7 | 1 | 0 | −1 | −1 | −4 |
| | | 1.7 | 2.89 | 3.213 | 3.7621 |
| | 1 | 1.7 | 1.89 | 2.213 | −0.2379 |

| | | | | | |
|---|---|---|---|---|---|
| 1.8 | 1 | 0 | −1 | −1 | −4 |
| | | 1.8 | 3.24 | 4.032 | 5.4576 |
| | 1 | 1.8 | 2.24 | 3.032 | 1.4526 |

$P(1.7)$ is negative; $P(1.8)$ is positive.
Therefore $P(x)$ has a zero between 1.7 and 1.8.

**30.** $P(x) = x^3 - x - 8$

| | | | | |
|---|---|---|---|---|
| 2.1 | 1 | 0 | −1 | −8 |
| | | 2.1 | 4.41 | 7.161 |
| | 1 | 2.1 | 3.41 | −0.839 |

| | | | | |
|---|---|---|---|---|
| 2.2 | 1 | 0 | −1 | −8 |
| | | 2.2 | 4.84 | 8.448 |
| | 1 | 2.2 | 3.84 | 0.448 |

$P(2.1)$ is negative; $P(2.2)$ is positive.
Therefore $P(x)$ has a zero between 2.1 and 2.2.

**31.** $P(x) = -x^4 + x^3 + 5x - 1$

| | | | | | |
|---|---|---|---|---|---|
| 0.1 | −1 | 1 | 0 | 5 | −1 |
| | | −0.1 | 0.09 | 0.009 | 0.5009 |
| | −1 | 0.9 | 0.09 | 5.009 | −0.4991 |

| | | | | | |
|---|---|---|---|---|---|
| 0.2 | −1 | 1 | 0 | 5 | −1 |
| | | −0.2 | 0.16 | 0.032 | 1.0064 |
| | −1 | 0.8 | 0.16 | 5.032 | 0.0064 |

$P(0.1)$ is negative; $P(0.2)$ is positive.
Therefore $P(x)$ has a zero between 0.1 and 0.2.

**32.** $P(x) = -x^3 - 2x^2 + x - 3$

| | | | | |
|---|---|---|---|---|
| −2.8 | −1 | −2 | 1 | −3 |
| | | 2.8 | −2.24 | 3.472 |
| | −1 | 0.8 | −1.24 | 0.472 |

| | | | | |
|---|---|---|---|---|
| −2.7 | −1 | −2 | 1 | −3 |
| | | 2.7 | −1.89 | 2.403 |
| | −1 | 0.7 | −0.89 | −0.597 |

$P(-2.8)$ is positive; $P(-2.7)$ is negative.
Therefore $P(x)$ has a zero between −2.8 and −2.7.

**33.** $P(x) = (x-1)(x+1)(x-3)$

$$0 = (x-1)(x+1)(x-3)$$

$x-1=0$ or $x+1=0$ or $x-3=0$

$x=1$ $x=-1$ $x=3$

The exponents on $(x + 1)$, $(x - 1)$, and $(x - 3)$ are odd integers. Therefore the graph of $P(x)$ will cross the $x$-axis at (–1, 0), (1, 0), and (3, 0).

**34.** $P(x) = (x+2)(x-6)^2$

$$0 = (x+2)(x-6)^2$$

$x+2=0$ or $x-6=0$

$x=-2$ $x=6$

The exponent on $(x + 2)$ is an odd integer. Therefore the graph of $P(x)$ will cross the $x$-axis at (–2, 0).

The exponent on $(x - 6)$ is an even integer. Therefore the graph of $P(x)$ will intersect but not cross the $x$-axis at (6, 0).

**35.** $P(x) = -(x-3)^2(x-7)^5$

$$0 = -(x-3)^2(x-7)^5$$

$x-3=0$ or $x-7=0$

$x=3$ $x=7$

The exponent on $(x - 7)$ is an odd integer. Therefore the graph of $P(x)$ will cross the $x$-axis at (7, 0).

The exponent on $(x - 3)$ is an even integer. Therefore the graph of $P(x)$ will intersect but not cross the $x$-axis at (3, 0).

**36.** $P(x) = (x+2)^3(x-6)^{10}$

$$0 = (x+2)^3(x-6)^{10}$$

$x+2=0$ or $x-6=0$

$x=-2$ $x=6$

The exponent on $(x + 2)$ is an odd integer. Therefore the graph of $P(x)$ will cross the $x$-axis at (–2, 0).

The exponent on $(x - 6)$ is an even integer. Therefore the graph of $P(x)$ will intersect but not cross the $x$-axis at (6, 0).

**37.** $P(x) = (2x-3)^4(x-1)^{15}$

$$0 = (2x-3)^4(x-1)^{15}$$

$2x-3=0$ or $x-1=0$

$x=\frac{3}{2}$ $x=1$

The exponent on $(x-1)$ is an odd integer. Therefore the graph of $P(x)$ will cross the $x$-axis at (1, 0).

The exponent on $(2x - 3)$ is an even integer. Therefore the graph of $P(x)$ will intersect but not cross the $x$-axis at $\left(\frac{3}{2}, 0\right)$.

**38.** $P(x) = (5x+10)^6(x-2.7)^5$

$$0 = (5x+10)^6(x-2.7)^5$$

$5x+10=0$ or $x-2.7=0$

$5x=-10$ $x=2.7$

$x=-2$

The exponent on $(x - 2.7)$ is an odd integer. Therefore the graph of $P(x)$ will cross the $x$-axis at (2.7, 0).

The exponent on $(5x + 10)$ is an even integer. Therefore the graph of $P(x)$ will intersect but not cross the $x$-axis at (–2, 0).

**39.** $P(x) = x^3 - 6x^2 + 9x$

$$0 = x(x^2 - 6x + 9)$$

$$0 = x(x-3)^2$$

$x=0$ or $x-3=0$

$x=3$

The exponent on $x$ is an odd integer. Therefore the graph of $P(x)$ will cross the $x$-axis at (0, 0)

The exponent on $(x - 3)$ is an even integer. Therefore the graph of $P(x)$ will intersect but not cross the $x$-axis at (3, 0).

**40.** $P(x) = x^4 + 3x^3 + 4x^2$

$$0 = x^2(x^2 + 3x + 4)$$

$x^2 = 0$ or $0 = x^2 + 3x + 4$

$x = 0$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(4)}}{2(2)} = \frac{-3 \pm i\sqrt{7}}{4}$$

Not a real number

The exponent on $x$ is an even integer. Therefore the graph of $P(x)$ will intersect but not cross the $x$-axis at (0, 0).

**41.** Let $P(x) = 0$.

$$x^3 - x^2 - 2x = 0$$
$$x(x^2 - x - 2) = 0$$
$$x(x-2)(x+1) = 0$$

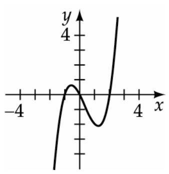
$x = 0, x = 2, x = -1$

The graph crosses the $x$-axis at (0, 0), (2, 0), and (–1, 0).

Let $x = 0$. $P(0) = 0^3 - 0^2 - 2(0) = 0$

The $y$-intercept is (0, 0).

$x^3$ has a positive coefficient and an odd exponent. Therefore, the graph goes down to the far left and up to the far right.




**42.** Let $P(x) = 0$.

$$x^3 + 2x^2 - 3x = 0$$
$$x(x^2 + 2x - 3) = 0$$
$$x(x+3)(x-1) = 0$$

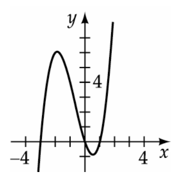
$x = 0, x = -3, x = 1$

The graph crosses the $x$-axis at (0, 0), (–3, 0), and (1, 0).

Let $x = 0$. $P(0) = 0^3 + 2(0)^2 - 3(0) = 0$

The $y$-intercept is (0, 0).

$x^3$ has a positive coefficient and an odd exponent. Therefore, the graph goes down to the far left and up to the far right.




**43.** Let $P(x) = 0$.

$$-x^3 - 2x^2 + 5x - 6 = 0$$
$$x^3 + 2x^2 - 5x + 6 = 0$$

| | | | | |
|---|---|---|---|---|
| 2 | 1 | 2 | –5 | –6 |
| | | 2 | 8 | 6 |
| | 1 | 4 | 3 | 0 |

$$(x-2)(x^2 + 4x + 3) = 0$$
$$(x-2)(x+3)(x+1) = 0$$

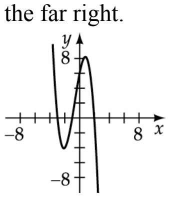
$x = 2, x = -3, x = -1$

The graph crosses the $x$-axis at (2, 0), (–3, 0), and (–1, 0).

Let $x = 0$. $P(0) = -(0)^3 - 2(0)^2 + 5(0) + 6 = 6$

The $y$-intercept is (0, 6).

$-x^3$ has a negative coefficient and an odd exponent. Therefore, the graph goes up to the far left and down to the far right.




**44.** Let $P(x) = 0$.

$$-x^3 - 3x^2 + x + 3 = 0$$
$$x^3 + 3x^2 - x - 3 = 0$$

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 3 | –1 | –3 |
| | | 1 | 4 | 3 |
| | 1 | 4 | 3 | 0 |

$$(x-1)(x^2 + 4x + 3) = 0$$
$$(x-1)(x+3)(x+1) = 0$$

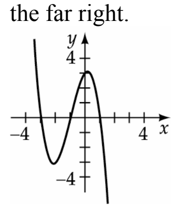
$x = 1, x = -3, x = -1$

The graph crosses the $x$-axis at (1, 0), (–3, 0), and (–1, 0).

Let $x = 0$. $P(0) = -(0)^3 - 3(0)^2 + (0) + 3 = 3$

The $y$-intercept is (0, 3).

$-x^3$ has a negative coefficient and an odd exponent. Therefore, the graph goes up to the far left and down to the far right.



**45.** Let $P(x) = 0$.

$x^4 - 4x^3 + 2x^2 + 4x - 3 = 0$

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1 | –4 | 2 | 4 | –3 |
| | | 1 | –3 | –1 | 3 |
| | 1 | –3 | -1 | 3 | 0 |

| | | | | |
|---|---|---|---|---|
| 3 | 1 | –3 | –1 | 3 |
| | | 3 | 0 | –3 |
| | 1 | 0 | –1 | 0 |

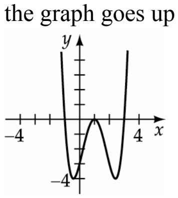
$$(x-1)(x-3)(x^2-1) = 0$$
$$(x-1)(x-3)(x+1)(x-1) = 0$$
$$(x-1)^2(x-3)(x+1) = 0$$

$x = 1, x = 3, x = -1$
The graph intersects the $x$-axis but does not cross it at (1, 0).
The graph crosses the $x$-axis at (3, 0) and (–1, 0).
Let $x = 0$.

$P(0) = (0)^4 - 4(0)^3 + 2(0)^2 + 4(0) - 3 = -3$

The $y$-intercept is (0, –3).

$x^4$ has a positive coefficient and an even exponent. Therefore, the graph goes up to the far left and up to the far right.

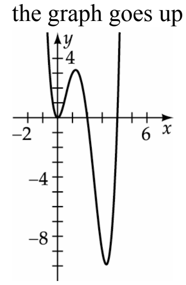



**46.** Let $P(x) = 0$.

$$x^4 - 6x^3 + 8x^2 = 0$$
$$x^2(x^2 - 6x + 8) = 0$$
$$x^2(x-2)(x-4) = 0$$

$x = 0, x = 2, x = 4$
The graph intersects the $x$-axis but does not cross it at (0, 0).
The graph crosses the $x$-axis at (0, 2) and (0, 4).
Let $x = 0$. $P(0) = (0)^4 - 6(0)^3 + 8(0)^2 = 0$
The $y$-intercept is (0, 0).

$x^4$ has a positive coefficient and an even exponent. Therefore, the graph goes up to the far left and up to the far right.

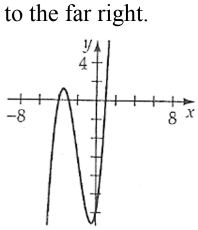



**47.** Let $P(x) = 0$.

$$x^3 + 6x^2 + 5x - 12 = 0$$
$$(x-1)(x+3)(x+4) = 0$$

$x = 1, x = -3, x = -4$
The graph crosses the $x$-axis at (1, 0), (–3, 0), and (–4, 0).
Let $x = 0$. $P(0) = (0)^3 + 6(0)^2 + 5(0) - 12 = -12$
The $y$-intercept is (0, –12).

$x^3$ has a positive coefficient and an odd exponent. Therefore, the graph goes down to the far left and up to the far right.

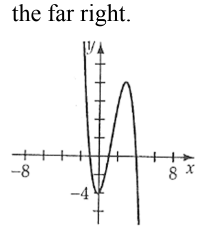



**48.** Let $P(x) = 0$.

$$-x^3 + 4x^2 + x - 4 = 0$$
$$-x^2(x-4) + (x-4) = 0$$
$$(-x^2 + 1)(x-4) = 0$$
$$(-x+1)(x+1)(x-4) = 0$$

$x = 1, x = -1, x = 4$
The graph crosses the $x$-axis at (1, 0), (–1, 0), and (4, 0).
Let $x = 0$. $P(0) = -(0)^3 + 4(0)^2 + 0 - 4 = -4$
The $y$-intercept is (0, –4).

$-x^3$ has a negative coefficient and an odd exponent. Therefore, the graph goes up to the far left and down to the far right.



**49.** Let $P(x) = 0$.

$$\begin{array}{r|rrrr} 1 & -1 & 0 & 7 & -6 \\ & & -1 & -1 & 6 \\ \hline & -1 & -1 & 6 & 0 \end{array}$$

$$-x^3 + 7x - 6 = 0$$
$$(x-1)(-x^2 - x + 6) = 0$$
$$(x-1)(-x-3)(x-2) = 0$$

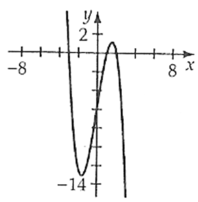
$x = 1, x = -3, x = 2$

The graph crosses the $x$-axis at (1, 0), (–3, 0), and (2, 0).

Let $x = 0$. $P(0) = -(0)^3 + 7(0) - 6 = -6$

The $y$-intercept is (0, –6).

$-x^3$ has a negative coefficient and an odd exponent. Therefore, the graph goes up to the far left and down to the far right.




**50.** Let $P(x) = 0$.

$$x^3 - 6x^2 + 9x = 0$$
$$x(x-3)^2 = 0$$

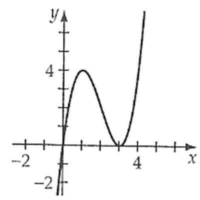
$x = 0, x = 3$

The graph crosses the $x$-axis at (0, 0), and (3, 0).

Let $x = 0$. $P(0) = (0)^3 - 6(0)^2 + 9(0) = 0$

The $y$-intercept is (0, 0).

$x^3$ has a positive coefficient and an odd exponent. Therefore, the graph goes down to the far left and up to the far right.




**51.** Let $P(x) = 0$.

$$-x^3 + 4x^2 - 4x = 0$$
$$-x(x-2)^2 = 0$$

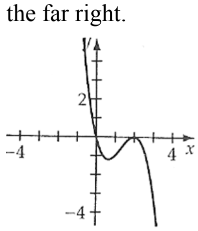
$x = 0, x = 2$

The graph crosses the $x$-axis at (0, 0), and (2, 0).

Let $x = 0$. $P(0) = -(0)^3 + 4(0)^2 - 4(0) = 0$

The $y$-intercept is (0, 0).

$-x^3$ has a negative coefficient and an odd exponent. Therefore, the graph goes up to the far left and down to the far right.




**52.** Let $P(x) = 0$.

$$-x^4 + 2x^3 + 3x^2 - 4x - 4 = 0$$
$$-(x-2)^2(x+1)^2 = 0$$

$x = 2, x = -1$

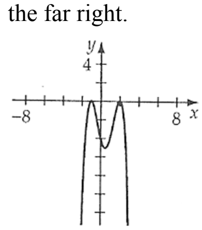
The graph intersects the $x$-axis but does not cross it at (2, 0) and (–1, 0).

Let $x = 0$.

$P(0) = -(0)^4 + 2(0)^3 + 3(0)^2 - 4(0) - 4 = -4$

The $y$-intercept is (0, –4).

$-x^4$ has a negative coefficient and an even exponent. Therefore, the graph goes down to the far left and down to the far right.



**53.** Let $P(x) = 0$.

$$-x^4 + 3x^3 + x^2 - 3x = 0$$
$$-x\left[x^3 - 3x^2 - x + 3\right] = 0$$
$$-x\left[x^2(x-3) - 1(x-3)\right] = 0$$
$$-x\left[(x-3)(x^2-1)\right] = 0$$
$$-x(x-3)(x+1)(x-1) = 0$$

$x = 0, x = 3, x = -1, x = 1$

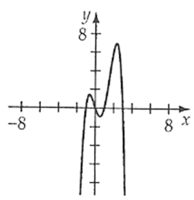
The graph crosses the $x$-axis at (0, 0), (3, 0), (–1, 0) and (1, 0).

Let $x = 0$.

$$P(0) = -(0)^4 + 3(0)^3 + 0^2 - 3(0) = 0$$

The $y$-intercept is (0, 0).

$-x^4$ has a negative coefficient and an even exponent. Therefore, the graph goes down to the far left and down to the far right.




**54.** Let $P(x) = 0$.

$$\frac{1}{2}x^4 + x^3 - 2x^2 - x + \frac{3}{2} = 0$$
$$\frac{1}{2}(x-1)^2(x+1)(x+3) = 0$$

$x = 1, x = -1, x = -3$

The graph intersects the $x$-axis but does not cross it at (1, 0).

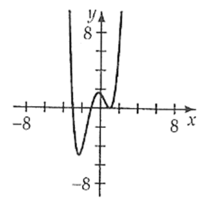
The graph crosses the $x$-axis at (–1, 0) and (–3, 0).

Let $x = 0$.

$$P(0) = \frac{1}{2}(0)^4 + (0)^3 - 2(0)^2 - 0 + \frac{3}{2} = \frac{3}{2}$$

The $y$-intercept is $\left(0, \frac{3}{2}\right)$.

$\frac{1}{2}x^4$ has a positive coefficient and an even exponent. Therefore, the graph goes up to the far left and up to the far right.




**55.** Let $P(x) = 0$.

$$x^5 - x^4 - 5x^3 + x^2 + 8x + 4 = 0$$
$$(x+1)^3(x-2)^2 = 0$$

$x = -1, x = 2$

The graph intersects the $x$-axis but does not cross it at (2, 0).

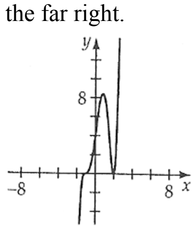
The graph crosses the $x$-axis at (–1, 0).

Let $x = 0$.

$$P(0) = 0^5 - 0^4 - 5(0)^3 + 0^2 + 8(0) + 4 = 4$$

The $y$-intercept is (0, 4).

$x^5$ has a positive coefficient and an odd exponent. Therefore, the graph goes down to the far left and up to the far right.




**56.** Let $P(x) = 0$.

$$2x^5 - 3x^4 - 4x^3 + 3x^2 + 2x = 0$$
$$x(2x^4 - 3x^3 - 4x^2 + 3x + 2) = 0$$

| | | | | | |
|---|---|---|---|---|---|
| –1 | 2 | –3 | –4 | –3 | 2 |
| | | –2 | 5 | –1 | –2 |
| | 2 | –5 | 1 | 2 | 0 |

| | | | | |
|---|---|---|---|---|
| 2 | 2 | –5 | 1 | 2 |
| | | 4 | –2 | –2 |
| | 2 | –1 | –1 | 0 |

$$x(2x^4 - 3x^3 - 4x^2 + 3x + 2) = 0$$
$$x(x+1)(x-2)(2x^2 - x - 1) = 0$$
$$x(x+1)(x-2)(2x+1)(x-1) = 0$$

$x = 0, x = -1, x = 2, \; x = -\frac{1}{2}, x = 1$

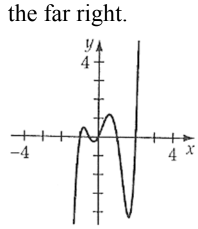
The graph crosses the $x$-axis at (–1, 0), $\left(-\frac{1}{2}, 0\right)$ (0, 0), (1, 0), and (2, 0).

Let $x = 0$.

$$P(0) = 2(0)^5 - 3(0)^4 - 4(0)^3 + 3(0)^2 + 2(0) = 0$$

The $y$-intercept is (0, 0).

$2x^5$ has a positive coefficient and an odd exponent. Therefore, the graph goes down to the far left and up to the far right.




**57.** Shift the graph of $P$ vertically upward 2 units.

**58.** Shift the graph of $P$ vertically downward 3 units.

**59.** Shift the graph of $P$ horizontally 1 unit to the right.

**60.** Shift the graph of $P$ horizontally 3 units to the left.

**61.** Shift the graph of *P* horizontally 2 units to the right and reflect this graph about the *x*-axis. Then shift the resulting graph vertically upward 3 units.

**62.** Shift the graph of *P* horizontally 4 units to the left and vertically downward 5 units.

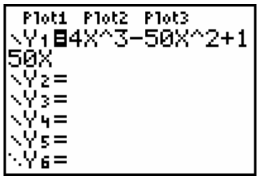
**63.** Since both $P(x)$ and $Q(x)$ have same value for $a_n$, they have the same far-left and far-right behavior.

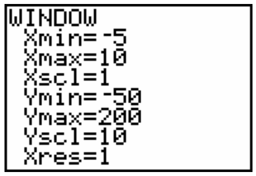
**64.** Since both $P(x)$ and $Q(x)$ have same value for $a_n$, they have the same far-left and far-right behavior.

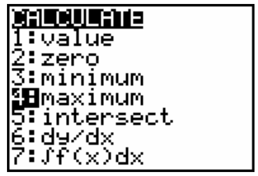
**65.** **a.** Volume = length × width × height

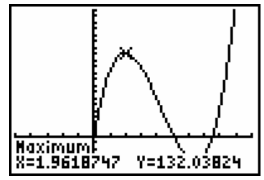
$$V(x) = (15-2x)(10-2x)x = [15(10-2x) - 2x(10-2x)]x = [150 - 30x - 20x + 4x^2]x$$
$$= [4x^2 - 50x + 150]x = 4x^3 - 50x^2 + 150x$$

**b.**





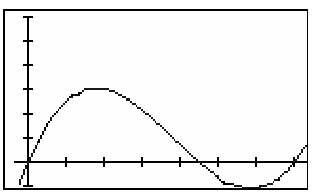





$x = 1.96$ inches (to the nearest 0.01 inch) maximizes the volume of the box.

**66.** $V = \left(\dfrac{42-3x}{2}\right)(18-2x)x = (21-1.5x)(18-2x)x$

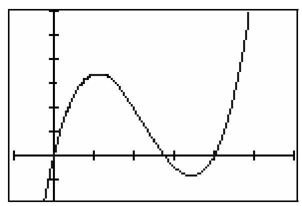
Use a graphing utility to graph
$y = (21-1.5x)(18-2x)x$
Then use the maximum feature of the graphing utility to determine the *x*- and *y*-coordinates of the relative maximum.



Xmin = –1, Xmax = 15, Xscl = 2
Ymin = –200, Ymax = 1200, Yscl = 200

The value of $x \approx 3.571$ inches will produce a box of maximum volume $V \approx 606.6$ cubic inches.

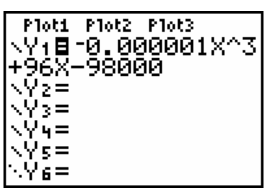
**67.** Use a graphing utility to graph
$y = (22-4x)(16-2x)x.$
Then use the maximum feature of the graphing utility to determine the *x*-and *y*-coordinates of the relative maximum.

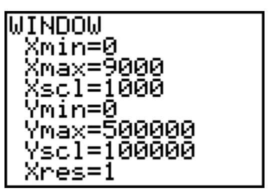


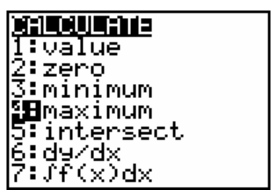
Xmin = –2, Xmax = 12, Xscl = 2
Ymin = –200, Ymax = 600, Yscl = 100

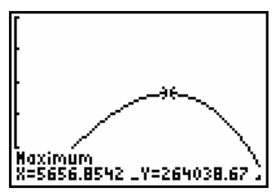
The value of $x \approx 2.137$ inches will produce a box of maximum volume $V \approx 337.1$ cubic inches.

**68.**



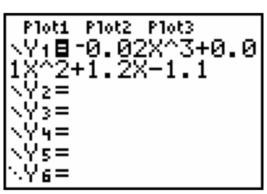


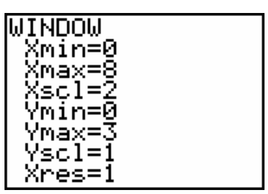


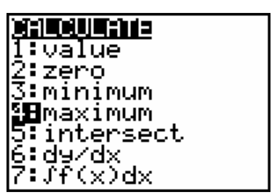


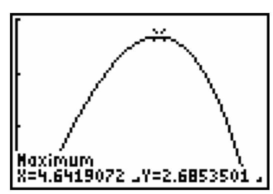

**a.** \$264,000
**b.** 5657 games

**69.**



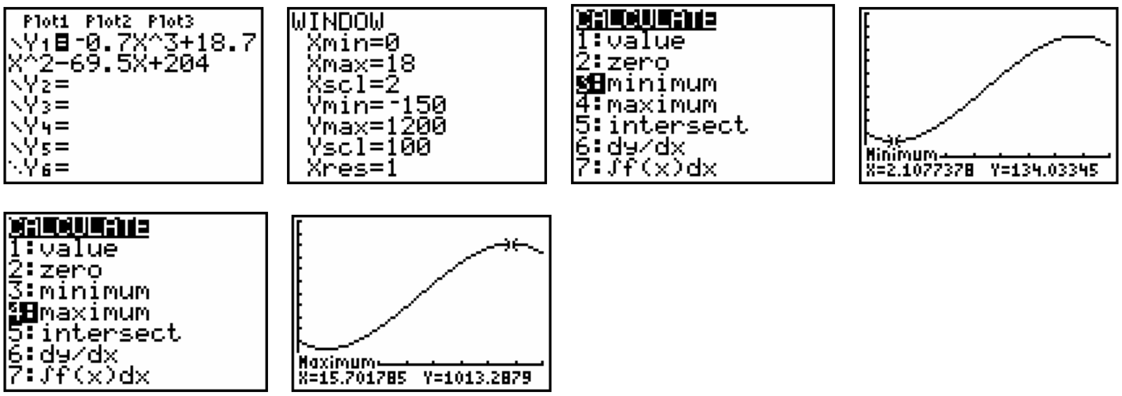







\$464,000

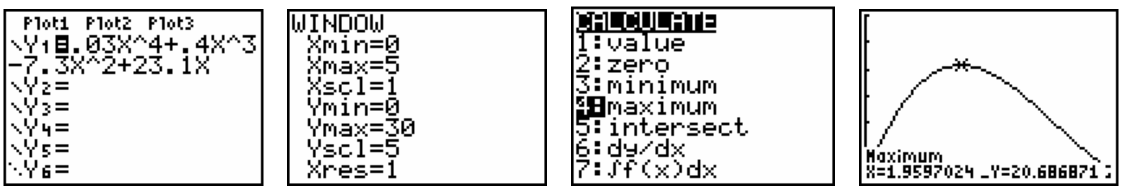
**70.**




**a.** 134 gazelles

**b.** 1013 gazelles

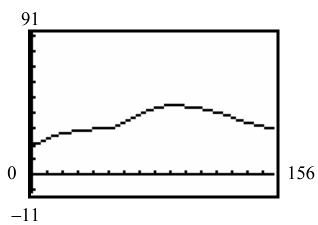
**71.**




**a.** 20.69 milligrams

**b.** 1.968 hours $\times 60$ minutes per hour $\approx 118$ minutes after taking the medication

**72.**

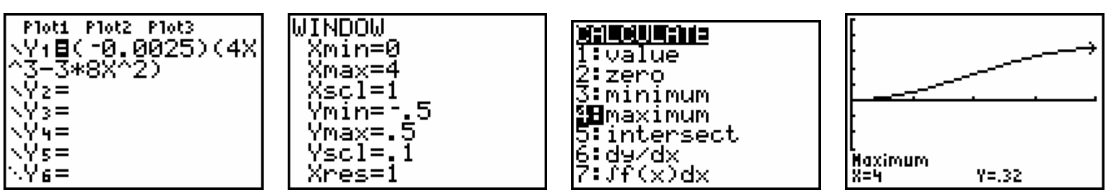



**73.** **a.** $D(x) = (-0.0025)(4x^3 - 3\cdot 8x^2)$

$D(3) = (-0.0025)[4(3)^3 - 3\cdot 8(3)^2] = (-0.0025)[4(27) - 3\cdot 8(9)] = (-0.0025)(108 - 216)$

$D(3) = (-0.0025)(-108) = 0.27$ foot $= 0.27$ foot $\times$ 12 inches per foot $= 3.24$ inches

**b.**



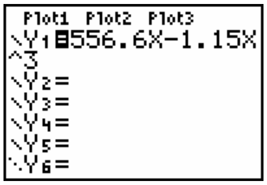

The beam achieves its maximum deflection, 4 feet from the end. The maximum deflection is 0.32 foot × 12 inches per foot = 3.84 inches.

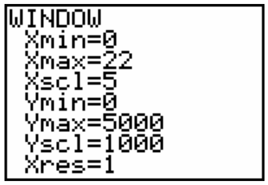
**c.** The formula is valid on the interval (0, 4]. Therefore, $D(5)$ cannot be determined by using the formula. However, 5 feet from one end of an 8-foot beam is 3 feet from the other end. Thus, the deflection at $x = 5$ is the same as the deflection where $x = 3$, which is 3.24 inches.

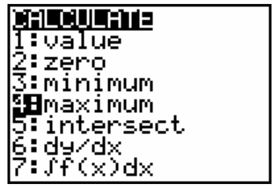
**74.** $w^2 + d^2 = 22^2$

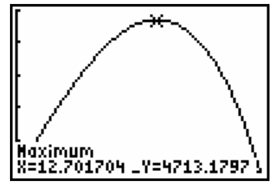
$d^2 = 22^2 - w^2 = 484 - w^2$

$S = 1.15wd^2 = 1.15w(484 - w^2) = 556.6w - 1.15w^3$



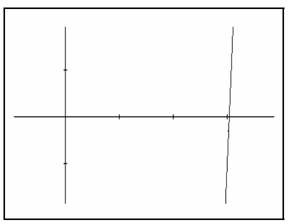

$w \approx 12.70$ inches

$d^2 = 484 - w^2 \approx 484 - (12.70)^2 \approx 484 - 161.29 \approx 322.71$

$d \approx \sqrt{322.71} \approx 17.96$ inches

## Connecting Concepts

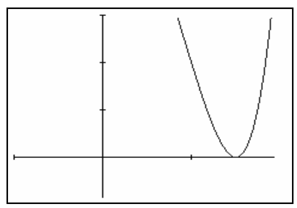
**75.**



Xmin = –1, Xmax = 4, Xscl = 1
Ymin = –2, Ymax = 2, Yscl = 1

There is a real zero between 3 and 4.

**76.**



Xmin = –1, Xmax = 2, Xscl = 1
Ymin = –1, Ymax = 4, Yscl = 1

There is a real zero between 1 and 2.

**77.** $P(x - 3)$ shifts the graph horizontally three points to the right.
$(2 + 3, 0) = (5, 0)$

**78.** $P(x + 1) - 2$ shifts the graph horizontally one points to the left and vertically two points down.
$(3 - 1, 5 - 2) = (2, 3)$

**79.** Shift the graph of $y = x^3$ horizontally two units to the right and vertically upward 1 unit.

**80.** False. Let $P(x) = x^2 - 2x - 8$, $a = -3$, and $b = -5$.
Then $P(a) = 7$ and $P(b) = 7$. However, $x = 4$ is a zero of $P$ and $-3 < 4 < 5$.

## Prepare for Section 3.3

**PS1.** $P(x) = 6x^2 - 25x + 14$

$0 = 6x^2 - 25x + 14$

$0 = (3x - 2)(2x - 7)$

$3x - 2 = 0$ or $2x - 7 = 0$

$x = \frac{2}{3}$ $\quad$ $x = \frac{7}{2}$

**PS2.**

| | | | | |
|---|---|---|---|---|
| –2 | 2 | 3 | 4 | –7 |
| | | –4 | 2 | –12 |
| | 2 | –1 | 6 | –19 |

$2x^2 - x + 6 - \frac{19}{x+2}$

**PS3.**

| | | | | | |
|---|---|---|---|---|---|
| 3 | 3 | 0 | –21 | –3 | –5 |
| | | 9 | 27 | 18 | 45 |
| | 3 | 9 | 6 | 15 | 40 |

$3x^3 + 9x^2 + 6x + 15 + \frac{40}{x-3}$

**PS4.** 1, 2, 3, 4, 6, 12

**PS5.** $\pm 1, \pm 3, \pm 9, \pm 27$

**PS6.** $P(x) = 4x^3 - 3x^2 - 2x + 5$

$P(-x) = 4(-x)^3 - 3(-x)^2 - 2(-x) + 5$

$P(-x) = -4x^3 - 3x^2 + 2x + 5$

# Section 3.3

**1.** $P(x) = (x-3)^2(x+5)$
The zeros are:
–5 (multiplicity 1), 3 (multiplicity 2).

**2.** $P(x) = (x+4)^3(x-1)^2$
The zeros are:
–4 (multiplicity 3), 1 (multiplicity 2).

**3.** $P(x) = x^2(3x+5)^2$
The zeros are:
$-\frac{5}{3}$ (multiplicity 2), 0 (multiplicity 2).

**4.** $P(x) = x^3(2x+1)(3x-12)^2$
$= x^3(2x+1)[3(x-4)]^2$
$= 9x^3(2x+1)(x-4)^2$
The zeros are:
$-\frac{1}{2}$ (multiplicity 1), 0 (multiplicity 3), 4 (multiplicity 2).

**5.** $P(x) = (x^2-4)(x+3)^2$
$= (x+2)(x-2)(x+3)^2$
The zeros are:
–3 (multiplicity 2), –2 (multiplicity 1), 2 (multiplicity 1).

**6.** $P(x) = (x+4)^3(x^2-9)^2$
$= (x+4)^3[(x+3)(x-3)]^2$
$= (x+4)^3(x+3)^2(x-3)^2$
The zeros are:
–4 (multiplicity 3), –3 (multiplicity 2), 3 (multiplicity 2).

**7.** $P(x) = x^3+3x^2-6x-8$
$p = \pm$ factors of $8 = \pm1,\ \pm2,\ \pm4,\ \pm8$
$q = \pm$ factors of $1 = \pm1$
$\frac{p}{q} =$ possible rational zeros $= \pm1,\ \pm2,\ \pm4,\ \pm8$

**8.** $P(x) = x^3-19x-30$
$p = \pm$ factors of $30 = \pm1,\ \pm2,\ \pm3,\ \pm5,\ \pm6,\ \pm10,\ \pm15,\ \pm30$
$q = \pm$ factors of $1 = \pm1$
$\frac{p}{q} =$ possible rational zeros $= \pm1,\ \pm2,\ \pm3,\ \pm5,\ \pm6,\ \pm10,\ \pm15,\ \pm30$

**9.** $P(x) = 2x^3+x^2-25x+12$
$p = \pm$ factors of $12 = \pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm6,\ \pm12$
$q = \pm$ factors of $2 = \pm1,\ \pm2$
$\frac{p}{q} =$ possible rational zeros $= \pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm6,\ \pm12,\ \pm\frac{1}{2},\ \pm\frac{3}{2}$

**10.** $P(x) = 3x^3+11x^2-6x-8$
$p = \pm$ factors of $8 = \pm1,\ \pm2,\ \pm4,\ \pm8$
$q = \pm$ factors of $3 = \pm1,\ \pm3,$
$\frac{p}{q} =$ possible rational zeros $= \pm1,\ \pm2,\ \pm4,\ \pm8,\ \pm\frac{1}{3},\ \pm\frac{2}{3},\ \pm\frac{4}{3},\ \pm\frac{8}{3}$

**11.** $P(x)=6x^4+23x^3+19x^2-8x-4$

$p=\pm$ factors of $4=\pm 1,\ \pm 2,\ \pm 4$

$q=\pm$ factors of $6=\pm 1,\ \pm 2,\ \pm 3,\ \pm 6$

$\frac{p}{q}=$ possible rational zeros $=\pm 1,\ \pm 2,\ \pm 4,\ \pm\frac{1}{2},\ \pm\frac{1}{3},\ \pm\frac{1}{6},\ \pm\frac{2}{3},\ \pm\frac{4}{3}$

**12.** $P(x)=2x^3+9x^2-2x-9$

$p=\pm$ factors of $9=\pm 1,\ \pm 3,\ \pm 9$

$q=\pm$ factors of $2=\pm 1,\ \pm 2$

$\frac{p}{q}=$ possible rational zeros $=\pm 1,\ \pm 3,\ \pm 9,\ \pm\frac{1}{2},\ \pm\frac{3}{2},\ \pm\frac{9}{2}$

**13.** $P(x)=4x^4-12x^3-3x^2+12x-7$

$p=\pm$ factors of $7=\pm 1,\ \pm 7$

$q=\pm$ factors of $4=\pm 1,\ \pm 2,\ \pm 4$

$\frac{p}{q}=$ possible rational zeros $=\pm 1,\ \pm 7,\ \pm\frac{1}{2},\ \pm\frac{7}{2},\ \pm\frac{1}{4},\ \pm\frac{7}{4}$

**14.** $P(x)=x^5-x^4-7x^3+7x^2-12x-12$

$p=\pm$ factors of $12=\pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 6,\ \pm 12$

$q=\pm$ factors of $1=\pm 1$

$\frac{p}{q}=$ possible rational zeros $=\pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 6,\ \pm 12$

**15.** $P(x)=x^5-32$

$p=\pm$ factors of $32=\pm 1,\ \pm 2,\ \pm 4,\ \pm 8,\ \pm 16,\ \pm 32$

$q=\pm$ factors of $1=\pm 1$

$\frac{p}{q}=$ possible rational zeros $=\pm 1,\ \pm 2,\ \pm 4,\ \pm 8,\ \pm 16,\ \pm 32$

**16.** $P(x)=x^4-1$

$p=\pm$ factors of $1=\pm 1$

$q=\pm$ factors of $1=\pm 1$

$\frac{p}{q}=$ possible rational zeros $=\pm 1$

**17.**

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 3 | −6 | −6 |
| | | 1 | 4 | |
| | 1 | 4 | −2 | |

Don't finish dividing. 1 is not an upper bound.

| | | | | |
|---|---|---|---|---|
| 2 | 1 | 3 | −6 | −6 |
| | | 2 | 10 | 8 |
| | 1 | 5 | 4 | 2 |

The smallest integer that is an upper bound is 2.

| | | | | |
|---|---|---|---|---|
| −1 | 1 | 3 | −6 | −6 |
| | | −1 | | |
| | 1 | 2 | | |

Don't finish dividing. −1 is not a lower bound.

| | | | | |
|---|---|---|---|---|
| −2 | 1 | 3 | −6 | −6 |
| | | −2 | | |
| | 1 | 1 | | |

Don't finish dividing. −2 is not a lower bound.

| | | | | |
|---|---|---|---|---|
| −3 | 1 | 3 | −6 | −6 |
| | | −3 | 0 | |
| | 1 | 0 | −6 | |

Don't finish dividing. −3 is not a lower bound.

| | | | | |
|---|---|---|---|---|
| −4 | 1 | 3 | −6 | −6 |
| | | −4 | 4 | |
| | 1 | −1 | −2 | |

Don't finish dividing. −4 is not a lower bound.

| | | | | |
|---|---|---|---|---|
| −5 | 1 | 3 | −6 | −6 |
| | | −5 | 10 | −20 |
| | 1 | −2 | 4 | −26 |

The largest integer that is a lower bound is −5.

**18.**

| | | | | |
|---|---|---|---|---|
| 3 | 1 | 0 | −19 | −28 |
| | | 3 | 9 | |
| | 1 | 3 | −10 | |

Don't finish dividing. 3 is not an upper bound.

| | | | | |
|---|---|---|---|---|
| 4 | 1 | 0 | −19 | −28 |
| | | 4 | 16 | |
| | 1 | 4 | −3 | |

Don't finish dividing. 3 is not an upper bound.

| | | | | |
|---|---|---|---|---|
| 5 | 1 | 0 | −19 | −28 |
| | | 5 | 25 | 30 |
| | 1 | 5 | 6 | 2 |

The smallest integer that is an upper bound is 5.

| | | | | |
|---|---|---|---|---|
| −3 | 1 | 0 | −19 | −28 |
| | | −3 | 9 | |
| | 1 | −3 | −10 | |

Don't finish dividing. −3 is not a lower bound.

| | | | | |
|---|---|---|---|---|
| −4 | 1 | 0 | −19 | −28 |
| | | −4 | 16 | |
| | 1 | −4 | | |

Don't finish dividing. −4 is not a lower bound.

| | | | | |
|---|---|---|---|---|
| −5 | 1 | 0 | −19 | −28 |
| | | −5 | 25 | −30 |
| | 1 | −5 | 6 | −40 |

The largest integer that is a lower bound is −5.

**19.**

| | | | | |
|---|---|---|---|---|
| 3 | 2 | 1 | −25 | 10 |
| | | 6 | 21 | |
| | 2 | 7 | −4 | |

Don't finish dividing. 3 is not an upper bound.

| | | | | |
|---|---|---|---|---|
| 4 | 2 | 1 | −25 | 10 |
| | | 8 | 36 | 44 |
| | 2 | 9 | 11 | 54 |

The smallest integer that is an upper bound is 4.

| | | | | |
|---|---|---|---|---|
| −3 | 2 | 1 | −25 | 10 |
| | | −6 | 15 | |
| | 2 | −5 | −10 | |

Don't finish dividing. −3 is not a lower bound.

| | | | | |
|---|---|---|---|---|
| −4 | 2 | 1 | −25 | 10 |
| | | −8 | 28 | −12 |
| | 2 | −7 | 3 | −2 |

The largest integer that is a lower bound is −4.

**20.**

| | | | | |
|---|---|---|---|---|
| 1 | 3 | 11 | −6 | −9 |
| | | 3 | 14 | 8 |
| | 3 | 14 | 8 | −1 |

1 is not an upper bound.

| | | | | |
|---|---|---|---|---|
| 2 | 3 | 11 | −6 | −9 |
| | | 6 | 34 | 56 |
| | 3 | 17 | 28 | 47 |

The smallest integer that is an upper bound is 2.

| | | | | |
|---|---|---|---|---|
| −4 | 3 | 11 | −6 | −9 |
| | | −12 | 4 | |
| | 3 | −1 | −2 | |

Don't finish dividing. −4 is not a lower bound.

| | | | | |
|---|---|---|---|---|
| −5 | 3 | 11 | −6 | −9 |
| | | −15 | 20 | −70 |
| | 3 | −4 | 14 | −79 |

The largest integer that is a lower bound is −5.

**21.**

| | | | | | |
|---|---|---|---|---|---|
| 1 | 6 | 23 | 19 | –8 | –4 |
| | | 6 | 29 | 48 | 40 |
| | 6 | 29 | 48 | 40 | 36 |

The smallest integer that is an upper bound is 1.

| | | | | | |
|---|---|---|---|---|---|
| –3 | 6 | 23 | 19 | –8 | –4 |
| | | –18 | | | |
| | 6 | 5 | | | |

Don't finish dividing. –3 is not a lower bound.

| | | | | | |
|---|---|---|---|---|---|
| –4 | 6 | 23 | 19 | –8 | –4 |
| | | –24 | 4 | –92 | 400 |
| | 6 | –1 | 23 | –100 | 396 |

The largest integer that is a lower bound is –4.

**22.**

| | | | | |
|---|---|---|---|---|
| 1 | –2 | –9 | 2 | 9 |
| | | –2 | –11 | –9 |
| | –2 | –11 | –9 | 0 |

The smallest integer that is an upper bound is 1.

| | | | | |
|---|---|---|---|---|
| –4 | –2 | –9 | 2 | 9 |
| | | 8 | | |
| | –2 | –1 | | |

Don't finish dividing. –4 is not a lower bound.

| | | | | |
|---|---|---|---|---|
| –5 | –2 | –9 | 2 | 9 |
| | | 10 | –5 | –15 |
| | –2 | 1 | 3 | –24 |

The largest integer that is a lower bound is –5.

**23.**

| | | | | | |
|---|---|---|---|---|---|
| 3 | –4 | 12 | 3 | –12 | 7 |
| | | –12 | 0 | | |
| | –4 | 0 | 3 | | |

Don't finish dividing. 3 is not an upper bound.

| | | | | | |
|---|---|---|---|---|---|
| 4 | –4 | 12 | 3 | –12 | 7 |
| | | –16 | –16 | –52 | –256 |
| | –4 | –4 | –13 | –64 | –249 |

The smallest integer that is an upper bound is 4.

| | | | | | |
|---|---|---|---|---|---|
| –1 | –4 | 12 | 3 | –12 | 7 |
| | | 4 | –16 | 13 | –1 |
| | –4 | 16 | –13 | 1 | 6 |

–1 is not a lower bound.

| | | | | | |
|---|---|---|---|---|---|
| –2 | –4 | 12 | 3 | –12 | 7 |
| | | 8 | –40 | 74 | –124 |
| | –4 | 20 | –37 | 62 | –117 |

The largest integer that is a lower bound is –2.

**24.**

| | | | | | | |
|---|---|---|---|---|---|---|
| 3 | 1 | –1 | –7 | 7 | –12 | –12 |
| | | 3 | 6 | | | |
| | 1 | 2 | –1 | | | |

Don't finish dividing. 3 is not an upper bound.

| | | | | | | |
|---|---|---|---|---|---|---|
| 4 | 1 | –1 | –7 | 7 | –12 | –12 |
| | | 4 | 12 | 20 | 108 | 384 |
| | 1 | 3 | 5 | 27 | 96 | 372 |

The smallest integer that is an upper bound is 4.

| | | | | | | |
|---|---|---|---|---|---|---|
| –2 | 1 | –1 | –7 | 7 | –12 | –12 |
| | | –2 | 6 | | | |
| | 1 | –3 | –1 | | | |

Don't finish dividing. –2 is not a lower bound.

| | | | | | | |
|---|---|---|---|---|---|---|
| –3 | 1 | –1 | –7 | 7 | –12 | –12 |
| | | –3 | 12 | –15 | 24 | –36 |
| | 1 | –4 | 5 | –8 | 12 | –48 |

The largest integer that is a lower bound is –3.

**25.**

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | –32 |
| | | 1 | 1 | 1 | 1 | 1 |
| | 1 | 1 | 1 | 1 | 1 | –31 |

1 is not an upper bound.

| | | | | | | |
|---|---|---|---|---|---|---|
| 2 | 1 | 0 | 0 | 0 | 0 | –32 |
| | | 2 | 4 | 8 | 16 | 32 |
| | 1 | 2 | 4 | 8 | 16 | 0 |

The smallest integer that is an upper bound is 2.

| | | | | | | |
|---|---|---|---|---|---|---|
| –1 | 1 | 0 | 0 | 0 | 0 | –32 |
| | | –1 | 1 | –1 | 1 | –1 |
| | 1 | –1 | 1 | –1 | 1 | –33 |

The largest integer that is a lower bound is –1.

**26.**

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | –1 |
| | | 1 | 1 | 1 | 1 |
| | 1 | 1 | 1 | 1 | 0 |

The smallest integer that is an upper bound is 1.

| | | | | | |
|---|---|---|---|---|---|
| –1 | 1 | 0 | 0 | 0 | –1 |
| | | –1 | 1 | –1 | 1 |
| | 1 | –1 | 1 | –1 | 0 |

The largest integer that is a lower bound is –1.

**27.** $P(x) = x^3 + 3x^2 - 6x - 8$ has 1 change in sign $\Rightarrow$ one positive zero.

$P(-x) = -x^3 + 3x^2 + 6x - 8$ has 2 changes in sign $\Rightarrow$ two or no negative zeros.

**28.** $P(x) = x^3 - 19x - 30$ has 1 change in sign $\Rightarrow$ one positive zero.

$P(-x) = -x^3 + 19x - 30$ has 2 changes in sign $\Rightarrow$ two or no negative zeros.

**29.** $P(x) = 2x^3 + x^2 - 25x + 12$ has 2 changes in sign $\Rightarrow$ two or no positive zeros.
$P(-x) = -2x^3 + x^2 + 25x + 12$ has 1 change in sign $\Rightarrow$ one negative zero.

**30.** $P(x) = 3x^3 + 11x^2 - 6x - 8$ has 1 change in sign $\Rightarrow$ one positive zero.
$P(-x) = -3x^3 + 11x^2 + 6x - 8$ has 2 changes in sign $\Rightarrow$ two or no negative zeros.

**31.** $P(x) = 6x^4 + 23x^3 + 19x^2 - 8x - 4$ has 1 change in sign $\Rightarrow$ one positive zero.
$P(-x) = 6x^4 - 23x^3 + 19x^2 + 8x - 4$ has 3 changes in sign $\Rightarrow$ three or one negative zero.

**32.** $P(x) = 2x^3 + 9x^2 - 2x - 9$ has 1 change in sign $\Rightarrow$ one positive zero.
$P(-x) = -2x^3 + 9x^2 + 2x - 9$ has 2 changes in sign $\Rightarrow$ two or no negative zeros.

**33.** $P(x) = 4x^4 - 12x^3 - 3x^2 + 12x - 7$ has 3 changes in sign $\Rightarrow$ three or one positive zeros.
$P(-x) = 4x^4 + 12x^3 - 3x^2 - 12x - 7$ has 1 change in sign $\Rightarrow$ one negative zero.

**34.** $P(x) = x^5 - x^4 - 7x^3 + 7x^2 - 12x - 12$ has 3 changes in sign $\Rightarrow$ three or one positive zeros.
$P(-x) = -x^5 - x^4 + 7x^3 + 7x^2 + 12x - 12$ has 2 changes in sign $\Rightarrow$ two or no negative zeros.

**35.** $P(x) = x^5 - 32$ has 1 change in sign $\Rightarrow$ one positive zero.
$P(-x) = -x^5 - 32$ has no changes in sign $\Rightarrow$ no negative zeros.

**36.** $P(x) = x^4 - 1$ has 1 change in sign $\Rightarrow$ one positive zero.
$P(-x) = x^4 - 1$ has 1 change in sign $\Rightarrow$ one negative zero.

**37.** $P(x) = 10x^6 - 9x^5 - 14x^4 - 8x^3 - 18x^2 + x + 6$ has 2 changes in sign $\Rightarrow$ two or no positive zeros.
$P(-x) = 10x^6 + 9x^5 - 14x^4 + 8x^3 - 18x^2 - x + 6$ has 4 change in sign $\Rightarrow$ four, two or no negative zeros.

**38.** $P(x) = 2x^6 - 5x^5 - 26x^4 + 76x^3 - 60x^2 - 255x + 700$ has 4 changes in sign $\Rightarrow$ four, two or no positive zeros.
$P(-x) = 2x^6 + 5x^5 - 26x^4 - 76x^3 - 60x^2 + 255x + 700$ has 2 changes in sign $\Rightarrow$ two or no negative zeros.

**39.** $P(x) = 12x^7 - 112x^6 + 421x^5 - 840x^4 + 1038x^3 - 938x^2 + 629x - 210$ has 7 changes in sign $\Rightarrow$ seven, five, three or one positive zeros.
$P(-x) = -12x^7 - 112x^6 - 421x^5 - 840x^4 - 1038x^3 - 938x^2 - 629x - 210$ has no changes in sign $\Rightarrow$ no negative zeros.

**40.** $P(x) = x^7 + 2x^5 + 3x^3 + x$ has no changes in sign $\Rightarrow$ no positive zeros.
$P(-x) = -x^7 - 2x^5 - 3x^3 - x$ has no changes in sign $\Rightarrow$ no negative zeros.

**41.** $P(x) = x^3 + 3x^2 - 6x - 8$
one positive and two or no negative real zeros
$\frac{p}{q} = \pm 1, \ \pm 2, \ \pm 4, \ \pm 8$

| | | | | |
|---|---|---|---|---|
| 2 | 1 | 3 | –6 | –8 |
| | | 2 | 10 | 8 |
| | 1 | 5 | 4 | 0 |

$x^2 + 5x + 4 = (x + 4)(x + 1) = 0 \Rightarrow x = -4, \ -1$
The zeros of $P(x)$ are 2, –4, and –1.

**42.** $P(x) = x^3 - 19x - 30$
one positive and two or no negative real zeros
$\frac{p}{q} = \pm 1, \ \pm 2, \ \pm 3, \ \pm 5, \ \pm 6, \ \pm 10, \ \pm 15, \ \pm 30$

| | | | | |
|---|---|---|---|---|
| 5 | 1 | 0 | –19 | –30 |
| | | 5 | 25 | 30 |
| | 1 | 5 | 6 | 0 |

$x^2 + 5x + 6 = (x + 3)(x + 2) = 0 \Rightarrow x = -3, \ -2$
The zeros of $P(x)$ are –3, –2, and 5.

**43.** $P(x) = 2x^3 + x^2 - 25x + 12$ has two or no positive and one negative real zero.

$\frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm\frac{1}{2}, \pm\frac{3}{2}$

| | | | | |
|---|---|---|---|---|
| 3 | 2 | 1 | –25 | 12 |
| | | 6 | 21 | –12 |
| | 2 | 7 | –4 | 0 |

$2x^2 + 7x - 4 = (2x-1)(x+4) = 0 \Rightarrow x = \frac{1}{2}, -4$. The zeros of $P(x)$ are 3, $\frac{1}{2}$, –4.

**44.** $P(x) = 3x^3 + 11x^2 - 6x - 8$

one positive and two or no negative real zeros

$\frac{p}{q} = \pm 1, \pm 2, \pm 4, \pm 8, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{8}{3}$

| | | | | |
|---|---|---|---|---|
| 1 | 3 | 11 | –6 | –8 |
| | | 3 | 14 | 8 |
| | 3 | 14 | 8 | 0 |

$3x^2 + 14x + 8 = (3x+2)(x+4) = 0$

$\Rightarrow x = -\frac{2}{3}, -4$

The zeros of $P(x)$ are $1, -\frac{2}{3}, -4$.

**45.** $P(x) = 6x^4 + 23x^3 + 19x^2 - 8x - 4$

one positive and three or one negative real zero

| | | | | | |
|---|---|---|---|---|---|
| –2 | 6 | 23 | 19 | –8 | –4 |
| | | –12 | –22 | 6 | 4 |
| | 6 | 11 | –3 | –2 | 0 |

| | | | | |
|---|---|---|---|---|
| –2 | 6 | 11 | –3 | –2 |
| | | –12 | 2 | 2 |
| | 6 | –1 | –1 | 0 |

$6x^2 - x - 1 = (3x+1)(2x-1) = 0 \Rightarrow x = -\frac{1}{3}, \frac{1}{2}$

The zeros of $P(x)$ are –2 (multiplicity 2), $-\frac{1}{3}, \frac{1}{2}$.

**46.** $P(x) = 2x^3 + 9x^2 - 2x - 9$

one positive and two or no negative real zeros

$\frac{p}{q} = \pm 1, \pm 3, \pm 9, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{9}{2}$

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 9 | –2 | –9 |
| | | 2 | 11 | 9 |
| | 2 | 11 | 9 | 0 |

$2x^2 + 11x + 9 = (2x+9)(x+1) = 0$

$\Rightarrow x = -\frac{9}{2}, -1$

The zeros of $P(x)$ are $1, -\frac{9}{2}, -1$.

**47.** $P(x) = 2x^4 - 9x^3 - 2x^2 + 27x - 12$

| | | | | | |
|---|---|---|---|---|---|
| 4 | 2 | –9 | –2 | 27 | –12 |
| | | 8 | –4 | –24 | 12 |
| | 2 | –1 | –6 | 3 | 0 |

| | | | | |
|---|---|---|---|---|
| $\frac{1}{2}$ | 2 | –1 | –6 | 3 |
| | | 1 | 0 | –3 |
| | 2 | 0 | –6 | 0 |

$2x^2 - 6 = 0 \Rightarrow 2(x^2 - 3) = 0 \Rightarrow x = \pm\sqrt{3}$

The zeros of $P(x)$ are $4, \frac{1}{2}, \sqrt{3}, -\sqrt{3}$.

**48.** $P(x) = 3x^3 - x^2 - 6x + 2$

two or no positive and one negative real zero

$\frac{p}{q} = \pm 1, \pm 2, \pm\frac{1}{3}, \pm\frac{2}{3}$

| | | | | |
|---|---|---|---|---|
| $\frac{1}{3}$ | 3 | –1 | –6 | 2 |
| | | 1 | 0 | –2 |
| | 3 | 0 | –6 | 0 |

$3x^2 - 6 = 0 \Rightarrow 3(x^2 - 2) = 0 \Rightarrow x = \pm\sqrt{2}$

The zeros of $P(x)$ are $\frac{1}{3}, \sqrt{2}, -\sqrt{2}$.

**49.** $P(x) = x^3 - 8x^2 + 8x + 24$

two or no positive and one negative real zero

$\frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$

| | | | | |
|---|---|---|---|---|
| 6 | 1 | –8 | 8 | 24 |
| | | 6 | –12 | –24 |
| | 1 | –2 | –4 | 0 |

$x^2 - 2x - 4 = 0$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-4)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{20}}{2} = \frac{2 \pm 2\sqrt{5}}{2} = 1 \pm \sqrt{5}$$

The zeros of $P(x)$ are $6, 1+\sqrt{5}, 1-\sqrt{5}$.

**50.** $P(x) = x^3 - 7x^2 - 7x + 69$

two or no positive and one negative real zero

$\frac{p}{q} = \pm 1, \ \pm 3, \ \pm 23, \ \pm 69$

$$\begin{array}{r|rrrr} -3 & 1 & -7 & -7 & 69 \\ & & -3 & 30 & -69 \\ \hline & 1 & -10 & 23 & 0 \end{array}$$

$x^2 - 10x + 23 = 0$

$$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(23)}}{2(1)} = \frac{10 \pm \sqrt{8}}{2} = \frac{10 \pm 2\sqrt{2}}{2} = 5 \pm \sqrt{2}$$

The zeros of $P(x)$ are $-3, \ 5+\sqrt{2}, \ 5-\sqrt{2}$.

**51.** $P(x) = 2x^4 - 19x^3 + 51x^2 - 31x + 5$

four, two or no positive and no negative real zeros

$\frac{p}{q} = \pm 1, \ \pm 5, \ \pm\frac{1}{2}, \ \pm\frac{5}{2}$

$$\begin{array}{r|rrrrr} 5 & 2 & -19 & 51 & -31 & 5 \\ & & 10 & -45 & 30 & -5 \\ \hline & 2 & -9 & 6 & -1 & 0 \end{array}$$

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -9 & 6 & -1 \\ & & 1 & -4 & 1 \\ \hline & 2 & -8 & 2 & 0 \end{array}$$

$2x^2 - 8x + 2 = 2(x^2 - 4x + 1) = 0$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)} = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

The zeros of $P(x)$ are $5, \ \frac{1}{2}, \ 2+\sqrt{3}, \ 2-\sqrt{3}$.

**52.** $P(x) = 4x^4 - 35x^3 + 71x^2 - 4x - 6$

three or one positive and one negative real zero

$\frac{p}{q} = \pm 1, \ \pm 2, \ \pm 3, \ \pm 6, \ \pm\frac{1}{2}, \ \pm\frac{3}{2}, \ \pm\frac{1}{4}, \ \pm\frac{3}{4}$

$$\begin{array}{r|rrrrr} 3 & 4 & -35 & 71 & -4 & -6 \\ & & 12 & -69 & 6 & 6 \\ \hline & 4 & -23 & 2 & 2 & 0 \end{array}$$

$$\begin{array}{r|rrrr} -\frac{1}{4} & 4 & -23 & 2 & 2 \\ & & -1 & 6 & -2 \\ \hline & 4 & -24 & 8 & 0 \end{array}$$

$4x^2 - 24x + 8 = 4(x^2 - 6x + 2) = 0$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(2)}}{2(1)} = \frac{6 \pm \sqrt{28}}{2} = \frac{6 \pm 2\sqrt{7}}{2} = 3 \pm \sqrt{7}$$

The zeros of $P(x)$ are $3, \ -\frac{1}{4}, \ 3+\sqrt{7}, \ 3-\sqrt{7}$.

**53.** $P(x) = 3x^6 - 10x^5 - 29x^4 + 34x^3 + 50x^2 - 24x - 24$

three or one positive and three or one negative real zeros

$$\begin{array}{r|rrrrrrr} 1 & 3 & -10 & -29 & 34 & 50 & -24 & -24 \\ & & 3 & -7 & -36 & -2 & 48 & 24 \\ \hline & 3 & -7 & -36 & -2 & 48 & 24 & 0 \end{array}$$

$$\begin{array}{r|rrrrrr} -1 & 3 & -7 & -36 & -2 & 48 & 24 \\ & & -3 & 10 & 26 & -24 & -24 \\ \hline & 3 & -10 & -26 & 24 & 24 & 0 \end{array}$$

$$\begin{array}{r|rrrrr} -2 & 3 & -10 & -26 & 24 & 24 \\ & & -6 & 32 & -12 & -24 \\ \hline & 3 & -16 & 6 & 12 & 9 \end{array}$$

$$\begin{array}{r|rrrr} -\frac{2}{3} & 3 & -16 & 6 & 12 \\ & & -2 & 12 & -12 \\ \hline & 3 & -18 & 18 & 0 \end{array}$$

$3x^2 - 18x + 18 = 3(x^2 - 6x + 6) = 0 \Rightarrow x^2 - 6x + 6 = 0$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(6)}}{2(1)} = \frac{6 \pm \sqrt{12}}{2} = \frac{6 \pm 2\sqrt{3}}{2} = 3 \pm \sqrt{3}$$

The zeros of $P(x)$ are $1, -1, -2, \ -\frac{2}{3}, \ 3+\sqrt{3}, \ 3-\sqrt{3}$.

**54.** $P(x) = 2x^4 + 3x^3 - 4x^2 - 3x + 2$
two or no positive and two or no negative real zeros
$\frac{p}{q} = \pm 1, \ \pm 2, \ \pm \frac{1}{2}$

| | | | | | |
|---|---|---|---|---|---|
| −2 | 2 | 3 | −4 | −3 | 2 |
| | | −4 | 2 | 4 | −2 |
| | 2 | −1 | −2 | 1 | 0 |

| | | | | |
|---|---|---|---|---|
| −1 | 2 | −1 | −2 | 1 |
| | | −2 | 3 | −1 |
| | 2 | −3 | 1 | 0 |

$2x^2 - 3x + 1 = (2x-1)(x-1) = 0 \Rightarrow x = \frac{1}{2}, \ 1$
The zeros of $P(x)$ are $-2, \ -1, \ \frac{1}{2}, \ 1$.

**55.** $P(x) = x^3 - 3x - 2$
one positive and two or no negative real zeros
$\frac{p}{q} = \pm 1, \ \pm 2$

| | | | | |
|---|---|---|---|---|
| 2 | 1 | 0 | −3 | −2 |
| | | 2 | 4 | 2 |
| | 1 | 2 | 1 | 0 |

$x^2 + 2x + 1 = (x+1)^2 = 0 \Rightarrow x = -1$
The zeros of $P(x)$ are 2, −1 (multiplicity 2).

**56.** $P(x) = 3x^4 - 4x^3 - 11x^2 + 16x - 4$
three or one positive and one negative real zeros
$\frac{p}{q} = \pm 1, \ \pm 2, \ \pm 4, \ \pm \frac{1}{3}, \ \pm \frac{2}{3}, \ \pm \frac{4}{3}$

| | | | | | |
|---|---|---|---|---|---|
| 1 | 3 | −4 | −11 | 16 | −4 |
| | | 3 | −1 | −12 | 4 |
| | 3 | −1 | −12 | 4 | 0 |

| | | | | |
|---|---|---|---|---|
| 2 | 3 | −1 | −12 | 4 |
| | | 6 | 10 | −4 |
| | 3 | 5 | −2 | 0 |

$3x^2 + 5x - 2 = (3x-1)(x+2) = 0 \Rightarrow x = \frac{1}{3}, \ -2$
The zeros of $P(x)$ are $1, 2, \ \frac{1}{3}, \ -2$.

**57.** $P(x) = x^4 - 5x^2 - 2x = x(x^3 - 5x - 2)$
one positive and two or no negative real zeros
$\frac{p}{q} = \pm 1, \ \pm 2$

| | | | | |
|---|---|---|---|---|
| −2 | 1 | 0 | −5 | −2 |
| | | −2 | 4 | 2 |
| | 1 | −2 | −1 | 0 |

$x^2 - 2x - 1 = 0$
$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$$
$$= \frac{2 \pm \sqrt{8}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$$
The zeros of $P(x)$ are $0, -2, \ 1+\sqrt{2}, \ 1-\sqrt{2}$.

**58.** $P(x) = x^3 - 2x + 1$
two or no positive and one negative real zeros
$\frac{p}{q} = \pm 1$

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 0 | −2 | 1 |
| | | 1 | 1 | −1 |
| | 1 | 1 | −1 | 0 |

$x^2 + x - 1 = 0$
$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}$$
The zeros of $P(x)$ are $1, \ \frac{-1+\sqrt{5}}{2}, \ \frac{-1-\sqrt{5}}{2}$.

**59.** $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$
one positive and three or one negative real zeros
$\frac{p}{q} = \pm 1, \ \pm 2$

| | | | | | |
|---|---|---|---|---|---|
| −1 | 1 | 1 | −3 | −5 | −2 |
| | | −1 | 0 | 3 | 2 |
| | 1 | 0 | −3 | −2 | 0 |

| | | | | |
|---|---|---|---|---|
| −1 | 1 | 0 | −3 | −2 |
| | | −1 | 1 | 2 |
| | 1 | −1 | −2 | 0 |

$x^2 - x - 2 = (x-2)(x+1) = 0 \Rightarrow x = 2, \ -1$
The zeros of $P(x)$ are 2, −1 (multiplicity 3).

**60.** $P(x)=6x^4-17x^3-11x^2+42x$

$=x(6x^3-17x^2-11x+42)$

two or no positive and one negative real zeros

$\frac{p}{q}=\pm1,\ \pm2,\ \pm3,\ \pm6,\ \pm7,\ \pm14,\ \pm21,\ \pm42,\ \pm\frac{1}{2},\ \pm\frac{3}{2},$

$\pm\frac{7}{2},\ \pm\frac{21}{2},\ \pm\frac{1}{3},\ \pm\frac{2}{3},\ \pm\frac{7}{3},\ \pm\frac{14}{3},\ \pm\frac{1}{6},\ \pm\frac{7}{6}$

| | | | | |
|---|---|---|---|---|
| 2 | 6 | −17 | −11 | 42 |
| | | 12 | −10 | −42 |
| | 6 | −5 | −21 | 0 |

$6x^2-5x-21=(3x-7)(2x+3)=0$

$\Rightarrow x=\frac{7}{3},\ -\frac{3}{2}$

The zeros of $P(x)$ are 0, 2, $\frac{7}{3},\ -\frac{3}{2}$.

**61.** $P(x)=2x^4-17x^3+4x^2+35x-24$

three or one positive and one negative real zeros

$\frac{p}{q}=\pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm6,\ \pm8,\ \pm12,\ \pm24,\ \pm\frac{1}{2},\ \pm\frac{3}{2}$

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | −17 | 4 | 35 | −24 |
| | | 2 | −15 | −11 | 24 |
| | 2 | −15 | −11 | 24 | 0 |

| | | | | |
|---|---|---|---|---|
| 1 | 2 | −15 | −11 | 24 |
| | | 2 | −13 | −24 |
| | 2 | −13 | −24 | 0 |

$2x^2-13x-24=(2x+3)(x-8)=0$

$\Rightarrow x=-\frac{3}{2},\ 8$

The zeros of $P(x)$ are 1(multiplicity 2), $-\frac{3}{2}$, 8.

**62.** $P(x)=x^5+5x^4+10x^3+10x^2+5x+1$

no positive and five, three or one negative real zeros

$\frac{p}{q}=\pm1$

| | | | | | | |
|---|---|---|---|---|---|---|
| −1 | 1 | 5 | 10 | 10 | 5 | 1 |
| | | −1 | −4 | −6 | −4 | −1 |
| | 1 | 4 | 6 | 4 | 1 | 0 |

| | | | | | |
|---|---|---|---|---|---|
| −1 | 1 | 4 | 6 | 4 | 1 |
| | | −1 | −3 | −3 | −1 |
| | 1 | 3 | 3 | 1 | 0 |

| | | | | |
|---|---|---|---|---|
| −1 | 1 | 3 | 3 | 1 |
| | | −1 | −2 | −1 |
| | 1 | 2 | 1 | 0 |

$x^2+2x+1=(x+1)^2\Rightarrow x=-1$

The zeros of $P(x)$ are −1 (multiplicity 5).

**63.** $P(x)=x^3-16x=x(x^2-16)$

one positive and one negative real zeros

$\frac{p}{q}=\pm1,\ \pm2,\ \pm4,\ \pm8,\ \pm16$

$x(x^2-16)=x(x+4)(x-4)\Rightarrow x=-4,0,4$

The zeros of $P(x)$ are −4, 0, and 4.

**64.** $P(x)=x^3-4x^2-3x=x(x^2-4x-3)$

one positive and one negative real zeros

$\frac{p}{q}=\pm3,\ \pm1$

$x^2-4x-3=0$

$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(-3)}}{2(1)}$

$=\frac{4\pm\sqrt{28}}{2}=\frac{4\pm2\sqrt{7}}{2}=2\pm\sqrt{7}$

The zeros of $P(x)$ are 0, $2+\sqrt{7},\ 2-\sqrt{7}$.

**65.** The original cube's dimensions are $n\times n\times n$.

The resulting solid measures $n\cdot n\cdot(n-2)$.

$n\cdot n\cdot(n-2)=567$

$n^2(n-2)=567$

$n^3-2n^2-567=0$

| | | | | |
|---|---|---|---|---|
| 9 | 1 | −2 | 0 | −567 |
| | | 9 | 63 | 567 |
| | 1 | 7 | 63 | 0 |

$n=9$ inches.

**66.** The new dimensions are $n\times(n-1)\times(n-3)$.

$n(n-1)(n-3)=1560$

$n(n^2-4n+3)=1560$

$n^3-4n^2+3n-1560=0$

| | | | | |
|---|---|---|---|---|
| 13 | 1 | −4 | 3 | −1560 |
| | | 13 | 117 | 1560 |
| | 1 | 9 | 120 | 0 |

The original cube was 13 inches on each edge.

**67.** $[(x)(x+1)(x+2)]-[(2)(1)(x)]=112$

$x(x^2+3x+2)-2x=112$

$x^3+3x^2+2x-2x=112$

$x^3+3x^2-112=0$

| | | | | |
|---|---|---|---|---|
| 4 | 1 | 3 | 0 | −112 |
| | | 4 | 28 | 112 |
| | 1 | 7 | 28 | 0 |

$x=4$ inches

**68.**

$$x(2x+1)(x+3)=126$$
$$x(2x^2+7x+3)=126$$
$$2x^3+7x^2+3x-126=0$$

| | | | | |
|---|---|---|---|---|
| 3 | 2 | 7 | 3 | −126 |
| | | 6 | 39 | 126 |
| | 2 | 13 | 42 | 0 |

$x=3$; $2x+1=2(3)+1=7$; $x+3=3+3=6$

The box is 3 inches by 7 inches by 6 inches.

**69.** **a.** $P(5)=\dfrac{5^3+5(5)+6}{6}=\dfrac{125+25+6}{6}$

$=\dfrac{156}{6}=26$ pieces

**b.** $\dfrac{n^3+5n+6}{6}=64$

$$n^3+5n+6=384$$
$$n^3+5n-372=0$$

| | | | | |
|---|---|---|---|---|
| 7 | 1 | 0 | 5 | −372 |
| | | 7 | 49 | 378 |
| | 1 | 7 | 54 | 0 |

At least 7 cuts are needed to produce 64 pieces.

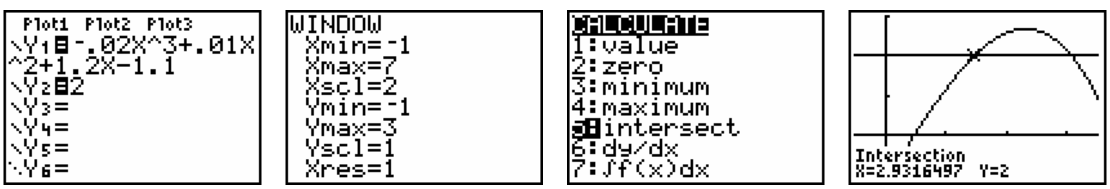
**70.** $x^3-(6^2+5^2+4^2)x-2(6)(5)(4)=0 \Rightarrow x^3-(36+25+16)x-(2(6)(5)(4)=0 \Rightarrow x^3-77x-240=0$


$x \approx 10.04$

**71.**

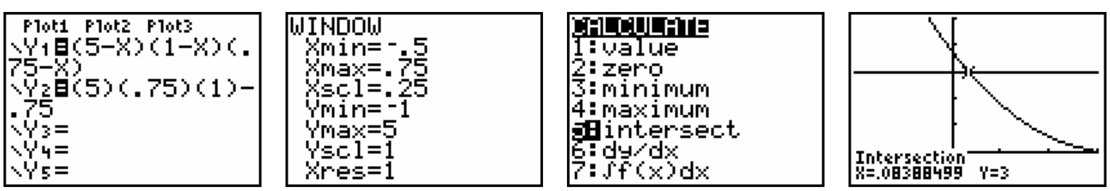

If 140 cannonballs are used, there are 7 rows in the pyramid.

**72.**



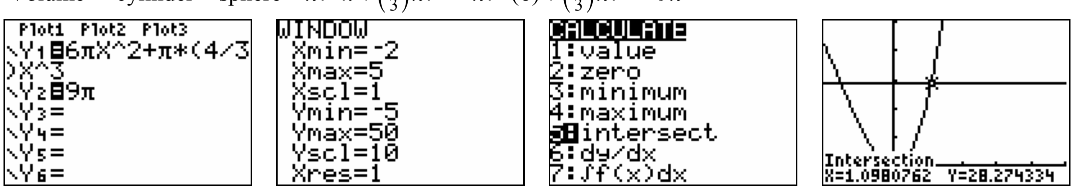

The minimum amount, to the nearest \$1000, the company needs to spend on advertising is \$293,000

**73.**




The company should decrease each dimension by 0.084 inch.

**74.** Volume = cylinder + sphere $=\pi r^2h+\left(\frac{4}{3}\right)\pi r^3=\pi r^2(6)+\left(\frac{4}{3}\right)\pi r^3=9\pi$

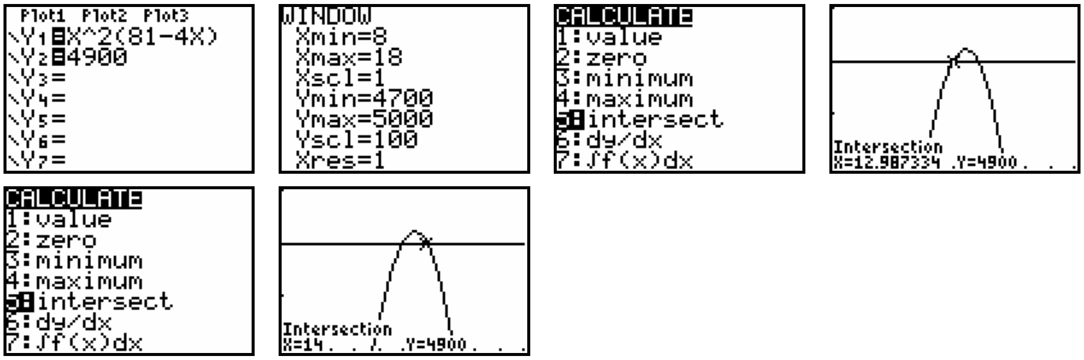



The radius is 1.098 feet.

**75.** $4w + l = 81 \Rightarrow l = 81 - 4w$

Volume = $w^2 l = 4900$

$w^2(81 - 4w) = 4900$

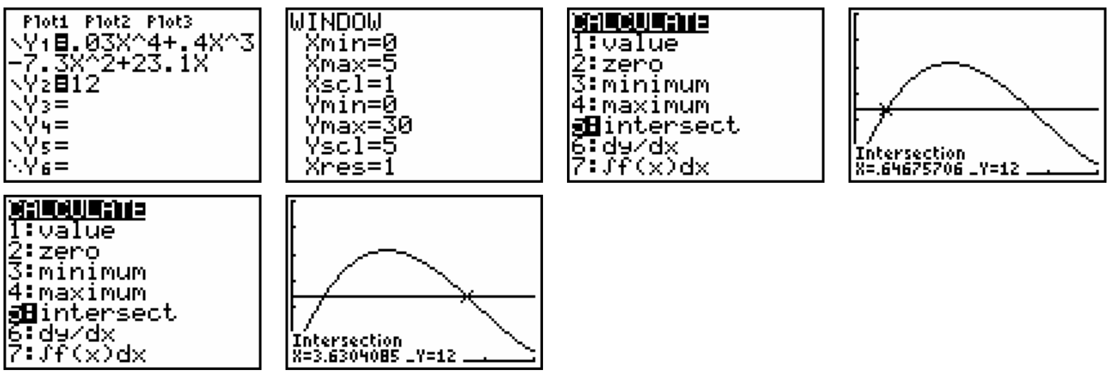



When $w$ = 12.9875, then $l = 81 - 4w = 81 - 4(12.9875) = 29.05$ in.

When $w$ = 14, then $l = 81 - 4w = 81 - 4(14) = 25$ in.

Thus, the lengths can be 25 in. or 29.05 in..

**76.**




0.65 hours × 60 minutes per hour = 39 minutes

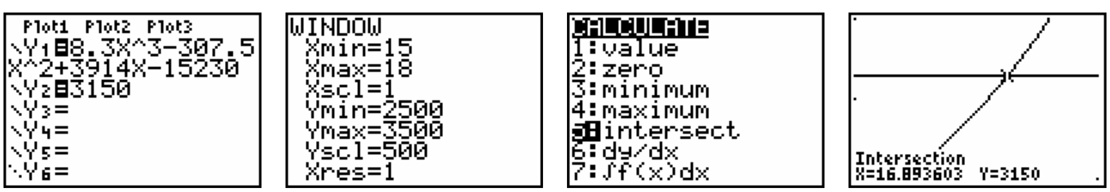
3.63 hours = 3 hours +0.63 hours

= 3 hours + 0.63(60) minutes

= 3 hours 38 minutes

After 39 minutes and after 3 hours 38 minutes.

**77.**



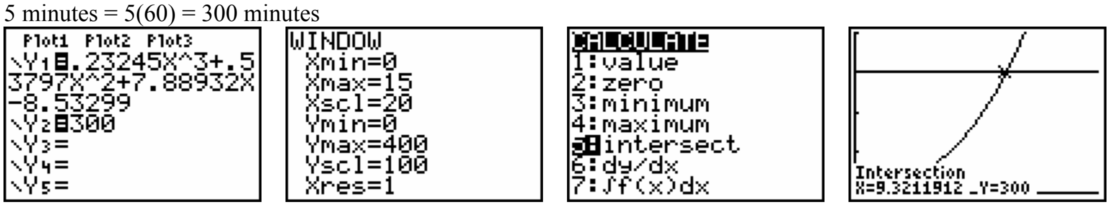

The giraffe is 16.9 feet tall.

**78.**


There are 9 cards in the group.

**79.** **a.** $T(5) = 0.23245(5)^3 + 0.53797(5)^2 + 7.88932(5) - 8.53299$
$= 0.23245(125) + 0.53797(25) + 7.88932(5) - 8.53299$
$= 29.05625 + 13.44925 + 39.4466 - 8.53299$
$= 73.41911 \approx 73$ seconds

**b.** 5 minutes = 5(60) = 300 minutes




Approximately 93,000 digits of $\pi$ can be computed in 5 minutes.

## Connecting Concepts

**80.**
$x=\sqrt{p}$
$x^2=p$
$x^2-p=0$
$P(x)=x^2-p$
$p=\pm p$
$q=\pm 1$
$\frac{p}{q}=\pm p$

Since, $P(p)=p^2-p\neq 0$
$P(-p)=p^2-p\neq 0$

There are no rational zeros.

**81.** $B=\left(\frac{\max\ \text{of}\ (|-5|,|-28|,|15|)}{|2|}+1\right)$

$B=\left(\frac{28}{2}+1\right)$

$B=15$

$|-3| = 3 < 15$

$\left|\frac{1}{2}\right|=\frac{1}{2}<15$

$|5| = 5 < 15$

The absolute value of each zero is less than $B$.

**82.** $B=\left(\frac{\max\ \text{of}\ (|-5|,|2|,|8|)}{|1|}+1\right)$

$B=\left(\frac{8}{1}+1\right)$

$B=9$

$|-1| = 1 < 9$

$|2| = 2 < 9$

$|4| = 4 < 9$

The absolute value of each zero is less than $B$.

**83.** $B=\left(\frac{\max\ \text{of}\ (|-2|,|9|,|2|,|-10|)}{|1|}+1\right)$

$B=\left(\frac{10}{1}+1\right)$

$B=11$

$|1 + 3i| = \sqrt{10} < 11$

$|1 - 3i| = \sqrt{10} < 11$

$|1| = 1 < 11$

$|-1| = 1 < 11$

The absolute value of each zero is less than $B$.

**84.** $B=\left(\frac{\max\ \text{of}\ (|-4|,|14|,|-4|,|13|)}{|1|}+1\right)$

$B=\left(\frac{14}{1}+1\right)$

$B=15$

$|2 + 3i| = \sqrt{13} < 15$

$|2 - 3i| = \sqrt{13}\ \sqrt{10} < 15$

$|i| = 1 < 15$

$|-i| = 1 < 15$

The absolute value of each zero is less than $B$.

## Prepare for Section 3.4

**PS1.** $3 + 2i$

**PS2.** $2 - i\sqrt{5}$

**PS3.**
$$(x-1)(x-3)(x-4)$$
$$(x-1)(x^2-7x+12)$$
$$x(x^2-7x+12)-1(x^2-7x+12)$$
$$x^3-7x^2+12x-x^2+7x-12$$
$$x^3-8x^2+19x-12$$

**PS4.**
$$(x-(2+i))(x-(2-i))$$
$$(x-2-i)(x-2+i)$$
$$((x-2)-i)((x-2)+i)$$
$$(x-2)^2-i^2$$
$$x^2-4x+4-(-1)$$
$$x^2-4x+4+1$$
$$x^2-4x+5$$

**PS5.**
$$x^2+9=0$$
$$x^2=-9$$
$$x=\pm\sqrt{-9}$$
$$x=\pm 3i$$
The solutions are $3i$ and $-3i$.

**PS6.**
$$x^2-x+5=0$$
$$x=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(5)}}{2(1)}$$
$$=\frac{1\pm\sqrt{1-20}}{2}=\frac{1\pm\sqrt{-19}}{2}$$
$$=\frac{1\pm i\sqrt{19}}{2}=\frac{1}{2}\pm\frac{\sqrt{19}}{2}i$$
The solutions are $\frac{1}{2}+\frac{\sqrt{19}}{2}i$ and $\frac{1}{2}-\frac{\sqrt{19}}{2}i$

## Section 3.4

**1.** $P(x)=x^4+x^3-2x^2+4x-24$

| | | | | | |
|---|---|---|---|---|---|
| 2 | 1 | 1 | −2 | 4 | −24 |
| | | 2 | 6 | 8 | 24 |
| | 1 | 3 | 4 | 12 | 0 |

| | | | | |
|---|---|---|---|---|
| −3 | 1 | 3 | 4 | 12 |
| | | −3 | 0 | −12 |
| | 1 | 0 | 4 | 0 |

$x^2+4=0 \Rightarrow x^2=-4 \Rightarrow x=\pm\sqrt{-4} \Rightarrow x=\pm 2i$

The zeros are $2,\ -3,\ 2i,\ -2i$.

$P(x)=(x-2)(x+3)(x-2i)(x+2i)$

**2.** $P(x)=x^3-3x^2+7x-5$

| | | | | |
|---|---|---|---|---|
| 1 | 1 | −3 | 7 | −5 |
| | | 1 | −2 | 5 |
| | 1 | −2 | 5 | 0 |

$$x^2-2x+5=0$$
$$x=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(5)}}{2(1)}$$
$$=\frac{2\pm\sqrt{-16}}{2}=\frac{2\pm 4i}{2}=1\pm 2i$$

The zeros are $1,\ 1+2i,\ 1-2i$.

$P(x)=(x-1)(x-1+2i)(x-1-2i)$

**3.** $P(x) = 2x^4 - x^3 - 4x^2 + 10x - 4$

$$\begin{array}{r|rrrrr} \frac{1}{2} & 2 & -1 & -4 & 10 & -4 \\ & & 1 & 0 & -2 & 4 \\ \hline & 2 & 0 & -4 & 8 & 0 \end{array}$$

$$\begin{array}{r|rrrr} -2 & 2 & 0 & -4 & 8 \\ & & -4 & 8 & -8 \\ \hline & 2 & -4 & 4 & 0 \end{array}$$

$2x^2 - 4x + 4 = 2(x^2 - 2x + 2) = 0$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

The zeros are $\frac{1}{2}, -2, 1+i, 1-i$.

$P(x) = 2\left(x - \frac{1}{2}\right)(x+2)(x-1-i)(x-1+i)$

**4.** $P(x) = x^3 - 13x^2 + 65x - 125$

$$\begin{array}{r|rrrr} 5 & 1 & -13 & 65 & -125 \\ & & 5 & -40 & 125 \\ \hline & 1 & -8 & 25 & 0 \end{array}$$

$x^2 - 8x + 25 = 0$

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(25)}}{2(1)} = \frac{8 \pm \sqrt{64 - 100}}{2}$$

$$= \frac{8 \pm \sqrt{-36}}{2} = \frac{8 \pm 6i}{2} = 4 \pm 3i$$

The zeros are $5, 4+3i, 4-3i$.

$P(x) = (x-5)(x-4-3i)(x-4+3i)$

**5.** $P(x) = x^5 - 9x^4 + 34x^3 - 58x^2 + 45x - 13$

$$\begin{array}{r|rrrrrr} 1 & 1 & -9 & 34 & -58 & 45 & -13 \\ & & 1 & -8 & 26 & -32 & 13 \\ \hline & 1 & -8 & 26 & -32 & 13 & 0 \end{array}$$

$$\begin{array}{r|rrrrr} 1 & 1 & -8 & 26 & -32 & 13 \\ & & 1 & -7 & 19 & -13 \\ \hline & 1 & -7 & 19 & -13 & 0 \end{array}$$

$$\begin{array}{r|rrrr} 1 & 1 & -7 & 19 & -13 \\ & & 1 & -6 & 13 \\ \hline & 1 & -6 & 13 & 0 \end{array}$$

$x^2 - 6x + 13 = 0$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(13)}}{2(1)} = \frac{6 \pm \sqrt{36 - 52}}{2}$$

$$= \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i$$

The zeros are 1 (multiplicity 3), $3+2i, 3-2i$.

$P(x) = (x-1)^3(x-3-2i)(x-3+2i)$

**6.** $P(x) = x^4 - 4x^3 + 53x^2 - 196x + 196$

$$\begin{array}{r|rrrrr} 2 & 1 & -4 & 53 & -196 & 196 \\ & & 2 & -4 & 98 & -196 \\ \hline & 1 & -2 & 49 & -98 & 0 \end{array}$$

$$\begin{array}{r|rrrr} 2 & 1 & -2 & 49 & -98 \\ & & 2 & 0 & 98 \\ \hline & 1 & 0 & 49 & 0 \end{array}$$

$x^2 + 49 = 0 \Rightarrow x^2 = -49 \Rightarrow x = \pm\sqrt{-49} = \pm 7i$

The zeros are 2 (multiplicity 2), $7i$, $-7$i.

$P(x) = (x-2)^2(x-7i)(x+7i)$

**7.** $P(x) = 2x^4 - x^3 - 15x^2 + 23x + 15$

$$\begin{array}{r|rrrrr} -3 & 2 & -1 & -15 & 23 & 15 \\ & & -6 & 21 & -18 & -15 \\ \hline & 2 & -7 & 6 & 5 & 0 \end{array}$$

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & -7 & 6 & 5 \\ & & -1 & 4 & -5 \\ \hline & 2 & -8 & 10 & 0 \end{array}$$

$2x^2 - 8x + 10 = 2(x^2 - 4x + 5) = 0$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

The zeros are $-3, -\frac{1}{2}, 2+i, 2-i$.

$P(x) = 2(x+3)\left(x + \frac{1}{2}\right)(x-2-i)(x-2+i)$

**8.** $P(x) = 3x^4 - 17x^3 - 39x^2 + 337x + 116$

$$\begin{array}{r|rrrrr} -4 & 3 & -17 & -39 & 337 & 116 \\ & & -12 & 116 & -308 & -116 \\ \hline & 3 & -29 & 77 & 29 & 0 \end{array}$$

$$\begin{array}{r|rrrr} -\frac{1}{3} & 3 & -29 & 77 & 29 \\ & & -1 & 10 & -29 \\ \hline & 3 & -30 & 87 & 0 \end{array}$$

$3x^2 - 30x + 87 = 3(x^2 - 10x + 29) = 0$

$$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(29)}}{2(1)}$$

$$= \frac{10 \pm \sqrt{-16}}{2} = \frac{10 \pm 4i}{2} = 5 \pm 2i$$

The zeros are $-4, -\frac{1}{3}, 5+2i, 5-2i$.

$P(x) = 3(x+4)\left(x + \frac{1}{3}\right)(x-5-2i)(x-5+2i)$

**9.** $P(x)=2x^4-14x^3+33x^2-46x+40$

| | | | | | |
|---|---|---|---|---|---|
| 4 | 2 | –14 | 33 | –46 | 40 |
| | | 8 | –24 | 36 | –40 |
| | 2 | –6 | 9 | –10 | 0 |

| | | | | |
|---|---|---|---|---|
| 2 | 2 | –6 | 9 | –10 |
| | | 4 | –4 | 10 |
| | 2 | –2 | 5 | 0 |

$2x^2-2x+5=0$

$x=\dfrac{-(-2)\pm\sqrt{(-2)^2-4(2)(5)}}{2(2)}$

$=\dfrac{2\pm\sqrt{-36}}{4}=\dfrac{2\pm 6i}{4}=\dfrac{1}{2}\pm\dfrac{3}{2}i$

The zeros are 4, 2, $\frac{1}{2}+\frac{3}{2}i,\ \frac{1}{2}-\frac{3}{2}i.$

$P(x)=(x-4)(x-2)\left(x-\frac{1}{2}-\frac{3}{2}i\right)\left(x-\frac{1}{2}+\frac{3}{2}i\right)$

**10.** $P(x)=3x^4-10x^3+15x^2+20x-8$

| | | | | | |
|---|---|---|---|---|---|
| –1 | 3 | –10 | 15 | 20 | –8 |
| | | –3 | 13 | –28 | 8 |
| | 3 | –13 | 28 | –8 | 0 |

| | | | | |
|---|---|---|---|---|
| $\frac{1}{3}$ | 3 | –13 | 28 | –8 |
| | | 1 | –4 | 8 |
| | 3 | –12 | 24 | 0 |

$3x^2-12x+24=3(x^2-4x+8)=0$

$x=\dfrac{-(-4)\pm\sqrt{(-4)^2-4(1)(8)}}{2(1)}$

$=\dfrac{4\pm\sqrt{-16}}{2}=\dfrac{4\pm 4i}{2}=2\pm 2i$

The zeros are $-1,\ \frac{1}{3},\ 2+2i,\ 2-2i.$

$P(x)=3(x+1)\left(x-\frac{1}{3}\right)(x-2-2i)(x-2+2i)$

**11.** $P(x)=2x^3-9x^2+18x-20$

| | | | | |
|---|---|---|---|---|
| $\frac{5}{2}$ | 2 | –9 | 18 | –20 |
| | | 5 | –10 | 20 |
| | 2 | –4 | 8 | 0 |

$2x^2-4x+8=2(x^2-2x+4)=0$

$x=\dfrac{-(-2)\pm\sqrt{(-2)^2-4(1)(4)}}{2(1)}$

$=\dfrac{2\pm\sqrt{-12}}{2}=\dfrac{2\pm 2i\sqrt{3}}{2}=1\pm i\sqrt{3}$

The zeros are $\frac{5}{2},\ 1+i\sqrt{3},\ 1-i\sqrt{3}.$

$P(x)=2\left(x-\frac{5}{2}\right)(x-1-i\sqrt{3})(x-1+i\sqrt{3})$

**12.** $P(x)=3x^4-19x^3+59x^2-79x+36$

| | | | | | |
|---|---|---|---|---|---|
| 1 | 3 | –19 | 59 | –79 | 36 |
| | | 3 | –16 | 43 | –36 |
| | 3 | –16 | 43 | –36 | 0 |

| | | | | |
|---|---|---|---|---|
| $\frac{4}{3}$ | 3 | –16 | 43 | –36 |
| | | 4 | –16 | 36 |
| | 3 | –12 | 27 | 0 |

$3x^2-12x+27=3(x^2-4x+9)=0$

$x=\dfrac{-(-4)\pm\sqrt{(-4)^2-4(1)(9)}}{2(1)}$

$=\dfrac{4\pm\sqrt{-20}}{2}=\dfrac{4\pm 2i\sqrt{5}}{2}=2\pm i\sqrt{5}$

The zeros are $1,\ \frac{4}{3},\ 2+i\sqrt{5},\ 2-i\sqrt{5}.$

$P(x)=3(x-1)\left(x-\frac{4}{3}\right)(x-2-i\sqrt{5})(x-2+i\sqrt{5})$

**13.** $P(x)=2x^4-x^3-2x^2+13x-6$

| | | | | | |
|---|---|---|---|---|---|
| –2 | 2 | –1 | –2 | 13 | –6 |
| | | –4 | 10 | –16 | 6 |
| | 2 | –5 | 8 | –3 | 0 |

| | | | | |
|---|---|---|---|---|
| $\frac{1}{2}$ | 2 | –5 | 8 | –3 |
| | | 1 | –2 | 3 |
| | 2 | –4 | 6 | 0 |

$2x^2-4x+6=2(x^2-2x+3)=0$

$x=\dfrac{-(-2)\pm\sqrt{(-2)^2-4(1)(3)}}{2(1)}$

$=\dfrac{2\pm\sqrt{-8}}{2}=\dfrac{2\pm 2i\sqrt{2}}{2}=1\pm i\sqrt{2}$

The zeros are $-1,\ \frac{1}{2},\ 1+i\sqrt{2},\ 1-i\sqrt{2}.$

$P(x)=2(x+2)\left(x-\frac{1}{2}\right)(x-1-i\sqrt{2})(x-1+i\sqrt{2})$

**14.** $P(x)=4x^4-4x^3+13x^2-12x+3$

| | | | | | |
|---|---|---|---|---|---|
| $\frac{1}{2}$ | 4 | –4 | 13 | –12 | 3 |
| | | 2 | –1 | 6 | –3 |
| | 4 | –2 | 12 | –6 | 0 |

| | | | | |
|---|---|---|---|---|
| $\frac{1}{2}$ | 4 | –2 | 12 | –6 |
| | | 2 | 0 | 6 |
| | 4 | 0 | 12 | 0 |

$4x^2+12=4(x^2+3)=0$

$x^2+3=0\Rightarrow x^2=-3\Rightarrow x=\pm\sqrt{-3}=\pm i\sqrt{3}$

The zeros are $\frac{1}{2}$, (multiplicity 2), $i\sqrt{3},\ -i\sqrt{3}.$

$P(x)=4\left(x-\frac{1}{2}\right)^2(x-i\sqrt{3})(x+i\sqrt{3})$

**15.** $P(x) = 3x^5 + 2x^4 + 10x^3 + 6x^2 - 25x - 20$

| | | | | | | |
|---|---|---|---|---|---|---|
| –1 | 3 | 2 | 10 | 6 | –25 | –20 |
| | | –3 | 1 | –11 | 5 | 20 |
| | 3 | –1 | 11 | –5 | –20 | 0 |

| | | | | | |
|---|---|---|---|---|---|
| –1 | 3 | –1 | 11 | –5 | –20 |
| | | –3 | 4 | –15 | 20 |
| | 3 | –4 | 15 | –20 | 0 |

| | | | | |
|---|---|---|---|---|
| $\frac{4}{3}$ | 3 | –4 | 15 | –20 |
| | | 4 | 0 | 20 |
| | 3 | 0 | 15 | 0 |

$3x^2 + 15 = 3(x^2 + 5) = 0$

$x^2 + 5 = 0 \Rightarrow x^2 = -5 \Rightarrow x = \pm\sqrt{-5} = \pm i\sqrt{5}$

The zeros are $-1$, (multiplicity 2), $\frac{4}{3}$, $i\sqrt{5}$, $-i\sqrt{5}$.

$P(x) = 3(x+1)^2\left(x - \frac{4}{3}\right)(x - i\sqrt{5})(x + i\sqrt{5})$

**16.** $P(x) = 2x^6 - 11x^5 + 5x^4 + 60x^3 - 62x^2 - 64x + 40$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| –2 | 2 | –11 | 5 | 60 | –62 | –64 | 40 |
| | | –4 | 30 | –70 | 20 | 84 | –40 |
| | 2 | –15 | 35 | –10 | –42 | 20 | 0 |

| | | | | | | |
|---|---|---|---|---|---|---|
| –1 | 2 | –15 | 35 | –10 | –42 | 20 |
| | | –2 | 17 | –52 | 62 | –20 |
| | 2 | –17 | 52 | –62 | 20 | 0 |

| | | | | | |
|---|---|---|---|---|---|
| 2 | 2 | –17 | 52 | –62 | 20 |
| | | 4 | –26 | 52 | –20 |
| | 2 | –13 | 26 | –10 | 0 |

| | | | | |
|---|---|---|---|---|
| $\frac{1}{2}$ | 2 | –13 | 26 | –10 |
| | | 1 | –6 | 10 |
| | 2 | –12 | 20 | 0 |

$2x^2 - 12x + 20 = 2(x^2 - 6x + 10) = 0$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(10)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{-4}}{2} = \frac{6 \pm 2i}{2} = 3 \pm i$$

The zeros are $-2$, $-1$, $2$, $\frac{1}{2}$, $3 + i$, $3 - i$.

$P(x) = 2(x+2)(x+1)(x-2)\left(x - \frac{1}{2}\right)(x - 3 - i)(x - 3 + i)$

**17.**

| | | | | |
|---|---|---|---|---|
| $1+i$ | 2 | –5 | 6 | –2 |
| | | $2+2i$ | $-5-i$ | 2 |
| | 2 | $-3+2i$ | $1-i$ | 0 |

| | | | |
|---|---|---|---|
| $1-i$ | 2 | $-3+2i$ | $1-i$ |
| | | $2-2i$ | $-1+i$ |
| | 2 | –1 | 0 |

$2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

The remaining zeros are $1 - i$, $\frac{1}{2}$.

**18.**

| | | | | |
|---|---|---|---|---|
| $5+3i$ | 3 | –29 | 92 | 34 |
| | | $15+9i$ | $-97+3i$ | –34 |
| | 3 | $-14+9i$ | $-5+3i$ | 0 |

| | | | |
|---|---|---|---|
| $5-3i$ | 3 | $-14+9i$ | $-5+3i$ |
| | | $15-9i$ | $5-3i$ |
| | 3 | 1 | 0 |

$3x + 1 = 0 \Rightarrow x = -\frac{1}{3}$

The remaining zeros are $5 - 3i$, $-\frac{1}{3}$.

**19.**

| | | | | |
|---|---|---|---|---|
| $-i$ | 1 | 3 | 1 | 3 |
| | | $-i$ | $-1-3i$ | –3 |
| | 1 | $3-i$ | $-3i$ | 0 |

| | | | |
|---|---|---|---|
| $i$ | 1 | $3-i$ | $-3i$ |
| | | $i$ | $3i$ |
| | 1 | 3 | 0 |

$x + 3 = 0 \Rightarrow x = -3$

The remaining zeros are $i$, $-3$.

**20.**

| | | | | | |
|---|---|---|---|---|---|
| $2+7i$ | 1 | –6 | 71 | –146 | 530 |
| | | $2+7i$ | $-57-14i$ | $126+70i$ | –530 |
| | 1 | $-4+7i$ | $14-14i$ | $-20+70i$ | 0 |

| | | | | |
|---|---|---|---|---|
| $2-7i$ | 1 | $-4+7i$ | $14-14i$ | $-20+70i$ |
| | | $2-7i$ | $-4+14i$ | $20-70i$ |
| | 1 | –2 | 10 | 0 |

$$x^2 - 2x + 10 = 0 \Rightarrow x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(10)}}{2(1)} = \frac{2 \pm \sqrt{-36}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$$

The remaining zeros are $2 - 7i$, $1 + 3i$, $1 - 3i$.

**21.**

| | | | | | |
|---|---|---|---|---|---|
| $2-3i$ | 1 | $-4$ | 14 | $-4$ | 13 |
| | | $2-3i$ | $-13$ | $2-3i$ | $-13$ |
| | 1 | $-2-3i$ | 1 | $-2-3i$ | 0 |

| | | | | |
|---|---|---|---|---|
| $2+3i$ | 1 | $-2-3i$ | 1 | $-2-3i$ |
| | | $2+3i$ | 0 | $2+3i$ |
| | 1 | 0 | 1 | 0 |

$x^2+1=0 \Rightarrow x^2=-1 \Rightarrow x=\pm i$

The remaining zeros are $2+3i,\ i,\ -i$.

**22.**

| | | | | | | |
|---|---|---|---|---|---|---|
| $3i$ | 1 | $-6$ | 22 | $-64$ | 117 | $-90$ |
| | | $3i$ | $-9-18i$ | $54+39i$ | $-117-30i$ | 90 |
| | 1 | $-6+3i$ | $13-18i$ | $-10+39i$ | $-30i$ | 0 |

| | | | | | |
|---|---|---|---|---|---|
| $-3i$ | 1 | $-6+3i$ | $13-18i$ | $-10+39i$ | $-30i$ |
| | | $-3i$ | $18i$ | $-39i$ | $30i$ |
| | 1 | $-6$ | 13 | $-10$ | 0 |

| | | | | |
|---|---|---|---|---|
| 2 | 1 | $-6$ | 13 | $-10$ |
| | | 1 | $-8$ | 10 |
| | 1 | $-4$ | 5 | 0 |

$$x^2-4x+5=0 \Rightarrow x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(5)}}{2(1)}=\frac{4\pm\sqrt{-4}}{2}=\frac{4\pm 2i}{2}=2\pm i$$

The remaining zeros are $-3i,\ 2,\ 2+i,\ 2-i$.

**23.**

| | | | | | |
|---|---|---|---|---|---|
| $1+3i$ | 1 | $-4$ | 19 | $-30$ | 50 |
| | | $1+3i$ | $-12-6i$ | $25+15i$ | $-50$ |
| | 1 | $-3+3i$ | $7-6i$ | $-5+15i$ | 0 |

| | | | | |
|---|---|---|---|---|
| $1-3i$ | 1 | $-3+3i$ | $7-6i$ | $-5+15i$ |
| | | $1-3i$ | $-2+6i$ | $5-15i$ |
| | 1 | $-2$ | 5 | 0 |

$$x^2-2x+5=0 \Rightarrow x=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(5)}}{2(1)}=\frac{2\pm\sqrt{-16}}{2}=\frac{2\pm 4i}{2}=1\pm 2i$$

The remaining zeros are $1-3i,\ 1+2i,\ 1-2i$.

**24.**

| | | | | | | |
|---|---|---|---|---|---|---|
| $i$ | 1 | $-1$ | $-4$ | $-4$ | $-5$ | $-3$ |
| | | $i$ | $-1-i$ | $1-5i$ | $5-3i$ | 3 |
| | 1 | $-1+i$ | $-5-i$ | $-3-5i$ | $-3i$ | 0 |

| | | | | | |
|---|---|---|---|---|---|
| $-i$ | 1 | $-1+i$ | $-5-i$ | $-3-5i$ | $-3i$ |
| | | $-i$ | $i$ | $5i$ | $3i$ |
| | 1 | $-1$ | $-5$ | $-3$ | 0 |

| | | | | |
|---|---|---|---|---|
| 3 | 1 | $-1$ | $-5$ | $-3$ |
| | | 3 | 6 | 3 |
| | 1 | 2 | 1 | 0 |

$x^2+2x+1=(x+1)^2=0 \Rightarrow x=-1$

The remaining zeros are $-i,\ 3,\ -1$ (multiplicity 2)

**25.**

| | | | | | | |
|---|---|---|---|---|---|---|
| $-2i$ | 1 | $-3$ | 7 | $-13$ | 12 | $-4$ |
| | | $-2i$ | $-4+6i$ | $12-6i$ | $-12+2i$ | 4 |
| | 1 | $-3-2i$ | $3+6i$ | $-1-6i$ | $2i$ | 0 |

| | | | | | |
|---|---|---|---|---|---|
| $2i$ | 1 | $-3-2i$ | $3+6i$ | $-1-6i$ | $2i$ |
| | | $2i$ | $-6i$ | $6i$ | $-2i$ |
| | 1 | $-3$ | 3 | $-1$ | 0 |

| | | | | |
|---|---|---|---|---|
| 1 | 1 | $-3$ | 3 | $-1$ |
| | | 1 | $-2$ | 1 |
| | 1 | $-2$ | 1 | 0 |

$x^2-2x+1=(x-1)^2=0 \Rightarrow x=1$

The remaining zeros are $2i$, 1 (multiplicity 3).

**26.**

$$\begin{array}{r|rrrrr} i & 1 & -8 & 18 & -8 & 17 \\ & & i & -1-8i & 8+17i & -17 \\ \hline & 1 & -8-i & 17-8i & 17i & 0 \end{array} \qquad \begin{array}{r|rrrr} -i & 1 & -8+i & 17-8i & 17i \\ & & -i & 8i & -17i \\ \hline & 1 & -8 & 17 & 0 \end{array}$$

$$x^2-8x+17=0 \Rightarrow x=\frac{-(-8)\pm\sqrt{(-8)^2-4(1)(17)}}{2(1)}=\frac{8\pm\sqrt{-4}}{2}=\frac{8\pm 2i}{2}=4\pm i$$

The remaining zeros are $-i,\ 4+i,\ 4-i.$

**27.**

$$\begin{array}{r|rrrrr} 5+2i & 1 & -17 & 112 & -333 & 337 \\ & & 5+2i & -64-14i & 268+26i & -337 \\ \hline & 1 & -12+2i & 48-14i & -65+26i & 0 \end{array}$$

$$\begin{array}{r|rrrr} 5-2i & 1 & -12+2i & 48-14i & -65+26i \\ & & 5-2i & -35+14i & 65-26i \\ \hline & 1 & -7 & 13 & 0 \end{array}$$

$$x^2-7x+13=0 \Rightarrow x=\frac{-(-7)\pm\sqrt{(-7)^2-4(1)(13)}}{2(1)}=\frac{7\pm\sqrt{-3}}{2}=\frac{7\pm i\sqrt{3}}{2}=\frac{7}{2}\pm\frac{\sqrt{3}}{2}i$$

The remaining zeros are $5-2i,\ \frac{7}{2}+\frac{\sqrt{3}}{2}i,\ \frac{7}{2}-\frac{\sqrt{3}}{2}i.$

**28.**

$$\begin{array}{r|rrrrrr} 1-5i & 2 & -8 & 61 & -99 & 12 & 182 \\ & & 2-10i & -56+20i & 105-5i & -19-35i & -182 \\ \hline & 2 & -6-10i & 5+20i & 6-5i & -7-35i & 0 \end{array}$$

$$\begin{array}{r|rrrrr} 1+5i & 2 & -6-10i & 5+20i & 6-5i & -7-35i \\ & & 2+10i & -4-20i & 1+5i & 7+35i \\ \hline & 2 & -4 & 1 & 7 & 0 \end{array}$$

$$\begin{array}{r|rrrr} -1 & 2 & -4 & 1 & 7 \\ & & -2 & 6 & -7 \\ \hline & 2 & -6 & 7 & 0 \end{array}$$

$$2x^2-6x+7=0 \Rightarrow x=\frac{-(-6)\pm\sqrt{(-6)^2-4(2)(7)}}{2(2)}=\frac{6\pm\sqrt{-20}}{4}=\frac{6\pm 2i\sqrt{5}}{4}=\frac{3}{2}\pm\frac{\sqrt{5}}{2}i$$

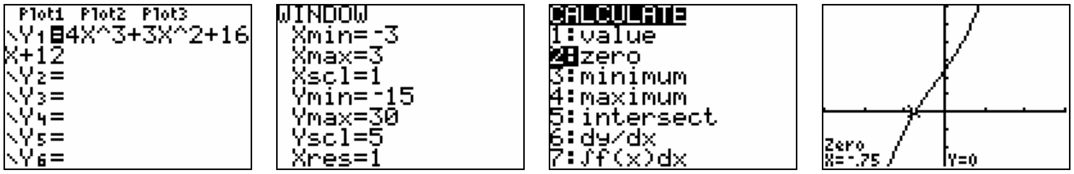
The remaining zeros are $1+5i,\ -1,\ \frac{3}{2}+\frac{\sqrt{5}}{2}i,\ \frac{3}{2}-\frac{\sqrt{5}}{2}i.$

**29.**


$$\begin{array}{r|rrrr} 1.5 & 2 & -1 & 1 & -6 \\ & & 3 & 3 & 6 \\ \hline & 2 & 2 & 4 & 0 \end{array}$$

$$2x^2+2x+4=2(x^2+x+2)=0 \Rightarrow x=\frac{-1\pm\sqrt{(1)^2-4(1)(2)}}{2(1)}=\frac{-1\pm\sqrt{-7}}{2}=\frac{-1\pm i\sqrt{7}}{2}=-\frac{1}{2}\pm\frac{\sqrt{7}}{2}i$$

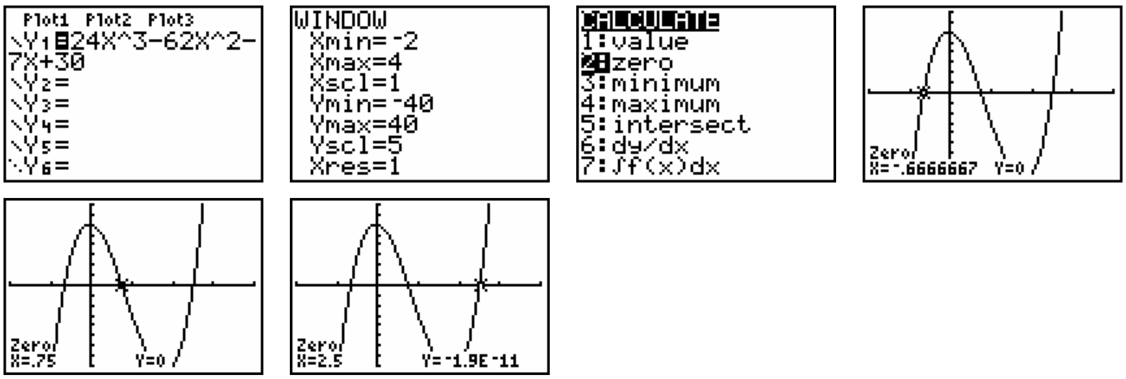
The solutions are $1.5,\ -\frac{1}{2}+\frac{\sqrt{7}}{2}i,\ -\frac{1}{2}-\frac{\sqrt{7}}{2}i.$

**30.** 

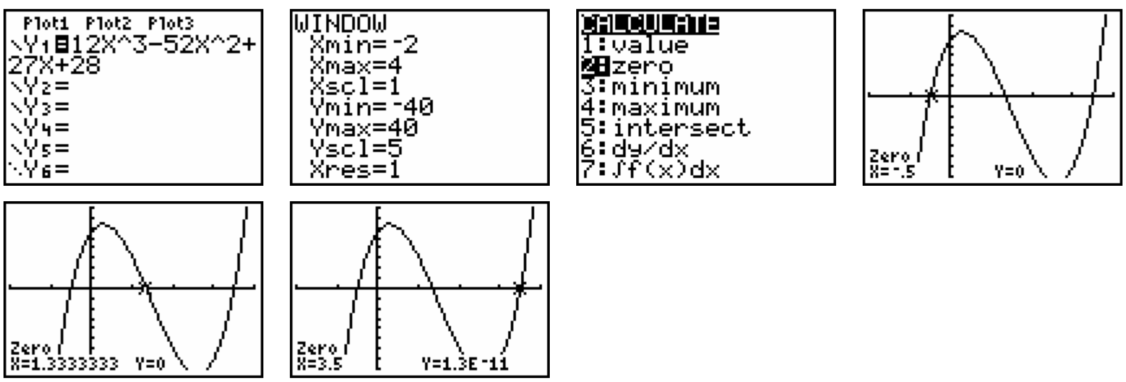

| | | | | |
|---|---|---|---|---|
| −0.75 | 4 | 3 | 16 | 12 |
| | | −3 | 0 | −12 |
| | 4 | 0 | 16 | 0 |

$4x^2+16=4(x^2+4)=0 \Rightarrow x^2=-4 \Rightarrow x=\pm\sqrt{-4}=\pm 2i$

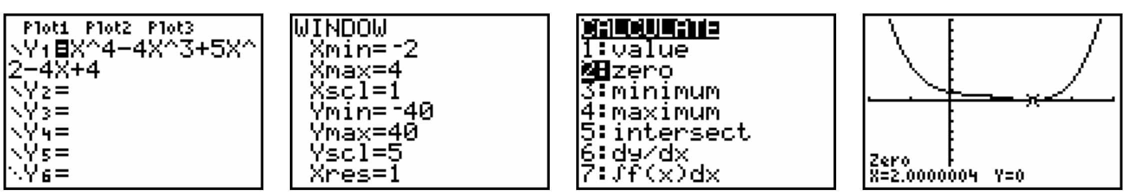
The solutions are $-0.75,\ 2i,\ -2i$.

**31.** 


The solutions are $-0.\overline{6},\ 0.75,\ 2.5$ or $-\frac{2}{3},\ \frac{3}{4},\ \frac{5}{2}$.

**32.** 


The solutions are $-0.5,\ 1.\overline{3},\ 3.5$ or $-\frac{1}{2},\ \frac{4}{3},\ \frac{7}{2}$.

**33.** 


| | | | | | |
|---|---|---|---|---|---|
| 2 | 1 | −4 | 5 | −4 | 4 |
| | | 2 | −4 | 2 | −4 |
| | 1 | −2 | 1 | −2 | 0 |

| | | | | |
|---|---|---|---|---|
| 2 | 1 | −2 | 1 | −2 |
| | | 2 | 0 | 2 |
| | 1 | 0 | 1 | 0 |

$x^2+1=0 \Rightarrow x^2=-1 \Rightarrow x=\pm\sqrt{-1}=\pm i$

The solutions are 2 (multiplicity 2), $i,\ -i$.

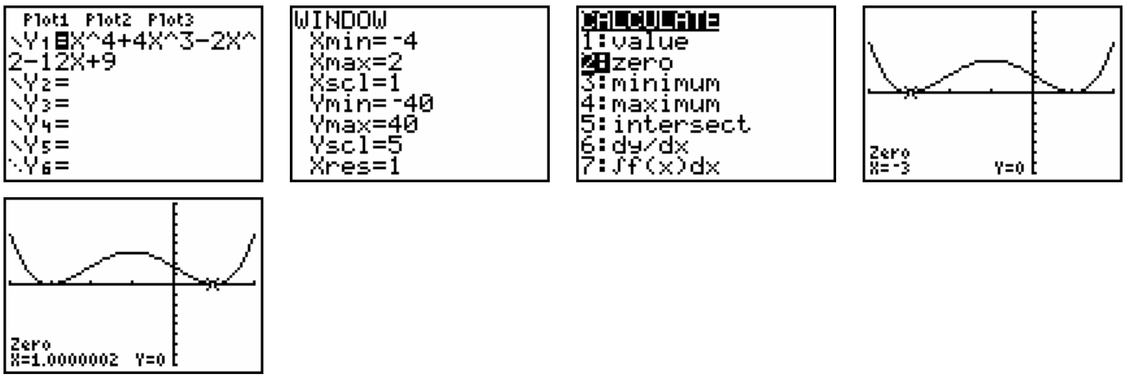
**34.** 


| | | | | | |
|---|---|---|---|---|---|
| –2 | 1 | 4 | 8 | 16 | 16 |
| | | –2 | –4 | –8 | –16 |
| | 1 | 2 | 4 | 8 | 0 |

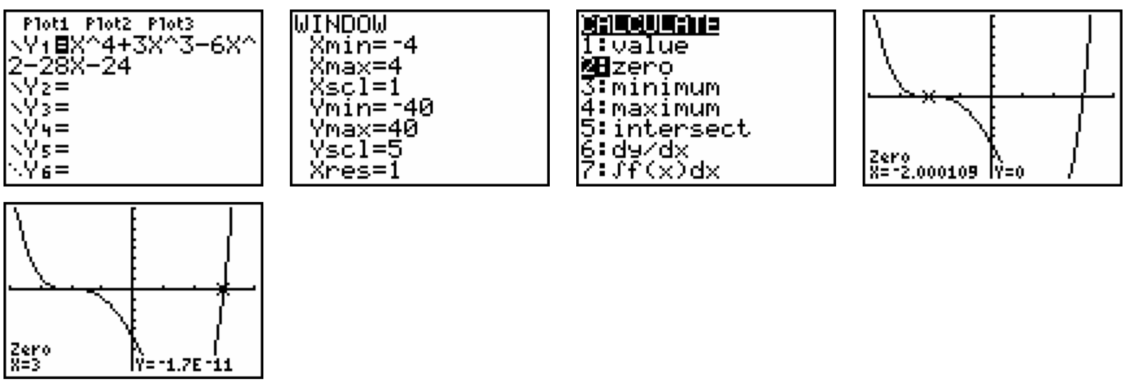
| | | | | |
|---|---|---|---|---|
| –2 | 1 | 2 | 4 | 8 |
| | | –2 | 0 | –8 |
| | 1 | 0 | 4 | 0 |

$x^2+4=0 \Rightarrow x^2=-4 \Rightarrow x=\pm\sqrt{-4}=\pm 2i$

The solutions are –2 (multiplicity 2), $2i$, $-2i$.

**35.** 


The solutions are –3 (multiplicity 2), 1 (multiplicity 2).

**36.** 


| | | | | | |
|---|---|---|---|---|---|
| 3 | 1 | 3 | –6 | –28 | –24 |
| | | 3 | 18 | 36 | 24 |
| | 1 | 6 | 12 | 8 | 0 |

| | | | | |
|---|---|---|---|---|
| –2 | 1 | 6 | 12 | 8 |
| | | –2 | –8 | –8 |
| | 1 | 4 | 4 | 0 |

$x^2+4x+4=(x+2)^2=0 \Rightarrow x=-2$ The solutions are 3, –2 (multiplicity 3).

**37.** $P(x)=(x-4)(x+3)(x-2)$

$P(x)=(x-4)(x^2+x-6)$

$P(x)=x(x^2+x-6)-4(x^2+x-6)$

$P(x)=x^3+x^2-6x-4x^2-4x+24$

$P(x)=x^3-3x^2-10x+24$

**38.** $P(x)=(x+1)(x-1)(x+5)$

$P(x)=(x^2-1)(x+5)$

$P(x)=x^2(x+5)-1(x+5)$

$P(x)=x^3+5x^2-x-5$

**39.** $P(x) = (x-3)(x-2i)(x+2i)$
$P(x) = (x-3)(x^2-[2i]^2)$
$P(x) = (x-3)(x^2-4i^2)$
$P(x) = (x-3)(x^2-4[-1])$
$P(x) = (x-3)(x^2+4)$
$P(x) = x(x^2+4)-3(x^2+4)$
$P(x) = x^3+4x-3x^2-12$
$P(x) = x^3-3x^2+4x-12$

**40.** $P(x) = x(x-i)(x+i)$
$P(x) = x(x^2-i^2)$
$P(x) = x(x^2-[-1])$
$P(x) = x(x^2+1)$
$P(x) = x^3+x$

**41.** $P(x) = [x-(3+i)][x-(3-i)][x-(2+5i)][x-(2-5i)]$
$P(x) = (x-3-i)(x-3+i)(x-2-5i)(x-2+5i)$
$P(x) = [(x-3)-i][(x-3)+i][(x-2)-5i][(x-2)+5i]$
$P(x) = [(x-3)^2-i^2][(x-2)^2-25i^2]$
$P(x) = [(x^2-6x+9)-(-1)][(x^2-4x+4)-(25[-1])]$
$P(x) = [(x^2-6x+9)+1][(x^2-4x+4)+25]$
$P(x) = (x^2-6x+9+1)(x^2-4x+4+25)$
$P(x) = (x^2-6x+10)(x^2-4x+29)$
$P(x) = x^2(x^2-4x+29)-6x(x^2-4x+29)+10(x^2-4x+29)$
$P(x) = x^4-4x^3+29x^2-6x^3+24x^2-174x+10x^2-40x+290$
$P(x) = x^4-10x^3+63x^2-214x+290$

**42.** $P(x) = [x-(2+3i)][x-(2-3i)](x+5)(x-2)$
$P(x) = [x-2-3i][x-2+3i](x+5)(x-2)$
$P(x) = [(x-2)^2-(3i)^2](x+5)(x-2)$
$P(x) = [(x^2-4x+4)-(9i^2)](x+5)(x-2)$
$P(x) = [(x^2-4x+4)-(9[-1])](x+5)(x-2)$
$P(x) = [(x^2-4x+4)+9](x+5)(x-2)$
$P(x) = (x^2-4x+13)(x^2+3x-10)$
$P(x) = x^2(x^2+3x-10)-4x(x^2+3x-10)+13(x^2+3x-10)$
$P(x) = x^4+3x^3-10x^2-4x^3-12x^2+40x+13x^2+39x-130$
$P(x) = x^4-x^3-9x^2+79x-130$

**43.** $P(x) = [x-(6+5i)][x-(6-5i)](x-2)(x-3)(x-5)$
$P(x) = [x-6-5i][x-6+5i](x-2)(x^2-8x+15)$
$P(x) = [(x-6)^2-(5i)^2][x(x^2-8x+15)-2(x^2-8x+15)]$
$P(x) = [(x^2-12x+36)-(25i^2)](x^3-8x^2+15x-2x^2+16x-30)$
$P(x) = [(x^2-12x+36)-(25[-1])](x^3-10x^2+31x-30)$
$P(x) = (x^2-12x+36+25)(x^3-10x^2+31x-30)$
$P(x) = (x^2-12x+61)(x^3-10x^2+31x-30)$
$P(x) = x^2(x^3-10x^2+31x-30)-12x(x^3-10x^2+31x-30)+61(x^3-10x^2+31x-30)$
$P(x) = x^5-10x^4+31x^3-30x^2-12x^4+120x^3-372x^2+360x+61x^3-610x^2+1891x-1830$
$P(x) = x^5-22x^4+212x^3-1012x^2+2251x-1830$

**44.** Note: $2x-1=0$ if and only if $x=\frac{1}{2}$. Therefore, if $x=\frac{1}{2}$ (that is, $\frac{1}{2}$ is a zero), then $2x-1=0$.

$P(x)=\left(x-\frac{1}{2}\right)[x-(4-i)][x-(4+i)]$

$P(x)=(2x-1)[x-4+i][x-4-i]$

$P(x)=(2x-1)[(x-4)^2-(i)^2]$

$P(x)=(2x-1)[x^2-8x+16-(-1)]$

$P(x)=(2x-1)[x^2-8x+16+1]$

$P(x)=(2x-1)(x^2-8x+17)$

$P(x)=2x(x^2-8x+17)-1(x^2-8x+17)$

$P(x)=2x^3-16x^2+34x-x^2+8x-17$

$P(x)=2x^3-17x^2+42x-17$

**45.** Note: $4x-3=0$ if and only if $x=\frac{3}{4}$. Therefore, if $x=\frac{3}{4}$ (that is , $\frac{3}{4}$ is a zero), then $4x-3=0$.

$P(x)=\left(x-\frac{3}{4}\right)[x-(2+7i)][x-(2-7i)]$

$P(x)=(4x-3)[x-2-7i][x-2+7i]$

$P(x)=(4x-3)[(x-2)^2-(7i)^2]$

$P(x)=(4x-3)[(x^2-4x+4)-49i^2]$

$P(x)=(4x-3)[(x^2-4x+4)-49(-1)]$

$P(x)=(4x-3)(x^2-4x+4+49)$

$P(x)=(4x-3)(x^2-4x+53)$

$P(x)=4x(x^2-4x+53)-3(x^2-4x+53)$

$P(x)=4x^3-16x^2+212x-3x^2+12x-159$

$P(x)=4x^3-19x^2+224x-159$

**46.** $P(x)=\left(x-\frac{1}{4}\right)\left(x+\frac{1}{5}\right)(x-i)(x+i)$

$P(x)=(4x-1)(5x+1)(x-i)(x+i)$

$P(x)=(20x^2-x-1)(x^2-i^2)$

$P(x)=(20x^2-x-1)(x^2-[-1])$

$P(x)=(20x^2-x-1)(x^2+1)$

$P(x)=20x^2(x^2+1)-x(x^2+1)-1(x^2+1)$

$P(x)=20x^4+20x^2-x^3-x-x^2-1$

$P(x)=20x^4-x^3+19x^2-x-1$

**47.** If 2 – 5*i* is a zero, then 2 + 5*i* is also a zero.

$P(x)=[x-(2-5i)][x-(2+5i)](x+4)$

$P(x)=[x-2+5i][x-2-5i](x+4)$

$P(x)=[(x-2)^2-(5i)^2](x+4)$

$P(x)=[x^2-4x+4-25i^2](x+4)$

$P(x)=[x^2-4x+4-25(-1)](x+4)$

$P(x)=[x^2-4x+4+25](x+4)$

$P(x)=(x^2-4x+29)(x+4)$

$P(x)=x^2(x+4)-4x(x+4)+29(x+4)$

$P(x)=x^3+4x^2-4x^2-16x+29x+116$

$P(x)=x^3+13x+116$

**48.** If 1 – 3*i* is a zero, then 1 + 3*i* is also a zero.

$P(x)=[x-(3+2i)][x-(3-2i)](x-7)$

$P(x)=(x^2-6x+13)(x-7)$

$P(x)=x^3-6x^2+13x-7x^2+42x-91$

$P(x)=x^3-13x^2+55x-91$

**49.** If 4 + 3*i*and 5 - *i* are zeros, then 4 – 3*i* and 5 + *i* are also zeros.

$P(x)=[x-(4+3i)][x-(4-3i)][x-(5-i)][x-(5+i)]$

$P(x)=(x^2-8x+25)(x^2-10x+26)$

$P(x)=x^4-10x^3+26x^2-8x^3+80x^2-208x+25x^2-250x+650$

$P(x)=x^4-18x^3+131x^2-458x+650$

**50.** If $i$ and $3 - 5i$ are zeros, then $-i$ and $3 + 5i$ are also zeros.

$P(x) = (x - i)(x + i)[x - (3 - 5i)][x - (3 + 5i)]$

$P(x) = (x^2 - i^2)[x - 3 + 5i][x - 3 - 5i]$

$P(x) = [x^2 - (-1)][(x - 3)^2 - (5i)^2]$

$P(x) = (x^2 + 1)[x^2 - 6x + 9 - 25i^2]$

$P(x) = (x^2 + 1)[x^2 - 6x + 9 - 25(-1)]$

$P(x) = (x^2 + 1)[x^2 - 6x + 9 + 25]$

$P(x) = (x^2 + 1)(x^2 - 6x + 34)$

$P(x) = x^2(x^2 - 6x + 34) + 1(x^2 - 6x + 34)$

$P(x) = x^4 - 6x^3 + 34x^2 + x^2 - 6x + 34$

$P(x) = x^4 - 6x^3 + 35x^2 - 6x + 34$

**51.** $P(x) = (x + 2)(x - 1)(x - 3)[x - (1 + 4i)][x - (1 - 4i)]$

$P(x) = (x^2 + x - 2)(x - 3)[x - 1 - 4i][x - 1 + 4i]$

$P(x) = [x^3 - 2x^2 - 5x + 6][(x - 1)^2 - (4i)^2]$

$P(x) = (x^3 - 2x^2 - 5x + 6)[x^2 - 2x + 1 - 16i^2]$

$P(x) = (x^3 - 2x^2 - 5x + 6)[x^2 - 2x + 1 + 16]$

$P(x) = (x^3 - 2x^2 - 5x + 6)[x^2 - 2x + 17]$

$P(x) = x^5 - 4x^4 + 16x^3 - 18x^2 - 97x + 102$

**52.** $P(x) = (x + 5)(x - 3)^2[x - (2 + i)][x - (2 - i)]$

$P(x) = (x + 5)(x^2 - 6x + 9)[x - 2 - i][x - 2 + i]$

$P(x) = [x^3 - x^2 - 21x + 45][(x - 2)^2 - (i)^2]$

$P(x) = (x^3 - x^2 - 21x + 45)[x^2 - 4x + 4 - i^2]$

$P(x) = (x^3 - x^2 - 21x + 45)[x^2 - 4x + 4 + 1]$

$P(x) = (x^3 - x^2 - 21x + 45)[x^2 - 4x + 5]$

$P(x) = x^5 - 5x^4 - 12x^3 + 124x^2 - 285x + 225$

## Connecting Concepts

**53.** $P(x) = a(x + 1)(x - 2)(x - 3)$

$P(1) = a(1 + 1)(1 - 2)(1 - 3)$

$12 = a(2)(-1)(-2)$

$12 = 4a$

$3 = a$

$P(x) = 3(x + 1)(x - 2)(x - 3)$

$P(x) = (3x + 3)(x^2 - 5x + 6)$

$P(x) = 3x(x^2 - 5x + 6) + 3(x^2 - 5x + 6)$

$P(x) = 3x^3 - 15x^2 + 18x + 3x^2 - 15x + 18$

$P(x) = 3x^3 - 12x^2 + 3x + 18$

**54.** $P(x) = a(x - 3i)(x + 3i)(x - 2)$

$P(x) = a(x^2 - 9i^2)(x - 2)$

$P(x) = a[x^2 - 9(i^2)](x - 2)$

$P(x) = a[x^2 - 9(-1)](x - 2)$

$P(x) = a(x^2 + 9)(x - 2)$

$P(3) = a(3^2 + 9)(3 - 2)$

$P(3) = a(9 + 9)(1)$

$P(3) = 18a$

$27 = 18a$

$\frac{3}{2} = a$

$P(x) = \frac{3}{2}(x^2 + 9)(x - 2)$

$P(x) = \frac{3}{2}(x^3 - 2x^2 + 9x - 18)$

$P(x) = \frac{3}{2}x^3 - 3x^2 + \frac{27}{2}x - 27$

**55.** $P(x) = a(x-3)(x+5)[x-(2+i)][x-(2-i)]$

$P(x) = a(x^2+2x-15)[x-20-i][x-2+i]$

$P(x) = a(x^2+2x-15)[(x-2)^2-i^2]$

$P(x) = a(x^2+2x-15)[x^2-4x+4-(-1)]$

$P(x) = a(x^2+2x-15)[x^2-4x+4+1]$

$P(x) = a(x^2+2x-15)(x^2-4x+5)$

$P(1) = a(1^2+2[1]-15)(1^2-4[1]+5)$

$48 = a(1+2-15)(1-4+5)$

$48 = a(-12)(2)$

$48 = -24a$

$-2 = a$

$P(x) = -2(x^2+2x-15)(x^2-4x+5)$

$P(x) = -2[x^2(x^2-4x+5)+2x(x^2-4x+5)-15(x^2-4x+5)]$

$P(x) = -2[x^4-4x^3+5x^2+2x^3-8x^2+10x-15x^2+60x-75)$

$P(x) = -2(x^4-2x^3-18x^2+70x-75)$

$P(x) = -2x^4+4x^3+36x^2-140x+150$

**56.** $P(x) = a(2x-1)[x-(1-i)][x-(1+i)]$

$P(x) = a(2x-1)[x-1+i][x-1-i]$

$P(x) = a(2x-1)[(x-1)^2-i^2]$

$P(x) = a(2x-1)[(x-1)^2-(-1)]$

$P(x) = a(2x-1)(x^2-2x+1+1)$

$P(x) = a(2x-1)(x^2-2x+2)$

$P(4) = a[2(4)-1][(4)^2-2(4)+2]$

$140 = a[8-1][16-8+2]$

$140 = a(7)(10)$

$140 = 70a$

$2 = a$

$P(x) = 2(2x-1)(x^2-2x+2)$

$P(x) = (4x-2)(x^2-2x+2)$

$P(x) = 4x(x^2-2x+2)-2(x^2-2x+2)$

$P(x) = 4x^3-8x^2+8x-2x^2+4x-4$

$P(x) = 4x^3-10x^2+12x-4$

**57.** $P(x) = x^3-x^2-ix^2-9x+9+9i = x^3+(-1-i)x^2-9x+(9+9i)$

| | | | | |
|---|---|---|---|---|
| $1+i$ | 1 | $-1-i$ | $-9$ | $9+9i$ |
| | | $1+i$ | 0 | $-9-9i$ |
| | 1 | 0 | $-9$ | 0 |

Zero remainder implies $1+i$ is a zero.

| | | | |
|---|---|---|---|
| $1-i$ | 1 | 0 | $-9$ |
| | | $1-i$ | $-2i$ |
| | 1 | $1-i$ | $-9-2i$ |

Non-zero remainder implies $1-i$ is not a zero.

The Conjugate Pair Theorem does not apply because some of the coefficients of the polynomial function are not real numbers.

**58.** **a.** Answers will vary.

**b.** Answers will vary.

**c.** No such polynomial function exists because the nonreal complex zeros must occur in pairs.

**d.** Answers will vary.

## Prepare for Section 3.5

**PS1.** $\dfrac{x^2-9}{x^2-2x-15} = \dfrac{(x+3)(x-3)}{(x+3)(x-5)} = \dfrac{x-3}{x-5},\ x \neq -3$

**PS2.** $\dfrac{-1+4}{(-1)^2-2(-1)-5} = \dfrac{3}{1+2-5} = \dfrac{3}{-2} = -\dfrac{3}{2}$

**PS3.** $\dfrac{2(-3)^2+4(-3)-5}{-3+6}=\dfrac{2(9)-12-5}{3}$
$=\dfrac{18-12-5}{3}=\dfrac{1}{3}$

**PS4.** $2x^3+x^2-15x=0$
$x(2x^2+x-15)=0$
$x(2x-5)(x+3)=0$
$x=0$ or $2x-5=0$ or $x+3=0$
$x=\frac{5}{2}$ $x=-3$

**PS5.** The degree of the numerator, $x^3+3x^2-5$, is 3; the degree of the denominator, $x^2-4$, is 2.

**PS6.**
$$\begin{array}{r} x+4+\dfrac{7x-11}{x^2-2x} \\ x^2-2x\overline{)x^3+2x^2-x-11} \\ \underline{x^3-2x^2} \\ 4x^2-x \\ \underline{4x^2-8x} \\ 7x-11 \end{array}$$

## Section 3.5

**1.** $F(x)=\dfrac{1}{x}\Rightarrow x\neq 0$
The domain is all real numbers except 0.

**2.** $F(x)=\dfrac{2}{x-3}\Rightarrow x\neq 3$
The domain is all real numbers except 3.

**3.** $F(x)=\dfrac{x^2-3}{x^2+1}\Rightarrow$ no restrictions on denominator.
The domain is all real numbers.

**4.** $F(x)=\dfrac{x^3+4}{x^2-25}=\dfrac{x^3+4}{(x-5)(x+5)}\Rightarrow x\neq 5,-5$
The domain is all real numbers except 5 and –5.

**5.** $F(x)=\dfrac{2x-1}{2x^2-15x+18}=\dfrac{2x-1}{(2x-3)(x-6)}\Rightarrow x\neq\frac{3}{2},\ 6$
The domain is all real numbers except $\frac{3}{2}$ and 6.

**6.** $F(x)=\dfrac{3x-2}{4x^2-27x+18}=\dfrac{3x-2}{(4x-3)(x-6)}\Rightarrow x\neq\frac{3}{4},\ 6$
The domain is all real numbers except $\frac{3}{4}$ and 6.

**7.** $F(x)=\dfrac{2x^2}{x^3-4x^2-12x}=\dfrac{2x^2}{x(x-6)(x+2)}\Rightarrow x\neq 0,\ 6,-2$
The domain is all real numbers except 0, 6 and –2.

**8.** $F(x)=\dfrac{3x^2}{x^2-5}=\dfrac{3x^2}{(x-\sqrt{5})(x+\sqrt{5})}\Rightarrow x\neq\sqrt{5},-\sqrt{5}$
The domain is all real numbers except $\sqrt{5}$ and $-\sqrt{5}$.

**9.** $x^2+3x=0$
$x(x+3)=0$
$x=0$ or $x+3=0$
$x=-3$
Vertical asymptotes: $x=0, x=-3$

**10.** $x^2-4=0$
$(x+2)(x-2)=0$
$x+2=0$ or $x-2=0$
$x=-2$ $x=2$
Vertical asymptotes: $x=2, x=-2$

**11.** $6x^2-5x-4=0$
$(3x-4)(2x+1)=0$
$3x-4=0$ or $2x+1=0$
$x=\frac{4}{3}$ $x=-\frac{1}{2}$
Vertical asymptotes: $x=-\frac{1}{2}$, $x=\frac{4}{3}$

**12.** $x^3-8=0$
$(x-2)(x^2+2x+4)=0$
Vertical asymptotes: $x=2$

**13.**
$$4x^3 - 25x^2 + 6x = 0$$
$$x(4x^2 - 25x + 6) = 0$$
$$x(4x-1)(x-6) = 0$$
$$x = 0 \quad \text{or} \quad 4x - 1 = 0 \quad \text{or} \quad x - 6 = 0$$
$$x = \frac{1}{4} \qquad x = 6$$
Vertical asymptotes: $x = 0,\ x = \frac{1}{4},\ x = 6$

**14.**
$$x^4 - 81 = 0$$
$$(x^2+9)(x^2-9) = 0$$
$$(x^2+9)(x-3)(x+3) = 0$$
$$x - 3 = 0 \quad \text{or} \quad x + 3 = 0$$
$$x = 3 \qquad x = -3$$
Vertical asymptotes: $x = 3,\ x = -3$

**15.** Horizontal asymptote: $y = \frac{4}{1} = 4$

**16.** Horizontal asymptote: $y = 0$

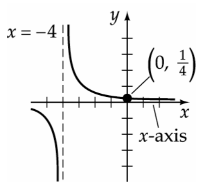
**17.** Horizontal asymptote: $y = \frac{15{,}000}{500} = 30$

**18.** Horizontal asymptote: $y = 6000\left(\frac{(x+5)^2 - 25}{(x+5)^2}\right)$
$$y = \frac{6000}{1} = 6000$$

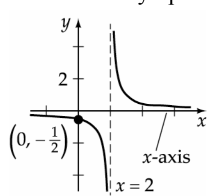
**19.** Horizontal asymptote: $y = \frac{4}{\frac{1}{3}} = 12$

**20.** $F(x) = \frac{(2x-3)(3x+4)}{(1-2x)(3-5x)} = \frac{6x^2 - x - 12}{10x^2 - 11x + 3}$
Horizontal asymptote: $y = \frac{6}{10} = \frac{3}{5}$

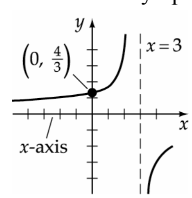
**21.** Vertical asymptote: $x = -4$
Horizontal asymptote: $y = 0$



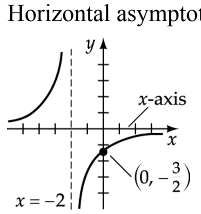

**22.** Vertical asymptote: $x = 2$
Horizontal asymptote: $y = 0$



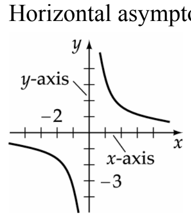

**23.** Vertical asymptote: $x = 3$
Horizontal asymptote: $y = 0$



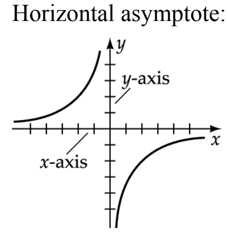

**24.** Vertical asymptote: $x = -2$
Horizontal asymptote: $y = 0$



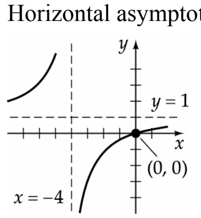

**25.** Vertical asymptote: $x = 0$
Horizontal asymptote: $y = 0$



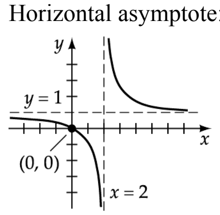

**26.** Vertical asymptote: $x = 0$
Horizontal asymptote: $y = 0$



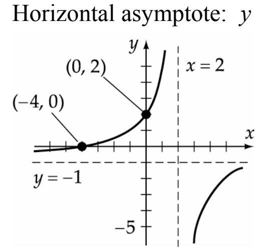

**27.** Vertical asymptote: $x = -4$
Horizontal asymptote: $y = 1$



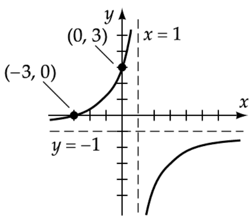

**28.** Vertical asymptote: $x = 2$
Horizontal asymptote: $y = 1$



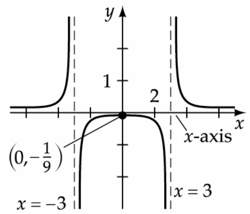

**29.** Vertical asymptote: $x = 2$
Horizontal asymptote: $y = -1$



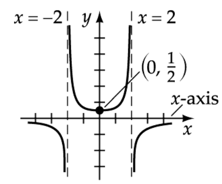
**30.** Vertical asymptote: $x = 1$
Horizontal asymptote: $y = -1$



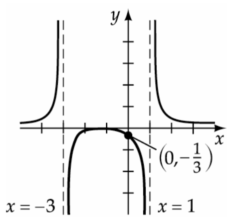

**31.** Vertical asymptotes: $x = 3, x = -3$
Horizontal asymptote: $y = 0$



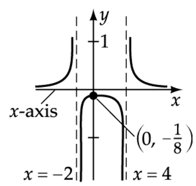

**32.** Vertical asymptotes: $x = 2, x = -2$
Horizontal asymptote: $y = 0$



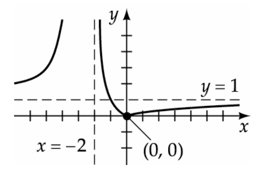

**33.** Vertical asymptotes: $x = -3, x = 1$
Horizontal asymptote: $y = 0$



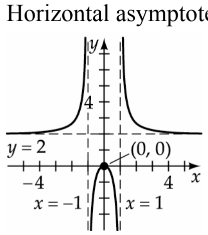

**34.** Vertical asymptotes: $x = 4, x = -2$
Horizontal asymptote: $y = 0$



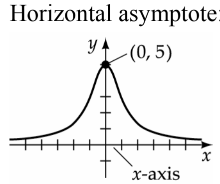

**35.** Vertical asymptote: $x = -2$
Horizontal asymptote: $y = 1$



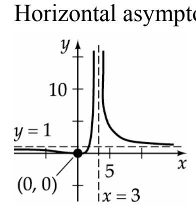

**36.** Vertical asymptote: $x = -1$, $x = 1$
Horizontal asymptote: $y = 2$



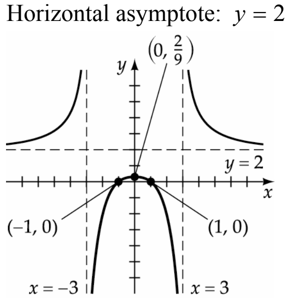

**37.** Vertical asymptote: none
Horizontal asymptote: $y = 0$



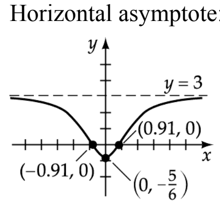

**38.** Vertical asymptote: $x = 3$
Horizontal asymptote: $y = 1$



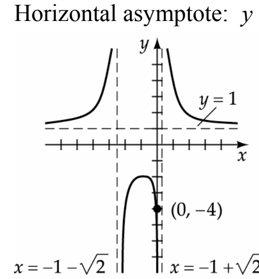

**39.** Vertical asymptotes: $x = 3, x = -3$
Horizontal asymptote: $y = 2$



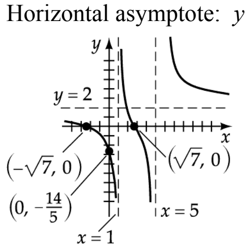

**40.** Vertical asymptote: none
Horizontal asymptote: $y = 3$




**41.** Vertical asymptotes:
$x = -1 + \sqrt{2}, x = -1 - \sqrt{2}$
Horizontal asymptote: $y = 1$




**42.** Vertical asymptotes: $x = 5, x = 1$
Horizontal asymptote: $y = 2$




**43.**

| | | | |
|---|---|---|---|
| −4 | 3 | 5 | −1 |
| | | −12 | 28 |
| | 3 | −7 | 27 |

$$F(x) = 3x - 7 + \frac{27}{x + 4}$$

Slant asymptote: $y = 3x - 7$

**44.**

$$\begin{array}{r} x+1+\dfrac{x-1}{x^2-3x+5} \\ x^2-3x+5\overline{)x^3-2x^2+3x+4} \\ \underline{x^3-3x^2+5x} \\ x^2-2x+4 \\ \underline{x^2-3x+5} \\ x-1 \end{array}$$

$$F(x)=x+1+\frac{x-1}{x^2-3x+5}$$

Slant asymptote: $y=x+1$

**45.** $\dfrac{x^3-1}{x^2}=\dfrac{x^3}{x^2}-\dfrac{1}{x^2}=x-\dfrac{1}{x^2}$

Slant asymptote: $y=x$

**46.** $F(x)=\dfrac{4000+20x+0.0001x^2}{x}=\dfrac{4000}{x}+20+0.0001x$

Slant Asymptote: $y=0.0001x+20$

**47.**

$$\begin{array}{r|rrr} 5 & -4 & 15 & 8 \\ & & -20 & -25 \\ \hline & -4 & -5 & -17 \end{array}$$

$$F(x)=-4x-5+\frac{-17}{x-5}$$

Slant asymptote: $y=-4x-5$

**48.**

$$\begin{array}{r} -x-2+\dfrac{-3x^2-3x+5}{x^3-1} \\ x^3-1\overline{)-x^4-2x^3-3x^2+4x-1} \\ \underline{-x^4 \qquad\qquad +x} \\ -2x^3-3x^2+3x-1 \\ \underline{-2x^3 \qquad\qquad +2} \\ -3x^2+3x-3 \end{array}$$

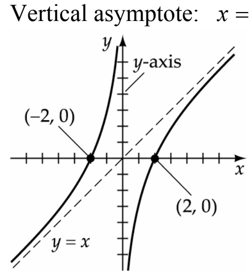
$$F(x)=-x-2+\frac{-3x^2-3x+5}{x^3-1}$$

Slant asymptote: $y=-x-2$

**49.** $F(x)=\dfrac{x^2-4}{x}=x-\dfrac{4}{x}$

Slant asymptote: $y=x$

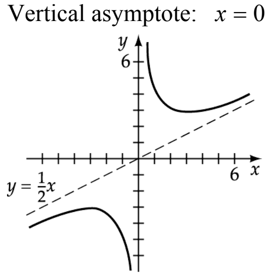
Vertical asymptote: $x=0$




**50.** $F(x)=\dfrac{x^2+10}{2x}=\dfrac{1}{2}x+\dfrac{5}{x}$

Slant asymptote: $y=\dfrac{1}{2}x$

Vertical asymptote: $x=0$



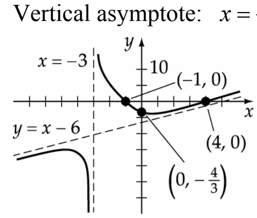

**51.**

$$\begin{array}{r|rrr} -3 & 1 & -3 & -4 \\ & & -3 & 18 \\ \hline & 1 & -6 & 14 \end{array}$$

$$F(x)=\frac{x^2-3x-4}{x+3}=x-6+\frac{14}{x+3}$$

Slant asymptote: $y=x-6$

Vertical asymptote: $x=-3$



**52.** $2x+5=0$

$2x=-5$

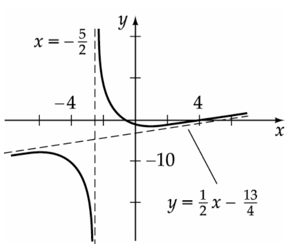
$x=-\frac{5}{2}$

$$\begin{array}{r} \frac{1}{2}x-\frac{13}{4}+\frac{45/4}{2x+5} \\ 2x+5\overline{)x^2-4x-5} \\ \underline{x^2+\frac{2}{5}x} \\ -\frac{13}{2}x-5 \\ \underline{-\frac{13}{2}x-\frac{65}{4}} \\ \frac{45}{4} \end{array}$$

$$F(x)=\frac{x^2-4x-5}{2x+5}=\frac{1}{2}x-\frac{13}{4}+\frac{45/4}{2x+5}$$

Slant asymptote: $y=\frac{1}{2}x-\frac{13}{4}$

Vertical asymptote: $x=-\frac{5}{2}$



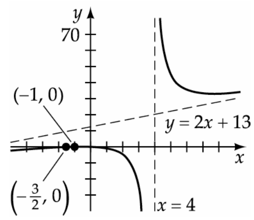

**53.**

$$\begin{array}{r|rrr} 4 & 2 & 5 & 3 \\ & & 8 & 52 \\ \hline & 2 & 13 & 55 \end{array}$$

$$F(x)=\frac{2x^2+5x+3}{x-4}=2x+13+\frac{55}{x-4}$$

Slant asymptote: $y=2x+13$

Vertical asymptote: $x=4$



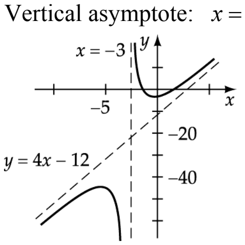

**54.**

$$\begin{array}{r|rrr} -3 & 4 & 0 & -9 \\ & & -12 & 36 \\ \hline & 4 & -12 & 27 \end{array}$$

$$F(x)=4x-12+\frac{27}{x+3}$$

Slant asymptote: $y=4x-12$

Vertical asymptote: $x=-3$



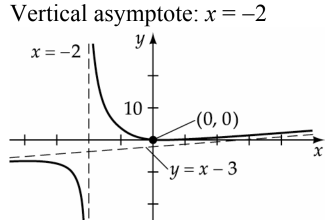

**55.**

$$\begin{array}{r|rrr} -2 & 1 & -1 & 0 \\ & & -2 & 6 \\ \hline & 1 & -3 & 6 \end{array}$$

$$F(x)=\frac{x^2-x}{x+2}=x-3+\frac{6}{x+2}$$

Slant asymptote: $y=x-3$

Vertical asymptote: $x=-2$

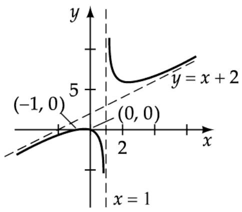


**56.** $\begin{array}{r|rrr} 1 & 1 & 1 & 0 \\ & & 1 & 2 \\ \hline & 1 & 2 & 2 \end{array}$

$F(x) = \dfrac{x^2 + x}{x-1} = x + 2 + \dfrac{2}{x-1}$

Slant asymptote: $y = x + 2$

Vertical asymptote: $x = 1$

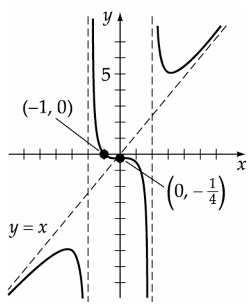



**57.** $x^2 - 4\overline{\smash{\big)}\,x^3 \qquad +1}$ with quotient $x$; $x^3 - 4x$; remainder $4x + 1$

$F(x) = \dfrac{x^3 + 1}{x^2 - 4} = x + \dfrac{4x + 1}{x^2 - 4}$

Slant asymptote: $y = x$

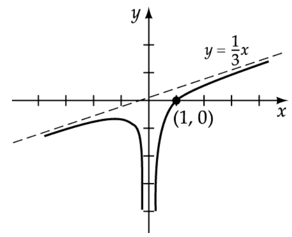
Vertical asymptotes: $x = 2, x = -2$



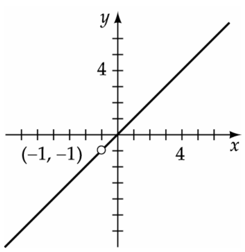

**58.** $F(x) = \dfrac{x^3}{3x^2} - \dfrac{1}{3x^2} = \dfrac{1}{3}x - \dfrac{1}{3x^2}$

Slant asymptote: $y = \dfrac{1}{3}x$

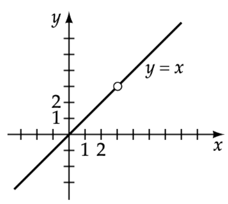
Vertical asymptote: $x = 0$



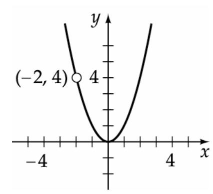

**59.**



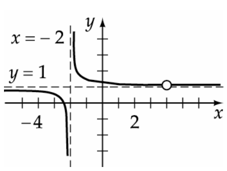

**60.**



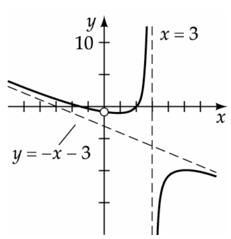

**61.**



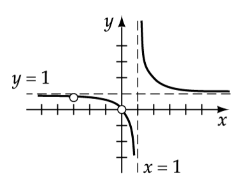

**62.**



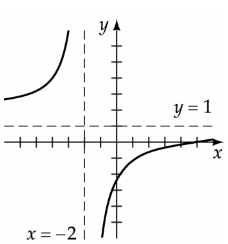

**63.**



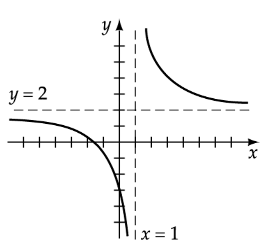

**64.**




**65.**




**66.**




**67.** **a.** $\overline{C}(1000) = \dfrac{0.43(1000) + 76{,}000}{1000} = \dfrac{430 + 76{,}000}{1000} = \dfrac{76{,}430}{1000} = \$76.43$

$\overline{C}(10{,}000) = \dfrac{0.43(10{,}000) + 76{,}000}{10{,}000} = \dfrac{4{,}300 + 76{,}000}{10{,}000} = \dfrac{80{,}300}{10{,}000} = \$8.03$

$\overline{C}(100{,}00) = \dfrac{0.43(100{,}000) + 76{,}000}{100{,}000} = \dfrac{43{,}000 + 76{,}000}{100{,}000} = \dfrac{119{,}000}{100{,}000} = \$1.19$

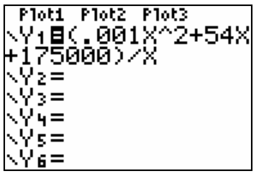
**b.** $y = 0.43$**.** As the number of golf balls that are produced increases, the average cost per golf ball approaches \$0.43.

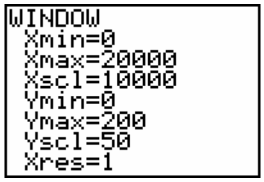
**68.** **a.** $\overline{C}(1000) = \dfrac{0.001(1000)^2 + 54(1000) + 175{,}000}{1000} = \dfrac{230{,}000}{1000} = \$230$

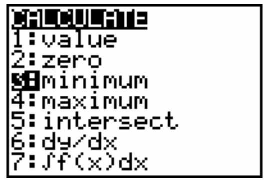
$\overline{C}(10{,}000) = \dfrac{0.001(10{,}000)^2 + 54(10{,}000) + 175{,}000}{10{,}000} = \dfrac{815{,}000}{10{,}000} = \$81.50$

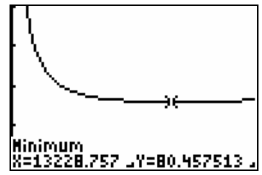
$\overline{C}(100{,}000) = \dfrac{0.001(100{,}000)^2 + 54(100{,}000) + 175{,}000}{100{,}000} = \dfrac{15{,}575{,}000}{100{,}000} = \$155.75$

**b.**



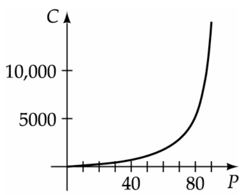




The minimum cost per CD player is approximately \$80.46. Producing approximately 13,229 CD players will minimize the average cost per CD player.

**69.** **a.** $C(40) = \dfrac{2000(40)}{100-40} = \dfrac{80{,}000}{60} = \$1{,}333.33$

**b.** $C(80) = \dfrac{2000(80)}{100-80} = \dfrac{160{,}000}{20} = \$8{,}000$

**c.**

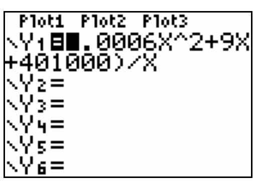


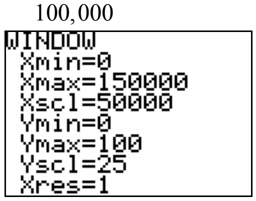

**70.** **a.** $\overline{C}(1000) = \dfrac{0.0006(1000)^2 + 9(1000) + 401{,}000}{1000} = \dfrac{410{,}060}{1000} = \$410.60$

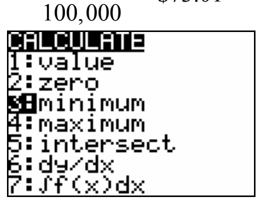
$\overline{C}(10{,}000) = \dfrac{0.0006(10{,}000)^2 + 9(10{,}000) + 401{,}000}{10{,}000} = \dfrac{551{,}000}{10{,}000} = \$55.10$

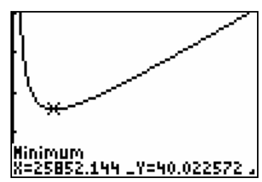
$\overline{C}(100{,}000) = \dfrac{0.0006(100{,}000)^2 + 9(100{,}000) + 401{,}000}{100{,}000} = \dfrac{7{,}301{,}000}{100{,}000} = \$73.01$

**b.**




The minimum cost per telephone is approximately \$40.02. Producing approximately 25,852 telephones will minimize the average cost per telephone.

**71.** **a.** $R(0) = \dfrac{22.8(0)^2 + 177(0) + 5900}{33.8(0)^2 + 266(0) + 15{,}200} = \dfrac{5900}{15{,}200} \approx 0.38816 \approx 38.8\%$

$R(7) = \dfrac{22.8(7)^2 + 177(7) + 5900}{33.8(7)^2 + 266(7) + 15{,}200} = \dfrac{8256.2}{18{,}718.2} \approx 0.44108 \approx 44.1\%$

$R(15) = \dfrac{22.8(15)^2 + 177(15) + 5900}{33.8(15)^2 + 266(15) + 15{,}200} = \dfrac{13{,}685}{26{,}795} \approx 0.51073 \approx 51.1\%$

**b.** $\dfrac{22.8}{33.8} \approx 67.5\%$

**72.** **a.**


$\approx 39.7\%$

**b.**


$\approx 53.5\%$

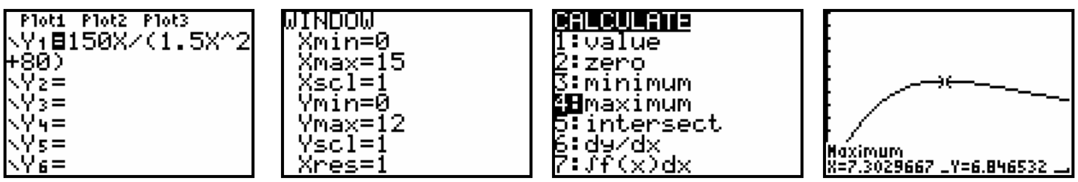
**73.** **a.**

$$S(2)=\frac{150(2)}{1.5(2)^2+80}=\frac{300}{86}\approx 3.488 \text{ thousand} \approx 3500$$

$$S(4)=\frac{150(4)}{1.5(4)^2+80}=\frac{600}{104}\approx 5.769 \text{ thousand} \approx 5800$$

$$S(10)=\frac{150(10)}{1.5(10)^2+80}=\frac{1500}{230}\approx 6.522 \text{ thousand} \approx 6500$$

**b.**




In the seventh month

**c.** $(n < m)$ The sales will approach zero.

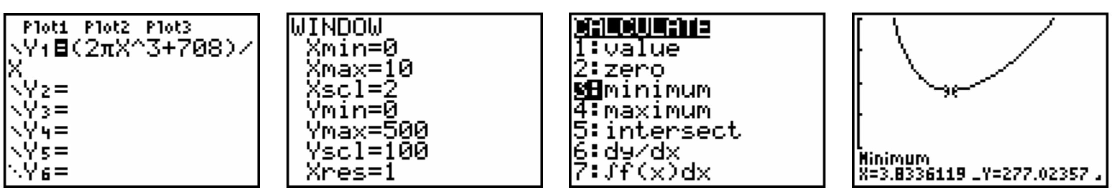
**74.** **a.**

$$M(5)=\frac{0.5(5)+400}{0.04(5)^2+1}=\frac{402.5}{2}\approx 201 \text{ milligrams}$$

$$M(10)=\frac{0.5(10)+400}{0.04(10)^2+1}=\frac{405}{5}=81 \text{ milligrams}$$

**b.** $(n < m) \Rightarrow$ as $t \to \infty$, $M \to 0$ milligrams.

**75.** **a.**




$r \approx 3.8$ centimeters

**b.** No. The degree of the numerator is not exactly one more than the degree of the denominator.

**c.** As the radius $r$ increases without bound, the surface area approaches twice the area of a circle with radius $r$. Since $V=\pi r^2 h$, then $h=\frac{V}{\pi r^2}$. $h \to 0$ as $r \to \infty$ so as the radius increases without bound, the surface area of the can approaches the area of the top and bottom of the can, two circles with radius $r$.

**76.** **a.** $R_T(2)=\dfrac{10(2)}{10+2}=\dfrac{20}{12}=\dfrac{5}{3}$ ohms

$R_T(20)=\dfrac{10(20)}{10+20}=\dfrac{200}{30}=\dfrac{20}{3}$ ohms

**b.** $R_T=\dfrac{10R_2}{R_2+10}\to\dfrac{10}{1}=10$ ohms

## Connecting Concepts

**77.** Horizontal asymptote: *y=2*

$$2=\frac{2x^2+3x+4}{x^2+4x+7}$$
$$2x^2+8x+14=2x^2+3x+4$$
$$5x+10=0$$
$$x=-2$$

The graph of *F* intersects its horizontal asymptote at (–2,2).

**78.**

$$\begin{array}{r} 3x+2\quad+\dfrac{-20x-20}{x^2+4}\\ x^2+4\overline{)3x^3+2x^2-8x-12}\\ \underline{3x^3\qquad\quad-12x}\\ 2x^2-20x-12\\ \underline{+\ 8}\\ -20x-20 \end{array}$$

Slant asymptote: $y=3x+2$

$$3x+2=\frac{3x^3+2x^2-8x-12}{x^2+4}$$
$$3x^3+2x^2+12x+8=3x^3+2x^2-8x-12$$
$$20x=-20$$
$$x=-1$$
$$y=3(-1)+2=-1$$

The graph of *F* intersects its slant asymptote at $(-1,-1)$.

**79.** Horizontal asymptote: $y=1$

$$1=\frac{x^3+x^2+4x+1}{x^3+1}$$
$$x^3+1=x^3+x^2+4x+1$$
$$x^2+4x=0$$
$$x(x+4)=0$$
$$x=0,\ \text{ or }\ x=-4$$

The graph of *F* intersects its horizontal asymptote at (0, 1) and $(-4,1)$.

**80.** Consider the rational function $f(x)=\dfrac{x^4+x^3+x^2+3x+2}{x^3}$

$$f(x)=\frac{x^4}{x^3}+\frac{x^3}{x^2}+\frac{x^2+3x+2}{x^3}$$
$$=x+1+\frac{(x+2)(x+1)}{x^3}$$
$$x+1=\frac{x^4+x^3+x^2+3x+2}{x^3}$$
$$x^4+x^3=x^4+x^3+x^2+3x+2$$
$$0=x^2+3x+2$$
$$0=(x+2)(x+1)$$
$$x=-2,-1$$

Thus, the graph of *f* intersects the slant asymptote $y=x+1$ when $x=-2$ and $-1$, that is, at the points $(-2,-1)$ and $(-1,0)$.

## Exploring Concepts with Technology

**Finding Zeros of a Polynomial Using *Mathematica***

**1.**
```
In(2):=
    NSolve[x^4-3x^3+x-5==0]
Out[2]=
    {{x -> -1.14039}, {x -> 0.536692 - 1.06842 I},
      {x -> 0.536692 + 1.06842 I}, {x -> 3.067}}
```

**2.**
```
In(3):=
    NSolve[3x^3-4x^2+x-3==0]
Out[3]=
    {{x -> -0.102814 - 0.799511 I},
      {x -> -0.1022814 + 0.799511 I}, {x -> 1.53896}}
```

**3.**
```
In(4):=
    NSolve[4x^5-3x^3+2x^2-x+2==0]
Out[4]=
    {{x -> -1.25095},, (x -> - 0.156173 - 0.685216 I},
      {x -> -0.156173 + 0.685216 I},
      {x -> 0.781647 - 0.445283 I}, {x -> 0.781647 + 0.445283 I}}
```

**4.**
```
In(5):=
    NSolve[-3x^4-6x^3+2x-8==0]
Out[5]=
    {{x -> -1.60199 - 0.623504 I}, {x -> -1.600199 + 0.623504 I}.
      {x -> 0.601988 - 0.743846 I}, {x -> 0.601988 + 0.7344846 I}}
```

## Assessing Concepts

**1.** True

**2.** True

**3.** False, Descartes' Rule of Signs indicates that $x^3 - x^2 + x - 1$ has three or one positive zeros.

**4.** $R(x) = \dfrac{3x^2}{(x-2)(x-5)}$ is one example.

**5.** $P(x) = x - 1 + i$ is one example.

**6.** c

**7.** d

**8.** f

**9.** a

**10.** b

**11.** e

**12.** f

## Chapter Review

**1.**
$$\begin{array}{r|rrrr} 3 & 4 & -11 & 5 & -2 \\ & & 12 & 3 & 24 \\ \hline & 4 & 1 & 8 & 22 \end{array} \qquad 4x^2 + x + 8 + \frac{22}{x-3}$$
[3.1]

**2.**
$$\begin{array}{r|rrrr} 1 & 5 & 0 & -18 & 2 \\ & & 5 & 5 & -13 \\ \hline & 5 & 5 & -13 & -11 \end{array} \qquad 5x^2 + 5x - 13 + \frac{-11}{x-1}$$
[3.1]

**3.**
$$\begin{array}{r|rrrr} -2 & 3 & 0 & -5 & 1 \\ & & -6 & 12 & -14 \\ \hline & 3 & -6 & 7 & -13 \end{array} \qquad 3x^2 - 6x + 7 + \frac{-13}{x+2}$$
[3.1]

**4.**
$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & 7 & 16 & -10 \\ & & 1 & 4 & 10 \\ \hline & 2 & 8 & 20 & 0 \end{array} \qquad 2x^2 + 8x + 20$$
[3.1]

**5.**

| 5 | 3 | −10 | −36 | 55 |
|---|---|---|---|---|
| | | 15 | 25 | −55 |
| | 3 | 5 | −11 | 0 |

$3x^2+5x-11$ [3.1]

**6.**

| −7 | 1 | 9 | 6 | −65 | −63 |
|---|---|---|---|---|---|
| | | −7 | −14 | 56 | 63 |
| | 1 | 2 | −8 | −9 | 0 |

$x^3+2x^2-8x-9$ [3.1]

**7.**

| 4 | 1 | 2 | −5 | 1 |
|---|---|---|---|---|
| | | 4 | 24 | 76 |
| | 1 | 6 | 19 | 77 |

$P(4)=77$ [3.1]

**8.**

| −1 | −4 | 0 | −10 | 8 |
|---|---|---|---|---|
| | | 4 | −4 | 14 |
| | −4 | 4 | −14 | 22 |

$P(-1)=22$ [3.1]

**9.**

| −2 | 6 | 0 | −12 | 8 | 1 |
|---|---|---|---|---|---|
| | | −12 | 24 | −24 | 32 |
| | 6 | −12 | 12 | −16 | 33 |

$P(-2)=33$ [3.1]

**10.**

| 3 | 5 | −8 | 2 | −6 | 0 | −9 |
|---|---|---|---|---|---|---|
| | | 15 | 21 | 69 | 189 | 567 |
| | 5 | 7 | 23 | 63 | 189 | 558 |

$P(3)=558$ [3.1]

**11.**

| 3 | 1 | 2 | −26 | 33 |
|---|---|---|---|---|
| | | 3 | 15 | −33 |
| | 1 | 5 | −11 | 0 |

[3.1]

**12.**

| −4 | 2 | 8 | −8 | −31 | 4 |
|---|---|---|---|---|---|
| | | −8 | 0 | 32 | −4 |
| | 2 | 0 | −8 | 1 | 0 |

[3.1]

**13.**

| 1 | 1 | −1 | 0 | −2 | 1 | 1 |
|---|---|---|---|---|---|---|
| | | 1 | 0 | 0 | −2 | −1 |
| | 1 | 0 | 0 | −2 | −1 | 0 |

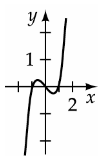
[3.1]

**14.**

| $\frac{1}{2}$ | 2 | 3 | −8 | 3 |
|---|---|---|---|---|
| | | 1 | 2 | −3 |
| | 2 | 4 | −6 | 0 |

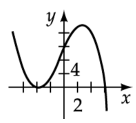
[3.1]

**15.**



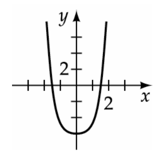

[3.2]

**16.**



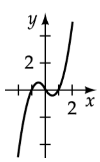

[3.2]

**17.**



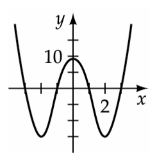

[3.2]

**18.**



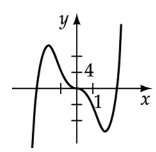

[3.2]

**19.**




[3.2]

**20.**




[3.2]

**21.** $\frac{p}{q}=\pm1,\pm2,\pm3,\pm6$ [3.3]

**22.** $\frac{p}{q}=\pm1,\ \pm2,\ \pm3,\ \pm5,\ \pm6,\ \pm10,\ \pm15,\ \pm30,\ \pm\frac{1}{2},\ \pm\frac{3}{2},\ \pm\frac{5}{2},\ \pm\frac{15}{2}$ [3.3]

**23.** $\frac{p}{q}=\pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm6,\ \pm12,\ \pm\frac{1}{3},\ \pm\frac{2}{3},\ \pm\frac{4}{3},\ \pm\frac{1}{5},\ \pm\frac{2}{5},\ \pm\frac{3}{5},\ \pm\frac{4}{5},\ \pm\frac{6}{5},\ \pm\frac{12}{5},\ \pm\frac{1}{15},\ \pm\frac{2}{15},\ \pm\frac{4}{15}$ [3.3]

**24.** $\frac{p}{q}=\pm1,\ \pm2,\ \pm4,\ \pm8,\ \pm16,\ \pm32,\ \pm64$ [3.3]

**25.** $\frac{p}{q}=\pm1$ [3.3]

**26.** $\frac{p}{q}=\pm1,\ \pm2,\ \pm\frac{1}{6},\ \pm\frac{1}{3},\ \pm\frac{1}{2},\ \pm\frac{2}{3}$ [3.3]

**27.** $P(x)=x^3+3x^2+x+3$
$P(-x)=-x^3+3x^2-x+3$
no positive and three or one negative real zeros [3.3]

**28.** $P(x)=x^4-6x^3-5x^2+74x-120$
$P(-x)=x^4+6x^3-5x^2-74x-120$
three or one positive and one negative real zeros [3.3]

**29.** $P(x)=x^4-x-1$
$P(-x)=x^4+x-1$
one positive and one negative real zeros [3.3]

**30.** $P(x) = x^5 - 4x^4 + 2x^3 - x^2 + x - 8$

$P(-x) = -x^5 - 4x^4 - 2x^3 - x^2 - x - 8$

five, three or one positive and no negative real zeros [3.3]

**31.**
$$\begin{array}{r|rrrr} 1 & 1 & 6 & 3 & -10 \\ & & 1 & 7 & 10 \\ \hline & 1 & 7 & 10 & 0 \end{array} \qquad \begin{aligned} x^2 + 7x + 10 &= 0 \\ (x+5)(x+2) &= 0 \\ x = -5 \quad &\text{or} \quad x = -2 \end{aligned}$$

The zeros of $x^3 + 6x^2 + 3x - 10$ are $1, -5$, and $-2$. [3.3]

**32.**
$$\begin{array}{r|rrrr} 2 & 1 & -10 & 31 & -30 \\ & & 2 & -16 & 30 \\ \hline & 1 & -8 & 15 & 0 \end{array} \qquad \begin{aligned} x^2 - 8x + 15 &= 0 \\ (x-5)(x-3) &= 0 \\ x = 5 \quad &\text{or} \quad x = 3 \end{aligned}$$

The zeros of $x^3 - 10x^2 + 31x - 30$ are $2, 5$, and $3$. [3.3]

**33.**
$$\begin{array}{r|rrrrr} -2 & 6 & 35 & 72 & 60 & 16 \\ & & -12 & -46 & -52 & -16 \\ \hline & 6 & 23 & 26 & 8 & 0 \end{array} \qquad \begin{array}{r|rrrr} -2 & 6 & 23 & 26 & 8 \\ & & -12 & -22 & -8 \\ \hline & 6 & 11 & 4 & 0 \end{array} \qquad \begin{aligned} 6x^2 + 11x + 4 &= 0 \\ (3x+4)(2x+1) &= 0 \\ x = -\frac{4}{3} \quad &\text{or} \quad x = -\frac{1}{2} \end{aligned}$$

The zeros of $6x^4 + 35x^3 + 72x^2 + 60x + 16$ are $-2$ (multiplicity 2), $-4/3$, and $-1/2$. [3.3]

**34.**
$$\begin{array}{r|rrrrr} -\frac{1}{2} & 2 & 7 & 5 & 7 & 3 \\ & & -1 & -3 & -1 & -3 \\ \hline & 2 & 6 & 2 & 6 & 0 \end{array} \qquad \begin{array}{r|rrrr} -3 & 2 & 6 & 2 & 6 \\ & & -6 & 0 & -6 \\ \hline & 2 & 0 & 2 & 0 \end{array} \qquad \begin{aligned} 2x^2 + 2 &= 0 \\ 2x^2 &= -2 \\ x^2 &= -1 \\ x &= \pm\sqrt{-1} \\ x &= \pm i \end{aligned}$$

The zeros of $2x^4 + 7x^3 + 5x^2 + 7x + 3$ are $-1/2, -3, i$, and $-i$. [3.4]

**35.**
$$\begin{array}{r|rrrrr} 1 & 1 & -4 & 6 & -4 & 1 \\ & & 1 & -3 & 3 & -1 \\ \hline & 1 & -3 & 3 & -1 & 0 \end{array} \qquad \begin{array}{r|rrrr} 1 & 1 & -3 & 3 & -1 \\ & & 1 & -2 & 1 \\ \hline & 1 & -2 & 1 & 0 \end{array} \qquad \begin{aligned} x^2 - 2x + 1 &= 0 \\ (x-1)(x-1) &= 0 \\ x = 1 \quad &\text{or} \quad x = 1 \end{aligned}$$

The zero of $x^4 - 4x^3 + 6x^2 - 4x + 1$ is $1$ (multiplicity 4). [3.3]

**36.**
$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & -7 & 22 & 13 \\ & & -1 & 4 & -13 \\ \hline & 2 & -8 & 26 & 0 \end{array} \qquad \begin{aligned} 2x^2 - 8x + 26 &= 0 \\ 2(x^2 - 4x + 13) &= 0 \end{aligned}$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)} = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

The zeros of $2x^3 - 7x^2 + 22x + 13$ are $-1/2, 2 + 3i$, and $2 - 3i$. [3.4]

**37.**
$$\begin{array}{r|rrrrr} 1-2i & 1 & -4 & 6 & -4 & 15 \\ & & 1-2i & -7+4i & 7+6i & 15 \\ \hline & 1 & -3-2i & -1+4i & 3+6i & 0 \end{array}$$

$$\begin{array}{r|rrrr} 1+2i & 1 & -3-2i & -1+4i & 3+6i \\ & & 1+2i & -2-4i & -3-6i \\ \hline & 1 & 2 & 3 & 0 \end{array}$$

$x^2 - 2x - 3 = (x-3)(x+1) = 0$

$x = 3,\ x = -1$

The remaining zeros are $1 + 2i$, $3$, and $-1$. [3.4]

**38.** 
$$\begin{array}{r|rrrrr} 2+i & 1 & -1 & -17 & 55 & -50 \\ & & 2+i & 1+3i & -35-10i & 50 \\ \hline & 1 & 1+i & -16+3i & 20-10i & 0 \end{array}$$

$$\begin{array}{r|rrrr} 2-i & 1 & 1+i & -16+3i & 20-10i \\ & & 2-i & 6-3i & -20+10i \\ \hline & 1 & 3 & -10 & 0 \end{array}$$

$x^2+3x-10=(x+5)(x-2)=0$

$x=-5,\ x=2$

The remaining zeros are $2-i$, $-5$, and 2. [3.4]

**39.** $(x-4)(x+3)(2x-1)=(x^2-x-12)(2x-1)$
$=2x^3-x^2-2x^2+x-24x+12$
$=2x^3-3x^2-23x+12$ [3.4]

**40.** $(x-2)(x+3)(x-i)(x+i)=(x^2+x-6)(x^2+1)$
$=x^4+x^2+x^3+x-6x^2-6$
$=x^4+x^3-5x^2+x-6$ [3.4]

**41.** $(x-1)(x-2)(x-5i)(x+5i)=(x^2-3x+2)(x^2+25)$
$=x^4+25x^2-3x^3-75x+2x^2+50$
$=x^4-3x^3+27x^2-75x+50$ [3.4]

**42.** $(x+2)(x+2)[x-(1+3i)][x-(1-3i)]=(x^2+4x+4)(x^2-2x+10)$
$=x^4-2x^3+10x^2+4x^3-8x^2+40x+4x^2-8x+40$
$=x^4+2x^3+6x^2+32x+40$ [3.4]

**43.** $F(x)=\dfrac{x^2}{x^2+7} \Rightarrow$ no restrictions on denominator.
The domain is all real numbers. [3.4]

**44.** $F(x)=\dfrac{3x^2+2x-5}{6x^2-25x+4}=\dfrac{3x^2+2x-5}{(6x-1)(x-4)} \Rightarrow x\neq\dfrac{1}{6},\ 4$
The domain is all real numbers except $\dfrac{1}{6}$ and 4. [3.4]

**45.** $x+2=0 \Rightarrow$ vertical asymptote : $x=-2$
$\dfrac{3}{1}=3 \Rightarrow$ horizontal asymptote : $y=3$ [3.5]

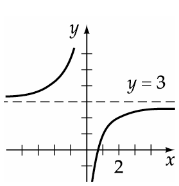
**46.** $x^2+2x-3=0$
$(x+3)(x-1)=0 \Rightarrow$ vertical asymptotes : $x=-3,\ x=1$
$\dfrac{2}{1}=2 \Rightarrow$ horizontal asymptote: $y=2$ [3.5]

**47.** $x+1=0 \Rightarrow$ vertical asymptote : $x=-1$

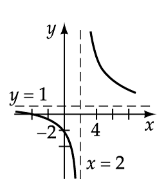
$$\begin{array}{r|rrr} -1 & 2 & 5 & 11 \\ & & -2 & -3 \\ \hline & 2 & 3 & 8 \end{array}$$

$f(x)=2x+3+\dfrac{8}{x+1}$
$\Rightarrow$ slant asymptote: $y=2x+3$ [3.5]

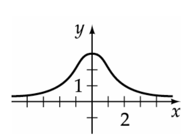
**48.** $2x^2+x+7=0$
$x=\dfrac{-1\pm\sqrt{1^2-4(1)(7)}}{2(1)}=\dfrac{-1\pm\sqrt{1-28}}{2}=\dfrac{-1\pm\sqrt{-27}}{2}$
$x$ is not a real number $\Rightarrow$ vertical asymptote : none
$\dfrac{6}{2}=3 \Rightarrow$ horizontal asymptote : $y=3$ [3.5]

**49.** 

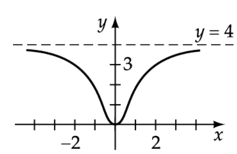
[3.5]

**50.** 

[3.5]

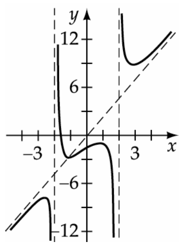
**51.** 

[3.5]

**52.** 

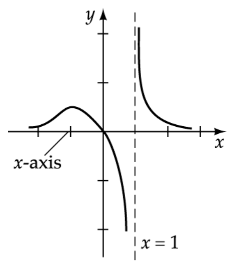
[3.5]

**53.**



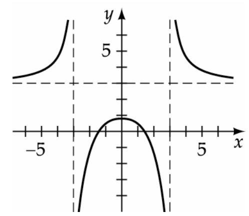

[3.5]

**54.**



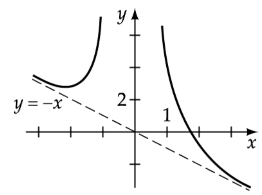

[3.5]

**55.**




[3.5]

**56.**




[3.5]

**57.** **a.** $C(5000) = \dfrac{5.75(5000) + 34,200}{5000} = \dfrac{62,950}{5000} = \$12.59$

$C(50,000) = \dfrac{5.75(50,000) + 34,200}{50,000} = \dfrac{321,700}{50,000} \approx \$6.43$

**b.** $y = 5.75$. As the number of skateboards that are produced increases, the average cost per skateboard approaches \$5.75. [3.5]

**58.** **a.** $F(1) = \dfrac{60}{1^2 + 2(1) + 1} = \dfrac{60}{4} = 15^\circ\text{F}$

**b.** $F(4) = \dfrac{60}{4^2 + 2(4) + 1} = \dfrac{60}{25} = 2.4^\circ\text{F}$

**c.** $F(t) \to 0^\circ\text{F}$ as $t \to \infty$. [3.5]

**59.** **a.** As the radius of the blood vessel gets smaller, the resistance gets larger.

**b.** As the radius of the blood vessel gets larger, the resistance gets smaller. [3.5]

## Quantitative Reasoning

**QR1.** $S(6) = 3.6(6)^3 - 36.8(6)^2 + 145.2(6) + 8 \approx 412$

$S(8) = 3.6(8)^3 - 36.8(8)^2 + 145.2(8) + 8 \approx 737.6$

There will be about 412,000 hybrid vehicles sold in 2010, and 737,600 hybrid vehicles sold in 2012.

**QR2.**

$$2.5\left(\frac{96,000}{18}\right) = 2.5\left(\frac{96,000}{18+x}\right) + 3600$$
$$\frac{240,000}{18} = \frac{240,000}{18+x} + 3600$$
$$240,000(18+x) = 18(240,000) + 3600(18)(18+x)$$
$$240,000x = 1,116,400 + 64,800x$$
$$175,200x = 1,166,400$$
$$x \approx 6.7 \text{ additional mpg}$$

**QR3.** $M(1) = 0.2416(1)^3 + 2.3106(1)^2 - 1.2373(1) + 25.00 = 26.2879$

$M(2) = 0.2416(2)^3 + 2.3106(2)^2 - 1.2373(2) + 25.00 = 33.4846$

$M(3) = 0.2416(3)^3 + 2.3106(3)^2 - 1.2373(3) + 25.00 = 47.8777$

$M(4) = 0.2416(4)^3 + 2.3106(4)^2 - 1.2373(4) + 25.00 = 70.7548$

$M(5) = 0.2416(5)^3 + 2.3106(5)^2 - 1.2373(5) + 25.00 = 103.4035$

$M(6) = 0.2416(6)^3 + 2.3106(6)^2 - 1.2373(6) + 25.00 = 147.1114$

$M(7) = 0.2416(7)^3 + 2.3106(7)^2 - 1.2373(7) + 25.00 = 203.1661$

$M(8) = 0.2416(8)^3 + 2.3106(8)^2 - 1.2373(8) + 25.00 = 272.8552$

$M(1) + M(2) + M(3) + M(4) + M(5) + M(6) + M(7) + M(8) = 904.9412 \approx \$900$

**QR4.**

$$3\left(\frac{96,000}{18}\right) = 3\left(\frac{96,000}{18+x}\right) + 3600 + 900$$
$$\frac{288,000}{18} = \frac{288,000}{18+x} + 4500$$
$$288,000(18+x) = 18(288,000) + 4500(18)(18+x)$$
$$288,000x = 1,458,000 + 81,000x$$
$$207,000x = 1,458,000$$
$$x \approx 7.0 \text{ additional mpg}$$

## Chapter Test

**1.**

| $-2$ | 3 | 5 | 4 | $-1$ |
|---|---|---|---|---|
| | | $-6$ | 2 | $-12$ |
| | 3 | $-1$ | 6 | $-13$ |

$3x^2 - x + 6 + \dfrac{-13}{x+2}$ [3.1]

**2.**

| $-2$ | $-3$ | 7 | 2 | $-5$ |
|---|---|---|---|---|
| | | 6 | $-26$ | 48 |
| | $-3$ | 13 | $-24$ | 43 |

$P(-2) = 43$ [3.1]

**3.**

| 1 | 1 | $-4$ | 7 | $-6$ | 2 |
|---|---|---|---|---|---|
| | | 1 | $-3$ | 4 | $-2$ |
| | 1 | $-3$ | 4 | $-2$ | 0 |

A remainder of 0 implies that $x-1$ is a factor of $x^4 - 4x^3 + 7x^2 - 6x + 2$. [3.1]

**4.** $P(x) = -3x^3 + 2x^2 - 5x + 2$ [3.2]

Since $A_n = -3$ is negative and $n = 3$ is odd, the graph of $P$ goes up to the far left and down to the far right.

**5.** $3x^3 + 7x^2 - 6x = 0$ [3.2]

$x(3x^2 + 7x - 6) = 0$

$x(3x - 2)(x + 3) = 0$

$x = 0, \quad 3x - 2 = 0, \text{ or } \quad x + 3 = 0$

$x = \dfrac{2}{3} \qquad x = -3$

The zeros of $3x^3 + 7x^2 - 6x = 0$ are $0, \dfrac{2}{3},$ and $-3$. [3.2]

**6.** $P(x) = 2x^3 - 3x^2 - x + 1$ [3.2]

| 1 | 2 | $-3$ | $-1$ | 1 |
|---|---|---|---|---|
| | | 2 | $-1$ | $-2$ |
| | 2 | $-1$ | $-2$ | $-1$ |

| 2 | 2 | $-3$ | $-1$ | 1 |
|---|---|---|---|---|
| | | 4 | 2 | 2 |
| | 2 | 1 | 1 | 3 |

Because $P(1)$ and $P(2)$ have different signs, $P$ must have a real zero between 1 and 2. [3.2]

**7.** $P(x) = (x^2 - 4)^2(2x - 3)(x + 1)^3$

$P(x) = (x - 2)^2(x + 2)^2(2x - 3)(x + 1)^3$

The zeros of $P$ are 2 (multiplicity 2), –2 (multiplicity 2), $\frac{3}{2}$ (multiplicity 1), and –1 (multiplicity 3). [3.3]

**8.** $p = \pm 1, \pm 3$

$q = \pm 1, \pm 2, \pm 3, \pm 6$

$\dfrac{p}{q} = \pm 1, \pm 3, \pm \dfrac{1}{2}, \pm \dfrac{3}{2}, \pm \dfrac{1}{3}, \pm \dfrac{1}{6}$ [3.3]

**9.**

| 4 | 2 | 5 | $-23$ | $-38$ | 24 |
|---|---|---|---|---|---|
| | | 8 | 52 | 116 | 312 |
| | 2 | 13 | 29 | 78 | 336 |

upper bound: 4

| $-5$ | 2 | 5 | $-23$ | $-38$ | 24 |
|---|---|---|---|---|---|
| | | $-10$ | 25 | $-10$ | 240 |
| | 2 | $-5$ | 2 | $-48$ | 264 |

lower bound: –5 [3.3]

**10.** $P(x) = x^4 - 3x^3 + 2x^2 - 5x + 1$

$P(-x) = x^4 + 3x^3 + 2x^2 + 5x + 1$

four, two, or no positive and no negative real zeros [3.3]

**11.**

| $\frac{1}{2}$ | 2 | $-3$ | $-11$ | 6 |
|---|---|---|---|---|
| | | 1 | $-1$ | $-6$ |
| | 2 | $-2$ | $-12$ | 0 |

$2x^2 - 2x - 12 = 0$

$2(x + 2)(x - 3) = 0$

$x = -2 \quad \text{or} \quad x = 3$

The zeros of $2x^3 - 3x^2 - 11x + 6$ are $1/2, -2,$ and $3$. [3.3]

**12.**

| $2 + 3i$ | 6 | $-5$ | 12 | 207 | 130 |
|---|---|---|---|---|---|
| | | $12 + 18i$ | $-40 + 57i$ | $-227 + 30i$ | $-130$ |
| | 6 | $7 + 18i$ | $-28 + 57i$ | $-20 + 30i$ | 0 |

| $2 - 3i$ | 6 | $7 + 18i$ | $-28 + 57i$ | $-20 + 30i$ |
|---|---|---|---|---|
| | | $12 - 18i$ | $38 - 57i$ | $20 - 30i$ |
| | 6 | 19 | 10 | 0 |

$6x^2 + 19x + 10 = 0$

$(3x + 2)(2x + 5) = 0$

$3x + 2 = 0 \qquad 2x + 5 = 0$

$x = -2/3 \qquad x = -5/2$

The zeros of $6x^4 - 5x^3 + 12x^2 + 207x + 130$ are $2+3i,\ 2-3i,\ -2/3,$ and $-5/2$. [3.4]

**13.** $P(x) = x(x^4 - 6x^3 + 14x^2 - 14x + 5)$

$$\begin{array}{r|rrrrr} 1 & 1 & -6 & 14 & -14 & 5 \\ & & 1 & -5 & 9 & -5 \\ \hline & 1 & -5 & 9 & -5 & 0 \end{array}$$

$$\begin{array}{r|rrrr} 1 & 1 & -5 & 9 & -5 \\ & & 1 & -4 & 5 \\ \hline & 1 & -4 & 5 & 0 \end{array}$$

$x^2 - 4x + 5 = 0$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)}$$

$$x = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm \sqrt{-4}}{2}$$

$$x = \frac{4 \pm 2i}{2} = 2 \pm i$$

The zeros of $x^5 - 6x^4 + 14x^3 - 14x^2 + 5x$ are 0, 1 (multiplicity 2), $2+i$, and $2-i$. [3.4]

**14.** $P(x) = [x-(1+i)][x-(1-i)](x-3)(x)$ [3.4]

$= (x^2 - 2x + 2)(x-3)(x)$

$= (x^3 - 5x^2 + 8x - 6)(x)$

$= x^4 - 5x^3 + 8x^2 - 6x$

**15.** $f(x) = \dfrac{3x^2 - 2x + 1}{x^2 - 5x + 6}$

$x^2 - 5x + 6 = 0$

$(x-3)(x-2) = 0$

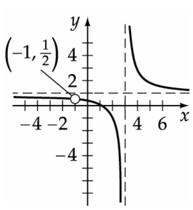
$x = 3 \quad x = 2$

vertical asymptotes: $x = 3$, $x = 2$ [3.5]

**16.** $f(x) = \dfrac{3x^2 - 2x + 1}{2x^2 - 1}$

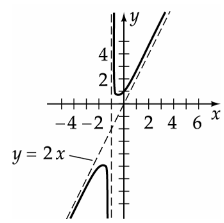
horizontal asymptote: $y = \dfrac{3}{2}$ [3.5]

**17.**




[3.5]

**18.**




[3.5]

**19.** **a.** $w(t) = \dfrac{70t + 120}{t + 40}$

$$w(1) = \frac{70(1)+120}{1+40} = \frac{70+120}{41} = \frac{190}{41} \approx 5 \text{ words per minute}$$

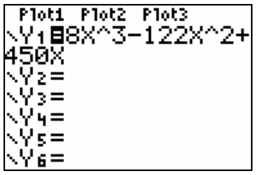
$$w(10) = \frac{70(10)+120}{10+40} = \frac{700+120}{50} = \frac{820}{50} \approx 16 \text{ words per minute}$$

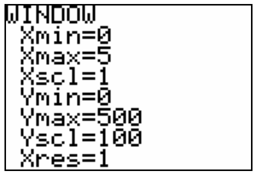
$$w(20) = \frac{70(20)+120}{20+40} = \frac{1400+120}{60} = \frac{1520}{60} \approx 25 \text{ words per minute}$$

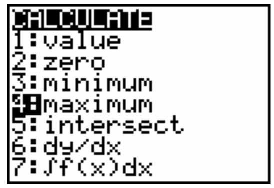
**b.** As $t \to \infty$, $w(t) \to \dfrac{70}{1} = 70$ words per minute [3.5]

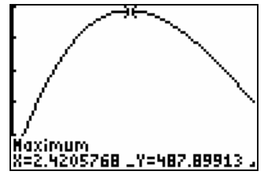
**20.** length = 25 – 2(2$x$) = 25 – 4$x$
width = 18 – 2$x$
height = $x$
volume = length × width × height

$$\begin{aligned}&= (25-4x)(18-2x)(x)\\&= (450-122x+8x^2)(x)\\&= 450x-122x^2+8x^3\\&= 8x^3-122x^2+450x\end{aligned}$$










The value of $x$ (to the nearest 0.001 inch) that will produce a box with the maximum volume is 2.42 inches. The maximum volume (to the nearest 0.1 cubic inch) is 487.9 cubic inches. [3.3]

## Cumulative Review

**1.** $$\frac{3+4i}{1-2i}=\frac{3+4i}{1-2i}\cdot\frac{1+2i}{1+2i}=\frac{3+10i+8i^2}{1^2-4i^2}\quad [\text{P.6}]$$
$$=\frac{3+10i+8(-1)}{1-4(-1)}=\frac{3+10i-8}{1+4}$$
$$=\frac{-5+10i}{5}=-1+2i$$

**2.** $$x^2-x-1=0$$
$$a=1,\ b=-1,\ c=-1$$
$$x=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-1)}}{2(1)}=\frac{1\pm\sqrt{5}}{2}$$
$$x=\frac{1-\sqrt{5}}{2},\ x=\frac{1+\sqrt{5}}{2}\quad [1.3]$$

**3.** $$\begin{aligned}\sqrt{2x+5}-\sqrt{x-1}&=2\\\sqrt{2x+5}&=2+\sqrt{x-1}\\\left(\sqrt{2x+5}\right)^2&=\left(2+\sqrt{x-1}\right)^2\\2x+5&=4+4\sqrt{x-1}+x-1\\2x+5&=3+4\sqrt{x-1}+x\\x+2&=4\sqrt{x-1}\\(x+2)^2&=\left(4\sqrt{x-1}\right)^2\\x^2+4x+4&=16(x-1)\\x^2+4x+4&=16x-16\\x^2-12x+20&=0\\(x-2)(x-10)&=0\\x=2,\ x&=10\end{aligned}$$

Check 2:
$$\begin{aligned}\sqrt{2(2)+5}-\sqrt{(2)-1}&=2\\\sqrt{4+5}-\sqrt{2-1}&=2\\\sqrt{9}-\sqrt{1}&=2\\3-1&=2\\2&=2\end{aligned}$$
Yes

Check 10:
$$\begin{aligned}\sqrt{2(10)+5}-\sqrt{(10)-1}&=2\\\sqrt{20+5}-\sqrt{10-1}&=2\\\sqrt{25}-\sqrt{9}&=2\\5-3&=2\\2&=2\end{aligned}$$
Yes

The solutions are $x = 2, x = 10$. [1.4]

**4.** $$|x-3|\le 11\quad [1.5]$$
$$-11\le x-3\le 11$$
$$-8\le x\le 14$$
$$\{x\,|\,-8\le x\le 14\}$$

**5.** $$d=\sqrt{(2-7)^2+[5-(-11)]^2}\quad [2.1]$$
$$=\sqrt{(2-7)^2+(5+11)^2}$$
$$=\sqrt{(-5)^2+(16)^2}$$
$$=\sqrt{25+256}$$
$$=\sqrt{281}$$

**6.** Shift the graph of $y=x^2$ two units to the right and four units up. [2.5]

**7.** $P(x) = x^2 - 2x - 3$

$$\frac{P(x+h)-P(x)}{h} = \frac{[(x+h)^2 - 2(x+h) - 3] - (x^2 - 2x - 3)}{h} = \frac{x^2 + 2xh + h^2 - 2x - 2h - 3 - x^2 + 2x + 3}{h}$$

$$= \frac{2xh + h^2 - 2h}{h} = 2x + h - 2$$ [2.6]

**8.**
$$\begin{aligned} f(x) &= 2x^2 + 5x - 3 \\ g(x) &= 4x - 7 \\ (f \circ g)(x) &= f[g(x)] \\ &= f(4x-7) \\ &= 2(4x-7)^2 + 5(4x-7) - 3 \\ &= 2(16x^2 - 56x + 49) + 5(4x-7) - 3 \\ &= 32x^2 - 112x + 98 + 20x - 35 - 3 \\ &= 32x^2 - 92x + 60 \end{aligned}$$ [2.6]

**9.**
$$\begin{aligned} (f-g)(x) &= f(x) - g(x) \\ &= x^3 - 2x + 7 - (x^2 - 3x - 4) \\ &= x^3 - 2x + 7 - x^2 + 3x + 4 \\ &= x^2 - x^2 + x + 11 \end{aligned}$$ [2.6]

**10.**

| | | | | | |
|---|---|---|---|---|---|
| –2 | 4 | 0 | –2 | –4 | –5 |
| | | –8 | 16 | –28 | 64 |
| | 4 | –8 | 14 | –32 | 59 |

$4x^3 - 8x^2 + 14x - 32 + \frac{59}{x+2}$ [3.1]

**11.**

| | | | | | |
|---|---|---|---|---|---|
| 3 | 2 | 0 | –3 | 4 | –6 |
| | | 6 | 18 | 45 | 147 |
| | 2 | 6 | 15 | 49 | 141 |

$P(3) = 141$ [3.1]

**12.** The leading term has a negative coefficient. The graph of $P(x)$ goes down to the far right. [3.2]

**13.**


The relative maximum (to the nearest 0.0001) is 0.3997. [3.2]

**14.** $P(x) = 3x^4 - 4x^3 - 11x^2 + 16x - 4$ [3.3]

$p = \pm$ factors of 4 $= \pm 1, \ \pm 2, \ \pm 4$

$q = \pm$ factors of 3 $= \pm 1, \ \pm 3$

$\frac{p}{q} = \pm 1, \ \pm 2, \ \pm 4, \ \pm\frac{1}{3}, \ \pm\frac{2}{3}, \ \pm\frac{4}{3}$

**15.** $P(x) = x^3 + x^2 + 2x + 4$ has no changes of sign. There are no positive real zeros.

$P(-x) = -x^3 + x^2 - 2x + 4$ has three changes of sign. There are three or one negative real zeros. [3.3]

**16.** $P(x)=x^3+x+10$

no positive and one negative real zeros

$\frac{p}{q}=\pm 1,\ \pm 2,\ \pm 5,\ \pm 10$

| | | | | |
|---|---|---|---|---|
| −2 | 1 | 0 | 1 | 10 |
| | | −2 | 4 | −10 |
| | 1 | −2 | 5 | 0 |

$$x=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(5)}}{2(1)}$$

$$=\frac{2\pm\sqrt{-16}}{2}=\frac{2\pm 4i}{2}=1\pm 2i$$

The zeros are −2, $1-2i$, $1+2i$. [3.4]

**17.** If $3+i$ is a zero of $P(x)$, then $3-i$ is also a zero.

$$\begin{aligned}P(x)&=[x-(3+i)][x-(3-i)](x+2)\\&=[x-3-i][x-3+i](x+2)\\&=[(x-3)^2-i^2](x+2)\\&=[x^2-6x+9-(-1)](x+2)\\&=[x^2-6x+9+1](x+2)\\&=(x^2-6x+10)(x+2)\\&=x^2(x+2)-6x(x+2)+10(x+2)\\&=x^3+2x^2-6x^2-12x+10x+20\\&=x^3-4x^2-2x+20\end{aligned}$$

[3.4]

**18.** $P(x)=x^3-2x^2+9x-18$

three or one positive and no negative real zeros

$\frac{p}{q}=\pm 1,\ \pm 2,\ \pm 3,\ \pm 6,\ \pm 9,\ \pm 18$

| | | | | |
|---|---|---|---|---|
| 2 | 1 | −2 | 9 | −18 |
| | | 2 | 0 | 18 |
| | 1 | 0 | 9 | 0 |

$$\begin{aligned}x^2+9&=0\\x^2&=-9\\x&=\pm\sqrt{-9}\\x&=\pm 3i\end{aligned}$$

$P(x)=(x-2)(x+3i)(x-3i)$ [3.4]

**19.** $F(x)=\frac{4x^2}{x^2+x-6}$

Vertical asymptotes:

$$\begin{aligned}x^2+x-6&=0\\(x+3)(x-2)&=0\end{aligned}$$

$x=-3,\ x=2$

Horizontal asymptote:

$y=\frac{4}{1}\Rightarrow y=4$ [3.5]

**20.** $F(x)=\frac{x^3+4x^2+1}{x^2+4}$

$$\begin{array}{r}x+4\phantom{xxxxxxx}\\ x^2+4\overline{)\,x^3+4x^2\phantom{-4x}+1}\\ \underline{x^3\phantom{+4x^2}+4x\phantom{+1}}\\ 4x^2-4x+1\\ \underline{4x^2\phantom{-4x}+16}\\ -4x-15\end{array}$$

The slant asymptote is $y=x+4$. [3.5]

# Chapter 4
# Exponential and Logarithmic Functions

## Section 4.1

**1.** If $f(3) = 7$, then $f^{-1}(7) = 3$.

**2.** If $g(-3) = 5$, then $g^{-1}(5) = -3$.

**3.** If $h^{-1}(-3) = -4$, then $h(-4) = -3$.

**4.** If $f^{-1}(7) = 0$, then $f(0) = 7$.

**5.** If 3 is in the domain of $f^{-1}$, then $f[f^{-1}(3)] = 3$.

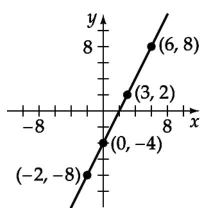
**6.** **a.** If $f$ is a one-to-one function and $f(0) = 5$, then $f^{-1}(5) = 0$.

**b.** If $f$ is a one-to-one function and $f(1) = 2$, then $f^{-1}(2) = 1$.

**7.** The domain of the inverse function $f^{-1}$ is the <u>range</u> of $f$.

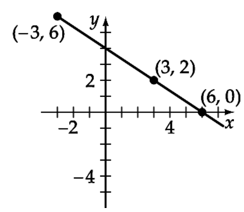
**8.** The range of the inverse function $f^{-1}$ is the <u>domain</u> of $f$.

**9.**



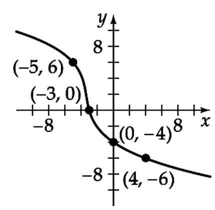

Yes, the inverse is a function.

**10.**



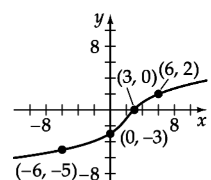

Yes, the inverse is a function.

**11.**



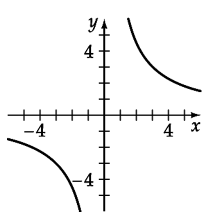

Yes, the inverse is a function.

**12.**



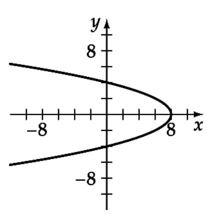

Yes, the inverse is a function.

**13.**



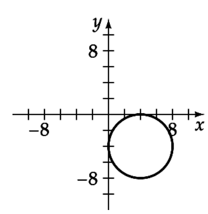

Yes, the inverse is a function.

**14.**



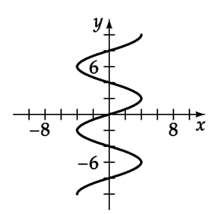

No, the inverse relation is not a function.

**15.**




No, the inverse relation is not a function.

**16.**




No, the inverse relation is not a function.

**17.** $f(x) = 4x;\ g(x) = \dfrac{x}{4}$

$f[g(x)] = f\left(\dfrac{x}{4}\right) = 4\left(\dfrac{x}{4}\right) = x$

$g[f(x)] = g(4x) = \dfrac{4x}{4} = x$

Yes, $f$ and $g$ are inverses of each other.

**18.** $f(x) = 3x;\ g(x) = \dfrac{1}{3x}$

$f[g(x)] = f\left(\dfrac{1}{3x}\right) = 3\left(\dfrac{1}{3x}\right) = \dfrac{1}{x} \neq x$

No, $f$ and $g$ are not inverses of each other.

**19.** $f(x)=4x-1;\ g(x)=\frac{1}{4}x+\frac{1}{4}$

$$f[g(x)]=f\left(\frac{1}{4}x+\frac{1}{4}\right)$$
$$=4\left(\frac{1}{4}x+\frac{1}{4}\right)-1=x+1-1$$
$$=x$$
$$g[f(x)]=g(4x-1)$$
$$=\frac{1}{4}(4x-1)+\frac{1}{4}=x-\frac{1}{4}+\frac{1}{4}$$
$$=x$$

Yes, $f$ and $g$ are inverses of each other.

**20.** $f(x)=\frac{1}{2}x-\frac{3}{2};\ g(x)=2x+3$

$$f[g(x)]=f(2x+3)$$
$$=\frac{1}{2}(2x+3)-\frac{3}{2}=x+\frac{3}{2}-\frac{3}{2}$$
$$=x$$
$$g[f(x)]=g\left(\frac{1}{2}x-\frac{3}{2}\right)$$
$$=2\left(\frac{1}{2}x-\frac{3}{2}\right)+3=x-3+3$$
$$=x$$

Yes, $f$ and $g$ are inverses of each other.

**21.** $f(x)=-\frac{1}{2}x-\frac{1}{2};\ g(x)=-2x+1$

$$f[g(x)]=f(-2x+1)$$
$$=-\frac{1}{2}(-2x+1)-\frac{1}{2}=x-\frac{1}{2}-\frac{1}{2}$$
$$=x-1$$
$$\neq x$$

No, $f$ and $g$ are not inverses of each other.

**22.** $f(x)=3x+2;\ g(x)=\frac{1}{3}x-\frac{2}{3}$

$$f[g(x)]=f\left(\frac{1}{3}x-\frac{2}{3}\right)$$
$$=3\left(\frac{1}{3}x-\frac{2}{3}\right)+2=x-2+2$$
$$=x$$
$$g[f(x)]=g(3x+2)$$
$$=\frac{1}{3}(3x+2)-\frac{2}{3}=x+\frac{2}{3}-\frac{2}{3}$$
$$=x$$

Yes, $f$ and $g$ are inverses of each other.

**23.** $f(x)=\frac{5}{x-3};\ g(x)=\frac{5}{x}+3$

$$f[g(x)]=f\left(\frac{5}{x}+3\right)$$
$$=\frac{5}{\frac{5}{x}+3-3}=\frac{5}{\frac{5}{x}}=5\cdot\frac{x}{5}$$
$$=x$$
$$g[f(x)]=g\left(\frac{5}{x-3}\right)$$
$$=\frac{5}{\frac{5}{x-3}}+3=x-3+3$$
$$=x$$

Yes, $f$ and $g$ are inverses of each other.

**24.** $f(x)=\frac{2x}{x-1};\ g(x)=\frac{x}{x-2}$

$$f[g(x)]=f\left(\frac{x}{x-2}\right)$$
$$=\frac{2\left(\frac{x}{x-2}\right)}{\frac{x}{x-2}-1}=\frac{\frac{2x}{x-2}}{\frac{x-(x-2)}{x-2}}=\frac{\frac{2x}{x-2}}{\frac{2}{x-2}}$$
$$=\frac{2x}{x-2}\cdot\frac{x-2}{2}$$
$$=x$$
$$g[f(x)]=g\left(\frac{2x}{x-1}\right)$$
$$=\frac{\frac{2x}{x-1}}{\frac{2x}{x-1}-2}=\frac{\frac{2x}{x-1}}{\frac{2x-2(x-1)}{x-1}}=\frac{\frac{2x}{x-1}}{\frac{2}{x-1}}$$
$$=\frac{2x}{x-1}\cdot\frac{x-1}{2}$$
$$=x$$

Yes, $f$ and $g$ are inverses of each other.

**25.** $f(x)=x^3+2;\ g(x)=\sqrt[3]{x-2}$

$$f[g(x)]=f\left(\sqrt[3]{x-2}\right)$$
$$=\left(\sqrt[3]{x-2}\right)^3+2=x-2+2$$
$$=x$$
$$g[f(x)]=g\left(x^3+2\right)$$
$$=\sqrt[3]{x^3+2-2}=\sqrt[3]{x^3}$$
$$=x$$

Yes, $f$ and $g$ are inverses of each other.

**26.** $f(x)=(x+5)^3;\ g(x)=\sqrt[3]{x}-5$

$$f[g(x)]=f\left(\sqrt[3]{x}-5\right)$$
$$=\left(\sqrt[3]{x}-5+5\right)^3=\sqrt[3]{x^3}$$
$$=x$$
$$g[f(x)]=g\left((x+5)^3\right)$$
$$=\sqrt[3]{(x+5)^3}-5=x+5-5$$
$$=x$$

Yes, $f$ and $g$ are inverses of each other.

**27.** The inverse of $\{(-3, 1), (-2, 2), (1, 5), (4, -7)\}$ is $\{(1, -3), (2, -2), (5, 1), (-7, 4)\}$.

**28.** The inverse of $\{(-5, 4), (-2, 3), (0, 1), (3, 2), (7, 11)\}$ is $\{(4, -5), (3, -2), (1, 0), (2, 3), (11, 7)\}$.

**29.** The inverse of $\{(0, 1), (1, 2), (2, 4), (3, 8), (4, 16)\}$ is $\{(1, 0), (2, 1), (4, 2), (8, 3), (16, 4)\}$.

**30.** The inverse of {(1, 0), (10, 1), (100, 2), (1000, 3), (10000, 4) is {(0, 1), (1, 10), (2, 100), (3, 1000), (4, 10,000).

**31.**
$$f(x) = 2x + 4$$
$$x = 2y + 4$$
$$x - 4 = 2y$$
$$\tfrac{1}{2}x - 2 = y$$
$$f^{-1}(x) = \tfrac{1}{2}x - 2$$

**32.**
$$f(x) = 4x - 8$$
$$x = 4y - 8$$
$$x + 8 = 4y$$
$$\tfrac{1}{4}x + 2 = y$$
$$f^{-1}(x) = \tfrac{1}{4}x + 2$$

**33.**
$$f(x) = 3x - 7$$
$$x = 3y - 7$$
$$x + 7 = 3y$$
$$\tfrac{1}{3}x + \tfrac{7}{3} = y$$
$$f^{-1}(x) = \tfrac{1}{3}x + \tfrac{7}{3}$$

**34.**
$$f(x) = -3x - 8$$
$$x = -3y - 8$$
$$x + 8 = -3y$$
$$-\tfrac{1}{3}x - \tfrac{8}{3} = y$$
$$f^{-1}(x) = -\tfrac{1}{3}x - \tfrac{8}{3}$$

**35.**
$$f(x) = -2x + 5$$
$$x = -2y + 5$$
$$x - 5 = -2y$$
$$-\tfrac{1}{2}x + \tfrac{5}{2} = y$$
$$f^{-1}(x) = -\tfrac{1}{2}x + \tfrac{5}{2}$$

**36.**
$$f(x) = -x + 3$$
$$x = -y + 3$$
$$y = -x + 3$$
$$f^{-1}(x) = -x + 3$$

**37.**
$$f(x) = \frac{2x}{x-1},\ x \neq 1$$
$$x = \frac{2y}{y-1}$$
$$x(y-1) = xy - x = 2y$$
$$xy - 2y = y(x-2) = x$$
$$y = \frac{x}{x-2}$$
$$f^{-1}(x) = \frac{x}{x-2},\ x \neq 2$$

**38.**
$$f(x) = \frac{x}{x-2},\ x \neq 2$$
$$x = \frac{y}{y-2}$$
$$x(y-2) = xy - 2x = y$$
$$xy - y = y(x-1) = 2x$$
$$y = \frac{2x}{x-1}$$
$$f^{-1}(x) = \frac{2x}{x-1},\ x \neq 1$$

**39.**
$$f(x) = \frac{x-1}{x+1},\ x \neq -1$$
$$x = \frac{y-1}{y+1}$$
$$x(y+1) = xy + x = y - 1$$
$$xy - y = -x - 1$$
$$y - xy = y(1-x) = x + 1$$
$$y = \frac{x+1}{1-x}$$
$$f^{-1}(x) = \frac{x+1}{1-x}, x \neq 1$$

**40.**
$$f(x) = \frac{2x-1}{x+3},\ x \neq -3$$
$$x = \frac{2y-1}{y+3}$$
$$xy + 3x = 2y - 1$$
$$xy - 2y = -3x - 1$$
$$y = \frac{3x+1}{2-x}$$
$$f^{-1}(x) = \frac{3x+1}{2-x},\ x \neq 2$$

**41.**
$$f(x) = x^2 + 1,\ x \geq 0$$
$$x = y^2 + 1$$
$$x - 1 = y^2$$
$$\sqrt{x-1} = y$$
$$f^{-1}(x) = \sqrt{x-1},\ x \geq 1$$

Note: Do not use $\pm$ with the radical because the domain of $f$, and thus the range of $f^{-1}$, is nonnegative.

**42.**
$$f(x) = x^2 - 4,\ x \geq 0$$
$$x = y^2 - 4$$
$$x + 4 = y^2$$
$$\sqrt{x+4} = y$$
$$f^{-1}(x) = \sqrt{x+4},\ x \geq -4$$

Note: Do not use $\pm$ with the radical because the domain of $f$, and thus the range of $f^{-1}$, is nonnegative.

**43.**
$$f(x) = \sqrt{x-2},\ x \geq 2$$
$$x = \sqrt{y-2}$$
$$x^2 = y - 2$$
$$x^2 + 2 = y$$
$$f^{-1}(x) = x^2 + 2,\ x \geq 0$$

Note: The range of $f$, is nonnegative, therefore the domain of $f^{-1}$ is also nonnegative.

**44.**
$$f(x) = \sqrt{4-x},\ x \leq 4$$
$$x = \sqrt{4-y}$$
$$x^2 = 4 - y$$
$$y = -x^2 + 4$$
$$f^{-1}(x) = -x^2 + 4,\ x \geq 0$$

Note: The range of $f$, is non-negative, therefore the domain of $f^{-1}$ is also non-negative.

**45.**
$$f(x)=x^2+4x,\ x\ge -2$$
$$x=y^2+4y$$
$$x+4=y^2+4y+4$$
$$x+4=(y+2)^2$$
$$\sqrt{x+4}=y+2$$
$$y=\sqrt{x+4}-2$$
$$f^{-1}(x)=\sqrt{x+4}-2,\ x\ge -4$$
Note: The range of $f$, is non-negative, therefore the domain of $f^{-1}$ is also non-negative.

**46.**
$$f(x)=x^2-6x,\ x\le 3$$
$$x=y^2-6y$$
$$x+9=y^2-6y+9$$
$$x+9=(y-3)^2$$
$$-\sqrt{x+9}=y-3$$
$$-\sqrt{x+9}+3=y$$
$$f^{-1}(x)=-\sqrt{x+9}+3,\ x\ge -9$$
Note: Because the range of $f$, is non-positive, the range of $f^{-1}$ must also be non-positive.

**47.**
$$f(x)=x^2+4x-1,\ x\le -2$$
$$x=y^2+4y-1$$
$$x+1=y^2+4y$$
$$x+1+4=y^2+4y+4$$
$$x+5=(y+2)^2$$
$$-\sqrt{x+5}=y+2$$
$$-\sqrt{x+5}-2=y$$
$$f^{-1}(x)=-\sqrt{x+5}-2,\ x\ge -5$$
Note: Because the range of $f$, is non-positive, the range of $f^{-1}$ must also be non-positive.

**48.**
$$f(x)=x^2-6x+1,\ x\ge 3$$
$$x=y^2-6y+1$$
$$x-1=y^2-6y$$
$$x-1+9=y^2-6y+9$$
$$x+8=(y-3)^2$$
$$\sqrt{x+8}=y-3$$
$$\sqrt{x+8}+3=y$$
$$f^{-1}(x)=\sqrt{x+8}+3,\ x\ge -8$$
Note: The range of $f$, is non-negative, therefore the domain of $f^{-1}$ is also non-negative.

**49.**
$$V(x)=x^3$$
$$x=y^3$$
$$\sqrt[3]{x}=y$$
$$V^{-1}(x)=\sqrt[3]{x}$$
$V^{-1}(x)$ finds the length of a side of a cube given the volume.

**50.**
$$f(x)=12x$$
$$x=12y$$
$$\frac{x}{12}=y$$
$$f^{-1}(x)=\frac{x}{12}$$
$f^{-1}(x)$ converts $x$ inches into feet.

**51.**
$$f(x)=\frac{5}{9}(x-32)$$
$$x=\frac{5}{9}(y-32)$$
$$\frac{9}{5}x=y-32$$
$$\frac{9}{5}x+32=y$$
$$f^{-1}(x)=\frac{9}{5}x+32$$
$f^{-1}(x)$ is used to convert $x$ degrees Celsius to an equivalent Fahrenheit temperature.

**52.** **a.** $S(96)=\frac{3}{2}(96)+18=\$162$

**b.**
$$S(x)=\frac{3}{2}x+18$$
$$x=\frac{3}{2}y+18$$
$$x-18=\frac{3}{2}y$$
$$\frac{2}{3}x-12=y$$
$$S^{-1}(x)=\frac{2}{3}x-12$$
$$S^{-1}(399)=\frac{2}{3}(399)-12=\$254$$

**53.**
$$s(x) = 2x + 24$$
$$x = 2y + 24$$
$$x - 24 = 2y$$
$$\frac{1}{2}x - 12 = y$$
$$s^{-1}(x) = \frac{1}{2}x - 12$$

**54.**
$$K(x) = 1.3x - 4.7$$
$$x = 1.3y - 4.7$$
$$x + 4.7 = 1.3y$$
$$\frac{x + 4.7}{1.3} = y$$
$$K^{-1}(x) = \frac{x + 4.7}{1.3}$$

**55.**
$$E(s) = 0.05s + 2500$$
$$s = 0.05y + 2500$$
$$s - 2500 = 0.05y$$
$$\frac{1}{0.05}s - \frac{2500}{0.05} = y$$
$$20s - 50{,}000 = y$$
$$E^{-1}(s) = 20s - 50{,}000$$

The executive can use the inverse function to determine the value of the software that must be sold in order to achieve a given monthly income.

**56.** No. It is not a one-to-one function. For a given cost, there is more than one weight that can be associated with that cost.

**57.** **a.** $p(10) \approx 0.12 = 12\%$ **;** $p(30) \approx 0.71 = 71\%$

**b.** The graph of $p$, for $1 \le n \le 60$, is an increasing function. Thus $p$ has an inverse that is a function.

**c.** $p^{-1}(0.223)$ represents the number of people required to be in the group for a 22.3% probability that at least two of the people will share a birthday.

**58.** **a.** Answers will vary.

**b.** No. $L$ is not a one-to-one function.

**59.** **a.**
D $f(13) = 2(13) - 1 = 25$
O $f(24) = 2(24) - 1 = 47$
(space) $f(36) = 2(36) - 1 = 71$
Y $f(34) = 2(34) - 1 = 67$
O $f(24) = 47$
U $f(30) = 2(30) - 1 = 59$
R $f(27) = 2(27) - 1 = 53$
(space) $f(36) = 71$
H $f(17) = 2(17) - 1 = 33$
O $f(24) = 47$
M $f(22) = 2(22) - 1 = 43$
E $f(14) = 2(14) - 1 = 27$
W $f(32) = 2(32) - 1 = 63$
O $f(24) = 47$
R $f(27) = 53$
K $f(20) = 2(20) - 1 = 39$

The code is
25 47 71 67 47 59 53 71 33 47 43 27 63 47 53 39.

**b.**
$f^{-1}(49) = \frac{49+1}{2} = 25$ P
$f^{-1}(33) = \frac{33+1}{2} = 17$ H
$f^{-1}(47) = \frac{47+1}{2} = 24$ O
$f^{-1}(45) = \frac{45+1}{2} = 23$ N
$f^{-1}(27) = \frac{27+1}{2} = 14$ E
$f^{-1}(71) = \frac{71+1}{2} = 36$ (space)
$f^{-1}(33) = 17$ H
$f^{-1}(47) = 24$ O
$f^{-1}(43) = \frac{43+1}{2} = 22$ M
$f^{-1}(27) = 14$ E

The message is PHONE HOME.

**c.** Answers will vary.

**60.**
$$g(x) = 2x + 3$$
$$x = 2y + 3$$
$$\frac{x-3}{2} = y$$
$$g^{-1}(x) = \frac{x-3}{2}$$

$g^{-1}(59) = \frac{59-3}{2} = 28$ S
$g^{-1}(31) = \frac{31-3}{2} = 14$ E
$g^{-1}(39) = \frac{39-3}{2} = 18$ I
$g^{-1}(73) = \frac{73-3}{2} = 35$ Z
$g^{-1}(31) = 14$ E
$g^{-1}(75) = \frac{75-3}{2} = 36$ (space)
$g^{-1}(61) = \frac{61-3}{2} = 29$ T
$g^{-1}(37) = \frac{37-3}{2} = 17$ H
$g^{-1}(31) = 14$ E
$g^{-1}(75) = 36$ (space)
$g^{-1}(29) = \frac{29-3}{2} = 13$ D
$g^{-1}(23) = \frac{23-3}{2} = 10$ A
$g^{-1}(71) = \frac{71-3}{2} = 34$ Y

The message is SEIZE THE DAY.

**61.** $f(2)=7, f(5)=12,$ and $f(4)=c$. Because $f$ is an increasing linear function, and 4 is between 2 and 5, then $f(4)$ is between $f(2)$ and $f(5)$. Thus, $c$ is between 7 and 12.

**62.** $f(1)=13, f(4)=9,$ and $f(3)=c$. Because $f$ is a decreasing linear function, and 3 is between 1 and 4, then $f(3)$ is between $f(1)$ and $f(4)$ Thus, $c$ is between 9 and 13.

**63.** $f$ is a linear function, therefore $f^{-1}$ is a linear function.

$f(2)=3 \Rightarrow f^{-1}(3)=2$

$f(5)=9 \Rightarrow f^{-1}(9)=5$

Since 6 is between 3 and 9, $f^{-1}(6)$ is between 2 and 5.

**64.** $f$ is a linear function, therefore $f^{-1}$ is a linear function.

$f(5)=-1 \Rightarrow f^{-1}(-1)=5$

$f(9)=-3 \Rightarrow f^{-1}(-3)=9$

Since –2 is between –1 and –3, $f^{-1}(-2)$ is between 5 and 9.

**65.** $g$ is a linear function, therefore $g^{-1}$ is a linear function.

$g^{-1}(3)=4 \Rightarrow g(4)=3$

$g^{-1}(7)=8 \Rightarrow g(8)=7$

Since 5 is between 4 and 8, $g(5)$ is between 3 and 7.

**66.** $g$ is a linear function, therefore $g^{-1}$ is a linear function.

$g^{-1}(-2)=5 \Rightarrow g(5)=-2$

$g^{-1}(0)=-3 \Rightarrow g(-3)=0$

Since 0 is between 5 and –3, then $g(0)$ is between –2 and 0.

## Connecting Concepts

**67.** $f(x)=mx+b, \quad m \neq 0$

$$y=mx+b$$
$$x=my+b$$
$$x-b=my$$
$$\frac{x-b}{m}=y$$
$$f^{-1}(x)=\frac{1}{m}x-\frac{b}{m}$$

The slope is $\frac{1}{m}$ and the $y$-intercept is $\left(0,-\frac{b}{m}\right)$.

**68.** $f(x)=ax^2+bx+c, \quad a>0,\ x>-\frac{b}{2a}$

Domain of $f$ is $\left\{x \middle| x>-\frac{b}{2a}\right\}$, Range of $f$ is $\left\{y \middle| y \geq \frac{4ac-b^2}{4a}\right\}$.

Domain of $f^{-1}$ is $\left\{x \middle| x \geq \frac{4ac-b^2}{4a}\right\}$, Range of $f^{-1}$ is $\left\{y \middle| y>-\frac{b}{2a}\right\}$.

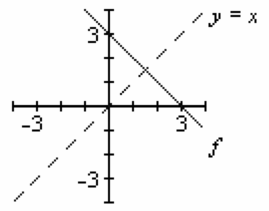
$$y=ax^2+bx+c$$
$$x=ay^2+by+c$$
$$x-c=a\left(y^2+\frac{b}{a}y\right)$$
$$\frac{b^2}{4a^2}+\frac{x-c}{a}=\left(y^2+\frac{b}{a}y+\frac{b^2}{4a^2}\right) \quad \text{Complete the square.}$$
$$\frac{b^2+4ax-4ac}{4a^2}=\left(y+\frac{b}{2a}\right)^2$$
$$a+\sqrt{\frac{b^2+4ax-4ac}{4a^2}}=\left(y+\frac{b}{2a}\right)$$

Choose the positive root, since the Range of $f^{-1}$ is $\left\{y \middle| y>-\frac{b}{2a}\right\}$.

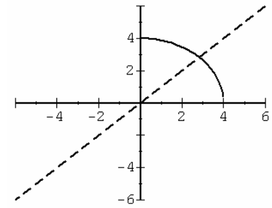
Thus $f^{-1}(x)=-\frac{b}{2a}+\sqrt{\frac{b^2+4ax-4ac}{4a^2}}$

$$f^{-1}(x)=\frac{-b+\sqrt{b^2+4ax-4ac}}{2a},\ a\neq 0,\ x\geq\frac{4ac-b^2}{4a}$$

**69.** The reflection of $f$ across the line given by $y=x$ yields $f$. Thus $f$ is its own inverse.




**70.** The reflection of $f$ across the line given by $y=x$ yields $f$. Thus $f$ is its own inverse.




**71.** There is at most one point where each horizontal line intersects the graph of the function. The function is a one-to-one function.

**72.** There is at most one point where each horizontal line intersects the graph of the function. The function is a one-to-one function.

**73.** A horizontal line intersects the graph of the function at more than one point. Thus, the function is not a one-to-one function.

**74.** A horizontal line intersects the graph of the function at more than one point. Thus, the function is not a one-to-one function.

## Prepare for Section 4.2

**PS1.** $2^3=2\cdot2\cdot2=8$

**PS2.** $3^{-4}=\frac{1}{3^4}=\frac{1}{3\cdot3\cdot3\cdot3}=\frac{1}{81}$

**PS3.** $\frac{2^2+2^{-2}}{2}=\frac{4+\frac{1}{4}}{2}=\frac{16+1}{8}=\frac{17}{8}$

**PS4.** $\frac{3^2-3^{-2}}{2}=\frac{9-\frac{1}{9}}{2}=\frac{81-1}{18}=\frac{80}{18}=\frac{40}{9}$

**PS5.**
$f(x)=10^x$
$f(-1)=10^{-1}=\frac{1}{10}$
$f(0)=10^0=1$
$f(1)=10^1=10$
$f(2)=10^2=100$

**PS6.**
$f(x)=\left(\frac{1}{2}\right)^x$
$f(-1)=\left(\frac{1}{2}\right)^{-1}=2$
$f(0)=\left(\frac{1}{2}\right)^0=1$
$f(1)=\left(\frac{1}{2}\right)^1=\frac{1}{2}$
$f(2)=\left(\frac{1}{2}\right)^2=\frac{1}{4}$

## Section 4.2

**1.** $f(0)=3^0=1$
$f(4)=3^4=81$

**2.** $f(3)=5^3=125$
$f(-2)=5^{-2}=\frac{1}{25}$

**3.** $g(-2)=10^{-2}=\frac{1}{100}$
$g(3)=10^3=1000$

**4.** $g(0)=4^0=1$
$g(-1)=4^{-1}=\frac{1}{4}$

**5.** $h(2)=\left(\frac{3}{2}\right)^2=\frac{9}{4}$
$h(-3)=\left(\frac{3}{2}\right)^{-3}=\frac{8}{27}$

**6.** $h(-1)=\left(\frac{2}{5}\right)^{-1}=\frac{5}{2}$
$h(3)=\left(\frac{2}{5}\right)^3=\frac{8}{125}$

**7.** $j(-2)=\left(\frac{1}{2}\right)^{-2}=4$
$j(4)=\left(\frac{1}{2}\right)^4=\frac{1}{16}$

**8.** $j(-1)=\left(\frac{1}{4}\right)^{-1}=4$
$j(5)=\left(\frac{1}{4}\right)^5=\frac{1}{1024}$

**9.** $f(3.2) = 2^{3.2} \approx 9.19$

**10.** $f(-1.5) = 3^{-1.5} \approx 0.19$

**11.** $g(2.2) = e^{2.2} \approx 9.03$

**12.** $g(-1.3) = e^{-1.3} \approx 0.27$

**13.** $h(\sqrt{2}) = 5^{\sqrt{2}} \approx 9.74$

**14.** $h(\pi) = 5^{\pi} \approx 0.11$ $h(\pi)=5^{\pi} \approx 0.11$

**15.** $f(x) = 5^x$ is a basic exponential graph.

$g(x) = 1 + 5^{-x}$ is the graph of $f(x)$ reflected across the $y$-axis and moved up 1 unit.

$h(x) = 5^{x+3}$ is the graph of $f(x)$ moved to the left 3 units.

$k(x) = 5^x + 3$ is the graph of $f(x)$ moved up 3 units.

**a.** $k(x)$ **b.** $g(x)$ **c.** $h(x)$ **d.** $f(x)$

**16.** $f(x) = \left(\frac{1}{4}\right)^x$ is an exponential function with a base between 0 and 1.

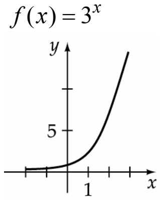
$g(x) = \left(\frac{1}{4}\right)^{-x}$ is the graph of $f(x)$ reflected across the $y$-axis.

$h(x) = \left(\frac{1}{4}\right)^{x-2}$ is the graph of $f(x)$ moved 2 units to the right.

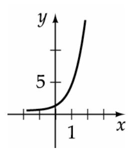
$k(x) = 3\left(\frac{1}{4}\right)^x$ is the graph of $f(x)$ stretched vertically by a factor of 3.

**a.** $k(x)$ **b.** $f(x)$ **c.** $g(x)$ **d.** $h(x)$

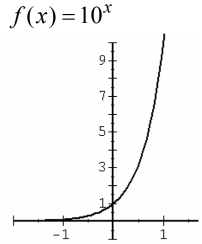
**17.** $f(x) = 3^x$



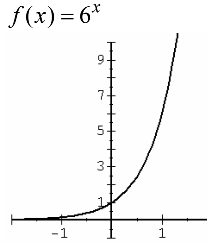

**18.** $f(x) = 4^x$



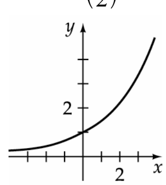

**19.** $f(x) = 10^x$



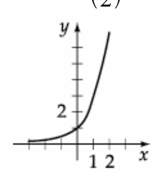

**20.** $f(x) = 6^x$



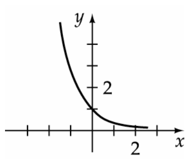

**21.** $f(x) = \left(\frac{3}{2}\right)^x$



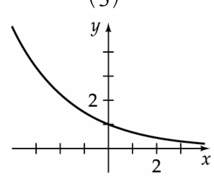

**22.** $f(x) = \left(\frac{5}{2}\right)^x$




**23.** $f(x) = \left(\frac{1}{3}\right)^x$




**24.** $f(x) = \left(\frac{2}{3}\right)^x$




**25.** Shift the graph of *f* vertically upward 2 units.

**26.** Shift the graph of *f* vertically downward 3 units.

**27.** Shift the graph of *f* horizontally to the right 2 units.

**28.** Shift the graph of *f* horizontally to the left 5 units.

**29.** Reflect the graph of *f* across the *y*-axis.

**30.** Reflect the graph of *f* across the *x*-axis.

**31.** Stretch the graph of $f$ vertically away from the $x$-axis by a factor of 2.

**32.** Shrink the graph of $f$ vertically towards the $x$-axis by a factor of $\frac{1}{2}$.

**33.** Reflect the graph of $f$ across the $y$-axis, and then shift this graph vertically upward 2 units.

**34.** Shift the graph of $f$ horizontally 3 units to the right, and then shift this graph vertically upward 1 unit.

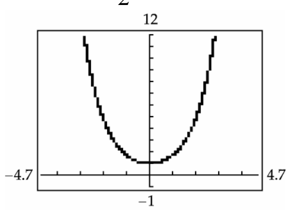
**35.** Shift the graph of $f$ horizontally 4 units to the right, and then reflect this graph across the $x$-axis.

**36.** Reflect the graph of $f$ across the $y$-axis, and then reflect this graph across the $x$-axis.

**37.** Reflect the graph of $f$ across the $y$-axis, and then shift this graph vertically upward 3 units.

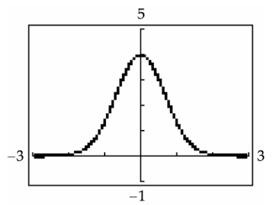
**38.** Shift the graph of $f$ horizontally 2 units to the left, stretch this graph away from the $x$-axis by a factor of 3, and then shift this graph vertically downward 1 unit.

**39.** $f(x) = \dfrac{3^x + 3^{-x}}{2}$



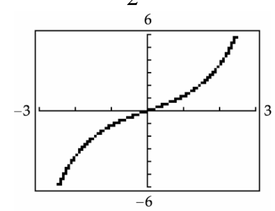

No horizontal asymptote

**40.** $f(x) = 4 \cdot 3^{-x^2}$



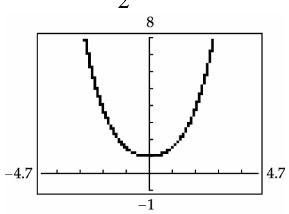

Horizontal asymptote: $y = 0$

**41.** $f(x) = \dfrac{e^x - e^{-x}}{2}$



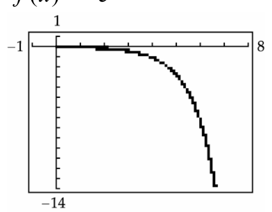

No horizontal asymptote.

**42.** $f(x) = \dfrac{e^x + e^{-x}}{2}$



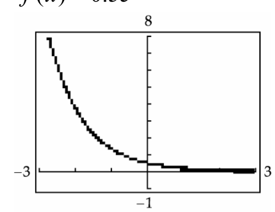

No horizontal asymptote

**43.** $f(x) = -e^{(x-4)}$



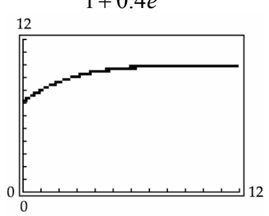

Horizontal asymptote: $y = 0$

**44.** $f(x) = 0.5e^{-x}$



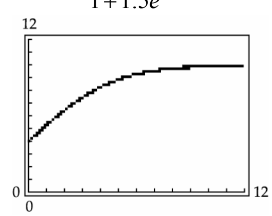

Horizontal asymptote: $y = 0$

**45.** $f(x) = \dfrac{10}{1 + 0.4e^{-0.5x}}$, with $x \ge 0$




Horizontal asymptote: $y = 10$

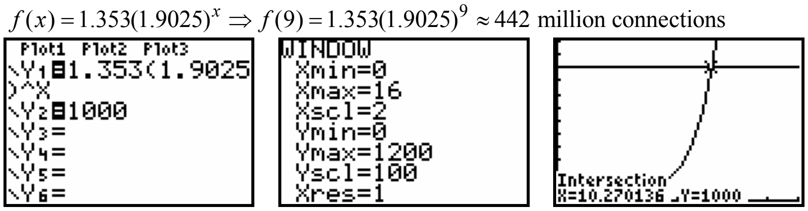
**46.** $f(x) = \dfrac{10}{1 + 1.5e^{-0.5x}}$, with $x \ge 0$




Horizontal asymptote: $y = 10$

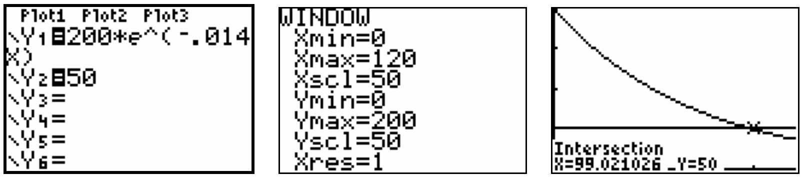
**47.** **a.** $f(x) = 1.353(1.9025)^x \Rightarrow f(9) = 1.353(1.9025)^9 \approx 442$ million connections

**b.**




10 years after January 1, 1998 is in 2008.

**48.** **a.** $A(t) = 200e^{-0.014t} \Rightarrow f(45) = 200e^{-0.014(45)} \approx 107$ mg

**b.** 


It will take 99 minutes.

**49.** **a.** $d(p) = 25 + 880e^{-0.18p} \Rightarrow d(8) = 25 + 880e^{-0.18(8)} \approx 233$ items per month

$d(p) = 25 + 880e^{-0.18p} \Rightarrow d(18) = 25 + 880e^{-0.18(18)} \approx 59$ items per month

**b.** As $p \to \infty$, $d(p) \to 25$. The demand will approach 25 items per month.

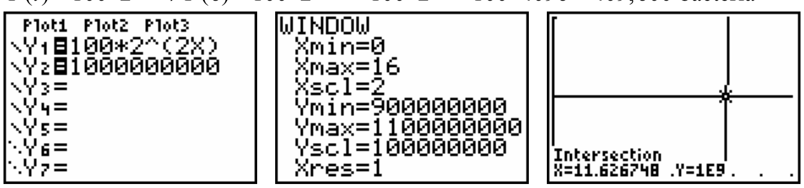
**50.** **a.** $I(t) = 24{,}000 - 22{,}000e^{-0.005t} \Rightarrow I(10) = 24{,}000 - 22{,}000e^{-0.005(10)} \approx \$3072.95$

$I(t) = 24{,}000 - 22{,}000e^{-0.005t} \Rightarrow I(100) = 24{,}000 - 22{,}000e^{-0.005(100)} \approx \$10{,}656.33$

**b.** As $t \to \infty$, $I(t) \to 24{,}000$. The monthly income will approach \$24,000.

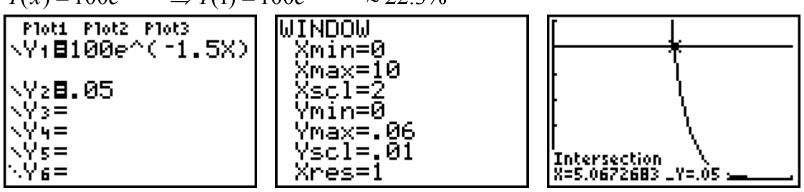
**51.** **a.** $P(t) = 100 \cdot 2^{2t} \Rightarrow P(3) = 100 \cdot 2^{2(3)} = 100 \cdot 2^6 = 100 \cdot 64 = 6400$ bacteria

$P(t) = 100 \cdot 2^{2t} \Rightarrow P(6) = 100 \cdot 2^{2(6)} = 100 \cdot 2^{12} = 100 \cdot 4096 = 409{,}600$ bacteria

**b.** 

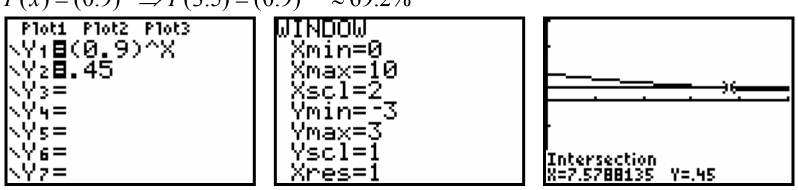

11.6 hours

**52.** **a.** $I(x) = 100e^{-1.5x} \Rightarrow I(1) = 100e^{-1.5(1)} \approx 22.3\%$

**b.** 


5 millimeters

**53.** **a.** $P(x) = (0.9)^x \Rightarrow P(3.5) = (0.9)^{3.5} \approx 69.2\%$

**b.** 


For a transparency of 45%, the UV index is 7.6.

**54.** **a.** $P(t) = \dfrac{3600}{1 + 7e^{-0.05t}} \Rightarrow P(0) = \dfrac{3600}{1 + 7e^{-0.05(0)}} = \dfrac{3600}{1 + 7e^0} = \dfrac{3600}{1 + 7(1)} = \dfrac{3600}{1 + 7} = \dfrac{3600}{8} = 450$ bass

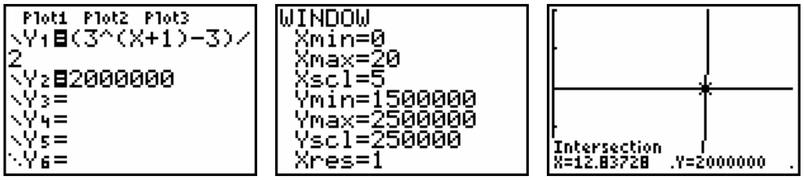
**b.** $P(t) = \dfrac{3600}{1 + 7e^{-0.05t}} \Rightarrow P(12) = \dfrac{3600}{1 + 7e^{-0.05(12)}} \approx 744$ bass

**c.** As $t \to \infty$, $P(t) \to 3600$. The bass population will increase, approaching 3600.

**55.** **a.** $B(n)=\dfrac{3^{n+1}-3}{2}\Rightarrow B(5)=\dfrac{3^{5+1}-3}{2}=\dfrac{3^6-3}{2}=\dfrac{729-3}{2}=\dfrac{726}{2}=363$ beneficiaries

$B(n)=\dfrac{3^{n+1}-3}{2}\Rightarrow B(10)=\dfrac{3^{10+1}-3}{2}=\dfrac{3^{11}-3}{2}=\dfrac{177147-3}{2}=\dfrac{177144}{2}=88{,}572$ beneficiaries

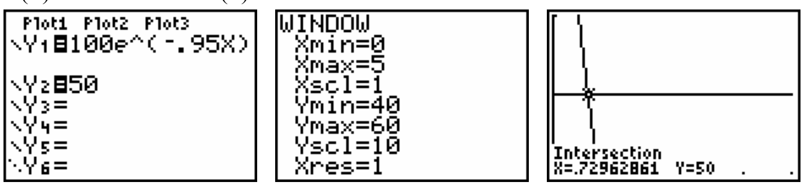
**b.**




13 rounds

**56.** **a.** $I(x)=100e^{-0.95x}\Rightarrow I(2)=100e^{-0.95(2)}=100e^{-1.9}\approx 15.0\%$

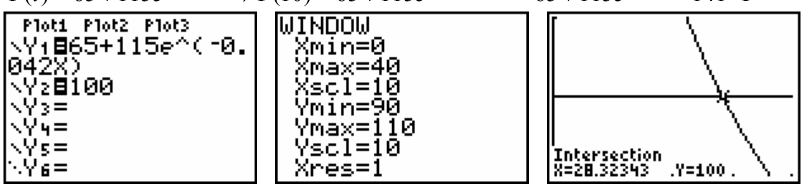
**b.**




about 0.73 foot

**57.** **a.** $T(t)=65+115e^{-0.042t}\Rightarrow T(10)=65+115e^{-0.042(10)}=65+115e^{-0.42}\approx 141^\circ$ F

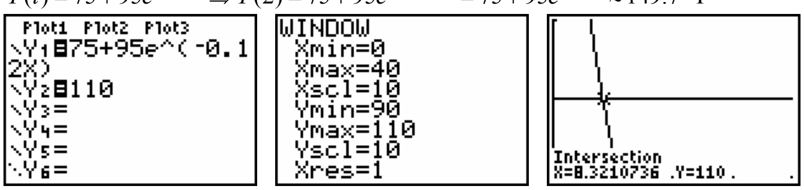
**b.**




28.3 minutes

**58.** **a.** $T(t)=75+95e^{-0.12t}\Rightarrow T(2)=75+95e^{-0.12(2)}=75+95e^{-0.24}\approx 149.7^\circ$ F

**b.**




8.3 minutes

**c.** As $t\to\infty$, $T(t)\to 75$. Therefore, room temperature is $75^\circ$ F.

**59.** **a.** $f(n)=(27.5)2^{(n-1)/12}\Rightarrow f(40)=(27.5)2^{(40-1)/12}=(27.5)2^{39/12}=(27.5)2^{3.25}\approx 261.63$ vibrations per second

**b.** No. The function $f(n)$ is not a linear function. Therefore, the graph of $f(n)$ does not increase at a constant rate.

**Connecting Concepts**

**60.** $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$ is an even function. That is, prove $\cosh(-x) = \cosh(x)$.

Proof: $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$

$\cosh(-x) = \dfrac{e^{-x} + e^x}{2}$

$\cosh(-x) = \dfrac{\left(e^x - e^{-x}\right)}{2}$

$\cosh(-x) = F(x)$

**61.** $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ is an odd function. That is, prove $\sinh(-x) = -\sinh(x)$.

Proof: $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$

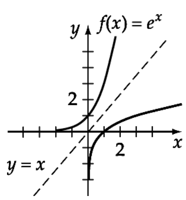
$\sinh(-x) = \dfrac{e^{-x} - e^x}{2}$

$\sinh(-x) = \dfrac{-e^{-x} + e^x}{2}$

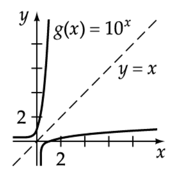
$\sinh(-x) = \dfrac{\left(e^x - e^{-x}\right)}{2}$

$\sinh(-x) = -F(x)$

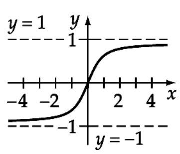
**62.**




**63.**

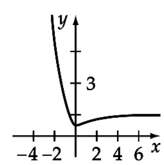



**64.**



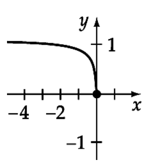

domain: $(-\infty, \infty)$

**65.**



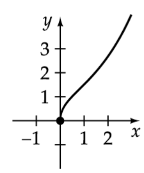

domain: $(-\infty, \infty)$

**66.**




domain: $(-\infty, 0]$

**67.**




domain: $[0, \infty)$

**68.** Let $f(x) = 2^x$ and $g(x) = x^2 + 4$. Then $h(x) = 2^{(x^2+4)} = f\left[x^2 + 4\right] = f[g(x)] = (f \circ g)(x)$.

**69.** Let $f(x) = e^x$ and $g(x) = 2x - 5$. Then $h(x) = e^{(2x-5)} = f[2x - 5] = f[g(x)] = (f \circ g)(x)$.

**70.** By definition the average of two numbers is their sum divided by 2. The expression $\dfrac{e^x + e^{-x}}{2}$ shows that *f* is the average of $g(x) = e^x$ and $h(x) = e^{-x}$.

## Prepare for Section 4.3

**PS1.** $2^x = 16$
$2^x = 2^4$
$x = 4$

**PS2.** $3^{-x} = \frac{1}{3^x} = \frac{1}{27}$
$\frac{1}{3^x} = \frac{1}{3^3}$
$x = 3$

**PS3.** $x^4 = 625$
$x^4 = 5^4$
$x = 5$

**PS4.** $f(x) = \frac{2x}{x+3}$
$x = \frac{2y}{y+3}$
$xy + 3x = 2y$
$3x = 2y - xy = y(2-x)$
$\frac{3x}{2-x} = y$
$f^{-1}(x) = \frac{3x}{2-x}$

**PS5.** $g(x) = \sqrt{x-2}$
$x - 2 \ge 0$
$x \ge 2$
The domain is $\{x \mid x \ge 2\}$.

**PS6.** The domain is the set of all positive real numbers.

## Section 4.3

**1.** $1 = \log 10 \Rightarrow 10^1 = 10$

**2.** $4 = \log 10{,}000 \Rightarrow 10^4 = 10{,}000$

**3.** $2 = \log_8 64 \Rightarrow 8^2 = 64$

**4.** $3 = \log_4 64 \Rightarrow 4^3 = 64$

**5.** $0 = \log_7 x \Rightarrow 7^0 = x$

**6.** $-4 = \log_3 \frac{1}{81} \Rightarrow 3^{-4} = \frac{1}{81}$

**7.** $\ln x = 4 \Rightarrow e^4 = x$

**8.** $\ln e^2 = 2 \Rightarrow e^2 = e^2$

**9.** $\ln 1 = 0 \Rightarrow e^0 = 1$

**10.** $\ln x = -3 \Rightarrow e^{-3} = x$

**11.** $2 = \log(3x+1) \Rightarrow 10^2 = 3x + 1$

**12.** $\frac{1}{3} = \ln\left(\frac{x+1}{x^2}\right) \Rightarrow e^{1/3} = \frac{x+1}{x^2}$

**13.** $3^2 = 9 \Rightarrow \log_3 9 = 2$

**14.** $5^3 = 125 \Rightarrow \log_5 125 = 3$

**15.** $4^{-2} = \frac{1}{16} \Rightarrow \log_4 \frac{1}{16} = -2$

**16.** $10^0 = 1 \Rightarrow \log 1 = 0$

**17.** $b^x = y \Rightarrow \log_b y = x$

**18.** $2^x = y \Rightarrow \log_2 y = x$

**19.** $y = e^x \Rightarrow \ln y = x$

**20.** $5^1 = 5 \Rightarrow \log_5 5 = 1$

**21.** $100 = 10^2 \Rightarrow \log 100 = 2$

**22.** $2^{-4} = \frac{1}{16} \Rightarrow \log_2 \frac{1}{16} = -4$

**23.** $e^2 = x + 5 \Rightarrow 2 = \ln(x+5)$

**24.** $3^x = 47 \Rightarrow \log_3 47 = x$

**25.** $\log_4 16 = 2$ because $4^2 = 16$

**26.** $\log_{3/2} \frac{8}{27} = -3$ because $\left(\frac{3}{2}\right)^{-3} = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$

**27.** $\log_3 \frac{1}{243} = -5$ because $3^{-5} = \left(\frac{1}{3}\right)^5 = \frac{1}{243}$

**28.** $\log_b 1 = 0$ because $\mathrm{b}^0 = 1$

**29.** $\ln e^3 = 3$ because $\mathrm{e}^3 = e^3$

**30.** $\log_b b = 1$ because $b^1 = b$

**31.** $\log \frac{1}{100} = -2$ because $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$

**32.** $\log 1{,}000{,}000 = 6$ because $10^6 = 1{,}000{,}000$

**33.** $\log_{0.5} 16 = \log_{1/2} 16 = -4$ because $\left(\frac{1}{2}\right)^{-4} = 2^4 = 16$

**34.** $\log_{0.3} \frac{100}{9} = \log_{3/10} \frac{100}{9} = -2$ because $\left(\frac{3}{10}\right)^{-2} = \left(\frac{10}{3}\right)^2 = \frac{100}{9}$

**35.** $4 \log 1000 = 12 \Rightarrow \log 1000^4 = 12$

because $10^{12} = (10^3)^4 = (1000)^4$

**36.** $\log_5 125^2 = 6$ because $5^6 = 125$

**37.** $2 \log_7 2401 = 8 \Rightarrow \log_7 2401^2 = 8$

because $7^8 = (7^4)^2 = (2401)^2$

**38.** $3 \log_{11} 161{,}051 = 15 \Rightarrow \log_{11} 161{,}051^3 = 15$

because $11^{15} = (11^5)^3 = (161{,}051)^3$

**39.** $\log_3 \sqrt[5]{9} = \frac{2}{5} \Rightarrow \log_3 9^{1/5} = \frac{2}{5}$

because $3^{2/5} = (3^2)^{1/5} = (9)^{1/5}$

**40.** $\log_6 \sqrt[3]{36} = \frac{2}{3} \Rightarrow \log_6 36^{1/3} = \frac{2}{3}$

because $6^{2/3} = (6^2)^{1/3} = (36)^{1/3}$

**41.** $5 \log_{13} \sqrt[3]{169} = \frac{10}{3} \Rightarrow \log_{13} 169^{5/3} = \frac{10}{3}$

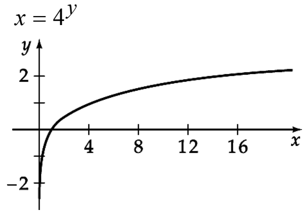
because $13^{10/3} = (13^2)^{5/3} = (169)^{5/3}$

**42.** $2 \log_7 \sqrt[7]{343} = \frac{6}{7} \Rightarrow \log_7 343^{2/7} = \frac{6}{7}$

because $7^{6/7} = (7^3)^{2/7} = (343)^{2/7}$

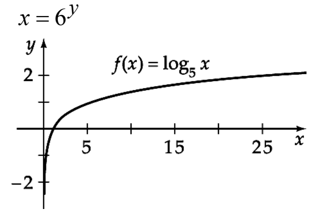
**43.** $y = \log_4 x$

$x = 4^y$



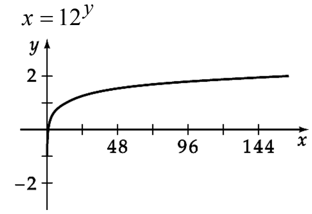

**44.** $y = \log_6 x$

$x = 6^y$



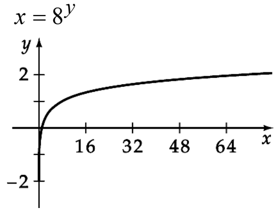

**45.** $y = \log_{12} x$

$x = 12^y$



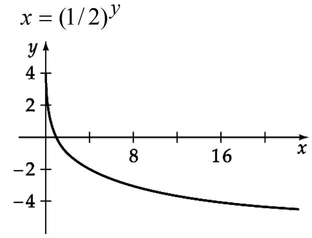

**46.** $y = \log_8 x$

$x = 8^y$



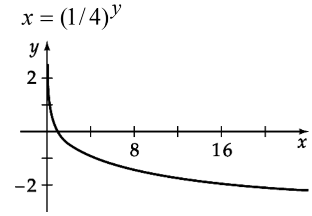

**47.** $y = \log_{1/2} x$

$x = (1/2)^y$



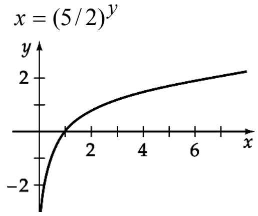

**48.** $y = \log_{1/4} x$

$x = (1/4)^y$



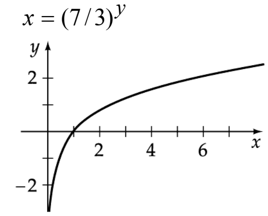

**49.** $y = \log_{5/2} x$

$x = (5/2)^y$




**50.** $y = \log_{7/3} x$

$x = (7/3)^y$



**51.** $f(x) = \log_5(x-3)$

$x - 3 > 0$

$x > 3$

The domain is $(3, \infty)$.

**52.** $k(x) = \log_4(5-x)$

$5 - x > 0$

$-x > -5$

$x < 5$

The domain is $(-\infty, 5)$.

**53.** $k(x) = \log_{2/3}(11-x)$

$11 - x > 0$

$-x > -11$

$x < 11$

The domain is $(-\infty, 11)$.

**54.** $H(x) = \log_{1/4}(x^2+1)$

$x^2 + 1 > 0$

$x^2 > -1$

True for all real numbers.

The domain is $(-\infty, \infty)$.

**55.** $P(x) = \ln(x^2-4)$

$x^2 - 4 > 0$

$(x+2)(x-2) > 0$

The critical values are –2 and 2.

The product is positive.

The domain is $(-\infty, -2) \cup (2, \infty)$.

**56.** $J(x) = \ln\left(\frac{x-3}{x}\right)$

$\frac{x-3}{x} > 0$

The critical values are 3 and 0.

The quotient is positive.

$x < 0$ or $x > 3$

The domain is $(-\infty, 0) \cup (3, \infty)$.

**57.** $h(x) = \ln\left(\frac{x^2}{x-4}\right)$

$\frac{x^2}{x-4} > 0$

The critical values are 0 and 4.

The quotient is positive.

$x > 4$

The domain is $(4, \infty)$.

**58.** $x^4 - x^2 > 0$

$x^2(x^2-1) > 0$

$x^2(x+1)(x-1) > 0$

Critical values are 0, −1 and 1

Product is positive.

$x < -1$ or $x > 1$

$(-\infty, -1) \cup (1, \infty)$

**59.** $x^3 - x > 0$

$x(x^2-1) > 0$

$x(x+1)(x-1) > 0$

Critical values are 0, −1 and 1.

Product is positive.

$-1 < x < 0$ or $x > 1$

$(-1, 0) \cup (1, \infty)$

**60.** $(x^2 + 7x + 10) > 0$

$(x+5)(x+2) > 0$

Critical values are –5 and –2.

Product is positive.

$(-\infty, -5) \cup (-2, \infty)$

**61.** $2x - 11 > 0$

$2x > 11$

$x > \frac{11}{2}$

The domain is $\left(\frac{11}{2}, \infty\right)$.

**62.** $4x - 8 = 0$

$4x = 8$

$x = 2$

The domain is $(-\infty, 2) \cup (2, \infty)$

**63.** $3x - 7 > 0$

$3x > 7$

$x > \frac{7}{3}$

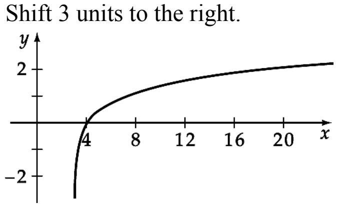
The domain is $\left(\frac{7}{3}, \infty\right)$.

**64.** $x - 4 = 0$

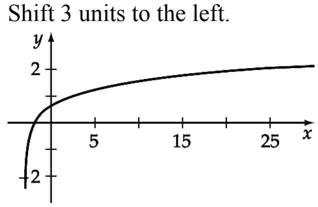
$x = 4$

The domain is $(-\infty, 4) \cup (4, \infty)$

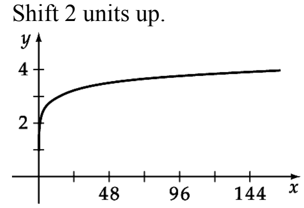
**65.** Shift 3 units to the right.



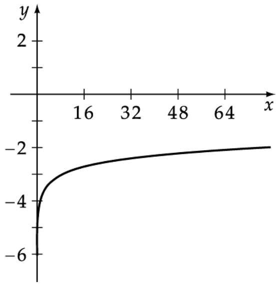

**66.** Shift 3 units to the left.



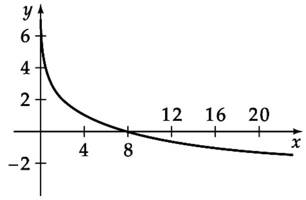

**67.** Shift 2 units up.



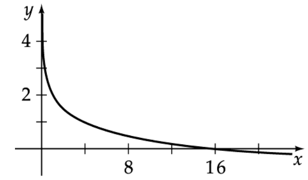
**68.** Shift 4 units down.



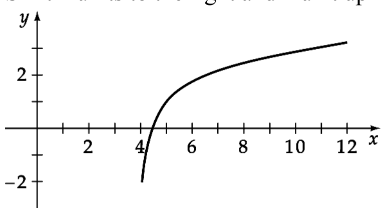

**69.** Shift 3 units up.



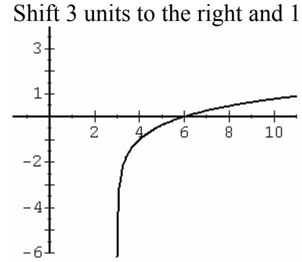

**70.** Shift 2 units up.




**71.** Shift 4 units to the right and 1 unit up




**72.** Shift 3 units to the right and 1 unit down



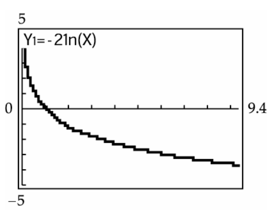

**73.** The graph of $f(x) = \log_5(x-2)$ is the graph of $y = \log_5 x$ shifted 2 units to the right.
The graph of $g(x) = 2 + \log_5 x$ is the graph of $y = \log_5 x$ shifted 2 units up.
The graph of $h(x) = \log_5(-x)$ is the graph of $y = \log_5 x$ reflected across the $y$-axis.
The graph of $k(x) = -\log_5(x+3)$ is the graph of $y = \log_5 x$ reflected across the $x$-axis and shifted left 3 units.

**a.** $k(x)$ **b.** $f(x)$ **c.** $g(x)$ **d.** $h(x)$

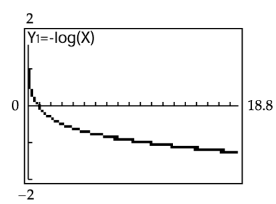
**74.** The graph of $f(x) = \ln x + 3$ is the graph of $y = \ln x$ shifted 3 units up.
The graph of $g(x) = \ln(x-3)$ is the graph of $y = \ln x$ shifted 3 units to the right.
The graph of $h(x) = \ln(3-x)$ is the graph of $y = \ln x$ shifted 3 units to the left and reflected across the $y$-axis.
The graph of $k(x) = -\log_5(x+3)$ is the graph of $y = \log_5 x$ reflected across the $x$-axis and shifted left 3 units.

**a.** $k(x)$ **b.** $h(x)$ **c.** $g(x)$ **d.** $f(x)$

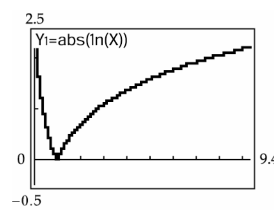
**75.**



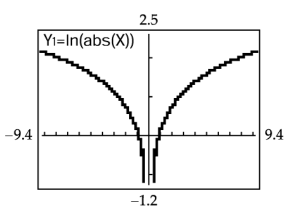

**76.**



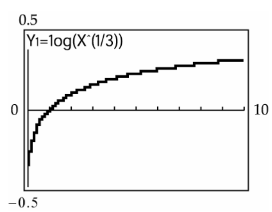

**77.**



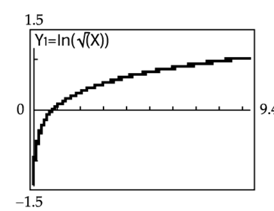

**78.**



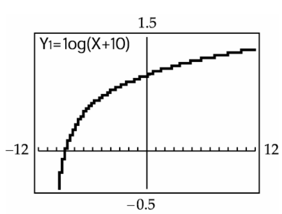

**79.**



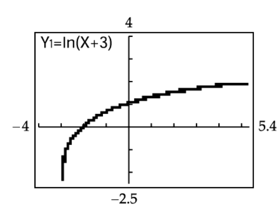

**80.**

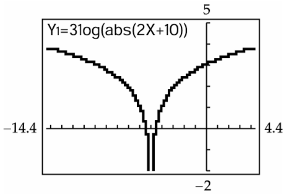


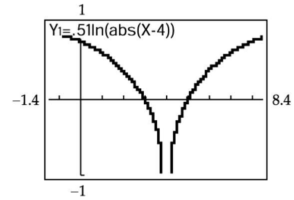

**81.**



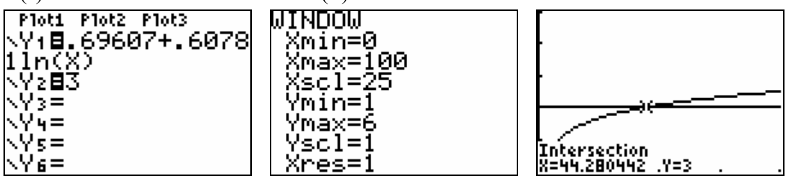

**82.**



**83.** 


**84.** 


**85.** **a.** $r(t) = 0.69607 + 0.60781\ln t \Rightarrow r(9) = 0.69607 + 0.60781\ln 9 \approx 2.0\%$

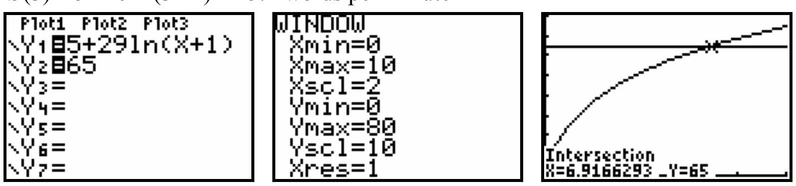
**b.** 


45 months

**86.** **a.** $S(t) = 5 + 29\ln(t+1)$

$S(0) = 5 + 29\ln(0+1) = 5$ words per minute

$S(3) = 5 + 29\ln(3+1) \approx 45.2$ words per minute

**b.** 


6.9 months

**87.** $N(x) = 2750 + 180\ln\left(\dfrac{x}{1000} + 1\right)$

**a.** $N(20{,}000) = 2750 + 180\ln\left(\dfrac{20{,}000}{1000} + 1\right) = 2750 + 180\ln(21) \approx 3298$ units

$N(40{,}000) = 2750 + 180\ln\left(\dfrac{40{,}000}{1000} + 1\right) = 2750 + 180\ln(41) \approx 3418$ units

$N(60{,}000) = 2750 + 180\ln\left(\dfrac{60{,}000}{1000} + 1\right) = 2750 + 180\ln(61) \approx 3490$ units

**b.** $N(0) = 2750 + 180\ln\left(\dfrac{0}{1000} + 1\right) = 2750 + 180\ln(1) = 2750 + 180(0) = 2750 + 0 = 2750$ units

**88.** $BSA = 0.0003207 \cdot H^{0.3} \cdot W^{(0.7285 - 0.0188\log W)}$

$BSA = 0.0003207 \cdot (162.56)^{0.3} \cdot (49{,}895.2)^{(0.7285 - 0.0188\log 49{,}895.2)} \approx 1.50$ square meters

**89.** $BSA = 0.0003207 \cdot H^{0.3} \cdot W^{(0.7285 - 0.0188\log W)}$

$BSA = 0.0003207 \cdot (185.42)^{0.3} \cdot (81{,}646.6)^{(0.7285 - 0.0188\log 81{,}646.6)} \approx 2.05$ square meters

**90.** $M(x) = -2.51\log x + 1,\ 0 < x \le 1$

**a.** $M\left(\dfrac{1}{10}\right) = -2.51\log\left(\dfrac{1}{10}\right) + 1 = 3.51$

**b.** $M\left(\dfrac{1}{400}\right) = -2.51\log\left(\dfrac{1}{400}\right) + 1 \approx 7.53$

**c.** The star with an apparent magnitude of 12 is brighter than a star with an apparent magnitude of 15.

**d.** $M(x)$ is a decreasing function.

**91.** $N = \text{int}(x \log b) + 1$

**a.** $N = \text{int}(10 \log 2) + 1 = 3 + 1 = 4$ digits

**b.** $N = \text{int}(200 \log 3) + 1 = 95 + 1 = 96$ digits

**c.** $N = \text{int}(4005 \log 7) + 1 = 3384 + 1 = 3385$ digits

**d.** $N = \text{int}(2{,}0996{,}001 \log 2) + 1 = 6{,}320{,}429 + 1 = 6{,}320{,}430$ digits

**92.** **a.** $9^9 = 387{,}420{,}489$

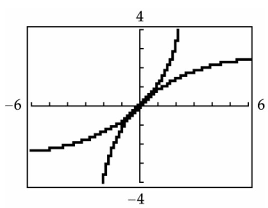
$9^{(9^9)} = 9^{387{,}420{,}489}$

$N = \text{int}(387{,}420{,}489 \log 9) + 1 = 369{,}693{,}100$ digits

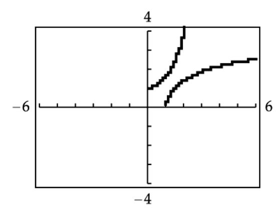
**b.** $369{,}693{,}100 \text{ digits} \times \dfrac{1 \text{ page}}{1000 \text{ digits}} \times \dfrac{1 \text{ ream}}{500 \text{ pages}} \approx 739.4$ reams

## Connecting Concepts

**93.** $f(x)$ and $g(x)$ are inverse functions

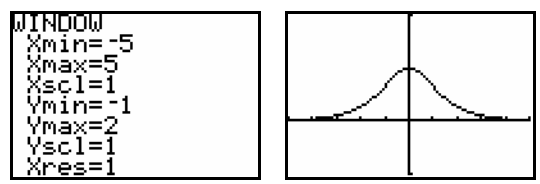



**94.** $f(x)$ and $g(x)$ are inverse functions




**95.** The domain of the inverse is the range of the function.
Range of $f$: $\{y \mid -1 < y \le 1\}$.
The domain of the function is the range of the inverse.
Range of $g$: all real numbers.

**96.** Domain: all real numbers; range $\{y \mid 0 < y \le 1\}$.




## Prepare for Section 4.4

**PS1.** $\log 3 + \log 2 \approx 0.77815$
$\log 6 \approx 0.77815$

**PS2.** $\ln 8 - \ln 3 \approx 0.98083$
$\ln\left(\dfrac{8}{3}\right) \approx 0.98083$

**PS3.** $3 \log 4 \approx 1.80618$
$\log(4^3) \approx 1.80618$

**PS4.** $2 \ln 5 \approx 3.21888$
$\ln(5^2) \approx 3.21888$

**PS5.** $\ln 5 \approx 1.60944$
$\dfrac{\log 5}{\log e} \approx 1.60944$

**PS6.** $\log 8 \approx 0.90309$
$\dfrac{\ln 8}{\ln 10} \approx 0.90309$

## Section 4.4

**1.** $\log_b(xyz) = \log_b x + \log_b y + \log_b z$

**2.** $\ln \dfrac{z^3}{\sqrt{xy}} = \ln z^3 - \ln\sqrt{x} - \ln\sqrt{y}$

$= 3\ln z - \dfrac{1}{2}\ln x - \dfrac{1}{2}\ln y$

**3.** $\ln \dfrac{x}{z^4} = \ln x - \ln z^4$

$= \ln x - 4\ln z$

**4.** $\log_5 \dfrac{xy^2}{z^4} = \log_5 x + \log_5 y^2 - \log_5 z^4$

$= \log_5 x + 2\log_5 y - 4\log_5 z$

**5.** $\log_2 \dfrac{\sqrt{x}}{y^3} = \log_2 \sqrt{x} - \log_2 y^3$

$= \log_2 x^{1/2} - \log_2 y^3$

$= \dfrac{1}{2}\log_2 x - 3\log_2 y$

**6.** $\log_b\left(x\sqrt[3]{y}\right) = \log_b x + \log_b y^{1/3}$

$= \log_b x + \dfrac{1}{3}\log_b y$

**7.** $\log_7 \dfrac{\sqrt{xz}}{y^2} = \log_7 \dfrac{(xz)^{1/2}}{y^2} = \log_7 \dfrac{x^{1/2}z^{1/2}}{y^2} = \log_7 x^{1/2} + \log_7 z^{1/2} - \log_7 y^2 = \dfrac{1}{2}\log_7 x + \dfrac{1}{2}\log_7 z - 2\log_7 y$

**8.** $\ln\sqrt[3]{x^2\sqrt{y}} = \ln\left(x^2y^{1/2}\right)^{1/3} = \ln\left(x^{2/3}y^{1/6}\right) = \ln x^{2/3} + \ln y^{1/6} = \dfrac{2}{3}\ln x + \dfrac{1}{6}\ln y$

**9.** $\ln\left(e^2z\right) = \ln e^2 + \ln z = 2\ln e + \ln z = 2 + \ln z$

**10.** $\ln\left(x^{1/2}y^{2/3}\right) = \ln x^{1/2} + \ln y^{2/3} = \dfrac{1}{2}\ln x + \dfrac{2}{3}\ln y$

**11.** $\log_4\left(\dfrac{\sqrt[3]{z}}{16y^3}\right) = \log_4 z^{1/3} - \log_4 4^2 - \log_4 y^3 = \dfrac{1}{3}\log_4 z - 2\log_4 4 - 3\log_4 y = \dfrac{1}{3}\log_4 z - 2 - 3\log_4 y$

**12.** $\log_5\left(\dfrac{\sqrt{x}z^4}{125}\right) = \log_5 x^{1/2} + \log_5 z^4 - \log_5 5^3 = \dfrac{1}{2}\log_5 x + 4\log_5 z - 3\log_5 5 = \dfrac{1}{2}\log_5 x + 4\log_5 z - 3$

**13.** $\log\sqrt{x\sqrt{z}} = \log\left(xz^{1/2}\right)^{1/2} = \log x^{1/2}z^{1/4} = \dfrac{1}{2}\log x + \dfrac{1}{4}\log z$

**14.** $\ln\left(\dfrac{\sqrt[3]{x^2}}{z^2}\right) = \ln x^{2/3} - \ln z^2 = \dfrac{2}{3}\ln x - 2\ln z$

**15.** $\ln\left(\sqrt[3]{z\sqrt{e}}\right) = \ln\left(ze^{1/2}\right)^{1/3} = \ln z^{1/3}e^{1/6} = \ln z^{1/3} + \ln e^{1/6} = \dfrac{1}{3}\ln z + \dfrac{1}{6}\ln e = \dfrac{1}{3}\ln z + \dfrac{1}{6}$

**16.** $\ln\left(\dfrac{x^2\sqrt{z}}{y^{-3}}\right) = \ln x^2 + \ln z^{1/2} - \ln y^{-3} = 2\ln x + \dfrac{1}{2}\ln z + 3\ln y$

**17.** $\log(x+5) + 2\log x = \log(x+5) + \log x^2 = \log[x^2(x+5)]$

**18.** $3\log_2 t - \frac{1}{3}\log_2 u + 4\log_2 v = \log_2 t^3 - \log_2 u^{1/3} + \log_2 v^4 = \log_2 t^3 + \log_2 \sqrt[3]{u} + \log_2 v^4 = \log_2 \frac{t^3 v^4}{\sqrt[3]{u}}$

**19.** $\ln(x^2 - y^2) - \ln(x - y) = \ln\frac{x^2 - y^2}{x - y} = \ln\frac{(x+y)(x-y)}{x-y} = \ln(x+y)$

**20.** $\frac{1}{2}\log_8(x+5) - 3\log_8 y = \log_8(x+5)^{1/2} - \log_8 y^3 = \log_8\sqrt{x+5} - \log_8 y^3 = \log_8\frac{\sqrt{x+5}}{y^3}$

**21.** $3\log x + \frac{1}{3}\log y + \log(x+1) = \log x^3 + \log y^{1/3} + \log(x+1) = \log x^3 + \log\sqrt[3]{y} + \log(x+1) = \log\left[x^3 \cdot \sqrt[3]{y}(x+1)\right]$

**22.** $\ln(xz) - \ln(x\sqrt{y}) + 2\ln\frac{y}{z} = \ln(xz) - \ln(x\sqrt{y}) + \ln\left(\frac{y}{z}\right)^2 = \ln\left(\frac{xz}{(x\sqrt{y})}\cdot\frac{y^2}{z^2}\right) = \ln\frac{y^2}{z\sqrt{y}} = \ln\frac{y^{2-1/2}}{z} = \ln\frac{y^{3/2}}{z}$

**23.** $\log\left(xy^2\right) - \log z = \log\left(\frac{xy^2}{z}\right)$

**24.** $\ln\left(y^{1/2}z\right) - \ln z^{1/2} = \ln\left(\frac{y^{1/2}z}{z^{1/2}}\right) = \ln\left(y^{1/2}z^{1/2}\right)$ or $\ln\sqrt{yz}$

**25.** $2\left(\log_6 x + \log_6 y^2\right) - \log_6(x+2) = \log_6 x^2 + \log_6 y^4 - \log_6(x+2) = \log_6\left(\frac{x^2y^4}{x+2}\right)$

**26.** $\frac{1}{2}\log_3 x - \log_3 y + 2\log_3(x+2) = \log_3 x^{1/2} - \log_3 y + \log_3(x+2)^2 = \log_3\left[\frac{\sqrt{x}(x+2)^2}{y}\right]$

**27.** $2\ln(x+4) - \ln x - \ln\left(x^2-3\right) = \ln(x+4)^4 - \ln x - \ln\left(x^2-3\right) = \ln\left[\frac{(x+4)^4}{x\left(x^2-3\right)}\right]$

**28.** $\log(3x) - (2\log x - \log y) = \log(3x) - \log x^2 + \log y = \log\left(\frac{3xy}{x^2}\right) = \log\left(\frac{3y}{x}\right)$

**29.** $\ln(2x+5) - \ln y - 2\ln z + \frac{1}{2}\ln w = \ln(2x+5) - \ln y - \ln z^2 + \ln w^{1/2} = \ln\left[\frac{(2x+5)\sqrt{w}}{yz^2}\right]$

**30.** $\log_b x + \log_b(y+3) + \log_b(y+2) - \log_b(y^2+5y+6) = \log_b\left[\frac{x(y+3)(y+2)}{(y+2)(y+3)}\right] = \log_b x$

**31.** $\ln\left(x^2-9\right) - 2\ln(x-3) + 3\ln y = \ln(x+3)(x-3) - \ln(x-3)^2 + \ln y^3 = \ln\left[\frac{(x+3)(x-3)y^3}{(x-3)^2}\right] = \ln\left[\frac{(x+3)y^3}{x-3}\right]$

**32.** $\log_b\left(x^2+7x+12\right) - 2\log_b(x+4) = \log_b(x+3)(x+4) - \log_b(x+4)^2 = \log_b\left[\frac{(x+3)(x+4)}{(x+4)^2}\right] = \log_b\left(\frac{x+3}{x+4}\right)$

**33.** $\log_7 20 = \frac{\log 20}{\log 7} \approx 1.5395$

**34.** $\log_5 37 = \frac{\log 37}{\log 5} \approx 2.2436$

**35.** $\log_{11} 8 = \frac{\log 8}{\log 11} \approx 0.8672$

**36.** $\log_{50} 22 = \dfrac{\log 22}{\log 50} \approx 0.7901$

**37.** $\log_6 \dfrac{1}{3} = \dfrac{\log \frac{1}{3}}{\log 6} \approx -0.6131$

**38.** $\log_3 \dfrac{7}{8} = \dfrac{\log \frac{7}{8}}{\log 3} \approx -0.1215$

**39.** $\log_9 \sqrt{17} = \dfrac{\log \sqrt{17}}{\log 9} \approx 0.6447$

**40.** $\log_4 \sqrt{7} = \dfrac{\log \sqrt{7}}{\log 4} \approx 0.7018$

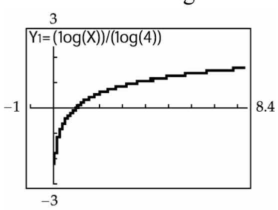
**41.** $\log_{\sqrt{2}} 17 = \dfrac{\log 17}{\log \sqrt{2}} \approx 8.1749$

**42.** $\log_{\sqrt{3}} 5.5 = \dfrac{\log 5.5}{\log \sqrt{3}} \approx 3.1035$

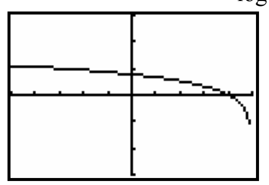
**43.** $\log_{\pi} e = \dfrac{\log e}{\log \pi} \approx 0.8735$

**44.** $\log_{\pi} \sqrt{15} = \dfrac{\log \sqrt{15}}{\log \pi} \approx 1.1828$

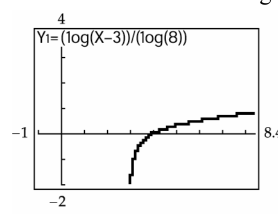
**45.** $f(x) = \log_4 x = \dfrac{\log x}{\log 4}$



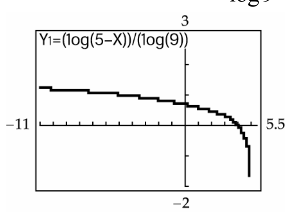

**46.** $g(x) = \log_8 (5-x) = \dfrac{\log(5-x)}{\log 8}$



**47.** $g(x) = \log_8 (x-3) = \dfrac{\log(x-3)}{\log 8}$



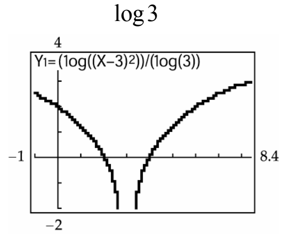

**48.** $t(x) = \log_9 (5-x) = \dfrac{\log(5-x)}{\log 9}$



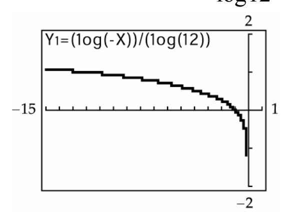

**49.** $h(x) = \log_3 (x-3)^2$

$= \dfrac{\log(x-3)^2}{\log 3}$

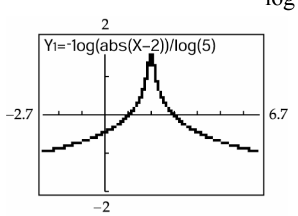
$= \dfrac{2\log(x-3)}{\log 3}$



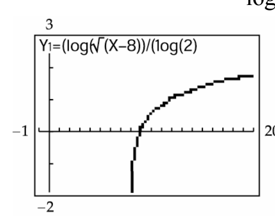

**50.** $J(x) = \log_{12} (-x) = \dfrac{\log(-x)}{\log 12}$




**51.** $F(x) = -\log_5 |x-2| = -\dfrac{\log|x-2|}{\log 5}$




**52.** $n(x) = \log_2 \sqrt{x-8} = \dfrac{\log \sqrt{x-8}}{\log 2}$




**53.** False. $\log 10 + \log 10 = 1 + 1 = 2$
but $\log(10+10) = \log 20 \neq 2$ .

**54.** False. $\log(10 \cdot 10) = \log(10^2) = 2$
but $\log 10 \cdot \log 10 = 1 \cdot 1 = 1$

**55.** True.

**56.** False. $\log 10 \cdot \log 10 = 1 \cdot 1 = 1$
but $\log 10 + \log 10 = 1 + 1 = 2$

**57.** False. $\log 100 - \log 10 = 2 - 1 = 1$
but $\log(100 - 10) = \log 90 \neq 1$

**58.** False. $\log \dfrac{100}{10} = \log 10 = 1$
but $\dfrac{\log 100}{\log 10} = \dfrac{2}{1} = 2$

**59.** False. $\dfrac{\log 100}{\log 10} = \dfrac{2}{1} = 2$
but $\log 100 - \log 10 = 2 - 1$
$= 1$

**60.** True.

**61.** False. $(\log 10)^2 = 1^2 = 1$ but $2\log 10 = 2(1) = 2$

**62.** True.

**63.** $\log_3 5 \cdot \log_5 7 \cdot \log_7 9 = \frac{\log 5}{\log 3} \cdot \frac{\log 7}{\log 5} \cdot \frac{\log 9}{\log 7} = \frac{\cancel{\log 5}}{\log 3} \cdot \frac{\cancel{\log 7}}{\cancel{\log 5}} \cdot \frac{\log 9}{\cancel{\log 7}} = \frac{\log 9}{\log 3} = \frac{\log 3^2}{\log 3} = \frac{2\log 3}{\log 3} = \frac{2\cancel{\log 3}}{\cancel{\log 3}} = 2$

**64.** 
$$\log_5 20 \cdot \log_{20} 60 \cdot \log_{60} 100 \cdot \log_{100} 125 = \frac{\log 20}{\log 5} \cdot \frac{\log 60}{\log 20} \cdot \frac{\log 100}{\log 60} \cdot \frac{\log 125}{\log 100}$$
$$= \frac{\cancel{\log 20}}{\log 5} \cdot \frac{\cancel{\log 60}}{\cancel{\log 20}} \cdot \frac{\cancel{\log 100}}{\cancel{\log 60}} \cdot \frac{\log 125}{\cancel{\log 100}} = \frac{\log 125}{\log 5} = \frac{\log 5^3}{\log 5}$$
$$= \frac{3\log 5}{\log 5} = \frac{3\cancel{\log 5}}{\cancel{\log 5}} = 3$$

**65.** $\ln 500^{501} = 501\ln 500 \approx 3113.52$

$\ln 506^{500} = 500\ln 506 \approx 3113.27$

$\ln 500^{501}$ is larger.

**66.** $\ln\left(\frac{1}{50^{300}}\right) = \ln 50^{-300} = -300\ln 50 \approx -1174$

$\ln\left(\frac{1}{151^{233}}\right) = \ln 151^{-233} = -233\ln 151 \approx -1169$

$\frac{1}{50^{300}}$ is smaller.

**67.** $S_n = S_0 \cdot 10^{\frac{n}{N}(\log S_f - \log S_0)}$

$S_1 = 1{,}000{,}000 \cdot 10^{\frac{1}{5}(\log 500{,}000 - \log 1{,}000{,}000)}$
$= 870{,}551$

$S_2 = 1{,}000{,}000 \cdot 10^{\frac{2}{5}(\log 500{,}000 - \log 1{,}000{,}000)}$
$= 757{,}858$

$S_3 = 1{,}000{,}000 \cdot 10^{\frac{3}{5}(\log 500{,}000 - \log 1{,}000{,}000)}$
$= 659{,}754$

$S_4 = 1{,}000{,}000 \cdot 10^{\frac{4}{5}(\log 500{,}000 - \log 1{,}000{,}000)}$
$= 574{,}349$

$S_5 = 1{,}000{,}000 \cdot 10^{\frac{5}{5}(\log 500{,}000 - \log 1{,}000{,}000)}$
$= 500{,}000$

The scales are 1:870,551; 1:757,858; 1:659,754; 1:574,349; 1:500,000.

**68.** $S_n = S_0 \cdot 10^{\frac{n}{N}(\log S_f - \log S_0)}$

$S_1 = 250{,}000 \cdot 10^{\frac{1}{4}(\log 100{,}000 - \log 250{,}000)}$
$= 198{,}818$

$S_2 = 250{,}000 \cdot 10^{\frac{2}{4}(\log 100{,}000 - \log 250{,}000)}$
$= 158{,}114$

$S_3 = 250{,}000 \cdot 10^{\frac{3}{4}(\log 100{,}000 - \log 250{,}000)}$
$= 125{,}743$

$S_4 = 250{,}000 \cdot 10^{\frac{4}{4}(\log 100{,}000 - \log 250{,}000)}$
$= 100{,}000$

The scales are 1:198,818; 1:158,114; 1:125,743; 1:100,000.

**69.** $\text{pH} = -\log[\text{H}^+]$

$\text{pH} = -\log[3.97 \times 10^{-11}]$

$\text{pH} = 10.4$

10.4 > 7; milk of magnesia is a base

**70.** $\text{pH} = -\log[\text{H}^+]$

$\text{pH} = -\log[1.26 \times 10^{-3}]$

$\text{pH} = 2.9$

2.9 < 7; vinegar is an acid.

**71.** $\text{pH} = -\log[\text{H}^+]$

$9.5 = -\log[\text{H}^+]$

$-9.5 = \log[\text{H}^+]$

$10^{-9.5} = 10^{\log[\text{H}^+]}$

$[\text{H}^+] = 3.16 \times 10^{-10}$ mole per liter

**72.** $\text{pH} = -\log[\text{H}^+]$

$5.6 = -\log[\text{H}^+]$

$\log[\text{H}^+] = -5.6$

$10^{\log[\text{H}^+]} = 10^{-5.6}$

$[\text{H}^+] = 2.51 \times 10^{-6}$ mole per liter

**73.** $dB(I) = 10\log\left(\frac{I}{I_0}\right)$

**a.** $dB(1.58\times10^8\cdot I_0) = 10\log\left(\frac{1.58\times10^8\cdot I_0}{I_0}\right)$
$= 10\log(1.58\times10^8)$
$\approx 82.0$ decibels

**b.** $dB(10{,}800\cdot I_0) = 10\log\left(\frac{10{,}800\cdot I_0}{I_0}\right)$
$= 10\log(10{,}800)$
$\approx 40.3$ decibels

**c.** $dB(3.16\times10^{11}\cdot I_0) = 10\log\left(\frac{3.16\times10^{11}\cdot I_0}{I_0}\right)$
$= 10\log(3.16\times10^{11})$
$\approx 115.0$ decibels

**d.** $dB(1.58\times10^{15}\cdot I_0) = 10\log\left(\frac{1.58\times10^{15}\cdot I_0}{I_0}\right)$
$= 10\log(1.58\times10^{15})$
$\approx 152.0$ decibels

**74.** $dB(I) = 10\log\left(\frac{I}{I_0}\right)$

$$175 = 10\log\left(\frac{I_{\text{Bronco}}}{I_0}\right)$$
$$17.5 = \log\left(\frac{I_{\text{Bronco}}}{I_0}\right)$$
$$10^{17.5} = \frac{I_{\text{Bronco}}}{I_0}$$
$$10^{17.5}\cdot I_0 = I_{\text{Bronco}}$$

$$125 = 10\log\left(\frac{I_{\text{pain}}}{I_0}\right)$$
$$12.5 = \log\left(\frac{I_{\text{pain}}}{I_0}\right)$$
$$10^{12.5} = \frac{I_{\text{pain}}}{I_0}$$
$$10^{12.5}\cdot I_0 = I_{\text{pain}}$$

$$\frac{I_{\text{Bronco}}}{I_{\text{pain}}} = \frac{10^{17.5}\cdot I_0}{10^{12.5}\cdot I_0} = \frac{10^{17.5}}{10^{12.5}} = 10^{17.5-12.5} = 10^5$$
= 100,000 times more intense

**75.** $dB(I) = 10\log\left(\frac{I}{I_0}\right)$

$$120 = 10\log\left(\frac{I_{120}}{I_0}\right)$$
$$12 = \log\left(\frac{I_{120}}{I_0}\right)$$
$$10^{12} = \frac{I_{120}}{I_0}$$
$$10^{12}\cdot I_0 = I_{120}$$

$$110 = 10\log\left(\frac{I_{110}}{I_0}\right)$$
$$11 = \log\left(\frac{I_{110}}{I_0}\right)$$
$$10^{11} = \frac{I_{110}}{I_0}$$
$$10^{11}\cdot I_0 = I_{110}$$

$$\frac{I_{120}}{I_{110}} = \frac{10^{12}\cdot I_0}{10^{11}\cdot I_0} = \frac{10^{12}}{10^{11}} = 10^{12-11} = 10^1$$
= 10 times more intense

**76.** $$dB(2I) - dB(I) = 10\log\left(\frac{2I}{I_0}\right) - 10\log\left(\frac{I}{I_0}\right) = \log\left(\frac{2I}{I_0}\right)^{10} - \log\left(\frac{I}{I_0}\right)^{10} = \log\frac{\left(\frac{2I}{I_0}\right)^{10}}{\left(\frac{I}{I_0}\right)^{10}} = \log\left[\left(\frac{2I}{I_0}\right)^{10}\cdot\left(\frac{I_0}{I}\right)^{10}\right]$$

$$= \log\left[\frac{(2I)^{10}}{(I_0)^{10}}\cdot\frac{(I_0)^{10}}{(I)^{10}}\right] = \log\left[\frac{(2)^{10}\cancel{(I)^{10}}}{\cancel{(I_0)^{10}}}\cdot\frac{\cancel{(I_0)^{10}}}{\cancel{(I)^{10}}}\right] = \log 2^{10} = 10\log 2 \approx 3.0103 \text{ decibels}$$

**77.** $M = \log\left(\frac{100{,}000 I_0}{I_0}\right) = \log 100{,}000 = \log 10^5 = 5$

**78.** $M = \log\left(\frac{I}{I_0}\right)$

$M = \log\left(\frac{398{,}107{,}000 I_0}{I_0}\right)$

$= \log 398{,}107{,}000$

$\approx 8.6$

**79.** $\log\left(\frac{I}{I_0}\right) = M$

$\log\left(\frac{I}{I_0}\right) = 6.5$

$\frac{I}{I_0} = 10^{6.5}$

$I = 10^{6.5} I_0$

$I \approx 3{,}162{,}277.7 I_0$

**80.** $l\,\text{og}\left(\frac{I}{I_0}\right) = M$

$\log\left(\frac{I}{I_0}\right) = 9.5$

$\frac{I}{I_0} = 10^{9.5}$

$I = 10^{9.5} I_0$

$I = 3{,}162{,}277{,}660 I_0$

**81.** $M = \log\left(\frac{I}{I_0}\right)$

$M_5 = \log\left(\frac{I_5}{I_0}\right) \Rightarrow 5 = \log\left(\frac{I_5}{I_0}\right) \Rightarrow 10^5 = \frac{I_5}{I_0} \Rightarrow 10^5 I_0 = I_5$

$M_3 = \log\left(\frac{I_3}{I_0}\right) \Rightarrow 3 = \log\left(\frac{I_3}{I_0}\right) \Rightarrow 10^3 = \frac{I_3}{I_0} \Rightarrow 10^3 I_0 = I_3$

$\frac{I_5}{I_3} = \frac{10^5 I_0}{10^3 I_0} = \frac{10^5}{10^3} = 10^{5-3} = 10^2 = 100$ to 1 • short cut: begin with this line

**82.** $\frac{10^{9.5}}{10^{8.3}} = 10^{9.5-8.3} = 10^{1.2} \approx 15.8$ times more intense

**83.** $\frac{10^{8.9}}{10^{7.1}} = \frac{10^{8.9-7.1}}{1} = 10^{1.8}$ to $1 \approx 63$ to 1

**84.** $\frac{10^{8.2}}{10^{6.9}} = \frac{10^{8.2-6.9}}{1} = 10^{1.3}$ to $1 \approx 20$ to 1

**85.** $M = \log A + 3\log 8t - 2.92$

$= \log 18 + 3\log[8(31)] - 2.92 \approx 5.5$

**86.** $M = \log A + 3\log 8t - 2.92$

$= \log 26 + 3\log[8(17)] - 2.92 \approx 4.9$

**87.** Let $r = \log_b M$ and $s = \log_b N$.

Then $M = b^r$ and $N = b^s$.

Consider the quotient of $M$ and $N$

$\frac{M}{N} = \frac{b^r}{b^s}$

$\frac{M}{N} = b^{r-s}$

$\log_b \frac{M}{N} = r - s$

$\log_b \frac{M}{N} = \log_b M - \log_b N$

**88.** Let $x = \log_b M$.

Then $M = b^x$.

$M = b^x$

$(M)^p = (b^x)^p$

$M^p = b^{xp}$

$\log_b M^p = xp$

$\log_b M^p = p\log_b M$

**89.** **a.** $M \approx 6$ **b.** $M \approx 4$

**c.** When $t = 40$, $M = \log A + 3\log 8t - 2.92 = \log 50 + 3\log[8(40)] - 2.92 \approx 6.3$

When $t = 30$, $M = \log A + 3\log 8t - 2.92 = \log 1 + 3\log[8(30)] - 2.92 \approx 4.2$

The results from parts **a.** and **b.** are close to the magnitudes of 6.3 and 4.2 produced by the amplitude time difference formula.

## Prepare for Section 4.5

**PS1.** $3^6 = 729 \Rightarrow \log_3 729 = 6$

**PS2.** $\log_5 625 = 4 \Rightarrow 5^4 = 625$

**PS3.** $a^{x+2} = b \Rightarrow \log_a b = x + 2$

**PS4.**
$$4a = 7bx + 2cx$$
$$7bx + 2cx = 4a$$
$$x(7b + 2c) = 4a$$
$$x = \frac{4a}{7b + 2c}$$

**PS5.**
$$165 = \frac{300}{1 + 12x}$$
$$165(1 + 12x) = 300$$
$$165 + 1980x = 300$$
$$1980x = 135$$
$$x = \frac{135}{1980} = \frac{3}{44}$$

**PS6.**
$$A = \frac{100 + x}{100 - x}$$
$$A(100 - x) = 100 + x$$
$$100A - Ax = 100 + x$$
$$100A - 100 = Ax + x$$
$$100(A - 1) = x(A + 1)$$
$$x = \frac{100(A - 1)}{A + 1}$$

## Section 4.5

**1.**
$$2^x = 64$$
$$2^x = 2^6$$
$$x = 6$$

**2.**
$$3^x = 243$$
$$3^x = 3^5$$
$$x = 5$$

**3.**
$$49^x = \frac{1}{343}$$
$$7^{2x} = 7^{-3}$$
$$2x = -3$$
$$x = -\frac{3}{2}$$

**4.**
$$9^x = \frac{1}{243}$$
$$3^{2x} = 3^{-5}$$
$$2x = -5$$
$$x = -\frac{5}{2}$$

**5.**
$$2^{5x+3} = \frac{1}{8}$$
$$2^{5x+3} = 2^{-3}$$
$$5x + 3 = -3$$
$$5x = -6$$
$$x = -\frac{6}{5}$$

**6.**
$$3^{4x-7} = \frac{1}{9}$$
$$3^{4x-7} = 3^{-2}$$
$$4x - 7 = -2$$
$$4x = 5$$
$$x = \frac{5}{4}$$

**7.**
$$\left(\frac{2}{5}\right)^x = \frac{8}{125}$$
$$\left(\frac{2}{5}\right)^x = \left(\frac{2}{5}\right)^3$$
$$x = 3$$

**8.**
$$\left(\frac{2}{5}\right)^x = \frac{25}{4}$$
$$\left(\frac{2}{5}\right)^x = \left(\frac{2}{5}\right)^{-2}$$
$$x = -2$$

**9.**
$$5^x = 70$$
$$\log(5^x) = \log 70$$
$$x \log 5 = \log 70$$
$$x = \frac{\log 70}{\log 5}$$

**10.**
$$6^x = 50$$
$$\log(6^x) = \log 50$$
$$x \log 6 = \log 50$$
$$x = \frac{\log 50}{\log 6}$$

**11.**
$$3^{-x} = 120$$
$$\log(3^{-x}) = \log 120$$
$$-x \log 3 = \log 120$$
$$-x = \frac{\log 120}{\log 3}$$
$$x = -\frac{\log 120}{\log 3}$$

**12.**
$$7^{-x} = 63$$
$$\log(7^{-x}) = \log 63$$
$$-x \log 7 = \log 63$$
$$-x = \frac{\log 63}{\log 7}$$
$$x = -\frac{\log 63}{\log 7}$$

**13.**
$$10^{2x+3} = 315$$
$$\log 10^{2x+3} = \log 315$$
$$(2x + 3)\log 10 = \log 315$$
$$2x + 3 = \log 315$$
$$x = \frac{\log 315 - 3}{2}$$

**14.**
$$10^{6-x} = 550$$
$$(6 - x)\log 10 = \log 550$$
$$6 - x = \log 550$$
$$x = 6 - \log 550$$

**15.**
$$e^x = 10$$
$$\ln e^x = \ln 10$$
$$x = \ln 10$$

**16.**

$$\begin{aligned} e^{x+1} &= 20 \\ \ln e^{x+1} &= \ln 20 \\ x+1 &= \ln 20 \\ x &= \ln 20 - 1 \end{aligned}$$

**17.**

$$\begin{aligned} 2^{1-x} &= 3^{x+1} \\ \log 2^{1-x} &= \log 3^{x+1} \\ (1-x)\log 2 &= (x+1)\log 3 \\ \log 2 - x\log 2 &= x\log 3 + \log 3 \\ \log 2 - x\log 2 - x\log 3 &= \log 3 \\ -x\log 2 - x\log 3 &= \log 3 - \log 2 \\ -x(\log 2 + \log 3) &= \log 3 - \log 2 \\ x &= -\frac{(\log 3 - \log 2)}{(\log 2 + \log 3)} \\ x &= \frac{\log 2 - \log 3}{\log 2 + \log 3} \text{ or } \frac{\log 2 - \log 3}{\log 6} \end{aligned}$$

**18.**

$$\begin{aligned} 3^{x-2} &= 4^{2x+1} \\ \log 3^{x-2} &= \log 4^{2x+1} \\ (x-2)\log 3 &= (2x+1)\log 4 \\ x\log 3 - 2\log 3 &= 2x\log 4 + \log 4 \\ x\log 3 - 2\log 3 - 2x\log 4 &= \log 4 \\ x(\log 3 - 2x\log 4 &= \log 4 + 2\log 3 \\ x &= \frac{\log 4 + 2\log 3}{\log 3 - 2\log 4} \end{aligned}$$

**19.**

$$\begin{aligned} 2^{2x-3} &= 5^{-x-1} \\ \log 2^{2x-3} &= \log 5^{-x-1} \\ (2x-3)\log 2 &= (-x-1)\log 5 \\ 2x\log 2 - 3\log 2 &= -x\log 5 - \log 5 \\ 2x\log 2 + x\log 5 - 3\log 2 &= -\log 5 \\ 2x\log 2 + x\log 5 &= 3\log 2 - \log 5 \\ x(2\log 2 + \log 5) &= 3\log 2 - \log 5 \\ x &= \frac{3\log 2 - \log 5}{2\log 2 + \log 5} \end{aligned}$$

**20.**

$$\begin{aligned} 5^{3x} &= 3^{x+4} \\ \log 5^{3x} &= \log 3^{x+4} \\ 3x\log 5 &= (x+4)\log 3 \\ 3x\log 5 &= x\log 3 + 4\log 3 \\ 3x\log 5 - x\log 3 &= 4\log 3 \\ x(3\log 5 - \log 3) &= 4\log 3 \\ x &= \frac{4\log 3}{3\log 5 - \log 3} \end{aligned}$$

**21.**

$$\begin{aligned} \log(4x-18) &= 1 \\ 4x-18 &= 10^1 \\ 4x-18 &= 10 \\ 4x &= 28 \\ x &= 7 \end{aligned}$$

**22.**

$$\begin{aligned} \log(x^2+19) &= 2 \\ x^2+19 &= 10^2 \\ x^2+19 &= 100 \\ x^2 &= 81 \\ x &= \pm\sqrt{81} \\ x &= 9,\ -9 \end{aligned}$$

**23.**

$$\begin{aligned} \ln(x^2-12) &= \ln x \\ x^2-12 &= x \\ x^2-x-12 &= 0 \\ (x-4)(x+3) &= 0 \end{aligned}$$

$x = 4$ or $x = -3$ (No; not in domain.)

$x = 4$

**24.** $\log(2x^2+3x)=\log(10x+30)$

$2x^2+3x=10x+30$

$2x^2-7x-30=0$

$(2x+5)(x-6)=0$

$x=-\frac{5}{2}$ or $x=6$

$-\frac{5}{2},\ 6$

**25.** $\log_2+\log_2(x-4)=2$

$\log_2 x(x-4)=2$

$\log_2(x^2-4x)=2$

$2^2=x^2-4x$

$0=x^2-4x-4$

$x=\dfrac{4\pm\sqrt{16-4(1)(-4)}}{2}$

$x=\dfrac{4\pm4\sqrt{2}^2}{2}$

$x=2\pm2\sqrt{2}$

$2-2\sqrt{2}$ is not a solution because the logarithm of a negative number is not defined. The solution is $x=2+2\sqrt{2}$ .

**26.** $\log_3 x+\log 3(x+6)=3$

$\log_3 x(x+6)=3$

$3^3=x(x+6)$

$27=x^2+6x$

$0=x^2+6x-27$

$0=(x+9)(x-3)$

$x=3$

$x=-9$

$\log_3(-9)$ is not defined. The solution is $x = 3$.

**27.** $\log(5x-1)=2+\log(x-2)$

$\log(5x-1)-\log(x-2)=2$

$\log\dfrac{(5x-1)}{(x-2)}=2$

$10^2=\dfrac{(5x-1)}{(x-2)}$

$100(x-2)=5x-1$

$100x-200=5x-1$

$95x=199$

$x=\dfrac{199}{95}$

**28.** $1+\log(3x-1)=\log(2x+1)$

$1=\log(2x+1)-\log(3x-1)$

$1=\log\dfrac{(2x+1)}{(3x-1)}$

$10=\dfrac{2x+1}{3x-1}$

$10(3x-1)=(2x+1)$

$30x-10=(2x+1)$

$28x=11$

$x=\dfrac{11}{28}$

**29.** $\ln(1-x)+\ln(3-x)=\ln 8$

$\ln[(1-x)(3-x)]=\ln 8$

$(1-x)(3-x)=8$

$3-4x+x^2=8$

$x^2-4x-5=0$

$(x+1)(x-5)=0$

$x=-1$ or $x=5$ (No; not in domain.)

The solution is $x = -1$.

**30.** $\log(4-x)=\log(x+8)+\log(2x+13)$

$\log(4-x)=\log[(x+8)(2x+13)]$

$4-x=(x+8)(2x+13)$

$4-x=2x^2+29x+104$

$0=2x^2+30x+100$

$0=2(x^2+15x+50)$

$0=2(x+5)(x+10)$

$x=-5$ or $x=-10$ (No; not in domain.)

The solution is $x = -5$.

**31.** $\log\sqrt{x^3-17}=\frac{1}{2}$

$\frac{1}{2}\log\left(x^3-17\right)=\frac{1}{2}$

$10^1=x^3-17$

$27=x^3$

$\sqrt[3]{27}=\sqrt[3]{x^3}$

$3=x$

The solution is $x = 3$.

**32.**
$$\log x^3 = (\log x)^2$$
$$3\log x = (\log x)^2$$
$$(\log x)^2 - 3\log x = 0$$
$$\log x(\log x - 3) = 0$$

$\log x = 0$ or $\log x - 3 = 0$
$x = 1$ $\quad \log x = 3$
$x = 1000$

**33.**
$$\log(\log x) = 1$$
$$10^1 = \log x$$
$$10^{10} = x$$

**34.**
$$\ln(\ln x) = 2$$
$$e^2 = \ln x$$
$$e^{e^2} = x$$

**35.**
$$\ln(e^{3x}) = 6$$
$$3x\ln e = 6$$
$$3x(1) = 6$$
$$3x = 6$$
$$x = 2$$

**36.**
$$\ln x = \frac{1}{2}\ln\left(2x + \frac{5}{2}\right) + \frac{1}{2}\ln 2$$
$$\ln x = \frac{1}{2}\ln 2\left(2x + \frac{5}{2}\right)$$
$$\ln x = \frac{1}{2}\ln(4x + 5)$$
$$\ln x = \ln(4x + 5)^{1/2}$$
$$x = \sqrt{4x + 5}$$
$$x^2 = 4x + 5$$
$$0 = x^2 - 4x - 5$$
$$0 = (x - 5)(x + 1)$$
$$x = 5,\ -1$$

Check: $\ln 5 = \frac{1}{2}\ln\left(10 + \frac{5}{2}\right) + \frac{1}{2}\ln 2$
$$1.609 = 1.2628 + 0.3465$$
$$1.609. = 1.609$$

$$\ln(-1) = \frac{1}{2}\ln\left(-2 + \frac{5}{2}\right) + \frac{1}{2}$$

$x = -1$ is not a solution because $\ln(-1)$ is not defined.

The solution is $x = 5$.

**37.**
$$\log_7(5x) - \log_7 3 = \log_7(2x + 1)$$
$$\log_7\left(\frac{5x}{3}\right) = \log_7(2x + 1)$$
$$\frac{5x}{3} = 2x + 1$$
$$5x = 6x + 3$$
$$-3 = x$$

$x = -3$ is not a solution because $\log_7(-15)$ is undefined.
No solution.

**38.**
$$\log_4 x + \log_4(x - 2) = \log_4 15$$
$$\log_4 x(x - 2) = \log_4 15$$
$$x^2 - 2x = 15$$
$$x^2 - 2x - 15 = 0$$
$$(x - 5)(x + 3) = 0$$
$$x = 5, -3$$

$x = -3$ is not a solution because $\log_4(-3)$ is undefined.
The solution is $x = 5$.

**39.**
$$e^{\ln(x-1)} = 4$$
$$\ln e^{\ln(x-1)} = \ln 4$$
$$\ln(x - 1)\ln e = \ln 4$$
$$\ln(x - 1)(1) = \ln 4$$
$$(x - 1) = 4$$
$$x = 5$$

**40.**
$$10^{\log(2x+7)} = 8$$
$$\log 10^{\log(2x+7)} = \log 8$$
$$\log(2x + 7) = \log 8$$
$$2x + 7 = 8$$
$$2x = 1$$
$$x = \frac{1}{2}$$

**41.**
$$\frac{10^x - 10^{-x}}{2} = 20$$
$$10^x\left(10^x - 10^{-x}\right) = 40\left(10^x\right)$$
$$10^{2x} - 1 = 40\left(10^x\right)$$
$$10^{2x} - 40(10)^x - 1 = 0$$
Let $u = 10^x$.
$$u^2 - 40u - 1 = 0$$
$$u = \frac{40 \pm \sqrt{40^2 - 4(1)(-1)}}{2}$$
$$= \frac{40 \pm \sqrt{1600 + 4}}{2}$$
$$= \frac{40 \pm \sqrt{1604}}{2}$$
$$= \frac{40 \pm 2\sqrt{401}}{2}$$
$$= 20 \pm \sqrt{401}$$
$$10^x = 20 + \sqrt{401}$$
$$\log 10^x = \log\left(20 + \sqrt{401}\right)$$
$$x = \log\left(20 + \sqrt{401}\right)$$

**42.**
$$\frac{10^x + 10^{-x}}{2} = 8$$
$$10^x\left(10^x + 10^{-x}\right) = (16)10^x$$
$$10^{2x} + 1 = 16(10^x)$$
$$10^{2x} - 16\left(10^x\right) + 1 = 0$$
Let $u = 10^x$.
$$u = \frac{16 \pm \sqrt{16^2 - 4(1)(1)}}{2}$$
$$u = \frac{16 \pm \sqrt{256 - 4}}{2}$$
$$u = \frac{16 \pm 6\sqrt{7}}{2}$$
$$u = 8 \pm 3\sqrt{7}$$
$$10^x = 8 \pm 3\sqrt{7}$$
$$x \log 10 = \log\left(8 \pm 3\sqrt{7}\right)$$
$$x = \log\left(8 \pm 3\sqrt{7}\right)$$

**43.**
$$\frac{10^x + 10^{-x}}{10^x - 10^{-x}} = 5$$
$$10^x + 10^{-x} = 5\left(10^x - 10^{-x}\right)$$
$$10^x\left(10^x + 10^{-x}\right) = 5\left(10^x - 10^{-x}\right)10^x$$
$$10^{2x} + 1 = 5\left(10^{2x} - 1\right)$$
$$4\left(10^{2x}\right) = 6$$
$$2\left(10^{2x}\right) = 3$$
$$\left(10^x\right)^2 = \frac{3}{2}$$
$$10^x = \sqrt{\frac{3}{2}}$$
$$x \log 10 = \log\sqrt{\frac{3}{2}}$$
$$x = \log\sqrt{\frac{3}{2}}$$
$$x = \frac{1}{2}\log\left(\frac{3}{2}\right)$$

**44.**
$$\frac{10^x - 10^{-x}}{10^x + 10^{-x}} = \frac{1}{2}$$
$$10^x\left(10^x - 10^{-x}\right) = \frac{1}{2}\left(10^x + 10^{-x}\right)\left(10^x\right)$$
$$10^{2x} - 1 = \frac{1}{2}\left(10^{2x} + 1\right)$$
$$10^{2x} - 1 = \frac{1}{2}\left(10^{2x}\right) + \frac{1}{2}$$
$$10^{2x} - \frac{1}{2}\left(10^{2x}\right) = \frac{3}{2}$$
$$\frac{1}{2}\left(10^{2x}\right) = \frac{3}{2}$$
$$10^{2x} = 3$$
$$2x \log 10 = \log 3$$
$$2x = \log 3$$
$$x = \frac{\log 3}{2}$$

**45.** $$\frac{e^x+e^{-x}}{2}=15$$
$$e^x\left(e^x+e^{-x}\right)=(30)e^x$$
$$e^{2x}+1=e^x(30)$$
$$e^{2x}-30e^x+1=0$$
Let $u=e^x$.
$$u^2-30u+1=0$$
$$u=\frac{30\pm\sqrt{900-4}}{2}$$
$$u=\frac{30\pm\sqrt{896}}{2}$$
$$u=\frac{30\pm8\sqrt{14}}{2}$$
$$u=15\pm4\sqrt{14}$$
$$e^x=15\pm4\sqrt{14}$$
$$x\ln e=\ln(15\pm4\sqrt{14})$$
$$x=\ln(15\pm4\sqrt{14})$$

**46.** $$\frac{e^x-e^{-x}}{2}=15$$
$$e^x(e^x-e^{-x})=(30)(e^x)$$
$$e^{2x}-1=30e^x$$
$$e^{2x}-30e^x-1=0$$
Let $u=e^x$.
$$u^2-30u-1=0$$
$$u=\frac{30\pm\sqrt{900-4(-1)}}{2}$$
$$u=\frac{30\pm\sqrt{904}}{2}=\frac{30\pm2\sqrt{226}}{2}$$
$$u=15\pm\sqrt{226}$$
$$e^x=15\pm\sqrt{226}$$
$$x\ln e=\ln(15\pm\sqrt{226})$$
$$x=\ln(15+\sqrt{226})$$

**47.** $$\frac{1}{e^x-e^{-x}}=4$$
$$1=4(e^x-e^{-x})$$
$$1(e^x)=4(e^x)(e^x-e^{-x})$$
$$e^x=4(e^{2x}-1)$$
$$e^x=4e^{2x}-4$$
$$0=4e^{2x}-e^x-4$$
Let $u=e^x$.
$$0=4u^2-u-4$$
$$u=\frac{1\pm\sqrt{1-4(4)(-4)}}{8}$$
$$u=\frac{1\pm\sqrt{65}}{8}$$
$$e^x=\frac{1+\sqrt{65}}{8}$$
$$x\ln e=\ln\left(\frac{1+\sqrt{65}}{8}\right)$$
$$x=\ln(1+\sqrt{65})-\ln 8$$

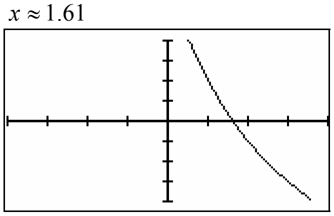
**48.** $$\frac{e^x+e^{-x}}{e^x-e^{-x}}=3$$
$$e^x(e^x+e^{-x})=3(e^x-e^{-x})e^x$$
$$e^{2x}+1=3e^{2x}-3$$
$$4=2e^{2x}$$
$$2=e^{2x}$$
$$\ln 2=2x\ln e$$
$$\frac{\ln 2}{2}=x$$

**49.** $2^{-x+3}=x+1$

Graph $f=2^{-x+3}-(x+1)$.

Its $x$-intercept is the solution.

$x\approx1.61$



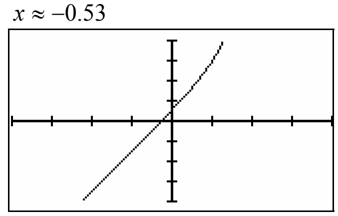
Xmin = –4, Xmax = 4, Xscl = 1,
Ymin = –4, Ymax = 4, Yscl = 1

**50.** $3^{x-2}=-2x-1$

Graph $f=3^{x-2}+2x+1$.

Its $x$-intercept is the solution.

$x\approx-0.53$



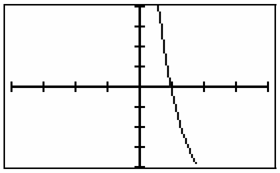
Xmin = –8, Xmax = 8, Xscl = 2,
Ymin = –8, Ymax = 8, Yscl = 2

**51.** $e^{3-2x} - 2x = 1$

Graph $f = e^{3-2x} - 2x - 1$.

Its $x$-intercept is the solution.

$x \approx 0.96$



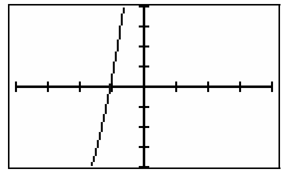
Xmin = –4, Xmax = 4, Xscl = 1,
Ymin = –4, Ymax = 4, Yscl = 1

**52.** $2e^{x+2} + 3x = 2$

Graph $f = 2e^{x+2} + 3x - 2$.

Its $x$-intercept is the solution.

$x \approx -1.05$



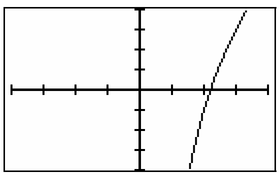
Xmin = –4, Xmax = 4, Xscl = 1,
Ymin = –4, Ymax = 4, Yscl = 1

**53.** $3\log_2(x-1) = -x + 3$

Graph $f = \dfrac{3\log(x-1)}{\log 2} + x - 3$.

Its $x$-intercept is the solution.

$x \approx 2.20$



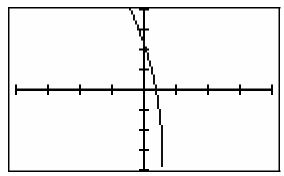
Xmin = –4, Xmax = 4, Xscl = 1,
Ymin = –4, Ymax = 4, Yscl = 1

**54.** $2\log_3(2-3x) = 2x - 1$

Graph $f = \dfrac{2\log(2-3x)}{\log 3} - 2x + 1$.

Its $x$-intercept is the solution.

$x \approx 0.38$



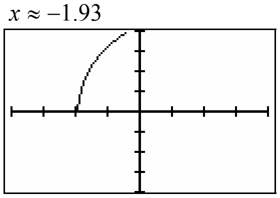
Xmin = –4, Xmax = 4, Xscl = 1,
Ymin = –4, Ymax = 4, Yscl = 1

**55.** $\ln(2x+4) + \frac{1}{2}x = -3$

Graph $f = \ln(2x+4) + \frac{1}{2}x + 3$.

Its $x$-intercept is the solution.

$x \approx -1.93$



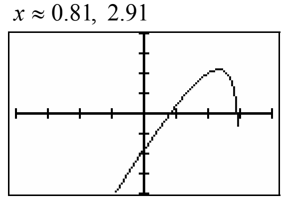
Xmin = –4, Xmax = 4, Xscl = 1,
Ymin = –4, Ymax = 4, Yscl = 1

**56.** $2\ln(3-x) + 3x = 4$

Graph $f = 2\ln(3\text{-}x) + 3x - 4$.

Its $x$-intercepts are the solutions.

$x \approx 0.81,\ 2.91$



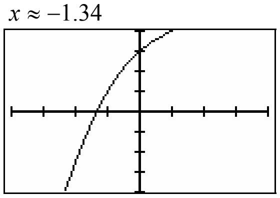
Xmin = –4, Xmax = 4, Xscl = 1,
Ymin = –4, Ymax = 4, Yscl = 1

**57.** $2^{x+1} = x^2 - 1$

Graph $f = 2^{x+1} - x^2 + 1$.

Its $x$-intercept is the solution.

$x \approx -1.34$



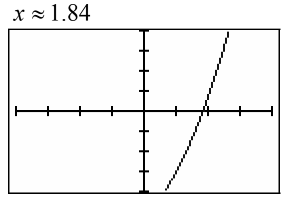
Xmin = –4, Xmax = 4, Xscl = 1,
Ymin = –4, Ymax = 4, Yscl = 1

**58.** $\ln(x) = -x^2 + 4$

Graph $f = \ln x + x^2 - 4$ .

Its $x$-intercept is the solution.

$x \approx 1.84$



Xmin = –4, Xmax = 4, Xscl = 1,
Ymin = –4, Ymax = 4, Yscl = 1

**59.** **a.** $P(0) = 8500(1.1)^0 = 8500(1) = 8500$

$P(2) = 8500(1.1)^2 = 10,285$

**b.**
$$15,000 = 8500(1.1)^t$$
$$\ln 15,000 = 8500(1.1)^t$$
$$\ln 51,000 = \ln 8500 + t\ln(1.1)$$
$$\frac{\ln 15,000 - \ln 8500}{\ln(1.1)} = t$$
$$6 \approx t$$
The population will reach 15,000 in 6 years.

**60.** **a.** $R(0) = 145e^0 = 145$ beats per minute

$R(1) = 145e^{-0.092} \approx 132$ beats per minute

**b.**
$$80 = 145e^{0.092t}$$
$$\ln 80 = \ln 145 - 0.092t\ln e$$
$$\frac{\ln 80 - \ln 145}{-0.092} = t$$
$$6 = t$$
6 minutes

**61.** **a.** $T(10) = 36 + 43e^{-0.058(10)} = 36 + 43e^{-0.58}$

$T \approx 60^\circ F$

**b.**
$$45 = 36 + 43e^{-0.058t}$$
$$\ln(45-36) = \ln 43 - 0.058t\ln e$$
$$\frac{\ln(45-36) - \ln 43}{-0.058} = t$$
$$t \approx 27 \text{ minutes}$$

**62.** **a.** $A\left(\frac{1}{2}\right) = 80(0.727)^{1/2} \approx 68$

$A = 68$ mg

**b.**
$$50 = 80(0.727)^t$$
$$\ln 50 = \ln 80 + t\ln 0.727$$
$$\frac{\ln 50 - \ln 80}{\ln 0.727} = t$$
$$t \approx 1.47 \text{ hours} \approx 88 \text{ minutes}$$

**63.**
$$114 = 198 - (198 - 0.9)e^{-0.23x}$$
$$-84 = -197.1e^{-0.23x}$$
$$\frac{84}{197.1} = e^{-0.23x}$$
$$\ln\left(\frac{84}{197.1}\right) = -0.23x$$
$$x = \frac{\frac{84}{197.1}}{-0.23} \approx 3.7 \text{ years}$$

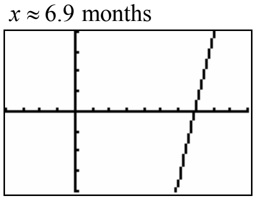
**64.**
$$21 = 94 - (94 - 0.6)e^{-0.21x}$$
$$-73 = -93.4e^{-0.21x}$$
$$\frac{73}{93.4} = e^{-0.21x}$$
$$\ln\left(\frac{73}{93.4}\right) = -0.21x$$
$$x = \frac{\frac{73}{93.4}}{-0.21} \approx 1.2 \text{ years}$$

**65.** $5 + 29\ln(t+1) = 65$

Graph $f = 29\ln(x+1) - 60$.

Its $x$-intercept is the solution.

$x \approx 6.9$ months



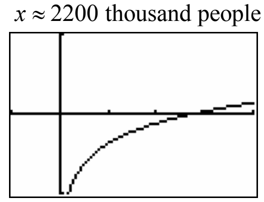
Xmin = −4, Xmax = 10, Xscl = 1,
Ymin = −4, Ymax = 4, Yscl = 1

**66.** $0.37\ln x + 0.05 = 2.9$

Graph $f = 0.37\ln x - 2.85$.

Its $x$-intercept is the solution.

$x \approx 2200$ thousand people or 2,200,000 people



Xmin = −800, Xmax = 3200, Xscl = 800,
Ymin = −1, Ymax = 1, Yscl = 1

**67.** Consider the first function for time less than 10 seconds.

$$275 = -2.25x^2 + 56.26x - 0.28$$
$$0 = -2.25x^2 + 56.26x - 275.28$$
$$x = \frac{-56.26 \pm \sqrt{(56.26)^2 - 4(-2.25)(-275.28)}}{2(-2.25)}$$
$$x = 6.67 \text{ or } 18.33$$

18.33 s > 10 s, so it is not a solution. The solution is 6.67 s.

Consider the second function for time greater than 10 seconds.

$$275 = 8320(0.73)^x$$
$$\frac{275}{8320} = (0.73)^x$$
$$\ln\left(\frac{275}{8320}\right) = \ln 0.73^x$$
$$\ln\left(\frac{275}{8320}\right) = x \ln 0.73$$
$$x = \frac{\ln\left(\frac{275}{8320}\right)}{\ln 0.73} \approx 10.83$$

The solutions are 6.67 s and 10.83 s.

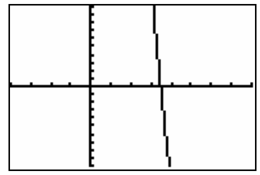
**68.** **a.** $363.4 - 88.4 \ln x = 50$

Graph $f = 313.4 - 88.4 \ln x$.

Its $x$-intercept is the solution.

$x \approx 34.65$

$2(34.65) = 69.3$ m



Xmin = −40, Xmax = 80, Xscl = 10,
Ymin = −10, Ymax = 10, Yscl = 1

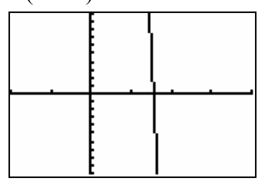
**b.** $568.2 - 161.5 \ln x = 125$

Graph $f = 443.2 - 161.5 \ln x$.

Its $x$-intercept is the solution.

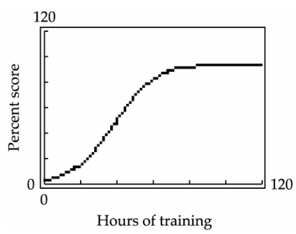
$x \approx 15.55$

$2(15.55) = 31.1$ m



Xmin = −20, Xmax = 40, Xscl = 10,
Ymin = −10, Ymax = 10, Yscl = 1

**69.** **a.**

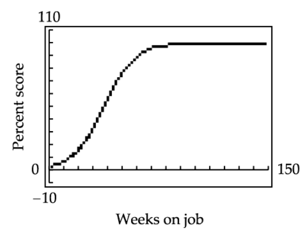



**b.** 48 hours

**c.** $P = 100$

**d.** As the number of hours of training increases, the test scores approach 100%.

**70.** **a.**

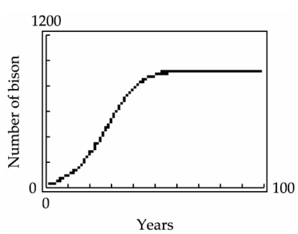



**b.** 45 weeks

**c.** $P = 100$

**d.** The more experience a person has, the closer the person's score is to 100%.

**71.** **a.**

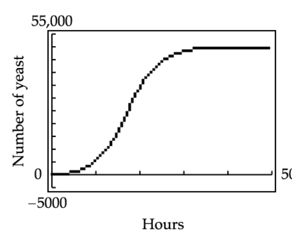



**b.** In 27 years or 2026

**c.** $B = 1000$

**d.** As the number of years increases, the bison population approaches but never exceeds 1000.

**72.** **a.**

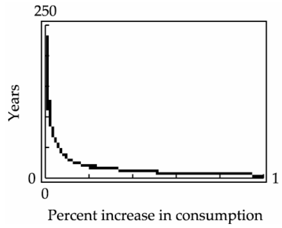



**b.** 21 hours

**c.** $Y = 50{,}000$

**d.** The number of yeast approaches but never exceeds 50,000.

**73.** **a.**




**b.** When $r$ = 3%, or 0.03, $T \approx 78$ years

**c.** When $T$ = 100, $r \approx 0.019$, or 1.9%

**74.** **a.**

$$t = -\frac{9}{24}\ln\left(\frac{24+v}{24-v}\right)$$
$$1.5 = -\frac{9}{24}\ln\left(\frac{24+v}{24-v}\right)$$
$$\frac{24(1.5)}{9} = \ln\left(\frac{24+v}{24-v}\right)$$
$$4 = \ln\left(\frac{24+v}{24-v}\right)$$
$$e^4 = \frac{24+v}{24-v}$$
$$e^4(24-v) = 24+v$$
$$e^4(24) - e^4 v = 24+v$$
$$-v - e^4 v = 24 - 24e^4$$
$$v\left(-1-e^4\right) = 24$$
$$v = \frac{24-24e^4}{-1-e^4}$$
$$v \approx 23.1367$$

The velocity after 1.5 seconds is approximately 23.14 feet per second.

**b.** The vertical asymptote occurs when the denominator of $\frac{24+v}{24-v}$ is zero, or when $v$ = 24.

**c.** The velocity of the object cannot reach or exceed 24 feet per second.

**75.** **a.**

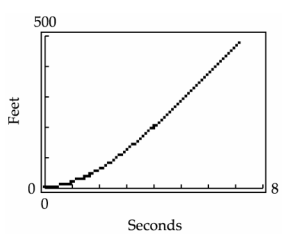
$$v = 100\left(\frac{e^{0.64t}-1}{e^{0.64t}+1}\right)$$
$$50 = 100\left(\frac{e^{0.64t}-1}{e^{0.64t}+1}\right)$$
$$\frac{50}{100} = \frac{e^{0.64t}-1}{e^{0.64t}+1}$$
$$0.5 = \frac{e^{0.64t}-1}{e^{0.64t}+1}$$
$$0.5(e^{0.64t}+1) = e^{0.64t}-1$$
$$0.5e0^{1.64t} + 0.5 = e^{0.64t}-1$$
$$0.5e^{0.64t} - e^{0.64t} = -1.5$$
$$-0.5e^{0.64t} = -1.5$$
$$e^{0.64t} = 3$$
$$0.64t = \ln 3$$
$$t = \frac{\ln 3}{0.64}$$
$$t \approx 1.72$$

In approximately 1.72 seconds, the velocity will be 50 feet per second.

**b.** The horizontal asymptote is the value of $100\left[\frac{e^{0.64t}-1}{e^{0.64t}+1}\right]$ as $t \to \infty$. Therefore, the horizontal asymptote is $v$ = 100 feet per second.

**c.** The object cannot fall faster than 100 feet per second.

**76.** **a.**




**b.** When $s = 100$, $t \approx 2.6$ seconds.

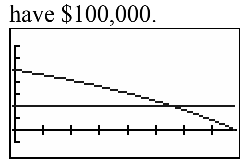
**77.** Graph $V = 400{,}000 - 150{,}000(1.005)^x$ and $V = 100{,}000$.

They intersect when $x \approx 138.97$.

After 138 withdrawals, the account has \$101,456.39.

After 139 withdrawals, the account has \$99,963.67.

The designer can make at most 138 withdrawals and still have \$100,000.



Xmin = 0,Xmax = 200,Xscl = 25

Ymin = −50000,Ymax = 350000,Yscl = 50000

**78.** **a.** $h(x) = 10(e^{x/20} + e^{-x/20})$ The lowest height of the cable is in the middle, where $x = 0$.

$h(0) = 10(e^{0/20} + e^{-0/20}) = 10(e^0 + e^0) = 10(1+1) = 10(2) = 20$ feet

**b.** $h(10) = 10(e^{10/20} + e^{-10/20}) = 10(e^{1/2} + e^{-1/2}) \approx 22.6$ feet

**c.** $24 = 10(e^{x/20} + e^{-x/20}) \Rightarrow 2.4 = e^{x/20} + e^{-x/20} \Rightarrow 2.4e^{x/20} = (e^{x/20} + e^{-x/20})e^{x/20}$

$2.4e^{x/20} = e^{2x/20} + e^0 = (e^{x/20})^2 + 1 \Rightarrow 2.4e^{x/20} = (e^{x/20})^2 1$. Let $u = e^{x/20}$. Then $2.4u = u^2 + 1$

$$0 = u^2 - 2.4u + 1 \Rightarrow u = \frac{-(-2.4) \pm \sqrt{(-2.4)^2 - 4(1)(1)}}{2(1)} = \frac{2.4 \pm \sqrt{1.76}}{2} \approx \frac{2.4 \pm 1.3266}{2}$$

$e^{x/20} = \dfrac{2.4 + \sqrt{1.76}}{2}$ or $e^{x/20} = \dfrac{2.4 - \sqrt{1.76}}{2}$

$x/20 = \ln \dfrac{2.4 + \sqrt{1.76}}{2}$ $\qquad x/20 = \ln \dfrac{2.4 - \sqrt{1.76}}{2}$

$x = 20 \ln \dfrac{2.4 + \sqrt{1.76}}{2} \approx 12.4$ feet $\qquad x = 20 \ln \dfrac{2.4 - \sqrt{1.76}}{2} \approx$ ~~−12.4~~ no negative height

## Connecting Concepts

**79.** The second step because $\log 0.5 < 0$. Thus the inequality sign must be reversed.

**80.** The third step. $\log_2(8+8)$ does not equal $\log_2 8 + \log_2 8$.

**81.**
$$\log(x+y) = \log x + \log y$$
$$\log(x+y) = \log xy$$
Therefore $x + y = xy$
$$x - xy = -y$$
$$x(1-y) = -y$$
$$x = \frac{-y}{1-y}$$
$$x = \frac{y}{y-1}$$

**82.** No. The domain of $g(x)$ includes negative numbers; the domain of $f(x)$ does not. Thus, for any negative value of $x$, $f(x) \neq g(x)$.

**83.** Since $e^{0.336} \approx 1.4$,

$F(x) = (1.4)^x \approx (e^{0.336})^x = e^{0.336x} = G(x)$

**84.**
$$2.2 = e^{-k}$$
$$\ln 2.2 = -k \ln e$$
$$-\ln 2.2 = k$$
$$-0.788 \approx k$$

## Prepare for Section 4.6

**PS1.** $A = 1000\left(1 + \dfrac{0.1}{12}\right)^{12(2)} = 1220.39$

**PS2.** $A = 600\left(1 + \dfrac{0.04}{4}\right)^{4(8)} = 824.96$

**PS3.**
$$0.5 = e^{14k}$$
$$\ln 0.5 = \ln e^{14k}$$
$$\ln 0.5 = 14k$$
$$\frac{\ln 0.5}{14} = k$$
$$-0.0495 \approx k$$

**PS4.**
$$0.85 = 0.5^{t/5730}$$
$$\ln 0.85 = \ln 0.5^{t/5730}$$
$$\ln 0.85 = \frac{t}{5730} \ln 0.5$$
$$\frac{5730 \ln 0.85}{\ln 0.5} = t$$
$$1340 \approx t$$

**PS5.**

$$
\begin{aligned}
6 &= \frac{70}{5+9e^{-12k}} \\
6(5+9e^{-12k}) &= 70 \\
30+54e^{-12k} &= 70 \\
54e^{-12k} &= 40 \\
e^{-12k} &= \frac{20}{27} \\
\ln e^{-12k} &= \ln\frac{20}{27} \\
-12k &= \ln\frac{20}{27} \\
k &= -\frac{1}{12}\ln\frac{20}{27} \\
k &\approx 0.025
\end{aligned}
$$

**PS6.**

$$
\begin{aligned}
2,000,000 &= \frac{3^{n+1}-3}{2} \\
4,000,000 &= 3^{n+1}-3 \\
3,999,997 &= 3^{n+1} \\
\ln 3,999,997 &= \ln 3^{n+1} \\
\ln 3,999,997 &= (n+1)\ln 3 \\
\frac{\ln 3,999,997}{\ln 3} &= n+1 \\
\frac{\ln 3,999,997}{\ln 3}-1 &= n \\
12.8 &\approx n
\end{aligned}
$$

## Section 4.6

**1.** **a.** $t = 0$ hours, $N(0) = 2200(2)^0 = 2200$ bacteria

**b.** $t = 3$ hours, $N(3) = 2200(2)^3 = 17,600$ bacteria

**2.** **a.** $t = 3$ years, $f(3) = 12,400(1.14)^3 \approx 18,400$

**b.** $t = 4.25$ years, $f(4.25) = 12,400(1.14)^{4.25} \approx 21,600$

**3.** **a.**

$$
\begin{aligned}
N(t) &= N_0 e^{kt} \text{ where } N_0 = 24600 \\
N(5) &= 22,600e^{k(5)} \\
24,200 &= 22,600e^{5k} \\
\frac{24,200}{22,600} &= e^{5k} \\
\ln\left(\frac{24,200}{22,600}\right) &= \ln\left(e^{5k}\right) \\
\ln\left(\frac{24,200}{22,600}\right) &= 5k \\
\frac{1}{5}\left[\ln\frac{24,200}{22,600}\right] &= k \\
0.01368 &\approx k \\
N(t) &= 22,600e^{0.01368t}
\end{aligned}
$$

**b.**

$$
\begin{aligned}
t &= 15 \\
N(15) &= 22,600e^{0.01368(15)} \\
&= 22,600e^{0.2052} \\
&\approx 27,700
\end{aligned}
$$

**4.** **a.**

$$
\begin{aligned}
N(t) &= N_0 e^{kt} \text{ where } N_0 = 53,700 \\
N(4) &= 53,700e^{k(4)} \\
58,100 &= 53,700e^{4k} \\
\frac{58,100}{53,700} &= e^{4k} \\
\ln\left(\frac{58,100}{53,700}\right) &= \ln\left(e^{4k}\right) \\
\ln\left(\frac{58,100}{53,700}\right) &= 4k \\
\frac{1}{4}\left[\ln\left(\frac{58,100}{53,700}\right)\right] &= k \\
0.01969 &\approx k \\
N(t) &= 53,700e^{0.01969t}
\end{aligned}
$$

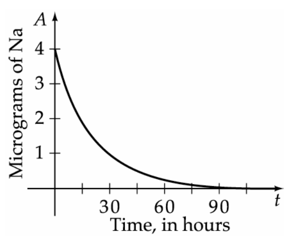
**b.**

$$
\begin{aligned}
t &= 12 \\
N(12) &= 53,700e^{0.01969(12)} \\
&= 53,700e^{0.23628} \\
&\approx 68,000
\end{aligned}
$$

**5.**
$$N(t) = N_0 e^{kt} \text{ where } N_0 = 362{,}300$$
$$N(4) = 362{,}300 e^{k(4)}$$
$$379{,}700 = 362{,}300 e^{4k}$$
$$\frac{379{,}700}{362{,}300} = e^{4k}$$
$$\ln\left(\frac{379{,}700}{362{,}300}\right) = \ln\left(e^{4k}\right)$$
$$\ln\left(\frac{379{,}700}{362{,}300}\right) = 4k$$
$$\frac{1}{4}\left[\ln\frac{379{,}700}{362{,}300}\right] = k$$
$$0.011727 \approx k$$
$$N(t) = 362{,}300 e^{0.011727\,t}$$
$$t = 9$$
$$N(9) = 362{,}300 e^{0.011727(9)}$$
$$= 362{,}300 e^{0.105543}$$
$$\approx 402{,}600$$

**6.**
$$N(t) = N_0 e^{kt} \text{ where } N_0 = 276{,}400$$
$$N(4) = 276{,}400 e^{k(4)}$$
$$291{,}800 = 276{,}400 e^{4k}$$
$$\frac{291{,}800}{276{,}400} = e^{4k}$$
$$\ln\left(\frac{291{,}800}{276{,}400}\right) = \ln\left(e^{4k}\right)$$
$$\ln\left(\frac{291{,}800}{276{,}400}\right) = 4k$$
$$\frac{1}{4}\left[\ln\frac{291{,}800}{276{,}400}\right] = k$$
$$0.013555 \approx k$$
$$N(t) = 276{,}400 e^{0.013555\,t}$$
$$t = 8$$
$$N(9) = 276{,}400 e^{0.013555(8)}$$
$$= 276{,}400 e^{0.10844}$$
$$\approx 308{,}100$$

**7.** **a.**




**b.** $A(5) = 4e^{-0.23} \approx 3.18$ micrograms

**c.** Since $A = 4$ micrograms are present when $t = 0$, find the time $t$ at which half remains—that is when $A = 2$.
$$2 = 4e^{-0.046t}$$
$$\frac{1}{2} = e^{-0.046t}$$
$$\ln\left(\frac{1}{2}\right) = -0.046t$$
$$\frac{\ln\left(\frac{1}{2}\right)}{-0.046} = t$$
$$15.07 \approx t$$
The half-life of sodium-24 is about 15.07 hours.

**d.**
$$1 = 4e^{-0.046t}$$
$$\frac{1}{4} = e^{-0.046t}$$
$$\ln\left(\frac{1}{4}\right) = -0.046t$$
$$\frac{\ln\left(\frac{1}{4}\right)}{-0.046t} = t$$
$$30.14 \approx t$$
The amount of sodium-24 will be 1 microgram after 30.14 hours.

**8.**
$$N(t) = N_0 e^{kt}$$
$$N(138) = N_0 e^{138k}$$
$$0.5N_0 = N_0 e^{138k}$$
$$0.5 = e^{138k}$$
$$\ln 0.5 = \ln e^{138k}$$
$$\ln 0.5 = 138k \ln e$$
$$\ln 0.5 = 138k$$
$$\frac{\ln 0.5}{138} = k$$
$$-0.005023 \approx k$$
$$N(t) = N0(0.5)^{t/138}$$
$$\approx N_0 e^{-0.005023t}$$

**9.**
$$N(t) = N_0 (0.5)^{t/5730}$$
$$N(t) = 0.45N_0$$
$$0.45N_0 = N_0 (0.5)^{t/5730}$$
$$\ln(0.45) = \frac{t}{5730}\ln 0.5$$
$$5730\frac{\ln 0.45}{\ln 0.5} = t$$
$$6601 \approx t$$
The bone is about 6601 years old.

**10.**
$$N(t) = N_0 (0.5)^{t/138}$$
$$N(730) = N_0(0.5)^{730/138} \approx 0.0256N_0$$
After 2 years (730 days), only 2.56% of the polonium sample will remain.

**11.**

$$N(t)=N_0(0.5)^{t/5730}$$
$$N(t)=0.75N_0$$
$$0.75N_0=N_0(0.5)^{t/5730}$$
$$\ln 0.75=\frac{t}{5730}\ln 0.5$$
$$5730\frac{\ln 0.75}{\ln 0.5}=t$$
$$2378\approx t$$

The Rhind papyrus is about 2378 years old.

**12.**

$$N(t)=N_0(0.5)^{t/5730}$$
$$N(t)=0.65N_0$$
$$0.65N_0=N_0(0.5)^{t/5730}$$
$$\ln 0.65=\frac{t}{5730}\ln 0.5$$
$$5730\frac{\ln 0.65}{\ln 0.5}=t$$
$$3561\approx t$$

The bone is about 3600 years old.

**13.** **a.** $P=8000,\ r=0.05,\ t=4,\ n=1$

$$B=8000\left(1+\frac{0.05}{1}\right)^4\approx \$9724.05$$

**b.** $t=7,\ B=8000\left(1+\frac{0.05}{1}\right)^7\approx \$11,256.80$

**14.** **a.** $P=22,000, r=0.045, n=1, t=2$

$$B=22,000\left(1+\frac{0.045}{1}\right)^2\approx \$24,024.55$$

**b.** $t=10, B=22,000\left(1+\frac{0.045}{1}\right)^{10}\approx \$34,165.33$

**15.** **a.** $P=38,000, r=0.065, t=4, n=1$

$$B=38,000\left(1+\frac{0.065}{1}\right)^4\approx \$48,885.72$$

**b.** $n=365, B=38,000\left(1+\frac{0.065}{365}\right)^{4(365)}\approx \$49,282.20$

**c.** $n=8760, B=38,000\left(1+\frac{0.065}{8760}\right)^{4(8760)}\approx \$49,283.30$

**16.** **a.** $P=12,500, r=0.08, t=10, n=1$

$$B=12,500\left(1+\frac{0.08}{1}\right)^{10}\approx \$26,986.56$$

**b.** $n=365, B=12,500\left(1+\frac{0.08}{365}\right)^{3650}\approx \$27,816.82$

**c.** $n=8760, B=12,500\left(1+\frac{0.08}{8760}\right)^{87,600}\approx \$27,819.16$

**17.** $P=15,000,\quad r=0.1, t=5$

$$B=15,000e^{5(0.1)}\approx \$24,730.82$$

**18.** $P=32,000,\quad r=0.08, t=3$

$$B=32,000e^{3(0.08)}\approx \$40,679.97$$

**19.** $t=\frac{\ln 2}{r}\qquad r=0.0784$

$$t=\frac{\ln 2}{0.0784}$$
$t\approx 8.8$ years

**20.** $t=\frac{\ln 2}{2}\qquad r=0.0588$

$$t=\frac{\ln 2}{0.0588}$$
$t\approx 11.8$ years

**21.** $B=Pe^{rt}\qquad$ Let $B=3P$

$$3P=Pe^{rt}$$
$$3=e^{rt}$$
$$\ln 3=rt\ln e$$
$$t=\frac{\ln 3}{r}$$

**22.** $t=\frac{\ln 3}{r}\qquad r=0.055$

$$t=\frac{\ln 3}{0.055}$$
$t\approx 20$ years

**23.** $t=\frac{\ln 3}{r}\qquad r=0.076$

$$t=\frac{\ln 3}{0.076}$$
$t\approx 14$ years

**24.** $t=\frac{\ln 3}{0.055}\qquad r=0.055$

$t=20$ years

**25.** **a.** 1900
**b.** 0.16
**c.** $P(0)=\dfrac{1900}{1+8.5e^{-0.16(0)}}=200$

**26.** **a.** 32,550
**b.** 0.08
**c.** $P(0)=\dfrac{32,550}{1+0.75e^{-0.08(0)}}=18,600$

**27.** **a.** 157,500
**b.** 0.04
**c.** $P(0)=\dfrac{157,500}{1+2.5e^{-0.04(0)}}=45,000$

**28.** **a.** 51
**b.** 0.03
**c.** $P(0)=\dfrac{51}{1+1.04e^{-0.03(0)}}=25$

**29.** **a.** 2400
**b.** 0.12
**c.** $P(0)=\dfrac{2400}{1+7e^{-0.12(0)}}=300$

**30.** **a.** 320
**b.** 0.12
**c.** $P(0)=\dfrac{320}{1+15e^{-0.12(0)}}=20$

**31.**

$$a=\frac{c-P_0}{P_0}=\frac{5500-400}{400}=12.75$$

$$P(t)=\frac{c}{1+ae^{-bt}}$$

$$P(2)=\frac{5500}{1+12.75e^{-b(2)}}$$

$$780=\frac{5500}{1+12.75e^{-2b}}$$

$$780(1+12.75e^{-2b})=5500$$

$$780+9945e^{-2b}=5500$$

$$9945e^{-2b}=4720$$

$$e^{-2b}=\frac{4720}{9945}$$

$$\ln e^{-2b}=\ln\frac{4720}{9945}$$

$$-2b=\ln\frac{4720}{9945}$$

$$b=-\frac{1}{2}\ln\frac{4720}{9945}$$

$$b\approx 0.37263$$

$$P(t)=\frac{5500}{1+12.75e^{-0.37263t}}$$

**32.**

$$a=\frac{c-P_0}{P_0}=\frac{9500-6200}{6200}=0.53226$$

$$P(t)=\frac{c}{1+ae^{-bt}}$$

$$P(8)=\frac{9500}{1+0.53226e^{-b(8)}}$$

$$7100=\frac{9500}{1+0.53226e^{-8b}}$$

$$7100(1+0.53226e^{-8b})=9500$$

$$7100+3779e^{-8b}=9500$$

$$3779e^{-8b}=2400$$

$$e^{-8b}=\frac{2400}{3779}$$

$$\ln e^{-8b}=\ln\frac{2400}{3779}$$

$$-8b=\ln\frac{2400}{3779}$$

$$b=-\frac{1}{8}\ln\frac{2400}{3779}$$

$$b\approx 0.05675$$

$$P(t)=\frac{9500}{1+0.53226e^{-0.05675t}}$$

**33.** $a = \dfrac{c - P_0}{P_0} = \dfrac{100 - 18}{18} = 4.55556$

$$\begin{aligned}
P(t) &= \frac{c}{1 + ae^{-bt}} \\
P(3) &= \frac{100}{1 + 4.55556e^{-b(3)}} \\
30 &= \frac{100}{1 + 4.55556e^{-3b}} \\
30(1 + 4.55556e^{-3b}) &= 100 \\
30 + 136.67e^{-3b} &= 100 \\
136.67e^{-3b} &= 70 \\
e^{-3b} &= \frac{70}{136.67} \\
\ln e^{-3b} &= \ln \frac{70}{136.67} \\
-3b &= \ln \frac{70}{136.67} \\
b &= -\frac{1}{3} \ln \frac{70}{136.67} \\
b &\approx 0.22302
\end{aligned}$$

$$P(t) = \frac{100}{1 + 4.55556e^{-0.22302t}}$$

**34.**

$$\begin{aligned}
a &= \frac{c - P_0}{P_0} \\
a &= \frac{c - 3200}{3200} \\
3200a + 3200 &= c \\
P(t) &= \frac{c}{1 + ae^{-bt}} \\
P(22) &= \frac{3200a + 3200}{1 + ae^{-0.056(22)}} \\
5565 &= \frac{3200a + 3200}{1 + ae^{-1.232}} \\
5565(1 + ae^{-1.232}) &= 3200a + 3200 \\
5565 + 5565ae^{-1.232} &= 3200a + 3200 \\
5565ae^{-1.232} - 3200a &= -2365 \\
a(5565e^{-1.232} - 3200) &= -2365 \\
a &= \frac{-2365}{5565e^{-1.232} - 3200} \\
a &= 1.5
\end{aligned}$$

$c = 3200a + 3200 = 3200(1.5) + 3200 = 8000$

$$P(t) = \frac{8000}{1 + 1.5e^{-0.056t}}$$

**35.** **a.** $R(t) = \dfrac{625{,}000}{1 + 3.1e^{-0.045t}}$

$R(1) = \dfrac{625{,}000}{1 + 3.1e^{-0.045(1)}} \approx \$158{,}000$

$R(2) = \dfrac{625{,}000}{1 + 3.1e^{-0.045(2)}} \approx \$163{,}000$

**b.** $R(t) = \dfrac{625{,}000}{1 + 3.1e^{-0.045t}}$, as $t \to \infty$, $R(t) \to \$625{,}000$

**36.** **a.** $A(t) = \dfrac{1650}{1 + 2.4e^{-0.055t}}$

$A(1) = \dfrac{1650}{1 + 2.4e^{-0.055(1)}} \approx 504$ cars

$A(2) = \dfrac{1650}{1 + 2.4e^{-0.055(2)}} \approx 524$ cars

**b.** $A(t) = \dfrac{1650}{1 + 2.4e^{-0.055t}}$, as $t \to \infty$, $A(t) \to 1650$ cars

**37.** **a.** $a = \dfrac{c - P_0}{P_0} = \dfrac{1600 - 312}{312} = 4.12821$

$$\begin{aligned}
P(t) &= \frac{c}{1 + ae^{-bt}} \\
P(6) &= \frac{1600}{1 + 4.12821e^{-b(6)}} \\
416 &= \frac{1600}{1 + 4.12821e^{-6b}} \\
416(1 + 4.12821e^{-6b}) &= 1600 \\
416 + 1717.34e^{-6b} &= 1600 \\
1717.34e^{-6b} &= 1184 \\
e^{-6b} &= \frac{1184}{1717.34} \\
\ln e^{-6b} &= \ln \frac{1184}{1717.34} \\
-6b &= \ln \frac{1184}{1717.34} \\
b &= -\frac{1}{6} \ln \frac{1184}{1717.34} \\
b &\approx 0.06198
\end{aligned}$$

$$P(t) = \frac{1600}{1 + 4.12821e^{-0.06198t}}$$

**b.** $P(10) = \dfrac{1600}{1 + 4.12821e^{-0.06198(10)}} \approx 497$ wolves

**38.** **a.** $a = \dfrac{c - P_0}{P_0} = \dfrac{3400 - 240}{240} = 13.167$

$$\begin{aligned}
P(t) &= \frac{c}{1 + ae^{-bt}} \\
P(1) &= \frac{3400}{1 + 13.16667e^{-b(1)}} \\
310 &= \frac{3400}{1 + 13.16667e^{-b}} \\
310(1 + 13.16667e^{-b}) &= 3400 \\
310 + 4081.6677e^{-b} &= 3400 \\
4081.6677e^{-b} &= 3090 \\
e^{-b} &= \frac{3090}{4081.6677} \\
\ln e^{-b} &= \ln \frac{3090}{4081.6677} \\
-b &= \ln \frac{3090}{4081.6677} \\
b &= -\ln \frac{3090}{4081.6677} \\
b &\approx 0.27833
\end{aligned}$$

$$P(t) = \frac{3400}{1 + 13.16667e^{-0.27833t}}$$

**b.** $P(7) = \dfrac{3400}{1 + 13.16667e^{-0.27833(7)}} \approx 1182$ groundhogs

**39.** **a.** $a = \dfrac{c - P_0}{P_0} = \dfrac{8500 - 1500}{1500} = 4.66667$

$$P(t) = \frac{c}{1 + ae^{-bt}}$$
$$P(2) = \frac{8500}{1 + 4.66667e^{-b(2)}}$$
$$1900 = \frac{8500}{1 + 4.66667e^{-2b}}$$
$$1900(1 + 4.66667e^{-2b}) = 8500$$
$$1900 + 8866.673e^{-2b} = 8500$$
$$8866.673e^{-2b} = 6600$$
$$e^{-2b} = \frac{6600}{8866.673}$$
$$\ln e^{-2b} = \ln \frac{6600}{8866.673}$$
$$-2b = \ln \frac{6600}{8866.673}$$
$$b = -\frac{1}{2}\ln \frac{6600}{8866.673}$$
$$b \approx 0.14761$$
$$P(t) = \frac{8500}{1 + 4.66667e^{-0.14761t}}$$

**b.**
$$4000 = \frac{8500}{1 + 4.66667e^{-0.14761t}}$$
$$4000(1 + 4.66667e^{-0.14761t}) = 8500$$
$$1 + 4.66667e^{-0.14761t} = 2.125$$
$$4.66667e^{-0.14761t} = 1.125$$
$$e^{-0.14761t} = \frac{1.125}{4.66667}$$
$$\ln e^{-0.14761t} = \ln \frac{1.125}{4.66667}$$
$$-0.14761t = \ln \frac{1.125}{4.66667}$$
$$t = -\frac{1}{0.14761}\ln \frac{1.125}{4.66667}$$
$$t \approx 9.6$$

The population will exceed 4000 in 2007 + 9 = 2016.

**40.** **a.** $a = \dfrac{c - P_0}{P_0} = \dfrac{5500 - 800}{800} = 5.875$

$$P(t) = \frac{c}{1 + ae^{-bt}}$$
$$P(1) = \frac{5500}{1 + 5.875e^{-b(1)}}$$
$$900 = \frac{5500}{1 + 5.875e^{-b}}$$
$$900(1 + 5.875e^{-b}) = 5500$$
$$900 + 5287.5e^{-b} = 5500$$
$$5287.5e^{-b} = 4600$$
$$e^{-b} = \frac{4600}{5287.5}$$
$$\ln e^{-b} = \ln \frac{4600}{5287.5}$$
$$-b = \ln \frac{4600}{5287.5}$$
$$b = -\ln \frac{4600}{5287.5}$$
$$b \approx 0.13929$$
$$P(t) = \frac{5500}{1 + 5.875e^{-0.13929t}}$$

**b.**
$$2000 = \frac{5500}{1 + 5.875e^{-0.13929t}}$$
$$2000(1 + 5.875e^{-0.13929t}) = 5500$$
$$1 + 5.875e^{-0.13929t} = 2.75$$
$$5.875e^{-0.13929t} = 1.75$$
$$e^{-0.13929t} = \frac{1.75}{5.875}$$
$$\ln e^{-0.13929t} = \ln \frac{1.75}{5.875}$$
$$-0.13929t = \ln \frac{1.75}{5.875}$$
$$t = -\frac{1}{0.13929}\ln \frac{1.75}{5.875}$$
$$t \approx 8.7$$

The population will exceed 2000 in 2003 + 8 = 2011.

**41.** **a.** $A = 34^\circ\text{F}, T_0 = 75^\circ\text{F}, T_t = 65^\circ\text{F}, t = 5$. Find $k$.

$$65 = 34 + (75 - 34)e^{-5k}$$
$$31 = 41e^{-5k}$$
$$\frac{31}{41} = e^{-5k}$$
$$\ln\left(\frac{31}{41}\right) = -5k$$
$$k = -\frac{1}{5}\ln\left(\frac{31}{41}\right)$$
$$k \approx 0.056$$

**b.** $A = 34^\circ\text{F}, k = 0.056,\ T_0 = 75^\circ\text{F}, t = 30$

$$T_t = 34 + (75 - 34)e^{-30(0.056)}$$
$$T_t = 34 + (41)e^{-1.68}$$
$$T_t \approx 42^\circ\text{F}$$

**c.** $T_t = 36^\circ\text{F}, k = 0.056,\ T_t = 75^\circ\text{F}, A = 34^\circ\text{F}$

$$36 = 34 + (75 - 34)e^{-0.056t}$$
$$2 = 41e^{-0.056t}$$
$$t \approx 54 \text{ minutes}$$

**42.** **a.**

$$N(2) = 100(1.04 - 0.99^2)$$
$$N(2) = 6 \text{ wpm}$$

**b.**

$$N(40) = 100(1.04 - 0.99^{40})$$
$$N(40) \approx 37 \text{ wpm}$$

**c.**

$$60 = 100(1.04 - 0.99^t)$$
$$0.6 = 1.04 - 0.99^t$$
$$0.44 = 0.99^t$$
$$\ln 0.44 = \ln 0.99^t$$
$$\frac{\ln 0.44}{\ln 0.99} = t$$
$$82 \approx t$$

After 82 hours of practice, a student can expect to type 60 wpm.

**43.** **a.** 10% of 80,000 is 8000.

$$8000 = 80,000(1 - e^{-0.0005t})$$
$$0.1 = 1 - e^{-0.0005t}$$
$$-0.9 = -e^{-0.0005t}$$
$$0.9 = e^{-0.0005t}$$
$$\ln 0.9 = -0.0005t \ln e$$
$$\ln 0.9 = -0.0005t$$
$$\frac{\ln 0.9}{-0.0005} = t$$
$$211 \text{ h} \approx t$$

**b.** 50% of 80,000 is 40,000.

$$40,000 = 80,000(1 - e^{-0.0005t})$$
$$0.5 = 1 - e^{-0.0005t}$$
$$-0.5 = -e^{-0.0005t}$$
$$0.5 = e^{-0.0005t}$$
$$\ln(0.5) = \ln(e^{-0.0005t})$$
$$\ln(0.5) = -0.0005t$$
$$\frac{\ln(0.5)}{-0.0005} = t$$
$$1386 \text{ h} \approx t$$

**44.** **a.** 40% of 1,200,000 is 480,000.

$$480,000 = 1,200,000(1 - e^{-0.03})$$
$$0.4 = 1 - e^{-0.03t}$$
$$-0.6 = -e^{-0.03t}$$
$$0.6 = e^{-0.03t}$$
$$\ln(0.6) = \ln e^{-0.03t}$$
$$\ln(0.6) = -0.03t$$
$$\frac{\ln(0.6)}{-0.03} = t$$
$$t \approx 17 \text{ days}$$

**b.**

$$960,000 = 1,200,000(1 - e^{-0.03t})$$
$$0.8 = 1 - e^{-0.03t}$$
$$0.2 = e^{-0.03t}$$
$$\frac{\ln 0.2}{-0.03} = t$$
$$t \approx 54 \text{ days}$$

**45.**

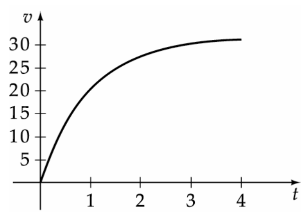
$$V(t) = V_0(1-r)^t$$
$$0.5V_0 = V_0(1-0.20)^t$$
$$0.5 = (1-0.20)^t$$
$$0.5 = 0.8^t$$
$$\ln 0.5 = \ln 0.8^t$$
$$\ln 0.5 = t \ln 0.8$$
$$\frac{\ln 0.5}{\ln 0.8} = t$$
$$3.1 \text{ years} \approx t$$

**46.** **a.** $I(0) = 6(1-e^0) = 0$ amps

**b.** $I(0.5) = 6(1-e^{-2.5(0.5)}) \approx 4.28$ amps

**c.**
$$I(t) = 6(1-e^{-2.56t})$$
$$\frac{I(t)}{6} - 1 = -e^{-2.5t}$$
$$1 - \frac{I(t)}{6} = e^{-2.5t}$$
$$\ln\left(1 - \frac{I(t)}{6}\right) = -2.5t$$
$$t = -\frac{2}{5}\ln\left(1 - \frac{I(t)}{6}\right)$$

**47.** **a.**

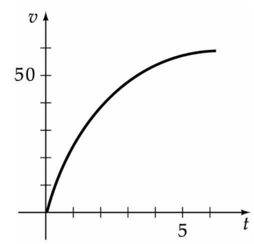



**b.**
$$20 = 64(1-e^{-t/2})$$
$$0.625 = 1 - e^{-t/2}$$
$$e^{-t/2} = 0.375$$
$$-t/2 = \ln 0.375$$
$$t \approx 0.98 \text{ seconds}$$

**c.** The horizontal asymptote is $v = 32$.

**d.** As time increases, the object's velocity approaches but never exceeds 32 ft/sec.

**48.** **a.**

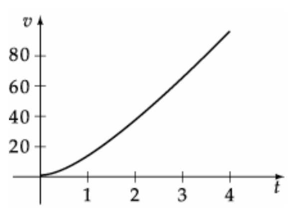



**b.**
$$50 = 64(1-e^{-t/2})$$
$$0.78125 = 1 - e^{-t/2}$$
$$e^{-t/2} = 0.21875$$
$$-t/2 = \ln 0.21875$$
$$t \approx 3.0 \text{ seconds}$$

**c.** The horizontal asymptote is $v = 64$.

**d.** As time increases, the object's velocity approaches but never exceeds 64 ft/sec.

**49.** **a.**

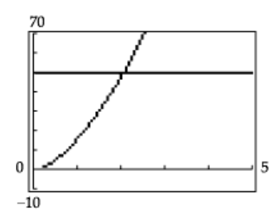



**b.** The graphs of $s = 32t + 32(e^{-t} - 1)$ and $s = 50$ intersect when $t \approx 2.5$ seconds.

**c.** The slope $m$ of the secant line containing $(1, s(1))$ and $(2, s(2))$ is $m = \frac{s(2) - s(1)}{2-1} \approx 24.56$ ft/sec

**d.** The average speed of the object was 24.56 feet per second between $t = 1$ and $t = 2$.

**50.** **a.**

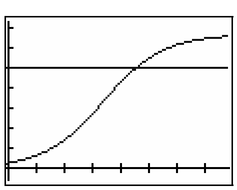



**b.** The graphs of $s = 64t + 128(e^{-t/2} - 1)$ and $s = 50$ intersect when $t \approx 2.1$ seconds.

**c.** The slope $m$ of the secant line containing $(1, s(1))$ and $(2, s(2))$ is $m = \frac{s(2) - s(1)}{2-1} \approx 33.5$ ft/sec

**d.** The average speed of the object was 33.5 feet per second between $t = 1$ and $t = 2$.

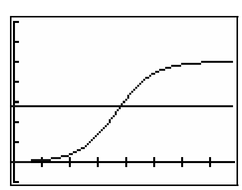
**51.**



Xmin = 0, Xmax = 80, Xscl = 10,
Ymin = −10, Ymax = 110, Yscl = 15

When $P = 75\%$, $t \approx 45$ hours.

**52.**



Xmin = 0, Xmax = 24, Xscl = 3,
Ymin = −50, Ymax = 350, Yscl = 50

When $N = 140$ circuits, $t \approx 11$ weeks.

## Connecting Concepts

**53.** **a.** $A(1) = 0.5^{1/2}$
$\approx 0.71$ gram

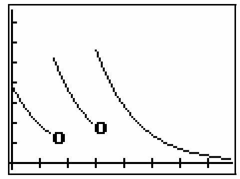
**b.** $A(4) = 0.5^{4/2} + 0.5^{(4-3)/2}$
$= 0.5^2 + 0.5^{1/2}$
$\approx 0.96$ gram

**c.** $A(9) = 0.5^{9/2} + 0.5^{(9-3)/2} + 0.5^{(9-6)/2}$
$= 0.5^{4.5} + 0.5^3 + 0.5^{1.5}$
$\approx 0.52$ gram

**54.** Use the TRACE feature and the graph of

$$y = \begin{cases} 0.5^{x/2} & 0 \le x < 3 \\ 0.5^{x/2} + 0.5^{(x-3)/2} & 3 \le x < 6 \\ 0.5^{x/2} + 0.5^{(x-3)/2} + 0.5^{(x-6)/2} & x \ge 6 \end{cases}$$

to determine the value of $x$ when $y = 0.25$ the first time.



Xmin = 0, Xmax = 16, Xscl = 2,
Ymin = 0, Ymax = 2, Yscl = 0.25

After 11.1 hours

**55.** $N(t) = 22,755e^{0.0287t}$
$= 22,755\left(e^{0.0287}\right)^t$
$\approx 22,755(1.0291)^t$

The annual growth rate is 2.91%.

**56.** **a.** $V(3) = 350,000\left(\frac{7}{8}\right)^{3/2} \approx 286,471$ gallons

**b.** $V(5) = 350,000\left(\frac{7}{8}\right)^{5/2} \approx 250,662$ gallons

**c.** $0.10(350,000) = 350,000(.875)^{t/2}$
$0.10 = 0.875^{t/2}$
$\ln 0.10 = \frac{t}{2}\ln 0.875$
$t = 2\frac{\ln 0.10}{\ln 0.875} \approx 34$
$t \approx 34$ hours

## Prepare for Section 4.7

**PS1.** Decreasing

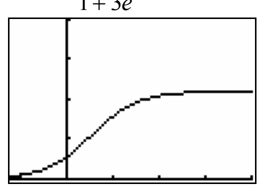
**PS2.** Decreasing

**PS3.** $P(0) = \frac{108}{1+2e^{-0.1(0)}} = \frac{108}{1+2} = 36$

**PS4.** $N(0) = 840e^{1.05(0)} = 840$

**PS5.**
$10 = \frac{20}{1+2.2e^{-0.05t}}$
$10(1+2.2e^{-0.05t}) = 20$
$10 + 22e^{-0.05t} = 20$
$e^{-0.05t} = \frac{10}{22}$
$\ln e^{-0.05t} = \ln\frac{10}{22}$
$-0.05t = \ln\frac{10}{22}$
$t = -20\ln\frac{10}{22}$
$t \approx 15.8$

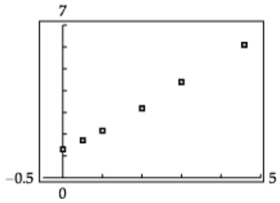
**PS6.** $P(t) = \frac{55}{1+3e^{-0.08t}}$



There is a horizontal asymptote at $P = 55$.

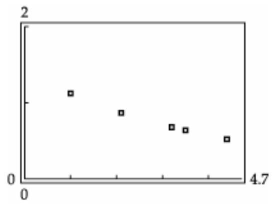
## Section 4.7

**1.**



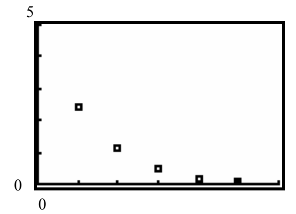

increasing exponential function

**2.**



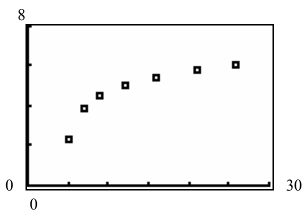

decreasing exponential function; decreasing logarithmic function

**3.**



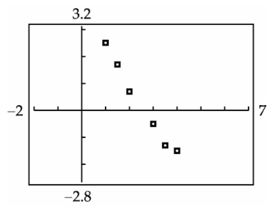

decreasing exponential function; decreasing logarithmic function

**4.**



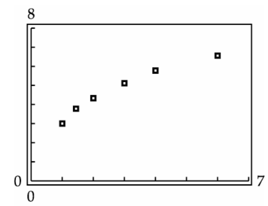

increasing logarithmic function

**5.**



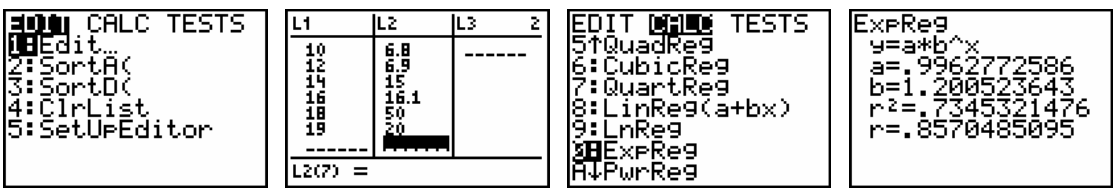

decreasing logarithmic function

**6.**



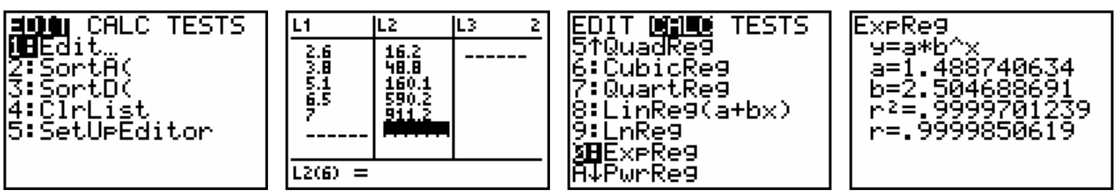

increasing logarithmic function

**7.**



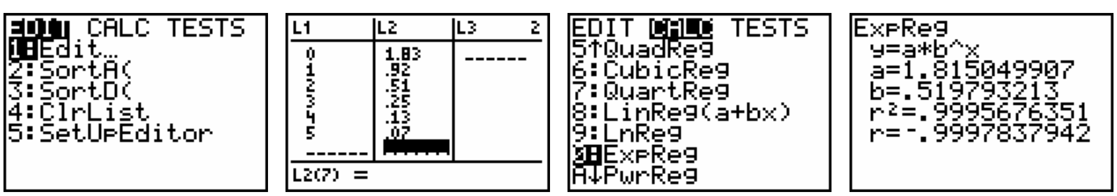

$y \approx 0.99628(1.20052)^x$; $r \approx 0.85705$

**8.**



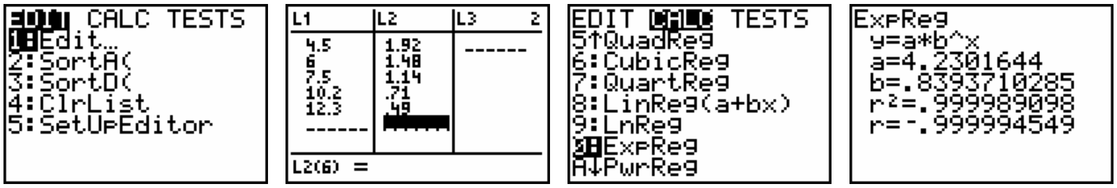

$y \approx 1.48874(2.50469)^x$; $r \approx 0.99999$

**9.**



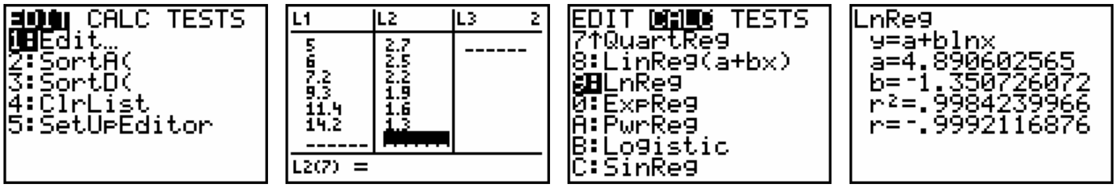

$y \approx 1.81505(0.51979)^x$; $r \approx -0.99978$

**10.**



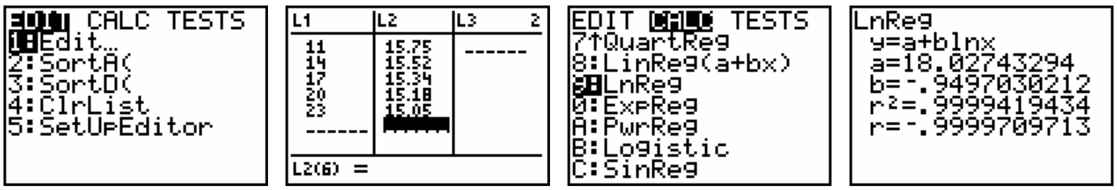

$y \approx 4.23016(0.83937)^x$; $r \approx -0.99999$

**11.**



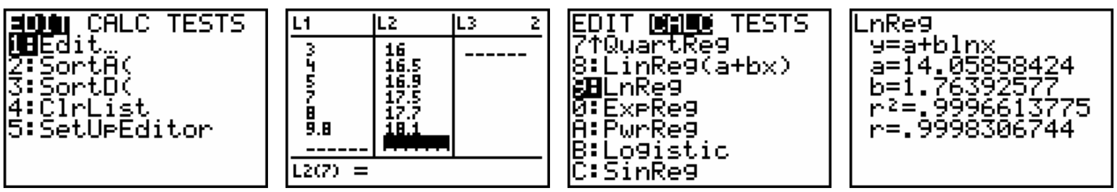

$y \approx 4.89060 - 1.35073 \ln x$; $r \approx -0.99921$

**12.**



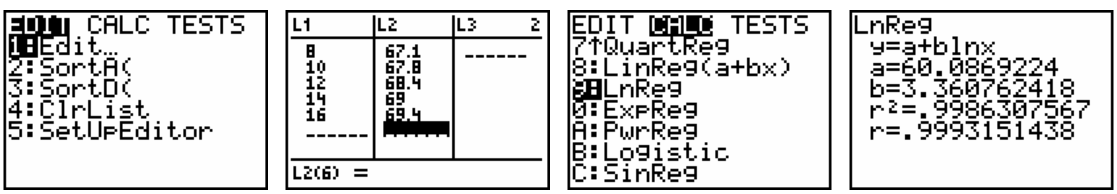

$y \approx 18.02743 - 0.94970 \ln x$; $r \approx -0.99997$

**13.**



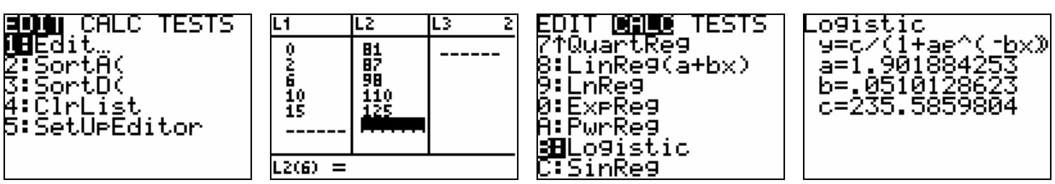

$y \approx 14.05858 + 1.76393 \ln x$; $r \approx 0.99983$

**14.**



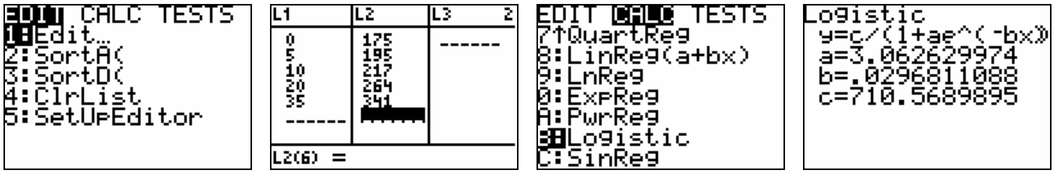

$y \approx 60.08692 + 3.36076 \ln x$; $r \approx 0.99932$

**15.**



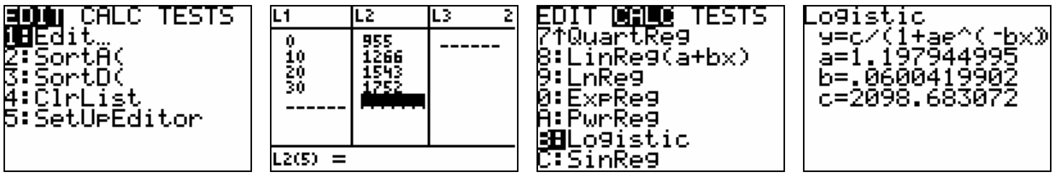

$$y \approx \frac{235.58598}{1 + 1.90188e^{-0.05101x}}$$

**16.**



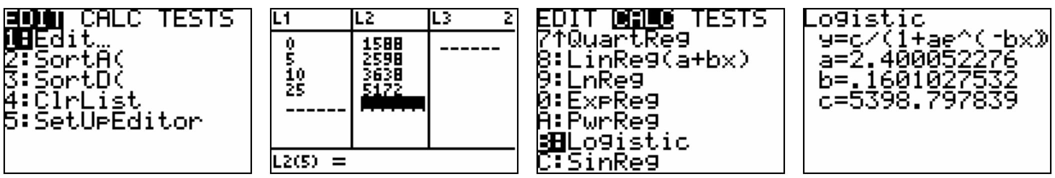

$$y \approx \frac{710.56899}{1+3.06263e^{-0.02968x}}$$

**17.**



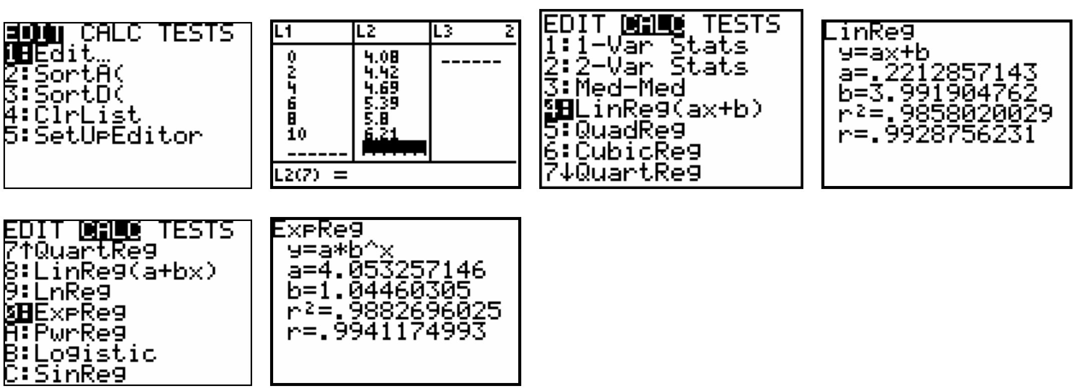

$$y \approx \frac{2098.68307}{1+1.19794e^{-0.06004x}}$$

**18.**




$$y \approx \frac{5398.79784}{1+2.40005e^{-0.16010x}}$$

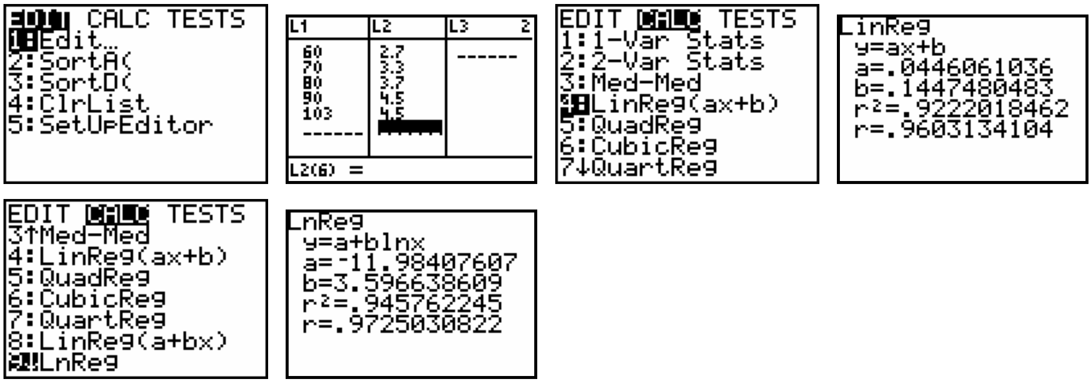
**19.**




**a.** Linear: $p \approx 0.22129t + 3.99190$; $r \approx 0.99288$ . Exponential: $p \approx 4.05326(1.04460)^t$ ; $r \approx 0.99412$

**b.** The exponential model's $r$ is closer to 1.

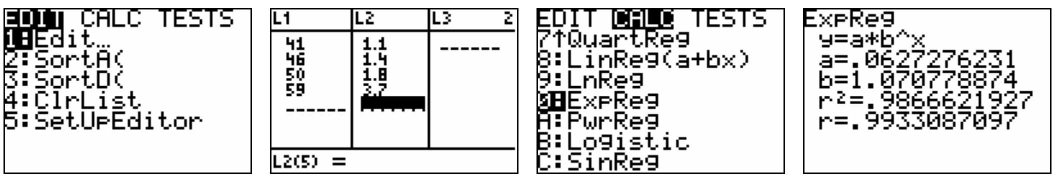
**c.** $p = 4.05326(1.04460)^{16} \approx \$8.15$

**20.** 


**a.** Linear: $p \approx 0.04461t + 0.14475$. Logarithmic: $p \approx -11.98408 + 3.59664 \ln t$

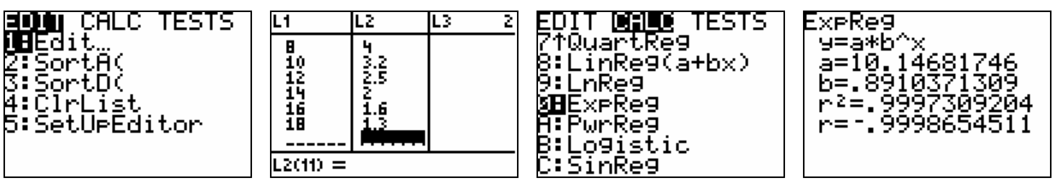
**b.** Linear: $r \approx 0.96031$. Logarithmic: $r \approx 0.97250$. The logarithmic model provides a slightly better fit.

**c.** $p \approx -11.98408 + 3.59664 \ln(109) \approx 4.89$ lbs per capita per day

**21.** 

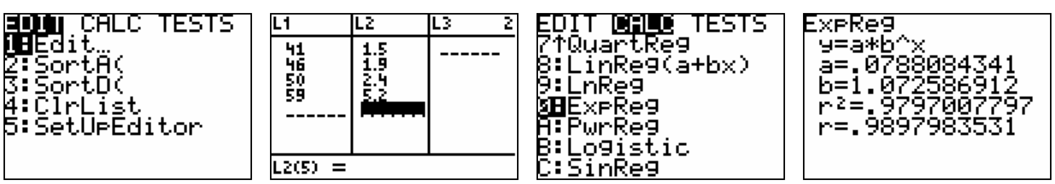

**a.** Exponential: $T \approx 0.06273(1.07078)^F$

**b.** $T \approx 0.06273(1.07078)^{65} \approx 5.3$ hours

**22.** 


$P \approx 10.147(0.89104)^x$

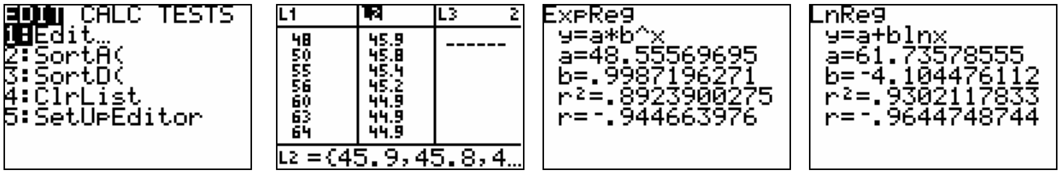
$P \approx 10.147(0.89104)^{24} \approx 0.6$ newtons/cm$^2$

**23.** 


**a.** $T \approx 0.7881(1.07259)^F$

**b.** $T \approx 0.07881(1.07259)^{65} \approx 7.5$ hours
7.5 – 5.3 = 2.2 hours.

**24.**




Exponential: $y \approx 48.5557(0.99872)^x$

Logarithmic: $y \approx 61.735786 - 4.104476 \ln x$

**a.** Exponential: $y \approx 48.5557(0.99872)^{48} \approx 45.7$

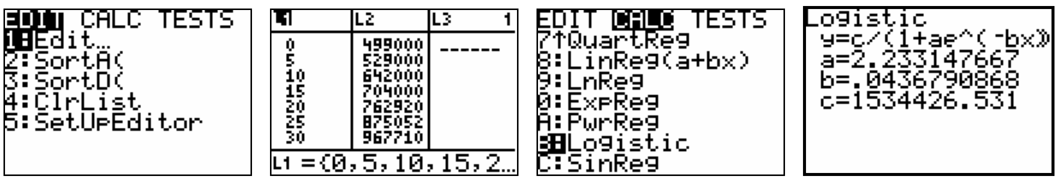
Logarithmic: $y \approx 61.735786 - 4.104476 \ln 48 \approx 45.8$

The decreasing logarithmic best models the data.

**b.** $y \approx 61.735786 - 4.104476 \ln 108 \approx 42.52$ s

**25.** An increasing logarithmic model provides a better fit because of the concave-downward nature of the graph.

**26.**

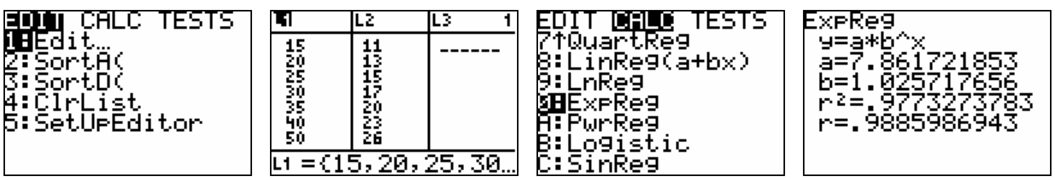



**a.** $P(t) \approx \dfrac{1{,}534{,}427}{1 + 2.233148e^{-0.043679\,t}}$

**b.** $P(60) \approx \dfrac{1{,}534{,}427}{1 + 2.233148e^{-0.043679\,(60)}} \approx 1{,}320{,}000$ people

**c.** $c \approx 1{,}534{,}000$

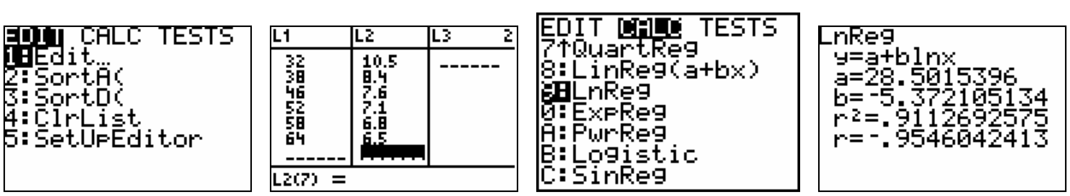
**27.**




**a.** $p \approx 7.862(1.026)^y$

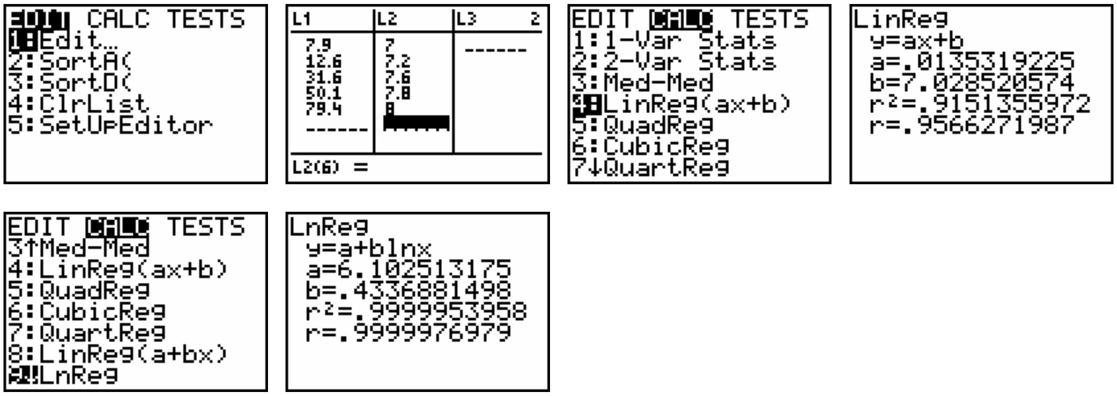
**b.** $p \approx 7.862(1.026)^{60} \approx 36$ cm

**28.**



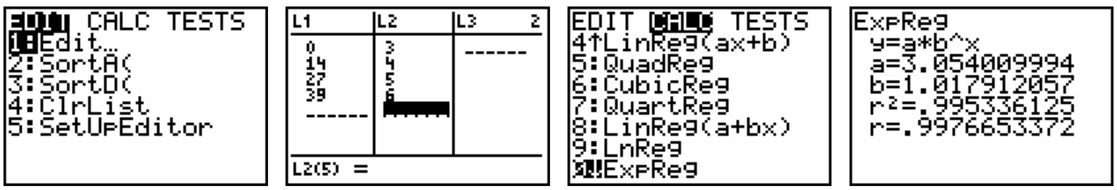

**a.** $T \approx 28.502 - 5.372 \ln x$

**b.** $T \approx 28.502 - 5.372 \ln(50) = 7.5$ ml/L

**29.** 


**a.** Linear: $\text{pH} \approx 0.01353q + 7.02852;\ r \approx 0.956627.$
Logarithmic: $\text{pH} \approx 6.10251 + 0.43369 \ln q;\ r \approx 0.999998.$ The logarithmic model provides the better fit.

**b.** $8.2 \approx 6.10251 + 0.43369 \ln q \Rightarrow 2.09749 \approx 0.43369 \ln q \Rightarrow \dfrac{2.09749}{0.43369} \approx \ln q \Rightarrow q \approx e^{\frac{2.09749}{0.43369}} \approx 126.0$

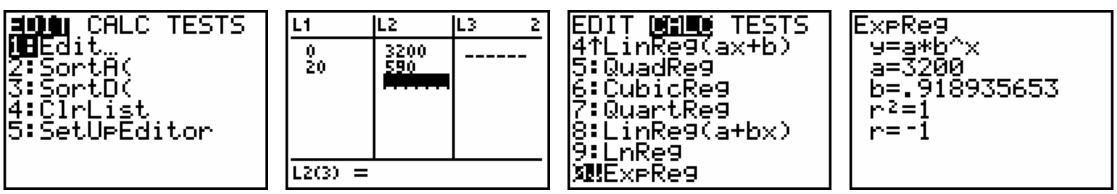
**30.** 


**a.** $y \approx 3.05401(1.0179)^x$

**b.** $8 \approx 3.05401(1.0179)^x \Rightarrow \dfrac{8}{3.05401} \approx (1.0179)^x \Rightarrow \ln \dfrac{8}{3.05401} \approx \ln(1.0179)^x$

$\Rightarrow \ln 8 - \ln 3.05401 \approx x \ln 1.0179 \Rightarrow x \approx \dfrac{\ln 8 - \ln 3.05401}{\ln 1.0179}$

$\Rightarrow x \approx 54.3$ years after 1960 $\Rightarrow$ in 2014

**31.** 

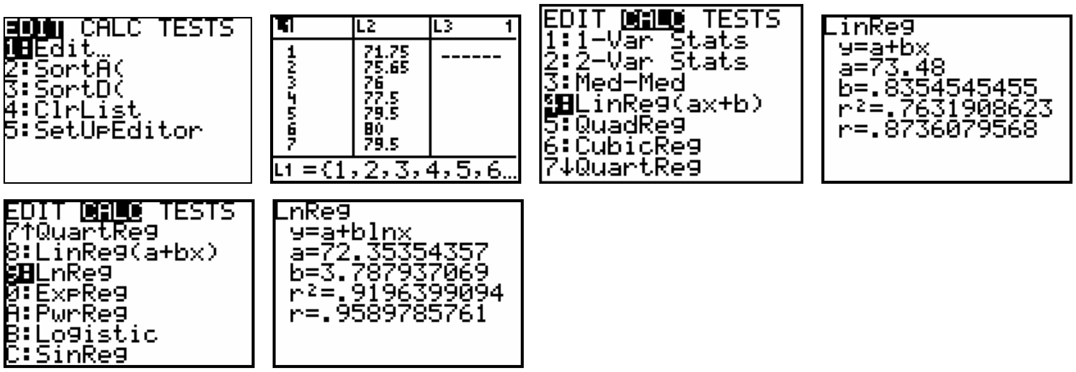

**a.** $p \approx 3200(0.91894)^t;\ \ 200 \approx 3200(0.91894)^t \Rightarrow \dfrac{1}{16} \approx (0.91894)^t \Rightarrow \ln \dfrac{1}{16} \approx \ln(0.91894)^t$

$\Rightarrow \ln 1 - \ln 16 \approx t \ln 0.91894 \Rightarrow t \approx \dfrac{-\ln 16}{\ln 0.91894}$

$\Rightarrow t \approx 32.8$ years after 1980 $\Rightarrow$ in 2012

**b.** No. The model fits the data perfectly because there are only two data points.

**32.**

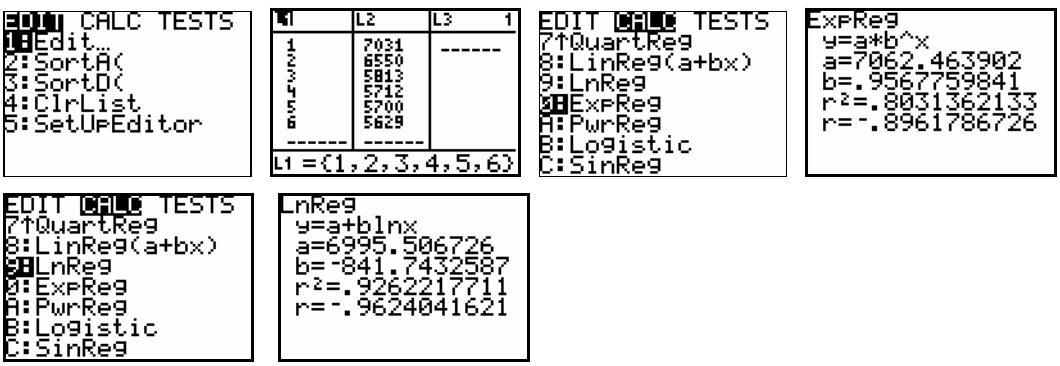



**a.** Linear: $y \approx 0.83545x + 73.48,\ r \approx 0.87361$; Logarithmic: $y \approx 72.35354 + 3.78794 \ln x,\ r \approx 0.95898$

**b.** The logarithmic regression provides the better fit.

**c.** $y \approx y \approx 72.35354 + 3.78794 \ln 12 \approx 81.8$ inches (6 feet 9.8 inches)

**33.**

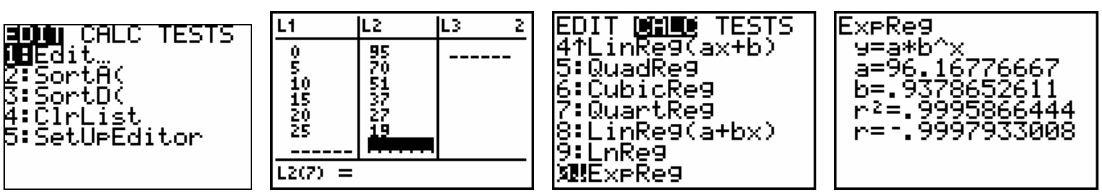



**a.** Exponential: $S \approx 7062.46390(0.956776)^t,\ r \approx -0.89618$; Logarithmic: $S \approx 6995.50673 - 841.74326 \ln t,\ r \approx -0.96240$

**b.** The logarithmic regression provides the better fit.

**c.** $S(11) \approx 6995.50673 - 841.74326 \ln 11 \approx 4977$ sites

**34.**



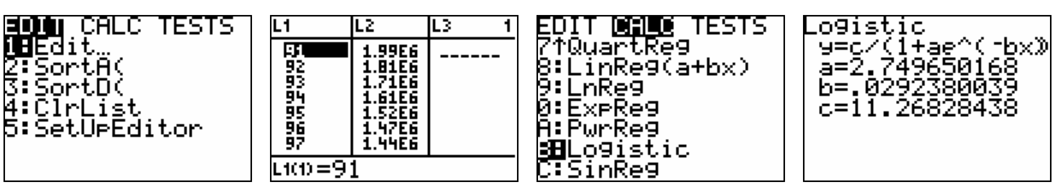

**a.** $T - 70^\circ \approx 96.16777(0.93787)^t$

**b.** $80 - 70 \approx 96.16777(0.93787)^t$

$$10 \approx 96.16777(0.93787)^t \Rightarrow \frac{10}{96.16777} \approx 0.93787^t \Rightarrow \ln\frac{10}{96.16777} \approx \ln 0.93787^t$$

$$\Rightarrow \ln 10 - \ln 96.16777 \approx t \ln 0.93787 \Rightarrow t \approx \frac{\ln 10 - \ln 96.16777}{\ln 0.93787} \approx 35 \text{ min}$$

**35.**



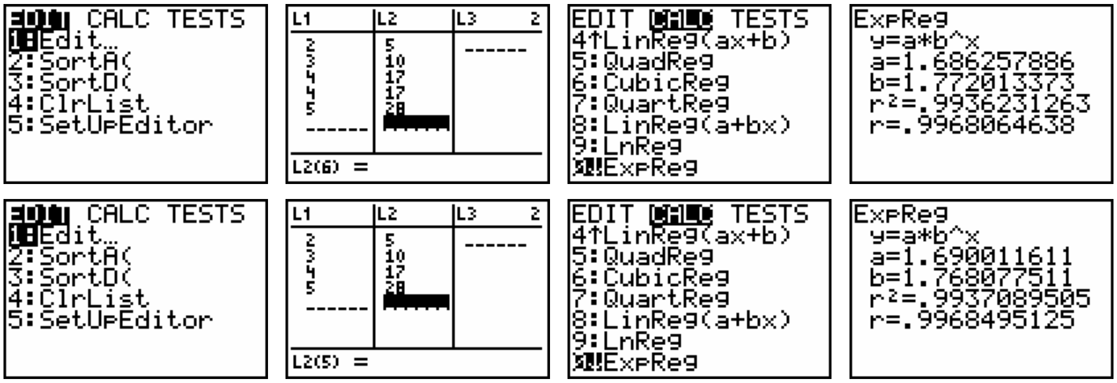

**a.** $P(t) \approx \dfrac{11.26828}{1 + 2.74965e^{-0.02924t}}$

**b.** As $t \to \infty,\ P(t) \to 11$ billion people $t \to \infty,\ P(t) \to 11$ billion people

**36.** The graph of the logarithmic regression equation passes through both of the data points.

## Connecting Concepts

**37.**

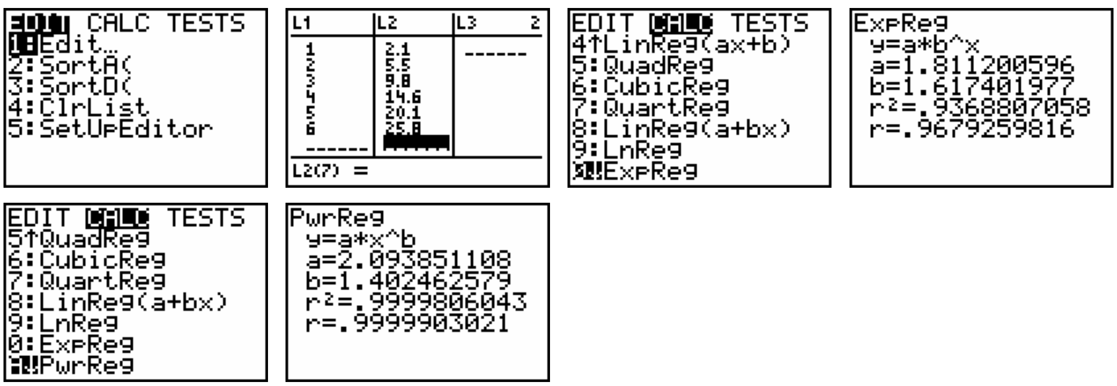



*A* and *B* have different exponential regression functions.

**38.** **a.** The $x$-coordinate of the first ordered pair is 0, and 0 is not in the domain of $y = \ln x$.

**b.** Use a horizontal translation. For instance, add 1 to each of the $x$-coordinates. Find the logarithmic regression function for this new data. Remember that each $x$-value in the regression represents $x - 1$ in the original data.

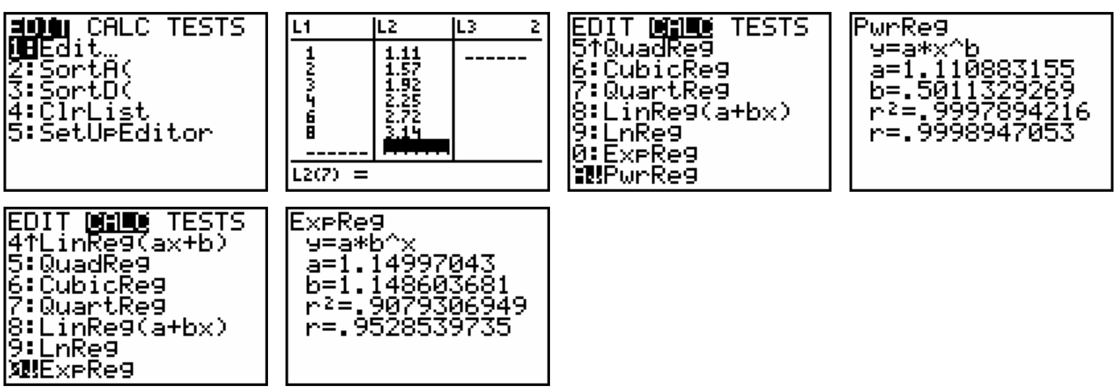
**39.**




**a.** Exponential: $y \approx 1.81120(1.61740)^x$; $r \approx 0.96793$. Power: $y \approx 2.09385(x^{1.40246})$; $r \approx 0.99999$.

**b.** The power regression provides the better fit.

**40.**



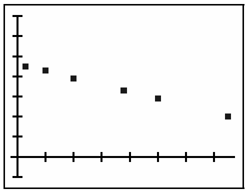

**a.** The power regression function, $t \approx 1.11088(l^{0.50113})$, provides the better fit.

**b.** $12 \approx 1.11088(l^{0.50113}) \Rightarrow \dfrac{12}{1.11088} \approx (l^{0.50113}) \Rightarrow \left(\dfrac{12}{1.11088}\right)^{(1/0.50113)} \approx (l^{0.50113})^{(1/0.50113)} \approx 115.4$ feet

## Exploring Concepts with Technology

**Using a Semilog Graph to Model Exponential Decay**

**1.** **a.**



Xmin = –1, Xmax = 31, Xscl = 4,
Ymin = –1, Ymax = 7, Yscl = 1

**b.** $m = \dfrac{4.3 - 3.3}{4 - 15} \approx -0.0909$

**c.** $\ln A - 4.3 = -0.0909(t - 4)$

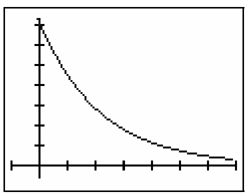
$\ln A = -0.0909t + 4.664$

**d.** $e^{\ln A} = e^{-0.0909\,t + 4.664}$

$A = e^{-0.0909\,t} e^{4.664}$

$A = e^{-0.0909\,t} e^{4.664}$

**e.**



Xmin = –5, Xmax = 35, Xscl = 5,
Ymin = –10, Ymax = 110, Yscl = 15

**f.** $A(t) = 106e^{-0.0909t}$

At $t = 0$ there is $A = 106$ mg present

We must find *t* where $A = \frac{1}{2}(106) = 53$ mg.

$53 = 106e^{-0.0909\,t}$

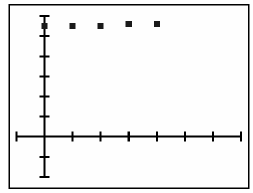
$\frac{53}{106} = e^{-0.0909\,t}$

$\frac{1}{2} = e^{-0.0909\,t}$

$\ln 0.5 = -0.0909t$

$t = \dfrac{\ln 0.5}{-0.0909} \approx 7.6$ days

**2.** **a.**



Xmin = –1, Xmax =7.1, Xscl = 1,
Ymin = –1, Ymax =3, Yscl = 0.5

**b.** $m = \dfrac{2.785 - 2.754}{3 - 1} = \dfrac{0.031}{2} \approx 0.0155$

**c.** $\ln B - 2.574 = 0.0155(t - 1)$

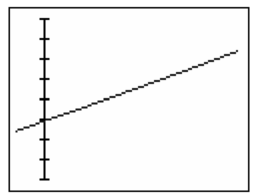
$\ln B = 0.0155t + 2.7385$

**d.** $e^{\ln B} = e^{0.0155\,t + 2.7385}$

$B = e^{0.0155\,t} e^{2.7385}$

$B = 15.46e^{0.0155\,t}$

**e.**



Xmin = –1, Xmax = 7, Xscl = 1,
Ymin = 14, Ymax = 18, Yscl = 0.5

**f.** If $B = 17.5$, use a graphing calculator to determine that the point of intersection of $y_1 = 15.46e^{0.0155\,t}$ and $y_2 = 17.5$ is $t \approx 8$ years.

$1986 + 8 = 1994$

Or algebraically, solve for *t*:

$17.5 = 15.46e^{0.0155\,t}$

$\frac{17.5}{15.46} = e^{0.0155\,t}$

$1.13195 = e^{0.0155\,t}$

$\ln(1.13195) = 0.0155\,t \ln e$

$\dfrac{\ln(1.13195)}{0.0155} = t$

$8 \approx t$

$1986 + 8 = 1994$

## Assessing Concepts

**1.** False; $f(x) = x^2$ does not have an inverse function.

**2.** True

**3.** True

**4.** False; $h(x)$ is not an increasing function for $0 < b < 1$.

**5.** False; $j(x)$ is not an increasing function for $0 < b < 1$.

**6.** c

**7.** b

**8.** f

**9.** a

**10.** e

**11.** d

**12.** g

## Chapter Review

**1.** $F[G(x)] = F\left(\frac{x+5}{2}\right) = 2\left(\frac{x+5}{2}\right) - 5 = x + 5 - 5 = x$ [4.1]

$G[F(x)] = G(2x-5) = \frac{2x-5+5}{2} = \frac{2x}{2} = x$

Yes, $F$ and $G$ are inverses.

**2.** $h[k(x)] = h\left(x^2\right) = \sqrt{x^2} = x$ [4.1]

$k[h(x)] = k\left(\sqrt{x}\right) = \left(\sqrt{x}\right)^2 = x$

Yes, $h$ and $k$ are inverses.

**3.** $l[m(x)] = l\left(\frac{3}{x-1}\right) = \frac{\frac{3}{x-1}+3}{\frac{3}{x-1}} = \frac{3+3(x-1)}{3} = \frac{3+3x-3}{3} = \frac{3x}{3} = x$ [4.1]

$m[l(x)] = m\left(\frac{x+3}{x}\right) = \frac{3}{\frac{x+3}{x}-1} = \frac{3x}{x+3-x} = \frac{3x}{3} = x$

Yes, $l$ and $m$ are inverses.

**4.** $p[q(x)] = p\left(\frac{2x}{x-5}\right) = \frac{\frac{2x}{x-5}-5}{2\left(\frac{2x}{x-5}\right)} = \frac{2x-5(x-5)}{2(2x)} = \frac{2x-5x+25}{4x} = \frac{-3x+25}{4x} \neq x$ [4.1]

No, $p$ and $q$ are not inverses.

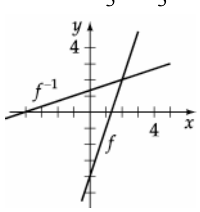
**5.** $y = 3x - 4$ [4.1]

$x = 3y - 4$

$x + 4 = 3y$

$\frac{x+4}{3} = y$

$f^{-1}(x) = \frac{1}{3}x + \frac{4}{3}$



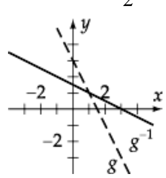

**6.** $y = -2x + 3$ [4.1]

$x = -2y + 3$

$x - 3 = -2y$

$\frac{x-3}{-2} = y$

$g^{-1}(x) = -\frac{1}{2}x + \frac{3}{2}$



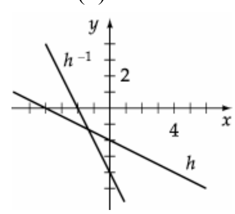

**7.** $y = -\frac{1}{2}x - 2$ [4.1]

$x = -\frac{1}{2}y - 2$

$x + 2 = -\frac{1}{2}y$

$-2(x+2) = y$

$h^{-1}(x) = -2x - 4$



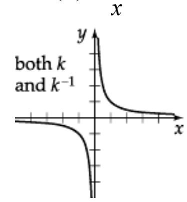

**8.** $y = \frac{1}{x}$ [4.1]

$x = \frac{1}{y}$

$xy = 1$

$y = \frac{1}{x}$

$k^{-1}(x) = \frac{1}{x}$




**9.** $\log_5 25 = x$ [4.3]

$5^x = 25$

$5^x = 5^2$

$x = 2$

**10.** $\log_3 81 = x$ [4.3]

$3^x = 81$

$3^x = 3^4$

$x = 4$

**11.** $\ln e^3 = x$ [4.3]

$e^x = e^3$

$x = 3$

**12.** $\ln e^{\pi} = x$ [4.3]

$e^x = e^{\pi}$

$x = \pi$

**13.** $3^{2x+7} = 27$ [4.5]

$3^{2x+7} = 3^3$

$2x + 7 = 3$

$2x = -4$

$x = -2$

**14.** $5^{x-4} = 625$ [4.5]

$5^{x-4} = 5^4$

$x - 4 = 4$

$x = 8$

**15.** $2^x = \frac{1}{8}$ [4.5]

$2^x = 2^{-3}$

$x = -3$

**16.** $27\left(3^x\right) = 3^{-1}$ [4.5]

$27\left(3^x\right) = \frac{1}{3}$

$3^x = \frac{1}{81}$

$3^x = 3^{-4}$

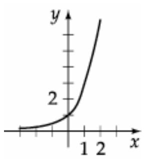
$x = -4$

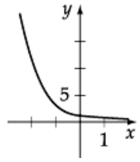
**17.** $\log x^2 = 6$ [4.5]
$10^6 = x^2$
$1{,}000{,}000 = x^2$
$\pm\sqrt{1{,}000{,}000} = x$
$\pm 1000 = x$

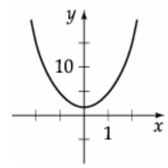
**18.** $\frac{1}{2}\log|x| = 5$ [4.5]
$\log|x| = 10$
$10^{10} = |x|$
$x = \pm 10^{10}$

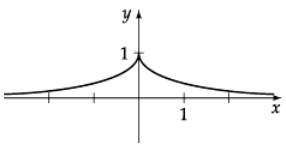
**19.** $10^{\log 2x} = 14$ [4.5]
$2x = 14$
$x = 7$

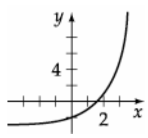
**20.** $e^{\ln x^2} = 64$ [4.5]
$x^2 = 64$
$x = \pm 8$

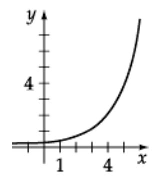
**21.**


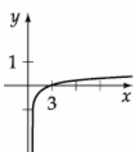

**22.**


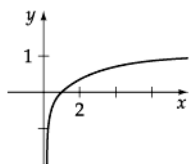

**23.**


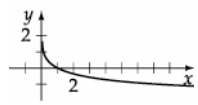

**24.**


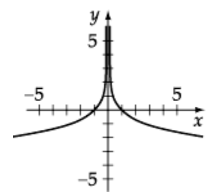

**25.**


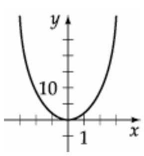

**26.**


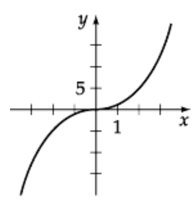

**27.**



**28.**



**29.**



**30.**



**31.**



**32.**



**33.** $\log_4 64 = 3$ [4.3]
$4^3 = 64$

**34.** $\log_{1/2} 8 = -3$ [4.3]
$\left(\frac{1}{2}\right)^{-3} = 8$

**35.** $\log_{\sqrt{2}} 4 = 4$ [4.3]
$\left(\sqrt{2}\right)^4 = 4$

**36.** $\ln 1 = 0$ [4.3]
$e^0 = 1$

**37.** $5^3 = 125$ [4.3]
$\log_5 125 = 3$

**38.** $2^{10} = 1024$ [4.3]
$\log_2 1024 = 10$

**39.** $10^0 = 1$ [4.3]
$\log_{10} 1 = 0$

**40.** $8^{1/2} = 2\sqrt{2}$ [4.3]
$\log_8 2\sqrt{2} = \frac{1}{2}$

**41.** $\log_b \frac{x^2 y^3}{z} = 2\log_b x + 3\log_b y - \log_b z$ [4.4]

**42.** $\log_b \frac{\sqrt{x}}{y^2 z} = \frac{1}{2}\log_b x - \left(2\log_b y + \log_b z\right)$ [4.4]
$= \frac{1}{2}\log_b x - 2\log_b y - \log_b z$

**43.** $\ln xy^3 = \ln x + 3\ln y$ [4.4]

**44.** $\ln \frac{\sqrt{xy}}{z^4} = \frac{1}{2}\left(\ln x + \ln y\right) - 4\ln z$ [4.4]
$= \frac{1}{2}\ln x + \frac{1}{2}\ln y - 4\ln z$

**45.** $2\log x + \frac{1}{3}\log(x+1) = \log\left(x^2\sqrt[3]{x+1}\right)$ [4.4]

**46.** $5\log x - 2\log(x+5) = \log\frac{x^5}{(x+5)^2}$ [4.4]

**47.** $\frac{1}{2}\ln 2xy - 3\ln z = \ln\frac{\sqrt{2xy}}{z^3}$ [4.4]

**48.** $\ln x - (\ln y - \ln z) = \ln\frac{x}{y/z} = \ln\frac{xz}{y}$ [4.4]

**49.** $\log_5 101 = \frac{\log 101}{\log 5} \approx 2.86754$ [4.4]

**50.** $\log_3 40 = \frac{\log 40}{\log 3} \approx 3.35776$ [4.4]

**51.** $\log_4 0.85 = \frac{\log 0.85}{\log 4} \approx -0.117233$ [4.4]

**52.** $\log_8 0.3 = \frac{\log 0.3}{\log 8} \approx -0.578989$ [4.4]

**53.** [4.5]

$$\begin{aligned} 4^x &= 30 \\ \log 4^x &= \log 30 \\ x\log 4 &= \log 30 \\ x &= \frac{\log 30}{\log 4} \end{aligned}$$

**54.** [4.5]

$$\begin{aligned} 5^{x+1} &= 41 \\ (x+1)\log 5 &= \log 41 \\ x+1 &= \frac{\log 41}{\log 5} \\ x &= \frac{\log 41}{\log 5} - 1 \end{aligned}$$

**55.** [4.5]

$$\begin{aligned} \ln(3x) - \ln(x-1) &= \ln 4 \\ \ln\frac{3x}{x-1} &= \ln 4 \\ \frac{3x}{x-1} &= 4 \\ 3x &= 4(x-1) \\ 3x &= 4x-4 \\ 4 &= x \end{aligned}$$

**56.** [4.5]

$$\begin{aligned} \ln(3x) + \ln 2 &= \ln 1 \\ \ln(3x\cdot 2) &= 1 \\ \ln(6x) &= 1 \\ e^1 &= 6x \\ \frac{e}{6} &= x \end{aligned}$$

**57.** [4.5]

$$\begin{aligned} e^{\ln(x+2)} &= 6 \\ (x+2) &= 6 \\ x+2 &= 6 \\ x &= 4 \end{aligned}$$

**58.** [4.5]

$$\begin{aligned} 10^{\log(2x+1)} &= 31 \\ 2x+1 &= 31 \\ 2x &= 30 \\ x &= 15 \end{aligned}$$

**59.** [4.5]

$$\begin{aligned} \frac{4^x + 4^{-x}}{4^x - 4^{-x}} &= 2 \\ 4^x\left(4^x + 4^{-x}\right) &= 2\left(4^x - 4^{-x}\right)4^x \\ 4^{2x} + 1 &= 2\left(4^{2x} - 1\right) \\ 4^{2x} + 1 &= 2\left(4^{2x} - 1\right) \\ 4^{2x} - 2\cdot 4^{2x} + 3 &= 0 \\ 4^{2x} &= 3 \\ 2^x \ln 4 &= \ln 3 \\ x &= \frac{\ln 3}{2\ln 4} \end{aligned}$$

**60.** [4.5]

$$\begin{aligned} \frac{5^x + 5^{-x}}{2} &= 8 \\ 5^x\left(5^x + 5^{-x}\right) &= 16\left(5^x\right) \\ 5^{2x} + 1 &= 16\left(5^x\right) \\ 5^{2x} - 16\left(5^x\right) + 1 &= 0 \end{aligned}$$

Let $5^x = u$

$$\begin{aligned} u^2 - 16u + 1 &= 0 \\ u &= \frac{16 \pm \sqrt{16^2 - 4(1)(1)}}{2} \\ u &= \frac{16 \pm \sqrt{252}}{2} \\ u &= \frac{16 \pm 6\sqrt{7}}{2} \\ u &= 8 \pm 3\sqrt{7} \\ 5^x &= 8 \pm 3\sqrt{7} \\ x &= \frac{\ln\left(8 \pm 3\sqrt{7}\right)}{\ln 5} \end{aligned}$$

**61.** [4.5]

$$\begin{aligned} \log(\log x) &= 3 \\ 10^3 &= \log x \\ 10^{(10^3)} &= x \\ 10^{1000} &= x \end{aligned}$$

**62.** [4.5]

$$\begin{aligned} \ln(\ln x) &= 2 \\ e^2 &= \ln x \\ e^{(e^2)} &= x \end{aligned}$$

**63.** [4.5]

$$\begin{aligned} \log\sqrt{x-5} &= 3 \\ 10^3 &= \sqrt{x-5} \\ 10^6 &= x-5 \\ 10^6 + 5 &= x \\ x &= 1{,}000{,}005 \end{aligned}$$

**64.** $\log x + \log(x-15) = 1$

$\log x(x-15) = 1$

$10 = x^2 - 15x$

$0 = x^2 - 15x - 10$

$x = \dfrac{15 \pm \sqrt{15^2 - 4(1)(-10)}}{2}$

$x = \dfrac{15 \pm \sqrt{265}}{2}$

$x = \dfrac{15 + \sqrt{265}}{2}$ [4.5]

**65.** $\log_4(\log_3 x) = 1$

$4 = \log_3 x$

$3^4 = x$

$81 = x$ [4.5]

**66.** $\log_7(\log_5 x^2) = 0$

$7^0 = \log_5 x^2$

$1 = \log_5 x^2$

$5 = x^2$

$\pm\sqrt{5} = x$ [4.5]

**67.** $\log_5 x^3 = \log_5 16x$ [4.5]

$x^3 = 16x$

$x^2 = 16$

$x = 4$

**68.** $25 = 16^{\log_4 x}$ [4.5]

$25 = 4^{2\log_4 x}$

$25 = 4^{\log_4 x^2}$

$25 = x^2$

$\pm 5 = x$

$5 = x$

**69.** $m = \log\left(\dfrac{I}{I_0}\right)$ [4.4]

$= \log\left(\dfrac{51{,}782{,}000 I_0}{I_0}\right)$

$= \log 51{,}782{,}000$

$\approx 7.7$

**70.** $M = \log A + 3\log 8t - 2.92$ [4.4]

$= \log 18 + 3\log 8(21) - 2.92$

$= \log 18 + 3\log 168 - 2.92$

$\approx 5.0$

**71.** $\log\left(\dfrac{I_1}{I_0}\right) = 7.2$ and $\log\left(\dfrac{I_2}{I_0}\right) = 3.7$ [4.4]

$\dfrac{I_1}{I_0} = 10^{7.2}$ $\qquad \dfrac{I_2}{I_0} = 10^{3.7}$

$I_1 = 10^{7.2} I_0$ $\qquad I_2 = 10^{3.7} I_0$

$\dfrac{I_1}{I_2} = \dfrac{10^{7.2} I_0}{10^{3.7} I_0} = \dfrac{10^{3.5}}{1} \approx \dfrac{3162}{1}$

3162 to 1

**72.** $\dfrac{I_1}{I_2} = 600 = 10^x$ [4.4]

$\log 600 = \log 10^x$

$\log 600 = x$

$2.8 \approx x$

**73.** $\text{pH} = -\log\left[H_3O^+\right]$ [4.4]

$= -\log\left[6.28 \times 10^{-5}\right]$

$\approx 4.2$

**74.** $5.4 = -\log\left[H_3O^+\right]$ [4.4]

$-5.4 = \log\left[H_3O^+\right]$

$10^{-5.4} = H_3O^+$

$0.00000398 \approx H_3O^+$

$H_3O^+ \approx 3.98 \times 10^{-6}$

**75.** $P = 16{,}000, r = 0.08, t = 3$ [4.6]

**a.** $B = 16{,}000\left(1 + \dfrac{0.08}{12}\right)^{36} \approx \$20{,}323.79$

**b.** $B = 16{,}000e^{0.08(3)}$

$B = 16{,}000e^{0.24} \approx \$20{,}339.99$

**76.** $P = 19{,}000, r = 0.06, t = 5$ [4.6]

**a.** $B = 19{,}000\left(1 + \dfrac{0.06}{365}\right)^{1825} \approx \$25{,}646.69$

**b.** $B = 19{,}000e^{0.3} \approx \$25{,}647.32$

**77.** $S(n) = P(1-r)^n, P = 12{,}400, r = 0.29, t = 3$ [4.6]

$S(n) = 12{,}400(1-0.29)^3 \approx \$4438.10$

**78.** **a.**

$$N(t) = N_0 e^{-0.12t} \quad [4.6]$$
$$N(10) = N_0 e^{-0.12(10)}$$
$$\frac{N(10)}{N_0} = e^{-1.2}$$
$$= .301$$
$$\frac{N(10)}{N_0} = 30.1\% \text{ healed}$$
$$100\% - 30.1\% = 69.9\% \text{ healed}$$

**b.**

$$\frac{N(t)}{N_0} = 0.5$$
$$0.5 = e^{-0.12t}$$
$$\ln 0.5 = -0.12t$$
$$\frac{\ln 0.5}{-0.12} = t$$
$$t \approx 6 \text{ days}$$

**c.**

$$\frac{N(t)}{N_0} = 0.1$$
$$0.1 = e^{-0.12t}$$
$$\ln 0.1 = -0.12t$$
$$\frac{\ln 0.1}{-0.12} = t$$
$$t \approx 19 \text{ days}$$

**79.**

$N(0) = 1$

$1 = N_0 e^{k(0)}$

$1 = N_0$

$N(2) = 5$

$5 = e^{2k}$

$\ln 5 = 2k$

$k = \frac{\ln 5}{2} \approx 0.8047$

Thus $N(t) = e^{0.8047t}$ [4.6]

**80.** $N(0) = N_0 = 2$ and $N(3) = N_0 e^{3k} = 2e^{3k} = 11$

$$e^{3k} = \frac{11}{2}$$
$$e^{3k} = \frac{11}{2}$$
$$3k = \ln\left(\frac{11}{2}\right)$$
$$k = \frac{1}{3}\ln\left(\frac{11}{2}\right)$$
$$\approx 0.5682$$

Thus $N(t) = 2e^{0.5682t}$ [4.6]

**81.** $4 = N(1) = N_0 e^k$ and thus $\frac{4}{N_0} = e^k$. Now, we also have $N(5) = 5 = N_0 e^{5k} = N_0\left(\frac{4}{N_0}\right)^5 = \frac{1024}{N_0^4}$.

$$N_0 = \sqrt[4]{\frac{1024}{5}} \approx 3.783$$

Thus $4 = 3.783e^k$.

$$\ln\left(\frac{4}{3.783}\right) = k$$
$$k \approx 0.0558$$

Thus $N_0 = 3.783e^{0.0558t}$. [4.6]

**82.** $1 = N(0) = N_0$ and $2 = N(-1) = N_0 e^{-k}$.

Since $N_0 = 1$, we have $2 = 1 \cdot e^{-k}$.

$\ln 2 = -k$

$k \approx -0.6931$

Thus $N(t) = e^{-0.6931t}$. [4.6]

**83.** **a.** $N(1) = 25{,}200e^{k(1)} = 26{,}800$ [4.6]

$$e^k = \frac{26{,}800}{25{,}200}$$

$$\ln e^k = \ln\left(\frac{26{,}800}{25{,}200}\right)$$

$$k \approx 0.061557893$$

$$N(t) = 25{,}200e^{0.061557893\,t}$$

**b.** $N(7) = 25{,}200e^{0.061557893(7)}$

$$= 25{,}200e^{0.430905251}$$

$$\approx 38{,}800$$

**84.** $P(t) = 0.5^{\,t/5730} = 0.96$ [4.6]

$$\log\left(0.5^{\,t/5730}\right) = \log 0.96$$

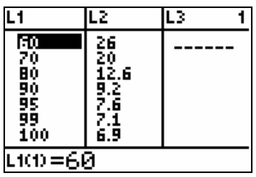
$$\frac{t}{5730}\log 0.5 = \log 0.96$$

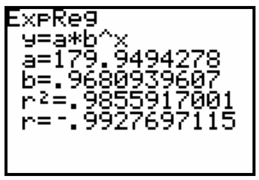
$$\frac{t}{5730} = \frac{\log 0.96}{\log 0.5}$$

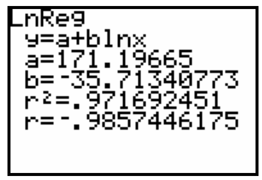
$$t = 5730\left(\frac{\log 0.96}{\log 0.5}\right) \approx 340 \text{ years}$$

**85.** Answers will vary.

**86.** **a.**




exponential: $R \approx 179.949\left(0.968094^t\right)$, $r \approx -0.99277$

logarithmic: $R \approx 171.19665 - 35.71341\ln t$, $r \approx -0.98574$

**b.** The exponential equation provides a better fit to the data.

**c.** $R \approx 179.949\left(0.968094^{108}\right) \approx 5.4$ per 1000 live births [4.7]

**87.** **a.**

$$P(t) = \frac{mP_0}{P_0 + (m - P_0)e^{-kt}}$$

$$P(3) = 360 = \frac{1400(210)}{210 + (1400 - 210)e^{-k(3)}}$$

$$360 = \frac{294000}{210 + 1190e^{-3k}}$$

$$360\left(210 + 1190e^{-3k}\right) = 294000$$

$$210 + 1190e^{-3k} = \frac{294000}{360}$$

$$1190e^{-3k} = \frac{29400}{36} - 210$$

$$e^{-3k} = \frac{29400/36 - 210}{1190}$$

$$\ln e^{-3k} = \ln\left(\frac{29400/36 - 210}{1190}\right)$$

$$-3k = \ln\left(\frac{29400/36 - 210}{1190}\right)$$

$$k = -\frac{1}{3}\ln\left(\frac{29400/36 - 210}{1190}\right)$$

$$k \approx 0.2245763649$$

$$P(t) = \frac{294000}{210 + 1190e^{-0.22458t}} = \frac{1400}{1 + \frac{17}{3}e^{-0.22458t}}$$

**b.** $P(13) = \dfrac{294000}{210 + 1190e^{-0.22458(13)}}$

$$= \frac{294000}{210 + 1190e^{-2.919492744}}$$

$$\approx 1070 \text{ coyotes}$$ [4.6]

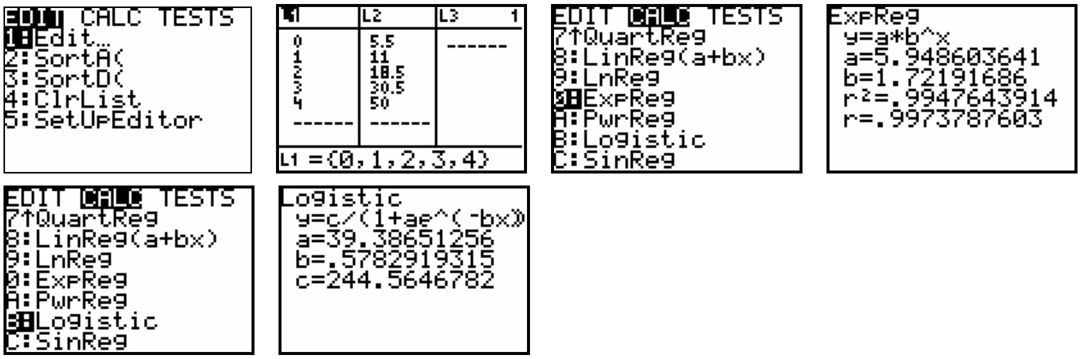
**88.** **a.** $P(0) = \dfrac{128}{1 + 5e^{-0.27(0)}} = \dfrac{128}{1 + 5e^0} = \dfrac{128}{1+5} = \dfrac{128}{6} = 21\frac{1}{3}$

**b.** As $t \to \infty$, $e^{-0.27t} \to 0$.

$$P(t) \to \frac{128}{1 + 5(0)} = \frac{128}{1} = 128$$ [4.6]

## Quantitative Reasoning

**QR1.**

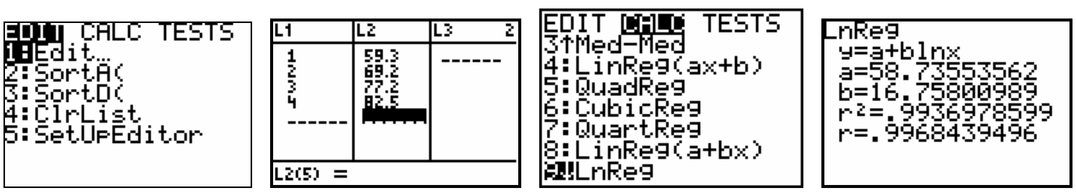



Exponential: $S \approx 5.94860(1.72192)^t$ Logistic: $S \approx \dfrac{244.56468}{1+39.38651e^{-0.57829\,t}}$

**QR2.** Exponential: $S \approx 5.94860(1.72192)^{10} \approx 1363.1$ million

Logistic: $S \approx \dfrac{244.56468}{1+39.38651e^{-0.57829\,(10)}} \approx 218.1$ million

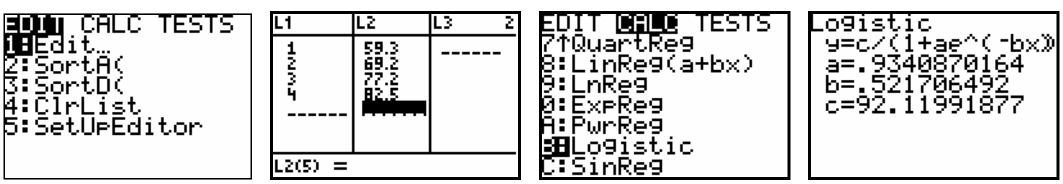
**QR3.**




$S \approx 58.73554 + 16.75801 \ln t$

**QR4.** $S \approx 58.73554 + 16.75801 \ln(6) \approx 88.8$ million

**QR5.**




$S \approx \dfrac{85.24460}{1+14.0040e^{-0.70591\,t}}$

**QR6.** Answers will vary.

## Chapter Test

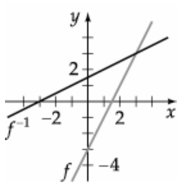
**1.** $y = 2x - 3$ [4.1]

$x = 2y - 3$

$x + 3 = 2y$

$\frac{1}{2}x + \frac{3}{2} = y$

$f^{-1}(x) = \frac{1}{2}x + \frac{3}{2}$




**2.** $f(x) = y = \dfrac{x}{4x-8}$ [4.1]

$x = \dfrac{y}{4y-8}$

$x(4y-8) = y$

$4xy - 8x = y$

$4xy - y = 8x$

$y(4x-1) = 8x$

$y = \dfrac{8x}{4x-1}$

$f^{-1}(x) = \dfrac{8x}{4x-1}$

$4x - 1 \neq 0 \Rightarrow 4x \neq 1 \Rightarrow x \neq \frac{1}{4}$

Domain of $f^{-1}$: all real numbers except $\frac{1}{4}$.

Range of $f^{-1}$ = domain of $f \Rightarrow 4x - 8 \neq 0 \Rightarrow x \neq 2$.

Range of $f^{-1}$: all real numbers except 2.

**3.** **a.** $\log_b(5x-3) = c$ [4.3]

$b^c = 5x - 3$

**b.** $3^{x/2} = y$

$\log_3 y = \dfrac{x}{2}$

**4.** $\log_b \dfrac{z^2}{y^3\sqrt{x}} = \log_b z^2 - \log_b y^3 - \log_b x^{1/2}$ [4.4]

$= 2\log_b z - 3\log_b y - \dfrac{1}{2}\log_b x$

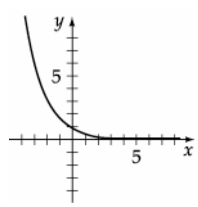
**5.** $\log_{10}(2x+3) - 3\log_{10}(x-2) = \log_{10}(2x+3) - \log_{10}(x-2)^3$

$= \log_{10}\dfrac{2x+3}{(x-2)^3}$ [4.4]

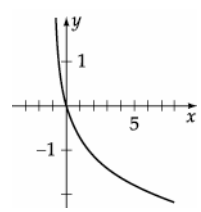
**6.** $\log_4 12 = \dfrac{\log 12}{\log 4}$ [4.4]

$\approx 1.7925$

**7.**




**8.**




**9.** $5^x = 22$ [4.5]

$x\log 5 = \log 22$

$x = \dfrac{\log 22}{\log 5}$

$x \approx 1.9206$

**10.** $4^{5-x} = 7^x$ [4.5]

$\ln 4^{5-x} = \ln 7^x$

$(5-x)\ln 4 = x\ln 7$

$5\ln 4 - x\ln 4 = x\ln 7$

$5\ln 4 = x\ln 7 + x\ln 4$

$5\ln 4 = x(\ln 7 + \ln 4)$

$\dfrac{5\ln 4}{\ln 28} = x$

**11.** $\log(x+99) - \log(3x-2) = 2$ [4.5]

$\log\dfrac{x+99}{3x-2} = 2$

$\dfrac{x+99}{3x-2} = 10^2$

$x + 99 = 100(3x-2)$

$x + 99 = 300x - 200$

$-299x = -299$

$x = 1$

**12.** $\ln(2-x) + \ln(5-x) = \ln(37-x)$

$\ln(2-x)(5-x) = \ln(37-x)$

$(2-x)(5-x) = (37-x)$

$10 - 7x + x^2 = 37 - x$

$x^2 - 6x - 27 = 0$

$(x-9)(x+3) = 0$

$x = 9$ (not in domain) or $x = -3$

$x = -3$ [4.5]

**13.** **a.** $A = P\left(1+\frac{r}{n}\right)^{n\,t}$ [4.6]

$= 20{,}000\left(1+\frac{0.078}{12}\right)^{12(5)}$

$= 20{,}000(1.0065)^{60}$

$= \$29{,}502.36$

**b.** $A = Pe^{rt}$

$= 20{,}000e^{0.078(5)}$

$= 20{,}000e^{0.39}$

$= \$29{,}539.62$

**14.** $A = P\left(1+\frac{r}{n}\right)^{nt}$ [4.6]

$2P = P\left(1+\frac{0.04}{12}\right)^{12t}$

$2 = \left(1+\frac{0.04}{12}\right)^{12t}$

$\ln 2 = \ln\left(1+\frac{0.04}{12}\right)^{12t}$

$\ln 2 = 12t\ln\left(1+\frac{0.04}{12}\right)$

$12t = \frac{\ln 2}{\ln\left(1+\frac{0.04}{12}\right)}$

$t = \frac{1}{12}\cdot\frac{\ln 2}{\ln\left(1+\frac{0.04}{12}\right)}$

$t \approx 17.36$ years

**15.** **a.** $M = \log\left(\frac{I}{I_0}\right)$ [4.4]

$= \log\left(\frac{42{,}304{,}000I_0}{I_0}\right)$

$= \log 42{,}304{,}000$

$\approx 7.6$

**b.** $\log\left(\frac{I_1}{I_0}\right) = 6.3$ and $\log\left(\frac{I_2}{I_0}\right) = 4.5$

$\frac{I_1}{I_0} = 10^{6.3}$ $\qquad \frac{I_2}{I_0} = 10^{4.5}$

$I_1 = 10^{6.3}I_0$ $\qquad I_2 = 10^{4.5}I_0$

$\frac{I_1}{I_2} = \frac{10^{6.3}I_0}{10^{4.5}I_0} = \frac{10^{1.8}}{1} \approx \frac{63}{1}$

Therefore the ratio is 63 to 1.

**16.** **a.** $N(3) = 34600e^{k(3)} = 39800$

$34600e^{3k} = 39800$

$e^{3k} = \frac{39800}{34600}$

$\ln e^{3k} = \ln\left(\frac{398}{346}\right)$

$3k = \ln\left(\frac{398}{346}\right)$

$k = \frac{1}{3}\ln\left(\frac{398}{346}\right)$

$k \approx 0.0466710767$

$N(t) = 34600e^{0.0466710767\,t}$ [4.6]

**b.** $N(10) = 34600e^{0.0466710767(10)}$

$= 34600e^{0.466710767}$

$\approx 55{,}000$

**17.** $P(t) = 0.5^{\,t/5730} = 0.92$ [4.6]

$\log 0.5^{\,t/5730} = \log 0.92$

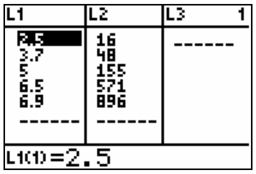
$\frac{t}{5730}\log 0.5 = \log 0.92$

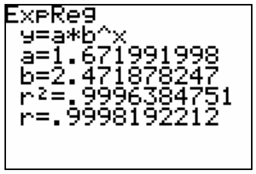
$\frac{t}{5730} = \frac{\log 0.92}{\log 0.5}$

$t = 5730\left(\frac{\log 0.92}{\log 0.5}\right)$

$t \approx 690$ years

**18.**



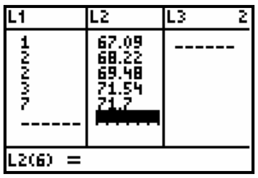



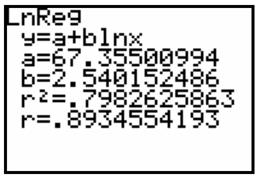

**a.** $y = 1.671991998(2.471878247)^{x}$

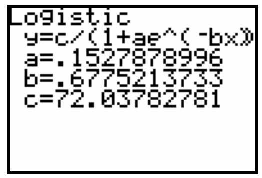
**b.** $y = 1.671991998(2.471878247)^{7.8}$ [4.7]

$\approx 1945$

**19.**




**a.** Logarithmic: $d \approx 67.35501 + 2.54015 \ln t$ ; Logistic: $d \approx \dfrac{72.03783}{1 + 0.15279e^{-0.67752\,t}}$

**b.** Logarithmic: $d \approx 67.35501 + 2.54015 \ln(12) \approx 73.67$ m ;

Logistic: $d \approx \dfrac{72.03783}{1 + 0.15279e^{-0.67752\,(12)}} \approx 72.03$ m [4.7]

**20.** **a.** $a = \dfrac{c - P_0}{P_0} = \dfrac{1100 - 160}{160} = 5.875$

$$P(t) = \frac{c}{1 + ae^{-bt}}$$

$$P(1) = \frac{1100}{1 + 5.875e^{-b(1)}}$$

$$190 = \frac{1100}{1 + 5.875e^{-b}}$$

$$190(1 + 5.875e^{-b}) = 1100$$

$$190 + 1116.25e^{-b} = 1100$$

$$1116.25e^{-b} = 910$$

$$e^{-b} = \frac{910}{1116.25}$$

$$\ln e^{-b} = \ln \frac{910}{1116.25}$$

$$-b = \ln \frac{910}{1116.25}$$

$$b = -\ln \frac{910}{1116.25}$$

$$b \approx 0.20429$$

$$P(t) = \frac{1100}{1 + 5.875e^{-0.20429\,t}} \quad [4.6]$$

**b.** $P(t) = \dfrac{1100}{1 + 5.875e^{-0.20429(7)}} \approx 457$ raccoons

## Cumulative Review

**1.** $|x - 4| \le 2 \Rightarrow -2 \le x - 4 \le 2 \Rightarrow 2 \le x \le 6$. The solution is [2, 6]. [1.5]

**2.** $\dfrac{x}{2x-6} \ge 1$ [1.5]

$\dfrac{x}{2x-6} - 1 \ge 0$

$\dfrac{x}{2x-6} - \dfrac{2x-6}{2x-6} \ge 0$

$\dfrac{x-2x+6}{2x-6} \ge 0$

$\dfrac{-x+6}{2x-6} \ge 0$

The critical values are:
$-x+6=0$ or $2x-6=0$
$x=6$ $\quad x=3$
The intervals are:
$(-\infty, 3), (3, 6)$, and $(6, \infty)$.
The quotient
$\dfrac{-x+6}{2x-6}$ is positive or zero.

$\dfrac{-x+6}{2x-6}$ – – – – | + + + + + | – –
(number line marked at 3 and 6)

The denominator $\ne 0 \Rightarrow x \ne 3$.
The solution is $\{x \mid 3 < x \le 6\}$.

**3.** $d = \sqrt{(11-5)^2 + (7-2)^2}$ [2.1]
$= \sqrt{6^2 + 5^2} = \sqrt{36+25}$
$= \sqrt{61} \approx 7.8$

**4.** Find the $y$-value of the vertex of [2.4]
$h(t) = -16t^2 + 44t + 8$.
$-\dfrac{b}{2a} = -\dfrac{44}{2(-16)} = 1.375$
$h(1.375) = -16(1.375)^2 + 44(1.375) + 8$
$= 38.25$ feet

**5.** $f(x) = 2x+1$ [2.6]
$g(x) = x^2 - 5$
$(g \circ f)(x) = g[f(x)]$
$= g(2x+1)$
$= (2x+1)^2 - 5$
$= 4x^2 + 4x + 1 - 5$
$= 4x^2 + 4x - 4$

**6.** $f(x) = 3x-5$ [4.1]
$x = 3y - 5$
$x + 5 = 3y$
$\frac{1}{3}x + \frac{5}{3} = y$
$f^{-1}(x) = \frac{1}{3}x + \frac{5}{3}$

**7.** $L = kwd^2$ [1.6]
$1500 = k(4)(8)^2$
$1500 = 256k$
$\dfrac{1500}{256} = \dfrac{375}{64} = k$
$L = \dfrac{375}{64}wd^2$
$L = \dfrac{375}{64}(6)(10)^2$
$L \approx 3500$ pounds

**8.** $P(x) = x^4 - 3x^3 + x^2 - x - 6$ has three changes of sign. There are three or one positive real zeros.
$P(-x) = x^4 + 3x^3 + x^2 + x - 6$ has one change of sign. There is one negative real zero. [3.3]

**9.** $P(x) = x^4 - 5x^3 + x^2 + 15x - 12$ has three or one positive and one negative real zeros. [3.3]
$\dfrac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$ are the possible rational zeros.

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1 | –5 | 1 | 15 | –12 |
| | | 1 | –4 | –3 | 12 |
| | 1 | –4 | –3 | 12 | 0 |

| | | | | |
|---|---|---|---|---|
| 4 | 1 | –4 | –3 | 12 |
| | | 4 | 0 | –12 |
| | 1 | 0 | –3 | 0 |

$x^2 - 3 = 0 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$ The zeros are $1, 4, -\sqrt{3}, \sqrt{3}$.

**10.** $P(x) = (x-2)[x-(1-i)][x-(1+i)] = (x-2)[x-1+i][x-1-i] = (x-2)[(x-1)+i][(x-1)-i]$ [3.4]
$= (x-2)[(x-1)^2 - i^2] = (x-2)[x^2-2x+1-(-1)] = (x-2)[x^2-2x+1+1] = (x-2)(x^2-2x+2)$
$= x(x^2-2x+2) - 2(x^2-2x+2) = x^3 - 2x^2 + 2x - 2x^2 + 4x - 4 = x^3 - 4x^2 + 6x - 4$

**11.** $r(x) = \dfrac{3x-5}{x-4}$ [3.5]

Vertical asymptote: $x - 4 = 0 \Rightarrow x = 4$

Horizontal asymptote:
$(n = m) \Rightarrow y = \dfrac{3}{1} \Rightarrow y = 3$

**12.** $R(x) = \dfrac{4}{x^2+1}$. The denominator of $R(x)$ will not equal zero for any real value of $x$. Thus, there are no restrictions on the domain. The domain is all real numbers. $R(x)$ is positive for all values of $x$. The smallest denominator value is 1, thus the highest $R(x)$ value is 4. The range is $\{y \mid 0 < y \le 4\}$. [3.5]

**13.** $f(x) = 0.4^x$ is a decreasing function since $0.4 < 1$. [4.2]

**14.** $\log_4 x = y \Rightarrow 4^y = x$ [4.3]

**15.** $5^3 = 125 \Rightarrow \log_5 125 = 3$ [4.3]

**16.** $M = \log\dfrac{I}{I_0} = \log\dfrac{11{,}650{,}600 I_0}{I_0} = \log 11{,}650{,}600 \approx 7.1$ [4.4]

**17.** $2e^x = 15 \Rightarrow e^x = 7.5 \Rightarrow \ln e^x = \ln 7.5 \Rightarrow x = \ln 7.5 \Rightarrow x \approx 2.0149$ [4.5]

**18.** $0.5N_0 = N_0 e^{5730k} \Rightarrow 0.5 = e^{5730k} \Rightarrow \ln 0.5 = \ln e^{5730k} \Rightarrow \ln 0.5 = 5730k \Rightarrow k = \dfrac{\ln 0.5}{5730} \approx -0.000121$ [4.6]

$N(t) = N_0 e^{-0.000121t} \Rightarrow 0.94N_0 = N_0 e^{-0.000121t} \Rightarrow 0.94 = e^{-0.000121t} \Rightarrow \ln 0.94 = \ln e^{-0.000121t}$

$\Rightarrow \ln 0.94 = -0.000121t \Rightarrow t = \dfrac{\ln 0.94}{-0.000121} \approx 510$ years old

**19.** $\dfrac{e^x - e^{-x}}{2} = 12$ [4.5]

$e^x\left(e^x - e^{-x}\right) = (24)e^x$

$e^{2x} - 1 = e^x(24)$

$e^{2x} - 24e^x - 1 = 0$

Let $u = e^x$.

$u^2 - 24u - 1 = 0$

$u = \dfrac{24 \pm \sqrt{576 - 4(-1)}}{2}$

$u = \dfrac{24 \pm \sqrt{580}}{2}$

$u = \dfrac{24 \pm 2\sqrt{145}}{2}$

$u = 12 \pm \sqrt{145}$

$e^x = 12 \pm \sqrt{145}$

$x \ln e = \ln(12 + \sqrt{145})$ cannot take ln of a negative

$x = \ln(12 + \sqrt{145})$

$x \approx 3.1798$

**20.** **a.** $a = \dfrac{c - P_0}{P_0} = \dfrac{450 - 160}{160} = \dfrac{290}{160} = 1.8125$ [4.7]

$P(3) = \dfrac{450}{1 + 1.8125e^{k(3)}} \Rightarrow 205 = \dfrac{450}{1 + 1.8125e^{3k}} \Rightarrow 205(1 + 1.8125e^{3k}) = 450 \Rightarrow 205 + 371.5625e^{3k} = 450$

$\Rightarrow 371.5625e^{3k} = 245 \Rightarrow e^{3k} = \dfrac{245}{371.5625} \Rightarrow \ln e^{3k} = \ln\dfrac{245}{371.5625}$

$\Rightarrow 3k = \ln 245 - \ln 371.5625 \Rightarrow k = \dfrac{\ln 245 - \ln 371.5625}{3} \Rightarrow k \approx -0.13882$

$P(t) \approx \dfrac{450}{1 + 1.8125e^{-0.13882t}}$

**b.** $P(10) \approx \dfrac{450}{1 + 1.8125e^{-0.13882(10)}} \approx 310$ wolves

# Chapter 5
# Trigonometric Functions

## Section 5.1

**1.** $90^\circ - 15^\circ = 75^\circ$

$180^\circ - 15^\circ = 165^\circ$

**2.** $90^\circ - 87^\circ = 3^\circ$

$180^\circ - 87^\circ = 93^\circ$

**3.** $\begin{array}{r} 90^\circ \\ \underline{-70^\circ 15'} \end{array} = \begin{array}{r} 89^\circ 60' \\ \underline{-70^\circ 15'} \\ 19^\circ 45' \end{array}$ $\qquad \begin{array}{r} 180^\circ \\ \underline{-70^\circ 15'} \end{array} = \begin{array}{r} 179^\circ 60' \\ \underline{-70^\circ 15'} \\ 109^\circ 45' \end{array}$

**4.** $\begin{array}{r} 90^\circ \\ \underline{-22^\circ 43'} \end{array} = \begin{array}{r} 89^\circ 60' \\ \underline{-22^\circ 43'} \\ 67^\circ 17' \end{array}$ $\qquad \begin{array}{r} 180^\circ \\ \underline{-22^\circ 43'} \end{array} = \begin{array}{r} 179^\circ 60' \\ \underline{-22^\circ 43'} \\ 157^\circ 17' \end{array}$

**5.** $\begin{array}{r} 90^\circ \\ \underline{-56^\circ 33' 15''} \end{array} = \begin{array}{r} 89^\circ 59' 60'' \\ \underline{-56^\circ 33' 15''} \\ 33^\circ 26' 45'' \end{array}$ $\qquad \begin{array}{r} 180^\circ \\ \underline{-56^\circ 33' 15''} \end{array} = \begin{array}{r} 179^\circ 59' 60'' \\ \underline{-56^\circ 33' 15''} \\ 123^\circ 26' 45'' \end{array}$

**6.** $\begin{array}{r} 90^\circ \\ \underline{-19^\circ 42' 05''} \end{array} = \begin{array}{r} 89^\circ 59' 60'' \\ \underline{-19^\circ 42' 05''} \\ 70^\circ 17' 55'' \end{array}$ $\qquad \begin{array}{r} 180^\circ \\ \underline{-19^\circ 42' 05''} \end{array} = \begin{array}{r} 179^\circ 59' 60'' \\ \underline{-19^\circ 42' 05''} \\ 160^\circ 17' 55'' \end{array}$

**7.** $\frac{\pi}{2} - 1$

$\pi - 1$

**8.** $\frac{\pi}{2} - 0.5$

$\pi - 0.5$

**9.** $\frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

$\pi - \frac{\pi}{4} = \frac{3\pi}{4}$

**10.** $\frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$

$\pi - \frac{\pi}{3} = \frac{2\pi}{3}$

**11.** $\frac{\pi}{2} - \frac{2\pi}{5} = \frac{\pi}{10}$

$\pi - \frac{2\pi}{5} = \frac{3\pi}{5}$

**12.** $\frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$

$\pi - \frac{\pi}{6} = \frac{5\pi}{6}$

**13.** $610^\circ = 250^\circ + 360^\circ$
$\alpha$ is a quadrant III angle coterminal with an angle of measure 250°.

**14.** $765^\circ = 45^\circ + 2 \cdot 360^\circ$
$\alpha$ is a quadrant I angle coterminal with an angle of measure 45°.

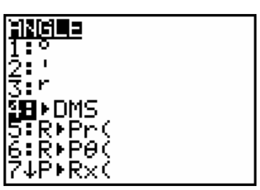
**15.** $-975^\circ = 105^\circ - 3 \cdot 360^\circ$
$\alpha$ is a quadrant II angle coterminal with an angle of measure 105°.

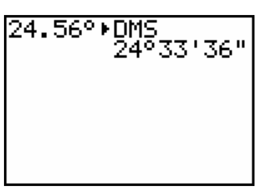
**16.** $-872^\circ = 208^\circ - 3 \cdot 360^\circ$
$\alpha$ is a quadrant III angle coterminal with an angle of measure 208°.

**17.** $2456^\circ = 296^\circ + 6 \cdot 360^\circ$
$\alpha$ is a quadrant IV angle coterminal with an angle of measure 296°.

**18.** $-3789^\circ = 171^\circ - 11 \cdot 360^\circ$
$\alpha$ is a quadrant II angle coterminal with an angle of measure 171°.

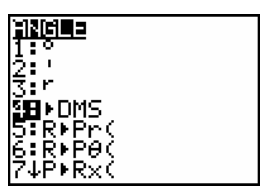
**19.** On a TI-83 graphing calculator, the degree symbol, °, and the DMS function are located in the ANGLE menu.

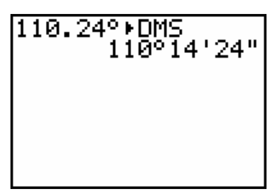



$24.56^\circ = 24^\circ 33' 36''$

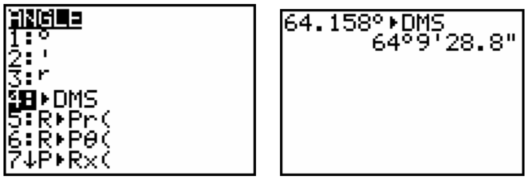
**20.** On a TI-83 graphing calculator, the degree symbol, °, and the DMS function are located in the ANGLE menu.



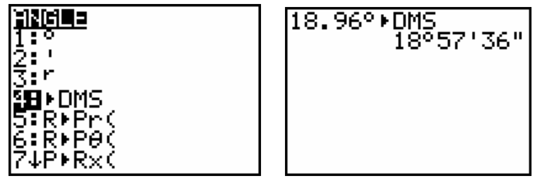

$110.24^\circ = 110^\circ 14' 24''$

**21.** On a TI-83 graphing calculator, the degree symbol, °, and the DMS function are located in the ANGLE menu.



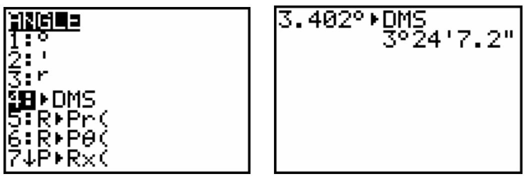

$64.158° = 64°9'28.8''$

**22.** On a TI-83 graphing calculator, the degree symbol, °, and the DMS function are located in the ANGLE menu.



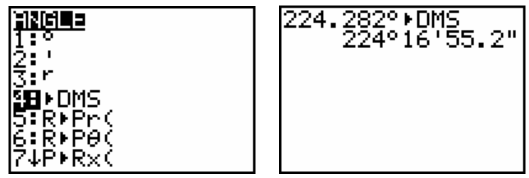

$18.96° = 18°57'36''$

**23.** On a TI-83 graphing calculator, the degree symbol, °, and the DMS function are located in the ANGLE menu.



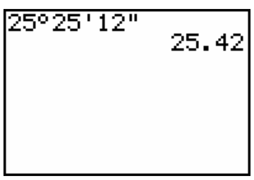

$3.402° = 3°24'7.2''$

**24.** On a TI-83 graphing calculator, the degree symbol, °, and the DMS function are located in the ANGLE menu.



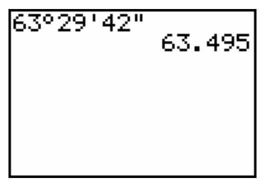

$224.282° = 224°16'55.2''$

**25.** A TI-83 calculator needs to be in degree mode to convert a DMS measure to its equivalent degree measure. On a TI-83 both the degree symbol, °, and the minute symbol, ', are located in the ANGLE menu. The second symbol, ", is entered by pressing ALPHA followed by [ " ] which is located on the plus sign, [+], key.



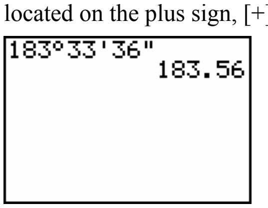

$25°25'12'' = 25.42°$

**26.** A TI-83 calculator needs to be in degree mode to convert a DMS measure to its equivalent degree measure. On a TI-83 both the degree symbol, °, and the minute symbol, ', are located in the ANGLE menu. The second symbol, ", is entered by pressing ALPHA followed by [ " ] which is located on the plus sign, [+], key.



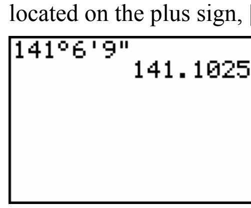

$63°29'42'' = 63.495°$

**27.** A TI-83 calculator needs to be in degree mode to convert a DMS measure to its equivalent degree measure. On a TI-83 both the degree symbol, °, and the minute symbol, ', are located in the ANGLE menu. The second symbol, ", is entered by pressing ALPHA followed by [ " ] which is located on the plus sign, [+], key.



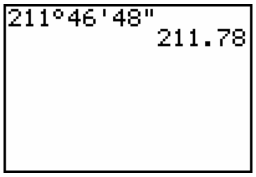

$183°33'36'' = 183.56°$

**28.** A TI-83 calculator needs to be in degree mode to convert a DMS measure to its equivalent degree measure. On a TI-83 both the degree symbol, °, and the minute symbol, ', are located in the ANGLE menu. The second symbol, ", is entered by pressing ALPHA followed by [ " ] which is located on the plus sign, [+], key.



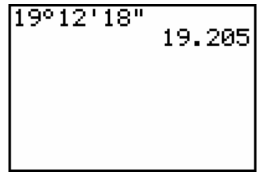

$141°6'9'' = 141.1025°$

**29.** A TI-83 calculator needs to be in degree mode to convert a DMS measure to its equivalent degree measure. On a TI-83 both the degree symbol, °, and the minute symbol, ', are located in the ANGLE menu. The second symbol, ", is entered by pressing ALPHA followed by [ " ] which is located on the plus sign, [+], key.




$211°46'48'' = 211.78°$

**30.** A TI-83 calculator needs to be in degree mode to convert a DMS measure to its equivalent degree measure. On a TI-83 both the degree symbol, °, and the minute symbol, ', are located in the ANGLE menu. The second symbol, ", is entered by pressing ALPHA followed by [ " ] which is located on the plus sign, [+], key.




$19°12'18'' = 19.205°$

**31.** $30° = 30°\left(\frac{\pi}{180°}\right) = \frac{\pi}{6}$

**32.** $-45° = -45°\left(\frac{\pi}{180°}\right) = -\frac{\pi}{4}$

**33.** $90° = 90°\left(\frac{\pi}{180°}\right) = \frac{\pi}{2}$

**34.** $15° = 15°\left(\frac{\pi}{180°}\right) = \frac{\pi}{12}$

**35.** $165° = 165°\left(\frac{\pi}{180°}\right) = \frac{11\pi}{12}$

**36.** $315° = 315°\left(\frac{\pi}{180°}\right) = \frac{7\pi}{4}$

**37.** $420° = 420°\left(\frac{\pi}{180°}\right) = \frac{7\pi}{3}$

**38.** $630° = 630°\left(\frac{\pi}{180°}\right) = \frac{7\pi}{2}$

**39.** $585° = 585°\left(\frac{\pi}{180°}\right) = \frac{13\pi}{4}$

**40.** $135° = 135°\left(\frac{\pi}{180°}\right) = \frac{3\pi}{4}$

**41.** $-9° = -9°\left(\frac{\pi}{180°}\right) = -\frac{\pi}{20}$

**42.** $-110° = -110°\left(\frac{\pi}{180°}\right) = -\frac{11\pi}{18}$

**43.** $\frac{7\pi}{3} = \frac{7\pi}{3}\left(\frac{180°}{\pi}\right) = 420°$

**44.** $\frac{\pi}{4} = \frac{\pi}{4}\left(\frac{180°}{\pi}\right) = 45°$

**45.** $\frac{\pi}{5} = \frac{\pi}{5}\left(\frac{180°}{\pi}\right) = 36°$

**46.** $-\frac{2\pi}{3} = -\frac{2\pi}{3}\left(\frac{180°}{\pi}\right) = -120°$

**47.** $\frac{\pi}{6} = \frac{\pi}{6}\left(\frac{180°}{\pi}\right) = 30°$

**48.** $\frac{\pi}{9} = \frac{\pi}{9}\left(\frac{180°}{\pi}\right) = 20°$

**49.** $\frac{3\pi}{8} = \frac{3\pi}{8}\left(\frac{180°}{\pi}\right) = 67.5°$

**50.** $\frac{11\pi}{18} = \frac{11\pi}{18}\left(\frac{180°}{\pi}\right) = 110°$

**51.** $\frac{11\pi}{3} = \frac{11\pi}{3}\left(\frac{180°}{\pi}\right) = 660°$

**52.** $\frac{6\pi}{5} = \frac{6\pi}{5}\left(\frac{180°}{\pi}\right) = 216°$

**53.** $-\frac{5\pi}{12} = -\frac{5\pi}{12}\left(\frac{180°}{\pi}\right) = -75°$

**54.** $-\frac{4\pi}{5} = -\frac{4\pi}{5}\left(\frac{180°}{\pi}\right) = -144°$

**55.** $1.5 = 1.5\left(\frac{180°}{\pi}\right) \approx 85.94°$

**56.** $-2.3 = -2.3\left(\frac{180°}{\pi}\right) \approx -131.78°$

**57.** $133° = 133°\left(\frac{\pi}{180°}\right) \approx 2.32$

**58.** $427° = 427°\left(\frac{\pi}{180°}\right) \approx 7.45$

**59.** $8.25 = 8.25\left(\frac{180°}{\pi}\right) \approx 472.69°$

**60.** $-90° = -90°\left(\frac{\pi}{180°}\right) \approx -1.57$

**61.**
$$\begin{aligned}\theta &= \frac{s}{r}\\ &= \frac{8}{2} = 4\\ &= 4\left(\frac{180°}{\pi}\right) \approx 229.18°\end{aligned}$$

**62.**
$$\begin{aligned}\theta &= \frac{s}{r}\\ &= \frac{4}{7} \approx 0.57\\ &= 0.57\left(\frac{180°}{\pi}\right) \approx 32.74°\end{aligned}$$

**63.**
$$\begin{aligned}\theta &= \frac{s}{r}\\ &= \frac{12.4}{5.2} \approx 2.38\\ &= 2.38\left(\frac{180°}{\pi}\right) \approx 136.63°\end{aligned}$$

**64.** $\theta = \frac{s}{r}$
$= \frac{84.3}{35.8} \approx 2.35$
$= \frac{84.3}{35.8}\left(\frac{180^\circ}{\pi}\right) \approx 134.92^\circ$

**65.** $s = r\theta$
$= (8)\frac{\pi}{4}$
$\approx 6.28$ in.

**66.** $s = r\theta$
$= 3\left(\frac{7\pi}{2}\right)$
$= 32.99$ ft

**67.** $s = r\theta$
$= 25 \cdot (42^\circ)\left(\frac{\pi}{180^\circ}\right)$
$\approx 18.33$ cm

**68.** $s = r\theta$
$= 5 \cdot (144^\circ)\left(\frac{\pi}{180^\circ}\right)$
$\approx 12.57$ m

**69.** $\theta = \frac{3}{2}(2\pi)$
$= 3\pi$

**70.** $\theta = \frac{3}{8}(2\pi)$
$= \frac{3\pi}{4}$

**71.** $\theta_2 = \frac{r_1}{r_2}\theta_1$
$= \frac{14}{28}(150^\circ)\left(\frac{\pi}{180^\circ}\right)$
$= \frac{5\pi}{12}$ radians or $75^\circ$

**72.** $\theta_2 = \frac{r_2}{r_1} \cdot \theta_2$
$= \frac{1.2}{0.8}(240^\circ)\left(\frac{\pi}{180^\circ}\right)$
$= 2\pi$ radians or $360^\circ$

**73.** $\omega = \frac{\theta}{t}$
$= \frac{2\pi}{60}$
$= \frac{\pi}{30}$ radian/sec

**74.** $\omega = \frac{\theta}{t}$
$= \frac{2\pi}{86,400}$
$\approx 7.27 \times 10^{-5}$ radian/sec

**75.** $\omega = \frac{\theta}{t}$
$= \frac{50(2\pi)}{60}$
$= \frac{5\pi}{3}$ radians/sec

**76.** $\omega = \frac{\theta}{t}$
$= \frac{200(2\pi)}{60}$
$= \frac{20\pi}{3}$ radians/sec

**77.** $\omega = \frac{\theta}{t}$
$= \frac{2\pi\left(33\frac{1}{3}\right)}{60}$
$= \frac{10\pi}{9}$ radians/sec
$\approx 3.49$ radians per second

**78.** $\omega = \frac{v}{r}$
$= 55 \cdot \frac{5280}{3600} \cdot \frac{12}{14}$
$\approx 69.14$ radians/sec

**79.** $v = \omega r$
$= \frac{450 \cdot 2\pi \cdot 60 \cdot 15}{12 \cdot 5280}$
$\approx 40$ mph

**80.** $v = \omega r$
$= \frac{500 \cdot 2\pi \cdot 60 \cdot 18}{12 \cdot 5280}$
$\approx 54$ mph

**81.** $r_1\theta_1 = r_2\theta_2$
$(3.5)(150 \cdot 2\pi) = (1.75)\theta_2$
$600\pi = \theta_2$
$300(2\pi) = \theta_2$
The rear gear is making 300 revolutions.
The rear gear and tire are making same number of revolutions
Tire is 12 inches = 1 ft
$s = 1 \text{ ft}(300)(2\pi) = 1885$ ft

**82.** **a.** $s = r\theta$
$= 6\left(\frac{5\pi}{6}\right)$
$= 5\pi$ in.

**b.** $s = r\theta$
$24 = 6\theta$
$\theta = 4$ radians

**83.** Speed for outer ring:
$v = \frac{s}{t}$
$= \frac{2\pi(32)}{3.75} \cdot \frac{60^2}{5280}$
$\approx 36.5567$ mph

Speed for inner ring:
$v = \frac{s}{t}$
$= \frac{2\pi(38)}{3.75} \cdot \frac{60^2}{5280}$
$\approx 43.4111$ mph

The outer swing has a greater speed of $43.4111 - 36.5567 \approx 6.9$ mph.

**84.** For the lead horse:

$d_1 = \pi(202) \approx 634.6$ ft

$t_1 = \dfrac{d_1}{r} = \dfrac{634.6}{24.4} \approx 26.008$ sec

For the second horse:

$d_2 = \pi(206.5) \approx 648.7$ ft

$r = \dfrac{d_2}{t} = \dfrac{648.7}{26.008} \approx 24.9$ ft/s

The second horse must go 24.9 ft/s.

**85.** $s = r\theta$

$= 93{,}000{,}000(31')\left(\dfrac{1^\circ}{60'}\right)\left(\dfrac{\pi}{180^\circ}\right)$

$\approx 840{,}000$ mi

**86.** **a.** $\theta = \dfrac{36}{60}(2\pi) = \dfrac{6\pi}{5}$ radians

**b.** $s = r\theta$

$s = 6.25\left(\dfrac{6\pi}{5}\right)$

$s \approx 23.6$ ft

**87.** **a.** $\omega = \dfrac{\theta}{t}$

$= \dfrac{2\pi}{1.61 \text{ hours}}$

$\approx 3.9$ radians per hour

**b.** $v = \dfrac{s}{t}$

$= \dfrac{2\pi r}{1.61 \text{ hours}}$

$= \dfrac{2\pi(625 \text{ km} + 6370 \text{ km})}{1.61 \text{ hours}}$

$= \dfrac{2\pi(6995 \text{ km})}{1.61 \text{ hours}}$

$\approx 27{,}300$ km per hour

**88.** $7.5^\circ = 7.5^\circ\left(\dfrac{\pi}{180^\circ}\right) = \dfrac{\pi}{24}$

Solving the formula $\theta = \dfrac{s}{r}$ for $r$,

we have $r = \dfrac{s}{\theta}$

$= \dfrac{520 \text{ miles}}{\pi/24}$

$\approx 3970$ miles

**89.** **a.** When the rear tire makes one revolution, the bicycle travels $s = r\theta = 2\pi r = 2\pi(30 \text{ inches}) = 60\pi$ inches.

The angular velocity of point $A$ is

$\omega = \dfrac{\theta}{t} = \dfrac{2\pi}{t} = 2\left(\dfrac{\pi}{t}\right).$

When the bicycle travels $60\pi$ inches, point $B$ on the front tire travels though an angle of

$\theta = \dfrac{s}{r} = \dfrac{60\pi \text{ inches}}{20 \text{ inches}} = 3\pi.$

The angular velocity of point $B$ is

$\omega = \dfrac{\theta}{t} = \dfrac{3\pi}{t} = 3\left(\dfrac{\pi}{t}\right).$

Thus, point $B$ has the greater angular velocity.

**b.** Point $A$ and point $B$ travel a linear distance of $60\pi$ inches in the same amount of time. Therefore, both points have the same linear velocity.

**90.** Solving the formula $\theta = \dfrac{s}{r}$ for $s$ yields $s = r\theta$.

Solving the formula $\theta = \dfrac{s}{r}$ for $r$ yields $r = \dfrac{s}{\theta}$.

$s = r\theta$

$\dfrac{s}{t} = \dfrac{r\theta}{t}$

$v = \dfrac{r\theta}{t}$

$v = \dfrac{r\theta}{t}$

$v = r\left(\dfrac{\theta}{t}\right)$

$v = r\omega$

$v = \dfrac{r\theta}{t}$

$v = \dfrac{s}{t}$

$v = \dfrac{r\theta}{t}$

$r\omega = \dfrac{r\theta}{t}$

$\omega = \dfrac{\theta}{t}$

Thus, all of the formulas are valid.

**91.** **a.** $1' = 1' \cdot \left(\dfrac{1^\circ}{60'}\right) \cdot \left(\dfrac{\pi}{180^\circ}\right) = \dfrac{\pi}{10{,}800}$ radians

1 nautical mile = $s = r\theta = (3960 \text{ statute miles})\left(\dfrac{\pi}{10{,}800}\right) \approx 1.15$ statute miles

**b.** Earth's circumference = $2\pi r \approx 2\pi(3960 \text{ statute miles})\left(\dfrac{1 \text{ nautical mile}}{1.15 \text{ statute miles}}\right) \approx \dfrac{7920\pi}{1.15}$ nautical miles

The question, then, is what percent of $\dfrac{7920\pi}{1.15}$ is 2217.

$\dfrac{2217}{7920\pi/1.15} \approx 0.10 = 10\%$

**92.** $12^\circ \cdot \dfrac{\pi}{180^\circ} \approx 0.067\pi$

$s = r\theta$

$s = 485(0.067\pi)$

$s \approx 102$ ft

## Connecting Concepts

**93.** $A = \frac{1}{2}r^2\theta = \frac{1}{2}(5^2)\left(\frac{\pi}{3}\right) \approx 13 \text{ in}^2$

**94.** $A = \frac{1}{2}r^2\theta = \frac{1}{2}(2.8)^2\left(\frac{5\pi}{2}\right) \approx 31 \text{ ft}^2$

**95.** $A = \frac{1}{2}r^2\theta = \frac{1}{2}(120)^2\, 0.65 = 4680 \text{ cm}^2$

**96.** $A = \frac{1}{2}r^2\theta = \frac{1}{2}(30)^2\, 62^\circ\left(\frac{\pi}{180^\circ}\right) = 487 \text{ ft}^2$

**97.** $25^\circ 47' = 25^\circ + 47'\left(\frac{1^\circ}{60'}\right) = 25\frac{47}{60}^\circ$

Convert to radians.

$25\frac{47}{60}^\circ \cdot \frac{\pi}{180^\circ} = \frac{1547\pi}{10{,}800}$ radians

$s = r\theta$

$s = 3960\left(\frac{1547\pi}{10{,}800}\right) \approx 1780$

To the nearest 10 miles, Miami is 1780 miles north of the equator.

**98.** $40^\circ 45' = 40^\circ + 45'\left(\frac{1^\circ}{60'}\right) = 40.75^\circ$

convert to radians.

$40.75^\circ \cdot \frac{\pi}{180^\circ} = \frac{4075\pi}{18{,}000} = \frac{163\pi}{720}$ radians

$s = r\theta$

$s = 3960\left(\frac{163\pi}{720}\right) \approx 2820$

To the nearest 10 miles, New York City is 2820 miles north of the equator.

## Prepare for Section 5.2

**PS1.** $\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

**PS2.** $\frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$

**PS3.** $a \div \left(\frac{a}{2}\right) = a \cdot \left(\frac{2}{a}\right) = 2$

**PS4.** $\left(\frac{a}{2}\right) \div \left(\frac{\sqrt{3}}{2}a\right) = \left(\frac{a}{2}\right) \cdot \left(\frac{2}{a\sqrt{3}}\right) = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

**PS5.** $\frac{\sqrt{2}}{2} = \frac{x}{5}$

$\frac{5\sqrt{2}}{2} = x$

$x \approx 3.54$

**PS6.** $\frac{\sqrt{3}}{3} = \frac{x}{18}$

$\frac{18\sqrt{3}}{3} = x$

$x = 6\sqrt{3}$

$x \approx 10.39$

# Section 5.2

**1.** $r = \sqrt{5^2 + 12^2}$

$r = \sqrt{25 + 144} + \sqrt{169}$

$r = 13$

$\sin\theta = \frac{y}{r} = \frac{12}{13}$ $\quad \csc\theta = \frac{r}{y} = \frac{13}{12}$

$\cos\theta = \frac{x}{r} = \frac{5}{13}$ $\quad \sec\theta = \frac{r}{x} = \frac{13}{5}$

$\tan\theta = \frac{y}{x} = \frac{12}{5}$ $\quad \cot\theta = \frac{x}{y} = \frac{5}{12}$

**2.** $r = \sqrt{3^2 + 7^2}$

$r = \sqrt{9+49} + \sqrt{58}$

$\sin\theta = \frac{y}{r} = \frac{7}{\sqrt{58}} = \frac{7\sqrt{58}}{58}$ $\quad \csc\theta = \frac{r}{y} = \frac{\sqrt{58}}{7}$

$\cos\theta = \frac{x}{r} = \frac{3}{\sqrt{58}} = \frac{3\sqrt{58}}{58}$ $\quad \sec\theta = \frac{r}{x} = \frac{\sqrt{58}}{3}$

$\tan\theta = \frac{y}{x} = \frac{7}{3}$ $\quad \cot\theta = \frac{x}{y} = \frac{3}{7}$

**3.** $x = \sqrt{7^2 - 4^2}$

$x = \sqrt{49-16} = \sqrt{33}$

$\sin\theta = \frac{y}{r} = \frac{4}{7}$ $\quad \csc\theta = \frac{r}{y} = \frac{7}{4}$

$\cos\theta = \frac{x}{r} = \frac{\sqrt{33}}{7}$ $\quad \sec\theta = \frac{r}{x} = \frac{7}{\sqrt{33}} = \frac{7\sqrt{33}}{33}$

$\tan\theta = \frac{y}{x} = \frac{4}{\sqrt{33}} = \frac{4\sqrt{33}}{33}$ $\quad \cot\theta = \frac{x}{y} = \frac{\sqrt{33}}{4}$

**4.** $x = \sqrt{9^2 - 3^2}$

$x = \sqrt{81-9} = \sqrt{72}$

$x = 6\sqrt{2}$

$\sin\theta = \frac{y}{r} = \frac{3}{9} = \frac{1}{3}$ $\quad \csc\theta = \frac{r}{y} = \frac{9}{3} = 3$

$\cos\theta = \frac{x}{r} = \frac{6\sqrt{2}}{9} = \frac{2\sqrt{2}}{3}$ $\quad \sec\theta = \frac{r}{x} = \frac{9}{6\sqrt{2}} = \frac{3\sqrt{2}}{4}$

$\tan\theta = \frac{y}{x} = \frac{3}{6\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$ $\quad \cot\theta = \frac{x}{y} = \frac{6\sqrt{2}}{3} = 2\sqrt{2}$

**5.** $r = \sqrt{2^2 + 5^2}$

$r = \sqrt{4+25} = \sqrt{29}$

$\sin\theta = \frac{y}{r} = \frac{5}{\sqrt{29}} = \frac{5\sqrt{29}}{29}$ $\quad \csc\theta = \frac{r}{y} = \frac{\sqrt{29}}{5}$

$\cos\theta = \frac{x}{r} = \frac{2}{\sqrt{29}} = \frac{2\sqrt{29}}{29}$ $\quad \sec\theta = \frac{r}{x} = \frac{\sqrt{29}}{2}$

$\tan\theta = \frac{y}{x} = \frac{5}{2}$ $\quad \cot\theta = \frac{x}{y} = \frac{2}{5}$

**6.** $x = \sqrt{8^2 - 5^2}$

$x = \sqrt{64-25} = \sqrt{39}$

$\sin\theta = \frac{y}{r} = \frac{5}{8}$ $\quad \csc\theta = \frac{r}{y} = \frac{8}{5}$

$\cos\theta = \frac{x}{r} = \frac{\sqrt{39}}{8}$ $\quad \sec\theta = \frac{r}{x} = \frac{8}{\sqrt{39}} = \frac{8\sqrt{39}}{39}$

$\tan\theta = \frac{y}{x} = \frac{5}{\sqrt{39}} = \frac{5\sqrt{39}}{39}$ $\quad \cot\theta = \frac{x}{y} = \frac{\sqrt{39}}{5}$

**7.** $x = \sqrt{2 + \left(\sqrt{3}\right)^2}$

$x = \sqrt{4+3} = \sqrt{7}$

$\sin\theta = \frac{y}{r} = \frac{\sqrt{3}}{\sqrt{7}} = \frac{\sqrt{21}}{7}$ $\quad \csc\theta = \frac{r}{y} = \frac{\sqrt{7}}{\sqrt{3}} = \frac{\sqrt{21}}{3}$

$\cos\theta = \frac{x}{r} = \frac{2}{\sqrt{7}} = \frac{2\sqrt{7}}{7}$ $\quad \sec\theta = \frac{r}{x} = \frac{\sqrt{7}}{2}$

$\tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{2}$ $\quad \cot\theta = \frac{x}{y} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

**8.** $x = \sqrt{\left(\sqrt{10}\right)^2 - \left(\sqrt{5}\right)^2}$

$x = \sqrt{10-5} = \sqrt{5}$

$\sin\theta = \frac{y}{r} = \frac{\sqrt{5}}{\sqrt{10}} = \frac{\sqrt{2}}{2}$ $\quad \csc\theta = \frac{r}{y} = \frac{\sqrt{10}}{\sqrt{5}} = \sqrt{2}$

$\cos\theta = \frac{x}{r} = \frac{\sqrt{5}}{\sqrt{10}} = \frac{\sqrt{2}}{2}$ $\quad \sec\theta = \frac{r}{x} = \frac{\sqrt{10}}{\sqrt{5}} = \sqrt{2}$

$\tan\theta = \frac{y}{x} = \frac{\sqrt{5}}{5} = 1$ $\quad \cot\theta = \frac{x}{y} = \frac{\sqrt{5}}{\sqrt{5}} = 1$

**9.** opposite side $= \sqrt{6^2 - 3^2} = \sqrt{36-9} = \sqrt{27} = 3\sqrt{3}$

$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2}$

$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{3}{6} = \frac{1}{2}$

$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{3\sqrt{3}}{3} = \sqrt{3}$

$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{6}{3\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

$\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{6}{3} = 2$

$\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

**10.** opposite side $= \sqrt{1^2 - 0.8^2} = \sqrt{1-0.64} = \sqrt{0.36} = 0.6$

$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{0.6}{1} = 0.6 = \frac{3}{5}$

$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{0.8}{1} = 0.8 = \frac{4}{5}$

$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{0.6}{0.8} = \frac{3}{4}$

$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{1}{0.6} = \frac{5}{3}$

$\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{1}{0.8} = \frac{5}{4}$

$\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{0.8}{0.6} = \frac{4}{3}$

**11.** hypotenuse $= \sqrt{5^2 + 6^2} = \sqrt{25+36} = \sqrt{61}$

$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{6}{\sqrt{61}} = \frac{6\sqrt{61}}{61}$

$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{5}{\sqrt{61}} = \frac{5\sqrt{61}}{61}$

$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{6}{5}$

$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{\sqrt{61}}{6}$

$\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{61}}{5}$

$\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{5}{6}$

**12.** opposite side $= \sqrt{\sqrt{2}^2 - 1^2} = \sqrt{2-1} = \sqrt{1} = 1$

$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{1}{1} = 1$

$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{\sqrt{2}}{1} = \sqrt{2}$

$\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{2}}{1} = \sqrt{2}$

$\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{1}{1} = 1$

For exercises 13 to 15, since $\sin\theta = \frac{y}{r} = \frac{3}{5}$, $y = 3$, $r = 5$, and $x = \sqrt{5^2 - 3^2} = 4$.

**13.** $\tan\theta = \frac{y}{x} = \frac{3}{4}$

**14.** $\sec\theta = \frac{r}{x} = \frac{5}{4}$

**15.** $\cos\theta = \frac{x}{r} = \frac{4}{5}$

For exercises 16 to 18, since $\tan\theta = \frac{y}{x} = \frac{4}{3}$, $y = 4$, $x = 3$, and $r = \sqrt{3^2 + 4^2} = 5$.

**16.** $\sin\theta = \frac{y}{r} = \frac{4}{5}$

**17.** $\cot\theta = \frac{x}{y} = \frac{3}{4}$

**18.** $\sec\theta = \frac{r}{x} = \frac{5}{3}$

For exercises 19 to 21, since $\sec\beta = \frac{r}{x} = \frac{13}{12}$, $r = 13$, $x = 12$, and $y = \sqrt{13^2 - 12^2} = \sqrt{25} = 5$.

**19.** $\cos\beta = \frac{x}{r} = \frac{12}{13}$

**20.** $\cot\beta = \frac{x}{y} = \frac{12}{5}$

**21.** $\csc\beta = \frac{r}{y} = \frac{13}{5}$

For exercises 22 to 24, since $\cos\theta \frac{x}{r} = \frac{2}{3}$, $x = 2$, $r = 3$, and $y = \sqrt{3^2 - 2^2} = \sqrt{9-4} = \sqrt{5}$.

**22.** $\sin\theta = \frac{y}{r} = \frac{\sqrt{5}}{3}$

**23.** $\sec\theta = \frac{r}{x} = \frac{3}{2}$

**24.** $\tan\theta = \frac{y}{x} = \frac{\sqrt{5}}{2}$

**25.** $\sin 45^\circ + \cos 45^\circ = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$

**26.** $\csc 45^\circ - \sec 45^\circ = \sqrt{2} - \sqrt{2} = 0$

**27.** $\sin 30^\circ \cos 60^\circ - \tan 45^\circ = \frac{1}{2}\cdot\frac{1}{2} - 1 = \frac{1}{4} - 1 = -\frac{3}{4}$

**28.** $\csc 60^\circ \sec 30^\circ + \cot 45^\circ = \frac{2\sqrt{3}}{3}\cdot\frac{2\sqrt{3}}{3} + 1 = \frac{4}{3} + 1 = \frac{7}{3}$

**29.** $\sin 30^\circ \cos 60^\circ + \tan 45^\circ = \frac{1}{2}\cdot\frac{1}{2} + 1 = \frac{1}{4} + 1 = \frac{5}{4}$

**30.** $\sec 30^\circ \cos 30^\circ - \tan 60^\circ \cot 60^\circ = \frac{2\sqrt{3}}{3}\cdot\frac{\sqrt{3}}{2} - \sqrt{3}\cdot\frac{\sqrt{3}}{3} = 1 - 1 = 0$

**31.** $\sin\frac{\pi}{3} + \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$

**32.** $\csc\frac{\pi}{6} - \sec\frac{\pi}{3} = 2 - 2 = 0$

**33.** $\sin\frac{\pi}{4} + \tan\frac{\pi}{6} = \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{3} = \frac{3\sqrt{2} + 2\sqrt{3}}{6}$

**34.** $\sin\frac{\pi}{3}\cos\frac{\pi}{4} - \tan\frac{\pi}{4} = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - 1 = \frac{\sqrt{6}}{4} - 1 = \frac{\sqrt{6} - 4}{4}$

**35.** $\sec\frac{\pi}{3}\cos\frac{\pi}{3} - \tan\frac{\pi}{6} = 2\cdot\frac{1}{2} - \frac{\sqrt{3}}{3} = 1 - \frac{\sqrt{3}}{3} = \frac{3 - \sqrt{3}}{3}$

**36.** $\cos\frac{\pi}{4}\tan\frac{\pi}{6} + 2\tan\frac{\pi}{3} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{3} + 2\cdot\sqrt{3} = \frac{\sqrt{6}}{6} + 2\sqrt{3} = \frac{\sqrt{6} + 12\sqrt{3}}{6}$

**37.** $2\csc\frac{\pi}{4} - \sec\frac{\pi}{3}\cos\frac{\pi}{6} = 2\cdot\sqrt{2} - 2\cdot\frac{\sqrt{3}}{2} = 2\sqrt{2} - \sqrt{3}$

**38.** $3\tan\frac{\pi}{4} + \sec\frac{\pi}{6}\sin\frac{\pi}{3} = 3\cdot 1 + \frac{2\sqrt{3}}{3}\cdot\frac{\sqrt{3}}{2} = 3 + 1 = 4$

**39.** $\tan 32^\circ \approx 0.6249$

**40.** $\sec 88^\circ \approx 28.6537$

**41.** $\cos 63^\circ 20' \approx 0.4488$

**42.** $\cot 55^\circ 50' \approx 0.6787$

**43.** $\cos 34.7^\circ \approx 0.8221$

**44.** $\tan 81.3^\circ \approx 6.5350$

**45.** $\sec 5.9^\circ \approx 1.0053$

**46.** $\sin\frac{\pi}{5} \approx 0.5878$

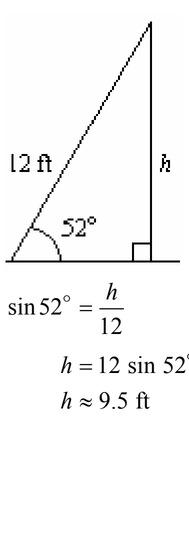
**47.** $\tan\frac{\pi}{7} \approx 0.4816$

**48.** $\sec\frac{3\pi}{8} \approx 2.6131$

**49.** $\csc 1.2 \approx 1.0729$

**50.** $\sin 0.45 \approx 0.4350$

**51.**

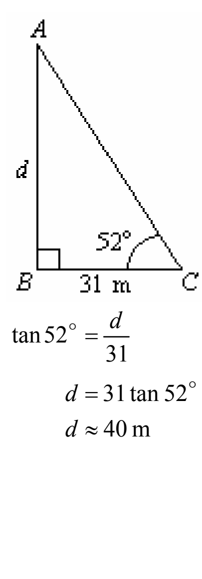



$\sin 52^\circ = \frac{h}{12}$

$h = 12\sin 52^\circ$

$h \approx 9.5$ ft

**52.**

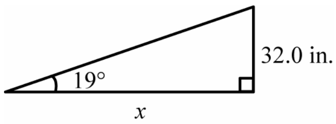



$\tan 52^\circ = \frac{d}{31}$

$d = 31\tan 52^\circ$

$d \approx 40$ m

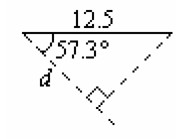
**53.**




$$\tan 19^\circ = \frac{32.0}{x}$$

$$x = \frac{32.0}{\tan 19^\circ} = 92.9 \text{ in.}$$

**54.**




$$d = rt$$

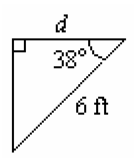
$$t = \frac{d}{r} \quad \cos 57.3^\circ = \frac{d}{12.5}, \quad d = 12.5\cos 57.3^\circ$$

$$t = \frac{12.5\cos 57.3^\circ}{11}$$

$$t \approx 0.61 \text{ h} \approx 37 \text{ min}$$

Time of closest approach; 3:00 P.M.+37 min is 3:37 P.M.

**55.**

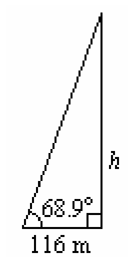



$$\cos 38^\circ = \frac{d + 0.33}{6}$$

$$d = 6\cos 38^\circ + 0.33$$

$$d \approx 5.1 \text{ ft}$$

**56.**

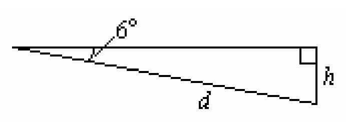



$$\tan 68.9^\circ = \frac{h}{116}$$

$$h = 116\tan 68.9^\circ$$

$$h \approx 301 \text{ m}$$

**57.**




$$d = \frac{240 \text{ mi}}{\text{hr}}\left(\frac{1 \text{ hr}}{60 \text{ min}}\right)4 \text{ min}$$

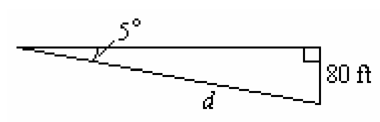
$$d = 16 \text{ mi}$$

$$\sin 6^\circ = \frac{h}{d}$$

$$h = 16\sin 6^\circ$$

$$h \approx 1.7 \text{ mi}$$

**58.**




$$\sin 5^\circ = \frac{80}{d}$$

$$d = \frac{80}{\sin 5^\circ} \text{ ft} = 917.9 \text{ ft}$$

$$t = \frac{d}{r}$$

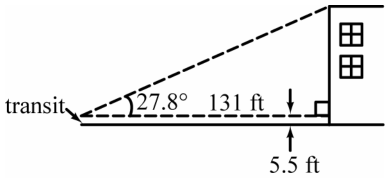
$$r = 9\frac{\text{mi}}{\text{h}} = 9\frac{\text{mi}}{\text{h}} \cdot 5280\frac{\text{ft}}{\text{mi}} \cdot \frac{1 \text{ h}}{60 \text{ min}}$$

$$= \frac{0(5280)}{60}\frac{\text{ft}}{\text{min}} = 792\frac{\text{ft}}{\text{min}}$$

$$t = \frac{917.9 \text{ ft}}{792 \text{ft/min}}$$

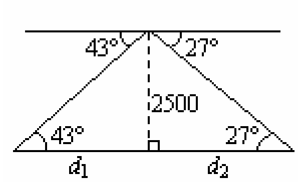
$$t \approx 1.2 \text{ min}$$

**59.**




$$\tan 27.8° = \frac{x}{131}$$
$$x = 131\tan 27.8° = 69.1 \text{ ft}$$
$$\text{height} = 69.1 + 5.5 = 74.6 \text{ ft}$$

**60.**




$$\tan 43° = \frac{2500}{d_1} \qquad \tan 27° = \frac{2500}{d_2}$$
$$d_1 = \frac{2500}{\tan 43°} \qquad d_2 = \frac{2500}{\tan 27°}$$
$$d = d_1 + d_2 = \frac{2500}{\tan 43°} + \frac{2500}{\tan 27°}$$
$$d \approx 2680.9 + 4906.5$$
$$d \approx 7600 \text{ ft}$$

**61.** $\sin 0.056° = \dfrac{670{,}900}{d}$

$$d = \frac{670{,}900}{\sin 0.056°} \text{ km}$$
$$\approx 686{,}000{,}000 \text{ km}$$

**62.** $\sin 46.5° = \dfrac{r}{149{,}000{,}000}$

$$r = 149{,}000{,}000 \sin 46.5° \text{ km}$$
$$\approx 108{,}000{,}000 \text{ km}$$

**63.** $A = \dfrac{1}{2}bh$

$$= \frac{1}{2}(2a\sin\theta)(a\cos\theta)$$
$$= a^2 \sin\theta\cos\theta$$

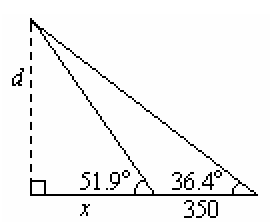
**64.** From exercise 63, find the area of one triangle

$$A = a^2 \sin\theta\cos\theta$$
$$= (4)^2 \sin 60° \cos 60°$$
$$= 16\frac{\sqrt{3}}{2}\cdot\frac{1}{2}$$
$$= 4\sqrt{3}$$

Find the area of the hexagon,

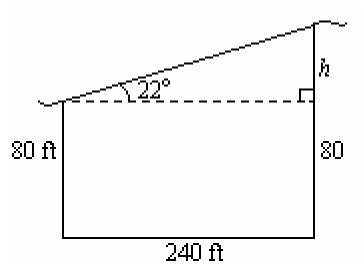
$$A = 6(4\sqrt{3}) = 24\sqrt{3} \text{ in.}^2$$

**65.**



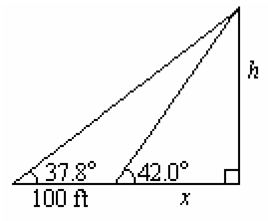

$$\tan 36.4 = \frac{d}{350 + x} \qquad \tan 51.9° = \frac{d}{x}$$
$$x = \frac{d}{\tan 51.9°}$$
$$\tan 36.4° = \frac{d}{350 + \frac{d}{\tan 51.9°}}$$
$$d = \frac{350\tan 36.4°}{1 - \frac{\tan 36.4°}{\tan 51.9°}}$$
$$d \approx 612 \text{ ft}$$

**66.**




$$\tan 22° = \frac{h}{240}$$
$$h = 240\tan 22°$$
$$d = 80 + h$$
$$d = 80 + 240\tan 22°$$
$$d \approx 180 \text{ ft}$$

**67.**




$\tan 42^\circ = \dfrac{h}{x}$

$x = \dfrac{h}{\tan 42^\circ}$

$\tan 37.8^\circ = \dfrac{h}{100 + x}$

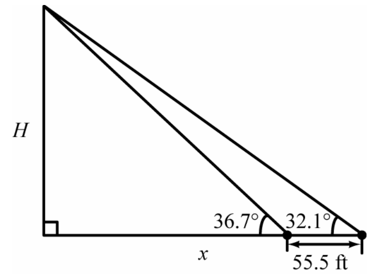
$\tan 37.8^\circ = \dfrac{h}{100 + \frac{h}{\tan 42^\circ}}$

$h = \dfrac{100 \tan 37.8^\circ}{1 - \frac{\tan 37.8^\circ}{\tan 42.0^\circ}}$

$h \approx 5.60 \times 10^2$ ft

$h \approx 560$ ft

**68.**




$x = \dfrac{H}{\tan 36.7^\circ}$ $\qquad x = \dfrac{H}{\tan 32.1^\circ} - 55.5$

$\dfrac{H}{\tan 36.7^\circ} = \dfrac{H}{\tan 32.1^\circ} - 55.5$

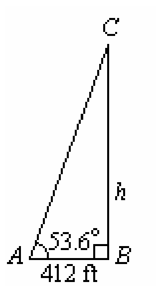
$H \tan 32.1 = H \tan 36.7 - 55.5(\tan 36.7)(\tan 32.1)$

$H(\tan 32.1 - \tan 36.7) = -55.5(\tan 36.7)(\tan 32.1)$

$H = \dfrac{-55.5(\tan 36.7)(\tan 32.1)}{\tan 32.1 - \tan 36.7}$

$H \approx 220$ ft

**69.** **a.**




$\tan 53.6^\circ = \dfrac{h}{412}$

$h = 412 \tan 53.6^\circ$

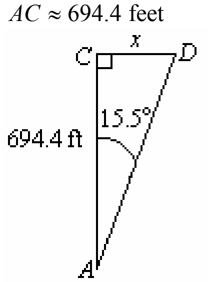
$h \approx 559$ feet

**b.** $(AC)^2 = 412^2 + 559^2$

$AC = \sqrt{412^2 + 559^2}$

$AC = \sqrt{482{,}225}$

$AC \approx 694.4$ feet

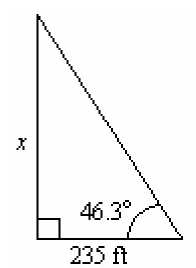



$\tan 15.5^\circ = \dfrac{x}{694.4}$

$x = 694.4 \tan 15.5^\circ$

$x \approx 193$ feet

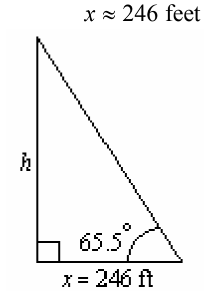
**70.**




$\tan 46.3^\circ = \dfrac{x}{235}$

$x = 235 \tan 46.3^\circ$

$x \approx 246$ feet



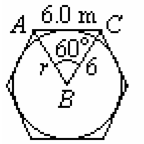

$\tan 65.5^\circ = \dfrac{h}{246}$

$h = 246 \tan 65.5^\circ$

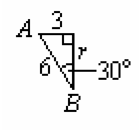
$h \approx 540$ feet

## Connecting Concepts

**71.**




Consider the right triangle formed by $A$, $B$ and the midpoint of $AC$.



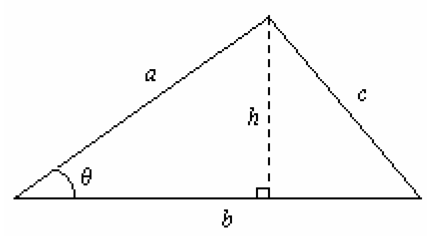

$r = \sqrt{6^2 - 3^2}$

$r = \sqrt{27}$

$r = 3\sqrt{3}$

$r \approx 5.2 \text{ m}$

**72.**




$\sin\theta = \dfrac{h}{a}$

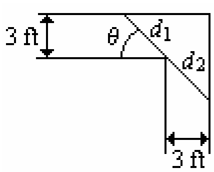
$h = a\sin\theta$

$A = \dfrac{1}{2}bh$

$A = \dfrac{1}{2}b(a\sin\theta)$

$A = \dfrac{1}{2}ab\sin\theta$

**73.**




if $\theta = 45^\circ$ $\quad d_1 = d_2$

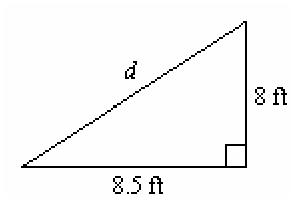
$\sin\theta = \dfrac{3}{d_1}$

$d_1 = \dfrac{3}{\sin 45^\circ}$

$d = 2d_1 = \dfrac{6}{\sin 45^\circ}$

$d \approx 8.5 \text{ ft}$

**74.**




$d = \sqrt{8^2 + 8.5^2}$

$d \approx 11.7 \text{ ft}$

## Prepare for Section 5.3

**PS1.** $-\dfrac{4}{3}$

**PS2.** $\dfrac{5}{2\sqrt{5}} \cdot \dfrac{\sqrt{5}}{\sqrt{5}} = \dfrac{5\sqrt{5}}{10} = \dfrac{\sqrt{5}}{2}$

**PS3.** $|120 - 180| = |-60| = 60$

**PS4.** $2\pi - \dfrac{9\pi}{5} = \dfrac{10\pi}{5} - \dfrac{9\pi}{5} = \dfrac{\pi}{5}$

**PS5.** $\dfrac{3}{2}\pi - \dfrac{1}{2}\pi = \dfrac{2}{2}\pi = \pi$

**PS6.** $\sqrt{(-3)^2 + (-5)^2} = \sqrt{9 + 25} = \sqrt{34}$

## Section 5.3

**1.** $x = 2,\ y = 3,\ r = \sqrt{2^2 + 3^2} = \sqrt{13}$

$\sin\theta = \frac{y}{r} = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}$ $\csc\theta = \frac{\sqrt{13}}{3}$

$\cos\theta = \frac{x}{r} = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13}$ $\sec\theta = \frac{\sqrt{13}}{2}$

$\tan\theta = \frac{y}{x} = \frac{3}{2}$ $\cot\theta = \frac{2}{3}$

**2.** $x = 3,\ y = 7,\ r = \sqrt{3^2 + 7^2} = \sqrt{58}$

$\sin\theta = \frac{y}{r} = \frac{7}{\sqrt{58}} = \frac{7\sqrt{58}}{58}$ $\csc\theta = \frac{\sqrt{58}}{7}$

$\cos\theta = \frac{x}{r} = \frac{3}{\sqrt{58}} = \frac{3\sqrt{58}}{58}$ $\sec\theta = \frac{\sqrt{58}}{3}$

$\tan\theta = \frac{y}{x} = \frac{7}{3}$ $\cot\theta = \frac{3}{7}$

**3.** $x = -2,\ y = 3,\ r = \sqrt{(-2)^2 + (3)^2} = \sqrt{13}$

$\sin\theta = \frac{y}{r} = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}$ $\csc\theta = \frac{\sqrt{13}}{3}$

$\cos\theta = \frac{x}{r} = \frac{-2}{\sqrt{13}} = -\frac{2\sqrt{13}}{13}$ $\sec\theta = -\frac{\sqrt{13}}{2}$

$\tan\theta = \frac{y}{x} = \frac{3}{-2} = -\frac{3}{2}$ $\cot\theta = -\frac{2}{3}$

**4.** $x = -3,\ y = 5,\ r = \sqrt{(-3)^2 + 5^2} = \sqrt{34}$

$\sin\theta = \frac{y}{r} = \frac{5}{\sqrt{34}} = \frac{5\sqrt{34}}{34}$ $\csc\theta = \frac{\sqrt{34}}{5}$

$\cos\theta = \frac{x}{r} = \frac{-3}{\sqrt{34}} = -\frac{3\sqrt{34}}{34}$ $\sec\theta = -\frac{\sqrt{34}}{3}$

$\tan\theta = \frac{y}{x} = \frac{5}{-3} = -\frac{5}{3}$ $\cot\theta = -\frac{3}{5}$

**5.** $x = -8,\ y = -5,\ \mathrm{c} = \sqrt{(-8)^2 + (-5)^2} = \sqrt{89}$

$\sin\theta = \frac{y}{r} = \frac{-5}{\sqrt{89}} = -\frac{5\sqrt{89}}{89}$ $\csc\theta = -\frac{\sqrt{89}}{5}$

$\cos\theta = \frac{x}{r} = \frac{-8}{\sqrt{89}} = -\frac{8\sqrt{89}}{89}$ $\sec\theta = -\frac{\sqrt{89}}{8}$

$\tan\theta = \frac{y}{x} = \frac{-5}{-8} = \frac{5}{8}$ $\cot\theta = \frac{8}{5}$

**6.** $x = -6,\ y = -9,\ r = \sqrt{(-6)^2 + (-9)^2} = \sqrt{117} = 3\sqrt{13}$

$\sin\theta = \frac{y}{r} = \frac{-9}{3\sqrt{13}} = -\frac{3}{\sqrt{13}} = -\frac{3\sqrt{13}}{13}$ $\csc\theta = -\frac{\sqrt{13}}{3}$

$\cos\theta = \frac{x}{r} = \frac{-6}{3\sqrt{13}} = -\frac{2}{\sqrt{13}} = -\frac{2\sqrt{13}}{13}$ $\sec\theta = -\frac{\sqrt{13}}{2}$

$\tan\theta = \frac{y}{x} = \frac{-9}{-6} = \frac{3}{2}$ $\cot\theta = \frac{2}{3}$

**7.** $x = -5,\ y = 0,\ r = \sqrt{(-5)^2 + (0)^2} = 5$

$\sin\theta = \frac{y}{r} = \frac{0}{5} = 0$ $\csc\theta$ is undefined

$\cos\theta = \frac{x}{r} = \frac{-5}{5} = -1$ $\sec\theta = -1$

$\tan\theta = \frac{y}{x} = \frac{0}{-5} = 0$ $\cot\theta$ is undefined

**8.** $x = 0,\ y = 2,\ r = \sqrt{0^2 + 2^2} = 2$

$\sin\theta = \frac{y}{r} = \frac{2}{2} = 1$ $\csc\theta = 1$

$\cos\theta = \frac{x}{r} = \frac{0}{2} = 0$ $\sec\theta$ is undefined

$\tan\theta = \frac{y}{x} = \frac{2}{0} \Rightarrow$ undefined $\cot\theta = 0$

**9.** $\sin 180^\circ = 0$

**10.** $\cos 270^\circ = 0$

**11.** $\tan 180^\circ = 0$

**12.** $\sec 90^\circ =$ undefined

**13.** $\csc 90^\circ = 1$

**14.** $\cot 90^\circ = 0$

**15.** $\cos\frac{\pi}{2} = 0$

**16.** $\sin\frac{3\pi}{2} = -1$

**17.** $\tan\frac{\pi}{2} =$ undefined

**18.** $\cot\pi =$ undefined

**19.** $\sin\frac{\pi}{2} = 1$

**20.** $\cos\pi = -1$

**21.** $\sin\theta > 0$ in quadrants I and II.
$\cos\theta > 0$ in quadrants I and IV.
quadrant I

**22.** $\tan\theta < 0$ in quadrants II and IV.
$\sin\theta < 0$ in quadrants III and IV.
quadrant IV

**23.** $\cos\theta > 0$ in quadrants I and IV.
$\tan\theta < 0$ in quadrants II and IV.
quadrant IV

**24.** $\sin\theta < 0$ in quadrants III and IV.
$\cos\theta > 0$ in quadrants I and IV.
quadrant IV

**25.** $\sin\theta < 0$ in quadrants III and IV.
$\cos\theta < 0$ in quadrants II and III.
quadrant III

**26.** $\tan\theta < 0$ in quadrants II and IV.
$\cos\theta < 0$ in quadrants II and III.
quadrant II

**27.** $\sin\theta = -\frac{1}{2} = \frac{y}{r}, y = -1, r = 2, x = \pm\sqrt{2^2 - (-1)^2} = \pm\sqrt{3}, x = -\sqrt{3}$ in quadrant III, $\tan\theta = \frac{y}{x} = \frac{-y}{-\sqrt{3}} = \frac{\sqrt{3}}{3}$

**28.** $\cot\theta = -1 = \frac{x}{y},\ x = -1,\ y = 1$ in quadrant II, $r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$, $\cos\theta = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$

**29.** $\csc\theta = \sqrt{2} = \frac{r}{y},\ r = \sqrt{2},\ y = 1,\ x = \pm\sqrt{(\sqrt{2})^2 - 1^2} = \pm 1$ , $x = -1$ in quadrant II, $\cot\theta = \frac{-1}{1} = -1$

**30.** $\sec\theta = \frac{2\sqrt{3}}{3} = \frac{r}{x},\ r = 2\sqrt{3}, x = 3,\ y = \pm\sqrt{(2\sqrt{3})^2 - 3^2} = \pm\sqrt{3},\ y = -\sqrt{3}$ in quadrant IV, $\sin\theta = \frac{-\sqrt{3}}{2\sqrt{3}} = -\frac{1}{2}$

**31.** $\theta$ is in quadrant IV, $\sin\theta = -\frac{1}{2} = \frac{y}{r},\ y = -1,\ r = 2,\ x = \sqrt{2^2 - 1^2} = \sqrt{3}$, $\tan\theta = \frac{-1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

**32.** $\theta$ is in quadrant III, $\tan\theta = 1 = \frac{y}{x}, y = -1, x = -1, r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$, $\cos\theta = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$

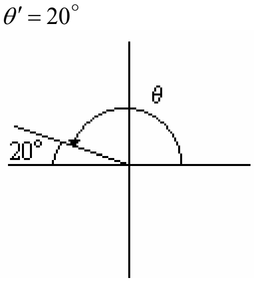
**33.** $\cos\theta = \frac{1}{2},\ \theta$ is in quadrant I or IV.
$\tan\theta = \sqrt{3}, \theta$ is in quadrant I or III.
$\theta$ is in quadrant I, $x = 1, y = \sqrt{3}, r = 2$
$\csc\theta = \frac{r}{y} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

**34.** $\tan\theta = 1$, $\theta$ is in quadrant I or III.
$\sin\theta = \frac{\sqrt{2}}{2},\ \theta$ is in quadrant III or IV.
$\theta$ is in quadrant III, $x = \sqrt{2},\ y = \sqrt{2},\ r = 2$
$\sec\theta = \frac{r}{x} = \frac{2}{\sqrt{2}} = \sqrt{2}$

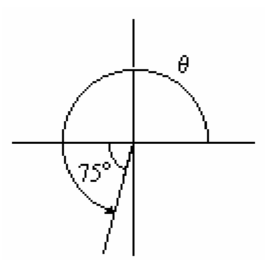
**35.** $\cos\theta = -\frac{1}{2}, \theta$ is in quadrant II or III.
$\sin\theta = \frac{\sqrt{3}}{2}, \theta$ is in quadrant I or II.
$\theta$ is in quadrant II, $x = -1, y = \sqrt{3}, r = 2$
$\cot\theta = \frac{x}{y} = \frac{-1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

**36.** $\sec\theta = \frac{2\sqrt{3}}{3},\ \theta$ is in quadrant I or IV.
$\sin\theta = -\frac{1}{2}, \theta$ is in quadrant II or IV.
$\theta$ is in quadrant IV, $x = \sqrt{3}, y = -1, r = 2$
$\cot\theta = \frac{x}{y} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$

**37.** $\theta = 160^\circ$
Since $90^\circ < \theta < 180^\circ$,
$\theta + \theta' = 180^\circ$
$\theta' = 20^\circ$




**38.** $\theta = 255^\circ$
Since $180^\circ < \theta < 270^\circ$,
$\theta' + 180^\circ = \theta$
$\theta' = 75^\circ$

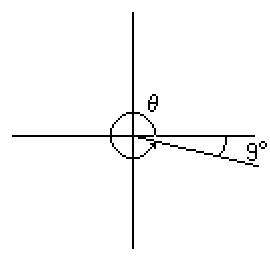


**39.** $\theta = 351^\circ$

Since $270^\circ < \theta < 360^\circ$,

$\theta = \theta' = 360^\circ$

$\theta' = 9^\circ$

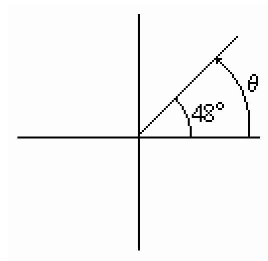



**40.** $\theta = 48^\circ$

Since $0^\circ < \theta < 48^\circ$,

$\theta' = \theta$

$\theta' = 48^\circ$




**41.** $\theta = \dfrac{11\pi}{5}$

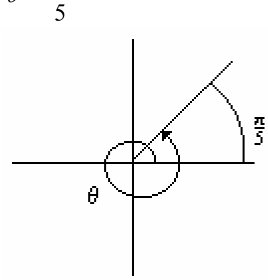
$\theta > 2\pi = \dfrac{10\pi}{5}$,

$\theta$ is coterminal with $\alpha = \dfrac{11\pi}{5} - \dfrac{10\pi}{5} = \dfrac{\pi}{5}$.

Since $0 < \alpha < \dfrac{\pi}{2}$,

$\alpha' = \alpha = \theta'$

$\theta' = \dfrac{\pi}{5}$



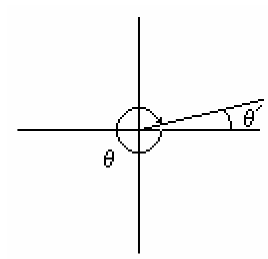

**42.** $\theta = -6$

Since $-\dfrac{3\pi}{2} < \theta < -2\pi$,

$\theta' + |\theta| = 2\pi$

$\theta' = 2\pi - 6$

$\approx 0.28$

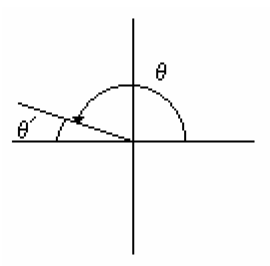


**43.** $\theta = \frac{8}{3}$

Since $\frac{\pi}{2} < \theta < \pi$,

$\theta + \theta' = \pi$

$\theta' = \pi - \frac{8}{3}$




**44.** $\theta = \frac{18\pi}{7}$

$\theta > 2\pi = \frac{14\pi}{7}$,

$\theta$ is coterminal with

$\alpha = \frac{18\pi}{7} - \frac{14\pi}{7} = \frac{4\pi}{7}$.

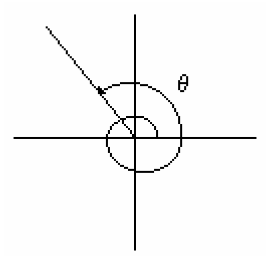
Since $\frac{\pi}{2} < \alpha < \frac{3\pi}{2}$,

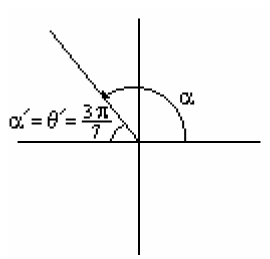
$\alpha + \alpha' = \pi$

$\alpha' = \pi - \frac{4\pi}{7}$

$= \frac{3\pi}{7}$

$\theta' = \frac{3\pi}{7}$




**45.** $\theta = 1406^\circ = 326^\circ + 3 \cdot 360^\circ$

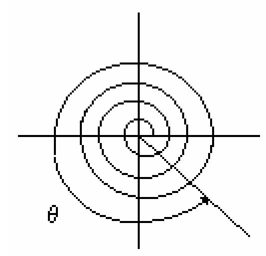
$\theta$ is coterminal with $\alpha = 326^\circ$.

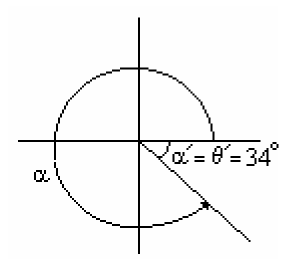
Since $270^\circ < \alpha < 360^\circ$,

$\alpha + \alpha' = 360^\circ$

$\alpha' = 34^\circ$

$\theta' = 34^\circ$



**46.** $\theta = 840^\circ = 120^\circ + 2 \cdot 360^\circ$

$\theta > 360^\circ$

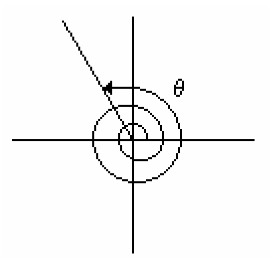
$\theta$ is coterminal with $\alpha = 120°$.

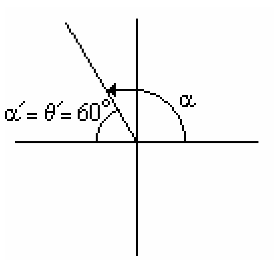
Since $90^\circ < \alpha < 180^\circ$,

$\alpha + \alpha' = 180^\circ$

$\alpha' = 60^\circ$

$\theta' = 60^\circ$




**47.** $\theta = -475^\circ = 245^\circ - 2 \cdot 360^\circ$

$\theta$ is coterminal with $\alpha = 245°$

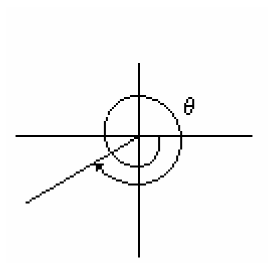
Since $180° < \alpha < 270°$,

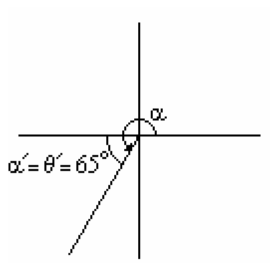
$\alpha' + 180° = \alpha$

$\alpha' = 245° - 180°$

$\alpha' = 65°$

$\theta' = 65°$



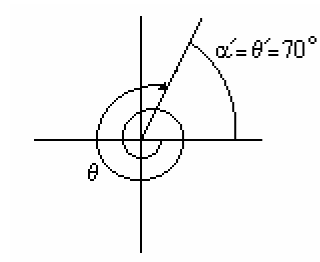

**48.** $\theta = -650^\circ = 70^\circ - 2 \cdot 360^\circ$

$\theta$ is coterminal with $\alpha = 70°$.

Since $0^\circ < \alpha < 90^\circ$,

$\alpha' = \alpha = 70^\circ$

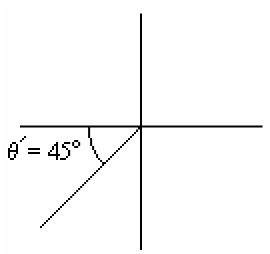
$\theta' = \alpha' = 70^\circ$




**49.** $\theta = 225°$ is in quadrant III.

$225° - 180° = 45°$ so $\theta' = 45°$.

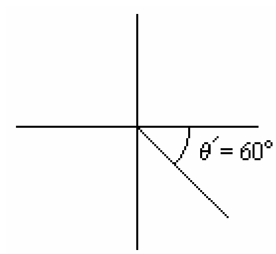
Thus, $\sin 225° = -\sin 45° = -\dfrac{\sqrt{2}}{2}$.




**50.** $\theta = 300°$ is in quadrant IV.

$360° - 300° = 60°$ so $\theta' = 60°$.

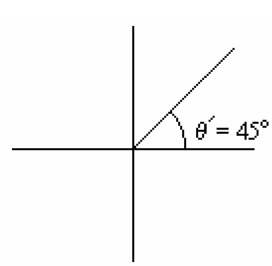
Thus, $\cos\ 300° = \cos\ 60° = \dfrac{1}{2}$.




**51.** $\theta = 405°$ is in quadrant I.

$405° - 360° = 45°$ so $\theta' = 45°$.

Thus, $\tan 405° = \tan 45° = 1$.

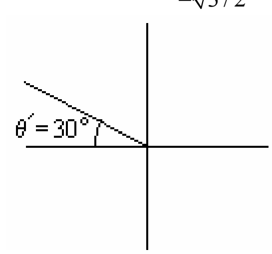



**52.** $\theta = 150°$ is in quadrant II.

$180° - 150° = 30°$ so $\theta' = 30°$.

Thus, $\sec 150° = \dfrac{1}{\cos 150°} = \dfrac{1}{-\cos 30°}$

$= \dfrac{1}{-\sqrt{3}/2} = \dfrac{-2}{\sqrt{3}} = -\dfrac{2\sqrt{3}}{3}$.

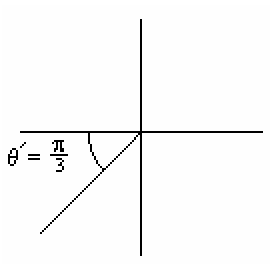


**53.** $\theta = \frac{4}{3}\pi$ is in quadrant III.

$\frac{4}{3}\pi - \pi = \frac{\pi}{3}$ so $\theta' = \frac{\pi}{3}$.

Thus, $\csc\frac{4\pi}{3} = \frac{1}{\sin 4\pi/3} = \frac{1}{-\sin\pi/3}$

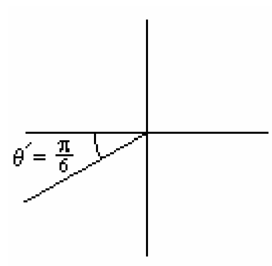
$= \frac{1}{-\sqrt{3}/2} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$.




**54.** $\theta = \frac{7\pi}{6}$ is in quadrant III.

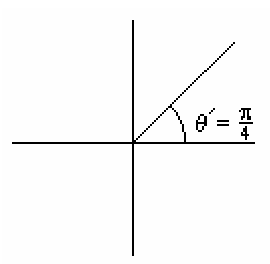
$\frac{7\pi}{6} - \pi = \frac{\pi}{6}$ so $\theta' = \frac{\pi}{6}$.

Thus, $\cot\frac{7\pi}{6} = \cot\frac{\pi}{6} = \frac{\cos\pi/6}{\sin\pi/6} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$.




**55.** $\theta = \frac{17\pi}{4} = \frac{16\pi}{4} + \frac{\pi}{4}$ is coterminal with $\frac{\pi}{4}$ in quadrant I and $\theta' = \frac{\pi}{4}$,

so $\cos\frac{17\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

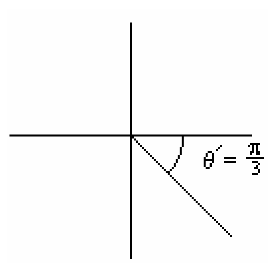



**56.** $\theta = \frac{-\pi}{3}$ is in quadrant IV.

$0 - \left(-\frac{\pi}{3}\right) = \frac{\pi}{3}$ so $\theta' = \frac{\pi}{3}$.

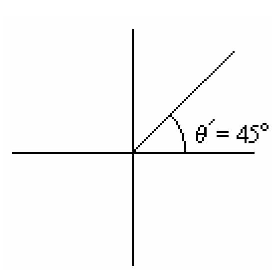
Thus, $\tan\left(\frac{-\pi}{3}\right) = -\tan\frac{\pi}{3} = \frac{-\sin\pi/3}{\cos\pi/3}$

$= \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3}$.




**57.** $\theta = 765^\circ = 720^\circ + 45^\circ$ is coterminal with $45^\circ$ in quadrant I and $\theta' = 45^\circ$,

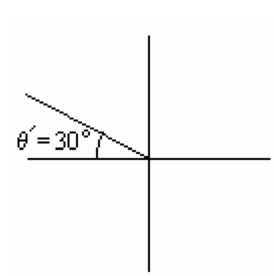
so $\sec 765^\circ = \sec 45^\circ = \frac{1}{\cos 45^\circ} = \frac{1}{\sqrt{2}/2} = \sqrt{2}$.



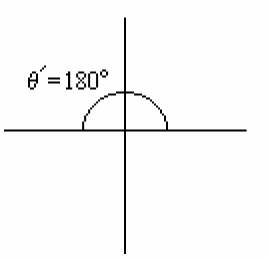

**58.** $\theta = -510^\circ = -720^\circ + 210^\circ$ is coterminal with $210^\circ$ in quadrant III and $\theta' = 30^\circ$,

so $\csc(-510^\circ) = \frac{1}{\sin(-510^\circ)}$

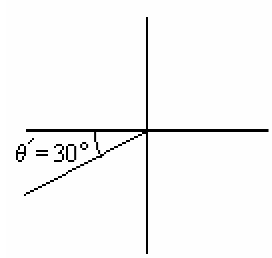
$\frac{1}{-\sin 30^\circ} = \frac{1}{-1/2} = -2$.



**59.** $\theta = 540^\circ = 360^\circ + 180^\circ$ is coterminal with $180^\circ$, so $\cot 540^\circ = \cot 180^\circ = \dfrac{\cos 180^\circ}{\sin 180^\circ} = \dfrac{-1}{0}$, which is undefined.




**60.** $\theta = 570^\circ = 360^\circ + 210^\circ$ is coterminal with $210^\circ$ in quadrant III and $\theta' = 30^\circ$, so $\cos 570^\circ = -\cos 30^\circ = -\dfrac{\sqrt{3}}{2}$.




**61.** $\sin 127^\circ \approx 0.798636$

**62.** $\sin(-257^\circ) \approx 0.974370$

**63.** $\cos(-116^\circ) \approx -0.438371$

**64.** $\cot 398^\circ \approx 1.27994$

**65.** $\sec 578^\circ \approx -1.26902$

**66.** $\sec 740^\circ \approx 1.06418$

**67.** $\sin\left(-\dfrac{\pi}{5}\right) \approx -0.587785$

**68.** $\cos\dfrac{3\pi}{7} \approx 0.222521$

**69.** $\csc\dfrac{9\pi}{5} \approx -1.70130$

**70.** $\tan(-4.12) \approx -1.48584$

**71.** $\sec(-4.45) \approx -3.85522$

**72.** $\csc 0.34 \approx 2.99862$

**73.** $\sin 210^\circ - \cos 330^\circ \tan 330^\circ = -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}\left(-\dfrac{\sqrt{3}}{3}\right) = -\dfrac{1}{2} + \dfrac{1}{2} = 0$

**74.** $\tan 225^\circ + \sin 240^\circ \cos 60^\circ = 1 + \left(-\dfrac{\sqrt{3}}{2}\right)\dfrac{1}{2} = 1 - \dfrac{\sqrt{3}}{4} = \dfrac{4-\sqrt{3}}{4}$

**75.** $\sin^2 30^\circ + \cos^2 30^\circ = \left(\dfrac{1}{2}\right)^\circ + \left(\dfrac{\sqrt{3}}{2}\right)^2 = \dfrac{1}{4} + \dfrac{3}{4} = 1$

**76.** $\cos \pi \sin\dfrac{7\pi}{4} - \tan\dfrac{11\pi}{6} = -1\left(-\dfrac{\sqrt{2}}{2}\right) - \left(-\dfrac{\sqrt{3}}{3}\right) = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{3}}{3} = \dfrac{3\sqrt{2}+2\sqrt{3}}{6}$

**77.** $\sin\dfrac{3\pi}{2}\tan\dfrac{\pi}{4} - \cos\dfrac{\pi}{3} = (-1)(1) - \dfrac{1}{2} = -1 - \dfrac{1}{2} = -\dfrac{3}{2}$

**78.** $\cos\dfrac{7\pi}{4}\tan\dfrac{4\pi}{3} + \cos\dfrac{7\pi}{6} = \dfrac{\sqrt{2}}{2}\left(\sqrt{3}\right) + \left(-\dfrac{\sqrt{3}}{2}\right) = \dfrac{\sqrt{6}}{2} - \dfrac{\sqrt{3}}{2} = \dfrac{\sqrt{6}-\sqrt{3}}{2}$

**79.** $\sin^2\dfrac{5\pi}{4} + \cos^2\dfrac{5\pi}{4} = \left(-\dfrac{\sqrt{2}}{2}\right)^2 + \left(-\dfrac{\sqrt{2}}{2}\right)^2 = \dfrac{1}{2} + \dfrac{1}{2} = 1$

**80.** $\tan^2\dfrac{7\pi}{4} - \sec^2\dfrac{7\pi}{4} = (-1)^2 - \left(\sqrt{2}\right)^2 = 1 - 2 = -1$

## Connecting Concepts

**81.** $\sin\theta = \frac{1}{2}$, $\theta$ is in quadrant I or quadrant II

$\theta = 30^\circ, 150^\circ$

**82.** $\tan\theta = -\sqrt{3}, \theta$ is in quadrant II or quadrant IV

$\theta = 120^\circ, 300^\circ$

**83.** $\cos\theta = \frac{-\sqrt{3}}{2}$, $\theta$ is in quadrant II or quadrant III

$\theta = 150^\circ, 120^\circ$

**84.** $\tan\theta = 1$, $\theta$ is in quadrant I or quadrant III

$\theta = 45^\circ, 225^\circ$

**85.** $\csc\theta = -\sqrt{2}$
$\theta$ is in quadrant III or IV
$\theta = 225^\circ, 315^\circ$

**86.** $\cot\theta = -1$
$\theta$ is in quadrant II or IV
$\theta = 135^\circ, 315^\circ$

**87.** $\tan\theta = -1$
$\theta$ is in quadrant II or IV
$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

**88.** $\cos\theta = \frac{1}{2}$
$\theta$ is in quadrant I or IV
$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

**89.** $\tan\theta = \frac{-\sqrt{3}}{3}$
$\theta$ is in quadrant II or IV
$\theta = \frac{5\pi}{6}, \frac{11\pi}{6}$

**90.** $\sec\theta = \frac{-2\sqrt{3}}{3}$
$\theta$ is in quadrant II or III
$\theta = \frac{5\pi}{6}, \frac{7\pi}{6}$

**91.** $\sin\theta = \frac{\sqrt{3}}{2}$
$\theta$ is in quadrant I or II
$\theta = \frac{\pi}{3}, \frac{2\pi}{3}$

**92.** $\cos\theta = -\frac{1}{2}$
$\theta$ is in quadrant II or III
$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

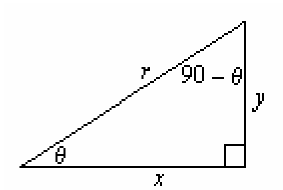
**93.** $1 + \tan^2\theta = \sec^2\theta$

$$1 + \frac{y^2}{x^2} = \frac{x^2 + y^2}{x^2} = \frac{r^2}{x^2} = \sec^2\theta$$

**94.** $\cot^2\theta + 1 = \csc^2\theta$

$$\frac{x^2}{y^2} + 1 = \frac{x^2 + y^2}{y^2} = \frac{r^2}{y^2} = \csc^2\theta$$

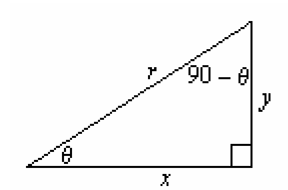
**95.**




$\cos(90^\circ - \theta) = \sin\theta$

$\frac{y}{r} = \frac{y}{r}$

**96.**




$\sin(90^\circ - \theta) = \cos\theta$

$\frac{x}{r} = \frac{x}{r}$

## Prepare for Section 5.4

**PS1.** $x^2 + y^2 = 1$
$(0)^2 + (1)^2 = 1$
Yes

**PS2.** $x^2 + y^2 = 1$
$\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$
$\frac{1}{4} + \frac{3}{4} = 1$
Yes

**PS3.** $x^2 + y^2 = 1$
$\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$
$\frac{2}{4} + \frac{3}{4} \neq 1$
No

**PS4.** $C = 2\pi r$
$= 2\pi(1)$
$= 2\pi$

**PS5.** even

**PS6.** neither

## Section 5.4

**1.** $t = \frac{\pi}{6}$

$y = \sin t = \sin\frac{\pi}{6} = \frac{1}{2}$ $\quad x = \cos t = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

The point on the unit circle corresponding to $t = \frac{\pi}{6}$ is $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

**2.** $t = \frac{\pi}{4}$

$y = \sin t = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\quad x = \cos t = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

The point on the unit circle corresponding to $t = \frac{\pi}{4}$ is $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

**3.** $t = \frac{7\pi}{6}$

$y = \sin t = \sin\frac{7\pi}{6} = -\frac{1}{2}$ $\quad x = \cos t = \cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$

The point on the unit circle corresponding to $t = \frac{7\pi}{6}$ is $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

**4.** $t = \frac{4\pi}{3}$

$y = \sin t = \sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$ $\quad x = \cos t = \cos\frac{4\pi}{3} = -\frac{1}{2}$

The point on the unit circle corresponding to $t = \frac{4\pi}{3}$ is $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

**5.** $t = \frac{5\pi}{3}$

$y = \sin t = \sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$ $\quad x = \cos t = \cos\frac{5\pi}{3} = \frac{1}{2}$

The point on the unit circle corresponding to $t = \frac{5\pi}{3}$ is $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

**6.** $t = -\frac{\pi}{6}$

$y = \sin t = \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ $\quad x = \cos t = \cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

The point on the unit circle corresponding to $t = -\frac{\pi}{6}$ is $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

**7.** $t = \frac{11\pi}{6}$

$y = \sin t = \sin\frac{11\pi}{6} = -\frac{1}{2}$ $\quad x = \cos t = \cos\frac{11\pi}{6} = \frac{\sqrt{3}}{2}$

The point on the unit circle corresponding to $t = \frac{11\pi}{6}$ is $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

**8.** $t = 0$

$y = \sin t = \sin 0 = 0$ $\quad x = \cos t = \cos 0 = 1$

The point on the unit circle corresponding to $t = 0$ is $(1, 0)$.

**9.** $t=\pi$

$y=\sin t \quad x=\cos t$

$=\sin\pi \quad =\cos\pi$

$=0 \quad =-1$

The point on the unit circle corresponding to $t=\pi$ is $(-1,0)$.

**10.** $t=-\dfrac{7\pi}{4}$

$y=\sin t \quad x=\cos t$

$=\sin\left(-\dfrac{7\pi}{4}\right) \quad =\cos\left(-\dfrac{7\pi}{4}\right)$

$=-\sin\dfrac{7\pi}{4} \quad =\cos\dfrac{7\pi}{4}$

$=\dfrac{\sqrt{2}}{2} \quad =\dfrac{\sqrt{2}}{2}$

The point on the unit circle corresponding to $t=-\frac{7\pi}{4}$ is $\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$.

**11.** $t=-\dfrac{2\pi}{3}$

$y=\sin t \quad x=\cos t$

$=\sin\left(-\dfrac{2\pi}{3}\right) \quad =\cos\left(-\dfrac{2\pi}{3}\right)$

$=-\sin\dfrac{2\pi}{3} \quad =\cos\dfrac{2\pi}{3}$

$=-\dfrac{\sqrt{3}}{2} \quad =-\dfrac{1}{2}$

The point on the unit circle corresponding to $t=-\frac{2\pi}{3}$ is $\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$.

**12.** $t=-\pi$

$y=\sin t \quad x=\cos t$

$=\sin(-\pi) \quad =\cos(-\pi)$

$=-\sin\pi \quad =\cos\pi$

$=0 \quad =-1$

The point on the unit circle corresponding to $t=-\pi$ is $(-1,0)$.

**13.** $\tan\dfrac{11\pi}{6}=-\tan\dfrac{\pi}{6}=-\dfrac{\sqrt{3}}{3}$

**14.** $\cot\dfrac{2\pi}{3}=-\cot\dfrac{\pi}{3}=-\dfrac{\sqrt{3}}{3}$

**15.** $\cos\left(-\dfrac{2\pi}{3}\right)=-\cos\dfrac{\pi}{3}=-\dfrac{1}{2}$

**16.** $\sec\left(-\dfrac{5\pi}{6}\right)=-\sec\dfrac{\pi}{6}=-\dfrac{2\sqrt{3}}{3}$

**17.** $\csc\left(-\dfrac{\pi}{3}\right)=-\csc\dfrac{\pi}{3}=-\dfrac{2\sqrt{3}}{3}$

**18.** $\tan(12\pi)=\tan 0=0$

**19.** $\sin\dfrac{3\pi}{2}=-\sin\dfrac{\pi}{2}=-1$

**20.** $\cos\dfrac{7\pi}{3}=\cos\dfrac{\pi}{3}=\dfrac{1}{2}$

**21.** $\sec\left(-\dfrac{7\pi}{6}\right)=-\sec\dfrac{\pi}{6}=-\dfrac{2\sqrt{3}}{3}$

**22.** $\sin\left(-\dfrac{5\pi}{3}\right)=\sin\dfrac{\pi}{3}=\dfrac{\sqrt{3}}{2}$

**23.** $\sin 1.22\approx 0.9391$

**24.** $\cos 4.22\approx -0.4727$

**25.** $\csc(-1.05)\approx -1.1528$

**26.** $\sin(-0.55)\approx -0.5227$

**27.** $\tan\dfrac{11\pi}{12}\approx -0.2679$

**28.** $\cos\dfrac{2\pi}{5}\approx 0.3090$

**29.** $\cos\left(-\dfrac{\pi}{5}\right)\approx 0.8090$

**30.** $\csc 8.2\approx 1.0630$

**31.** $\sec 1.55\approx 48.0889$

**32.** $\cot 2.11\approx -0.5983$

**33.** **a.** $\sin 2\approx 0.9$
**b.** $\cos 2\approx -0.4$

**34.** **a.** $\sin 3\approx 0.1$
**b.** $\cos 3\approx -1.0$

**35.** **a.** $\sin 5.4\approx -0.8$
**b.** $\cos 5.4\approx 0.6$

**36.** **a.** $\sin 4.1\approx -0.8$
**b.** $\cos 4.1\approx -0.6$

**37.** $\sin t=0.4$ when $t=0.4$ or $2.7$

**38.** $\cos t=0.8$ when $t=0.6$ or $5.6$

**39.** $\sin t=-0.3$ when $t=3.4$ or $6.0$

**40.** $\cos t = -0.7$ when $t = 2.3$ or $3.9$

**41.** $f(-x) = -4\sin(-x)$
$= 4\sin x$
$= -f(x)$

The function defined by $f(x) = -4\sin x$ is an odd function.

**42.** $f(-x) = -2\cos(-x)$
$= -2\cos x$
$= f(x)$

The function defined by $f(x) = -2\cos x$ is an even function.

**43.** $G(-x) = \sin(-x) + \cos(-x)$
$= -\sin x + \cos x$

The function defined by $G(x) = \sin x + \cos x$ is neither an even nor an odd function.

**44.** $F(-x) = \tan(-x) + \sin(-x)$
$= -\tan x - \sin x$
$= -F(x)$

The function defined by $F(x) = \tan x + \sin x$ is an odd function.

**45.** $S(-x) = \dfrac{\sin(-x)}{-x}$
$= -\dfrac{\sin x}{-x} = \dfrac{\sin x}{x}$
$= S(x)$

The function defined by $S(x) = \dfrac{\sin(x)}{x}$ is an even function.

**46.** $C(-x) = \dfrac{\cos(-x)}{-x}$
$= -\dfrac{\cos x}{x}$
$= -C(x)$

The function defined by $C(x) = \dfrac{\cos x}{x}$ is an odd function.

**47.** $v(-x) = 2\sin(-x)\cos(-x)$
$= -2\sin x\cos x$
$= -v(x)$

The function defined by $v(x) = 2\sin x\cos x$ is an odd function.

**48.** $w(-x) = -x\tan(-x)$
$= x\tan x$
$= w(x)$

The function defined by $w(x) = x\tan x$ is an even function.

**49.** $2\pi$

**50.** $2\pi$

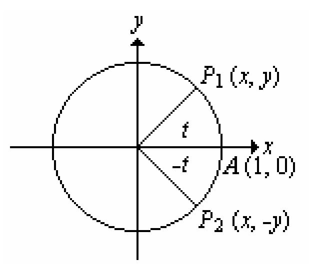
**51.** $\pi$

**52.** $\pi$

**53.** $2\pi$

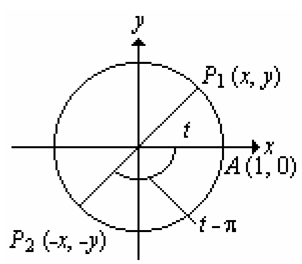
**54.** $2\pi$

**55.**



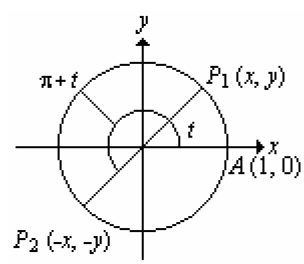

$\cos t = x$
$\cos(-t) = x$
$\cos(-t) = \cos t$

**56.**



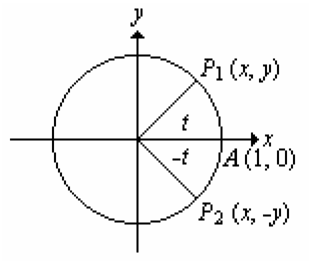

$\tan t = \dfrac{y}{x}$
$\tan(t - \pi) = \dfrac{-y}{-x}$
$\tan(t - \pi) = \tan t$

**57.**



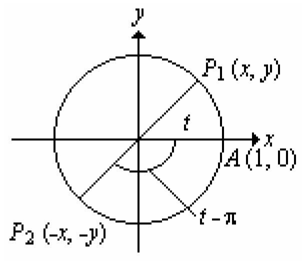

$\cos t = x$
$\cos(\pi + t) = -x$
$\cos t = -\cos(\pi + t)$

**58.**



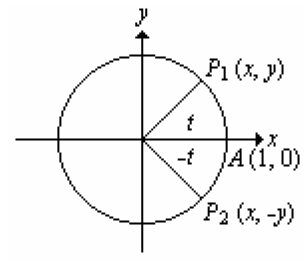

$$\sin(-t) = -y$$
$$\sin t = y$$
$$\sin(-t) = -\sin t$$

**59.**




$$\sin(t-\pi) = -y$$
$$\sin t = y$$
$$\sin(t-\pi) = -\sin t$$

**60.**




$$\sec t = \frac{1}{x}$$
$$\sec(-t) = \frac{1}{x}$$
$$\sec(-t) = \sec t$$

**61.** $$\tan t \cos t = \frac{\sin t}{\cos t} \cdot \cos t = \sin t$$

**62.** $$\cot t \sin t = \frac{\cos t}{\sin t} \cdot \sin t = \cos t$$

**63.** $$\frac{\csc t}{\cot t} = \frac{\frac{1}{\sin t}}{\frac{\cos t}{\sin t}} = \frac{1}{\sin t} \cdot \frac{\sin t}{\cos t} = \frac{1}{\cos t} = \sec t$$

**64.** $$\frac{\sec t}{\tan t} = \frac{\frac{1}{\cos t}}{\frac{\sin t}{\cos t}} = \frac{1}{\cos t} \cdot \frac{\cos t}{\sin t} = \frac{1}{\sin t} = \csc t$$

**65.** $$1 - \sec^2 t = 1 - \frac{1}{\cos^2 t} = \frac{\cos^2 t - 1}{\cos^2 t} = \frac{-\sin^2 t}{\cos^2 t} = -\tan^2 t$$

**66.** $$1 - \csc^2 t = -(\csc^2 t - 1) = -\cot^2 t$$

**67.** $$\tan t - \frac{\sec^2 t}{\tan t} = \tan t - \frac{1 + \tan^2 t}{\tan t} = \tan t - \frac{1}{\tan t} - \frac{\tan^2 t}{\tan t} = \tan t - \cot t - \tan t = -\cot t$$

**68.** $$\frac{\csc^2 t}{\cot t} - \cot t = \frac{1 + \cot^2 t}{\cot t} - \cot t = \frac{1}{\cot t} + \frac{\cot^2 t}{\cot t} - \cot t = \tan t + \cot t - \cot t = \tan t$$

**69.** $$\frac{1 - \cos^2 t}{\tan^2 t} = \frac{\sin^2 t}{\frac{\sin^2 t}{\cos^2 t}} = \sin^2 t \cdot \frac{\cos^2 t}{\sin^2 t} = \cos^2 t$$

**70.** $$\frac{1 - \sin^2 t}{\cot^2 t} = \frac{\cos^2 t}{\frac{\cos^2 t}{\sin^2 t}} = \cos^2 t \cdot \frac{\sin^2 t}{\cos^2 t} = \sin^2 t$$

**71.** $$\frac{1}{1 - \cos t} + \frac{1}{1 + \cos t} = \frac{1 + \cos t + 1 - \cos t}{(1 - \cos t)(1 + \cos t)} = \frac{2}{1 - \cos^2 t} = \frac{2}{\sin^2 t} = 2\csc^2 t$$

**72.** $$\frac{1}{1-\sin t}+\frac{1}{1+\sin t}=\frac{1+\sin t+1-\sin t}{(1-\sin t)(1+\sin t)}=\frac{2}{1-\sin^2 t}=\frac{2}{\cos^2 t}=2\sec^2 t$$

**73.** $$\frac{\tan t+\cot t}{\tan t}=\frac{\frac{\sin t}{\cos t}+\frac{\cos t}{\sin t}}{\frac{\sin t}{\cos t}}=\left(\frac{\sin^2 t+\cos^2 t}{\sin t\cdot\cos t}\right)\frac{\cos t}{\sin t}=\frac{\sin^2 t+\cos^2 t}{\sin^2 t}=\frac{1}{\sin^2 t}=\csc^2 t$$

**74.** $$\frac{\csc t-\sin t}{\csc t}=\frac{\frac{1}{\sin t}-\sin t}{\frac{1}{\sin t}}=\frac{1-\sin^2 t}{1}=\cos^2 t$$

**75.** $$\sin^2 t(1+\cot^2 t)=\sin^2 t(\csc^2 t)=\sin^2 t\cdot\frac{1}{\sin^2 t}=1$$

**76.** $$\cos^2 t(1+\tan^2 t)=\cos^2 t\sec^2 t=\cos^2 t\cdot\frac{1}{\cos^2 t}=1$$

**77.** $$\sin^2 t+\cos^2 t=1$$
$$\sin^2 t=1-\cos^2 t$$
$$\sin t=\pm\sqrt{1-\cos^2 t}$$

Because $0<t<\frac{\pi}{2}$, $\sin t$ is positive.

Thus, $\sin t=\sqrt{1-\cos^2 t}$.

**78.** $$1+\tan^2 t=\sec^2 t$$
$$\tan^2 t=\sec^2 t-1$$
$$\tan t=\pm\sqrt{\sec^2 t-1}$$

Because $\frac{3\pi}{2}<t<2\pi$, tan $t$ is negative.

Thus, $\tan t=-\sqrt{\sec^2 t-1}$.

**79.** $$\csc^2 t=1+\cot^2 t$$
$$\csc t=\pm\sqrt{1+\cot^2 t}$$

Because $\frac{\pi}{2}<t<\pi$, csc $t$ is positive.

Thus, $\csc t=\sqrt{1+\cot^2 t}$.

**80.** $$\sec^2 t=1+\tan^2 t$$
$$\sec t=\pm\sqrt{1+\tan^2 t}$$

Because $\pi<t<\frac{3\pi}{2}$, sec $t$ is negative.

Thus, $\sec t=-\sqrt{1+\tan^2 t}$.

**81.** $$d(t)=1970\cos\left(\frac{\pi}{64}t\right)$$
$$d(24)=1970\cos\left(\frac{\pi}{64}\cdot 24\right)=1970\cos\left(\frac{3\pi}{8}\right)\approx 750 \text{ miles}$$

**82.** $T(t) = -41\cos\left(\frac{\pi}{6}t\right) + 36$

March 5:

$$\begin{aligned} T(2) &= -41\cos\left(\frac{\pi}{6}\cdot 2\right) + 36 \\ &= -41\cos\left(\frac{\pi}{3}\right) + 36 \\ &= -41(0.5) + 36 \\ &= 15.5° \text{ F} \end{aligned}$$

July 20:

$$\begin{aligned} T(6.5) &= -41\cos\left(\frac{\pi}{6}\cdot 6.5\right) + 36 \\ &= -41\cos\left(\frac{13\pi}{12}\right) + 36 \\ &\approx -41(-0.9659258263) + 36 \\ &\approx 75.6° \text{ F} \end{aligned}$$

**83.** $\cos t - \frac{1}{\cos t} = \frac{\cos^2 t - 1}{\cos t} = -\frac{\sin^2 t}{\cos t}$

**84.** $\tan t + \frac{1}{\tan t} = \frac{\tan^2 t + 1}{\tan t} = \frac{\sec^2 t}{\tan t} = \frac{\frac{1}{\cos^2 t}}{\frac{\sin t}{\cos t}} = \frac{1}{\cos^2 t}\cdot\frac{\cos t}{\sin t} = \frac{1}{\sin t\cos t} = \frac{1}{\sin t}\cdot\frac{1}{\cos t} = \csc t\sec t$

**85.** $\cot t + \frac{1}{\cot t} = \frac{\cot^2 t + 1}{\cot t} = \frac{\csc^2 t}{\cot t} = \frac{\frac{1}{\sin^2 t}}{\frac{\cos t}{\sin t}} = \frac{1}{\sin^2 t}\cdot\frac{\sin t}{\cos t} = \frac{1}{\sin t\cos t} = \frac{1}{\sin t}\cdot\frac{1}{\cos t} = \csc t\sec t$

**86.** $\sin t - \frac{1}{\sin t} = \frac{\sin^2 t - 1}{\sin t} = -\frac{\cos^2 t}{\sin t}$

**87.** $(1-\sin t)^2 = 1 - 2\sin t + \sin^2 t$

**88.** $(1-\cos t)^2 = 1 - 2\cos t + \cos^2 t$

**89.** $$\begin{aligned}(\sin t - \cos t)^2 &= \sin^2 t - 2\sin t\cos t + \cos^2 t \\ &= 1 - 2\sin t\cos t\end{aligned}$$

**90.** $$\begin{aligned}(\sin t + \cos t)^2 &= \sin^2 t + 2\sin t\cos t + \cos^2 t \\ &= 1 + 2\sin t\cos t\end{aligned}$$

**91.** $$\begin{aligned}(1-\sin t)(1+\sin t) &= 1 - \sin^2 t \\ &= \cos^2 t\end{aligned}$$

**92.** $$\begin{aligned}(1-\cos t)(1+\cos t) &= 1 - \cos^2 t \\ &= \sin^2 t\end{aligned}$$

**93.** $$\begin{aligned}\frac{\sin t}{1+\cos t} + \frac{1+\cos t}{\sin t} &= \frac{(\sin t)(\sin t) + (1+\cos t)(1+\cos t)}{\sin t(1+\cos t)} \\ &= \frac{\sin^2 t + 1 + 2\cos t + \cos^2 t}{\sin t(1+\cos t)} \\ &= \frac{2+2\cos t}{\sin t(1+\cos t)} = \frac{2(1+\cos t)}{\sin t(1+\cos t)} \\ &= \frac{2}{\sin t} = 2\csc t\end{aligned}$$

**94.** $$\begin{aligned}\frac{1-\sin t}{\cos t} - \frac{1}{\tan t + \sec t} &= \frac{1-\sin t}{\cos t} - \frac{1}{\frac{\sin t}{\cos t} + \frac{1}{\cos t}} \\ &= \frac{1-\sin t}{\cos t} - \frac{\cos t}{\sin t + 1} \\ &= \frac{(1-\sin t)(1+\sin t) - (\cos t)(\cos t)}{\cos t(1+\sin t)} \\ &= \frac{1-\sin^2 t - \cos^2 t}{\cos t(1+\sin t)} = \frac{1-1}{\cos t(1+\sin t)} \\ &= 0\end{aligned}$$

**95.** $\cos^2 t - \sin^2 t = (\cos t - \sin t)(\cos t + \sin t)$

**96.** $\sec^2 t - \csc^2 t = (\sec t - \csc t)(\sec t + \csc t)$

**97.** $\tan^2 t - \tan t - 6 = (\tan t + 2)(\tan t - 3)$

**98.** $\cos^2 t + 3\cos t - 4 = (\cos t - 1)(\cos t + 4)$

**99.** $2\sin^2 t - \sin t - 1 = (2\sin t + 1)(\sin t - 1)$

**100.** $4\cos^2 t + 4\cos t + 1 = (2\cos t + 1)(2\cos t + 1)$
$= (2\cos t + 1)^2$

## Connecting Concepts

**101.** $\csc t = \sqrt{2}, \quad 0 < t < \frac{\pi}{2}$

$\sin t = \frac{1}{\csc t} = \frac{\sqrt{2}}{2}$

$\cos^2 t + \sin^2 t = 1$

$\cos t = \pm\sqrt{1 - \sin^2 t}$

$\cos t$ is positive in quadrant I.

$\cos t = \sqrt{1 - \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{1}{2}}$

$\cos t = \frac{\sqrt{2}}{2}$

**102.** $\cos t = \frac{1}{2}, \quad \frac{3\pi}{2} < t < 2\pi$

$\cos^2 t + \sin^2 t = 1$

$\sin t = \pm\sqrt{1 - \cos^2 t}$

Because $\frac{3\pi}{2} < t < 2\pi$, $\sin t$ is negative.

$\sin t = -\sqrt{1 - \left(\frac{1}{2}\right)^2}$

$\sin t = -\frac{\sqrt{3}}{2}$

**103.** $\sin t = \frac{1}{2}, \frac{\pi}{2} < t < \pi$

$\tan t = \frac{\sin t}{\cos t}$

$\tan t = \frac{\sin t}{\pm\sqrt{1 - \sin^2 t}}$

Because $\frac{\pi}{2} < t < \pi$, $\tan t$ is negative.

$\tan t = -\frac{\frac{1}{2}}{\sqrt{1 - \left(\frac{1}{2}\right)^2}}$

$\tan t = -\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$

$\tan t = -\frac{\sqrt{3}}{3}$

**104.** $\cot t = \frac{\sqrt{3}}{3}, \pi < t < \frac{3\pi}{2}$

$\cot t = \frac{\cos t}{\sin t}$

$\cos t = \cot t \sin t$

$1 + \cot^2 t = \csc^2 t = \frac{1}{\sin^2 t}$

$\sin t = \pm\frac{1}{\sqrt{1 + \cot^2 t}}$

Because $\pi < t < \frac{3\pi}{2}$, $\sin t$ is negative.

$\cos t = -\cot t \frac{1}{\sqrt{1 + \cot^2 t}}$

$= -\frac{\sqrt{3}}{3} \frac{1}{\sqrt{1 + \left(\frac{\sqrt{3}}{3}\right)^2}} = \frac{-\sqrt{3}}{3} \cdot \frac{\sqrt{3}}{2}$

$\cos t = -\frac{1}{2}$

**105.** $\frac{\sin^2 t + \cos^2 t}{\sin^2 t} = \frac{1}{\sin^2 t}$
$= \csc^2 t$

**106.** $\frac{\sin^2 t + \cos^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}$
$= \sec^2 t$

**107.** $(\cos t - 1)(\cos t + 1) = \cos^2 t - 1$
$= -(1 - \cos^2 t)$
$= -\sin^2 t$

**108.** $(\sec t - 1)(\sec t + 1) = \sec^2 t - 1$
$= 1 + \tan^2 t - 1$
$= \tan^2 t$

## Prepare for Section 5.5

**PS1.** $\sin\frac{3\pi}{4} \approx 0.7$

**PS2.** $\cos\frac{5\pi}{4} \approx -0.7$

**PS3.** Reflect the graph of $y = f(x)$ across the $x$-axis to produce $y = -f(x)$.

**PS4.** Compress each point on the graph of $y = f(x)$ toward the $y$-axis by a factor of $\frac{1}{2}$.

**PS5.** $\frac{2\pi}{\frac{1}{3}} = \frac{2\pi}{1} \cdot \frac{3}{1} = 6\pi$

**PS6.** $\frac{2\pi}{\frac{2}{5}} = \frac{2\pi}{1} \cdot \frac{5}{2} = 5\pi$

## Section 5.5

**1.** $y = 2\sin x$
$a = 2,\ p = 2\pi$

**2.** $y = -\frac{1}{2}\sin x$
$a = \left|-\frac{1}{2}\right| = \frac{1}{2},\ p = 2\pi$

**3.** $y = \sin 2x$
$a = 1,\ p = \frac{2\pi}{2} = \pi$

**4.** $y = \sin\frac{2\pi}{3}$
$a = 1,\ p = \frac{2\pi}{2/3} = 3\pi$

**5.** $y = \frac{1}{2}\sin 2\pi x$
$a = \frac{1}{2},\ p = \frac{2\pi}{2\pi} = 1$

**6.** $y = 2\sin\frac{\pi x}{3}$
$a = 2,\ p = \frac{2\pi}{\pi/3} = 6$

**7.** $y = -2\sin\frac{x}{2}$
$a = |-2| = 2,\ p = \frac{2\pi}{1/2} = 4\pi$

**8.** $y = -\frac{1}{2}\sin\frac{x}{2}$
$a = \left|-\frac{1}{2}\right| = \frac{1}{2},\ p = \frac{2\pi}{1/2} = 4\pi$

**9.** $y = \frac{1}{2}\cos x$
$a = \frac{1}{2},\ p = 2\pi$

**10.** $y = -3\cos x$
$a = |-3| = 3,\ p = 2\pi$

**11.** $y = \cos\frac{x}{4}$
$a = 1,\ p = \frac{2\pi}{1/4} = 8\pi$

**12.** $y = \cos 3x$
$a = 1,\ p = \frac{2\pi}{3}$

**13.** $y = 2\cos\frac{\pi x}{3}$
$a = 2,\ p = \frac{2\pi}{\pi/3} = 6$

**14.** $y = \frac{1}{2}\cos 2\pi x$
$a = \frac{1}{2},\ p = \frac{2\pi}{2\pi} = 1$

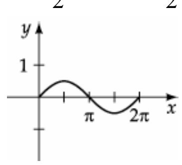
**15.** $y = -3\cos\frac{2x}{3}$
$a = |-3| = 3,\ p = \frac{2\pi}{2/3} = 3\pi$

**16.** $y = \frac{3}{4}\cos 4\pi$
$a = \frac{3}{4},\ p = \frac{2\pi}{4} = \frac{\pi}{2}$

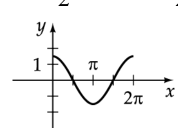
**17.** $y = 4.7\sin 0.8\pi t$
$a = 4.7,\ p = \frac{2\pi}{0.8\pi} = 2.5$

**18.** $y = 2.3\cos 0.005\pi t$
$a = 2.3,\ p = \frac{2\pi}{0.005\pi} = 400$

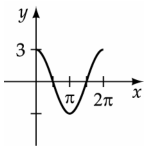
**19.** $y = \frac{1}{2}\sin x,\ a = \frac{1}{2},\ p = 2\pi$



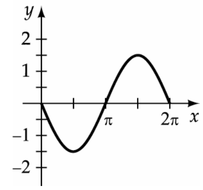

**20.** $y = \frac{3}{2}\cos x,\ a = \frac{3}{2},\ p = 2\pi$



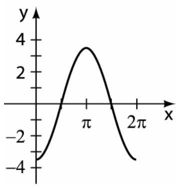
**21.** $y = 3\cos x,\ a = 3,\ p = 2\pi$



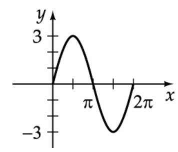

**22.** $y = -\frac{3}{2}\sin x,\ a = \left|-\frac{3}{2}\right| = \frac{3}{2},\ p = 2\pi$



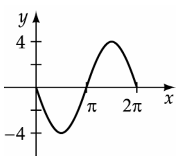

**23.** $y = -\frac{7}{2}\cos x,\ a = \left|-\frac{7}{2}\right| = \frac{7}{2},\ p = 2\pi$



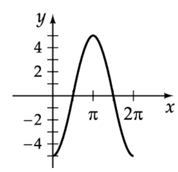

**24.** $y = 3\sin x,\ a = 3,\ p = 2\pi$



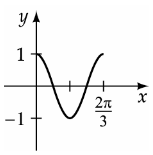

**25.** $y = -4\sin x,\ a = |-4| = 4,\ p = 2\pi$



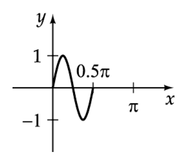

**26.** $y = -5\cos x,\ a = |-5| = 5,\ p = 2\pi$



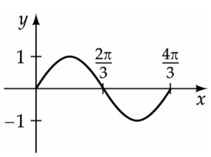

**27.** $y = \cos 3x,\ a = 1,\ p = \frac{2\pi}{3}$



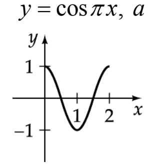

**28.** $y = \sin 4x,\ a = 1,\ p = \frac{\pi}{2}$



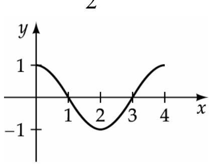

**29.** $y = \sin\frac{3x}{2},\ a = 1,\ p = \frac{4\pi}{3}$



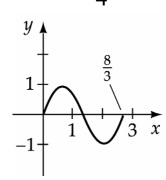

**30.** $y = \cos \pi x,\ a = 1,\ p = 2$



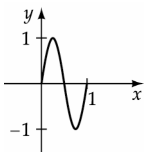

**31.** $y = \cos\frac{\pi}{2}x,\ a = 1,\ p = 4$



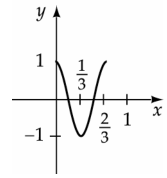

**32.** $y = \sin\frac{3\pi}{4}x,\ a = 1,\ p = \frac{8}{3}$



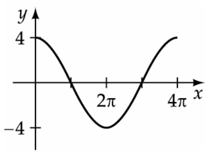
**33.** $y = \sin 2\pi x,\ a = 1,\ p = 1$



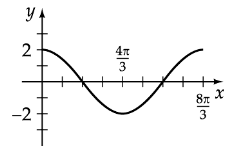

**34.** $y = \cos 3\pi x,\ a = 1,\ p = \dfrac{2}{3}$



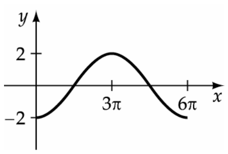

**35.** $y = 4\cos\dfrac{x}{2},\ a = 4,\ p = \dfrac{2\pi}{1/2} = 4\pi$



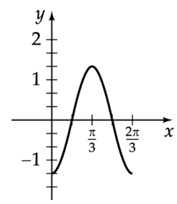

**36.** $y = 2\cos\dfrac{3x}{4},\ a = 2,\ p = \dfrac{2\pi}{3/4} = \dfrac{8\pi}{3}$



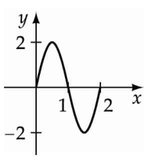

**37.** $y = -2\cos\dfrac{x}{3},\ a = |-2| = 2,\ p = \dfrac{2\pi}{1/3} = 6\pi$



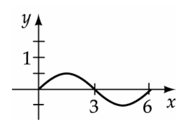

**38.** $y = -\dfrac{4}{3}\cos 3x,\ a = \left|-\dfrac{4}{3}\right| = \dfrac{4}{3},\ p = \dfrac{2\pi}{3}$



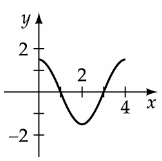

**39.** $y = 2\sin \pi x,\ a = 2,\ p = \dfrac{2\pi}{\pi} = 2$



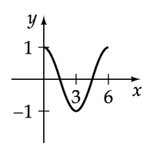

**40.** $y = \dfrac{1}{2}\sin\dfrac{\pi x}{3},\ a = \dfrac{1}{2},\ p = \dfrac{2\pi}{\pi/3} = 6$



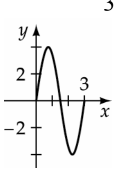

**41.** $y = \dfrac{3}{2}\cos\dfrac{\pi x}{2},\ a = \dfrac{3}{2},\ p = \dfrac{2\pi}{\pi/2} = 4$



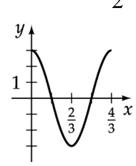

**42.** $y = \cos\dfrac{\pi x}{3},\ a = 1,\ p = \dfrac{2\pi}{\pi/3} = 6$



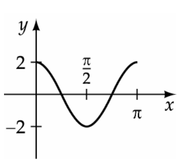

**43.** $y = 4\sin\dfrac{2\pi}{3}x,\ a = |4| = 4,\ p = \dfrac{2\pi}{2\pi/3} = 3$



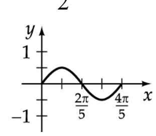

**44.** $y = 3\cos\dfrac{3\pi}{2}x,\ a = 3,\ p = \dfrac{2\pi}{3\pi/2} = \dfrac{4}{3}$



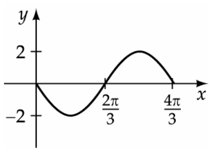

**45.** $y = 2\cos 2x,\ a = 2,\ p = \dfrac{2\pi}{2} = \pi$



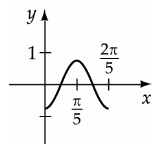

**46.** $y = \dfrac{1}{2}\sin 2.5x,\ a = \dfrac{1}{2},\ p = \dfrac{2\pi}{2.5} = \dfrac{4\pi}{5}$



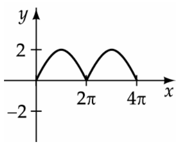
**47.** $y = -2\sin 1.5x,\ a = |-2| = 2,\ p = \dfrac{2\pi}{1.5} = \dfrac{4\pi}{3}$



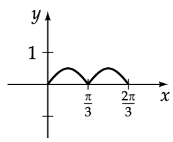

**48.** $y = -\dfrac{3}{4}\cos 5x,\ a = \left|-\dfrac{3}{4}\right| = \dfrac{3}{4},\ p = \dfrac{2\pi}{5}$



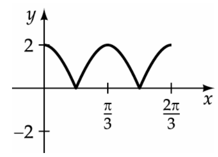

**49.** $y = \left|2\sin\dfrac{x}{2}\right|$



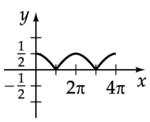

**50.** $y = \left|\dfrac{1}{2}\sin 3x\right|$



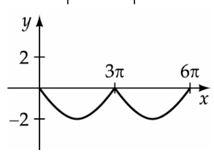

**51.** $y = |-2\cos 3x|$



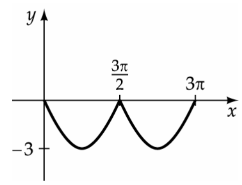

**52.** $y = \left|-\dfrac{1}{2}\cos\dfrac{x}{2}\right|$



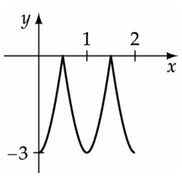

**53.** $y = -\left|2\sin\dfrac{x}{3}\right|$



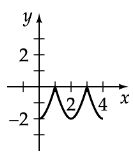

**54.** $y = -\left|3\sin\dfrac{2}{3}x\right|$




**55.** $y = -|3\cos \pi x|$




**56.** $y = -\left|2\cos\dfrac{\pi}{2}x\right|$




**57.** $\dfrac{2\pi}{b} = \pi,\ b = 2,\ a = 1$

$y = \cos 2x$

**58.** $\dfrac{2\pi}{b} = 6\pi, b = \dfrac{1}{3}, a = 2$

$y = 2\cos\dfrac{1}{3}x$

**59.** $\dfrac{2x}{b} = 3\pi, b = \dfrac{2}{3}, a = 2$

$y = 2\sin\dfrac{2}{3}x$

**60.** $\dfrac{2\pi}{b} = \dfrac{4\pi}{3},\ b = \dfrac{3}{2},\ a = \dfrac{3}{2}$

$y = \dfrac{3}{2}\sin\dfrac{3}{2}x$

**61.** $\dfrac{2\pi}{b} = 2, b = \pi, a = 2$

$y = -2\cos \pi x$

**62.** $\dfrac{2\pi}{b} = 1, b = 2\pi, a = 3$

$y = -3\sin 2\pi x$

**63.** **a.** Amplitude, $a = 4$, period, $b = \dfrac{2\pi}{2} = \pi$

$V = 4\sin \pi t,\ 0 \le t \le 8$ ms

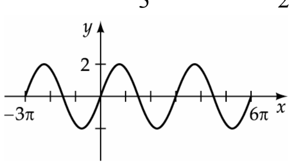
**b.** Frequency $= \dfrac{4 \text{ cycles}}{8 \text{ ms}} = \dfrac{1}{2}$ cycle/ms

**64.** **a.** Amplitude, $\dfrac{3/5}{2.5} = 1.5$, period, $b = \dfrac{2\pi}{2.5} = 0.8\pi$

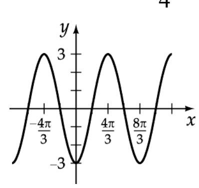
$V = 1.5\sin 0.8\pi t,\ 0 \le t \le 5$ ms

**b.** Frequency $= \dfrac{2 \text{ cycles}}{5 \text{ ms}} = 0.4$ cycle/ms

**65.** $f(x) = 2\sin\dfrac{2x}{3},\ a = 2,\ p = \dfrac{2\pi}{2/3} = 3\pi$

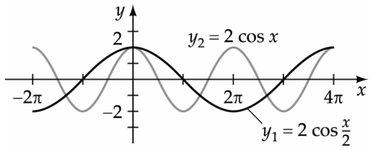



**66.** $f(x) = -3\cos\dfrac{3x}{4},\ a = |-3| = 3,\ b = \dfrac{2\pi}{3/4} = \dfrac{8\pi}{3}$



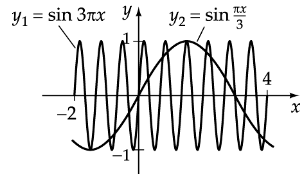
**67.** $y_1 = 2\cos\frac{x}{2},\ a = 2,\ p = \frac{2\pi}{1/2} = 4\pi$

$y_2 = 2\cos x,\ a = 2,\ p = 2\pi$

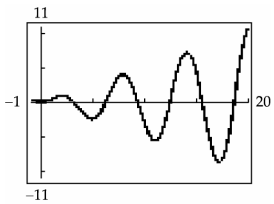



**68.** $y_1 = \sin 3\pi x,\ p = 1,\ p = \frac{2\pi}{3\pi} = \frac{2}{3}$

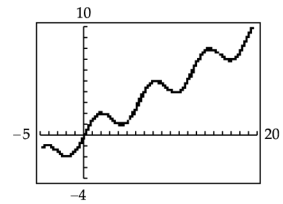
$y_2 = \sin\frac{\pi x}{3},\ p = 1,\ p = \frac{2\pi}{\pi/3} = 6$



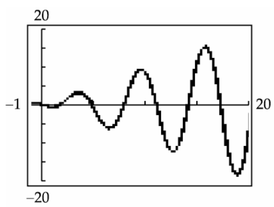

**69.** $y = \frac{1}{2}x\sin x$



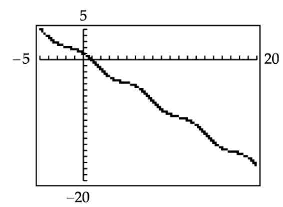

**70.** $y = \frac{1}{2}x + \sin x$



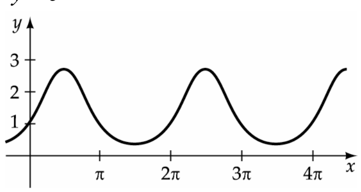

**71.** $y = -x\cos x$




**72.** $y = -x + \cos x$




**73.** $y = e^{\sin x}$

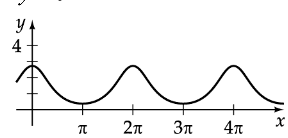



The maximum value of $e^{\sin x}$ is $e$.

The minimum value of $e^{\sin x}$ is $\frac{1}{e} \approx 0.3679$.

The function defined by $y = e^{\sin x}$ is periodic with a period of $2\pi$.

**74.** $y = e^{\cos x}$




The maximum value of $e^{\cos x}$ is $e$.

The minimum value of $e^{\cos x}$ is $\frac{1}{e} \approx 0.3679$.

The function defined by $y = e^{\cos x}$ is periodic with a period of $2\pi$.

## Connecting Concepts

**75.** $a = 2$

$p = \frac{2\pi}{b} = 3\pi$

$b = \frac{2}{3}$

$y = 2\sin\frac{2}{3}x$

**76.** $a = 4$

$p = \frac{2\pi}{b} = 2$

$b = \pi$

$y = 4\sin \pi x$

**77.** $a = 2.5$

$p = \frac{2\pi}{b} = 3.2$

$b = \frac{5\pi}{8}$

$y = 2.5\sin\frac{5\pi}{8}x$

**78.** $a = 3$

$p = \dfrac{2\pi}{b} = \dfrac{\pi}{2}$

$b = 4$

$y = 3\cos 4x$

**79.** $a = 3$

$p = \dfrac{2\pi}{b} = 2.5$

$b = \dfrac{4\pi}{5}$

$y = 3\cos\dfrac{4\pi}{5}x$

**80.** $a = 4.2$

$p = \dfrac{2\pi}{b} = 1$

$b = 2\pi$

$y = 4.2\cos 2\pi x$

## Prepare for Section 5.6

**PS1.** $\tan\dfrac{\pi}{3} \approx 1.7$

**PS2.** $\cot\dfrac{\pi}{3} \approx 0.6$

**PS3.** Stretch each point on the graph of $y = f(x)$ away from the $x$-axis by a factor of 2 to produce $y = 2f(x)$.

**PS4.** Shift the graph of $y = f(x)$ 2 units to the right and up 3 units.

**PS5.** $\dfrac{\pi}{\frac{1}{2}} = \dfrac{\pi}{1}\cdot\dfrac{2}{1} = 2\pi$

**PS6.** $\dfrac{\pi}{|-3/4|} = \dfrac{\pi}{1}\cdot\left|-\dfrac{4}{3}\right| = \dfrac{\pi}{1}\cdot\dfrac{4}{3} = \dfrac{4}{3}\pi$

## Section 5.6

**1.** $y = \tan x$ is undefined for $\dfrac{\pi}{2} + k\pi$, $k$ an integer.

**2.** $y = \cot x$ is undefined for $k\pi$, $k$ an integer.

**3.** $y = \sec x$ is undefined for $\dfrac{\pi}{2} + k\pi$, $k$ an integer.

**4.** $y = \csc x$ is undefined for $k\pi$, $k$ an integer.

**5.** $p = 2\pi$

**6.** $p = \pi$

**7.** $p = \pi$

**8.** $p = 2\pi$

**9.** $p = \dfrac{\pi}{1/2} = 2\pi$

**10.** $p = \dfrac{\pi}{2}$

**11.** $p = \dfrac{2\pi}{3}$

**12.** $p = \dfrac{2\pi}{1/2} = 4\pi$

**13.** $p = \dfrac{\pi}{3}$

**14.** $p = \dfrac{\pi}{2/3} = \dfrac{3\pi}{2}$

**15.** $p = \dfrac{2\pi}{1/4} = 8\pi$

**16.** $p = \dfrac{2\pi}{2} = \pi$

**17.** $p = \dfrac{\pi}{\pi} = 1$

**18.** $p = \dfrac{\pi}{\pi/3} = 3$

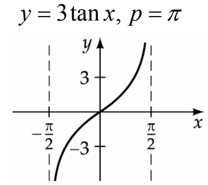
**19.** $p = \dfrac{\pi}{\pi/5} = 5$

**20.** $p = \dfrac{\pi}{\pi/2} = 2$

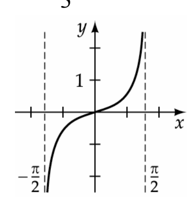
**21.** $p = \dfrac{2\pi}{\pi/4.25} = 8.5$

**22.** $p = \dfrac{2\pi}{\pi/2.5} = 5$

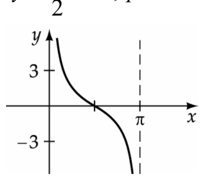
**23.** $y = 3\tan x,\ p = \pi$



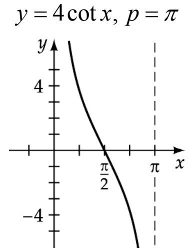

**24.** $y = \dfrac{1}{3}\tan x,\ p = \pi$



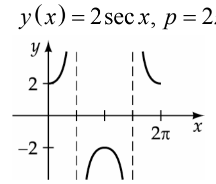

**25.** $y = \dfrac{3}{2}\cot x,\ p = \pi$



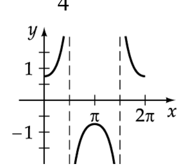

**26.** $y = 4\cot x,\ p = \pi$



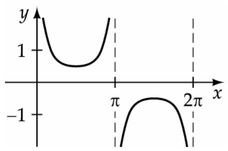

**27.** $y(x) = 2\sec x,\ p = 2\pi$



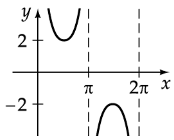

**28.** $y = \dfrac{3}{4}\sec x,\ p = 2\pi$



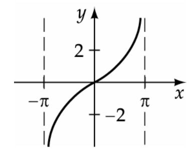
**29.** $y = \frac{1}{2}\csc x,\ p = 2\pi$



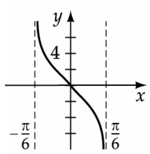

**30.** $y = 2\csc x,\ p = 2\pi$



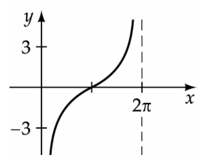

**31.** $y = 2\tan\frac{x}{2},\ p = \frac{\pi}{1/2} = 2\pi$



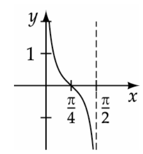

**32.** $y = -3\tan 3x,\ p = \frac{\pi}{3}$



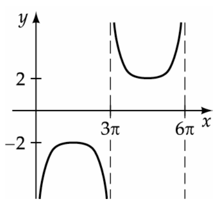

**33.** $y = -3\cot\frac{x}{2},\ p = \frac{\pi}{1/2} = 2\pi$



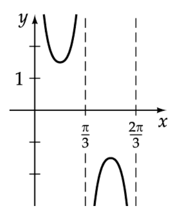

**34.** $y = \frac{1}{2}\cot 2x, p = \frac{\pi}{2}$



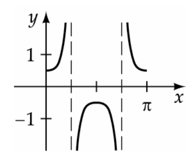

**35.** $y = -2\csc\frac{x}{3},\ p = \frac{2\pi}{1/3} = 6\pi$



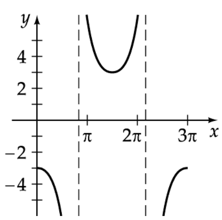

**36.** $y = \frac{3}{2}\csc 3x,\ p = \frac{2\pi}{3}$



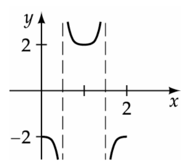

**37.** $y = \frac{1}{2}\sec 2x,\ p = \frac{2\pi}{2} = \pi$



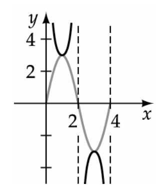

**38.** $y = -3\sec\frac{2x}{3},\ p = \frac{2\pi}{2/3} = 3\pi$



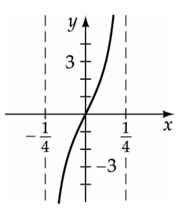

**39.** $y = -2\sec\pi x,\ p = \frac{2\pi}{\pi} = 2$



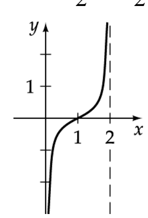

**40.** $y = 3\csc\frac{\pi x}{2},\ p = \frac{2\pi}{\pi/2} = 4$



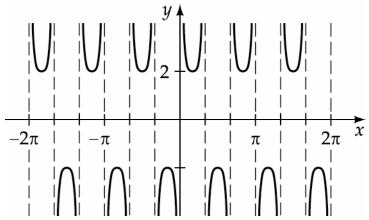

**41.** $y = 3\tan 2\pi x,\ p = \frac{\pi}{2\pi} = \frac{1}{2}$



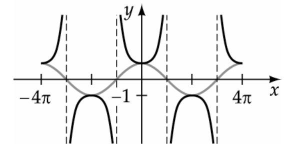

**42.** $y = -\frac{1}{2}\cot\frac{\pi x}{2},\ p = \frac{\pi}{\pi/2} = 2$



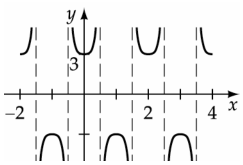

**43.** $y = 2\csc 3x,\ p = \frac{2\pi}{3}$



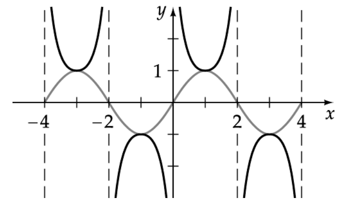

**44.** $y = \sec\frac{x}{2},\ p = \frac{2\pi}{1/2} = 4\pi$



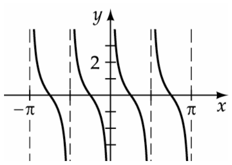
**45.** $y = 3\sec \pi x,\ p = \dfrac{2\pi}{\pi} = 2$



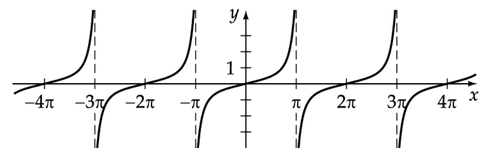

**46.** $y = \csc \dfrac{\pi x}{2},\ p = \dfrac{2\pi}{\pi/2} = 4$



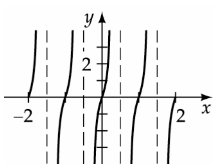

**47.** $y = 2\cot 2x,\ p = \dfrac{\pi}{2}$



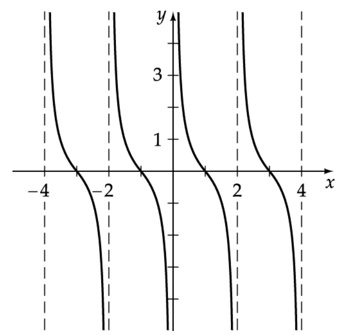

**48.** $y = \dfrac{1}{2}\tan \dfrac{x}{2},\ p = \dfrac{\pi}{1/2} = 2\pi$




**49.** $y = 3\tan \pi x,\ p = \dfrac{\pi}{\pi} = 1$




**50.** $y = \cot \dfrac{\pi x}{2},\ p = \dfrac{\pi}{\pi/2} = 2$




**51.** $\dfrac{\pi}{b} = \dfrac{2\pi}{3},\ b = \dfrac{3}{2}$

$y = \cot \dfrac{3}{2}x$

**52.** $\dfrac{\pi}{b} = 2\pi,\ b = \dfrac{1}{2}$

$y = \tan \dfrac{1}{2}x$

**53.** $\dfrac{2\pi}{b} = 3\pi,\ b = \dfrac{2}{3}$

$y = \csc \dfrac{2}{3}x$

**54.** $\dfrac{2\pi}{b} = 2,\ b = \pi$

$y = \csc \pi x$

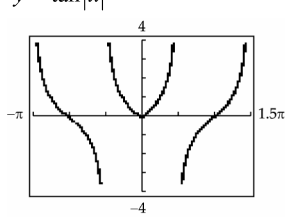
**55.** $\dfrac{2\pi}{b} = \dfrac{8\pi}{3},\ b = \dfrac{3}{4}$

$y = \sec \dfrac{3}{4}x$

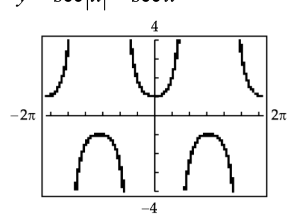
**56.** $\dfrac{2\pi}{b} = 1,\ b = 2\pi$

$y = \sec 2\pi x$

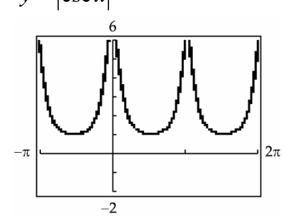
**57.** $y = \tan|x|$



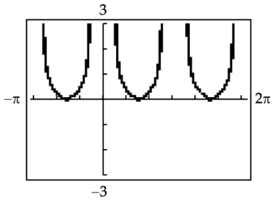

**58.** $y = \sec|x| = \sec x$




**59.** $y = |\csc x|$



**60.** $y = |\cot x|$



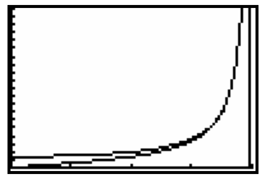

**61.** **a.** $\tan x = \dfrac{h}{1.4}$

$h = 1.4 \tan x$

**b.** $\cos x = \dfrac{1.4}{d}$

$d = \dfrac{1.4}{\cos x} = 1.4 \sec x$

**c.**



**d.** The graph of $d$ is above the graph of $h$, however, the distance between the graphs approaches 0 as $x$ approaches $\dfrac{\pi}{2}$.

**62.** **a.** $\sin x = \dfrac{3.5}{d}$

$d = \dfrac{3.5}{\sin x} = 3.5 \csc x$

**b.** $d = 3.5 \csc x$

$d(1) = 3.5 \csc(1) \approx 4.16$ mi

$d(1.2) = 3.5 \csc(1.2) \approx 3.76$ mi

## Connecting Concepts

**63.** $\dfrac{\pi}{b} = \dfrac{\pi}{3}, b = 3$

$y = \tan 3x$

**64.** $\dfrac{\pi}{b} = \dfrac{\pi}{2}, b = 2$

$y = \cot 2x$

**65.** $\dfrac{2\pi}{b} = \dfrac{3\pi}{4}, b = \dfrac{8}{3}$

$y = \sec \dfrac{8}{3} x$

**66.** $\dfrac{2\pi}{b} = \dfrac{5\pi}{2}, b = \dfrac{4}{5}$

$y = \csc \dfrac{4}{5} x$

**67.** $\dfrac{\pi}{b} = 2, b = \dfrac{\pi}{2}$

$y = \cot \dfrac{\pi}{2} x$

**68.** $\dfrac{\pi}{b} = 0.5, b = 2\pi$

$y = \tan 2\pi x$

**69.** $\dfrac{2\pi}{b} = 1.5, b = \dfrac{4}{3}\pi$

$y = \csc \dfrac{4\pi}{3} x$

**70.** $\dfrac{2\pi}{b} = 3, b = \dfrac{2}{3}\pi$

$y = \sec \dfrac{2\pi}{3} x$

## Prepare for Section 5.7

**PS1.** $y = 2 \sin 2x$

amplitude = 2, period = $\pi$

**PS2.** $y = \dfrac{2}{3} \cos \dfrac{x}{3}$

amplitude = $\dfrac{2}{3}$, period = $6\pi$

**PS3.** $y = -4 \sin 2\pi x$

amplitude = 4, period = 1

**PS4.** maximum at 2

**PS5.** minimum at –3

**PS6.** $f(x) = \cos x$ is symmetric to $y$–axis.

## Section 5.7

**1.** $a = 2,\ p = 2\pi,$ phase shift $= \dfrac{\pi}{2}$

**2.** $a = |-3| = 3,\ p = 2\pi,$ phase shift $= -\pi$

**3.** $a = 1,\ p = \dfrac{2\pi}{2} = \pi,$ phase shift $= \dfrac{\pi/4}{2} = \dfrac{\pi}{8}$

**4.** $a = \dfrac{3}{4},\ p = \dfrac{2\pi}{1/2} = 4\pi,$ phase shift $= \dfrac{-\pi/3}{1/2} = -\dfrac{2\pi}{3}$

**5.** $a = |-4| = 4,\ p = \dfrac{2\pi}{2/3} = 3\pi,$ phase shift $= \dfrac{-\pi/6}{2/3} = -\dfrac{\pi}{4}$

**6.** $a = \left|\dfrac{3}{2}\right| = \dfrac{3}{2},\ p = \dfrac{2\pi}{1/4} = 8\pi,$ phase shift $= \dfrac{3\pi/4}{1/4} = 3\pi$

**7.** $a = \dfrac{5}{4},\ p = \dfrac{2\pi}{3},$ phase shift $= \dfrac{2\pi}{3}$

**8.** $a = 6,\ p = \dfrac{2\pi}{1/3} = 6\pi,$ phase shift $= \dfrac{\pi/6}{1/3} = \dfrac{\pi}{2}$

**9.** $p = \dfrac{\pi}{2},$ phase shift $= \dfrac{\pi/4}{2} = \dfrac{\pi}{8}$

**10.** $p = \dfrac{\pi}{1/2} = 2\pi,$ phase shift $= -\dfrac{\pi}{1/2} = 2\pi$

**11.** $p = \dfrac{2\pi}{1/3} = 6\pi,$ phase shift $= \dfrac{-\pi}{1/3} = -3\pi$

**12.** $p = \dfrac{2\pi}{3},$ phase shift $= \dfrac{\pi/6}{3} = \dfrac{\pi}{18}$

**13.** $p = \dfrac{2\pi}{2} = \pi,$ phase shift $= \dfrac{\pi/8}{2} = \dfrac{\pi}{16}$

**14.** $p = \dfrac{2\pi}{1/4} = 8\pi,$ phase shift $= \dfrac{\pi/2}{1/4} = 2\pi$

**15.** $p = \dfrac{\pi}{1/4} = 4\pi,$ phase shift $= \dfrac{-3\pi}{1/4} = -12\pi$

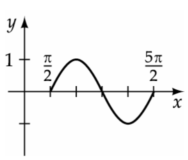
**16.** $p = \dfrac{\pi}{2},$ phase shift $= \dfrac{\pi/4}{2} = \dfrac{\pi}{8}$

**17.** $y = \sin\left(x - \dfrac{\pi}{2}\right)$

$0 \le x - \dfrac{\pi}{2} \le 2\pi$

$\dfrac{\pi}{2} \le x \le \dfrac{5\pi}{2}$

period $= 2\pi,$ phase shift $\dfrac{\pi}{2}$

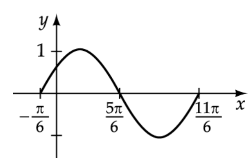



**18.** $y = \sin\left(x + \dfrac{\pi}{6}\right)$

$0 \le x + \dfrac{\pi}{6} \le 2\pi$

$-\dfrac{\pi}{6} \le x \le \dfrac{11\pi}{6}$

period $= 2\pi,$ phase shift $= -\dfrac{\pi}{6}$



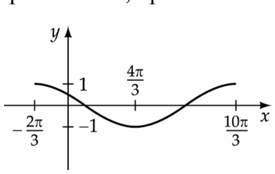

**19.** $y = \cos\left(\dfrac{x}{2} + \dfrac{\pi}{3}\right)$

$0 \le \dfrac{x}{2} + \dfrac{\pi}{3} \le 2\pi$

$-\dfrac{\pi}{3} \le \dfrac{x}{2} \le \dfrac{5\pi}{3}$

$-\dfrac{2\pi}{3} \le x \le \dfrac{10\pi}{3}$

period $= 4\pi,$ phase shift $= -\dfrac{2\pi}{3}$



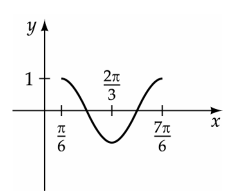

**20.** $y = \cos\left(2x - \dfrac{\pi}{3}\right)$

$0 \le 2x - \dfrac{\pi}{3} \le 2\pi$

$\dfrac{\pi}{3} \le 2x \le \dfrac{7\pi}{3}$

$\dfrac{\pi}{6} \le x \le \dfrac{7\pi}{6}$

period $= \pi,$ phase shift $= \dfrac{\pi}{6}$

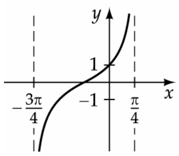


**21.** $y = \tan\left(x + \frac{\pi}{4}\right)$

$-\frac{\pi}{2} < x + \frac{\pi}{4} < \frac{\pi}{2}$

$-\frac{3\pi}{4} < x < \frac{\pi}{4}$

period $= \pi$, phase shift $= -\frac{\pi}{4}$

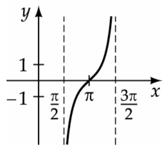



**22.** $y = \tan(x - \pi)$

$-\frac{\pi}{2} < x - \pi < \frac{\pi}{2}$

$\frac{\pi}{2} < x < \frac{3\pi}{2}$

period $= \pi$, phase shift $= \pi$



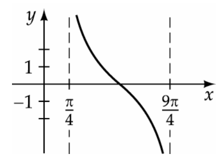

**23.** $y = 2\cot\left(\frac{x}{2} - \frac{\pi}{8}\right)$

$0 < \frac{x}{2} - \frac{\pi}{8} < \pi$

$\frac{\pi}{8} < \frac{x}{2} < \frac{9\pi}{8}$

$\frac{\pi}{4} < x < \frac{9\pi}{4}$

period $= 2\pi$, phase shift $= \frac{\pi}{4}$



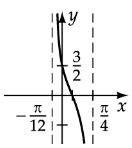

**24.** $y = \frac{3}{2}\cot\left(3x + \frac{\pi}{4}\right)$

$0 < 3x + \frac{\pi}{4} < \pi$

$-\frac{\pi}{4} < 3x < \frac{3\pi}{4}$

$-\frac{\pi}{12} < x < \frac{\pi}{4}$

period $= \frac{\pi}{3}$, phase shift $= -\frac{\pi}{12}$

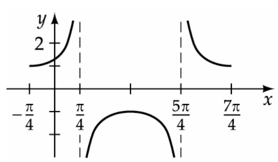



**25.** $y = \sec\left(x + \frac{\pi}{4}\right)$

$0 \le x + \frac{\pi}{4} \le 2\pi$

$-\frac{\pi}{4} \le x \le \frac{7\pi}{4}$

period $= 2\pi$, phase shift $= -\frac{\pi}{4}$



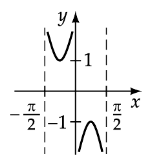

**26.** $y = \csc(2x + \pi)$

$0 \le 2x + \pi \le 2\pi$

$-\pi \le 2x \le \pi$

$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

period $= \pi$, phase shift $= -\frac{\pi}{2}$



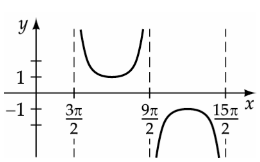

**27.** $y = \csc\left(\frac{x}{3} - \frac{\pi}{2}\right)$

$0 \le \frac{x}{3} - \frac{\pi}{2} \le 2\pi$

$\frac{\pi}{2} \le \frac{x}{3} \le \frac{5\pi}{2}$

$\frac{3\pi}{2} \le x \le \frac{15\pi}{2}$

period $= 6\pi$, phase shift $= \frac{3\pi}{2}$



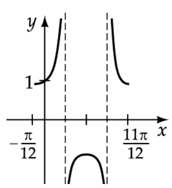

**28.** $y = \sec\left(2x + \frac{\pi}{6}\right)$

$0 \le 2x + \frac{\pi}{6} \le 2\pi$

$-\frac{\pi}{6} \le 2x \le \frac{11\pi}{6}$

$-\frac{\pi}{12} \le x \le \frac{11\pi}{12}$

period $= \pi$, phase shift $= -\frac{\pi}{12}$



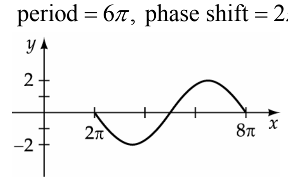

**29.** $y = -2\sin\left(\frac{x}{3} - \frac{2\pi}{3}\right)$

$0 \le \frac{x}{3} - \frac{2\pi}{3} \le 2\pi$

$\frac{2\pi}{3} \le \frac{x}{3} \le \frac{8\pi}{3}$

$2\pi \le x \le 8\pi$

period $= 6\pi$, phase shift $= 2\pi$



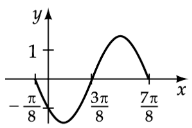
**30.** $y = -\frac{3}{2}\sin\left(2x + \frac{\pi}{4}\right)$

$0 \le 2x + \frac{\pi}{4} \le 2\pi$

$-\frac{\pi}{4} \le 2x \le \frac{7\pi}{4}$

$-\frac{\pi}{8} \le x \le \frac{7\pi}{8}$

period $= \pi$, phase shift $= -\frac{\pi}{8}$



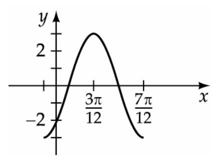

**31.** $y = -3\cos\left(3x + \frac{\pi}{4}\right)$

$0 \le 3x + \frac{\pi}{4} \le 2\pi$

$-\frac{\pi}{4} \le 3x \le \frac{7\pi}{4}$

$-\frac{\pi}{12} \le x \le \frac{7\pi}{12}$

period $= \frac{2\pi}{3}$, phase shift $= -\frac{\pi}{12}$



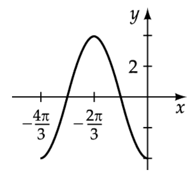

**32.** $y = -4\cos\left(\frac{3x}{2} + 2\pi\right)$

$0 \le \frac{3x}{2} + 2\pi \le 2\pi$

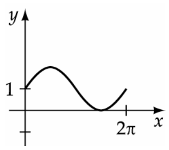
$-2\pi \le \frac{3x}{2} \le 0$

$-\frac{4}{3}\pi \le x \le 0$

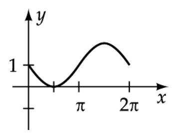
period $= \frac{4}{3}\pi$, phase shift $= -\frac{4}{3}\pi$



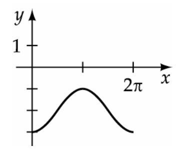

**33.** $y = \sin x + 1,\ p = 2\pi$



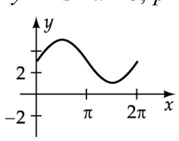

**34.** $y = -\sin x + 1,\ p = 2\pi$



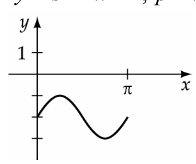

**35.** $y = -\cos x - 2,\ p = 2\pi$



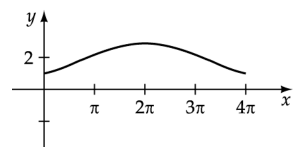

**36.** $y = 2\sin x + 3,\ p = 2\pi$




**37.** $y = \sin 2x - 2,\ p = \pi$

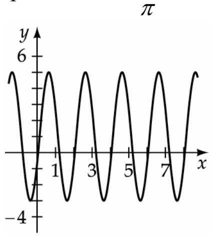



**38.** $y = -\cos\frac{x}{2} + 2,\ p = 4\pi$



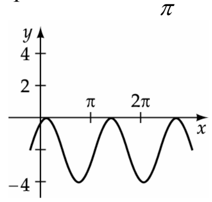

**39.** $y = 4\cos(\pi x - 2) + 1,\ p = 2$

phase shift $= \frac{2}{\pi}$



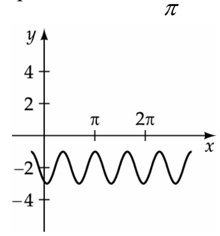

**40.** $y = 2\sin\left(\frac{\pi x}{2} + 1\right) - 2,\ p = 4$

phase shift $= -\frac{2}{\pi}$



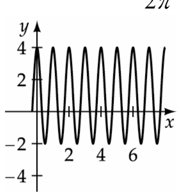

**41.** $y = -\sin(\pi x + 1) - 2,\ p = 2$

phase shift $= -\frac{1}{\pi}$



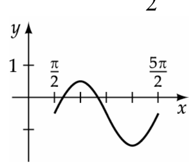

**42.** $y = -3\cos(2\pi x - 3) + 1,\ p = 1$

phase shift $= \frac{3}{2\pi}$



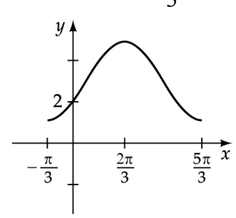

**43.** $y = \sin\left(x - \frac{\pi}{2}\right) - \frac{1}{2},\ p = 2\pi$

phase shift $= \frac{\pi}{2}$

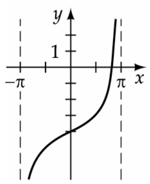



**44.** $y = -2\cos\left(x + \frac{\pi}{3}\right) + 3,\ p = 2\pi$

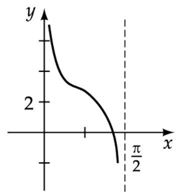
phase shift $= -\frac{\pi}{3}$



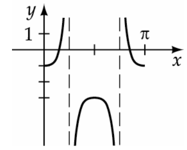
**45.** $y = \tan\frac{x}{2} - 4,\ p = 2\pi$



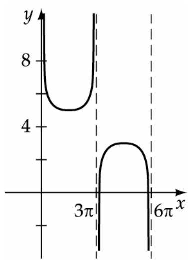

**46.** $y = \cot 2x + 3,\ p = \frac{\pi}{2}$



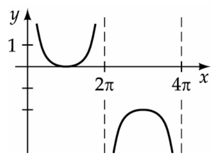

**47.** $y = \sec 2x - 2,\ p = \pi$




**48.** $y = \csc\frac{x}{3} + 4,\ p = 6\pi$

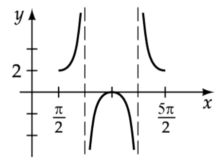



**49.** $y = \csc\frac{x}{2} - 1,\ p = 4\pi$

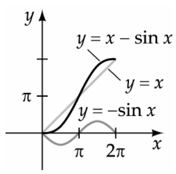



**50.** $y = \sec\left(x - \frac{\pi}{2}\right) + 1,\ p = 2\pi$

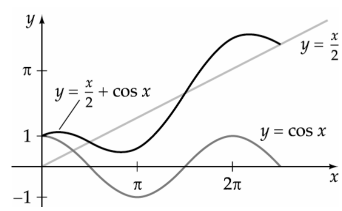
phase shift $= \frac{\pi}{2}$



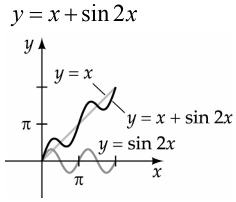

**51.** $y = x - \sin x$



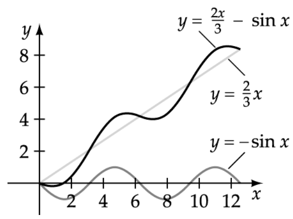

**52.** $y = \frac{x}{2} + \cos x$



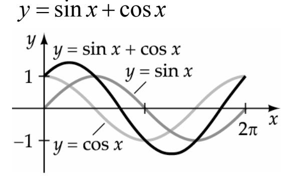

**53.** $y = x + \sin 2x$



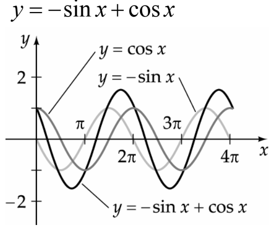

**54.** $y = \frac{2x}{3} - \sin x$




**55.** $y = \sin x + \cos x$




**56.** $y = -\sin x + \cos x$




**57.** sine curve, $a = 1,\ \frac{2\pi}{b} = \pi,\ b = 2$

phase shift $= -\frac{c}{b} = \frac{\pi}{6},\ c = -\frac{\pi}{3}$

$y = \sin\left(2x - \frac{\pi}{3}\right)$

**58.** cosine curve, $a = 1,\ \frac{2\pi}{b} = 2\pi,\ b = 1$

phase shift $= -\frac{c}{b} = \pi,\ c = -\pi$

translation one unit upward

$y = \cos(x - \pi) + 1$

**59.** cosecant curve, $\frac{2\pi}{b} = 4\pi,\ b = \frac{1}{2}$

phase shift $= -\frac{c}{b} = 2\pi,\ c = -\pi$

$y = \csc\left(\frac{x}{2} - \pi\right)$

**60.** tangent curve, $\frac{\pi}{b} = \pi,\ b = 1$

phase shift $= -\frac{c}{b} = \frac{\pi}{2},\ c = -\frac{\pi}{2}$

$y = \tan\left(x - \frac{\pi}{2}\right)$

**61.** secant curve, $\frac{2\pi}{b} = 2\pi,\ b = 1$

phase shift $= -\frac{c}{b} = \frac{\pi}{2},\ c = -\frac{\pi}{2}$

$y = \sec\left(x - \frac{\pi}{2}\right)$

**62.** sine curve, $\frac{2\pi}{b} = 2\pi,\ b = 1$

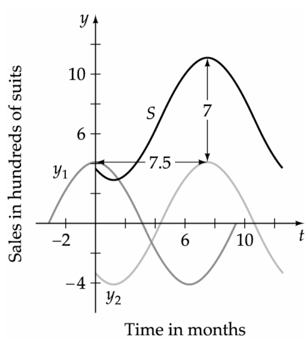
phase shift $= -\frac{c}{b} = -\frac{\pi}{2},\ c = \frac{\pi}{2}$

$y = \sin\left(x + \frac{\pi}{2}\right)$ or $y = \cos x$

**63.** **a.** phase shift: $-\frac{c}{b} = -\frac{-1.25\pi}{\pi/6} = 1.25(6) = 7.5$ months

period: $\frac{2\pi}{b} = \frac{2\pi}{\pi/6} = 2(6) = 12$ months

**b.** First graph $y_1 = 4.1\cos\left(\frac{\pi}{6}t\right)$. Because the phase shift is 7.5 months, shift the graph of $y_1$ 7.5 units to the right to produce the graph of $y_2$. Now shift the graph of $y_2$ upward 7 units to product the graph of *S*.



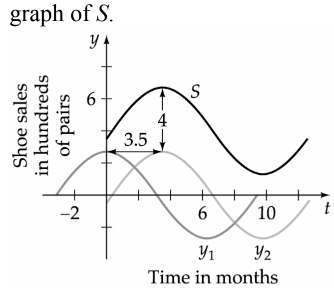

**c.** 7.5 months after January 1 is the middle of August.

**64.** **a.** phase shift: $-\frac{c}{b} = \frac{\left(-\frac{7}{12}\pi\right)}{\left(\frac{\pi}{6}\right)}$

$= \left(-\frac{7\pi}{12}\right)\cdot\left(\frac{6}{\pi}\right) = 3.5$ months

period: $\frac{2\pi}{b} = \frac{2\pi}{\pi/6} = 2(6) = 12$ months

**b.** First graph $y_1 = 2.7\cos\left(\frac{\pi}{6}t\right)$. Because the phase shift is 3.5 months, shift the graph of $y_1$ 3.5 units to the right to produce the graph of $y_2$. Now shift the graph of $y_2$ upward 4 units to product the graph of *S*.




**c.** 3.5 months after January 1 is the middle of April.

**65.** $y = 2.3\sin 2\pi t + 1.25t + 315$

$t = 16$ between 1990 and 2006

$y = 2.3\sin 2\pi(16) + 1.25(16) + 315$

$y \approx 335$ ppm

difference $\approx 335 - 315$

$\approx 20$ ppm

**66.** **a.** $(\cos t,\ \sin t)$

**b.** $(\cos t + 5,\ 0)$

**c.** As *A* rotates counterclockwise from (1, 0) to (–1, 0), point *B* moves horizontally from (6, 0) to (4, 0). As *A* continues to rotate counterclockwise from (–1, 0) back to (1, 0), point *B* moves horizontally from (4, 0) back to (6, 0).

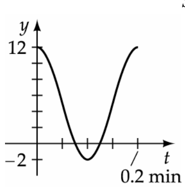
**67.** 5 rpm $a = 7$

$p = 0.2$ min $\quad p = \frac{2\pi}{b}$

$0.2 = \frac{2\pi}{b}$

$b = 10\pi$

$s = 7\cos 10\pi t + 5$



**68.** current:

$a = 5$

$\frac{1}{60} = \frac{2\pi}{b} \Rightarrow b = 120\pi$

no phase shift

$i = 5\cos(120\pi t)$

voltage:

$a = 180$

$\frac{1}{60} = \frac{2\pi}{b} \Rightarrow b = 120\pi$

phase shift $= = -\frac{c}{b}$

$= -\frac{c}{120\pi} = -0.005 \Rightarrow c = 0.6\pi$ or $\frac{3\pi}{5}$

$V = 180\cos\left(120\pi t + \frac{3\pi}{5}\right)$

**69.** Change 6 rpm to radians/second.

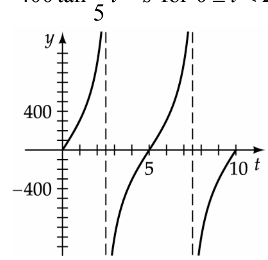
$6 \text{ rpm} = \frac{6 \cdot 2\pi \text{ radians}}{1 \text{ minute}} \cdot \frac{1 \text{ minute}}{60 \text{ sec}}$

$= \frac{\pi}{5}$ radians/sec

in $t$ seconds, $\theta$ increased by $\frac{\pi}{5}t$ radians.

$\tan\frac{\pi}{5}t = \frac{s}{400}$

$400\tan\frac{\pi}{5}t = s$ for $0 \le t < 2.5$ or $7.5 < t \le 10$




**70.** $p = 12,\ \frac{2\pi}{b} = 12,\ b = \frac{\pi}{6}$

$-\frac{c}{b} = 3 \quad c = -3 \cdot \frac{\pi}{6} = -\frac{\pi}{2}$

Curve 1. $y = 0.6\sin\left(\frac{\pi}{6}x - \frac{\pi}{2}\right) + 12$

Curve 2. $y = 1.3\sin\left(\frac{\pi}{6}x - \frac{\pi}{2}\right) + 12$

Curve 3. $y = 2\sin\left(\frac{\pi}{6}x - \frac{\pi}{2}\right) + 12$

Curve 4. $y = 2.7\sin\left(\frac{\pi}{6}x - \frac{\pi}{2}\right) + 12$

Domain: $0 \le x \le 12$

**71.** amplitude $A = 3$

$k = 9$

24 = 2 cycles

12 = 1 cycle

$\frac{2\pi}{\text{B}} = 12 \Rightarrow B = \frac{\pi}{6}$

$y = 3\cos\frac{\pi}{6}t + 9$

At 6:00 P.M., $t = 12$.

$y = 3\cos\frac{\pi}{6} \cdot 12 + 9$

$y = 3\cos 2\pi + 9$

$y = 3 + 9$

$y = 12$ ft

**72.** amplitude $= 30$

$p = 24,\ \frac{2\pi}{b} = 24,\ b = \frac{\pi}{12}$

phase shift $= -\frac{c}{b} = -\frac{c}{\pi/12} = -15$

$c = \frac{5\pi}{4}$

$f(t) = 30\cos\left(\frac{\pi}{12}t + \frac{5\pi}{4}\right) + 90$

At 1:00 P.M., $t = 7$.

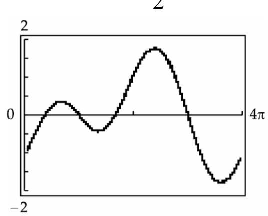
$f(7) = 30\cos\left(\frac{7\pi}{12} + \frac{5\pi}{4}\right) + 90$

$= 30\cos\frac{11\pi}{6} + 90$

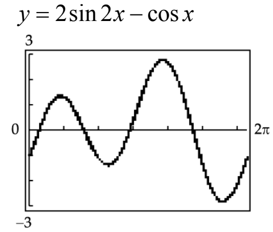
$= 30\left(\frac{\sqrt{3}}{2}\right) + 90 = 15\sqrt{3} + 90$

$f(7) \approx 116°$

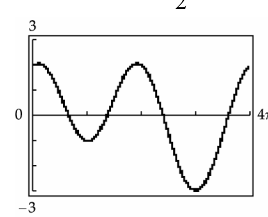
**73.** $y = \sin x - \cos\frac{x}{2}$



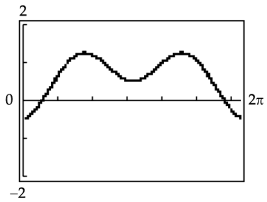

**74.** $y = 2\sin 2x - \cos x$



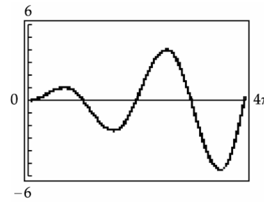

**75.** $y = 2\cos x + \sin\frac{x}{2}$



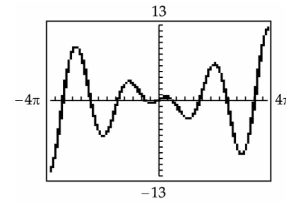
**76.** $y = -\frac{1}{2}\cos 2x + \sin\frac{x}{2}$



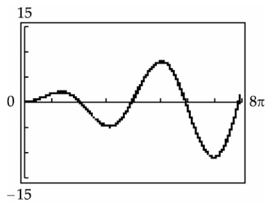

**77.** $y = \frac{x}{2}\sin x$



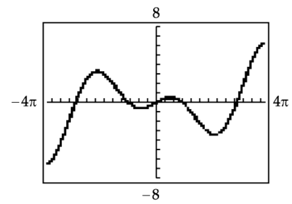

**78.** $y = x\cos x$



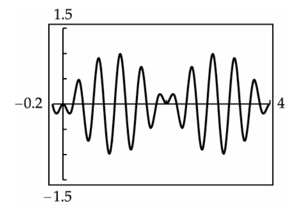

**79.** $y = x\sin\frac{x}{2}$



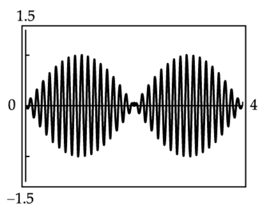

**80.** $y = \frac{x}{2}\cos\frac{x}{2}$




**81.**




**82.**




## Connecting Concepts

**83.** sine function, $a = 2,\ p = \pi$

$\frac{2\pi}{b} = \pi,\ b = 2$

phase shift $= -\frac{c}{b} = \frac{\pi}{3},\ c = -\frac{2\pi}{3}$

$y = 2\sin\left(2x - \frac{2\pi}{3}\right)$

**84.** cosine function, $a = 3,\ p = 3\pi$

$\frac{2\pi}{b} = 3\pi,\ b = \frac{2}{3}$

phase shift $= -\frac{c}{b} = -\frac{\pi}{4},\ c = \frac{\pi}{6}$

$y = 3\cos\left(\frac{2}{3}x + \frac{\pi}{6}\right)$

**85.** tangent function, $p = 2\pi$

$\frac{\pi}{b} = 2\pi \Rightarrow b = \frac{1}{2}$

phase shift $= -\frac{c}{b} = \frac{\pi}{2} \Rightarrow c = -\frac{\pi}{4}$

$y = \tan\left(\frac{x}{2} - \frac{\pi}{4}\right)$

**86.** cotangent function, $p = \frac{\pi}{2}$

$\frac{\pi}{b} = \frac{\pi}{2},\ b = 2$

phase shift $= -\frac{c}{b} = -\frac{\pi}{4},\ c = \frac{\pi}{2}$

$y = \cot\left(2x + \frac{\pi}{2}\right)$

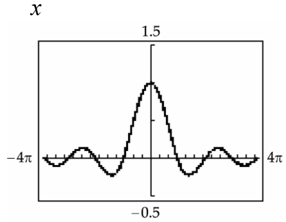
**87.** $g[h(x)] = (\cos x)^2 + 2$

$= \cos^2 x + 2$

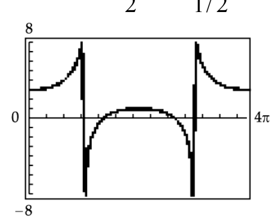
**88.** $h[g(x)] = (\sin x)^2 + 2(\sin x) + 1$

$= \sin^2 x + 2\sin x + 1$

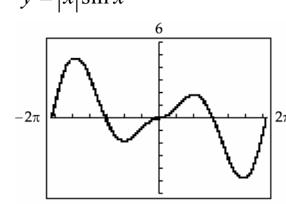
**89.** $\frac{\sin x}{x} \to 1$ as $x \to 0$



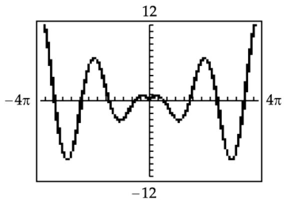

**90.** $y = 2 + \sec\frac{x}{2},\ p = \frac{2\pi}{1/2} = 4\pi$




**91.** $y = |x|\sin x$



**92.** $y = |x|\cos x$




## Prepare for Section 5.8

**PS1.** $\frac{3}{2\pi}$

**PS2.** $\frac{5}{2}$

**PS3.** $4\cos 2\pi(0) = 4\cos 0 = 4$

**PS4.** $\sqrt{\frac{18}{2}} = \sqrt{9} = 3$

**PS5.** $4\cos\left(\sqrt{\frac{16}{4}} \cdot 2\pi\right) = 4\cos(4 \cdot 2\pi)$
$= 4\cos 8\pi = 4$

**PS6.** $y = 4\cos \pi x$

## Section 5.8

**1.** $y = 2\sin 2t$

amplitude = 2

$p = \frac{2\pi}{2} = \pi$

frequency $= \frac{1}{p} = \frac{1}{\pi}$

**2.** $y = \frac{2}{3}\cos\frac{t}{3}$

amplitude $= \frac{2}{3}$

$p = \frac{2\pi}{1/3} = 6\pi$

frequency $= \frac{1}{p} = \frac{1}{6\pi}$

**3.** $y = 3\cos\frac{2t}{3}$

amplitude = 3

$p = \frac{2\pi}{2/3} = 3\pi$

frequency $= \frac{1}{p} = \frac{1}{3\pi}$

**4.** $y = 4\sin 3t$

amplitude = 4

$p = \frac{2\pi}{3}$

frequency $= \frac{1}{\text{p}} = \frac{1}{2\pi/3} = \frac{3}{2\pi}$

**5.** $y = 4\cos \pi t$

amplitude = 4

$p = \frac{2\pi}{\pi} = 2$

frequency $= \frac{1}{p} = \frac{1}{2}$

**6.** $y = 2\sin\frac{\pi t}{3}$

amplitude = 2

$p = \frac{2\pi}{\pi/3} = 6$

frequency $\frac{1}{\text{p}} = \frac{1}{6}$

**7.** $y = \frac{3}{4}\sin\frac{\pi t}{2}$

amplitude $= \frac{3}{4}$

$p = \frac{2\pi}{\pi/2} = 4$

frequency $= \frac{1}{p} = \frac{1}{4}$

**8.** $y = 5\cos 2\pi t$

amplitude = 5

$p = \frac{2\pi}{2\pi} = 1$

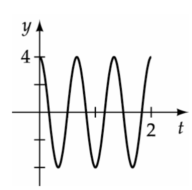
frequency $= \frac{1}{p} = \frac{1}{1} = 1$

**9.** $a = 4$

$p = \frac{1}{\text{frequency}} = \frac{1}{1.5} = \frac{2}{3}$

$\frac{2\pi}{b} = \frac{2}{3} \Rightarrow b = 3\pi$

$y = 4\cos 3\pi t$



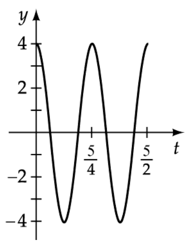
**10.** frequency = 0.8

$$p = \frac{1}{\text{frequency}} = \frac{1}{0.8} = \frac{5}{4}$$

$$\frac{2\pi}{b} = \frac{5}{4} \Rightarrow b = \frac{8}{5}\pi$$

amplitude = 4

$$y = 4\cos\frac{8}{5}\pi t$$

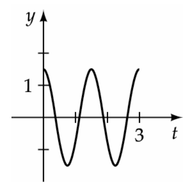



**11.** $p = 1.5$

$$\text{amplitude} = \frac{3}{2}$$

$$\frac{2\pi}{b} = 1.5 \Rightarrow b = \frac{4\pi}{3}$$

$$y = \frac{3}{2}\cos\frac{4\pi}{3}t$$

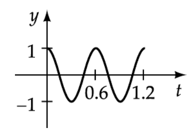



**12.** $p = 0.6$

$a = 1$

$$\frac{2\pi}{b} = 0.6 \Rightarrow b = \frac{10}{3}\pi$$

$$y = \cos\frac{10}{3}\pi t$$

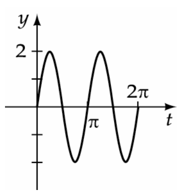



**13.** amplitude = 2

$p = \pi$

$$\frac{2\pi}{b} = \pi \Rightarrow b = 2$$

$$y = 2\sin 2t$$

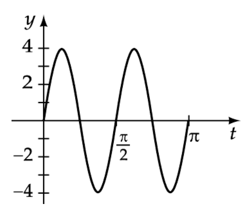



**14.** amplitude = 4

$$p = \frac{\pi}{2}$$

$$\frac{2\pi}{b} = \frac{\pi}{2} \Rightarrow b = 4$$

$$y = 4\sin 4t$$

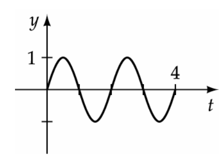



**15.** amplitude = 1

$p = 2$

$$\frac{2\pi}{b} = 2 \Rightarrow b = \pi$$

$$y = \sin \pi t$$

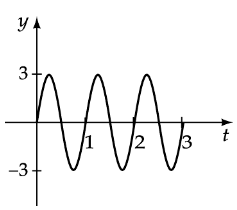



**16.** amplitude = 3

$p = 1$

$$\frac{2\pi}{b} = 1 \Rightarrow b = 2\pi$$

$$y = 3\sin 2\pi t$$



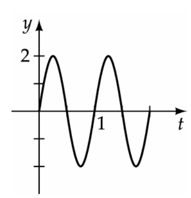

**17.** amplitude = 2

frequency = 1

$$p = \frac{1}{\text{frequency}} = \frac{1}{1} = 1$$

$$\frac{2\pi}{b} = 1 \Rightarrow b = 2\pi$$

$$y = 2\sin 2\pi t$$



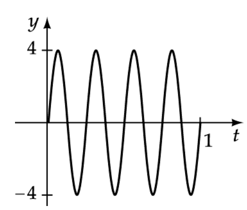

**18.** amplitude = 4

frequency = 4

$$p = \frac{1}{\text{frequency}} = \frac{1}{4}$$

$$\frac{2\pi}{b} = \frac{1}{4} \Rightarrow b = 8\pi$$

$$y = 4\sin 8\pi t$$



**19.** amplitude $= \frac{1}{2}$

frequency $= \frac{2}{\pi} \Rightarrow p = \frac{\pi}{2}$

$\frac{2\pi}{b} = \frac{\pi}{2} \Rightarrow b = 4$

$y = \frac{1}{2}\cos 4t$

**20.** amplitude $= 3$

frequency $\frac{1}{\pi} \Rightarrow p = \pi$

$\frac{2\pi}{b} = \pi \Rightarrow b = 2$

$y = 3\cos 2t$

**21.** amplitude $= 2.5$

frequency $= 0.5 \Rightarrow p = 2$

$\frac{2\pi}{b} = 2 \Rightarrow b = \pi$

$y = 2.5\cos \pi t$

**22.** amplitude $= 5$

frequency $= \frac{1}{8} \Rightarrow p = 8$

$\frac{2\pi}{b} = 8 \Rightarrow b = \frac{\pi}{4}$

$y = 5\cos \frac{\pi}{4} t$

**23.** amplitude $= \frac{1}{2}$

$p = 3$

$\frac{2\pi}{b} = 3 \Rightarrow b = \frac{2}{3}\pi$

$y = \frac{1}{2}\cos \frac{2\pi}{3} t$

**24.** amplitude $= 5$

$p = 5$

$\frac{2\pi}{b} = 5 \Rightarrow b = \frac{2\pi}{5}$

$y = 5\cos \frac{2\pi}{5} t$

**25.** amplitude $= 4$

$p = \frac{\pi}{2}$

$\frac{2\pi}{b} = \frac{\pi}{2} \Rightarrow b = 4$

$y = 4\cos 4t$

**26.** amplitude $= 2$

$p = \pi$

$\frac{2\pi}{b} = \pi \Rightarrow b = 2$

$y = 2\cos 2t$

**27.** amplitude $= 2$ feet, $a = -2$

frequency $= f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{8}{32}} = \frac{1}{2\pi}\cdot\frac{1}{2} = \frac{1}{4\pi}$

period $= p = \frac{1}{f} = 4\pi$

$y = a\cos 2\pi ft = -2\cos 2\pi\left(\frac{1}{4\pi}\right)t$

$= -2\cos \frac{1}{2} t$

**28.** amplitude $= 1.5$ feet, $a = -1.5$

frequency $= f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{3}{27}} = \frac{1}{2\pi}\cdot\frac{1}{3} = \frac{1}{6\pi}$

period $= p = \frac{1}{f} = \frac{1}{1/6\pi} = 6\pi$

$y = -1.5\cos 2\pi ft = -1.5\cos 2\pi\left(\frac{1}{6\pi}\right)t$

$= -1.5\cos \frac{1}{3} t$

**29.** **a.** frequency $= f = \frac{392\pi}{2\pi} = 196$ cycles/s

period $= p = \frac{1}{196}$ s

**b.** The amplitude needs to increase.

**30.** amplitude $= 9.3$ ft, $a = 9.3$

period $= p = \frac{2\pi}{12.5} = 0.16$

$h = 9.3\sin 0.16\pi t$

**31.** amplitude $= a = \frac{78-4}{2} = 37$

period $= p = \frac{2\pi}{45} = \frac{\pi}{22.5}$

$h = -37\cos\left(\frac{\pi}{22.5}t\right) + 41$

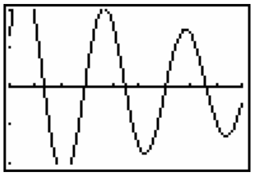
**32.** amplitude $= a = \frac{6.5-4.5}{2} = 1$

period $= p = \frac{2\pi}{60} = \frac{\pi}{30}$

$d = \cos \frac{\pi}{30} t + 5.5$

**33.** **a.** The pseudoperiod is $\frac{2\pi}{2} = \pi$. There are $\frac{10}{\pi} \approx 3$ complete oscillations of length $\pi$ in $0 \le t \le 10$.

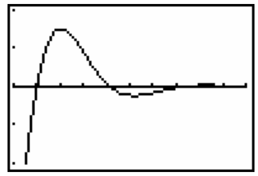
**b.** $|f(t)| < 0.01$ for all $t > 59.8$ (nearest tenth).



Xmin = 56, Xmax = 65, Xscl = 1,
Ymin = −0.01, Ymax = 0.01, Yscl = 0.005

**34.** **a.** The pseudoperiod is $\frac{2\pi}{1} = 2\pi$. There are $\frac{10}{2\pi} \approx 1$ complete oscillations of length $2\pi$ in $0 \le t \le 10$.

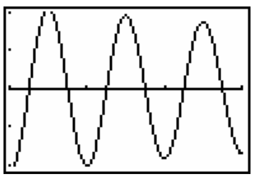
**b.** $|f(t)| < 0.01$ for all $t > 10.5$ (nearest tenth).



Xmin = 10, Xmax = 20, Xscl = 1,
Ymin = −0.01, Ymax = 0.01, Yscl = 0.005

**35.** **a.** The pseudoperiod is $\frac{2\pi}{2\pi} = 1$. There are 10 complete oscillations of length 1 in $0 \le t \le 10$.

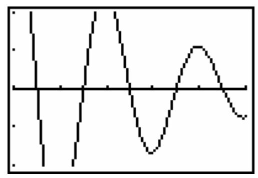
**b.** $|f(t)| < 0.01$ for all $t > 71.0$ (nearest tenth).



Xmin = 70, Xmax = 73, Xscl = 1
Ymin = −0.01, Ymax = 0.01, Yscl = 0.005

**36.** **a.** The pseudoperiod is $\frac{2\pi}{\pi} = 2$. There are $\frac{10}{2} = 5$ complete oscillations of length 2 in $0 \le t \le 10$.

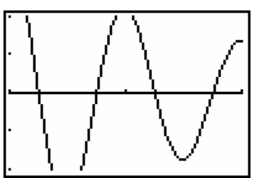
**b.** $|f(t)| < 0.01$ for all $t > 17.2$ (nearest tenth).



Xmin = 15, Xmax = 20, Xscl = 1,
Ymin = −0.01, Ymax = −.01, Yscl = 0.005

**37.** **a.** The pseudoperiod is $\frac{2\pi}{2\pi} = 1$. There are 10 complete oscillations of length 1 in $0 \le t \le 10$.

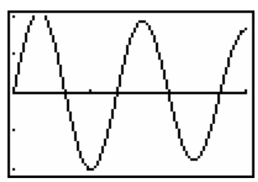
**b.** $|f(t)| < 0.01$ for all $t > 9.1$ (nearest tenth).



Xmin = 8, Xmax = 10, Xscl = 1,
Ymin = −0.01, Ymax = 0.01, Yscl = 0.005

**38.** **a.** The pseudoperiod is $\frac{2\pi}{3\pi} = \frac{2}{3}$. There are $\left(\frac{10}{2/3}\right) = 15$ complete oscillations of length $\frac{2}{3}$ in $0 \le t \le 10$.

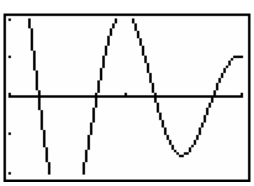
**b.** $|f(t)| < 0.01$ for all $t > 23.0$ (nearest tenth).



Xmin = 22.5, Xmax = 24, Xscl = 1,
Ymin = −0.01, Ymax = 0.01, Yscl = 0.005

**39.** **a.** The pseudoperiod is $\frac{2\pi}{2\pi} = 1$. There are 10 complete oscillations of length 1 in $0 \le t \le 10$.

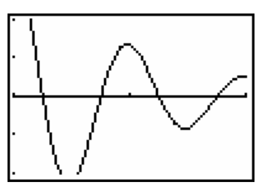
**b.** $|f(t)| < 0.01$ for all $t > 6.1$ (nearest tenth).



Xmin = 5, Xmax = 7, Xscl = 1,
Ymin = −0.01, Ymax = .01, Yscl = 0.005

**40.** **a.** The pseudoperiod is $\frac{2\pi}{2\pi} = 1$. There are 10 complete oscillations of length 1 in $0 \le t \le 10$.

**b.** $|f(t)| < 0.01$ for all $t > 4.6$ (nearest tenth).



Xmin = 4, Xmax = 6, Xscl = 1,
Ymin = −0.01, Ymax = 0.01, Yscl = 0.005

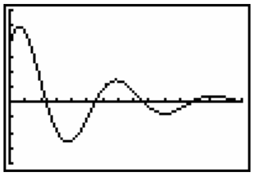
## Connecting Concepts

**41.** $p_1 = 2\pi\sqrt{\dfrac{m}{k}},\ p_2 = 2\pi\sqrt{\dfrac{9m}{k}} = 3p_1$

Increasing the main mass to $9m$ will triple the period.

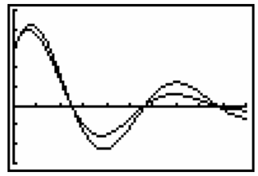
**42.** The frequency will double.

**43.** yes



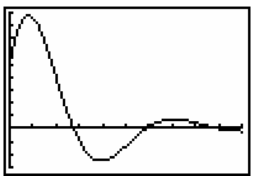
Xmin = 0, Xmax = 15, Xscl = 1,
Ymin = −1, Ymax = 1.5, Yscl = 0.25

**44.** no



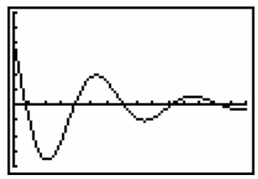
Xmin = 0, Xmax = 10, Xscl = 1,
Ymin = −3, Ymax = 5, Yscl = 1

**45.** yes



Xmin = 0, Xmax = 10, Xscl = 1,
Ymin = −3, Ymax = 9, Yscl = 1

**46.** yes

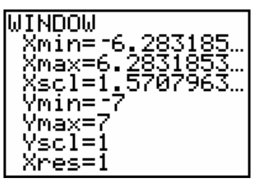


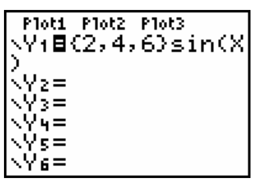
Xmin = 0, Xmax = 15, Xscl = 1,
Ymin = −1, Ymax = 1.5, Yscl = 0.25

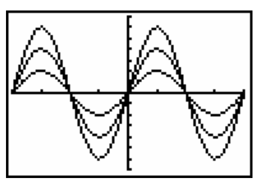
## Exploring Concepts with Technology

**Sinusoidal Families**

**1.**

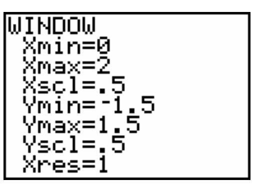


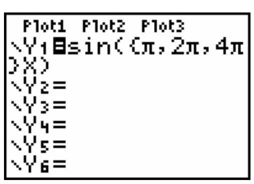



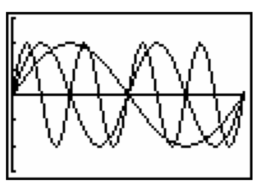



All three sine graphs have, a period of $2\pi$, $x$-intercepts at $n\pi$, and no phase shift, but their amplitudes are 2, 4, and 6 respectively.

**2.**

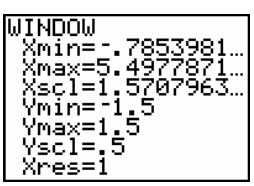


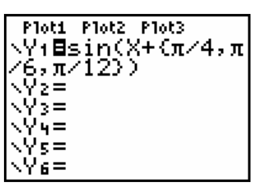



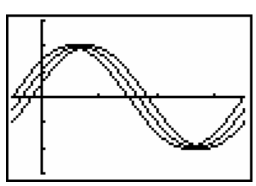



All three sine graphs have $x$-intercepts at $n$, an amplitude of 1, and no phase shift, but their periods are 2, 1, and 0.5 respectively, and $y = \sin 2\pi x$ and $y = 4\pi x$ have additional $x$-intercepts at $0.5n$ and $0.25n$ respectively.

**3.**



All three sine graphs have a period of $2\pi$ and an amplitude of 1, but their phase shifts are $-\pi/4$, $-\pi/6$, and $-\pi/12$, respectively.

**4.** Yes, the calculator has displayed all three graphs. All three sine graphs have an amplitude of 1, a period of $2\pi$, and a phase shift of $-(2n-1)\pi$.

## Assessing Concepts

**1.** True

**2.** False; $\sec^2\theta - \tan^2\theta = 1$ is an identity.

**3.** False; 1 rad $\approx 57.3°$.

**4.** True

**5.** $\frac{\pi}{4}$

**6.** (0, 1)

**7.** The period is $\frac{2\pi}{3\pi/4} = \frac{8}{3}$.

**8.** Shift the graph of $y_1$ to the left $\frac{\pi}{2}$ units.

**9.** All real numbers except multiples of $\pi$.

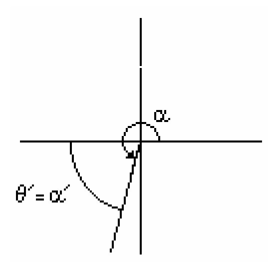
**10.** The vertical asymptotes are $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.

## Chapter Review

**1.** complement: $90° - 65° = 25°$ [5.1]
supplement: $180° - 65° = 115°$

**2.** $\theta = 980° = 260° + 2\cdot 360°$ [5.3]
$\theta$ is coterminal with $\alpha = 260°$ and $\theta' = \alpha'$.
Since $180° < \alpha < 270°$,

$$180° + \alpha' = \alpha$$
$$180° + a' = 260°$$
$$\alpha' = 80°$$
$$\theta = 80°$$




**3.** $2 = 2\left(\frac{180°}{\pi}\right)$ [5.1]
$= 114.59°$

**4.** $315° = 315°\left(\frac{\pi}{180°}\right)$ [5.1]
$= \frac{7\pi}{4}$

**5.** $s = r\theta = 3(75°)\left(\frac{\pi}{180°}\right)$ [5.1]
$= 3.93$ m

**6.** $\theta = \frac{s}{r} = \frac{12}{40}$ [5.1]
$= 0.3$

**7.** $w = \frac{V}{r} = \frac{50}{16}\cdot\frac{63360}{3600}$ [5.1]
$\approx 55$ rad/sec

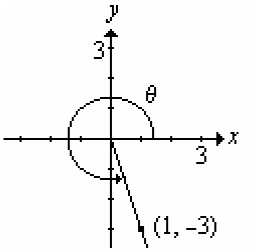
For exercises 8 to 11, $\csc\theta = \frac{3}{2} = \frac{r}{y}$, $r = 3$, $y = 2$, and $x = \sqrt{3^2 - 2^2} = \sqrt{5}$.

**8.** $\cos\theta = \frac{x}{r} = \frac{\sqrt{5}}{3}$ [5.2]

**9.** $\cot\theta = \frac{x}{y} = \frac{\sqrt{5}}{2}$ [5.2]

**10.** $\sin\theta = \frac{y}{r} = \frac{2}{3}$ [5.2]

**11.** $\sec\theta = \frac{r}{x} = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$ [5.2]

**12.** 


$x = 1,\ y = -3,\ r = \sqrt{1^2 + (-3)^2} = \sqrt{10}$

$\sin\theta = -\dfrac{3}{\sqrt{10}} = -\dfrac{3\sqrt{10}}{10}$ $\quad\cos\theta = \dfrac{1}{\sqrt{10}} = \dfrac{\sqrt{10}}{10}$ $\quad\tan\theta = \dfrac{-3}{1} = -3$

$\csc\theta = -\dfrac{\sqrt{10}}{3}$ $\quad\sec\theta = \sqrt{10}$ $\quad\cot\theta = -\dfrac{1}{3}$

**13.** **a.** $\sec 150° = \dfrac{2}{-\sqrt{3}} = -\dfrac{2\sqrt{3}}{3}$

**b.** $\tan\left(-\dfrac{3\pi}{4}\right) = 1$

**c.** $\cot(-225°) = -1$

**d.** $\cos\dfrac{2\pi}{3} = -\dfrac{1}{2}$ [5.3]

**14.** **a.** $\cos 123° \approx -0.5446$

**b.** $\cot 4.22 \approx 0.5365$

**c.** $\sec 612° \approx -3.2361$

**d.** $\tan\dfrac{2\pi}{5} \approx 3.0777$ [5.3]

**15.** $\cos\phi = -\dfrac{\sqrt{3}}{2} = \dfrac{x}{r},\ x = -\sqrt{3},\ r = 2,\ y = -\sqrt{2^2 - \left(-\sqrt{3}\right)^2} = -1$ [5.3]

**a.** $\sin\phi = \dfrac{y}{r} = -\dfrac{1}{2}$

**b.** $\tan\phi = \dfrac{y}{x} = \dfrac{-1}{-\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

**16.** $\tan\phi = -\dfrac{\sqrt{3}}{3} = \dfrac{y}{x},\ y = \sqrt{3},\ x = -3,\ r = \sqrt{(-3)^2 + \left(\sqrt{3}\right)^2} = 2\sqrt{3}$ [5.3]

**a.** $\sec\phi = \dfrac{r}{x} = \dfrac{2\sqrt{3}}{-3} = -\dfrac{2\sqrt{3}}{3}$

**b.** $\csc\phi = \dfrac{r}{y} = \dfrac{2\sqrt{3}}{\sqrt{3}} = 2$

**17.** $\sin\phi = -\dfrac{\sqrt{2}}{2},\ y = -\sqrt{2},\ r = 2,\ x = -\sqrt{2^2 - \left(-\sqrt{2}\right)^2} = \sqrt{2}$ [5.3]

**a.** $\cos\phi = \dfrac{x}{r} = \dfrac{\sqrt{2}}{2}$

**b.** $\cot\phi = \dfrac{x}{y} = \dfrac{\sqrt{2}}{-\sqrt{2}} = -1$

**18.** **a.** $W(\pi)=(-1,0)$ [5.4]

**b.** $W\left(-\frac{\pi}{3}\right)=\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$

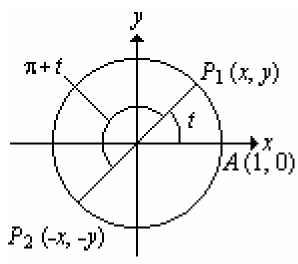
**c.** $W\left(\frac{5\pi}{4}\right)=\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$

**19.** $f(x)=\sin(x)\tan(x)$ [5.4]

$$f(-x)=\sin(-x)\tan(-x)=(-\sin x)(-\tan x)$$
$$=\sin x\tan x$$
$$=f(x)$$

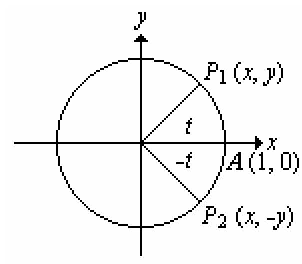
The function defined by $f(x)=\sin(x)\tan(x)$ is an even function.

**20.**




$$\cos(\pi+t)=-x$$
$$\cos t=x$$
$$\cos(\pi+t)=-\cos t$$

**21.**




$$\tan(-t)=-\frac{y}{x}$$
$$\tan t=\frac{y}{x}$$
$$\tan(-t)=-\tan t$$

**22.** $1+\frac{\sin^2\phi}{\cos^2\phi}=1+\tan^2\phi$ [5.4]

$$=\sec^2\phi$$

**23.** $\frac{\tan\phi+1}{\cot\phi+1}=\frac{\frac{\sin\phi}{\cos\phi}+1}{\frac{\cos\phi}{\sin\phi}+1}$ [5.4]

$$=\frac{\frac{\sin\phi+\cos\phi}{\cos\phi}}{\frac{\cos\phi+\sin\phi}{\sin\phi}}$$
$$=\frac{\sin\phi(\sin\phi+\cos\phi)}{\cos\phi(\cos\phi+\sin\phi)}$$
$$=\tan\phi$$

**24.** $\frac{\cos^2\phi+\sin^2\phi}{\csc\phi}=\frac{1}{\csc\phi}$ [5.4]

$$=\sin\phi$$

**25.** $\sin^2\phi(\tan^2\phi+1)=\sin^2\phi\sec^2\phi$ [5.4]

$$=\frac{\sin^2\phi}{\cos^2\phi}$$
$$=\tan^2\phi$$

**26.** $1+\dfrac{1}{\tan^2\phi}=\dfrac{\tan^2\phi+1}{\tan^2\phi}$ [5.4]

$$=\frac{\sec^2\phi}{\tan^2\phi}$$

$$=\frac{\frac{1}{\cos^2\phi}}{\frac{\sin^2\phi}{\cos^2\phi}}$$

$$=\frac{1}{\sin^2\phi}$$

$$=\csc^2\phi$$

**27.** $\dfrac{\cos^2\phi}{1-\sin^2\phi}-1=\dfrac{1-\sin^2\phi}{1-\sin^2\phi}-1$ [5.4]

$$=1-1$$

$$=0$$

**28.** $y=3\cos(2x-\pi)$ [5.5]

$a=|3|=3$; period $=\dfrac{2\pi}{b}=\dfrac{2\pi}{2}=\pi$

phase shift $=-\dfrac{c}{b}=-\dfrac{-\pi}{2}=\dfrac{\pi}{2}$

**29.** $y=2\tan 3x$ [5.6]

no amplitude; period $=\dfrac{\pi}{b}=\dfrac{\pi}{3}$

phase shift = 0

**30.** $y=-2\sin\left(3x+\dfrac{\pi}{3}\right)$ [5.5]

$a=|-2|=2$; period $=\dfrac{2\pi}{b}=\dfrac{2\pi}{3}$

phase shift $=-\dfrac{\mathrm{c}}{\mathrm{b}}=-\dfrac{\pi/3}{3}=-\dfrac{\pi}{9}$

**31.** $y=\cos\left(2x-\dfrac{2\pi}{3}\right)+2$ [5.5]

$a=|1|=1$; period $=\dfrac{2\pi}{b}=\dfrac{2\pi}{2}=\pi$

phase shift $=-\dfrac{c}{b}=-\dfrac{-2\pi/3}{2}=\dfrac{\pi}{3}$

**32.** $y=-4\sec\left(4x-\dfrac{3\pi}{2}\right)$ [5.6]

no amplitude; period $=\dfrac{2\pi}{b}=\dfrac{2\pi}{4}=\dfrac{\pi}{2}$

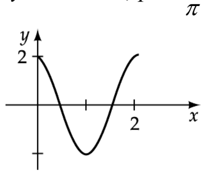
phase shift $=-\dfrac{c}{b}=-\dfrac{-3\pi/2}{4}=\dfrac{3\pi}{8}$

**33.** $y=2\csc\left(x-\dfrac{\pi}{4}\right)-3$ [5.6]

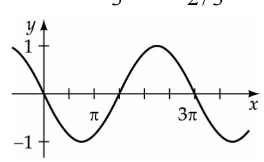
no amplitude; period $=\dfrac{2\pi}{b}=\dfrac{2\pi}{1}=2\pi$

phase shift $=-\dfrac{c}{b}=-\dfrac{-\pi/4}{1}=\dfrac{\pi}{4}$

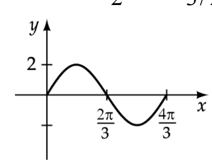
**34.** $y=2\cos\pi x,\ p=\dfrac{2\pi}{\pi}=2$



**35.** $y=-\sin\dfrac{2x}{3},\ p=\dfrac{2\pi}{2/3}=3\pi$

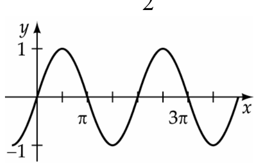


**36.** $y=2\sin\dfrac{3x}{2},\ p=\dfrac{2\pi}{3/2}=\dfrac{4\pi}{3}$



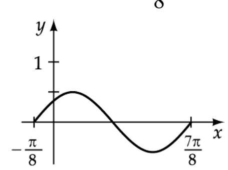
**37.** $y=\cos\left(x-\dfrac{\pi}{2}\right),\ p=2\pi$

phase shift $=\dfrac{\pi}{2}$



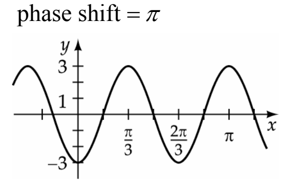
**38.** $y=\dfrac{1}{2}\sin\left(2x+\dfrac{\pi}{4}\right),\ p=\dfrac{2\pi}{2}=\pi$

phase shift $-\dfrac{\pi}{8}$

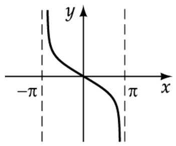


**39.** $y=3\cos 3(x-\pi),\ p=\dfrac{2\pi}{3}$

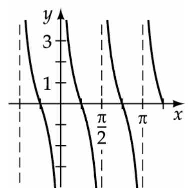
phase shift $=\pi$



**40.** $y = -\tan\dfrac{x}{2},\ p = \dfrac{\pi}{1/2} = 2\pi$

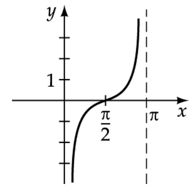



**41.** $y = 2\cot 2x,\ p = \dfrac{\pi}{2}$



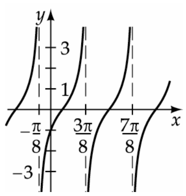

**42.** $y = \tan\left(x - \dfrac{\pi}{2}\right),\ p = \pi$

phase shift $= \dfrac{\pi}{2}$



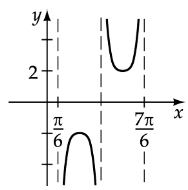

**43.** $y = -\cot\left(2x + \dfrac{\pi}{4}\right),\ p = \dfrac{\pi}{2}$

phase shift $= -\dfrac{\pi}{8}$



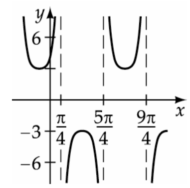

**44.** $y = -2\csc\left(2x - \dfrac{\pi}{3}\right),\ p = \dfrac{2\pi}{2} = \pi$

phase shift $= \dfrac{\pi}{6}$

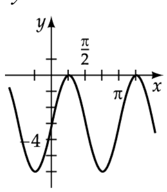



**45.** $y = 3\sec\left(x + \dfrac{\pi}{4}\right),\ p = 2\pi$

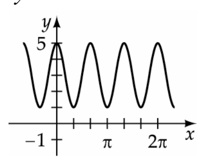
phase shift $= -\dfrac{\pi}{4}$



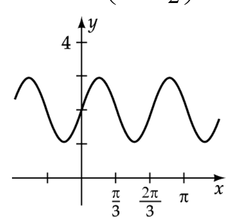

**46.** $y = 3\sin 2x - 3$



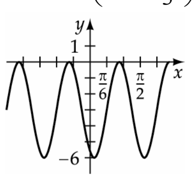

**47.** $y = 2\cos 3x + 3$



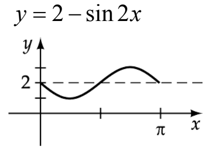

**48.** $y = -\cos\left(3x + \dfrac{\pi}{2}\right) + 2$



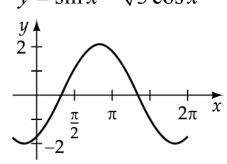

**49.** $y = 3\sin\left(4x - \dfrac{2\pi}{3}\right) - 3$



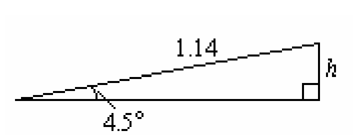

**50.** $y = 2 - \sin 2x$




**51.** $y = \sin x - \sqrt{3}\cos x$




**52.**

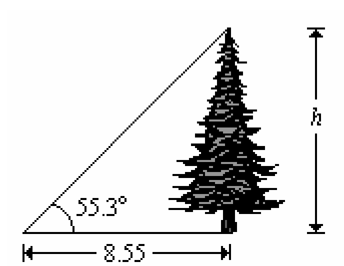



$\sin 4.5° = \dfrac{h}{1.14}$

$h = 1.14\sin 4.5° \approx 0.089$ mi

[5.2]

**53.**




$\tan 55.3° = \dfrac{h}{8.55}$

$h = 8.55\tan 55.3 \approx 12.3$ feet

[5.2]

**54.** Speed for inner ring:

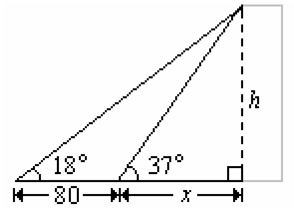
$$v = \frac{s}{t} = \frac{2\pi(14.5)}{24} \approx 3.79609 \text{ ft/s}$$

Speed for outer ring:

$$v = \frac{s}{t} = \frac{2\pi(21)}{24} \approx 5.497787 \text{ ft/s}$$

The outer swing has a greater speed of $5.497787 - 3.79609 \approx 1.7$ ft/s. [5.1]

**55.**




(1) $\cot 18° = \dfrac{80+x}{h} = \dfrac{80}{h} + \dfrac{x}{h}$

(2) $\cot 37° = \dfrac{x}{h}$

Substitute for $\dfrac{x}{h}$ in equation (1).

$$\cot 18° = \frac{80}{h} + \cot 37°$$

Solve for $h$. $\dfrac{80}{h} = \cot 18° - \cot 37°$

$$\frac{h}{80} = \frac{1}{\cot 18° - \cot 37°}$$

$$h = \frac{80}{\cot 18° - \cot 37°} \approx 46 \text{ ft}$$

[5.2]

**56.** $y = 2.5 \sin 50t$ [5.8]

amplitude = 2.5

$p = \dfrac{2\pi}{b} = \dfrac{2\pi}{50} = \dfrac{\pi}{25}$

frequency $= \dfrac{1}{p} = \dfrac{25}{\pi}$

**57.** amplitude = 0.5 [5.8]

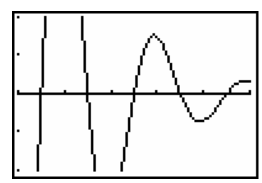
$f = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}} = \dfrac{1}{2\pi}\sqrt{\dfrac{20}{5}} = \dfrac{1}{\pi}$

$p = \pi$

$y = -0.5\cos 2\pi f t = -0.5 \cos 2\pi\left(\dfrac{1}{\pi}\right)t$

$y = -0.5 \cos 2t$

**58.** $|f(t)| < 0.01$ for all $t > 7.2$ [5.8]



Xmin = 5, Xmax = 10, Xcsl = 1
Ymin = −.01, Ymax = .01, Yscl = .005

## Quantitative Reasoning

**QR1. a.** $\dfrac{2\pi}{3\pi} = \dfrac{m}{n} \Rightarrow 2\pi n = 3\pi m \Rightarrow$ period $= 6\pi$
$2\pi(3) = 3\pi(2)$

**b.** $\dfrac{2/3}{4} = \dfrac{m}{n} \Rightarrow \dfrac{2}{3}n = 4m \Rightarrow$ period $= 4$
$\dfrac{2}{3}(6) = 4(1)$

**c.** $\dfrac{\pi/2}{2\pi/3} = \dfrac{m}{n} \Rightarrow \dfrac{\pi}{2}n = \dfrac{2\pi}{3}m \Rightarrow$ period $= 2\pi$
$\dfrac{\pi}{2}(4) = \dfrac{2\pi}{3}(3)$

**d.** $\dfrac{3\pi/2}{8\pi/3} = \dfrac{m}{n} \Rightarrow \dfrac{3\pi}{2}n = \dfrac{8\pi}{3}m \Rightarrow$ period $= 24\pi$
$\dfrac{3\pi}{2}(16) = \dfrac{8\pi}{3}(9)$

**e.** $\dfrac{5/2}{3/2} = \dfrac{m}{n} \Rightarrow \dfrac{5}{2}n = \dfrac{3}{2}m \Rightarrow$ period $= 7.5$
$\dfrac{5}{2}(3) = \dfrac{3}{2}(5)$

**f.** $\dfrac{4\pi/5}{4\pi} = \dfrac{m}{n} \Rightarrow \dfrac{4\pi}{5}n = 4\pi m \Rightarrow$ period $= 4\pi$
$\dfrac{4\pi}{5}(5) = 4\pi(1)$

**QR2.** $\dfrac{3}{2.5} = \dfrac{m}{n} \Rightarrow 3n = 2.5m \Rightarrow$ period = 15 s
$3(5) = 2.5(6)$

**QR3.** $\dfrac{1.25}{2.25} = \dfrac{m}{n} \Rightarrow 1.25n = 2.25m \Rightarrow$ period = 11.25 s
$1.25(9) = 2.25(5)$

**QR4.** $6n = 4.5m = 27w \quad \Rightarrow \quad \text{period} = 54 \text{ s}$

$6(9) = 4.5(12) = 27(2)$

## Chapter Test

**1.** $150° = 150°\left(\frac{\pi}{180°}\right) = \frac{5\pi}{6}$ [5.1]

**2.** $\pi - \frac{11}{12}\pi = \frac{\pi}{12}$ [5.1]

**3.** $s = r\theta$ [5.1]

$s = 10(75°)\left(\frac{\pi}{180°}\right)$

$s \approx 13.1 \text{ cm}$

**4.** $w = 6\frac{rev}{\sec}$ [5.1]

$w = 6\frac{rev}{\sec}\left(\frac{2\pi \ rad}{rev}\right)$

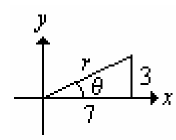
$w = 12\pi$ rad/sec

**5.** $v = rw$ [5.1]

$= 8 \cdot 10$

$= 80$ cm/sec

**6.**




$r = \sqrt{7^2 + 3^2}$

$r = \sqrt{58}$

$\sec\theta = \frac{\sqrt{58}}{7}$ [5.2]

**7.** $\csc 67° \approx 1.0864$ [5.2]

**8.** $\tan\frac{\pi}{6}\cos\frac{\pi}{3} - \sin\frac{\pi}{2} = \frac{1}{\sqrt{3}} \cdot \frac{1}{2} - 1$ [5.3]

$= \frac{1}{2\sqrt{3}} - 1$

$= \frac{\sqrt{3}}{6} - 1$

$= \frac{\sqrt{3} - 6}{6}$

**9.** $t = \frac{11\pi}{6}$ [5.4]

$x = \cos t = \frac{\sqrt{3}}{2}$ $\quad y = \sin t = -\frac{1}{2}$

$W(x, y) = W\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

**10.** $\frac{\sec^2 t - 1}{\sec^2 t} = \frac{\frac{1}{\cos^2 t} - 1}{\frac{1}{\cos^2 t}}$ [5.4]

$= \frac{\frac{1 - \cos^2 t}{\cos^2 t}}{\frac{1}{\cos^2 t}}$

$= 1 - \cos^2 t$

$= \sin^2 t$

**11.** $\text{period} = \frac{\pi}{b} = \frac{\pi}{3}$ [5.6]

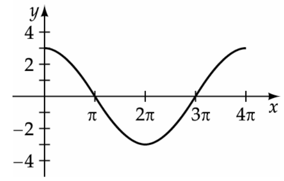
**12.** $a = |-3| = 3$; $\text{period} = \frac{2\pi}{b} = \frac{2\pi}{2} = \pi$ [5.7]

$\text{phase shift} = -\frac{\pi}{4}$

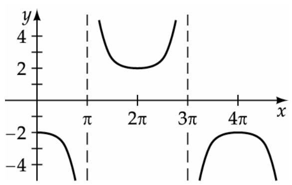
**13.** period = $\frac{\pi}{\pi/3} = 3$ [5.7]

phase shift = $-\frac{c}{b} = -\frac{\pi/6}{\pi/3} = -\frac{1}{2}$

**14.** $y = 3\cos\frac{1}{2}x,\ p = 4\pi$



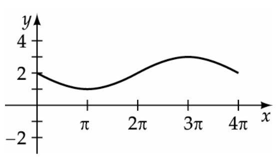
**15.** $y = -2\sec\frac{1}{2}x, \quad p = 4\pi$



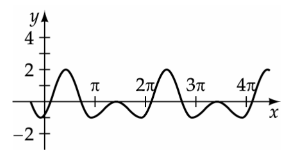
**16.** Shift the graph [of $y = 2\sin(2x)$] [5.7]

$\frac{\pi}{4}$ units to the right and down 1 unit.

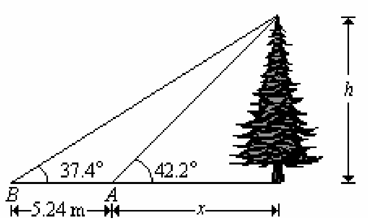
**17.** $y = 2 - \sin\frac{x}{2}$



**18.** $y = \sin x - \cos 2x$



**19.**




$$\tan 42.2° = \frac{h}{x}$$

$$x = \frac{h}{\tan 42.2°}$$

$$= h\cot 422.2°$$

$$\tan 37.4° = \frac{h}{5.24x}$$

$$= \frac{h}{5.24 + h\cot 42.2°}$$

Solve for $h$.

$$\tan 37.4° = \frac{h}{5.24 + h\cot 42.2°}$$

$$\tan 37.4°(5.24 + h\cot 42.2°) = h$$

$$5.24\tan 37.4° + h\tan 37.4°\cot 42.2° = h$$

$$h - h\tan 37.4°\cot 42.2° = 5.24\tan 37.4°$$

$$h(1 - \tan 37.4°\cot 42.2°) = 5.24\tan 37.4°$$

$$h = \frac{5.24\tan 37.4°}{1 - \tan 37.4°\cot 42.2°}$$

$$h \approx 25.5 \text{ meters}$$

The height of the tree is approximately 25.5 meters. [5.2]

**20.** $p = 5 \qquad 5 = \frac{2\pi}{b}, b = \frac{2\pi}{5}$

$a = 13$

$y = 13\cos\frac{2\pi}{5}t$ or $y = 13\sin\frac{2\pi}{5}t$ [5.8]

## Cumulative Review

**1.** $x^2 - y^2 = (x+y)(x-y)$ [P.4]

**2.** $\frac{\sqrt{3}}{2} \div \frac{1}{2} = \frac{\sqrt{3}}{2} \cdot \frac{2}{1} = \sqrt{3}$ [P.5]

**3.** $A = \frac{1}{2}bh$ [P.4]
$= \frac{1}{2}(4)(6)$
$= 12 \text{ in}^2$

**4.** $f(-x) = \frac{-x}{(-x)^2+1} = \frac{-x}{x^2+1} = -f(x)$ [2.5]
Odd function

**5.** $f(x) = \frac{x}{2x-3}$ [4.1]
$x = \frac{y}{2y-3}$
$x(2y-3) = 2xy - 3x = y$
$2xy - y = y(2x-1) = 3x$
$y = \frac{3x}{2x-1}$
$f^{-1}(x) = \frac{3x}{2x-1}$

**6.** Domain: $(-\infty, 4) \cup (4, \infty)$ [2.2/3.5]

**7.** Range: [0, 2]
[2.2]

**8.** Shift the graph of $y = f(x)$ horizontally 3 units to the right.
[2.5]

**9.** Reflect the graph of $y = f(x)$ across the $y$-axis. [2.5]

**10.** $300^\circ = 300^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{5\pi}{3}$ [5.1]

**11.** $\frac{5\pi}{4} = \frac{5\pi}{4}\left(\frac{180^\circ}{\pi}\right) = 225^\circ$ [5.1]

**12.** $f\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{2}\right) = 1$ [5.3]

**13.** $f\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{\sqrt{3}+1}{2}$ [5.2]

**14.** $\cos^2 45^\circ + \sin^2 60^\circ + = \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{2}{4} + \frac{3}{4} = \frac{5}{4}$ [5.2]

**15.** negative [5.3]

**16.** $\theta = 210^\circ$ [5.3]
Since $180^\circ < \theta < 270^\circ$,
$\theta' + 180^\circ = \theta$
$\theta' = 30^\circ$

**17.** $\theta = \frac{2\pi}{3}$ [5.3]
Since $\frac{\pi}{2} < \theta < \pi$,
$\theta + \theta' = \pi$
$\theta' = \frac{\pi}{3}$

**18.** Domain: $(-\infty, \infty)$ [5.4]

**19.** Range: [–1, 1] [5.4]

**20.** $\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{3}{4}$

hypotenuse $= \sqrt{3^2 + 4^2}$
$= \sqrt{9+16}$
$= \sqrt{25}$
$= 5$

$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{3}{5}$ [5.2]

# Chapter 6
# Trigonometric Identities and Equations

## Section 6.1

**1.** $\tan x \csc x \cos x = \dfrac{\sin x}{\cos x} \cdot \dfrac{1}{\sin x} \cdot \cos x = 1$

**2.** $$\begin{aligned}\tan x \sec x \sin x &= \tan x \cdot \frac{1}{\cos x} \cdot \sin x\\ &= \tan x \cdot \frac{\sin x}{\cos x}\\ &= \tan x \cdot \tan x\\ &= \tan^2 x\end{aligned}$$

**3.** $\dfrac{4\sin^2 x - 1}{2\sin x + 1} = \dfrac{(2\sin x - 1)(2\sin x + 1)}{2\sin x + 1} = 2\sin x - 1$

**4.** $\dfrac{\sin^2 x - 2\sin x + 1}{\sin x - 1} = \dfrac{(\sin x - 1)^2}{\sin x - 1} = \sin x - 1$

**5.** $$\begin{aligned}(\sin x - \cos x)(\sin x + \cos x) &= \sin^2 x - \cos^2 x\\ &= 1 - \cos^2 x - \cos^2 x\\ &= 1 - 2\cos^2 x\end{aligned}$$

**6.** $$\begin{aligned}(\tan x)(1 - \cot x) &= \tan x - \tan x \cot x\\ &= \tan x - 1\end{aligned}$$

**7.** $$\begin{aligned}\frac{1}{\sin x} - \frac{1}{\cos x} &= \frac{\cos x}{\sin x \cos x} - \frac{\sin x}{\sin x \cos x}\\ &= \frac{\cos x - \sin x}{\sin x \cos x}\end{aligned}$$

**8.** $$\begin{aligned}\frac{1}{\sin x} + \frac{3}{\cos x} &= \frac{\cos x}{\sin x \cos x} + \frac{3\sin x}{\sin x \cos x}\\ &= \frac{\cos x + 3\sin x}{\sin x \cos x}\end{aligned}$$

**9.** $$\begin{aligned}\frac{\cos x}{1 - \sin x} &= \frac{\cos x(1 + \sin x)}{(1 - \sin x)(1 + \sin x)}\\ &= \frac{\cos x(1 + \sin x)}{1 - \sin^2 x}\\ &= \frac{\cos x(1 + \sin x)}{\cos^2 x}\\ &= \frac{(1 + \sin x)}{\cos x} = \frac{1}{\cos x} + \frac{\sin x}{\cos x}\\ &= \sec x + \tan x\end{aligned}$$

**10.** $$\begin{aligned}\frac{\sin x}{1 - \cos x} &= \frac{\sin x(1 + \cos x)}{(1 - \cos x)(1 + \cos x)}\\ &= \frac{\sin x(1 + \cos x)}{1 - \cos^2 x}\\ &= \frac{\sin x(1 + \cos x)}{\sin^2 x}\\ &= \frac{1 + \cos x}{\sin x}\\ &= \frac{1}{\sin x} + \frac{\cos x}{\sin x}\\ &= \csc x + \cot x\end{aligned}$$

**11.** $$\begin{aligned}\frac{1 - \tan^4 x}{\sec^2 x} &= \frac{(1 + \tan^2 x)(1 - \tan^2 x)}{\sec^2 x}\\ &= \frac{\sec^2 x(1 - \tan^2 x)}{\sec^2 x}\\ &= 1 - \tan^2 x\end{aligned}$$

**12.** $$\begin{aligned}\sin^4 x - \cos^4 x &= (\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)\\ &= 1(\sin^2 x - \cos^2 x)\\ &= \sin^2 x - \cos^2 x\end{aligned}$$

**13.** $$\begin{aligned}\frac{1 + \tan^3 x}{1 + \tan x} &= \frac{(1 + \tan x)(1 - \tan x + \tan^2 x)}{1 + \tan x}\\ &= 1 - \tan x + \tan^2 x\end{aligned}$$

**14.** $$\begin{aligned}\frac{\cos x \tan x - \sin x}{\cot x} &= \frac{\cos x\left(\frac{\sin x}{\cos x}\right) - \sin x}{\cot x}\\ &= \frac{\sin x - \sin x}{\cot x}\\ &= 0\end{aligned}$$

**15.** $$\frac{\sin x - 2 + \frac{1}{\sin x}}{\sin x - \frac{1}{\sin x}} = \frac{\sin x - 2 + \frac{1}{\sin x}}{\sin x - \frac{1}{\sin x}} \cdot \frac{\sin x}{\sin x} = \frac{\sin^2 x - 2\sin x + 1}{\sin^2 x - 1} = \frac{(\sin x - 1)(\sin x - 1)}{(\sin x - 1)(\sin x + 1)} = \frac{\sin x - 1}{\sin x + 1}$$

**16.** $$\frac{\sin x}{1 - \cos x} - \frac{\sin x}{1 + \cos x} = \frac{\sin x(1 + \cos x) - \sin x(1 - \cos x)}{(1 - \cos x)(1 + \cos x)} = \frac{\sin x + \sin x \cos x - \sin x + \sin x \cos x}{1 - \cos^2 x} = \frac{2\sin x \cos x}{\sin^2 x} = \frac{2\cos x}{\sin x} = 2\cot x$$

**17.** $$(\sin x + \cos x)^2 = \sin^2 x + 2\sin x\cos x + \cos^2 x = \sin^2 x + \cos^2 x + 2\sin x \cos x = 1 + 2\sin x\cos x$$

**18.** $$(\tan x + 1)^2 = \tan^2 x + 2\tan x + 1 = 1 + \tan^2 x + 2\tan x = \sec^2 x + 2\tan x$$

**19.** $$\frac{\cos x}{1 + \sin x} = \frac{\cos x(1 - \sin x)}{(1 + \sin x)(1 - \sin x)} = \frac{\cos x(1 - \sin x)}{1 - \sin^2 x} = \frac{\cos x(1 - \sin x)}{\cos^2 x} = \frac{1 - \sin x}{\cos x} = \frac{1}{\cos x} - \frac{\sin x}{\cos x} = \sec x - \tan x$$

**20.** $$\frac{\sin x}{1 + \cos x} = \frac{\sin x(1 - \cos x)}{(1 + \cos x)(1 - \cos x)} = \frac{\sin x(1 - \cos x)}{1 - \cos^2 x} = \frac{\sin x(1 - \cos x)}{\sin^2 x} = \frac{1 - \cos x}{\sin x} = \frac{1}{\sin x} - \frac{\cos x}{\sin x} = \csc x - \cot x$$

**21.** $$\frac{\cot x + \tan x}{\sec x} = \frac{\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}}{\frac{1}{\cos x}} = \frac{\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}}{\frac{1}{\cos x}} \cdot \frac{\sin x \cos x}{\sin x \cos x} = \frac{\cos^2 x + \sin^2 x}{\sin x} = \frac{1}{\sin x} = \csc x$$

**22.** $$\frac{\cot x + \tan x}{\csc x} = \frac{\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}}{\frac{1}{\sin x}} = \frac{\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}}{\frac{1}{\sin x}} \cdot \frac{\sin x \cos x}{\sin x \cos x} = \frac{\cos^2 x + \sin^2 x}{\cos x} = \frac{1}{\cos x} = \sec x$$

**23.** $$\frac{\cos x \tan x + 2\cos x - \tan x - 2}{\tan x + 2} = \frac{\cos x(\tan x + 2) - (\tan x + 2)}{\tan x + 2} = \frac{(\tan x + 2)(\cos x - 1)}{\tan x + 2} = \cos x - 1$$

**24.** $$\frac{2\sin x \cot x + \sin x - 4\cot x - 2}{2\cot x + 1} = \frac{\sin x(2\cot x + 1) - 2(2\cot x + 1)}{2\cot x + 1} = \frac{(2\cot x + 1)(\sin x - 2)}{2\cot x + 1} = \sin x - 2$$

**25.** $\frac{1-\sin x}{\cos x}=\frac{1}{\cos x}-\frac{\sin x}{\cos x}=\sec x-\tan x$

**26.** $\frac{\cos x-1}{\sin x}=\frac{\cos x}{\sin x}-\frac{1}{\sin x}=\cot x-\csc x$

**27.** $$\begin{aligned}\sin^2 x-\cos^2 x&=\sin^2 x-(1-\sin^2 x)\\&=\sin^2 x-1+\sin^2 x\\&=2\sin^2 x-1\end{aligned}$$

**28.** $$\begin{aligned}\sin^2 x-\cos^2 x&=1-\cos^2 x-\cos^2 x\\&=1-2\cos^2 x\end{aligned}$$

**29.** $$\begin{aligned}\frac{1}{\sin^2 x}+\frac{1}{\cos^2 x}&=\frac{\cos^2 x+\sin^2 x}{\sin^2 x\cos^2 x}\\&=\frac{1}{\sin^2 x\cos^2 x}\\&=\csc^2 x\sec^2 x\end{aligned}$$

**30.** $$\begin{aligned}\frac{1}{\tan^2 x}-\frac{1}{\cot^2 x}&=\frac{\cot^2 x-\tan^2 x}{\tan^2 x\cot^2 x}\\&=\frac{(\csc^2 x-1)-(\sec^2 x-1)}{1}\\&=\csc^2 x-1-\sec^2 x+1\\&=\csc^2 x-\sec^2 x\end{aligned}$$

**31.** $$\begin{aligned}\sec x-\cos x&=\frac{1}{\cos x}-\cos x\\&=\frac{1-\cos^2 x}{\cos x}\\&=\frac{\sin^2 x}{\cos x}\\&=\sin x\tan x\end{aligned}$$

**32.** $$\begin{aligned}\tan x+\cot x&=\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}\\&=\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}\\&=\frac{1}{\sin x\cos x}\\&=\csc x\sec x\end{aligned}$$

**33.** $$\begin{aligned}\frac{\frac{1}{\sin x}+1}{\frac{1}{\sin x}-1}&=\frac{\frac{1}{\sin x}+1}{\frac{1}{\sin x}-1}\cdot\frac{\sin x}{\sin x}\\&=\frac{1+\sin x}{1-\sin x}\\&=\frac{(1+\sin x)}{1-\sin x}\cdot\frac{(1+\sin x)}{1+\sin x}\\&=\frac{1+2\sin x+\sin^2 x}{1-\sin^2 x}\\&=\frac{1+2\sin x+\sin^2 x}{\cos^2 x}\\&=\frac{1}{\cos^2 x}+\frac{2\sin x}{\cos^2 x}+\frac{\sin^2 x}{\cos^2 x}\\&=\sec^2 x+2\tan x\sec x+\tan^2 x\end{aligned}$$

**34.** $$\begin{aligned}\frac{\frac{1}{\sin x}+\frac{1}{\cos x}}{\frac{1}{\sin x}-\frac{1}{\cos x}}&=\frac{\frac{1}{\sin x}+\frac{1}{\cos x}}{\frac{1}{\sin x}-\frac{1}{\cos x}}\cdot\frac{\sin x\cos x}{\sin x\cos x}\\&=\frac{\cos x+\sin x}{\cos x-\sin x}\\&=\frac{\cos x+\sin x}{\cos x-\sin x}\cdot\frac{\cos x-\sin x}{\cos x-\sin x}\\&=\frac{\cos^2 x-\sin^2 x}{\cos^2 x-2\sin x\cos x+\sin^2 x}\\&=\frac{\cos^2 x-\sin^2 x}{1-2\sin x\cos x}\end{aligned}$$

**35.** $$\begin{aligned}\sin^4 x-\cos^4 x&=(\sin^2 x+\cos^2 x)(\sin^2 x-\cos^2 x)\\&=1(\sin^2 x-\cos^2 x)\\&=\sin^2 x-(1-\sin^2 x)\\&=\sin^2 x-1+\sin^2 x\\&=2\sin^2 x-1\end{aligned}$$

**36.** $$\begin{aligned}\sin^6 x+\cos^6 x&=(\sin^2 x+\cos^2 x)(\sin^4 x-\sin^2 x\cos^2 x+\cos^4 x)\\&=\sin^4 x-\sin^2 x\cos^2 x+\cos^4 x\end{aligned}$$

**37.** $$\frac{1}{1-\cos x}=\frac{1}{1-\cos x}\cdot\frac{1+\cos x}{1+\cos x}=\frac{1+\cos x}{1-\cos^2 x}=\frac{1+\cos x}{\sin^2 x}$$

**38.** $$\frac{\cos^2 x}{1-\sin x}=\frac{1-\sin^2 x}{1-\sin x}=\frac{(1-\sin x)(1+\sin x)}{1-\sin x}=1+\sin x$$

**39.** $$\frac{\sin x}{1-\sin x}-\frac{\cos x}{1-\sin x}=\frac{\sin x-\cos x}{1-\sin x}=\frac{\frac{\sin x}{\sin x}-\frac{\cos x}{\sin x}}{\frac{1}{\sin x}-\frac{\sin x}{\sin x}}=\frac{1-\cot x}{\csc x-1}$$

**40.** $$\frac{\tan x}{1+\tan x}-\frac{\cot x}{1+\tan x}=\frac{\tan x-\cot x}{1+\tan x}=\frac{\frac{\tan x}{\tan x}-\frac{\cot x}{\tan x}}{\frac{1}{\tan x}+\frac{\tan x}{\tan x}}=\frac{1-\cot^2 x}{\cot x+1}=\frac{(1-\cot x)(1+\cot x)}{\cot x+1}=1-\cot x$$

**41.** $$\frac{1}{1+\cos x}-\frac{1}{1-\cos x}=\frac{(1-\cos x)-(1+\cos x)}{(1+\cos x)(1-\cos x)}=\frac{1-\cos x-1-\cos x}{1-\cos^2 x}=\frac{-2\cos x}{\sin^2 x}=-2\cot x\csc x$$

**42.** $$\frac{1}{1-\sin x}-\frac{1}{1+\sin x}=\frac{(1+\sin x)-(1-\sin x)}{(1-\sin x)(1+\sin x)}=\frac{1+\sin x-1+\sin x}{1-\sin^2 x}=\frac{2\sin x}{\cos^2 x}=2\tan x\sec x$$

**43.** $$\frac{\frac{1}{\sin x}+\csc x}{\frac{1}{\sin x}-\sin x}=\frac{\frac{1}{\sin x}+\csc x}{\frac{1}{\sin x}-\sin x}\cdot\frac{\sin x}{\sin x}=\frac{1+1}{1-\sin^2 x}=\frac{2}{\cos^2 x}$$

**44.** $$\frac{2\cot x}{\cot x+\tan x}=\frac{\tan x}{\tan x}\left(\frac{\frac{2}{\tan x}}{\frac{1}{\tan x}+\tan x}\right)=\frac{2}{1+\tan^2 x}=\frac{2}{\sec^2 x}=2\cos^2 x$$

**45.** $$\frac{\cot x}{1+\csc x}+\frac{1+\csc x}{\cot x}=\frac{\frac{\cos x}{\sin x}}{1+\frac{1}{\sin x}}+\frac{1+\frac{1}{\sin x}}{\frac{\cos x}{\sin x}}=\frac{\cos x}{\sin x+1}+\frac{\sin x+1}{\cos x}=\frac{\cos^2 x+(\sin x+1)^2}{\cos x(\sin x+1)}=\frac{\cos^2 x+\sin^2 x+2\sin x+1}{\cos x(\sin x+1)}=\frac{1+2\sin x+1}{\cos x(\sin x+1)}=\frac{2(1+\sin x)}{\cos x(\sin x+1)}=2\sec x$$

**46.** $$\sec^2 x-\csc^2 x=\frac{1}{\cos^2 x}-\frac{1}{\sin^2 x}=\frac{\sin^2 x-\cos^2 x}{\sin^2 x\cos^2 x}=\frac{\frac{\sin^2 x}{\sin x\cos x}-\frac{\cos^2 x}{\sin x\cos x}}{\frac{\sin^2 x\cos^2 x}{\sin x\cos x}}=\frac{\frac{\sin x}{\cos x}-\frac{\cos x}{\sin x}}{\sin x\cos x}=\frac{\tan x-\cot x}{\sin x\cos x}$$

**47.** $$\sqrt{\frac{1+\sin x}{1-\sin x}} = \sqrt{\frac{1+\sin x}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x}} = \sqrt{\frac{(1+\sin x)^2}{1-\sin^2 x}} = \sqrt{\frac{(1+\sin x)^2}{\cos^2 x}} = \frac{1+\sin x}{\cos x},\ \cos x > 0$$

**48.** $$\frac{\cos x + \cot x \sin x}{\cot x} = \frac{\cos x + \frac{\cos x}{\sin x}\cdot \sin x}{\cot x} = \frac{2\cos x}{\cos x/\sin x} = 2\sin x$$

**49.** $$\frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = \frac{(\sin x + \cos x)(\sin^2 x - \sin x\cos x + \cos^2 x)}{\sin x + \cos x} = \sin^2 x - \sin x \cos x + \cos^2 x = 1 - \sin x \cos x$$

**50.** $$\frac{1-\sin x}{1+\sin x} - \frac{1+\sin x}{1-\sin x} = \frac{(1-\sin x)^2 - (1-\sin x)^2}{(1+\sin x)(1-\sin x)} = \frac{1-2\sin x + \sin^2 x - 1 - 2\sin x - \sin^2 x}{1-\sin^2 x} = \frac{-4\sin x}{\cos^2 x} = -4\tan x \sec x$$

**51.** $$\frac{\sec x - 1}{\sec x + 1} - \frac{\sec x + 1}{\sec x - 1} = \frac{(\sec x - 1)^2 - (\sec x + 1)^2}{(\sec x - 1)(\sec x + 1)} = \frac{\sec^2 x - 2\sec x + 1 - \sec^2 x - 2\sec x - 1}{\sec^2 x - 1} = \frac{-4\sec x}{\tan^2 x} = \frac{-4}{\cos x}\cdot\frac{\cos^2 x}{\sin^2 x} = -4\csc x \cot x$$

**52.** $$\frac{1}{1-\cos x} - \frac{\cos x}{1+\cos x} = \frac{1+\cos x - \cos x(1-\cos x)}{(1-\cos x)(1+\cos x)} = \frac{1+\cos x - \cos x + \cos^2 x}{1-\cos^2 x} = \frac{1+\cos^2 x}{\sin^2 x} = \frac{1+1-\sin^2 x}{\sin^2 x} = \frac{2}{\sin^2 x} - \frac{\sin^2 x}{\sin^2 x} = 2\csc^2 x - 1$$

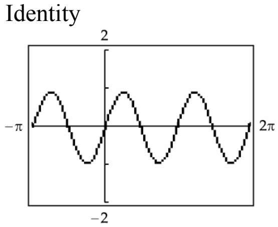
**53.** $$\frac{1+\sin x}{\cos x} - \frac{\cos x}{1-\sin x} = \frac{(1+\sin x)(1-\sin x) - \cos x(\cos x)}{\cos x(1-\sin x)} = \frac{1-\sin^2 x - \cos^2 x}{\cos x(1-\sin x)} = \frac{\cos^2 x - \cos^2 x}{\cos x(1-\sin x)} = 0$$

**54.** $$(\sin x + \cos x + 1)^2 = \sin^2 x + \sin x\cos x + \sin x + \cos x \sin x + \cos^2 x + \cos x + \sin x + \cos x + 1 = 1 + 2\sin x\cos x + 2\sin x + 2\cos x + 1 = 2(\sin x\cos x + \cos x + \sin x + 1) = 2(\sin x + 1)(\cos x + 1)$$

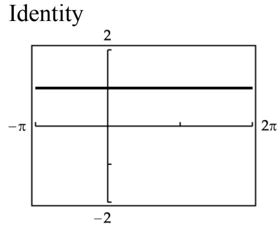
**55.** $$\frac{\sec x + \tan x}{\sec x - \tan x} = \frac{\frac{1}{\cos x} + \frac{\sin x}{\cos x}}{\frac{1}{\cos x} - \frac{\sin x}{\cos x}}\cdot\frac{\cos x}{\cos x} = \frac{1+\sin x}{1-\sin x} = \frac{1+\sin x}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x} = \frac{(1+\sin x)^2}{1-\sin^2 x} = \frac{(1+\sin x)^2}{\cos^2 x}$$

**56.** $$\begin{aligned}\frac{\sin^3 x-\cos^3 x}{\sin x+\cos x}&=\frac{(\sin x-\cos x)(\sin^2 x+\sin x\cos x+\cos^2 x)}{\sin x+\cos x}\\&=\frac{\sin x-\cos x}{\sin x-\cos x}\cdot\frac{(\sin x-\cos x)(1+\sin x\cos x)}{\sin x+\cos x}\\&=\frac{(\sin^2 x-2\sin x\cos x+\cos^2 x)(1+\sin x\cos x)}{\sin^2 x-\cos^2 x}\\&=\frac{(1-2\sin x\cos x)(1+\sin x\cos x)}{\sin^2 x-\cos^2 x}\\&=\frac{1-\sin x\cos x-2\sin^2 x\cos^2 x}{\sin^2 x-\cos^2 x}\\&=\frac{\frac{1}{\sin^2 x}-\frac{\sin x\cos x}{\sin^2 x}-\frac{2\sin^2 x\cos^2 x}{\sin^2 x}}{\frac{\sin^2 x}{\sin^2 x}-\frac{\cos^2 x}{\sin^2 x}}\\&=\frac{\csc^2 x-\cot x-2\cos^2 x}{1-\cot^2 x}\end{aligned}$$

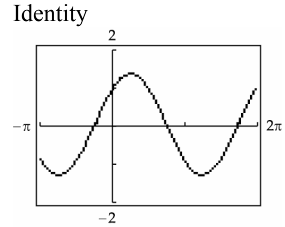
**57.** Identity



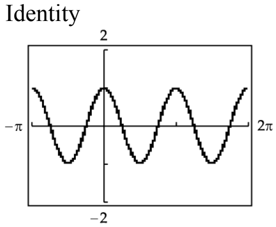

**58.** Identity



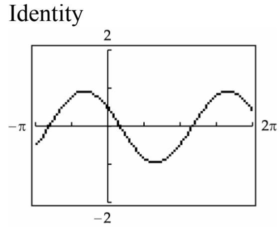

**59.** Identity



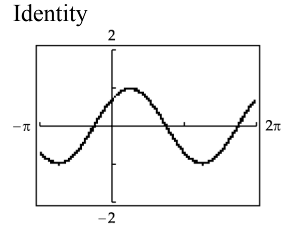

**60.** Identity



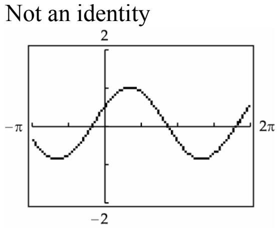

**61.** Identity



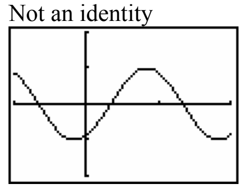

**62.** Identity




**63.** Not an identity




**64.** Not an identity



**65.** Not an identity. If $x=\pi/4$, the left side is 2 and the right side is 1.

**66.** Not an identity. If $x=\pi/6$, the left side is $\sqrt{3}$ and the right side is $2\sqrt{3}/3$.

**67.** Not an identity. If $x=0°$, the left side is $\sqrt{3}/2$ and the right side is $(2+\sqrt{3})/2$.

**68.** Not an identity. If $x=\pi$, the left side is 1 and the right side is –1.

**69.** Not an identity. If $x=0$, the left side is –1 and the right side is 1.

**70.** Not an identity. If $x=\pi$, the left side is 1 and the right side is –1.

## Connecting Concepts

**71.**
$$\frac{1-\sin x+\cos x}{1+\sin x+\cos x}=\frac{1-\sin x+\cos x}{1+\sin x+\cos x}\cdot\frac{1+\sin x+\cos x}{1+\sin x-\cos x}$$
$$=\frac{1-\sin^2 x+2\sin x\cos x-\cos^2 x}{1+2\sin x+\sin^2 x-\cos^2 x}$$
$$=\frac{1-(\sin^2 x+\cos^2 x)+2\sin x\cos x}{1+2\sin x+\sin^2 x-(1-\sin^2 x)}$$
$$=\frac{1-1+2\sin x\cos x}{1+2\sin x+\sin^2 x-1+\sin^2 x}$$
$$=\frac{2\sin x\cos x}{2\sin x+2\sin^2 x}=\frac{2\sin x\cos x}{2\sin x(1+\sin x)}$$
$$=\frac{\cos x}{1+\sin x}$$

**72.**
$$\frac{1-\tan x+\sec x}{1+\tan x-\sec x}=\frac{(1-\tan x+\sec x)(1+\tan x+\sec x)}{(1+\tan x-\sec x)(1+\tan x+\sec x)}$$
$$=\frac{1+2\sec x-\tan^2 x+\sec^2 x}{1+2\tan x+\tan^2 x-\sec^2 x}$$
$$=\frac{1+2\sec x-(\sec^2 x-1)+\sec^2 x}{1+2\tan x+\tan^2 x-(\tan^2 x+1)}$$
$$=\frac{2+2\sec x}{2\tan x}$$
$$=\frac{1+\sec x}{\tan x}$$

**73.**
$$\frac{2\sin^4 x+2\sin^2 x\cos^2 x-3\sin^2 x-3\cos^2 x}{2\sin^2 x}=\frac{2\sin^2 x(\sin^2 x+\cos^2 x)-3(\sin^2 x+\cos^2 x)}{2\sin^2 x}$$
$$=\frac{(2\sin^2 x-3)(\sin^2 x+\cos^2 x)}{2\sin^2 x}$$
$$=\frac{2\sin^2 x-3}{2\sin^2 x}$$
$$=\frac{2\sin^2 x}{2\sin^2 x}-\frac{3}{2\sin^2 x}$$
$$=1-\frac{3}{2}\csc^2 x$$

**74.**
$$\frac{4\tan x\sec^2 x-4\tan x-\sec^2 x+1}{4\tan^3 x-\tan^2 x}=\frac{4\tan x(\sec^2 x-1)-(\sec^2 x-1)}{4\tan^3 x-\tan^2 x}$$
$$=\frac{(4\tan x-1)(\sec^2 x-1)}{\tan^2 x(4\tan x-1)}$$
$$=\frac{\tan^2 x}{\tan^2 x}$$
$$=1$$

**75.**
$$\frac{\sin x(\tan x+1)-2\tan x\cos x}{\sin x-\cos x}=\frac{\sin x\tan x+\sin x-2\frac{\sin x}{\cos x}\cos x}{\sin x-\cos x}$$
$$=\frac{\sin x\tan x+\sin x-2\sin x}{\sin x-\cos x}$$
$$=\frac{\sin x(\tan x-1)}{\sin x-\cos x}$$
$$=\frac{\frac{\sin x(\tan x-1)}{\cos x}}{\frac{\sin x}{\cos x}-\frac{\cos x}{\cos x}}$$
$$=\frac{\tan x(\tan x-1)}{\tan x-1}$$
$$=\tan x$$

**76.** $$\frac{\sin^2 x\cos x+\cos^3 x-\sin^3 x\cos x-\sin x\cos^3 x}{1-\sin^2 x}=\frac{\cos x(\sin^2 x+\cos^2 x)-\sin x\cos x(\sin^2 x+\cos^2 x)}{1-\sin^2 x}$$
$$=\frac{\cos x-\sin x\cos x}{1-\sin^2 x}$$
$$=\frac{\cos x(1-\sin x)}{(1-\sin x)(1+\sin x)}$$
$$=\frac{\cos x}{1+\sin x}$$

**77.** $$\sin^4 x+\cos^4 x=\sin^4 x+2\sin^2 x\cos^2 x+\cos^4 x-2\sin^2 x\cos^2 x$$
$$=(\sin^2 x+\cos^2 x)^2-2\sin^2 x\cos^2 x$$
$$=1-2\sin^2 x\cos^2 x$$

**78.** $$\tan^4 x+\sec^4 x=\tan^4 x-2\tan^2 x\sec^2 x+\sec^4 x+2\tan^2 x\sec^2 x$$
$$=(\tan^2 x-\sec^2 x)^2+2\tan^2 x\sec^2 x$$
$$=1+2\tan^2 x\sec^2 x$$

## Prepare for Section 6.2

**PS1.** $$\cos\left(\frac{\pi}{2}-\frac{\pi}{6}\right)=\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$$
$$\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{6}\right)+\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{6}\right)=0\cdot\frac{\sqrt{3}}{2}+1\cdot\frac{1}{2}=\frac{1}{2}$$
Both functional values equal $\frac{1}{2}$.

**PS2.** $$\sin\left(\frac{\pi}{2}+\frac{\pi}{3}\right)=\sin\left(\frac{5\pi}{6}\right)=\frac{1}{2}$$
$$\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{3}\right)+\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{3}\right)=1\cdot\frac{1}{2}+0\cdot\frac{\sqrt{3}}{2}=\frac{1}{2}$$
Both functional values equal to $\frac{1}{2}$.

**PS3.** $$\sin(90^\circ-30^\circ)=\sin(60^\circ)=\frac{\sqrt{3}}{2}=\cos(30^\circ)$$
$$\sin(90^\circ-45^\circ)=\sin(45^\circ)=\frac{\sqrt{2}}{2}=\cos(45^\circ)$$
$$\sin(90^\circ-120^\circ)=\sin(-30^\circ)=-\frac{1}{2}=\cos(120^\circ)$$
For each of the given values of $\theta$, the functional values are equal.

**PS4.** $$\tan\left(\frac{\pi}{2}-\frac{\pi}{6}\right)=\tan\left(\frac{\pi}{3}\right)=\cot\left(\frac{\pi}{6}\right)$$
$$\tan\left(\frac{\pi}{2}-\frac{\pi}{4}\right)=\tan\left(\frac{\pi}{4}\right)=\cot\left(\frac{\pi}{4}\right)$$
$$\tan\left(\frac{\pi}{2}-\frac{4\pi}{3}\right)=\tan\left(-\frac{5\pi}{6}\right)=\cot\left(\frac{4\pi}{3}\right)$$
For each of the given values of $\theta$, the functional values are equal.

**PS5.** $$\tan\left(\frac{\pi}{3}-\frac{\pi}{6}\right)=\tan\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{3}$$
$$\frac{\tan\left(\frac{\pi}{3}\right)-\tan\left(\frac{\pi}{6}\right)}{1+\tan\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{6}\right)}=\frac{\sqrt{3}-\frac{\sqrt{3}}{3}}{1+\sqrt{3}\cdot\frac{\sqrt{3}}{3}}=\frac{\frac{3\sqrt{3}}{3}-\frac{\sqrt{3}}{3}}{1+1}=\frac{2\frac{\sqrt{3}}{3}}{2}=\frac{\sqrt{3}}{3}$$
Both functional values equal $\frac{\sqrt{3}}{3}$.

**PS6.** For $k$ is any integer, the value of $(2k + 1)$ will result in odd integers.
Thus $\sin[(2k+1)\pi]$ will be 0.

# Section 6.2

**1.** $\sin(45°+30°) = \sin 45° \cos 30° + \cos 45° \sin 30°$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$$

$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

**2.** $\sin(330°+45°) = \sin 330° \cos 45° + \cos 330° \sin 45°$

$$= -\frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}$$

$$= -\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{-\sqrt{2}+\sqrt{6}}{4}$$

**3.** $\cos(45°-30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$$

$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

**4.** $\cos(120°-45°) = \cos 120° \cos 45° + \sin 120° \sin 45°$

$$= -\frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}$$

$$= -\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{-\sqrt{2}+\sqrt{6}}{4}$$

**5.** $\tan(45°-30°) = \dfrac{\tan 45° - \tan 30°}{1+\tan 45° \tan 30°}$

$$= \frac{1-\frac{\sqrt{3}}{3}}{1+1\left(\frac{\sqrt{3}}{3}\right)} = \frac{\frac{3-\sqrt{3}}{3}}{\frac{3+\sqrt{3}}{3}} = \frac{3-\sqrt{3}}{3+\sqrt{3}} = 2-\sqrt{3}$$

**6.** $\tan(240°-45°) = \dfrac{\tan 240° - \tan 45°}{1+\tan 240° \tan 45°}$

$$= \frac{\sqrt{3}-1}{1+(\sqrt{3})(1)} = \frac{\sqrt{3}-1}{1+\sqrt{3}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$= \frac{(\sqrt{3}-1)^2}{3-1} = \frac{4-2\sqrt{3}}{2} = 2-\sqrt{3}$$

**7.** $\sin\left(\dfrac{5\pi}{4}-\dfrac{\pi}{6}\right) = \sin\dfrac{5\pi}{4}\cos\dfrac{\pi}{6} - \cos\dfrac{5\pi}{4}\sin\dfrac{\pi}{6}$

$$= -\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{2}}{2}\right)\cdot\frac{1}{2}$$

$$= -\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{-\sqrt{6}+\sqrt{2}}{4}$$

**8.** $\sin\left(\dfrac{4\pi}{3}+\dfrac{\pi}{4}\right) = \sin\dfrac{4\pi}{3}\cos\dfrac{\pi}{4} + \cos\dfrac{4\pi}{3}\sin\dfrac{\pi}{4}$

$$= -\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \left(-\frac{1}{2}\right)\cdot\frac{\sqrt{2}}{2}$$

$$= -\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = -\frac{\sqrt{6}+\sqrt{2}}{4}$$

**9.** $\cos\left(\dfrac{3\pi}{4}+\dfrac{\pi}{6}\right) = \cos\dfrac{3\pi}{4}\cos\dfrac{\pi}{6} - \sin\dfrac{3\pi}{4}\sin\dfrac{\pi}{6}$

$$= -\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$$

$$= -\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = -\frac{\sqrt{6}+\sqrt{2}}{4}$$

**10.** $\cos\left(\dfrac{\pi}{4}-\dfrac{\pi}{3}\right) = \cos\dfrac{\pi}{4}\cos\dfrac{\pi}{3} + \sin\dfrac{\pi}{4}\sin\dfrac{\pi}{3}$

$$= \frac{\sqrt{2}}{2}\cdot\frac{1}{2} + \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

**11.** $\tan\left(\dfrac{\pi}{6}+\dfrac{\pi}{4}\right) = \dfrac{\tan\frac{\pi}{6}+\tan\frac{\pi}{4}}{1-\tan\frac{\pi}{6}\tan\frac{\pi}{4}}$

$$= \frac{\frac{\sqrt{3}}{3}+1}{1-\frac{\sqrt{3}}{3}\cdot 1} = \frac{\frac{\sqrt{3}+3}{3}}{\frac{3-\sqrt{3}}{3}}$$

$$= \frac{\sqrt{3}+3}{3-\sqrt{3}}\cdot\frac{3+\sqrt{3}}{3+\sqrt{3}} = \frac{9+6\sqrt{3}+3}{9-3}$$

$$= \frac{12+6\sqrt{3}}{6} = 2+\sqrt{3}$$

**12.** $\tan\left(\dfrac{11\pi}{6}-\dfrac{\pi}{4}\right) = \dfrac{\tan\frac{11\pi}{6}-\tan\frac{\pi}{4}}{1+\tan\frac{11\pi}{6}\tan\frac{\pi}{4}}$

$$= \frac{-\frac{\sqrt{3}}{3}-1}{1+\left(-\frac{\sqrt{3}}{3}\right)(1)} = \frac{-\frac{\sqrt{3}}{3}-1}{1-\frac{\sqrt{3}}{3}} = \frac{-\sqrt{3}-3}{3-\sqrt{3}}$$

$$= \frac{(-3-\sqrt{3})(3+\sqrt{3})}{(3-\sqrt{3})(3+\sqrt{3})}$$

$$= \frac{-9-6\sqrt{3}-3}{9-3} = \frac{-12-6\sqrt{3}}{6}$$

$$= -2-\sqrt{3}$$

**13.** $\cos 212°\cos 122° + \sin 212°\sin 122° = \cos(212° - 122°) = \cos 90° = 0$

**14.** $\sin 167°\cos 107° - \cos 167°\sin 107° = \sin(167° - 107°) = \sin 60° = \dfrac{\sqrt{3}}{2}$

**15.** $\sin\dfrac{5\pi}{12}\cos\dfrac{\pi}{4} - \cos\dfrac{5\pi}{12}\sin\dfrac{\pi}{4} = \sin\left(\dfrac{5\pi}{12} - \dfrac{\pi}{4}\right) = \sin\dfrac{\pi}{6} = \dfrac{1}{2}$

**16.** $\cos\dfrac{\pi}{12}\cos\dfrac{\pi}{4} - \sin\dfrac{\pi}{12}\sin\dfrac{\pi}{4} = \cos\left(\dfrac{\pi}{12} + \dfrac{\pi}{4}\right) = \cos\dfrac{\pi}{3} = \dfrac{1}{2}$

**17.** $\dfrac{\tan\frac{7\pi}{12} - \tan\frac{\pi}{4}}{1 + \tan\frac{7\pi}{12}\tan\frac{\pi}{4}} = \tan\left(\dfrac{7\pi}{12} - \dfrac{\pi}{4}\right) = \tan\dfrac{\pi}{3} = \sqrt{3}$

**18.** $\dfrac{\tan\frac{\pi}{6} + \tan\frac{\pi}{3}}{1 - \tan\frac{\pi}{6}\tan\frac{\pi}{3}} = \tan\left(\dfrac{\pi}{6} + \dfrac{\pi}{3}\right) = \tan\dfrac{\pi}{2} = \text{undefined}$

**19.** $\sin 42^\circ = \cos(90^\circ - 42^\circ)$
$= \cos 48^\circ$

**20.** $\cos 80^\circ = \sin(90^\circ - 80^\circ)$
$= \sin 10^\circ$

**21.** $\tan 15^\circ = \cot(90^\circ - 15^\circ)$
$= \cot 75^\circ$

**22.** $\cot 2^\circ = \tan(90^\circ - 2^\circ)$
$= \tan 88^\circ$

**23.** $\sec 25^\circ = \csc(90^\circ - 25^\circ)$
$= \csc 65^\circ$

**24.** $\csc 84^\circ = \sec(90^\circ - 84^\circ)$
$= \sec 6^\circ$

**25.** $\sin 7x\cos 2x - \cos 7x\sin 2x = \sin(7x - 2x) = \sin 5x$

**26.** $\sin x\cos 3x + \cos x\sin 3x = \sin(x + 3x) = \sin 4x$

**27.** $\cos x\cos 2x + \sin x\sin 2x = \cos(x - 2x) = \cos(-x) = \cos x$

**28.** $\cos 4x\cos 2x - \sin 4x\sin 2x = \cos(4x + 2x) = \cos 6x$

**29.** $\sin 7x\cos 3x - \cos 7x\sin 3x = \sin(7x - 3x) = \sin 4x$

**30.** $\cos x\cos 5x - \sin x\sin 5x = \cos(x + 5x) = \cos 6x$

**31.**
$$\begin{aligned}\cos 4x\cos(-2x) - \sin 4x\sin(-2x) &= \cos 4x\cos 2x + \sin 4x\sin 2x \\ &= \cos(4x - 2x) \\ &= \cos 2x\end{aligned}$$

**32.**
$$\begin{aligned}\sin(-x)\cos 3x - \cos(-x)\sin 3x &= -\sin x\cos 3x - \cos x\sin 3x \\ &= -(\sin x\cos 3x + \cos x\sin 3x) \\ &= -\sin(x + 3x) \\ &= -\sin 4x\end{aligned}$$

**33.** $\sin\dfrac{x}{3}\cos\dfrac{2x}{3} + \cos\dfrac{x}{3}\sin\dfrac{2x}{3} = \sin\left(\dfrac{x}{3} + \dfrac{2x}{3}\right) = \sin x$

**34.** $\cos\dfrac{3x}{4}\cos\dfrac{x}{4} + \sin\dfrac{3x}{4}\sin\dfrac{x}{4} = \cos\left(\dfrac{3x}{4} - \dfrac{x}{4}\right) = \cos\dfrac{x}{2}$

**35.** $\dfrac{\tan 3x + \tan 4x}{1 - \tan 3x\tan 4x} = \tan(3x + 4x) = \tan 7x$

**36.** $\dfrac{\tan 2x - \tan 3x}{1 + \tan 2x\tan 3x} = \tan(2x - 3x) = \tan(-x) = -\tan x$

**37.** $\tan\alpha = -\frac{4}{3},\ \sin\alpha = \frac{4}{5},\ \cos\alpha = -\frac{3}{5},$

$\tan\beta = \frac{15}{8},\ \sin\beta = -\frac{15}{17},\ \cos\beta = -\frac{8}{17}$

**a.** $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$$= \left(\frac{4}{5}\right)\left(-\frac{8}{17}\right) - \left(-\frac{3}{5}\right)\left(-\frac{15}{17}\right)$$

$$= -\frac{32}{85} - \frac{45}{85} = -\frac{77}{85}$$

**b.** $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= \left(-\frac{3}{5}\right)\left(-\frac{8}{17}\right) - \left(\frac{4}{5}\right)\left(-\frac{15}{17}\right)$$

$$= \frac{24}{85} + \frac{60}{85} = \frac{84}{85}$$

**c.** $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

$$= \frac{-\frac{4}{3} - \frac{15}{8}}{1 + \left(-\frac{4}{3}\right)\left(\frac{15}{8}\right)} = \frac{-\frac{4}{3} - \frac{15}{8}}{1 - \frac{60}{24}} \cdot \frac{24}{24}$$

$$= \frac{-32 - 45}{24 - 60} = \frac{77}{36}$$

**38.** $\tan\alpha = \frac{24}{7},\ \sin\alpha = \frac{24}{25},\ \cos\alpha = \frac{7}{25},$

$\sin\beta = -\frac{8}{17},\ \cos\beta = -\frac{15}{17},\ \tan\beta = \frac{8}{15}$

**a.** $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \left(\frac{24}{25}\right)\left(-\frac{15}{17}\right) + \left(\frac{7}{25}\right)\left(-\frac{8}{17}\right)$$

$$= -\frac{360}{425} - \frac{56}{425} = -\frac{416}{425}$$

**b.** $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= \left(\frac{7}{25}\right)\left(-\frac{15}{17}\right) - \left(\frac{24}{25}\right)\left(-\frac{8}{17}\right)$$

$$= -\frac{105}{425} + \frac{192}{425} = \frac{87}{425}$$

**c.** $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

$$= \frac{\frac{24}{7} - \frac{8}{15}}{1 + \left(\frac{24}{7}\right)\left(\frac{8}{15}\right)} = \frac{\frac{24}{7} - \frac{8}{15}}{1 + \frac{192}{105}} \cdot \frac{105}{105}$$

$$= \frac{360 - 56}{105 + 192} = \frac{304}{297}$$

**39.** $\sin\alpha = \frac{3}{5},\ \cos\alpha = \frac{4}{5},\ \tan\alpha = \frac{3}{4},$

$\cos\beta = -\frac{5}{13},\ \sin\beta = \frac{12}{13},\ \tan\beta = -\frac{12}{5}$

**a.** $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$$= \frac{3}{5}\left(-\frac{5}{13}\right) - \frac{4}{5}\left(\frac{12}{13}\right)$$

$$= -\frac{15}{65} - \frac{48}{65}$$

$$= -\frac{63}{65}$$

**b.** $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= \frac{4}{5}\left(-\frac{5}{13}\right) - \frac{3}{5}\left(\frac{12}{13}\right)$$

$$= -\frac{20}{65} - \frac{36}{65}$$

$$= -\frac{56}{65}$$

**c.** $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

$$= \frac{\frac{3}{4} - \left(-\frac{12}{5}\right)}{1 + \left(\frac{3}{4}\right)\left(-\frac{12}{5}\right)}$$

$$= \frac{\frac{3}{4} + \frac{12}{5}}{1 - \frac{36}{20}} \cdot \frac{20}{20}$$

$$= \frac{15 + 48}{20 - 36} = -\frac{63}{16}$$

**40.** $\sin\alpha = \frac{24}{25},\ \cos\alpha = -\frac{7}{25},\ \tan\alpha = -\frac{24}{7},$

$\cos\beta = -\frac{4}{5},\ \sin\beta = -\frac{3}{5},\ \tan\beta = \frac{3}{4}$

**a.** $\cos(\beta - \alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha$

$$= -\frac{4}{5}\left(-\frac{7}{25}\right) + \left(-\frac{3}{5}\right)\frac{24}{25}$$

$$= \frac{28}{125} - \frac{72}{125} = -\frac{44}{125}$$

**b.** $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \frac{24}{25}\left(-\frac{4}{5}\right) + \left(-\frac{7}{25}\right)\left(-\frac{3}{5}\right)$$

$$= -\frac{96}{125} + \frac{21}{125}$$

$$= -\frac{75}{125} = -\frac{3}{5}$$

**c.** $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$$= \frac{-\frac{24}{7} + \frac{3}{4}}{1 - \left(-\frac{24}{7}\right)\left(\frac{3}{4}\right)}$$

$$= \frac{-\frac{24}{7} + \frac{3}{4}}{1 + \frac{72}{28}} \cdot \frac{28}{28}$$

$$= \frac{-96 + 21}{28 + 72}$$

$$= -\frac{75}{100} = -\frac{3}{4}$$

**41.** $\sin\alpha = -\frac{4}{5},\ \cos\alpha = -\frac{3}{5},\ \tan\alpha = \frac{4}{3},$

$\cos\beta = -\frac{12}{13},\ \sin\beta = \frac{5}{13},\ \tan\beta = -\frac{5}{12}$

**a.** $$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta\\ &= \left(-\frac{4}{5}\right)\left(-\frac{12}{13}\right) - \left(-\frac{3}{5}\right)\frac{5}{13}\\ &= \frac{48}{65} + \frac{15}{65} = \frac{63}{65}\end{aligned}$$

**b.** $$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\\ &= \left(-\frac{3}{5}\right)\left(-\frac{12}{13}\right) - \left(-\frac{4}{5}\right)\frac{5}{13}\\ &= \frac{36}{65} + \frac{20}{65} = \frac{56}{65}\end{aligned}$$

**c.** $$\begin{aligned}\tan(\alpha+\beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\\ &= \frac{\frac{4}{3} + \left(-\frac{5}{12}\right)}{1 - \left(\frac{4}{3}\right)\left(-\frac{5}{12}\right)}\\ &= \frac{\frac{4}{3} - \frac{5}{12}}{1 + \frac{29}{36}} = \frac{\frac{4}{3} - \frac{5}{12}}{1 + \frac{20}{36}} \cdot \frac{36}{36}\\ &= \frac{48 - 15}{36 + 20} = \frac{33}{56}\end{aligned}$$

**42.** $\sin\alpha = -\frac{7}{25},\ \cos\alpha = \frac{24}{25},\ \tan\alpha = -\frac{7}{24},$

$\cos\beta = \frac{8}{17},\ \sin\beta = -\frac{15}{17},\ \tan\beta = -\frac{15}{8}$

**a.** $$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\\ &= \left(-\frac{7}{25}\right)\left(\frac{8}{17}\right) + \frac{24}{25}\left(-\frac{15}{17}\right)\\ &= -\frac{56}{425} - \frac{360}{425} = -\frac{416}{425}\end{aligned}$$

**b.** $$\begin{aligned}\cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta\\ &= \frac{24}{25}\left(\frac{8}{17}\right) + \left(-\frac{7}{25}\right)\left(-\frac{15}{17}\right)\\ &= \frac{192}{425} + \frac{105}{425} = \frac{297}{425}\end{aligned}$$

**c.** $$\begin{aligned}\tan(\alpha+\beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\\ &= \frac{-\frac{7}{24} + \left(-\frac{15}{8}\right)}{1 - \left(-\frac{7}{24}\right)\left(-\frac{15}{8}\right)}\\ &= \frac{-\frac{7}{24} - \frac{15}{8}}{1 - \frac{105}{192}} = \frac{-\frac{7}{24} - \frac{15}{8}}{1 - \frac{105}{192}} \cdot \frac{192}{192}\\ &= \frac{-56 - 360}{192 - 105} = -\frac{416}{87}\end{aligned}$$

**43.** $\cos\alpha = \frac{15}{17},\ \sin\alpha = \frac{8}{17},\ \tan\alpha = \frac{8}{15},$

$\sin\beta = -\frac{3}{5},\ \cos\beta = -\frac{4}{5},\ \tan\beta = \frac{3}{4}$

**a.** $$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\\ &= \frac{8}{17}\left(-\frac{4}{5}\right) + \frac{15}{17}\left(-\frac{3}{5}\right)\\ &= -\frac{32}{85} - \frac{45}{85} = -\frac{77}{85}\end{aligned}$$

**b.** $$\begin{aligned}\cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta\\ &= \frac{15}{17}\left(-\frac{4}{5}\right) + \frac{8}{17}\left(-\frac{3}{5}\right)\\ &= -\frac{60}{85} - \frac{24}{85} = -\frac{84}{85}\end{aligned}$$

**c.** $$\begin{aligned}\tan(\alpha-\beta) &= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}\\ &= \frac{\frac{8}{15} - \frac{3}{4}}{1 + \frac{8}{15}\left(\frac{3}{4}\right)} = \frac{\frac{8}{15} - \frac{3}{4}}{1 + \frac{24}{60}} = \frac{\frac{8}{15} - \frac{3}{4}}{1 + \frac{24}{60}} \cdot \frac{60}{60}\\ &= \frac{32 - 45}{60 + 24} = -\frac{13}{84}\end{aligned}$$

**44.** $\cos\alpha = -\frac{7}{25},\ \sin\alpha = \frac{24}{25},\ \tan\alpha = -\frac{24}{7},$

$\sin\beta = -\frac{12}{13},\ \cos\beta = \frac{5}{13},\ \tan\beta = -\frac{12}{5}$

**a.** $$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\\ &= \frac{24}{25}\left(\frac{5}{12}\right) + \left(-\frac{7}{25}\right)\left(-\frac{12}{13}\right)\\ &= \frac{120}{325} + \frac{84}{325} = \frac{204}{325}\end{aligned}$$

**b.** $$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\\ &= \left(-\frac{7}{25}\right)\frac{5}{13} - \frac{24}{25}\left(-\frac{12}{13}\right)\\ &= -\frac{35}{325} + \frac{288}{325} = \frac{253}{325}\end{aligned}$$

**c.** $$\begin{aligned}\tan(\alpha-\beta) &= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}\\ &= \frac{-\frac{24}{7} - \left(-\frac{12}{5}\right)}{1 + \left(-\frac{24}{7}\right)\left(-\frac{12}{5}\right)} = \frac{-\frac{24}{7} + \frac{12}{5}}{1 + \frac{288}{35}} = \frac{-\frac{24}{7} + \frac{12}{5}}{1 + \frac{288}{35}} \cdot \frac{35}{35}\\ &= \frac{-120 + 84}{35 + 288} = -\frac{36}{323}\end{aligned}$$

**45.** $\cos\alpha = -\frac{3}{5},\ \sin\alpha = -\frac{4}{5},\ \tan\alpha = \frac{4}{3},$

$\sin\beta = \frac{5}{13},\ \cos\beta = \frac{12}{13},\ \tan\beta = \frac{5}{12}$

**a.** $$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \left(-\frac{4}{5}\right)\frac{12}{13} - \left(-\frac{3}{5}\right)\left(\frac{5}{13}\right) = -\frac{48}{65} + \frac{15}{65} = -\frac{33}{65}$$

**b.** $$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \left(-\frac{3}{5}\right)\frac{12}{13} - \left(-\frac{4}{5}\right)\frac{5}{13} = -\frac{36}{65} + \frac{20}{65} = -\frac{16}{65}$$

**c.** $$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{\frac{4}{3} + \frac{5}{12}}{1 - \frac{4}{3}\left(\frac{5}{12}\right)} = \frac{\frac{4}{3} + \frac{5}{12}}{1 - \frac{20}{36}} = \frac{\frac{4}{3} + \frac{5}{12}}{1 - \frac{20}{36}} \cdot \frac{36}{36} = \frac{48+15}{36-20} = \frac{63}{16}$$

**46.** $\cos\alpha = \frac{8}{17},\ \sin\alpha = -\frac{15}{17},\ \tan\alpha = -\frac{15}{8},$

$\sin\beta = -\frac{24}{25},\ \cos\beta = -\frac{7}{25},\ \tan\beta = \frac{24}{7}$

**a.** $$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \left(-\frac{15}{17}\right)\left(-\frac{7}{25}\right) - \frac{8}{17}\left(-\frac{24}{25}\right) = \frac{105}{425} + \frac{192}{425} = \frac{297}{425}$$

**b.** $$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{8}{17}\left(-\frac{7}{25}\right) - \left(-\frac{15}{17}\right)\left(-\frac{24}{25}\right) = -\frac{56}{425} - \frac{360}{425} = -\frac{416}{425}$$

**c.** $$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{-\frac{15}{8} + \frac{24}{7}}{1 - \left(-\frac{15}{8}\right)\left(\frac{24}{7}\right)} = \frac{-\frac{15}{8} + \frac{24}{7}}{1 + \frac{360}{56}} \cdot \frac{56}{56} = \frac{-105+192}{56+360} = \frac{87}{416}$$

**47.** $\sin\alpha = \frac{3}{5},\ \cos\alpha = \frac{4}{5},\ \tan\alpha = \frac{3}{4},$

$\tan\beta = \frac{5}{12},\ \sin\beta = -\frac{5}{13},\ \cos\beta = -\frac{12}{13}$

**a.** $$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right) + \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) = -\frac{36}{65} - \frac{20}{65} = -\frac{56}{65}$$

**b.** $$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) + \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right) = -\frac{48}{65} - \frac{15}{65} = -\frac{63}{65}$$

**c.** $$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \frac{\frac{3}{4} - \frac{5}{12}}{1 + \left(\frac{3}{4}\right)\left(\frac{5}{12}\right)} = \frac{\frac{3}{4} - \frac{5}{12}}{1 + \frac{15}{48}} \cdot \frac{48}{48} = \frac{36-20}{48+15} = \frac{16}{63}$$

**48.** $\tan\alpha = \frac{15}{8},\ \sin\alpha = \frac{15}{17},\ \cos\alpha = \frac{8}{17},$

$\tan\beta = -\frac{7}{24},\ \sin\beta = -\frac{7}{25},\ \cos\beta = \frac{24}{25}$

**a.** $$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \frac{15}{17}\left(\frac{24}{25}\right) - \frac{8}{17}\left(-\frac{7}{25}\right) = \frac{360}{425} + \frac{56}{425} = \frac{416}{425}$$

**b.** $$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \frac{8}{7}\left(\frac{24}{25}\right) + \frac{15}{17}\left(-\frac{7}{25}\right) = \frac{192}{425} - \frac{105}{425} = \frac{87}{425}$$

**c.** $$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{\frac{15}{8} + \left(-\frac{7}{24}\right)}{1 - \frac{15}{8}\left(-\frac{7}{24}\right)} = \frac{\frac{15}{8} - \frac{7}{24}}{1 + \frac{105}{192}} \cdot \frac{192}{192} = \frac{360-56}{192+105} = \frac{304}{297}$$

**49.** $$\cos\left(\frac{\pi}{2} - \theta\right) = \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta = 0\cdot\cos\theta + 1\cdot\sin\theta = \sin\theta$$

**50.** $$\cos(\theta+\pi) = \cos\theta\cos\pi - \sin\theta\sin\pi = \cos\theta(-1) - \sin\theta(0) = -\cos\theta$$

**51.** $$\sin\left(\theta+\frac{\pi}{2}\right)=\sin\theta\cos\frac{\pi}{2}+\cos\theta\sin\frac{\pi}{2}$$
$$=\sin\theta(0)+\cos\theta(1)$$
$$=\cos\theta$$

**52.** $$\sin(\theta+\pi)=\sin\theta\cos\pi+\cos\theta\sin\pi$$
$$=\sin\theta(-1)+\cos\theta(0)$$
$$=-\sin\theta$$

**53.** $$\tan\left(\theta+\frac{\pi}{4}\right)=\frac{\tan\theta+\tan\frac{\pi}{4}}{1-\tan\theta\tan\frac{\pi}{4}}$$
$$=\frac{\tan\theta+1}{1-\tan\theta}$$

**54.** $$\tan 2\theta=\tan(\theta+\theta)$$
$$=\frac{\tan\theta+\tan\theta}{1-\tan\theta\tan\theta}$$
$$=\frac{2\tan\theta}{1-\tan^2\theta}$$

**55.** $$\cos\left(\frac{3\pi}{2}-\theta\right)=\cos\frac{3\pi}{2}\cos\theta+\sin\frac{3\pi}{2}\sin\theta$$
$$=0(\cos\theta)+(-1)\sin\theta$$
$$=-\sin\theta$$

**56.** $$\sin\left(\frac{3\pi}{2}+\theta\right)=\sin\frac{3\pi}{2}\cos\theta+\cos\frac{3\pi}{2}\sin\theta$$
$$=(-1)\cos\theta+(0)\sin\theta$$
$$=-\cos\theta$$

**57.** $$\cot\left(\frac{\pi}{2}-\theta\right)=\frac{\cos(\pi/2-\theta)}{\sin(\pi/2-\theta)}$$
$$=\frac{\left(\cos\frac{\pi}{2}\right)\cos\theta+\left(\sin\frac{\pi}{2}\right)\sin\theta}{\left(\sin\frac{\pi}{2}\right)\cos\theta-\left(\cos\frac{\pi}{2}\right)\sin\theta}$$
$$=\frac{(0)\cos\theta+(1)\sin\theta}{(1)\cos\theta-(0)\sin\theta}$$
$$=\frac{\sin\theta}{\cos\theta}$$
$$=\tan\theta$$

**58.** $$\cot(\pi+\theta)=\frac{\cos(\pi+\theta)}{\sin(\pi+\theta)}$$
$$=\frac{\cos\pi\cos\theta-\sin\pi\sin\theta}{\sin\pi\cos\theta+\cos\pi\sin\theta}$$
$$=\frac{(-1)\cos\theta-(0)\sin\theta}{(0)\cos\theta+(-1)\sin\theta}$$
$$=\frac{-\cos\theta}{-\sin\theta}$$
$$=\cot\theta$$

**59.** $$\csc(\pi-\theta)=\frac{1}{\sin(\pi-\theta)}$$
$$=\frac{1}{\sin\pi\cos\theta-\cos\pi\sin\theta}$$
$$=\frac{1}{(0)\cos\theta-(-1)\sin\theta}$$
$$=\frac{1}{\sin\theta}$$
$$=\csc\theta$$

**60.** $$\sec\left(\frac{\pi}{2}-\theta\right)=\frac{1}{\cos\left(\frac{\pi}{2}-\theta\right)}$$
$$=\frac{1}{\cos\frac{\pi}{2}\cos\theta+\sin\frac{\pi}{2}\sin\theta}$$
$$=\frac{1}{(0)\cos\theta+(1)\sin\theta}$$
$$=\frac{1}{\sin\theta}$$
$$=\csc\theta$$

**61.** $$\sin 6x\cos 2x-\cos 6x\sin 2x=\sin(6x-2x)$$
$$=\sin 4x$$
$$=\sin(2x+2x)$$
$$=\sin 2x\cos 2x+\cos 2x\sin 2x$$
$$=2\sin 2x\cos 2x$$

**62.** $$\cos 5x\cos 3x+\sin 5x\sin 3x=\cos(5x-3x)$$
$$=\cos 2x$$
$$=\cos(x+x)$$
$$=\cos x\cos x-\sin x\sin x$$
$$=\cos^2 x-\sin^2 x$$

**63.** $$\cos(\alpha+\beta)+\cos(\alpha-\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta+\cos\alpha\cos\beta+\sin\alpha\sin\beta$$
$$=2\cos\alpha\cos\beta$$

**64.** $$\cos(\alpha-\beta)-\cos(\alpha+\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta-\cos\alpha\cos\beta+\sin\alpha\sin\beta$$
$$=2\sin\alpha\sin\beta$$

**65.** $$\sin(\alpha+\beta)+\sin(\alpha-\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta+\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$=2\sin\alpha\cos\beta$$

**66.** $\sin(\alpha-\beta)-\sin(\alpha+\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta-\sin\alpha\cos\beta-\cos\alpha\sin\beta$
$=-2\cos\alpha\sin\beta$

**67.** $$\frac{\cos(\alpha-\beta)}{\sin(\alpha+\beta)}=\frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\sin\alpha\cos\beta+\cos\alpha\sin\beta}=\frac{\frac{\cos\alpha\cos\beta}{\sin\alpha\cos\beta}+\frac{\sin\alpha\sin\beta}{\sin\alpha\cos\beta}}{\frac{\sin\alpha\cos\beta}{\sin\alpha\cos\beta}+\frac{\cos\alpha\sin\beta}{\sin\alpha\cos\beta}}=\frac{\cot\alpha+\tan\beta}{1+\cot\alpha\tan\beta}$$

**68.** $$\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\alpha\cos\beta-\cos\alpha\sin\beta}=\frac{\frac{\sin\alpha\cos\beta}{\sin\alpha\cos\beta}+\frac{\cos\alpha\sin\beta}{\sin\alpha\cos\beta}}{\frac{\sin\alpha\cos\beta}{\sin\alpha\cos\beta}-\frac{\cos\alpha\sin\beta}{\sin\alpha\cos\beta}}=\frac{1+\cot\alpha\tan\beta}{1-\cot\alpha\tan\beta}$$

**69.** $$\frac{\sin(x+h)-\sin x}{h}=\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}=\frac{\sin x(\cos h-1)}{h}+\frac{\cos x\sin h}{h}=\sin x\frac{\cos h-1}{h}+\cos x\frac{\sin h}{h}$$

**70.** $$\frac{\cos(x+h)-\cos x}{h}=\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}=\frac{\cos x(\cos h-1)}{h}-\frac{\sin x\sin h}{h}=\cos x\frac{\cos h-1}{h}-\sin x\frac{\sin h}{h}$$

**71.** $$\begin{aligned}\sin\left(\frac{\pi}{2}+\alpha-\beta\right)&=\sin\left[\frac{\pi}{2}+(\alpha-\beta)\right]\\&=\sin\frac{\pi}{2}\cos(\alpha-\beta)+\cos\frac{\pi}{2}\sin(\alpha-\beta)\\&=(1)\cos(\alpha-\beta)+(0)\sin(\alpha-\beta)\\&=\cos(\alpha-\beta)\\&=\cos\alpha\cos\beta+\sin\alpha\sin\beta\end{aligned}$$

**72.** $$\begin{aligned}\cos\left(\frac{\pi}{2}+\alpha+\beta\right)&=\cos\left[\frac{\pi}{2}+(\alpha+\beta)\right]\\&=\cos\frac{\pi}{2}\cos(\alpha+\beta)-\sin\frac{\pi}{2}\sin(\alpha+\beta)\\&=(0)\cos(\alpha+\beta)-(1)\sin(\alpha+\beta)\\&=-\sin(\alpha+\beta)\\&=-(\sin\alpha\cos\beta+\cos\alpha\sin\beta)\end{aligned}$$

**73.** $$\begin{aligned}\sin 3x&=\sin(2x+x)\\&=\sin 2x\cos x+\cos 2x\sin x\\&=\sin(x+x)\cos x+\cos(x+x)\sin x\\&=(\sin x\cos x+\cos x\sin x)\cos x+(\cos x\cos x-\sin x\sin x)\sin x\\&=2\sin x\cos^2 x+\sin x\cos^2 x-\sin^3 x\\&=3\sin x\cos^2 x-\sin^3 x\\&=3\sin x(1-\sin^2 x)-\sin^3 x\\&=3\sin x-3\sin^3 x-\sin^3 x\\&=3\sin x-4\sin^3 x\end{aligned}$$

**74.** $$\begin{aligned}\cos 3x&=\cos(2x+x)\\&=\cos 2x\cos x-\sin 2x\sin x\\&=\cos(x+x)\cos x-\sin(x+x)\sin x\\&=(\cos x\cos x-\sin x\sin x)\cos x-(\sin x\cos x+\cos x\sin x)\sin x\\&=\cos^3 x-\cos x\sin^2 x-2\cos x\sin^2 x\\&=\cos^3 x-3\cos x\sin^2 x\\&=\cos^3 x-3\cos x(1-\cos^2 x)\\&=\cos^3 x-3\cos x+3\cos^3 x\\&=4\cos^3 x-3\cos x\end{aligned}$$

**75.** $$\begin{aligned}\cos(\theta+3\pi) &= \cos\theta\cos 3\pi - \sin\theta\sin 3\pi\\ &= (\cos\theta)(-1)-(\sin\theta)(0)\\ &= -\cos\theta\end{aligned}$$

**76.** $$\begin{aligned}\sin(\theta+2\pi) &= \sin\theta\cos 2\pi + \cos\theta\sin 2\pi\\ &= (\sin\theta)(1)+(\cos\theta)(0)\\ &= \sin\theta\end{aligned}$$

**77.** $$\begin{aligned}\tan(\theta+\pi) &= \frac{\tan\theta+\tan\pi}{1-\tan\theta\tan\pi}\\ &= \frac{\tan\theta+0}{1-(\tan\theta)(0)}\\ &= \tan\theta\end{aligned}$$

**78.** $$\begin{aligned}\cos[\theta+(2k+1)\pi] &= \cos\theta\cos[(2k+1)\pi]\\ &\quad -\sin\theta\sin[(2k+1)\pi]\\ &= (\cos\theta)(-1)-(\sin\theta)(0)\\ &= -\cos\theta\end{aligned}$$

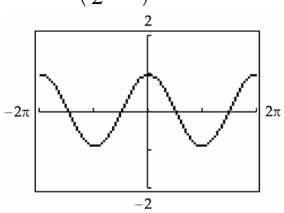
**79.** $$\begin{aligned}\sin(\theta+2k\pi) &= \sin\theta\cos(2k\pi)+\cos\theta\sin 2k\pi\\ &= (\sin\theta)(1)+(\cos\theta)(0)\\ &= \sin\theta\end{aligned}$$

**80.** We consider two cases, (1) $k$ an odd and (2) $k$ an even integer.

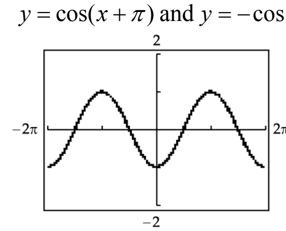
(1) $$\begin{aligned}\sin(\theta-k\pi) &= \sin\theta\cos(k\pi)-\cos\theta\sin(k\pi)\\ &= (\sin\theta)(-1)-(\cos\theta)(0)\\ &= -\sin\theta, \text{ provided } k \text{ is odd}\end{aligned}$$

(2) $$\begin{aligned}\sin(\theta-k\pi) &= \sin\theta\cos(k\pi)-\cos\theta\sin(k\pi)\\ &= (\sin\theta)(1)-(\cos\theta)(0)\\ &= \sin\theta, \text{ provided } k \text{ is even}\end{aligned}$$

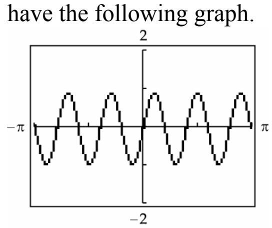
**81.** $y=\sin\left(\frac{\pi}{2}-x\right)$ and $y=\cos x$ both have the following graph.



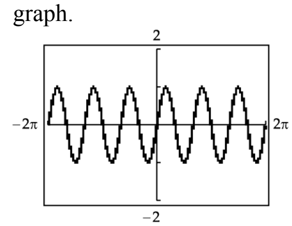

**82.** $y=\cos(x+\pi)$ and $y=-\cos x$ both have the following graph.




**83.** $y=\sin 7x\ \cos 2x-\cos 7x\ \sin 2x$ and $y=\sin 5x$ both have the following graph.




**84.** $y=\sin 3x$ and $y=3\sin x-4\sin^3 x$ both have the following graph.




## Connecting Concepts

**85.** $$\begin{aligned}\sin(x-y)\cdot\sin(x+y) &= (\sin x\cos y-\cos x\sin y)(\sin x\cos y+\cos x\sin y)\\ &= \sin^2 x\cos^2 y-\cos^2 x\sin^2 y\end{aligned}$$

**86.** $$\begin{aligned}\sin(x+y+z) &= \sin[x+(y+z)]\\ &= \sin x\cos(y+z)+\cos x\sin(y+z)\\ &= \sin x(\cos y\cos z-\sin y\sin z)+\cos x(\sin y\cos z+\cos y\sin z)\\ &= \sin x\cos y\cos z-\sin x\sin y\sin z+\cos x\sin y\cos z+\cos x\cos y\sin z\end{aligned}$$

**87.** $\cos(x+y+z) = \cos[x+(y+z)]$
$= \cos x\cos(y+z) - \sin x\sin(y+z)$
$= \cos x[\cos y\cos z - \sin y\sin z] - \sin x[\sin y\cos z + \cos y\sin z]$
$= \cos x\cos y\cos z - \cos x\sin y\sin z - \sin x\sin y\cos z - \sin x\cos y\sin z$

**88.** $\dfrac{\sin(x+y)}{\sin x\sin y} = \dfrac{\sin x\cos y + \cos x\sin y}{\sin x\sin y}$
$= \dfrac{\sin x\cos y}{\sin x\sin y} + \dfrac{\cos x\sin y}{\sin x\sin y}$
$= \cot y + \cot x$

**89.** $\dfrac{\cos(x-y)}{\cos x\sin y} = \dfrac{\cos x\cos y + \sin x\sin y}{\cos x\sin y}$
$= \dfrac{\cos x\cos y}{\cos x\sin y} + \dfrac{\sin x\sin y}{\cos x\sin y}$
$= \cot y + \tan x$

**90.** $E_R = \dfrac{2\sin 10° + \sin 20°}{2\sin(10° + 20°)}$
$= \dfrac{2\sin 10° + \sin 20°}{2\sin 30°}$
$E_R \approx 0.69$

## Prepare for Section 6.3

**PS1.** $\sin 2\alpha = \sin(\alpha+\alpha)$
$= \sin\alpha\cos\alpha + \cos\alpha\sin\alpha$
$= 2\sin\alpha\cos\alpha$

**PS2.** $\cos 2\alpha = \cos(\alpha+\alpha)$
$= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$
$= \cos^2\alpha - \sin^2\alpha$

**PS3.** $\tan 2\alpha = \tan(\alpha+\alpha)$
$= \dfrac{\tan\alpha + \tan\alpha}{1-\tan\alpha\tan\alpha}$
$= \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

**PS4.** $\tan\left(\dfrac{60^\circ}{2}\right) = \tan(30^\circ) = \dfrac{\sqrt{3}}{3}$ $\quad \dfrac{\sin(60^\circ)}{1+\cos(60^\circ)} = \dfrac{\frac{\sqrt{3}}{2}}{1+\frac{1}{2}} = \dfrac{\frac{\sqrt{3}}{2}}{\frac{3}{2}} = \dfrac{\sqrt{3}}{3}$

$\tan\left(\dfrac{90^\circ}{2}\right) = \tan(45^\circ) = 1$ $\quad \dfrac{\sin(90^\circ)}{1+\cos(90^\circ)} = \dfrac{1}{1+0} = 1$

$\tan\left(\dfrac{120^\circ}{2}\right) = \tan(60^\circ) = \sqrt{3}$ $\quad \dfrac{\sin(120^\circ)}{1+\cos(120^\circ)} = \dfrac{\frac{\sqrt{3}}{2}}{1-\frac{1}{2}} = \dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$

For each of the given values of $\alpha$, the functional values are equal.

**PS5.** Let $\alpha = 45^\circ$; then the left side of the equation is 1, and the right side of the equation is $\sqrt{2}$.

**PS6.** Let $\alpha = 60^\circ$; then the left side of the equation is $\dfrac{\sqrt{3}}{2}$, and the right side of the equation is $\dfrac{1}{4}$.

## Section 6.3

**1.** $2\sin 2\alpha\cos 2\alpha = \sin 2(2\alpha)$
$= \sin 4\alpha$

**2.** $2\sin 3\theta\cos 3\theta = \sin 2(3\theta)$
$= \sin 6\theta$

**3.** $1-2\sin^2 5\beta = \cos 2(5\beta)$
$= \cos 10\beta$

**4.** $2\cos^2 2\beta - 1 = \cos 2(2\beta)$
$= \cos 4\beta$

**5.** $\cos^2 3\alpha - \sin^2 3\alpha = \cos 2(3\alpha)$
$= \cos 6\alpha$

**6.** $\cos^2 6\alpha - \sin^2 6\alpha = \cos 2(6\alpha)$
$= \cos 12\alpha$

**7.** $\dfrac{2\tan 3\alpha}{1-\tan^2 3\alpha} = \tan 2(3\alpha) = \tan 6\alpha$

**8.** $\dfrac{2\tan 4\theta}{1-\tan^2 4\theta} = \tan 2(4\theta) = \tan 8\theta$

**9.** $\cos\alpha = -\dfrac{4}{5},\ \sin\alpha = \sqrt{1-\left(\dfrac{4}{5}\right)^2} = \dfrac{3}{5},\ \tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{3/5}{-4/5} = -\dfrac{3}{4}$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) = -\frac{24}{25}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\left(-\frac{3}{4}\right)}{1-\left(-\frac{3}{4}\right)^2} = \frac{-\frac{6}{4}}{1-\frac{9}{16}} = \frac{-\frac{6}{4}}{1-\frac{9}{16}}\cdot\frac{16}{16} = \frac{-24}{16-9} = -\frac{24}{7}$$

**10.** $\cos\alpha = \dfrac{24}{25},\ \sin\alpha = -\sqrt{1-\left(\dfrac{24}{25}\right)^2} = -\dfrac{7}{25},\ \tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{-7/25}{24/25} = -\dfrac{7}{24}$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\left(-\frac{7}{25}\right)\left(\frac{24}{25}\right) = -\frac{336}{625}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \left(\frac{24}{25}\right)^2 - \left(-\frac{7}{25}\right)^2 = \frac{576}{625} - \frac{49}{625} = \frac{527}{625}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\left(-\frac{7}{24}\right)}{1-\left(-\frac{7}{24}\right)^2} = \frac{-\frac{7}{12}}{1-\frac{49}{576}}\cdot\frac{576}{576} = -\frac{336}{576-49} = -\frac{336}{527}$$

**11.** $\sin\alpha = \dfrac{8}{17},\ \cos\alpha = -\sqrt{1-\left(\dfrac{8}{17}\right)^2} = -\dfrac{15}{17},\ \tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{8/17}{-15/17} = -\dfrac{8}{15}$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\left(\frac{8}{17}\right)\left(-\frac{15}{17}\right) = -\frac{240}{289}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \left(-\frac{15}{17}\right)^2 - \left(\frac{8}{17}\right)^2 = \frac{225}{289} - \frac{64}{289} = \frac{161}{289}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\left(-\frac{8}{15}\right)}{1-\left(-\frac{8}{15}\right)^2} = \frac{-\frac{16}{15}}{1-\frac{64}{225}} = \frac{-\frac{16}{15}}{1-\frac{64}{225}}\cdot\frac{225}{225} = \frac{-240}{225-64} = -\frac{240}{161}$$

**12.** $\sin\alpha = -\frac{9}{41},\ \cos\alpha = -\sqrt{1-\left(-\frac{9}{41}\right)^2} = -\frac{40}{41},\ \tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{-9/41}{-40/41} = \frac{9}{40}$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\left(-\frac{9}{41}\right)\left(-\frac{40}{41}\right) = \frac{720}{1681}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \left(-\frac{40}{41}\right)^2 - \left(-\frac{9}{41}\right)^2 = \frac{1600}{1681} - \frac{81}{1681} = \frac{1519}{1681}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\left(\frac{9}{40}\right)}{1-\left(\frac{9}{40}\right)^2} = \frac{\frac{9}{20}}{1-\frac{81}{1600}} = \frac{\frac{9}{20}}{1-\frac{81}{1600}}\cdot\frac{1600}{1600} = \frac{720}{1600-81} = \frac{720}{1519}$$

**13.** $\tan\alpha = -\frac{24}{7},\ r = \sqrt{24^2+7^2} = \sqrt{576+49} = \sqrt{625} = 25,\ \sin\alpha = -\frac{24}{25},\ \cos\alpha = \frac{7}{25}$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\left(-\frac{24}{25}\right)\left(\frac{7}{25}\right) = -\frac{336}{625}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \left(\frac{7}{25}\right)^2 - \left(-\frac{24}{25}\right)^2 = \frac{49}{625} - \frac{576}{625} = -\frac{527}{625}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\left(-\frac{24}{7}\right)}{1-\left(-\frac{24}{7}\right)^2} = \frac{-\frac{48}{7}}{1-\frac{576}{49}}\cdot\frac{49}{49} = \frac{-336}{49-576} = \frac{336}{527}$$

**14.** $\tan\alpha = \frac{4}{3},\ r = \sqrt{4^2+3^2} = \sqrt{16+9} = \sqrt{25} = 5,\ \sin\alpha = \frac{4}{5},\ \cos\alpha = \frac{3}{5}$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) = \frac{24}{25}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\left(\frac{4}{3}\right)}{1-\left(\frac{4}{3}\right)^2} = \frac{\frac{8}{3}}{1-\frac{16}{9}} = \frac{\frac{8}{3}}{1-\frac{16}{9}}\cdot\frac{9}{9} = \frac{24}{9-16} = -\frac{24}{7}$$

**15.** $\sin\alpha = \frac{15}{17},\ \cos\alpha = \sqrt{1-\left(\frac{15}{17}\right)^2} = \sqrt{1-\frac{225}{289}} = \sqrt{\frac{289-225}{289}} = \sqrt{\frac{64}{289}} = \frac{8}{17},\ \tan\alpha = \frac{15/17}{8/17} = \frac{15}{8}$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\left(\frac{15}{17}\right)\left(\frac{8}{17}\right) = \frac{240}{289}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \left(\frac{8}{17}\right)^2 - \left(\frac{15}{17}\right)^2 = \frac{64}{289} - \frac{225}{289} = -\frac{161}{289}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\left(\frac{15}{8}\right)}{1-\left(\frac{15}{8}\right)^2} = \frac{\frac{15}{4}}{1-\frac{225}{64}} = \frac{\frac{15}{4}}{1-\frac{225}{64}}\cdot\frac{64}{64} = \frac{240}{64-225} = -\frac{240}{161}$$

**16.** $\sin\alpha = -\frac{3}{5},\ \cos\alpha = -\sqrt{1-\left(-\frac{3}{5}\right)^2} = -\sqrt{1-\frac{9}{25}} = -\frac{4}{5},\ \tan\alpha = \frac{-\frac{3}{5}}{-\frac{4}{5}} = \frac{3}{4}$

$$\begin{aligned}\sin 2\alpha &= 2\sin\alpha\cos\alpha\\ &= 2\left(-\frac{3}{5}\right)\left(-\frac{4}{5}\right)\\ &= \frac{24}{25}\end{aligned}$$

$$\begin{aligned}\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha\\ &= \left(-\frac{4}{5}\right)^2 - \left(-\frac{3}{5}\right)^2\\ &= \frac{16}{25} - \frac{9}{25}\\ &= \frac{7}{25}\end{aligned}$$

$$\begin{aligned}\tan 2\alpha &= \frac{2\tan\alpha}{1-\tan^2\alpha}\\ &= \frac{2\left(\frac{3}{4}\right)}{1-\left(\frac{3}{4}\right)^2} = \frac{\frac{3}{2}}{1-\frac{9}{16}}\\ &= \frac{\frac{3}{2}}{1-\frac{9}{16}}\cdot\frac{16}{16} = \frac{24}{16-9}\\ &= \frac{24}{7}\end{aligned}$$

**17.** $\cos\alpha = \frac{40}{41},\ \sin\alpha = -\sqrt{1-\left(\frac{40}{41}\right)^2} = -\sqrt{1-\frac{1600}{1681}} = -\frac{9}{41},\ \tan\alpha = \frac{-9/41}{40/41} = -\frac{9}{40}$

$$\begin{aligned}\sin 2\alpha &= 2\sin\alpha\cos\alpha\\ &= 2\left(-\frac{9}{41}\right)\left(\frac{40}{41}\right)\\ &= -\frac{720}{1681}\end{aligned}$$

$$\begin{aligned}\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha\\ &= \left(\frac{40}{41}\right)^2 - \left(-\frac{9}{41}\right)^2\\ &= \frac{1600}{1681} - \frac{81}{1681}\\ &= \frac{1519}{1681}\end{aligned}$$

$$\begin{aligned}\tan 2\alpha &= \frac{2\tan\alpha}{1-\tan^2\alpha}\\ &= \frac{2\left(-\frac{9}{40}\right)}{1-\left(-\frac{9}{40}\right)^2} = \frac{-\frac{9}{20}}{1-\frac{81}{1600}}\\ &= \frac{-\frac{9}{20}}{1-\frac{81}{1600}}\cdot\frac{1600}{1600}\\ &= \frac{-720}{1600-81} = -\frac{720}{1519}\end{aligned}$$

**18.** $\cos\alpha = \frac{4}{5},\ \sin\alpha = -\sqrt{1-\left(\frac{4}{5}\right)^2} = -\sqrt{1-\frac{16}{25}} = -\frac{3}{5},\ \tan\alpha = \frac{-3/5}{4/5} = -\frac{3}{4}$

$$\begin{aligned}\sin 2\alpha &= 2\sin\alpha\cos\alpha\\ &= 2\left(-\frac{3}{5}\right)\left(\frac{4}{5}\right)\\ &= -\frac{24}{25}\end{aligned}$$

$$\begin{aligned}\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha\\ &= \left(\frac{4}{5}\right)^2 - \left(-\frac{3}{5}\right)^2\\ &= \frac{16}{25} - \frac{9}{25}\\ &= \frac{7}{25}\end{aligned}$$

$$\begin{aligned}\tan 2\alpha &= \frac{2\tan\alpha}{1-\tan^2\alpha}\\ &= \frac{2\left(-\frac{3}{4}\right)}{1-\left(-\frac{3}{4}\right)^2} = \frac{-\frac{3}{2}}{1-\frac{9}{16}}\\ &= \frac{-\frac{3}{2}}{1-\frac{9}{16}}\cdot\frac{16}{16} = \frac{-24}{16-9}\\ &= -\frac{24}{7}\end{aligned}$$

**19.** $6\cos^2 x = 6\left(\frac{1+\cos 2x}{2}\right) = 3(1+\cos 2x)$

**20.** $\sin^4 x\ \cos^4 x = \left(\frac{1-\cos 2x}{2}\right)^2\left(\frac{1+\cos 2x}{2}\right)^2$

$$= \frac{1}{16}\left(1-\cos 2x+\cos^2 2x\right)\left(1+\cos 2x+\cos^2 2x\right)$$
$$= \frac{1}{16}\left(1-2\cos^2 2x+\cos^4 2x\right)$$
$$= \frac{1}{16}\left(1-2\left(\frac{1+\cos 4x}{2}\right)+\left(\frac{1+\cos 4x}{2}\right)^2\right)$$
$$= \frac{1}{16}\left(1-1-\cos 4x+\frac{1}{4}+\frac{1}{2}\cos 4x+\frac{1}{4}\left(\frac{1+\cos 8x}{2}\right)\right)$$
$$= \frac{1}{16}\left(\frac{3}{8}-\frac{1}{2}\cos 4x+\frac{1}{8}\cos 8x\right)$$
$$= \frac{1}{32}\left(\frac{3}{4}-\cos 4x+\frac{1}{4}\cos 8x\right)$$

**21.** $\cos^4 x = \left(\frac{1+\cos 2x}{2}\right)^2$

$$= \frac{1}{4}\left(1+2\cos 2x+\cos^2 2x\right)$$
$$= \frac{1}{4}\left(1+2\cos 2x+\frac{1+\cos 4x}{2}\right)$$
$$= \frac{1}{8}(3+4\cos 2x+\cos 4x)$$

**22.** $\sin^2 x\ \cos^4 x = \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2$

$$= \frac{1}{8}\left(1-\cos^2 2x\right)(1+\cos 2x)$$
$$= \frac{1}{8}\left(1-\frac{1+\cos 4x}{2}\right)(1+\cos 2x)$$
$$= \frac{1}{8}\left(\frac{1}{2}-\frac{1}{2}\cos 4x\right)(1+\cos 2x)$$
$$= \frac{1}{16}(1+\cos 2x-\cos 4x-\cos 2x\cos 4x)$$

**23.** $\sin^4 x\ \cos^2 x = \left(\frac{1-\cos 2x}{2}\right)^2\left(\frac{1+\cos 2x}{2}\right)$

$$= \frac{1}{8}\left(1-\cos^2 2x\right)(1-\cos 2x)$$
$$= \frac{1}{8}\left(1-\frac{1+\cos 4x}{2}\right)(1-\cos 2x)$$
$$= \frac{1}{8}\left(\frac{1}{2}-\frac{1}{2}\cos 4x\right)(1-\cos 2x)$$
$$= \frac{1}{16}(1-\cos 2x-\cos 4x+\cos 4x\cos 2x)$$

**24.** $\sin^6 x = \left(\frac{1-\cos 2x}{2}\right)^3$

$$= \frac{1}{8}\left(1-3\cos 2x+3\cos^2 2x-\cos^3 2x\right)$$
$$= \frac{1}{8}\left(1-3\cos 2x+\frac{3}{2}+\frac{3}{2}\cos 4x-\cos 2x\left(\frac{1+\cos 4x}{2}\right)\right)$$
$$= \frac{1}{16}(5-7\cos 2x+3\cos 4x-\cos 2x\cos 4x)$$

**25.** $\sin 75° = \sin\frac{150°}{2}$

$$= +\sqrt{\frac{1-\cos 150°}{2}}$$
$$= \sqrt{\frac{1-(-\sqrt{3}/2)}{2}}$$
$$= \sqrt{\frac{2+\sqrt{3}}{4}}$$
$$= \frac{\sqrt{2+\sqrt{3}}}{2}$$

**26.** $\cos 105° = \cos\frac{210°}{2}$

$$= -\sqrt{\frac{1+\cos 210°}{2}}$$
$$= -\sqrt{\frac{1+(-\sqrt{3}/2)}{2}}$$
$$= -\sqrt{\frac{2-\sqrt{3}}{4}}$$
$$= -\frac{\sqrt{2-\sqrt{3}}}{2}$$

**27.** $\tan 67.5° = \tan\frac{135°}{2}$

$$= \frac{1-\cos 135°}{\sin 135°}$$
$$= \frac{1-(-\sqrt{2}/2)}{\sqrt{2}/2}$$
$$= \frac{2+\sqrt{2}}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{2\sqrt{2}+2}{2}$$
$$= \sqrt{2}+1$$

**28.** $\cos 165° = \cos\dfrac{330°}{2} = -\sqrt{\dfrac{1+\cos 330°}{2}} = -\sqrt{\dfrac{1+\sqrt{3}/2}{2}} = -\dfrac{\sqrt{2+\sqrt{3}}}{2}$

**29.** $\cos 157.5° = \cos\dfrac{315°}{2} = -\sqrt{\dfrac{1+\cos 315°}{2}} = -\sqrt{\dfrac{1+(\sqrt{2}/2)}{2}} = -\sqrt{\dfrac{2+\sqrt{2}}{4}} = -\dfrac{\sqrt{2+\sqrt{2}}}{2}$

**30.** $\sin 112.5° = \sin\dfrac{225°}{2} = +\sqrt{\dfrac{1-\cos 225°}{2}} = \sqrt{\dfrac{1-(-\sqrt{2}/2)}{2}} = \sqrt{\dfrac{2+\sqrt{2}}{4}} = \dfrac{\sqrt{2+\sqrt{2}}}{2}$

**31.** $\sin 22.5° = \sin\dfrac{45°}{2} = +\sqrt{\dfrac{1-\cos 45°}{2}} = \sqrt{\dfrac{1-\sqrt{2}/2}{2}} = \sqrt{\dfrac{2-\sqrt{2}}{4}} = \dfrac{\sqrt{2-\sqrt{2}}}{2}$

**32.** $\cos 67.5° = \cos\dfrac{135°}{2} = +\sqrt{\dfrac{1+\cos 135°}{2}} = \sqrt{\dfrac{1+(-\sqrt{2}/2)}{\sqrt{2}/2}} = \sqrt{\dfrac{2-\sqrt{2}}{4}} = \dfrac{\sqrt{2-\sqrt{2}}}{2}$

**33.** $\sin\dfrac{7\pi}{8} = \sin\dfrac{7\pi/4}{2} = +\sqrt{\dfrac{1-\cos 7\pi/4}{2}} = \sqrt{\dfrac{1-\sqrt{2}/2}{2}} = \sqrt{\dfrac{2-\sqrt{2}}{4}} = \dfrac{\sqrt{2-\sqrt{2}}}{2}$

**34.** $\cos\dfrac{5\pi}{8} = \cos\dfrac{5\pi/4}{2} = -\sqrt{\dfrac{1+\cos 5\pi/4}{2}} = -\sqrt{\dfrac{1-\sqrt{2}/2}{2}} = -\sqrt{\dfrac{2-\sqrt{2}}{4}} = -\dfrac{\sqrt{2-\sqrt{2}}}{2}$

**35.** $\cos\dfrac{5\pi}{12} = \cos\dfrac{5\pi/6}{2} = +\sqrt{\dfrac{1+\cos 5\pi/6}{2}} = \sqrt{\dfrac{1-\sqrt{3}/2}{2}} = \sqrt{\dfrac{2-\sqrt{3}}{4}} = \dfrac{\sqrt{2-\sqrt{3}}}{2}$

**36.** $\sin\dfrac{3\pi}{8} = \sin\dfrac{3\pi/4}{2} = +\sqrt{\dfrac{1-\cos 3\pi/4}{2}} = \sqrt{\dfrac{1-\left(-\sqrt{2}/2\right)}{2}} = \sqrt{\dfrac{2+\sqrt{2}}{4}} = \dfrac{\sqrt{2+\sqrt{2}}}{2}$

**37.** $\sin\alpha = \dfrac{5}{13},\ \cos\alpha = -\sqrt{1-\left(\dfrac{5}{13}\right)^2} = -\sqrt{1-\dfrac{25}{169}} = -\dfrac{12}{13}.$

$\sin\dfrac{\alpha}{2} = \sqrt{\dfrac{1-\cos\alpha}{2}} = \sqrt{\dfrac{1-(-12/13)}{2}} = \sqrt{\dfrac{13+12}{26}} = \sqrt{\dfrac{25}{26}} = \dfrac{5}{\sqrt{26}} = \dfrac{5\sqrt{26}}{26}$

$\cos\dfrac{\alpha}{2} = \sqrt{\dfrac{1+\cos\alpha}{2}} = \sqrt{\dfrac{1-12/13}{2}} = \sqrt{\dfrac{13-12}{26}} = \sqrt{\dfrac{1}{26}} = \dfrac{1}{\sqrt{26}} = \dfrac{\sqrt{26}}{26}$

$\tan\dfrac{\alpha}{2} = \dfrac{1-\cos\alpha}{\sin\alpha} = \dfrac{1+\frac{12}{13}}{\frac{5}{13}} = \dfrac{13+12}{5} = 5$

**38.** $\sin\alpha = -\frac{7}{25},\ \cos\alpha = -\sqrt{1-\left(-\frac{7}{25}\right)^2} = -\sqrt{1-\frac{49}{625}} = -\frac{24}{25}$

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1-(-24/25)}{2}} = \sqrt{\frac{25+24}{50}} = \sqrt{\frac{49}{50}} = \frac{7\sqrt{2}}{10}$$

$$\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}} = -\sqrt{\frac{1-24/25}{2}} = -\sqrt{\frac{25-24}{50}} = -\sqrt{\frac{1}{50}} = -\frac{\sqrt{2}}{10}$$

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{1+\frac{24}{25}}{-\frac{7}{25}} = \frac{25+24}{-7} = -7$$

**39.** $\cos\alpha = -\frac{8}{17},\ \sin\alpha = -\sqrt{1-\left(-\frac{8}{17}\right)^2} = -\sqrt{1-\frac{64}{289}} = -\frac{15}{17}$

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1-(-8/17)}{2}} = \sqrt{\frac{17+8}{34}} = \sqrt{\frac{25}{34}} = \frac{5\sqrt{34}}{34}$$

$$\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}} = -\sqrt{\frac{1-8/17}{2}} = -\sqrt{\frac{17-8}{34}} = -\sqrt{\frac{9}{34}} = -\frac{3\sqrt{34}}{34}$$

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{1+\frac{18}{17}}{-\frac{15}{17}} = \frac{17+8}{-15} = -\frac{5}{3}$$

**40.** $\cos\alpha = \frac{12}{13},\ \sin\alpha = \sqrt{1-\left(\frac{12}{13}\right)^2} = \sqrt{1-\frac{144}{169}} = \frac{5}{13}$

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1-12/13}{2}} = \sqrt{\frac{13-12}{26}} = \sqrt{\frac{1}{26}} = \frac{\sqrt{26}}{26}$$

$$\cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}} = \sqrt{\frac{1+12/13}{2}} = \sqrt{\frac{13+12}{26}} = \sqrt{\frac{25}{26}} = \frac{5\sqrt{26}}{26}$$

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{1-\frac{12}{13}}{\frac{5}{13}} = \frac{13-12}{5} = \frac{1}{5}$$

**41.** $\tan\alpha = \frac{4}{3},\ r = \sqrt{3^2+4^2} = \sqrt{25} = 5,\ \sin\alpha = \frac{4}{5},\ \cos = \frac{3}{5}$

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1-3/5}{2}} = \sqrt{\frac{5-3}{10}} = \sqrt{\frac{1}{5}} = \frac{\sqrt{5}}{5}$$

$$\cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}} = \sqrt{\frac{1+3/5}{2}} = \sqrt{\frac{5+3}{10}} = \sqrt{\frac{4}{5}} = \frac{2\sqrt{5}}{5}$$

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{1-\frac{3}{5}}{\frac{4}{5}} = \frac{5-3}{4} = \frac{1}{2}$$

**42.** $\tan\alpha = -\frac{8}{15},\ r = \sqrt{8^2+15^2} = \sqrt{64+225} = 17,\ \sin\alpha = \frac{8}{17},\ \cos = -\frac{15}{17}$

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1-(-15/17)}{2}} = \sqrt{\frac{17+15}{34}} = \sqrt{\frac{32}{34}} = \sqrt{\frac{16}{17}} = \frac{4}{\sqrt{17}} = \frac{4\sqrt{17}}{17}$$

$$\cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}} = \sqrt{\frac{1+(-15/17)}{2}} = \sqrt{\frac{17-15}{34}} = \sqrt{\frac{2}{34}} = \sqrt{\frac{1}{17}} = \frac{\sqrt{17}}{17}$$

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{1-\left(-\frac{15}{17}\right)}{\frac{8}{17}} = \frac{17+15}{8} = 4$$

**43.** $\cos\alpha = \frac{24}{25},\ \sin\alpha = -\sqrt{1-\left(\frac{24}{25}\right)^2} = -\sqrt{1-\frac{576}{625}} = -\frac{7}{25}$

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1-24/25}{2}} = \sqrt{\frac{25-24}{50}} = \sqrt{\frac{1}{50}} = \frac{\sqrt{2}}{10}$$

$$\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}} = -\sqrt{\frac{1+24/25}{2}} = -\sqrt{\frac{25+24}{50}} = -\sqrt{\frac{49}{50}} = -\frac{7\sqrt{2}}{10}$$

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{1-\frac{24}{25}}{-\frac{7}{25}} = \frac{25-24}{-7} = -\frac{1}{7}$$

**44.** $\sin\alpha = -\frac{9}{41},\ \cos\alpha = \sqrt{1-\left(-\frac{9}{41}\right)^2} = \sqrt{1-\frac{81}{1681}} = \frac{40}{41}$

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1-40/41}{2}} = \sqrt{\frac{41-40}{82}} = \sqrt{\frac{1}{82}} = \frac{\sqrt{82}}{82}$$

$$\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}} = -\sqrt{\frac{1+40/41}{2}} = -\sqrt{\frac{41+40}{82}} = -\sqrt{\frac{81}{82}} = -\frac{9\sqrt{82}}{82}$$

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{1-\frac{40}{41}}{-\frac{9}{41}} = \frac{41-40}{-9} = -\frac{1}{9}$$

**45.** $$\sin 3x\cos 3x = \frac{1}{2}(2\sin 3x\cos 3x) = \frac{1}{2}\sin 2(3x) = \frac{1}{2}\sin 6x$$

**46.** $$\cos 8x = \cos 2(4x) = \cos^2 4x - \sin^2 4x$$

**47.** $$\sin^2 x + \cos 2x = \sin^2 x + \cos^2 x - \sin^2 x = \cos^2 x$$

**48.** $$\frac{\cos 2x}{\sin^2 x} = \frac{\cos^2 x - \sin^2 x}{\sin^2 x} = \frac{\cos^2 x}{\sin^2 x} - \frac{\sin^2 x}{\sin^2 x} = \cot^2 x - 1$$

**49.** $$\frac{1+\cos 2x}{\sin 2x} = \frac{1+2\cos^2 x - 1}{2\sin x\cos x} = \frac{2\cos^2 x}{2\sin x\cos x} = \cot x$$

**50.** $$\frac{1}{1-\cos 2x} = \frac{1}{1-1+2\sin^2 x} = \frac{1}{2\sin^2 x} = \frac{1}{2}\csc^2 x$$

**51.** $$\frac{\sin 2x}{1-\sin^2 x} = \frac{2\sin x\cos x}{\cos^2 x} = 2\tan x$$

**52.** $$\frac{\cos^2 x - \sin^2 x}{2\sin x\cos x} = \frac{\cos 2x}{\sin 2x} = \cot 2x$$

**53.** $$\frac{\cos 2x}{\cos^2 x} = \frac{\cos^2 x - \sin^2 x}{\cos^2 x} = \frac{\cos^2 x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} = 1 - \tan^2 x$$

**54.** $$\frac{2\sin x\cos x}{\cos^2 x - \sin^2 x} = \frac{\sin 2x}{\cos 2x} = \tan 2x$$

**55.** $$\sin 2x - \tan x = 2\sin x\cos x - \frac{\sin x}{\cos x} = \frac{2\sin x\cos^2 x - \sin x}{\cos x} = \frac{\sin x(2\cos^2 x - 1)}{\cos x} = \tan x\cos 2x$$

**56.** $$\sin 2x - \cot x = 2\sin x\cos x - \frac{\cos x}{\sin x} = \frac{2\sin^2 x\cos x - \cos x}{\sin x} = \frac{\cos x(2\sin^2 x - 1)}{\sin x} = \cot x(-\cos 2x) = -\cot x\cos 2x$$

**57.** $$\cos^4 x - \sin^4 x = (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x) = \cos^2 x - \sin^2 x = \cos 2x$$

**58.** $$\sin 4x = 2\sin 2x\cos 2x = 2(2\sin x\cos x)(\cos^2 x - \sin^2 x) = 4\sin x\cos^3 x - 4\sin^3 x\cos x$$

**59.** $$\cos^2 x - 2\sin^2 x\cos^2 x - \sin^2 x + 2\sin^4 x = \cos^2 x(1-2\sin^2 x) - \sin^2 x(1-2\sin^2 x) = (1-2\sin^2 x)(\cos^2 x - \sin^2 x) = \cos 2x\cos 2x = \cos^2 2x$$

**60.** $$2\cos^4 x - \cos^2 x - 2\sin^2 x\cos^2 x + \sin^2 x = \cos^2 x(2\cos^2 x - 1) - \sin^2 x(2\cos^2 x - 1) = (2\cos^2 x - 1)(\cos^2 x - \sin^2 x) = \cos 2x\cdot\cos 2x = \cos^2 2x$$

**61.** $$\cos 4x = \cos 2(2x) = 2\cos^2 2x - 1 = 2(2\cos^2 x - 1)^2 - 1 = 2(4\cos^4 x - 4\cos^2 x + 1) - 1 = 8\cos^4 x - 8\cos^2 x + 1$$

**62.** $$\sin 4x = \sin 2(2x) = 2\sin 2x\cos 2x = 2(2\sin x\cos x)(1-2\sin^2 x) = 4\sin x\cos x - 8\sin^3 x\cos x$$

**63.** $\cos 3x - \cos x = \cos(2x + x) - \cos x$
$= \cos 2x \cos x - \sin 2x \sin x - \cos x$
$= (2\cos^2 x - 1)\cos x - 2\sin x \cos x \cdot \sin x - \cos x$
$= 2\cos^3 x - \cos x - 2\sin^2 x \cos x - \cos x$
$= 2\cos^3 x - 2\cos x - 2\sin^2 x \cos x$
$= 2\cos^3 x - 2\cos x - 2(1 - \cos^2 x)\cos x$
$= 2\cos^3 x - 2\cos x - 2\cos x + 2\cos^3 x$
$= 4\cos^3 x - 4\cos x$

**64.** $\sin 3x + \sin x = \sin(2x + x) + \sin x$
$= \sin 2x \cos x + \cos 2x \sin x + \sin x$
$= (2\sin x \cos x)\cos x + (1 - 2\sin^2 x)\sin x + \sin x$
$= 2\sin x \cos^2 x + \sin x - 2\sin^3 x + \sin x$
$= 2\sin x(1 - \sin^2 x) + 2\sin x - 2\sin^3 x$
$= 2\sin x - 2\sin^3 x + 2\sin x - 2\sin^3 x$
$= 4\sin x - 4\sin^3 x$

**65.** $\sin^3 x + \cos^3 x = (\sin x + \cos x)(\sin^2 x - \sin x \cos x + \cos^2 x)$
$= (\sin x + \cos x)\left(\sin^2 x + \cos^2 x - \frac{2\sin x \cos x}{2}\right)$
$= (\sin x + \cos x)\left(1 - \frac{1}{2}\sin 2x\right)$

**66.** $\cos^3 x - \sin^3 x = (\cos x - \sin x)(\cos^2 x + \sin x \cos x + \sin^2 x)$
$= (\cos x - \sin x)\left(\cos^2 x + \sin^2 x + \frac{2\sin x \cos x}{2}\right)$
$= (\cos x - \sin x)\left(1 + \frac{1}{2}\sin 2x\right)$

**67.** $\sin^2 \frac{x}{2} = \left[\pm\sqrt{\frac{1 - \cos x}{2}}\right]^2$
$= \frac{1 - \cos x}{2}$
$= \frac{1 - \cos x}{2} \cdot \frac{\sec x}{\sec x}$
$= \frac{\sec x - 1}{2\sec x}$

**68.** $\cos^2 \frac{x}{2} = \left[\pm\sqrt{\frac{1 + \cos x}{2}}\right]^2$
$= \frac{1 + \cos x}{2}$
$= \frac{1 + \cos x}{2} \cdot \frac{\sec x}{\sec x}$
$= \frac{\sec x + 1}{2\sec x}$

**69.** $\tan \frac{x}{2} = \frac{1 - \cos x}{\sin x}$
$= \frac{1}{\sin x} - \frac{\cos x}{\sin x}$
$= \csc x - \cot x$

**70.** $\tan \frac{x}{2} = \frac{\sin x}{1 + \cos x}$
$= \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \frac{\cos x}{\cos x}}$
$= \frac{\tan x}{\sec x + 1}$

**71.** $2\sin \frac{x}{2} \cos \frac{x}{2} = \sin 2\left(\frac{x}{2}\right)$
$= \sin x$

**72.** $\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \cos 2\left(\frac{x}{2}\right)$
$= \cos x$

**73.** $$\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2 = \cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}+\sin^2\frac{x}{2}$$
$$= \cos^2\frac{x}{2}+\sin^2\frac{x}{2}+\sin 2\left(\frac{x}{2}\right)$$
$$= 1+\sin x$$

**74.** $$\tan^2\frac{x}{2} = \left(\frac{1-\cos x}{\sin x}\right)^2$$
$$= \frac{(1-\cos x)^2}{\sin^2 x}$$
$$= \frac{(1-\cos x)^2}{1-\cos^2 x}$$
$$= \frac{(1-\cos x)^2}{(1-\cos x)(1+\cos x)}$$
$$= \frac{1-\cos x}{1+\cos x}$$
$$= \frac{\frac{1}{\cos x}-\frac{\cos x}{\cos x}}{\frac{1}{\cos x}+\frac{\cos x}{\cos x}}$$
$$= \frac{\sec x-1}{\sec x+1}$$

**75.** $$\sin^2\frac{x}{2}\sec x = \left(\pm\sqrt{\frac{1-\cos x}{2}}\right)^2\sec x$$
$$= \frac{1-\cos x}{2}\cdot\sec x$$
$$= \frac{1}{2}(\sec x-1)$$

**76.** $$\cos^2\frac{x}{2}\sec x = \left(\pm\sqrt{\frac{1+\cos x}{2}}\right)^2\sec x$$
$$= \frac{1+\cos x}{2}\cdot\sec x$$
$$= \frac{1}{2}(\sec x+1)$$

**77.** $$\cos^2\frac{x}{2}-\cos x = \left(\pm\sqrt{\frac{1+\cos x}{2}}\right)^2-\cos x$$
$$= \frac{1+\cos x}{2}-\cos x$$
$$= \frac{1+\cos x-2\cos x}{2}$$
$$= \frac{1-\cos x}{2}$$
$$= \sin^2\frac{x}{2}$$

**78.** $$\sin^2\frac{x}{2}+\cos x = \left(\pm\sqrt{\frac{1-\cos x}{2}}\right)^2+\cos x$$
$$= \frac{1-\cos x}{2}+\cos x$$
$$= \frac{1-\cos x+2\cos x}{2}$$
$$= \frac{1+\cos x}{2}$$
$$= \cos^2\frac{x}{2}$$

**79.** $$\sin^2\frac{x}{2}-\cos^2\frac{x}{2} = -\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)$$
$$= -\cos 2\left(\frac{x}{2}\right)$$
$$= -\cos x$$

**80.** $$\cos^2\frac{x}{2}-\sin^2\frac{x}{2} = \cos x$$
$$= \frac{2\sin x\cos x}{2\sin x}$$
$$= \frac{1}{2}\csc x\sin 2x$$

**81.** $$\sin 2x-\cos x = 2\sin x\cos x-\cos x$$
$$= (\cos x)(2\sin x-1)$$

**82.** $$\frac{\cos 2x}{\sin^2 x} = \frac{1-2\sin^2 x}{\sin^2 x}$$
$$= \frac{1}{\sin^2 x}-\frac{2\sin^2 x}{\sin^2 x}$$
$$= \csc^2 x-2$$

**83.** $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$
$= \dfrac{\frac{2\tan x}{\tan x}}{\frac{1}{\tan x} - \frac{\tan^2 x}{\tan x}}$
$= \dfrac{2}{\cot x - \tan x}$

**84.** $\dfrac{2\cos 2x}{\sin 2x} = \dfrac{2\left(\cos^2 x - \sin^2 x\right)}{2\sin x\cos x}$
$= \dfrac{\cos^2 x}{\sin x\cos x} - \dfrac{\sin^2 x}{\sin x\cos x}$
$= \cot x - \tan x$

**85.** $\dfrac{\sin^2 x + 1 - \cos^2 x}{\sin x(1+\cos x)} = \dfrac{1-\cos^2 x + 1 - \cos^2 x}{\sin x(1+\cos x)}$
$= \dfrac{2(1-\cos^2 x)}{\sin x(1+\cos x)}$
$= \dfrac{2(1-\cos x)(1+\cos x)}{\sin x(1+\cos x)}$
$= \dfrac{2(1-\cos x)}{\sin x}$
$= 2\tan\dfrac{x}{2}$

**86.** $\dfrac{1}{2}\csc^2\dfrac{x}{2} = \dfrac{1}{2\sin^2\frac{x}{2}}$
$= \dfrac{1}{2\left(\pm\sqrt{\frac{1-\cos x}{2}}\right)^2}$
$= \dfrac{1}{2\left(\frac{1-\cos x}{2}\right)} = \dfrac{1}{1-\cos x}$
$= \dfrac{1}{1-\cos x}\cdot\dfrac{1+\cos x}{1+\cos x}$
$= \dfrac{1+\cos x}{1-\cos^2 x} = \dfrac{1+\cos x}{\sin^2 x}$
$= \dfrac{1}{\sin^2 x} + \dfrac{\cos x}{\sin^2 x}$
$= \csc^2 x + \cot x\csc x$

**87.** $\csc 2x = \dfrac{1}{\sin 2x}$
$= \dfrac{1}{2\sin x\cos x}$
$= \dfrac{1}{2}\csc x\sec x$

**88.** $\sec 2x = \dfrac{1}{\cos 2x}$
$= \dfrac{1}{2\cos^2 x - 1}$
$= \dfrac{1}{2\cos^2 x - 1}\cdot\dfrac{\sec^2 x}{\sec^2 x}$
$= \dfrac{\sec^2 x}{2-\sec^2 x}$

**89.** $\cos\dfrac{x}{5} = \cos 2\left(\dfrac{x}{10}\right)$
$= 1 - 2\sin^2\dfrac{x}{10}$

**90.** $\sec^2\dfrac{x}{2} = \dfrac{1}{\cos^2\frac{x}{2}}$
$= \dfrac{1}{\left(\pm\sqrt{\frac{1+\cos x}{2}}\right)^2}$
$= \dfrac{1}{\frac{1+\cos x}{2}}$
$= \dfrac{2}{1+\cos x}$

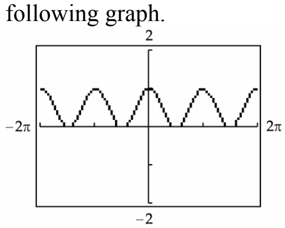
**91.** **a.**
$$M = \frac{1}{\sin\frac{(\pi/4)}{2}} = 1 \div \sqrt{\frac{1-\cos\frac{\pi}{4}}{2}}$$
$$= 1 \div \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}$$
$$= 1 \div \sqrt{\frac{2-\sqrt{2}}{4}}$$
$$= \frac{2}{\sqrt{2-\sqrt{2}}}$$
$$\approx 2.61$$

**c.** As $M$ increases $\frac{1}{M}$ decreases. So if $\sin^{-1}\left(\frac{1}{M}\right)$ decreases, then $\alpha$ decreases.

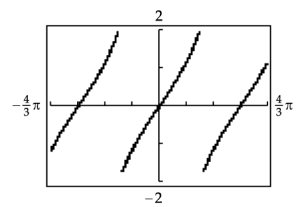
**b.**
$$M\sin\frac{\alpha}{2} = 1$$
$$\sin\frac{\alpha}{2} = \frac{1}{M}$$
$$\frac{\alpha}{2} = \sin^{-1}\left(\frac{1}{M}\right)$$
$$\alpha = 2\sin^{-1}\left(\frac{1}{M}\right)$$

## Connecting Concepts

**92.** $y = \sin^2 x + \cos 2x$ and $y = \cos^2 x$ both have the following graph.

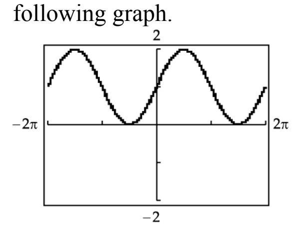



**93.** $y = \frac{\sin 2x}{1-\sin^2 x}$ and $y = 2\tan x$ both have the following graph.




**94.** $y = \sin\frac{x}{2}\cos\frac{x}{2}$ and $y = \sin x$ do not have the same graph.

**95.** $y = \left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2$ and $y = 1 + \sin x$ both have the following graph.




**96.**
$$\frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = \frac{(\sin x + \cos x)(\sin^2 x - \sin x\cos x + \cos^2 x)}{\sin x + \cos x}$$
$$= \sin^2 x + \cos^2 x - \frac{2\sin x\cos x}{2}$$
$$= 1 - \tfrac{1}{2}\sin 2x$$

**97.**
$$\cos^4 x = \cos^2 x \cdot \cos^2 x$$
$$= \frac{\cos 2x + 1}{2} \cdot \frac{\cos 2x + 1}{2}$$
$$= \tfrac{1}{4}\left(\cos^2 2x + 2\cos 2x + 1\right)$$
$$= \tfrac{1}{4}\left(\frac{\cos 4x + 1}{2} + 2\cos 2x + 1\right)$$
$$= \tfrac{1}{8}\cos 4x + \tfrac{1}{8} + \tfrac{1}{2}\cos 2x + \tfrac{1}{4}$$
$$= \tfrac{1}{8}\cos 4x + \tfrac{1}{2}\cos 2x + \tfrac{3}{8}$$

**98.** $\dfrac{\sin x-\sin 2x}{\cos x+\cos 2x}=\dfrac{\sin x-2\sin x\cos x}{\cos x+2\cos^2 x-1}=\dfrac{\sin x-2\sin x\cos x}{2\cos^2 x+\cos x-1}=\dfrac{\sin x(1-2\cos x)}{(2\cos x-1)(\cos x+1)}=\dfrac{-\sin x}{\cos x+1}=-\tan\dfrac{x}{2}$

## Prepare for Section 6.4

**PS1.** $\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$
$=\frac{1}{2}[\sin\alpha\cos\beta+\cos\alpha\sin\beta+\sin\alpha\cos\beta-\cos\alpha\sin\beta]$
$=\sin\alpha\cos\beta$

**PS2.** $\frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$
$=\frac{1}{2}[\cos\alpha\cos\beta-\sin\alpha\sin\beta+\cos\alpha\cos\beta-\sin\alpha\sin\beta]$
$=\cos\alpha\cos\beta$

**PS3.** $\sin\pi-\sin\frac{\pi}{6}=0-\frac{1}{2}=-\frac{1}{2}$

$$2\cos\left(\frac{\pi+\frac{\pi}{6}}{\frac{2}{1}}\right)\sin\left(\frac{\pi+\frac{\pi}{6}}{2}\right)=2\cos\left(\frac{7\pi}{6}\right)\cos\left(\frac{5\pi}{6}\right)$$
$$2\left(-\sqrt{\frac{1+\cos\left(\frac{7\pi}{6}\right)}{2}}\right)+\left(\sqrt{\frac{1-\cos\left(\frac{5\pi}{6}\right)}{2}}\right)$$
$$=-2\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$$
$$=-\sqrt{1-\frac{3}{4}}=-\sqrt{\frac{1}{4}}$$
$$=-\frac{1}{2}$$

Both functional values equal $-\frac{1}{2}$.

**PS4.** $\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)=\sqrt{2}\left[\sin x\cos\left(\frac{\pi}{4}\right)+\cos x\sin\left(\frac{\pi}{4}\right)\right]$
$=\sqrt{2}\left[(\sin x)\left(\frac{\sqrt{2}}{2}\right)+(\cos x)\left(\frac{\sqrt{2}}{2}\right)\right]$
$=\sin x+\cos x$

**PS5.** Answers will vary.

**PS6.** $\sqrt{(-1)^2+\left(\sqrt{3}\right)^2}=\sqrt{1+3}=\sqrt{4}=2$

## Section 6.4

**1.** $2\sin x\cos 2x=2\cdot\frac{1}{2}[\sin(x+2x)+\sin(x-2x)]$
$=\sin 3x+\sin(-x)$
$=\sin 3x-\sin x$

**2.** $2\sin 4x\sin 2x=2\cdot\frac{1}{2}[\cos(4x-2x)-\cos(4x+2x)]$
$=\cos 2x-\cos 6x$

**3.** $\cos 6x\sin 2x=\frac{1}{2}[\sin(6x+2x)-\sin(6x-2x)]$
$=\frac{1}{2}[\sin 8x-\sin 4x]$

**4.** $\cos 3x\cos 5x=\frac{1}{2}[\cos(3x+5x)+\cos(3x-5x)]$
$=\frac{1}{2}[\cos 8x+\cos(-2x)]$
$=\frac{1}{2}(\cos 8x+\cos 2x)$

**5.** $2\sin 5x\cos 3x=\sin(5x+3x)+\sin(5x-3x)$
$=\sin 8x+\sin 2x$

**6.** $2\sin 2x\cos 6x=\sin(2x+6x)+\sin(2x-6x)$
$=\sin 8x+\sin(-4x)$
$=\sin 8x-\sin 4x$

**7.** $$\sin x\cos 5x = \frac{1}{2}\left[\cos(x-5x)-\cos(x+5x)\right] = \frac{1}{2}\left[\cos(-4x)-\cos 6x\right] = \frac{1}{2}(\cos 4x-\cos 6x)$$

**8.** $$\cos 3x\sin x = \frac{1}{2}\left[\sin(3x+x)-\sin(3x-x)\right] = \frac{1}{2}(\sin 4x-\sin 2x)$$

**9.** $$\cos 75^\circ\cos 15^\circ = \frac{1}{2}\left[\cos(75^\circ+15^\circ)+\cos(75^\circ-15^\circ)\right] = \frac{1}{2}(\cos 90^\circ+\cos 60^\circ) = \frac{1}{2}\left(0+\frac{1}{2}\right) = \frac{1}{4}$$

**10.** $$\sin 105^\circ\cos 15^\circ = \frac{1}{2}\left[\sin(105^\circ+15^\circ)+\sin(105^\circ-15^\circ)\right] = \frac{1}{2}(\sin 120^\circ+\sin 90^\circ) = \frac{1}{2}\left(\frac{\sqrt{3}}{2}+1\right) = \frac{\sqrt{3}+2}{4}$$

**11.** $$\cos 157.5^\circ\sin 22.5^\circ = \frac{1}{2}\left[\sin(157.5^\circ+22.5^\circ)-\sin(157.5^\circ-22.5^\circ)\right] = \frac{1}{2}(\sin 180^\circ-\sin 135^\circ) = \frac{1}{2}\left(0-\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{4}$$

**12.** $$\sin 195^\circ\cos 15^\circ = \frac{1}{2}\left[\sin(195^\circ+15^\circ)+\sin(195^\circ-15^\circ)\right] = \frac{1}{2}(\sin 210^\circ+\sin 180^\circ) = \frac{1}{2}\left(-\frac{1}{2}+0\right) = -\frac{1}{4}$$

**13.** $$\sin\frac{13\pi}{12}\cos\frac{\pi}{12} = \frac{1}{2}\left[\sin\left(\frac{13\pi}{12}+\frac{\pi}{12}\right)+\sin\left(\frac{13\pi}{12}-\frac{\pi}{12}\right)\right] = \frac{1}{2}\left(\sin\frac{7\pi}{6}+\sin\pi\right) = \frac{1}{2}\left(-\frac{1}{2}+0\right) = -\frac{1}{4}$$

**14.** $$\sin\frac{11\pi}{12}\sin\frac{7\pi}{12} = \frac{1}{2}\left[\cos\left(\frac{11\pi}{12}-\frac{7\pi}{12}\right)-\cos\left(\frac{11\pi}{12}+\frac{7\pi}{12}\right)\right] = \frac{1}{2}\left(\cos\frac{\pi}{3}-\cos\frac{3\pi}{2}\right) = \frac{1}{2}\left(\frac{1}{2}-0\right) = \frac{1}{4}$$

**15.** $$\sin\frac{\pi}{12}\cos\frac{7\pi}{12} = \frac{1}{2}\left[\sin\left(\frac{\pi}{12}+\frac{7\pi}{12}\right)+\sin\left(\frac{\pi}{12}-\frac{7\pi}{12}\right)\right] = \frac{1}{2}\left[\sin\frac{2\pi}{3}+\sin\left(-\frac{\pi}{2}\right)\right] = \frac{1}{2}\left(\sin\frac{2\pi}{3}-\sin\frac{\pi}{2}\right) = \frac{1}{2}\left(\frac{\sqrt{3}}{2}-1\right) = \frac{\sqrt{3}-2}{4}$$

**16.** $$\cos\frac{17\pi}{12}\sin\frac{7\pi}{12} = \frac{1}{2}\left[\sin\left(\frac{17\pi}{12}+\frac{7\pi}{12}\right)-\sin\left(\frac{17\pi}{12}-\frac{7\pi}{12}\right)\right] = \frac{1}{2}\left(\sin 2\pi-\sin\frac{5\pi}{6}\right) = \frac{1}{2}\left(0-\frac{1}{2}\right) = -\frac{1}{4}$$

**17.** $$\sin 4\theta+\sin 2\theta = 2\sin\frac{4\theta+2\theta}{2}\cos\frac{4\theta-2\theta}{2} = 2\sin 3\theta\cos\theta$$

**18.** $\cos 5\theta - \cos 3\theta = -2\sin\frac{5\theta+3\theta}{2}\sin\frac{5\theta-3\theta}{2}$
$= -2\sin 4\theta \sin\theta$

**19.** $\cos 3\theta + \cos\theta = 2\cos\frac{3\theta+\theta}{2}\cos\frac{3\theta-\theta}{2}$
$= 2\cos 2\theta\cos\theta$

**20.** $\sin 7\theta - \sin 3\theta = 2\cos\frac{7\theta+3\theta}{2}\sin\frac{7\theta-3\theta}{2}$
$= 2\cos 5\theta\sin 2\theta$

**21.** $\cos 6\theta - \cos 2\theta = -2\sin\frac{6\theta+2\theta}{2}\sin\frac{6\theta-2\theta}{2}$
$= -2\sin 4\theta\sin 2\theta$

**22.** $\cos 3\theta + \cos 5\theta = 2\cos\frac{3\theta+5\theta}{2}\cos\frac{3\theta-5\theta}{2}$
$= 2\cos 4\theta\cos(-\theta)$
$= 2\cos 4\theta\cos\theta$

**23.** $\cos\theta + \cos 7\theta = 2\cos\frac{\theta+7\theta}{2}\cos\frac{\theta-7\theta}{2}$
$= 2\cos 4\theta\cos(-3\theta)$
$= 2\cos 4\theta\cos 3\theta$

**24.** $\sin 3\theta + \sin 7\theta = 2\sin\frac{3\theta+7\theta}{2}\cos\frac{3\theta-7\theta}{2}$
$= 2\sin 5\theta\cos(-2\theta)$
$= 2\sin 5\theta\cos 2\theta$

**25.** $\sin 5\theta + \sin 9\theta = 2\sin\frac{5\theta+9\theta}{2}\cos\frac{5\theta-9\theta}{2}$
$= 2\sin 7\theta\cos(-2\theta)$
$= 2\sin 7\theta\cos 2\theta$

**26.** $\cos 5\theta - \cos\theta = -2\sin\frac{5\theta+\theta}{2}\sin\frac{5\theta-\theta}{2}$
$= -2\sin 3\theta\sin 2\theta$

**27.** $\cos 2\theta - \cos\theta = -2\sin\frac{2\theta+\theta}{2}\sin\frac{2\theta-\theta}{2}$
$= -2\sin\frac{3}{2}\theta\sin\frac{1}{2}\theta$

**28.** $\sin 2\theta + \sin 6\theta = 2\sin\frac{2\theta+6\theta}{2}\cos\frac{2\theta-6\theta}{2}$
$= 2\sin 4\theta\cos(-2\theta)$
$= 2\sin 4\theta\cos 2\theta$

**29.** $\cos\frac{\theta}{2} - \cos\theta = -2\sin\frac{\frac{\theta}{2}+\theta}{2}\sin\frac{\frac{\theta}{2}-\theta}{2}$
$= -2\sin\frac{3}{4}\theta\sin\left(-\frac{1}{4}\theta\right)$
$= 2\sin\frac{3}{4}\theta\sin\frac{1}{4}\theta$

**30.** $\sin\frac{3\theta}{4} + \sin\frac{\theta}{2} = 2\sin\frac{\frac{3\theta}{4}+\frac{\theta}{2}}{2}\cos\frac{\frac{3\theta}{4}-\frac{\theta}{2}}{2}$
$= 2\sin\frac{5}{8}\theta\cos\frac{1}{8}\theta$

**31.** $\sin\frac{\theta}{2} - \sin\frac{\theta}{3} = 2\cos\frac{\frac{\theta}{2}+\frac{\theta}{3}}{2}\sin\frac{\frac{\theta}{2}-\frac{\theta}{3}}{2}$
$= 2\cos\frac{5}{12}\theta\sin\frac{1}{12}\theta$

**32.** $\cos\theta + \cos\frac{\theta}{2} = 2\cos\frac{\theta+\frac{\theta}{2}}{2}\cos\frac{\theta-\frac{\theta}{2}}{2}$
$= 2\cos\frac{3}{4}\theta\cos\frac{1}{4}\theta$

**33.** $\cos(\alpha+\beta) + \cos(\alpha-\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta + \cos\alpha\cos\beta + \sin\alpha\sin\beta$
$= 2\cos\alpha\cos\beta$

**34.** $\cos(\alpha-\beta) - \cos(\alpha+\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta - \cos\alpha\cos\beta + \sin\alpha\sin\beta$
$= 2\sin\alpha\sin\beta$

**35.** $$\begin{aligned}2\cos 3x\sin x &= 2\cdot\frac{1}{2}\left[\sin(3x+x)-\sin(3x-x)\right]\\ &= \sin 4x-\sin 2x\\ &= 2\sin 2x\cos 2x-\sin 2x\\ &= \sin 2x(2\cos 2x-1)\\ &= 2\sin x\cos x\left[2(1-2\sin^2 x)-1\right]\\ &= 4\sin x\cos x-8\sin^3 x\cos x-2\sin x\cos x\\ &= 2\sin x\cos x-8\cos x\sin^3 x\end{aligned}$$

**36.** $$\begin{aligned}\sin 5x\cos 3x &= \frac{1}{2}\left[\sin(5x+3x)+\sin(5x-3x)\right]\\ &= \frac{1}{2}(\sin 8x+\sin 2x)\\ &= \frac{1}{2}(2\sin 4x\cos 4x+2\sin x\cos x)\\ &= \sin 4x\cos 4x+\sin x\cos x\end{aligned}$$

**37.** $$\begin{aligned}2\cos 5x\cos 7x &= 2\cdot\frac{1}{2}\left[\cos(5x+7x)+\cos(5x-7x)\right]\\ &= \cos 12x+\cos(-2x)\\ &= \cos 12x+\cos 2x\\ &= \cos^2 6x-\sin^2 6x+2\cos^2 x-1\end{aligned}$$

**38.** $$\begin{aligned}\sin 3x\cos x &= \frac{1}{2}\left[\sin(3x+x)+\sin(3x-x)\right]\\ &= \frac{1}{2}(\sin 4x+\sin 2x)\\ &= \frac{1}{2}(2\sin 2x\cos 2x+\sin 2x)\\ &= \frac{1}{2}\left[(\sin 2x(2\cos 2x+1)\right]\\ &= \frac{1}{2}\cdot 2\sin x\cos x\left[2(1-2\sin^2 x)+1\right]\\ &= \sin x\cos x(2-4\sin^2 x+1)\\ &= \sin x\cos x(3-4\sin^2 x)\end{aligned}$$

**39.** $$\begin{aligned}\sin 3x-\sin x &= 2\cos\frac{3x+x}{2}\sin\frac{3x-x}{2}\\ &= 2\cos 2x\sin x\\ &= 2(1-2\sin^2 x)\sin x\\ &= 2\sin x-4\sin^3 x\end{aligned}$$

**40.** $$\begin{aligned}\cos 5x-\cos 3x &= -2\sin\frac{5x+3x}{2}\sin\frac{5x-3x}{2}\\ &= 2\sin 4x\sin x\\ &= -2(2\sin 2x\cos 2x\sin x)\\ &= -4\left[2\sin x\cos x(2\cos^2 x-1)\sin x\right]\\ &= -8\sin^2 x(2\cos^3 x-\cos x)\end{aligned}$$

**41.** $$\begin{aligned}\sin 2x+\sin 4x &= 2\cos\frac{2x+4}{2}\cos\frac{2x-x4}{2}\\ &= 2\cos 3x\cos(-x)\\ &= 2\cos 3x\cos x\\ &= 2\cos x\sin(2x+x)\\ &= 2\cos x(\sin 2x\cos x+\cos 2x\sin x)\\ &= 2\cos x[(2\sin x\cos x)\cos x+(2\cos^2 x-1)\sin x]\\ &= 2\cos x\sin x(4\cos^2 x-1)\end{aligned}$$

**42.** $$\begin{aligned}\cos 3x+\cos x &= 2\cos\frac{3x+x}{2}\cos\frac{3x-x}{2}\\ &= 2\cos 2x\cos x\\ &= 2(2\cos^2 x-1)\cos x\\ &= 4\cos^3 x-2\cos x\end{aligned}$$

**43.** $$\begin{aligned}\frac{\sin 3x-\sin x}{\cos 3x-\cos x} &= \frac{2\cos\frac{3x+x}{2}\sin\frac{3x-x}{2}}{-2\sin\frac{3x+x}{2}\sin\frac{3x-x}{2}}\\ &= -\frac{\cos 2x}{\sin 2x}\\ &= -\cot 2x\end{aligned}$$

**44.** $\dfrac{\cos 5x - \cos 3x}{\sin 5x + \sin 3x} = \dfrac{-2\sin\frac{5x+3x}{2}\sin\frac{5x-3x}{2}}{2\sin\frac{5x+3x}{2}\cos\frac{5x-3x}{2}}$
$= -\dfrac{\sin 4x \sin x}{\sin 4x \cos x}$
$= -\tan x$

**45.** $\dfrac{\sin 5x + \sin 3x}{4\sin x\cos^3 x - 4\sin^3 x\cos x} = \dfrac{2\sin\frac{5x+3x}{2}\cos\frac{5x-3x}{2}}{4\sin x\cos x(\cos^2 x - \sin^2 x)}$
$= \dfrac{\sin 4x \cos x}{2\sin x\cos x\cos 2x}$
$= \dfrac{2\sin 2x\cos 2x\cos x}{\sin 2x\cos 2x}$
$= 2\cos x$

**46.** $\dfrac{\cos 4x - \cos 2x}{\sin 2x - \sin 4x} = \dfrac{-2\sin\frac{4x+2x}{2}\sin\frac{4x-2x}{2}}{2\cos\frac{2x+4x}{2}\sin\frac{2x-4x}{2}}$
$= \dfrac{-\sin 3x\sin x}{\cos 3x\sin(-x)}$
$= \dfrac{-\sin 3x\sin x}{-\cos 3x\sin x}$
$= \tan 3x$

**47.** $\sin(x+y)\cos(x-y) = \frac{1}{2}\left[\sin(x+y+x-y)+\sin(x+y-x+y)\right]$
$= \frac{1}{2}\left[\sin 2x + \sin 2y\right]$
$= \frac{1}{2}\left[2\sin x\cos x + 2\sin y\cos y\right]$
$= \sin x\cos x + \sin y\cos y$

**48.** $\sin(x+y)\sin(x-y) = \frac{1}{2}\left[\cos(x+y-x+y)-\cos(x+y+x-y)\right]$
$= \frac{1}{2}\left[\cos 2y - \cos 2x\right]$
$= \frac{1}{2}\left[1-2\sin^2 y - 1 + 2\sin^2 x\right]$
$= \sin^2 x - \sin^2 y$

**49.** $a = -1,\ b = -1,\ k = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2},$
$\alpha$ is a third quadrant angle.
$\sin\beta = \left|\dfrac{-1}{\sqrt{2}}\right| = \dfrac{1}{\sqrt{2}}$
$\beta = 45°$
$\alpha = -180° + 45° = -135°$
$y = \sqrt{2}\sin(x - 135°)$

**50.** $a = \sqrt{3},\ b = -1,\ k = \sqrt{\left(\sqrt{-3}\right)^2 + (-1)^2} = 2,$
$\alpha$ is a fourth quadrant angle.
$\sin\beta = \left|-\dfrac{1}{2}\right| = \dfrac{1}{2}$
$\beta = 30°$
$\alpha = -30°$
$y = 2\sin(x - 30°)$

**51.** $a = \frac{1}{2},\ b = -\frac{\sqrt{3}}{2},\ k = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = 1,$

$\alpha$ is a fourth quadrant angle.

$$\sin\beta = \left|\frac{-\frac{\sqrt{3}}{2}}{1}\right| = \frac{\sqrt{3}}{2}$$
$$\beta = 60°$$
$$\alpha = -60°$$
$$y = \sin(x - 60°)$$

**52.** $a = \frac{\sqrt{3}}{2},\ b = -\frac{1}{2},\ k = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = 1,$

$\alpha$ is a fourth quadrant angle.

$$\sin\beta = \left|\frac{-1/2}{1}\right| = \frac{1}{2}$$
$$\beta = 30°$$
$$\alpha = -30°$$
$$y = \sin(x - 30°)$$

**53.** $a = \frac{1}{2},\ b = -\frac{1}{2},\ k = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2},$

$\alpha$ is a fourth quadrant angle.

$$\sin\beta = \left|\frac{-1/2}{\sqrt{2}/2}\right| = \frac{\sqrt{2}}{2}$$
$$\beta = 45°$$
$$\alpha = -45°$$
$$y = \frac{\sqrt{2}}{2}\sin(x - 45°)$$

**54.** $a = -\frac{\sqrt{3}}{2},\ b = -\frac{1}{2},\ k = \sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = 1,$

$\alpha$ is a third quadrant angle.

$$\sin\beta = \left|\frac{-1/2}{1}\right| = \frac{1}{2}$$
$$\beta = 30°$$
$$\alpha = 30° - 180° = -150°$$
$$y = \sin(x - 150°)$$

**55.** $a = -3,\ b = 3,\ k = \sqrt{(-3)^2 + 3^2} = 3\sqrt{2},$

$\alpha$ is a second quadrant angle.

$$\sin\beta = \left|\frac{3}{3\sqrt{2}}\right| = \frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{2}$$
$$\beta = 45°$$
$$\alpha = 180° - 45° = 135°$$
$$y = 3\sqrt{2}\sin(x + 135°)$$

**56.** $a = \frac{\sqrt{2}}{2},\ b = \frac{\sqrt{2}}{2},\ k = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = 1,$

$\alpha$ is a first quadrant angle.

$$\sin\beta = \left|\frac{\sqrt{2}/2}{1}\right| = \frac{\sqrt{2}}{2}$$
$$\beta = 45°$$
$$\alpha = 45°$$
$$y = \sin(x + 45°)$$

**57.** $a = \pi,\ b = -\pi,\ k = \sqrt{\pi^2 + (-\pi)^2} = \pi\sqrt{2},$

$\alpha$ is a fourth quadrant angle.

$$\sin\beta = \left|\frac{-\pi}{\pi\sqrt{2}}\right| = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\beta = 45°$$
$$\alpha = -45°$$
$$y = \pi\sqrt{2}\sin(x - 45°)$$

**58.** $a = -0.4,\ b = 0.4,\ k = \sqrt{(-0.4)^2 + 0.4^2} = 0.4\sqrt{2},$

$\alpha$ is a second quadrant angle.

$$\sin\beta = \left|\frac{0.4}{0.4\sqrt{2}}\right| = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\beta = 45°$$
$$\alpha = 180° - 45° = 135°$$
$$y = 0.4\sqrt{2}\sin(x + 135°)$$

**59.** $a = -1,\ b = 1,\ k = \sqrt{(-1)^2 + 1^2} = \sqrt{2},$

$\alpha$ is a second quadrant angle.

$$\sin\beta = \left|\frac{1}{\sqrt{2}}\right| = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\beta = \frac{\pi}{4}$$
$$\alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$
$$y = \sqrt{2}\sin\left(x + \frac{3\pi}{4}\right)$$

**60.** $a = -\sqrt{3},\ b = -1,\ k = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2,$

$\alpha$ is a third quadrant angle.

$$\sin\beta = \left|-\frac{1}{2}\right| = \frac{1}{2}$$
$$\beta = \frac{\pi}{6}$$
$$\alpha = \frac{\pi}{6} - \pi = -\frac{5\pi}{6}$$
$$y = 2\sin\left(x - \frac{5\pi}{6}\right)$$

**61.** $a=\frac{\sqrt{3}}{2},\ b=\frac{1}{2},\ k=\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2}=1,$

$\alpha$ is a first quadrant angle.

$$\sin\beta=\left|\frac{1/2}{1}\right|=\frac{1}{2}$$
$$\beta=\frac{\pi}{6}$$
$$\alpha=\frac{\pi}{6}$$
$$y=\sin\left(x+\frac{\pi}{6}\right)$$

**62.** $a=1,\ b=\sqrt{3},\ k=\sqrt{(\sqrt{3})^2+(1)^2}=2,$

$\alpha$ is a first quadrant angle.

$$\sin\beta=\left|\frac{\sqrt{3}}{2}\right|=\frac{\sqrt{3}}{2}$$
$$\beta=\frac{\pi}{3}$$
$$\alpha=\frac{\pi}{3}$$
$$y=2\sin\left(x+\frac{\pi}{3}\right)$$

**63.** $a=-10,\ b=10\sqrt{3},\ k=\sqrt{(-10)^2+(10\sqrt{3})^2}=20,$

$\alpha$ is a second quadrant angle.

$$\sin\beta=\frac{10\sqrt{3}}{20}=\frac{\sqrt{3}}{2}$$
$$\beta=\frac{\pi}{3}$$
$$\alpha=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$$
$$y=20\sin\left(x+\frac{2\pi}{3}\right)$$

**64.** $a=3,\ b=-3\sqrt{3},\ k=\sqrt{(3)^2+(-3\sqrt{3})^2}=6,$

$\alpha$ is a fourth quadrant angle.

$$\sin\beta=\left|\frac{-3\sqrt{3}}{6}\right|=\frac{\sqrt{3}}{2}$$
$$\beta=\frac{\pi}{3}$$
$$\alpha=-\frac{\pi}{3}$$
$$y=6\sin\left(x-\frac{\pi}{3}\right)$$

**65.** $a=-5,\ b=5,\ k=\sqrt{(-5)^2+5^2}=5\sqrt{2},$

$\alpha$ is a second quadrant angle.

$$\sin\beta=\left|\frac{5}{5\sqrt{2}}\right|=\frac{\sqrt{2}}{2}$$
$$\beta=\frac{\pi}{4}$$
$$\alpha=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$$
$$y=5\sqrt{2}\sin\left(x+\frac{3\pi}{4}\right)$$

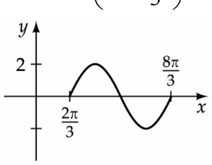
**66.** $a=3,\ b=-3,\ k=\sqrt{3^2+(-3)^2}=3\sqrt{2},$

$\alpha$ is a fourth quadrant angle.

$$\sin\beta=\left|\frac{-3}{3\sqrt{2}}\right|=\frac{\sqrt{2}}{2}$$
$$\beta=\frac{\pi}{4}$$
$$\alpha=-\frac{\pi}{4}$$
$$y=3\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)$$

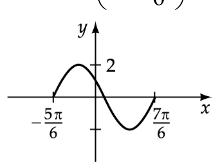
**67.** $y=-\sin x-\sqrt{3}\cos x$

$y=2\sin\left(x-\frac{2\pi}{3}\right)$



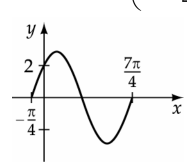

**68.** $y=-\sqrt{3}\sin x+\cos x$

$y=2\sin\left(x+\frac{5\pi}{6}\right)$



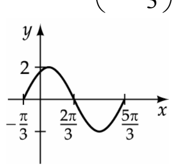

**69.** $y=2\sin x+2\cos x$

$y=2\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$



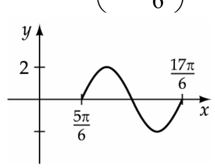

**70.** $y=\sin x+\sqrt{3}\cos x$

$y=2\sin\left(x+\frac{\pi}{3}\right)$



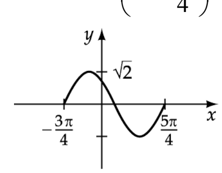

**71.** $y=-\sqrt{3}\sin x-\cos x$

$y=2\sin\left(x-\frac{5\pi}{6}\right)$



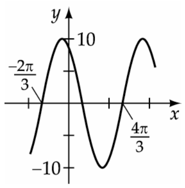

**72.** $y=-\sin x+\cos x$

$y=\sqrt{2}\sin\left(x+\frac{3\pi}{4}\right)$



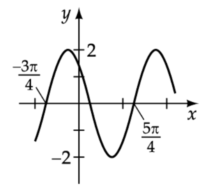
**73.** $y = -5\sin x + 5\sqrt{3}\cos x$

$y = 10\sin\left(x + \dfrac{2\pi}{3}\right)$



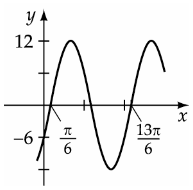

**74.** $y = -\sqrt{2}\sin x + \sqrt{2}\cos x$

$y = 2\sin\left(x + \dfrac{3\pi}{4}\right)$



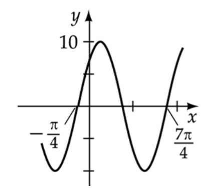

**75.** $y = 6\sqrt{3}\sin x - 6\cos x$

$y = 12\sin\left(x - \dfrac{\pi}{6}\right)$




**76.** $y = 5\sqrt{2}\sin x + 5\sqrt{2}\cos x$

$y = 10\sin\left(x + \dfrac{\pi}{4}\right)$




**77.** **a.** $p(t) = \sin(2\pi \cdot 1336t) + \sin(2\pi \cdot 770t)$

**b.** $p(t) = 2\sin\left(\dfrac{2\pi \cdot 1336t + 2\pi \cdot 770t}{2}\right)\sin\left(\dfrac{2\pi \cdot 1336t - 2\pi \cdot 770t}{2}\right)$

$= 2\sin(2106\pi t)\sin(556\pi t)$

**c.** $\dfrac{1336 + 770}{2} = \dfrac{2106}{2} = 1053$ cycles per second

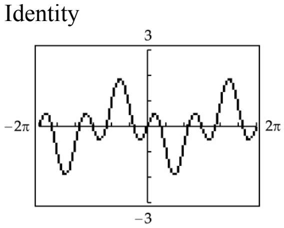
**78.** **a.** $p(t) = \sin(2\pi \cdot 1336t) + \sin(2\pi \cdot 852t)$

**b.** $p(t) = 2\sin\left(\dfrac{2\pi \cdot 1336t + 2\pi \cdot 852t}{2}\right)\sin\left(\dfrac{2\pi \cdot 1336t - 2\pi \cdot 852t}{2}\right)$

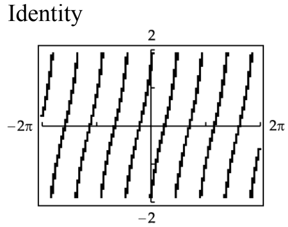
$= 2\sin(2188\pi t)\sin(484\pi t)$

**c.** $\dfrac{1336 + 852}{2} = \dfrac{2188}{2} = 1094$ cycles per second

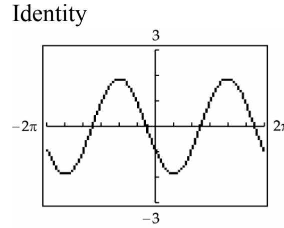
**79.** Identity



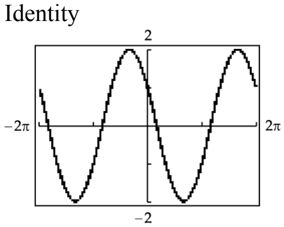

**80.** Identity



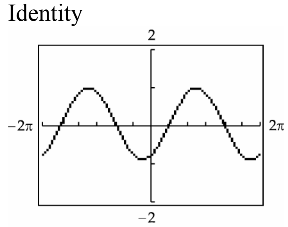

**81.** Identity



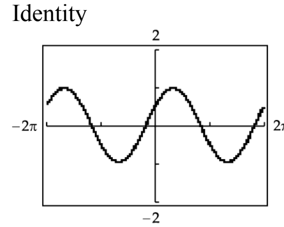

**82.** Identity




**83.** Identity




**84.** Identity



## Connecting Concepts

**85.** Let $x=\alpha+\beta$ and $y=\alpha-\beta$.

$x+y=\alpha+\beta+\alpha-\beta$ and $x-y=\alpha+\beta-(\alpha-\beta)$

$x+y=2\alpha$ $\qquad x-y=2\beta$

$\alpha=\dfrac{x+y}{2}$ $\qquad 2\beta=\dfrac{x-y}{2}$

$$\cos(\alpha-\beta)+\cos(\alpha+\beta)=2\cos\alpha\cos\beta$$

$$\cos\left[\frac{x+y}{2}-\frac{x-y}{2}\right]+\cos\left[\frac{x+y}{2}+\frac{x-y}{2}\right]=2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\cos y+\cos x=2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$

**86.** $\cos(x-y)=\cos x\cos y+\sin x\sin y$

$\cos(x+y)=\cos x\cos y-\sin x\sin y$

$$(\cos x\cos y+\sin x\sin y)-(\cos x\cos y-\sin x\sin y)=\cos(x-y)-\cos(x+y)$$
$$\cos x\cos y+\sin x\sin y-\cos x\cos y+\sin x\sin y=\cos(x-y)-\cos(x+y)$$
$$2\sin x\sin y=\cos(x-y)-\cos(x+y)$$
$$\sin x\sin y=\frac{1}{2}[\cos(x-y)-\cos(x+y)]$$

**87.** $x+y=180°$

$y=180°-x$

$$\begin{aligned}\sin x+\sin y&=\sin x+\sin(180°-x)\\&=\sin x+\sin 180°\cos x-\cos 180°\sin x\\&=\sin x+0(\cos x)-(-1)\sin x\\&=2\sin x\end{aligned}$$

**88.** $x+y=360°$

$y=360°-x$

$$\begin{aligned}\cos x+\cos y&=\cos x+\cos(360°-x)\\&=\cos x+\cos 360°\cos x+\sin 360°\sin x\\&=\cos x+(1)\cos x+(0)\sin x\\&=2\cos x\end{aligned}$$

**89.**
$$\begin{aligned}\sin 2x+\sin 4x+\sin 6x&=2\sin\frac{2x+4x}{2}\cos\frac{2x-4x}{2}+2\sin 3x\cos 3x\\&=2\sin 3x\cos x+2\sin 3x\cos 3x\\&=2\sin 3x(\cos x+\cos 3x)\\&=2\sin 3x\left(2\cos\frac{x+3x}{2}\cos\frac{x-3x}{2}\right)\\&=4\sin 3x\cos 2x\cos x\end{aligned}$$

**90.**
$$\begin{aligned}\sin 4x-\sin 2x+\sin 6x&=2\sin\frac{4x+2x}{2}\sin\frac{4x-2x}{2}+2\sin 3x\cos 3x\\&=2\cos 3x\sin x+2\sin 3x\cos 3x\\&=2\cos 3x(\sin x+\sin 3x)\\&=2\cos 3x\left(2\sin\frac{x+3x}{2}\cos\frac{x-3x}{2}\right)\\&=2\cos 3x(2\sin 2x\cos x)\\&=4\cos 3x\sin 2x\cos x\end{aligned}$$

**91.**
$$\begin{aligned}\frac{\cos 10x+\cos 8x}{\sin 10x-\sin 8x}&=\frac{2\cos\frac{10x+8x}{2}\cos\frac{10x-8x}{2}}{2\cos\frac{10x+8x}{2}\sin\frac{10x-8x}{2}}\\&=\frac{2\cos 9x\cos x}{2\cos 9x\sin x}\\&=\cot x\end{aligned}$$

**92.**
$$\begin{aligned}\frac{\sin 10x+\sin 2x}{\cos 10x+\cos 2x}&=\frac{2\sin\frac{10x+2x}{2}\cos\frac{10x-2x}{2}}{2\cos\frac{10x+2x}{2}\cos\frac{10x-2x}{2}}\\&=\frac{\sin 6x\cos 4x}{\cos 6x\cos 4x}\\&=\tan 6x\\&=\frac{2\tan 3x}{1-\tan^2 3x}\end{aligned}$$

**93.** 
$$\frac{\sin 2x+\sin 4x+\sin 6x}{\cos 2x+\cos 4x+\cos 6x}=\frac{\sin 2x+\sin 6x+\sin 4x}{\cos 2x+\cos 6x+\cos 4x}$$
$$=\frac{2\sin\frac{2x+6x}{2}\cos\frac{2x-6x}{2}+\sin 4x}{2\cos\frac{2x+6x}{2}\cos\frac{2x-6x}{2}+\cos 4x}$$
$$=\frac{2\sin 4x\cos 2x+\sin 4x}{2\cos 4x\cos 2x+\cos 4x}$$
$$=\frac{\sin 4x(2\cos 2x+1)}{\cos 4x(2\cos 2x+1)}$$
$$=\frac{\sin 4x}{\cos 4x}$$
$$=\tan 4x$$

**94.** 
$$\frac{\sin 2x+\sin 6x}{\cos 6x-\cos 2x}=\frac{2\sin\frac{2x+6x}{2}\cos\frac{2x-6x}{2}}{-2\sin\frac{6x+2x}{2}\sin\frac{6x-2x}{2}}$$
$$=-\frac{2\sin 4x\cos 2x}{2\sin 4x\sin 2x}$$
$$=-\frac{\cos 2x}{\sin 2x}$$
$$=-\cot 2x$$

**95.** 
$$\cos^2 x-\sin^2 x=\cos x\cdot\cos x-\sin x\cdot\sin x$$
$$=\frac{1}{2}\left[\cos(x+x)+\cos(x-x)\right]-\frac{1}{2}\left[\cos(x-x)-\cos(x+x)\right]$$
$$=\frac{1}{2}\cos 2x+\frac{1}{2}\cos 0-\frac{1}{2}\cos 0+\frac{1}{2}\cos 2x$$
$$=\cos 2x$$

**96.** 
$$2\sin x\cos x=2\cdot\frac{1}{2}\left[\sin(x+x)+\sin(x-x)\right]$$
$$=\sin 2x+\sin 0$$
$$=\sin 2x$$

**97.** Let $k=\sqrt{a^2+b^2}$, $\tan\alpha=\dfrac{a}{b}$

$$a\sin x+b\cos x=\frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}}(a\sin x+b\cos x)$$
$$=\sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\sin x+\frac{b}{\sqrt{a^2+b^2}}\cos x\right)$$
$$=k(\sin\alpha\sin x+\cos\alpha\cos x)\text{ because }\sin\alpha=\frac{a}{\sqrt{a^2+b^2}}\text{ and }\cos\alpha=\frac{b}{\sqrt{a^2+b^2}}$$
$$=k(\cos x\cos\alpha+\sin x\sin\alpha)=k\cos(x-\alpha)$$

**98.** Let $k=\sqrt{a^2+b^2}$, $\tan\alpha=\dfrac{b}{a}$

$$a\sin cx+b\cos cx=\frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}}(a\sin cx+b\cos cx)$$
$$=\sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\sin cx+\frac{b}{\sqrt{a^2+b^2}}\cos cx\right)$$
$$=k(\cos\alpha\sin cx+\sin\alpha\cos cx)\text{ because }\cos\alpha=\frac{a}{\sqrt{a^2+b^2}}\text{ and }\sin\alpha=\frac{b}{\sqrt{a^2+b^2}}$$
$$=k(\sin cx\cos\alpha+\cos cx\sin\alpha)=k\sin(cx+\alpha)$$

## Prepare for Section 6.5

**PS1.** A one-to-one function is a function for which each range value ($y$-value) is paired with one and only one domain value ($x$-value).

**PS2.** If every horizontal line intersects the graph of a function at most once, then the function is a one-to-one function.

**PS3.**
$$\begin{aligned} f[g(x)] &= f\left[\frac{1}{2}x-2\right] \\ &= 2\left(\frac{1}{2}x-2\right)+4 \\ &= x-4+4 \\ &= x \end{aligned}$$

**PS4.** $f[f^{-1}(x)] = x$

**PS5.** The graph of $f^{-1}$ is the reflection of the graph of $f$ across the line given by $y = x$.

**PS6.** No, it does not pass the horizontal line test.

## Section 6.5

**1.** $y = \sin^{-1} 1$

$\sin y = 1$ with $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$y = \frac{\pi}{2}$

**2.** $y = \sin^{-1} \frac{\sqrt{2}}{2}$

$\sin y = \frac{\sqrt{2}}{2} \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$y = \frac{\pi}{4}$

**3.** $y = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$\cos y = -\frac{\sqrt{3}}{2} \qquad 0 \le y \le \pi$

$y = \frac{5\pi}{6}$

**4.** $y = \cos^{-1}\left(-\frac{1}{2}\right)$

$\cos y = -\frac{1}{2} \qquad 0 \le y \le \pi$

$y = \frac{2\pi}{3}$

**5.** $y = \tan^{-1}(1)$

$\tan y = -1 \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$

$y = -\frac{\pi}{4}$

**6.** $y = \tan^{-1} \sqrt{3}$

$\tan y = \sqrt{3} \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$

$y = \frac{\pi}{3}$

**7.** $y = \cot^{-1} \frac{\sqrt{3}}{3}$

$\cot y = \frac{\sqrt{3}}{3} \qquad 0 < y < \pi$

$y = \frac{\pi}{3}$

**8.** $y = \cot^{-1} 1$

$\cot y = 1 \qquad 0 < y < \pi$

$y = \frac{\pi}{4}$

**9.** $y = \sec^{-1} 2$

$\sec y = 2 \qquad 0 \le y \le \pi$

$y = \frac{\pi}{3}$

**10.** $y = \sec^{-1} \frac{2\sqrt{3}}{3}$

$\sec y = \frac{2\sqrt{3}}{3} \qquad 0 \le y \le \pi$

$y = \frac{\pi}{6}$

**11.** $y = \csc^{-1}\left(-\sqrt{2}\right)$

$\csc y = -\sqrt{2} \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$y = -\frac{\pi}{4}$

**12.** $y = \csc^{-1}(-2)$

$\csc y = -2 \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$y = -\frac{\pi}{6}$

**13.** $y = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$\sin y = -\frac{\sqrt{3}}{2} \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$y = -\frac{\pi}{3}$

**14.** $y = \sin^{-1} \frac{1}{2}$

$\sin y = \frac{1}{2} \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$y = \frac{\pi}{6}$

**15.** $y = \cos^{-1}\left(-\frac{1}{2}\right)$

$\cos y = -\frac{1}{2} \qquad 0 \le y \le \pi$

$y = \frac{2\pi}{3}$

**16.** $y = \cos^{-1}\dfrac{\sqrt{3}}{2}$

$\cos y = \dfrac{\sqrt{3}}{2} \qquad 0 \le y \le \pi$

$y = \dfrac{\pi}{6}$

**17.** $y = \tan^{-1}\dfrac{\sqrt{3}}{3}$

$\tan y = \dfrac{\sqrt{3}}{3} \qquad -\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$

$y = \dfrac{\pi}{6}$

**18.** $y = \tan^{-1}(1)$

$\tan y = 1 \qquad -\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$

$y = \dfrac{\pi}{4}$

**19.** **a.** $\sin^{-1}(0.8422) \approx 1.0014$

**b.** $\tan^{-1}(0.2385) \approx 0.2341$

**20.** **a.** $\cos^{-1}(-0.0356) \approx 1.6064$

**b.** $\tan^{-1}(3.7555) \approx 1.3106$

**21.** **a.** $\sec^{-1}(2.2500) = \cos^{-1}\left(\dfrac{1}{2.2500}\right) \approx 1.1102$

**b.** $\cot^{-1}(3.4545) = \tan^{-1}\left(\dfrac{1}{3.4545}\right) \approx 0.2818$

**22.** **a.** $\csc^{-1}(1.3465) = \sin^{-1}\left(\dfrac{1}{1.3465}\right) \approx 0.8370$

**b.** $\cot^{-1}(0.1274) = \tan^{-1}\left(\dfrac{1}{0.1274}\right) \approx 1.4441$

**23.** $\cos\theta = \dfrac{x}{7}$ or $\theta = \cos^{-1}\left(\dfrac{x}{7}\right)$

**24.** $\tan\theta = \dfrac{5}{x}$ or $\theta = \tan^{-1}\left(\dfrac{5}{x}\right)$

**25.** $y = \cos\left(\cos^{-1}\dfrac{1}{2}\right)$

$y = \cos\dfrac{\pi}{3}$

$y = \dfrac{1}{2}$

**26.** $y = \cos(\cos^{-1} 2)$

$y$ is not defined.

**27.** $y = \tan(\tan^{-1} 2)$

$y = 2$

**28.** $y = \tan\left(\tan^{-1}\dfrac{1}{2}\right)$

$y = \dfrac{1}{2}$

**29.** $y = \sin\left(\tan^{-1}\dfrac{3}{4}\right)$

$y = \dfrac{3}{5}$

**30.** $y = \cos\left(\sin^{-1}\dfrac{5}{13}\right)$

$y = \dfrac{12}{13}$

**31.** $y = \tan\left(\sin^{-1}\dfrac{\sqrt{2}}{2}\right)$

$y = 1$

**32.** $y = \sin\left[\cos^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)\right]$

$y = \dfrac{1}{2}$

**33.** $y = \cos(\sec^{-1} 2)$

$y = \dfrac{1}{2}$

**34.** $y = \sin^{-1}(\sin 2)$

$\approx \sin^{-1}(0.9093)$

$y \approx 1.1416$

**35.** $y = \sin^{-1}\left(\sin\dfrac{\pi}{6}\right)$

$= \sin^{-1}\dfrac{1}{2}$

$y = \dfrac{\pi}{6}$

**36.** $y = \sin^{-1}\left(\sin\dfrac{5\pi}{6}\right)$

$= \sin^{-1}\dfrac{1}{2}$

$y = \dfrac{\pi}{6}$

**37.** $y = \cos^{-1}\left(\sin\dfrac{\pi}{4}\right)$

$= \cos^{-1}\dfrac{\sqrt{2}}{2}$

$y = \dfrac{\pi}{4}$

**38.** $y = \cos^{-1}\left(\cos\dfrac{5\pi}{4}\right)$

$= \cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$

$y = \dfrac{3\pi}{4}$

**39.** $y = \sin^{-1}\left(\tan\dfrac{\pi}{3}\right)$

$= \sin^{-1}\sqrt{3}$

$y$ is not defined.

**40.** $y = \cos^{-1}\left(\tan\dfrac{2\pi}{3}\right)$

$= \cos^{-1}\left(-\sqrt{3}\right)$

$y$ is not defined.

**41.** $y = \tan^{-1}\left(\sin\frac{\pi}{6}\right)$

$= \tan^{-1}\frac{1}{2}$

$y \approx 0.4636$

**42.** $y = \cot^{-1}\left(\cos\frac{2\pi}{3}\right)$

$= \tan^{-1}\left(\frac{-1}{0.5}\right) + \pi$

$y \approx -1.1071 + \pi$

$y \approx 2.0344$

**43.** $y = \sin^{-1}\left(\cos\left[-\frac{2\pi}{3}\right]\right)$

$= \sin^{-1}\left(-\frac{1}{2}\right)$

$y = -\frac{\pi}{6}$

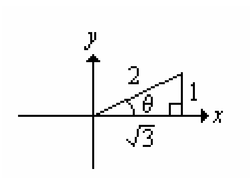
**44.** $y = \cos^{-1}\left[\tan\left(-\frac{\pi}{3}\right)\right]$

$= \cos^{-1}\left(-\sqrt{3}\right)$

$y$ is not defined.

**45.** Let $\theta = \sin^{-1}\frac{1}{2}$ and find $y = \tan\theta$.

Then $\sin\theta = \frac{1}{2}$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

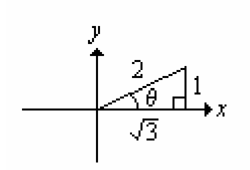



Thus $\tan\theta = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

$y = \frac{\sqrt{3}}{3}$

**46.** Let $\theta = \csc^{-1} 2$ and find $y = \cot\theta$.

Then $\csc\theta = 2$, $\sin\theta = \frac{1}{2}$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

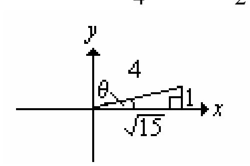



Thus $\cot\theta = \sqrt{3}$.

$y = \sqrt{3}$

**47.** Let $\theta = \sin^{-1}\frac{1}{4}$ and find $y = \sec\theta$.

Then $\sin\theta = \frac{1}{4}$, and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

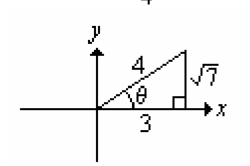



Thus $\sec\theta = \frac{4}{\sqrt{15}} = \frac{4\sqrt{15}}{15}$.

$y = \frac{4\sqrt{15}}{15}$

**48.** Let $\theta = \cos^{-1}\frac{3}{4}$ and find $y = \csc\theta$.

Then $\cos\theta = \frac{3}{4}$, and $0 \le \theta \le \pi$.

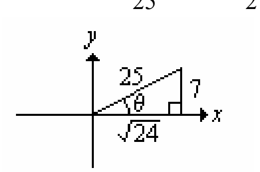



Thus $\csc\theta = \frac{4}{\sqrt{7}} = \frac{4\sqrt{7}}{7}$.

$y = \frac{4\sqrt{7}}{7}$

**49.** Let $\theta = \sin^{-1}\frac{7}{25}$ and find $y = \cos\theta$.

Then $\sin\theta = \frac{7}{25}$, and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

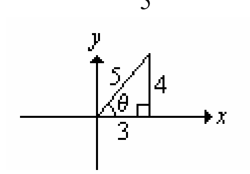



Thus $\cos\theta = \frac{24}{25}$.

$y = \frac{24}{25}$

**50.** Let $\theta = \cos^{-1}\frac{3}{5}$ and find $y = \tan\theta$.

Then $\cos\theta = \frac{3}{5}$, and $0 \le \theta \le \pi$.




Thus $\tan\theta = \frac{4}{3}$.

$y = \frac{4}{3}$

**51.** Let $\alpha = \sin^{-1}\frac{\sqrt{2}}{2},\ \alpha = \frac{\pi}{4},\ \sin\alpha = \frac{\sqrt{2}}{2},\ \cos\alpha = \frac{\sqrt{2}}{2}$.

$$\begin{aligned} y &= \cos\left(2\sin^{-1}\frac{\sqrt{2}}{2}\right) \\ &= \cos 2\alpha \\ &= \cos^2\alpha - \sin^2\alpha \\ &= \left(\frac{\sqrt{2}}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2 \\ &= 0 \end{aligned}$$

**52.** Let $\alpha = \sin^{-1}\frac{\sqrt{3}}{2}$.

Then, $\alpha = \frac{\pi}{3},\ \sin\alpha = \frac{\sqrt{3}}{2},\ \cos\alpha = \frac{1}{2},\ \tan\alpha = \sqrt{3}$.

$$\begin{aligned} y &= \tan\left(2\sin^{-1}\frac{\sqrt{3}}{2}\right) \\ &= \tan 2\alpha \\ &= \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\left(\sqrt{3}\right)}{1-\left(\sqrt{3}\right)^2} = \frac{2\sqrt{3}}{1-3} = \frac{2\sqrt{3}}{-2} \\ &= -\sqrt{3} \end{aligned}$$

**53.** Let $\alpha = \sin^{-1}\frac{4}{5},\ \sin\alpha = \frac{4}{5},\ \cos\alpha = \sqrt{1-\left(\frac{4}{5}\right)^2} = \frac{3}{5}$.

$$\begin{aligned} y &= \sin\left(2\sin^{-1}\frac{4}{5}\right) \\ &= \sin 2\alpha = 2\sin\alpha\cos\alpha \\ &= 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) = \frac{24}{25} \end{aligned}$$

**54.** Let $\alpha = \tan^{-1}1,\ \alpha = \frac{\pi}{4},\ \sin\alpha = \frac{\sqrt{2}}{2},\ \cos\alpha = \frac{\sqrt{2}}{2}$.

$$\begin{aligned} y &= \cos(2\tan^{-1}1) \\ &= \cos 2\alpha = \cos^2\alpha - \sin^2\alpha \\ &= \left(\frac{\sqrt{2}}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2 = 0 \end{aligned}$$

**55.** $y = \sin\left(\sin^{-1}\frac{2}{3} + \cos^{-1}\frac{1}{2}\right)$

Let $\alpha = \sin^{-1}\frac{2}{3},\ \sin\alpha = \frac{2}{3},\ \cos\alpha = \sqrt{1-\left(\frac{2}{3}\right)^2} = \frac{\sqrt{5}}{3}$.

$\beta = \cos^{-1}\frac{1}{2},\ \cos\beta = \frac{1}{2},\ \sin\beta = \sqrt{1-\left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$.

$$\begin{aligned} y &= \sin(\alpha+\beta) \\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \frac{2}{3}\left(\frac{1}{2}\right) + \frac{\sqrt{5}}{3}\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{1}{3} + \frac{\sqrt{15}}{6} = \frac{2+\sqrt{15}}{6} \end{aligned}$$

**56.** $y = \cos\left(\sin^{-1}\frac{3}{4} + \cos^{-1}\frac{5}{13}\right)$

Let $\alpha = \sin^{-1}\frac{3}{4},\ \sin\alpha = \frac{3}{4},\ \cos\alpha = \sqrt{1-\left(\frac{3}{4}\right)^2} = \frac{\sqrt{7}}{4}$.

$\beta = cos^{-1}\frac{5}{13},\ \cos\beta = \frac{5}{13},\ \sin\beta = \sqrt{1-\left(\frac{5}{13}\right)^2} = \frac{12}{13}$.

$$\begin{aligned} y &= \cos(\alpha+\beta) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \frac{\sqrt{7}}{4}\left(\frac{5}{13}\right) - \frac{3}{4}\left(\frac{12}{13}\right) \\ &= \frac{5\sqrt{7}}{52} - \frac{36}{52} = \frac{5\sqrt{7}-36}{52} \end{aligned}$$

**57.** $y = \tan\left(\cos^{-1}\frac{1}{2} - \sin^{-1}\frac{3}{4}\right)$

Let $\alpha = cos^{-1}\frac{1}{2},\ \cos\alpha = \frac{1}{2},\ \sin\alpha = \sqrt{1-\left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2},\ \tan\alpha = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$.

$\beta = \sin^{-1}\frac{3}{4},\ \sin\beta = \frac{3}{4},\ \cos\beta = \sqrt{1-\left(\frac{3}{4}\right)^2} = \frac{\sqrt{7}}{4},\ \tan\beta = \frac{\frac{3}{4}}{\frac{\sqrt{7}}{4}} = \frac{3}{\sqrt{7}} = \frac{3\sqrt{7}}{7}$.

$$\begin{aligned} y &= \tan(\alpha-\beta) \\ &= \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} \\ &= \frac{\sqrt{3}-\frac{3\sqrt{7}}{7}}{1+\sqrt{3}\cdot\frac{3\sqrt{7}}{7}} = \frac{\sqrt{3}-\frac{3\sqrt{7}}{7}}{1+\sqrt{3}\cdot\frac{3\sqrt{7}}{7}}\cdot\frac{7}{7} = \frac{7\sqrt{3}-3\sqrt{7}}{7+3\sqrt{21}} = \frac{7\sqrt{3}-3\sqrt{7}}{7+3\sqrt{21}}\cdot\frac{7-3\sqrt{21}}{7-3\sqrt{21}} = \frac{112\sqrt{3}-84\sqrt{7}}{-140} = \frac{3\sqrt{7}-4\sqrt{3}}{5} = \frac{1}{5}\left(3\sqrt{7}-4\sqrt{3}\right) \end{aligned}$$

**58.** $y = \sec\left(\cos^{-1}\frac{2}{3} + \sin^{-1}\frac{2}{3}\right)$

Let $\alpha = cos^{-1}\frac{2}{3},\ \cos\alpha = \frac{2}{3},\ \sin\alpha = \sqrt{1-\left(\frac{2}{3}\right)^2} = \frac{\sqrt{5}}{3}.$

$\beta = \sin^{-1}\frac{2}{3},\ \sin\beta = \frac{2}{3},\ \cos\beta = \sqrt{1-\left(\frac{2}{3}\right)^2} = \frac{\sqrt{5}}{3}.$

$$y = \sec(\alpha+\beta)$$
$$= \frac{1}{\cos(\alpha+\beta)} = \frac{1}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$
$$= \frac{1}{\frac{2}{3}\cdot\frac{\sqrt{5}}{3} - \frac{\sqrt{5}}{3}\cdot\frac{2}{3}} = \frac{1}{\frac{2\sqrt{5}}{9} - \frac{2\sqrt{5}}{9}}$$

$y$ is undefined.

**59.**
$$\sin^{-1}x = \cos^{-1}\tfrac{5}{13}$$
$$\sin(\sin^{-1}x) = \sin\left(\cos^{-1}\tfrac{5}{13}\right)$$
$$x = \tfrac{12}{13}$$

**60.**
$$\tan^{-1}x = \sin^{-1}\tfrac{24}{25}$$
$$\tan(\tan^{-1}x) = \tan\left(\sin^{-1}\tfrac{24}{25}\right)$$
$$x = \tan\left(\sin^{-1}\tfrac{24}{25}\right)$$
$$x = \tfrac{24}{7}$$

**61.**
$$\sin^{-1}(x-1) = \frac{\pi}{2}$$
$$(x-1) = \sin\frac{\pi}{2}$$
$$(x-1) = 1$$
$$x = 2$$

**62.**
$$\cos^{-1}\left(x-\frac{1}{2}\right) = \frac{\pi}{3}$$
$$\left(x-\frac{1}{2}\right) = \cos\frac{\pi}{3}$$
$$x = \frac{1}{2}+\frac{1}{2}$$
$$x = 1$$

**63.**
$$\tan^{-1}\left(x+\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$
$$\left(x+\frac{\sqrt{2}}{2}\right) = \tan\frac{\pi}{4}$$
$$x = 1-\frac{\sqrt{2}}{2}$$
$$= \frac{2-\sqrt{2}}{2}$$

**64.**
$$\sin^{-1}(x-2) = -\frac{\pi}{6}$$
$$(x-2) = \sin\left(-\frac{\pi}{6}\right)$$
$$x = -\frac{1}{2}+2$$
$$x = \frac{3}{2}$$

**65.**
$$\sin^{-1}\frac{3}{5} + \cos^{-1}x = \frac{\pi}{4}$$
$$\cos^{-1}x = \frac{\pi}{4} - \sin^{-1}\frac{3}{5}$$
$$x = \cos\left(\frac{\pi}{4} - \sin^{-1}\frac{3}{5}\right)$$

Let $\alpha = \sin^{-1}\frac{3}{5},\ \sin\alpha = \frac{3}{5},\ \cos\alpha = \frac{4}{5}.$

$$x = \cos\left(\frac{\pi}{4} - \alpha\right)$$
$$x = \cos\frac{\pi}{4}\cos\alpha + \sin\frac{\pi}{4}\sin\alpha$$
$$x = \frac{\sqrt{2}}{2}\cdot\frac{4}{5} + \frac{\sqrt{2}}{2}\cdot\frac{3}{5}$$
$$= \frac{4\sqrt{2}}{10} + \frac{3\sqrt{2}}{10} = \frac{7\sqrt{2}}{10}$$

**66.**
$$\sin^{-1}x + \cos^{-1}\frac{4}{5} = \frac{\pi}{6}$$
$$\sin^{-1}x = \frac{\pi}{6} - \cos^{-1}\frac{4}{5}$$
$$x = \sin\left(\frac{\pi}{6} - \cos^{-1}\frac{4}{5}\right)$$

Let $\alpha = \cos^{-1}\frac{4}{5},\ \cos\alpha = \frac{4}{5}, \sin\alpha = \frac{3}{5}.$

$$x = \sin\left(\frac{\pi}{6} - \alpha\right)$$
$$x = \sin\frac{\pi}{6}\cos\alpha - \cos\frac{\pi}{6}\sin\alpha$$
$$x = \frac{1}{2}\cdot\frac{4}{5} - \frac{\sqrt{3}}{2}\cdot\frac{3}{5}$$
$$= \frac{4}{10} - \frac{3\sqrt{3}}{10} = \frac{4-3\sqrt{3}}{10}$$

**67.** $\sin^{-1}\frac{\sqrt{2}}{2}+\cos^{-1}x=\frac{2\pi}{3}$

$\cos^{-1}x=\frac{2\pi}{3}-\sin^{-1}\frac{\sqrt{2}}{2}$

$x=\cos\left(\frac{2\pi}{3}-\sin^{-1}\frac{\sqrt{2}}{2}\right)$

Let $\alpha=\sin^{-1}\frac{\sqrt{2}}{2}, \sin\alpha=\frac{\sqrt{2}}{2}, \cos\alpha=\frac{\sqrt{2}}{2}$.

$x=\cos\left(\frac{2\pi}{3}-\alpha\right)$

$x=\cos\frac{2\pi}{3}\cos\alpha+\sin\frac{2\pi}{3}\sin\alpha$

$x=-\frac{1}{2}\cdot\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}$

$=-\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}=\frac{\sqrt{6}-\sqrt{2}}{4}\approx 0.2588$

Note: Since $\sin\alpha=\frac{\sqrt{2}}{2}$ and $\cos\alpha=\frac{\sqrt{2}}{2}$, then $\alpha=\frac{\pi}{4}$.

Thus, $\cos\left(\frac{2\pi}{3}-\alpha\right)=\cos\left(\frac{2\pi}{3}-\frac{\pi}{4}\right)=\cos\left(\frac{5\pi}{12}\right)\approx 0.2588$.

**68.** $\cos^{-1}x+\sin^{-1}\frac{\sqrt{3}}{2}=\frac{\pi}{2}$

$\cos^{-1}x=\frac{\pi}{2}-\sin^{-1}\frac{\sqrt{3}}{2}$

$x=\cos\left(\frac{\pi}{2}-\sin^{-1}\frac{\sqrt{3}}{2}\right)$

Let $\alpha=\sin^{-1}\frac{\sqrt{3}}{2}, \sin\alpha=\frac{\sqrt{3}}{2}, \cos\alpha=\frac{1}{2}$.

$x=\cos\left(\frac{\pi}{2}-\alpha\right)$

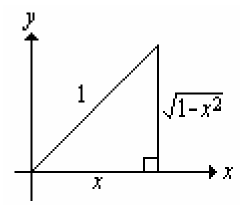
$x=\cos\frac{\pi}{2}\cos\alpha+\sin\frac{\pi}{2}\sin\alpha$

$x=0\cdot\frac{1}{2}+1\cdot\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{2}$

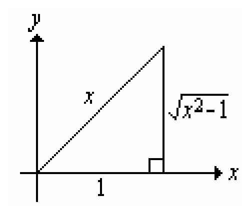
Note: Since $\sin\alpha=\frac{\sqrt{3}}{2}$ and $\cos\alpha=\frac{1}{2}$, then $\alpha=\frac{\pi}{3}$.

Thus, $\cos\left(\frac{\pi}{2}-\alpha\right)=\cos\left(\frac{\pi}{2}-\frac{\pi}{3}\right)=\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$.

**69.** $\tan(\cos^{-1}x)=\frac{\sqrt{1-x^2}}{x}$




**70.** $\sin(\sec^{-1}x)=\frac{\sqrt{x^2-1}}{|x|}$




**71.** Let $\alpha=\sin^{-1}x,\ \sin\alpha=x,\ \cos\alpha=\sqrt{1-x^2}$.

Let $\beta=\sin^{-1}(-x),\ \sin\beta=-x,\ \cos\beta=\sqrt{1-x^2}$.

$$\begin{aligned}\sin^{-1}x+\sin^{-1}(-x)&=\alpha+\beta\\&=\sin^{-1}\left[\sin(\alpha+\beta)\right]\\&=\sin^{-1}(\sin\alpha\cos\beta+\cos\alpha\sin\beta)\\&=\sin^{-1}\left[x\sqrt{1-x^2}+\sqrt{1-x^2}(-x)\right]\\&=\sin^{-1}0\\&=0\end{aligned}$$

**72.** Let $\alpha=\cos^{-1}x,\ \cos\alpha=x,\ \sin\alpha=\sqrt{1-x^2}$.

Let $\beta=\cos^{-1}(-x),\ \cos\beta=-x,\ \sin\beta=\sqrt{1-x^2}$.

$$\begin{aligned}\cos^{-1}x+\cos^{-1}(-x)&=\alpha+\beta\\&=\cos^{-1}\left[\cos(\alpha+\beta)\right]\\&=\cos^{-1}(\cos\alpha\cos\beta-\sin\alpha\sin\beta)\\&=\cos^{-1}\left[x(-x)-\sqrt{1-x^2}\cdot\sqrt{1-x^2}\right]\\&=\cos^{-1}\left(-x^2-1+x^2\right)\\&=\cos^{-1}(-1)\\&=\pi\end{aligned}$$

**73.** Let $\alpha = \tan^{-1}x,\ \tan\alpha = x,\ \beta = \tan^{-1}\frac{1}{x},\ \tan\beta = \frac{1}{x}$.

$$\begin{aligned}\tan^{-1}x + \tan^{-1}\frac{1}{x} &= \alpha + \beta \\ &= \tan^{-1}\left[\tan(\alpha+\beta)\right] \\ &= \tan^{-1}\left[\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\right] \\ &= \tan^{-1}\left[\frac{x + \frac{1}{x}}{1 - x\cdot\frac{1}{x}}\right] \\ &= \tan^{-1}\frac{\frac{x^2+1}{x}}{1-1}, \text{ which is undefined}\end{aligned}$$

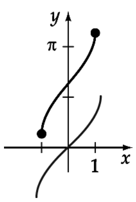
Thus $x = \frac{\pi}{2}$

**74.** Let $\sec^{-1}\frac{1}{x} = \alpha,\ \sec\alpha = \frac{1}{x},\ \cos\alpha = x,\ \sin\alpha = \sqrt{1-x^2}$.

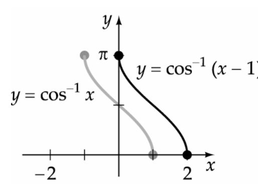
Let $\csc^{-1}\frac{1}{x} = \beta,\ \csc\beta = \frac{1}{x},\ \sin\beta = x,\ \cos\beta = \sqrt{1-x^2}$.

$$\begin{aligned}\sec^{-1}\frac{1}{x} + \csc^{-1}\frac{1}{x} &= \alpha + \beta \\ &= \sin^{-1}\left[\sin(\alpha+\beta)\right] \\ &= \sin^{-1}\left[\sin\alpha\cos\beta + \cos\alpha\sin\beta\right] \\ &= \sin^{-1}\left[\sqrt{1-x^2}\cdot\sqrt{1-x^2} + x\cdot x\right] \\ &= \sin^{-1}\left(1 - x^2 + x^2\right) \\ &= \sin^{-1}1 \\ &= \frac{\pi}{2}\end{aligned}$$

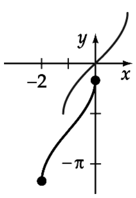
**75.** The graph of $y = \sin^{-1}(x) + 2$ (shown as a black graph) is the graph of $y = \sin^{-1}x$ (shown as a gray graph) moved two units up.



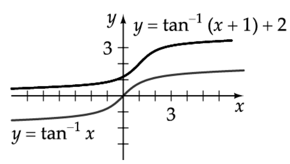

**76.** The graph of $y = \cos^{-1}(x-1)$ (shown as a black graph) is the graph of $y = \cos^{-1}x$ (shown as a gray graph) moved one unit to the right.



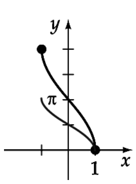

**77.** The graph of $y = \sin^{-1}(x+1) - 2$ (shown as a black graph) is the graph of $y = \sin^{-1}x$ (shown as a gray graph) moved one unit to the left and two units down.



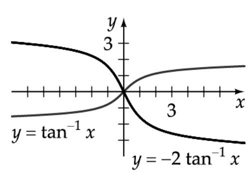

**78.** The graph of $y = \tan^{-1}(x-1) + 2$ (shown as a black graph) is the graph of $y = \tan^{-1}x$ (shown as a gray graph) moved one unit to the right and two units up.



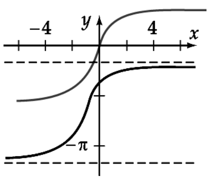

**79.** The graph of $y = 2\cos^{-1}x$ (shown as a black graph) is the graph of $y = \cos^{-1}x$ (shown as a gray graph) stretched.



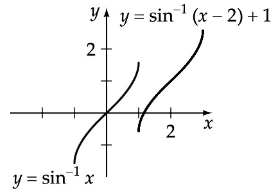

**80.** The graph of $y = -2\tan^{-1}x$ (shown as a black graph) is the graph of $y = \tan^{-1}x$ (shown as a gray graph) stretched and reflected through the $x$-axis.



**81.** The graph of $y = \tan^{-1}(x+1) - 2$ (shown as a black graph) is the graph of $y = \tan^{-1} x$ (shown as a gray graph) moved one unit to the left and two units down.




**82.** The graph of $y = \sin^{-1}(x-2) + 1$ (shown as a black graph) is the graph of $y = \sin^{-1} x$ (shown as a gray graph) moved two units to the right and one unit up.



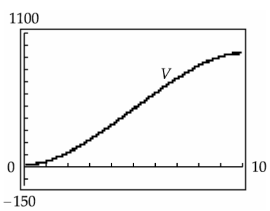

**83. a.**

$$s = 3960\theta$$
$$\cos\theta = \frac{3960}{a+3960}$$
$$\theta = \cos^{-1}\left(\frac{3960}{a+3960}\right)$$
$$s = 3960\cos^{-1}\left(\frac{3960}{a+3960}\right)$$

**b.**

$$5500 = 3960\cos^{-1}\left(\frac{3960}{a+3960}\right)$$
$$\frac{5500}{3960} = \cos^{-1}\left(\frac{3960}{a+3960}\right)$$
$$\cos\left(\frac{5500}{3960}\right) = \frac{3960}{a+3960}$$
$$a + 3960 = \frac{3960}{\cos\left(\frac{5500}{3960}\right)}$$
$$a = \frac{3960}{\cos\left(\frac{5500}{3960}\right)} - 3960$$
$$a \approx 17{,}930 \text{ mi}$$

**84. a.**



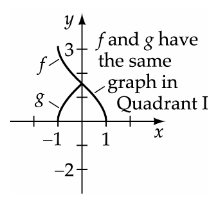

(Make sure you are in "radian" mode.)

**b.** Although the water rises 0.1 ft in each case, there is more surface area, and thus more volume of water, at the 4.9 to 5-foot level near the diameter of the cylinder than at the 0.1 to 0.2-foot level near the bottom.

**c.** $V(4) \approx 352.04 \text{ ft}^3$.

**d.** If $V = 288 \text{ ft}^3$, $x \approx 3.45$ ft.

**85.**

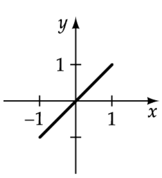



[Note: $f(x) = \cos^{-1} x$ is neither odd nor even.

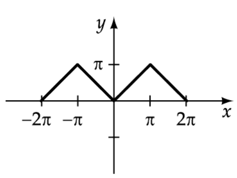
$g(x) = \sin^{-1}\sqrt{1-x^2}$ is an even function.]

No, $f(x) \neq g(x)$ on the interval $[-1, 1]$.

**86.**




$y = \cos(\cos^{-1} x)$



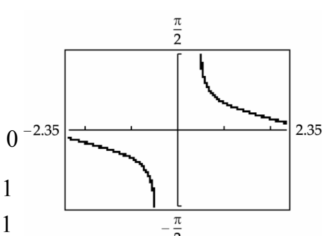

$y(x) = \cos^{-1}(\cos x)$

[Note: The domain of $\cos(\cos^{-1} x)$ is $-1 \le x \le 1$.

The domain of $\cos^{-1}(\cos x)$ is all the real numbers.]

**87.**

$$y = \csc^{-1} 2x$$
$$\csc y = 2x$$
$$-\frac{\pi}{2} \le y \le \frac{\pi}{2},\ y \neq 0$$
$$2x \le -1 \quad \text{or} \quad 2x \ge 1$$
$$x \le -\frac{1}{2} \quad \text{or} \quad x \ge \frac{1}{2}$$



**88.** $y = 0.5\sec^{-1}\frac{x}{2}$

$2y = \sec^{-1}\frac{x}{2}$

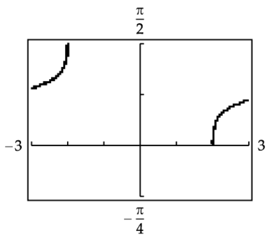
$\sec 2y = \frac{x}{2}$

$0 < 2y < \pi,\ y \neq \frac{\pi}{2}$

$0 < y < \frac{\pi}{2},\ y \neq \frac{\pi}{4}$

$\frac{x}{2} \le -1 \qquad \frac{x}{2} \ge 1$

$x \le -2 \qquad x \ge 2$




**89.** $y = \sec^{-1}(x-1)$

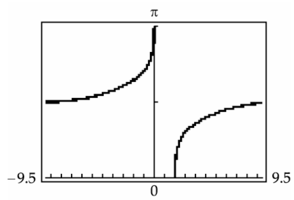
$\sec y = x - 1$

$0 < y < \pi$

$y \neq \frac{\pi}{2}$

$x - 1 \le -1 \qquad x - 1 \ge 1$

$x \le 0 \qquad x \ge 2$




**90.** $y = \sec^{-1}(x+\pi)$

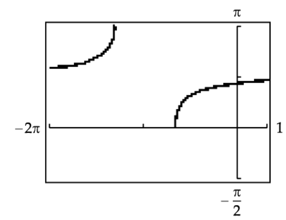
$\sec y = x + \pi$

$0 < y < \pi$

$y \neq \frac{\pi}{2}$

$x + \pi \le -1 \qquad x + \pi \ge 1$

$x \le -\pi - 1 \qquad x \ge 1 - \pi$




**91.** $y = 2\tan^{-1} 2x$

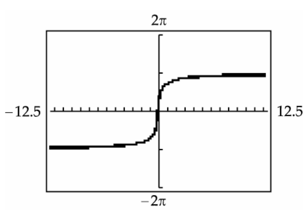
$\frac{y}{2} = \tan^{-1} 2x$

$\tan y = 2x$

$-\frac{\pi}{2} < \frac{y}{2} < \frac{\pi}{2}$

$-\pi < y < \pi$

$-\infty < 2x < \infty$

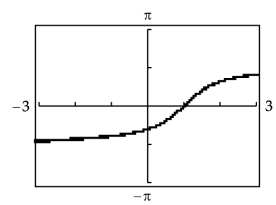



**92.** $y = \tan^{-1}(x-1)$

$\tan y = x - 1$

$-\frac{\pi}{2} < y < \frac{\pi}{2}$

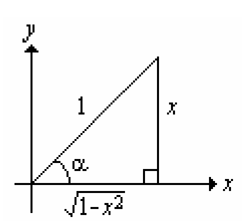
$x - 1 \rightarrow$ Graph is displaced one unit to the right.




## Connecting Concepts

**93.** Let $\alpha = \sin^{-1} x$

$\sin\alpha = x$



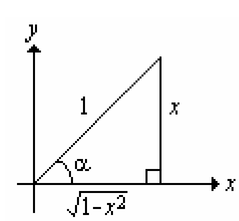

$\cos\left(\sin^{-1} x\right) = \cos\alpha$

$= \frac{\sqrt{1-x^2}}{1}$

$= \sqrt{1-x^2}$

**94.** Let $\alpha = \sin^{-1} x$

$\sin\alpha = x$



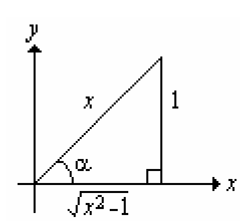

$\sec(\sin^{-1} x) = \sec\alpha$

$= \frac{1}{\sqrt{1-x^2}}$

$= \frac{\sqrt{1-x^2}}{1-x^2}$

**95.** Let $\alpha = \csc^{-1} x$

$\csc\alpha = x$



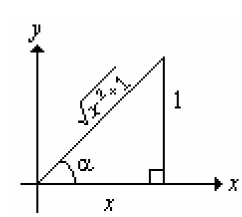

$\tan(\csc^{-1} x) = \tan\alpha$

$= \frac{1}{\sqrt{x^2-1}}$

$= \frac{\sqrt{x^2-1}}{x^2-1}$

**96.** Let $\alpha = \cot^{-1} x$

$\cot\alpha = x$




$\sin(\cot^{-1} x) = \sin\alpha$

$= \frac{1}{\sqrt{x^2+1}}$

$= \frac{\sqrt{x^2+1}}{x^2+1}$

**97.** $5x = \tan^{-1} 3y$
$\tan 5x = 3y$
$y = \frac{1}{3}\tan 5x$

**98.** $2x = \frac{1}{2}\sin^{-1} 2y$
$4x = \sin^{-1} 2y$
$\sin 4x = 2y$
$y = \frac{1}{2}\sin 4x$

**99.** $x - \frac{\pi}{3} = \cos^{-1}(y-3)$
$\cos\left(x - \frac{\pi}{3}\right) = y - 3$
$y = 3 + \cos\left(x - \frac{\pi}{3}\right)$

**100.** $x + \frac{\pi}{2} = \tan^{-1}(2y-1)$
$\tan\left(x + \frac{\pi}{2}\right) = 2y - 1$
$2y = 1 + \tan\left(x + \frac{\pi}{2}\right)$
$y = \frac{1}{2} + \frac{1}{2}\tan\left(x + \frac{\pi}{2}\right)$

## Prepare for Section 6.6

**PS1.** $x = \frac{5 \pm \sqrt{(-5)^2 - 4(3)(-4)}}{2(3)} = \frac{5 \pm \sqrt{73}}{6}$

**PS2.** $\sin^2 x + \cos^2 x = 1$
$\sin^2 x = 1 - \cos^2 x$

**PS3.** $\frac{\pi}{2} + 2(1)\pi = \frac{5}{2}\pi$
$\frac{\pi}{2} + 2(2)\pi = \frac{9}{2}\pi$
$\frac{\pi}{2} + 2(3)\pi = \frac{13}{2}\pi$

**PS4.** $x^2 - \frac{\sqrt{3}}{2}x + x - \frac{\sqrt{3}}{2} = x\left(x - \frac{\sqrt{3}}{2}\right) + 1\left(x - \frac{\sqrt{3}}{2}\right)$
$= (x+1)\left(x - \frac{\sqrt{3}}{2}\right)$

**PS5.**

**PS6.** $2x^2 - 2x = 0$
$2x(x-1) = 0$
$x = 0 \quad x - 1 = 0$
$x = 1$
The solutions are 0, 1.

# Section 6.6

**1.** $\sec x - \sqrt{2} = 0$
$\sec x = \sqrt{2}$
$x = \frac{\pi}{4}, \frac{7\pi}{4}$

**2.** $2\sin x = \sqrt{3}$
$\sin x = \frac{\sqrt{3}}{2}$
$x = \frac{\pi}{3}, \frac{2\pi}{3}$

**3.** $\tan x - \sqrt{3} = 0$
$\tan x = \sqrt{3}$
$x = \frac{\pi}{3}, \frac{4\pi}{3}$

**4.** $\cos x - 1 = 0$
$\cos x = 1$
$x = 0$

**5.** $2\sin x \cos x = \sqrt{2}\cos x$
$2\sin x \cos x - \sqrt{2}\cos x = 0$
$\cos x(2\sin x - \sqrt{2}) = 0$
$\cos x = 0 \qquad 2\sin x - \sqrt{2} = 0$
$\sin x = \frac{\sqrt{2}}{2}$
$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad x = \frac{\pi}{4}, \frac{3\pi}{4}$
The solutions are $\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{2}$.

**6.** $2\sin x \cos x = \sqrt{3}\sin x$
$2\sin x \cos x - \sqrt{3}\sin x = 0$
$\sin x(2\cos x - \sqrt{3}) = 0$
$\sin x = 0 \qquad 2\cos x - \sqrt{3} = 0$
$\cos x = \frac{\sqrt{3}}{2}$
$x = 0, \pi \qquad x = \frac{\pi}{6}, \frac{11\pi}{6}$
The solutions are $0, \frac{\pi}{6}, \pi, \frac{11\pi}{6}$.

**7.**
$$\sin^2 x - 1 = 0$$
$$\sin^2 x = 1$$
$$\sin x = \pm\sqrt{1}$$
$$\sin x = \pm 1$$
$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

**8.**
$$\cos^2 x - 1 = 0$$
$$\cos^2 x = 1$$
$$\cos x = \pm\sqrt{1}$$
$$\cos x = \pm 1$$
$$x = 0, \pi$$

**9.**
$$4\sin x\cos x - 2\sqrt{3}\sin x - 2\sqrt{2}\cos x + \sqrt{6} = 0$$
$$2\sin x(2\cos x - \sqrt{3}) - \sqrt{2}(2\cos x - \sqrt{3}) = 0$$
$$(2\cos x - \sqrt{3})(2\sin x - \sqrt{2}) = 0$$
$$2\cos x - \sqrt{3} = 0 \qquad 2\sin x - \sqrt{2} = 0$$
$$\cos x = \frac{\sqrt{3}}{2} \qquad \sin x = \frac{\sqrt{2}}{2}$$
$$x = \frac{\pi}{6}, \frac{11\pi}{6} \qquad x = \frac{\pi}{4}, \frac{3\pi}{4}$$
The solutions are $\frac{\pi}{6}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{11\pi}{6}$.

**10.**
$$\sec^2 x + \sqrt{3}\sec x - \sqrt{2}\sec x - \sqrt{6} = 0$$
$$\sec x(\sec x + \sqrt{3}) - \sqrt{2}(\sec x + \sqrt{3}) = 0$$
$$(\sec x + \sqrt{3})(\sec x - \sqrt{2}) = 0$$
$$\sec x + \sqrt{3} = 0 \qquad \sec x - \sqrt{2} = 0$$
$$\sec x = -\sqrt{3} \qquad \sec x = \sqrt{2}$$
$$x \approx 2.1863,\ 4.0969 \qquad x = \frac{\pi}{4}, \frac{7\pi}{4}$$
The solutions are $\frac{\pi}{4}$, 2.1863, 4.0960, $\frac{7\pi}{4}$.

**11.**
$$\csc x - \sqrt{2} = 0$$
$$\csc x = \sqrt{2}$$
$$x = \frac{\pi}{4}, \frac{3\pi}{4}$$

**12.**
$$3\cot x + \sqrt{3} = 0$$
$$3\cot x = -\sqrt{3}$$
$$\cot x = -\frac{\sqrt{3}}{3}$$
$$x = \frac{2\pi}{3}, \frac{5\pi}{3}$$

**13.**
$$2\sin^2 x + 1 = 3\sin x$$
$$2\sin^2 x - 3\sin x + 1 = 0$$
$$(2\sin x - 1)(\sin x - 1) = 0$$
$$2\sin x - 1 = 0 \qquad \sin x - 1 = 0$$
$$\sin x = \tfrac{1}{2} \qquad \sin x = 1$$
$$x = \frac{\pi}{6}, \frac{5\pi}{6} \qquad x = \frac{\pi}{2}$$
The solutions are $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$.

**14.**
$$2\cos^2 x + 1 = -3\cos x$$
$$2\cos^2 x + 3\cos x + 1 = 0$$
$$(2\cos x + 1)(\cos x + 1) = 0$$
$$2\cos x + 1 = 0 \qquad \cos x + 1 = 0$$
$$\cos x = -\tfrac{1}{2} \qquad \cos x = -1$$
$$x = \frac{2\pi}{3}, \frac{4\pi}{3} \qquad x = \pi$$
The solutions are $\frac{2\pi}{3}, \pi, \frac{4\pi}{3}$.

**15.**
$$4\cos^2 x - 3 = 0$$
$$\cos^2 x = \tfrac{3}{4}$$
$$\cos x = \pm\tfrac{\sqrt{3}}{2}$$
$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

**16.**
$$2\sin^2 x - 1 = 0$$
$$\sin^2 x = \tfrac{1}{2}$$
$$\sin x = \pm\tfrac{\sqrt{2}}{2}$$
$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

**17.**
$$2\sin^3 x = \sin x$$
$$2\sin^3 x - \sin x = 0$$
$$\sin x(2\sin^2 x - 1) = 0$$
$\sin x = 0$ $\quad x = 0,\ \pi$

$2\sin^2 x = 1$ $\quad \sin x = \pm\frac{\sqrt{2}}{2}$ $\quad x = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{7\pi}{4}$

The solutions are $0,\ \frac{\pi}{4},\ \frac{3\pi}{4},\ \pi,\ \frac{5\pi}{4},\ \frac{7\pi}{4}.$

**18.**
$$4\cos^3 x = 3\cos x$$
$$4\cos^3 x - 3\cos x = 0$$
$$\cos x(4\cos^2 x - 3) = 0$$
$\cos x = 0$ $\quad x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\cos^2 x = \frac{3}{4}$ $\quad \cos x = \pm\frac{\sqrt{3}}{2}$ $\quad x = \frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{7\pi}{6},\ \frac{11\pi}{6}$

The solutions are $\frac{\pi}{6},\ \frac{\pi}{2},\ \frac{5\pi}{6},\ \frac{7\pi}{6},\ \frac{3\pi}{2},\ \frac{11\pi}{6}.$

**19.**
$$4\sin^2 x + 2\sqrt{3}\sin x - \sqrt{3} = 2\sin x$$
$$4\sin^2 x + 2\sqrt{3}\sin x - 2\sin x - \sqrt{3} = 0$$
$$2\sin x(2\sin x + \sqrt{3}) - (2\sin x + \sqrt{3}) = 0$$
$$(2\sin x + \sqrt{3})(2\sin x - 1) = 0$$
$2\sin x + \sqrt{3} = 0$ $\quad \sin x = -\frac{\sqrt{3}}{2}$ $\quad x = \frac{4\pi}{3},\ \frac{5\pi}{3}$

$2\sin x - 1 = 0$ $\quad \sin x = \frac{1}{2}$ $\quad x = \frac{\pi}{6},\ \frac{5\pi}{6}$

The solutions are $\frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{4\pi}{3},\ \frac{5\pi}{3}.$

**20.**
$$\tan^2 x + \tan x - \sqrt{3} = \sqrt{3}\tan x$$
$$\tan^2 x + \tan x - \sqrt{3}\tan x - \sqrt{3} = 0$$
$$\tan x(\tan x + 1) - \sqrt{3}(\tan x + 1) = 0$$
$$(\tan x + 1)(\tan x - \sqrt{3}) = 0$$
$\tan x + 1 = 0$ $\quad \tan x = -1$ $\quad x = \frac{3\pi}{4},\ \frac{7\pi}{4}$

$\tan x - \sqrt{3} = 0$ $\quad \tan x = \sqrt{3}$ $\quad x = \frac{\pi}{3},\ \frac{4\pi}{3}$

The solutions are $\frac{\pi}{3},\ \frac{3\pi}{4},\ \frac{4\pi}{3},\ \frac{7\pi}{4}.$

**21.**
$$\sin^4 x = \sin^2 x$$
$$\sin^4 x - \sin^2 x = 0$$
$$\sin^2 x(\sin^2 x - 1) = 0$$
$\sin^2 x = 0$ $\quad \sin x = 0$ $\quad x = 0,\ \pi$

$\sin^2 x - 1 = 0$ $\quad \sin x = \pm 1$ $\quad x = \frac{\pi}{2},\ \frac{3\pi}{2}$

The solutions are $0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}.$

**22.**
$$\cos^4 x = \cos^2 x$$
$$\cos^4 x - \cos^2 x = 0$$
$$\cos^2 x(\cos^2 x - 1) = 0$$
$\cos^2 x = 0$ $\quad \cos x = 0$ $\quad x = \frac{\pi}{2},\ \frac{3\pi}{2}$

$\cos^2 x - 1 = 0$ $\quad \cos x = \pm 1$ $\quad x = 0,\ \pi$

The solutions are $0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}.$

**23.**
$$\cos x - 0.75 = 0$$
$$\cos x = 0.75$$
$$x \approx 41.4°,\ 318.6°$$

**24.**
$$\sin x + 0.432 = 0$$
$$\sin x = -0.432$$
$$x \approx 205.6°,\ 334.4°$$

**25.**
$$3\sin x - 5 = 0$$
$$3\sin x = 5$$
$$\sin x = \frac{5}{3}$$
no solution

**26.**
$$4\cos x - 1 = 0$$
$$4\cos x = 1$$
$$\cos x = \frac{1}{4}$$
$$x \approx 75.5°,\ 284.5°$$

**27.**
$$3\sec x - 8 = 0$$
$$3\sec x = 8$$
$$\sec x = \frac{8}{3}$$
$$\frac{1}{\cos x} = \frac{8}{3}$$
$$\cos x = \frac{3}{8}$$
$$x \approx 68.0°,\ 292.0°$$

**28.**
$$4\csc x + 9 = 0$$
$$\csc x = -\frac{9}{4}$$
$$\sin x = -\frac{4}{9}$$
$$x \approx 206.4°,\ 333.6°$$

**29.** $\cos x + 3 = 0$
$\cos x = -3$
no solution

**30.** $\sin x - 4 = 0$
$\sin x = 4$
no solution

**31.** $3 - 5\sin x = 4\sin x + 1$
$-9\sin x = -2$
$\sin x = \frac{2}{9}$
$x \approx 12.8°,\ 167.2°$

**32.** $4\cos x - 5 = \cos x - 3$
$3\cos x = 2$
$\cos x = \frac{2}{3}$
$x \approx 48.2°,\ 311.8°$

**33.** $\frac{1}{2}\sin x + \frac{2}{3} = \frac{3}{4}\sin x + \frac{3}{5}$
$-\frac{1}{4}\sin x = \frac{3}{5} - \frac{2}{3}$
$-\frac{1}{4}\sin x = -\frac{1}{15}$
$\sin x = \frac{4}{15}$
$x \approx 15.5°,\ 164.5°$

**34.** $\frac{2}{5}\cos x - \frac{1}{2} = \frac{1}{3} - \frac{1}{2}\cos x$
$\frac{9}{10}\cos x = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$
$\cos x = \frac{25}{27}$
$x \approx 22.2°,\ 337.8°$

**35.** $3\tan^2 x - 2\tan x = 0$
$\tan x(3\tan x - 2) = 0$

$\tan x = 0$
$x = 0,\ 180°$

$3\tan x - 2 = 0$
$\tan x = \frac{2}{3}$
$x \approx 33.7°,\ 213.7°$

The solutions are 0°, 33.7°, 180°, 213.7°.

**36.** $4\cot^2 x + 3\cot x = 0$
$\cot x(4\cot x + 3) = 0$

$\cot x = 0$
$\cot x = 0$
$x = 90°,\ 270°$

$4\cot x + 3 = 0$
$\cot x = -\frac{3}{4}$
$x \approx 126.9°,\ 306.9°$

The solutions are 90°, 126.9°, 270°, 306.9°

**37.** $3\cos x + \sec x = 0$
$3\cos x + \frac{1}{\cos x} = 0$
$3\cos^2 x + 1 = 0$
$\cos^2 x = -\frac{1}{3}$
no solution

**38.** $5\sin x - \csc x = 0$
$5\sin x - \frac{1}{\sin x} = 0$
$5\sin^2 x - 1 = 0$
$\sin^2 x = \frac{1}{5}$
$\sin x = \pm\sqrt{\frac{1}{5}}$
$x = 26.6°,\ 153.4°,\ 206.6°,\ 333.4°$

**39.** $\tan^2 x = 3\sec^2 x - 2$
$\tan^2 x = 3(1 + \tan^2 x) - 2$
$\tan^2 x = 3 + 3\tan^2 x - 2$
$-2\tan^2 x = 1$
$\tan^2 x = -\frac{1}{2}$
no solution

**40.** $\csc^2 x - 1 = 3\cot^2 x + 2$
$\cot^2 x = 3\cot^2 x + 2$
$-2\cot^2 x = 2$
$\cot^2 x = -1$
no solution.

**41.** $2\sin^2 x = 1 - \cos x$
$2(1 - \cos^2 x) = 1 - \cos x$
$2 - 2\cos^2 x = 1 - \cos x$
$0 = 2\cos^2 x - \cos x - 1$
$0 = (2\cos x + 1)(\cos x - 1)$

$2\cos x + 1 = 0$
$\cos x = -\frac{1}{2}$
$x = 120°,\ 240°$

$\cos x - 1 = 0$
$\cos x = 1$
$x = 0°$

The solutions are 0°, 120°, 240°.

**42.**

$$\cos^2 x + 4 = 2\sin x - 3$$
$$1 - \sin^2 x + 4 = 2\sin x - 3$$
$$0 = \sin^2 x + 2\sin x - 8$$
$$0 = (\sin x + 4)(\sin x - 2)$$

$\sin x + 4 = 0$ $\quad$ $\sin x - 2 = 0$

$\sin x = -4$ $\quad$ $\sin x = 2$

no solution $\quad$ no solution

There is no solution.

**43.**

$$3\cos^2 x + 5\cos x - 2 = 0$$
$$\cos x = \frac{-5 \pm \sqrt{5^2 - 4(3)(-2)}}{2 \cdot 3} = \frac{-5 \pm \sqrt{49}}{6} = \frac{-5 \pm 7}{6}$$

$\cos x = \frac{1}{3}$ $\quad$ $\cos x = -2$

$x \approx 70.5°, 289.5°$ $\quad$ no solution

The solutions are 70.5°, 289.5°.

**44.**

$$2\sin^2 x + 5\sin x + 3 = 0$$
$$\sin x = \frac{-5 \pm \sqrt{5^2 - 4(2)(3)}}{2 \cdot 2} = \frac{-5 \pm \sqrt{1}}{4} = \frac{-5 \pm 1}{4}$$

$\sin x = -1$ $\quad$ $\sin x = -\frac{3}{2}$

$x = 270°$ $\quad$ no solution

The solution is 270°.

**45.**

$$2\tan^2 x - \tan x - 10 = 0$$
$$(\tan x + 2)(2\tan x - 5) = 0$$

$\tan x + 2 = 0$ $\quad$ $2\tan x - 5 = 0$

$\tan x = -2$ $\quad$ $\tan x = \frac{5}{2}$

$x \approx 116.6°, 296.6°$ $\quad$ $x \approx 68.2°, 248.2°$

The solutions are 68.2°, 116.6°, 248.2°, 296.6°.

**46.**

$$2\cot^2 x - 7\cot x + 3 = 0$$
$$(2\cot x - 1)(\cot x - 3) = 0$$

$2\cot x - 1 = 0$ $\quad$ $\cot x - 3 = 0$

$\cot x = \frac{1}{2}$ $\quad$ $\cot x = 3$

$x \approx 63.4°, 243.4°$ $\quad$ $x \approx 18.4°, 198.4°$

The solutions are 18.4°, 63.4°, 198.4°, 243.4°.

**47.**

$$3\sin x\cos x - \cos x = 0$$
$$\cos x(3\sin x - 1) = 0$$

$\cos x = 0$ $\quad$ $3\sin x - 1 = 0$

$x = 90°, 270°$ $\quad$ $\sin x = \frac{1}{3}$

$x \approx 19.5°, 160.5°$

The solutions are 19.5°, 90°, 160.5°, 270°.

**48.**

$$\tan x \sin x - \sin x = 0$$
$$\sin x(\tan x - 1) = 0$$

$\sin x = 0$ $\quad$ $\tan x - 1 = 0$

$x = 0°, 180°$ $\quad$ $x = 1$

$x = 45°, 225°$

The solutions are 0°, 45°, 180°, 225°.

**49.**

$$2\sin x\cos x - \sin x - 2\cos x + 1 = 0$$
$$\sin x(2\cos x - 1) - (2\cos x - 1) = 0$$
$$(2\cos x - 1)(\sin x - 1) = 0$$

$2\cos x - 1 = 0$ $\quad$ $\sin x - 1 = 0$

$\cos x = \frac{1}{2}$ $\quad$ $\sin x = 1$

$x = 60°, 300°$ $\quad$ $x = 90°$

The solutions are 60°, 90°, 300°.

**50.**

$$6\cos x\sin x - 3\cos x - 4\sin x + 2 = 0$$
$$3\cos x(2\sin x - 1) - 2(2\sin x - 1) = 0$$
$$(2\sin x - 1)(3\cos x - 2) = 0$$

$2\sin x - 1 = 0$ $\quad$ $3\cos x - 2 = 0$

$\sin x = \frac{1}{2}$ $\quad$ $\cos x = \frac{2}{3}$

$x = 30°, 150°$ $\quad$ $x \approx 48.2°, 311.8°$

The solutions are 30°, 48.2°, 150°, 311.8°.

**51.**

$$2\sin x - \cos x = 1$$
$$2\sin x - 1 = \cos x$$
$$(2\sin x - 1)^2 = (\cos x)^2$$
$$4\sin^2 x - 4\sin x + 1 = \cos^2 x$$
$$4\sin^2 x - 4\sin x + 1 = 1 - \sin^2 x$$
$$5\sin^2 x - 4\sin x = 0$$
$$\sin x(5\sin x - 4) = 0$$

$\sin x = 0$ $\quad$ $5\sin x - 4 = 0$

$x = 180°$ $\quad$ $\sin x = \frac{4}{5}$

$x \approx 53.1°$ or $126.9°$

126.9° does not check.
The solutions are 53.1°, 180°.

**52.** $\sin x + 2\cos x = 1$

$\sin x = 1 - 2\cos x$

$(\sin x)^2 = (1 - 2\cos x)^2$

$\sin^2 x = 1 - 4\cos x + 4\cos^2 x$

$1 - \cos^2 x = 1 - 4\cos x + 4\cos^2 x$

$0 = 5\cos^2 x - 4\cos x$

$0 = \cos x(5\cos x - 4)$

$\cos x = 0$ $\quad$ $5\cos x - 4 = 0$

$x = 90°$ $\quad$ $\cos x = \frac{4}{5}$

$x \approx 323.1°$ or $36.9°$

36.9° does not check.

The solutions are 90°, 323.1°.

**53.** $2\sin x - 3\cos x = 1$

$2\sin x = 3\cos x + 1$

$(2\sin x)^2 = (3\cos x + 1)^2$

$4\sin^2 x = 9\cos^2 x + 6\cos x + 1$

$4(1 - \cos^2 x) = 9\cos^2 x + 6\cos x + 1$

$0 = 13\cos^2 x + 6\cos x - 3$

$$\cos x = \frac{-6 \pm \sqrt{6^2 - 4(13)(-3)}}{2(13)}$$

$$= \frac{-6 \pm \sqrt{192}}{26}$$

$\cos x \approx 0.3022$ $\quad$ $\cos x \approx -0.7637$

$x \approx 72.4°$ or $287.6°$ $\quad$ $x \approx 139.8°$ or $220.2°$

287.6° and 139.8° do not check.
The solutions are 72.4°, 220.2°.

**54.** $\sqrt{3}\sin x + \cos x = 1$

$\sqrt{3}\sin x = 1 - \cos x$

$(\sqrt{3}\sin x)^2 = (1 - \cos x)^2$

$3\sin^2 x = 1 - 2\cos x + \cos^2 x$

$3(1 - \cos^2 x) = 1 - 2\cos x + \cos^2 x$

$0 = 4\cos^2 x - 2\cos x - 2$

$0 = 2(2\cos^2 x - \cos x - 1)$
$0 = 2(2\cos x + 1)(\cos x - 1)$
$2\cos x + 1 = 0$ $\quad$ $\cos x - 1 = 0$

$\cos x = -\frac{1}{2}$ $\quad$ $\cos x = 1$

$x = 120°$ or $240°$ $\quad$ $x = 0°$

240° does not check.
The solutions are 0°, 120°.

**55.** $3\sin^2 x - \sin x - 1 = 0$

$$\sin x = \frac{1 \pm \sqrt{(-1)^2 - 4(3)(-1)}}{2(3)}$$

$$= \frac{1 \pm \sqrt{13}}{6}$$

$\sin x = 0.7676$ $\quad$ $\sin x = -0.4343$

$x = 50.1°, 129.9°$ $\quad$ $x = 205.7°, 334.3°$

The solutions are 50.1°, 129.9°, 205.7°, 334.3°.

**56.** $2\cos^2 x - 5\cos x - 5 = 0$

$$\cos x = \frac{5 \pm \sqrt{(-5)^2 - 4(2)(-5)}}{2(2)}$$

$$= \frac{5 \pm \sqrt{65}}{4}$$

$\cos x = 3.26$ $\quad$ $\cos x = -0.7656$

no solution $\quad$ $x = 140.0°, 220.0°$

The solutions are 140.0°, 220.0°.

**57.** $2\cos x - 1 + 3\sec x = 0$

$2\cos x - 1 + \frac{3}{\cos x} = 0$

$2\cos^2 x - \cos x + 3 = 0$

$$\cos x = \frac{1 \pm \sqrt{(-1)^2 - 4(2)(3)}}{2(2)}$$

$$= \frac{1 \pm \sqrt{-23}}{4}$$

no solution

**58.** $3\sin x - 5 + \csc x = 0$

$3\sin x - 5 + \frac{1}{\sin x} = 0$

$3\sin^2 x - 5\sin x + 1 = 0$

$\sin x = \frac{5 \pm \sqrt{(-5)^2 - 4(3)(1)}}{2(3)}$

$= \frac{5 \pm \sqrt{13}}{6}$

$\sin x = 1.4343$ no solution

$\sin x = 0.2324$, $x = 13.4°, 166.6°$

The solutions are $13.4°, 166.6°$.

**59.** $\cos^2 x - 3\sin x + 2\sin^2 x = 0$

$1 - \sin^2 x - 3\sin x + 2\sin^2 x = 0$

$\sin^2 x - 3\sin x + 1 = 0$

$\sin x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(1)}}{2(1)}$

$= \frac{3 \pm \sqrt{5}}{2}$

$\sin x = 2.6180$ no solution

$\sin x = 0.3820$, $x = 22.5°, 157.5°$

The solutions are $22.5°, 157.5°$.

**60.** $\sin^2 x = 2\cos x + 3\cos^2 x$

$1 - \cos^2 x = 2\cos x + 3\cos^2 x$

$0 = 4\cos^2 x + 2\cos x - 1$

$\cos x = \frac{-2 \pm \sqrt{2^2 - 4(4)(-1)}}{2(4)}$

$= \frac{-2 \pm \sqrt{20}}{8}$

$\cos x = 0.3090$, $x = 72°, 288°$

$\cos x = -0.8090$, $x = 144°, 216°$

The solutions are $72.0°, 144.0°, 216.0°, 288.0°$.

**61.** $\tan 2x - 1 = 0$

$\tan 2x = 1$

$2x = \frac{\pi}{4} + k\pi$

$x = \frac{\pi}{8} + \frac{k\pi}{2}$, where $k$ is an integer

**62.** $\sec 3x - \frac{2\sqrt{3}}{3} = 0$

$\sec 3x = \frac{2\sqrt{3}}{3}$

$3x = \frac{\pi}{6} + 2k\pi$ or $3x = \frac{11\pi}{6} + 2k\pi$

$x = \frac{\pi}{18} + \frac{2}{3}k\pi$ or $x = \frac{11\pi}{18} + \frac{2}{3}k\pi$

The solutions are $\frac{\pi}{18} + \frac{2}{3}k\pi$, $\frac{11\pi}{18} + \frac{2}{3}k\pi$, where $k$ is an integer.

**63.** $\sin 5x = 1$

$5x = \frac{\pi}{2} + 2k\pi$

$x = \frac{\pi}{10} + \frac{2}{5}k\pi$, where $k$ is an integer

**64.** $\cos 4x = -\frac{\sqrt{2}}{2}$

$4x = \frac{3\pi}{4} + 2k\pi$ or $4x = \frac{5\pi}{4} + 2k\pi$

$x = \frac{3\pi}{16} + \frac{1}{2}k\pi$ , $x = \frac{5\pi}{16} + \frac{1}{2}k\pi$

The solutions are $\frac{3\pi}{16} + \frac{1}{2}k\pi$ , $\frac{5\pi}{16} + \frac{1}{2}k\pi$ where $k$ is an integer.

**65.** $\sin 2x - \sin x = 0$

$2\sin x\cos x - \sin x = 0$

$\sin x(2\cos x - 1) = 0$

$\sin x = 0$

$x = 0 + 2k\pi$ or $x = \pi + 2k\pi$

$2\cos x - 1 = 0$

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3} + 2k\pi$ or $x = \frac{5\pi}{3} + 2k\pi$

The solutions are $0 + 2k\pi$, $\frac{\pi}{3} + 2k\pi$, $\pi + 2k\pi$, $\frac{5\pi}{3} + 2k\pi$ where $k$ is an integer.

**66.** $\cos 2x = -\dfrac{\sqrt{3}}{2}$

$2x = \dfrac{5\pi}{6} + 2k\pi$ or $2x = \dfrac{7\pi}{6} + 2k\pi$

$x = \dfrac{5\pi}{12} + k\pi$ $\quad x = \dfrac{7\pi}{12} + k\pi$

The solutions are $\dfrac{5\pi}{12} + k\pi$, $\dfrac{7\pi}{12} + k\pi$ where $k$ is an integer.

**67.** $\sin\left(2x + \dfrac{\pi}{6}\right) = -\dfrac{1}{2}$

$2x + \dfrac{\pi}{6} = \dfrac{7\pi}{6} + 2k\pi$ or $2x + \dfrac{\pi}{6} = \dfrac{11\pi}{6} + 2k\pi$

$2x = \pi + 2k\pi$ $\quad 2x = \dfrac{5\pi}{3} + 2k\pi$

$x = \dfrac{\pi}{2} + k\pi$ $\quad x = \dfrac{5\pi}{6} + k\pi$

The solutions are $\dfrac{\pi}{2} + k\pi$, $\dfrac{5\pi}{6} + k\pi$ where $k$ is an integer.

**68.** $\cos\left(2x - \dfrac{\pi}{4}\right) = -\dfrac{\sqrt{2}}{2}$

$2x - \dfrac{\pi}{4} = \dfrac{3\pi}{4} + 2k\pi$ or $2x - \dfrac{\pi}{4} = \dfrac{5\pi}{4} + 2k\pi$

$2x = \pi + 2k\pi$ $\quad 2x = \dfrac{3\pi}{2} + 2k\pi$

$x = \dfrac{\pi}{2} + k\pi$ $\quad x = \dfrac{3\pi}{4} + k\pi$

The solutions are $\dfrac{\pi}{2} + k\pi$, $\dfrac{3\pi}{4} + k\pi$ where $k$ is an integer.

**69.** $\sin^2 \dfrac{x}{2} + \cos x = 1$

$\left(\pm\sqrt{\dfrac{1-\cos x}{2}}\right)^2 + \cos x = 1$

$\dfrac{1-\cos x}{2} + \cos x = 1$

$1 - \cos x + 2\cos x = 2$

$\cos x = 1$

$x = 0 + 2k\pi$ where $k$ is an integer

**70.** $\cos^2 \dfrac{x}{2} - \cos x = 1$

$\left(\pm\sqrt{\dfrac{1+\cos x}{2}}\right)^2 - \cos x = 1$

$\dfrac{1+\cos x}{2} - \cos x = 1$

$1 + \cos x - 2\cos x = 2$

$-\cos x = 1$

$\cos x = -1$

$x = \pi + 2k\pi$ where $k$ is an integer

**71.** $\cos 2x = 1 - 3\sin x$

$1 - 2\sin^2 x = 1 - 3\sin x$

$0 = 2\sin^2 x - 3\sin x$

$0 = \sin x(2\sin x - 3)$

$\sin x = 0$ $\quad 2\sin x - 3 = 0$

$x = 0, \pi$ $\quad \sin x = \dfrac{3}{2}$

no solution.

The solutions are $0, \pi$.

**72.** $\cos 2x = 2\cos x - 1$

$2\cos^2 x - 1 = 2\cos x - 1$

$2\cos^2 x - 2\cos x = 0$

$2\cos x(\cos x - 1) = 0$

$\cos x = 0$ $\quad \cos x - 1 = 0$

$x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ $\quad \cos x = 1$

$x = 0$

The solutions are $0, \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

**73.** $\sin 4x - \sin 2x = 0$

$2\sin 2x\cos 2x - \sin 2x = 0$

$\sin 2x(2\cos 2x - 1) = 0$

$\sin 2x = 0$ $\quad 2\cos 2x - 1 = 0$

$2x = 0 + 2k\pi$ or $2x = \pi + 2k\pi$

$\cos 2x = \dfrac{1}{2}$

$2x = \dfrac{\pi}{3} + 2k\pi$ or $2x = \dfrac{5\pi}{3} + 2k\pi$

$x = 0 + k\pi, \dfrac{\pi}{2} + k\pi, \dfrac{\pi}{6} + k\pi, \dfrac{5\pi}{6} + k\pi$

The solutions are $0, \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}, \pi, \dfrac{7\pi}{6}, \dfrac{3\pi}{2}, \dfrac{11\pi}{6}$.

**74.** $\sin 4x - \cos 2x = 0$

$2\sin 2x\cos 2x - \cos 2x = 0$

$\cos 2x(2\sin 2x - 1) = 0$

$\cos 2x = 0$ $\quad 2\sin 2x - 1 = 0$

$2x = \dfrac{\pi}{2} + 2k\pi$ or $2x = \dfrac{3\pi}{2} + 2k\pi$

$\sin 2x = \dfrac{1}{2}$

$2x = \dfrac{\pi}{6} + 2k\pi$ or $2x = \dfrac{5\pi}{6} + 2k\pi$

$x = \dfrac{\pi}{12} + k\pi, \dfrac{\pi}{4} + k\pi, \dfrac{5\pi}{12} + k\pi, \dfrac{3\pi}{4} + k\pi$

The solutions are $\dfrac{\pi}{12}, \dfrac{\pi}{4}, \dfrac{5\pi}{12}, \dfrac{3\pi}{4}, \dfrac{13\pi}{12}, \dfrac{5\pi}{4}, \dfrac{17\pi}{12}, \dfrac{7\pi}{4}$.

**75.** $\tan\frac{\pi}{2} = \sin x$

$\frac{1-\cos x}{\sin x} = \sin x$

$1-\cos x = \sin^2 x$

$1-\cos x = 1-\cos^2 x$

$\cos^2 x - \cos x = 0$

$\cos x(\cos x - 1) = 0$

$\cos x = 0 \qquad \cos x = 1$

$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad x = 0$

The solutions are $0, \frac{\pi}{2}, \frac{3\pi}{2}$.

**76.** $\tan\frac{\pi}{2} = 1-\cos x$

$\frac{\sin x}{1+\cos x} = 1-\cos x$

$\sin x = 1-\cos^2 x$

$\sin x = \sin^2 x$

$0 = \sin^2 x - \sin x$

$0 = \sin x(\sin x - 1)$

$\sin x = 0 \qquad \sin x - 1 = 0$

$x = 0, \pi \qquad \sin x = 1$

$x = \frac{\pi}{2}$

$\pi$ does not check.

The solutions are $0, \frac{\pi}{2}$.

**77.** $\sin 2x\cos x + \cos 2x\sin x = 0$

$\sin(2x + x) = 0$

$\sin 3x = 0$

$3x = 0 + 2k\pi \quad \text{or} \quad 3x = \pi + 2k\pi$

$x = 0 + \frac{2}{3}k\pi \quad \text{or} \quad x = \frac{\pi}{3} + \frac{2}{3}k\pi$

The solutions are $0, \frac{\pi}{3}, \frac{2}{3}\pi, \pi, \frac{4}{3}\pi\ \frac{5}{3}\pi$.

**78.** $\cos 2x\cos x - \sin 2x\sin x = 0$

$\cos(2x + x) = 0$

$\cos 3x = 0$

$3x = \frac{\pi}{2} + 2k\pi \quad \text{or} \quad 3x = \frac{3\pi}{2} + 2k\pi$

$x = \frac{\pi}{6} + \frac{2}{3}k\pi \quad \text{or} \quad x = \frac{\pi}{2} + \frac{2}{3}k\pi$

The solutions are $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$.

**79.** $\sin x\cos 2x - \cos x\sin 2x = \frac{\sqrt{3}}{2}$

$\sin(x - 2x) = \frac{\sqrt{3}}{2}$

$\sin(-x) = \frac{\sqrt{3}}{2}$

$\sin x = -\frac{\sqrt{3}}{2}$

$x = \frac{4\pi}{3}, \frac{5\pi}{3}$

**80.** $\cos 2x\cos x + \sin 2x\sin x = -1$

$\cos(2x - x) = -1$

$\cos x = -1$

$x = \pi$

**81.** $\sin 3x - \sin x = 0$

$2\cos\frac{3x + x}{2}\sin\frac{3x - 2}{2} = 0$

$2\cos 2x\sin x = 0$

$2(1 - 2\sin^2 x)\sin x = 0$

$\sin x = 0 \qquad 1 - 2\sin^2 x = 0$

$x = 0, \pi \qquad \sin^2 x = \frac{1}{2}$

$\sin x = \pm\frac{\sqrt{2}}{2}$

$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

The solutions are $0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}$.

**82.** $\cos 3x + \cos x = 0$

$2\cos\frac{3x + x}{2}\cos\frac{3x - x}{2} = 0$

$2\cos 2x\cos x = 0$

$2(2\cos^2 x - 1)\cos x = 0$

$\cos x = 0 \qquad 2\cos^2 x - 1 = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad \cos x = \pm\frac{\sqrt{2}}{2}$

$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

The solutions are $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$.

**83.** $2\sin x\cos x + 2\sin x - \cos x - 1 = 0$

$2\sin x(\cos x + 1) - (\cos x + 1) = 0$

$(\cos x + 1)(2\sin x - 1) = 0$

$\cos x + 1 = 0$     $2\sin x - 1 = 0$

$\cos x = -1$     $\sin x = \frac{1}{2}$

$x = \pi$     $x = \frac{\pi}{6}, \frac{5\pi}{6}$

The solutions are $\frac{\pi}{6}, \frac{5\pi}{6}, \pi$.

**84.** $2\sin x\cos x - 2\sqrt{2}\sin x - \sqrt{3}\cos x + \sqrt{6} = 0$

$2\sin x(\cos x - \sqrt{2}) - \sqrt{3}(\cos x - \sqrt{2}) = 0$

$(\cos x - \sqrt{2})(2\sin x - \sqrt{3}) = 0$

$\cos x = \sqrt{2}$     $\sin x = \frac{\sqrt{3}}{2}$

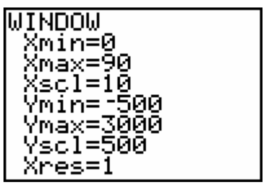
no solution     $x = \frac{\pi}{3}, \frac{2\pi}{3}$

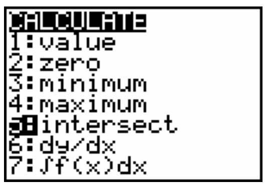
The solutions are $\frac{\pi}{3}, \frac{2\pi}{3}$.

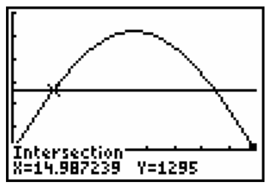
**85.** 0.7391     **86.** 0, 1.8955     **87.** –3.2957, 3.2957

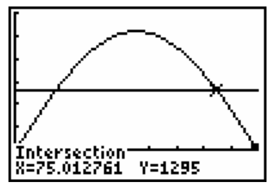
**88.** 4.9172     **89.** 1.16     **90.** 0.5

**91.** Set your graphing utility to "degree" mode and graph $d = \frac{(288)^2}{16}\sin\theta\cos\theta$ and $d = 1295$ for $0° \le \theta \le 90°$, $-500 \le d \le 3000$.
Use the TRACE or INTERSECT feature of your graphing utility to determine the intersection of the two graphs.



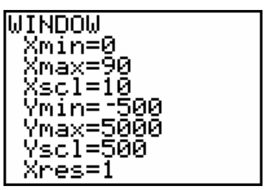


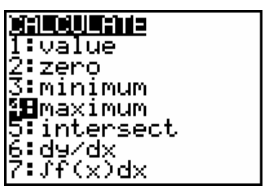


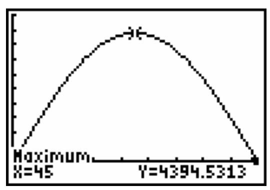



Thus, $d = 1295$ for $\theta \approx 14.99°$ and $\theta \approx 75.01°$.

**92.** Set your graphing utility to "degree" mode and graph $d = \frac{(375)^2}{16}\sin\theta\cos\theta$. Use the MAXIMUM feature of your graphing utility to determine the maximum horizontal range.



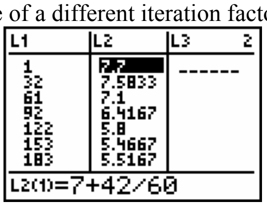


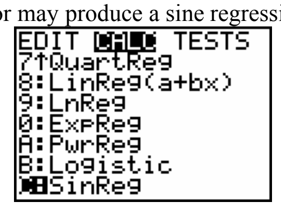


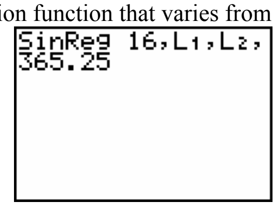

The maximum horizontal range is about 4394.5 ft., and it is produced when $\theta = 45°$.

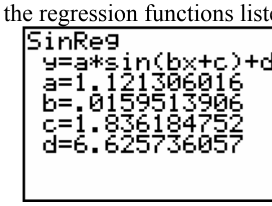
The sine regression functions in Exercises 93 –98 were obtained on a TI-83/TI-83 Plus/TI-84 Plus calculator by using an iteration value of 16. The use of a different iteration factor may produce a sine regression function that varies from the regression functions listed below.

**93. a.**










$f(x) = y \approx 1.1213\sin(0.01595x + 1.8362) + 6.6257$

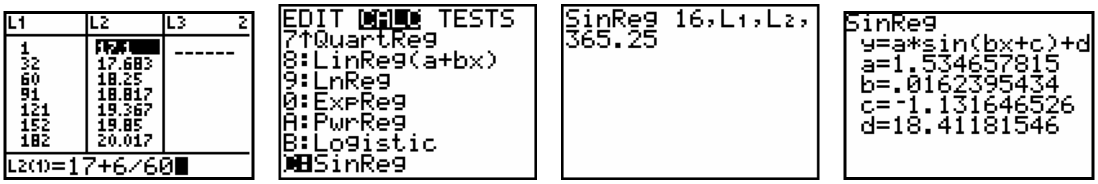
**b.** Set your graphing utility to "radian" mode.

$f(71) \approx 1.1213\sin(0.01595(71) + 1.8362) + 6.6257$

$\approx 6.8186$ hours

$\approx 6 + .8186(60) \to 6:49$

**94.** **a.**




$f(x)=y\approx 1.5347\sin(0.01624x-1.1316)+18.4118$

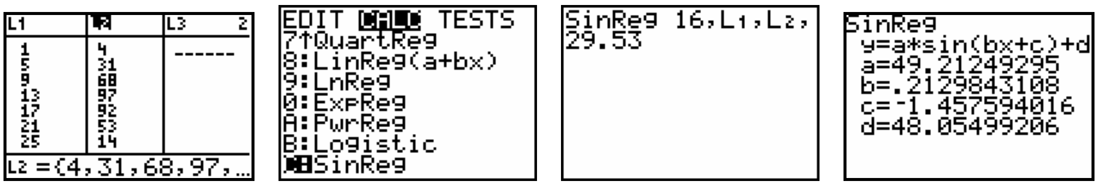
**b.** Set your graphing utility to "radian" mode.

$f(73)\approx 1.5347\sin(0.01624(73)-1.1316)+18.4118$

$\approx 18.495$ hours

$\approx 18+0.495(60)\to 18:30$

**95.** **a.**



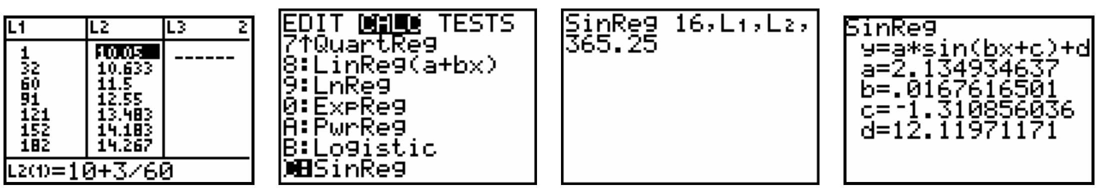

$f(x)=y\approx 49.2125\sin(0.2130x-1.4576)+48.0550$

**b.** Set your graphing utility to "radian" mode.

$f(31)\approx 49.2125\sin(0.2130(31)-1.4576)+48.0550$

$\approx 3\%$

**96.** **a.**




$f(x)=y\approx 2.1350\sin(0.01676x-1.3109)+12.1197$

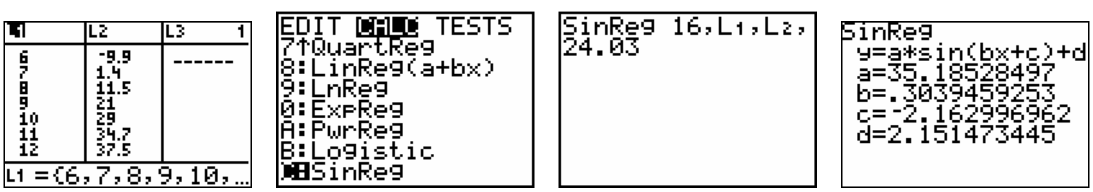
**b.** Set your graphing utility to "radian" mode.

$f(132)\approx 2.1350\sin(0.01676(132)-1.3109)+12.1197$

$\approx 13.794$ hours

$\approx 13+0.794(60)\to 13$ hours 48 minutes

**97.** **a.**



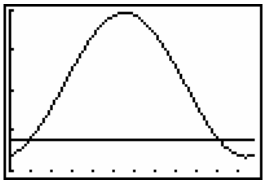

$f(x)=y\approx 35.185\sin(0.30395x-2.1630)+2.1515$

**b.** Set your graphing utility to "radian" mode.

$f\left(9\frac{25}{60}\right)\approx 35.185\sin\left(0.30395\left(9\frac{25}{60}\right)-2.1630\right)+2.1515$

$\approx 24.8°$

**98.** $d(t) \geq 10.75$ for $28 \leq t \leq 314$, so New Orleans will have at least 10.75 hours of daylight about $314 - 28 = 286$ days of the year.



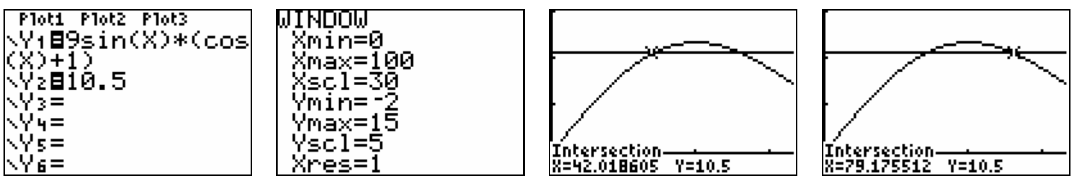
Xmin = 0, Xmax = 365, Xscl = 31,
Ymin = 10, Ymax = 14, Yscl = 1

**99.** **a.** $\sin\theta = \frac{y}{3}$, so $y = 3\sin\theta$

$\cos\theta = \frac{x}{3}$, so $x = 3\cos\theta$

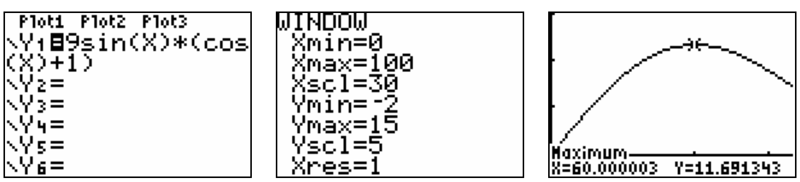
$$\begin{aligned} A &= 2\left(\frac{1}{2}bh\right) + 3y \\ &= (3\sin\theta)(3\cos\theta) + 3(3\sin\theta) \\ &= 9\sin\theta\cos\theta + 9\sin\theta \\ &= 9\sin\theta(\cos\theta + 1) \end{aligned}$$

**b.** Make sure your calculator is in degree mode.



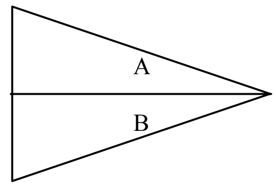

The solutions are $42^\circ$ and $79^\circ$.

**c.** Make sure your calculator is in degree mode.




The value of $\theta$ that would maximize the area is $60^\circ$.

**100.** **a.**




First separate $\theta$ into A + B, where A is the angle above the horizontal for the observer and B is the angle below the horizontal.

$$\tan A = \frac{3.5}{d}$$
$$\tan B = \frac{2.5}{d}$$
$$\tan\theta = \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$
$$\tan\theta = \frac{\frac{3.5}{d} + \frac{2.5}{d}}{1 - \left(\frac{3.5}{d}\right)\left(\frac{2.5}{d}\right)}$$
$$\tan\theta = \frac{3.5d + 2.5d}{d^2 - 8.75}$$
$$\theta = \tan^{-1}\left(\frac{6d}{d^2 - 8.75}\right)$$

**b.**

$$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$
$$\frac{1}{\sqrt{3}} = \frac{6d}{d^2 - 8.75}$$
$$d^2 - 8.75 = 6\sqrt{3}d$$
$$d^2 - 6\sqrt{3}d - 8.75 = 0$$

Solve for $d$ using quadratic formula.

$$d = \frac{6\sqrt{3} \pm \sqrt{(-6\sqrt{3})^2 - 4(1)(-8.75)}}{2(1)}$$
$$= \frac{6\sqrt{3} \pm \sqrt{108 + 35}}{2}$$
$$\approx 11.2 \text{ ft}$$

**101.** $\theta = 20°$

$$x = \sqrt{(4 + 18\cot 20°)^2 + 100} - (4 + 18\cot 20°)$$
$$x \approx 0.93 \text{ ft.}$$

$\theta = 30°$

$$x = \sqrt{(4 + 18\cot 30°)^2 + 100} - (4 + 18\cot 30°)$$
$$x \approx 1.39 \text{ ft.}$$

**102.** $\cos\theta = 2^{(x-4)/(x+4)}$

$$\cos^{-1} 2^{(2-4)/(2+4)} \le \theta \le \cos^{-1} 2^{(1-4)/(1+4)}$$
$$\cos^{-1} 2^{-1/3} \le \theta \le \cos^{-1} 2^{-3/5}$$
$$37.5° \le \theta \le 48.7°$$

## Connecting Concepts

**103.** $\sqrt{3}\sin x + \cos x = \sqrt{3}$

$a = \sqrt{3},\ b = 1,\ k = \sqrt{(\sqrt{3})^2 + 1} = 2,\ \alpha$ is in first quadrant

$$\tan\beta = \frac{1}{\sqrt{3}}$$
$$\beta = \frac{\pi}{6}$$
$$\alpha = \frac{\pi}{6}$$
$$2\sin\left(x + \frac{\pi}{6}\right) = \sqrt{3}$$
$$\sin\left(x + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$x + \frac{\pi}{6} = \frac{\pi}{3}$, $x = \frac{\pi}{6}$     $x + \frac{\pi}{6} = \frac{2\pi}{3}$, $x = \frac{\pi}{2}$

**104.** $\sin x - \cos x = 1$

$k = \sqrt{1^2 + 1^2} = \sqrt{2}$   $\tan\beta = \left|\frac{-1}{1}\right| = 1$

$$\beta = \frac{\pi}{4}$$
$$\alpha = -\frac{\pi}{4} \text{ fourth quadrant}$$
$$\sqrt{2}\sin\left(x - \frac{\pi}{4}\right) = 1$$
$$\sin\left(x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$x - \frac{\pi}{4} = \frac{\pi}{4}$, $x = \frac{\pi}{2}$     $x - \frac{\pi}{4} = \frac{3\pi}{4}$, $x = \pi$

**105.** $-\sin x + \sqrt{3}\cos x = \sqrt{3}$

$k = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$ $\quad \tan\beta = \left|\frac{\sqrt{3}}{-1}\right| = \sqrt{3}$

$\beta = \frac{\pi}{3}$

$\alpha = \frac{2\pi}{3}$ second quadrant

$2\sin\left(x + \frac{2\pi}{3}\right) = \sqrt{3}$

$\sin\left(x + \frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$x + \frac{2\pi}{3} = \frac{\pi}{3}$ $\qquad x + \frac{2\pi}{3} = \frac{2\pi}{3}$

$x = -\frac{\pi}{3}$ $\qquad x = 0$

$-\frac{\pi}{3} + 2\pi = \frac{5\pi}{3}$

The solutions are 0 and $\frac{5\pi}{3}$.

**106.** $-\sqrt{3}\sin x - \cos x = 1$

$k = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2$

$\tan\beta = \left|\frac{-1}{-\sqrt{3}}\right| = \frac{1}{\sqrt{3}}$

$\beta = \frac{\pi}{6}$

$\alpha = -\frac{5\pi}{6}$ third quadrant

$2\sin\left(x - \frac{5\pi}{6}\right) = 1$

$\sin\left(x - \frac{5\pi}{6}\right) = \frac{1}{2}$

$x - \frac{5\pi}{6} = \frac{\pi}{6}$ $\qquad x - \frac{5\pi}{6} = \frac{5\pi}{6}$

$x = \pi$ $\qquad x = \frac{5\pi}{3}$

**107.** $\cos 5x - \cos 3x = 0$

$-2\sin\frac{5x+3x}{2}\sin\frac{5x-3x}{2} = 0$

$-2\sin 4x \sin x = 0$

$\sin 4x = 0$ $\qquad \sin x = 0$

$4x = 0 + 2k\pi$ $\qquad 4x = \pi + 2k\pi$ $\qquad x = 0,\ \pi$

$x = 0 + \frac{1}{2}k\pi$ $\qquad x = \frac{\pi}{4} + \frac{1}{2}k\pi$

$x = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}$ $\qquad x = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{7\pi}{4}$

The solutions are 0, $\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$.

**108.** $\cos 5x - \cos x - \sin 3x = 0$

$-2\sin\frac{5x+x}{2}\sin\frac{5x-x}{2} - \sin 3x = 0$

$-2\sin 3x \sin 2x - \sin 3x = 0$

$\sin 3x(-2\sin 2x - 1) = 0$

$\sin 3x = 0$ $\qquad \sin 2x = -\frac{1}{2}$

$3x = 0 + 2k\pi$ $\qquad 3x = \pi + 2k\pi$ $\qquad 2x = \frac{7\pi}{6} + 2k\pi$ $\qquad 2x = \frac{11\pi}{6} + 2k\pi$

$x = 0 + \frac{2}{3}k\pi$ $\qquad x = \frac{\pi}{3} + \frac{2}{3}k\pi$ $\qquad x = \frac{7\pi}{12} + k\pi$ $\qquad x = \frac{11\pi}{12} + k\pi$

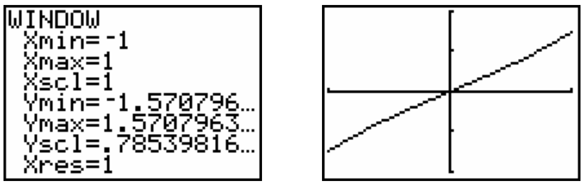
$x = 0,\ \frac{2}{3}\pi,\ \frac{4\pi}{3}$ $\qquad x = \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3}$ $\qquad x = \frac{7\pi}{12},\ \frac{19\pi}{12}$ $\qquad x = \frac{11\pi}{12},\ \frac{23\pi}{12}$

The solutions are 0, $\frac{\pi}{3}, \frac{7\pi}{12}, \frac{2\pi}{3}, \frac{11\pi}{12}, \pi, \frac{4\pi}{3}, \frac{19\pi}{12}, \frac{5\pi}{3}, \frac{23\pi}{12}$.

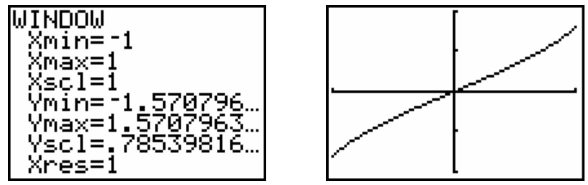
## Exploring Concepts with Technology

**Approximate an Inverse Trigonometric Function with Polynomials**

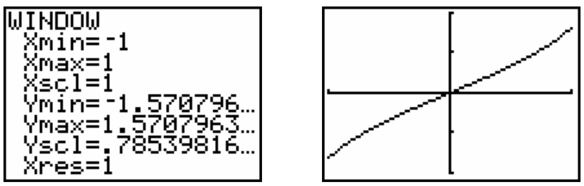
**1.** $y = f_1 = x + \dfrac{x^3}{2\cdot 3}$ where $-1 \le x \le 1$




$y = f_2 = x + \dfrac{x^3}{2\cdot 3} + \dfrac{1\cdot 3x^5}{2\cdot 4\cdot 5}$ where $-1 \le x \le 1$

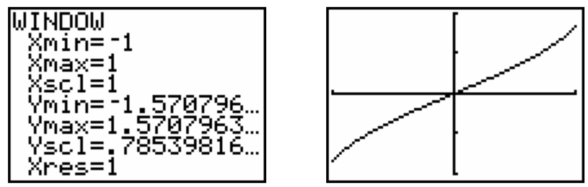



$y = f_3 = x + \dfrac{x^3}{2\cdot 3} + \dfrac{1\cdot 3x^5}{2\cdot 4\cdot 5} + \dfrac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7}$ where $-1 \le x \le 1$




$y = f_4 = x + \dfrac{x^3}{2\cdot 3} + \dfrac{1\cdot 3x^5}{2\cdot 4\cdot 5} + \dfrac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7} + \dfrac{1\cdot 3\cdot 5\cdot 7x^9}{2\cdot 4\cdot 6\cdot 8\cdot 9}$

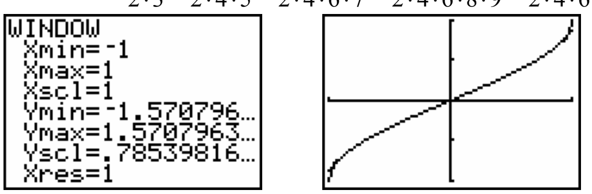
where $-1 \le x \le 1$




**2.** $\left|f_3(x) - \sin^{-1} x\right| < 0.001$ for $-0.6552 < x < 0.6552$

**3.** $\left|f_4(x) - \sin^{-1} x\right| < 0.001$ for $-0.7186 < x < 0.7186$

**4.** $f_6(x) = x + \dfrac{x^3}{2\cdot 3} + \dfrac{1\cdot 3x^5}{2\cdot 4\cdot 5} + \dfrac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7} + \dfrac{1\cdot 3\cdot 5\cdot 7x^9}{2\cdot 4\cdot 6\cdot 8\cdot 9} + \dfrac{1\cdot 3\cdot 5\cdot 7\cdot 9x^{11}}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 11} + \dfrac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11x^{13}}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 13}$




**5.** $f_6(1) = 1 + \dfrac{1}{6} + \dfrac{3}{40} + \dfrac{5}{112} + \dfrac{35}{1152} + \dfrac{63}{2816} + \dfrac{231}{13312}$

$\approx 1 + 0.167 + 0.075 + 0.045 + 0.030 + 0.022 + 0.017$

Each term is much smaller than the previous term.

**6.** The largest-degree term in $f_{10}(x)$ is $\dfrac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 15\cdot 17\cdot 19x^{21}}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 14\cdot 16\cdot 18\cdot 20\cdot 21}$

## Assessing Concepts

**1.** True

**2.** False. $\cos^{-1}[\cos(3\pi/2)] = \cos^{-1}(0) = \pi/2 \ne 3\pi/2$.

**3.** False. $\cos(\cos^{-1} 2) \ne 2$ because $\cos^{-1} 2$ is undefined.

**4.** True

**5.** 4

**6.** Domain is $-\dfrac{1}{2} \le x \le \dfrac{1}{2}$.

**7.** Range is $0 \le y \le \pi$.

**8.** $\sin^{-1}\left(\sin\left(\frac{7\pi}{3}\right)\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

**9.** $\sqrt{2}$

**10.** $\tan 75° = \tan(45° + 30°) = \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°} = \frac{1 + \frac{\sqrt{3}}{3}}{1 - 1\left(\frac{\sqrt{3}}{3}\right)} = \frac{3+\sqrt{3}}{3-\sqrt{3}} \cdot \frac{3+\sqrt{3}}{3+\sqrt{3}} = \frac{9 + 6\sqrt{3} + 3}{9 - 3} = 2 + \sqrt{3}$

## Chapter Review

**1.** $$\begin{aligned}\cos(45° + 30°) &= \cos 45° \cos 30° - \sin 45° \sin 30° \quad [6.2]\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

**2.** $$\begin{aligned}\tan(210° - 45°) &= \frac{\tan 210° - \tan 45°}{1 + \tan 210° \tan 45°} \quad [6.2]\\ &= \frac{\frac{1}{\sqrt{3}} - 1}{1 + \frac{1}{\sqrt{3}}\cdot 1} = \frac{\frac{1}{\sqrt{3}} - 1}{1 + \frac{1}{\sqrt{3}}}\cdot\frac{\sqrt{3}}{\sqrt{3}}\\ &= \frac{1-\sqrt{3}}{\sqrt{3}+1}\cdot\frac{1-\sqrt{3}}{1-\sqrt{3}} = \frac{1 - 2\sqrt{3} + 3}{1-3}\\ &= \frac{4 - 2\sqrt{3}}{-2} = \sqrt{3} - 2\end{aligned}$$

**3.** $$\begin{aligned}\sin\left(\frac{2\pi}{3} + \frac{\pi}{4}\right) &= \sin\frac{2\pi}{3}\cos\frac{\pi}{4} + \cos\frac{2\pi}{3}\sin\frac{\pi}{4} \quad [6.2]\\ &= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

**4.** $$\begin{aligned}\sec\left(\frac{4\pi}{3} - \frac{\pi}{4}\right) &= \frac{1}{\cos\left(\frac{4\pi}{3} - \frac{\pi}{4}\right)} \quad [6.2]\\ &= \frac{1}{\cos\frac{4\pi}{3}\cos\frac{\pi}{4} + \sin\frac{4\pi}{3}\sin\frac{\pi}{4}}\\ &= \frac{1}{-\frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \left(-\frac{\sqrt{3}}{2}\right)\cdot\frac{\sqrt{2}}{2}}\cdot\frac{-4}{-4}\\ &= \frac{-4}{\sqrt{2}+\sqrt{6}}\cdot\frac{\sqrt{2}-\sqrt{6}}{\sqrt{2}-\sqrt{6}}\\ &= \frac{-4\left(\sqrt{2}-\sqrt{6}\right)}{2-6} = \sqrt{2} - \sqrt{6}\end{aligned}$$

**5.** $$\begin{aligned}\sin(60° - 135°) &= \sin 60° \cos 135° - \cos 60° \sin 135° \quad [6.2]\\ &= \frac{\sqrt{3}}{2}\left(-\frac{\sqrt{2}}{2}\right) - \frac{1}{2}\cdot\frac{\sqrt{2}}{2}\\ &= -\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = -\frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

**6.** $$\begin{aligned}\cos\left(\frac{5\pi}{3} - \frac{7\pi}{4}\right) &= \cos\frac{5\pi}{3}\cos\frac{7\pi}{4} + \sin\frac{5\pi}{3}\sin\frac{7\pi}{4} \quad [6.2]\\ &= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)\\ &= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{2}+\sqrt{6}}{4}\end{aligned}$$

**7.** $\sin 22.5° = \sin\frac{45°}{2}$ [6.3]

$$= \sqrt{\frac{1-\cos 45°}{2}} = \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{2-\sqrt{2}}{4}} = \frac{\sqrt{2-\sqrt{2}}}{2}$$

**8.** $\cos 105° = \cos\frac{210°}{2}$ [6.3]

$$= -\sqrt{\frac{1+\cos 210°}{2}} = -\sqrt{\frac{1+\left(-\frac{\sqrt{3}}{2}\right)}{2}} = -\sqrt{\frac{2-\sqrt{3}}{4}} = -\frac{\sqrt{2-\sqrt{3}}}{2}$$

**9.** $\tan 67.5° = \tan\frac{135°}{2}$ [6.3]

$$= \frac{1-\cos 135°}{\sin 135°} = \frac{1-\left(-\frac{1}{\sqrt{2}}\right)}{\frac{1}{\sqrt{2}}} = \frac{1+\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \sqrt{2}+1$$

**10.** $\sin 112.5° = \sin\frac{225°}{2}$ [6.3]

$$= \sqrt{\frac{1-\cos 225°}{2}} = \sqrt{\frac{1-\left(-\frac{\sqrt{2}}{2}\right)}{2}} = \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}$$

**11.** $\sin\alpha = \frac{1}{2}, \cos\alpha = \frac{\sqrt{3}}{2}$, quadrant I, $\tan\alpha = \frac{\sqrt{3}}{3}$ [6.2/6.3]

$\cos\beta = \frac{1}{2}, \sin\beta = -\frac{\sqrt{3}}{2}$, quadrant IV, $\tan\beta = -\sqrt{3}$

**a.** $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$= \frac{\sqrt{3}}{2}\cdot\frac{1}{2} + \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = 0$$

**b.** $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$

$$= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{\sqrt{3}}{3}\right)^2} = \frac{\frac{2\sqrt{3}}{3}}{1-\frac{1}{3}}\cdot\frac{3}{3} = \frac{2\sqrt{3}}{3-1} = \sqrt{3}$$

**c.** $\sin\frac{\beta}{2} = \sqrt{\frac{1-\cos\beta}{2}}$

$$= \sqrt{\frac{1-\frac{1}{2}}{2}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

**12.** $\sin\alpha = \frac{\sqrt{3}}{2}, \cos\alpha = -\frac{1}{2}$, quadrant II [6.2/6.3]

$\cos\beta = -\frac{1}{2}, \sin\beta = -\frac{\sqrt{3}}{2}$, quadrant III

**a.** $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \frac{\sqrt{3}}{2}\cdot\left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = 0$$

**b.** $\sec 2\beta = \frac{1}{\cos 2\beta} = \frac{1}{\cos^2\beta - \sin^2\beta}$

$$= \frac{1}{\left(-\frac{1}{2}\right)^2 - \left(-\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{\frac{1}{4}-\frac{3}{4}} = \frac{1}{-\frac{2}{4}} = -2$$

**c.** $\cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}}$

$$= \sqrt{\frac{1+\left(-\frac{1}{2}\right)}{2}} = \sqrt{\frac{\frac{1}{2}}{2}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

**13.** $\sin\alpha = -\frac{1}{2}$, $\cos\alpha = \frac{\sqrt{3}}{2}$, quadrant IV, $\tan\alpha = -\frac{\sqrt{3}}{3}$ [6.2/6.3]

$\cos\beta = -\frac{\sqrt{3}}{2}$, $\sin\beta = -\frac{1}{2}$, quadrant III

**a.** $$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\ &= -\frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{2}\left(-\frac{1}{2}\right) \\ &= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = \frac{2\sqrt{3}}{4} \\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

**b.** $$\begin{aligned}\tan 2\alpha &= \frac{2\tan\alpha}{1-\tan^2\alpha} \\ &= \frac{2\left(-\frac{\sqrt{3}}{3}\right)}{1-\left(-\frac{\sqrt{3}}{3}\right)^2} \\ &= \frac{-\frac{2\sqrt{3}}{3}}{1-\frac{1}{3}}\cdot\frac{3}{3} \\ &= \frac{-2\sqrt{3}}{3-1} \\ &= -\sqrt{3}\end{aligned}$$

**c.** $$\begin{aligned}\cos\frac{\beta}{2} &= -\sqrt{\frac{1+\cos\beta}{2}} \\ &= -\sqrt{\frac{1+\left(-\frac{\sqrt{3}}{2}\right)}{2}} \\ &= -\sqrt{\frac{2-\sqrt{3}}{4}} \\ &= -\frac{\sqrt{2-\sqrt{3}}}{2}\end{aligned}$$

**14.** $\sin\alpha = \frac{\sqrt{2}}{2}, \cos\alpha = \frac{\sqrt{2}}{2}$, quadrant I [6.2/6.3]

$\cos\beta = \frac{\sqrt{3}}{2}, \sin\beta = -\frac{1}{2}$, quadrant IV, $\tan\beta = -\frac{\sqrt{3}}{3}$

**a.** $$\begin{aligned}\cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right) \\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

**b.** $$\begin{aligned}\tan 2\beta &= \frac{2\tan\beta}{1-\tan^2\beta} \\ &= \frac{2\left(-\frac{\sqrt{3}}{3}\right)}{1-\left(-\frac{\sqrt{3}}{3}\right)^2} \\ &= \frac{-2\left(\frac{\sqrt{3}}{3}\right)}{1-\frac{1}{3}}\cdot\frac{3}{3} \\ &= \frac{2\sqrt{3}}{3-1} \\ &= -\sqrt{3}\end{aligned}$$

**c.** $$\begin{aligned}\sin 2\alpha &= 2\sin\alpha\cos\alpha \\ &= 2\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} \\ &= 1\end{aligned}$$

**15.** $$\begin{aligned}2\sin 3x\cos 3x &= \sin 2(3x) \quad [6.3] \\ &= \sin 6x\end{aligned}$$

**16.** $$\begin{aligned}\frac{\tan 2x + \tan x}{1-\tan 2x\tan x} &= \tan(2x+x) \quad [6.2] \\ &= \tan 3x\end{aligned}$$

**17.** $$\begin{aligned}\sin 4x\cos x - \cos 4x\sin x &= \sin(4x-x) \quad [6.2] \\ &= \sin 3x\end{aligned}$$

**18.** $$\begin{aligned}\cos^2 2\theta - \sin^2 2\theta &= \cos 2(2\theta) \quad [6.3] \\ &= \cos 4\theta\end{aligned}$$

**19.** $\frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$ [6.1]

**20.** [6.3] $$\begin{aligned}\frac{1-\cos 2\theta}{\sin 2\theta} &= \frac{1-(\cos^2\theta - \sin^2\theta)}{2\sin\theta\cos\theta} \\ &= \frac{\sin^2\theta + \cos^2\theta - \cos^2\theta + \sin^2\theta}{2\sin\theta\cos\theta} \\ &= \frac{2\sin^2\theta}{2\sin\theta\cos\theta} \\ &= \frac{\sin\theta}{\cos\theta} \\ &= \tan\theta\end{aligned}$$

**21.** $\cos 2\theta - \cos 4\theta = -2\sin\frac{2\theta+4\theta}{2}\sin\frac{2\theta-4\theta}{2}$ [6.4]
$$= -2\sin 3\theta \sin(-\theta)$$
$$= 2\sin 3\theta \sin\theta$$

**22.** $\sin 3\theta - \sin 5\theta = 2\cos\frac{3\theta+5\theta}{2}\sin\frac{3\theta-5\theta}{2}$ [6.4]
$$= 2\cos 4\theta \sin(-\theta)$$
$$= -2\cos 4\theta \sin\theta$$

**23.** $\sin 6\theta + \sin 2\theta = 2\sin\frac{6\theta+2\theta}{2}\cos\frac{6\theta-2\theta}{2}$ [6.4]
$$= 2\sin 4\theta \cos 2\theta$$

**24.** $\sin 5\theta - \sin\theta = 2\cos\frac{5\theta+\theta}{2}\sin\frac{5\theta-\theta}{2}$ [6.4]
$$= 2\cos 3\theta \sin 2\theta$$

**25.** $\frac{1}{\sin x - 1} + \frac{1}{\sin x + 1} = \frac{(\sin x + 1) + (\sin x - 1)}{(\sin x - 1)(\sin x + 1)}$
$$= \frac{2\sin x}{\sin^2 x - 1}$$
$$= \frac{2\sin x}{-\cos^2 x}$$
$$= -2\tan x \sec x$$

**26.** $\frac{\sin x}{1-\cos x} = \frac{\sin x(1+\cos x)}{(1-\cos x)(1+\cos x)}$
$$= \frac{\sin x + \sin x \cos x}{1-\cos^2 x}$$
$$= \frac{\sin x + \sin x \cos x}{\sin^2 x}$$
$$= \frac{\sin x}{\sin^2 x} + \frac{\sin x \cos x}{\sin^2 x}$$
$$= \csc x + \cot x,\ 0 < x < \frac{\pi}{2}$$

**27.** $\frac{1+\sin x}{\cos^2 x} = \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}$
$$= \sec^2 x + \tan x \sec x$$
$$= \tan^2 x + 1 + \tan x \sec x$$

**28.** $\frac{\cos^2 2x - \sin^2 2x}{\cos 2x + \sin 2x} = \frac{(\cos 2x - \sin 2x)(\cos 2x + \sin 2x)}{\cos 2x + \sin 2x}$
$$= \cos 2x - \sin 2x$$
$$= \cos^2 x - \sin^2 x - \sin 2x$$

**29.** $\frac{1}{\cos x} - \cos x = \frac{1-\cos^2 x}{\cos x}$
$$= \frac{\sin^2 x}{\cos x}$$
$$= \tan x \sin x$$

**30.** $\sin(270° - \theta) - \cos(270° - \theta) = \sin 270° \cos\theta - \cos 270° \sin\theta - \cos 270° \cos\theta - \sin 270° \sin\theta$
$$= (-1)\cos\theta - 0 - 0 - (-1)\sin\theta$$
$$= -\cos\theta + \sin\theta$$
$$= \sin\theta - \cos\theta$$

**31.** $\sin\left(\frac{\pi}{4} - \alpha\right) = \sin\frac{\pi}{4}\cos\alpha - \cos\frac{\pi}{4}\sin\alpha$
$$= \frac{\sqrt{2}}{2}\cos\alpha - \frac{\sqrt{2}}{2}\sin\alpha$$
$$= \frac{\sqrt{2}}{2}(\cos\alpha - \sin\alpha)$$

**32.** $\sin(180° - \alpha + \beta) = \sin\left[180° - (\alpha - \beta)\right]$
$$= \sin 180° \cos(\alpha - \beta) - \cos 180° \sin(\alpha - \beta)$$
$$= 0\left[\cos(\alpha - \beta)\right] - (-1)\left[\sin\alpha\cos\beta - \cos\alpha\sin\beta\right]$$
$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

**33.** $\frac{\sin 4x - \sin 2x}{\cos 4x - \cos 2x} = \frac{2\cos\frac{4x+2x}{2}\sin\frac{4x-2x}{2}}{-2\sin\frac{4x+2x}{2}\sin\frac{4x-2x}{2}}$
$$= -\frac{\cos 3x \sin x}{\sin 3x \sin x}$$
$$= -\cot 3x$$

**34.** $2\sin x \sin 3x = \cos(x - 3x) - \cos(x + 3x)$
$$= \cos 2x - \cos 4x$$
$$= \cos 2x - (2\cos^2 2x - 1)$$
$$= 1 + \cos 2x - 2\cos^2 2x$$
$$= (1 - \cos 2x)(1 + 2\cos 2x)$$

**35.** $\sin x - \cos 2x = \sin x - (1 - 2\sin^2 x)$
$$= 2\sin^2 x + \sin x - 1$$
$$= (2\sin x - 1)(\sin x + 1)$$

**36.** $\cos 4x = 1 - 2\sin^2 2x$
$$= 1 - 2(2\sin x \cos x)^2$$
$$= 1 - 2(4\sin^2 x \cos^2 x)$$
$$= 1 - 8\sin^2 x \cos^2 x$$
$$= 1 - 8\sin^2 x(1 - \sin^2 x)$$
$$= 1 - 8\sin^2 x + 8\sin^4 x$$

**37.** $\tan 4x = \dfrac{2\tan 2x}{1 - \tan^2 2x}$
$$= \frac{2\left(\frac{2\tan x}{1 - \tan^2 x}\right)}{1 - \left(\frac{2\tan x}{1 - \tan^2 x}\right)^2}$$
$$= \frac{\frac{4\tan x}{1 - \tan^2 x}}{\frac{(1 - \tan^2 x)^2 - (2\tan x)^2}{(1 - \tan^2 x)^2}} \cdot \frac{(1 - \tan^2 x)^2}{(1 - \tan^2 x)^2}$$
$$= \frac{4\tan x(1 - \tan^2 x)}{(1 - \tan^2 x)^2 - 4\tan^2 x}$$
$$= \frac{4\tan x - 4\tan^3 x}{1 - 2\tan^2 x + \tan^4 x - 4\tan^2 x}$$
$$= \frac{4\tan x - 4\tan^3 x}{1 - 6\tan^2 x + \tan^4 x}$$

**38.** $\dfrac{\sin 2x - \sin x}{\cos 2x + \cos x} = \dfrac{2\cos\frac{2x + x}{2}\sin\frac{2x - x}{2}}{2\cos\frac{2x + x}{2}\cos\frac{2x - x}{2}}$
$$= \frac{\sin\frac{\pi}{2}}{\cos\frac{\pi}{2}}$$
$$= \tan\frac{\pi}{2}$$
$$= \frac{1 - \cos x}{\sin x}$$

**39.** $2\sin 3x\cos 3x - 2\sin x\cos x = \sin 6x - \sin 2x$
$$= 2\cos\frac{6x + 2x}{2}\sin\frac{6x - 2x}{2}$$
$$= 2\cos 4x\sin 2x$$

**40.** $2\sin x\sin 2x = 2\sin x(2\sin x\cos x)$
$$= 4\cos x\sin^2 x$$

**41.** $\cos(x + y)\cos(x - y) = \frac{1}{2}\left[\cos(x + y + x - y) + \cos(x + y - x + y)\right]$
$$= \frac{1}{2}(\cos 2x + \cos 2y)$$
$$= \frac{1}{2}\left[2\cos^2 x - 1 + 2\cos^2 y - 1\right]$$
$$= \cos^2 x + \cos^2 y - 1$$

**42.** $\cos(x + y)\sin(x - y) = \frac{1}{2}\left[\sin(x + y + x - y) - \sin(x + y - x + y)\right]$
$$= \frac{1}{2}(\sin 2x - \sin 2y)$$
$$= \frac{1}{2}\left[2\sin x\cos x - 2\sin y\cos y\right]$$
$$= \sin x\cos x - \sin y\cos y$$

**43.** $y = \sec\left(\sin^{-1}\frac{12}{13}\right),\ \alpha = \sin^{-1}\frac{12}{13},\ \sin\alpha = \frac{12}{13},\ \cos\alpha = \frac{5}{13},\ \sec\alpha = \frac{13}{5}$ [6.5]

$y = \sec\alpha = \frac{13}{5}$

**44.** $y=\cos\left(\sin^{-1}\frac{3}{5}\right)$ [6.5]

$\alpha=\sin^{-1}\frac{3}{5},\ \sin\alpha=\frac{3}{5},\ y=\cos\alpha=\frac{4}{5}$

**45.** $\alpha=\sin^{-1}\left(-\frac{3}{5}\right)$ $\beta=\cos^{-1}\frac{5}{13}$ [6.5]

$\sin\alpha=-\frac{3}{5}$ $\cos\beta=\frac{5}{13}$

$\cos\alpha=\frac{4}{5}$ $\sin\beta=\frac{12}{13}$

$$\begin{aligned}y&=\cos\left[\sin^{-1}\left(-\frac{3}{5}\right)+\cos^{-1}\frac{5}{13}\right]\\&=\cos(\alpha+\beta)\\&=\cos\alpha\cos\beta-\sin\alpha\sin\beta\\&=\frac{4}{5}\cdot\frac{5}{13}-\left(-\frac{3}{5}\right)\cdot\frac{12}{13}\\&=\frac{20}{65}+\frac{36}{65}\\&=\frac{56}{65}\end{aligned}$$

**46.** $y=\cos\left(2\sin^{-1}\frac{3}{5}\right)$ $\alpha=\sin^{-1}\frac{3}{5}$ [6.5]

$\sin\alpha=\frac{3}{5}$

$\cos\alpha=\frac{4}{5}$

$$\begin{aligned}y&=\cos2\alpha\\&=\cos^2\alpha-\sin^2\alpha\\&=\left(\frac{4}{5}\right)^2-\left(\frac{3}{5}\right)^2\\&=\frac{16}{25}-\frac{9}{25}\\&=\frac{7}{25}\end{aligned}$$

**47.** $2\sin^{-1}(x-1)=\frac{\pi}{3}$ [6.6]

$$\begin{aligned}\sin^{-1}(x-1)&=\frac{\pi}{6}\\x-1&=\sin\frac{\pi}{6}\\x-1&=\frac{1}{2}\\x&=\frac{3}{2}\end{aligned}$$

**48.** $\sin^{-1}x+\cos^{-1}\frac{4}{5}=\frac{\pi}{2}$ $\alpha=\cos^{-1}\frac{4}{5}$ [6.6]

$\sin^{-1}x+\alpha=\frac{\pi}{2}$ $\cos\alpha=\frac{4}{5}$

$\sin^{-1}x=\frac{\pi}{2}-\alpha$ $\sin\alpha=\frac{3}{5}$

$$\begin{aligned}x&=\sin\left(\frac{\pi}{2}-\alpha\right)\\&=\sin\frac{\pi}{2}\cos\alpha-\cos\frac{\pi}{2}\sin\alpha\\&=1\cdot\frac{4}{5}-0\cdot\frac{3}{5}\\&=\frac{4}{5}\end{aligned}$$

**49.** $4\sin^2x+2\sqrt{3}\sin x-2\sin x-\sqrt{3}=0$ [6.6]

$2\sin x\left(2\sin x+\sqrt{3}\right)-\left(2\sin x+\sqrt{3}\right)=0$

$\left(2\sin x+\sqrt{3}\right)\left(2\sin x-1\right)=0$

$2\sin x+\sqrt{3}=0$ $2\sin x-1=0$

$\sin x=-\frac{\sqrt{3}}{2}$ $\sin x=\frac{1}{2}$

$x=240°,300°$ $x=30°,150°$

The solutions are 30°, 150°, 240°, 300°.

**50.** $2\sin x\cos x-\sqrt{2}\cos x-2\sin x+\sqrt{2}=0$ [6.6]

$\cos x(2\sin x-\sqrt{2})-(2\sin x-\sqrt{2})=0$

$(2\sin x-\sqrt{2})(\cos x-1)=0$

$2\sin x-\sqrt{2}=0$ $\cos x-1=0$

$\sin x=\frac{\sqrt{2}}{2}$ $\cos x=1$

$x=45°,135°$ $x=0°$

The solutions are 0°, 45°, 135°.

**51.** $3\cos^2x+\sin x=1$ [6.6]

$3(1-\sin^2x)+\sin x=1$

$0=3\sin^2x-\sin x-2$

$0=(3\sin x+2)(\sin x-1)$

$3\sin x+2=0$ $\sin x-1=0$

$\sin x=-\frac{2}{3}$ $\sin x=1$

$x=3.8713\text{ or }5.553$ $x=\frac{\pi}{2}$

The solutions are $\frac{\pi}{2}+2k\pi,\ 3.8713+2k\pi,\ 5.553+2k\pi$ where $k$ is an integer.

**52.** $\tan^2 x - 2\tan x - 3 = 0$ [6.6]

$(\tan x + 1)(\tan x - 3) = 0$

$\tan x = -1$ $\qquad$ $\tan x = 3$

$x = -\frac{\pi}{4} + k\pi$ $\qquad$ $x = 1.2490 + k\pi$

The solutions are $-\frac{\pi}{4} + k\pi,\ 1.2490 + k\pi$ where $k$ is an integer.

**53.** $\sin 3x \cos x - cox3x \sin x = \frac{1}{2}$ [6.6]

$\sin(3x - x) = \frac{1}{2}$

$\sin 2x = \frac{1}{2}$

$2x = \frac{\pi}{6} + 2k\pi$ $\qquad$ $2x = \frac{5\pi}{6} + 2k\pi$

$x = \frac{\pi}{12} + k\pi$ $\qquad$ $x = \frac{5\pi}{12} + k\pi$

The solutions are $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$.

**54.** $\cos\left(2x - \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ [6.6]

$2x - \frac{\pi}{3} = \frac{5\pi}{6} + 2k\pi$ $\qquad$ $2x - \frac{\pi}{3} = \frac{7\pi}{6} + 2k\pi$

$2x = \frac{7\pi}{6} + 2k\pi$ $\qquad$ $2x = \frac{3\pi}{2} + 2k\pi$

$x = \frac{7\pi}{12} + k\pi$ $\qquad$ $x = \frac{3\pi}{4} + k\pi$

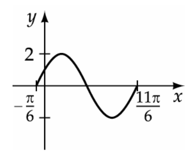
The solutions are $\frac{7\pi}{12}, \frac{19\pi}{12}, \frac{3\pi}{4}, \frac{7\pi}{4}$.

**55.** $f(x) = \sqrt{3}\sin x + \cos x$ [6.4]

$f(x) = 2\sin\left(x + \frac{\pi}{6}\right)$

amplitude = 2

phase shift = $-\frac{\pi}{6}$

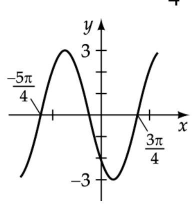



**56.** $f(x) = -2\sin x - 2\cos x$ [6.4]

$f(x) = 2\sqrt{2}\sin\left(x + \frac{5\pi}{4}\right)$

amplitude = $2\sqrt{2}$

phase shift = $-\frac{5\pi}{4}$

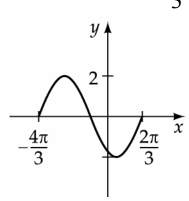



**57.** $f(x) = -\sin x - \sqrt{3}\cos x$ [6.4]

$f(x) = 2\sin\left(x + \frac{4\pi}{3}\right)$

amplitude = 2

phase shift = $-\frac{4\pi}{3}$

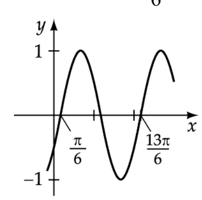



**58.** $f(x) = \frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x$ [6.4]

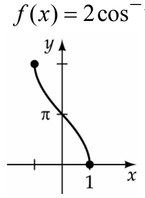
$f(x) = \sin\left(x + \frac{11\pi}{6}\right)$ or $\sin\left(x - \frac{\pi}{6}\right)$

amplitude = 1

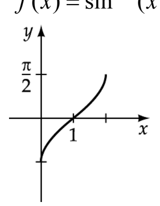
phase shift = $\frac{\pi}{6}$



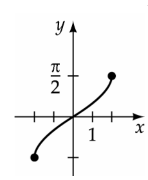

**59.** $f(x) = 2\cos^{-1} x$



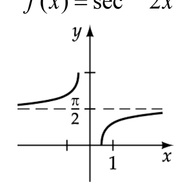

**60.** $f(x) = \sin^{-1}(x - 1)$



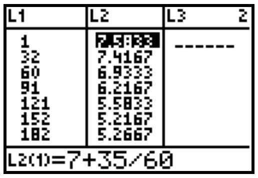

**61.** $f(x) = \sin^{-1}\frac{x}{2}$

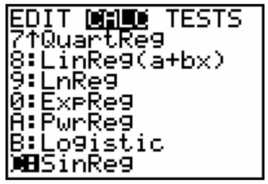


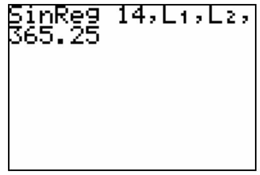

**62.** $f(x) = \sec^{-1} 2x$

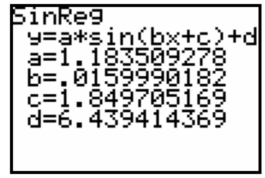


**63.** **a.**




$y \approx 1.1835\sin(0.01600x + 1.8497) + 6.4394$

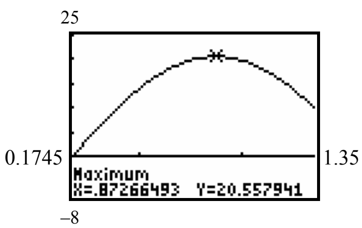
**b.** $y \approx 1.1835\sin(0.01600(104) + 1.8497) + 6.4394$ [6.6]

$\approx 6.009$

$= 6:01$

## Quantitative Reasoning

**QR1.** **a.** $s = \dfrac{(28)^2 \cos\alpha \sec\left(\frac{\pi}{18}\right)\sin\left(\alpha - \frac{\pi}{18}\right)}{16}$



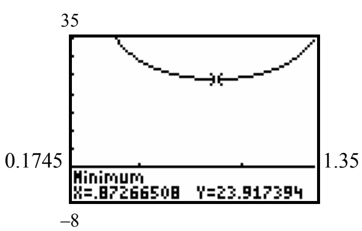

$\alpha \approx 0.87266$

**b.** $\dfrac{\pi}{4} + \dfrac{\frac{\pi}{18}}{2} = \dfrac{\pi}{4} + \dfrac{\pi}{36} = \dfrac{9\pi + \pi}{36} = \dfrac{5\pi}{18} \approx 0.8727$

**QR2.** 
$$\cos\alpha\sec\beta\sin(\alpha-\beta) = \left(\cos\alpha \cdot \frac{1}{\cos\beta}\right)(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$$
$$= \cos\alpha\sin\alpha - \cos^2\alpha \cdot \frac{\sin\beta}{\cos\beta}$$
$$= \frac{\cos\alpha}{\cos\alpha} \cdot \cos\alpha\sin\alpha - \cos^2\alpha \cdot \tan\beta$$
$$= \cos^2\alpha \cdot \frac{\sin\alpha}{\cos\alpha} - \cos^2\alpha\tan\beta$$
$$= \cos^2\alpha(\tan\alpha - \tan\beta)$$

$$s = \frac{v^2\cos\alpha\sec\beta\sin(\alpha-\beta)}{16}$$
$$s = \frac{v^2\cos^2\alpha(\tan\alpha - \tan\beta)}{16}$$
$$16s = v^2\cos^2\alpha(\tan\alpha - \tan\beta)$$
$$v^2 = \frac{16s}{\cos^2\alpha(\tan\alpha - \tan\beta)}$$
$$v = \sqrt{\frac{16s}{\cos^2\alpha(\tan\alpha - \tan\beta)}}$$
$$v = 4\sec\alpha\sqrt{\frac{s}{\tan\alpha - \tan\beta}}$$

**a.** $v = 4\sec\alpha\sqrt{\dfrac{15}{\tan\alpha - \tan\frac{\pi}{18}}}$




$\alpha \approx 0.87266$

**b.** $\dfrac{\pi}{4} + \dfrac{\frac{\pi}{18}}{2} = \dfrac{\pi}{4} + \dfrac{\pi}{36} = \dfrac{9\pi + \pi}{36} = \dfrac{5\pi}{18} \approx 0.8727$

## Chapter Test

**1.** $1+\sin^2 x\sec^2 x = 1+\sin^2 x\dfrac{1}{\cos^2 x}$
$= 1+\tan^2 x$
$= \sec^2 x$

**2.** $\dfrac{1}{\sec x-\tan x}-\dfrac{1}{\sec x+\tan x}=\dfrac{\sec x+\tan x-\sec x+\tan x}{\sec^2 x-\tan^2 x}$
$=\dfrac{2\tan x}{1}$
$=2\tan x$

**3.** $\cos^3 x+\cos x\sin^2 x=\cos x(\cos^2 x+\sin^2 x)$
$=\cos x$

**4.** $\csc x-\cot x=\dfrac{1}{\sin x}-\dfrac{\cos x}{\sin x}$
$=\dfrac{1-\cos x}{\sin x}$

**5.** $\sin 195° = \sin(150°+45°)$ [6.2]
$= \sin 150°\cos 45° + \cos 150°\sin 45°$
$=\dfrac{1}{2}\left(\dfrac{\sqrt{2}}{2}\right)+\left(-\dfrac{\sqrt{3}}{2}\right)\left(\dfrac{\sqrt{2}}{2}\right)$
$=\dfrac{\sqrt{2}-\sqrt{6}}{4}$

**6.** $\sin\alpha=-\dfrac{3}{5}, \cos\alpha=-\dfrac{4}{5}, \cos\beta=-\dfrac{\sqrt{2}}{2}, \sin\beta\dfrac{\sqrt{2}}{2}$ [6.2]
$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$
$=\left(-\dfrac{3}{5}\right)\left(-\dfrac{\sqrt{2}}{2}\right)+\left(-\dfrac{4}{5}\right)\left(\dfrac{\sqrt{2}}{2}\right)$
$=\dfrac{3\sqrt{2}-4\sqrt{2}}{10}$
$=-\dfrac{\sqrt{2}}{10}$

**7.** $\sin\left(\theta-\dfrac{3\pi}{2}\right)=\sin\theta\cos\dfrac{3\pi}{2}-\cos\theta\sin\dfrac{3\pi}{2}$
$=\sin\theta(0)-\cos\theta(-1)$
$=\cos\theta$

**8.** $\cos 6x\sin 3x+\sin 6x\cos 3x=\sin(6x+3x)$ [6.2]
$=\sin 9x$

**9.** $\sin\theta=\dfrac{4}{5}, \cos\theta=-\dfrac{3}{5}$ [6.3]
$\cos 2\theta = 2\cos^2\theta-1$
$=2\left(-\dfrac{3}{5}\right)^2-1$
$=\dfrac{18}{25}-1=-\dfrac{7}{25}$

**10.** $\tan\dfrac{\theta}{2}+\dfrac{\cos\theta}{\sin\theta}=\dfrac{1-\cos\theta}{\sin\theta}+\dfrac{\cos\theta}{\sin\theta}$
$=\dfrac{1-\cos\theta+\cos\theta}{\sin\theta}$
$=\csc\theta$

**11.** $\sin^2 2x+4\cos^4 x=(2\sin x\cos x)^2+4\cos^4 x$
$=4\sin^2 x\cos^2 x+4\cos^4 x$
$=4\cos^2 x(\sin^2 x+\cos^2 x)$
$=4\cos^2 x$

**12.** $\sin 15°\cos 75° = \dfrac{1}{2}[\sin(15°+75°)+\sin(15°-75°)]$ [6.4]
$=\dfrac{1}{2}(\sin 90°-\sin 60°)$
$=\dfrac{1}{2}\left(1-\dfrac{\sqrt{3}}{2}\right)$
$=\dfrac{1}{2}-\dfrac{\sqrt{3}}{4}$
$=\dfrac{2-\sqrt{3}}{4}$

**13.** $y = -\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x$ [6.4]

$a = -\frac{\sqrt{3}}{2},\ b = \frac{1}{2}$

$k = \sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 1$

$\sin\beta = \left|\frac{\frac{1}{2}}{1}\right| = \frac{1}{2}$

$\beta = \frac{\pi}{6}$

$\alpha = \pi - \frac{\pi}{6}$

$= \frac{5\pi}{6}$

$y = \sin\left(x + \frac{5\pi}{6}\right)$

**14.** $\theta = \cos^{-1}(0.7644)$ [6.5]

$\theta = 0.701$

**15.** $\sin\left(\cos^{-1}\frac{12}{13}\right)$ [6.5]

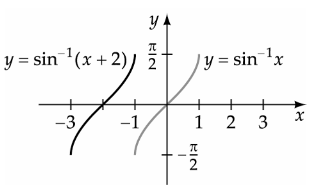
Let $\theta = \cos^{-1}\frac{12}{13}$ and find $\sin\theta$.

Then $\cos\theta = \frac{12}{13}$ and $0 \le \theta \le \pi$.

$\sin\theta = \frac{5}{13}$

$\sin\left(\cos^{-1}\frac{12}{13}\right) = \frac{5}{13}$

**16.** The graph of $y = \sin^{-1}(x+2)$ is the graph of $y = \sin^{-1}x$ moved two units to the left.




**17.** $3\sin x - 2 = 0$ [6.6]

$\sin x = \frac{2}{3}$

$x = 41.8°,\ 138.2°$

**18.** $\sin x\cos x - \frac{\sqrt{3}}{2}\sin x = 0$ [6.6]

$\sin x\left(\cos x - \frac{\sqrt{3}}{2}\right) = 0$

$\sin x = 0$

$x = 0,\ \pi$

$\cos x - \frac{\sqrt{3}}{2} = 0$

$\cos x = \frac{\sqrt{3}}{2}$

$x = \frac{\pi}{6},\ \frac{11\pi}{6}$

The solutions are $0,\ \frac{\pi}{6},\ \pi,\ \frac{11\pi}{6}$.

**19.** $\sin 2x + \sin x - 2\cos x - 1 = 0$

$2\sin x\cos x + \sin x - 2\cos x - 1 = 0$

$\sin x(2\cos x + 1) - (2\cos x + 1) = 0$

$(2\cos x + 1)(\sin x - 1) = 0$

$\cos x = -\frac{1}{2}$

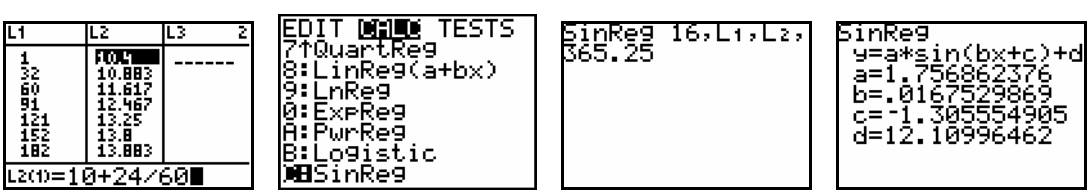
$x = \frac{2\pi}{3},\ \frac{4\pi}{3}$

$\sin x = 1$

$x = \frac{\pi}{2}$

The solutions are $\frac{\pi}{2},\ \frac{2\pi}{3},\ \frac{4\pi}{3}$.

**20.** **a.**




$f(x) = y \approx 1.7569\sin(0.0168x - 1.3056) + 12.1100$

**b.** $f(75) \approx 1.7569\sin(0.01675(75) - 1.3056) + 12.1100$ [6.6]

$\approx 12.0233$

$\approx 12$ hr 1 min

## Cumulative Review

**1.** $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$ [P.4]

**2.** $|x-5| = 3$ [1.1]

$x - 5 = -3 \quad x - 5 = 3$

$x = 2 \quad x = 8$

**3.** Shift the graph of $y = f(x)$ horizontally 1 unit to the left and up 2 units. [2.5]

**4.** Reflect the graph of $y = f(x)$ across the $x$-axis. [2.5]

**5.** $x - 2 = 0$ [3.5]

$x = 2$

**6.** $f(-x) = -x - \sin(-x)$ [2.5/5.5]

$= -x + \sin x$

$= -(x - \sin x)$

$= -f(x)$

odd function

**7.**

$$f(x) = \frac{5x}{x-1}$$
$$y = \frac{5x}{x-1}$$
$$x = \frac{5y}{y-1}$$
$$x(y-1) = 5y$$
$$xy - 5y = x$$
$$y(x-5) = x$$
$$y = \frac{x}{x-5}$$
$$f^{-1}(x) = \frac{x}{x-5} \quad [4.1]$$

**8.** $x = 2^5$ [4.3]

$\log_2 x = 5$

**9.** $\log_{10} 1000 = \log_{10} 10^3$ [4.3]

$= 3\log_{10} 10 = 3$

**10.** $240^\circ = 240^\circ\left(\dfrac{\pi}{180^\circ}\right) = \dfrac{4\pi}{3}$ [5.1]

**11.** $\dfrac{5\pi}{3} = \dfrac{5\pi}{3}\left(\dfrac{180^\circ}{\pi}\right) = 300^\circ$ [5.1]

**12.** $\sin\theta = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{2}{3}$ [5.2]

$\text{adjacent side} = \sqrt{3^2 - 2^2}$

$= \sqrt{9-4}$

$= \sqrt{5}$

$\tan\theta = \dfrac{\text{opp}}{\text{adj}} = \dfrac{2}{\sqrt{5}} = \dfrac{2\sqrt{5}}{5}$

**13.** $\cot\theta > 0$ in quadrant III

Positive [5.3]

**14.** $\theta = 310^\circ$ [5.3]

Since $270^\circ < \theta < 360^\circ$,

$\theta = \theta' = 360^\circ$

$\theta' = 50^\circ$

**15.** $\theta = \dfrac{5\pi}{3}$ [5.3]

Since $\dfrac{3\pi}{2} < \theta < 2\pi$,

$\theta = \theta' = 2\pi$

$\theta' = \dfrac{\pi}{3}$

**16.** $t = \dfrac{\pi}{3}$ [5.4]

$y = \sin t = \sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$

$x = \cos t = \cos\dfrac{\pi}{3} = \dfrac{1}{2}$

The point on the unit circle corresponding to $t = \dfrac{\pi}{3}$ is $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$.

**17.** $y = 0.43\cos\left(2x - \frac{\pi}{6}\right)$ [5.7]

amplitude: 0.43

$0 \le 2x - \frac{\pi}{6} \le 2\pi$

$\frac{\pi}{6} \le 2x \le \frac{13\pi}{6}$

$\frac{\pi}{12} \le x \le \frac{13\pi}{12}$

period $= \pi$, phase shift $= \frac{\pi}{12}$

**18.** $y = \sin^{-1}\frac{1}{2}$ [6.5]

$\sin y = \frac{1}{2} \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$y = \frac{\pi}{6}$

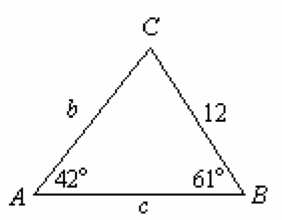
**19.** Domain: [–1, 1]. [6.5]

**20.** Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ [6.5]

# Chapter 7
# Applications of Trigonometry

## Section 7.1

**1.**



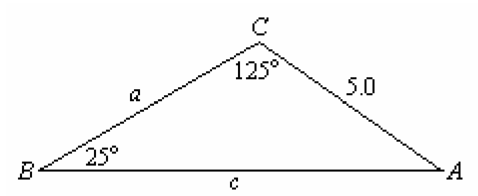

$C = 180° - 42° - 61°$

$C = 77°$

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$
$$\frac{b}{\sin 61°} = \frac{12}{\sin 42°}$$
$$b = \frac{12\sin 61°}{\sin 42°}$$
$$b \approx 16$$

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$
$$\frac{c}{\sin 77°} = \frac{12}{\sin 42°}$$
$$c = \frac{12\sin 77°}{\sin 42°}$$
$$c \approx 17$$

**2.**



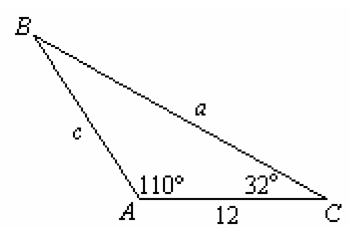

$A = 180° - 125° - 25°$

$A = 30°$

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$
$$\frac{c}{\sin 125°} = \frac{5.0}{\sin 25°}$$
$$c = \frac{5.0\sin 125°}{\sin 25°}$$
$$c \approx 9.7$$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{a}{\sin 30°} = \frac{5.0}{\sin 25°}$$
$$a = \frac{5.0\sin 30°}{\sin 25°}$$
$$a \approx 5.9$$

**3.**



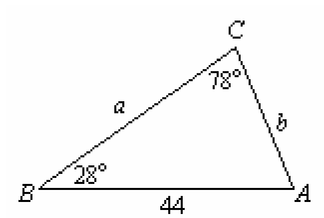

$B = 180° - 110° - 32°$

$B = 38°$

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$
$$\frac{12}{\sin 38°} = \frac{a}{\sin 110°}$$
$$a = \frac{12\sin 110°}{\sin 38°}$$
$$a \approx 18$$

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$
$$\frac{c}{\sin 32°} = \frac{12}{\sin 38°}$$
$$c = \frac{12\sin 32°}{\sin 38°}$$
$$c \approx 10$$

**4.**



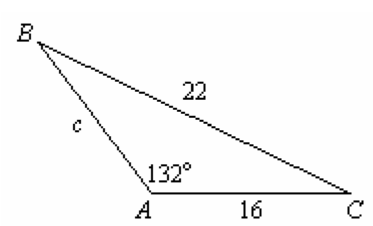

$A = 180° - 78° - 28°$

$A = 74°$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$
$$\frac{b}{\sin 28°} = \frac{44}{\sin 78°}$$
$$b = \frac{44\sin 28°}{\sin 78°}$$
$$b \approx 21$$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$
$$\frac{a}{\sin 74°} = \frac{44}{\sin 78°}$$
$$a = \frac{44\sin 74°}{\sin 78°}$$
$$a \approx 43$$

**5.**



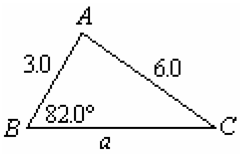

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{22}{\sin 132°} = \frac{16}{\sin B}$$
$$\sin B = \frac{16\sin 132°}{22}$$
$$\sin B \approx 0.5405$$
$$B \approx 33°$$

$C \approx 180° - 33° - 132°$

$C \approx 15°$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$
$$\frac{22}{\sin 132°} \approx \frac{c}{\sin 15°}$$
$$c \approx \frac{22\sin 15°}{\sin 132°}$$
$$c \approx 7.7$$

**6.**

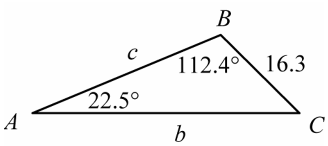



$$\frac{b}{\sin B} = \frac{c}{\sin C}$$
$$\frac{6.0}{\sin 82^\circ} = \frac{3.0}{\sin C}$$
$$\sin C = \frac{3 \sin 82.0^\circ}{6.0}$$
$$\sin C \approx 0.4951$$
$$C \approx 29.7^\circ$$

$$A \approx 180.0^\circ - 82.0^\circ - 29.7^\circ$$
$$A \approx 68.3^\circ$$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{a}{\sin 68.3^\circ} \approx \frac{6.0}{\sin 82.0^\circ}$$
$$a = \frac{6.0 \sin 68.3^\circ}{\sin 82.0^\circ}$$
$$a \approx 5.6$$

**7.**

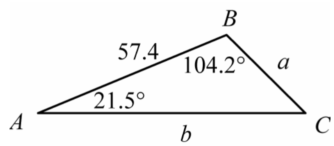



$$C = 180^\circ - 22.5^\circ - 112.4^\circ$$
$$C = 45.1^\circ$$

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$
$$\frac{b}{\sin 112.4^\circ} = \frac{16.3}{\sin 22.5^\circ}$$
$$b = \frac{16.3 \sin 112.4^\circ}{\sin 22.5^\circ}$$
$$b \approx 39.4$$

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$
$$\frac{c}{\sin 45.1^\circ} = \frac{16.3}{\sin 22.5^\circ}$$
$$c = \frac{16.3 \sin 45.1^\circ}{\sin 22.5^\circ}$$
$$c \approx 30.2$$

**8.**

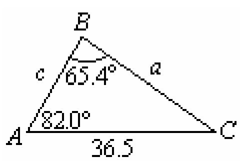



$$C = 180^\circ - 21.5^\circ - 104.2^\circ$$
$$C = 54.3^\circ$$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$
$$\frac{a}{\sin 21.5^\circ} = \frac{57.4}{\sin 54.3^\circ}$$
$$a = \frac{57.4 \sin 21.5^\circ}{\sin 54.3^\circ}$$
$$a \approx 25.9$$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$
$$\frac{b}{\sin 104.2^\circ} = \frac{57.4}{\sin 54.3^\circ}$$
$$b = \frac{57.4 \sin 104.2^\circ}{\sin 54.3^\circ}$$
$$b \approx 68.5$$

**9.**

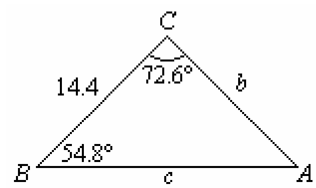



$$C = 180^\circ - 65.4^\circ - 82.0^\circ$$
$$C = 32.6^\circ$$

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$
$$\frac{c}{\sin 32.6^\circ} = \frac{36.5}{\sin 65.4^\circ}$$
$$c = \frac{36.5 \sin 32.6^\circ}{\sin 65.4^\circ}$$
$$c \approx 21.6$$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{a}{\sin 82.0^\circ} = \frac{36.5}{\sin 65.4^\circ}$$
$$a = \frac{36.5 \sin 82.0^\circ}{\sin 65.4^\circ}$$
$$a \approx 39.8$$

**10.**

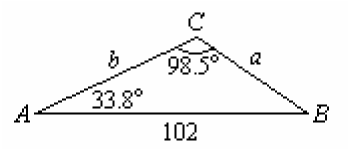



$$A = 180^\circ - 72.6^\circ - 54.8^\circ$$
$$A = 52.6^\circ$$

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$
$$\frac{b}{54.8^\circ} = \frac{14.4}{\sin 52.6^\circ}$$
$$b = \frac{14.4 \sin 54.8^\circ}{\sin 52.6^\circ}$$
$$b \approx 14.8$$

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$
$$\frac{c}{\sin 72.6^\circ} = \frac{14.4}{\sin 52.6^\circ}$$
$$a = \frac{14.4 \sin 72.6^\circ}{\sin 52.6^\circ}$$
$$a \approx 17.3$$

**11.**



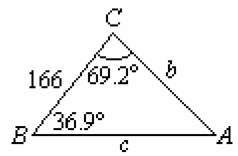

$B = 180^\circ - 98.5^\circ - 33.8^\circ$

$B = 47.7^\circ$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$
$$\frac{a}{\sin 33.8^\circ} = \frac{102}{\sin 98.5^\circ}$$
$$a = \frac{102\sin 33.8^\circ}{\sin 98.5^\circ}$$
$$a \approx 57.4$$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$
$$\frac{b}{\sin 47.7^\circ} = \frac{102}{\sin 98.5^\circ}$$
$$b = \frac{102\sin 47.7^\circ}{\sin 98.5^\circ}$$
$$b \approx 76.3$$

**12.**



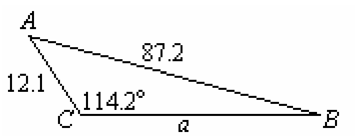

$A = 180^\circ - 69.2^\circ - 36.9^\circ$

$A = 73.9^\circ$

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$
$$\frac{b}{\sin 36.9^\circ} = \frac{166}{\sin 73.9^\circ}$$
$$b = \frac{166\sin 36.9^\circ}{\sin 73.9^\circ}$$
$$b \approx 104$$

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$
$$\frac{c}{\sin 69.2^\circ} = \frac{166}{\sin 73.9^\circ}$$
$$c = \frac{166\sin 69.2^\circ}{\sin 73.9^\circ}$$
$$c \approx 162$$

**13.**



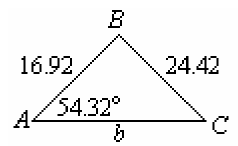

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$
$$\frac{87.2}{\sin 114.2^\circ} = \frac{12.1}{\sin B}$$
$$\sin B = \frac{12.1\sin 114.2^\circ}{87.2} \approx 0.1266$$
$$B \approx 7.3^\circ$$

$A \approx 180^\circ - 114.2^\circ - 7.3^\circ$

$A \approx 58.5^\circ$

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$
$$\frac{87.2}{\sin 114.2^\circ} \approx \frac{a}{\sin 58.5^\circ}$$
$$a = \frac{87.2\sin 58.5^\circ}{\sin 114.2^\circ} \approx 81.5$$

**14.**



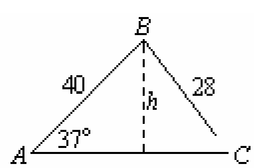

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$
$$\frac{24.42}{\sin 54.32^\circ} = \frac{16.92}{\sin C}$$
$$\sin C = \frac{16.92\sin 54.32^\circ}{24.42} \approx 0.5628$$
$$C \approx 34.25^\circ$$

$B = 180^\circ - 54.32^\circ - 34.25^\circ$

$B = 91.43^\circ$

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$
$$\frac{b}{\sin 91.43^\circ} \approx \frac{24.42}{\sin 54.32^\circ}$$
$$b = \frac{24.42\sin 91.43^\circ}{\sin 54.32^\circ} \approx 30.05$$

**15.**




$$\frac{c}{\sin C} = \frac{a}{\sin A}$$
$$\frac{40}{\sin C} = \frac{28}{\sin 37^\circ}$$
$$\sin C = \frac{40\sin 37^\circ}{28} = 0.8597$$
$$C \approx 59^\circ \text{ or } 121^\circ$$

$$\sin 37^\circ = \frac{h}{40}$$
$$h = 40\sin 37^\circ$$
$$h \approx 24$$

Since $h < 28$, two triangles exist.

$C = 59^\circ$

$B = 180^\circ - 37^\circ - 59^\circ = 84^\circ$

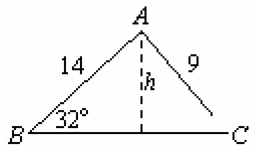
$$\frac{b}{\sin 84^\circ} = \frac{28}{\sin 37^\circ}$$
$$b = \frac{28\sin 84^\circ}{\sin 37^\circ} = 46$$

$C = 121^\circ$

$B = 180^\circ - 121^\circ - 37^\circ = 22^\circ$

$$\frac{b}{\sin 22^\circ} = \frac{28}{\sin 37^\circ}$$
$$b = \frac{28\sin 22^\circ}{\sin 37^\circ} \approx 17$$

**16.**




$$\sin 32^\circ = \frac{h}{14}$$
$$h = 14\sin 32^\circ$$
$$h \approx 7.4$$

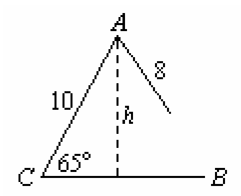
Since $h < 9.0$, two triangles exist.

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$
$$\frac{9.0}{\sin 32^\circ} = \frac{14}{\sin C}$$
$$\sin C = \frac{14\sin 32^\circ}{9} = 0.8243$$
$$C = 56^\circ \text{ or } 124^\circ$$

$$C = 56^\circ$$
$$A = 180^\circ - 32^\circ - 56^\circ = 92^\circ$$
$$\frac{a}{\sin 92^\circ} = \frac{9.0}{\sin 32^\circ}$$
$$a = \frac{9\sin 92^\circ}{\sin 32^\circ} \approx 17$$

$$C = 124^\circ$$
$$A = 180^\circ - 124^\circ - 32^\circ = 24^\circ$$
$$\frac{a}{\sin 24^\circ} = \frac{9.0}{\sin 32^\circ}$$
$$a = \frac{9\sin 24^\circ}{\sin 32^\circ} \approx 6.9$$

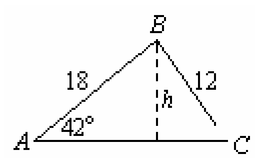
**17.**




$$\sin 65^\circ = \frac{h}{10}$$
$$h = 10\sin 65^\circ$$
$$h \approx 9.06$$

Since $h > 8$, no triangle is formed.

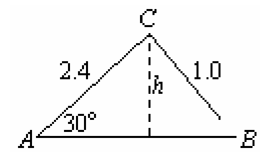
**18.**




$$\sin 42^\circ = \frac{h}{18}$$
$$h = 18\sin 42^\circ$$
$$h \approx 12.04$$

Since $h > 12$, no triangle is formed.

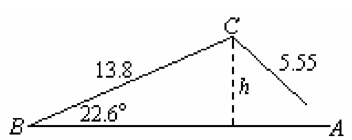
**19.**




$$\sin 30^\circ = \frac{h}{2.4}$$
$$h = 2.4\sin 30^\circ$$
$$h \approx 1.2$$

Since $h > 1$, no triangle is formed.

**20.**




$$\sin 22.6^\circ = \frac{h}{13.8}$$
$$h = 13.8\sin 22.6^\circ$$
$$h \approx 5.30$$

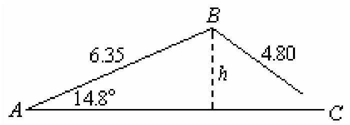
Since $h < 5.55$, two solutions exist.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{13.8}{\sin A} = \frac{5.55}{\sin 22.6^\circ}$$
$$\sin A = \frac{13.8\sin 22.6^\circ}{5.55}$$
$$\sin A = 0.9555$$
$$A \approx 72.9^\circ \text{ or } 107.1^\circ$$

$$A = 72.9^\circ$$
$$C = 180^\circ - 72.9^\circ - 22.6^\circ$$
$$C = 84.5^\circ$$
$$\frac{c}{\sin 84.5^\circ} = \frac{5.55}{\sin 22.6^\circ}$$
$$c = \frac{5.55\sin 84.5^\circ}{\sin 22.6^\circ}$$
$$c \approx 14.4$$

$$A = 107.1^\circ$$
$$C = 180^\circ - 107.1^\circ - 22.6^\circ$$
$$C = 50.3^\circ$$
$$\frac{c}{\sin 50.3^\circ} = \frac{5.55}{\sin 22.6^\circ}$$
$$c = \frac{5.55\sin 50.3^\circ}{\sin 22.6^\circ}$$
$$c \approx 11.1$$

**21.**




$$\sin 14.8^\circ = \frac{h}{6.35}$$
$$h = 6.35\sin 14.8^\circ$$
$$h \approx 1.62$$

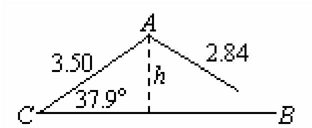
Since $h < 4.80$, two solutions exist.

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$
$$\frac{6.35}{\sin C} = \frac{4.80}{\sin 14.8^\circ}$$
$$\sin C = \frac{6.35\sin 14.8^\circ}{4.80}$$
$$\sin C = 0.3379$$
$$C \approx 19.8^\circ \text{ or } 160.2^\circ$$

$$C = 19.8^\circ$$
$$B = 180^\circ - 19.8^\circ - 14.8^\circ$$
$$= 145.4^\circ$$
$$\frac{b}{\sin 145.4^\circ} = \frac{4.80}{\sin 14.8^\circ}$$
$$b = \frac{4.80\sin 145.4^\circ}{\sin 14.8^\circ}$$
$$b \approx 10.7$$

$$C = 160.2^\circ$$
$$B = 180^\circ - 160.2^\circ - 14.8^\circ$$
$$= 5.0^\circ$$
$$\frac{b}{\sin 5.0^\circ} = \frac{4.80}{\sin 14.8^\circ}$$
$$b = \frac{4.80\sin 5.0^\circ}{\sin 14.8^\circ}$$
$$b \approx 1.64$$

**22.**




$$\sin 37.9^\circ = \frac{h}{3.50}$$
$$h = 3.50\sin 37.9^\circ$$
$$h \approx 2.15$$

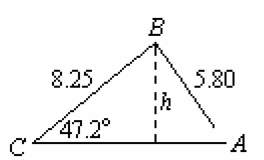
Since $h < 2.84$, two solutions exist.

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$
$$\frac{3.50}{\sin B} = \frac{2.84}{\sin\ 37.9^\circ}$$
$$\sin B = \frac{3.50\sin 37.9^\circ}{2.84}$$
$$\sin B = 0.7570$$
$$B \approx 49.2^\circ \quad \text{or} \quad 130.8^\circ$$

$$B = 49.2^\circ$$
$$A = 180^\circ - 49.2^\circ - 37.9^\circ$$
$$A = 92.9^\circ$$
$$\frac{a}{\sin 92.9^\circ} = \frac{2.84}{\sin 37.9^\circ}$$
$$a = \frac{2.84\sin 92.9^\circ}{\sin 37.9^\circ}$$
$$a \approx 4.62$$

$$B = 130.8^\circ$$
$$A = 180^\circ - 130.8^\circ - 37.9^\circ$$
$$A = 11.3^\circ$$
$$\frac{a}{\sin 11.3^\circ} = \frac{2.84}{\sin 37.9^\circ}$$
$$a = \frac{2.84\sin 11.3^\circ}{\sin 37.9^\circ}$$
$$a \approx 0.906$$

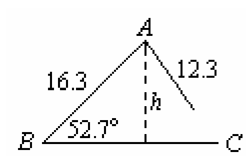
**23.**




$$\sin 47.2^\circ = \frac{h}{8.25}$$
$$h = 8.25\sin 47.2^\circ$$
$$h \approx 6.05$$

Since $h > 5.80$, no triangle is formed.

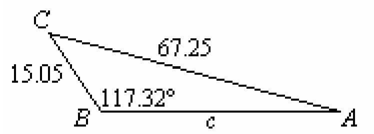
**24.**




$$\sin 52.7^\circ = \frac{h}{16.3}$$
$$h = 16.3\sin 57.2^\circ$$
$$h \approx 13.0$$

Since $h > 12.3$, no triangle is formed.

**25.**




Since b > a, one triangle exists.

$$\frac{a}{\sin A}=\frac{b}{\sin B}$$

$$\frac{15.05}{\sin A}=\frac{67.25}{\sin 117.32°}$$

$$\sin A=\frac{15.04\sin 117.32°}{67.25}\approx 0.1988$$

$$A=11.47°$$

$$C=180°-11.47°-117.32°$$

$$C=51.21°$$

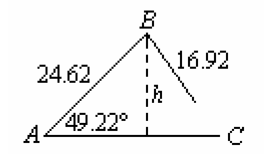
$$\frac{c}{\sin C}=\frac{b}{\sin B}$$

$$\frac{c}{\sin 51.21°}=\frac{67.25}{\sin 117.32°}$$

$$c=\frac{67.25\sin 51.21°}{\sin 117.32°}$$

$$c\approx 59.00$$

**26.**



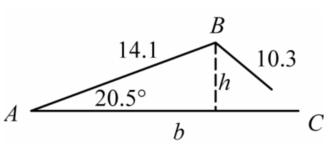

$$\sin 49.22°=\frac{h}{24.62}$$

$$h=24.62\sin 49.22°$$

$$h\approx 18.64$$

Since $h > 16.92$, no triangle is formed.

**27.**




$$\sin 20.5^\circ=\frac{h}{14.1}$$

$$h=14.1\sin 20.5^\circ$$

$$h\approx 4.9$$

Since $h<10.3$, two solutions exist.

$$\frac{a}{\sin A}=\frac{c}{\sin C}$$

$$\frac{10.3}{\sin 20.5^\circ}=\frac{14.1}{\sin C}$$

$$\sin C=\frac{14.1\sin 20.5^\circ}{10.3}$$

$$\sin C=0.4794$$

$$C\approx 28.6^\circ \text{ or } 151.4^\circ$$

$$A=20.5^\circ$$

$$B=180^\circ-28.6^\circ-20.5^\circ$$

$$B=130.9^\circ$$

$$\frac{b}{\sin 130.9^\circ}=\frac{10.3}{\sin 20.5^\circ}$$

$$b=\frac{10.3\sin 130.9^\circ}{\sin 20.5^\circ}$$

$$b\approx 22.2$$

$$A=20.5^\circ$$

$$B=180^\circ-151.4^\circ-20.5^\circ$$

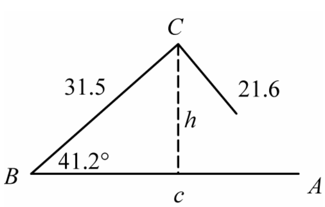
$$B=8.1^\circ$$

$$\frac{b}{\sin 8.1^\circ}=\frac{10.3}{\sin 20.5^\circ}$$

$$b=\frac{10.3\sin 8.1^\circ}{\sin 20.5^\circ}$$

$$b\approx 4.1$$

**28.**




$$\sin 41.2^\circ=\frac{h}{31.5}$$

$$h=31.5\sin 41.2^\circ$$

$$h\approx 20.7$$

Since $h<21.6$, two solutions exist.

$$\frac{a}{\sin A}=\frac{b}{\sin B}$$

$$\frac{31.5}{\sin A}=\frac{21.6}{\sin 41.2^\circ}$$

$$\sin A=\frac{31.5\sin 41.2^\circ}{21.6}$$

$$\sin A=0.9606$$

$$A\approx 73.9^\circ \text{ or } 106.1^\circ$$

$$A=73.9^\circ$$

$$C=180^\circ-73.9^\circ-41.2^\circ$$

$$C=64.9^\circ$$

$$\frac{c}{\sin 64.9^\circ}=\frac{21.6}{\sin 41.2^\circ}$$

$$c=\frac{21.6\sin 64.9^\circ}{\sin 41.2^\circ}$$

$$c\approx 29.7$$

$$A=106.1^\circ$$

$$C=180^\circ-106.1^\circ-41.2^\circ$$

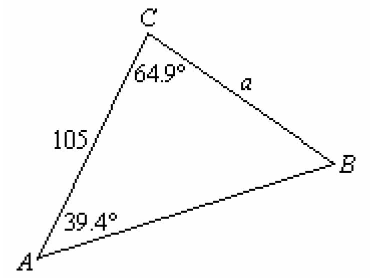
$$C=32.7^\circ$$

$$\frac{c}{\sin 32.7^\circ}=\frac{21.6}{\sin 41.2^\circ}$$

$$c=\frac{21.6\sin 32.7^\circ}{\sin 41.2^\circ}$$

$$c\approx 17.7$$

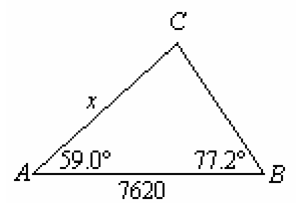
**29.**




$\angle B = 180° - (39.4° + 64.9°)$
$\angle B = 75.7°$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{a}{\sin 39.4°} = \frac{105}{\sin 75.7°}$$
$$a = \frac{105 \sin 39.4°}{\sin 75.7°}$$
$$a \approx 68.8 \text{ miles}$$

**30.**




$\angle C = 180° - (59.0° + 77.2°)$
$\angle C = 43.8°$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$
$$\frac{x}{\sin 77.2°} = \frac{7620}{\sin 43.8°}$$
$$x = \frac{7620 \sin 77.2°}{\sin 43.8°}$$
$$x \approx 10{,}700 \text{ feet}$$

**31.** $a = 155$ yd, $c = 165$ yd, $A = 42.0^\circ$

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$
$$\frac{165}{\sin C} = \frac{155}{\sin 42.0^\circ}$$
$$\sin C = \frac{155}{165 \sin 42.0^\circ}$$
$$C = \sin^{-1}\left(\frac{155}{165 \sin 42.0^\circ}\right) = 45.4^\circ$$

$B = 180^\circ - 42.0^\circ - 45.4^\circ = 92.6^\circ$

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$
$$\frac{b}{\sin 92.6^\circ} = \frac{155}{\sin 42.0^\circ}$$
$$b = \frac{155 \sin 92.6^\circ}{165 \sin 42.0^\circ} = 231 \text{ yd}$$

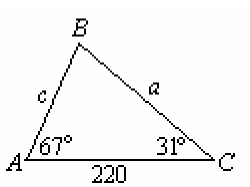
**32.** $b = 365$ yd, $A = 11.2^\circ$, $C = 22.9^\circ$

$B = 180^\circ - 11.2^\circ - 22.9^\circ = 145.9^\circ$

**a.**
$$\frac{c}{\sin C} = \frac{b}{\sin B}$$
$$\frac{c}{\sin 22.9^\circ} = \frac{365}{\sin 145.9^\circ}$$
$$c = \frac{365 \sin 22.9^\circ}{\sin 145.9^\circ} = 253 \text{ yd}$$

**b.**
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{a}{\sin 11.2^\circ} = \frac{365}{\sin 145.9^\circ}$$
$$a = \frac{365 \sin 11.2^\circ}{\sin 145.9^\circ} = 126 \text{ yd}$$

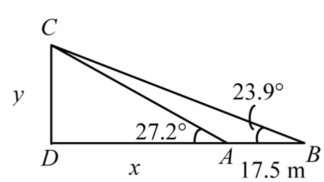
**33.**




$B = 180° - 67° - 31°$
$B = 82°$

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$
$$\frac{c}{\sin 31°} = \frac{220}{\sin 82°}$$
$$c = \frac{220 \sin 31°}{\sin 82°}$$
$$c \approx 110 \text{ feet}$$

**34.**



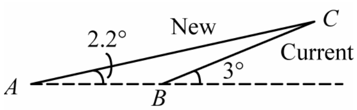

$CAB = 180° - 27.2° = 152.8°$
$ACB = 180° - 152.8° - 23.9° = 3.3°$

$$\frac{AC}{\sin ABC} = \frac{AB}{\sin ACB}$$
$$\frac{AC}{\sin 23.9°} = \frac{17.5}{\sin 3.3°}$$
$$AC = \frac{17.5 \sin 23.9°}{\sin 3.3°}$$
$$\approx 123.2 \text{ m}$$

$$\frac{CD}{\sin CAD} = \frac{AC}{\sin ADC}$$
$$\frac{CD}{\sin 27.2°} = \frac{123.2}{\sin 90°}$$
$$CD = \frac{123.2 \sin 27.2°}{\sin 90°}$$
$$\approx 56.3 \text{ m}$$

Responses will vary.

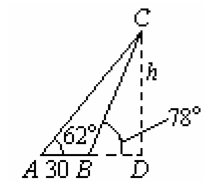
**35.**




$ABC = 180° - 3°$
$= 177°$

$$\frac{AC}{\sin ABC} = \frac{BC}{\sin CAB}$$
$$\frac{AC}{\sin 177°} = \frac{3550}{\sin 2.2°}$$
$$AC = \frac{3550 \sin 177°}{\sin 2.2°} \approx 4840 \text{ ft}$$

**36.**

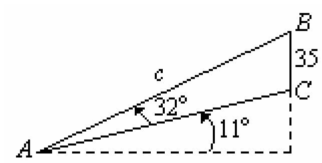



$ABC = 180° - 78°$
$= 102°$
$ACB = 180° - 102° - 62°$
$= 16°$

$$\frac{AC}{\sin ABC} = \frac{AB}{\sin ACB}$$
$$\frac{AC}{\sin 102°} = \frac{30}{\sin 16°}$$
$$AC = \frac{30 \sin 102°}{\sin 16°}$$

$$\sin 62° = \frac{h}{AC}$$
$$h = AC \sin 62°$$
$$h = \frac{30 \sin 102°}{\sin 16°} \sin 62°$$
$h \approx 94$ feet

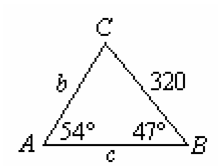
**37.**




$A = 32° - 11°$
$A = 21°$
$B = 180° - 90° - 32°$
$B = 58°$
$C = 180° - 58° - 21°$
$C = 101°$

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$
$$\frac{c}{\sin 101°} = \frac{35}{\sin 21°}$$
$$c = \frac{35 \sin 101°}{\sin 21°}$$
$c \approx 96$ feet

**38.**

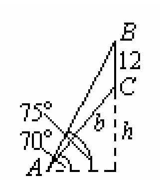



$C = 180° - 54° - 47°$
$C = 79°$

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$
$$\frac{b}{\sin 47°} = \frac{320}{\sin 54°}$$
$$b = \frac{320 \sin 47°}{\sin 54°}$$
$b \approx 290$ feet

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$
$$\frac{c}{\sin 79°} = \frac{320}{\sin 54°}$$
$$c = \frac{320 \sin 79°}{\sin 54°}$$
$c \approx 390$ feet

**39.**

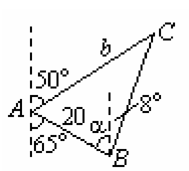



$A = 5°$
$B = 180° - 90° - 75°$
$B = 15°$
$C = 180° - 15° - 5°$
$C = 160°$

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$
$$\frac{b}{\sin 15°} = \frac{12}{\sin 5°}$$
$$b = \frac{12 \sin 15°}{\sin 5°}$$

$$\sin 70° = \frac{h}{b}$$
$$h = b \sin 70°$$
$$h = \left(\frac{12 \sin 15°}{\sin 5°}\right) \sin 70°$$
$h \approx 33$ feet

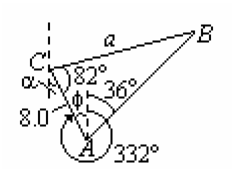
**40.**




$\alpha = 65°$
$B = 65° + 8°$
$B = 73°$
$A = 180° - 50° - 65°$
$A = 65°$
$C = 180° - 65° - 73°$
$C = 42°$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$
$$\frac{b}{\sin 73°} = \frac{20}{\sin 42°}$$
$$b = \frac{20 \sin 73°}{\sin 42°}$$
$b \approx 29$ miles

**41.**

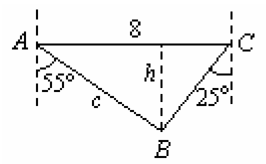



$\phi = 360° - 332°$
$\phi = 28°$
$\alpha = 28°$

$C = 82° - 28°$
$C = 54°$
$A = 28° + 36°$
$A = 64°$
$B = 180° - 64° - 54°$
$B = 62°$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{a}{\sin 64°} = \frac{8.0}{\sin 62°}$$
$$a = \frac{8.0 \sin 64°}{\sin 62°}$$
$a \approx 8.1$ miles

**42.**

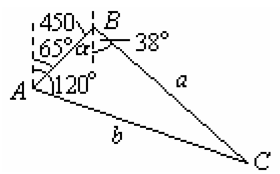



$A = 90° - 55°$
$A = 35°$
$C = 90° - 25°$
$C = 65°$
$B = 180° - 35° - 65°$
$B = 80°$

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$
$$\frac{c}{\sin 65°} = \frac{8}{\sin 80°}$$
$$c = \frac{8 \sin 65°}{\sin 80°}$$

$$\sin 35° = \frac{h}{c}$$
$h = c \sin 35°$
$$h = \frac{88 \sin 65°}{\sin 80°} \sin 35°$$
$h \approx 4.2$ miles

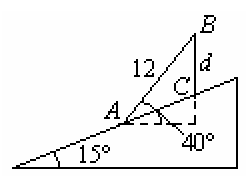
**43.**




$A = 120° - 65°$
$A = 55°$
$\alpha = 65°$
$B = 38° + 65°$
$B = 103°$
$C = 180° - 103° - 55°$
$C = 22°$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$
$$\frac{b}{\sin 103°} = \frac{450}{\sin 22°}$$
$$b = \frac{450 \sin 103°}{\sin 22°}$$
$b \approx 1200$ miles

**44.**

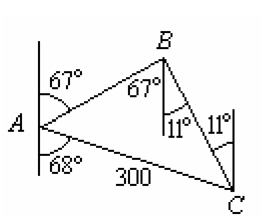



$A = 40° - 15°$
$A = 25°$
$B = 180° - 90° - 40°$
$B = 50°$
$C = 180° - 25° - 50°$
$C = 105°$

$$\frac{d}{\sin A} = \frac{12}{\sin C}$$
$$d = \frac{12 \sin 25°}{\sin 105°}$$
$d \approx 5.3$ feet

## Connecting Concepts

**45.**




$A = 180° - 67° - 68°$
$A = 45°$
$B = 67° + 11°$
$B = 78°$
$C = 180° - 45° - 78°$
$C = 57°$

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$
$$\frac{c}{\sin 57°} = \frac{300}{\sin 78°}$$
$$c = \frac{300 \sin 57°}{\sin 78°}$$
$c \approx 260$ meters

**46.**

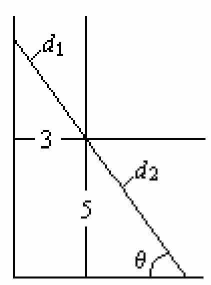
$$\frac{a}{\sin A} = \frac{b}{siinB}$$
$$\frac{a}{b} = \frac{\sin A}{\sin B}$$
$$\frac{a}{b} - 1 = \frac{\sin A}{\sin B} - 1$$
$$\frac{a-b}{b} = \frac{\sin A - \sin B}{\sin B}$$

**47.** $$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{a}{b} = \frac{\sin A}{\sin B}$$
$$\frac{a}{b} + 1 = \frac{\sin A}{\sin B} + 1$$
$$\frac{a+b}{b} = \frac{\sin A + \sin B}{\sin B}$$

**48.** Use the results of Problems 46 and 47.

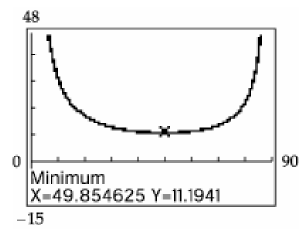
$$\frac{\frac{a-b}{b}}{\frac{a+b}{b}} = \frac{\frac{\sin A - \sin B}{\sin B}}{\frac{\sin A + \sin B}{\sin B}}$$
$$\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}$$

**49.**




$$\frac{3}{d_1} = \cos\theta, \ \frac{5}{d_2} = \sin\theta$$
$$d_1 = \frac{3}{\cos\theta}, \ d_2 = \frac{5}{\sin\theta}$$
$$L = d_1 + d_2$$
$$L(\theta) = \frac{3}{\cos\theta} + \frac{5}{\sin\theta}$$

The graph of $L$ is shown.
The minimum value of $L$ is approximately 11.19 m.




## Prepare for Section 7.2

**PS1.** $\sqrt{(10.0)^2 + (15.0)^2 - 2(10.0)(15.0)\cos 110.0^\circ} \approx 20.7$

**PS2.** $A = \frac{1}{2}bh = \frac{1}{2}(6)(8.5) = 22.5 \text{ in.}^2$

**PS3.**
$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$c^2 - a^2 - b^2 = -2ab\cos C$$
$$\cos C = \frac{c^2 - a^2 - b^2}{-2ab}$$
$$C = \cos^{-1}\left(\frac{c^2 - a^2 - b^2}{-2ab}\right) = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$$

**PS4.** $P = 6 + 9 + 10 = 25$
semiperimeter $= \frac{1}{2}(25) = 12.5$ m

**PS5.** $s = \frac{a+b+c}{2} = \frac{3+4+5}{2} = 6$
$$\sqrt{6(6-3)(6-4)(6-5)} = \sqrt{6(3)(2)(1)} = 6$$

**PS6.** $c^2 = a^2 + b^2$

## Section 7.2

**1.**
$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$c^2 = 12^2 + 18^2 - 2(12)(18)\cos 44^\circ$$
$$c^2 = 468 - 432\cos 44^\circ$$
$$c = \sqrt{468 - 432\cos 44^\circ}$$
$$c \approx 13$$

**2.**
$$a^2 = b^2 + c^2 - bc\cos A$$
$$a^2 = 30^2 + 24^2 - 2(30)(24)\cos 120^\circ$$
$$a^2 = 1476 - 1440\cos 120^\circ$$
$$a = \sqrt{1476 - 1440\cos 120^\circ}$$
$$a \approx 47$$

**3.**
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$b^2 = 120^2 + 180^2 - 2(120)(180)\cos 56^\circ$$
$$b^2 = 46{,}800 - 43{,}200\cos 56^\circ$$
$$b = \sqrt{46{,}800 - 43{,}200\cos 56^\circ}$$
$$b \approx 150$$

**4.**
$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$c^2 = 400^2 + 620^2 - 2(400)(620)\cos 116^\circ$$
$$c^2 = 544{,}400 - 496{,}000\cos 116^\circ$$
$$c = \sqrt{544{,}400 - 496{,}000\cos 116^\circ}$$
$$c \approx 870$$

**5.** $a^2 = b^2 + c^2 - 2bc\cos A$
$a^2 = 60^2 + 84^2 - 2(60)(84)\cos 13°$
$a^2 = 10,656 - 10,080\cos 13°$
$a = \sqrt{10,656 - 10,080\cos 13°}$
$a \approx 29$

**6.** $b^2 = a^2 + c^2 - 2ac\cos B$
$b^2 = 122^2 + 144^2 - 2(122)(144)\cos 48°$
$b^2 = 35,620 - 35,136\cos 48°$
$b = \sqrt{35,620 - 35,136\cos 48°}$
$b \approx 110$

**7.** $c^2 = a^2 + b^2 - 2ab\cos C$
$c^2 = 9.0^2 + 7.0^2 - 2(9.0)(7.0)\cos 72°$
$c^2 = 130 - 126\cos 72°$
$c = \sqrt{130 - 126\cos 72°}$
$c \approx 9.5$

**8.** $a^2 = b^2 + c^2 - 2bc\cos A$
$a^2 = 12^2 + 22^2 - 2(12)(22)\cos 55°$
$a^2 = 628 - 528\cos 55°$
$a = \sqrt{628 - 528\cos 55°}$
$a \approx 18$

**9.** $c^2 = a^2 + b^2 - 2ab\cos C$
$c^2 = 4.6^2 + 7.2^2 - 2(4.6)(7.2)\cos 124°$
$c^2 = 73 - 66.24\cos 124°$
$c = \sqrt{73 - 66.24\cos 124°}$
$c \approx 10$

**10.** $a^2 = b^2 + c^2 - 2bc\cos A$
$a^2 = 12.3^2 + 14.5^2 - 2(12.3)(14.5)\cos 6.5°$
$a^2 = 361.54 - 356.7\cos 6.5°$
$a = \sqrt{361.54 - 356.7\cos 6.5°}$
$a \approx 2.67$

**11.** $b^2 = a^2 + c^2 - 2ac\cos B$
$b^2 = 25.9^2 + 33.4^2 - 2(25.9)(33.4)\cos 84.0°$
$b^2 = 1786.37 - 1730.12\cos 84.0°$
$b = \sqrt{1786.37 - 1730\cos 84.0°}$
$b \approx 40.1$

**12.** $c^2 = a^2 + b^2 - 2ab\cos C$
$c^2 = 14.2^2 + 9.30^2 - 2(14.2)(9.30)\cos 9.20°$
$c^2 = 288.13 - 264.12\cos 9.20°$
$c = \sqrt{288.13 - 264.12\cos 9.20°}$
$c \approx 5.24$

**13.** $b^2 = a^2 + c^2 - 2ac\cos B$
$b^2 = 122^2 + 55.9^2 - 2(122)(55.9)\cos 44.2°$
$b^2 = 18,008.81 - 13,639.6\cos 44.2°$
$b = \sqrt{18,008.81 - 13,639.6\cos 44.2°}$
$b \approx 90.7$

**14.** $a^2 = b^2 + c^2 - 2bc\cos A$
$a^2 = 444.8^2 + 389.6^2 - 2(444.8)(389.6)\cos 78.44°$
$a^2 = 349,635.2 - 346,588.16\cos 78.44°$
$a = \sqrt{349,635.2 - 346,588.16\cos 78.44°}$
$a \approx 529.3$

**15.** $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$
$\cos A = \dfrac{32^2 + 40^2 - 25^2}{2(32)(40)}$
$\cos A = \dfrac{1999}{2560}$
$A = \cos^{-1}\left(\dfrac{1999}{2560}\right) \approx 39°$

**16.** $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$
$\cos B = \dfrac{60^2 + 120^2 - 88^2}{2(60)(120)}$
$\cos B = \dfrac{10256}{14400}$
$B = \cos^{-1}\left(\dfrac{10256}{14400}\right) \approx 45°$

**17.** $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$
$\cos C = \dfrac{8.0^2 + 9.0^2 - 12^2}{2(8.0)(9.0)}$
$\cos C = \dfrac{1}{144}$
$C = \cos^{-1}\left(\dfrac{1}{144}\right) \approx 90°$

**18.** $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$
$\cos A = \dfrac{132^2 + 160^2 - 108^2}{2(132)(160)}$
$\cos A = \dfrac{31,360}{42,240}$
$A = \cos^{-1}\left(\dfrac{31360}{42240}\right) \approx 42.1°$

**19.** $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

$\cos B = \dfrac{80^2 + 124^2 - 92^2}{2(80)(124)}$

$\cos B = \dfrac{13{,}312}{19{,}840}$

$B = \cos^{-1}\left(\dfrac{13312}{19840}\right) \approx 47.9°$

**20.** $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

$\cos B = \dfrac{166^2 + 139^2 - 124^2}{2(166)(139)}$

$\cos B = \dfrac{31{,}501}{46{,}148}$

$B = \cos^{-1}\left(\dfrac{31501}{46148}\right) \approx 47.0°$

**21.** $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$\cos C = \dfrac{1025^2 + 625^2 - 1420^2}{2(1025)(625)}$

$\cos C = \dfrac{-575{,}150}{1{,}281{,}250}$

$C = \cos^{-1}\left(\dfrac{-575{,}150}{1{,}281{,}250}\right) \approx 116.67°$

**22.** $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\cos A = \dfrac{3.2^2 + 5.9^2 - 4.7^2}{2(3.2)(5.9)}$

$\cos A = \dfrac{22.96}{37.76}$

$A = \cos^{-1}\left(\dfrac{22.96}{37.76}\right) \approx 53°$

**23.** $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

$\cos B = \dfrac{32.5^2 + 29.6^2 - 40.1^2}{2(32.5)(29.6)}$

$\cos B = \dfrac{324.4}{1924}$

$B = \cos^{-1}\left(\dfrac{324.4}{1924}\right) \approx 80.3°$

**24.** $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$\cos C = \dfrac{112.4^2 + 96.80^2 - 129.2^2}{2(112.40)(96.80)}$

$\cos C \approx 0.2441$

$C \approx 75.87°$

**25.** $a^2 = b^2 + c^2 - 2bc\cos A$

$a = \sqrt{15.5^2 + 17.2^2 - 2(15.5)(17.2)\cos 39.4°}$

$a \approx 11.13860218$

$a \approx 11.1$

$\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

$\cos B = \dfrac{11.13860218^2 + 17.2^2 - 15.5^2}{2(11.13860218)(17.2)}$

$B = \cos^{-1}\left(\dfrac{179.6584585}{383.167915}\right)$

$B \approx 62.0°$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$\cos C = \dfrac{11.13860218^2 + 15.5^2 - 17.2^2}{2(11.13860218)(15.5)}$

$C = \cos^{-1}\left(\dfrac{68.47845852}{345.2966676}\right)$

$C \approx 78.6°$

**26.** $c^2 = a^2 + b^2 - 2ab\cos C$

$c = \sqrt{141^2 + 92.3^2 - 2(141)(92.3)\cos 98.4°}$

$c \approx 179.4509034$

$c \approx 179$

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\cos A = \dfrac{92.3^2 + 179.4509034^2 - 141^2}{2(92.3)(179.4509034)}$

$A = \cos^{-1}\left(\dfrac{20{,}840.91673}{33{,}126.64242}\right)$

$A \approx 51.0°$

$\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

$\cos B = \dfrac{141^2 + 179.4509034^2 - 92.3^2}{2(141)(179.4509034)}$

$B = \cos^{-1}\left(\dfrac{43{,}564.33673}{50{,}605.15476}\right)$

$B \approx 30.6°$

**27.** $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\cos A = \dfrac{144^2 + 98.1^2 - 83.6^2}{2(144)(98.1)}$

$A = \cos^{-1}\left(\dfrac{23{,}370.65}{28{,}252.8}\right)$

$A \approx 34.2°$

$\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

$\cos B = \dfrac{83.6^2 + 98.1^2 - 144^2}{2(83.6)(98.1)}$

$B = \cos^{-1}\left(\dfrac{-4123.43}{16{,}402.32}\right)$

$B \approx 104.6°$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$\cos C = \dfrac{83.6^2 + 144^2 - 98.1^2}{2(83.6)(144)}$

$C = \cos^{-1}\left(\dfrac{18{,}101.35}{24{,}076.8}\right)$

$C \approx 41.3°$

**28.** $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\cos A = \dfrac{36.3^2 + 38.2^2 - 25.4^2}{2(36.3)(38.2)}$

$A = \cos^{-1}\left(\dfrac{2131.77}{2773.32}\right)$

$A \approx 39.8°$

$\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

$\cos B = \dfrac{25.4^2 + 38.2^2 - 36.3^2}{2(25.4)(38.2)}$

$B = \cos^{-1}\left(\dfrac{786.71}{1940.56}\right)$

$B \approx 66.1°$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$\cos C = \dfrac{25.4^2 + 36.3^2 - 38.2^2}{2(25.4)(36.3)}$

$C = \cos^{-1}\left(\dfrac{503.61}{1844.04}\right)$

$C \approx 74.2°$

**29.** $K = \dfrac{1}{2}bc\sin A$

$K = \dfrac{1}{2}(12)(24)\sin 105°$

$K \approx 140$ square units

**30.** $K = \dfrac{1}{2}ac\sin B$

$K = \dfrac{1}{2}(32)(25)\sin 127°$

$K \approx 320$ square units

**31.** $C = 180° - 42° - 76°$

$C = 62°$

$K = \dfrac{c^2 \sin A \sin B}{2\sin C}$

$K = \dfrac{12^2 \sin 42° \sin 76°}{2\sin 62°}$

$K \approx 53$ square units

**32.** $A = 180° - 102° - 27°$

$A = 51°$

$K = \dfrac{a^2 \sin B \sin C}{2\sin A}$

$K = \dfrac{8.5^2 \sin 102° \sin 27°}{2\sin 51°}$

$K \approx 21$ square units

**33.** $s = \dfrac{1}{2}(a+b+c)$

$s = \dfrac{1}{2}(16+12+14)$

$s = 21$

$K = \sqrt{s(s-a)(s-b)(s-c)}$

$K = \sqrt{21(21-16)(21-12)(21-14)}$

$K = \sqrt{21(5)(9)(7)}$

$K \approx 81$ square units

**34.** $s = \dfrac{1}{2}(a+b+c)$

$s = \dfrac{1}{2}(24+32+36)$

$s = 46$

$K = \sqrt{s(s-a)(s-b)(s-c)}$

$K = \sqrt{46(46-24)(46-32)(46-36)}$

$K = \sqrt{46(22)(14)(10)}$

$K \approx 380$ square units

**35.** $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

$\dfrac{22.4}{\sin A} = \dfrac{26.9}{\sin 54.3°}$

$\sin A = \dfrac{22.4 \sin 54.3°}{26.9}$

$\sin A \approx 0.6762$

$A \approx 42.5°$

$C = 180° - 42.5° - 54.3°$

$C = 83.2°$

$K = \dfrac{1}{2}ab\sin C$

$K = \dfrac{1}{2}(22.4)(26.9)\sin 83.2°$

$K \approx 299$ square units

**36.** $K = \dfrac{1}{2}ab\sin C$

$K = \dfrac{1}{2}(9.84)(13.4)\sin 18.2°$

$K \approx 20.6$ square units

**37.** $C = 180° - 116° - 34°$

$C = 30°$

$K = \dfrac{c^2 \sin A \sin B}{2\sin C}$

$K = \dfrac{8.5^2 \sin 116° \sin 34°}{2\sin 30°}$

$K \approx 36$ square units

**38.** $A = 180° - 76.3° - 42.8°$

$A = 60.9°$

$K = \dfrac{c^2 \sin A \sin B}{2 \sin C}$

$K = \dfrac{17.9^2 \sin 60.9° \sin 42.8°}{2 \sin 76.3°}$

$K \approx 97.9$ square units

**39.** $s = \dfrac{1}{2}(a + b + c)$

$s = \dfrac{1}{2}(3.6 + 4.2 + 4.8)$

$s = 6.3$

$K = \sqrt{s(s-a)(s-b)(s-c)}$

$K = \sqrt{6.3(6.3-3.6)(6.3-4.2)(6.3-4.8)}$

$K = \sqrt{6.3(2.7)(2.1)(1.5)}$

$K \approx 7.3$ square units

**40.** $s = \dfrac{1}{2}(a + b + c)$

$s = \dfrac{1}{2}(10.2 + 13.3 + 15.4)$ $s = \dfrac{1}{2}(a + b + c)$

$s = 19.45$

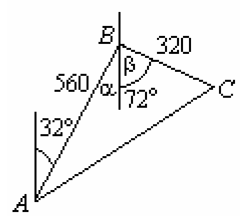
$K = \sqrt{s(s-a)(s-b)(s-c)}$

$K = \sqrt{19.45(19.45-10.2)(19.45-13.3)(19.45-15.4)}$

$K = \sqrt{19.45(9.25)(6.15)(4.05)}$

$K \approx 66.9$ square units

**41.**




$\alpha = 32°$

$\beta = 72°$

$B = 72° + 32°$

$B = 104°$

$b^2 = a^2 + c^2 - 2ac \cos 104°$

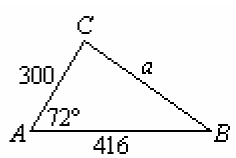
$b^2 = 320^2 + 560^2 - 2(320)(560)\cos 104°$

$b^2 = 416{,}000 - 358{,}400 \cos 104°$

$b = \sqrt{416{,}000 - 358{,}400 \cos 104°}$

$b \approx 710$ miles

**42.**




$a^2 = b^2 + c^2 - 2bc \cos A$

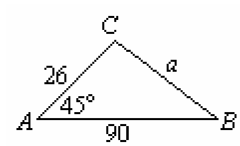
$a^2 = 300^2 + 416^2 - 2(300)(416)\cos 72°$

$a^2 = 236{,}056 - 249{,}600 \cos 72°$

$a = \sqrt{236{,}056 - 249{,}600 \cos 72°}$

$a \approx 430$ feet

**43.**




$a^2 = b^2 + c^2 - 2bc \cos A$

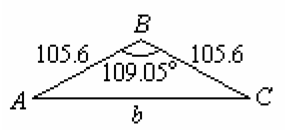
$a^2 = 26^2 + 90^2 - 2(26)(90)\cos 45°$

$a^2 = 8776 - 4680 \cos 45°$

$a = \sqrt{8776 - 4680 \cos 45°}$

$a \approx 74$ feet

**44.**



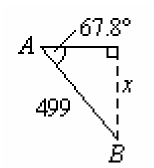

$b^2 = a^2 + c^2 - 2ac\cos B$

$b^2 = (105.6)^2 + (105.6)^2 - 2(105.6)(105.6)\cos 109.05°$

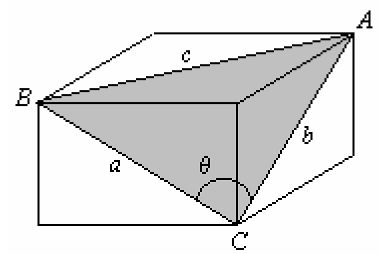
$b^2 = 22302.72 - 22302.72\cos 109.05°$

$b = \sqrt{22302.72 - 22302.72\cos 109.05°}$

$b \approx 172.0$ feet




**45.**




Let $a$ = the length of the diagonal on the front of the box.
Let $b$ = the length of the diagonal on the right side of the box.
Let $c$ = the length of the diagonal on the top of the box.

$a^2 = (4.75)^2 + (6.50)^2 = 64.8125$

$a = \sqrt{64.8125}$

$b^2 = (3.25)^2 + (4.75)^2 = 33.125$

$b = \sqrt{33.125}$

$c^2 = (6.50)^2 + (3.25)^2 = 52.8125$

$\theta = C$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

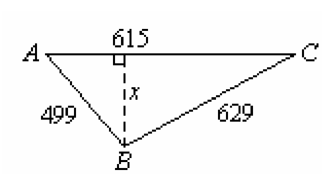
$$\cos\theta = \frac{64.8125 + 33.125 - 52.8125}{2\sqrt{64.8125}\sqrt{33.125}}$$

$$\cos\theta = \frac{45.125}{2\sqrt{64.8125}\sqrt{33.125}}$$

$$\theta = \cos^{-1}\left(\frac{45.125}{2\sqrt{64.8125}\sqrt{33.125}}\right)$$

$\theta \approx 60.9°$

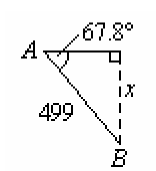
**46.**




$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{(615)^2 + (499)^2 - (629)^2}{2(615)(499)} = \frac{231585}{613770}$$

$$A = \cos^{-1}\left(\frac{231585}{613770}\right) \approx 67.8°$$



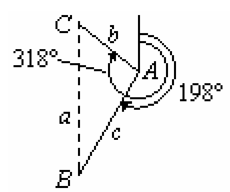

$$\sin A = \frac{x}{499}$$

$$\sin 67.8° = \frac{x}{499}$$

$499\ \sin 67.8° = x$

$x \approx 462$ feet

**47.**




$b = (18 \text{ mph})(10 \text{ hours}) = 180$ miles

$c = (22 \text{ mph})(10 \text{ hours}) = 220$ miles

$A = 318° - 198°$

$A = 120°$

$a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 180^2 + 220^2 - 2(180)(220)\cos 120°$

$a^2 = 120,400$

$a \approx 350$ miles

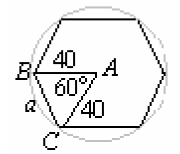
**48.** $d^2 = 136^2 + 162^2 - 2(136)(162)\cos 78°$

$d^2 = 44,740 - 44,064\cos 78°$

$d = \sqrt{44,740 - 44,064\cos 78°}$

$d \approx 189$ miles

**49.**




$A = \dfrac{360°}{6}$

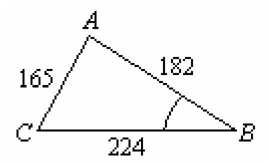
$A = 60°$

$a^2 = 40^2 + 40^2 - 2(40)(40)\cos 60°$

$a^2 = 1600$

$a = 40$ cm

**50.**

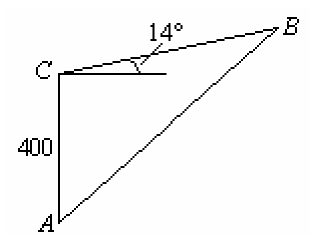



$\cos B = \dfrac{224^2 + 182^2 - 165^2}{2(224)(182)}$

$\cos B \approx 0.6877$

$B \approx 46.5°$

**51.**




$C = 90° + 14°$

$C = 104°$

$a = \dfrac{180(5280)}{3600} \cdot 10$

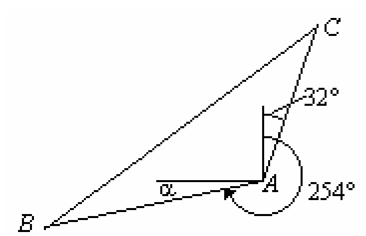
$a = 2640$ feet

$c^2 = 2640^2 + 400^2 - 2(2640)(400)(\cos 104°)$

$c^2 \approx 7,640,539$

$c \approx 2800$ feet

**52.**




$\alpha = 270° - 254°$

$\alpha = 16°$

$A = 16° + 90° + 32°$

$A = 138°$

$b = 4 \cdot 16 = 64$ miles

$c = 3 \cdot 22 = 66$ miles

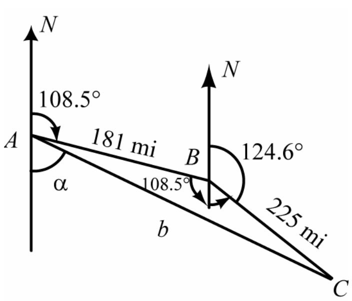
$a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 64^2 + 66^2 - 2(64)(66)\cos 138°$

$a^2 = 8452 - 8448\cos 138°$

$a \approx 120$ miles

**53.**




$B = 108.5 + (180 - 124.6)$

$B = 163.9°$

$b^2 = a^2 + c^2 - 2ac\cos B$

$b^2 = 225^2 + 181^2 - 2(225)(181)\cos 163.9°$

$b^2 = 83,386 - 81,450\cos 163.9°$

$b = \sqrt{83,386 - 81,450\cos 163.9°}$

$b \approx 402.046592$

$b \approx 402$ mi

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\cos A = \dfrac{402.046592^2 + 181^2 - 225^2}{2(402.046592)(181)}$

$A = \cos^{-1}\left(\dfrac{143,777.4621}{145,540.8663}\right)$

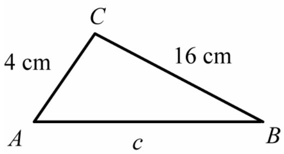
$A \approx 8.9°$

$\alpha = 180 - (108.5 + 8.9)$

$\alpha = 62.6°$

The distance is 402 mi and the bearing is S62.6°E.

**54.**



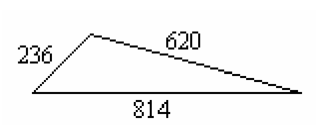

**a.** $16^2 = 4^2 + c^2 - 2(4)(c)\cos A$

**b.**
$$16^2 = 4^2 + c^2 - 2(4)(c)\cos A$$
$$0 = c^2 - (8\cos A)c - 240$$
$$c = \frac{8\cos A \pm \sqrt{(-8\cos A)^2 - 4(1)(-240)}}{2(1)}$$
$$c = \frac{8\cos A + \sqrt{64\cos^2 A + 960}}{2}$$

**c.**
$$c = \frac{8\cos(55) + \sqrt{64\cos^2(55) + 960}}{2}$$
$c \approx 18$ cm

**d.**
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{16}{\sin 55} = \frac{4}{\sin B}$$
$$B = \sin^{-1}\left(\frac{4\sin 55}{16}\right) \approx 11.8°$$
$$C = 180 - 55 - 11.8 = 113.2°$$
$$\frac{a}{\sin A} = \frac{c}{\sin C}$$
$$\frac{16}{\sin 55} = \frac{c}{\sin 113.2}$$
$$c = \frac{16\sin 113.2}{\sin 55} \approx 18 \text{ cm}$$
They are the same.

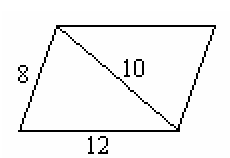
**55.**




$$s = \frac{1}{2}(a+b+c)$$
$$s = \frac{1}{2}(236+620+814)$$
$$s = 835$$

$$K = \sqrt{s(s-a)(s-b)(s-c)}$$
$$K = \sqrt{835(835-236)(835-620)(835-814)}$$
$$K = \sqrt{835(599)(215)(21)}$$
$$K \approx \sqrt{2{,}258{,}240{,}000}$$
$K \approx 47{,}500$ square meters

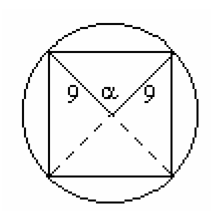
**56.**




$$s = \frac{1}{2}(8+10+12)$$
$$s = 15$$

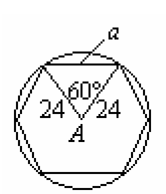
$$K = 2\sqrt{s(s-a)(s-b)(s-c)}$$
$$K = 2\sqrt{15(15-8)(15-10)(15-12)}$$
$K = 30\sqrt{7}$ square feet

**57.**




$$\alpha = 90°$$
$$K = 4\left[\frac{1}{2}(9)(9)\sin 90°\right]$$
$$K = 162 \text{ in}^2$$

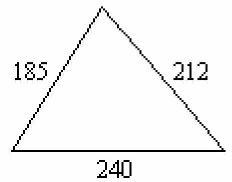
**58.**




$$a = 24$$
$$s = \frac{1}{2}(24+24+24)$$
$$s = 36$$

$$K = 6\sqrt{36(36-24)(36-24)(36-24)}$$
$$K = 6(144)\sqrt{3}$$
$$K = 864\sqrt{3} \text{ cm}^2$$

**59.**




$$s = \frac{1}{2}(185 + 212 + 240)$$
$$s = 318.5$$

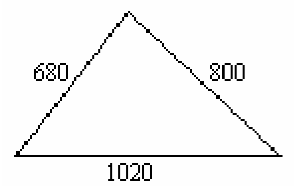
$$K = \sqrt{318.5(318.5 - 185)(318.5 - 212)(318.5 - 240)}$$
$$K \approx 18{,}854 \text{ ft}^2$$
$$\text{cost} = 2.20(18{,}854)$$
$$\text{cost} \approx \$41{,}000$$

**60.**

$$s = \frac{1}{2}(324 + 412 + 516)$$
$$s = 626$$

$$K = \sqrt{626(626 - 324)(626 - 412)(626 - 516)}$$
$$K \approx 66{,}710 \text{ ft}^2$$
$$\text{cost} = 4.15(66{,}710)$$
$$\text{cost} \approx \$277{,}000$$

**61.**




$$s = \frac{1}{2}(680 + 800 + 1020)$$
$$s = 1250$$

$$K = \sqrt{1250(1250 - 680)(1250 - 800)(1250 - 1020)}$$
$$K \approx 271{,}558 \text{ ft}^2$$
$$\text{Acres} = \frac{271{,}558}{43{,}560}$$
$$\text{Acres} \approx 6.23$$

**62.**

$$s = \frac{1}{2}(420 + 500 + 540)$$
$$s = 730$$

$$K = \sqrt{730(730 - 420)(730 - 500)(730 - 540)}$$
$$K \approx 99{,}445$$
$$\text{Acres} = \frac{99{,}445}{4840}$$
$$\text{Acres} \approx 20.5$$

**63.** For *ABC*,

$$\frac{13.0 - 16.1}{10.4} \approx -0.2981$$
$$\frac{\sin\left(\frac{53.5 - 86.5}{2}\right)}{\cos\left(\frac{40.0}{2}\right)} \approx -0.3022$$

Triangle *ABC* has correct dimensions.

For *DEF*,

$$\frac{17.2 - 21.3}{22.8} \approx -0.1798$$
$$\frac{\sin\left(\frac{52.1 - 59.9}{2}\right)}{\cos\left(\frac{68.0}{2}\right)} \approx -0.0820$$

Triangle *DEF* has an incorrect dimension.

**64.** For *ABC*,

$$\frac{9.23 - 15.1}{16.2} \approx -0.3623$$
$$\frac{\sin\left(\frac{34.1 - 66.2}{2}\right)}{\cos\left(\frac{79.7}{2}\right)} \approx -0.3601$$

Triangle *ABC* has correct dimensions.

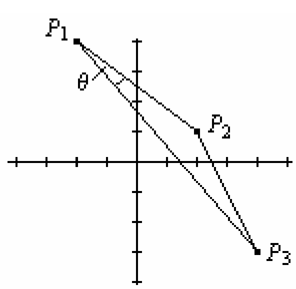
For *DEF*,

$$\frac{13.6 - 16.0}{18.9} \approx -0.1270$$
$$\frac{\sin\left(\frac{45.0 - 56.2}{2}\right)}{\cos\left(\frac{78.8}{2}\right)} \approx -0.1263$$

Triangle *DEF* has correct dimensions.

## Connecting Concepts

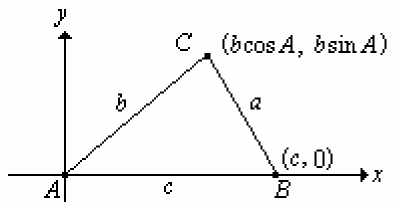
**65.**




$$d(P_1, P_2) = \sqrt{[2 - (-2)]^2 + (1 - 4)^2} = 5$$
$$d(P_1, P_3) = \sqrt{(-2 - 4)^2 + (4 - (-3))^2} = \sqrt{85}$$
$$d(P_2, P_3) = \sqrt{(2 - 4)^2 + (1 - (-3))^2} = 2\sqrt{5}$$

$$\cos\theta = \frac{5^2 + \left(\sqrt{85}\right)^2 - \left(2\sqrt{5}\right)^2}{2 \cdot 5 \cdot \sqrt{85}}$$
$$\cos\theta \approx 0.9762$$
$$\theta \approx 12.5°$$

**66.**

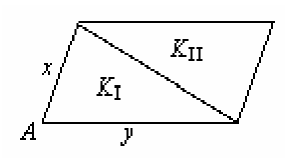



$$a = \sqrt{(b\cos A - c)^2 + (b\sin A - 0)^2}$$
$$a^2 = b^2\cos^2 A - 2bc\cos A + c^2 + b^2\sin^2 A$$
$$a^2 = b^2(\cos^2 A + \sin^2 A) + c^2 - 2bc\cos A$$
$$a^2 = b^2 + c^2 - 2bc\cos A$$

**67.**

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{b^2+2bc+c^2-a^2-2bc}{2bc} = \frac{(b+c)^2-a^2}{2bc} - \frac{2bc}{2bc} = \frac{(b+c-a)(b+c+a)}{2bc} - 1$$

**68.**

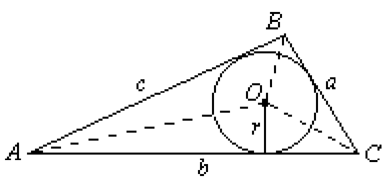



$$K_I = \frac{1}{2}xy\sin A$$
$$K_I = K_{II}$$
$$K = 2K_I$$
$$K = xy\sin A$$

**69.** $V = 18K$, where $K$ = area of triangular base

$$V = \frac{1}{2}(4)(4)(\sin 72°)(18)$$
$$V \approx 140 \text{ in}^3$$

**70.**




$$K_{BOC} = \frac{1}{2}ar$$
$$K_{AOC} = \frac{1}{2}br$$
$$K_{AOB} = \frac{1}{2}cr$$

$$K = K_{BOC} + K_{AOC} + K_{AOB}$$
$$K = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr$$
$$K = \frac{1}{2}r(a+b+c)$$
$$K = \frac{1}{2}rs \text{ where } s = \frac{1}{2}(a+b+c)$$

## Prepare for Section 7.3

**PS1.** $\sqrt{\left(\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25}+\frac{16}{25}} = \sqrt{\frac{25}{25}} = 1$

**PS2.** $10\cos 228° \approx -6.691$

**PS3.**

$$\tan\alpha = \left|\frac{-\sqrt{3}}{3}\right|$$
$$\tan\alpha = \frac{\sqrt{3}}{3}$$
$$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = 30°$$

**PS4.**

$$\cos\alpha = \frac{-17}{\sqrt{338}}$$
$$\alpha = \cos^{-1}\left(\frac{-17}{\sqrt{338}}\right) \approx 157.6°$$

**PS5.** $\frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5}$

**PS6.** $\frac{28}{\sqrt{68}} = \frac{28}{2\sqrt{17}} = \frac{14}{\sqrt{17}} \cdot \frac{\sqrt{17}}{\sqrt{17}} = \frac{14\sqrt{17}}{17}$

# Section 7.3

**1.** $a = 5 - 1 = 4$
$b = 4 - 2 = 2$

A vector equivalent to $\mathbf{P_1P_2}$ is $\mathbf{v} = \langle 4,\ 2 \rangle$.

**2.** $a = 3 - 4 = -1$
$b = -2 - 2 = -4$

A vector equivalent to $\mathbf{P_1P_2}$ is $\mathbf{v} = \langle -1,\ -4 \rangle$.

**3.** $a = -3 - 2 = -5$
$b = 5 - 1 = 4$

A vector equivalent to $\mathbf{P_1P_2}$ is $\mathbf{v} = \langle -5,\ 4 \rangle$.

**4.** $a = 3 - (-1) = 4$
$b = 3 - 4 = -1$

A vector equivalent to $\mathbf{P_1P_2}$ is $\mathbf{v} = \langle 4,\ -1 \rangle$.

**5.** $a = 4 - (-3) = 7$
$b = -1 - 0 = -1$

A vector equivalent to $\mathbf{P_1P_2}$ is $\mathbf{v} = \langle 7,\ -1 \rangle$.

**6.** $a = 3 - 5 = -2$
$b = 1 - (-1) = 2$

A vector equivalent to $\mathbf{P_1P_2}$ is $\mathbf{v} = \langle -2,\ 2 \rangle$.

**7.** $a = -3 - 4 = -7$
$b = -3 - 2 = -5$

A vector equivalent to $\mathbf{P_1P_2}$ is $\mathbf{v} = \langle -7,\ -5 \rangle$.

**8.** $a = 0 - 0 = 0$
$b = 4 - (-3) = 7$

A vector equivalent to $\mathbf{P_1P_2}$ is $\mathbf{v} = \langle 0,\ 7 \rangle$.

**9.** $a = 2 - 2 = 0$
$b = 3 - (-5) = 8$

A vector equivalent to $\mathbf{P_1P_2}$ is $\mathbf{v} = \langle 0,\ 8 \rangle$.

**10.** $a = 3 - 3 = 0$
$b = 0 - (-2) = 2$

A vector equivalent to $\mathbf{P_1P_2}$ is $\mathbf{v} = \langle 0,\ 2 \rangle$.

**11.** $\|\mathbf{v}\| = \sqrt{(-3)^2 + 4^2}$ $\quad \alpha = \tan^{-1}\left|\frac{4}{-3}\right| = \tan^{-1}\frac{4}{3}$
$\|\mathbf{v}\| = \sqrt{9 + 16}$ $\quad \alpha \approx 53.1°$
$\|\mathbf{v}\| = 5$ $\quad \theta = 180° - \alpha$
$\theta \approx 180° - 53.1°$
$\theta \approx 126.9°$

$$\mathbf{u} = \left\langle \frac{-3}{5},\ \frac{4}{5} \right\rangle$$

A unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \left\langle -\frac{3}{5},\ \frac{4}{5} \right\rangle$.

**12.** $\|\mathbf{v}\| = \sqrt{6^2 + 10^2}$ $\quad \alpha = \tan^{-1}\left|\frac{10}{6}\right| = \tan^{-1}\frac{5}{3}$
$\|\mathbf{v}\| = \sqrt{36 + 100}$ $\quad \alpha \approx 59.0°$
$\|\mathbf{v}\| = 2\sqrt{34}$ $\quad \theta = 59.0°$
$\approx 11.7$

$$\mathbf{u} = \left\langle \frac{6}{2\sqrt{34}},\ \frac{10}{2\sqrt{34}} \right\rangle = \left\langle \frac{3\sqrt{34}}{34},\ \frac{5\sqrt{34}}{34} \right\rangle$$

A unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \left\langle \frac{3\sqrt{34}}{34},\ \frac{5\sqrt{34}}{34} \right\rangle$.

**13.** $\|\mathbf{v}\| = \sqrt{20^2 + (-40)^2}$ $\quad \alpha = \tan^{-1}\left|\frac{-40}{20}\right| = \tan^{-1} 2$
$\|\mathbf{v}\| = \sqrt{400 + 1600}$ $\quad \alpha \approx 63.4°$
$\|\mathbf{v}\| = \sqrt{2000} = 20\sqrt{5}$ $\quad \theta = 360° - \alpha$
$\approx 44.7$ $\quad \theta \approx 360° - 63.4°$
$\theta \approx 296.6°$

$$\mathbf{u} = \left\langle \frac{20}{20\sqrt{5}},\ \frac{-40}{20\sqrt{5}} \right\rangle = \left\langle \frac{\sqrt{5}}{5},\ \frac{-2\sqrt{5}}{5} \right\rangle$$

A unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \left\langle \frac{\sqrt{5}}{5},\ -\frac{2\sqrt{5}}{5} \right\rangle$.

**14.** $\|\mathbf{v}\| = \sqrt{(-50)^2 + 30^2}$ $\quad \alpha = \tan^{-1}\left|\frac{30}{-50}\right| = \tan^{-1}\frac{3}{5}$
$\|\mathbf{v}\| = \sqrt{2500 + 900}$ $\quad \alpha \approx 31.0°$
$\|\mathbf{v}\| = \sqrt{3400} = 10\sqrt{34}$ $\quad \theta = 180° - \alpha$
$\approx 58.3$ $\quad \theta \approx 180° - 31°$
$\theta \approx 149°$

$$\mathbf{u} = \left\langle \frac{-50}{10\sqrt{34}},\ \frac{30}{10\sqrt{34}} \right\rangle = \left\langle -\frac{5\sqrt{34}}{34},\ \frac{3\sqrt{34}}{34} \right\rangle$$

A unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \left\langle -\frac{5\sqrt{34}}{34},\ \frac{3\sqrt{34}}{34} \right\rangle$.

**15.** $\|\mathbf{v}\| = \sqrt{2^2 + (-4)^2}$ $\alpha = \tan^{-1}\left|\frac{-4}{2}\right| = \tan^{-1} 2$

$\|\mathbf{v}\| = \sqrt{4+16}$ $\alpha \approx 63.4°$

$\|\mathbf{v}\| = \sqrt{20} = 2\sqrt{5}$ $\theta = 360° - \alpha$

$\approx 4.5$ $\theta \approx 360° - 63.4°$

$\approx 296.6°$

$$\mathbf{u} = \left\langle \frac{2}{2\sqrt{5}}, \frac{-4}{2\sqrt{5}} \right\rangle = \left\langle \frac{\sqrt{5}}{5}, -\frac{2\sqrt{5}}{5} \right\rangle$$

A unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \left\langle \frac{\sqrt{5}}{5}, -\frac{2\sqrt{5}}{5} \right\rangle$.

**16.** $\|\mathbf{v}\| = \sqrt{(-5)^2 + 6^2}$ $\alpha = \tan^{-1}\left|\frac{6}{-5}\right| = \tan^{-1}\frac{6}{5}$

$\|\mathbf{v}\| = \sqrt{25+36}$ $\alpha \approx 50.2°$

$\|\mathbf{v}\| = \sqrt{61}$ $\theta = 180° - \alpha$

$\approx 7.8$ $\theta \approx 180° - 50.2°$

$\theta \approx 129.8°$

$$\mathbf{u} = \left\langle \frac{-5}{\sqrt{61}}, \frac{6}{\sqrt{61}} \right\rangle = \left\langle -\frac{5\sqrt{61}}{61}, \frac{6\sqrt{61}}{61} \right\rangle$$

A unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \left\langle -\frac{5\sqrt{61}}{61}, \frac{6\sqrt{61}}{61} \right\rangle$.

**17.** $\|\mathbf{v}\| = \sqrt{42^2 + (-18)^2}$ $\alpha = \tan^{-1}\left|\frac{-18}{42}\right| = \tan^{-1}\frac{3}{7}$

$\|\mathbf{v}\| = \sqrt{1764+324}$ $\alpha \approx 23.2°$

$\|\mathbf{v}\| = \sqrt{2088}$ $\theta = 360° - \alpha$

$= 6\sqrt{58}$ $\theta \approx 360° - 23.2°$

$\approx 45.7$ $\theta \approx 336.8°$

$$\mathbf{u} = \left\langle \frac{42}{6\sqrt{58}}, \frac{-18}{6\sqrt{58}} \right\rangle = \left\langle \frac{7\sqrt{58}}{58}, -\frac{3\sqrt{58}}{58} \right\rangle$$

A unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \left\langle \frac{7\sqrt{58}}{58}, -\frac{3\sqrt{58}}{58} \right\rangle$.

**18.** $\|\mathbf{v}\| = \sqrt{(-22)^2 + (-32)^2}$ $\alpha = \tan^{-1}\left|\frac{-32}{-22}\right| = \tan^{-1}\frac{16}{11}$

$\|\mathbf{v}\| = \sqrt{484+1024}$ $\alpha \approx 55.5°$

$\|\mathbf{v}\| = \sqrt{1508}$ $\theta = 180° - \alpha$

$= 2\sqrt{377}$ $\theta \approx 180° + 55.5°$

$\approx 38.8$ $\theta \approx 235.5°$

$$\mathbf{u} = \left\langle \frac{-22}{2\sqrt{377}}, \frac{-32}{2\sqrt{377}} \right\rangle = \left\langle -\frac{11\sqrt{377}}{377}, -\frac{16\sqrt{377}}{377} \right\rangle$$

A unit vector in the direction of $\mathbf{v}$ is

$\mathbf{u} = \left\langle -\frac{11\sqrt{377}}{377}, -\frac{16\sqrt{377}}{377} \right\rangle$.

**19.** $3\mathbf{u} = 3\langle -2, 4\rangle = \langle -6, 12\rangle$

**20.** $-4\mathbf{v} = -4\langle -3, -2\rangle = \langle 12, 8\rangle$

**21.** $2\mathbf{u} - \mathbf{v} = 2\langle -2, 4\rangle - \langle -3, -2\rangle$

$= \langle -4, 8\rangle - \langle -3, -2\rangle$

$= \langle -1, 10\rangle$

**22.** $4\mathbf{v} - 2\mathbf{u} = 4\langle -3, -2\rangle - 2\langle -2, 4\rangle$

$= \langle -12, -8\rangle - \langle -4, 8\rangle$

$= \langle -8, -16\rangle$

**23.** $\frac{2}{3}\mathbf{u} + \frac{1}{6}\mathbf{v} = \frac{2}{3}\langle -2, 4\rangle + \frac{1}{6}\langle -3, -2\rangle$

$= \left\langle -\frac{4}{3}, \frac{8}{3}\right\rangle + \left\langle -\frac{1}{2}, -\frac{1}{3}\right\rangle$

$= \left\langle -\frac{11}{6}, \frac{7}{3}\right\rangle$

**24.** $\frac{3}{4}\mathbf{u} - 2\mathbf{v} = \frac{3}{4}\langle -2, 4\rangle - 2\langle -3, -2\rangle$

$= \left\langle -\frac{2}{3}, 3\right\rangle - \langle -6, -4\rangle$

$= \left\langle \frac{9}{2}, 7\right\rangle$

**25.** $\|\mathbf{u}\| = \sqrt{(-2)^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$

**26.** $\mathbf{v} + 2\mathbf{u} = \langle -3, -2\rangle + 2\langle -2, 4\rangle$

$= \langle -3, -2\rangle + \langle -4, 8\rangle$

$= \langle -7, 6\rangle$

$\|\mathbf{v} + 2\mathbf{u}\| = \sqrt{(-7)^2 + 6^2} = \sqrt{49+36} = \sqrt{85}$

**27.** $3\mathbf{u} - 4\mathbf{v} = 3\langle -2, 4\rangle - 4\langle -3, -2\rangle$

$= \langle -6, 12\rangle - \langle -12, -8\rangle$

$= \langle 6, 20\rangle$

$\|3\mathbf{u} - 4\mathbf{v}\| = \sqrt{6^2 + 20^2} = \sqrt{436} = 2\sqrt{109}$

**28.** $-2\mathbf{u} = -2(3\mathbf{i} - 2\mathbf{j})$

$= -6\mathbf{i} + 4\mathbf{j}$

**29.** $4\mathbf{v} = 4(-2\mathbf{i} + 3\mathbf{j})$
$= -8\mathbf{i} + 12\mathbf{j}$

**30.** $3\mathbf{u} + 2\mathbf{v} = 3(3\mathbf{i} - 2\mathbf{j}) + 2(-2\mathbf{i} + 3\mathbf{j})$
$= (9\mathbf{i} - 6\mathbf{j}) + (-4\mathbf{i} + 6\mathbf{j})$
$= (9 - 4)\mathbf{i} + (-6 + 6)\mathbf{j}$
$= 5\mathbf{i} + 0\mathbf{j}$
$= 5\mathbf{i}$

**31.** $6\mathbf{u} + 2\mathbf{v} = 6(3\mathbf{i} - 2\mathbf{j}) + 2(-2\mathbf{i} + 3\mathbf{j})$
$= (18\mathbf{i} - 12\mathbf{j}) + (-4\mathbf{i} + 6\mathbf{j})$
$= (18 - 4)\mathbf{i} + (-12 + 6)\mathbf{j}$
$= 14\mathbf{i} - 6\mathbf{j}$

**32.** $\frac{1}{2}\mathbf{u} - \frac{3}{4}\mathbf{v} = \frac{1}{2}(3\mathbf{i} - 2\mathbf{j}) - \frac{3}{4}(-2\mathbf{i} + 3\mathbf{j})$
$= \left(\frac{3}{2}\mathbf{i} - \mathbf{j}\right) - \left(-\frac{3}{2}\mathbf{i} + \frac{9}{4}\mathbf{j}\right)$
$= \left(\frac{3}{2} + \frac{3}{2}\right)\mathbf{i} + \left(-1 - \frac{9}{4}\right)\mathbf{j}$
$= 3\mathbf{i} - \frac{13}{4}\mathbf{j}$

**33.** $\frac{2}{3}\mathbf{v} + \frac{3}{4}\mathbf{u} = \frac{2}{3}(-2\mathbf{i} + 3\mathbf{j}) + \frac{3}{4}(3\mathbf{i} - 2\mathbf{j})$
$= \left(-\frac{4}{3}\mathbf{i} + 2\mathbf{j}\right) + \left(\frac{9}{4}\mathbf{i} - \frac{3}{2}\mathbf{j}\right)$
$= \left(-\frac{4}{3} + \frac{9}{4}\right)\mathbf{i} + \left(2 - \frac{3}{2}\right)\mathbf{j}$
$= \frac{11}{12}\mathbf{i} + \frac{1}{2}\mathbf{j}$

**34.** $\|\mathbf{v}\| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$

**35.** $\mathbf{u} - 2\mathbf{v} = (3\mathbf{i} - 2\mathbf{j}) - 2(-2\mathbf{i} + 3\mathbf{j})$
$= (3\mathbf{i} - 2\mathbf{j}) - (-4\mathbf{i} + 6\mathbf{j})$
$= (3 + 4)\mathbf{i} + (-2 - 6)\mathbf{j}$
$= 7\mathbf{i} - 8\mathbf{j}$

$\|\mathbf{u} - 2\mathbf{v}\| = \sqrt{7^2 + (-8)^2} = \sqrt{113}$

**36.** $2\mathbf{v} + 3\mathbf{u} = 2(-2\mathbf{i} + 3\mathbf{j}) + 3(3\mathbf{i} - 2\mathbf{j})$
$= (-4\mathbf{i} + 6\mathbf{j}) + (9\mathbf{i} - 6\mathbf{j})$
$= (-4 + 9)\mathbf{i} + (6 - 6)\mathbf{j}$
$= 5\mathbf{i}$

$\|\mathbf{u} + 2\mathbf{v}\| = \sqrt{5^2 + 0^2} = 5$

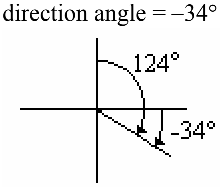
**37.** $a_1 = 5\cos 27° \approx 4.5$
$a_2 = 5\sin 27° \approx 2.3$
$\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j} \approx 4.5\mathbf{i} + 2.3\mathbf{j}$

**38.** $a_1 = 4\cos 127° \approx -2.4$
$a_2 = 4\sin 127° \approx 3.2$
$\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j} \approx -2.4\mathbf{i} + 3.2\mathbf{j}$

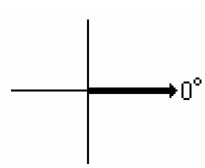
**39.** $a_1 = 4\cos\frac{\pi}{4} \approx 2.8$
$a_2 = 4\sin\frac{\pi}{4} \approx 2.8$
$\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j} \approx 2.8\mathbf{i} + 2.8\mathbf{j}$

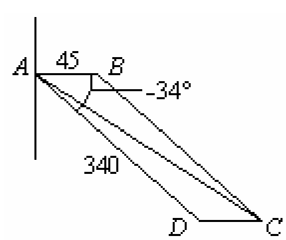
**40.** $a_1 = 2\cos\frac{8\pi}{7} \approx -1.8$
$a_2 = 2\sin\frac{8\pi}{7} \approx -0.9$
$\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j} \approx -1.8\mathbf{i} - 0.9\mathbf{j}$

**41.** heading = 124° ⇒
direction angle = –34°



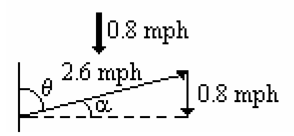

wind from the west ⇒
direction angle = 0°




$\mathbf{AB} = 45\mathbf{i}$
$\mathbf{AD} = 340\cos(-34°)\mathbf{i} + 340\sin(-34°)\mathbf{j}$
$\mathbf{AD} \approx 281.9\mathbf{i} - 190.1\mathbf{j}$
$\mathbf{AC} = \mathbf{AB} + \mathbf{AD}$
$\mathbf{AC} = 45\mathbf{i} + 281.9\mathbf{i} - 190.1\mathbf{j}$
$\mathbf{AC} \approx 327\mathbf{i} - 190\mathbf{j}$
$\|\mathbf{AC}\| = \sqrt{327^2 + (-190)^2}$
$\|\mathbf{AC}\| \approx 380$ mph

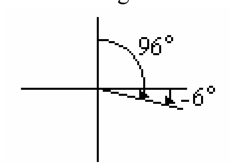
The ground speed of the plane is approximately 380 mph.

**42.** 

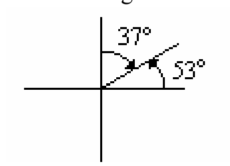

$$\alpha = \sin^{-1}\left|\frac{0.8}{2.6}\right| = \sin^{-1}\frac{0.8}{2.6}$$
$$\alpha \approx 17.9°$$

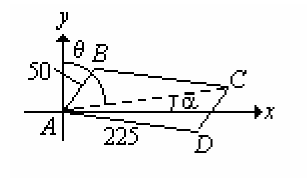
$$\text{heading} = \theta = 90° - \alpha$$
$$\theta \approx 90° - 17.9°$$
$$\theta \approx 72.1°$$

**43.** heading = 96° ⇒ direction angle = −6°




heading = 37° ⇒ direction angle = 53°



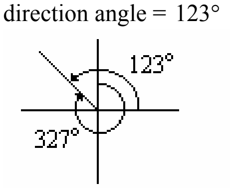




$$\mathbf{AB} = 50\cos 53°\mathbf{i} + 50\sin 53°\mathbf{j}$$
$$\approx 30.1\mathbf{i} + 39.9\mathbf{j}$$
$$\mathbf{AD} = 225\cos(-6°)\mathbf{i} + 225\sin(-6°)\mathbf{j}$$
$$\approx 223.8\mathbf{i} - 23.5\mathbf{j}$$
$$\mathbf{AC} = \mathbf{AB} + \mathbf{AD}$$
$$\approx 30.1\mathbf{i} + 39.9\mathbf{j} + 223.8\mathbf{i} - 23.5\mathbf{j}$$
$$\approx 253.9\mathbf{i} + 16.4\mathbf{j}$$
$$\|\mathbf{AC}\| = \sqrt{(253.9)^2 + (16.4)^2} \approx 250$$

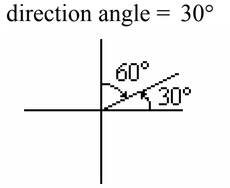
$$\alpha = \tan^{-1}\left|\frac{16.4}{253.9}\right|$$
$$= \tan^{-1}\frac{16.4}{253.9}$$
$$\alpha \approx 4°$$
$$\theta = 90° - \alpha$$
$$\theta \approx 90° - 4°$$
$$\theta \approx 86°$$

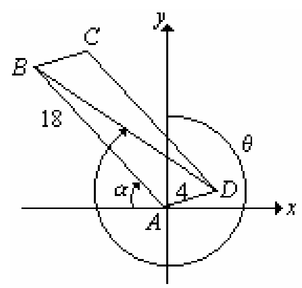
The ground speed of the plane is about 250 mph at a heading of approximately 86° .

**44.** heading = 327° ⇒ direction angle = 123°




heading = 60° ⇒ direction angle = 30°

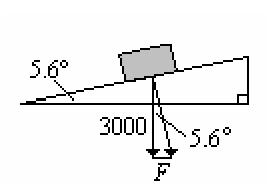


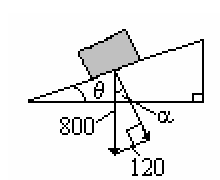

$$\mathbf{AB} = 18\cos 123°\mathbf{i} + 18\sin 123°\mathbf{j}$$
$$\mathbf{AB} \approx -9.8\mathbf{i} + 15.1\mathbf{j}$$
$$\mathbf{AD} = 4\cos 30°\mathbf{i} + 4\sin 30°\mathbf{j}$$
$$\mathbf{AD} = 3.5\mathbf{i} + 2\mathbf{j}$$
$$\mathbf{AC} = \mathbf{AB} + \mathbf{AD}$$
$$= -9.8\mathbf{i} + 15.1\mathbf{j} + 3.5\mathbf{i} + 2\mathbf{j}$$
$$= -6.3\mathbf{i} + 17.1\mathbf{j}$$
$$\|\mathbf{AC}\| = \sqrt{(-6.3)^2 + (17.1)^2} \approx 18$$

$$\alpha = \tan^{-1}\left|\frac{17.1}{-6.3}\right|$$
$$= \tan^{-1}\frac{17.1}{6.3}$$
$$\alpha \approx 70°$$
$$\theta = 270° + 70°$$
$$\theta = 340°$$

The course of the boat is about 18 mph at a heading of approximately 340°.

**45.** 


$$\sin 5.6° = \frac{F}{3000}$$
$$F = 3000\sin 5.6°$$
$$F \approx 293 \text{ lb}$$

**46.** 


$$\alpha = \theta$$
$$\sin\alpha = \frac{120}{800}$$
$$\alpha \approx 8.6°$$

**47.** **a.** $\sin 22.4° = \frac{\|\mathbf{F_1}\|}{345}$

$\|\mathbf{F_1}\| = 345\sin 22.4°$

$\|\mathbf{F_1}\| \approx 131$ lb

**b.** $\cos 22.4° = \frac{\|\mathbf{F_2}\|}{345}$

$\|\mathbf{F_2}\| = 345\cos 22.4°$

$\|\mathbf{F_2}\| \approx 319$ lb

**48.** **a.** $\sin 31.8° = \frac{\|\mathbf{F_1}\|}{345}$

$\|\mathbf{F_1}\| = 811\sin 31.8°$

$\|\mathbf{F_1}\| \approx 427$ lb

**b.** $\cos 31.8° = \frac{\|\mathbf{F_2}\|}{345}$

$\|\mathbf{F_2}\| = 811\cos 31.8°$

$\|\mathbf{F_2}\| \approx 689$ lb

**49.** $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (18\mathbf{i} + 13.1\mathbf{j}) + (-12.4\mathbf{i} + 3.8\mathbf{j}) + (-5.8\mathbf{i} - 16.9\mathbf{j})$
$= (18 - 12.4 - 5.8)\mathbf{i} + (13.1 + 3.8 - 16.9)\mathbf{j}$
$= 0$

The forces are in equilibrium.

**50.** $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (-4.6\mathbf{i} + 5.3\mathbf{j}) + (6.2\mathbf{i} + 4.9\mathbf{j}) + (-1.6\mathbf{i} - 10.2\mathbf{j})$
$= (-4.6 + 6.2 - 1.6)\mathbf{i} + (5.3 + 4.9 - 10.2)\mathbf{j}$
$= 0$

The forces are in equilibrium.

**51.** $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (155\mathbf{i} - 257\mathbf{j}) + (-124\mathbf{i} + 149\mathbf{j}) + (-31\mathbf{i} + 98\mathbf{j})$
$= (155 - 124 - 31)\mathbf{i} + (-257 + 149 + 98)\mathbf{j}$
$= 0\mathbf{i} - 10\mathbf{j}$

The forces are not in equilibrium. $\mathbf{F}_4 = 0\mathbf{i} + 10\mathbf{j}$

**52.** $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (23.5\mathbf{i} + 18.9\mathbf{j}) + (-18.7\mathbf{i} + 2.5\mathbf{j}) + (-5.6\mathbf{i} - 15.6\mathbf{j})$
$= (23.5 - 18.7 - 5.6)\mathbf{i} + (18.9 + 2.5 - 15.6)\mathbf{j}$
$= -0.8\mathbf{i} + 5.8\mathbf{j}$

The forces are not in equilibrium. $\mathbf{F}_4 = 0.8\mathbf{i} - 5.8\mathbf{j}$

**53.** $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (189.3\mathbf{i} + 235.7\mathbf{j}) + (45.8\mathbf{i} - 205.6\mathbf{j})$
$+ (-175.2\mathbf{i} - 37.7\mathbf{j}) + (-59.9\mathbf{i} + 7.6\mathbf{j})$
$= (189.3 + 45.8 - 175.2 - 59.9)\mathbf{i}$
$+ (235.7 - 205.6 - 37.7 + 7.6)\mathbf{j}$
$= 0$

The forces are in equilibrium.

**54.** $\mathbf{F}_1 = \left(\|\mathbf{F_1}\|\cos 144°\right)\mathbf{i} + \left(\|\mathbf{F_1}\|\sin 144°\right)\mathbf{j}$
$= (6223\cos 144°)\mathbf{i} + (6223\sin 144°)\mathbf{j}$
$\approx -5034\mathbf{i} + 3658\mathbf{j}$

$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0$
$\mathbf{F}_2 = -\mathbf{F}_1 - \mathbf{F}_3$
$= -(5034\mathbf{i} + 3658\mathbf{j}) - (0\mathbf{i} - 9450\mathbf{j})$
$= 5034\mathbf{i} + 5792\mathbf{j}$

$\|\mathbf{F_2}\| = \mathbf{F_2}\cos 49° + \mathbf{F_2}\sin 49°$
$\|\mathbf{F_2}\| = 5034\cos 49° + 5034\sin 49°$
$\approx 3300 + 4370$
$\approx 7670$ lbs

**55.** $\mathbf{v} \cdot \mathbf{w} = \langle 3, -2\rangle \cdot \langle 1, 3\rangle$
$= 3(1) + (-2)3$
$= 3 - 6$
$= -3$

**56.** $\mathbf{v} \cdot \mathbf{w} = \langle 2, 4\rangle \cdot \langle 0, 2\rangle$
$= 2(0) + (4)2$
$= 0 + 8$
$= 8$

**57.** $\mathbf{v} \cdot \mathbf{w} = \langle 4, 1\rangle \cdot \langle -1, 4\rangle$
$= 4(-1) + 1(4)$
$= -4 + 4$
$= 0$

**58.** $\mathbf{v} \cdot \mathbf{w} = \langle 2, -3\rangle \cdot \langle 3, 2\rangle$
$= 2(3) + (-3)2$
$= 6 - 6$
$= 0$

**59.** $\mathbf{v} \cdot \mathbf{w} = (\mathbf{i} + 2\mathbf{j}) \cdot (-\mathbf{i} + \mathbf{j})$
$= 1(-1) + 2(1)$
$= -1 + 2$
$= 1$

**60.** $\mathbf{v} \cdot \mathbf{w} = (5\mathbf{i} + 3\mathbf{j}) \cdot (4\mathbf{i} - 2\mathbf{j})$
$= 5(4) + 3(-2)$
$= 20 - 6$
$= 14$

**61.** $\mathbf{v}\cdot\mathbf{w} = (6\mathbf{i}-4\mathbf{j})\cdot(-2\mathbf{i}-3\mathbf{j})$
$= 6(-2)+(-4)(-3)$
$= -12+12$
$= 0$

**62.** $\mathbf{v}\cdot\mathbf{w} = (-4\mathbf{i}+2\mathbf{j})\cdot(-2\mathbf{i}-4\mathbf{j})$
$= (-4)(-2)+2(-4)$
$= 8-8$
$= 0$

**63.** $\cos\theta = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\ \|\mathbf{w}\|}$
$\cos\theta = \dfrac{\langle 2,\ -1\rangle\cdot\langle 3,\ 4\rangle}{\sqrt{2^2+(-1)^2}\sqrt{3^2+4^2}}$
$\cos\theta = \dfrac{2(3)+(-1)4}{\sqrt{5}\sqrt{25}}$
$\cos\theta = \dfrac{2}{5\sqrt{5}} \approx 0.1789$
$\theta \approx 79.7^\circ$

**64.** $\cos\theta = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\ \|\mathbf{w}\|}$
$\cos\theta = \dfrac{\langle 1,\ -5\rangle\cdot\langle -2,\ 3\rangle}{\sqrt{1^2+(-5)^2}\sqrt{(-2)^2+3^2}}$
$\cos\theta = \dfrac{1(-2)+(-5)(3)}{\sqrt{26}\sqrt{13}}$
$\cos\theta = \dfrac{-17}{\sqrt{26}\sqrt{13}} \approx -0.9247$
$\theta = 157.6°$

**65.** $\cos\theta = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\ \|\mathbf{w}\|}$
$\cos\theta = \dfrac{\langle 0,\ 3\rangle\cdot\langle 2,\ 2\rangle}{\sqrt{0^2+3^2}\sqrt{2^2+2^2}}$
$\cos\theta = \dfrac{0(2)+3(2)}{\sqrt{9}\sqrt{8}}$
$\cos\theta = \dfrac{6}{6\sqrt{2}} \approx 0.7071$
$\theta = 45^\circ$

**66.** $\cos\theta = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\ \|\mathbf{w}\|}$
$\cos\theta = \dfrac{\langle -1,\ 7\rangle\cdot\langle 3,\ -2\rangle}{\sqrt{(-1)^2+7^2}\sqrt{3^2+(-2)^2}}$
$\cos\theta = \dfrac{(-1)3+7(-2)}{\sqrt{50}\sqrt{13}}$
$\cos\theta = \dfrac{-17}{5\sqrt{2}\sqrt{13}} = -0.6668$
$\theta = 131.8^\circ$

**67.** $\cos\theta = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\ \|\mathbf{w}\|}$
$\cos\theta = \dfrac{(5\mathbf{i}-2\mathbf{j})\cdot(2\mathbf{i}+5\mathbf{j})}{\sqrt{5^2+(-2)^2}\sqrt{2^2+5^2}}$
$\cos\theta = \dfrac{5(2)+(-2)(5)}{\sqrt{29}\sqrt{29}}$
$\cos\theta = \dfrac{0}{\sqrt{29}\sqrt{29}} = 0$
$\theta = 90^\circ$

Thus, the vectors are orthogonal.

**68.** $\cos\theta = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\ \|\mathbf{w}\|}$
$\cos\theta = \dfrac{(8\mathbf{i}+\mathbf{j})\cdot(-\mathbf{i}+8\mathbf{j})}{\sqrt{8^2+1^2}\sqrt{(-1)^2+8^2}}$
$\cos\theta = \dfrac{8(-1)+(1)(8)}{\sqrt{65}\sqrt{65}}$
$\cos\theta = \dfrac{0}{\sqrt{65}\sqrt{65}} = 0$
$\theta = 90^\circ$

Thus, the vectors are orthogonal.

**69.** $\cos\theta = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\ \|\mathbf{w}\|}$
$\cos\theta = \dfrac{(5\mathbf{i}+2\mathbf{j})\cdot(-5\mathbf{i}-2\mathbf{j})}{\sqrt{5^2+2^2}\sqrt{(-5)^2+(-2)^2}}$
$\cos\theta = \dfrac{5(-5)+2(-2)}{\sqrt{29}\sqrt{29}}$
$\cos\theta = \dfrac{-29}{\sqrt{29}\sqrt{29}} = -1$
$\theta = 180^\circ$

**70.** $\cos\theta = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\ \|\mathbf{w}\|}$
$\cos\theta = \dfrac{(3\mathbf{i}-4\mathbf{j})\cdot(6\mathbf{i}-12\mathbf{j})}{\sqrt{3^2+(-4)^2}\sqrt{6^2+(-12)^2}}$
$\cos\theta = \dfrac{3(6)+(-4)(-12)}{\sqrt{25}\sqrt{180}}$
$\cos\theta = \dfrac{66}{5\sqrt{180}} = 0.9839$
$\theta = 10.3^\circ$

**71.** $\text{proj}_{\mathbf{w}}\mathbf{v} = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|}$
$\text{proj}_{\mathbf{w}}\mathbf{v} = \dfrac{\langle 6,\ 7\rangle\cdot\langle 3,\ 4\rangle}{\sqrt{3^2+4^2}} = \dfrac{18+28}{\sqrt{25}} = \dfrac{46}{5}$

**72.** $\text{proj}_{\mathbf{w}}\mathbf{v} = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|}$
$\text{proj}_{\mathbf{w}}\mathbf{v} = \dfrac{\langle -7,\ 5\rangle\cdot\langle -4,\ 1\rangle}{\sqrt{(-4)^2+1^2}} = \dfrac{28+5}{\sqrt{17}} = \dfrac{33}{\sqrt{17}} = \dfrac{33\sqrt{17}}{17} \approx 8.0$

**73.** $\text{proj}_{\mathbf{w}}\mathbf{v} = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|}$
$\text{proj}_{\mathbf{w}}\mathbf{v} = \dfrac{\langle -3,\ 4\rangle\cdot\langle 2,\ 5\rangle}{\sqrt{2^2+5^2}} = \dfrac{-6+20}{\sqrt{29}} = \dfrac{14}{\sqrt{29}} = \dfrac{14\sqrt{29}}{29} \approx 2.6$

**74.** $\text{proj}_{\mathbf{w}}\mathbf{v} = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|}$

$\text{proj}_{\mathbf{w}}\mathbf{v} = \dfrac{\langle 2,\ 4\rangle\cdot\langle -1,\ 5\rangle}{\sqrt{(-1)^2+5^2}} = \dfrac{-2+20}{\sqrt{26}} = \dfrac{18}{\sqrt{26}} = \dfrac{9\sqrt{26}}{13} \approx 3.5$

**75.** $\text{proj}_{\mathbf{w}}\mathbf{v} = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|}$

$\text{proj}_{\mathbf{w}}\mathbf{v} = \dfrac{(2\mathbf{i}+\mathbf{j})\cdot(6\mathbf{i}+3\mathbf{j})}{\sqrt{6^2+3^2}} = \dfrac{12+3}{\sqrt{45}} = \dfrac{5}{\sqrt{5}} = \sqrt{5} \approx 2.2$

**76.** $\text{proj}_{\mathbf{w}}\mathbf{v} = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|}$

$\text{proj}_{\mathbf{w}}\mathbf{v} = \dfrac{(5\mathbf{i}+2\mathbf{j})\cdot(-5\mathbf{i}+-2\mathbf{j})}{\sqrt{(-5)^2+(-2)^2}} = \dfrac{-25-4}{\sqrt{29}} = \dfrac{-29}{\sqrt{29}}$

$= -\sqrt{29} \approx -5.4$

**77.** $\text{proj}_{\mathbf{w}}\mathbf{v} = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|}$

$\text{proj}_{\mathbf{w}}\mathbf{v} = \dfrac{(3\mathbf{i}-4\mathbf{j})\cdot(3\mathbf{i}-4\mathbf{j})}{\sqrt{(-6)^2+12^2}} = \dfrac{-18-48}{\sqrt{180}} = -\dfrac{11}{\sqrt{5}}$

$= -\dfrac{11\sqrt{5}}{5} \approx -4.9$

**78.** $\text{proj}_{\mathbf{w}}\mathbf{v} = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|}$

$\text{proj}_{\mathbf{w}}\mathbf{v} = \dfrac{(2\mathbf{i}+2\mathbf{j})\cdot(-4\mathbf{i}-2\mathbf{j})}{\sqrt{(-4)^2+(-2)^2}} = \dfrac{-8-4}{\sqrt{20}} = \dfrac{-6}{\sqrt{5}}$

$= -\dfrac{6\sqrt{5}}{5} \approx -2.7$

**79.** $W = \mathbf{F}\cdot\mathbf{s}$

$W = \|\mathbf{F}\|\ \|\mathbf{s}\|\cos\alpha$

$W = (75)(15)(\cos 32°)$

$W \approx 954$ foot-pounds

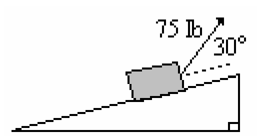
**80.** $W = \mathbf{F}\cdot\mathbf{s}$

$W = \|\mathbf{F}\|\ \|\mathbf{s}\|\cos\alpha$

$W = (100)(25)(\cos 42°)$

$W \approx 1858$ foot-pounds

**81.**



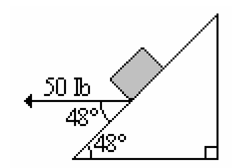

$W = \mathbf{F}\cdot\mathbf{s}$

$W = \|\mathbf{F}\|\ \|\mathbf{s}\|\cos\alpha$

$W = (75)(12)(\cos 30°)$

$W \approx 779$ foot-pounds

**82.**




$W = \mathbf{F}\cdot\mathbf{s}$

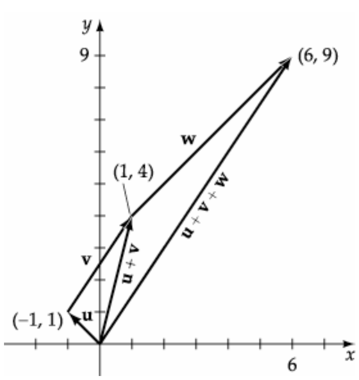
$W = \|\mathbf{F}\|\ \|\mathbf{s}\|\cos\alpha$

$W = (50)(6)(\cos 48°)$

$W \approx 201$ foot-pounds

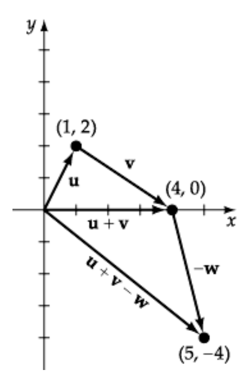
## Connecting Concepts

**83.**



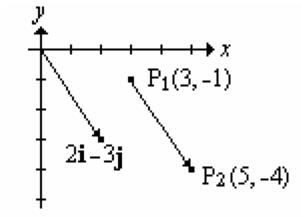

Thus, the sum is $\langle 6,\ 9\rangle$.

**84.**



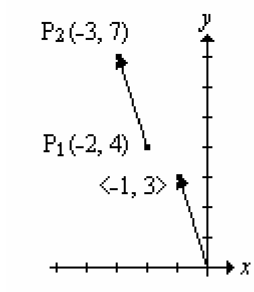

Thus, $\mathbf{u}+\mathbf{v}-\mathbf{w} = \langle 5,\ -4\rangle$.

**85.**



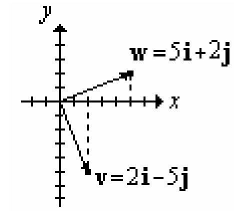

The vector from $P_1(3, -1)$ to $P_2(5, -4)$ is equivalent to $2\mathbf{i} - 3\mathbf{j}$.

**86.**




The vector from $P_1(-2, 4)$ to $P_2(-3, 7)$ is equivalent to $\langle -1, 3 \rangle$.

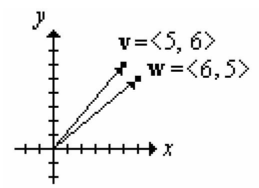
**87.**




$$\begin{aligned}\mathbf{v} \cdot \mathbf{w} &= (2\mathbf{i} - 5\mathbf{j}) \cdot (5\mathbf{i} + 2\mathbf{j}) \\ &= 10 - 10 \\ &= 0\end{aligned}$$

The two vectors are perpendicular.

**88.**




$$\begin{aligned}\mathbf{v} \cdot \mathbf{w} &= \langle 5,\ 6 \rangle \cdot \langle 6,\ 5 \rangle \\ &= 30 + 30 \\ &= 60 \neq 0\end{aligned}$$

The vectors are not perpendicular.

**89.** $\mathbf{v} = \langle -2,\ 7 \rangle$

$$\begin{aligned}\langle -2,\ 7 \rangle \cdot \langle a,\ b \rangle &= 0 \\ -2a + 7b &= 0 \\ a &= \frac{7}{2}b \\ \text{Let } b &= 2 \\ a &= 7\end{aligned}$$

Thus, $\mathbf{u} = \langle 7,\ 2 \rangle$ is one example.

**90.** $\mathbf{w} = 4\mathbf{i} + \mathbf{j}$

$$\begin{aligned}\langle 4,\ 1 \rangle \cdot \langle a,\ b \rangle &= 0 \\ 4a + b &= 0 \\ a &= -\frac{1}{4}b \\ \text{Let } b &= 4 \\ a &= -1\end{aligned}$$

Thus, $\mathbf{u} = \langle -1,\ 4 \rangle$ is one example.

**91.** Let $\mathbf{u} = c\mathbf{i} + b\mathbf{j}$, $\mathbf{v} = c\mathbf{i} + d\mathbf{j}$, and $\mathbf{w} = e\mathbf{i} + f\mathbf{j}$.

$$\begin{aligned}(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w} &= \left[(a\mathbf{i} + b\mathbf{j}) \cdot (c\mathbf{i} + d\mathbf{j})\right] \cdot (e\mathbf{i} + f\mathbf{j}). \\ &= (ac + bd) \cdot (e\mathbf{i} + f\mathbf{j})\end{aligned}$$

$ac + bd$ is a scalar quantity. The product of a scalar and a vector is not defined. Therefore, no, $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$ does not equal $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$.

**92.** Let $\mathbf{v} = \langle a,\ b \rangle$ and $\mathbf{w} = \langle c,\ d \rangle$.

$$\mathbf{v} \cdot \mathbf{w} = \langle a,\ b \rangle \cdot \langle c,\ d \rangle = ac + bd$$
$$\mathbf{w} \cdot \mathbf{v} = \langle c,\ d \rangle \cdot \langle a,\ b \rangle = ca + db = ac + bd$$

Therefore, $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$

**93.** Let $\mathbf{v} = \langle a, b \rangle$ and $\mathbf{w} = \langle d, e \rangle$

$c\mathbf{v} = \langle ca, ab \rangle$

$$c(\mathbf{v} \cdot \mathbf{w}) = c\langle a, b \rangle \cdot \langle d, e \rangle = c(ad + be) = cad + cbe$$
$$(c\mathbf{v} \cdot \mathbf{w}) = \langle ca, ab \rangle \cdot \langle d, e \rangle = cad + cbe$$

Therefore, $c(\mathbf{v} \cdot \mathbf{w}) = (c\mathbf{v}) \cdot \mathbf{w}$.

**94.** Let $\theta$ be the angle between vectors **v** and **w**.

$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\ \|\mathbf{w}\| \cos\theta$

$\mathbf{v} \cdot \mathbf{w}$ is positive if $\cos\theta$ is positive.

$\cos\theta$ is positive when $0° < \theta < 90°$.

This is an acute angle.

$\mathbf{v} \cdot \mathbf{w}$ is negative if $\cos\theta$ is negative.

$\cos\theta$ is negative when $90° < \theta < 180°$.

This is an obtuse angle.

**95.** Neither. If the force and the distance are the same, the work will be the same.

## Prepare for Section 7.4

**PS1.** $(1 + i)(2 + i) = 2 + 3i + i^2 = 1 + 3i$

**PS2.** $\dfrac{2+i}{3-i} \cdot \dfrac{3+i}{3+i} = \dfrac{6+5i+i^2}{9-i^2} = \dfrac{5+5i}{10} = \dfrac{1}{2} + \dfrac{1}{2}i$

**PS3.** $2 - 3i$

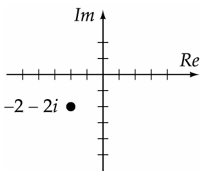
**PS4.** $3 + 5i$

**PS5.** $x = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \dfrac{-1 \pm \sqrt{-3}}{2} = -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$

**PS6.**
$$\begin{aligned}x^2 + 9 &= 0 \\ x^2 &= -9 \\ x &= \pm\ 3i\end{aligned}$$

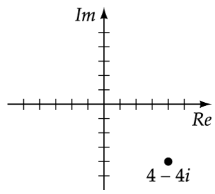
# Section 7.4

**1.**



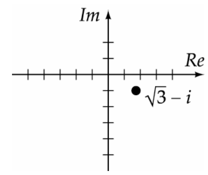

$|z|=\sqrt{(-2)^2+(-2)^2}$
$=\sqrt{8}=2\sqrt{2}$

**2.**



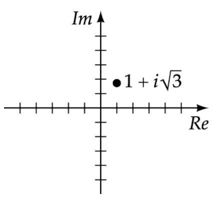

$|z|=\sqrt{4^2+(-4)^2}$
$=\sqrt{32}$
$=4\sqrt{2}$

**3.**



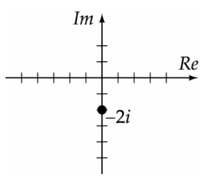

$|z|=\sqrt{(\sqrt{3})^2+(-1)^2}$
$=\sqrt{3+1}=\sqrt{4}$
$=2$

**4.**



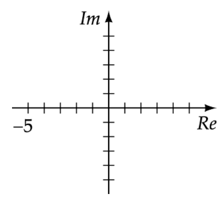

$|z|=\sqrt{1^2+(\sqrt{3})^2}=2$

**5.**



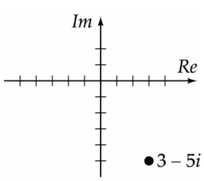

$|z|=\sqrt{0^2+(-2)^2}=2$

**6.**



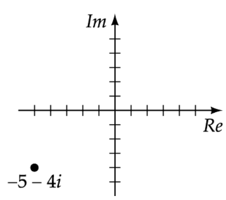

$|z|=\sqrt{(-5)^2+0^2}=5$

**7.**




$|z|=\sqrt{3^2+(-5)^2}$
$=\sqrt{34}$

**8.**




$|z|=\sqrt{(-5)^2+(-4)^2}$
$=\sqrt{41}$

**9.** $r=\sqrt{1^2+(-1)^2}$
$r=\sqrt{2}$
$\alpha=\tan^{-1}\left|\frac{-1}{1}\right|$
$=\tan^{-1}1$
$=45°$
$\theta=360°-45°=315°$
$z=\sqrt{2}\text{ cis }315°$

**10.** $r=\sqrt{(-4)^2+(-4)^2}$
$r=\sqrt{32}$
$r=4\sqrt{2}$
$\alpha=\tan^{-1}\left|\frac{-4}{-4}\right|$
$=\tan^{-1}1=45°$
$\theta=180°+45°=225°$
$z=4\sqrt{2}\text{ cis }225°$

**11.** $r=\sqrt{(\sqrt{3})^2+(-1)^2}$
$r=2$
$\alpha=\tan^{-1}\left|\frac{-1}{\sqrt{3}}\right|$
$=\tan^{-1}\frac{1}{\sqrt{3}}=30°$
$\alpha=360°-30°=330°$
$z=2\text{ cis }330°$

**12.** $r=\sqrt{1^2+(\sqrt{3})^2}$
$r=2$
$\alpha=\tan^{-1}\left|\frac{\sqrt{3}}{1}\right|$
$=\tan^{-1}\sqrt{3}=60°$
$\theta=60°$
$z=2\text{ cis }60°$

**13.** $r=\sqrt{0^2+3^2}$
$r=3$
$\theta=90°$
$z=3\text{ cis }90°$

**14.** $r=\sqrt{0^2+(-2)^2}$
$r=2$
$\theta=270°$
$z=3\text{ cis }270°$

**15.** $r=\sqrt{(-5)^2+0^2}$
$r=5$
$\theta=180°$
$z=5\text{ cis }180°$

**16.** $r = \sqrt{3^2 + 0^2}$
$r = 3$
$\theta = 0^\circ$
$z = 3 \text{ cis } 0^\circ$

**17.** $r = \sqrt{(-8)^2 + (8\sqrt{3})^2}$
$r = 16$
$\alpha = \tan^{-1}\left|\frac{8\sqrt{3}}{-8}\right|$
$= \tan^{-1} \sqrt{3} = 60^\circ$
$\theta = 180^\circ - 60^\circ = 120^\circ$
$z = 16 \text{ cis } 120^\circ$

**18.** $r = \sqrt{(-2\sqrt{2})^2 + (2\sqrt{2})^2}$
$r = 4$
$\alpha = \tan^{-1}\left|\frac{2\sqrt{2}}{-2\sqrt{2}}\right|$
$= \tan^{-1} 1 = 45^\circ$
$\theta = 180^\circ - 45^\circ = 135^\circ$
$z = 4 \text{ cis } 135^\circ$

**19.** $r = \sqrt{(-2)^2 + (-2\sqrt{3})^2}$
$r = 4$
$\alpha = \tan^{-1}\left|\frac{-2\sqrt{3}}{-2}\right|$
$= \tan^{-1} \sqrt{3} = 60^\circ$
$\theta = 180^\circ + 60^\circ = 240^\circ$
$z = 4 \text{ cis } 240^\circ$

**20.** $r = \sqrt{(\sqrt{2})^2 + (-\sqrt{2})^2}$
$r = 2$
$\alpha = \tan^{-1}\left|\frac{-\sqrt{2}}{\sqrt{2}}\right|$
$= \tan^{-1} 1 = 45^\circ$
$\theta = 360^\circ - 45^\circ = 315^\circ$
$z = 2 \text{ cis } 315^\circ$

**21.** $z = 2(\cos 45^\circ + i \sin 45^\circ)$
$z = 2\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)$
$z = \sqrt{2} + i\sqrt{2}$

**22.** $z = 3(\cos 240^\circ + i \sin 240^\circ)$
$z = 3\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$
$z = -\frac{3}{2} - \frac{3\sqrt{3}}{2} i$

**23.** $z = \cos 315^\circ + i \sin 315^\circ$
$z = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$

**24.** $z = 5(\cos 120^\circ + i \sin 120^\circ)$
$z = 5\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$
$z = -\frac{5}{2} + \frac{5\sqrt{3}}{2} i$

**25.** $z = 6 \text{ cis } 135^\circ$
$z = 6(\cos 135^\circ + i \sin 135^\circ)$
$z = 6\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)$
$z = -3\sqrt{2} + 3i\sqrt{2}$

**26.** $z = \text{cis } 315°$
$z = \cos 315° + i \sin 315°$
$z = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$

**27.** $z = 8 \text{ cis } 0°$
$z = 8(\cos 0^\circ + i \sin 0^\circ)$
$z = 8(1 + 0i)$
$z = 8$

**28.** $z = 5 \text{ cis } 90°$
$z = 5(\cos 90° + i \sin 90°)$
$z = 5(0 + i)$
$z = 5i$

**29.** $z = 2\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)$
$z = 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2} i\right)$
$z = -\sqrt{3} + i$

**30.** $z = 4\left(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}\right)$
$z = 4\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)$
$z = 2 - 2i\sqrt{3}$

**31.** $z = 3\left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}\right)$
$z = 3(0 - i)$
$z = -3i$

**32.** $z = 5(\cos \pi + i \sin \pi)$
$z = 5(-1 + 0i)$
$z = -5$

**33.** $z = 8 \text{ cis } \frac{3\pi}{4}$
$= 8\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right)$
$z = 8\left(-\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right)$
$z = -4\sqrt{2} + 4i\sqrt{2}$

**34.** $z = 9 \text{ cis } \frac{4\pi}{3}$
$= 9\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$
$z = 9\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$
$z = -\frac{9}{2} - \frac{9\sqrt{3}}{2}i$

**35.** $z = 9 \text{ cis } \frac{11\pi}{6}$
$z = 9\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$
$z = 9\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$
$z = \frac{9\sqrt{3}}{2} - \frac{9}{2}i$

**36.** $z = \text{cis } \frac{3\pi}{2}$
$z = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$
$z = 0 - i$
$z = -i$

**37.** $z = 2 \text{ cis } 2$
$z = 2(\cos 2 + i \sin 2)$
$z \approx 2(-0.4161 + 0.9093i)$
$z \approx -0.832 + 1.819i$

**38.** $z = 5 \text{ cis } 4$
$z = 5(\cos 4 + i \sin 4)$
$z \approx 5(-0.6536 - 0.7568i)$
$z \approx -3.268 - 3.784i$

**39.** $z_1z_2 = 2 \text{ cis } 30° \cdot 3 \text{ cis } 225°$
$z_1z_2 = 6 \text{ cis}(30° + 225°)$
$z_1z_2 = 6 \text{ cis } 255°$

**40.** $z_1z_2 = 4 \text{ cis } 120° \cdot 6 \text{ cis } 315°$
$z_1z_2 = 24 \text{ cis}(120° + 315°)$
$z_1z_2 = 24 \text{ cis } 435°$
$z_1z_2 = 24 \text{ cis } 75°$

**41.** $z_1z_2 = 3(\cos 122° + i \sin 122°) \cdot 4(\cos 213° + i \sin 213°)$
$z_1z_2 = 12[\cos(122° + 213°) + i \sin(122° + 213°)]$
$z_1z_2 = 12(\cos 335° + i \sin 335°)$
$z_1z_2 = 12 \text{ cis } 335°$

**42.** $z_1z_2 = 8(\cos 88° + i \sin 88°) \cdot 12(\cos 112° + i \sin 112°)$
$= 96[\cos(88° + 112°) + i \sin(88° + 112°)]$
$= 96[\cos 200° + i \sin 200°]$
$= 96 \text{ cis } 200°$

**43.** $z_1z_2 = 5\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) \cdot 2\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)$
$z_1z_2 = 10\left[\cos\left(\frac{2\pi}{3} + \frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{3} + \frac{2\pi}{5}\right)\right]$
$z_1z_2 = 10\left(\cos\frac{16\pi}{15} + i\sin\frac{16\pi}{15}\right)$
$z_1z_2 = 10 \text{ cis } \frac{16\pi}{15}$

**44.** $z_1z_2 = 5 \text{ cis } \frac{11\pi}{12} \cdot 3 \text{ cis } \frac{4\pi}{3}$
$z_1z_2 = 15 \text{ cis } \left(\frac{11\pi}{4} + \frac{4\pi}{3}\right)$
$z_1z_2 = 15 \text{ cis } \frac{49\pi}{12}$
$z_1z_2 = 15 \text{ cis } \frac{\pi}{12}$

**45.** $z_1z_2 = 4 \text{ cis } 2.4 \cdot 6 \text{ cis } 4.1$
$z_1z_2 = 24 \text{ cis } (2.4 + 4.1)$
$z_1z_2 = 24 \text{ cis } 6.5$

**46.** $z_1z_2 = 7 \text{ cis } 0.88 \cdot 5 \text{ cis } 1.32$
$z_1z_2 = 35 \text{ cis } (0.88 + 1.32)$
$z_1z_2 = 35 \text{ cis } 2.2$

**47.** $\frac{z_1}{z_2} = \frac{32 \text{ cis } 30°}{4 \text{ cis } 150°}$
$\frac{z_1}{z_2} = 8 \text{ cis}(30° - 150°)$
$\frac{z_1}{z_2} = 8 \text{ cis}(-120°)$
$\frac{z_1}{z_2} = 8(\cos 120° - i \sin 120°)$
$\frac{z_1}{z_2} = 8\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) = -4 - 4i\sqrt{3}$

**48.** $\frac{z_1}{z_2} = \frac{15 \text{ cis } 240°}{3 \text{ cis } 135°}$
$\frac{z_1}{z_2} = 5 \text{ cis } (240° - 135°)$
$\frac{z_1}{z_2} = 5 \text{ cis } 105°$
$\frac{z_1}{z_2} = 5(\cos 105° + i \sin 105°)$
$\frac{z_1}{z_2} \approx 5(-0.2588 + 0.9659i)$
$\frac{z_1}{z_2} \approx -1.294 + 4.830i$

**49.** $\frac{z_1}{z_2} = \frac{27(\cos 315^\circ + i \sin 315^\circ)}{9(\cos 225^\circ + i \sin 225^\circ)}$

$\frac{z_1}{z_2} = 3\,[\cos(315^\circ - 225^\circ) + i \sin(315^\circ - 225^\circ)]$

$\frac{z_1}{z_2} = 3(\cos 90^\circ + i \sin 90^\circ) = 3(0 + i) = 3i$

**50.** $\frac{z_1}{z_2} = \frac{9(\cos 25^\circ + i \sin 25^\circ)}{3(\cos 175^\circ + i \sin 175^\circ)}$

$\frac{z_1}{z_2} = 3\,[\cos(25^\circ - 175^\circ) + i \sin(25^\circ - 175^\circ)]$

$\frac{z_1}{z_2} = 3(\cos 150^\circ - i \sin 150^\circ)$

$\frac{z_1}{z_2} = 3\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$

**51.** $\frac{z_1}{z_2} = \frac{12\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)}{4\left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right)}$

$\frac{z_1}{z_2} = 3\left[\cos\left(\frac{2\pi}{3} - \frac{11\pi}{6}\right) + i \sin\left(\frac{2\pi}{3} - \frac{11\pi}{6}\right)\right]$

$\frac{z_1}{z_2} = 3\left(\cos \frac{7\pi}{6} - i \sin \frac{7\pi}{6}\right)$

$\frac{z_1}{z_2} = 3\left[-\frac{\sqrt{3}}{2} - \left(-\frac{1}{2}i\right)\right]$

$\frac{z_1}{z_2} = -\frac{3\sqrt{3}}{2} + \frac{3}{2}i$

**52.** $\frac{z_1}{z_2} = \frac{10(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})}{5(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})}$

$\frac{z_1}{z_2} = 2\left[\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\right]$

$\frac{z_1}{z_2} = 2\left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12}\right)$

$\frac{z_1}{z_2} \approx 2(0.9659 + 0.2588i)$

$\frac{z_1}{z_2} \approx 1.932 + 0.518i$

**53.** $\frac{z_1}{z_2} = \frac{25 \operatorname{cis} 3.5}{5 \operatorname{cis} 1.5}$

$\frac{z_1}{z_2} = 5 \operatorname{cis} (3.5 - 1.5)$

$\frac{z_1}{z_2} = 5 \operatorname{cis} 2$

$\frac{z_1}{z_2} = 5\,(\cos 2 + i \sin 2)$

$\frac{z_1}{z_2} \approx 5\,(-0.4161 + 0.9093i)$

$\frac{z_1}{z_2} \approx -2.081 + 4.546i$

**54.** $\frac{z_1}{z_2} = \frac{18 \operatorname{cis} 0.56}{6 \operatorname{cis} 1.22}$

$\frac{z_1}{z_2} = 3 \operatorname{cis} (0.56 - 1.22)$

$\frac{z_1}{z_2} = 3 \operatorname{cis} (-0.66)$

$\frac{z_1}{z_2} = 3\,(\cos 0.66 - i \sin 0.66)$

$\frac{z_1}{z_2} \approx 3\,(0.7900 - 0.6131i)$

$\frac{z_1}{z_2} \approx 2.370 - 1.839i$

**55.** $z_1 = 1 - i\sqrt{3}$

$r_1 = \sqrt{1^2 + (\sqrt{3})^2}$ $\quad \alpha = \tan^{-1}\left|\frac{-\sqrt{3}}{1}\right| = 60^\circ$

$r_1 = 2$ $\quad \theta_1 = 300^\circ$

$z_1 = 2(\cos 300^\circ + i \sin 300^\circ)$

$z_2 = 1 + i$

$r_2 = \sqrt{1^2 + 1^2}$ $\quad \alpha = \tan^{-1}\left|\frac{1}{1}\right| = 45^\circ$

$r_2 = \sqrt{2}$ $\quad \theta_2 = 45^\circ$

$z_2 = \sqrt{2}(\cos 45^\circ + i \sin 45^\circ)$

$z_1 z_2 = 2(\cos 300^\circ + i \sin 300^\circ) \cdot \sqrt{2}(\cos 45^\circ + i \sin 45^\circ)$

$z_1 z_2 = 2\sqrt{2}[\cos(300^\circ + 45^\circ) + i \sin(300^\circ + 45^\circ)]$

$z_1 z_2 = 2\sqrt{2}(\cos 345^\circ + i \sin 345^\circ)$

$z_1 z_2 \approx 2.732 - 0.732i$

**56.** $z_1 = \sqrt{3} - i$

$r_1 = \sqrt{(\sqrt{3})^2 + (-1)^2}$ $\quad \alpha = \tan^{-1}\left|\frac{-1}{\sqrt{3}}\right| = 30°$

$r_1 = 2$ $\quad \theta_1 = 330°$

$z_1 = 2(\cos 330° + i \sin 330°)$

$z_2 = 1 + i\sqrt{3}$

$r_2 = \sqrt{1^2 + (\sqrt{3})^2}$ $\quad \alpha = \tan^{-1}\left|\frac{\sqrt{3}}{1}\right| = 60°$

$r_2 = 2$ $\quad \theta_2 = 60°$

$z_2 = 2(\cos 60° + i \sin 60°)$

$z_1z_2 = 2(\cos 330° + i \sin 330°) \cdot 2(\cos 60° + i \sin 60°)$

$z_1z_2 = 4[\cos(330° + 60°) + i \sin(330° + 60°)]$

$z_1z_2 = 4(\cos 390° + i \sin 390°)$

$z_1z_2 = 4\left[\frac{\sqrt{3}}{2} + \frac{i}{2}\right] = 2\sqrt{3} + 2i$

**57.** $z_1 = 3 - 3i$

$r_1 = \sqrt{3^2 + (-3)^2}$ $\quad \alpha = \tan^{-1}\left|\frac{-3}{3}\right| = 45°$

$r_1 = 3\sqrt{2}$ $\quad \theta_1 = 315°$

$z_1 = 3\sqrt{2}(\cos 315° + i \sin 315°)$

$z_2 = 1 + i$

$r_2 = \sqrt{1^2 + 1^2}$ $\quad \alpha = \tan^{-1}\left|\frac{1}{1}\right| = 45°$

$r_2 = \sqrt{2}$ $\quad \theta_2 = 45°$

$z_2 = \sqrt{2}(\cos 45° + i \sin 45°)$

$z_1z_2 = 3\sqrt{2}(\cos 315° + i \sin 315°) \cdot \sqrt{2}(\cos 45° + i \sin 45°)$

$z_1z_2 = 6[\cos(315° + 45°) + i \sin(315° + 45°)]$

$z_1z_2 = 6(\cos 360° + i \sin 360°)$

$z_1z_2 = 6 + 0i$

$z_1z_2 = 6$

**58.** $z_1 = 2 + 2i$

$r_1 = \sqrt{2^2 + 2^2}$ $\quad \alpha = \tan^{-1}\left|\frac{2}{2}\right| = 45°$

$r_1 = 2\sqrt{2}$ $\quad \theta_1 = 45°$

$z_1 = 2\sqrt{2}(\cos 45° + i \sin 45°)$

$z_2 = \sqrt{3} - i$

$r_2 = \sqrt{(\sqrt{3})^2 + (-1)^2}$ $\quad \alpha = \tan^{-1}\left|\frac{-1}{\sqrt{3}}\right| = 30°$

$r_2 = 2$ $\quad \theta_2 = 330°$

$z_2 = 2(\cos 330° + i \sin 330°)$

$z_1z_2 = 2\sqrt{2}(\cos 45° + i \sin 45°) \cdot 2(\cos 330° + i \sin 330°)$

$z_1z_2 = 4\sqrt{2}[\cos(45° + 330°) + i \sin(45° + 330°)]$

$z_1z_2 = 4\sqrt{2}(\cos 375° + i \sin 375°)$

$z_1z_2 \approx 5.4641 + 1.4641i$

**59.** $z_1 = 1 + i\sqrt{3}$

$r_1 = \sqrt{1^2 + (\sqrt{3})^2}$ $\quad \alpha_1 = \tan^{-1}\left|\frac{\sqrt{3}}{1}\right| = 60°$

$r_1 = 2$ $\quad \theta_1 = 60°$

$z_1 = 2(\cos 60° + i \sin 60°)$

$z_2 = 1 - i\sqrt{3}$

$r_2 = \sqrt{1^2 + (\sqrt{3})^2}$ $\quad \alpha = \tan^{-1}\left|\frac{-\sqrt{3}}{1}\right| = 60°$

$r_2 = 2$ $\quad \theta_2 = 300°$

$z_2 = 2(\cos 300° + i \sin 300°)$

$$\frac{z_1}{z_2} = \frac{2(\cos 60° + i \sin 60°)}{2(\cos 300° + i \sin 300°)}$$

$$\frac{z_1}{z_2} = \cos(60° - 300°) + i \sin(60° - 300°)$$

$$\frac{z_1}{z_2} = \cos 240° - i \sin 240° = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

**60.** $z_1 = 1 + i$

$r_1 = \sqrt{1^2 + 1^2}$ $\quad \alpha_1 = \tan^{-1}\left|\frac{1}{1}\right| = 45^\circ$

$r_1 = \sqrt{2}$ $\quad \theta_1 = 45^\circ$

$z_1 = \sqrt{2}(\cos 45^\circ + i \sin 45^\circ)$

$z_2 = 1 - i$

$r_2 = \sqrt{1^2 + (-1)^2}$ $\quad \alpha_2 = \tan^{-1}\left|\frac{-1}{1}\right| = 45^\circ$

$r_2 = \sqrt{2}$ $\quad \theta_2 = 315^\circ$

$z_2 = \sqrt{2}(\cos 315^\circ + i \sin 315^\circ)$

$$\frac{z_1}{z_2} = \frac{\sqrt{2}(\cos 45^\circ + i \sin 45^\circ)}{\sqrt{2}(\cos 315^\circ + i \sin 315^\circ)}$$

$$\frac{z_1}{z_2} = \cos(45^\circ - 315^\circ) + i \sin(45^\circ - 315^\circ)$$

$$\frac{z_1}{z_2} = \cos 270^\circ - i \sin 270^\circ = 0 - i(-1) = 0 + 1i = i$$

**61.** $z_1 = \sqrt{2} - i\sqrt{2}$

$r_1 = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2}$ $\quad \alpha_1 = \tan^{-1}\left|\frac{-\sqrt{2}}{2}\right| = 45^\circ$

$r_1 = 2$ $\quad \theta_1 = 315^\circ$

$z_1 = 2(\cos 315^\circ + i \sin 315^\circ)$

$z_2 = 1 + i$

$r_2 = \sqrt{1^2 + 1^2}$ $\quad \alpha_2 = \tan^{-1}\left|\frac{1}{1}\right| = 45^\circ$

$r_2 = \sqrt{2}$ $\quad \theta_2 = 45^\circ$

$z_2 = \sqrt{2}(\cos 45^\circ + i \sin 45^\circ)$

$$\frac{z_1}{z_2} = \frac{2(\cos 315^\circ + i \sin 315^\circ)}{\sqrt{2}(\cos 45^\circ + i \sin 45^\circ)}$$

$$\frac{z_1}{z_2} = \sqrt{2}[\cos(315^\circ - 45^\circ) + i \sin(315^\circ - 45^\circ)]$$

$$\frac{z_1}{z_2} = \sqrt{2}(\cos 270^\circ + i \sin 270^\circ)$$

$$\frac{z_1}{z_2} = \sqrt{2}[0 + i(-1)] = \sqrt{2}(0 - 1i) = 0 - \sqrt{2}i = -\sqrt{2}i \text{ or } -i\sqrt{2}$$

**62.** $z_1 = 1 + i\sqrt{3}$

$r_1 = \sqrt{1^2 + (\sqrt{3})^2}$ $\quad \alpha_1 = \tan^{-1}\left|\frac{\sqrt{3}}{1}\right| = 60^\circ$

$r_1 = 2$ $\quad \theta_1 = 60^\circ$

$z_1 = 2(\cos 60^\circ + i \sin 60^\circ)$

$z_2 = 4 - 4i$

$r_2 = \sqrt{4^2 + (-4)^2}$ $\quad \alpha_2 = \tan^{-1}\left|\frac{-4}{4}\right| = 45^\circ$

$r_2 = 4\sqrt{2}$ $\quad \theta_2 = 315^\circ$

$z_2 = 4\sqrt{2}(\cos 315^\circ + i \sin 315^\circ)$

$$\frac{z_1}{z_2} = \frac{2(\cos 60^\circ + i \sin 60^\circ)}{4\sqrt{2}(\cos 315^\circ + i \sin 315^\circ)}$$

$$\frac{z_1}{z_2} = \frac{\sqrt{2}}{4}[\cos(60^\circ - 315^\circ) + i \sin(60^\circ - 315^\circ)]$$

$$\frac{z_1}{z_2} = \frac{\sqrt{2}}{4}[\cos(-255^\circ) + i \sin(-255^\circ)]$$

$$\frac{z_1}{z_2} \approx -0.0915 + 0.3415i$$

**Connecting Concepts**

**63.** $z_1 = \sqrt{3} - 1$

$r_1 = \sqrt{(\sqrt{3})^2 + (-1)^2}$

$r_1 = 2$

$\alpha_1 = \tan^{-1}\left|\frac{-1}{\sqrt{3}}\right| = 30°$

$\theta_1 = 330°$

$z_2 = 2 + 2i$

$r_2 = \sqrt{2^2 + 2^2}$

$r_2 = 2\sqrt{2}$

$\alpha_2 = \tan^{-1}\left|\frac{2}{2}\right| = 45°$

$\theta_2 = 45°$

$z_3 = 2 - 2i\sqrt{3}$

$r_3 = \sqrt{2^2 + (-2\sqrt{3})^2}$

$r_3 = 4$

$\alpha_3 = \tan^{-1}\left|\frac{-2\sqrt{3}}{2}\right| = 60°$

$\theta_3 = 300°$

$z_1 = 2(\cos 330° + i \sin 330°)$ $\quad z_2 = 2\sqrt{2}(\cos 45° + i \sin 45°)$ $\quad z_3 = 4(\cos 300° + i \sin 300°)$

$z_1z_2z_3 = 2(\cos 330° + i \sin 330°) \cdot 2\sqrt{2}(\cos 45° + i \sin 45°) \cdot 4(\cos 300° + i \sin 300°)$

$z_1z_2z_3 = 16\sqrt{2}[\cos(330° + 45° + 300°) + i \sin(330° + 45° + 300°)]$

$z_1z_2z_3 = 16\sqrt{2}(\cos 675° + i \sin 675°)$

$z_1z_2z_3 = 16\sqrt{2}(\cos 315° + i \sin 315°)$

$z_1z_2z_3 = 16\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = 16 - 16i$

**64.** $z_1 = 1 - i$

$r_1 = \sqrt{1^2 + (-1)^2}$

$r_1 = \sqrt{2}$

$\alpha_1 = \tan^{-1}\left|\frac{-1}{1}\right| = 45°$

$\theta_1 = 315°$

$z_2 = 1 + i\sqrt{3}$

$r_2 = \sqrt{1^2 + (\sqrt{3})^2}$

$r_2 = 2$

$\alpha_2 = \tan^{-1}\left|\frac{\sqrt{3}}{1}\right| = 60°$

$\theta_2 = 60°$

$z_3 = \sqrt{3} - i$

$r_3 = \sqrt{(\sqrt{3})^2 + (-1)^2}$

$r_3 = 2$

$\alpha_3 = \tan^{-1}\left|\frac{-1}{\sqrt{3}}\right| = 30°$

$\theta_3 = 330°$

$z_1 = \sqrt{2}(\cos 315° + i \sin 315°)$ $\quad z_2 = 2(\cos 60° + i \sin 60°)$ $\quad z_3 = 2(\cos 330° + i \sin 330°)$

$z_1z_2z_3 = \sqrt{2}(\cos 315° + i \sin 315°) \cdot 2(\cos 60° + i \sin 60°) \cdot 2(\cos 330° + i \sin 330°)$

$z_1z_2z_3 = 4\sqrt{2}[\cos(315° + 60° + 330°) + i \sin(315° + 60° + 330°)]$

$z_1z_2z_3 = 4\sqrt{2}(\cos 705° + i \sin 705°)$

$z_1z_2z_3 = 4\sqrt{2}(\cos 345° + i \sin 345°)$

$z_1z_2z_3 \approx 5.4641 - 1.4641i$

**65.**

$z_1 = \sqrt{3} + i\sqrt{3}$

$r_1 = \sqrt{(\sqrt{3})^2 + (\sqrt{3})^2}$

$r_1 = \sqrt{6}$

$\alpha_1 = \tan^{-1}\left|\frac{\sqrt{3}}{\sqrt{3}}\right| = 45^\circ$

$\theta_1 = 45^\circ$

$z_1 = \sqrt{6}(\cos 45^\circ + i \sin 45^\circ)$

$z_2 = 1 - i\sqrt{3}$

$r_2 = \sqrt{1^2 + (-\sqrt{3})^2}$

$r_2 = 2$

$\alpha_2 = \tan^{-1}\left|\frac{-\sqrt{3}}{1}\right| = 60^\circ$

$\theta_2 = 300^\circ$

$z_2 = 2(\cos 300^\circ + i \sin 300^\circ)$

$z_3 = 2 - 2i$

$r_3 = \sqrt{2^2 + (-2)^2}$

$r_3 = 2\sqrt{2}$

$\alpha_3 = \tan^{-1}\left|\frac{-2}{2}\right| = 45^\circ$

$\theta_3 = 315^\circ$

$z_3 = 2\sqrt{2}(\cos 315^\circ + i \sin 315^\circ)$

$$\frac{z_1}{z_2 z_3} = \frac{\sqrt{6}(\cos 45^\circ + i \sin 45^\circ)}{2(\cos 300^\circ + i \sin 300^\circ) \cdot 2\sqrt{2}(\cos 315^\circ + i \sin 315^\circ)}$$

$$\frac{z_1}{z_2 z_3} = \frac{\sqrt{6}(\cos 45^\circ + i \sin 45^\circ)}{4\sqrt{2}[\cos(300^\circ + 315^\circ) + i \sin(300^\circ + 315^\circ)]}$$

$$\frac{z_1}{z_2 z_3} = \frac{\sqrt{6}(\cos 45^\circ + i \sin 45^\circ)}{4\sqrt{2}(\cos 255^\circ + i \sin 255^\circ)}$$

$$\frac{z_1}{z_2 z_3} = \frac{\sqrt{3}}{4}[\cos(45^\circ - 255^\circ) + i \sin(45^\circ - 255^\circ)]$$

$$\frac{z_1}{z_2 z_3} = \frac{\sqrt{3}}{4}(\cos 210^\circ - i \sin 210^\circ) = \frac{\sqrt{3}}{4}\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = -\frac{3}{8} + \frac{\sqrt{3}}{8}i$$

**66.**

$z_1 = 2 - 2i\sqrt{3}$

$r_1 = \sqrt{2^2 + (-2\sqrt{3})^2}$

$r_1 = 4$

$\alpha_1 = \tan^{-1}\left|\frac{-2\sqrt{3}}{2}\right| = 60^\circ$

$\theta_1 = 300^\circ$

$z_1 = 4(\cos 300^\circ + i \sin 300^\circ)$

$z_2 = 1 - i\sqrt{3}$

$r_2 = \sqrt{1^2 + (-\sqrt{3})^2}$

$r_2 = 2$

$\alpha_2 = \tan^{-1}\left|\frac{-\sqrt{3}}{1}\right| = 60^\circ$

$\theta_2 = 300^\circ$

$z_2 = 2(\cos 300^\circ + i \sin 300^\circ)$

$z_3 = 4\sqrt{3} + 4i$

$r_3 = \sqrt{(4\sqrt{3})^2 + 4^2}$

$r_3 = 8$

$\alpha_3 = \tan^{-1}\left|\frac{1}{\sqrt{3}}\right| = 30^\circ$

$\theta_3 = 30^\circ$

$z_3 = 8(\cos 30^\circ + i \sin 30^\circ)$

$$\frac{z_1}{z_2 z_3} = \frac{4(\cos 300^\circ + i \sin 300^\circ) \cdot 2(\cos 300^\circ + i \sin 300^\circ)}{8(\cos 30^\circ + i \sin 30^\circ)}$$

$$\frac{z_1}{z_2 z_3} = \frac{4 \cdot 2}{8}[\cos(300^\circ + 300^\circ - 30^\circ) + i \sin(300^\circ + 300^\circ - 30^\circ)]$$

$$\frac{z_1}{z_2 z_3} = (\cos 210^\circ + i \sin 210^\circ) = -\frac{\sqrt{3}}{2} - \frac{i}{2}$$

**67.**

$z_1 = 1 - 3i$

$r_1 = \sqrt{1^2 + (-3)^2}$

$r_1 = \sqrt{10}$

$\alpha_1 = \tan^{-1}\left|\frac{-3}{1}\right| \approx 71.57^\circ$

$\theta_1 = 288.43^\circ$

$z_1 = \sqrt{10}(\cos 288.4^\circ + i \sin 288.4^\circ)$

$z_2 = 2 + 3i$

$r_2 = \sqrt{2^2 + 3^2}$

$r_2 = \sqrt{13}$

$\alpha_2 = \tan^{-1}\left|\frac{3}{2}\right| \approx 56.31^\circ$

$\theta_2 = 56.31^\circ$

$z_2 = \sqrt{13}(\cos 56.3^\circ + i \sin 56.3^\circ)$

$z_3 = 4 + 5i$

$r_3 = \sqrt{4^2 + 5^2}$

$r_3 = \sqrt{41}$

$\alpha_3 = \tan^{1}\left|\frac{5}{4}\right| \approx 51.34^\circ$

$\theta_3 = 51.34^\circ$

$z_3 = \sqrt{41}(\cos 51.3^\circ + i \sin 51.3^\circ)$

$z_1 z_2 z_3 = \sqrt{10}(\cos 288.4^\circ + i \sin 288.4^\circ) \cdot \sqrt{13}(\cos 56.3^\circ + i \sin 56.3^\circ) \cdot \sqrt{41}(\cos 51.3^\circ + i \sin 51.3^\circ)$

$z_1 z_2 z_3 = \sqrt{10} \cdot \sqrt{13} \cdot \sqrt{41}[\cos(288.43^\circ + 56.31^\circ + 51.34^\circ) + i \sin(288.43^\circ + 56.31^\circ + 51.34^\circ)]$

$z_1 z_2 z_3 \approx 73.0(\cos 396.08^\circ + i \sin 396.08^\circ)$

$z_1 z_2 z_3 = 73.0(\cos 36.08^\circ + i \sin 36.08^\circ)$

$z_1 z_2 z_3 \approx 59.0 + 43.0i$

**68.** $z_1 = 2 - 5i$

$r_1 = \sqrt{2^2 + (-5)^2}$

$r_1 = \sqrt{29}$

$\alpha_1 = \tan^{-1}\left|\frac{-5}{2}\right| \approx 68.1986°$

$\theta_1 = 291.8014°$

$z_1 = \sqrt{29}\ \text{cis}\ 291.8014°$

$z_2 = 1 - 6i$

$r_2 = \sqrt{1^2 + (-6)^2}$

$r_2 = \sqrt{37}$

$\alpha_2 = \tan^{-1}\left|\frac{-6}{1}\right| \approx 80.5377°$

$\theta_2 = 279.4623°$

$z_2 = \sqrt{37}\ \text{cis}\ 279.4623°$

$z_3 = 3 + 4i$

$r_3 = \sqrt{3^2 + 4^2}$

$r_3 = 5$

$\alpha_3 = \tan^{-1}\left|\frac{4}{3}\right| \approx 53.1301°$

$\theta_3 = 53.1301°$

$z_3 = 5\ \text{cis}\ 53.1301°$

$$\begin{aligned}\frac{z_1 z_2}{z_3} &= \frac{\sqrt{29}\ \text{cis}\ 291.8014° \cdot \sqrt{37}\ \text{cis}\ 279.4623°}{5\ \text{cis}\ 53.1301°}\\ &= \frac{\sqrt{29}\cdot\sqrt{37}}{5}\ \text{cis}\ (291.8014° + 279.4623° - 53.1301°)\\ &= \frac{\sqrt{29}\cdot\sqrt{37}}{5}(\cos 518.1336° + i\sin 518.1336°)\\ &\approx -6.0800 + 2.4400i\end{aligned}$$

**69.** $z = r(\cos\theta + i\sin\theta)$ $\quad \bar{z} = r(\cos\theta - i\sin\theta)$

$$\begin{aligned} z\cdot\bar{z} &= r(\cos\theta + i\sin\theta)\cdot r(\cos\theta - i\sin\theta)\\ z\cdot\bar{z} &= r(\cos\theta + i\sin\theta)\cdot r[\cos(-\theta) + i\sin(-\theta)]\\ z\cdot\bar{z} &= r^2[\cos(\theta-\theta) + i\sin(\theta-\theta)]\\ z\cdot\bar{z} &= r^2(\cos 0 + i\sin 0)\\ z\cdot\bar{z} &= r^2 \text{ or } a^2 + b^2\end{aligned}$$

**70.** $z = r(\cos\theta + i\sin\theta)$ $\quad \bar{z} = r(\cos\theta - i\sin\theta)$

$$\begin{aligned}\frac{z}{\bar{z}} &= \frac{r(\cos\theta + i\sin\theta)}{r(\cos\theta - i\sin\theta)}\\ &= \frac{\cos\theta + i\sin\theta}{\cos(-\theta) + i\sin(-\theta)}\\ &= \cos(\theta+\theta) + i\sin(\theta+\theta)\\ &= \cos 2\theta + i\sin 2\theta\end{aligned}$$

## Prepare for Section 7.5

**PS1.** $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2 = \frac{2}{4} + 2\frac{2}{4}i + \frac{2}{4}i^2 = i$

**PS2.** $x^3 - 8 = (x-2)(x^2 + 2x + 4)$

$(x^2 + 2x + 4)$ yields 2 complex solutions

$(x-2)$ yields 1 real solution

The real root is 2.

**PS3.** $x^5 - 243 = (x-3)(3x^4 + 3x^3 + 9x^2 + 27x + 81)$

$(3x^4 + 3x^3 + 9x^2 + 27x + 81)$ yields 4 complex solutions

$(x-3)$ yields 1 real solution

The real root is 3.

**PS4.** $r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$

$$\begin{aligned}\alpha &= \tan^{-1}\left|\frac{2}{2}\right|\\ &= \tan^{-1} 1 = 45°\end{aligned}$$

$\theta = 45°$

$z = 2\sqrt{2}\ \text{cis}\ 45°$ or $2\sqrt{2}\ \text{cis}\ \frac{\pi}{4}$

**PS5.** $2(\cos 150° + i\sin 150°) = 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = -\sqrt{3} + i$

**PS6.** 
$$\begin{aligned}|z| &= \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2}\\ &= \sqrt{\frac{2}{4} + \frac{2}{4}}\\ &= 1\end{aligned}$$

## Section 7.5

**1.** $[2(\cos 30° + i\sin 30°)]^8 = 2^8[\cos(8\cdot 30°) + i\sin(8\cdot 30°)]$
$= 256(\cos 240° + i\sin 240°)$
$= -128 - 128i\sqrt{3}$

**2.** $(\cos 240° + i\sin 240°)^{12} = \cos(12\cdot 240°) + i\sin(12\cdot 240°)$
$= \cos 2880° + i\sin 2880°$
$= \cos 0° + i\sin 0°$
$= 1 + 0i$
$= 1$

**3.** $[2(\cos 240° + i\sin 240°)]^5 = 2^5[\cos(5\cdot 240°) + i\sin(5\cdot 240°)]$
$= 32[\cos 1200° + i\sin 1200°]$
$= 32(\cos 120° + i\sin 120°)$
$= -16 + 16i\sqrt{3}$

**4.** $[2(\cos 45° + i\sin 45°)]^{10} = 2^{10}[\cos(10\cdot 45°) + i\sin(10\cdot 45°)]$
$= 1024(\cos 450° + i\sin 450°)$
$= 1024(\cos 90° + i\sin 90°)$
$= 0 + 1024i$
$= 1024i$

**5.** $[2\text{cis}(225°)]^5 = 2^5\text{cis}(5\cdot 225°)$
$= 32(\cos 1125° + i\sin 1125°)$
$= 32(\cos 45° + i\sin 45°)$
$= 16\sqrt{2} + 16i\sqrt{2}$

**6.** $[2\text{cis}(330°)]^4 = 2^4\text{cis}(4\cdot 330°)$
$= 16(\cos 1320° + i\sin 1320°)$
$= 16(\cos 240° + i\sin 240°)$
$= -8 - 8i\sqrt{3}$

**7.** $[2\text{cis } (120°)]^6 = 2^6\text{cis}(6\cdot 2\pi/3)$
$= 64(\cos 720° + i\sin 720°)$
$= 64(\cos 0° + i\sin 0°)$
$= 64$

**8.** $[4\text{cis}(150°)]^3 = 4^3\text{cis}(3\cdot 5\pi/6)$
$= 64(\cos 450° + i\sin 450°)$
$= 64(\cos 90° + i\sin 90°)$
$= 64(0 + 1i) = 0 + 64i$
$= 64i$

**9.** $z = 1 - i$
$r = \sqrt{1^2 + (-1)^2}$ $\quad \alpha = \tan^{-1}\left|\frac{-1}{1}\right| = 45°$
$r = \sqrt{2}$
$\theta = 315°$
$z = \sqrt{2}(\cos 315° + i\sin 315°)$
$(1-i)^{10} = [\sqrt{2}(\cos 315° + i\sin 315°)]^{10}$
$= (\sqrt{2})^{10}[\cos(10\cdot 315°) + i\sin(10\cdot 315°)]$
$= 32(\cos 3150° + i\sin 3150°)$
$= 32(\cos 270° + i\sin 270°)$
$= 0 - 32i = -32i$

**10.** $z = 1 + i\sqrt{3}$
$r = \sqrt{1^2 + (\sqrt{3})^2}$ $\quad \alpha = \tan^{-1}\left|\frac{\sqrt{3}}{1}\right| = 60°$
$r = 2$
$\theta = 60°$
$z = 2(\cos 60° + i\sin 60°)$
$(1 + i\sqrt{3})^8 = [2(\cos 60° + i\sin 60°)]^8$
$= 2^8[\cos(8\cdot 60°) + i\sin(8\cdot 60°)]$
$= 256(\cos 480° + i\sin 480°)$
$= 256(\cos 120° + i\sin 120°)$
$= -128 + 128i\sqrt{3}$

**11.** $z = 1 + i$
$r = \sqrt{1^2 + 1^2}$ $\quad \alpha = \tan^{-1}\left|\frac{1}{1}\right| = 45°$
$r = \sqrt{2}$
$\theta = 45°$
$z = \sqrt{2}(\cos 45° + i\sin 45°)$
$(1+i)^4 = [\sqrt{2}(\cos 45° + i\sin 45°)]^4$
$= (\sqrt{2})^4[\cos(4\cdot 45°) + i\sin(4\cdot 45°)]$
$= 4(\cos 180° + i\sin 180°)$
$= -4 + 0i = -4$

**12.** $z = 2 - 2i\sqrt{3}$
$r = \sqrt{2^2 + (-2\sqrt{3})^2}$ $\quad \alpha = \tan^{-1}\left|\frac{-2\sqrt{3}}{2}\right| = 60°$
$r = 4$
$\theta = 300°$
$z = 4(\cos 300° + i\sin 300°)$
$(2 - 2i\sqrt{3})^3 = [4(\cos 300° + i\sin 300°)]^3$
$= 4^3[\cos(3\cdot 300°) + i\sin(3\cdot 300°)]$
$= 64(\cos 900° + i\sin 900°)$
$= 64(\cos 180° + i\sin 180°)$
$= -64 + 0i = -64$

**13.** $z = 2 + 2i$

$r = \sqrt{2^2 + 2^2}$ $\alpha = \tan^{-1}\left|\frac{2}{2}\right| = 45^\circ$

$r = 2\sqrt{2}$ $\theta = 45^\circ$

$$z = 2\sqrt{2}(\cos 45^\circ + i\sin 45^\circ)$$

$$\begin{aligned}(2+2i)^7 &= [2\sqrt{2}(\cos 45^\circ + i\sin 45^\circ)]^7\\ &= 1024\sqrt{2}[\cos(7\cdot 45^\circ) + i\sin(7\cdot 45^\circ)]\\ &= 1024\sqrt{2}(\cos 315^\circ + i\sin 315^\circ)\\ &= 1024 - 1024i\end{aligned}$$

**14.** $z = 2\sqrt{3} - 2i$

$r = \sqrt{(2\sqrt{3})^2 + (-2)^2}$ $\alpha = \tan^{-1}\left|\frac{-2}{2\sqrt{3}}\right| = 30^\circ$

$r = 4$ $\theta = 330^\circ$

$$z = 4(\cos 330^\circ + i\sin 330^\circ)$$

$$\begin{aligned}(2\sqrt{3} - 2i)^5 &= [4(\cos 330^\circ + i\sin 330^\circ)]^5\\ &= 4^5[\cos(5\cdot 330^\circ) + i\sin(5\cdot 330^\circ)]\\ &= 1024(\cos 1650^\circ + i\sin 1650^\circ)\\ &= 1024(\cos 210^\circ + i\sin 210^\circ)\\ &= -512\sqrt{3} - 512i\end{aligned}$$

**15.** $z = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$r = \sqrt{(\sqrt{2}/2)^2 + (\sqrt{2}/2)^2}$ $\alpha = \tan^{-1}\left|\frac{\sqrt{2}/2}{\sqrt{2}/2}\right| = 45^\circ$

$r = 1$ $\theta = 45^\circ$

$$z = \cos 45^\circ + i\sin 45^\circ$$

$$\begin{aligned}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)^6 &= (\cos 45^\circ + i\sin 45^\circ)^6\\ &= \cos(6\cdot 45^\circ) + i\sin(6\cdot 45^\circ)\\ &= \cos 270^\circ + i\sin 270^\circ\\ &= 0 - 1i = -i\end{aligned}$$

**16.** $z = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$r = \sqrt{(-\sqrt{2}/2)^2 + (\sqrt{2}/2)^2}$ $\alpha = \tan^{-1}\left|\frac{\sqrt{2}/2}{-\sqrt{2}/2}\right| = 45^\circ$

$r = 1$ $\theta = 135^\circ$

$$z = \cos 135^\circ + i\sin 135^\circ$$

$$\begin{aligned}\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)^{12} &= (\cos 135^\circ + i\sin 135^\circ)^{12}\\ &= \cos(12\cdot 135^\circ) + i\sin(12\cdot 135^\circ)\\ &= \cos 1620^\circ + i\sin 1620^\circ\\ &= \cos 180^\circ + i\sin 180^\circ\\ &= -1 + 0i = -1\end{aligned}$$

**17.** $9 = 9(\cos 0^\circ + i\sin 0^\circ)$

$w_k = 9^{1/2}\left(\cos\frac{0^\circ + 360^\circ k}{2} + i\sin\frac{0^\circ + 360^\circ k}{2}\right)$ $k = 0,1$

$w_0 = 3(\cos 0^\circ + i\sin 0^\circ)$

$w_0 = 3 + 0i = 3$

$w_1 = 3\left(\frac{\cos 0^\circ + 360^\circ}{2} + i\sin\frac{0^\circ + 360^\circ}{2}\right)$

$w_1 = 3(\cos 180^\circ + i\sin 180^\circ)$

$w_1 = -3 + 0i = -3$

**18.** $16 = 16(\cos 0^\circ + i\sin 0^\circ)$

$w_k = 16^{1/2}\left(\cos\frac{0^\circ + 360^\circ k}{2} + i\sin\frac{0^\circ + 360^\circ k}{2}\right)$ $k = 0,1$

$w_0 = 4(\cos 0^\circ + i\sin 0^\circ)$

$w_0 = 4 + 0i = 4$

$w_1 = 4\left(\cos\frac{0^\circ + 360^\circ}{2} + i\sin\frac{0^\circ + 360^\circ}{2}\right)$

$w_1 = 4(\cos 180^\circ + i\sin 180^\circ)$

$w_1 = -4 + 0i = -4$

**19.** $64 = 64(\cos 0° + i\sin 0°)$

$w_k = 64^{1/6}\left(\cos\frac{0° + 360°k}{6} + i\sin\frac{0° + 360°k}{6}\right) \quad k = 0, 1, 2, 3, 4, 5$

$w_0 = 2(\cos 0° + i\sin 0°)$
$w_0 = 2 + 0i = 2$

$w_1 = 2\left(\cos\frac{0 + 360°}{6} + i\sin\frac{0 + 360°}{6}\right)$
$w_1 = 2(\cos 60° + i\sin 60°)$
$w_1 = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$
$w_1 = 1 + i\sqrt{3}$

$w_2 = 2\left(\cos\frac{0° + 360°\cdot 2}{6} + i\sin\frac{0° + 360°\cdot 2}{6}\right)$
$w_2 = 2(\cos 120° + i\sin 120°)$
$w_2 = 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$
$w_2 = -1 + i\sqrt{3}$

$w_3 = 2\left(\cos\frac{0° + 360°\cdot 3}{6} + i\sin\frac{0° + 360°\cdot 3}{6}\right)$
$w_3 = 2(\cos 180° + i\sin 180°)$
$w_3 = 2(-1 + 0i)$
$w_3 = -2 + 0i = -2$

$w_4 = 2\left(\cos\frac{0° + 360°\cdot 4}{6} + i\sin\frac{0° + 360°\cdot 4}{6}\right)$
$w_4 = 2(\cos 240° + i\sin 240°)$
$w_4 = 2\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$
$w_4 = -1 - i\sqrt{3}$

$w_5 = 2\left(\cos\frac{0° + 360°\cdot 5}{6} + i\sin\frac{0° + 360°\cdot 5}{6}\right)$
$w_5 = 2(\cos 300° + i\sin 300°)$
$w_5 = 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$
$w_5 = 1 - i\sqrt{3}$

**20.** $32 = 32(\cos 0° + i\sin 0°)$

$w_k = 32^{1/5}\left(\cos\frac{0° + 360°k}{5} + i\sin\frac{0° + 360°k}{5}\right) \quad k = 0, 1, 2, 3, 4$

$w_0 = 2(\cos 0° + i\sin 0°)$
$w_0 = 2 + 0i = 2$

$w_1 = 2\left(\cos\frac{0° + 360°}{5} + i\sin\frac{0° + 360°}{5}\right)$
$w_1 = 2(\cos 72° + i\sin 72°)$
$w_1 \approx 2(0.3090 + 0.9511i)$
$w_1 = 0.6180 + 1.9021i$

$w_2 = 2\left(\cos\frac{0° + 360°\cdot 2}{5} + i\sin\frac{0° + 360°\cdot 2}{5}\right)$
$w_2 = 2(\cos 144° + i\sin 144°)$
$w_2 \approx 2(-0.8090 + 0.5878i)$
$w_2 = -1.6180 + 1.1756i$

$w_3 = 2\left(\cos\frac{0° + 360°°\cdot 3}{5} + i\sin\frac{0° + 360°\cdot 3}{5}\right)$
$w_3 = 2(\cos 216° + i\sin 216°)$
$w_3 \approx 2(-0.8090 - 0.5878i)$
$w_3 = -1.6180 - 1.1756i$

$w_4 = 2\left(\cos\frac{0° + 360°\cdot 4}{5} + i\sin\frac{0° + 360°\cdot 4}{5}\right)$
$w_4 = 2(\cos 288° + i\sin 288°)$
$w_4 \approx 2(0.3090 - 0.9511i)$
$w_4 = 0.6180 - 1.9021i$

**21.** $-1 = 1(\cos 180° + i\sin 180°)$

$$w_k = 1^{1/5}\left(\cos\frac{180° + 360°k}{5} + i\sin\frac{180° + 360°k}{5}\right) \qquad k = 0, 1, 2, 3, 4$$

$w_0 = 1(\cos 36° + i\sin 36°)$
$w_0 \approx 0.809 + 0.588i$

$w_1 = \cos\frac{180° + 360°}{5} + i\sin\frac{180° + 360°}{5}$
$w_1 = \cos 108° + i\sin 108°$
$w_1 \approx -0.309 + 0.951i$

$w_2 = \cos\frac{180° + 360° \cdot 2}{5} + i\sin\frac{180° + 360° \cdot 2}{5}$
$w_2 = \cos 180° + i\sin 180°$
$w_2 = -1 + 0i = -1$

$w_3 = \cos\frac{180° + 360° \cdot 3}{5} + i\sin\frac{180° + 360° \cdot 3}{5}$
$w_3 = \cos 252° + i\sin 252°$
$w_3 \approx -0.309 - 0.951i$

$w_4 = \cos\frac{180° + 360° \cdot 4}{5} + i\sin\frac{180° + 360° \cdot 4}{5}$
$w_4 = \cos 324° + i\sin 324°$
$w_4 \approx 0.809 - 0.588i$

**22.** $-16 = 16(\cos 180° + i\sin 180°)$

$$w_k = 16^{1/4}\left(\cos\frac{180° + 360°k}{4} + i\sin\frac{180° + 360°k}{4}\right) \qquad k = 0, 1, 2, 3$$

$w_0 = 2(\cos 45° + i\sin 45°)$
$w_0 = 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$
$w_0 = \sqrt{2} + i\sqrt{2}$

$w_1 = 2\left(\cos\frac{180° + 360°}{4} + i\sin\frac{180° + 360°}{4}\right)$
$w_1 = 2(\cos 135° + i\sin 135°)$
$w_1 = 2\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$
$w_1 = -\sqrt{2} + i\sqrt{2}$

$w_2 = 2\left(\cos\frac{180° + 360° \cdot 2}{4} + i\sin\frac{180° + 360° \cdot 2}{4}\right)$
$w_2 = 2(\cos 225° + i\sin 225°)$
$w_2 = 2\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$
$w_2 = -\sqrt{2} - i\sqrt{2}$

$w_3 = 2\left(\cos\frac{180° + 360° \cdot 3}{4} + i\sin\frac{180° + 360° \cdot 3}{4}\right)$
$w_3 = 2(\cos 315° + i\sin 315°)$
$w_3 \approx 2\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$
$w_3 = \sqrt{2} - i\sqrt{2}$

**23.** $1 = \cos 0° + i\sin 0°$

$$w_k = \cos\frac{0° + 360°k}{3} + i\sin\frac{0° + 360°k}{3} \qquad k = 0, 1, 2$$

$w_0 = \cos\frac{0°}{3} + i\sin\frac{0°}{3}$
$w_0 = \cos 0° + i\sin 0°$
$w_0 = 1 + 0i = 1$

$w_1 = \cos\frac{0° + 360°}{3} + i\sin\frac{0° + 360°}{3}$
$w_1 = \cos 120° + i\sin 120°$
$w_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$w_2 = \cos\frac{0° + 360° \cdot 2}{3} + i\sin\frac{0° + 360° \cdot 2}{3}$
$w_2 = \cos 240° + i\sin 240°$
$w_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

**24.** $i = \cos 90° + i\sin 90°$

$$w_k = \cos\frac{90° + 360°k}{3} + i\sin\frac{90° + 360°k}{3} \qquad k = 0, 1, 2$$

$$w_0 = \cos\frac{90^\circ}{3} + i\sin\frac{90^\circ}{3}$$
$$w_0 = \cos 30° + i\sin 30°$$
$$w_0 = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$w_1 = \cos\frac{90^\circ + 360^\circ}{3} + i\sin\frac{90^\circ + 360^\circ}{3}$$
$$w_1 = \cos 150° + i\sin 150°$$
$$w_1 = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$w_2 = \cos\frac{90^\circ + 360^\circ \cdot 2}{3} + i\sin\frac{90^\circ + 360^\circ \cdot 2}{3}$$
$$w_2 = \cos 270° + i\sin 270°$$
$$w_2 = 0 - i = -i$$

**25.** $1 + i = \sqrt{2}(\cos 45° + i\sin 45°)$

$$w_k = \left(\sqrt{2}\right)^{1/4}\left(\cos\frac{45° + 360°k}{4} + i\sin\frac{45° + 360°k}{4}\right) \qquad k = 0, 1, 2, 3$$

$$w_0 = 2^{1/8}\left(\cos\frac{45°}{4} + i\sin\frac{45°}{4}\right)$$
$$w_0 = 2^{1/8}(\cos 11.25° + i\sin 11.25°)$$
$$w_0 \approx 1.070 + 0.213i$$

$$w_1 = 2^{1/8}\left(\cos\frac{45^\circ + 360^\circ}{4} + i\sin\frac{45^\circ + 360^\circ}{4}\right)$$
$$w_1 = 2^{1/8}(\cos 101.25° = i\sin 101.25°)$$
$$w_1 \approx -0.213 - 1.070i$$

$$w_2 = 2^{1/8}\left(\cos\frac{45^\circ + 360^\circ \cdot 2}{4} + i\sin\frac{45^\circ + 360^\circ \cdot 2}{4}\right)$$
$$w_2 = 2^{1/8}(\cos 191.25° + i\sin 191.25°)$$
$$w_2 \approx -1.070 - 0.213i$$

$$w_3 = 2^{1/8}\left(\cos\frac{45^\circ + 360^\circ \cdot 3}{4} + i\sin\frac{45^\circ + 360^\circ \cdot 3}{4}\right)$$
$$w_3 = 2^{1/8}(\cos 281.25° + i\sin 281.25°)$$
$$w_3 \approx 0.213 - 1.070i$$

**26.** $-1 + i = \sqrt{2}(\cos 135° + i\sin 135°)$

$$w_k = \left(\sqrt{2}\right)^{1/5}\left(\cos\frac{135° + 360°k}{5} + i\sin\frac{135° + 360°k}{5}\right) \qquad k = 0, 1, 2, 3, 4$$

$$w_0 = 2^{1/10}\left(\cos\frac{135^\circ}{5} + i\sin\frac{135^\circ}{5}\right)$$
$$w_0 = 2^{1/10}(\cos 27° + i\sin 27°)$$
$$w_0 \approx 0.955 + 0.487i$$

$$w_1 = 2^{1/10}\left(\cos\frac{135^\circ + 360^\circ}{5} + i\sin\frac{135^\circ + 360^\circ}{5}\right)$$
$$w_1 = 2^{1/10}(\cos 99° + i\sin 99°)$$
$$w_1 \approx -0.168 + 1.059i$$

$$w_2 = 2^{1/10}\left(\cos\frac{135^\circ + 360^\circ \cdot 2}{5} + i\sin\frac{135^\circ + 360^\circ \cdot 2}{5}\right)$$
$$w_2 = 2^{1/10}(\cos 171° + i\sin 171°)$$
$$w_2 \approx -1.059 + 0.168i$$

$$w_3 = 2^{1/10}\left(\cos\frac{135° + 360° \cdot 3}{5} + i\sin\frac{135° + 360° \cdot 3}{5}\right)$$
$$w_3 = 2^{1/10}(\cos 243° + i\sin 243°)$$
$$w_3 \approx -0.487 - 0.955i$$

$$w_4 = 2^{1/10}\left(\cos\frac{135^\circ + 360^\circ \cdot 4}{5} + i\sin\frac{135^\circ + 360^\circ \cdot 4}{5}\right)$$
$$w_4 = 2^{1/10}(\cos 315° + i\sin 315°)$$
$$w_4 \approx 0.758 - 0.758i$$

**27.** $2-2i\sqrt{3}=4(\cos 300°+i\sin 300°)\quad k=0,1,2$

$$w_k=4^{1/3}\left(\cos\frac{300°+360°k}{3}+i\sin\frac{300°+360°k}{3}\right)$$

$$w_0=4^{1/3}\left(\cos\frac{300°}{3}+i\sin\frac{300°}{3}\right)$$
$$w_0=4^{1/3}(\cos 100°+i\sin 100°)$$
$$w_0\approx -0.276+1.563i$$

$$w_1=4^{1/3}\left(\cos\frac{300°+360°}{3}+i\sin\frac{300°+360°}{3}\right)$$
$$w_1=4^{1/3}(\cos 220°+i\sin 220°)$$
$$w_1\approx -1.216-1.020i$$

$$w_2=4^{1/3}\left(\cos\frac{300°+360°\cdot 2}{3}+i\sin\frac{300°+360°\cdot 2}{3}\right)$$
$$w_2=4^{1/3}(\cos 340°+i\sin 340°)$$
$$w_2\approx 1.492-0.543i$$

**28.** $-2+2i\sqrt{3}=4(\cos 120°+i\sin 120°)$

$$w_k=4^{1/3}\left(\cos\frac{120°+360°k}{3}+i\sin\frac{120°+360°k}{3}\right)\quad k=0,1,2$$

$$w_0=4^{1/3}\left(\cos\frac{120°}{3}+i\sin\frac{120°}{3}\right)$$
$$w_0=4^{1/3}(\cos 40°+i\sin 40°)$$
$$w_0\approx 1.216+1.020i$$

$$w_1=4^{1/3}\left(\cos\frac{120°+360°}{3}+i\sin\frac{120°+360°}{3}\right)$$
$$w_1=4^{1/3}(\cos 160°+i\sin 160°)$$
$$w_1\approx -1.492+0.543i$$

$$w_2=4^{1/3}\left(\cos\frac{120°+360°\cdot 2}{3}+i\sin\frac{120°+360°\cdot 2}{3}\right)$$
$$w_2=4^{1/3}(\cos 280°+i\sin 280°)$$
$$w_2\approx 0.276-1.563i$$

**29.** $-16+16i\sqrt{3}=32(\cos 120°+i\sin 120°)$

$$w_k=32^{1/2}\left(\cos\frac{120°+360°k}{2}+i\sin\frac{120°+360°k}{2}\right)\quad k=0,1$$

$$w_0=4\sqrt{2}\left(\cos\frac{120°}{2}+i\sin\frac{120°}{2}\right)$$
$$w_0=4\sqrt{2}(\cos 60°+i\sin 60°)$$
$$w_0\approx 2\sqrt{2}+2i\sqrt{6}$$

$$w_1=4\sqrt{2}\left(\cos\frac{120°+360°}{2}+i\sin\frac{120°+360°}{2}\right)$$
$$w_1=4\sqrt{2}(\cos 240°+i\sin 240°)$$
$$w_1\approx -2\sqrt{2}-2i\sqrt{6}$$

**30.** $-1+\sqrt{3}i=2(\cos 120°+i\sin 120°)$

$$w_k=2^{1/2}\left(\cos\frac{120°+360°k}{2}+i\sin\frac{120°+360°k}{2}\right)\quad k=0,1$$

$$w_0=2^{1/2}\left(\cos\frac{120°}{2}+i\sin\frac{120°}{2}\right)$$
$$w_0=2^{1/2}(\cos 60°+i\sin 60°)$$
$$w_0=\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2}i$$

$$w_1=2^{1/2}\left(\cos\frac{120°+360°}{2}+i\sin\frac{120°+360°}{2}\right)$$
$$w_1=2^{1/2}(\cos 240°+i\sin 240°)$$
$$w_1=-\frac{\sqrt{2}}{2}-\frac{\sqrt{6}}{2}i$$

**31.** $x^3 + 8 = 0$

$x^3 = -8$

Find the three cube roots of –8.

$-8 = 8(\cos 180° + i\sin 180°)$

$$x_k = 8^{1/3}\left(\cos\frac{180^\circ + 360°k}{3} + i\sin\frac{180^\circ + 360°k}{3}\right) \quad k = 0,1,2$$

$$w_0 = 2\left(\cos\frac{180^\circ}{3} + i\sin\frac{180^\circ}{3}\right)$$
$w_0 = 2(\cos 60° + i\sin 60°)$
$w_0 = 2 \text{ cis } 60°$

$$w_1 = 2\left(\cos\frac{180° + 360°}{3} + i\sin\frac{180° + 360°}{3}\right)$$
$w_1 = 2(\cos 180° + i\sin 180°)$
$w_1 = 2 \text{ cis } 180°$

$$w_2 = 2\left(\cos\frac{180° + 360° \cdot 2}{3} + i\sin\frac{180° + 360° \cdot 2}{3}\right)$$
$w_2 = 2(\cos 300° + i\sin 300°)$
$w_2 = 2 \text{ cis } 300°$

**32.** $x^5 - 32 = 0$

$x^5 = 32$

Find the five fifth roots of 32.

$32 = 32(\cos 0° + i\sin 0°)$

$$w_k = 32^{1/5}\left(\cos\frac{0° + 360°k}{5} + i\sin\frac{0° + 360°k}{5}\right) \quad k = 0, 1, 2, 3, 4$$

$w_0 = 2 \text{ cis } \dfrac{0°}{5}$
$w_0 = 2 \text{ cis } 0°$

$w_1 = 2 \text{ cis } \dfrac{0^\circ + 360^\circ}{5}$
$w_1 = 2 \text{ cis } 72^\circ$

$w_2 = 2 \text{ cis } \dfrac{0^\circ + 360^\circ \cdot 2}{5}$
$w_2 = 2 \text{ cis } 144°$

$w_3 = 2 \text{ cis } \dfrac{0^\circ + 360^\circ \cdot 3}{5}$
$w_3 = 2 \text{ cis } 216°$

$w_4 = 2 \text{ cis } \dfrac{0^\circ + 360^\circ \cdot 4}{5}$
$w_4 = 2 \text{ cis } 288°$

**33.** $x^4 + i = 0$

$x^4 = -i$

Find the four fourth roots of –*i*.

$-i = (\cos 270^\circ + i\sin 270^\circ)$

$$w_k = \cos\frac{270° + 360°k}{4} + i\sin\frac{270° + 360°k}{4} \quad k = 0, 1, 2, 3$$

$w_0 = \text{ cis } \dfrac{270°}{4}$
$w_0 = \text{ cis } 67.5°$

$w_1 = \text{ cis } \dfrac{270° + 360°}{4}$
$w_1 = \text{cis } 157.5°$

$w_2 = \text{cis } \dfrac{270° + 360° \cdot 2}{4}$
$w_2 = \text{cis } 247.5°$

$w_3 = \text{cis } \dfrac{270° + 360° \cdot 3}{4}$
$w_3 = \text{cis } 337.5°$

**34.** $x^3 - 2i = 0$

$x^3 = 2i$

Find the three cube roots of 2*i*.

$2i = 2(\cos 90° + i\sin 90°)$

$w_k = 2^{1/3}\left(\cos\dfrac{90° + 360°k}{3} + i\sin\dfrac{90° + 360°k}{3}\right) = \sqrt[3]{2}\left(\cos\dfrac{90° + 360°k}{3} + i\sin\dfrac{90° + 360°k}{3}\right) \quad k = 0, 1, 2$

$w_0 = \sqrt[3]{2}\ \text{cis}\dfrac{90°}{3}$ $\quad w_1 = \sqrt[3]{2}\ \text{cis}\ \dfrac{90° + 360°}{3}$ $\quad w_2 = \sqrt[3]{2}\ \text{cis}\ \dfrac{90°+360° \cdot 2}{3}$

$w_0 = \sqrt[3]{2}\ \text{cis}\ 30°$ $\quad w_1 = \sqrt[3]{2}\ \text{cis}\ 150°$ $\quad w_2 = \sqrt[3]{2}\ \text{cis}\ 270°$

**35.** $x^3 - 27 = 0$

$x^3 = 27$

Find the three cube roots of 27.

$27 = 27(\cos 0° + i\sin 0°)$

$w_k = 3\left(\cos\dfrac{0° + 360°k}{3} + i\sin\dfrac{0° + 360°k}{3}\right) \quad k = 0, 1, 2$

$w_0 = 3\ \text{cis}\ \dfrac{0°}{3}$ $\quad w_1 = 3\ \text{cis}\ \dfrac{0° + 360°}{3}$ $\quad w_2 = 3\ \text{cis}\ \dfrac{0° + 360° \cdot 2}{3}$

$w_0 = 3\ \text{cis}\ 0°$ $\quad w_1 = 3\ \text{cis}\ 120°$ $\quad w_2 = 3\ \text{cis}\ 240°$

**36.** $x^5 + 32i = 0$

$x^5 = -32i$

Find the five fifth roots of –32*i*.

$-32i = 32(\cos 270° + i\sin 270°)$

$w_k = 32^{1/5}\left(\cos\dfrac{270° + 360°k}{5} + i\sin\dfrac{270° + 360°k}{5}\right) \quad k = 0, 1, 2, 3, 4$

$w_0 = 2\ \text{cis}\ \dfrac{270°}{5}$ $\quad w_1 = 2\ \text{cis}\ \dfrac{270° + 360°}{5}$ $\quad w_2 = 2\ \text{cis}\ \dfrac{270° + 360° \cdot 2}{5}$

$w_0 = 2\ \text{cis}\ 54°$ $\quad w_1 = 2\ \text{cis}\ 126°$ $\quad w_2 = 2\ \text{cis}\ 198°$

$w_3 = 2\ \text{cis}\ \dfrac{270° + 360° \cdot 3}{5}$ $\quad w_4 = 2\ \text{cis}\ \dfrac{270° + 360° \cdot 4}{5}$

$w_3 = 2\ \text{cis}\ 270°$ $\quad w_4 = 2\ \text{cis}\ 342°$

**37.** $x^4 + 81 = 0$

$x^4 = -81$

Find the four fourth roots of –81.

$-81 = 81(\cos 180° + i\sin 180°)$

$w_k = 81^{1/4}\left(\cos\dfrac{180° + 360°k}{4} + i\sin\dfrac{180° + 360°k}{4}\right) \quad k = 0, 1, 2, 3$

$w_0 = 3\ \text{cis}\ \dfrac{180°}{4}$ $\quad w_1 = 3\ \text{cis}\ \dfrac{180° + 360°}{4}$ $\quad w_2 = 2\ \text{cis}\ \dfrac{0° + 360° \cdot 2}{5}$ $\quad w_3 = 3\ \text{cis}\ \dfrac{180° + 360° \cdot 3}{4}$

$w_0 = 3\ \text{cis}\ 45°$ $\quad w_1 = 3\ \text{cis}\ 135°$ $\quad w_2 = 2\ \text{cis}\ 225°$ $\quad w_3 = 3\ \text{cis}\ 315°$

**38.** $x^3 - 64i = 0$

$x^3 = 64i$

Find the three cube roots of $64i$.

$64i = 64(\cos 90° + i\sin 90°)$

$w_k = 64^{1/3}\left(\cos\frac{90° + 360°k}{3} + i\sin\frac{90° + 360°k}{3}\right) \qquad k = 0, 1, 2$

$w_0 = 4 \text{ cis } \frac{90°}{3}$ $\qquad w_1 = 4 \text{ cis } \frac{90° + 360°}{3}$ $\qquad w_2 = 4 \text{ cis } \frac{90° + 360° \cdot 2}{3}$

$w_0 = 4 \text{ cis } 30°$ $\qquad w_1 = 4 \text{ cis } 150°$ $\qquad w_2 = 4 \text{ cis } 270°$

**39.** $x^4 - (1 - i\sqrt{3}) = 0$

$x^4 = 1 - i\sqrt{3}$

Find the four fourth roots of $1 - i\sqrt{3}$.

$1 - i\sqrt{3} = 2(\cos 300° + i\sin 300°)$

$w_k = 2^{1/4}\left(\cos\frac{300° + 360°k}{4} + i\sin\frac{300° + 360°k}{4}\right) \qquad k = 0, 1, 2, 3$

$w_0 = \sqrt[4]{2} \text{ cis } \frac{300°}{4}$ $\qquad w_1 = \sqrt[4]{2} \text{ cis } \frac{300° + 360°}{4}$ $\qquad w_2 = \sqrt[4]{2} \text{ cis } \frac{300° + 360° \cdot 2}{4}$ $\qquad w_3 = \sqrt[4]{2} \text{ cis } \frac{300° + 360° \cdot 3}{4}$

$w_0 = \sqrt[4]{2} \text{ cis } 75°$ $\qquad w_1 = \sqrt[4]{2} \text{ cis } 165°$ $\qquad w_2 = \sqrt[4]{2} \text{ cis } 255°$ $\qquad w_3 = \sqrt[4]{2} \text{ cis } 345°$

**40.** $x^3 + (2\sqrt{3} - 2i) = 0$

$x^3 = -2\sqrt{3} + 2i$

Find the three cube roots of $-2\sqrt{3} + 2i$.

$-2\sqrt{3} + 2i = 4(\cos 150° + i\sin 150°)$

$w_k = 4^{1/3}\left(\cos\frac{150° + 360°k}{3} + i\sin\frac{150° + 360°k}{3}\right) \qquad k = 0, 1, 2$

$w_0 = \sqrt[3]{4} \text{ cis } \frac{150°}{3}$ $\qquad w_1 = \sqrt[3]{4} \text{ cis } \frac{150° + 360°}{3}$ $\qquad w_2 = \sqrt[3]{4} \text{ cis } \frac{150° + 360° \cdot 2}{3}$

$w_0 = \sqrt[3]{4} \text{ cis } 50°$ $\qquad w_1 = \sqrt[3]{4} \text{ cis } 170°$ $\qquad w_2 = \sqrt[3]{4} \text{ cis } 290°$

**41.** $x^3 + (1 + i\sqrt{3}) = 0$

$x^3 = -1 - i\sqrt{3}$

Find the three cube roots of $-1 - i\sqrt{3}$.

$-1 - i\sqrt{3} = 2(\cos 240° + i\sin 240°)$

$w_k = 2^{1/3}\left(\cos\frac{240° + 360°k}{3} + i\sin\frac{240° + 360°k}{3}\right) \qquad k = 0, 1, 2$

$w_0 = \sqrt[3]{2} \text{ cis } \frac{240°}{3}$ $\qquad w_1 = \sqrt[3]{2} \text{ cis } \frac{240° + 360°}{3}$ $\qquad w_2 = \sqrt[3]{2} \text{ cis } \frac{240° + 360° \cdot 2}{3}$

$w_0 = \sqrt[3]{2} \text{ cis } 80°$ $\qquad w_1 = \sqrt[3]{2} \text{ cis } 200°$ $\qquad w_2 = \sqrt[3]{2} \text{ cis } 320°$

**42.** $x^6 - (4-4i) = 0$

$$x^6 = 4-4i$$

Find the six sixth roots of $4-4i$.

$$4-4i = 4\sqrt{2}(\cos 315° + i\sin 315°)$$

$$w_k = \left(4\sqrt{2}\right)^{1/6}\left(\cos\frac{315° + 360°k}{6} + i\sin\frac{315° + 360°k}{6}\right) \qquad k = 0, 1, 2, 3, 4, 5$$

$$w_0 = \left(4\sqrt{2}\right)^{1/6} \text{ cis } \frac{315°}{6}$$
$$w_0 = \left(4\sqrt{2}\right)^{1/6} \text{ cis } 52.5°$$

$$w_1 = \left(4\sqrt{2}\right)^{1/6} \text{ cis } \frac{315° + 360°}{6}$$
$$w_1 = \left(4\sqrt{2}\right)^{1/6} \text{ cis } 112.5°$$

$$w_2 = \left(4\sqrt{2}\right)^{1/6} \text{ cis } \frac{315° + 360° \cdot 2}{6}$$
$$w_2 = \left(4\sqrt{2}\right)^{1/6} \text{ cis } 172.5°$$

$$w_3 = \left(4\sqrt{2}\right)^{1/6} \text{ cis } \frac{315° + 360° \cdot 3}{6}$$
$$w_3 = \left(4\sqrt{2}\right)^{1/6} \text{ cis } 232.5°$$

$$w_4 = \left(4\sqrt{2}\right)^{1/6} \text{ cis } \frac{315° + 360° \cdot 4}{6}$$
$$w_4 = \left(4\sqrt{2}\right)^{1/6} \text{ cis } 292.5°$$

$$w_5 = \left(4\sqrt{2}\right)^{1/6} \text{ cis } \frac{315° + 360° \cdot 5}{6}$$
$$w_5 = \left(4\sqrt{2}\right)^{1/6} \text{ cis } 352.5°$$

## Connecting Concepts

**43.** Let $z = a + bi$. Then $\bar{z} = a - bi$ by definition.
Substitute $a = r\cos\theta$ and $b = r\sin\theta$.
Thus $\bar{z} = r\cos\theta - ri\sin\theta = r(\cos\theta - i\sin\theta)$

**44.** $w_5 = 1.33(\cos 352.5° + i\sin 352.5°)$

$$\frac{1}{z} = \frac{1}{r(\cos\theta + i\sin\theta)} = \frac{\cos\theta - i\sin\theta}{r(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$$
$$= \frac{\cos\theta - i\sin\theta}{r(\cos^2\theta + i^2\sin^2\theta)} = \frac{\cos\theta - i\sin\theta}{r(\cos^2\theta + \sin^2\theta)}$$
$$z^{-1} = r^{-1}(\cos\theta - i\sin\theta)$$

**45.**

$$z = r(\cos\theta + i\sin\theta)$$
$$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$$
$$\frac{1}{z^2} = \frac{1}{r^2(\cos 2\theta + i\sin 2\theta)}$$
$$= \frac{\cos 2\theta - i\sin 2\theta}{r^2(\cos 2\theta + i\sin 2\theta)(\cos 2\theta - i\sin 2\theta)}$$
$$= \frac{\cos 2\theta - i\sin 2\theta}{r^2(\cos^2 2\theta - i^2\sin^2 2\theta)} = \frac{\cos 2\theta - i\sin 2\theta}{r^2(\cos^2 2\theta + \sin^2 2\theta)}$$
$$z^{-2} = r^{-2}(\cos 2\theta - i\sin 2\theta)$$

**46.** Exercises 42 and 43 imply that
$z^{-n} = r^{-n}(\cos n\theta - i\sin n\theta)$.

$$z = 1 - i\sqrt{3} = 2(\cos 300° + i\sin 300°)$$
$$z^{-4} = 2^{-4}[\cos 4(300°) - i\sin 4(300°)]$$
$$z^{-4} = \tfrac{1}{16}(\cos 1200° - i\sin 1200°)$$
$$z^{-4} = \tfrac{1}{16}(\cos 120° - i\sin 120°)$$
$$z^{-4} = \tfrac{1}{16}[\cos(-120°) + i\sin(-120°)]$$
$$z^{-4} = \tfrac{1}{16} \text{ cis } (-120°)$$

**47.** For $n = 2$, the two square roots of 1 are
1 and –1.
The sum of these roots is
$1 + (-1) = 0$.

For $n = 3$, the three cube roots of 1 are (from exercise 23)
$1,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i$ and $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$.
The sum of these roots is
$1-\frac{1}{2}+\frac{\sqrt{3}}{2}i-\frac{1}{2}-\frac{\sqrt{3}}{2}i=0$.

For $n = 4$, the four fourth roots of 1 are
1, –1, $i$, and $-i$.
The sum of these roots is
$1-1+i-i=0$

For $n = 5$, the five fifth roots of 1 are
1, cis 72°, cis 144°, cis 216°, cis 288°
The sum of these roots is
$1+\text{cis } 72°+\text{cis } 144°+\text{cis } 216°+\text{cis } 288°=0$

For $n = 6$, the six sixth roots of 1 are
$1, -1,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i,\ \frac{1}{2}+\frac{\sqrt{3}}{2}i,\ \frac{1}{2}-\frac{\sqrt{3}}{2}i$
The sum of these roots is
$1-1-\frac{1}{2}+\frac{\sqrt{3}}{2}i-\frac{1}{2}-\frac{\sqrt{3}}{2}i+\frac{1}{2}+\frac{\sqrt{3}}{2}i+\frac{1}{2}-\frac{\sqrt{3}}{2}i=0$

For $n \ge 2$, the sum of the $n$th roots of 1 is 0.

**48.** For $n = 2$, the two square roots of 1 are
1 and –1.
The product of these roots is
$1\cdot(-1)=-1$.

For $n = 3$, the three cube roots of 1 are (from exercise 23)
$1,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i$ and $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$.
The product of these roots is
$1\cdot\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\cdot\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=1$.

For $n = 4$, the four fourth roots of 1 are
1, –1, $i$, and $-i$.
The product of these roots is
$1\cdot(-1)\cdot(i)\cdot(-i)=-1$

For $n = 5$, the five fifth roots of 1 are
1, cis 72°, cis 144°, cis 216°, cis 288°
The product of these roots is
$1\cdot(\text{cis } 72°)\cdot(\text{cis } 144°)\cdot(\text{cis } 216°)\cdot(\text{cis } 288°)=1$

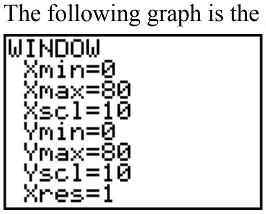
For $n = 6$, the six sixth roots of 1 are
$1, -1,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i,\ \frac{1}{2}+\frac{\sqrt{3}}{2}i,\ \frac{1}{2}-\frac{\sqrt{3}}{2}i$
The product of these roots is
$1\cdot(-1)\cdot\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\cdot\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\cdot\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\cdot\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=-1$

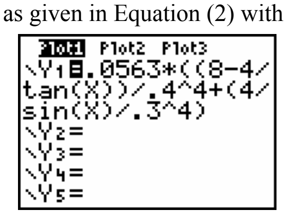
For $n \ge 2$, the sum of the $n$th roots of 1 is –1 if $n$ is even and 1 if $n$ is odd.

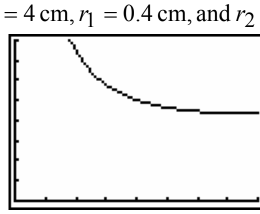
## Exploring Concepts with Technology

**Optimal Branching of Arteries**

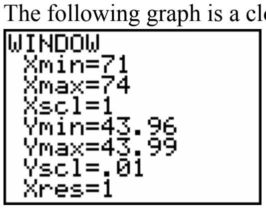
The following graph is the graph of $R$ as given in Equation (2) with $a = 8$ cm, $b = 4$ cm, $r_1 = 0.4$ cm, and $r_2 = 0.3$ cm.

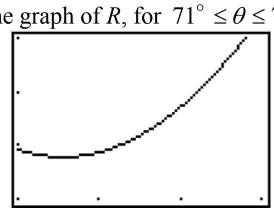


The following graph is a close-up of the graph of $R$, for $71^\circ \le \theta \le 74^\circ$.



According to this graph $R$ is a minimum when $\theta = 72^\circ$ (to the nearest degree).
Using Equation (3) yields

$$\cos\theta = \left(\frac{\frac{3}{4}r_1}{r_1}\right)^4$$

$$\cos\theta = \frac{81}{256}$$

$$\theta = \cos^{-1}\left(\frac{81}{256}\right) \approx 72^\circ$$

## Assessing Concepts

**1.** An oblique triangle that does not contain a right angle.

**2.** The Law of Cosines

**3.** SSA

**4.** A scalar

**5.** A scalar

**6.** Yes

**7.** Yes

**8.** One

**9.** Five

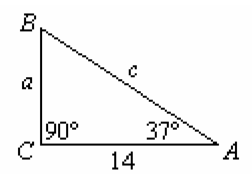
**10.** Yes

**11.** No

**12.** 3 + 5$i$

## Chapter Review

**1.**



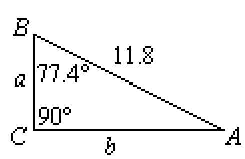

$B = 180° - 92° - 37°$
$B = 51°$

$\tan A = \dfrac{a}{4}$
$a = 14 \tan 37°$
$a \approx 11$

$\cos A = \dfrac{14}{c}$
$c = \dfrac{14}{\cos\ 37°}$
$c \approx 18$

[7.1]

**2.**



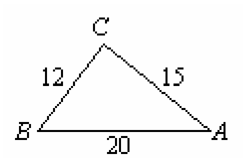

$A = 180° - 94.0° - 77.4°$
$A = 8.6°$

$\sin\ B = \dfrac{b}{11.8}$
$b = 11.8\ \sin\ 77.4°$
$b \approx 11.5$

$\cos\ B = \dfrac{a}{11.8}$
$a = 11.8\ \cos\ 77.4°$
$a \approx 2.57$

[7.1]

**3.**

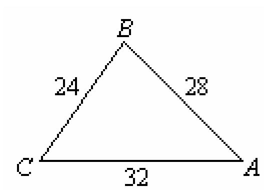



$\cos\ B = \dfrac{12^2 + 20^2 - 15^2}{2(12)(20)}$
$\cos\ B \approx 0.6646$
$B \approx 48°$

$\cos C = \dfrac{12^2 + 15^2 - 20^2}{2(12)(15)}$
$\cos C \approx -0.0861$
$C \approx 95°$

$A = 180° - 48° - 95°$ [7.2]
$A = 37°$

**4.**

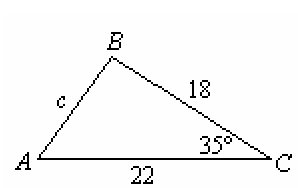



$\cos C = \dfrac{24^2 + 32^2 - 28^2}{2(24)(32)}$
$\cos C \approx 0.5313$
$C \approx 58°$

$\cos A = \dfrac{32^2 + 28^2 - 24^2}{2(32)(28)}$
$\cos A \approx 0.6875$
$A \approx 47°$

$B \approx 180° - 58° - 47°$ [7.2]
$B \approx 75°$

**5.**

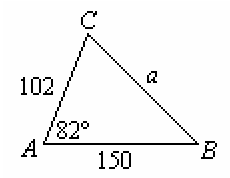



$c^2 = 22^2 + 18^2 - 2(22)(18)\cos 35°$
$c^2 \approx 159$
$c = \sqrt{159}$
$c \approx 13$

$\dfrac{18}{\sin A} = \dfrac{\sqrt{159}}{\sin 35°}$
$\sin A = \dfrac{18 \sin 35°}{\sqrt{159}}$
$\sin A \approx 0.8188$
$A \approx 55°$

$B \approx 180 - 35° - 55°$ [7.2]
$B \approx 90°$

**6.**

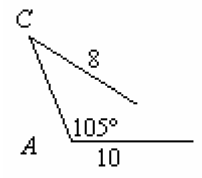



$$a^2 = 102^2 + 150^2 - 2(102)(150)\cos 82°$$
$$a^2 \approx 28645$$
$$a = \sqrt{28645}$$
$$a \approx 169$$

$$\frac{150}{\sin C} = \frac{\sqrt{28645}}{\sin 82°}$$
$$\sin C = \frac{150 \sin 82°}{\sqrt{28645}}$$
$$\sin C \approx 0.8776$$
$$C \approx 61°$$

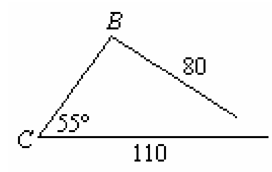
$$B \approx 180° - 61° - 82° \quad [7.2]$$
$$B \approx 37°$$

**7.**



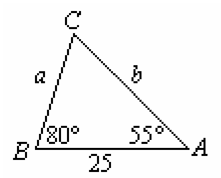

$$\frac{10}{\sin C} = \frac{8}{\sin 105°} \quad [7.1]$$
$$\sin C = \frac{10 \sin 105°}{8}$$
$$\sin C \approx 1.207$$
No triangle is formed.

**8.**




$$\frac{110}{\sin B} = \frac{80}{\sin 55°} \quad [7.1]$$
$$\sin B = \frac{110 \sin 55°}{80}$$
$$\sin B \approx 1.1263$$
No triangle is formed.

**9.**

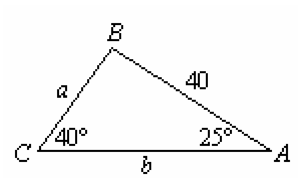



$$C = 180° - 80° - 55°$$
$$C = 45°$$

$$\frac{25}{\sin 45°} = \frac{a}{\sin 55°}$$
$$a = \frac{25 \sin 55°}{\sin 45°}$$
$$a \approx 29$$

$$\frac{25}{\sin 45°} = \frac{b}{\sin 80°} \quad [7.1]$$
$$b = \frac{25 \sin 80°}{\sin 45°}$$
$$b \approx 35$$

**10.**




$$A = 180° - 40° - 25°$$
$$A = 115°$$

$$\frac{a}{\sin 115°} = \frac{40}{\sin 40°}$$
$$a = \frac{40 \sin 115°}{\sin 40°}$$
$$a \approx 56$$

$$\frac{b}{\sin 25°} = \frac{40}{\sin 40°} \quad [7.1]$$
$$b = \frac{40 \sin 25°}{\sin 40°}$$
$$b \approx 26$$

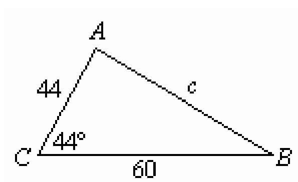
**11.** $s = \frac{1}{2}(a+b+c)$ [7.2]

$$s = \frac{1}{2}(24+30+36)$$
$$s = 45$$
$$K = \sqrt{s(s-a)(s-b)(s-c)}$$
$$K = \sqrt{45(45-24)(45-30)(45-36)}$$
$$K = \sqrt{127,575}$$
$K \approx 360$ square units

**12.** $s = \frac{1}{2}(a+b+c)$ [7.2]

$$s = \frac{1}{2}(9.0+7.0+12)$$
$$s = 14$$
$$K = \sqrt{s(s-a)(s-b)(s-c)}$$
$$K = \sqrt{14(14-9.0)(14-7.0)(14-12)}$$
$$K = \sqrt{980}$$
$K \approx 31$ square units

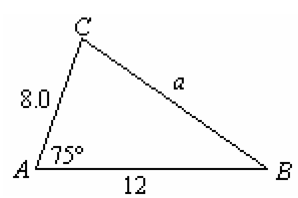
**13.**




$$K = \frac{1}{2}ab\sin C$$
$$K = \frac{1}{2}(60)(44)\sin 44°$$
$K \approx 920$ square units

[7.2]

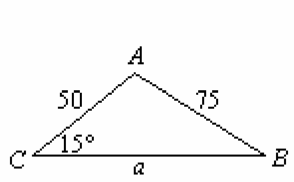
**14.**




$$K = \frac{1}{2}bc\sin A$$
$$K = \frac{1}{2}(8.0)(12)\sin 75°$$
$K \approx 46$ square units

[7.2]

**15.**

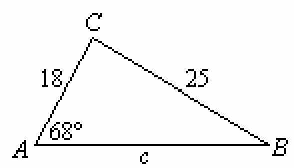



$$\frac{50}{\sin B}=\frac{75}{\sin 15°}$$
$$\sin B=\frac{50\sin 15°}{75}$$
$$\sin B\approx 0.1725$$
$$B\approx 10°$$

$$A\approx 180°-10°-15°$$
$$A\approx 155°$$

$$K\approx\frac{1}{2}(50)(75)\sin 155°$$ [7.2]
$K\approx 790$ square units

**16.**

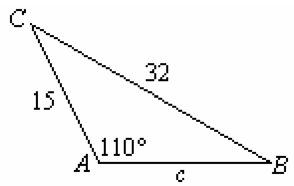



$$\frac{18}{\sin B}=\frac{25}{\sin 68°}$$
$$\sin B=\frac{18\sin 68°}{25}$$
$$\sin B\approx 0.6676$$
$$B\approx 42°$$

$$C\approx 180°-42°-68°$$
$$C\approx 70°$$

$$K\approx\frac{1}{2}(18)(25)\sin 70°$$ [7.2]
$K\approx 210$ square units

**17.**

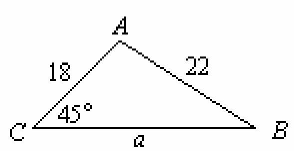



$$\frac{15}{\sin B}=\frac{32}{\sin 110°}$$
$$\sin B=\frac{15\sin 110°}{32}$$
$$\sin B\approx 0.4405$$
$$B\approx 26°$$

$$C\approx 180°-110°-26°$$
$$C\approx 44°$$

$$K\approx\frac{1}{2}(15)(32)\sin 44°$$ [7.2]
$K\approx 170$ square units

**18.**




$$\frac{18}{\sin B}=\frac{22}{\sin 45°}$$
$$\sin B\approx\frac{18\sin 45°}{22}$$
$$\sin B\approx 0.5785$$
$$B\approx 35°$$

$$A\approx 180°-45°-35°$$
$$A\approx 100°$$

$$K\approx\frac{1}{2}(18)(22)\sin 100°$$ [7.2]
$K\approx 190$ square units

**19.** Let $\mathbf{P_1P_2}=a_1\mathbf{i}+a_2\mathbf{j}$. [7.3]

$$a_1=3-(-2)=5$$
$$a_2=7-4=3$$

A vector equivalent to $\mathbf{P_1P_2}$ is $\mathbf{v}=\langle 5,\ 3\rangle$.

**20.** Let $\mathbf{P_1P_2}=a_1\mathbf{i}+a_2\mathbf{j}$. [7.3]

$$a_1=-3-(-4)=1$$
$$a_2=6-0=6$$

A vector equivalent to $\mathbf{P_1P_2}$ is $\mathbf{v}=\langle 1,\ 6\rangle$.

**21.**

$$\|\mathbf{v}\|=\sqrt{(-4)^2+2^2}$$
$$\|\mathbf{v}\|=\sqrt{16+4}$$
$$\|\mathbf{v}\|\approx 4.5$$

$$\alpha\approx\tan^{-1}\left|\frac{2}{-4}\right|=\tan^{-1}\frac{1}{2}$$ [7.3]
$$\alpha\approx 26.6°$$
$$\theta\approx 180°-26.6°$$
$$\theta\approx 153.4°$$

**22.**

$$\|\mathbf{v}\|=\sqrt{6^2+(-3)^2}$$
$$\|\mathbf{v}\|=\sqrt{36+9}$$
$$\|\mathbf{v}\|\approx 6.7$$

$$\alpha\approx\tan^{-1}\left|\frac{-3}{6}\right|=\tan^{-1}\frac{1}{2}$$ [7.3]
$$\alpha\approx 26.6°$$
$$\theta\approx 360°-26.6°$$
$$\theta\approx 333.4°$$

**23.**

$$\|\mathbf{u}\|=\sqrt{(-2)^2+3^2}$$
$$\|\mathbf{u}\|=\sqrt{4+9}$$
$$\|\mathbf{u}\|\approx 3.6$$

$$\alpha=\tan^{-1}\left|\frac{3}{-2}\right|=\tan^{-1}\frac{3}{2}$$ [7.3]
$$\alpha\approx 56.3°$$
$$\theta\approx 180°-56.3°$$
$$\theta\approx 123.7°$$

**24.**

$$\|\mathbf{u}\|=\sqrt{(-4)^2+(-7)^2}$$
$$\|\mathbf{u}\|=\sqrt{16+49}$$
$$\|\mathbf{u}\|\approx 8.1$$

$$\alpha=\tan^{-1}\left|\frac{-7}{-4}\right|=\tan^{-1}\frac{7}{4}$$ [7.3]
$$\alpha\approx 60.3°$$
$$\theta\approx 180°+60.3°$$
$$\theta\approx 240.3°$$

**25.**

$$\|\mathbf{w}\|=\sqrt{(-8)^2+5^2}$$
$$\|\mathbf{w}\|=\sqrt{89}$$

$$\|\mathbf{u}\|=\left\langle\frac{-8}{\sqrt{89}},\ \frac{5}{\sqrt{89}}\right\rangle=\left\langle-\frac{8\sqrt{89}}{89},\ \frac{5\sqrt{89}}{89}\right\rangle$$ [7.3]

A unit vector in the direction of $\|\mathbf{w}\|$ is $\|\mathbf{u}\|=\left\langle-\frac{8\sqrt{89}}{89},\ \frac{5\sqrt{89}}{89}\right\rangle$.

**26.** $\|\mathbf{w}\| = \sqrt{7^2 + (-12)^2}$

$\|\mathbf{w}\| = \sqrt{193}$

$\mathbf{u} = \left\langle \frac{7}{\sqrt{193}}, \frac{-12}{\sqrt{193}} \right\rangle = \left\langle \frac{7\sqrt{193}}{193}, -\frac{12\sqrt{193}}{193} \right\rangle$ [7.3]

A unit vector in the direction of $\mathbf{w}$ is $\mathbf{u} = \left\langle \frac{7\sqrt{193}}{193}, -\frac{12\sqrt{193}}{193} \right\rangle$.

**27.** $\|\mathbf{v}\| = \sqrt{5^2 + 1^2}$

$\|\mathbf{v}\| = \sqrt{26}$

$\mathbf{u} = \frac{5}{\sqrt{26}}\mathbf{i} + \frac{1}{\sqrt{26}}\mathbf{j} = \frac{5\sqrt{26}}{26}\mathbf{i} + \frac{\sqrt{26}}{26}\mathbf{j}$ [7.3]

A unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{5\sqrt{26}}{26}\mathbf{i} + \frac{\sqrt{26}}{26}\mathbf{j}$.

**28.** $\|\mathbf{v}\| = \sqrt{3^2 + (-5)^2}$

$\|\mathbf{v}\| = \sqrt{34}$

$\mathbf{u} = \frac{3}{\sqrt{34}}\mathbf{i} - \frac{5}{\sqrt{34}}\mathbf{j} = \frac{3\sqrt{34}}{34}\mathbf{i} - \frac{5\sqrt{34}}{34}\mathbf{j}$ [7.3]

A unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{3\sqrt{34}}{34}\mathbf{i} - \frac{5\sqrt{34}}{34}\mathbf{j}$.

**29.** $\mathbf{v} - \mathbf{u} = \langle -4, -1 \rangle - \langle 3, 2 \rangle$ [7.3]

$= \langle -7, -3 \rangle$

**30.** $2\mathbf{u} - 3\mathbf{v} = 2\langle 3, 2 \rangle - 3\langle -4, -1 \rangle$ [7.3]

$= \langle 6, 4 \rangle - \langle -12, -3 \rangle$

$= \langle 18, 7 \rangle$

**31.** $-\mathbf{u} + \frac{1}{2}\mathbf{v} = -(10\mathbf{i} + 6\mathbf{j}) + \frac{1}{2}(8\mathbf{i} - 5\mathbf{j})$ [7.3]

$= (-10\mathbf{i} - 6\mathbf{j}) + \left(4\mathbf{i} - \frac{5}{2}\mathbf{j}\right)$

$= (-10 + 4)\mathbf{i} + \left(-6 - \frac{5}{2}\right)\mathbf{j}$

$= -6\mathbf{i} - \frac{17}{2}\mathbf{j}$

**32.** $\frac{2}{3}\mathbf{v} - \frac{3}{4}\mathbf{u} = \frac{2}{3}(8\mathbf{i} - 5\mathbf{j}) - \frac{3}{4}(10\mathbf{i} + 6\mathbf{j})$ [7.3]

$= \left(\frac{16}{3}\mathbf{i} - \frac{10}{3}\mathbf{j}\right) - \left(\frac{15}{2}\mathbf{i} + \frac{9}{2}\mathbf{j}\right)$

$= \left(\frac{16}{3} - \frac{15}{2}\right)\mathbf{i} + \left(-\frac{10}{3} - \frac{9}{2}\right)\mathbf{j}$

$= \frac{32 - 45}{6}\mathbf{i} + \frac{-20 = 27}{6}\mathbf{j}$

$= -\frac{13}{6}\mathbf{i} - \frac{47}{6}\mathbf{j}$

**33.** $\mathbf{v} = 400\sin 204°\mathbf{i} + 400\cos 204°\mathbf{j}$

$\mathbf{v} \approx -162.7\mathbf{i} - 365.4\mathbf{j}$

$\mathbf{w} = -45\mathbf{i}$

$\mathbf{R} = \mathbf{v} + \mathbf{w}$

$\mathbf{R} \approx -162.7\mathbf{i} - 365.4\mathbf{j} - 45\mathbf{i}$

$\mathbf{R} \approx -207.7\mathbf{i} - 365.4\mathbf{j}$

$\|\mathbf{R}\| \approx \sqrt{(-207.7)^2 + (-365.4)^2}$

$\|\mathbf{R}\| \approx 420$ mph

$\alpha = \tan^{-1}\left|\frac{-365.4}{-207.7}\right| = \tan^{-1}\frac{365.4}{207.7}$

$\alpha \approx 60°$

$\theta \approx 180° + 60°$

$\theta \approx 240°$

The ground speed is approximately 420 mph at a heading of 240° [7.3]

**34.** $\theta = \sin^{-1}\frac{40}{320}$ [7.3]

$\theta \approx 7°$

**35.** $\mathbf{u} \cdot \mathbf{v} = \langle 3, 7 \rangle \cdot \langle -1, 3 \rangle$ [7.3]

$= (3)(-1) + (7)(3)$

$= 18$

**36.** $\mathbf{v} \cdot \mathbf{u} = \langle -8, 5 \rangle \cdot \langle 2, -1 \rangle$ [7.3]

$= (-8)(2) + (5)(-1)$

$= -21$

**37.** $\mathbf{v}\cdot\mathbf{u} = (-4\mathbf{i}-\mathbf{j})\cdot(2\mathbf{i}+\mathbf{j})$ [7.3]
$= (-4)(2)+(-1)(1)$
$= -9$

**38.** $\mathbf{u}\cdot\mathbf{v} = (-3\mathbf{i}+7\mathbf{j})\cdot(-2\mathbf{i}+2\mathbf{j})$ [7.3]
$= (-3)(-2)+(7)(2)$
$= 20$

**39.** $\cos\alpha = \dfrac{\langle 7,-4\rangle\cdot\langle 2,3\rangle}{\sqrt{7^2+(-4)^2}\sqrt{2^2+3^2}}$ [7.3]
$\cos\alpha = \dfrac{14+(-12)}{\sqrt{65}\sqrt{13}}$
$\cos\alpha \approx 0.0688$
$\alpha \approx 86°$

**40.** $\cos\alpha = \dfrac{\langle -5,2\rangle\cdot\langle 2,-4\rangle}{\sqrt{(-5)^2+2^2}\sqrt{2^2+(-4)^2}}$
$\cos\alpha = \dfrac{-10-8}{\sqrt{29}\sqrt{20}}$
$\cos\alpha \approx -0.7474$ [7.3]
$\alpha = 138°$

**41.** $\cos\alpha = \dfrac{(6\mathbf{i}-11\mathbf{j})\cdot(2\mathbf{i}-4\mathbf{j})}{\sqrt{6^2+(-11)^2}\sqrt{2^2+4^2}}$
$\cos\alpha = \dfrac{12-44}{\sqrt{157}\sqrt{20}}$
$\cos\alpha \approx -0.5711$ [7.3]
$\cos\alpha \approx 125°$

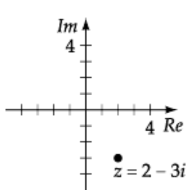
**42.** $\cos\alpha = \dfrac{(\mathbf{i}-5\mathbf{j})\cdot(\mathbf{i}+5\mathbf{j})}{\sqrt{1^2+(-5)^2}\sqrt{1^2+5^2}}$
$\cos\alpha = \dfrac{1-25}{\sqrt{26}\sqrt{26}}$
$\cos\alpha \approx -0.9231$ [7.3]
$\alpha \approx 157°$

**43.** $\text{proj}_{\mathbf{w}}\mathbf{v} = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|}$ [7.3]
$\text{proj}_{\mathbf{w}}\mathbf{v} = \dfrac{\langle -2,5\rangle\cdot\langle 5,4\rangle}{\sqrt{5^2+4^2}}$
$= \dfrac{-10+20}{\sqrt{41}}$
$= \dfrac{10}{\sqrt{41}}$
$= \dfrac{10\sqrt{41}}{41}$

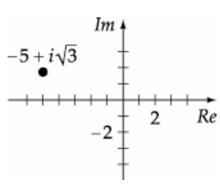
**44.** $\text{proj}_{\mathbf{w}}\mathbf{v} = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|}$ [7.3]
$\text{proj}_{\mathbf{w}}\mathbf{v} = \dfrac{(4\mathbf{i}-7\mathbf{j})\cdot(-2\mathbf{i}-5\mathbf{j})}{\sqrt{(-2)^2+(-5)^2}}$
$= \dfrac{-8+35}{\sqrt{29}}$
$= \dfrac{27}{\sqrt{29}}$
$= \dfrac{27\sqrt{29}}{29}$

**45.** $\mathbf{w} = \|\mathbf{F}\|\ \|\mathbf{S}\|\cos\theta$ [7.3]
$\mathbf{w} = 60\cdot 14\cos 38°$
$\mathbf{w} \approx 662$ foot-pounds

**46.** $r = \sqrt{2^2+(-3)^2}$ [7.4]
$r = \sqrt{13}$
$\alpha = \tan^{-1}\left|\dfrac{-3}{2}\right| = \tan^{-1}\dfrac{3}{2}$
$\alpha \approx 56°$
$\theta \approx 360° - 56°$
$\theta \approx 304°$




**47.** $r = \sqrt{(-5)^2+(\sqrt{3})^2}$ [7.4]
$r = \sqrt{28} \approx 5.29$
$\alpha = \tan^{-1}\left|\dfrac{\sqrt{3}}{-5}\right| = \tan^{-1}\dfrac{\sqrt{3}}{5}$
$\alpha \approx 19°$
$\theta \approx 180° - 19°$
$\theta \approx 161°$




**48.** $z = 2-2i$ [7.4]
$r = \sqrt{2^2+(-2)^2}$
$r = \sqrt{8} = 2\sqrt{2}$
$\alpha = \tan^{-1}\left|\dfrac{-2}{2}\right| = \tan^{-1}1$
$\alpha = 45°$
$\theta = 360° - 45°$
$\theta = 315°$
$z = 2\sqrt{2}\ \text{cis}\ 315°$

**49.** $z = -\sqrt{3} + 3i$ [7.4]

$r = \sqrt{\left(-\sqrt{3}\right)^2 + 3^2}$

$r = \sqrt{12} = 2\sqrt{3}$

$\alpha = \tan^{-1}\left|\frac{3}{-\sqrt{3}}\right| = \tan^{-1}\frac{3}{\sqrt{3}}$

$\alpha = 60°$

$\theta = 180° - 60°$

$\theta = 120°$

$z = 2\sqrt{3}$ cis $120°$

**50.** $z = 5(\cos 315° + i\sin 315°)$ [7.4]

$z = 5\left(\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}\right)$

$z = \frac{5\sqrt{2}}{2} - \frac{5\sqrt{2}}{2}i$

**51.** $z = 6\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$ [7.4]

$z = 6\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$

$z = -3 - 3i\sqrt{3}$

**52.** $z_1z_2 = 5 \text{ cis } 162° \cdot 2 \text{ cis } 63°$ [7.4]

$z_1z_2 = 10 \text{ cis}(162° + 63°)$

$z_1z_2 = 10 \text{ cis } 225°$

$z_1z_2 = 10(\cos 225° + i\sin 225°)$

$z_1z_2 = 10\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$

$z_1z_2 = -5\sqrt{2} - 5\sqrt{2}i$

or $-5\sqrt{2} - 5i\sqrt{2}$

**53.** $z_1z_2 = 3 \text{ cis } 12° \cdot 4 \text{ cis } 126°$ [7.4]

$z_1z_2 = 12 \text{ cis}(12° + 126°)$

$z_1z_2 = 12 \text{ cis } 138°$

$z_1z_2 = 12(\cos 138° + i\sin 138°)$

$z_1z_2 \approx -8.918 + 8.030i$

**54.** $z_1z_2 = 7 \text{ cis } \frac{2\pi}{3} \cdot 4 \text{ cis } \frac{\pi}{4}$ [7.4]

$z_1z_2 = 28 \text{ cis } \left(\frac{2\pi}{3} + \frac{\pi}{4}\right)$

$z_1z_2 = 28 \text{ cis } \frac{11\pi}{12}$

$z_1z_2 = 28\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$

$z_1z_2 \approx -27.046 + 7.247i$

**55.** $z_1z_2 = (3 \text{ cis } 1.8) \cdot (5 \text{ cis } 2.5)$ [7.4]

$z_1z_2 = 15 \text{ cis}(1.8 + 2.5)$

$z_1z_2 = 15 \text{ cis } 4.3$

$z_1z_2 = 15(\cos 4.3 + i\sin 4.3)$

$z_1z_2 \approx -6.012 - 13.742i$

**56.** $\frac{z_1}{z_2} = \frac{6 \text{ cis } 50°}{2 \text{ cis } 150°}$ [7.4]

$\frac{z_1}{z_2} = 3 \text{ cis } 9\ (50° - 150°)$

$\frac{z_1}{z_2} = 3 \text{ cis } (-100°)$ or $3 \text{ cis } 260°$

**57.** $\frac{z_1}{z_2} = \frac{30 \text{ cis } 165°}{10 \text{ cis } 55°}$ [7.4]

$\frac{z_1}{z_2} = 3 \text{ cis}(165° - 55°)$

$\frac{z_1}{z_2} = 3 \text{ cis } 110°$

**58.** $\frac{z_1}{z_2} = \frac{40 \text{ cis } 66°}{8 \text{ cis } 125°} = 5 \text{ cis } (66° - 125°)$ [7.4]

$\frac{z_1}{z_2} = 5 \text{ cis } (-59°)$ or $5 \text{ cis } 301°$

**59.** $\frac{z_1}{z_2} = \frac{\sqrt{3} - i}{1 + i}$ $\quad\sqrt{3} - i = 2 \text{ cis } 330° \quad 1 + \text{i} = \sqrt{2} \text{ cis } 45°$ [7.4]

$= \frac{2 \text{ cis } 330°}{\sqrt{2} \text{ cis } 45°}$

$= \sqrt{2} \text{ cis}(330° - 45°)$

$= \sqrt{2} \text{ cis } 285°$

**60.** $(3 \text{ cis } 45°)^6 = 3^6 \text{cis } 6 \cdot 45°$ [7.5]

$= 729 \text{ cis } 270° = 729(\cos 270° + i\sin 270°)$

$= 729(0 - 1i) = 0 - 729i$

**61.** $\left(\text{cis } \frac{11\pi}{6}\right)^8 = \text{cis } 8 \cdot \frac{11\pi}{6} = \text{cis } \frac{44\pi}{3} = \text{cis } \frac{2\pi}{3}$ [7.5]

$= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ or $-\frac{1}{2} + \frac{i\sqrt{3}}{2}$

**62.** $\left(1-i\sqrt{3}\right)^7=\left(2\text{ cis }300°\right)^7=2^7\text{cis }7\cdot 300°$ [7.5]

$=128\text{ cis }2100°=128\text{ cis }300°$

$=128\left(\cos 300°+i\sin 300°\right)=128\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$

$=64-64i\sqrt{3}$

**63.** $\left(-2-2i\right)^{10}=\left(2\sqrt{2}\text{ cis }225°\right)^{10}=32,768\text{ cis }10\cdot 225°$ [7.5]

$=32,768\text{ cis }2250°=32,768\left(\cos 2250°+i\sin 2250°\right)$

$=0+32,768i$

$=32,768i$

**64.** $27i=27\text{ cis }90°$ [7.5]

$w_k=27^{1/3}\text{cis}\dfrac{90°+360°k}{3}\quad k=0,\ 1,\ 2$

$w_0=3\text{ cis }\dfrac{90°}{3}$ $\quad w_1=3\text{ cis }\dfrac{90°+360°}{3}$ $\quad w_2=3\text{ cis }\dfrac{90°+300°\cdot 2}{3}$

$w_0=3\text{ cis }30°$ $\quad w_1=3\text{ cis }150°$ $\quad w_2=3\text{ cis }270°$

**65.** $8i=8\text{ cis }90°$ [7.5]

$w_k=8^{1/4}\text{cis}\dfrac{90°+360°k}{4}=\sqrt[4]{8}\text{ cis}\dfrac{90°+360°k}{4}\quad k=0,\ 1,\ 2,\ 3$

$w_0=\sqrt[4]{8}\text{ cis }\dfrac{90°}{4}$ $\quad w_1=\sqrt[4]{8}\text{ cis }\dfrac{90°+360°}{4}$ $\quad w_2=\sqrt[4]{8}\text{ cis }\dfrac{90°+360°\cdot 2}{4}$ $\quad w_3=\sqrt[4]{8}\text{ cis }\dfrac{90°+360°\cdot 3}{4}$

$w_0=\sqrt[4]{8}\text{ cis }22.5°$ $\quad w_1=\sqrt[4]{8}\text{ cis }112.5°$ $\quad w_2=\sqrt[4]{8}\text{ cis }202.5°$ $\quad w_3=\sqrt[4]{8}\text{ cis }292.5°$

**66.** $w_k=256^{1/4}\text{cis}\dfrac{120°+360°k}{4}\quad k=0,1,2,3$ [7.5]

$w_0=4\text{ cis }\dfrac{120°}{4}$ $\quad w_1=4\text{ cis }\dfrac{120°+360°}{4}$ $\quad w_2=4\text{ cis }\dfrac{120°+360°\cdot 2}{4}$ $\quad w_3=4\text{ cis }\dfrac{120°+360°\cdot 3}{4}$

$w_0=4\text{ cis }30°$ $\quad w_1=4\text{ cis }120°$ $\quad w_2=4\text{ cis }210°$ $\quad w_3=4\text{ cis }300°$

**67.** $\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\sqrt{\frac{1}{4}+\frac{3}{4}}=1$ [7.5]

$z=1\text{ cis }60^\circ$

$w_k=(1)^{1/5}\text{ cis }\dfrac{60°+360°k}{5}=1\text{ cis }\dfrac{60°+360°k}{5}\quad k=0,\ 1,\ 2,\ 3,\ 4$

$w_0=\text{cis }\frac{60°}{5}$ $\quad w_1=\text{cis }\frac{60°+360°}{5}$ $\quad w_2=\text{cis }\frac{60°+360°\cdot 2}{5}$

$w_0=\text{cis }12°$ $\quad w_1=\text{cis }84°$ $\quad w_2=\text{cis }156°$

$w_3=\text{cis }\frac{60°+360°\cdot 3}{5}$ $\quad w_4=\text{cis }\frac{60°+360°\cdot 4}{5}$

$w_3=\text{cis }228°$ $\quad w_4=\text{cis }300°$

## Quantitative Reasoning

**QR1.** The distance between MCO and LAX is

$d = \cos^{-1}\left[\cos(\text{lat1})\cos(\text{lat2})\cos(\text{lon1} - \text{lon2}) + \sin(\text{lat1})\sin(\text{lat2})\right]$

$d = \cos^{-1}\left[\cos(0.496187)\cos(0.592409)\cos(-1.419110 - (-2.066611)) + \sin(0.496187)\sin(0.592409)\right]$

$d \approx 0.559146$ radian

The great circle distance between MCO and LAX is

$d \approx 0.559146 \times 3960$

$d \approx 2210$ mi

**QR2.** Since $\sin(\text{lon2} - \text{lon1}) = \sin(-2.066611 - (-1.419110)) < 0$, we use Formula (3) to find the initial heading.

$$\begin{aligned} \text{h1} &= 2\pi - \cos^{-1}\left[\frac{\sin(\text{lat2}) - \sin(\text{lat1})\cos(d)}{\sin(d)\cos(\text{lat1})}\right] \\ &= 2\pi - \cos^{-1}\left[\frac{\sin(0.592409) - \sin(0.496187)\cos(0.559146)}{\sin(0.559146)\cos(0.496187)}\right] \\ &\approx 5.050614 \text{ radians} \\ &= 5.050614\left(\frac{180^\circ}{\pi}\right) \\ &\approx 289^\circ \end{aligned}$$

**QR3.** The distance between JFK and DEN is

$d = \cos^{-1}\left[\cos(\text{lat1})\cos(\text{lat2})\cos(\text{lon1} - \text{lon2}) + \sin(\text{lat1})\sin(\text{lat2})\right]$

$d = \cos^{-1}\left[\cos(0.709476)\cos(0.695717)\cos(-1.287756 - (-1.826892)) + \sin(0.709476)\sin(0.695717)\right]$

$d \approx 0.409558$ radian

The great circle distance between JFK and DEN is

$d \approx 0.409558 \times 3960$

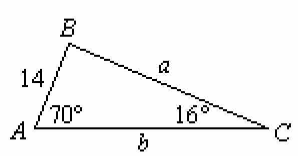
$d \approx 1620$ mi

**QR4.** Flying from DEN to JFK, $\sin(\text{lon2} - \text{lon1}) = \sin(-1.287756 - (-1.826892)) > 0$, we use Formula (2) to find the initial heading.

$$\begin{aligned} \text{h1} &= \cos^{-1}\left[\frac{\sin(\text{lat2}) - \sin(\text{lat1})\cos(d)}{\sin(d)\cos(\text{lat1})}\right] \\ &= \cos^{-1}\left[\frac{\sin(0.709476) - \sin(0.695717)\cos(0.409558)}{\sin(0.409558)\cos(0.695717)}\right] \\ &\approx 1.361499 \text{ radians} \\ &= 1.361499\left(\frac{180^\circ}{\pi}\right) \\ &\approx 78^\circ \end{aligned}$$

## Chapter Test

**1.**

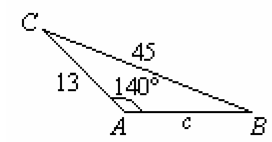



$B = 180° - 70° - 16°$
$B = 94°$

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$
$$a = \frac{14 \sin 70°}{\sin 16°}$$
$$a \approx 48$$

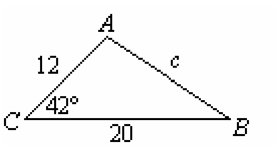
$$\frac{c}{\sin C} = \frac{b}{\sin B} \qquad [7.1]$$
$$b = \frac{14 \sin 94°}{\sin 16°}$$
$$b \approx 51$$

**2.**



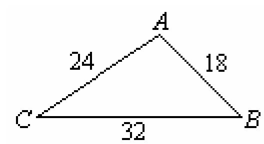

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$
$$B = \sin^{-1}\left(\frac{b \sin A}{a}\right) \qquad [7.1]$$
$$B = \sin^{-1}\left(\frac{13 \sin 140°}{45}\right)$$
$$B \approx 11°$$

**3.**



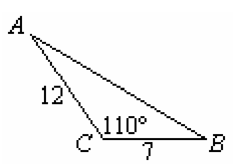

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$c^2 = 20^2 + 12^2 - 2(20)(12)\cos 42° \quad [7.2]$$
$$c \approx 14$$

**4.**



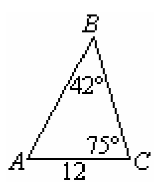

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$B = \cos^{-1}\left(\frac{32^2 + 18^2 - 24^2}{2(32)(18)}\right) \quad [7.2]$$
$$B \approx 48°$$

**5.**



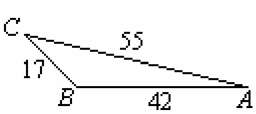

$$K = \frac{1}{2}ab\sin C$$
$$K = \frac{1}{2}(7)(12)(\sin 110°)$$
$K \approx 39$ square units
[7.2]

**6.**




$A = 180° - 42° - 75°$
$A = 63°$
$$K = \frac{b^2 \sin A \sin C}{2\sin B}$$
$$K = \frac{12^2 \sin 63° \sin 75°}{2\sin 42°}$$
$K \approx 93$ square units
[7.2]

**7.**




$$s = \frac{1}{2}(a + b + c)$$
$$s = \frac{1}{2}(17 + 55 + 42) = 57$$
$$K = \sqrt{s(s-a)(s-b)(s-c)}$$
$$K = \sqrt{57(57-17)(57-55)(57-42)}$$
$K \approx 260$ square units [7.2]

**8.**

$a_1 = 12\cos 220° \approx -9.2$ [7.3]
$a_2 = 12\sin 220° \approx -7.7$
$\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j}$
$\mathbf{v} = -9.2\mathbf{i} - 7.7\mathbf{j}$

**9.** $3\mathbf{u}-5\mathbf{v}=3(2\mathbf{i}-3\mathbf{j})-5(5\mathbf{i}+4\mathbf{j})$ [7.3]
$$=(6\mathbf{i}-9\mathbf{j})-(25\mathbf{i}+20\mathbf{j})$$
$$=(6-25)\mathbf{i}+(-9-20)\mathbf{j}$$
$$=-19\mathbf{i}-29\mathbf{j}$$

**10.** $\mathbf{u}\cdot\mathbf{v}=(-2\mathbf{i}+3\mathbf{j})\cdot(5\mathbf{i}+3\mathbf{j})$ [7.3]
$$=(-2\cdot 5)+(3\cdot 3)$$
$$=-10+9$$
$$=-1$$

**11.** $\cos\theta=\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\ \|\mathbf{v}\|}=\dfrac{\langle 3,5\rangle\cdot\langle -6,2\rangle}{\sqrt{3^2+5^2}\sqrt{(-6)^2+2^2}}$ [7.3]
$$\cos\theta=\frac{-18+10}{\sqrt{34}\sqrt{40}}=\frac{-8}{\sqrt{34}\sqrt{40}}$$
$$\theta\approx 103°$$

**12.** $z=-3\sqrt{2}+3i$
$$|z|=\sqrt{(-3\sqrt{2})^2+3^2}$$
$$|z|=3\sqrt{3}$$
$$\alpha=\tan^{-1}\left|\frac{3}{-3\sqrt{2}}\right|=\tan^{-1}\frac{\sqrt{2}}{2}$$
$$\alpha\approx 35°$$
$$\theta\approx 180°-35°$$
$$\theta\approx 145°$$
$z\approx 3\sqrt{3}\text{ cis }145°$ [7.4]

**13.** $z=5\text{ cis }315°$ [7.4]
$$z=5(\cos 315°+i\sin 315°)$$
$$z=5\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)$$
$$z=\frac{5\sqrt{2}}{2}-\frac{5\sqrt{2}}{2}i$$

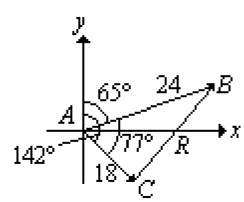
**14.** $z=\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i$ [7.5]
$$r=\sqrt{(1/2)^2+(\sqrt{3}/2)^2}\qquad \alpha=\tan^{-1}\left|\frac{\sqrt{3}/2}{1/2}\right|=60°$$
$$r=1$$
$$z=\cos 60°+i\sin 60°$$
$$\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3=(\cos 60°+i\sin 60°)^3$$
$$=\cos(3\cdot 60°)+i\sin(3\cdot 600°)$$
$$=\cos 180°+i\sin 180°$$
$$=-1+0i=-1$$

**15.** $\dfrac{z_1}{z_2}=\dfrac{25\text{ cis }115°}{10\text{ cis }210°}$ [7.4]
$$\frac{z_1}{z_2}=2.5\text{ cis }(115°-210°)$$
$$\frac{z_1}{z_2}=2.5\text{ cis }(-95°)\text{ or }2.5\text{ cis }265°$$

**16.** $z=\sqrt{2}-i$
$$|z|=\sqrt{\sqrt{2}^2+(-1)^2}$$
$$|z|=\sqrt{3}$$
$$\alpha=\tan^{-1}\left|\frac{-1}{\sqrt{2}}\right|=\tan^{-1}\frac{\sqrt{2}}{2}$$
$$\alpha\approx 35.2644°$$
$$\theta\approx 360°-35.2644°$$
$$\theta\approx 324.7356°$$
$$z\approx\sqrt{3}\text{ cis }324.7356°$$
$$z^5\approx(\sqrt{3})^5\text{ cis }(5\cdot 324.7356°)$$
$$z^5\approx 9\sqrt{3}(\cos 1623.678°+i\sin 1623.678°)$$
$z^5\approx -15.556-1.000i$ [7.5]

**17.** $27i=27(\cos 90°+i\sin 90°)=27\text{ cis }90°$
$$w_k=27^{1/3}\text{cis}\frac{90°+360°k}{3}\qquad k=0,\ 1,\ 2$$
$$w_0=3\text{ cis }\frac{90°}{3}=3\text{ cis }30°$$
$$w_0=\frac{3\sqrt{3}}{2}+\frac{3}{2}i$$
$$w_1=3\text{ cis }150°$$
$$w_1=-\frac{3\sqrt{3}}{2}+\frac{3}{2}i$$
$$w_2=3\text{ cis }270°$$
$w_2=0-3i$ or $-3i$ [7.5]

**18.**




$$A=142°-65°=77°$$
$$R^2=24^2+18^2-2(24)(18)\cos 77°$$
$R\approx 27$ miles [7.3]

**19.** $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i = 1(\cos 45° + i\sin 45°)$

$w_k = \cos\frac{45°+360°k}{5} + i\sin\frac{45°+360°k}{5} \quad k = 0, 1, 2, 3, 4$

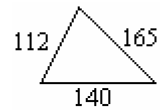
$w_0 = \cos\frac{45°}{5} + i\sin\frac{45°}{5}$
$w_0 = (\cos 9° + i\sin 9°)$
$w_0 = \text{cis } 9°$

$w_1 = \cos\frac{45°+360°}{5} + i\sin\frac{45°+360°}{5}$
$w_1 = (\cos 81° + i\sin 81°)$
$w_1 = \text{cis } 81°$

$w_2 = \cos\frac{45°+360°\cdot 2}{5} + i\sin\frac{45°+360°\cdot 2}{5}$
$w_2 = \cos 153° + i\sin 153°$
$w_2 = \text{cis } 153°$

$w_3 = \cos\frac{45°+360°\cdot 3}{5} + i\sin\frac{45°+360°\cdot 3}{5}$
$w_3 = \cos 225° + i\sin 225°$
$w_3 = \text{cis } 225°$

$w_4 = \cos\frac{45°+360°\cdot 4}{5} + i\sin\frac{45°+360°\cdot 4}{5}$
$w_4 = \cos 297° + i\sin 297°$
$w_4 = \text{cis } 297°$

**20.** 


$S = \frac{1}{2}(112+165+140) = 208.5$
$K = \sqrt{208.5(208.5-112)(208.5-165)(208.5-140)}$
$K \approx 7743$
cost $\approx 8.50(7743)$
cost $\approx \$66,000$
[7.2]

## Cumulative Review

**1.** $(f\circ g)(x) = f[g(x)]$ [2.6]
$= f[x^2+1]$
$= \cos(x^2+1)$

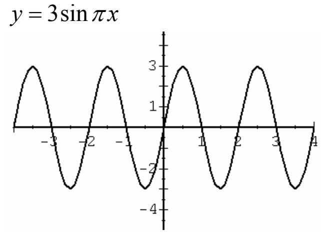
**2.** $f(x) = 2x+8$ [4.1]
$y = 2x+8$
$x = 2y+8$
$x-8 = 2y$
$\frac{x-8}{2} = y$
$f^{-1}(x) = \frac{1}{2}x - 4$

**3.** $\frac{3\pi}{2}\left(\frac{180°}{\pi}\right) = 270°$ [5.1]

**4.** hyp $= \sqrt{3^2+4^2} = \sqrt{25} = 5$ [5.2]
$\sin\theta = \frac{3}{5},\ \cos\theta = \frac{4}{5},\ \tan\theta = \frac{3}{4}$

**5.** $\cos 26.0° = \frac{15.0}{c}$ [5.2]
$c = \frac{15.0}{\cos 26.0°} \approx 16.7$ cm

**6.** $y = 3\sin\pi x$



**7.** $y = 4\cos\left(2x - \frac{\pi}{2}\right)$ [5.7]

$0 \le 2x - \frac{\pi}{2} \le 2\pi$

$\frac{\pi}{2} \le 2x \le \frac{5\pi}{2}$

$\frac{\pi}{4} \le x \le \frac{5\pi}{4}$

amplitude = 4, period $= \pi$, phase shift $= \frac{\pi}{4}$

**8.** $\sin x - \cos x$ [6.4]

$a = 1,\ b = -1$

$k = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\sin\alpha = \frac{1}{\sqrt{2}},\ \cos\alpha = \frac{-1}{\sqrt{2}},\ \alpha = \frac{7\pi}{4}$ or $-\frac{\pi}{4}$

$\sin x - \cos x = \sqrt{2}\sin\left(x + \frac{7\pi}{4}\right)$ or $\sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$

**9.** $y = \sin x$ is an odd function. [2.5]

**10.** $\frac{1}{\sin x} - \sin x = \frac{1 - \sin^2 x}{\sin x}$ [6.1]

$= \frac{\cos^2 x}{\sin x}$

$= \cos x \frac{\cos x}{\sin x}$

$= \cos x \cot x$

**11.** $\tan\left(\sin^{-1}\left(\frac{12}{13}\right)\right) = \tan(67.38^\circ) = \frac{12}{5}$ [6.5]

**12.** $\sin 2x \cos 3x - \cos 2x \sin 3x = \sin(2x - 3x)$ [6.2]

$= \sin(-x)$ or $-\sin x$

**13.** $2\cos^2 x + \sin x - 1 = 0$ [6.6]

$2(1 - \sin^2 x) + \sin x - 1 = 0$

$2\sin^2 - \sin x + 1 = 0$

$(2\sin x + 1)(\sin x - 1) = 0$

$2\sin x + 1 = 0$ $\qquad$ $\sin x - 1 = 0$

$\sin x = -\frac{1}{2}$ $\qquad$ $\sin x = 1$

$x = \frac{7\pi}{6}, \frac{11\pi}{6}$ $\qquad$ $x = \frac{\pi}{2}$

The solutions are $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

**14.** $\sin 2x = \sqrt{3}\sin x$ [6.6]

$2\sin x\cos x - \sqrt{3}\sin x = 0$

$\sin x(2\cos x - \sqrt{3}) = 0$

$\sin x = 0$ $\qquad$ $2\cos x - \sqrt{3} = 0$

$x = 0,\ \pi$ $\qquad$ $\cos x = \frac{\sqrt{3}}{2}$

$x = \frac{\pi}{6}, \frac{11\pi}{6}$

The solutions are $0, \frac{\pi}{6}, \pi, \frac{11\pi}{6}$.

**15.** $\|\mathbf{v}\| = \sqrt{(-3)^2 + 4^2}$ $\qquad$ $\alpha = \tan^{-1}\left|\frac{4}{-3}\right| = \tan^{-1}\frac{4}{3}$ [7.3]

$\|\mathbf{v}\| = \sqrt{9 + 16}$ $\qquad$ $\alpha \approx 53.1^\circ$

$\|\mathbf{v}\| = 5$ $\qquad$ $\theta = 180^\circ - \alpha$

$\theta \approx 180^\circ - 53.1^\circ$

$\theta \approx 126.9^\circ$

magnitude: 5, angle: 126.9°

**16.** $\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\ \|\mathbf{w}\|}$ [7.3]

$\cos\theta = \frac{\langle 2, -3\rangle \cdot \langle -3, 4\rangle}{\sqrt{2^2 + (-3)^2}\sqrt{(-3)^2 + 4^2}}$

$\cos\theta = \frac{2(-3) + (-3)(4)}{\sqrt{13}\sqrt{25}}$

$\cos\theta = \frac{-18}{5\sqrt{13}} \approx -0.9985$

$\theta = 176.8^\circ$

**17.** $\mathbf{AB} = 415(\cos 42\mathbf{i} + \sin 42\mathbf{j}) \approx 308.4\mathbf{i} + 277.6\mathbf{j}$ [7.3]

$\mathbf{AD} = 55[\cos(-25°)\mathbf{i} + \sin(-25°)\mathbf{j}] \approx 49.8\mathbf{i} - 23.2\mathbf{j}$

$\mathbf{AC} = \mathbf{AB} + \mathbf{AD}$

$\mathbf{AC} = 308.4\mathbf{i} + 277.6\mathbf{j} + 49.8\mathbf{i} - 23.2\mathbf{j}$

$\mathbf{AC} \approx 358.2\mathbf{i} + 254.4\mathbf{j}$

$\|\mathbf{AC}\| = \sqrt{358.2^2 + (254.4)^2}$

$\|\mathbf{AC}\| \approx 439$ mph

$\alpha = 90^\circ - \theta = 90^\circ - \tan^{-1}\left(\frac{254.4}{358.2}\right) \approx 54.6^\circ$

**18.** $\frac{a}{\sin A} = \frac{b}{\sin B}$ [7.1]

$\sin A = \frac{a \sin B}{b}$

$= \frac{42 \sin 32^\circ}{51} = 0.4364041$

$A = \sin^{-1}(0.4364041) \approx 26^\circ$

**19.** $z = 1 - i$ [7.5]

$r = \sqrt{1^2 + (-1)^2}$ $\quad \alpha = \tan^{-1}\left|\frac{-1}{1}\right| = 45^\circ$

$r = \sqrt{2}$ $\quad \theta = 315^\circ$

$z = \sqrt{2}(\cos 315° + i \sin 315°)$

$(1-i)^8 = [\sqrt{2}(\cos 315° + i \sin 315°)]^8$

$= (\sqrt{2})^8[\cos(8 \cdot 315°) + i \sin(8 \cdot 315°)]$

$= 16(\cos 2520° + i \sin 2520°)$

$= 16(\cos 0° + i \sin 0°)$

$= 16 - 0i = 16$

**20.** $i = \cos 90° + i \sin 90°$ [7.5]

$w_k = \cos\frac{90° + 360°k}{2} + i \sin\frac{90° + 360°k}{2}$ $\quad k = 0, 1$

$w_0 = \cos\frac{90^\circ}{2} + i\sin\frac{90^\circ}{2}$ $\quad w_1 = \cos\frac{90^\circ + 360^\circ}{2} + i\sin\frac{90^\circ + 360^\circ}{2}$

$w_0 = \cos 45° + i \sin 45°$ $\quad w_1 = \cos 225° + i \sin 225°$

$w_0 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ $\quad w_1 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

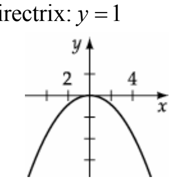
# Chapter 8
# Topics in Analytic Geometry

## Section 8.1

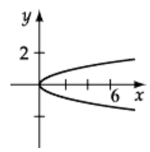
**1.** **a.** iii **b.** i **c.** iv **d.** ii

**2.** **a.** ii **b.** iii **c.** i **d.** iv

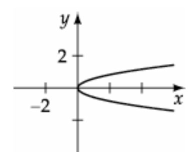
**3.** $x^2 = -4y$
$4p = -4$
$p = -1$
vertex $= (0,\ 0)$
focus $= (0,\ -1)$
directrix: $y = 1$



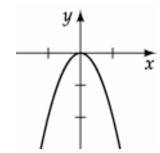

**4.** $y^2 = \frac{1}{2}x$
$4p = \frac{1}{2}$
$p = \frac{1}{8}$
vertex $= (0,\ 0)$
focus $= \left(\frac{1}{8},\ 0\right)$
directix: $x = -\frac{1}{8}$



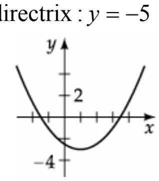

**5.** $y^2 = \frac{1}{3}x$
$4p = \frac{1}{3}$
$p = \frac{1}{12}$
vertex $= (0,\ 0)$
focus $= \left(\frac{1}{12},\ 0\right)$
directrix: $x = -\frac{1}{12}$



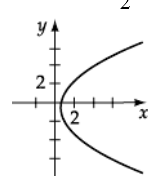

**6.** $x^2 = -\frac{1}{4}y$
$4p = -\frac{1}{4}$
$p = -\frac{1}{16}$
vertex $= (0,\ 0)$
focus $= \left(0,\ -\frac{1}{16}\right)$
directrix: $y = \frac{1}{16}$



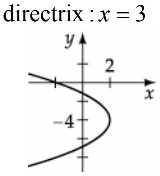

**7.** $(x-2)^2 = 8(y+3)$
vertex $= (2, -3)$
$4p = 8 \quad p = 2$
$(h, k+p) = (2, -3+2) = (2, -1)$
focus $= (2, -1)$
$k - p = -3 - 2 = -5$
directrix : $y = -5$




**8.** $(y+1)^2 = 6(x-1)$
vertex $= (1,\ -1)$, $4p = 6 \quad p = \frac{3}{2}$
$(h+p, k) = \left(1 + \frac{3}{2},\ -1\right) = \left(\frac{5}{2},\ -1\right)$
focus $= \left(\frac{5}{2},\ -1\right)$
$h - p = 1 - \frac{3}{2} = -\frac{1}{2}$
directrix: $x = -\frac{1}{2}$




**9.** $(y+4)^2 = -4(x-2)$
vertex $= (2, -4)$
$4p = -4 \quad p = -1$
$(h+p, k) = (2-1, -4) = (1, -4)$
focus $= (1, -4)$
$h - p = 2 + 1 = 3$
directrix : $x = 3$



**10.** $(x-3)^2 = -(y+2)$

vertex $= (3, -2)$

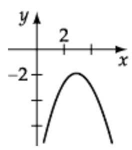
$4p = -1 \quad p = -\frac{1}{4}$

$(h, k+p) = \left(3, -2-\frac{1}{4}\right) = \left(3, -\frac{9}{4}\right)$

focus $= \left(3, -\frac{9}{4}\right)$

$k - p = -2 + \frac{1}{4} = -\frac{7}{4}$

directrix : $y = -\frac{7}{4}$




**11.** $(y-1)^2 = 2(x+4)$

vertex $= (-4, 1)$

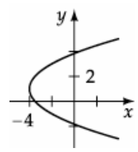
$4p = 2 \quad p = \frac{1}{2}$

$(h+p, k) = \left(-4+\frac{1}{2}, 1\right) = \left(-\frac{7}{2}, 1\right)$

focus $= \left(-\frac{7}{2}, 1\right)$

$h - p = -4 - \frac{1}{2} = -\frac{9}{2}$

directrix : $x = -\frac{9}{2}$




**12.** $(x+2)^2 = 3(y-2)$

vertex $= (-2, 2)$

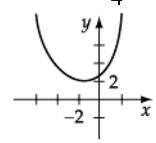
$4p = 3 \quad p = \frac{3}{4}$

$(h, k+p) = \left(-2, 2+\frac{3}{4}\right) = \left(-2, \frac{11}{4}\right)$

focus $= \left(-2, \frac{11}{4}\right)$

$k - p = 2 - \frac{3}{4} = \frac{5}{4}$

directrix : $y = \frac{5}{4}$




**13.** $(2x-4)^2 = 8y - 16$

$4(x-2)^2 = 8(y-2)$

$(x-2)^2 = 2(y-2)$

vertex $= (2,\ 2)$

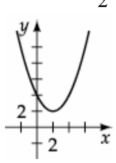
$4p = 2 \quad p = \frac{1}{2}$

$(h, k+p) = \left(2,\ 2+\frac{1}{2}\right) = \left(2,\ \frac{5}{2}\right)$

focus $= \left(2,\ \frac{5}{2}\right)$

$k - p = 2 - \frac{1}{2} = \frac{3}{2}$

directrix: $y = \frac{3}{2}$




**14.** $(3x+6)^2 = 18y - 36$

$9(x+2)^2 = 18(y-2)$

$(x+2)^2 = 2(y-2)$

vertex $= (-2,\ 2)$

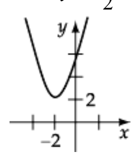
$4p = 2 \quad p = \frac{1}{2}$

$(h, k+p) = \left(-2,\ 2+\frac{1}{2}\right) = \left(-2,\ \frac{5}{2}\right)$

focus $= \left(-2,\ \frac{5}{2}\right)$

$k - p = 2 - \frac{1}{2} = \frac{3}{2}$

directrix: $y = \frac{3}{2}$




**15.** $x^2 + 8x - y + 6 = 0$

$x^2 + 8x = y - 6$

$x^2 + 8x + 16 = y - 6 + 16$

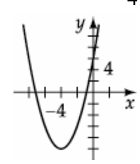
$(x+4)^2 = y + 10$

vertex $= (-4,\ -10)$

$4p = 1, \ \ p = \frac{1}{4}$

focus $= \left(-4,\ -\frac{39}{4}\right)$

directrix: $y = -\frac{41}{4}$



**16.** $x^2 - 6x + y + 10 = 0$

$x^2 - 6x = -y - 10$

$x^2 - 6x + 9 = -y - 10 + 9$

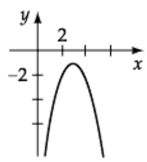
$(x-3)^3 = -(y+1)$

vertex $= (3, \ -1)$

$4p = -1, \ p = -\frac{1}{4}$

focus $= \left(3, \ -\frac{5}{4}\right)$

directrix: $y = -\frac{3}{4}$




**17.** $x + y^2 - 3y + 4 = 0$

$y^2 - 3y = -x - 4$

$y^2 - 3y + 9/4 = -x - 4 + 9/4$

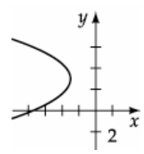
$\left(y - \frac{3}{2}\right)^2 = -\left(x + \frac{7}{4}\right)$

vertex $= \left(-\frac{7}{4}, \ \frac{3}{2}\right)$

$4p = -1, \ p = -\frac{1}{4}$

focus $= \left(-2, \ \frac{3}{2}\right)$

directrix: $x = -\frac{3}{2}$




**18.** $x - y^2 - 4y + 9 = 0$

$-y^2 - 4y = -x - 9$

$y^2 + 4y = x + 9$

$y^2 + 4y + 4 = x + 9 + 4$

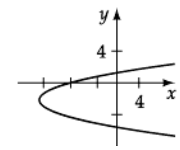
$(y+2)^2 = (x+13)$

vertex $= (-13, \ -2)$

$4p = 1, \ p = \frac{1}{4}$

focus $= \left(-\frac{51}{4}, \ -2\right)$

directrix: $x = -\frac{53}{4}$




**19.** $2x - y^2 - 6y + 1 = 0$

$-y^2 - 6y = -2x - 1$

$y^2 + 6y = 2x + 1$

$y^2 + 6y + 9 = 2x + 1 + 9$

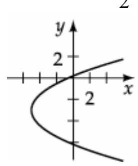
$(y+3)^2 = 2(x+5)$

vertex $= (-5, \ -3)$

$4p = 2, \ p = \frac{1}{2}$

focus $= \left(-\frac{9}{2}, \ -3\right)$

directrix: $x = -\frac{11}{2}$




**20.** $3x + y^2 + 8y + 4 = 0$

$y^2 + 8y = -3x - 4$

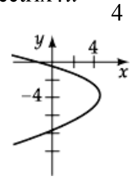
$y^2 + 8y + 16 = -3x - 4 + 16$

$(y+4)^2 = -3(x-4)$

vertex $= (4, -4)$, $4p = -3, \ p = -\dfrac{3}{4}$

focus $= \left(\dfrac{13}{4}, -4\right)$

directrix : $x = \dfrac{19}{4}$




**21.** $x^2 + 3x + 3y - 1 = 0$

$x^2 + 3x = -3y + 1$

$x^2 + 3x + 9/4 = -3y + 1 + 9/4$

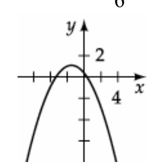
$\left(x + \frac{3}{2}\right)^2 = -3 \left(y - \frac{13}{12}\right)$

vertex $\left(-\frac{3}{2}, \ \frac{13}{12}\right)$

$4p = -3, \ p = -\frac{3}{4}$

focus $= \left(-\frac{3}{2}, \ \frac{1}{3}\right)$

directrix: $y = \frac{11}{6}$



**22.** $x^2 + 5x - 4y - 1 = 0$

$x^2 + 5x = 4y + 1$

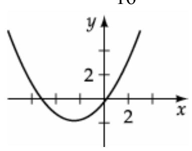
$x^2 + 5x + \frac{25}{4} = 4y + 1 + \frac{25}{4}$

$\left(x + \frac{5}{2}\right)^2 = 4\left(y + \frac{29}{16}\right)$

vertex $= \left(-\frac{5}{2}, -\frac{29}{16}\right)$, $4p = 4$, $p = 1$

focus $= \left(-\frac{5}{2}, -\frac{13}{16}\right)$

directrix: $y = -\frac{45}{16}$




**23.** $2x^2 - 8x - 4y + 3 = 0$

$2(x^2 - 4x) = 4y - 3$

$2(x^2 - 4x + 4) = 4y - 3 + 8$

$2(x - 2)^2 = 4y + 5$

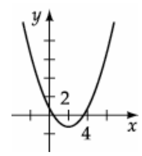
$(x - 2)^2 = 2y + \frac{5}{2}$

$(x - 2)^2 = 2\left(y + \frac{5}{4}\right)$

vertex $= \left(2, -\frac{5}{4}\right)$, $4p = 2$, $p = \frac{1}{2}$

focus $= \left(2, -\frac{3}{4}\right)$

directrix $y = -\frac{7}{4}$




**24.** $6x - 3y^2 - 12y + 4 = 0$

$-3y^2 - 12y = -6x - 4$

$-3(y^2 - 4y) = -6x - 4$

$-3(y^2 - 4y + 4) = -6x - 4 - 12$

$-3(y - 2)^2 = -6x - 16$

$(y - 2)^2 = 2x + \frac{16}{3}$

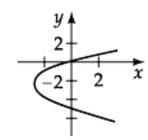
$(y + 2)^2 = 2\left(x + \frac{8}{3}\right)$

vertex $= \left(-\frac{8}{3}, -2\right)$

$4p = 2$, $p = \frac{1}{2}$

focus $= \left(-\frac{13}{6}, -2\right)$

directrix $x = -\frac{19}{6}$




**25.** $2x + 4y^2 + 8y - 5 = 0$

$4y^2 + 8y = -2x + 5$

$4(y^2 + 2y) = -2x + 5$

$4(y^2 + 2y + 1) = -2x + 5 + 4$

$4(y + 1)^2 = -2x + 9$

$(y + 1)^2 = -\frac{1}{2}x + \frac{9}{4}$

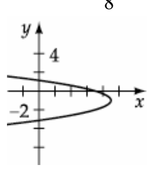
$(y + 1)^2 = -\frac{1}{2}\left(x - \frac{9}{2}\right)$

vertex $= \left(\frac{9}{2}, -1\right)$

$4p = -\frac{1}{2}$, $p = -\frac{1}{8}$

focus $= \left(\frac{35}{8}, -1\right)$

directrix $x = \frac{37}{8}$




**26.** $4x^2 - 12x + 12y + 7 = 0$

$4x^2 - 12x = -12y - 7$

$4(x^2 - 3x) = -12y - 7$

$4\left(x^2 - 3x + \frac{9}{4}\right) = -12y - 7 + 9$

$4\left(x - \frac{3}{2}\right)^2 = -12y + 2$

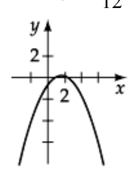
$\left(x - \frac{3}{2}\right)^2 = -3y + \frac{1}{2}$

$\left(x - \frac{3}{2}\right)^2 = -3\left(y - \frac{1}{6}\right)$

vertex $= \left(\frac{3}{2}, \frac{1}{6}\right)$, $4p = -3$, $p = -\frac{3}{4}$

focus $= \left(\frac{3}{2}, -\frac{7}{12}\right)$

directrix $y = \frac{11}{12}$




**27.** $3x^2 - 6x - 9y + 4 = 0$

$3(x^2 - 2x) = 9y - 4$

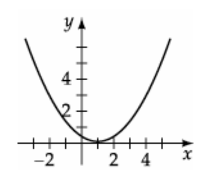
$3(x^2 - 2x + 1) = 9y - 1$

$(x - 1)^2 = 3\left(y - \frac{1}{9}\right)$

vertex $= \left(1, \frac{1}{9}\right)$, $4p = 3$, $p = \frac{3}{4}$

focus $= \left(1, \frac{31}{36}\right)$

directrix $y = -\frac{23}{36}$

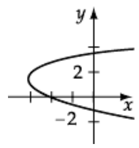


**28.** $\left(y-\frac{3}{2}\right)^2=\frac{2}{3}\left(x+\frac{47}{8}\right)$

vertex $=\left(-\frac{47}{8},\ \frac{3}{2}\right)$, $4p=\frac{2}{3}$, $p=\frac{1}{6}$

focus $=\left(-\frac{137}{24},\ \frac{3}{2}\right)$

directrix $x=-\frac{145}{24}$




**29.** vertex $(0,0)$, focus $(0,-4)$

$x^2=4py$

$p=-4$ since focus is $(0,\ p)$

$x^2=4(-4)y$

$x^2=-16y$

**30.** vertex $(0,0)$, focus $(5,0)$

$y^2=4px$

$p=5$ since focus is $(p,0)$

$y^2=4(5)x$

$y^2=20x$

**31.** vertex $(-1,2)$, focus $(-1,3)$

$(x-h)^2=4p(y-k)$

$h=-1,\ k=2.$

The distance $p$ from the vertex to the focus is 1.

$(x+1)^2=4(1)(y-2)$

$(x+1)^2=4(y-2)$

**32.** vertex $(2,\ -3)$, focus $(0,\ -3)$

$(y-k)^2=4p(x-h)$

$h=2,\ k=-3.$

Since the focus is $(h+p,\ k)$, $2+p=0$ and $p=-2$.

$(y+3)^2=4(-2)(x-2)$

$(y+3)^2=-8(x-2)$

**33.** focus $(3,\ -3)$, directrix $y=-5$

The vertex is the midpoint of the line segment joining (3,–3) and the point (3, –5) on the directrix.

$$(h,\ k)=\left(\frac{3+2}{2},\ \frac{-3+(-5)}{2}\right)=(3,\ -4)$$

The distance $p$ from the vertex to the focus is 1.

$4p=4(1)=4$

$(x-h)^2=4p(y-k)$

$(x-3)^2=4(y+4)$

**34.** focus $(-2,4)$, directrix $x=4$

The vertex is the midpoint of the line segment joining (–2,4) and the point (4, 4) on the directrix.

$$(h,\ k)=\left(\frac{-2+4}{2},\ \frac{4+4}{2}\right)=(1,\ 4)$$

The distance $p$ from the vertex to the focus is –3.

$4p=4(-3)=-12$

$(y-k)^2=4p(x-h)$

$(y-4)^2=-12(x-1)$

**35.** vertex $=(-4,\ 1)$, point: $(-2,\ 2)$ on the parabola.

Axis of symmetry $x=-4$.

If $P_1=(-2,\ 2)$ $(x+4)^2=4p(y-1)$. Since (–2, 2) is on the curve, we get

$(-2+4)^2=4p(2-1)$

$4=4p\Rightarrow p=1$

Thus, the equation in standard form is

$(x+4)^2=4(y-1)$

**36.** vertex $=(3,\ -5)$ and the point $(4,3)$ is on the parabola.

The equation of the parabola in standard form must be

$(y+5)^2=4p(x-3)$

Since $(4,3)$ is on the curve, we get

$(3+5)^2=4p(4-3)$

$8^2=4p(1)$

$64=4p$

$p=16$

Thus, the equation of the parabola in standard form is

$(y+5)^2=64(x-3)$

**37.** Find the vertex.

$$x = -0.325y^2 + 13y + 120$$
$$x - 120 = -0.325y^2 + 13y$$
$$x - 120 = -0.325(y^2 - 40y)$$
$$x - 120 - 130 = -0.325(y^2 - 40y + 400)$$
$$x - 250 = -0.325(y - 20)^2$$
$$-\frac{40}{13}(x - 250) = (y - 20)^2$$

The vertex is (250, 20).

Find the focus.

$$4p = -\frac{40}{13},\ \ p = -\frac{10}{13}$$

The focus is $\left(\frac{3240}{13}, 20\right)$.

**38.** To find where the fountains intersect, set the equations equal and solve for $x$.

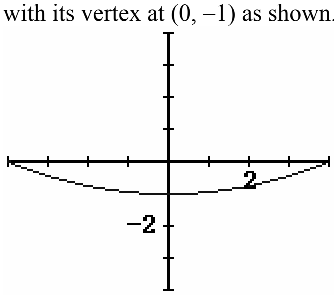
$$-0.25x^2 + 2x = -0.25x^2 + 4.5x - 16.25$$
$$2x = 4.5x - 16.25$$
$$-2.5x = -16.25$$
$$x = 6.5$$

Substitute the value of $x$ into one equation and solve for $y$.

$$y = -0.25x^2 + 2x$$
$$y = -0.25(6.5)^2 + 2(6.5)$$
$$y = 2.4375$$

The fountains of water intersect 2.4375 feet above the base.

**39.** Place the satellite dish on an $xy$-coordinate system with its vertex at (0, −1) as shown.



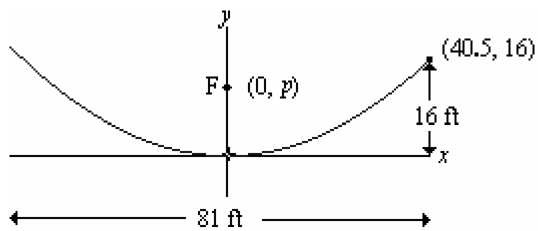

The equation of the parabola is $x^2 = 4p(y+1) \quad -1 \le y \le 0$

Because (4, 0) is a point on this graph, (4, 0) must be a solution of the equation of the parabola. Thus,

$$16 = 4p(0+1)$$
$$16 = 4p$$
$$4 = p$$

Because $p$ is the distance from the vertex to the focus, the focus is on the $x$-axis 4 feet above the vertex.

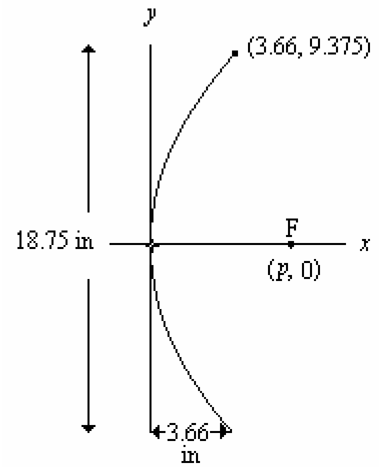
**40.**




The focus of the parabola is (0, $p$) where $x^2 = 4py$. Half of 81 feet = 40.5 feet. Therefore, the point (40.5, 16) is on the parabola.

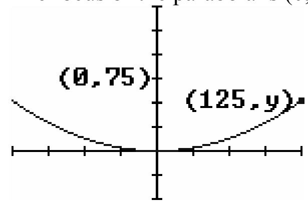
$$(40.5)^2 = 4p(16)$$
$$1640.25 = 64p$$
$$\frac{1640.25}{64} = p$$
$$p \approx 25.6 \text{ feet}$$

**41.** The focus of the parabola is $(p, 0)$ where $y^2 = 4px$.
Half of 18.75 inches is 9.375 inches.
Therefore, the point (3.66, 9.375) is on the parabola.




$$(9.375)^2 = 4p(3.66)$$
$$87.890625 = 14.64p$$
$$\frac{87.890625}{14.64} = p$$
$$p \approx 6.0 \text{ inches}$$

**42.** **a.** The focus of the parabola is (0, 75).




$$x^2 = 4py$$
$$x^2 = 4(75)y$$
$$x^2 = 300y \quad \text{or} \quad y = \frac{1}{300}x^2$$

**b.** Half of 250 feet is 125 feet. Therefore, $(125, y)$ is a point on the parabola, where $y$ is the depth of the dish.

$$x^2 = 300y$$
$$(125)^2 = 300y$$
$$15625 = 300y$$
$$\frac{15625}{300} = y$$
$$y \approx 52 \text{ feet}$$

**43.** $$S = \frac{\pi r}{6d^2}\left[\left(r^2 + 4d^2\right)^{3/2} - r^3\right]$$

**a.** $r = 40.5$ feet
$d = 16$ feet

$$S = \frac{\pi(40.5)}{6(16)^2}\left[\left([40.5]^2 + 4[16]^2\right)^{3/2} - (40.5)^3\right]$$
$$= \frac{40.5\pi}{1536}\left[(266.25)^{3/2} - 66430.125\right]$$
$$= \frac{40.5\pi}{1536}[137518.9228 - 66430.125]$$
$$= \frac{40.5\pi}{1536}[71088.79775]$$
$$\approx 5900 \text{ square feet}$$

**b.** $r = 125$ feet
$d = 52$ feet

$$S = \frac{\pi(125)}{6(52)^2}\left[\left([125]^2 + 4[52]^2\right)^{3/2} - (125)^3\right]$$
$$= \frac{125\pi}{16224}\left[(26441)^{3/2} - 1953125\right]$$
$$= \frac{125\pi}{16224}[4299488.724 - 1953125]$$
$$= \frac{125\pi}{16224}[2346363.724]$$
$$\approx 56{,}800 \text{ square feet}$$

**44.** The equation of the mirror is
$x^2 = 4py \quad -100 \le x \le 100$

Because $(100, 3.75375)$ is a point on the parabola, $(100, 3.75375)$ must be a solution of the equation. Thus

$$100^2 = 4p(3.75375)$$
$$10000 = 15.015p$$
$$666 \approx p$$

The focus is approximately 666 inches above the vertex.

**45.** The equation of the mirror is given by

$x^2 = 4py \quad -60 \le x \le 60$

Because $p$ is the distance from the vertex to the focus and the coordinates of the focus are (0, 600), $p$ = 600. Therefore,

$x^2 = 4(600)y$

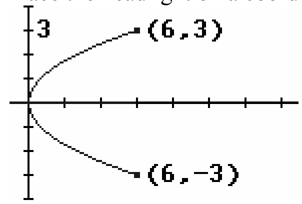
$x^2 = 2400y$

To determine $a$, substitute (60, $a$) into the equation $x^2 = 2400y$ and solve for a.

$$x^2 = 2400y$$
$$60^2 = 2400a$$
$$3600 = 2400a$$
$$1.5 = a.$$

The concave depth of the mirror is 1.5 inches.

**46.** Place the headlight on a coordinate grid as shown.




The equation of the parabola is $y^2 = 4px$.

Because (6, 3) is on the graph of the parabola, the coordinates must be a solution of the equation $y^2 = 4px$. Thus,

$$y^2 = 4px$$
$$3^2 = 4p(6)$$
$$9 = 24p$$
$$p = 0.375$$

The value $p$ is the distance from the vertex to the focus. Therefore, the focus is 0.375 inches to the right of the vertex.

**47.** **a.** The equation of the parabola is

$x^2 = 4p(y-32)$

Because (–800, 53) is a point on this graph, (–800, 53) must be a solution of the equation of the parabola. Thus,

$$(-800)^2 = 4p(53-32)$$
$$800^2 = 84p$$
$$\frac{800^2}{84} = p$$

The equation of the parabola is $x^2 = 4\left(\frac{800^2}{84}\right)(y-32)$.

**b.** To find the width use $x$ = 900, and solve for $y$ and then multiply by 2

$$x^2 = 4\left(\frac{800^2}{84}\right)(y-32)$$
$$900^2 = 4\left(\frac{800^2}{84}\right)(y-32)$$
$$\left(\frac{900^2}{800^2}\right)\frac{84}{4} + 32 = y$$
$$y \approx 58.58$$
$$2y \approx 117$$

The width of the ski is 117 mm.

**48.** An infinite number of parabolas pass through the points (2, 3) and (–2, 3).

## Connecting Concepts

**49.** $x^2 = 4y$

$4p = 4$

$p = 1$

focus = (0, 1)

Substituting the vertical coordinate of the focus for $y$ to obtain $x$-coordinates of endpoints $(x_1, y_1)$, $(x_2, y_2)$, we have

$$x^2 = 4(1), \text{ or } x^2 = 4$$
$$x = \pm\sqrt{4}$$
$$x_1 = -2 \quad x_2 = 2$$

Length of latus rectum $= |x_2 - x_1|$
$= 2-(-2) = 4.$

**50.** $y^2 = -8x$

$4p = -8$

$p = -2$

focus $= (-2, 0)$

Substituting the horizontal coordinate of the focus for $x$ to obtain the $y$-coordinates of the endpoints $(x_1, y_1)$, $(x_2, y_2)$, we have

$$y^2 = -8(-2) = 16$$
$$y = \pm\sqrt{16}$$
$$y_1 = -4 \quad y_2 = 4$$

Length of latus rectum $= |y_2 - y_1|$
$= 4-(-4) = 8$

**51.** $(x-h)^2 = 4p(y-k)$

focus $= (h, k+p)$

Substituting the vertical coordinate of the focus for $y$ to obtain $x$-coordinates of endpoints $(x_1, y_1)$, $(x_2, y_2)$, we have

$$(x-h)^2 = 4p(k+p-k)$$
$$(x-h)^2 = 4p^2$$
$$x-h = \pm 2p$$
$$x_1 = h-2p \quad x_2 = h+2p$$

Solving for $|x_2 - x_1|$, we obtain

$$\Delta x = |x_2 - x_1| = |h+2p-h+2p| = 4|p|$$

or $(y-k)^2 = 4p(x-h)$

focus = $(h+p, k)$

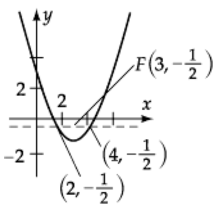
Substituting the horizontal coordinate of the focus for $x$ to obtain the $y$-coordinates of the endpoints $(x_1, y_1)$, $(x_2, y_2)$, we have

$$(y-k)^2 = 4p(h+p-h)$$
$$(y-k)^2 = 4p^2$$
$$y-k = \pm 2p$$
$$y_1 = k-2p \quad y_2 = k+2p$$

Solving for $|y_2 - y_1|$, we obtain $\Delta y = |y_2 - y_1|$
$= |k+2p-k+2p|$
$= 4|p|$

Thus, the length of the latus rectum is $4|p|$.

**52.**




$4p = 2$

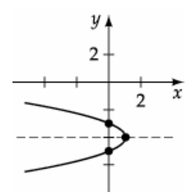
$p = \frac{1}{2}$

focus $\left(3, -\frac{1}{2}\right)$

one point: $\left(h+2p, -\frac{1}{2}\right) = \left(4, -\frac{1}{2}\right)$

one point: $\left(h-2p, -\frac{1}{2}\right) = \left(2, -\frac{1}{2}\right)$

**53.**




$4p = -1$

$p = -\frac{1}{4}$

focus $\left(\frac{3}{4}, -4\right)$

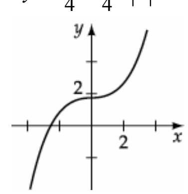
one point: $\left(\frac{3}{4}, k+2p\right) = \left(\frac{3}{4}, -\frac{9}{2}\right)$

one point: $\left(\frac{3}{4}, k-2p\right) = \left(\frac{3}{4}, -\frac{7}{2}\right)$

**54.** By definition, the point $(x, y)$ on the curve must be equidistant from the focus $(-c, 0)$ and the directrix $(x = c)$. So,

$$\sqrt{\left(x+c^2\right)+(y-0)^2} = \sqrt{(x-c)^2}$$
$$(x+c)^2 + (y-0)^2 = (x-c)^2$$
$$x^2 + 2cx + c^2 + y^2 = x^2 - 2cx + c^2$$
$$y^2 = -4cx$$

**55.** Graph $y = \frac{7}{4} + \frac{1}{4}x|x|$.



**56.** Since the axis of symmetry passes through the vertex (0, 0) and focus (1, 1), its equation is given by $y = x$.

Because the directrix is perpendicular to the axis of symmetry, its slope $m$ must be $-\frac{1}{1}$ or $-1$.

Since the vertex (0, 0) is the midpoint of the line segment connecting the focus (1, 1) and the directrix, the distance from the vertex to the focus $\left(\sqrt{(1-0)^2+(1-0)^2} \text{ or } \sqrt{2}\right)$ must equal the distance from the vertex to the directrix [at point $(x_1, y_1)$] along the axis of symmetry. Therefore, $y_1 = x_1$ (since the point is also on the axis of symmetry) and $\sqrt{(x_1-0)^2+(y_1-0)^2} = \sqrt{2}$.

Thus, by substituting $y_1$ for $x_1$, we obtain:

$$\sqrt{(y_1-0)^2+(y_1-0)^2} = \sqrt{2}$$
$$\sqrt{y_1^2+y_1^2} = \sqrt{2}$$
$$\sqrt{2y_1^2} = \sqrt{2}$$

Thus, $y_1^2 = 1$ and $y_1 = \pm 1$. If $y_1 = 1$ (and $x_1 = 1$), the directrix would pass through the focus, which is an impossibility. Thus, $y_1 = -1$, and $x_1 = -1$.

The equation of the directrix is derived by substituting $y_1$, $x_1$, and $m$ in the point slope form of the straight line.

$$y = y_1 = m(x - x_1)$$
$$y-(-1) = (-1)(x-(-1))$$
$$y+1 = -(x+1)$$
$$y+1 = -x-1$$
$$y = -x-2$$

**57.** By definition, any point on the curve $(x, y)$ will be equidistant from both the focus (1, 1) and the directrix, $(y_2 = -x_2 - 2)$.

If we let $d_1$ equal the distance from the focus to the point $(x, y)$, we get $d_1 = \sqrt{(x-1)^2+(y-1)^2}$

To determine the distance $d_2$ from the point $(x, y)$ to the line $y = -x - 2$, draw a line segment from $(x, y)$ to the directrix so as to meet the directrix at a $90°$ angle.

Now drop a line segment parallel to the $y$-axis from $(x, y)$ to the directrix. This segment will meet the directrix at a 45° angle, thus forming a right isosceles triangle with the directrix and the line segment perpendicular to the directrix from $(x, y)$. The length of this segment, which is the hypotenuse of the triangle, is the difference between $y$ and the $y$-value of the directrix at $x$, or $-x - 2$. Thus, the hypotenuse has a length of $y + x + 2$, and since the right triangle is also isosceles, each leg has a length of $\frac{y+x+2}{\sqrt{2}}$.

But since $d_2$ is the length of the leg drawn from $(x, y)$ to the directrix, $d_2 = \frac{y+x+2}{\sqrt{2}}$.

Thus, $d_1 = \sqrt{(x-1)^2+(y-1)^2}$ and $d_2 = \frac{x+y+2}{\sqrt{2}}$.

By definition, $d_1 = d_2$. So, by substitution,

$$\sqrt{(x-1)^2+(y-1)^2} = \frac{x+y+2}{\sqrt{2}}$$
$$\sqrt{2}\sqrt{(x-1)^2+(y-1)^2} = x+y+2$$
$$2\left[(x-1)^2+(y-1)^2\right] = x^2+y^2+4x+4y+2xy+4$$
$$2\left(x^2-2x+1+y^2-2y+1\right) = x^2+y^2+4x+4y+2xy+4$$
$$2x^2-4x+2y^2-4y+4 = x^2+y^2+4x+4y+2xy+4$$
$$x^2+y^2-8x-8y-2xy = 0$$

## Prepare for Section 8.2

**PS1.** midpoint: $\dfrac{x_1+x_2}{2}$ $\quad \dfrac{y_1+y_2}{2}$

$\dfrac{5+-1}{2}=2 \quad \dfrac{1+5}{2}=3$

The midpoint is (2, 3).

length: $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$\sqrt{(-1-5)^2+(5-1)^2}=\sqrt{36+16}=\sqrt{52}=2\sqrt{13}$

The length is $2\sqrt{13}$.

**PS2.** $x^2+6x-16=0$

$(x+8)(x-2)=0$

$x+8=0 \quad x-2=0$

$x=-8 \quad x=2$

The solutions are –8, 2.

**PS3.**

$x^2-2x=2$

$x^2-2x+1=2+1$

$(x-1)^2=3$

$x-1=\pm\sqrt{3}$

$x=1\pm\sqrt{3}$

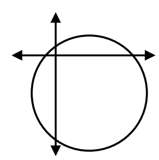
**PS4.** $x^2-8x+16=(x-4)^2$

**PS5.** $(x-2)^2+y^2=4$

$y^2=4-(x-2)^2$

$y=\pm\sqrt{4-(x-2)^2}$

**PS6.**



$(x-2)^2+(y+3)^2=16$

Center: (2, –3), radius 4

## Section 8.2

**1.** **a.** iv **b.** i **c.** ii **d.** iii

**2.** **a.** iii **b.** i **c.** iv **d.** ii

**3.** $\dfrac{x^2}{16}+\dfrac{y^2}{25}=1$

$a^2=25\to a=5$

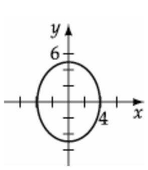
$b^2=16\to b=4$

$c=\sqrt{a^2-b^2}$
$=\sqrt{25-16}$
$=\sqrt{9}$
$=3$

Center $(0, 0)$

Vertices $(0, \pm 5)$

Foci $(0, \pm 3)$




**4.** $\dfrac{x^2}{49}+\dfrac{y^2}{36}=1$

$a^2=49\to a=7$

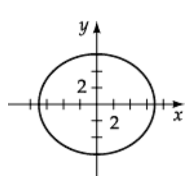
$b^2=36\to b=6$

$c=\sqrt{a^2-b^2}$
$=\sqrt{49-36}$
$=\sqrt{13}$

Center $(0, 0)$

Vertices $(\pm 7, 0)$

Foci $(\pm\sqrt{13}, 0)$




**5.** $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$

$a^2=9\to a=3$

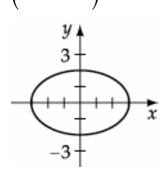
$b^2=4\to b=2$

$c=\sqrt{a^2-b^2}$
$=\sqrt{9-4}$
$=\sqrt{5}$

Center $(0, 0)$

Vertices $(\pm 3, 0)$

Foci $(\pm\sqrt{5}, 0)$




**6.** $\dfrac{x^2}{64}+\dfrac{y^2}{25}=1$

$a^2=64\to a=8$

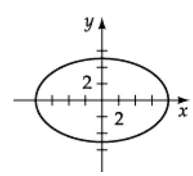
$b^2=25\to b=5$

$c=\sqrt{a^2-b^2}$
$=\sqrt{64-25}$
$=\sqrt{39}$

Center $(0, 0)$

Vertices $(\pm 8, 0)$

Foci $(\pm\sqrt{39}, 0)$



**7.** $\dfrac{x^2}{9}+\dfrac{y^2}{7}=1$

$a^2=9\to a=3$

$b^2=7\to b=\sqrt{7}$

$c=\sqrt{a^2-b^2}$

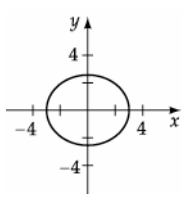
$=\sqrt{9-7}$

$=\sqrt{2}$

Center $(0,0)$

Vertices $(0,\ \pm 3)$

Foci $\left(0,\ \pm\sqrt{2}\right)$




**8.** $\dfrac{x^2}{5}+\dfrac{y^2}{4}=1$

$a^2=5\to a=\sqrt{5}$

$b^2=4\to b=2$

$c=\sqrt{a^2-b^2}$

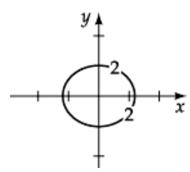
$=\sqrt{5-4}$

$=\sqrt{1}=1$

Center $(0,0)$

Vertices $\left(\pm\sqrt{5},\ 0\right)$

Foci $(\pm 1,\ 0)$




**9.** $\dfrac{4x^2}{9}+\dfrac{y^2}{16}=1$

Rewrite as

$\dfrac{x^2}{9/4}+\dfrac{y^2}{16}=1$

$a^2=16\to a=4$

$b^2=9/4\to b=3/2$

$c=\sqrt{a^2-b^2}$

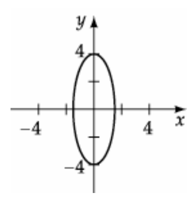
$=\sqrt{16-9/4}$

$=\sqrt{55}/2$

Center $(0,0)$

Vertices $(0,\ \pm 4)$

Foci $\left(0,\ \pm\dfrac{\sqrt{55}}{2}\right)$




**10.** $\dfrac{x^2}{9}+\dfrac{9y^2}{16}=1$

Rewrite as

$\dfrac{x^2}{9}+\dfrac{y^2}{16/9}=1$

$a^2=9\to a=3$

$b^2=16/9\to b=4/3$

$c=\sqrt{a^2-b^2}$

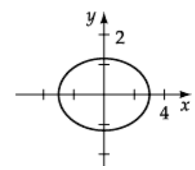
$=\sqrt{9-16/9}$

$=\sqrt{65}/3$

Center $(0,0)$

Vertices $(\pm 3,\ 0)$

Foci $\left(\pm\dfrac{\sqrt{65}}{3},\ 0\right)$

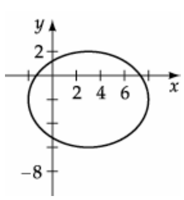



**11.** $\dfrac{(x-3)^2}{25}+\dfrac{(y+2)^2}{16}=1$

Center $(3,\ -2)$

Vertices $(3\pm 5,\ -2)=(8,\ -2),\ (-2,\ -2)$

Foci $(3\pm 3,\ -2)=(6,\ -2),\ (0,\ -2)$

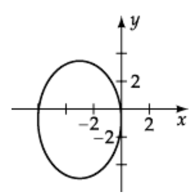



**12.** $\dfrac{(x+3)^2}{9}+\dfrac{(y+1)^2}{16}=1$

Center $(-3,-1)$

Vertices $(-3,-1,\pm 4)=(-3,3),(-3,-5)$

Foci $\left(-3,-1\pm\sqrt{7}\right)$

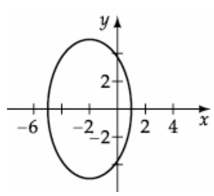


**13.** $\dfrac{(x+2)^2}{9}+\dfrac{y^2}{25}=1$

Center $(-2, 0)$

Vertices $(-2, 5), (-2, -5)$

Foci $(-2, 4), (-2, -4)$

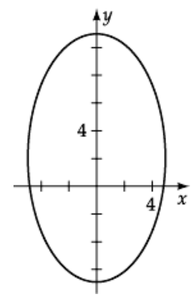



**14.** $\dfrac{x^2}{25}+\dfrac{(y-2)^2}{81}=1$

Center $(0, 2)$

Vertices $(0, 2 \pm 9) = (0, 11), (0, -7)$

Foci $\left(0, 2 \pm \sqrt{56}\right) = \left(0, 2 \pm 2\sqrt{14}\right)$

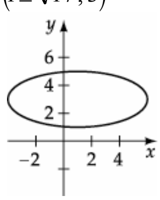



**15.** $\dfrac{(x-1)^2}{21}+\dfrac{(y-3)^2}{4}=1$

Center $(1, 3)$

Vertices $\left(1 \pm \sqrt{21}, 3\right)$

Foci $\left(1 \pm \sqrt{17}, 3\right)$

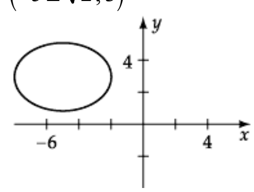



**16.** $\dfrac{(x+5)^2}{9}+\dfrac{(y-3)^2}{7}=1$

Center $(-5, 3)$

Vertices $(-5 \pm 3, 3) = (-2, 3), (-8, 3)$

Foci $\left(-5 \pm \sqrt{2}, 3\right)$

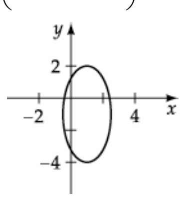



**17.** $\dfrac{9(x-1)^2}{16}+\dfrac{(y+1)^2}{9}=1$

Center $(1, -1)$

Vertices $(1, -1 \pm 3) = (1, 2), (1, -4)$

Foci $\left(1, -1 \pm \dfrac{\sqrt{65}}{3}\right)$

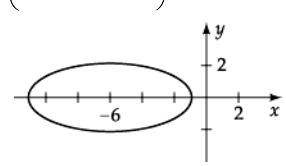



**18.** $\dfrac{(x+6)^2}{25}+\dfrac{25y^2}{144}=1$

Center $(-6, 0)$

Vertices $(-6 \pm 5, 0) = (-1, 0), (-11, 0)$

Foci $\left(-6 \pm \dfrac{\sqrt{481}}{5}, 0\right)$



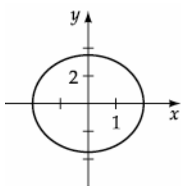
**19.** $3x^2 + 4y^2 = 12$

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

Center $(0, 0)$

Vertices $(\pm 2, 0)$

Foci $(\pm 1, 0)$



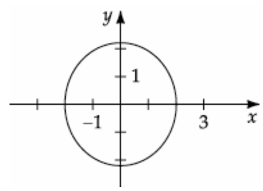

**20.** $5x^2 + 4y^2 = 20$

$$\frac{x^2}{4} + \frac{y^2}{5} = 1$$

Center $(0, 0)$

Vertices $\left(0, \pm\sqrt{5}\right)$

Foci $(0, \pm 1)$



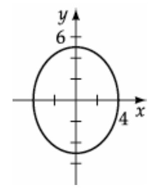

**21.** $25x^2 + 16y^2 = 400$

$$\frac{x^2}{16} + \frac{y^2}{25} = 1$$

Center $(0, 0)$

Vertices $(0, \pm 5)$

Foci $(0, \pm 3)$



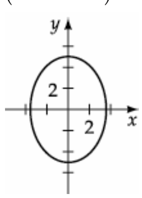

**22.** $25x^2 + 12y^2 = 300$

$$\frac{x^2}{12} + \frac{y^2}{25} = 1$$

Center $(0, 0)$

Vertices $(0, \pm 5)$

Foci $\left(0, \pm\sqrt{13}\right)$



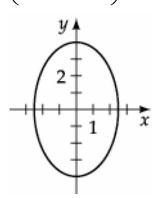

**23.** $64x^2 + 25y^2 = 400$

$$\frac{x^2}{\frac{25}{4}} + \frac{y^2}{16} = 1$$

Center $(0, 0)$

Vertices $(0, \pm 4)$

Foci $\left(0, \pm\frac{\sqrt{39}}{2}\right)$



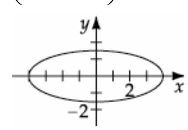

**24.** $9x^2 + 64y^2 = 144$

$$\frac{x^2}{16} + \frac{y^2}{\frac{9}{4}} = 1$$

Center $(0, 0)$

Vertices $(\pm 4, 0)$

Foci $\left(\pm\frac{\sqrt{55}}{2}, 0\right)$



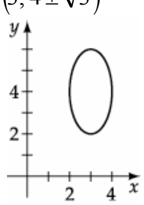

**25.**

$$4x^2 + y^2 - 24x - 8y + 48 = 0$$
$$4\left(x^2 - 6x\right) + \left(y^2 - 8y\right) = -48$$
$$4\left(x^2 - 6x + 9\right) + \left(y^2 - 8y + 16\right) = -48 + 36 + 16$$
$$4(x-3)^3 + (y-4)^2 = 4$$
$$\frac{(x-3)^2}{1} + \frac{(y-2)^2}{4} = 1$$

Center $(3, 4)$

Vertices $(3, 4 \pm 2) = (3, 6), (3, 2)$

Foci $\left(3, 4 \pm \sqrt{3}\right)$



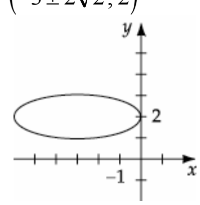

**26.**

$$x^2 + 9y^2 + 6x - 36y + 36 = 0$$
$$\left(x^2 + 6x\right) + 9\left(y^2 - 4y\right) = -36$$
$$\left(x^2 + 6x + 9\right) + 9\left(y^2 - 4y + 4\right) = -36 + 9 + 36$$
$$(x+3)^2 + 9(y-2)^2 = 9$$
$$\frac{(x+3)^2}{9} + \frac{(y-2)^2}{1} = 1$$

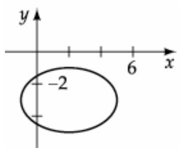
Center $(-3, 2)$

Vertices $(-3 \pm 3, 2) = (0, 2), (-6, 2)$

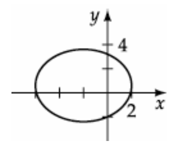
Foci $\left(-3 \pm 2\sqrt{2}, 2\right)$



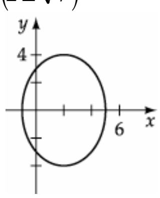
**27.**
$$5x^2+9y^2-20x+54y+56=0$$
$$5(x^2-4x)+9(y^2+6y)=-56$$
$$5(x^2-4x+4)+9(y^2+6y+9)=-56+20+81$$
$$5(x-2)^2+9(y+3)^2=45$$
$$\frac{(x-2)^2}{9}+\frac{(y+3)^2}{5}=1$$
Center $(2,-3)$
Vertices $(2\pm 3,-3)=(-1,-3),(5,-3)$
Foci $(2\pm 2,3)=(0,-3),(4,-3)$



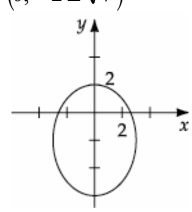

**28.**
$$9x^2+16y^2+36x-16y-104=0$$
$$9(x^2+4x)+16(y^2-y)=104$$
$$9(x^2+4x+4)+16\left(y^2-y+\tfrac{1}{4}\right)=104+36+4$$
$$9(x+2)^2+16\left(y-\tfrac{1}{2}\right)^2=144$$
$$\frac{(x+2)^2}{16}+\frac{\left(y-\frac{1}{2}\right)^2}{9}=1$$
Center $\left(-2,\ \frac{1}{2}\right)$
Vertices $\left(-2\pm 4,\ \frac{1}{2}\right)=\left(2,\ \frac{1}{2}\right),\ \left(-6,\ \frac{1}{2}\right)$
Foci $\left(-2\pm\sqrt{7},\ \frac{1}{2}\right)$




**29.**
$$16x^2+9y^2-64x-80=0$$
$$16(x^2-4x)+9y^2=80$$
$$16(x^2-4x+4)+9y^2=80+64$$
$$16(x-2)^2+9y^2=144$$
$$\frac{(x-2)^2}{9}+\frac{y^2}{16}=1$$
Center $(2,0)$
Vertices $(2,\pm 4)=(2,4),(2,-4)$
Foci $(2\pm\sqrt{7})$




**30.**
$$16x^2+9y^2+36y-108=0$$
$$16x^2+9(y^2+4y)=108$$
$$16x^2+9(y^2+4y+4)=108+36$$
$$16x^2+9(y-+2)^2=144$$
$$\frac{x^2}{9}+\frac{(y+2)^2}{16}=1$$
Center $(0,-2)$
Vertices $(0,-2\pm 4)=(0,2),(0,-6)$
Foci $(0,-2\pm\sqrt{7})$



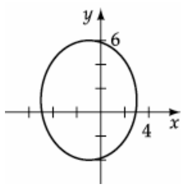
**31.** $25x^2 + 16y^2 + 50x - 32y - 359 = 0$

$$25(x^2 + 2x) + 16(y^2 - 2y) = 359$$
$$25(x^2 + 2x + 1) + 16(y^2 - 2y + 1) = 359 + 25 + 16$$
$$25(x+1)^2 + 16(y-1)^2 = 400$$
$$\frac{(x+1)^2}{16} + \frac{(y-1)^2}{25} = 1$$

Center $(-1, 1)$

Vertices $(-1, 1 \pm 5) = (-1, 6), (-1, -4)$

Foci $(-1, 1 \pm 3) = (-1, 4), (-1, -2)$



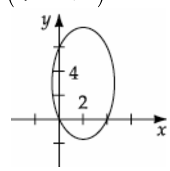

**32.** $16x^2 + 9y^2 - 64x - 54y + 1 = 0$

$$16(x^2 - 4x) + 9(y^2 - 6y) = -1$$
$$16(x^2 - 4x + 4) + 9(y^2 - 6y + 9) = -1 + 64 + 81$$
$$16(x-2)^2 + 9(y-3)^2 = 144$$
$$\frac{(x-2)^2}{9} + \frac{(y-3)^2}{16} = 1$$

Center $(2, 3)$

Vertices $(2, 3 \pm 4) = (2, 7), (2, -1)$

Foci $\left(2, 3 \pm \sqrt{7}\right)$



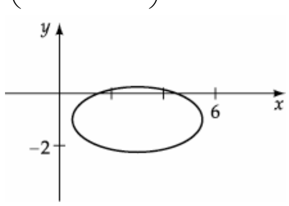

**33.** $8x^2 + 25y^2 - 48x + 50y + 47 = 0$

$$8(x^2 - 6x) + 25(y^2 + 2y) = -47$$
$$8(x^2 - 6x + 9) + 25(y^2 + 2y + 1) = -47 + 72 + 25$$
$$8(x-3)^2 + 25(y+1)^2 = 50$$
$$\frac{(x-3)^2}{25/4} + \frac{(y+1)^2}{2} = 1$$

Center: $(3, \ -1)$

Vertices: $\left(3 \pm \frac{5}{2}, \ -1\right) = \left(\frac{11}{2}, \ -1\right), \left(\frac{1}{2}, \ -1\right)$

Foci: $\left(3 \pm \frac{\sqrt{17}}{2}, \ -1\right)$



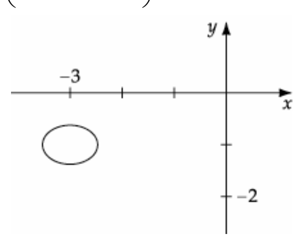

**34.** $4x^2 + 9y^2 + 24x + 18y + 44 = 0$

$$4(x^2 + 6x) + 9(y^2 + 2y) = -44$$
$$4(x^2 + 6x + 9) + 9(y^2 + 2y + 1) = -44 + 36 + 9$$
$$4(x+3)^2 + 9(y+1)^2 = 1$$
$$\frac{(x+3)^2}{1/4} + \frac{(y+1)^2}{1/9} = 1$$

Center $(-3, -1)$

Vertices $\left(-3 \pm \frac{1}{2}, -1\right) = \left(-\frac{5}{2}, -1\right), \left(-\frac{7}{2}, -1\right)$

Foci $\left(-3 \pm \frac{\sqrt{5}}{6}, -1\right)$




**35.**
$$2a = 10$$
$$a = 5$$
$$a^2 = 25$$
$$c = 4$$
$$c^2 = a^2 - b^2$$
$$16 = 25 - b^2$$
$$b^2 = 9$$
$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

**36.**
$$2b = 6$$
$$b = 3$$
$$b^2 = 9$$
$$c = 4$$
$$c^2 = a^2 - b^2$$
$$16 = a^2 - 9$$
$$25 = a^2$$
$$\frac{x^2}{9} + \frac{y^2}{25} = 1$$

**37.**
$$a = 6$$
$$a^2 = 36$$
$$b = 4$$
$$b^2 = 16$$
$$\frac{x^2}{36} + \frac{y^2}{16} = 1$$

**38.**
$$a = 7$$
$$a^2 = 49$$
$$b = 5$$
$$b^2 = 25$$
$$\frac{x^2}{49} + \frac{y^2}{25} = 1$$

**39.**

$$2a = 12$$
$$a = 6$$
$$a^2 = 36$$
$$\frac{x^2}{36} + \frac{y^2}{b^2} = 1$$
$$\frac{(2)^2}{36} + \frac{(-3)^2}{b^2} = 1$$
$$\frac{4}{36} + \frac{9}{b^2} = 1$$
$$\frac{9}{b^2} = \frac{8}{9}$$
$$8b^2 = 81$$
$$b^2 = \frac{81}{8}$$
$$\frac{x^2}{36} + \frac{y^2}{81/8} = 1$$

**40.**

$$2a = 8$$
$$a = 4$$
$$a^2 = 16$$

$$\frac{x^2}{16} + \frac{y^2}{b^2} = 1 \qquad \frac{x^2}{b^2} + \frac{y^2}{16} = 1$$
$$\frac{(-2)^2}{16} + \frac{(2)^2}{b^2} = 1 \qquad \frac{(-2)^3}{b^2} + \frac{(2)^2}{16} = 1$$
$$\frac{4}{16} + \frac{4}{b^2} = 1 \qquad \frac{4}{b^2} + \frac{4}{16} = 1$$
$$\frac{4}{b^2} = \frac{3}{4} \qquad \frac{4}{b^2} = \frac{3}{4}$$
$$3b^2 = 16 \qquad 3b^2 = 16$$
$$b^2 = \frac{16}{3} \qquad b^2 = \frac{16}{3}$$

$$\frac{x^2}{16} + \frac{y^2}{16/3} = 1 \text{ or } \frac{x^2}{16/3} + \frac{y^2}{16} = 1$$

**41.**

$$c = 3$$
$$2a = 8$$
$$a = 4$$
$$a^2 = 16$$
$$c^2 = a^2 - b^2$$
$$9 = 16 - b^2$$
$$b^2 = 7$$
$$\frac{(x+2)^2}{16} + \frac{(y-4)^2}{7} = 1$$

**42.**

$$c = 3$$
$$2b = 4$$
$$2b = 4$$
$$b = 2$$
$$b^2 = 4$$
$$c^2 = a^2 - b^2$$
$$9 = a^2 - 4$$
$$a^2 = 13$$
$$\frac{x^2}{4} + \frac{(y-3)^2}{13} = 1$$

**43.**

$$2a = 10$$
$$a = 5$$
$$a^2 = 25$$

Since the center of the ellipse is (2, 4) and the point (3, 3) is on the ellipse, we have

$$\frac{(x-2)^2}{b^2} + \frac{(y-4)^2}{a^2} = 1$$
$$\frac{(3-2)^2}{b^2} + \frac{(3-4)^2}{25} = 1$$
$$\frac{1}{b^2} = 1 - \frac{1}{25}$$
$$b^2 = \frac{25}{24}$$
$$\frac{(x-2)^2}{\frac{25}{24}} + \frac{(y-4)^2}{25} = 1$$

**44.**

$$2b = 8$$
$$b = 4$$
$$b^2 = 16$$
$$\frac{(x+4)^2}{a^2} + \frac{(y-1)^2}{16} = 1$$
$$\frac{(0+4)^2}{a^2} + \frac{(4-1)^2}{16} = 1$$
$$\frac{16}{a^2} + \frac{9}{16} = 1$$
$$\frac{16}{a^2} = \frac{7}{16}$$
$$7a^2 = 256$$
$$a^2 = \frac{256}{7}$$
$$\frac{(x+4)^2}{256/7} + \frac{(y-1)^2}{16} = 1$$

**45.** center $(5, 1)$

$$c = 3$$
$$2a = 10$$
$$a = 5$$
$$a^2 = 25$$
$$c^2 = a^2 - b^2$$
$$9 = 25 - b^2$$
$$b^2 = 16$$
$$\frac{(x-5)^2}{16} + \frac{(y-1)^2}{25} = 1$$

**46.** center $(-1, -1)$

$$c = 4$$
$$2a = 12$$
$$a = 6$$
$$a^2 = 36$$
$$c^2 = a^2 - b^2$$
$$16 = 36 - b^2$$
$$b^2 = 20$$
$$\frac{(x+1)^2}{36} + \frac{(y+1)^2}{20} = 1$$

**47.**

$$2a = 10$$
$$a = 5$$
$$a^2 = 25$$
$$\frac{c}{a} = \frac{2}{5}$$
$$\frac{c}{5} = \frac{2}{5}$$
$$c = 2$$
$$c^2 = a^2 - b^2$$
$$4 = 25 - b^2$$
$$b^2 = 21$$
$$\frac{x^2}{25} + \frac{y^2}{21} = 1$$

**48.** center $(0, 0)$

$$c = 9$$
$$\frac{c}{a} = \frac{3}{4}$$
$$\frac{9}{a} = \frac{3}{4}$$
$$a = 12$$
$$c^2 = a^2 - b^2$$
$$81 = 144 - b^2$$
$$b^2 = 63$$
$$\frac{x^2}{144} + \frac{y^2}{63} = 1$$

**49.** center $(0, 0)$

$$c = 4$$
$$\frac{c}{a} = \frac{2}{3}$$
$$\frac{4}{a} = \frac{2}{3}$$
$$a = 6$$
$$c^2 = a^2 - b^2$$
$$16 = 36 - b^2$$
$$b^2 = 20$$
$$\frac{x^2}{20} + \frac{y^2}{36} = 1$$

**50.** center $(0, 0)$

$$c = 3$$
$$\frac{c}{a} = \frac{1}{4}$$
$$\frac{3}{a} = \frac{1}{4}$$
$$a = 12$$
$$c^2 = a^2 - b^2$$
$$9 = 144 - b^2$$
$$b^2 = 135$$
$$\frac{x^2}{135} + \frac{y^2}{144} = 1$$

**51.** center $(1, 3)$

$$c = 2$$
$$\frac{c}{a} = \frac{2}{5}$$
$$\frac{2}{a} = \frac{2}{5}$$
$$a = 5$$
$$c^2 = a^2 - b^2$$
$$4 = 25 - b^2$$
$$b^2 = 21$$
$$\frac{(x-1)^2}{25} + \frac{(y-3)^2}{21} = 1$$

**52.** center $(-2, 1)$

$$c = 3$$
$$\frac{c}{a} = \frac{1}{4}$$
$$\frac{3}{a} = \frac{1}{4}$$
$$a = 12$$
$$c^2 = a^2 - b^2$$
$$9 = 144 - b^2$$
$$b^2 = 135$$
$$\frac{(x+2)^2}{135} + \frac{(y-1)^2}{144} = 1$$

**53.**

$$2a = 24$$
$$a = 12$$
$$\frac{c}{a} = \frac{2}{3}$$
$$\frac{c}{12} = \frac{2}{3}$$
$$c = 8$$
$$c^2 = a^2 - b^2$$
$$64 = 144 - b^2$$
$$b^2 = 80$$
$$\frac{x^2}{80} + \frac{y^2}{144} = 1$$

**54.**
$$2a = 15$$
$$a = \frac{15}{2}$$
$$a^2 = \frac{225}{4}$$
$$\frac{c}{a} = \frac{3}{5}$$
$$\frac{2c}{15} = \frac{3}{5}$$
$$c = \frac{9}{2}$$
$$c^2 = a^2 - b^2$$
$$\frac{81}{4} = \frac{225}{4} - b^2$$
$$b^2 = \frac{144}{4} = 36$$
$$\frac{x^2}{225/4} + \frac{y^2}{36} = 1$$

**55.**
$$484 = 64 + c^2$$
$$c^2 = 420$$
$$c = 20.494$$
$$2c = 40.9878 \approx 41$$
The emitter should be placed 41 cm away.

**56.** **a.**
$$\frac{4}{5} = \frac{h}{4.5}$$
$$h = \frac{18}{5} = 3.6$$
The value of $h$ is 3.6 in.

**b.** Major axis:
$$\sqrt{(4.5)^2 + (3.6)^2} = 5.76 \text{ in.}$$
Minor axis:
Diameter of vent pipe = 4.50 in.

**c.** From part b,
$2a = 5.763 \quad 2b = 4.5$
$a = 2.88 \quad b = 2.25$
$a^2 = 8.3025 \quad b^2 = 5.0625$

The equation is $\frac{x^2}{8.3025} + \frac{y^2}{5.0625} = 1$.

**57.** Aphelion = $2a$ − perihelion
$$934.34 = 2a - 835.14$$
$$a = 884.74 \text{ million miles}$$
Aphelion = $a + c = 934.34$
$$884.74 + c = 934.34$$
$$c = 49.6 \text{ million miles}$$

$$b = \sqrt{a^2 - c^2}$$
$$= \sqrt{884.74^2 - 49.6^2}$$
$$\approx 883.35 \text{ million miles}$$

An equation of the orbit of Saturn is
$$\frac{x^2}{884.74^2} + \frac{y^2}{883.35^2} = 1$$

**58.** The mean distance is $a = 67.08$ million miles.

Aphelion = $a + c$
$= 67.58$ million miles.

Thus $c = 67.58 - a$
$= 0.50$ million miles.

$$b = \sqrt{a^2 - c^2}$$
$$= \sqrt{67.08^2 - 0.50^2}$$
$$\approx 67.078$$

An equation of the orbit of Venus is
$$\frac{x^2}{67.08^2} + \frac{y^2}{67.078^2} = 1$$

**59.** $a$ = semimajor axis = 50 feet
$b$ = height = 30 feet
$$c^2 = a^2 - b^2$$
$$c^2 = 50^2 - 30^2$$
$$c = \sqrt{1600} = 40$$
The foci are located 40 feet to the right and to the left of center.

**60.** The length of the semimajor axis is 50 feet. Thus

$$c^2 = a^2 - b^2$$
$$32^2 = 50^2 - b^2$$
$$b^2 = 50^2 - 32^2$$
$$b = \sqrt{50^2 - 32^2}$$
$$b \approx 38.4 \text{ feet}$$

**61.** $2a = 36 \qquad 2b = 9$

$a = 18 \qquad b = \frac{9}{2}$

$$c^2 = a^2 - b^2$$
$$c^2 = 18^2 - \left(\frac{9}{2}\right)^2$$
$$c^2 = 324 - \frac{81}{4}$$
$$c^2 = \frac{1215}{4}$$
$$c = \frac{9\sqrt{15}}{2}$$

Since one focus is at (0, 0), the center of the ellipse is at $(9\sqrt{15}/2,\ 0)$
(17.43, 0). The equation of the path of Halley's Comet in astronomical units is

$$\frac{\left(x - 9\sqrt{15}/2\right)^2}{324} + \frac{y^2}{81/4} = 1$$

**62.** The reflective property of an ellipse.

**63.** $\frac{x^2}{75^2} + \frac{y^2}{34^2} = 1$

Solve for $y$, where $x = 55$.

$$\frac{55^2}{75^2} + \frac{y^2}{34^2} = 1$$
$$\frac{y^2}{34^2} = 1 - \frac{55^2}{75^2}$$
$$y^2 = 34^2\left(1 - \frac{55^2}{75^2}\right)$$
$$y = \sqrt{34^2\left(1 - \frac{55^2}{75^2}\right)}$$
$$y \approx 23 \text{ ft}$$
$$h = y + 1 = 23 + 1 = 24 \text{ ft}$$

**64.** The gear on the left speeds up and slows down twice as the gear on the right makes one complete revolution at a constant angular speed.

**65.** **a.**

$$c^2 = a^2 - b^2$$
$$c^2 = 4^2 - 3^2$$
$$c^2 = 7$$
$$c = \sqrt{7}$$

$\sqrt{7}$ ft to the right and left of $O$.

**b.** $2a = 2(4) = 8$ ft

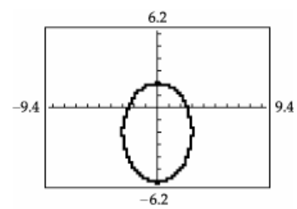
**66.** $2a = 3.04 \Rightarrow a = 1.52$

$2b = 2.99 \Rightarrow b = 1.495$

$$p = \pi\sqrt{2\left(a^2 + b^2\right)}$$
$$= \pi\sqrt{2\left(1.52^2 + 1.495^2\right)}$$
$$= \pi\sqrt{2(2.3104 + 2.235025)}$$
$$= \pi\sqrt{2(4.545425)}$$
$$= \pi\sqrt{9.09085}\ \text{AU} \cdot (92.96 \text{ million miles per AU})$$
$$\approx 881 \text{ million miles}$$

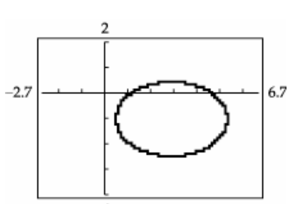
**67.** $9y^2 + 36y + 16x^2 - 108 = 0$

$$y = \frac{-36 \pm \sqrt{36^2 - 4(9)(16x^2 - 108)}}{2(9)}$$
$$= \frac{-36 \pm \sqrt{1296 - 36(16x^2 - 108)}}{18}$$
$$= \frac{-36 \pm \sqrt{1296 - 576x^2 + 3888}}{18}$$
$$= \frac{-36 \pm \sqrt{-576x^2 + 5184}}{18}$$
$$= \frac{-36 \pm \sqrt{576(-x^2 + 9)}}{18}$$
$$= \frac{-36 \pm 24\sqrt{(-x^2 + 9)}}{18}$$
$$= \frac{-6 \pm 4\sqrt{(-x^2 + 9)}}{3}$$



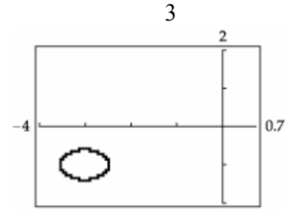

**68.** $25y^2 + 50y + 8x^2 - 48x + 47 = 0$

$$y = \frac{-50 \pm \sqrt{50^2 - 4(25)(8x^2 - 48x + 47)}}{2(25)}$$
$$= \frac{-50 \pm \sqrt{2500 - 100(8x^2 - 48x + 47)}}{50}$$
$$= \frac{-50 \pm \sqrt{2500 - 800x^2 + 4800x - 4700}}{50}$$
$$= \frac{-50 \pm \sqrt{-800x^2 + 4800x - 2200}}{50}$$
$$= \frac{-50 \pm \sqrt{100(-8x^2 + 48x - 22)}}{50}$$
$$= \frac{-50 \pm 10\sqrt{-8x^2 + 48x - 22}}{50}$$
$$= \frac{-5 \pm \sqrt{-8x^2 + 48x - 22}}{5}$$




**69.** $9y^2 + 18y + 4x^2 + 24x + 44 = 0$

$$y = \frac{-18 \pm \sqrt{18^2 - 4(9)(4x^2 + 24x + 44)}}{2(9)}$$
$$= \frac{-18 \pm \sqrt{324 - 36(4x^2 + 24x + 44)}}{18}$$
$$= \frac{-18 \pm \sqrt{324 - 144x^2 - 864x - 1584}}{18}$$
$$= \frac{-18 \pm \sqrt{-144x^2 - 864x - 1260}}{18}$$
$$= \frac{-18 \pm \sqrt{36(-4x^2 - 24x - 35)}}{18}$$
$$= \frac{-18 \pm 6\sqrt{-4x^2 - 24x - 35}}{18}$$
$$= \frac{-3 \pm \sqrt{-4x^2 - 24x - 35}}{3}$$

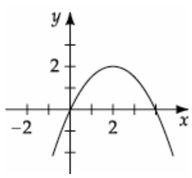


**Connecting Concepts**

**70.** $4x^2 + 9y - 16x - 2 = 0$

This is not the equation of an ellipse because there is no $y^2$ term. It is a quadratic equation.

$9y = -4x^2 + 16x + 2$




**71.** The sum of the distances between the two foci and a point on the ellipse is $2a$.

$$\begin{aligned} 2a &= \sqrt{\left(\frac{9}{2}-0\right)^2 + (3-3)^2} + \sqrt{\left(\frac{9}{2}-0\right)^2 + (3+3)^2} \\ &= \sqrt{\left(\frac{9}{2}\right)^2} + \sqrt{\frac{225}{4}} \\ &= \frac{9}{2} + \frac{15}{2} \\ &= 12 \end{aligned}$$

$$a = 6$$
$$c = 3$$
$$c^2 = a^2 - b^2$$
$$9 = 36 - b^2$$
$$b^2 = 27$$

$$\frac{x^2}{36} + \frac{y^2}{27} = 1$$

**72.** The sum of the distances between the two foci and a point on the ellipse is $2a$.

$$\begin{aligned} 2a &= \sqrt{(4-4)^2 + \left(\frac{9}{5}-0\right)^2} + \sqrt{(4+4)^2 + \left(\frac{9}{5}-0\right)^2} \\ &= \sqrt{\left(\frac{9}{5}\right)^2} + \sqrt{\frac{1681}{25}} \\ &= \frac{9}{5} + \frac{41}{5} \\ &= 10 \end{aligned}$$

$$5 = a$$
$$c = 4$$
$$c^2 = a^2 - b^2$$
$$16 = 25 - b^2$$
$$b^2 = 9$$

$$\frac{x^2}{9} + \frac{y^2}{25} = 1$$

**73.** The sum of the distances between the two foci and a point on the ellipse is $2a$.

$$\begin{aligned} 2a &= \sqrt{(5-2)^2 + (3+1)^2} + \sqrt{(5-2)^2 + (3-3)^2} \\ &= \sqrt{25} + \sqrt{3^2} \\ &= 5 + 3 \\ &= 8 \end{aligned}$$

$$a = 4$$
$$c = 2$$

$$c^2 = a^2 - b^2$$
$$4 = 16 - b^2$$
$$b^2 = 12$$

$$\frac{(x-1)^2}{16} + \frac{(y-2)^2}{12} = 1$$

**74.** The sum of the distances between the two foci and a point on the ellipse is $2a$.

$$2a = \sqrt{(1-7)^2 + \left(\frac{3}{4}+1\right)^2} + \sqrt{(1-1)^2 + \left(\frac{3}{4}+1\right)^2}$$
$$= \sqrt{\frac{625}{16}} + \sqrt{\left(\frac{7}{4}\right)^2}$$
$$= \frac{25}{4} + \frac{7}{4}$$
$$= 8$$

$$a = 4$$
$$c = 3$$
$$c^2 = a^2 - b^2$$
$$9 = 16 - b^2$$
$$b^2 = 7$$
$$\frac{(x+1)^2}{7} + \frac{(y-4)^2}{16} = 1$$

**75.** Center $(1,-1)$

$$c^2 = a^2 - b^2$$
$$c^2 = 16 - 9$$
$$c^2 = 7$$
$$c = \sqrt{7}$$

The latus rectum is on the graph of $y = -1 + \sqrt{7}$, or $y = -1 - \sqrt{7}$

$$\frac{(x-1)^2}{9} + \frac{(y+1)^2}{16} = 1$$
$$\frac{(x-1)^2}{9} + \frac{(-1+\sqrt{7}+1)^2}{16} = 1 \text{ or } \frac{(x-1)^2}{9} + \frac{(-1-\sqrt{7}+1)^2}{16} = 1$$
$$\frac{(x-1)^2}{9} + \frac{7}{16} = 1$$
$$\frac{(x-1)^2}{9} = \frac{9}{16}$$
$$16(x-1)^2 = 81$$
$$(x-1)^2 = \frac{81}{16}$$
$$x - 1 = \pm\sqrt{\frac{81}{16}}$$
$$x - 1 = \pm\frac{9}{4}$$
$$x = \frac{13}{4} \text{ and } -\frac{5}{4}$$

The $x$-coordinates of the endpoints of the latus rectum are $\frac{13}{4}$ and $-\frac{5}{4}$.

$$\left|\frac{13}{4} - \left(-\frac{5}{4}\right)\right| = \frac{9}{2}$$

The length of the latus rectum is $\frac{9}{2}$.

**76.**
$$9x^2 + 16y - 36x + 96y + 36 = 0$$
$$9x^2 - 36x + 16y^2 + 96y + 36 = 0$$
$$9\left(x^2 - 4x\right) + 16\left(y^2 + 6y\right) = -36$$
$$9\left(x^2 - 4x + 4\right) + 16\left(y^2 + 6y + 9\right) = -36 + 36 + 144$$
$$9(x-2)^2 + 16(y+3)^2 = 144$$
$$\frac{(x-2)^2}{16} + \frac{(y+3)^2}{9} = 1$$

Center $(2,\ -3)$

$c^2 = a^2 - b^2$

$c^2 = 16 - 9$

$c^2 = 7$

$c = \sqrt{7}$

The latus rectum is on the graph of $x = 2 + \sqrt{7}$ or $x = 2 - \sqrt{7}$.

$$\frac{(x-2)^2}{16} + \frac{(y+3)^2}{9} = 1$$
$$\frac{\left(2+\sqrt{7}-2\right)^2}{16} + \frac{(y+3)^2}{9} = 1 \text{ or } \frac{\left(2-\sqrt{7}-2\right)^2}{16} + \frac{(y+3)^2}{9} = 1$$
$$\frac{7}{16} + \frac{(y+3)^2}{9} = 1$$
$$\frac{(y+3)^2}{9} = \frac{9}{16}$$
$$16(y+3)^2 = 81$$
$$(y+3)^2\ \frac{81}{16}$$
$$y + 3 = \pm\sqrt{\frac{81}{16}}$$
$$y + 3 = \pm\frac{9}{4}$$
$$y = -\frac{3}{4} \text{ and } -\frac{21}{4}$$

The $y$-coordinates of the endpoints of the latus rectum are $-\frac{3}{4}$ and $-\frac{21}{4}$ .

$$\left|-\frac{3}{4} - \left(-\frac{21}{4}\right)\right| = \frac{9}{2}$$

The length of the latus rectum is $\frac{9}{2}$.

**77.** Let us transform the general equation of an ellipse into an $x'y'$ - coordinate system where the center is at the origin by replacing $(x-h)$ by $x'$ and $(y-k)$ by $y'$.

We have $\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1.$

Letting $x'=c$ and solving for $y'$ yields

$$\frac{(c)^2}{a^2}+\frac{y'^2}{b^2}=1$$
$$b^2c^2+a^2y'^2=a^2b^2$$
$$a^2y'^2=a^2b^2-b^2c^2$$
$$a^2y'^2=b^2(a^2-c^2)$$

But since $c^2=a^2-b^2$, $b^2=a^2-c^2$, we can substitute to obtain

$$a^2y'^2=b^2(b^2)$$
$$y'^2=\frac{b^4}{a^2}$$
$$y'=\pm\sqrt{\frac{b^4}{a^2}}=\pm\frac{b^2}{a}$$

The endpoints of the latus rectum, then, are $\left(c,\ \frac{b^2}{a}\right)$ and $\left(c,\ -\frac{b^2}{a}\right)$.

The distance between these points is $\frac{2b^2}{a}$.

**78.** Let $P(x,y)$ be a point on the ellipse and let $F_1(0,c)$ and $F_2(0,-c)$ be the foci. By the definition of an ellipse,

$$d(P,\ F_1)+d(P,\ F_2)=2a$$
$$\sqrt{(x-0)^2+(y-c)^2}+\sqrt{(x-0)^2+(y+c)^2}=2a$$
$$\sqrt{x^2+(y+c)^2}=2a-\sqrt{x^2+(y-c)^2}$$
$$x^2+(y+c)^2=4a^2-4a\sqrt{x^2+(y-c)^2}+x^2+(y-c)^2 \qquad \text{Square each side.}$$
$$x^2+y^2+2cy+c^2=4a^2-4a\sqrt{x^2+(y-c)}+x^2+y^2-2cy+c^2$$
$$4cy=4a^2-4a\sqrt{x^2+(y-c)^2}$$
$$cy=a^2-a\sqrt{x^2+(y-c)^2}$$
$$cy-a^2=-a\sqrt{x^2+(y-c)^2}$$
$$c^2y^2-2cya^2+a^4=a^2(x^2+(y-c)^2) \qquad \text{Square each side.}$$
$$c^2y^2-2cya^2+a^4=a^2x^2+a^2y^2-2cya^2+a^2c^2$$
$$a^4-a^2c^2=a^2x^2+a^2y^2-c^2y^2$$
$$a^2(a^2-c^2)=a^2x^2+(a^2-c^2)y^2$$
$$a^2b^2=a^2x^2+b^2y^2 \qquad \text{Let } b^2=a^2-c^2.$$
$$1=\frac{x^2}{b^2}+\frac{y^2}{a^2}$$

## Prepare for Section 8.3

**PS1.** $\dfrac{4+-2}{2}=1$

$\dfrac{-3+1}{2}=-1$

Midpoint: (1, –1)

$\sqrt{(-2-4)^2+(1--3)^2}=\sqrt{52}=2\sqrt{13}$

Length: $2\sqrt{13}$

**PS2.** $(x-1)(x+3)=5$

$x^2+2x-3=5$

$x^2+2x-8=0$

$(x+4)(x-2)=0$

$x+4=0 \quad x-2=0$

$x=-4 \quad x=2$

**PS3.** $\dfrac{4}{\sqrt{8}}=\dfrac{4\sqrt{8}}{8}=\dfrac{8\sqrt{2}}{8}=\sqrt{2}$

**PS4.** $4x^2+24x=4(x^2+6x)$

$=4(x^2+6x+9)$

$=4(x+3)^2$

**PS5.** $\dfrac{x^2}{4}-\dfrac{y^2}{9}=1$

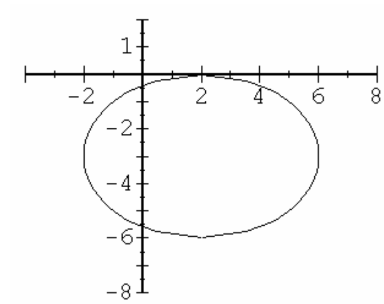
$-\dfrac{y^2}{9}=1-\dfrac{x^2}{4}$

$y^2=\dfrac{9x^2}{4}-9$

$y=\pm\sqrt{\dfrac{9x^2}{4}-9}$

$y=\pm\dfrac{3}{2}\sqrt{x^2-4}$

**PS6.**




## Section 8.3

**1.** **a.** iii **b.** ii **c.** i **d.** iv

**2.** **a.** iii **b.** i **c.** iv **d.** ii

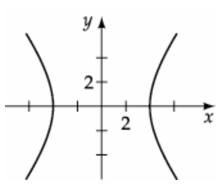
**3.** $\dfrac{x^2}{16}-\dfrac{y^2}{25}=1$

Center $(0, 0)$

Vertices $(\pm 4, 0)$

Foci $(\pm\sqrt{41},\ 0)$

Asymptotes $y=\pm\dfrac{5}{4}x$



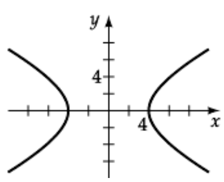

**4.** $\dfrac{x^2}{16}-\dfrac{y^2}{9}=1$

Center $(0, 0)$

Vertices $(\pm 4, 0)$

Foci $(\pm 5,\ 0)$

Asymptotes $y=\pm\dfrac{3}{4}x$



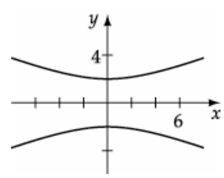

**5.** $\dfrac{y^2}{4}-\dfrac{x^2}{25}=1$

Center $(0, 0)$

Vertices $(0,\ \pm 2)$

Foci $(0,\ \pm\sqrt{29})$

Asymptotes $y=\pm\dfrac{2}{5}x$



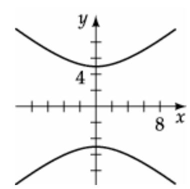

**6.** $\dfrac{y^2}{25}-\dfrac{x^2}{36}=1$

Center $(0, 0)$

Vertices $(0,\ \pm 5)$

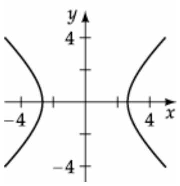
Foci $(0,\ \pm\sqrt{61})$

Asymptotes $y=\pm\dfrac{5}{6}x$



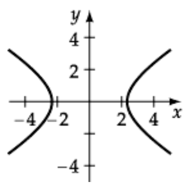
**7.** $\dfrac{x^2}{7}-\dfrac{y^2}{9}=1$

Center $(0, 0)$
Vertices $\left(\pm\sqrt{7}, 0\right)$
Foci $(\pm 4,\ 0)$
Asymptotes $y=\pm\frac{3\sqrt{7}}{7}x$



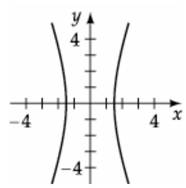

**8.** $\dfrac{x^2}{5}-\dfrac{y^2}{4}=1$

Center $(0, 0)$
Vertices $\left(\pm\sqrt{5}, 0\right)$
Foci $(\pm 3,\ 0)$
Asymptotes $y=\pm\frac{2\sqrt{5}}{5}x$



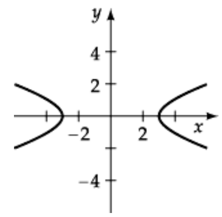

**9.** $\dfrac{4x^2}{9}-\dfrac{y^2}{16}=1$

Center $(0, 0)$
Vertices $\left(\pm\dfrac{3}{2}, 0\right)$
Foci $\left(\pm\dfrac{\sqrt{73}}{2},\ 0\right)$
Asymptotes $y=\pm\dfrac{8}{3}x$



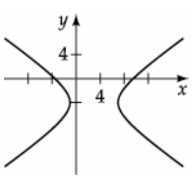

**10.** $\dfrac{x^2}{9}-\dfrac{9y^2}{16}=1$

Center $(0, 0)$
Vertices $(\pm 3, 0)$
Foci $\left(\pm\dfrac{\sqrt{97}}{3}, 0\right)$
Asymptotes $y=\pm\dfrac{4}{9}x$



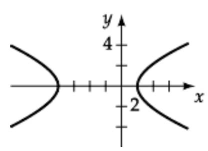

**11.** $\dfrac{(x-3)^2}{16}-\dfrac{(y+4)^2}{9}=1$

Center $(3,\ -4)$
Vertices $(3\pm 4,\ -4)=(7,\ -4),\ (-1,\ -4)$
Foci $(3\pm 5,\ -4)=(8,\ -4),\ (-2,\ -4)$
Asymptotes $y+4=\pm\dfrac{3}{4}(x-3)$



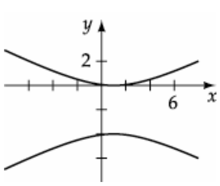

**12.** $\dfrac{(x+3)^2}{25}-\dfrac{y^2}{4}=1$

Center $(-3, 0)$
Vertices $(-3\pm 5, 0)=(2,\ 0),\ (-8,\ 0)$
Foci $\left(-3\pm\sqrt{29},\ 0\right)=\left(-3+\sqrt{29},\ 0\right),\ \left(-3-\sqrt{29},\ 0\right)$
Asymptotes $y=\pm\dfrac{2}{5}(x+3)$



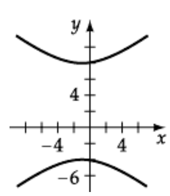

**13.** $\dfrac{(y+2)^2}{4}-\dfrac{(x-1)^2}{16}=1$

Center $(1,\ -2)$
Vertices $(1,\ -2\pm 2)=(1,\ 0),\ (1,\ -4)$
Foci $\left(1,\ -2\pm 2\sqrt{5}\right)=\left(1,\ -2+2\sqrt{5}\right),\ \left(1,\ -2-2\sqrt{5}\right)$
Asymptotes $y+2=\pm\dfrac{1}{2}(x-1)$




**14.** $\dfrac{(y-2)^2}{36}-\dfrac{(x+1)^2}{49}=1$

Center $(-1, 2)$
Vertices $(-1,\ 2\pm 6)=(-1,\ 8),\ (-1,\ -4)$
Foci $\left(-1,\ 2\pm\sqrt{85},\ 0\right)=\left(-1,\ 2+\sqrt{85}\right),\ \left(-1,\ 2-\sqrt{85}\right)$
Asymptotes $(y-2)=\pm\dfrac{6}{7}(x+1)$



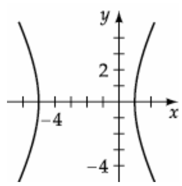
**15.** $\dfrac{(x+2)^2}{9}-\dfrac{y^2}{25}=1$

Center $(-2, 0)$

Vertices $(-2\pm 3, 0)=(1, 0), (-5, 0)$

Foci $\left(-2\pm\sqrt{34}, 0\right)$

Asymptotes $y=\pm\dfrac{5}{3}(x+2)$



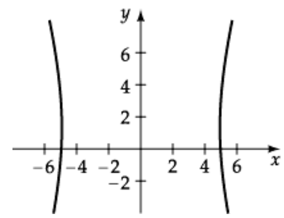

**16.** $\dfrac{x^2}{25}-\dfrac{(y-2)^2}{81}=1$

Center $(0, 2)$

Vertices $(0\pm 5, 2)=(5, 2), (-5, 2)$

Foci $\left(\pm\sqrt{106}, 2\right)=\left(\sqrt{106}, 2\right), \left(-\sqrt{106}, 2\right)$

Asymptotes $(y-2)=\pm\dfrac{9}{5}x$




**17.** $\dfrac{9(x-1)^2}{16}-\dfrac{(y+1)^2}{9}=1$

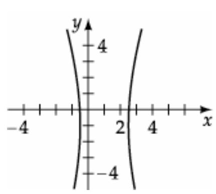
$\dfrac{(x-1)^2}{16/9}-\dfrac{(y+1)^2}{9}$

Center $(1, -1)$

Vertices $\left(1\pm\dfrac{4}{3}, -1\right)=\left(\dfrac{7}{3}, -1\right), \left(-\dfrac{1}{3}, -1\right)$

Foci $\left(1\pm\dfrac{\sqrt{97}}{3}, -1\right)$

Asymptotes $(y+1)=\pm\dfrac{9}{4}(x-1)$




**18.** $\dfrac{(x+6)^2}{25}-\dfrac{25y^2}{144}=1$

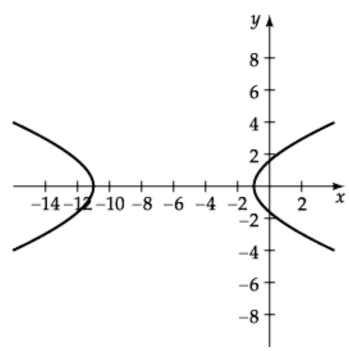
$\dfrac{(x+6)^2}{25}-\dfrac{y^2}{144/25}=1$

Center $(-6, 0)$

Vertices $(-6\pm 5, 0)=(-1, 0), (-11, 0)$

Foci $\left(-6\pm\dfrac{\sqrt{769}}{5}, 0\right)$

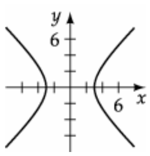
Asymptotes $y=\pm\dfrac{12}{25}(x+6)$



**19.** $x^2 - y^2 = 9$

$\frac{x^2}{9} - \frac{y^2}{9} = 1$

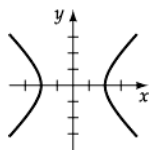
Center $(0, 0)$
Vertices $(\pm 3, 0)$
Foci $(\pm 3\sqrt{2}, 0)$
Asymptotes $y = \pm x$




**20.** $4x^2 - y^2 = 16$

$\frac{x^2}{4} - \frac{y^2}{16} = 1$

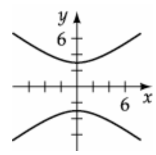
Center $(0, 0)$
Vertices $(\pm 2, 0)$
Foci $(\pm 2\sqrt{5}, 0)$
Asymptotes $y = \pm 2x$




**21.** $16y^2 - 9x^2 = 144$

$\frac{y^2}{9} - \frac{x^2}{16} = 1$

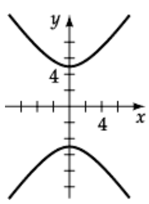
Center $(0, 0)$
Vertices $(0, \pm 3)$
Foci $(0 \pm 5)$
Asymptotes $y = \pm \frac{3}{4}x$




**22.** $9y^2 - 25x^2 = 225$

$\frac{y^2}{25} - \frac{x^2}{9} = 1$

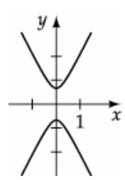
Center $(0, 0)$
Vertices $(0, \pm 5)$
Foci $(0, \pm\sqrt{34})$
Asymptotes $y = \pm \frac{5}{3}x$




**23.** $9y^2 - 36x^2 = 4$

$\frac{y^2}{4/9} - \frac{x^2}{1/9} = 1$

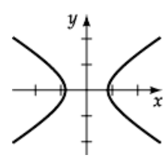
Center $(0, 0)$
Vertices $\left(0, \pm\frac{2}{3}\right)$
Foci $\left(0, \pm\frac{\sqrt{5}}{3}\right)$
Asymptotes $y = \pm 2x$




**24.** $16x^2 - 25y^2 = 9$

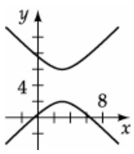
$\frac{x^2}{9/16} - \frac{y^2}{9/25} = 1$

Center $(0, 0)$
Vertices $\left(\pm\frac{3}{4}, 0\right)$
Foci $\left(\pm\frac{3\sqrt{41}}{20}, 0\right)$
Asymptotes $y = \pm \frac{4}{5}x$



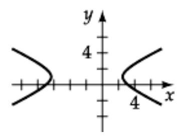
**25.**
$$x^2-y^2-6x+8y=3$$
$$(x^2-6x)-(y^2-8y)=3$$
$$(x^2-6x+9)-(y^2-8y+16)=3+9-16$$
$$(x-3)^2-(y-4)^2=-4$$
$$\frac{(y-4)^2}{4}-\frac{(x-3)^2}{4}=1$$

Center $(3,4)$
Vertices $(3,4\pm 2)=(3,6),(3,2)$
Foci $\left(3,4\pm 2\sqrt{2}\right)=\left(3,4+2\sqrt{2}\right),\left(3,4-2\sqrt{2}\right)$
Asymptotes $y-4=\pm(x-3)$



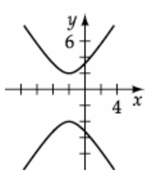

**26.**
$$4x^2-25y^2+16x+50y-109=0$$
$$4(x^2+4x)-25(y^2-2y)=109$$
$$4(x^2+4x+4)-25(y^2-2y+1)=109+16-25$$
$$4(x+2)^2-25(y-1)^2=100$$
$$\frac{(x+2)^2}{25}-\frac{(y-1)^2}{4}=1$$

Center $(-2,1)$
Vertices $(-2\pm 5,1)=(-7,1),(3,1)$
Foci $\left(-2\pm\sqrt{29},1\right)=\left(-2+\sqrt{29},1\right),\left(-2-\sqrt{29},1\right)$
Asymptotes $y-1=\pm\frac{2}{5}(x+2)$



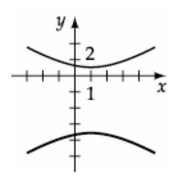

**27.**
$$9x^2-4y^2+36x-8y+68=0$$
$$9x^2+36x-4y^2-8y=-68$$
$$9(x^2+4x)-4(y^2+2y)=-68$$
$$9(x^2+4x+4)-4(y^2+2y+1)=-68+36-4$$
$$9(x+2)^2-4(y+1)^2=-36$$
$$\frac{(y+1)^2}{9}-\frac{(x+2)^2}{4}=1$$

Center $(-2,-1)$
Vertices $(-2,-1\pm 3)=(-2,2),(-2,-4)$
Foci $\left(-2,-1\pm\sqrt{13}\right)=\left(-2,-1+\sqrt{13}\right),\left(-2,-1-\sqrt{13}\right)$
Asymptotes $y+1=\pm\frac{3}{2}(x+2)$




**28.**
$$16x^2-9y^2-32x-54y+79=0$$
$$16(x^2-2x)-9(y^2+6y)=-79$$
$$16(x^2-2x+1)-9(y^2+6y+9)=-79+16-81$$
$$16(x-1)^2-9(y+3)^2=-144$$
$$\frac{(y+3)^2}{16}-\frac{(x-1)^2}{9}=1$$

Center $(1,-3)$
Vertices $(1,-3\pm 4)=(1,1),(1,-7)$
Foci $(1,-3\pm 5)=(1,2),(1,-8)$
Asymptotes $(y+3)=\pm\frac{4}{3}(x-1)$



**29.** $y = \dfrac{-6 \pm \sqrt{6^2 - 4(-1)\left(4x^2 + 32x + 39\right)}}{2(-1)}$

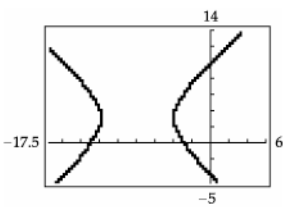
$= \dfrac{-6 \pm \sqrt{36 + 4\left(4x^2 + 32x + 39\right)}}{-2}$

$= \dfrac{-6 \pm \sqrt{16x^2 + 128x + 192}}{-2}$

$= \dfrac{-6 \pm \sqrt{16(x^2 + 8x + 12)}}{-2}$

$= \dfrac{-6 \pm 4\sqrt{x^2 + 8x + 12}}{-2}$

$= 3 \pm 2\sqrt{x^2 + 8x + 12}$



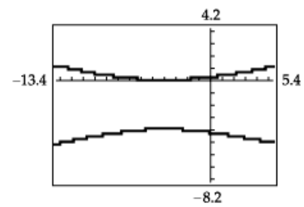

**30.** $y = \dfrac{64 \pm \sqrt{(-64)^2 - 4(-16)\left(x^2 + 8x + 16\right)}}{2(-16)}$

$= \dfrac{64 \pm \sqrt{4096 + 64\left(x^2 + 8x + 16\right)}}{-32}$

$= \dfrac{64 \pm \sqrt{64\left(x^2 + 8x + 16 + 64\right)}}{-32}$

$= \dfrac{64 \pm 8\sqrt{\left(x^2 + 8x + 80\right)}}{-32}$

$= \dfrac{-8 \pm \sqrt{\left(x^2 + 8x + 80\right)}}{4}$




**31.** $y = \dfrac{64 \pm \sqrt{(-64)^2 - 4(-16)\left(9x^2 - 36x + 116\right)}}{2(-16)}$

$= \dfrac{64 \pm \sqrt{4096 + 64(9x^2 - 36x + 116)}}{-32}$

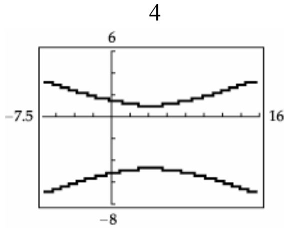
$= \dfrac{64 \pm \sqrt{64(9x^2 - 36x + 116 + 64)}}{-32}$

$= \dfrac{64 \pm 8\sqrt{(9x^2 - 36x + 180)}}{-32}$

$= \dfrac{64 \pm 8\sqrt{9(x^2 - 4x + 20)}}{-32}$

$= \dfrac{64 \pm 24\sqrt{x^2 - 4x + 20}}{-32}$

$= \dfrac{-8 \pm 3\sqrt{x^2 - 4x + 20}}{4}$

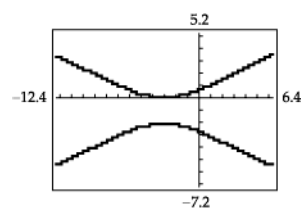



**32.** $y = \dfrac{18 \pm \sqrt{(-18)^2 - 4(-9)(2x^2 + 12x + 18)}}{2(-9)}$

$= \dfrac{18 \pm \sqrt{324 + 36(2x^2 + 12x + 18)}}{-18}$

$= \dfrac{18 \pm \sqrt{36(2x^2 + 12x + 18 + 9)}}{-18}$

$= \dfrac{-3 \pm \sqrt{2x^2 + 12x + 27}}{3}$



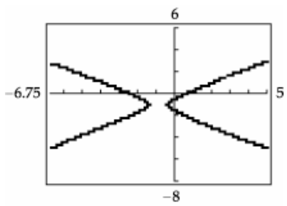
**33.** $y = \dfrac{18 \pm \sqrt{(-18)^2 - 4(-9)\left(4x^2 + 8x - 6\right)}}{2(-9)}$

$= \dfrac{18 \pm \sqrt{324 + 36\left(4x^2 + 8x - 6\right)}}{-18}$

$= \dfrac{18 \pm \sqrt{36\left(4x^2 + 8x - 6 + 9\right)}}{-18}$

$= \dfrac{18 \pm 6\sqrt{\left(4x^2 + 8x + 3\right)}}{-18}$

$= \dfrac{-3 \pm \sqrt{\left(4x^2 + 8x + 3\right)}}{3}$



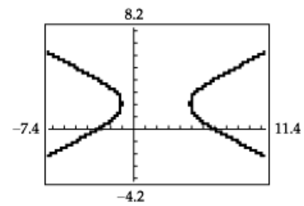

**34.** $y = \dfrac{-36 \pm \sqrt{36^2 - 4(-9)\left(2x^2 - 8x - 46\right)}}{2(-9)}$

$= \dfrac{-36 \pm \sqrt{1296 + 36\left(2x^2 - 8x - 46\right)}}{-18}$

$= \dfrac{-36 \pm \sqrt{36\left(2x^2 - 8x - 46 + 36\right)}}{-18}$

$= \dfrac{-36 \pm 6\sqrt{\left(2x^2 - 8x - 10\right)}}{-18}$

$= \dfrac{6 \pm \sqrt{2x^2 - 8x - 10}}{3}$




**35.** vertices $(3,\ 0)$ and $(-3,\ 0)$, foci $(4,\ 0)$ and $(-4,\ 0)$

Traverse axis is on *x*-axis. For a standard hyperbola, the vertices are at $(h + a, k)$ and $(h - a, k)$, $h + a = 3$, $h - a = -3$, and $k = 0$.**.**

If $h + a = 3$ and $h - a = -3$, then $h = 0$ and $a = 3$.

The foci are located at (4, 0) and (–4, 0). Thus, $h = 0$ and $c = 4$.

Since $c^2 = a^2 + b^2$, $b^2 = c^2 - a^2$

$b^2 = (4)^2 - (3)^2 = 16 - 9 = 7$

$\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$

$\dfrac{(x-0)^2}{(3)^2} - \dfrac{(y-0)^2}{7} = 1$

$\dfrac{x^2}{9} - \dfrac{y^2}{7} = 1$

**36.** vertices $(0,\ 2)$ and $(0,\ -2)$, foci $(0,\ 3)$ and $(0,\ -3)$

Transverse axis is on *y*-axis. Since vertices are at $(h,\ k + a)$ and $(h,\ k - a)$, $k + a = 2$, $k - a = -2$, and $h = 0$.

Therefore, $k = 0$ and $a = 2$.

The foci are located at $(h,\ k + c)$ and $(h,\ k - c)$, or specifically at $(0,\ 3)$ and $(0,\ -3)$.

Since $k = 0$, $c = 3$.

$b^2 = c^2 - a^2 = (3)^2 - (2)^2 = 9 - 4 = 5$

$\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$

$\dfrac{y^2}{4} - \dfrac{x^2}{5} = 1$

**37.** foci $(0, 5)$ and $(0, -5)$, asymptotes $y = 2x$ and $y = -2x$

Transverse axis is on $y$-axis. Since foci are at $(h, k+c)$ and $(h, k-c)$, $k+c=5$, $k-c=-5$, and $h=0$.

Therefore, $k=0$ and $c=5$.

Since one of the asymptotes is $y=\frac{a}{b}x$, $\frac{a}{b}=2$ and $a=2b$.

$a^2+b^2=c^2$; then substituting $a=2b$ and $c=5$ yields $(2b)^2+b^2=(5)^2$, or $5b^2=25$.

Therefore, $b^2=5$ and $b=\sqrt{5}$.
Since $a=2b$, $a=2\left(\sqrt{5}\right)=2\sqrt{5}$.

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$
$$\frac{y^2}{\left(2\sqrt{5}\right)^2}-\frac{x^2}{5}=1$$
$$\frac{y^2}{20}-\frac{x^2}{5}=1$$

**38.** foci $(4, 0)$ and $(-4, 0)$, asymptotes $y=x$ and $y=-x$

Transverse axis is on $x$-axis. Since foci are at $(h+c, k)$ and $(h-c, \text{k})$, $h+c=4$, $h-c=-4$, and $k=0$.

Therefore, $h=0$ and $c=4$. Since the asymptotes are $y=\frac{b}{a}x$, and $y=-\frac{b}{a}x$, $\frac{b}{a}=1$ and $b=a$.

$a^2+b^2=c^2$; then substituting $b=a$ and $c=4$ yields $a^2+a^2=4^2$, or $2a^2=16$.

Therefore, $a^2=8$ and $a=2\sqrt{2}$.
Since $b=a$, $b=2\sqrt{2}$.

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$
$$\frac{x^2}{8}-\frac{y^2}{\left(2\sqrt{2}\right)^2}=1$$
$$\frac{x^2}{8}-\frac{y^2}{8}=1$$

**39.** vertices (0, 3) and $(0,-3)$, point (2, 4)

The distance between the two vertices is the length of the transverse axis, which is $2a$.

$2a=\left|3-(-3)\right|=6$ or $a=3$.

Since the midpoint of the transverse axis is the center of the hyperbola, the center is given by
$\left(\frac{0+0}{2}, \frac{3+(-3)}{2}\right)$, or (0, 0)

Since both vertices lie on the $y$-axis, the transverse axis must be on the $y$-axis.

Taking the standard form of the hyperbola, we have
$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$
Substituting the point (2, 4) for $x$ and $y$, and 3 for $a$, we have
$$\frac{16}{9}-\frac{4}{b^2}=1$$
Solving for $b^2$ yields $b^2=\frac{36}{7}$.

Therefore, the equation is
$$\frac{y^2}{9}-\frac{x^2}{36/7}=1$$

**40.** vertices (5, 0) and $(-5, 0)$, point $(-1, 3)$

The length of the transverse axis, $5-(-5)$, or 10, is equal to 2*a*. Therefore, *a* = 5.

The midpoint of the transverse axis, or the center of the hyperbola, is given by
$\left(\frac{5+(-5)}{2}, \frac{0+0}{2}\right)$, or (0, 0)

Since both vertices are on the *x*-axis, the transverse axis must lie on the *x*-axis. Therefore, we have

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Substituting the point $(-1, 3)$ for *x* and *y*, and 5 for *a*, we have

$$\frac{1}{25} - \frac{9}{b^2} = 1$$

Solving for $b^2$ yields $b^2 = -\frac{225}{24}$.

However, $b^2$ must be positive. Therefore, no such hyperbola exists.

**41.** vertices $(0, 4)$ and $(0, -4)$, asymptotes $y = \frac{1}{2}x$ and $y = -\frac{1}{2}x$.

The length of the transverse axis, or the distance between the vertices, is equal to 2*a* .

$2a = 4-(-4) = 8$, or $a = 4$

The center of the hyperbola, or the midpoint of the line segment joining the vertices, is
$\left(\frac{0+0}{2}, \frac{4+(-4)}{2}\right)$, or (0, 0)

Since both vertices lie on the *y*-axis, the transverse axis must lie on the *y*-axis. Therefore, the asymptotes are given by $y = \frac{a}{b}x$ and $y = -\frac{a}{b}x$. One asymptote is $y = \frac{1}{2}x$. Thus $\frac{a}{b} = \frac{1}{2}$ or $b = 2a$.

Since *b* = 2*a* and *a* = 4, *b* = 2(4) = 8.

Thus, the equation is

$$\frac{y^2}{4^2} - \frac{x^2}{8^2} = 1 \text{ or } \frac{y^2}{16} - \frac{x^2}{64} = 1$$

**42.** vertices (6, 0) and $(-6, 0)$, asymptotes $y = \frac{2}{3}x$ and $y = -\frac{2}{3}x$

Length of transverse axis = distance between vertices

$$2a = 6-(-6)$$
$$2a = 12$$
$$a = 6$$

The center of the hyperbola is given by the midpoint of the line segment joining the vertices, or
$\left(\frac{6+(-6)}{2}, \frac{0+0}{2}\right)$, which is (0, 0).

Since both vertices lie on the *x*-axis, the transverse axis must lie on the *x*-axis.

Thus, the asymptotes must be given by $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$.

Since $y = \frac{2}{3}x$ and $y = -\frac{2}{3}x$ define the actual asymptotes, $\frac{b}{a} = \frac{2}{3}$ and $b = \frac{2a}{3}$.

Since $a = 6, b = \frac{2(6)}{3} = 4$.

If *a* = 6 and *b* = 4.

$$\frac{x^2}{6^2} - \frac{y^2}{4^2} = 1$$

$$\frac{x^2}{36} - \frac{y^2}{16} = 1$$

**43.** vertices (6, 3) and (2, 3), foci (7, 3) and (1, 3)

Length of transverse axis = distance between vertices

$$2a = |6-2|$$
$$a = 2$$

The center of the hyperbola (*h*, *k*) is the midpoint of the line segment joining the vertices, or the point $\left(\frac{6+2}{2}, \frac{3+3}{2}\right)$.

Thus, $h = \frac{6+2}{2}$, or 4, and $k = \frac{3+3}{3}$, or 3 .

Since both vertices lie on the horizontal line *y* = 3, the transverse axis is parallel to the *x*-axis. The location of the foci is given by (*h* + *c*, *k*) and (*h* – *c*, *k*), or specifically (7, 3) and (1, 3). Thus *h* + *c* = 7, *h* – *c* = 1, and *k* = 3. Solving for *h* and *c* simultaneously yields *h* = 4 and *c* = 3.

Since $c^2 = a^2 + b^2, b^2 = c^2 - a^2$.

Substituting, we have $b^2 = 3^2 - 2^2 = 9 - 4 = 5$.

Substituting $a = 2, b^2 = 5, h = 4,$ and $k = 3$ in the standard equation $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ yields $\frac{(x-4)^2}{4} - \frac{(y-3)^2}{5} = 1$ .

**44.** vertices $(-1, 5)$ and $(-1, -1)$, foci $(-1, 7)$ and $(-1, -3)$

Length of transverse axis = distance between vertices

$$2a = |5-(-1)| = 6$$
$$a = 3$$

Center (midpoint of transverse axis) is

$\left(\frac{-1-1}{2}, \frac{5-1}{2}\right)$, or (-1, 2)

Thus, $h = -1$ and $k = 2$.

The foci, $(h, k+c)$ and $(h, k-c)$, are $(-1, 7)$ and $(-1, -3)$.

Thus, since *k* + *c* = 7 and *k* = 2, *c* = 5.

$b^2 = c^2 - a^2 = 5^2 - 3^2 = 25 - 9 = 16$

Thus, we obtain $\frac{(y-2)^2}{9} - \frac{(x+1)^2}{16} = 1$

**45.** foci (1, –2) and (7, –2) , slope of an asymptote = $\frac{5}{4}$

Both foci lie on the horizontal line *y* = –2 ; therefore, the transverse axis is parallel to the *x*-axis.

The foci are given by (*h* + *c*, *k*) and (*h* –*c*, *k*) .

Thus, $h - c = 1, h + c = 7,$ and $k = -2$ . Solving simultaneously for *h* and *c* yields *h* = 4 and *c* = 3.

Since $y - k = \frac{b}{a}(x - h)$ is the equation for an asymptote, and the slope of an asymptote is given as $\frac{5}{4}, \frac{b}{a} = \frac{5}{4}, b = \frac{5a}{4}$, and $b^2 = \frac{25a^2}{16}$.

Because $a^2 + b^2 = c^2$, substituting $c = 3$ and $b^2 = \frac{25a^2}{16}$ yields $a^2 = \frac{144}{41}$ .

Therefore, $b^2 = \frac{3600}{656} = \frac{225}{41}$.

Substituting in the standard equation for a hyperbola yields $\frac{(x-4)^2}{144/41} - \frac{(y+2)^2}{225/41} = 1$

**46.** foci $(-3,-6)$ and $(-3,-2)$, slope of an asymptote =1

Both foci are on the vertical line $x=-3$; therefore, the transverse axis is parallel to the $y$-axis.

The foci are given by $(h, k+c)$ and $(h, k-c)$.

Thus, $k-c=-6, k+c=-2,$ and $h=-3$. Solving simultaneously for $k$ and $c$ yields $k=-4$ and $c=2$.

Since $y-k=\frac{a}{b}(x-h)$ is the equation for one asymptote, and the slope of an asymptote is given as 1, $\frac{a}{b}=1, b=a,$ and $b^2=a^2$.

Because $a^2+b^2=c^2$, substituting $c=2$ and $b^2=a^2$ yields $a^2=2$; therefore, $b^2=2$. Substituting in the standard equation of a hyperbola yields $\frac{(y+4)^2}{2}-\frac{(x+3)^2}{2}=1$

**47.** Because the transverse axis is parallel to the $y$-axis and the center is (7, 2), the equation of the hyperbola is

$$\frac{(y-2)^2}{a^2}-\frac{(x-7)^2}{b^2}=1$$

Because (9, 4) is a point on the hyperbola,

$$\frac{(4-2)^2}{a^2}-\frac{(9-7)^2}{b^2}=1$$

The slope of the asymptote is $\frac{1}{2}$. Therefore $\frac{1}{2}=\frac{a}{b}$ or $b=2a$.

Substituting, we have

$$\frac{4}{a^2}-\frac{4}{4a^2}=1$$

$$\frac{4}{a^2}-\frac{1}{a^2}=1, \text{ or } a^2=3$$

Since $b=2a$, $b^2=4a^2$, or $b^2=12$. The equation is $\frac{(y-2)^2}{3}-\frac{(x-7)^2}{12}=1$.

**48.** Because the transverse axis is parallel to the $x$-axis, and the center is (3, 3), the equation of the hyperbola is

$$\frac{(x-3)^2}{a^2}-\frac{(y-3)^2}{b^2}=1$$

Because (6, 1) is a point on the hyperbola,

$$\frac{(6-3)^2}{a^2}-\frac{(1-3)^2}{b^2}=1$$

The slope of an asymptote is 2. Therefore, $2=\frac{b}{a}$, or $2a=b$.

Substituting, we have

$$\frac{9}{a^2}-\frac{4}{4a^2}=\frac{9}{a^2}-\frac{1}{a^2}=\frac{8}{a^2}=1$$

Thus, $a^2=8$. Since $2a=b$, $4a^2=b^2$, or $b^2=32$.

The equation of the hyperbola is $\frac{(x-3)^2}{8}-\frac{(y-3)^2}{32}=1$.

**49.** vertices (1, 6) and (1, 8), eccentricity = 2

Length of transverse axis = distance between vertices

$$2a = |6-8| = 2$$

$$a = 1 \text{ and } a^2 = 1$$

Center (midpoint of transverse axis) is $\left(\frac{1+1}{2}, \frac{6+8}{2}\right)$, or $(1, 7)$.

Therefore, $h = 1$ and $k = 7$.

Since both vertices lie on the vertical line $x = 1$, the transverse axis is parallel to the $y$-axis.

Since $e = \frac{c}{a}$, $c = ae = (1)(2) = 2$.

Because $b^2 = c^2 - a^2$, $b^2 = (2)^2 - (1)^2 = 4 - 1 = 3$.

Substituting $h$, $k$, $a^2$, and $b^2$ into the standard equation yields $\frac{(y-7)^2}{1} - \frac{(x-1)^2}{3} = 1$

**50.** vertices (2, 3) and $(-2, 3)$, eccentricity $= \frac{5}{2}$

Length of transverse axis = distance between vertices

$$2a = |2-(-2)| = 4$$

$$a = 2 \text{ and } a^2 = 4$$

Center (midpoint of transverse axis) is $\left(\frac{2-2}{2}, \frac{3+3}{2}\right)$, or $(0, 3)$.

Thus, $h = 0$ and $k = 3$.

Since both vertices lie on the horizontal line $y = 3$, the transverse axis is parallel to the $x$-axis.

Since $e = \frac{c}{a}$, $c = ae = (2)\left(\frac{5}{2}\right) = 5$.

Because $b^2 = c^2 - a^2$, $b^2 = 5^2 - 2^2 = 25 - 4 = 21$.

Substituting $h$, $k$, $a^2$, and $b^2$ into the standard equation yields $\frac{x^2}{4} - \frac{(y-3)^2}{21} = 1$

**51.** foci (4, 0) and $(-4, 0)$, eccentricity $= 2$

Center (midpoint of line segment joining foci) is $\left(\frac{4+(-4)}{2}, \frac{0+0}{2}\right)$, or $(0, 0)$

Thus, $h = 0$ and $k = 0$.

Since both foci lie on the horizontal line $y = 0$, the transverse axis is parallel to the $x$-axis. The locations of the foci are given by $(h + c, k)$ and $(h - c, k)$, or specifically $(4, 0)$ and $(-4, 0)$

Since $h = 0$, $c = 4$.

Because $e = \frac{c}{a}$, $a = \frac{c}{e} = \frac{4}{2} = 2$ and $a^2 = 4$.

Because $b^2 = c^2 - a^2$, $b^2 = 4^2 - 2^2 = 16 - 4 = 12$.

Substituting $h$, $k$, $a^2$ and $b^2$ into the standard formula for a hyperbola yields $\frac{x^2}{4} - \frac{y^2}{12} = 1$

**52.** foci (0, 6) and $(0,-6)$, eccentricity $=\frac{4}{3}$

Center (midpoint of the line segment joining foci) is

$\left(\dfrac{0+0}{2}, \dfrac{6-6}{2}\right)$, or (0, 0)

Thus, $h = 0$ and $k = 0$.

Since both foci lie on the vertical line $x = 0$, the transverse axis is parallel to the $y$-axis. The location of the foci are given by $(h, k + c)$ and $(h, k-c)$, or specifically (0, 6) and $(0,-6)$. Since $k = 0$, $c = 6$.

Because $e=\dfrac{c}{a}$, $a=\dfrac{c}{e}=\dfrac{6}{4/3}=\dfrac{9}{2}$ and $a^2=\dfrac{81}{4}$.

Because $b^2=c^2-a^2$, $b^2=6^2-\left(\dfrac{9}{2}\right)^2=36-\dfrac{81}{4}=\dfrac{63}{4}$.

Substituting $h$, $k$, $a^2$, and $b^2$ into the standard formula for a hyperbola yields

$$\frac{y^2}{81/4}-\frac{x^2}{63/4}=1$$

**53.** conjugate axis length = 4, center (4, 1), eccentricity = $\frac{4}{3}$

$2b$ = conjugate axis length = 4

$b = 2$ and $b^2 = 4$

Since

$e=\dfrac{c}{a}=\dfrac{4}{3}$, $c=\dfrac{4a}{3}$ and $c^2=\dfrac{16a^2}{9}$. Since $a^2+b^2=c^2$, substituting $b^2=4$ and $c^2=\dfrac{16a^2}{9}$ and solving for $a^2$ yields $a^2=\frac{36}{7}$.

Substituting into the two standard equations of a hyperbola yields $\dfrac{(x-4)^2}{36/7}-\dfrac{(y-1)^2}{4}=1$ and $\dfrac{(y-1)^2}{36/7}-\dfrac{(x-4)^2}{4}=1$

**54.** conjugate axis length = 6, center $(-3,-3)$, eccentricity $=2$

$2b$ = conjugate axis length = 6

$b = 3$ and $b^2 = 9$

Since $e=\dfrac{c}{a}=2$, $c=2a$ and $c^2=4a^2$. Since $a^2+b^2=c^2$, substituting $b^2=9$ and $c^2=4a^2$ and solving for $a^2$ yields $a^2=3$.

Substituting into the two standard equations for a hyperbola yields $\dfrac{(x+3)^2}{3}-\dfrac{(y+3)^2}{9}=1$ and $\dfrac{(y+3)^2}{3}-\dfrac{(x+3)^2}{9}=1$

**55.** **a.** Because the transmitters are 250 miles apart,

$2c = 250$ and $c = 125$.

$2a$ = rate $\times$ time

$2a = 0.186 \times 500 = 93$

Thus, $a = 46.5$ miles.

$b=\sqrt{c^2-a^2}=\sqrt{125^2-46.5^2}=\sqrt{13{,}462.75}$ miles

The ship is located on the hyperbola given by

$$\frac{x^2}{2{,}162.25}-\frac{y^2}{13{,}462.75}=1$$

**b.** $x = 100$

$$\frac{10{,}000}{2{,}162.25}-\frac{y^2}{13{,}462.75}=1$$

$$\frac{-y^2}{13{,}462.75}\approx -3.6248121$$

$$y^2\approx 48{,}799.939$$

$$y\approx 221$$

The ship is 221 miles from the coastline.

**56.** **a.** Because the transmitters are 300 miles apart,

$2c = 300$ and $c = 150$.

$2a = \text{rate} \times \text{time}$

$2a = 0.186 \times 800 = 148.8$ miles

Thus, $a = 74.4$ miles.

$b = \sqrt{c^2 - a^2} = \sqrt{150^2 - 74.4^2} \approx 130.25$ miles

The ship is located on the hyperbola given by

$$\frac{x^2}{74.4^2} - \frac{y^2}{130.25^2} = 1$$

**b.** The ship will reach the coastline when $y = 0$. Thus,

$$\frac{x^2}{74.4^2} - \frac{0^2}{130.25^2} = 1$$
$$\frac{x^2}{74.4^2} = 1$$
$$x^2 = 74.4^2$$
$$x = 74.4 \text{ miles}$$

The ship reaches the coastline 74.4 miles to the left of the origin at the point (–74.4, 0).

**57.** When the wave hits Earth, $z = 0$.

$$y^2 = x^2 + (z - 10{,}000)^2$$
$$y^2 = x^2 + (0 - 10{,}000)^2$$
$$y^2 - x^2 = 10{,}000^2$$

It is a hyperbola.

**58.** **a.** At the top of the tower, $y = 380$.

$$\frac{x^2}{80^2} - \frac{(380-220)^2}{180^2} = 1$$
$$\frac{x^2}{80^2} = 1 + \frac{160^2}{180^2}$$
$$x^2 = 80^2\left(1 + \frac{160^2}{180^2}\right)$$
$$x = \sqrt{80^2\left(1 + \frac{160^2}{180^2}\right)}$$
$$x \approx 107 \text{ ft}$$

At the bottom of the tower, $y = 0$.

$$\frac{x^2}{80^2} - \frac{(0-220)^2}{180^2} = 1$$
$$\frac{x^2}{80^2} = 1 + \frac{220^2}{180^2}$$
$$x^2 = 80^2\left(1 + \frac{220^2}{180^2}\right)$$
$$x = \sqrt{80^2\left(1 + \frac{220^2}{180^2}\right)}$$
$$x \approx 126 \text{ ft}$$

**b.** From the equation, $a = 80$ ft.

**59.** **a.** Using the eccentricity, and $a = 2$,

$$\frac{c}{2} = \frac{\sqrt{17}}{4} \Rightarrow c = \frac{\sqrt{17}}{2}$$

Solve for $b$.

$$a^2 + b^2 = c^2$$
$$b^2 = c^2 - a^2$$
$$b^2 = \left(\frac{\sqrt{17}}{2}\right)^2 - 2^2$$
$$b^2 = \frac{17}{4} - \frac{16}{4}$$
$$b^2 = \frac{1}{4}$$
$$b = \frac{1}{2} = 0.5$$
$$\frac{x^2}{2^2} - \frac{y^2}{0.5^2} = 1$$

**b.** For $FG$, $y = 0.6$.

$$\frac{x^2}{2^2} - \frac{0.6^2}{0.5^2} = 1$$
$$\frac{x^2}{2^2} = 1 + \frac{0.6^2}{0.5^2}$$
$$x^2 = 2^2\left(1 + \frac{0.6^2}{0.5^2}\right)$$
$$x = \sqrt{2^2\left(1 + \frac{0.6^2}{0.5^2}\right)}$$
$$x \approx 3.1241$$
$$FG = 2x \approx 6.25 \text{ in.}$$

**60.** **a.** hyperbola

**b.** Foci: (–2, 0), (2, 0)
Center (0, 0)
$c = 2$

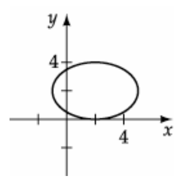
$$2a = |F_1P - F_2P|$$
$$|F_1P - F_2P| = 2$$
$$a = 1$$
$$c^2 = a^2 + b^2$$
$$2^2 = 1^2 + b^2$$
$$b^2 = 3$$
$$\frac{x^2}{1} - \frac{y^2}{3} = 1$$

**61.**
$$4x^2 + 9y^2 - 16x - 36y + 16 = 0$$
$$4(x^2 - 4x) + 9(y^2 - 4y) = -16$$
$$4(x^2 - 4x + 4) + 9(y^2 - 4y + 4) = -16 + 16 + 36$$
$$4(x-2)^2 + 9(y-2)^2 = 36$$
$$\frac{(x-2)^2}{9} + \frac{(y-2)^2}{4} = 1$$

ellipse

center (2, 2)

vertices $(2 \pm 3, 2) = (5, 2), (-1, 2)$
foci $(2 \pm \sqrt{5}, 2) = (2 + \sqrt{5}, 2), (2 - \sqrt{5}, 2)$



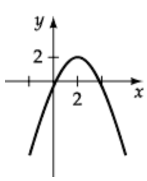

**62.**
$$2x^2 + 3y - 8x + 2 = 0$$
$$2(x^2 - 4x) = -3y - 2$$
$$2(x^2 - 4x + 4) = -3y - 2 + 8$$
$$2(x-2)^2 = -3y + 6$$
$$2(x-2)^2 = -3(y-2)$$
$$(x-2)^2 = -\frac{3}{2}(y-2)$$

parabola

vertex (2, 2)

focus $\left(2, 2 - \frac{3}{8}\right) = \left(2, \frac{13}{8}\right)$

directrix $y = 2 + \frac{3}{8}$, or $y = \frac{19}{8}$



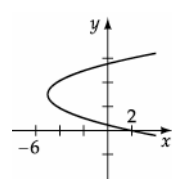

**63.**
$$5x - 4y^2 + 24y - 11 = 0$$
$$-4(y^2 - 6y) = -5x + 11$$
$$-4(y^2 - 6y + 9) = -5x + 11 - 36$$
$$-4(y-3)^2 = -5(x - 25)$$
$$-4(y-3)^2 = -5(x+5)$$
$$(y-3)^2 = \frac{5}{4}(x+5)$$

parabola

vertex $(-5, 3)$

focus $\left(-5 + \frac{5}{16}, 3\right) = \left(-\frac{75}{16}, 3\right)$

directrix $x = -5 - \frac{5}{16}$, or $x = \frac{-85}{16}$



**64.**
$$9x^2 - 25y^2 - 18x + 50y = 0$$
$$9(x^2 - 2x) - 25(y^2 - 2y) = 0$$
$$9(x^2 - 2x + 1) - 25(y^2 - 2y + 1) = 9 - 25$$
$$9(x-1)^2 - 25(y-1)^2 = -16$$
$$\frac{(y-1)^2}{16/25} - \frac{(x-1)^2}{16/9} = 1$$

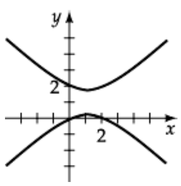
hyperbola

center (1, 1)

vertices $\left(1, 1 \pm \frac{4}{5}\right) = \left(1, \frac{9}{5}\right), \left(1, \frac{1}{5}\right)$

foci $\left(1, 1 \pm \frac{4\sqrt{34}}{15}\right) = \left(1, 1 + \frac{4\sqrt{34}}{15}\right), \left(1, 1 - \frac{4\sqrt{34}}{15}\right)$

asymptotes $y - 1 = \pm\frac{3}{5}(x-1)$



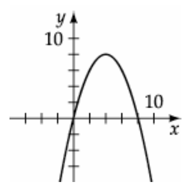

**65.**
$$x^2 + 2y - 8x = 0$$
$$x^2 - 8x = -2y$$
$$x^2 - 8x + 16 = -2y + 16$$
$$(x-4)^2 = -2(y-8)$$

parabola

vertex (4, 8)

foci $\left(4, 8 - \frac{1}{2}\right) = \left(4, \frac{15}{2}\right)$

directrix $y = 8 + \frac{1}{2}$, or $y = \frac{17}{2}$



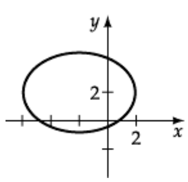

**66.**
$$9x^2 + 16y^2 + 36x - 64y - 44 = 0$$
$$9(x^2 + 4x) + 16(y^2 - 4y) = 44$$
$$9(x^2 + 4x + 4) + 16(y^2 - 4y + 4) = 44 + 36 + 64$$
$$9(x+2)^2 + 16(y-2)^2 = 144$$
$$\frac{(x+2)^2}{16} + \frac{(y-2)^2}{9} = 1$$

ellipse

center $(-2, 2)$

vertices $(-2 \pm 4, 2) = (2, 2), (-6, 2)$

foci $\left(-2 \pm \sqrt{7}, 2\right) = \left(-2 + \sqrt{7}, 2\right), \left(-2 - \sqrt{7}, 2\right)$



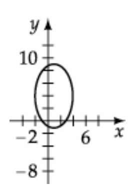

**67.**
$$25x^2 + 9y^2 - 50x - 72y - 56 = 0$$
$$25(x^2 - 2x) + 9(y^2 - 8y) = 56$$
$$25(x^2 - 2x + 1) + 9(y^2 - 8y + 16) = 56 + 25 + 144$$
$$25(x-1)^2 + 9(y-4)^2 = 225$$
$$\frac{(x-1)^2}{9} + \frac{(y-4)^2}{25} = 1$$

ellipse

center (1, 4)

vertices $(1, 4 \pm 5) = (1, 9), (1, -1)$

foci $(1, 4 \pm 4) = (1, 8), (1, 0)$



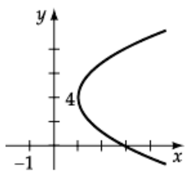
**68.** $$(x-3)^2+(y-4)^2=(x+1)^2$$
$$x^2-6x+9+y^2-8y+16=x^2+2x+1$$
$$-8x+8=-y^2+8y-16$$
$$-8(x-1)=-(y-4)^2$$
$$8(x-1)=(y-4)^2$$

parabola

vertex (1, 4)

focus $(1+2, 4)=(3, 4)$

directrix $x=1-2$, or $x=-1$




**Connecting Concepts**

**69.** foci $F_1(2, 0)$, $F_2(-2, 0)$ passing through $P_1(2, 3)$

$$d(P_1, F_2)-d(P_1, F_1)=\sqrt{(2+2)^2+3^2}-\sqrt{(2-2)^2+3^2}=5-3=2$$

Let $P(x, y)$ be any point on the hyperbola. Since the difference between $F_1P$ and $F_2P$ is the same as the difference between $F_1P_1$ and
$F_2P_1$, we have

$$\sqrt{(x-2)^2+y^2}-\sqrt{(x+2)^2+y^2}=2$$
$$\sqrt{(x-2)^2+y^2}=2+\sqrt{(x+2)^2+y^2}$$
$$x^2-4x+4+y^2=4+4\sqrt{(x+2)^2+y^2}+x^2+4x+4+y^2$$
$$-8x-4=4\sqrt{(x+2)^2+y^2}$$
$$-2x-1=\sqrt{(x+2)^2+y^2}$$
$$4x^2+4x+1=x^2+4x+4+y^2$$
$$3x^2-y^2=3$$
$$\frac{x^2}{1}-\frac{y^2}{3}=1$$

**70.** foci $(0,\ 3)$ and $(0,\ -3)$ , point $\left(\frac{5}{2},\ 3\right)$

Difference of distances from $(x, y)$ to foci = difference of distances from $\left(\frac{5}{2},\ 3\right)$ to foci

$$\sqrt{(x-0)^2+(y-3)^2}-\sqrt{(x-0)^2+(y+3)^2}=\sqrt{\left(\frac{5}{2}-0\right)^2+(3-3)^2}-\sqrt{\left(\frac{5}{2}-0\right)^2+(3+3)^2}$$

$$\sqrt{x^2+y^2-6y+9}-\sqrt{x^2+y^2+6y+9}=\frac{5}{2}-\frac{13}{2}=-\ 4$$

$$\sqrt{x^2+y^2-6y+9}=\sqrt{x^2+y^2+6y+9}=-\ 4$$

$$x^2+y^2-6y+9=x^2+y^2+6y+9-8\sqrt{x^2+y^2+6y+9}+16$$

$$-12y-16=-8\sqrt{x^2+y^2+6y+9}$$

$$3y+4=2\sqrt{x^2+y^2+6y+9}$$

$$9y^2+24y+16=4x^2+4y^2+24y+36$$

$$5y^2-4x^2=20$$

$$\frac{y^2}{4}-\frac{x^2}{5}=1$$

**71.** foci $(0,4)$ and $(0,-4)$, point $\left(\frac{7}{3},4\right)$

Difference in distances from $(x, y)$ to foci = difference of distances from $\left(\frac{7}{3},4\right)$ to foci

$$\sqrt{(x-0)^2+(y-4)^2}-\sqrt{(x-0)^2+(y+4)^2}=\sqrt{\left(\frac{7}{3}-0\right)^2+(4-4)^2}-\sqrt{\left(\frac{7}{3}-0\right)^2+(4+4)^2}$$

$$\sqrt{x^2+y^2-8y+16}-\sqrt{x^2+y^2+8y+16}=\frac{7}{3}-\frac{25}{3}=-\ 6$$

$$\sqrt{x^2+y^2-8y+16}=\sqrt{x^2+y^2+8y+16}-6$$

$$x^2+y^2-8y+16=x^2+y^2+8y+16-12\sqrt{x^2+y^2+8y+16}+36$$

$$-16y-36=-12\sqrt{x^2+y^2+8y+16}$$

$$4y+9=3\sqrt{x^2+y^2+8y+16}$$

$$16y^2+72y+81=9x^2+9y^2+72y+144$$

$$7y^2-9x^2=63$$

$$\frac{y^2}{9}-\frac{x^2}{7}=1$$

**72.** foci $(5,0)$ and $(-5,0)$, point $\left(5,\frac{9}{4}\right)$

Difference of distances from $(x, y)$ to foci = difference of distances from $\left(5,\frac{9}{4}\right)$ to foci

$$\sqrt{(x-5)^2+(y-0)^2}-\sqrt{(x+5)^2+(y-0)^2}=\sqrt{(5-5)^2+\left(\frac{9}{4}-0\right)^2}-\sqrt{(5+5)^2+\left(\frac{9}{4}-0\right)^2}$$

$$\sqrt{x^2-10x+25+y^2}-\sqrt{x^2+10x+25+y^2}=\frac{9}{4}-\frac{41}{4}=-8$$

$$\sqrt{x^2-10x+25+y^2}=\sqrt{x^2+10x+25+y^2}-8$$

$$x^2+10x+25+y^2=x^2+10x+25+y^2-16\sqrt{x^2+10x+25+y^2}+64$$

$$-20x-64=-16\sqrt{x^2+10x+25+y^2}$$

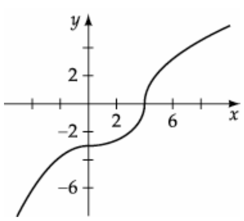
$$5x+16=4\sqrt{x^2+10x+25+y^2}$$

$$25x^2+160x+256=16x^2+160x+400+16y^2$$

$$9x^2-16y^2=144$$

$$\frac{x^2}{16}-\frac{y^2}{9}=1$$

**73.**




**74.** The hyperbola in **a.** has the larger eccentricity.

## Prepare for Section 8.4

**PS1.** $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

**PS2.** $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

**PS3.** $\cot 2\alpha=\frac{\sqrt{3}}{3}$

$\tan 2\alpha=\frac{3}{\sqrt{3}}$

$2\alpha=\tan^{-1}\left(\frac{3}{\sqrt{3}}\right)=\frac{\pi}{3}$

$\alpha=\frac{\pi}{6}$

**PS4.** $\sin\alpha=\frac{1}{2},\ \alpha=30^\circ$ or $150^\circ$

$\cos\alpha=-\frac{\sqrt{3}}{2}\ \ \alpha=150^\circ$ or $210^\circ$

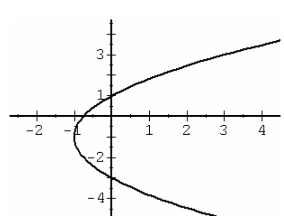
$\alpha=150^\circ$

**PS5.** $4x^2-6y^2+9x+16y-8=0$

$A=4, B=0, C=-6$

Since $B^2-4AC=0^2-4(4)(-6)=96>0$,
the graph is a hyperbola.

**PS6.**



## Section 8.4

**1.** $xy = 3$

$A = 0, B = 1, C = 0$

$\cot 2\alpha = \dfrac{A-C}{B}$

$\cot 2\alpha = \dfrac{0-0}{1}$

$\cot 2\alpha = 0$

$2\alpha = 90°$

$\alpha = 45°$

**2.** $5x^2 - 3xy - 5y^2 - 1 = 0$

$A = 5, B = -3, C = -5$

$\cot 2\alpha = \dfrac{A-C}{B}$

$\cot 2\alpha = \dfrac{5-(-5)}{-3}$

$\cot 2\alpha = -\dfrac{10}{3}$

$2\alpha \approx 163.3$

$\alpha \approx 81.7°$

**3.** $9x^2 - 24xy + 16y^2 - 320x - 240y = 0$

$A = 9, B = -24, C = 16$

$\cot 2\alpha = \dfrac{A-C}{B}$

$\cot 2\alpha = \dfrac{9-16}{-24}$

$\cot 2\alpha = \dfrac{-7}{-24}$

$\cot 2\alpha = \dfrac{7}{24}$

$2\alpha \approx 73.74°$

$\alpha \approx 36.9°$

**4.** $x^2 + 4xy + 4y^2 - 6x - 5 = 0$

$A = 1, B = 4, C = 4$

$\cot 2a = \dfrac{A-C}{B}$

$\cot 2a = \dfrac{1-4}{4}$

$\cot 2a = -\dfrac{3}{4}$

$2\alpha \approx 126.9°$

$\alpha \approx 63.4°$

**5.** $5x^2 - 6\sqrt{3}xy - 11y^2 + 4x - 3y + 2 = 0$

$A = 5, B = -6\sqrt{3}, C = -11$

$\cot 2a = \dfrac{A-C}{B}$

$\cot 2a = \dfrac{5-(-11)}{-6\sqrt{3}}$

$\cot 2a = \dfrac{5+11}{-6\sqrt{3}}$

$\cot 2a = \dfrac{16}{-6\sqrt{3}}$

$\cot 2a = -\dfrac{8}{3\sqrt{3}}$

$\cot 2a = -\dfrac{8\sqrt{3}}{9}$

$2\alpha \approx 147°$

$\alpha \approx 73.5°$

**6.** $5x^2 + 4xy + 8y^2 - 6x + 3y - 12 = 0$

$A = 5, B = 4, C = 8$

$\cot 2a = \dfrac{A-C}{B}$

$\cot 2a = \dfrac{5-8}{4}$

$\cot 2a = -\dfrac{3}{4}$

$2\alpha \approx 126.9°$

$\alpha \approx 63.4°$

**7.** $2x^2 + xy + y^2 - 4 = 0$

$A = 2, B = 1, C = 1$

$\cot 2a = \dfrac{A-C}{B}$

$\cot 2a = \dfrac{2-1}{1}$

$\cot 2a = 1$

$2\alpha \approx 45°$

$\alpha \approx 22.5°$

**8.** $-2x^2 + \sqrt{3}xy - 3y^2 + 2x + 6y + 36 = 0$

$A = -2, B = \sqrt{3}, C = -3$

$\cot 2a = \dfrac{A-C}{B}$

$\cot 2a = \dfrac{-2-(-3)}{\sqrt{3}}$

$\cot 2a = \dfrac{\sqrt{3}}{3}$

$2\alpha \approx 60.0°$

$\alpha \approx 30.0°$

**9.** $xy = 4$

$xy - 4 = 0$

$A = 0, B = 1, C = 0, F = -4$

$\cot 2\alpha = \dfrac{A-C}{B} = \dfrac{0-0}{1} = 0$

$\csc^2 2\alpha = \cot^2 2\alpha + 1$

$\csc^2 2\alpha = 0^2 + 1 = 1$

$\csc 2\alpha = +1$ ($2\alpha$ is in the first quadrant.)

$\sin 2\alpha = \dfrac{1}{\csc 2\alpha} = \dfrac{1}{1} = 1$

$\sin^2 2\alpha + \cos^2 2\alpha = 1$

$\cos^2 2\alpha = 1 - \sin^2 2\alpha$

$\cos^2 2\alpha = 1 - (1)^2$

$\cos^2 2\alpha = 0$

$\cos 2\alpha = 0$

$\sin\alpha = \sqrt{\dfrac{1-(0)}{2}} = \dfrac{\sqrt{2}}{2} \qquad \cos\alpha = \sqrt{\dfrac{1+(0)}{2}} = \dfrac{\sqrt{2}}{2}$

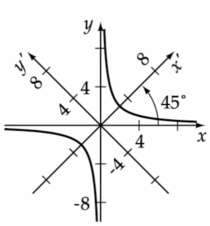
$\alpha = 45°$

$A' = A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha = 0\left(\dfrac{\sqrt{2}}{2}\right)^2 + 1\left(\dfrac{\sqrt{2}}{2}\right)\left(\dfrac{\sqrt{2}}{2}\right) + 0\left(\dfrac{\sqrt{2}}{2}\right)^2 = \dfrac{1}{2}$

$C' = A\sin^2\alpha - B\cos\alpha\sin\alpha + C\cos^2\alpha = 0\left(\dfrac{\sqrt{2}}{2}\right)^2 - 1\left(\dfrac{\sqrt{2}}{2}\right)\left(\dfrac{\sqrt{2}}{2}\right) + 0\left(\dfrac{\sqrt{2}}{2}\right)^2 = -\dfrac{1}{2}$

$F' = F = -4$

$\dfrac{1}{2}x'^2 - \dfrac{1}{2}y'^2 - 4 = 0$ or $\dfrac{x'^2}{8} - \dfrac{y'^2}{8} = 1$



**10.** $xy = -10$

$xy + 10 = 0$

$A = 0,\ B = 1,\ C = 0,\ F = 10$

$$\cot 2\alpha = \frac{A-C}{B} = \frac{0-0}{1} = 0$$

$\csc^2 2\alpha = \cot^2 2\alpha + 1$

$\csc^2 2\alpha = 0^2 + 1 = 1$

$\csc 2\alpha = +1$ ($2\alpha$ is in the first quadrant.)

$$\sin 2\alpha = \frac{1}{\csc 2\alpha} = \frac{1}{1} = 1$$

$\sin^2 2\alpha + \cos^2 2\alpha = 1$

$\cos^2 2\alpha = 1 - \sin^2 2\alpha$

$\cos^2 2\alpha = 1 - (1)^2$

$\cos^2 2\alpha = 0$

$\cos 2\alpha = 0$

$$\sin\alpha = \sqrt{\frac{1-(0)}{2}} = \frac{\sqrt{2}}{2} \qquad \cos\alpha = \sqrt{\frac{1+(0)}{2}} = \frac{\sqrt{2}}{2}$$

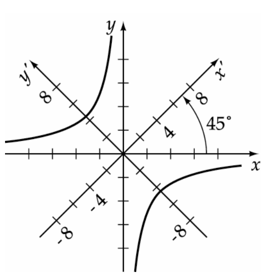
$\alpha = 45°$

$$A' = A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha = 0\left(\frac{\sqrt{2}}{2}\right)^2 + 1\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + 0\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$$

$$C' = A\sin^2\alpha - B\cos\alpha\sin\alpha + C\cos^2\alpha = 0\left(\frac{\sqrt{2}}{2}\right)^2 - 1\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + 0\left(\frac{\sqrt{2}}{2}\right)^2 = -\frac{1}{2}$$

$F' = F = 10$

$$\frac{1}{2}x'^2 - \frac{1}{2}y'^2 + 10 = 0 \text{ or } \frac{(y')^2}{20} - \frac{(x')^2}{20} = 1$$



**11.** $6x^2 - 6xy + 14y^2 - 45 = 0$

$A = 6, B = -6, C = 14, F = -45$

$$\cot 2\alpha = \frac{A-C}{B} = \frac{6-14}{-6} = \frac{4}{3}$$
$$\csc^2 2\alpha = \cot^2 2\alpha + 1$$
$$\csc^2 2\alpha = \left(\frac{4}{3}\right)^2 + 1 = \frac{25}{9}$$
$$\csc 2\alpha = +\sqrt{\frac{25}{9}} = \frac{5}{3} \qquad (2\alpha \text{ is in the first quadrant.})$$

$$\sin 2\alpha = \frac{1}{\csc 2\alpha} = \frac{3}{5}$$
$$\sin^2 \alpha + \cos^2 2\alpha = 1$$
$$\cos^2 2\alpha = 1 - \sin^2 2\alpha$$
$$\cos^2 2\alpha = 1 - \left(\frac{3}{5}\right)^2 = \frac{16}{25}$$
$$\cos 2\alpha = +\sqrt{\frac{16}{25}} = \frac{4}{5} \qquad (2\alpha \text{ is in the first quadrant.})$$

$$\sin\alpha = \sqrt{\frac{1-\left(\frac{4}{5}\right)}{2}} = \frac{\sqrt{10}}{10} \qquad \cos\alpha = \sqrt{\frac{1+\left(\frac{4}{5}\right)}{2}} = \frac{3\sqrt{10}}{10}$$

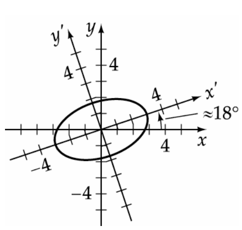
$\alpha = 18.4°$

$$A' = A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha = 6\left(\frac{3\sqrt{10}}{10}\right)^2 - 6\left(\frac{3\sqrt{10}}{10}\right)\left(\frac{\sqrt{10}}{10}\right) + 14\left(\frac{\sqrt{10}}{10}\right)^2 = 5$$

$$C' = A\sin^2\alpha - B\cos\alpha\sin\alpha + C\cos^2\alpha = 6\left(\frac{\sqrt{10}}{10}\right)^2 + 6\left(\frac{3\sqrt{10}}{10}\right)\left(\frac{\sqrt{10}}{10}\right) + 14\left(\frac{3\sqrt{10}}{10}\right)^2 = 15$$

$F' = F = -45$

$5x'^2 + 15y'^2 - 45 = 0$ or $\frac{(x')^2}{9} + \frac{(y')^2}{3} = 1$



**12.** $11x^2 - 10\sqrt{3}xy + y^2 - 20 = 0$

$A = 11,\ B = -10\sqrt{3},\ C = 1,\ F = -20$

$$\cot 2\alpha = \frac{A-C}{B} = \frac{11-1}{-10\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\csc^2 2\alpha = \cot^2 2\alpha + 1$$

$$\csc^2 2\alpha = \left(-\frac{\sqrt{3}}{3}\right)^2 + 1 = \frac{4}{3}$$

$$\csc 2\alpha = +\sqrt{\frac{4}{3}} = \frac{2\sqrt{3}}{3} \qquad (2\alpha \text{ is in second quadrant.})$$

$$\sin 2\alpha = \frac{1}{\csc 2\alpha} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$\sin^2 2\alpha + \cos^2 2\alpha = 1$$

$$\cos^2 2\alpha = 1 - \sin^2 2\alpha$$

$$\cos^2 2\alpha = 1 - \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4}$$

$$\cos 2\alpha = -\sqrt{\frac{1}{4}} = -\frac{1}{2} \qquad (2\alpha \text{ is in second quadrant.})$$

$$\sin\alpha = \sqrt{\frac{1-\left(-\frac{1}{2}\right)}{2}} = \frac{\sqrt{3}}{2} \qquad \cos\alpha = \sqrt{\frac{1+\left(-\frac{1}{2}\right)}{2}} = \frac{1}{2}$$

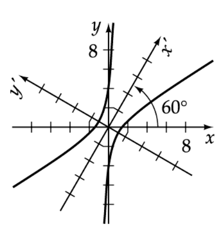
$\alpha = 60°$

$$A' = A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha = 11\left(\frac{1}{2}\right)^2 - 10\sqrt{3}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + 1\left(\frac{\sqrt{3}}{2}\right)^2 = -4$$

$$C' = A\sin^2\alpha - B\cos\alpha\sin\alpha + C\cos^2\alpha = 11\left(\frac{\sqrt{3}}{2}\right)^2 + 10\sqrt{3}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + 1\left(\frac{1}{2}\right)^2 = 16$$

$F' = F = -20$

$$-4x'^2 + 16y'^2 - 20 = 0 \text{ or } \frac{4(y')^2}{5} - \frac{(x')^2}{5} = 1$$



**13.** $x^2 - 4xy + 2y^2 - 1 = 0$

$A = 1, B = 4, C = -2, F = -1$

$$\cot 2\alpha = \frac{A-C}{B} = \frac{1-(-2)}{4} = \frac{3}{4}$$

$$\csc^2 2\alpha = \cot^2 2\alpha + 1$$

$$\csc^2 2\alpha = \left(\frac{3}{4}\right)^2 + 1 = \frac{25}{16}$$

$$\csc 2\alpha = +\sqrt{\frac{25}{16}} = \frac{5}{4} \qquad (2\alpha \text{ is in the first quadrant.})$$

$$\sin 2\alpha = \frac{1}{\csc 2\alpha} = \frac{4}{5}$$

$$\sin^2 \alpha + \cos^2 2\alpha = 1$$

$$\cos^2 2\alpha = 1 - \sin^2 \alpha$$

$$\cos^2 2\alpha = 1 - \left(\frac{4}{5}\right)^2 = \frac{9}{25}$$

$$\cos 2\alpha = +\sqrt{\frac{9}{25}} = \frac{3}{5} \qquad (2\alpha \text{ is in first quadrant.})$$

$$\sin \alpha = \sqrt{\frac{1-\left(\frac{3}{5}\right)}{2}} = \frac{\sqrt{5}}{5} \qquad \cos \alpha = \sqrt{\frac{1+\left(\frac{3}{5}\right)}{2}} = \frac{2\sqrt{5}}{5}$$

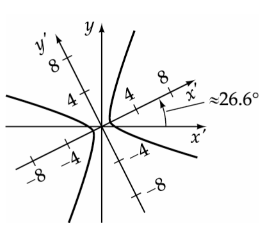
$\alpha \approx 26.6°$

$$A' = A\cos^2 \alpha + B\cos \alpha \sin \alpha + C\sin^2 \alpha = 1\left(\frac{2\sqrt{5}}{5}\right)^2 + 4\left(\frac{2\sqrt{5}}{5}\right)\left(\frac{\sqrt{5}}{5}\right) - 2\left(\frac{\sqrt{5}}{5}\right)^2 = 2$$

$$C' = A\sin^2 \alpha - B\cos \alpha \sin \alpha + C\cos^2 \alpha = 1\left(\frac{\sqrt{5}}{5}\right)^2 - 4\left(\frac{2\sqrt{5}}{5}\right)\left(\frac{\sqrt{5}}{5}\right) - 2\left(\frac{2\sqrt{5}}{5}\right)^2 = -3$$

$F' = F = -1$

$2(x')^2 + 3(y')^2 = 1$



**14.** $9x^2 - 24xy + 16y^2 + 100 = 0$

$A = 9,\ B = -24,\ C = 16,\ F = 100$

$$\cot 2\alpha = \frac{A - C}{B} = \frac{9 - 16}{-24} = \frac{7}{24}$$

$$\csc^2 2\alpha = \cot^2 2\alpha + 1$$

$$\csc^2 2\alpha = \left(\frac{7}{24}\right)^2 + 1 = \frac{625}{576}$$

$$\csc 2\alpha = +\sqrt{\frac{625}{576}} = \frac{25}{24}$$ ($2\alpha$ is in the first quadrant.)

$$\sin 2\alpha = \frac{1}{\csc 2\alpha} = \frac{24}{25}$$

$$\sin^2 2\alpha + \cos^2 2\alpha = 1$$

$$\cos^2 2\alpha = 1 - \sin^2 2\alpha$$

$$\cos^2 2\alpha = 1 - \left(\frac{24}{25}\right)^2 = \frac{49}{625}$$

$$\cos 2\alpha = +\sqrt{\frac{49}{625}} = \frac{7}{25}$$ ($2\alpha$ is in the first quadrant.)

$$\sin\alpha = \sqrt{\frac{1 - \left(\frac{7}{25}\right)}{2}} = \frac{3}{5} \qquad \cos\alpha = \sqrt{\frac{1 + \left(\frac{7}{25}\right)}{2}} = \frac{4}{5}$$

$\alpha \approx 36.9°$

$$A' = A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha = 9\left(\frac{4}{5}\right)^2 - 24\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) + 16\left(\frac{3}{5}\right)^2 = 0$$

$$C' = A\sin^2\alpha - B\cos\alpha\sin\alpha + C\cos^2\alpha = 9\left(\frac{3}{5}\right)^2 + 24\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) + 16\left(\frac{4}{5}\right)^2 = 25$$

$F' = F = 100$

$25(y')^2 + 100 = 0$

$(y')^2 = -4$ This equation has no real solutions, so there is no graph.

**15.** $3x^2 + 2\sqrt{3}xy + y^2 + 2x - 2\sqrt{3}y + 16 = 0$

$A = 3, B = 2\sqrt{3}, C = 1, D = 2, E = -2\sqrt{3}, F = 16$

$$\cot 2\alpha = \frac{A-C}{B} = \frac{3-1}{2\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\csc^2 2\alpha = \cot^2 2\alpha + 1$$

$$\csc^2 2\alpha = \left(\frac{\sqrt{3}}{3}\right)^2 + 1 = \frac{4}{3}$$

$$\csc 2\alpha = +\sqrt{\frac{4}{3}} = \frac{2\sqrt{3}}{3} \qquad (2\alpha \text{ is in the first quadrant.})$$

$$\sin 2\alpha = \frac{1}{\csc 2\alpha} = \frac{\sqrt{3}}{2}$$

$$\sin^2 \alpha + \cos^2 2\alpha = 1$$

$$\cos^2 2\alpha = 1 - \sin^2 2\alpha$$

$$\cos^2 2\alpha = 1 - \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4}$$

$$\cos 2\alpha = +\sqrt{\frac{1}{4}} = \frac{1}{2} \qquad (2\alpha \text{ is in the first quadrant.})$$

$$\sin\alpha = \sqrt{\frac{1-\left(\frac{1}{2}\right)}{2}} = \frac{1}{2} \qquad \cos\alpha = \sqrt{\frac{1+\left(\frac{1}{2}\right)}{2}} = \frac{\sqrt{3}}{2}$$

$\alpha = 30°$

$$A' = A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha = 3\left(\frac{\sqrt{3}}{2}\right)^2 + 2\sqrt{3}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right)^2 = 4$$

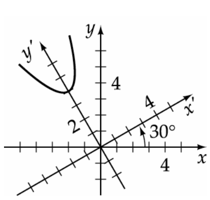
$$C' = A\sin^2\alpha - B\cos\alpha\sin\alpha + C\cos^2\alpha = 3\left(\frac{1}{2}\right)^2 - 2\sqrt{3}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) + 1\left(\frac{\sqrt{3}}{2}\right)^2 = 0$$

$$D' = D\cos\alpha + E\sin\alpha = 2\left(\frac{\sqrt{3}}{2}\right) - 2\sqrt{3}\left(\frac{1}{2}\right) = 0$$

$$E' = -D\sin\alpha + E\cos\alpha = -2\left(\frac{1}{2}\right) - 2\sqrt{3}\left(\frac{\sqrt{3}}{2}\right) = -4$$

$$F' = F = 16$$

$4(x')^2 - 4y' + 16 = 0$ or $y' = (x')^2 + 4$



**16.** $x^2+2xy+y^2+2\sqrt{2}x-2\sqrt{2}y=0$

$A=1, B=2, C=1, D=\sqrt{8}, E=-\sqrt{8}$

$\cot 2\alpha=\dfrac{A-C}{B}=\dfrac{1-1}{2}=0$

$\csc^2 2\alpha=\cot^2 2\alpha+1$

$\csc^2 2\alpha=0^2+1=1$

$\csc 2\alpha=1$

$\sin 2\alpha=\dfrac{1}{\csc 2\alpha}=\dfrac{1}{1}=1$

$\sin^2\alpha+\cos^2 2\alpha=1$

$\cos^2 2\alpha=1-\sin^2 2\alpha$

$\cos^2 2\alpha=1-(1)^2=0$

$\cos 2\alpha=0$

$\sin\alpha=\sqrt{\dfrac{1-(0)}{2}}=\dfrac{\sqrt{2}}{2}$ $\qquad\cos\alpha=\sqrt{\dfrac{1+(0)}{2}}=\dfrac{\sqrt{2}}{2}$

$\alpha=45°$

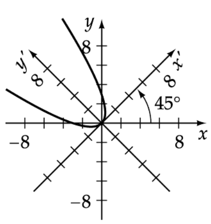
$A'=A\cos^2\alpha+B\cos\alpha\sin\alpha+C\sin^2\alpha=1\left(\dfrac{\sqrt{2}}{2}\right)^2+2\left(\dfrac{\sqrt{2}}{2}\right)\left(\dfrac{\sqrt{2}}{2}\right)+1\left(\dfrac{\sqrt{2}}{2}\right)^2=2$

$C'=A\sin^2\alpha-B\cos\alpha\sin\alpha+C\cos^2\alpha=1\left(\dfrac{\sqrt{2}}{2}\right)^2-2\left(\dfrac{\sqrt{2}}{2}\right)\left(\dfrac{\sqrt{2}}{2}\right)+1\left(\dfrac{\sqrt{2}}{2}\right)^2=0$

$D'=D\cos\alpha+E\sin\alpha=\sqrt{8}\left(\dfrac{\sqrt{2}}{2}\right)-\sqrt{8}\left(\dfrac{\sqrt{2}}{2}\right)=0$

$E'=-D\sin\alpha+E\cos\alpha=-\sqrt{8}\left(\dfrac{\sqrt{2}}{2}\right)-\sqrt{8}\left(\dfrac{\sqrt{2}}{2}\right)=-4$

$2(x')^2-4y'=0$ or $y'=\dfrac{1}{2}(x')^2$



**17.** $9x^2+24xy+16y^2-40x-30y+100=0$

$A=9,\ B=-24,\ C=16,\ D=-40,\ E=-30\ F=100$

$$\cot 2\alpha=\frac{A-C}{B}=\frac{9-16}{-24}=\frac{7}{24}$$

$$\csc^2 2\alpha=\cot^2 2\alpha+1$$

$$\csc^2 2\alpha=\left(\frac{7}{24}\right)^2+1=\frac{625}{576}$$

$$\csc 2\alpha=+\sqrt{\frac{625}{576}}=\frac{25}{24}\qquad (2\alpha \text{ is in the first quadrant.})$$

$$\sin 2\alpha=\frac{1}{\csc 2\alpha}=\frac{24}{25}$$

$$\sin^2\alpha+\cos^2 2\alpha=1$$

$$\cos^2 2\alpha=1-\sin^2 2\alpha$$

$$\cos^2 2\alpha=1-\left(\frac{24}{25}\right)^2=\frac{49}{625}$$

$$\cos 2\alpha=+\sqrt{\frac{49}{625}}=\frac{7}{25}\qquad (2\alpha \text{ is in first quadrant.})$$

$$\sin\alpha=\sqrt{\frac{1-\left(\frac{7}{25}\right)}{2}}=\frac{3}{5}\qquad \cos\alpha=\sqrt{\frac{1+\left(\frac{7}{25}\right)}{2}}=\frac{4}{5}$$

$\alpha\approx 36.9°$

$$A'=A\cos^2\alpha+B\cos\alpha\sin\alpha+C\sin^2\alpha=9\left(\frac{4}{5}\right)^2-24\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)+16\left(\frac{3}{5}\right)^2=0$$

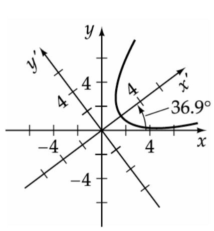
$$C'=A\sin^2\alpha-B\cos\alpha\sin\alpha+C\cos^2\alpha=9\left(\frac{3}{5}\right)^2+24\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)+16\left(\frac{4}{5}\right)^2=25$$

$$D'=D\cos\alpha+E\sin\alpha=-40\left(\frac{4}{5}\right)-30\left(\frac{3}{5}\right)=-50$$

$$E'=-D\sin\alpha+E\cos\alpha=40\left(\frac{3}{5}\right)-30\left(\frac{4}{5}\right)=0$$

$$F'=F=100$$

$25(x')^2-50x'+100=0$ or $(y')^2=2(x-2)$




9

**18.** $24x^2 + 16\sqrt{3}xy + 8y^2 - x + \sqrt{3}y - 8 = 0$

$A = 24,\ B = 16\sqrt{3},\ C = 8,\ D = -1,\ E = \sqrt{3},\ F = -8$

$$\cot 2\alpha = \frac{A-C}{B} = \frac{24-8}{16\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\csc^2 2\alpha = \cot^2 2\alpha + 1$$

$$\csc^2 2\alpha = \left(\frac{\sqrt{3}}{3}\right)^2 + 1 = \frac{4}{3}$$

$$\csc 2\alpha = +\sqrt{\frac{4}{3}} = \frac{2\sqrt{3}}{3}$$ ($2\alpha$ is in the first quadrant.)

$$\sin 2\alpha = \frac{1}{\csc 2\alpha} = \frac{\sqrt{3}}{2}$$

$$\sin^2 2\alpha + \cos^2 2\alpha = 1$$

$$\cos^2 2\alpha = 1 - \sin^2 2\alpha$$

$$\cos^2 2\alpha = 1 - \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4}$$

$$\cos 2\alpha = +\sqrt{\frac{1}{4}} = \frac{1}{2}$$ ($2\alpha$ is in the first quadrant.)

$$\sin\alpha = \sqrt{\frac{1-\left(\frac{1}{2}\right)}{2}} = \frac{1}{2} \qquad \cos\alpha = \sqrt{\frac{1+\left(\frac{1}{2}\right)}{2}} = \frac{\sqrt{3}}{2}$$

$\alpha = 30°$

$$A' = A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha = 24\left(\frac{\sqrt{3}}{2}\right)^2 + 16\sqrt{3}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) + 8\left(\frac{1}{2}\right)^2 = 32$$

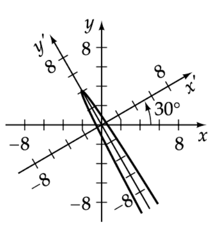
$$C' = A\sin^2\alpha - B\cos\alpha\sin\alpha + C\cos^2\alpha = 24\left(\frac{1}{2}\right)^2 - 16\sqrt{3}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) + 8\left(\frac{\sqrt{3}}{2}\right)^2 = 0$$

$$D' = D\cos\alpha + E\sin\alpha = -1\left(\frac{\sqrt{3}}{2}\right) + \sqrt{3}\left(\frac{1}{2}\right) = 0$$

$$E' = -D\sin\alpha + E\cos\alpha = 1\left(\frac{1}{2}\right) + \sqrt{3}\left(\frac{\sqrt{3}}{2}\right) = 2$$

$$F' = F = -8$$

$32(x')^2 + 2y' - 8 = 0$ or $y' = -16(x')^2 + 4$



**19.** $6x^2 + 24xy - y^2 - 12x + 26y + 11 = 0$

$A = 6, B = 24, C = -1, D = -12, E = 26\ F = 11$

$$\cot 2\alpha = \frac{A - C}{B} = \frac{6 - (-1)}{24} = \frac{7}{24}$$

$$\csc^2 2\alpha = \cot^2 2\alpha + 1$$

$$\csc^2 2\alpha = \left(\frac{7}{24}\right)^2 + 1 = \frac{625}{576}$$

$$\csc 2\alpha = +\sqrt{\frac{625}{576}} = \frac{25}{24} \qquad (2\alpha \text{ is in the first quadrant.})$$

$$\sin 2\alpha = \frac{1}{\csc 2\alpha} = \frac{24}{25}$$

$$\sin^2 2\alpha + \cos^2 2\alpha = 1$$

$$\cos^2 2\alpha = 1 - \sin^2 2\alpha$$

$$\cos^2 2\alpha = 1 - \left(\frac{24}{25}\right)^2 = \frac{49}{625}$$

$$\cos 2\alpha = +\sqrt{\frac{49}{625}} = \frac{7}{25} \qquad (2\alpha \text{ is in the first quadrant.})$$

$$\sin \alpha = \sqrt{\frac{1 - \left(\frac{7}{25}\right)}{2}} = \frac{3}{5} \qquad \cos \alpha = \sqrt{\frac{1 + \left(\frac{7}{25}\right)}{2}} = \frac{4}{5}$$

$\alpha \approx 36.9°$

$$A' = A\cos^2 \alpha + B\cos \alpha \sin \alpha + C\sin^2 \alpha = 6\left(\frac{4}{5}\right)^2 + 24\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) - 1\left(\frac{3}{5}\right)^2 = 15$$

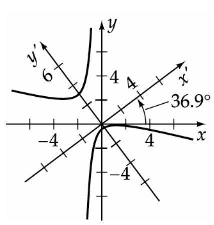
$$C' = A\sin^2 \alpha - B\cos \alpha \sin \alpha + C\cos^2 \alpha = 6\left(\frac{3}{5}\right)^2 - 24\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) - 1\left(\frac{4}{5}\right)^2 = -10$$

$$D' = D\cos \alpha + E\sin \alpha = -12\left(\frac{4}{5}\right) + 26\left(\frac{3}{5}\right) = 6$$

$$E' = -D\sin \alpha + E\cos \alpha = 12\left(\frac{3}{5}\right) + 26\left(\frac{4}{5}\right) = 28$$

$$F' = F = 11$$

$$15(x')^2 - 10(y')^2 + 6x' + 28y' + 11 = 0$$



**20.** $x^2 + 4xy + 4y^2 - 2\sqrt{5}x + \sqrt{5}y = 0$

$A = 1, B = 4, C = 4, D = -2\sqrt{5}, E = \sqrt{5}, F = 0$

$$\cot 2\alpha = \frac{A - C}{B} = \frac{1-4}{4} = -\frac{3}{4}$$
$$\csc^2 2\alpha = \cot^2 2\alpha + 1$$
$$\csc^2 2\alpha = \left(-\frac{3}{4}\right)^2 + 1 = \frac{25}{16}$$
$$\csc 2\alpha = +\sqrt{\frac{25}{16}} = \frac{5}{4} \qquad (2\alpha \text{ is in the second quadrant.})$$
$$\sin 2\alpha = \frac{1}{\csc 2\alpha} = \frac{4}{5}$$

$$\sin^2 2\alpha + \cos^2 2\alpha = 1$$
$$\cos^2 2\alpha = 1 - \sin^2 2\alpha$$
$$\cos^2 2\alpha = 1 - \left(\frac{4}{5}\right)^2 = \frac{9}{25}$$
$$\cos 2\alpha = -\sqrt{\frac{9}{25}} = -\frac{3}{5} \qquad (2\alpha \text{ is in second quadrant.})$$

$$\sin\alpha = \sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}} = \frac{2\sqrt{5}}{5} \qquad \cos\alpha = \sqrt{\frac{1+\left(-\frac{3}{5}\right)}{2}} = \frac{\sqrt{5}}{5}$$

$\alpha \approx 63.43°$

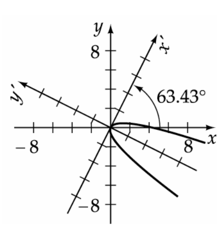
$$A' = A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha = 1\left(\frac{\sqrt{5}}{5}\right)^2 + 4\left(\frac{\sqrt{5}}{5}\right)\left(\frac{2\sqrt{5}}{5}\right) + 4\left(\frac{2\sqrt{5}}{5}\right)^2 = 5$$

$$C' = A\sin^2\alpha - B\cos\alpha\sin\alpha + C\cos^2\alpha = 1\left(\frac{2\sqrt{5}}{5}\right)^2 - 4\left(\frac{\sqrt{5}}{5}\right)\left(\frac{2\sqrt{5}}{5}\right) + 4\left(\frac{\sqrt{5}}{5}\right)^2 = 0$$

$$D' = D\cos\alpha + E\sin\alpha = -2\sqrt{5}\left(\frac{\sqrt{5}}{5}\right) + \sqrt{5}\left(\frac{2\sqrt{5}}{5}\right) = 0$$

$$E' = -D\sin\alpha + E\cos\alpha = 2\sqrt{5}\left(\frac{2\sqrt{5}}{5}\right) - \sqrt{5}\left(\frac{\sqrt{5}}{5}\right) = 5$$

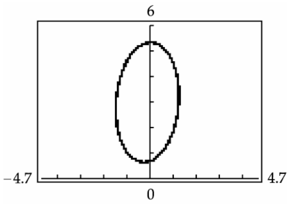
$5(x')^2 + 5y' = 0$ or $y' = -(x')^2$



**21.** $A = 6, B = -1, C = 2, D = 4, E = -12, F = 7$

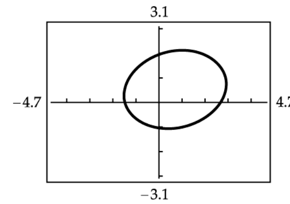
Graph $y = \dfrac{-(-x-12) \pm \sqrt{(-x-12)^2 - 8(6x^2 + 4x + 7)}}{4}$

The graph will appear disconnected at the endpoints of the minor axes on a graphing utility.



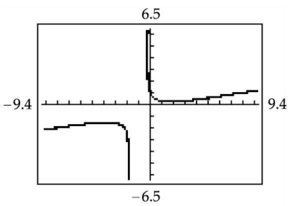

**22.** $A = 5, B = -2, C = 10, D = -6, E = -9, F = -20$

Graph $y = \dfrac{-(-2x-9) \pm \sqrt{(-2x-9)^2 - 40(5x^2 - 6x - 20)}}{20}$



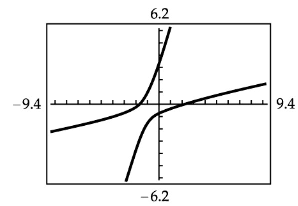

**23.** $A = 1, B = -6, C = 1, D = -2, E = -5, F = 4$

Graph $y = \dfrac{-(-6x-5) \pm \sqrt{(-6x-5)^2 - 4(x^2 - 2x + 4)}}{2}$



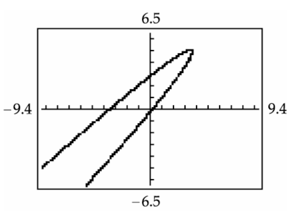

**24.** $A = 2, B = -10, C = 3, D = -1, E = -8, F = -7$

Graph $y = \dfrac{-(-10x-8) \pm \sqrt{(-10x-8)^2 - 12(2x^2 - x - 7)}}{6}$



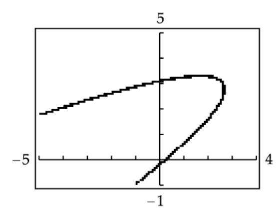

**25.** $A = 3, B = -6, C = 3, D = 10, E = -8, F = -2$

Graph $y = \dfrac{-(-6x-8) \pm \sqrt{(-6x-8)^2 - 12(3x^2 + 10x - 2)}}{6}$




**26.** $A = 2, B = -8, C = 8, D = 20, E = -24, F = -3$

Graph $y = \dfrac{-(-8x-24) \pm \sqrt{(-8x-24)^2 - 32(2x^2 + 20x - 3)}}{16}$



**27.** $\frac{2x'^2}{1} - \frac{3y'^2}{1} = 1 \quad \sin\alpha = \frac{\sqrt{5}}{5} \quad \cos\alpha = \frac{2\sqrt{5}}{5}$

$a^2 = \frac{1}{2}; \quad a = \frac{\sqrt{2}}{2}$

$b^2 = \frac{1}{3}; \quad b = \frac{\sqrt{3}}{3}$

Asymptotes: $y' = \pm\frac{b}{a}x'$ or $y' = \pm\frac{\sqrt{6}}{3}x'$

Using the transformation formulas for $x'$ and $y'$ yields

$$y\cos\alpha - x\sin\alpha = \pm\frac{\sqrt{6}}{3}(x\cos\alpha + y\sin\alpha)$$

$$\frac{2\sqrt{5}}{5}y - \frac{\sqrt{5}}{5}x = \pm\frac{\sqrt{6}}{3}\left(\frac{2\sqrt{5}}{5}x + \frac{\sqrt{5}}{5}y\right)$$

$$\frac{2\sqrt{5}}{5}y - \frac{\sqrt{5}}{5}x = \pm\left(\frac{2\sqrt{30}}{15} + \frac{\sqrt{30}}{15}y\right)$$

Multiplying both sides of the equation by $15/\sqrt{5}$ yields

$$6y - 3x = \pm\left(2\sqrt{6}x + \sqrt{6}y\right)$$

$$6y - 3x = 2\sqrt{6}x + \sqrt{6}y \quad \text{and} \quad 6y - 3x = -(2\sqrt{6}x + \sqrt{6}y)$$

$$6y - \sqrt{6}y = 3x + 2\sqrt{6}x \qquad 6y + \sqrt{6}y = 3x - 2\sqrt{6}x$$

$$y = \frac{3 + 2\sqrt{6}}{6 - \sqrt{6}}x \qquad y = \frac{3 - 2\sqrt{6}}{6 + \sqrt{6}}x$$

Rationalizing the denominators, we obtain

$$y = \frac{2 + \sqrt{6}}{2}x \quad \text{and} \quad y = \frac{2 - \sqrt{6}}{2}x$$

**28.** From Exercise 16, $y' = \frac{1}{2}x'^2$. Relative to $x'y'$-coordinates, the focus is $\left(0, \frac{1}{2}\right)$ and $\alpha = 45°$ and the directrix is $y' = -\frac{1}{2}$.

Using the transformation equation

$x = x'\cos\alpha - y'\sin\alpha$

$y = y'\cos\alpha + x'\sin\alpha$

we have

$$x = 0\cdot\cos 45° - \frac{1}{2}\sin 45° = -\frac{\sqrt{2}}{4}$$

$$y = \frac{1}{2}\cos 45° + 0\cdot\sin 45° = \frac{\sqrt{2}}{4}$$

The coordinates of the focus in the $xy$-coordinate system are $\left(-\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right)$.

The equation of the directrix is given by

$$y' = -\frac{1}{2}$$

$$y\cos 45° - x\sin 45° = -\frac{1}{2}$$

$$y\frac{\sqrt{2}}{2} - x\frac{\sqrt{2}}{2} = -\frac{1}{2}$$

$\sqrt{2}y - \sqrt{2}x = -1$ or $\sqrt{2}x - \sqrt{2}y = 1$.

The equation of the directrix is $\sqrt{2}x - \sqrt{2}y = 1$

**29.** From Exercise 11, $\frac{x'^2}{9}+\frac{y'^2}{3}=1, \sin\alpha=\frac{\sqrt{10}}{10},\ \cos\alpha=\frac{3\sqrt{10}}{10}, a^2=9, b^2=3, c^2=9-3=6.$

Thus $c=\sqrt{6}$.

Foci in $x'y'$-coordinates are $(\pm\sqrt{6}, 0)$. Thus $x'=\pm\sqrt{6}, y'=0$.

$$x = x'\cos\alpha - y'\sin\alpha = \pm\sqrt{6}\left(\frac{3\sqrt{10}}{10}\right)-0\cdot\frac{\sqrt{10}}{10} = \pm\frac{3\sqrt{15}}{5}$$

$$y = y'\cos\alpha + x'\sin\alpha = 0\cdot\cos\alpha \pm \sqrt{6}\left(\frac{\sqrt{10}}{10}\right) = \pm\frac{\sqrt{15}}{5}$$

Foci in the $xy$-coordinate system are $\left(\frac{3\sqrt{15}}{5}, \frac{\sqrt{15}}{5}\right)$ and $\left(-\frac{3\sqrt{15}}{5}, -\frac{\sqrt{15}}{5}\right)$.

**30.** $xy=4$
$xy-4=0$
$A=0, B=1, C=0$
Since $B^2-4AC=1^2-4(0)(0)=1>0$,
the graph is a hyperbola.

**31.** $x^2+xy-y^2-40=0$
$A=1, B=1, C=-1$
Since $B^2-4AC=1^2-4(1)(-1)=5>0$,
the graph is a hyperbola.

**32.** $11x^2-10\sqrt{3}xy+y^2-20=0$
$A=11, B=-10\sqrt{3}, C=1$
Since $B^2-4AC=(-10\sqrt{3})^2-4(11)(1)=256>0$,
the graph is a hyperbola.

**33.** $3x^2+2\sqrt{3}xy+y^2-3x+2y+20=0$
$A=3, B=2\sqrt{3}, C=1$
Since $B^2-4AC=(2\sqrt{3})^2-4(3)(1)=0$,
the graph is a parabola.

**34.** $9x^2-24xy+16y^2+8x-12y-20=0$
$A=9, B=-24, C=16$
Since $B^2-4AC=(-24)^2-4(9)(16)=0$,
the graph is a parabola.

**35.** $4x^2-4xy+y^2-12y-20=0$
$A=4, B=-4, C=1$
Since $B^2-4AC=(-4)^2-4(4)(1)=0$, the graph is a parabola.

**36.** $5x^2-4xy+8y^2-6x+3y-12=0$
$A=5, B=4, C=8$
Since $B^2-4AC=(4)^2-4(5)(8)=-144<0$,
the graph is an ellipse.

**37.** $5x^2-6\sqrt{3}xy-11y^2+4x-3y+2=0$
$A=5, B=-6\sqrt{3}, C=-11$
Since $B^2-4AC=(-6\sqrt{3})^2-4(5)(-11)=328>0$,
the graph is a hyperbola.

**38.** $6x^2-6xy+14y^2-14x+12y-60=0$
$A=6, B=-6, C=14$
Since $B^2-4AC=(-6)^2-4(6)(14)=-300<0$,
the graph is an ellipse.

**39.** $6x^2+2\sqrt{3}xy+5y^2-3x+2y-20=0$
$A=6, B=2\sqrt{3}, C=5$
Since $B^2-4AC=(2\sqrt{3})^2-4(6)(5)=-108<0$,
the graph is an ellipse.

**40.** $5x^2-2\sqrt{3}xy+3y^2-x+y-12=0$
$A=5, B=-2\sqrt{3}, C=3$
Since $B^2-4AC=\left(-2\sqrt{3}\right)^2-4(5)(3)=-48<0$,
the graph is an ellipse.

## Connecting Concepts

**41.** $x^2 + y^2 = r^2$

Substitute $x = x'\cos\alpha - y'\sin\alpha$ and $y = y'\cos\alpha + x'\sin\alpha$.

$$(x'\cos\alpha - y'\sin\alpha)^2 + (y'\cos\alpha + x'\sin\alpha)^2 = r^2$$
$$x'^2\cos^2\alpha - 2x'y'\cos\alpha\sin\alpha + y'^2\sin^2\alpha + y'^2\cos^2\alpha + 2x'y'\cos\alpha\sin\alpha + x'^2\sin^2\alpha = r^2$$
$$x'^2(\cos^2\alpha + \sin^2\alpha) + x'y'(2\cos\alpha\sin\alpha - 2\cos\alpha\sin\alpha) + y'^2(\sin^2\alpha + \cos^2\alpha) = r^2$$
$$x'^2(1) + x'y'(0) + y'^2(1) = r^2$$
$$x'^2 + y'^2 = r^2$$

**42.** Because the vertices are (1, 1) and $(-1, -1)$, the transverse axis is on the line $y = x$. Consider an $x'y'$-coordinate system rotated $45°$ ($\tan\alpha = 1$ implies $\alpha = 45°$) from an *xy*-coordinate system. The equation of the hyperbola is

$$\frac{x'^2}{2} - \frac{y'^2}{2} = 1 \qquad (1)$$

This follows from the fact that (1, 1) in *xy*-coordinates is $(\sqrt{2}, 0)$ in $x'y'$-coordinates.

Therefore, $a = \sqrt{2}$. Also, $(\sqrt{2}, \sqrt{2})$ in $x'y'$-coordinates is $(2, 0)$ in $x'y'$-coordinates.

Therefore, $c = 2$. Since $a = \sqrt{2}$ and $c = 2$, then $b = \sqrt{2}$.

Now let $x' = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y$ and $y' = \frac{\sqrt{2}}{2}y - \frac{\sqrt{2}}{2}x$ and substitute into Equation (1):

$$\frac{\left(\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right)^2}{2} - \frac{\left(\frac{\sqrt{2}}{2}y - \frac{\sqrt{2}}{2}x\right)^2}{2} = 1$$

Simplifying, we have $\dfrac{\left(\frac{1}{2}x^2 + xy + \frac{1}{2}y^2\right)}{2} - \dfrac{\left(\frac{1}{2}x^2 - xy + \frac{1}{2}y^2\right)}{2} = 1$

or $xy = 1$.

**43.** Vertices (2, 4) and $(-2,\ -4)$ imply that the major axis is on the line $y = 2x$. Consider an $x'y'$-coordinate system rotated an angle $\alpha$,

where $\tan\alpha = 2$. From this equation, using identities, $\cos\alpha = \frac{\sqrt{5}}{5}$ and $\sin\alpha = \frac{2\sqrt{5}}{5}$.

The equation of the ellipse in the $x'y'$-coordinate system is

$$\frac{x'^2}{20} + \frac{y'^2}{10} = 1 \qquad (1)$$

This follows from the fact that $(2, 4)$ in $xy$-coordinates is $(\sqrt{20},\ 0)$ in $x'y'$-coordinates.

Therefore, $\alpha = \sqrt{20}$. Also, $(\sqrt{2},\ 2\sqrt{2})$ in $xy$-coordinates is $(\sqrt{10},\ 0)$ in $x'y'$-coordinates.

Therefore, $c = \sqrt{10}$. Thus $b = \sqrt{10}$.

Now let $x' = \frac{\sqrt{5}}{5}x + \frac{2\sqrt{5}}{5}y$ and $y' = \frac{\sqrt{5}}{5}y - \frac{2\sqrt{5}}{5}x$ and substitute into Equation $(1)$:

$$\frac{\left(\frac{\sqrt{5}}{5}x + \frac{2\sqrt{5}}{5}y\right)^2}{20} + \frac{\left(\frac{\sqrt{5}}{5}y - \frac{2\sqrt{5}}{5}x\right)^2}{10} = 1$$

Simplifying, we have

$$\frac{\left(\frac{1}{5}x^2 + \frac{4}{5}xy + \frac{4}{5}y^2\right)}{20} + \frac{\left(\frac{4}{5}x^2 - \frac{4}{5}xy + \frac{1}{5}y^2\right)}{10} = 1$$

$$\frac{\frac{9}{5}x^2 - \frac{4}{5}xy + \frac{6}{5}y^2}{20} = 1$$

$$\frac{9x^2 - 4xy + 6y^2}{100} = 1$$

$$\text{or } 9x^2 - 4xy + 6y^2 = 100$$

**44.** Vertex: $(0, 0)$ Focus: $(1, 3)$

Since both the vertex and the focus lie on the axis of symmetry, we can find the equation of the line by using these two points. We obtain

$$y = 3x$$

The slope of the line $m$ is equal to 3. But $\theta$, the angle the line makes with the $x$-axis, is equal to the arctangent of $m$. In other words, $\tan\theta = m = 3$. But rotating the axes upward through $\theta$ $(\alpha = \theta)$, we can place the vertex and focus on the $x'$-axis.

Since $\tan\alpha = 3, \sin\alpha = \frac{3\sqrt{10}}{10}$ and $\cos\alpha = \frac{\sqrt{10}}{10}$.

Vertex: $(0, 0)$

Since the origins for both the $xy$- and $x'y'$-systems are coincident, the $x'y'$ vertex is $(0, 0)$.

Focus: $(1, 3)$

$x = 1,\ y = 3$

$x' = x\cos\alpha + y\sin\alpha$

$y' = y\cos\alpha - x\sin\alpha$

$x' = 1\left(\frac{\sqrt{10}}{10}\right) + 3\left(\frac{3\sqrt{10}}{10}\right) = \sqrt{10}$

$y' = 3\left(\frac{\sqrt{10}}{10}\right) - 1\left(\frac{3\sqrt{10}}{10}\right) = 0$

$x'y'$ Focus: $(\sqrt{10}, 0)$

Since the vertex is at the origin at the focus at $(\sqrt{10},\ 0)$, $p = \sqrt{10}$. Therefore, $y'^2 = 4\sqrt{10}x'$.

Substituting $x' = x\cos\alpha + y\sin\alpha$ and $y' = y\cos\alpha - x\sin\alpha$, we have

$(y\cos\alpha - x\sin\alpha)^2 = 4\sqrt{10}\,(x\cos\alpha + y\sin\alpha)$

Substituting $\sin\alpha = \frac{3\sqrt{10}}{10}$ and $\cos\alpha = \frac{\sqrt{10}}{10}$ yields

$$\left(\frac{\sqrt{10}}{10}y - \frac{3\sqrt{10}}{10}x\right)^2 = 4\sqrt{10}\left(\frac{\sqrt{10}}{10}x + \frac{3\sqrt{10}}{10}y\right)$$

$$\frac{1}{10}y^2 - \frac{3}{5}xy + \frac{9}{10}x^2 = 4x + 12y$$

$9x^2 - 6xy + y^2 - 40x - 120y = 0$

**45.** $A' + C' = A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha + A\sin^2\alpha - B\cos\alpha\sin\alpha + C\cos^2\alpha$

$= A(\cos^2\alpha + \sin^2\alpha) + B(\cos\alpha\sin\alpha - \cos\alpha\sin\alpha) + C(\sin^2\alpha + \cos^2\alpha)$

$= A + C$

**46.** Begin by using double-angle formula to rewrite $A'$, $B'$, and $C'$. Recall $\cos^2\alpha = \frac{1+2\cos 2\alpha}{2}$, $\sin^2 a = \frac{1-2\cos 2\alpha}{2}$, and $\sin 2\alpha = 2\cos\alpha\sin\alpha$.

$$\begin{aligned} A' &= A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha \\ &= A\left(\frac{1+\cos 2\alpha}{2}\right) + \frac{B}{2}\sin 2\alpha + C\left(\frac{1-\cos 2\alpha}{2}\right) \\ &= \frac{1}{2}[(A+C) + B\sin 2\alpha + (A-C)\cos 2\alpha] \end{aligned}$$

$$\begin{aligned} B' &= B(\cos^2\alpha - \sin^2\alpha) + (C-A)2\sin\alpha\cos\alpha \\ &= B\cos 2\alpha - (A-C)\sin 2\alpha \end{aligned}$$

$$\begin{aligned} C' &= A\sin^2\alpha - B\cos\alpha\sin\alpha + C\cos^2\alpha \\ &= A\left(\frac{1-\cos 2\alpha}{2}\right) - \frac{B}{2}\sin 2\alpha + C\left(\frac{1+\cos 2\alpha}{2}\right) \\ &= \frac{1}{2}[(A+C) - B\sin 2\alpha - (A-C)\cos 2\alpha] \end{aligned}$$

Using these expressions, we have

$$\begin{aligned} B'^2 - 4A'C' &= [B\cos 2\alpha - (A-C)\sin 2\alpha]^2 - 4\left(\frac{1}{2}[(A+C) + B\sin 2\alpha + (A-C)\cos 2\alpha]\right) \\ &\quad \times\left(\frac{1}{2}[(A+C) - B\sin 2\alpha - (A-C)\cos 2\alpha]\right) \\ &= [B^2\cos^2 2\alpha - 2B(A-C)\cos 2\alpha\sin 2\alpha + (A-C)^2\sin^2 2\alpha] \\ &\quad -[(A+C)^2 - B^2\sin^2\alpha - (A-C)^2\cos^2 2\alpha - 2B(A-C)\cos 2\alpha\sin 2\alpha] \\ &= B^2\cos^2 2\alpha - 2B(A-C)\cos 2\alpha\sin 2\alpha + (A-C)^2\sin^2 2\alpha - (A+C)^2 \\ &\quad + B^2\sin^2 2\alpha + (A-C)^2\cos^2 2\alpha + 2B(A-C)\cos 2\alpha\sin 2\alpha \\ &= B^2(\cos^2 2\alpha + \sin^2 2\alpha) + (A-C)^2(\sin^2 2\alpha + \cos^2 2\alpha) - (A+C)^2 \\ &= B^2 + (A-C)^2 - (A+C)^2 \\ &= B^2 + A^2 - 2AC + C^2 - A^2 - 2AC - C^2 \\ &= B^2 - 4AC \end{aligned}$$

**47.** Ellipse with major axis parallel to $x$-axis:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

$$b^2(x-h)^2+a^2(y-k)^2=a^2b^2$$
$$b^2(x^2-2hx+h^2)+a^2(y^2-2ky+k^2)=a^2b^2$$
$$b^2x^2-2b^2hx+b^2h^2+a^2y^2-2a^2ky+a^2k^2=a^2b^2$$
$$b^2x^2+a^2y^2-2b^2hx-2a^2ky+b^2h^2+a^2k^2-a^2b^2=0$$

$A=b^2,\ B=0,\ C=a^2$

$B^2-4AC=0^2-4b^2a^2=4a^2b^2<0$

$B^2-4AC<0$ for an ellipse whose major axis is parallel to the $x$-axis.

Ellipse with major axis parallel to $y$-axis:

$$\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1$$

$$b^2(y-k)^2+a^2(x-h)^2=a^2b^2$$
$$b^2(y^2-2ky+k^2)+a^2(x^2-2hx+h^2)=a^2b^2$$
$$b^2y^2-2b^2ky+b^2k^2+a^2x^2-2a^2hx+a^2h^2=a^2b^2$$
$$b^2y^2+a^2x^2-2b^2ky-2a^2hx+b^2k^2+a^2h^2-a^2b^2=0$$

$A=b^2,\ B=0,\ C=a^2$

$B^2-4AC=0^2-4b^2a^2=-4a^2b^2<0$

$B^2-4AC<0$ for an ellipse whose major axis is parallel to the $y$-axis.

Parabola with axis of symmetry parallel to $y$-axis:

$$\left(x-h\right)^2=4p\left(y-k\right)$$
$$x^2-2hx+h^2=4py-4pk$$

$x^2-2hx-4py+h^2+4pk=0$

$A=1,\ B=0,\ C=0$

$B^2-4AC=0^2-4(1)(0)=0$

$B^2-4AC=0$ for a parabola with axis of symmetry parallel to the $y$-axis.

Parabola with axis of symmetry parallel to $x$-axis:

$$\left(y-k\right)^2=4p\left(x-h\right)$$
$$y^2-2ky+k^2=4px-4ph$$

$y^2-2ky-4px+k^2+4ph=0$

$A=1,\ B=0,\ C=0$

$B^2-4AC=0^2-4\left(1\right)\left(0\right)=0$

$B^2-4AC=0$ for a parabola with axis of symmetry parallel to the $x$-axis.

Hyperbola with the transverse axis parallel to $x$-axis:

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

$$b^2(x-h)^2-a^2(y-k)^2=a^2b^2$$
$$b^2(x^2-2hx+h^2)-a^2(y^2-2ky+k^2)=a^2b^2$$
$$b^2x^2-2b^2hx+b^2h^2-a^2y^2+2a^2ky-a^2k^2=a^2b^2$$
$$b^2x^2-a^2y^2-2b^2hx+2a^2ky+b^2h^2-a^2k^2-a^2b^2=0$$

$A=b^2,\ B=0,\ C=-a^2$

$B^2-4AC=0^2-4b^2(-a^2)=4a^2b^2>0$

$B^2-4AC>0$ for a hyperbola whose transverse axis is parallel to the $x$-axis.

Hyperbola with transverse axis parallel to $y$-axis:

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

$$b^2(y-k)^2-a^2(x-h)^2=a^2b^2$$
$$b^2(y^2-2ky+k^2)-a^2(x^2-2hx+h^2)=a^2b^2$$
$$b^2y^2-2b^2ky+b^2k^2-a^2x^2+2a^2hx-a^2h^2=a^2b^2$$
$$b^2y^2-a^2x^2-2b^2ky+2a^2hx+b^2k^2-a^2h^2-a^2b^2=0$$

$A=-a^2$, $B=0$, $C=b^2$

$B^2-4AC=0^2-4b^2(-a^2)=4a^2b^2>0$

$B^2-4AC>0$ for a hyperbola whose transverse axis is parallel to the $y$-axis.

**48.** $x=x'\cos\alpha-y'\sin\alpha$

$y=y'\cos\alpha+x'\sin\alpha$

If this represents a system where the $xy$-axes have been rotated through $\alpha$ to create $x'y'$ axes, then we can rotate the new system backward through $\alpha$ (that is, the angle of rotation is $-\alpha$ ) and create an $x''y''$ system that is consistent with the original $xy$-system.

Thus, we can use the original formulas with $x$ replaced by $x'$, $y$ replaced by $y'$, $x'$ replaced by $x''$, $y'$ replaced by $y''$, and $\alpha$ replaced by $-\alpha$ to rotate the $x'y'$-system backward through $\alpha$ to an $x''y''$-system.

$x'=x''\cos(-\alpha)-y''\sin(-\alpha)$

$y'=y''\cos(-\alpha)+x''\sin(-\alpha)$

Since $\cos(-\alpha)=\cos(\alpha)$ and $\sin(-\alpha)=-\sin(\alpha)$, we can say

$x'=x''\cos(\alpha)+y''\sin(\alpha)$

$y'=y''\cos(\alpha)-x''\sin(\alpha)$

But since the $x''y''$-system is coincident with the $xy$-system, $x''$ can be replaced by $x$ and $y''$ by $y$.
Hence

$x'=x\cos\alpha+y\sin\alpha$

$y'=y\cos\alpha-x\sin\alpha$

## Prepare for Section 8.5

**PS1.** $\sin(-x)=-\sin x$ odd function

**PS2.** $\cos(-x)=\cos x$ even function

**PS3.** $\tan\alpha=-\sqrt{3}$

$\alpha=\frac{2\pi}{3},\frac{5\pi}{3}$

**PS4.** $\sin\alpha=-\frac{\sqrt{3}}{2}$, $\alpha=240^\circ$ or $300^\circ$

$\cos\alpha=-\frac{1}{2}$ $\alpha=120^\circ$ or $240^\circ$

$\alpha=240^\circ$

**PS5.** $(r\cos\theta)^2+(r\sin\theta)^2=r^2\cos^2\theta+r^2\sin^2\theta$

$=r^2(\cos^2\theta+\sin^2\theta)$

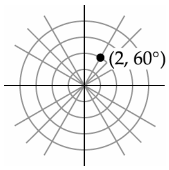
$=r^2$

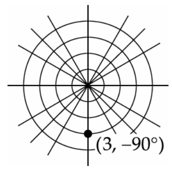
**PS6.** $\sin 32^\circ=\frac{y}{5}$ $y=5\sin 32^\circ\approx 2.6$

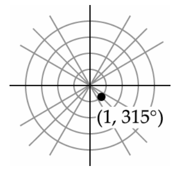
$\cos 32^\circ=\frac{x}{5}$ $x=5\cos 32^\circ\approx 4.2$

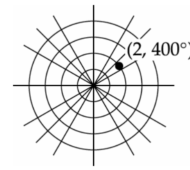
(4.2, 2.6)

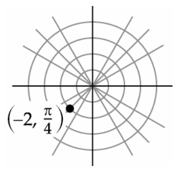
## Section 8.5

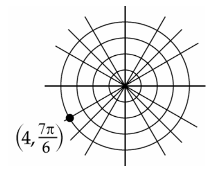
**1.**


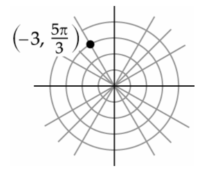

**2.**


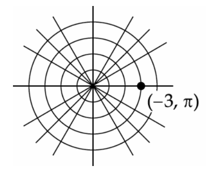

**3.**


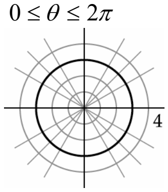

**4.**


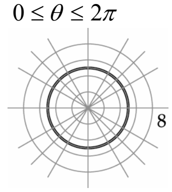

**5.**


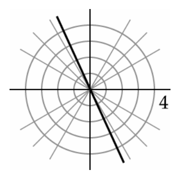

**6.**


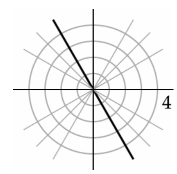

**7.**


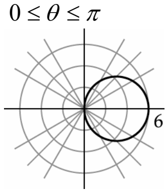

**8.**


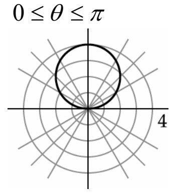

**9.** $0 \le \theta \le 2\pi$


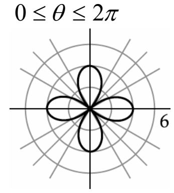

**10.** $0 \le \theta \le 2\pi$


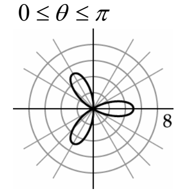

**11.**


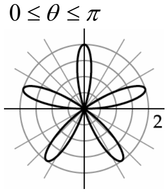

**12.**


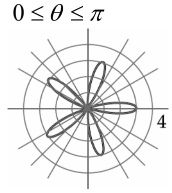

**13.** $0 \le \theta \le \pi$


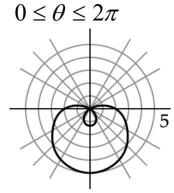

**14.** $0 \le \theta \le \pi$


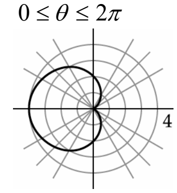

**15.** $0 \le \theta \le 2\pi$


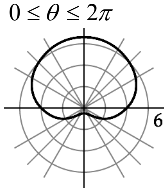

**16.** $0 \le \theta \le \pi$


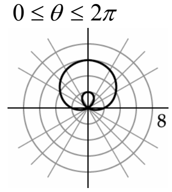

**17.** $0 \le \theta \le \pi$


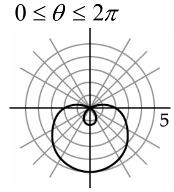

**18.** $0 \le \theta \le \pi$


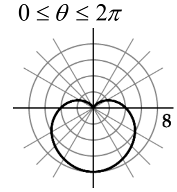

**19.** $0 \le \theta \le 2\pi$


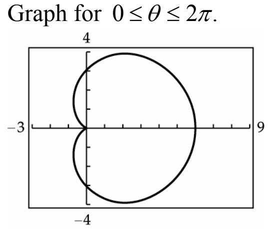

**20.** $0 \le \theta \le 2\pi$


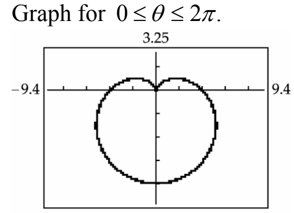

**21.** $0 \le \theta \le 2\pi$


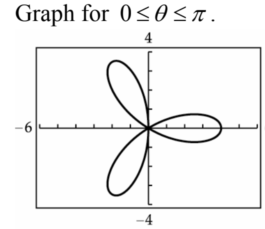

**22.** $0 \le \theta \le 2\pi$



**23.** $0 \le \theta \le 2\pi$


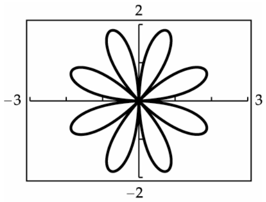

**24.** $0 \le \theta \le 2\pi$



**25.** Graph for $0 \le \theta \le 2\pi$.


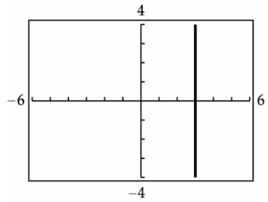

**26.** See Exercise 24.
Graph for $0 \le \theta \le 2\pi$.



**27.** Graph for $0 \le \theta \le \pi$.


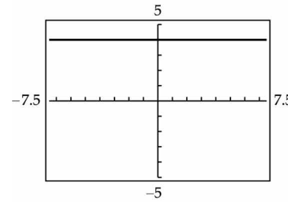
**28.** Graph for $0 \le \theta \le 2\pi$.



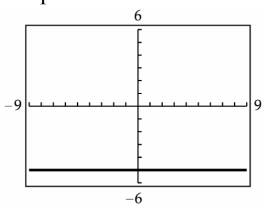

**29.** Graph for $0 \le \theta \le \pi$.



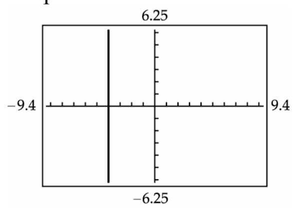

**30.** Graph for $0 \le \theta \le \pi$.



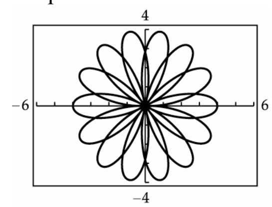

**31.** Graph for $0 \le \theta \le \pi$.



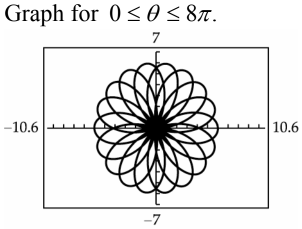

**32.** Graph for $0 \le \theta \le \pi$.



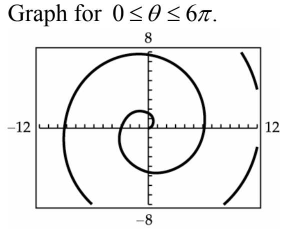

**33.** Graph for $0 \le \theta \le 4\pi$.



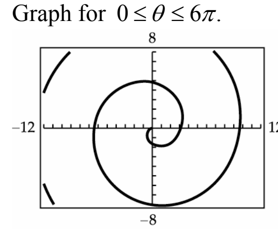

**34.** Graph for $0 \le \theta \le 8\pi$.



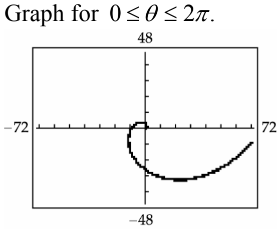

**35.** Graph for $0 \le \theta \le 6\pi$.



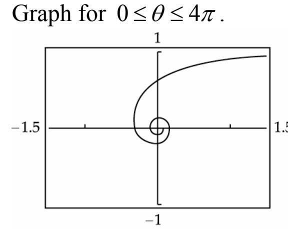

**36.** Graph for $0 \le \theta \le 6\pi$.




**37.** Graph for $0 \le \theta \le 2\pi$.

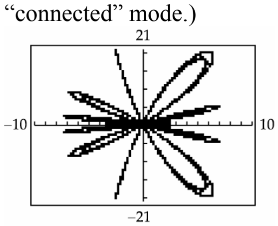



**38.** Graph for $0 \le \theta \le 4\pi$.

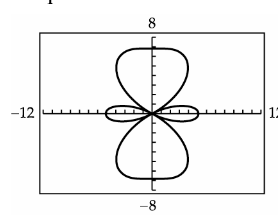



**39.** Graph for $0 \le \theta \le 2\pi$ with $\theta_{\text{step}} = \pi/200$.

(Some graphing utilities may draw a false asymptote in "connected" mode.)




**40.** Graph for $0 \le \theta \le 2\pi$ with $\theta_{\text{step}} = \pi/200$.




**41.**

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{1^2 + (-\sqrt{3})^2} \\ &= \sqrt{1+3} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1}\frac{y}{x} \\ &= \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) \\ &= \tan^{-1}\left(-\frac{\sqrt{3}}{1}\right) \\ &= -60^\circ \end{aligned}$$

The polar coordinates of the point are $(2, -60^\circ)$.

**42.**

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-2\sqrt{3})^2 + (2)^2} \\ &= \sqrt{12+4} \\ &= \sqrt{16} \\ &= 4 \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1}\frac{y}{x} \\ &= \tan^{-1}\left(\frac{2}{-2\sqrt{3}}\right) \\ &= \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) \\ &= 150^\circ \end{aligned}$$

The polar coordinates of the point are $(4, 150^\circ)$.

**43.**
$$\begin{aligned} x &= r\cos\theta \\ &= (-3)\left(\cos\frac{2\pi}{3}\right) \\ &= (-3)\left(-\frac{1}{2}\right) \\ &= \frac{3}{2} \end{aligned} \qquad \begin{aligned} y &= r\sin\theta \\ &= (-3)\left(\sin\frac{2\pi}{3}\right) \\ &= (-3)\left(\frac{\sqrt{3}}{2}\right) \\ &= -\frac{3\sqrt{3}}{2} \end{aligned}$$

The rectangular coordinates of the point are $\left(\frac{3}{2}, -\frac{3\sqrt{3}}{2}\right)$.

**44.**
$$\begin{aligned} x &= r\cos\theta \\ &= (2)\left[\cos\left(-\frac{\pi}{3}\right)\right] \\ &= (2)\left(\frac{1}{2}\right) \\ &= 1 \end{aligned} \qquad \begin{aligned} y &= r\sin\theta \\ &= (2)\left[\sin\left(-\frac{\pi}{3}\right)\right] \\ &= (2)\left(-\frac{\sqrt{3}}{2}\right) \\ &= -\sqrt{3} \end{aligned}$$

The rectangular coordinates of the point are $\left(1, -\sqrt{3}\right)$.

**45.**
$$\begin{aligned} x &= r\cos\theta \\ &= 0\cos\left(-\frac{\pi}{2}\right) \\ &= 0 \end{aligned} \qquad \begin{aligned} y &= r\sin\theta \\ &= 0\sin\left(-\frac{\pi}{2}\right) \\ &= 0 \end{aligned}$$

The rectangular coordinates of the point are (0, 0)

**46.**
$$\begin{aligned} x &= r\cos\theta \\ &= (3)\left(\cos\frac{5\pi}{6}\right) \\ &= (3)\left(-\frac{\sqrt{3}}{2}\right) \\ &= -\frac{3\sqrt{3}}{2} \end{aligned} \qquad \begin{aligned} y &= r\sin\theta \\ &= (3)\left(\sin\frac{5\pi}{6}\right) \\ &= (3)\left(\frac{1}{2}\right) \\ &= \frac{3}{2} \end{aligned}$$

The rectangular coordinates of the point are $\left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$.

**47.**
$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ &= \sqrt{(3)^2+(4)^2} \\ &= \sqrt{9+16} \\ &= \sqrt{25} \\ &= 5 \end{aligned} \qquad \begin{aligned} \theta &= \tan^{-1}\frac{y}{x} \\ &= \tan^{-1}\frac{4}{3} \\ &\approx 53.1^\circ \end{aligned}$$

The approximate polar coordinates of the point are (5, 53.1°).

**48.**
$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ &= \sqrt{(12)^2+(-5)^2} \\ &= \sqrt{144+25} \\ &= \sqrt{169} \\ &= 13 \end{aligned} \qquad \begin{aligned} \theta &= \tan^{-1}\frac{y}{x} \\ &= \tan^{-1}\left(\frac{-5}{12}\right) \\ &= \tan^{-1}\left(-\frac{5}{12}\right) \\ &\approx 337.4^\circ \end{aligned}$$

The approximate polar coordinates of the point are (13, 337.4°).

**49.**
$$\begin{aligned} r &= 3\cos\theta \\ r - 3\cos\theta &= 0 \\ r^2 - 3r\cos\theta &= 0 \\ x^2 + y^2 - 3x &= 0 \end{aligned}$$

**50.**
$$\begin{aligned} r &= 2\sin\theta \\ r - 2\sin\theta &= 0 \\ r^2 - 2r\sin\theta &= 0 \\ x^2 + y^2 - 2y &= 0 \end{aligned}$$

**51.**
$$\begin{aligned} r &= 3\sec\theta \\ r &= \frac{3}{\cos\theta} \\ r\cos\theta &= 3 \\ x &= 3 \end{aligned}$$

**52.**
$$\begin{aligned} r &= 4\csc\theta \\ r &= \frac{4}{\sin\theta} \\ r\sin\theta &= 4 \\ y &= 4 \end{aligned}$$

**53.**
$$\begin{aligned} r &= 4 \\ \sqrt{x^2+y^2} &= 4 \\ x^2+y^2 &= 16 \end{aligned}$$

**54.**
$$\begin{aligned} \theta &= \frac{\pi}{4} \\ \tan\theta &= \frac{y}{x} \\ \tan\frac{\pi}{4} &= \frac{y}{x} \\ 1 &= \frac{y}{x} \\ y &= x \end{aligned}$$

**55.**
$$\theta = \frac{\pi}{6}$$
$$\tan\theta = \frac{y}{x}$$
$$\tan\frac{\pi}{6} = \frac{y}{x}$$
$$\frac{\sqrt{3}}{3} = \frac{y}{x}$$
$$y = \frac{\sqrt{3}}{3}x$$

**56.**
$$r\cos\theta = -4$$
$$x = -4$$

**57.**
$$r = \tan\theta$$
$$r = \frac{\sin\theta}{\cos\theta}$$
$$r\cos\theta = \sin\theta$$
$$r\cos\theta - \sin\theta = 0$$
$$r^2\cos\theta - r\sin\theta = 0$$
$$\sqrt{x^2+y^2}(x) - y = 0$$
$$\sqrt{x^2+y^2} = \frac{y}{x}$$
$$x^2+y^2 = \frac{y^2}{x^2}$$
$$x^4 - y^2 + x^2y^2 = 0$$

**58.**
$$r = \cot\theta$$
$$r = \frac{\cos\theta}{\sin\theta}$$
$$r\sin\theta = \cos\theta$$
$$r\sin\theta - \cos\theta = 0$$
$$r^2\sin\theta - r\cos\theta = 0$$
$$\sqrt{x^2+y^2}(y) - x = 0$$
$$\sqrt{x^2+y^2} = \frac{x}{y}$$
$$x^2+y^2 = \frac{x^2}{y^2}$$
$$y^4 - x^2 + x^2y^2 = 0$$

**59.**
$$r = \frac{2}{1+\cos\theta}$$
$$r + r\cos\theta = 2$$
$$\sqrt{x^2+y^2} + x = 2$$
$$\sqrt{x^2+y^2} = 2 - x$$
$$x^2+y^2 = 4 - 4x + x^2$$
$$y^2 + 4x - 4 = 0$$

**60.**
$$r = \frac{2}{1-\sin\theta}$$
$$r - r\sin\theta = 2$$
$$\sqrt{x^2+y^2} - y = 2$$
$$\sqrt{x^2+y^2} = 2 + y$$
$$x^2+y^2 = 4 + 4y + y^2$$
$$x^2 - 4y - 4 = 0$$
$$x^2 = 4y + 4$$

**61.**
$$r(\sin\theta - 2\cos\theta) = 6$$
$$r\sin\theta - 2r\cos\theta = 6$$
$$y - 2x = 6$$
$$y = 2x + 6$$

**62.**
$$r(2\cos\theta + \sin\theta) = 3$$
$$2r\cos\theta + r\sin\theta = 3$$
$$2x + y = 3$$
$$y = -2x + 3$$

**63.**
$$y = 2$$
$$r\sin\theta = 2$$
$$r = 2\csc\theta$$

**64.**
$$x = -4$$
$$r\cos\theta = -4$$
$$r = -4\sec\theta$$

**65.**
$$y = \sqrt{3}x$$
$$r\sin\theta = \sqrt{3}(r\cos\theta)$$
$$\tan\theta = \sqrt{3}$$
$$\theta = \frac{\pi}{3}$$

**66.**
$$y = x^2$$
$$\frac{y}{x} = x$$
$$\tan\theta = r\cos\theta$$
$$\tan\theta\sec\theta = r$$

**67.**
$$x = 3$$
$$r\cos\theta = 3$$
$$r = 3\sec\theta$$

**68.**
$$xy = 4$$
$$(r\cos\theta)(r\sin\theta) = 4$$
$$r^2\sin\theta\cos\theta = 4$$

**69.**
$$x^2 + y^2 = 4$$
$$r^2 = 4$$
$$r = 2$$

**70.**
$$2x - 3y = 6$$
$$2r\cos\theta - 3r\sin\theta = 6$$
$$r(2\cos\theta - 3\sin\theta) = 6$$
$$r = \frac{6}{2\cos\theta - 3\sin\theta}$$

**71.**
$$x^2 = 8y$$
$$r^2\cos^2\theta = 8r\sin\theta$$
$$r\cos^2\theta = 8\sin\theta$$

**72.**

$$y^2 = 4y$$
$$r^2\sin^2\theta = 4r\sin\theta$$
$$r^2\sin^2\theta - 4r\sin\theta = 0$$
$$r\sin\theta(r\sin\theta - 4) = 0$$

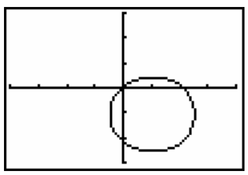
**73.**

$$x^2 - y^2 = 25$$
$$r^2\cos^2\theta - r^2\sin^2\theta = 25$$
$$r^2(\cos^2\theta - \sin^2\theta = 25$$
$$r^2(\cos 2\theta) = 25$$

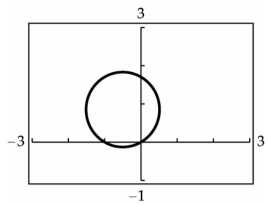
**74.**

$$x^2 + 4y^2 = 16$$
$$r^2\cos^2\theta + 4r^2\sin^2\theta = 16$$
$$r^2(\cos^2\theta + 4\sin^2\theta) = 16$$
$$r^2(\cos^2\theta + \sin^2\theta + 3\sin^2\theta = 16$$
$$r^2(1 + 3\sin^2\theta) = 16$$
$$r^2 + 3r^2\sin^2\theta = 16$$
$$3r^2\sin^2\theta + r^2 = 16$$

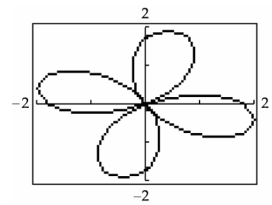
**75.**



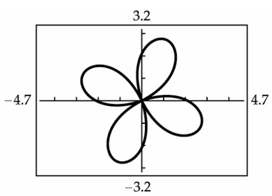
**76.**



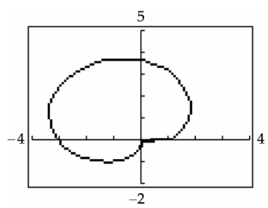

**77.**



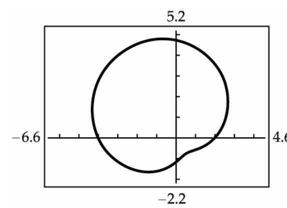

**78.**



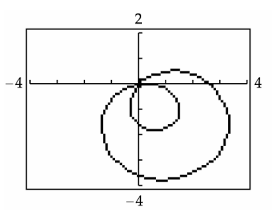

**79.**



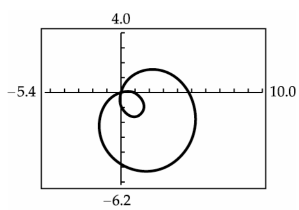

**80.**




**81.**




**82.**

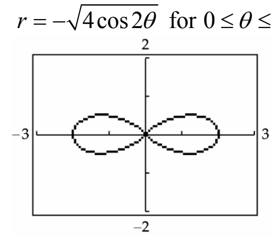



## Connecting Concepts

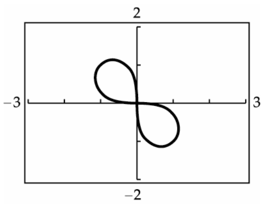
**83.** $\cos\theta = \pm\sqrt{\cos^2\theta}$ is *not* an identity.

**84.** $\cos 2\theta = 2\cos^2\theta - 1$ is an identity.

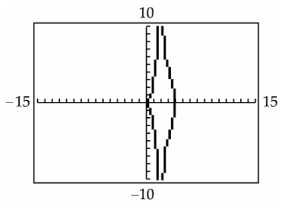
**85.** Enter as $r = \sqrt{4\cos 2\theta}$ and $r = -\sqrt{4\cos 2\theta}$ for $0 \le \theta \le 4\pi$.



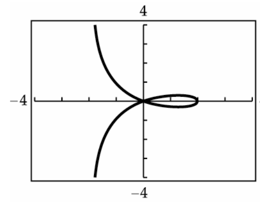
**86.** Enter as $r = \sqrt{-2\sin 2\theta}$ and $r = -\sqrt{-2\sin 2\theta}$ for $0 \le \theta \le \pi$.



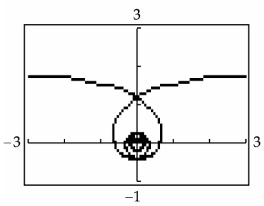

**87.** Graph for $0 \le \theta \le 2\pi$ with $\theta_{\text{step}} = \pi/200$.
(Some graphing utilities may produce a false asymptote in "connected" mode.)



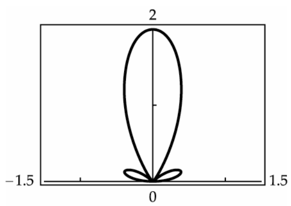

**88.** Graph for $\pi/2 \le \theta \le 3\pi/2$.
(Some graphing utilities may produce a false asymptote in "connected" mode.)



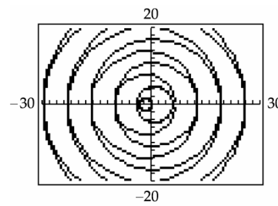

**89.** Graph $r = 2/\theta$ for $-4\pi < \theta < 4\pi$.



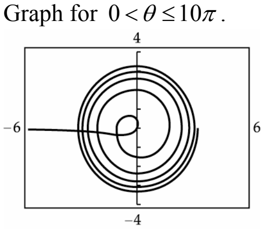

**90.** Graph for $0 \le \theta \le 2\pi$.



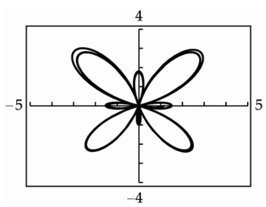

**91.** Graph for $-30 \le \theta \le 30$.



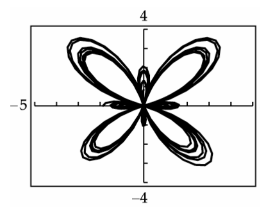

**92.** Graph for $0 < \theta \le 10\pi$.




**93. a.** $0 \le \theta \le 5\pi$




**b.** $0 \le \theta \le 20\pi$




## Prepare for Section 8.6

**PS1.** $\frac{x^2}{25} + \frac{y^2}{16} = 1$
$a^2 = 25,\ a = 5$
$b^2 = 16$
$c^2 = a^2 - b^2 = 25 - 16 = 9$
$c = 3$
$e = \frac{c}{a} = \frac{3}{5}$
The eccentricity is $\frac{3}{5}$.

**PS2.** $y^2 = 4x$
$4p = 4$
$p = 1$
directrix: $x = -1$

**PS3.**
$$\begin{aligned} y &= 2(1 + yx) \\ y &= 2 + 2yx \\ y - 2yx &= 2 \\ y(1 - 2x) &= 2 \\ y &= \frac{2}{1 - 2x} \end{aligned}$$

**PS4.** $1 + \sin x = 0$
$\sin x = -1$
$x = \frac{3\pi}{2}$

**PS5.** For a hyperbola, $e > 1$.

**PS6.**
$$\frac{4\sec x}{2\sec x - 1} = \frac{4\frac{1}{\cos x}}{2\frac{1}{\cos x} - 1} = \frac{\cos x}{\cos x} \cdot \frac{4\frac{1}{\cos x}}{2\frac{1}{\cos x} - 1} = \frac{4}{2 - \cos x}$$

## Section 8.6

**1.** $r = \dfrac{12}{3-6\cos\theta} = \dfrac{4}{1-2\cos\theta}$

$e = 2$ The graph is a hyperbola.
The transverse axis is on the polar axis.

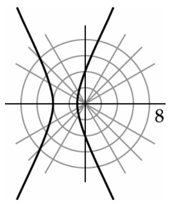
Let $\theta = 0$.

$r = \dfrac{12}{3-6\cos 0} = \dfrac{12}{3-6} = -4$

Let $\theta = \pi$.

$r = \dfrac{12}{3-6\cos\pi} = \dfrac{12}{3+6} = \dfrac{4}{3}$

The vertices are at $(-4, 0)$ and $\left(\frac{4}{3}, \pi\right)$.




**2.** $r = \dfrac{8}{2-4\cos\theta} = \dfrac{4}{1-2\cos\theta}$

$e = 2$ The graph is a hyperbola.
The transverse axis is on the polar axis.

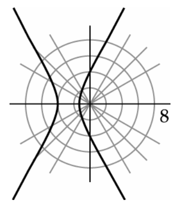
Let $\theta = 0$.

$r = \dfrac{8}{2-4\cos 0} = \dfrac{8}{2-4} = -4$

Let $\theta = \pi$.

$r = \dfrac{8}{2-4\cos\pi} = \dfrac{8}{2+4} = \dfrac{4}{3}$

The vertices are at $(-4, 0)$ and $\left(\frac{4}{3}, \pi\right)$.




**3.** $r = \dfrac{8}{4+3\sin\theta} = \dfrac{2}{1+\frac{3}{4}\sin\theta}$

$e = \frac{3}{4}$ The graph is an ellipse.

The major axis is on the line $\theta = \frac{\pi}{2}$.

Let $\theta = \dfrac{\pi}{2}$.

$r = \dfrac{8}{4+3\sin\frac{\pi}{2}} = \dfrac{8}{4+3} = \dfrac{8}{7}$

Let $\theta = \dfrac{3\pi}{2}$.

$r = \dfrac{8}{4+3\sin\frac{3\pi}{2}} = \dfrac{8}{4-3} = 8$

Vertices on major axis are at $\left(\frac{8}{7}, \frac{\pi}{2}\right)$ and $\left(8, \frac{3\pi}{2}\right)$.

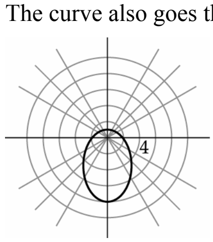
Let $\theta = 0$.

$r = \dfrac{8}{4+3\sin 0} = \dfrac{8}{4+0} = 2$

Let $\theta = \pi$.

$r = \dfrac{8}{4+3\sin\pi} = \dfrac{8}{4+0} = 2$

The curve also goes through $(2, 0)$ and $(2, \pi)$.




**4.** $r = \dfrac{6}{3+2\cos\theta} = \dfrac{2}{1+\frac{2}{3}\cos\theta}$

$e = \frac{2}{3}$ The graph is an ellipse.

The major axis is on the polar axis.

Let $\theta = 0$.

$r = \dfrac{6}{3+2\cos 0} = \dfrac{6}{3+2} = \dfrac{6}{5}$

Let $\theta = \pi$.

$r = \dfrac{6}{3+2\cos\pi} = \dfrac{6}{3-2} = 6$

Vertices on major axis are at $\left(\frac{6}{5}, 0\right)$ and $(6, \pi)$.

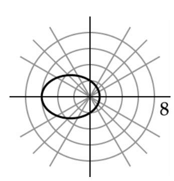
Let $\theta = \frac{\pi}{2}$.

$r = \dfrac{6}{3+2\cos\frac{\pi}{2}} = \dfrac{6}{3+0} = 2$

Let $\theta = \dfrac{3\pi}{2}$.

$r = \dfrac{6}{3+2\cos\frac{3\pi}{2}} = \dfrac{6}{3+0} = 2$

The curve also goes through $\left(2, \frac{\pi}{2}\right)$ and $\left(2, \frac{3\pi}{2}\right)$.



**5.** $r = \dfrac{9}{3 - 3\sin\theta} = \dfrac{3}{1 - \sin\theta}$

$e = 1$ The graph is a parabola.

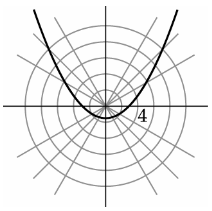
The axis of symmetry is $\theta = \frac{\pi}{2}$.

When $\theta = \frac{\pi}{2}$, $r$ is undefined.

Let $\theta = \dfrac{3\pi}{2}$.

$$r = \frac{9}{3 - 3\sin\frac{3\pi}{2}} = \frac{9}{3+3} = \frac{3}{2}$$

Vertex is at $\left(\frac{3}{2}, \frac{3\pi}{2}\right)$.




**6.** $r = \dfrac{5}{2 - 2\sin\theta} = \dfrac{\frac{5}{2}}{1 - \sin\theta}$

$e = 1$ The graph is a parabola.

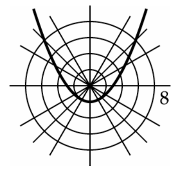
The axis of symmetry is $\theta = \frac{\pi}{2}$.

When $\theta = \frac{\pi}{2}$, $r$ is undefined.

Let $\theta = \dfrac{3\pi}{2}$.

$$r = \frac{5}{2 - 2\sin\frac{3\pi}{2}} = \frac{5}{2+2} = \frac{5}{4}$$

Vertex is at $\left(\frac{5}{4}, \frac{3\pi}{2}\right)$.




**7.** $r = \dfrac{10}{5 + 6\cos\theta} = \dfrac{2}{1 + \frac{6}{5}\cos\theta}$

$e = \frac{6}{5}$ The graph is a hyperbola.

The transverse axis is on the polar axis.

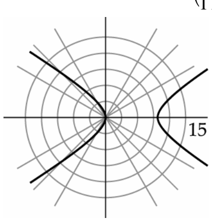
Let $\theta = 0$.

$$r = \frac{10}{5 + 6\cos 0} = \frac{10}{5+6} = \frac{10}{11}$$

Let $\theta = \pi$.

$$r = \frac{10}{5 + 6\cos\pi} = \frac{10}{5-6} = -10$$

The vertices are at $\left(\frac{10}{11}, 0\right)$ and $(-10, \pi)$.




**8.** $r = \dfrac{8}{2 + 4\cos\theta} = \dfrac{4}{1 + 2\cos\theta}$

$e = 2$ The graph is a hyperbola.

The transverse axis is on the polar axis.

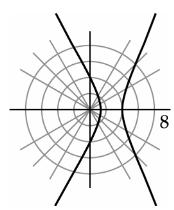
Let $\theta = 0$.

$$r = \frac{8}{2 + 4\cos 0} = \frac{8}{2+4} = \frac{4}{3}$$

Let $\theta = \pi$.

$$r = \frac{8}{2 + 4\cos\pi} = \frac{8}{2-4} = -4$$

The vertices are at $\left(\frac{4}{3}, 0\right)$ and $(-4, \pi)$.



**9.** $r=\dfrac{4\sec\theta}{2\sec\theta-1}$

$=\dfrac{\frac{4}{\cos\theta}}{\frac{2}{\cos\theta}-1}=\dfrac{4}{2-\cos\theta}$

$=\dfrac{2}{1-\frac{1}{2}\cos\theta}$

$e=\frac{1}{2}$ The graph is an ellipse.
The major axis is on the polar axis.

However, the original equation is undefined at $\frac{\pi}{2}$ and at $\frac{3\pi}{2}$.

Thus, the ellipse will have holes at those angles.

Let $\theta=0$.

$r=\dfrac{4}{2-\cos 0}=\dfrac{4}{2-1}=4$

Let $\theta=\pi$.

$r=\dfrac{4}{2-\cos\pi}=\dfrac{4}{2+1}=\dfrac{4}{3}$

Vertices on major axis are at (4, 0) and $\left(\frac{4}{3},\pi\right)$.

$\theta=\dfrac{\pi}{2}$.

$r=\dfrac{4}{2-\cos\frac{\pi}{2}}=\dfrac{4}{2-0}=2$

Let $\theta=\dfrac{3\pi}{2}$.

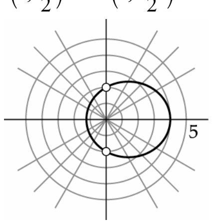
$r=\dfrac{4}{2-\cos\frac{3\pi}{2}}=\dfrac{4}{2-0}=2$

Vertices on minor axis of $\frac{2}{1-\frac{1}{2}\cos\theta}$ are at

$\left(2,\frac{\pi}{2}\right)$ and $\left(2,\frac{3\pi}{2}\right)$.

Thus, the equation $r=\frac{4\sec\theta}{2\sec\theta-1}$ will have holes at

$\left(2,\frac{\pi}{2}\right)$ and $\left(2,\frac{3\pi}{2}\right)$




**10.** $r=\dfrac{3\sec\theta}{2\sec\theta+2}$

$=\dfrac{\frac{3}{\cos\theta}}{\frac{2}{\cos\theta}+2}=\dfrac{3}{2+2\cos\theta}$

$e=1$ The graph is a parabola.
The axis of symmetry is the polar axis.

Let $\theta=0$. Vertex of $\dfrac{\frac{3}{2}}{1+\cos\theta}$ is at $\left(\dfrac{3}{4},0\right)$.

$r=\dfrac{3}{2+2\cos 0}=\dfrac{3}{4}$

Let $\theta=\dfrac{\pi}{2}$.

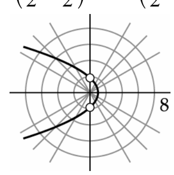
$r=\dfrac{3}{2+2\cos\frac{\pi}{2}}=\dfrac{3}{2+0}=\dfrac{3}{2}$

Let $\theta=\dfrac{3\pi}{2}$.

$r=\dfrac{3}{2+2\cos\frac{3\pi}{2}}=\dfrac{3}{2}$

Thus, the equation $r=\frac{3\sec\theta}{2\sec\theta+1}$ has holes at

$\left(\frac{3}{2},\frac{\pi}{2}\right)$ and $\left(\frac{3}{2},\frac{3\pi}{2}\right)$.



**11.** $r = \dfrac{12\csc\theta}{6\csc\theta - 2}$

$= \dfrac{\frac{12}{\sin\theta}}{\frac{6}{\sin\theta} - 2} = \dfrac{12}{6 - 2\sin\theta}$

$= \dfrac{2}{1 - \frac{1}{3}\sin\theta}$

$e = \frac{1}{3}$ The graph is an ellipse.

The major axis is on $\theta = \frac{\pi}{2}$.

Let $\theta = \dfrac{\pi}{2}$.

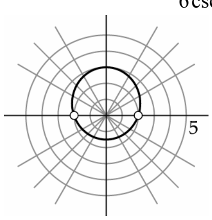
$r = \dfrac{12}{6 - 2\sin\frac{\pi}{2}} = \dfrac{12}{6-2} = 3$

Let $\theta = \frac{3\pi}{2}$.

$r = \dfrac{12}{6 - 2\sin\frac{3\pi}{2}} = \dfrac{12}{6+2} = \dfrac{3}{2}$

Vertices on major axis are at $\left(3, \frac{\pi}{2}\right)$ and $\left(\frac{3}{2}, \frac{3\pi}{2}\right)$.

The equation $r = \frac{12\csc\theta}{6\csc\theta - 2}$ has holes at (2, 0) and $(2, \pi)$.




**12.** $r = \dfrac{3\csc\theta}{2\csc\theta + 2}$

$= \dfrac{\frac{3}{\sin\theta}}{\frac{2}{\sin\theta} + 2} = \dfrac{3}{2 + 2\sin\theta}$

$= \dfrac{\frac{3}{2}}{1 + \sin\theta}$

$e = 1$ The graph is a parabola.

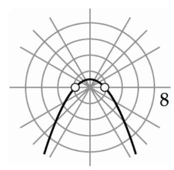
The axis of symmetry is $\theta = \frac{\pi}{2}$.

Let $\theta = \dfrac{\pi}{2}$.

$r = \dfrac{3}{2 + 2\sin\frac{\pi}{2}} = \dfrac{3}{2+2} = \dfrac{3}{4}$

Vertex is at $\left(\frac{3}{4}, \frac{\pi}{2}\right)$.

The parabola has holes at $\left(\frac{3}{2}, 0\right)$ and $\left(\frac{3}{2}, \pi\right)$.



**13.** $r = \dfrac{3}{\cos\theta - 1}$

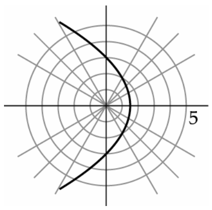
$= \dfrac{-3}{1-\cos\theta}$

$e = 1$ The graph is a parabola.
The axis of symmetry is the polar axis.

Let $\theta = \pi$.

$r = \dfrac{-3}{1-\cos\pi} = \dfrac{-3}{1-(-1)} = \dfrac{-3}{1+1} = -\dfrac{3}{2}$

Vertex is at $\left(-\frac{3}{2}, \pi\right)$.




**14.** $r = \dfrac{2}{\sin\theta + 2}$

$= \dfrac{1}{1+\frac{1}{2}\sin\theta}$

$e = \frac{1}{2}$ The graph is an ellipse.

The major axis is on $\theta = \frac{\pi}{2}$.

Let $\theta = \frac{\pi}{2}$.

$r = \dfrac{2}{\sin\frac{\pi}{2}+2} = \dfrac{2}{1+2} = \dfrac{2}{3}$

Let $\theta = \frac{3\pi}{2}$.

$r = \dfrac{2}{\sin\frac{3\pi}{2}+2} = \dfrac{2}{-1+2} = 2$

Vertices of major axis are at $\left(\frac{2}{3}, \frac{\pi}{2}\right)$ and $\left(2, \frac{3\pi}{2}\right)$.

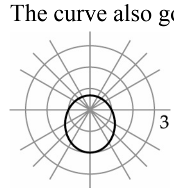
Let $\theta = 0$.

$r = \dfrac{2}{\sin\theta+2} = \dfrac{2}{0+2} = 1$

Let $\theta = \pi$.

$r = \dfrac{2}{\sin\pi+2} = \dfrac{2}{0+2} = 1$

The curve also goes through $(1, 0)$ and $(1, \pi)$.




**15.**
$$
\begin{aligned}
r &= \frac{12}{3-6\cos\theta} \\
r(3-6\cos\theta) &= 12 \\
3r - 6r\cos\theta &= 12 \\
3\sqrt{x^2+y^2} - 6x &= 12 \\
3\sqrt{x^2+y^2} &= 6x+12 \\
\sqrt{x^2+y^2} &= 2x+4 \\
x^2+y^2 &= 4x^2+16x+16 \\
3x^2 - y^2 + 16x + 16 &= 0
\end{aligned}
$$

**16.**
$$
\begin{aligned}
r &= \frac{8}{2-4\cos\theta} \\
r(2-4\cos\theta) &= 8 \\
2r - 4r\cos\theta &= 8 \\
2\sqrt{x^2+y^2} - 4x &= 8 \\
2\sqrt{x^2+y^2} &= 4x+8 \\
\sqrt{x^2+y^2} &= 2x+4 \\
x^2+y^2 &= 4x^2+16x+16 \\
3x^2 - y^2 + 16x + 16 &= 0
\end{aligned}
$$

**17.**
$$
\begin{aligned}
r &= \frac{8}{4+3\sin\theta} \\
r(4+3\sin\theta) &= 8 \\
4r + 3r\sin\theta &= 8 \\
4\sqrt{x^2+y^2} + 3y &= 8 \\
4\sqrt{x^2+y^2} &= -3y+8 \\
16x^2+16y^2 &= 9y^2-48y+64 \\
16x^2 + 7y^2 + 48y - 64 &= 0
\end{aligned}
$$

**18.**
$$
\begin{aligned}
r &= \frac{6}{3+2\cos\theta} \\
r(3+2\cos\theta) &= 6 \\
3r + 2r\cos\theta &= 6 \\
3\sqrt{x^2+y^2} + 2x &= 6 \\
3\sqrt{x^2+y^2} &= -2x+6 \\
9x^2+9y^2 &= 4x^2-24x+36 \\
5x^2 + 9y^2 + 24x - 36 &= 0
\end{aligned}
$$

**19.** $r = \dfrac{9}{3 - 3\sin\theta}$

$r(3 - 3\sin\theta) = 9$

$3r - 3r\sin\theta = 9$

$3\sqrt{x^2 + y^2} - 3y = 9$

$3\sqrt{x^2 + y^2} = 3y + 9$

$3\sqrt{x^2 + y^2} = 3(y + 3)$

$\sqrt{x^2 + y^2} = y + 3$

$x^2 + y^2 = y^2 + 6y + 9$

$x^2 - 6y - 9 = 0$

**20.** $r = \dfrac{5}{2 - 2\sin\theta}$

$r(2 - 2\sin\theta) = 5$

$2r - 2r\sin\theta = 5$

$2\sqrt{x^2 + y^2} - 2y = 5$

$2\sqrt{x^2 + y^2} = 2y + 5$

$4x^2 + 4y^2 = 4y^2 + 20y + 25$

$4x^2 - 20y - 25 = 0$

**21.** $e = 2,\ r\cos\theta = -1,$

$d = |-1| = 1$

$r = \dfrac{ed}{1 - e\cos\theta}$

$= \dfrac{(2)(1)}{1 - (2)\cos\theta}$

$= \dfrac{2}{1 - 2\cos\theta}$

**22.** $e = \dfrac{3}{2},\ r\sin\theta = 1,\ d = |1| = 1$

$r = \dfrac{ed}{1 + e\sin\theta}$

$= \dfrac{\left(\frac{3}{2}\right)(1)}{1 + \left(\frac{3}{2}\right)\sin\theta}$

$= \dfrac{\frac{3}{2}}{1 + \frac{3}{2}\sin\theta}$

$= \dfrac{3}{2 + 3\sin\theta}$

**23.** $e = 1,\ r\sin\theta = 2,\ d = |2| = 2$

$r = \dfrac{ed}{1 + e\sin\theta}$

$= \dfrac{(1)(2)}{1 + (1)\sin\theta}$

$= \dfrac{2}{1 + \sin\theta}$

**24.** $e = 1,\ r\cos\theta = -2,$

$d = |-2| = 2$

$r = \dfrac{ed}{1 - e\cos\theta}$

$= \dfrac{(1)(2)}{1 - (1)\cos\theta}$

$= \dfrac{2}{1 - \cos\theta}$

**25.** $e = \dfrac{2}{3},\ r\sin\theta = -4,$

$d = |-4| = 4$

$r = \dfrac{ed}{1 - e\sin\theta}$

$= \dfrac{\left(\frac{2}{3}\right)(4)}{1 - \left(\frac{2}{3}\right)\sin\theta}$

$= \dfrac{\frac{8}{3}}{1 - \frac{2}{3}\sin\theta}$

$= \dfrac{8}{3 - 2\sin\theta}$

**26.** $e = \dfrac{1}{2},\ r\cos\theta = 2,\ d = |2| = 2$

$r = \dfrac{ed}{1 + e\cos\theta}$

$= \dfrac{\left(\frac{1}{2}\right)(2)}{1 + \left(\frac{1}{2}\right)\cos\theta}$

$= \dfrac{1}{1 + \frac{1}{2}\cos\theta}$

$= \dfrac{2}{2 + \sin\theta}$

**27.** $e=\frac{3}{2}, r=2\sec\theta$

$r=\frac{2}{\cos\theta}$

$r\cos\theta=2, \quad d=|2|=2$

$$r=\frac{ed}{1+e\cos\theta}=\frac{\left(\frac{3}{2}\right)(2)}{1+\left(\frac{3}{2}\right)\cos\theta}=\frac{3}{1+\frac{3}{2}\cos\theta}=\frac{6}{2+3\cos\theta}$$

**28.** $e=\frac{3}{4}, r=2\csc\theta$

$r=\frac{2}{\sin\theta}$

$r\sin\theta=2, \quad d=|2|=2$

$$r=\frac{ed}{1+e\sin\theta}=\frac{\left(\frac{3}{4}\right)(2)}{1+\left(\frac{3}{4}\right)\sin\theta}=\frac{\frac{3}{2}}{1+\frac{3}{4}\sin\theta}=\frac{6}{4+3\sin\theta}$$

**29.** vertex: $(2, \pi)$, curve: parabola

$r=\frac{ed}{1-e\cos\theta}$ $\quad e=1$ (by definition of a parabola)

When $\theta=\pi, r=2$. Substituting into

$r=\frac{ed}{1-e\cos\theta}$,

we have

$$2=\frac{1\cdot d}{1-1\cdot\cos(\pi)}=\frac{d}{2}$$

Therefore, $d=4$. Substituting $e=1$ and $d=4$ yields

$$r=\frac{(1)\ (4)}{1-(1)\cos\theta} \text{ or } r=\frac{4}{1-\cos\theta}$$

**30.** vertex: $(4, 0)$, $e=\frac{1}{2}$

$r=\frac{ed}{1+e\cos\theta}$

When $\theta=0, r=4$. Substituting into

$r=\frac{ed}{1+e\cos\theta}$,

we have

$$4=\frac{\frac{1}{2}d}{1+\frac{1}{2}\cdot\cos(0)}=\frac{\frac{1}{2}d}{\frac{3}{2}}=\frac{d}{3}$$

Therefore, $d=12$. Substituting $e=\frac{1}{2}$ and $d=12$ yields

$$r=\frac{\left(\frac{1}{2}\right)(12)}{1+\left(\frac{1}{2}\right)\cos\theta}=\frac{6}{1+\frac{1}{2}\cos\theta}=\frac{12}{2+\cos\theta}$$

**31.** vertex: $(1, 3\pi/2)$, $e=2$

$r=\frac{ed}{1-e\sin\theta}$

When $\theta=\frac{3\pi}{2}, r=1$. Substituting into

$r=\frac{ed}{1-e\sin\theta}$,

we have

$$1=\frac{2d}{1-2\sin\left(\frac{3\pi}{2}\right)}=\frac{2d}{3}$$

Therefore $d=\frac{3}{2}$. Substituting $e=2$ and $d=\frac{3}{2}$ yields

$$r=\frac{(2)\left(\frac{2}{3}\right)}{1-(2)\sin\theta}=\frac{3}{1-2\sin\theta}$$

**32.** vertex: $(2, 3\pi/2)$, $e=\frac{2}{3}$

$r=\frac{ed}{1-e\sin\theta}$

When $\theta=\frac{3\pi}{2}$, $r=2$. Substituting into

$r=\frac{ed}{1-e\sin\theta}$,

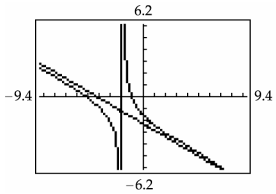
we have

$$2=\frac{(2/3)d}{1-(2/3)\sin(3\pi/2)}=\frac{2d/3}{5/3}$$

Therefore, $d=5$. Substituting $e=2/3$ and $d=5$ yields

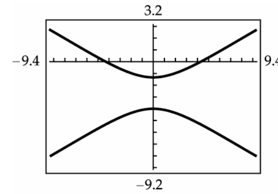
$$r=\frac{(2/3)\ (5)}{1-(2/3)\sin\theta}=\frac{(10/3)}{1-(2/3)\sin\theta}=\frac{10}{3-2\sin\theta}$$

**33.**



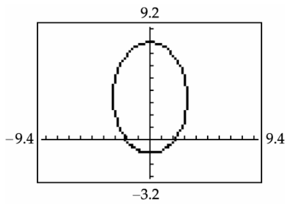

Rotate the graph of Exercise 1 $\frac{\pi}{6}$ radians counterclockwise about the pole.

**34.**



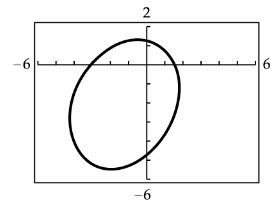

Rotate the graph of Exercise 2 $\frac{\pi}{2}$ radians counterclockwise about the pole.

**35.**



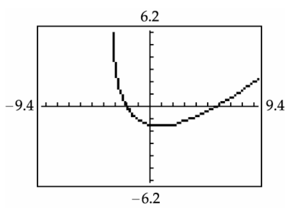

Rotate the graph of Exercise 3 $\pi$ radians counterclockwise about the pole.

**36.**



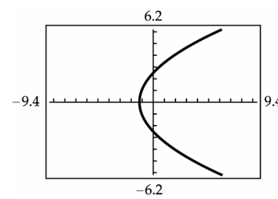

Rotate the graph of Exercise 4 $\frac{\pi}{3}$ radians counterclockwise about the pole.

**37.**



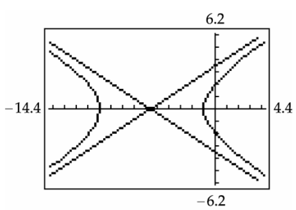

Rotate the graph of Exercise 5 $\frac{\pi}{6}$ radians clockwise about the pole.

**38.**



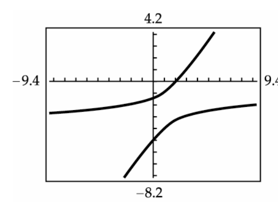

Rotate the graph of Exercise 6 $\frac{\pi}{2}$ radians clockwise about the pole.

**39.**




Rotate the graph of Exercise 7 $\pi$ radians clockwise about the pole.

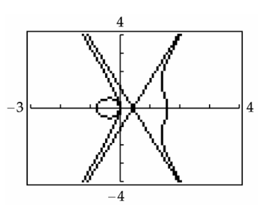
**40.**



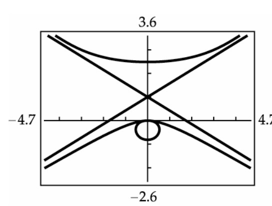

Rotate the graph of Exercise 8 $\frac{\pi}{3}$ radians clockwise about the pole.

## Connecting Concepts

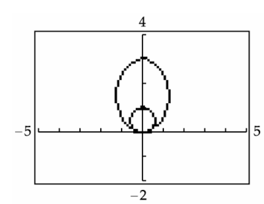
**41.**



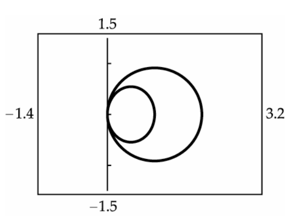

**42.**



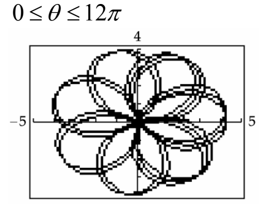

**43.**



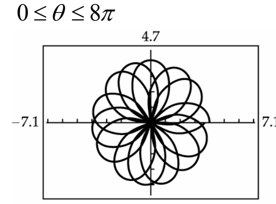

**44.**




**45.** $0 \le \theta \le 12\pi$




**46.** $0 \le \theta \le 8\pi$



**47.** Convert the equations for the conic and the directrix to rectangular form.

$$r = \frac{ed}{1-e\cos\theta} \qquad d = -r\cos\theta \text{ (by definition)}$$
$$r(1-e\cos\theta) = ed \qquad = -x$$
$$r - er\cos\theta = ed \qquad x = -d$$
$$\sqrt{x^2+y^2} - ex = ed$$
$$\sqrt{x^2+y^2} = ex + ed = e(x+d)$$
$$x^2+y^2 = e^2(x^2+2dx+d^2)$$
$$x^2+y^2-e^2x^2-2e^2dx-e^2d^2=0$$
$$(1-e^2)x^2+y^2-(2e^2d)x-(e^2d^2)=0$$

Solving for $y^2$ yields $y^2=(e^2-1)x^2+(2ed)x+(e^2d^2)$.

Now, with $y^2=(e^2-1)x^2+(2ed)x+(e^2d^2)$ and a directrix of $x=-d$, let $k=\dfrac{d(P,F)}{d(P,D)}$, where the focus is at the origin (by definition).

$$k=\frac{d(P,F)}{d(P,D)}=\frac{\sqrt{x^2+y^2}}{|x+d|}$$

We can substitute $y^2=(e^2-1)x^2+(2ed)x+(e^2d^2)$ to obtain

$$k=\frac{\sqrt{x^2+(e^2-1)x^2+(2ed)x+(e^2d^2)}}{|x+d|}=\frac{\sqrt{e^2x^2+2e^2dx+e^2d^2}}{|x+d|}$$

Solving for $k^2$ gives us $k^2=\dfrac{e^2x^2+2e^2dx+e^2d^2}{x^2+2dx+d^2}=\dfrac{e^2(x^2+2dx+d^2)}{x^2+2dx+d^2}=e^2$.

Since $k^2=e^2$, $k=\pm\sqrt{e^2}=\pm e$. But, since $k=\dfrac{d(P,F)}{d(P,D)}$, and the ratio of two distances must be positive, $k$ cannot be negative.

Therefore, $k=e$, or $\dfrac{d(P,F)}{d(P,D)}=e$.

**48.** Convert the polar equation of the directrix into the *xy*-coordinate system:

$$r\sin\theta = d$$
$$y = d$$

Since we have shown (in Exercise 47) that $\dfrac{d(P,F)}{d(P,D)}=e$ and since the focus is at the origin (or pole), we can say

$$\frac{PF}{PD}=e$$
$$\frac{\sqrt{x^2+y^2}}{|y-d|}=e$$
$$\frac{x^2+y^2}{(y-d)^2}=e^2$$
$$x^2+y^2=e^2(d-y)^2$$

Converting back into polar coordinates yields

$$r^2=e^2(d-r\sin\theta)^2$$
$$r=e(d-r\sin\theta)$$
$$r=ed-er\sin\theta$$
$$r+er\sin\theta=ed$$
$$r(1+e\sin\theta)=ed$$
$$r=\frac{ed}{1+e\sin\theta}$$

## Prepare for Section 8.7

**PS1.** $y^2 + 3y + \left(\frac{3}{2}\right)^2 = \left(y + \frac{3}{2}\right)^2$

**PS2.** $y = x^2 = (2t+1)^2 = 4t^2 + 4t + 1$

**PS3.** ellipse

**PS4.** $x^2 + y^2 = (\sin t)^2 + (\cos t)^2 = 1$

**PS5.** $y = \ln t$
$e^y = t$

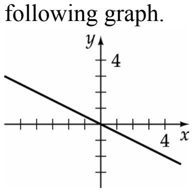
**PS6.** Domain: $(-\infty, \infty)$
Range: $[-3, 3]$

# Section 8.7

**1.** A table of five arbitrarily chosen values of $t$ and the corresponding values of $x$ and $y$ are shown in the table below.

| $t$ | $x = 2t$ | $y = -t$ | $(x, y)$ |
|---|---|---|---|
| −2 | −4 | 2 | (−4, 2) |
| −1 | −2 | 1 | (−2, 1) |
| 0 | 0 | 0 | (0, 0) |
| 1 | 2 | −1 | (2, −1) |
| 2 | 4 | −2 | (4, −2) |

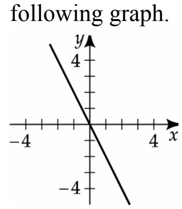
Plotting points for several values of $t$ yields the following graph.




**2.** A table of five arbitrarily chosen values of $t$ and the corresponding values of $x$ and $y$ are shown the table below.

| $t$ | $x = -3t$ | $y = 6t$ | $(x, y)$ |
|---|---|---|---|
| −2 | 6 | −12 | (6, −12) |
| −1 | 3 | −6 | (3, −6) |
| 0 | 0 | 0 | (0, 0) |
| 1 | −3 | 6 | (−3, 6) |
| 2 | −6 | 12 | (−6, 12) |

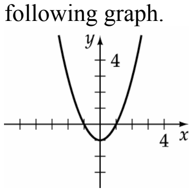
Plotting points for several values of $t$ yields the following graph.




**3.** A table of five arbitrarily chosen values of $t$ and the corresponding values of $x$ and $y$ are shown in the table below.

| $t$ | $x = -t$ | $y = t^2 - 1$ | $(x, y)$ |
|---|---|---|---|
| −2 | 2 | 3 | (2, 3) |
| −1 | 1 | 0 | (1, 0) |
| 0 | 0 | −1 | (0, −1) |
| 1 | −1 | 0 | (−1, 0) |
| 2 | −2 | 3 | (−2, 3) |

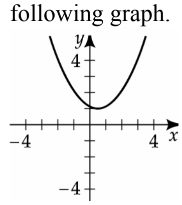
Plotting points for several values of $t$ yields the following graph.




**4.** A table of five arbitrarily chosen values of $t$ and the corresponding values of $x$ and $y$ are shown the table below.

| $t$ | $x = 2t$ | $y = 2t^2 - t + 1$ | $(x, y)$ |
|---|---|---|---|
| −2 | −4 | 11 | (−4, 11) |
| −1 | −2 | 4 | (−2, 4) |
| 0 | 0 | 1 | (0, 1) |
| 1 | 2 | 2 | (2, 2) |
| 2 | 4 | 7 | (4, 7) |

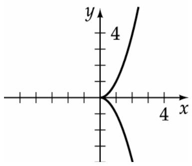
Plotting points for several values of $t$ yields the following graph.



**5.** A table of five arbitrarily chosen values of $t$ and the corresponding values of $x$ and $y$ are shown the table below.

| $t$ | $x = t^2$ | $y = t^3$ | $(x, y)$ |
|---|---|---|---|
| –2 | 4 | –8 | (4, –8) |
| –1 | 1 | –1 | (1, –1) |
| 0 | 0 | 0 | (0, 0) |
| 1 | 1 | 1 | (1, 1) |
| 2 | 4 | 8 | (4, 8) |

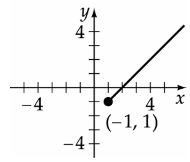
Plotting points for several values of $t$ yields the following graph.




**6.** A table of five arbitrarily chosen values of $t$ and the corresponding values of $x$ and $y$ are shown the table below.

| $t$ | $x = t^2 + 1$ | $y = t^2 - 1$ | $(x, y)$ |
|---|---|---|---|
| –2 | 5 | 3 | (5, 3) |
| –1 | 2 | 0 | (2, 0) |
| 0 | 1 | –1 | (1, –1) |
| 1 | 2 | 0 | (2, 0) |
| 2 | 5 | 3 | (5, 3) |

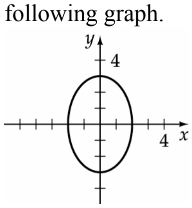
Plotting points for several values of $t$ yields the following graph.




**7.** A table of eight values of $t$ in the specified interval and the corresponding values of $x$ and $y$ are shown the table below.

| $t$ | $x = 2\cos t$ | $y = 3\sin t$ | $(x, y)$ |
|---|---|---|---|
| 0 | 2 | 0 | (2, 0) |
| $\pi/4$ | $\sqrt{2}$ | $3\sqrt{2}/2$ | $(\sqrt{2}, 3\sqrt{2}/2)$ |
| $\pi/2$ | 0 | 3 | (0, 3) |
| $3\pi/4$ | $-\sqrt{2}$ | $3\sqrt{2}/2$ | $(-\sqrt{2}, 3\sqrt{2}/2)$ |
| $\pi$ | –2 | 0 | (–2, 0) |
| $5\pi/4$ | $-\sqrt{2}$ | $-3\sqrt{2}/2$ | $(-\sqrt{2}, -3\sqrt{2}/2)$ |
| $3\pi/2$ | 0 | –3 | (0, –3) |
| $7\pi/4$ | $\sqrt{2}$ | $-3\sqrt{2}/2$ | $(\sqrt{2}, -3\sqrt{2}/2)$ |

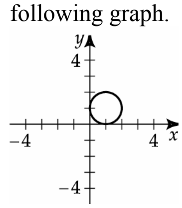
Plotting points for several values of $t$ yields the following graph.




**8.** A table of eight values of $t$ in the specified interval and the corresponding values of $x$ and $y$ are shown the table below.

| $t$ | $x = 1 - \sin t$ | $y = 1 + \cos t$ | $(x, y)$ |
|---|---|---|---|
| 0 | 1 | 2 | (1, 2) |
| $\pi/4$ | $\approx 0.29$ | $\approx 1.7$ | $(\approx 0.29, \approx 1.7)$ |
| $\pi/2$ | 0 | 1 | (0, 1) |
| $3\pi/4$ | $\approx -0.29$ | $\approx 0.29$ | $(\approx -0.29, \approx 0.29)$ |
| $\pi$ | 1 | 0 | (1, 0) |
| $5\pi/4$ | $\approx 1.7$ | $\approx 0.29$ | $(\approx 1.7, \approx 0.29)$ |
| $3\pi/2$ | 2 | 1 | (0, –3) |
| $7\pi/4$ | $\approx 1.7$ | $\approx 1.7$ | $(\approx 1.7, \approx 1.7)$ |

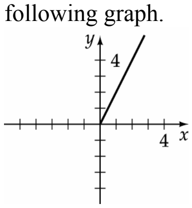
Plotting points for several values of $t$ yields the following graph.




**9.** A table of five arbitrarily chosen values of $t$ and the corresponding values of $x$ and $y$ are shown the table below.

| $t$ | $x = 2^t$ | $y = 2^{t+1}$ | $(x, y)$ |
|---|---|---|---|
| –2 | 1/4 | 1/2 | (1/4, 1/2) |
| –1 | 1/2 | 1 | (1/2, 1) |
| 0 | 1 | 2 | (1, 2) |
| 1 | 2 | 4 | (2, 4) |
| 2 | 4 | 8 | (4, 8) |

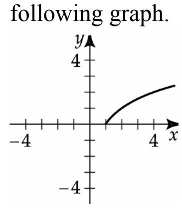
Plotting points for several values of $t$ yields the following graph.




**10.** A table of five values of $t$ in the domain of the parameter and the corresponding values of $x$ and $y$ are shown the table below.

| $t$ | $x = t^2$ | $y = 2\log_2 t$ | $(x, y)$ |
|---|---|---|---|
| 1 | 1 | 0 | (1, 0) |
| 2 | 4 | 2 | (4, 2) |
| 3 | 9 | $\approx 3.17$ | $(9, \approx 3.17)$ |
| 4 | 16 | 4 | (16, 4) |
| 5 | 25 | $\approx 4.64$ | $(25, \approx 4.64)$ |

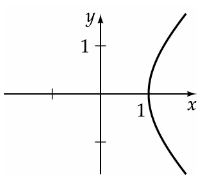
Plotting points for several values of $t$ yields the following graph.



**11.** $x = \sec t \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}$

$y = \tan t$

$$\tan^2 t + 1 = \sec^2 t$$
$$y^2 + 1 = x^2$$
$$x^2 - y^2 - 1 = 0 \qquad x \ge 1,\ y \in R$$

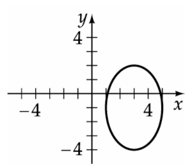



**12.** $x = 3 + 2\cos t \qquad 0 \le t < 2\pi$

$y = -1 - 3\sin t$

$$\cos t = \frac{x-3}{2}$$
$$\sin t = \frac{y+1}{3}$$

$$\cos^2 t + \sin^2 t = 1$$
$$\left(\frac{x-3}{2}\right)^2 + \left(-\frac{y+1}{3}\right)^2 = 1$$
$$\frac{(x-3)^2}{4} + \frac{(y+1)^2}{9} = 1$$

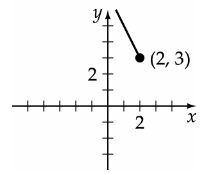



**13.** $x = 2 - t^2 \qquad t \in R$

$y = 3 + 2t^2$

$$x = 2 - t^2 \to t^2 = 2 - x$$
$$y = 3 + 2(2 - x)$$
$$y = -2x + 7$$

Because $x = 2 - t^2$ and $t^2 \ge 0$ for all real numbers $t$, $x \le 2$ for all $t$. Similarly, $y \ge 3$ for all $t$.

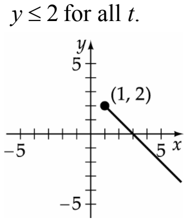



**14.** $x = 1 + t^2 \qquad t \in R$

$y = 2 - t^2$

$$x = 1 + t^2 \to t^2 = x - 1$$
$$y = 2 - (x - 1)$$
$$y = -x + 3$$

Because $x = 1 + t^2$ and $t^2 \ge 0$ for all real numbers $t$, $x \ge 1$ for all $t$. Similarly, $y \le 2$ for all $t$.

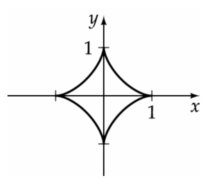



**15.** $x = \cos^3 t \qquad 0 \le t < 2\pi$

$y = \sin^3 t$

$$\cos^2 t = x^{2/3}$$
$$\sin^2 t = y^{2/3}$$

$$\cos^2 t + \sin^2 t = 1 \qquad -1 \le x \le 1$$
$$x^{2/3} + y^{2/3} = 1 \qquad -1 \le y \le 1$$

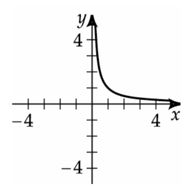



**16.** $x = e^{-t} \qquad t \in R,\ x > 0$

$y = e^{t} \qquad y > 0$

$e^{-t} \cdot e^{t} = e^{(t-t)} = e^0 = 1$

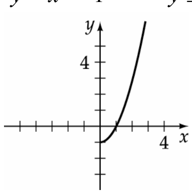
$xy = 1$ for $x > 0$ and $y > 0$




**17.** $x = \sqrt{t+1} \qquad t \ge -1$

$y = t$

$$x = \sqrt{y+1} \qquad x \ge 0$$
$$y = x^2 - 1 \qquad y \ge -1$$

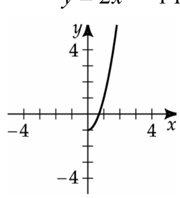



**18.** $x = \sqrt{t} \qquad t \ge 0,\ x \ge 0$

$y = 2t - 1 \qquad y \ge -1$

$$y = 2t - 1 \to t = \frac{y+1}{2}$$
$$x = \sqrt{\frac{y+1}{2}}$$

$y = 2x^2 - 1$ for $x \ge 0$ and $y \ge -1$

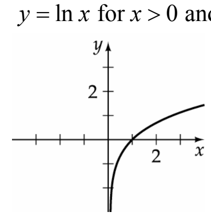



**19.** $x = t^3 \qquad t > 0,\ x > 0$

$y = 3 \ln t \qquad y \in R$

$$x = t^3 \to t = x^{1/3}$$
$$y = 3 \ln x^{1/3}$$

$y = \ln x$ for $x > 0$ and $y \in R$

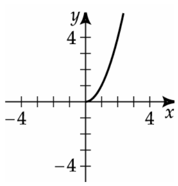


**20.** $x = e^t \quad t \in R,\ x > 0$

$y = e^{2t} \quad y > 0$

$x = e^t \to x^2 = e^{2t} \to x^2 = y$

$y = x^2$ for $x > 0$ and $y > 0$




**21.** $x = 2 + 3\cos t \qquad y = 3 + 2\sin t$

$x - 2 = 3\cos t \qquad y - 3 = 2\sin t$

$\frac{x-2}{3} = \cos t \qquad \frac{y-3}{2} = \sin t$

$$\left(\frac{x-2}{3}\right)^2 + \left(\frac{y-3}{2}\right)^2 = \cos^2 t + \sin^2 t$$

$$\left(\frac{x-2}{3}\right)^2 + \left(\frac{y-3}{2}\right)^2 = 1$$

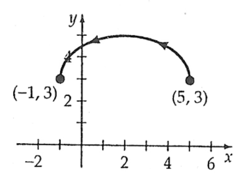
At $t = 0$,

$x = 2 + 3\cos 0 = 5 \qquad y = 3 + 2\sin 0 = 3$

At $t = \pi$,

$x = 2 + 3\cos\pi = -1 \qquad y = 3 + 2\sin\pi = 3$

The point traces the top half of the ellipse $\left(\frac{x-2}{3}\right)^2 + \left(\frac{y-3}{2}\right)^2 = 1$, as shown in the figure. The point starts at (5, 3) and moves counterclockwise along the ellipse until it reaches the point (–1, 3) at time $t = \pi$.




**22.** $x = \sin t \qquad y = -\cos t$

$$x^2 + y^2 = \sin^2 t + \cos^2 t$$

$$x^2 + y^2 = 1$$

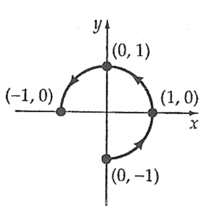
At $t = 0$,

$x = \sin 0 = 0 \qquad y = -\cos 0 = -1$

At $t = \frac{3\pi}{2}$,

$x = \sin\frac{3\pi}{2} = -1 \qquad y = -\cos\frac{3\pi}{2} = 0$

The point traces a portion of the circle $x^2 + y^2 = 1$, as shown in the figure. The point starts at (0, –1) and moves counter clockwise along the circle until it reaches the point (–1, 0) at time $t = \frac{3\pi}{2}$.




**23.** $y = t + 1 \ \Rightarrow\ t = y - 1$

$x = 2t - 1$

$x = 2(y-1) - 1$

$x = 2y - 3$

$x + 3 = 2y$

$y = \frac{1}{2}x + 3$

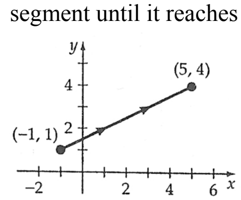
At $t = 0$,

$x = 2(0) - 1 = -1 \qquad y = 0 + 1 = 1$

At $t = 3$,

$x = 2(3) - 1 = 5 \qquad y = 3 + 1 = 4$

The point traces a line segment, as shown in the figure. The point starts at (–1, 1) and moves along the line segment until it reaches the point (5, 4) at time $t = 3$.



**24.** $x = t + 1 \Rightarrow t = x - 1$

$y = \sqrt{t}$

$y = \sqrt{x-1}$ or $y^2 = x - 1$

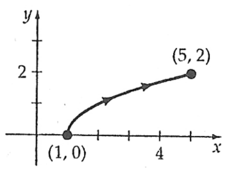
At $t = 0$,

$x = 0 + 1 = 1 \qquad y = \sqrt{0} = 0$

At $t = 4$,

$x = 4 + 1 = 5 \qquad y = \sqrt{4} = 2$

The point traces a portion of the parabola $y^2 = x - 1$, as shown in the figure. The point starts at (1, 0) and moves along the parabola until it reaches the point (5, 2) at time $t = 4$.




**25.** $x = \tan\left(\frac{\pi}{4} - t\right) \qquad y = \sec\left(\frac{\pi}{4} - t\right)$

$y^2 - x^2 = \sec^2\left(\frac{\pi}{4} - t\right) - \tan^2\left(\frac{\pi}{4} - t\right)$

$y^2 - x^2 = 1 \quad$ Since $1 + \tan^2\theta = \sec^2\theta$

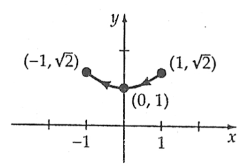
At $t = 0$,

$x = \tan\left(\frac{\pi}{4} - 0\right) = 1 \qquad y = \sec\left(\frac{\pi}{4} - 0\right) = \sqrt{2}$

At $t = \frac{\pi}{2}$,

$x = \tan\left(\frac{\pi}{4} - \frac{\pi}{2}\right) = -1 \qquad y = \sec\left(\frac{\pi}{4} - \frac{\pi}{2}\right) = \sqrt{2}$

The point traces a portion of the top branch of the hyperbola $y^2 - x^2 = 1$, as shown in the figure. The point starts at $(1, \sqrt{2})$ and moves along the hyperbola until it reaches the point $(-1, \sqrt{2})$ at time $t = \frac{\pi}{2}$.




**26.** $x = 1 - t \Rightarrow t = 1 - x$

$y = t^2$

$y = (1 - x)^2$ or $y = (x - 1)^2$

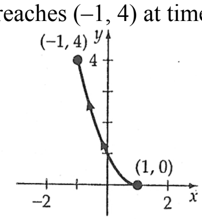
At $t = 0$,

$x = 1 - 0 = 1 \qquad y = (0)^2 = 0$

At $t = 2$,

$x = 1 - 2 = -1 \qquad y = (2)^2 = 4$

The point traces a portion of the parabola given by $y = (x-1)^2$, as shown in the figure. The point starts at (1, 0) and moves along the parabola until it reaches (–1, 4) at time $t = 2$.




**27.** $C_1: x = 2 + t^2$

$y = 1 - 2t^2$

$x = 2 + t^2 \rightarrow t^2 = x - 2$

$y = 1 - 2(x - 2)$

$y = -2x + 5 \quad x \geq 2, y \leq 1$

$C_2: x = 2 + t$

$y = 1 - 2t$

$x = 2 + t \rightarrow t = x - 2$

$y = 1 - 2(x - 2)$

$y = -2x + 5 \quad x \in R, y \in R$

The graph of $C_1$ is a ray beginning at (2, 1) with slope –2.
The graph of $C_2$ is a line passing through (2, 1) with slope –2.

**28.** $C_1: \; x = \sec^2 t$

$y = \tan^2 t$

$\tan^2 t + 1 = \sec^2 t$
$y + 1 = x$
$y = x - 1$

Because $0 \le t < \dfrac{\pi}{2}$,

$1 \le \sec^2 t < \infty$ and $0 \le \tan^2 t < \infty$
$1 \le \quad x \quad < \infty$ and $0 \le \quad y \quad < \infty$
Thus, $x \ge 1$ and $y \ge 0$.

$C_2: \; x = 1 + t^2$

$y = t^2$
$x = 1 + y$
$y = x - 1$

Because $0 < t < \dfrac{\pi}{2}$,

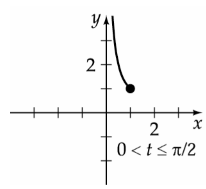
$0 \le t^2 < \dfrac{\pi^2}{4}$ and $1 \le t^2 + 1 < 1 + \dfrac{\pi^2}{4}$.
Thus, $1 \le x < 1 + \dfrac{\pi^2}{4}$ and $0 \le y < \dfrac{\pi^2}{4}$.

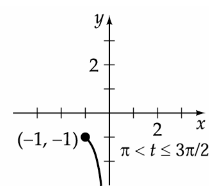
$C_1$ is a ray from (1, 0) in the direction of $\left(1 + \dfrac{\pi^2}{4}, \; \dfrac{\pi^2}{4}\right)$.

$C_2$ is the points on the line $y = x - 1$ between (1, 0) and $\left(1 + \dfrac{\pi^2}{4}, \; \dfrac{\pi^2}{4}\right)$. $C_2$ includes (1, 0) but not $\left(1 + \dfrac{\pi^2}{4}, \; \dfrac{\pi^2}{4}\right)$.

**29.** $x = \sin t$
$y = \csc t$

$\csc t = \dfrac{1}{\sin t}$
$y = \dfrac{1}{x}$




Range for graph 1: $0 < t \le \dfrac{\pi}{2}$
$0 < x \le 1$
$y \ge 1$

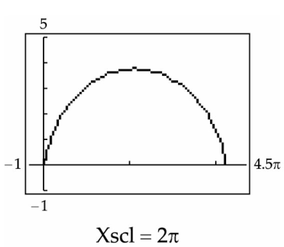
Range for graph 2: $\pi \le t \le \dfrac{3\pi}{2}$
$-1 \le x \le 0$
$y \le -1$

**30.** $C_1: \; x = \cos t \qquad 0 \le t \le \pi, \; -1 \le x \le 1$
$y = \cos^2 t \qquad 0 \le y \le 1$
$(\cos t)^2 = \cos^2 t$
$(x)^2 = y$
$y = x^2$ for $-1 \le x \le 1$ and $0 \le y \le 1$

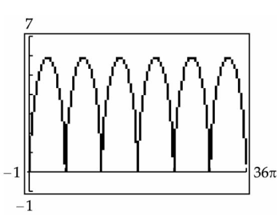
$C_2: \; x = \sin t \qquad 0 \le t \le \pi, \; 0 \le x \le 1$
$y = \sin^2 t \qquad 0 \le y \le 1$
$(\sin t)^2 = \sin^2 t$
$(x)^2 = y$
$y = x^2$ for $0 \le x \le 1$ and $0 \le y \le 1$

$C_1$ is the graph of the parabola $y = x^2$ for $-1 \le x \le 1$, while $C_2$ is the graph of the parabola $y = x^2$ for $0 \le x \le 1$.

**31.**




**32.**



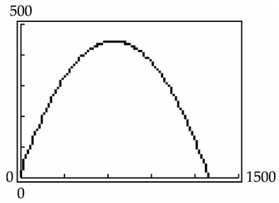

**33.** **a.** For the Hummer,
$x = 6$
$y = 60t$ for $t \ge 0$

**b.** Using the graphing calculator in SIMUL and PAR mode, the Hummer is the first to reach the intersection.

**34.** **a.** For the Learjet,
$x = 300 - 420t$
$y = 200$ for $t \ge 0$

**b.** Using the graphing calculator in SIMUL and PAR mode, the Piper Seneca is the first to reach the intersection point.

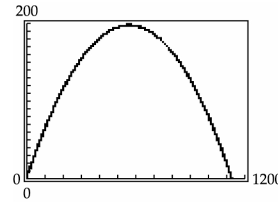
**35.**




Maximum height (to the nearest foot) of 462 feet is attained when $t \approx 5.38$ seconds.

The projectile has a range (to the nearest foot) of 1295 feet and hits the ground in about 10.75 seconds.

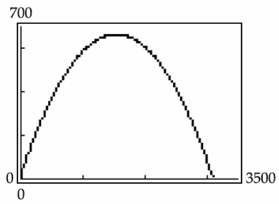
**36.**




Maximum height (to the nearest foot) of 195 feet is attained when $t \approx 3.50$ seconds.

The projectile has a range (to the nearest foot) of 1117 feet and hits the ground in about 6.99 seconds.

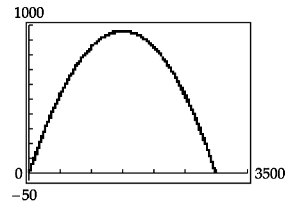
**37.**




Maximum height (to the nearest foot) of 694 feet is attained when $t \approx 6.59$ seconds.

The projectile has a range (to the nearest foot) of 3084 feet and hits the ground in about 13.17 seconds.

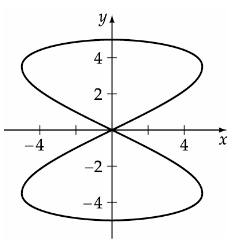
**38.**



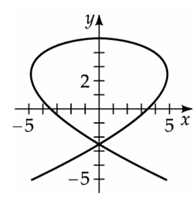

Maximum height: 963 ft at time $t \approx 7.76$ sec

Range: 3009 ft at time $t \approx 15.51$ sec

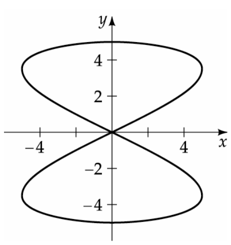
**39.**



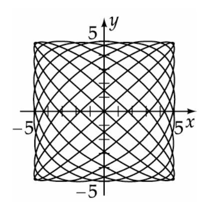

**40.**




**41.**




**42.**




## Connecting Concepts

**43.** Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two distinct points on a line.

If $P(x, y)$ is any other point on the line, then

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$ (Slope is constant along entire line.) This equation can be rewritten as

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$ Let this value equal $t$.

Thus, $\dfrac{x - x_1}{x_2 - x_1} = t$ and $\dfrac{y - y_1}{y_2 - y_1} = t$.

Solving for $x$ and $y$, respectively, we have

$x = x_1 + t(x_2 - x_1)$ and $y = y_1 + t(y_2 - y_1)$

**44.** $x = h + a\sin t \quad a > 0, x \in R,\ 0 \le t < 2\pi$

$y = k + b\cos t \quad b > 0, y \in R,\ 0 \le t < 2\pi$

$x = h + a\sin t \to \sin t = \dfrac{x - h}{a}$

$y = k + b\cos t \to \cos t = \dfrac{y - k}{b}$

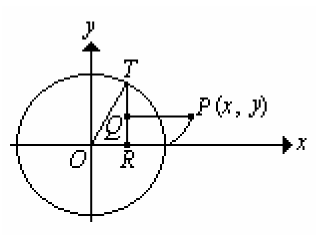
$\sin^2 t + \cos^2 t = 1$

$$\left(\frac{x - h}{a}\right)^2 + \left(\frac{y - k}{b}\right)^2 = 1$$

$\dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1$, which is the standard equation for an ellipse at $(h, k)$.

**45.** radius $= a$, $\theta = \angle TOR$

The $x$-coordinate of $P(x, y)$ is given by $x = OR + QP$.
The $y$-coordinate is given by $y = TR - TQ$.




From the figure,

$OR = a\cos\theta$ and $QP = a\theta\sin\theta$. Thus,

$x = a\cos\theta + a\theta\sin\theta$

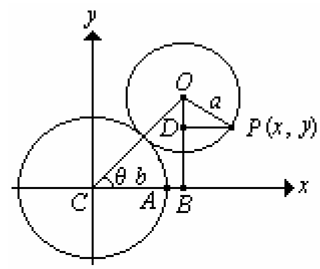
$TR = a\sin\theta$ and $TQ = a\theta\cos\theta$. Thus,

$y = a\sin\theta - a\theta\cos\theta$

The parametric equations are

$$x = a\cos\theta + a\theta\sin\theta$$
$$y = a\sin\theta - a\theta\cos\theta$$

**46.** Let $\alpha = \angle COP$. Because the smaller circle does not slip, $b\theta = a\alpha$ or $\alpha = \frac{b}{a}\theta$.




The coordinates of $P(x, y)$ are given by

$x = BC + DP$

$y = OB - OD$

Thus, $x = (a+b)\cos\theta + a\sin\left(\frac{a+b}{a}\theta - \frac{\pi}{2}\right)$

$= (a+b)\cos\theta - a\cos\left(\frac{a+b}{a}\theta\right)$

$y = (a+b)\sin\theta - a\cos\left(\frac{a+b}{a}\theta - \frac{\pi}{2}\right)$

$= (a+b)\sin\theta - a\cos\left(\frac{a+b}{a}\theta\right)$

The parametric equations are

$$x = (a+b)\cos\theta - a\cos\left(\frac{a+b}{a}\theta\right)$$
$$y = (a+b)\sin\theta - a\sin\left(\frac{a+b}{a}\theta\right)$$

**47.** Because the circle moves without slipping, $b\theta = a\alpha$ .

Therefore, $\alpha = \frac{b\theta}{a}$. Let $P(x, y)$ be the coordinates of the moving point.

Angle $\phi = \frac{\pi}{2} - \left(\frac{b-a}{a}\right)\theta$

Thus, $x = (b-a)\cos\theta + a\sin\left[\frac{\pi}{2} - \left(\frac{b-a}{a}\right)\theta\right]$

$y = (b-a)\sin\theta - a\cos\left[\frac{\pi}{2} - \left(\frac{b-a}{a}\right)\theta\right]$

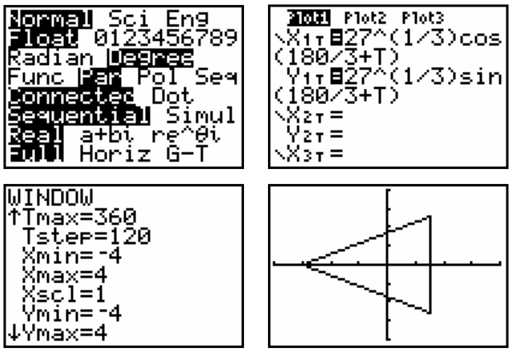
Simplifying, we have

$$x = (b-a)\cos\theta + a\cos\left(\frac{b-a}{a}\theta\right)$$
$$y = (b-a)\sin\theta - a\sin\left(\frac{b-a}{a}\theta\right)$$

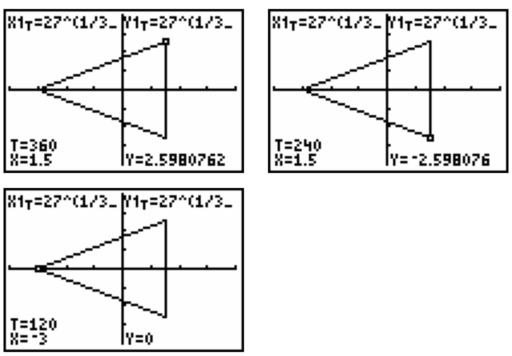
## Exploring Concepts with Technology

**Using a Graphing Calculator to Find the *n*th Roots of *z***

**1.** The procedure for a TI-83 calculator is illustrated below.
$z = -27 = 27(\cos 180° + i \sin 180°)$, thus, $r = 27$, $\theta = 180°$.
cube roots $\Rightarrow n = 3$

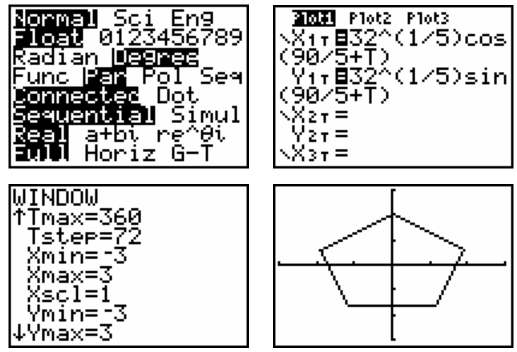



Use the TRACE feature and the arrow keys to display and move to each of the vertices of the polygon.

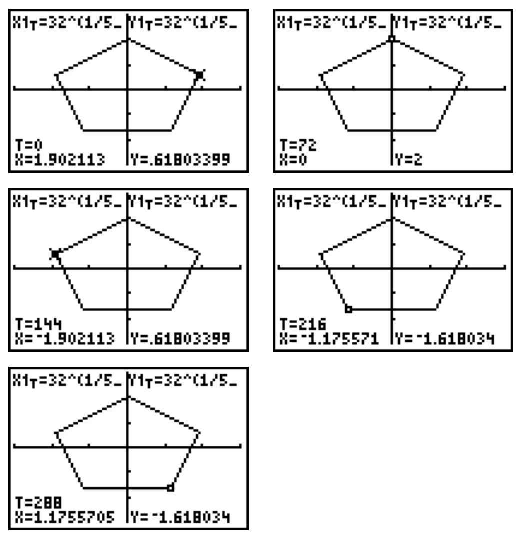



Thus, the three cube roots of –27 are
$1.5 + 2.598076i$, $1.5 - 2.598076i$, and $-3$.

**2.** The procedure for a TI-83 calculator is illustrated below.
$z = 32i = 32(\cos 90° + i \sin 90°)$, thus, $r = 32$, $\theta = 90°$.
fifth roots $\Rightarrow n = 5$




Use the TRACE feature and the arrow keys to display and move to each of the vertices of the polygon.




Thus, the five fifth roots of 32*i* are
1.902113 + 0.61803399*i*, –1.902113 + 0.61803399*i*,
–1.1755705 – 1.618034*i*, 1.1755705 – 1.618034*i*, and 2*i*.

**3.** Here is the procedure for a TI-83 graphing calculator. Be sure the calculator is in parametric and degree mode. In the `WINDOW` menu, set `Tmin=0`, `Tmax=360`, and, since $360/4=90$, set `Tstep=90`. Set `Xmin`, `Xmax`, `Ymin`, and `Ymax` to appropriate values that will allow the roots to be seen. Since $z=\sqrt{8}+\sqrt{8}i=4(\cos 45°+i\sin 45°)$, in the `Y=` menu, enter `X1T=4^(1/4)cos(45/4+T)` and `Y1T=4^(1/4)sin(45/4+T)`. Press `GRAPH` to display a polygon. The *x*- and *y*-coordinates of each vertex represent a root of *z* in the rectangular form $x+yi$. Use the `TRACE` feature and the arrow keys to display and move to each of the vertices of the polygon.

The fourth roots of $\sqrt{8}+\sqrt{8}i$ are $1.38704+0.2758994i$, $-0.2758994+1.38704i$, $-1.38704-0.2758994i$, and $0.2758994-1.38704i$.

**4.** Here is the procedure for a TI-83 graphing calculator. Be sure the calculator is in parametric and degree mode. In the `WINDOW` menu, set `Tmin=0`, `Tmax=360`, and, since $360/6=60$, set `Tstep=60`. Set `Xmin`, `Xmax`, `Ymin`, and `Ymax` to appropriate values that will allow the roots to be seen. Since $z=-64i=64(\cos 270°+i\sin 270°)$, in the `Y=` menu, enter `X1T=64^(1/6)cos(270/6+T)` and `Y1T=64^(1/6)sin(270/6+T)`. Press `GRAPH` to display a polygon. The *x*- and *y*-coordinates of each vertex represent a root of *z* in the rectangular form $x+yi$. Use the `TRACE` feature and the arrow keys to display and move to each of the vertices of the polygon.

The sixth roots of $-64i$ are $1.414214+1.414214i$, $-0.5176381+1.931852i$, $-1.931852+0.5176381i$, $-1.414214-1.414214i$, $0.5176381-1.931852i$, and $1.931852-0.5176381i$.

## Assessing Concepts

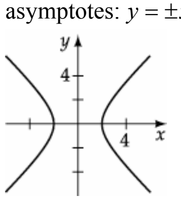
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **1.** | d | **2.** | b | **3.** | e | **4.** | c |
| **5.** | a | **6.** | f | **7.** | g | **8.** | i |
| **9.** | h | **10.** | j | **11.** | k | **12.** | k |

## Chapter Review

**1.** $x^2-y^2=4$ [8.3]

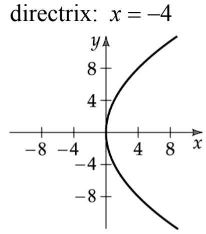
$$\frac{x^2}{4}-\frac{y^2}{4}=1$$

hyperbola
center: (0, 0)
vertices: $(\pm 2, 0)$
foci: $(\pm 2\sqrt{2}, 0)$
asymptotes: $y=\pm x$




**2.** $y^2=16x$ [8.1]

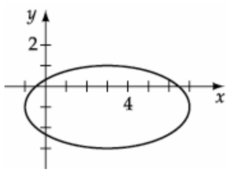
parabola
vertex: (0, 0)
focus: (4, 0)
directrix: $x=-4$



**3.** [8.2]

$$x^2+4y^2-6x+8y-3=0$$
$$x^2-6x+4(y^2+2y)=3$$
$$(x^2-6x+9)+4(y^2+2y+1)=3+9+4$$
$$(x-3)^2+4(y+1)^2=16$$
$$\frac{(x-3)^2}{16}+\frac{(y+1)^2}{4}=1$$

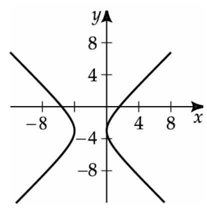
ellipse
center: $(3,\ -1)$
vertices: $(3\pm 4,-1)=(7,-1),(-1,-1)$
foci: $(3\pm 2\sqrt{3},\ -1)=(3+2\sqrt{3},\ -1),\ (3-2\sqrt{3},\ -1)$




**4.** [8.3]

$$3x^2-4y^2+12x-24y-36=0$$
$$3(x^2+4x)-4(y^2+6y)=36$$
$$3(x^2+4x+4)-4(y^2+6y+9)=36+12-36$$
$$3(x+2)^2-4(y+3)^2=12$$
$$\frac{(x+2)^2}{4}-\frac{(y+3)^2}{3}=1$$

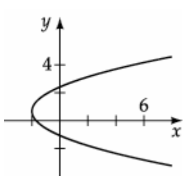
hyperbola
center: $(-2,\ -3)$
vertices: $(-2\pm 2,\ -3)=(0,\ -3),\ (-4,\ -3)$
foci: $(-2\pm\sqrt{7},\ -3)=(-2+\sqrt{7},\ -3),\ (-2-\sqrt{7},\ -3)$
asymptotes: $y+3=\pm\frac{\sqrt{3}}{2}(x+2)$




**5.** [8.1]

$$3x-4y^2+8y+2=0$$
$$-4(y^2-2y)=-3x-2$$
$$-4(y^2-2y+1)=-3x-2-4$$
$$-4(y-1)^2=-3(x+2)$$
$$(y-1)^2=\frac{3}{4}(x+2)$$

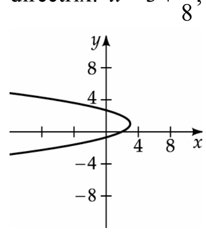
parabola
vertex: $(-2,\ 1)$
focus: $\left(-2+\frac{3}{16},1\right)=\left(-\frac{29}{16},1\right)$
directrix: $x=-2-\frac{3}{16}$, or $x=-\frac{35}{16}$




**6.** [8.1]

$$3x+2y^2-4y-7=0$$
$$2(y^2-2y)=-3x+7$$
$$2(y^2-2y+1)=-3x+7+2$$
$$2(y-1)^2=-3(x-3)$$
$$(y-1)^2=-\frac{3}{2}(x-3)$$

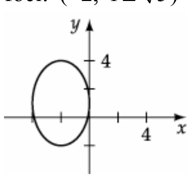
parabola
vertex: $(3,\ 1)$
focus: $\left(3-\frac{3}{8},1\right)=\left(\frac{21}{8},1\right)$
directrix: $x=3+\frac{3}{8}$, or $x=\frac{27}{8}$



**7.** [8.2]

$$
\begin{aligned}
9x^2+4y^2+36x-8y+4&=0\\
9(x^2+4x)+4(y^2-2y)&=-4\\
9(x^2+4x+4)+4(y^2-2y+1)&=-4+36+4\\
9(x+2)^2+4(y-1)^2&=36\\
\frac{(x+2)^2}{4}+\frac{(y-1)^2}{9}&=1
\end{aligned}
$$

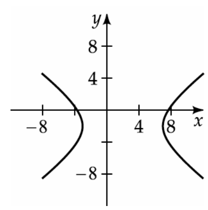
ellipse
center: $(-2,\ 1)$
vertices: $(-2,\ 1\pm 3)=(-2,\ 4),\ (-2,\ -2)$
foci: $(-2,\ 1\pm\sqrt{5})=(-2,\ 1+\sqrt{5}),\ (-2,\ 1-\sqrt{5})$




**8.** [8.3]

$$
\begin{aligned}
11x^2-25y^2-44x-50y-256&=0\\
11(x^2-4x)-25(y^2+2y)&=256\\
11(x^2-4x+4)-25(y^2+2y+1)&=256+44-25\\
11(x-2)^2-25(y+1)^2&=275\\
\frac{(x-2)^2}{25}-\frac{(y+1)^2}{11}&=1
\end{aligned}
$$

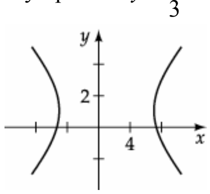
hyperbola
center: $(2,\ -1)$
vertices: $(2\pm 5,\ -1)=(7,\ -1),\ (-3,\ -1)$
foci: $(2\pm 6,\ -1)=(8,\ -1),\ (-4,\ -1)$
asymptotes: $y+1=\pm\dfrac{\sqrt{11}}{5}(x-2)$




**9.** [8.3]

$$
\begin{aligned}
4x^2-9y^2-8x+12y-144&=0\\
4(x^2-2x)-9\left(y^2-\frac{4}{3}y\right)&=144\\
4(x^2-2x+1)-\left(9y^2-\frac{4}{3}y+\frac{4}{9}\right)&=144+4-4\\
4(x-1)^2-9\left(y-\frac{2}{3}\right)^2&=144\\
\frac{(x-1)^2}{36}-\frac{\left(y-\frac{2}{3}\right)^2}{16}&=1
\end{aligned}
$$

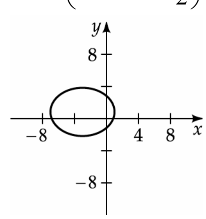
hyperbola
center: $\left(1,\dfrac{2}{3}\right)$
vertices: $\left(1\pm 6,\ \dfrac{2}{3}\right)=\left(7,\ \dfrac{2}{3}\right),\ \left(-5,\ \dfrac{2}{3}\right)$
foci: $\left(1\pm 2\sqrt{13},\ \dfrac{2}{3}\right)=\left(1+2\sqrt{13},\ \dfrac{2}{3}\right),\ \left(1-2\sqrt{13},\ \dfrac{2}{3}\right)$
asymptotes: $y-\dfrac{2}{3}=\pm\dfrac{2}{3}(x-1)$




**10.** [8.2]

$$
\begin{aligned}
9x^2+16y^2+36x-16y-104&=0\\
9(x^2+4x)+16(y^2-y)&=104\\
9(x^2+4x+4)+16\left(y^2-y+\frac{1}{4}\right)&=104+36+4\\
9(x+2)^2+16\left(y-\frac{1}{2}\right)^2&=144\\
\frac{(x+2)^2}{16}+\frac{\left(y-\frac{1}{2}\right)^2}{9}&=1
\end{aligned}
$$

ellipse
center: $\left(-2,\dfrac{1}{2}\right)$
vertices: $\left(-2\pm 4,\ \dfrac{1}{2}\right)=\left(2,\ \dfrac{1}{2}\right),\ \left(-6,\ \dfrac{1}{2}\right)$
foci: $\left(-2\pm\sqrt{7},\ \dfrac{1}{2}\right)=\left(-2+\sqrt{7},\ \dfrac{1}{2}\right),\ \left(-2-\sqrt{7},\ \dfrac{1}{2}\right)$



**11.** $4x^2 + 28x + 32y + 81 = 0$ [8.1]

$$4(x^2 + 7x) = -32y - 81$$
$$4\left(x^2 + 7x + \frac{49}{4}\right) = -32y - 81 + 49$$
$$4\left(x + \frac{7}{2}\right)^2 = -32(y+1)$$
$$\left(x + \frac{7}{2}\right)^2 = -8(y+1)$$

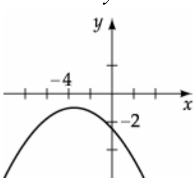
parabola

$4p = -8 \Rightarrow p = -2$

vertex: $\left(-\frac{7}{2}, -1\right)$

focus: $\left(-\frac{7}{2}, -1-2\right) = \left(-\frac{7}{2}, -3\right)$

directrix: $y = 1$




**12.** $x^2 - 6x - 9y + 27 = 0$ [8.1]

$$x^2 - 6x = 9y - 27$$
$$x^2 - 6x + 9 = 9y - 27 + 9$$
$$(x-3)^2 = 9(y-2)$$

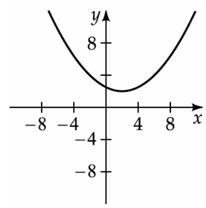
parabola

$4p = 9 \Rightarrow p = \frac{9}{4}$

vertex: (3, 2)

focus: $\left(3,\ 2 + \frac{9}{4}\right) = \left(3,\ \frac{17}{4}\right)$

directrix: $y = 3 - \frac{9}{4}$, or $y = -\frac{1}{4}$




**13.** $2a = |7 - (-3)| = 10$ [8.2]

$a = 5$

$a^2 = 25$

$2b = 8$

$b = 4$

$b^2 = 16$

center (2, 3)

$$\frac{(x-2)^2}{25} + \frac{(y-3)^2}{16} = 1$$

**14.** $2a = |4 - (-2)| = 6$ [8.3]

$a = 3$

$a^2 = 9$

$e = \frac{c}{a} = \frac{4}{3}$

$\frac{c}{3} = \frac{4}{3}$

$c = 4$

$c^2 = a^2 + b^2$

$16 = 9 + b^2$

$b^2 = 7$

center (1, 1)

$$\frac{(x-1)^2}{9} - \frac{(y-1)^2}{7} = 1$$

**15.** center $(-2,\ 2)$, $c = 3$ [8.3]

$2a = 4$

$a = 2$

$a^2 = 4$

$c^2 = a^2 + b^2$

$9 = 4 + b^2$

$b^2 = 5$

$$\frac{(x+2)^2}{4} - \frac{(y-2)^2}{5} = 1$$

**16.** $(h,\ k) = \left(\frac{6+2}{2}, \frac{-3-3}{2}\right) = (4,\ -3)$ [8.1]

$p = 2 - 4$

$p = -2$

$4p = -8$

$(y+3)^2 = -8(x-4)$

**17.** $(x-h)^2 = 4p(y-k)$ or $(y-k)^2 = 4p(x-h)$

$(3-0)^2 = 4p(4+2)$ $\qquad$ $(4+2)^2 = 4p(3-0)$

$9 = 4p(6)$ $\qquad$ $36 = 4p(3)$

$p = \frac{3}{8}$ $\qquad$ $p = 3$

Thus, there are two parabolas that satisfy the given conditions:

$x^2 = \frac{3}{2}(y+2)$ or $(y+2)^2 = 12x$ [8.1]

**18.** center $(-2,\ -1)$

$c = 2$

$e = \frac{c}{a} = \frac{2}{3}$

$\frac{2}{a} = \frac{2}{3}$

$a = 3$

$c^2 = a^2 - b^2$

$4 = 9 - b^2$

$b^2 = 5$

$\frac{(x+2)^2}{9} + \frac{(y+1)^2}{5} = 1$ [8.2]

**19.** $a = 6$ and the transverse axis is on the $x$-axis.

$\pm\frac{b}{a} = \pm\frac{1}{9}$

$\frac{b}{6} = \frac{1}{9}$

$b = \frac{2}{3}$

$\frac{x^2}{36} - \frac{y^2}{4/9} = 1$ [8.3]

**20.** $(x-h)^2 = 4p(y-k)$
$(1-h)^2 = 4p(0-k)$
$(2-h)^2 = 4p(1-k)$
$(0-h)^2 = 4p(1-k)$

In the last two equations, by substitution:

$(2-h)^2 = (0-h)^2$

$4 - 4h + h^2 = h^2$

$4 - 4h = 0$

$4h = 4$

$h = 1$

Thus:

$(1-1)^2 = 4p(0-k)$

$0 = -4pk$

$k = 0$

$(2-1)^2 = 4p(1-k)$

$1 = 4p(1)$

$p = \frac{1}{4}$

The equation is $y = (x-1)^2$ [8.1]

**21.** $A = 11,\ B = -6,\ C = 19,\ D = 0,\ E = 0,\ F = -40$

$\cot 2\alpha = \frac{11-19}{-6} = \frac{-8}{-6} = \frac{4}{3}$ ($2\alpha$ is in quadrant I.)

$\csc^2 2\alpha = 1 + \cot^2 2\alpha = 1 + \frac{16}{9} = \frac{25}{9}$

$\csc 2\alpha = \frac{5}{3}$

Thus, $\sin 2\alpha = \frac{3}{5}$ and $\cos 2\alpha = \frac{4}{5}$.

$\sin\alpha = \sqrt{\frac{1-\frac{4}{5}}{2}} = \frac{\sqrt{10}}{10}$ $\cos\alpha = \frac{3\sqrt{10}}{10}$

$$A' = 11\left(\frac{3\sqrt{10}}{10}\right)^2 - 6\left(\frac{\sqrt{10}}{10}\right)\left(\frac{3\sqrt{10}}{10}\right) + 19\left(\frac{\sqrt{10}}{10}\right)^2$$
$$= \frac{99}{10} - \frac{18}{10} + \frac{19}{10} = \frac{100}{10} = 10$$

$B' = 0$

$$C' = 11\left(\frac{\sqrt{10}}{10}\right)^2 + 6\left(\frac{\sqrt{10}}{10}\right)\left(\frac{3\sqrt{10}}{10}\right) + 19\left(\frac{3\sqrt{10}}{10}\right)^2$$
$$= \frac{11}{10} + \frac{18}{10} + \frac{171}{10} = \frac{200}{10} = 20$$

$F' = F$

$10(x')^2 + 20(y')^2 - 40 = 0$ or $(x')^2 + 2(y')^2 - 4 = 0$

The graph is an ellipse. [8.4]

**22.** $A = 3, B = 6, C = 3, D = -4, E = 5, F = -12$

$\cot 2\alpha = \frac{3-3}{6} = 0$

$1 + \cot^2 2\alpha = \csc^2 2\alpha$

$1 = \csc^2 2\alpha$

$\csc 2\alpha = 1$

Thus, $\sin 2\alpha = 1$ so $2\alpha = 90°$ or $\alpha = 45°$.

Therefore, $\cos\alpha = \frac{\sqrt{2}}{2}$, $\sin\alpha = \frac{\sqrt{2}}{2}$.

$$A' = 3\left(\frac{\sqrt{2}}{2}\right)^2 + 6\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + 3\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{3}{2} + \frac{6}{2} + \frac{3}{2} = 6$$

$B' = 0$

$$C' = 3\left(\frac{\sqrt{2}}{2}\right)^2 - 6\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + 3\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{3}{2} - \frac{6}{2} + \frac{3}{2} = 0$$

$$D' = -4\left(\frac{\sqrt{2}}{2}\right) + 5\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$$

$$E' = 4\left(\frac{\sqrt{2}}{2}\right) + 5\left(\frac{\sqrt{2}}{2}\right) = \frac{9\sqrt{2}}{2}$$

$F' = -12$

$6(x')^2 + \frac{\sqrt{2}}{2}x' + \frac{9\sqrt{2}}{2}y' - 12 = 0$

The graph is a parabola. [8.4]

**23.** $A=1, B=2\sqrt{3}, C=3, D=8\sqrt{3}, E=-8, F=32$

$\cot 2\alpha = \dfrac{1-3}{2\sqrt{3}} = -\dfrac{1}{\sqrt{3}}$. Thus $90° < 2\alpha < 180°$.

$\cot^2 2\alpha + 1 = \csc^2 2\alpha$

$\frac{1}{3} + 1 = \csc^2 2\alpha$

$\frac{4}{3} = \csc^2 2\alpha$, or $\csc 2\alpha = \frac{2}{\sqrt{3}}$

Therefore, $\sin 2\alpha = \frac{\sqrt{3}}{2}$ and $\cos 2\alpha = -\frac{1}{2}$.

Since $2\alpha$ is in quadrant II,

$\cos\alpha = \sqrt{\dfrac{1+(-1/2)}{2}} = \dfrac{1}{2}$ $\qquad \sin\alpha = \sqrt{\dfrac{1-(-1/2)}{2}} = \dfrac{\sqrt{3}}{2}$

$A' = 1\left(\frac{1}{2}\right)^2 + 2\sqrt{3}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + 3\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{6}{4} + \frac{9}{4} = 4$

$B' = 0$

$C' = 1\left(\frac{\sqrt{3}}{2}\right)^2 - 2\sqrt{3}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + 3\left(\frac{1}{2}\right)^2 = \frac{3}{4} - \frac{6}{4} + \frac{3}{4} = 0$

$D' = 8\sqrt{3}\left(\frac{1}{2}\right) - 8\left(\frac{\sqrt{3}}{2}\right) = 0$

$E' = 8\sqrt{3}\left(\frac{\sqrt{3}}{2}\right) - 8\left(\frac{1}{2}\right) = -12 - 4 = -16$

$F' = 32$

$4(x')^2 - 16y' + 32 = 0$ or $(x')^2 - 4y' + 8 = 0$

The graph is a parabola. [8.4]

**24.** $A = 0, B = 1, C = 0, D = -1, E = -1, F = -1$

$\cot 2\alpha = \dfrac{0-0}{1} = 0$. Thus $\alpha = 45°$.

Therefore, $\sin\alpha = \dfrac{\sqrt{2}}{2}$ and $\cos\alpha = \dfrac{\sqrt{2}}{2}$.

$A' = 0 + 1\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + 0 = \frac{1}{2}$

$B' = 0$

$C' = 0 - \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} + 0 = -\frac{1}{2}$

$D' = -1\left(\frac{\sqrt{2}}{2}\right) - 1\left(\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$

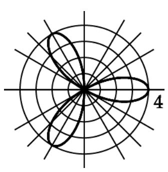
$E' = \frac{\sqrt{2}}{2} + (-1)\frac{\sqrt{2}}{2} = 0$

$F' = -1$

$\frac{1}{2}(x')^2 - \frac{1}{2}(y')^2 - \sqrt{2}x' - 1 = 0$

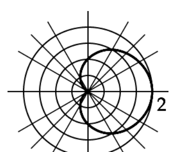
The equation is a hyperbola. [8.4]

**25.**



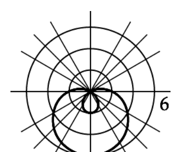
[8.5]

**26.**



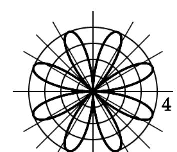
[8.5]

**27.**



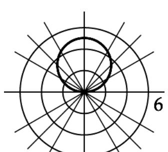
[8.5]

**28.**



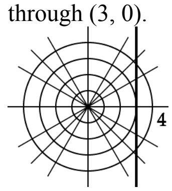
[8.5]

**29.**



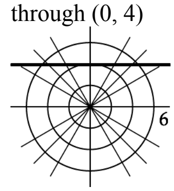
[8.5]

**30.** vertical line through (3, 0).



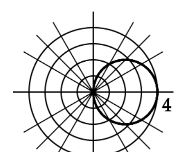
[8.5]

**31.** horizontal line through (0, 4)



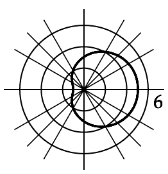
[8.5]

**32.**



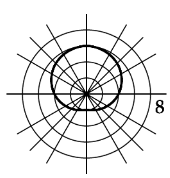
[8.5]

**33.**



[8.5]

**34.**



[8.5]

**35.** $y^2 = 16x$ [8.5]

$(r\sin\theta)^2 = 16(r\cos\theta)$

$r^2\sin^2\theta = 16r\cos\theta$

$r\sin^2\theta = 16\cos\theta$

**36.** $x^2 + y^2 + 4x + 3y = 0$ [8.5]

$(r\cos\theta)^2 + (r\sin\theta)^2 + 4(r\cos\theta) + 3(r\sin\theta) = 0$

$r^2(\cos^2\theta + \sin^2\theta) + 4r\cos\theta + 3r\sin\theta = 0$

$r + 4\cos\theta + 3\sin\theta = 0$

**37.** $3x - 2y = 6$ [8.6]

$3r\cos\theta - 2r\sin\theta = 6$

**38.**
$$xy = 4 \quad [8.5]$$
$$(r\cos\theta)(r\sin\theta) = 4$$
$$r^2\cos\theta\sin\theta = 4$$
$$r^2(2)\cos\theta\sin\theta = 4(2)$$
$$r^2\sin 2\theta = 8$$

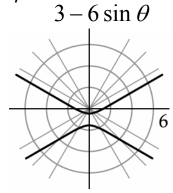
**39.**
$$r = \frac{4}{1-\cos\theta} \quad [8.5]$$
$$r - r\cos\theta = 4$$
$$\sqrt{x^2+y^2} - x = 4$$
$$\sqrt{x^2+y^2} = x + 4$$
$$x^2 + y^2 = x^2 + 8x + 16$$
$$y^2 = 8x + 16$$

**40.**
$$r^2 = 3r\cos\theta - 4r\sin\theta \quad [8.5]$$
$$x^2 + y^2 = 3x - 4y$$
$$x^2 - 3x + y^2 + 4y = 0$$

**41.**
$$r^2 = \cos 2\theta \quad [8.5]$$
$$r^2 = \cos^2\theta - \sin^2\theta$$
$$r^4 = r^2\cos^2\theta - r^2\sin^2\theta$$
$$(r^2)^2 = r^2\cos^2\theta - r^2\sin^2\theta$$
$$(x^2+y^2)^2 = x^2 - y^2$$
$$x^4 + 2x^2y^2 + y^4 = x^2 - y^2$$
$$x^4 + y^4 + 2x^2y^2 - x^2 + y^2 = 0$$

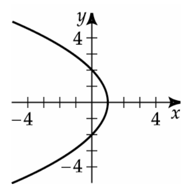
**42.**
$$\theta = 1 \quad [8.5]$$
$$\tan\theta = \tan 1$$
$$\frac{y}{x} \approx 1.5574$$
$$y = 1.5574x$$

**43.** $r = \dfrac{4}{3 - 6\sin\theta}$



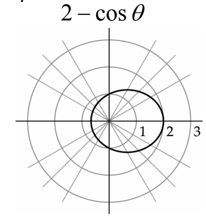

[8.6]

**44.** $r = \dfrac{2}{1 + \cos\theta}$



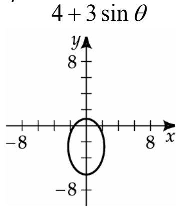

[8.6]

**45.** $r = \dfrac{2}{2 - \cos\theta}$



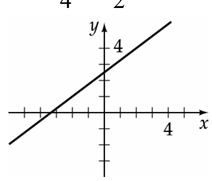

[8.6]

**46.** $r = \dfrac{6}{4 + 3\sin\theta}$



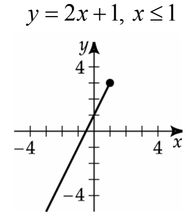

[8.6]

**47.** $x = 4t - 2,\ y = 3t + 1,\ t \in R$
$$4t = x + 2$$
$$t = \frac{x+2}{4}$$
$$y = 3t + 1$$
$$y = 3\left(\frac{x+2}{4}\right) + 1$$
$$y = \frac{3}{4}x + \frac{5}{2}$$




[8.7]

**48.** $x = 1 - t^2,\ y = 3 - 2t^2,\ t \in R$
$$t^2 = -x + 1$$
$$y = 3 - 2t^2$$
$$y = 3 - 2(-x + 1)$$
$$y = 3 + 2x - 2$$
$$y = 2x + 1,\ x \le 1$$




[8.7]

**49.** $x = 4\sin t \qquad y = 3\cos t \qquad 0 \le t < 2\pi$

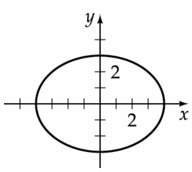
$\frac{1}{4}x = \sin t \qquad \frac{1}{3}y = \cos t$

$\frac{1}{16}x^2 = \sin^2 t \qquad \frac{1}{9}y^2 = \cos t^2$

Using the trignometric identity $\sin^2 t + \cos^2 t = 1$, we have

$\frac{1}{16}x^2 + \frac{1}{9}y^2 = 1$

$\frac{x^2}{19} + \frac{y^2}{9} = 1$



[8.7]

**50.** $x = \sec t \qquad y = 4\tan t \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}$

$x^2 = \sec^2 t \qquad \frac{1}{4}y = \tan t$

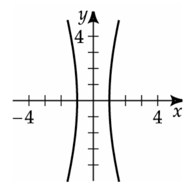
$\left(\frac{1}{4}y\right)^2 = \tan^2 t$

$\frac{1}{16}y^2 = \tan^2 t$

Using the trignometric identity $1 + \tan^2 t = \sec^2 t$, we have

$1 + \frac{y^2}{16} = x^2$

$\frac{x^2}{1} - \frac{y^2}{16} = 1$

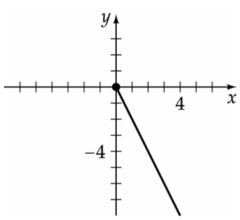


[8.7]

**51.** $x = \frac{1}{t} \qquad y = -\frac{2}{t} \qquad t > 0$

$y = -2\left(\frac{1}{t}\right)$

$y = -2x,\ x > 0$



[8.7]

**52.** $x = 1 + \cos t \qquad y = 2 - \sin t \qquad 0 \le t < 2\pi$

$x - 1 = \cos t \qquad 2 - y = \sin t$

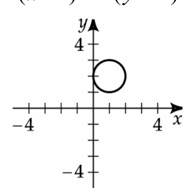
$-(y-2) = \sin t$

$(x-1)^2 = \cos^2 t \qquad (-(y-2))^2 = \sin^2 t$

$(y-2)^2 = \sin^2 t$

Using the trigonometric identity $\cos^2 t + \sin^2 t = 1$ we have

$(x-1)^2 + (y-2)^2 = 1$

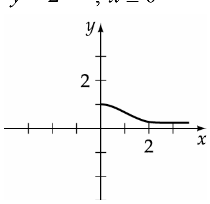


[8.7]

**53.** $x = \sqrt{t},\ y = 2^{-t}, t \ge 0$

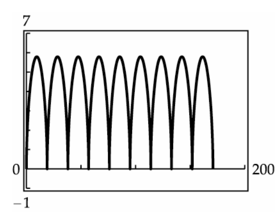
$t = x^2$

$y = 2^{-x^2},\ x \ge 0$



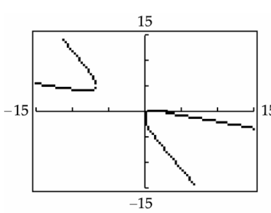
[8.7]

**54.**



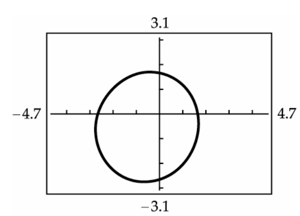
[8.7]

**55.** Graph $y = \frac{-(4x+5) \pm \sqrt{(4x+5)^2 - 8(x^2 - 2x + 1)}}{4}$.

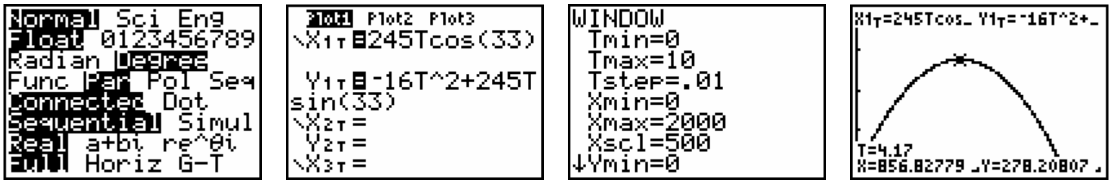


[8.4]

**56.**



[8.5]

**57.** 


Graph in parametric mode. Use the TRACE feature to determine that the maximum height (to the nearest foot) of 278 feet is attained when $t \approx 4.17$ seconds. [8.7]

## Quantitative Reasoning

**QR1.** They appear to be the same.

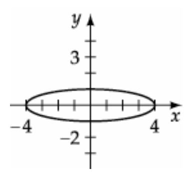
**QR2.** They appear to be the same.

**QR3.** 5.2 units

## Chapter Test

**1.**
$$y = \frac{1}{8}x^2$$
$$x^2 = 8y$$
$$4p = 8$$
$$p = 2$$

vertex: (0, 0) [8.1]
focus: (0, 2)
directrix: $y = -2$

**2.** 


[8.2]

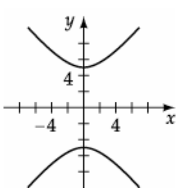
**3.**
$$25x^2 - 150x + 9y^2 + 18y + 9 = 0 \quad [8.2]$$
$$25(x^2 - 6x + 9) + 9(y^2 + 2y + 1) = -9 + 255 + 9$$
$$25(x-3)^2 + 9(y+1)^2 = 225$$
$$\frac{(x-3)^2}{9} + \frac{(y+1)^2}{25} = 1$$
$$a = 5 \quad b = 3 \quad c = 4$$

vertices: (3, 4), (3, –6)
foci: (3, 3), (3, –5)

**4.**
$$2b = 6 \qquad c = 6 \quad [8.2]$$
$$b = 3$$
$$a^2 = 9 + 36 = 45$$

center = (0, –3)

$$\frac{x^2}{45} + \frac{(y+3)^2}{9} = 1$$

**5.** 


[8.3]

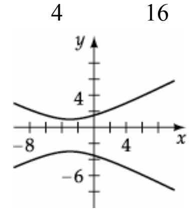
**6.**
$$\frac{x^2}{36} - \frac{y^2}{64} = 1$$

vertices: (6, 0), (–6, 0)

foci: (10, 0), (–10, 0)

asymptotes: $y = \pm\frac{4}{3}x$ [8.3]

**7.**
$$\frac{(y+1)^2}{4} - \frac{(x+3)^2}{16} = 1$$




[8.3]

**8.**
$$x^2 - 4xy - 5y^2 + 3x - 5y - 20 = 0 \quad [8.4]$$
$A = 1 \quad B = -4 \quad C = -5 \quad D = 3 \quad E = -5 \quad F = -20$

$\cot 2\alpha = \frac{A - C}{B} = \frac{1-(-5)}{-4} = -\frac{3}{2}$ $\quad 2\alpha$ is in quadrant II.

$$\tan 2\alpha = -\frac{2}{3}$$
$$2\alpha = \tan^{-1}\left(-\frac{2}{3}\right)$$
$$2\alpha \approx (-33.69° + 180°)$$
$$\alpha \approx 73.15°$$

**9.** $A = 8 \quad B = 5 \quad C = 2 \quad D = -10 \quad E = 5 \quad F = 4$

Since $B^2 - 4AC = (5)^2 - 4(8)(2) = -39 < 0$, the graph is an ellipse. [8.4]

**10.** $P(1, -\sqrt{3})$

$r = \sqrt{x^2 + y^2}$
$r = \sqrt{1^2 + (-\sqrt{3})^2}$
$r = 2$

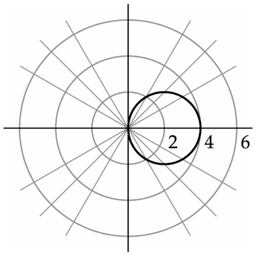
$r\cos\theta = x$
$2\cos\theta = 1$
$\cos\theta = \frac{1}{2}$

$r\sin\ \theta = y$
$2\sin\theta = -\sqrt{3}$
$\sin\theta = -\frac{\sqrt{3}}{2}$

$\theta$ is in quadrant IV.
$\theta = 300°$

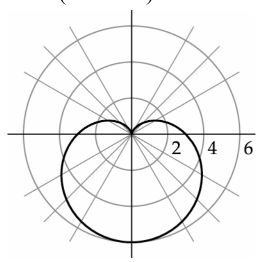
$P(1, -\sqrt{3}) = P(2,\ 300°)$ [8.5]

**11.** $r = 4\cos\theta$



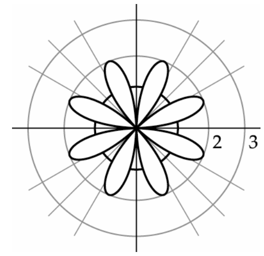

[8.5]

**12.** $r = 3(1 - \sin\ \theta)$




[8.5]

**13.** $r = 2\sin 4\theta$




[8.5]

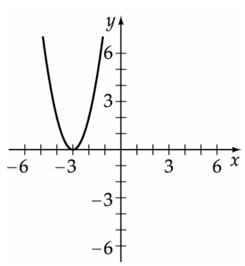
**14.**
$x = r\ \cos\ \theta$ $\quad$ $y = r\ \sin\ \theta$
$x = 5\cos\frac{7\pi}{3}$ $\quad$ $y = 5\sin\frac{7\pi}{3}$
$x = \frac{5}{2}$ $\quad$ $y = \frac{5\sqrt{3}}{2}$

The rectangular coordinates of the point are $(5/2,\ 5\sqrt{3}/2)$. [8.5]

**15.**
$r - r\cos x = 4$ [8.5]
$\sqrt{x^2 + y^2} - x = 4$
$\sqrt{x^2 + y^2} = x + 4$
$x^2 + y^2 = x^2 + 8x + 16$
$y^2 - 8x - 16 = 0$

**16.**
$r = \frac{4}{1 + \sin\theta}$ [8.6]
$r + r\sin\theta = 4$
$\sqrt{x^2 + y^2} + y = 4$
$x^2 + y^2 = 16 - 8y + y^2$
$x^2 + 8y - 16 = 0$

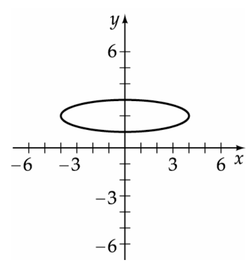
**17.**
$x = t - 3$
$x + 3 = t$
$(x+3)^2 = t^2$
$2(x+3)^2 = 2t^2$
$2(x+3)^2 = y$
$(x+3)^2 = \frac{1}{2}y$




[8.7]

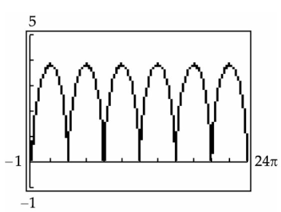
**18.**
$x = 4\sin\theta$ $\quad$ $y = \cos\theta + 2$
$\sin\ \theta = x/4$ $\quad$ $\cos\theta = y - 2$

$\sin^2\theta + \cos^2\theta = 1$
$\left(\frac{x}{4}\right)^2 + (y-2)^2 = 1$
$\frac{x^2}{16} + \frac{(y-2)^2}{1} = 1$



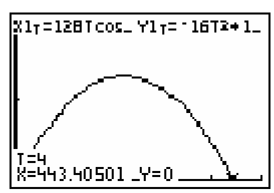

[8.7]

**19.**




Xscl = 2π [8.7]

**20.**




The projectile will travel $256\sqrt{3}$ feet $\approx$ 443 feet. [8.7]

## Cumulative Review

**1.** $x^4 - 2x^2 - 8 = 0$ [1.4]

Let $u = x^2$.

$u^2 - 2u - 8 = 0$

$(u - 4)(u + 2) = 0$

$u = 4$ or $u = -2$

$x^2 = 4$ $\quad x^2 = -2$

$x = \pm 2$ $\quad x = \pm i\sqrt{2}$

The solutions are 2, $-2$, $i\sqrt{2}$, $-i\sqrt{2}$.

**2.** $\frac{2}{x-1} - \frac{3}{x+2} = \frac{2(x+2) - 3(x-1)}{(x-1)(x+2)}$ [P.5]

$= \frac{2x + 4 - 3x + 3}{(x-1)(x+2)}$

$= \frac{-x + 7}{(x-1)(x+2)}$

**3.** $\frac{f(2+h) - f(2)}{h} = \frac{\left[1-(2+h)^2\right] - \left[1-(2)^2\right]}{h}$ [2.6]

$= \frac{1 - 4 - 4h - (h)^2 - 1 + 4}{h}$

$= \frac{-4h - h^2}{h}$

$= -4 - h$

**4.** $(f \circ g)(x) = f[g(x)]$ [2.6]

$= f[2 - x^2]$

$= 3(2 - x^2) + 2$

$= 6 - 3x^2 + 2$

$= -3x^2 + 8$

$(f \circ g)(-3) = -3(-3)^2 + 8$

$= -27 + 8$

$= -19$

**5.** By the Linear Factor Theorem, since the polynomial is of degree 6, there are 6 complex number solutions to $x^6 + 2x^4 - 3x^3 - x^2 + 5x - 7 = 0$. [3.4]

**6.** $m = \frac{2 - (-4)}{-3 - 1} = \frac{6}{-4} = -\frac{3}{2}$ [2.3]

$y - (-4) = -\frac{3}{2}(x - 1)$

$y + 4 = -\frac{3}{2}x + \frac{3}{2}$

$y = -\frac{3}{2}x - \frac{5}{2}$

**7.** $x = -3, y = 2$ [3.5]

**8.** $d = \sqrt{(-5-(-3))^2 + (4-(-1))^2}$

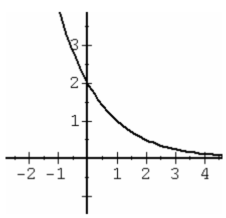
$= \sqrt{(-2)^2 + (5)^2}$

$= \sqrt{4 + 25}$

$= \sqrt{29}$

[2.1]

**9.**




[4.2]

**10.** $\log_2(x+3) - \log_2(x) = 2$ [4.5]

$\log_2 \frac{(x+3)}{x} = 2$

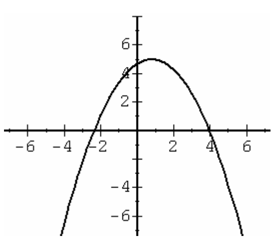
$2^2 = \frac{(x+3)}{x}$

$4x = x + 3$

$3x = 3$

$x = 1$

**11.**




[2.5]

**12.** $f(x) = 2x - 8$ [4.1]

$x = 2y - 8$

$x + 8 = 2y$

$\frac{1}{2}x + 4 = y$

$f^{-1}(x) = \frac{1}{2}x + 4$

**13.**

$$\begin{array}{r|rrrrr} 2i & 1 & 1 & -8 & 4 & -48 \\ & & 2i & -4+2i & -4-24i & 48 \\ \hline & 1 & 1+2i & -12+2i & -24i & 0 \end{array}$$

$$\begin{array}{r|rrrr} -2i & 1 & 1+2i & -12+2i & -24i \\ & & -2i & -2i & 24i \\ \hline & 1 & 1 & -12 & 0 \end{array}$$

$x^2 + x - 12 = (x-3)(x+4) = 0$

$x = 3,\ x = -4$

The remaining zeros are $-2i$ and $-4$. [3.4]

**14.** $f(-x) = \dfrac{3(-x)}{(-x)^2+1} = \dfrac{-3x}{x^2+1} = -f(x)$

odd [2.5]

**15.** $120^\circ = 120^\circ\left(\dfrac{\pi}{180^\circ}\right) = \dfrac{2\pi}{3}$ [5.1]

**16.** $\tan 40^\circ = \dfrac{a}{15}$ [5.2]

$a = 15\sin 40^\circ$

$a = 12.6$ cm

**17.** $f(x) = 3\cos 4x$ [5.5]

period: $\dfrac{\pi}{2}$

amplitude: 3

**18.** $\sin 2\alpha \tan\alpha = (2\sin\alpha\cos\alpha)\tan\alpha$ [6.3]

$= 2\sin\alpha\cos\alpha\dfrac{\sin\alpha}{\cos\alpha}$

$= 2\sin^2\alpha$

**19.** $\sin x\cos x - \dfrac{1}{2}\cos x = 0$ [6.6]

$\cos x\left(\sin x - \dfrac{1}{2}\right) = 0$

$\cos x = 0$

$x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

$\sin x - \dfrac{1}{2} = 0$

$\sin x = \dfrac{1}{2}$

$x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

**20.** $\mathbf{v}\cdot\mathbf{w} = (3\mathbf{i} - 4\mathbf{j})\cdot(4\mathbf{i} + \mathbf{j})$ [7.3]

$= 3(4) + (-4)(1)$

$= 12 - 4$

$= 8$

# Chapter 9
# Systems of Equations and Inequalities

## Section 9.1

**1.** $\begin{cases} 2x - 3y = 16 \\ x = 2 \end{cases}$

$$2(2) - 3y = 16$$
$$-3y = 12$$
$$y = -4$$

The solution is (2, –4).

**2.** $\begin{cases} 3x - 2y = -11 \\ y = 1 \end{cases}$

$$3x - 2(1) = -11$$
$$3x = -9$$
$$x = -3$$

The solution is (–3, 1).

**3.** $\begin{cases} 3x + 4y = 18 \\ y = -2x + 3 \end{cases}$

$$3x + 4(-2x + 3) = 18$$
$$3x - 8x + 12 = 18$$
$$-5x = 6$$
$$x = -\frac{6}{5}$$
$$y = -2\left(-\frac{6}{5}\right) + 3$$
$$y = \frac{27}{5}$$

The solution is $\left(-\frac{6}{5}, \frac{27}{5}\right)$.

**4.** $\begin{cases} 5x - 4y = -22 \\ y = 5x - 2 \end{cases}$

$$5x - 4(5x - 2) = -22$$
$$5x - 20x + 8 = -22$$
$$-15x = -30$$
$$x = 2$$
$$y = 5(2) - 2$$
$$y = 8$$

The solution is (2, 8).

**5.** $\begin{cases} -2x + 3y = 6 \\ x = 2y - 5 \end{cases}$

$$-2(2y - 5) + 3y = 6$$
$$-4y + 10 + 3y = 6$$
$$-y = -4$$
$$y = 4$$

$$x = 2(4) - 5$$
$$x = 3$$

The solution is (3, 4).

**6.** $\begin{cases} 8x + 3y = -7 \\ x = 3y + 15 \end{cases}$

$$8(3y + 15) + 3y = -7$$
$$24y + 120 + 3y = -7$$
$$27y = -127$$
$$y = -\frac{127}{27}$$
$$x = 3\left(-\frac{127}{27}\right) + 15$$
$$x = \frac{8}{9}$$

The solution is $\left(\frac{8}{9}, -\frac{127}{27}\right)$.

**7.** $\begin{cases} 6x + 5y = 1 & (1) \\ x - 3y = 4 & (2) \end{cases}$

Solve (2) for $x$ : $x = 3y + 4$

$$6(3y + 4) + 5y = 1$$
$$18y + 24 + 5y = 1$$
$$23y = -23$$
$$y = -1$$
$$x = 3(-1) + 4$$
$$x = 1$$

The solution is (1, –1).

**8.** $\begin{cases} -3x + 7y = 14 & (1) \\ 2x - y = -13 & (2) \end{cases}$

Solve (2) for $y$ : $y = 2x + 13$

$$-3x + 7(2x + 13) = 14$$
$$-3x + 14x + 91 = 14$$
$$11x = -77$$
$$x = -7$$
$$y = 2(-7) + 13$$
$$y = -1$$

The solution is (–7, –1).

**9.** $\begin{cases} 7x + 6y = -3 & (1) \\ y = \frac{2}{3}x - 6 & (2) \end{cases}$

$$7x + 6\left(\frac{2}{3}x - 6\right) = -3$$
$$7x + 4x - 36 = -3$$
$$11x = 33$$
$$x = 3$$
$$y = \frac{2}{3}(3) - 6$$
$$y = -4$$

The solution is (3, –4).

**10.** $\begin{cases} 9x - 4y = 3 & (1) \\ x = \frac{4}{3}y + 3 & (2) \end{cases}$

$$9\left(\frac{4}{3}y + 3\right) - 4y = 3$$
$$12y + 27 - 4y = 3$$
$$8y = -24$$
$$y = -3$$

$$x = \frac{4}{3}(-3) + 3$$
$$x = -1$$

The solution is (–1, –3).

**11.** $\begin{cases} y = 4x - 3 \\ y = 3x - 1 \end{cases}$

$$4x - 3 = 3x - 1$$
$$x = 2$$

$$y = 4(2) - 3$$
$$y = 5$$

The solution is (2, 5).

**12.** $\begin{cases} y = 5x + 1 \\ y = 4x - 2 \end{cases}$

$$5x + 1 = 4x - 2$$
$$x = -3$$

$$y = 5(-3) + 1$$
$$y = -14$$

The solution is (–3, –14).

**13.** $\begin{cases} y = 5x + 4 \\ x = -3y - 4 \end{cases}$

$$y = 5(-3y - 4) + 4$$
$$y = -15y - 20 + 4$$
$$16y = -16$$
$$y = -1$$

$$x = -3(-1) - 4$$
$$x = -1$$

The solution is (–1, –1).

**14.** $\begin{cases} y = -2x - 6 \\ x = -2y - 2 \end{cases}$

$$y = -2(-2y - 2) - 6$$
$$y = 4y + 4 - 6$$
$$-3y = -2$$
$$y = \frac{2}{3}$$

$$x = -2\left(\frac{2}{3}\right) - 2$$
$$x = -\frac{10}{3}$$

The solution is $\left(-\frac{10}{3}, \frac{2}{3}\right)$.

**15.** $\begin{cases} 3x - 4y = 2 & (1) \\ 4x + 3y = 14 & (2) \end{cases}$

Solve (1) for $x$ and substitute into (2).

$$3x = 4y + 2$$
$$x = \frac{4y + 2}{3}$$

$$4\left(\frac{4y + 2}{3}\right) + 3y = 14$$
$$16y + 8 + 9y = 42$$
$$25y = 34$$
$$y = \frac{34}{25}$$

$$x = \frac{4}{3}\left(\frac{34}{25}\right) + \frac{2}{3}$$
$$x = \frac{62}{25}$$

The solution is $\left(\frac{62}{25}, \frac{34}{25}\right)$.

**16.** $\begin{cases} 6x + 7y = -4 & (1) \\ 2x + 5y = 4 & (2) \end{cases}$

Solve (2) for $x$ and substitute into (1).

$$2x + 5y = 4$$
$$x = \frac{4 - 5y}{2}$$

$$6\left(\frac{4 - 5y}{2}\right) + 7y = -4$$
$$3(4 - 5y) + 7y = -4$$
$$12 - 15y + 7y = -4$$
$$-8y = -16$$
$$y = 2$$

$$x = \frac{4 - 5(2)}{2}$$
$$x = -3$$

The solution is (–3, 2).

**17.** $\begin{cases} 3x - 3y = 5 & (1) \\ 4x - 4y = 9 & (2) \end{cases}$

Solve (1) for $x$ and substitute into (2).

$$3x - 3y = 5$$
$$x = \frac{3y + 5}{3}$$

$$4\left(\frac{3y + 5}{3}\right) - 4y = 9$$
$$12y + 20 - 12y = 27$$
$$20 = 27$$

The system of equations is inconsistent and has no solution.

**18.** $\begin{cases} 3x - 4y = 8 & (1) \\ 6x - 8y = 9 & (2) \end{cases}$

Solve (1) for $x$ and substitute into (2).

$$3x - 4y = 8$$
$$x = \frac{4y + 8}{3}$$

$$6\left(\frac{4y + 8}{3}\right) - 8y = 9$$
$$8y + 16 - 8y = 9$$
$$16 = 9$$

The system of equations is inconsistent and has no solution.

**19.** $\begin{cases} 4x+3y=6 \\ y=-\frac{4}{3}x+2 \end{cases}$

$$4x+3\left(-\frac{4}{3}x+2\right)=6$$
$$4x-4x+6=6$$
$$0=0$$

The system of equations is dependent.
Let $x=c$ and $y=-\frac{4}{3}c+2$.
The solutions are $\left(c,\ -\frac{4}{3}c+2\right)$.

**20.** $\begin{cases} 5x+2y=2 \\ y=-\frac{5}{2}x+1 \end{cases}$

$$5x+2\left(-\frac{5}{2}x+1\right)=2$$
$$5x-5x+2=2$$
$$0=0$$

The system of equations is dependent.
Let $x=c$ and $y=-\frac{5}{2}c+1$.
The solutions are $\left(c,\ -\frac{5}{2}c+1\right)$.

**21.** $\begin{cases} 3x-y=10 & (1) \\ 4x+3y=-4 & (2) \end{cases}$

$9x-3y=30$ 3 times (1)
$4x+3y=-4$ (2)
$13x=26$
$x=2$
$3(2)-y=10$
$6-y=10$
$y=-4$
The solution is (2, –4).

**22.** $\begin{cases} 3x+4y=-5 & (1) \\ x-5y=-8 & (2) \end{cases}$

$3x+4y=-5$ (1)
$-3x+15y=24$ –3 times (2)
$19y=19$
$y=1$

$x-5(1)=-8$
$x=-3$
The solution is (–3, 1).

**23.** $\begin{cases} 4x+7y=21 & (1) \\ 5x-4y=-12 & (2) \end{cases}$

$20x+35y=105$ 5 times (1)
$-20x+16y=48$ –4 times (2)
$51y=153$
$y=3$

$4x+7(3)=21$
$x=0$
The solution is (0, 3).

**24.** $\begin{cases} 3x-8y=-6 & (1) \\ -5x+4y=10 & (2) \end{cases}$

$3x-8y=-6$ (1)
$-10x+8y=20$ 2 times (2)
$-7x=14$
$x=-2$
$3(-2)-8y=-6$
$-8y=0$
$y=0$
The solution is (–2, 0).

**25.** $\begin{cases} 5x-3y=0 & (1) \\ 10x-6y=0 & (2) \end{cases}$

$-10x+6y=0$ –2 times (1)
$10x-6y=0$ (2)
$0=0$

$5x-3c=0$
$x=\frac{3c}{5}$

The solution is $\left(\frac{3c}{5},\ c\right)$.

**26.** $\begin{cases} 3x+2y=0 & (1) \\ 2x+3y=0 & (2) \end{cases}$

$6x+4y=0$ 2 times (1)
$-6x-9y=0$ –3 times (2)
$-5y=0$
$y=0$
$3x+2(0)=0$
$x=0$

The solution is (0, 0).

**27.** $\begin{cases} 6x+6y=1 & (1) \\ 4x+9y=4 & (2) \end{cases}$

$12x+12y=2$ 2 times (1)
$-12x-27y=-12$ –3 times (2)
$-15y=-10$
$y=\frac{2}{3}$
$6x+6\left(\frac{2}{3}\right)=1$
$6x=-3$
$x=-\frac{1}{2}$

The solution is $\left(-\frac{1}{2},\ \frac{2}{3}\right)$.

**28.** $\begin{cases} 4x+5y=2 & (1) \\ 8x-15y=9 & (2) \end{cases}$

$$\begin{array}{rl} 12x+15y=6 & \text{3 times (1)} \\ 8x-15y=9 & (2) \\ \hline 20x \quad =15 & \\ x=\frac{3}{4} & \end{array}$$

$$4\left(\frac{3}{4}\right)+5y=2$$
$$5y=-1$$
$$y=-\frac{1}{5}$$

The solution is $\left(\frac{3}{4},\ -\frac{1}{5}\right)$.

**29.** $\begin{cases} 3x+6y=11 & (1) \\ 2x+4y=9 & (2) \end{cases}$

$$\begin{array}{rl} 6x+12y=\ \ 22 & \text{2 times (1)} \\ -6x-12y=-27 & -3\text{ times (2)} \\ \hline 0=-5 & \end{array}$$

The system of equations is inconsistent and has no solution.

**30.** $\begin{cases} 4x-2y=9 & (1) \\ 2x-\ \ y=3 & (2) \end{cases}$

$$\begin{array}{rl} 4x-2y=9 & (1) \\ -4x+2y=3 & -2\text{ times (2)} \\ \hline 0=3 & \end{array}$$

The system of equations is inconsistent and has no solution.

**31.** $\begin{cases} \frac{5}{6}x-\frac{1}{3}y=-6 & (1) \\ \frac{1}{6}x+\frac{2}{3}y=1 & (2) \end{cases}$

$$\begin{array}{rl} \frac{5}{3}x-\frac{2}{3}y=-12 & \text{2 times (1)} \\ \frac{1}{6}x+\frac{2}{3}y=\ \ 1 & (2) \\ \hline \frac{11}{6}x=-11 & \\ x=-6 & \end{array}$$

$$\frac{1}{6}x+\frac{2}{3}y=1$$
$$\frac{1}{6}(-6)+\frac{2}{3}y=1$$
$$\frac{2}{3}y=2$$
$$y=3$$

The solution is (–6, 3).

**32.** $\begin{cases} \frac{3}{4}x+\frac{2}{5}y=1 & (1) \\ \frac{1}{2}x-\frac{3}{5}y=-1 & (2) \end{cases}$

$\begin{cases} 15x+8y=20 & \text{20 times (1)} \\ 5x-6y=-10 & \text{10 times (2)} \end{cases}$

$$\begin{array}{r} 15x+8y=20 \\ -15x+18y=30 \\ \hline 26y=50 \\ y=\frac{25}{13} \end{array}$$

$$5x-6\left(\frac{25}{13}\right)=-10$$
$$5x=-10+\frac{150}{13}$$
$$5x=-10+\frac{150}{13}$$
$$5x=\frac{20}{13}$$
$$x=\frac{4}{13}$$

The solution is $\left(\frac{4}{13},\ \frac{25}{13}\right)$.

**33.** $\begin{cases} \frac{3}{4}x+\frac{1}{3}y=1 \\ \frac{1}{2}x+\frac{2}{3}y=0 \end{cases}$

$\begin{cases} 9x+4y=12 & (1) \\ 3x+4y=0 & (2) \end{cases}$

$$\begin{array}{rl} 9x+4y=12 & (1) \\ -3x-4y=0 & -1\text{ times (2)} \\ \hline 6x \quad =12 & \\ x=2 & \end{array}$$

$$3(2)+4y=0$$
$$4y=-6$$
$$y=-\frac{3}{2}$$

The solution is $\left(2,-\frac{3}{2}\right)$.

**34.** $\begin{cases} \frac{3}{5}x - \frac{2}{3}y = 7 \\ \frac{2}{5}x - \frac{5}{6}y = 7 \end{cases}$

$\begin{cases} 9x - 10y = 105 & (1) \\ 12x - 25y = 210 & (2) \end{cases}$

$$\begin{array}{rll} 36x - 40y = & 420 & \text{4 times (1)} \\ -36x + 75y = & -630 & -3 \text{ times (2)} \\ \hline 35y = & -210 & \end{array}$$

$$\begin{aligned} y &= -6 \\ 9x - 10(-6) &= 105 \\ 9x &= 45 \\ x &= 5 \end{aligned}$$

The solution is (5, –6).

**35.** $\begin{cases} 2\sqrt{3}x - 3y = 3 & (1) \\ 3\sqrt{3}x + 2y = 24 & (2) \end{cases}$

$$\begin{array}{rll} 6\sqrt{3}x - 9y = & 9 & \text{3 times (1)} \\ -6\sqrt{3}x - 4y = & -48 & -2 \text{ times (2)} \\ \hline -13y = & -39 & \end{array}$$

$$\begin{aligned} y &= 3 \\ 2\sqrt{3}x - 3(3) &= 3 \\ 2\sqrt{3}x &= 12 \\ \sqrt{3}x &= 6 \\ x &= 2\sqrt{3} \end{aligned}$$

The solution is $\left(2\sqrt{3},\ 3\right)$.

**36.** $\begin{cases} 4x - 3\sqrt{5}y = -19 & (1) \\ 3x + 4\sqrt{5}y = 17 & (2) \end{cases}$

$$\begin{array}{rll} 16x - 12\sqrt{5}y = & -76 & \text{4 times (1)} \\ 9x + 12\sqrt{5}y = & 51 & \text{3 times (2)} \\ \hline 25x \qquad = & -25 & \end{array}$$

$$\begin{aligned} x &= -1 \\ 3(-1) + 4\sqrt{5}y &= 17 \\ 4\sqrt{5}y &= 20 \\ \sqrt{5}y &= 5 \\ y &= \frac{5}{\sqrt{5}} \\ y &= \sqrt{5} \end{aligned}$$

The solution is $\left(-1,\ \sqrt{5}\right)$.

**37.** $\begin{cases} 3\pi x - 4y = 6 & (1) \\ 2\pi x + 3y = 5 & (2) \end{cases}$

$$\begin{array}{rll} 6\pi x - 8y = & 12 & \text{2 times (1)} \\ -6\pi x - 9y = & -15 & -3 \text{ times (2)} \\ \hline -17y = & -3 & \end{array}$$

$$y = \frac{3}{17}$$

$$\begin{array}{rll} 9\pi x - 12y = & 18 & \text{3 times (1)} \\ 8\pi x + 12y = & 20 & \text{4 times (2)} \\ \hline 17\pi x \qquad = & 38 & \end{array}$$

$$x = \frac{38}{17\pi}$$

The solution is $\left(\frac{38}{17\pi},\ \frac{3}{17}\right)$.

**38.** $\begin{cases} 2x - 5\pi y = 3 & (1) \\ 3x + 4\pi y = 2 & (2) \end{cases}$

$$\begin{array}{rll} 8x - 20\pi y = & 12 & \text{4 times (1)} \\ 15x + 20\pi y = & 10 & \text{5 times (2)} \\ \hline 23x \qquad = & 22 & \end{array}$$

$$x = \frac{22}{23}$$

$$\begin{array}{rll} 6x - 15\pi y = & 9 & \text{3 times (1)} \\ -6x - \ 8\pi y = & -4 & -2 \text{ times (2)} \\ -23\pi y = & 5 & \end{array}$$

$$y = -\frac{5}{23\pi}$$

The solution is $\left(\frac{22}{23},\ -\frac{5}{23\pi}\right)$.

**39.** $\begin{cases} 3\sqrt{2}x - 4\sqrt{3}y = -6 & (1) \\ 2\sqrt{2}x + 3\sqrt{3}y = 13 & (2) \end{cases}$

$$\begin{array}{rll} 6\sqrt{2}x - 8\sqrt{3}y = & -12 & \text{2 times (1)} \\ -6\sqrt{2}x - 9\sqrt{3}y = & -39 & -3 \text{ times (2)} \\ \hline -17\sqrt{3}y = & -51 & \end{array}$$

$$\begin{aligned} y &= \frac{3}{\sqrt{3}} \\ y &= \sqrt{3} \end{aligned}$$

$$\begin{array}{rl} 9\sqrt{2}x - 12\sqrt{3}y = & -18 \\ 8\sqrt{2}x + 12\sqrt{3}y = & 52 \\ \hline 17\sqrt{2}x \qquad = & 34 \end{array}$$

$$\begin{aligned} x &= \frac{2}{\sqrt{2}} \\ x &= \sqrt{2} \end{aligned}$$

The solution is $\left(\sqrt{2},\ \sqrt{3}\right)$.

**40.** $\begin{cases} 2\sqrt{2}x + 3\sqrt{5}y = 7 & (1) \\ 3\sqrt{2}x - \sqrt{5}y = -17 & (2) \end{cases}$

$$\begin{aligned} 6\sqrt{2}x + 9\sqrt{5}y &= 21 && \text{3 times (1)} \\ -6\sqrt{2}x + 2\sqrt{5}y &= 34 && -2 \text{ times (2)} \\ \hline 11\sqrt{5}y &= 55 \\ y &= \frac{5}{\sqrt{5}} \\ y &= \sqrt{5} \end{aligned}$$

$$\begin{aligned} 2\sqrt{2}x + 3\sqrt{5}y &= 7 && (1) \\ 9\sqrt{2}x - 3\sqrt{5}y &= -51 && \text{3 times (2)} \\ \hline 11\sqrt{2}x &= -44 \\ x &= \frac{-4}{\sqrt{2}} \\ x &= -2\sqrt{2} \end{aligned}$$

The solution is $\left(-2\sqrt{2},\ \sqrt{5}\right)$.

**41.** Solve the system by substitution.

$$\begin{aligned} 20p - 2000 &= -4p + 1000 \\ 24p &= 3000 \\ p &= 125 \end{aligned}$$

The solution is \$125.

**42.** Solve the system by substitution.

$$\begin{aligned} 25p - 500 &= -7p + 1100 \\ 32p &= 1600 \\ p &= 50 \end{aligned}$$

The solution is \$50.

**43.** Rate of plane with the wind: $r + w$
Rate of plane against the wind: $r - w$

$$r \cdot t = d$$

$$\begin{cases} (r + w) \cdot 3 = 450 \\ (r - w) \cdot 5 = 450 \end{cases}$$

$$\begin{aligned} r + w &= 150 \\ r - w &= 90 \\ \hline 2r \quad &= 240 \\ r &= 120 \\ 120 + w &= 150 \\ w &= 30 \end{aligned}$$

Rate of plane = 120 mph.
Rate of wind = 30mph.

**44.** Rate of plane with the wind: $r + w$
Rate of plane against the wind: $r - w$

$$r \cdot t = d$$

$$\begin{cases} (r + w) \cdot 4 = 800 \\ (r - w) \cdot 5 = 800 \end{cases}$$

$$\begin{aligned} r + w &= 200 \\ r - w &= 160 \\ \hline 2r \quad &= 360 \\ r &= 180 \\ 180 + w &= 200 \\ w &= 20 \end{aligned}$$

Rate of plane = 180 mph.
Rate of wind = 20 mph.

**45.** Rate of boat with the current: $r + w$
Rate of boat against the wind: $r - w$

$$r \cdot t = d$$

$$\begin{cases} (r + w) \cdot 4 = 120 \\ (r - w) \cdot 6 = 120 \end{cases}$$

$$\begin{aligned} r + w &= 30 \\ r - w &= 20 \\ \hline 2r \quad &= 50 \\ r &= 25 \\ 25 + w &= 30 \\ w &= 5 \end{aligned}$$

Rate of boat = 25 mph.
Rate of current = 5 mph.

**46.** Rate of canoeist with the current: $r + w$
Rate of canoeist against the current: $r - w$

$$r \cdot t = d$$
$$\begin{cases} (r+w)\cdot 2 = 12 \\ (r-w)\cdot 4 = 12 \end{cases}$$
$$\begin{aligned} r + w &= 6 \\ \underline{r - w} &\underline{= 3} \\ 2r \quad &= 9 \\ r &= 4.5 \\ 4.5 + w &= 6 \\ w &= 1.5 \end{aligned}$$

Rate of canoeist = 4.5 mph.
Rate of current = 1.5 mph.

**47.** $x$ = cost per kilogram of iron alloy
$y$ = cost per kilogram of lead alloy

$$\begin{cases} 30x + 45y = 1080 & (1) \\ 15x + 12y = 372 & (2) \end{cases}$$
$$\begin{cases} 30x + 45y = 1080 & (1) \\ \underline{-30x - 24y = -744} & -2 \text{ times } (2) \end{cases}$$
$$\begin{aligned} 21y &= 336 \\ y &= 16 \\ 15x + 12(16) &= 372 \\ 15x &= 180 \\ x &= 12 \end{aligned}$$

Cost of iron alloy: \$12 per kilogram
Cost of lead alloy: \$16 per kilogram

**48.** $x$ = cost of hydrochloric acid
$y$ = cost of silver nitrate

$$\begin{cases} 10x + 15y = 14.10 & (1) \\ 12x + 20y = 18.16 & (2) \end{cases}$$
$$\begin{aligned} 40x + 60y &= 56.40 && 4 \text{ times } (1) \\ \underline{-36x - 60y} &\underline{= -54.48} && -3 \text{ times } (2) \\ 4x \quad &= 1.92 \\ x &= 0.48 \\ 10(0.48) + 15y &= 14.10 \\ 15y &= 9.30 \\ y &= 0.62 \end{aligned}$$

Cost of hydrochloric acid: \$0.48/liter
Cost of silver nitrate: \$0.62/liter

**49.** $x$ = amount of 40% gold
$y$ = amount of 60% gold

$$\begin{cases} x + y = 20 & (1) \\ 0.40x + 0.60y = (0.52)(20) & (2) \end{cases}$$
$$\begin{aligned} -0.40x - 0.40y &= -8 && -0.40 \text{ times } (1) \\ \underline{0.40 + 0.60y} &\underline{= 10.4} && (2) \\ 0.20y &= 2.4 \\ y &= 12 \end{aligned}$$

$$\begin{aligned} x + 12 &= 20 \\ x &= 8 \end{aligned}$$

Amount of 40% gold: 8 g
Amount of 60% gold: 12 g

**50.** $x$ = amount of 70% solution
$y$ = amount of 30% solution

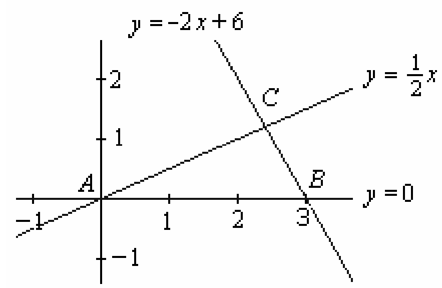
$$\begin{cases} x + y = 20 & (1) \\ 0.70x + 0.30y = 0.40(20) & (2) \end{cases}$$

$$\begin{aligned} -0.30x - 0.30y &= -6 \quad -0.30 \text{ times } (1) \\ 0.70x + 0.30y &= 8 \\ \hline 0.40x &= 2 \\ x &= 5 \end{aligned}$$

$$\begin{aligned} 5 + y &= 20 \\ y &= 15 \end{aligned}$$

Amount of 70% solution: 5 liters.
Amount of 30% solution: 15 liters.

**51.** Sketch a graph to visualize the right triangle.




To find the coordinates of point $A$, solve the system

$$\begin{cases} y = 0 \\ y = \frac{1}{2}x \end{cases}$$

By substitution, $\frac{1}{2}x = 0$

$x = 0$ Thus $A$ is (0, 0).

To find the coordinates of point $B$, solve the system

$$\begin{cases} y = 0 \\ y = -2x + 6 \end{cases}$$

By substitution, $-2x + 6 = 0$

$$\begin{aligned} -2x &= -6 \\ x &= 3 \end{aligned}$$

Thus $B$ is (3, 0).

To find the coordinates of the point $C$, solve the system

$$\begin{cases} y = -2x + 6 & (1) \\ y = \frac{1}{2}x & (2) \end{cases}$$

By substitution, $\frac{1}{2}x = -2x + 6$

$$\begin{aligned} \frac{5}{2}x &= 6 \\ x &= \frac{12}{5} \end{aligned}$$

Substituting $\frac{12}{5}$ for $x$ in Equation (2), we have

$y = \frac{1}{2}\left(\frac{12}{5}\right) = \frac{6}{5}$. Thus $C$ is $\left(\frac{12}{5}, \frac{6}{5}\right)$.

From the graph, $\angle C$ is the right angle.

Use the distance formula to find $AC$ and $BC$.

$$\begin{aligned} AC &= \sqrt{\left(\frac{12}{5} - 0\right)^2 + \left(\frac{6}{5} - 0\right)^2} \\ &= \sqrt{\frac{144}{25} + \frac{36}{25}} = \sqrt{\frac{180}{25}} = \frac{6}{5}\sqrt{5} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{\left(3 - \frac{12}{5}\right)^2 + \left(0 - \frac{6}{5}\right)^2} \\ &= \sqrt{\left(\frac{3}{5}\right)^2 + \left(-\frac{6}{5}\right)^2} = \sqrt{\frac{45}{25}} = \frac{3}{5}\sqrt{5} \end{aligned}$$

$$\begin{aligned} \text{Area} &= \frac{1}{2}(\text{base})(\text{height}) \\ &= \frac{1}{2}\left(\frac{6}{5}\sqrt{5}\right)\left(\frac{3}{5}\sqrt{5}\right) \\ &= \frac{9}{25}(5) \\ &= \frac{9}{5} \text{ square units} \end{aligned}$$

**52.** Solve the system

$$\begin{cases} 2x+3y=1 & (1) \\ 3x-4y=10 & (2) \end{cases}$$

$$\begin{array}{rl} 6x+9y=3 & \text{3 times (1)} \\ \underline{-6x+8y=-20} & -2 \text{ times (2)} \\ 17y=-17 & \\ y=-1 & \\ 2x+3(-1)=1 & (1) \\ 2x-3=1 & \\ 2x=4 & \\ x=2 & \end{array}$$

Thus the point is (2, –1).
Substitute this point into $4x+ky=5$.

$$\begin{aligned} 4(2)+k(-1)&=5 \\ 8-k&=5 \\ -k&=-3 \\ k&=3 \end{aligned}$$

**53.**

$$\begin{array}{r} 5Z7 \\ +\ \underline{256} \\ XY3 \end{array}$$

Case 1: $Z+5+1\le 9$

$$\begin{cases} Z+5+1=Y \\ 5+2=X \end{cases}$$

$$\begin{cases} Z+6=Y \\ 7=X \end{cases}$$

$$X+Y=7+Z+6$$
$$X+Y=Z+13$$

Case 2: $Z+5+1>9$

$$\begin{cases} Z+5+1=10+Y \\ 5+2+1=X \end{cases}$$

$$\begin{cases} Z-4=Y \\ 8=X \end{cases}$$

$$X+Y=8+Z-4$$
$$X+Y=Z+4$$

$XY3$ is divisible by 3 $\Rightarrow X+Y$ is divisible by 3.

If $Z$ + 13 is divisible by 3, then $Z$ = 2, 5, or 8.
If Z + 4 is divisible by 3, then $Z$ = 2, 5, or 8.

In both cases, the largest digit $Z$ can be is 8.

**54.** $2x+5=6x+k=4x-7$

$$2x+5=4x-7 \Rightarrow -2x=-12 \Rightarrow x=6$$

$$\begin{aligned} 2(6)+5&=6(6)+k \\ 12+5&=36+k \\ 17&=36+k \\ -19&=k \end{aligned}$$

**55.**

$$\begin{aligned} 14&=c-b \\ 126&=c+b \\ 140&=2c \\ 70&=c,\ b=56 \end{aligned}$$

$$\begin{aligned} 18&=c-b \\ 98&=c+b \\ 116&=2c \\ 58&=c,\ b=40 \end{aligned}$$

$$\begin{aligned} 294&=c-b \\ 6&=c+b \\ 300&=2c \\ 150&=c,\ b=144 \end{aligned}$$

$$\begin{aligned} 2&=c-b \\ 882&=c+b \\ 884&=2c \\ 442&=c,\ b=440 \end{aligned}$$

The Pythagorean triples are: 42, 56, 70; 42, 40, 58; 42, 144, 150; 42, 440, 442.

**56.** If $a = 30$, then $a^2 = 900$

$18 = c - b$, $50 = c + b$

$68 = 2c$

$34 = c, b = 16$

$90 = c - b$, $10 = c + b$

$100 = 2c$

$50 = c, b = 40$

$150 = c - b$, $6 = c + b$

$156 = 2c$

$78 = c, b = 72$

$2 = c - b$, $450 = c + b$

$452 = 2c$

$226 = c, b = 224$

The Pythagorean triples are: 30, 16, 34; 30, 40, 50; 30, 72, 78; 30, 224, 226.

**57.** $x$ = people who like lip balm but do not like skin cream
$y$ = people who like lip balm and skin cream
$z$ = people who do not like lip balm but do like skin cream
$w$ = people who do not like lip balm nor skin cream

$$x + y + z + w = 100$$
$$0.80(y + z) = y$$
$$0.50(x + w) = w$$
$$x + y = 77$$

Rewrite the system by solving eq (2) for $z$, eq (3) for $w$, and eq (4) for $x$.

$$x + y + z + w = 100$$
$$z = 0.25y$$
$$w = x$$
$$x = -y + 77$$

Substitute the values from equations (2), (3), and (4) into equation (1) and solve for $y$.

$$(-y + 77) + y + 0.25y + (-y + 77) = 100$$
$$-0.75y + 154 = 100$$
$$-0.75y = -54$$
$$y = 72$$

$z = 0.25(72) = 18$
$x = -72 + 77 = 5$
$w = 5$
Find the number of people who like skin cream ($y + z$)

$$y + z = 72 + 18 = 90$$

90 people liked the skin cream.

**58.** $x$ = people who pass the fire exam but not the chemical exam
$y$ = people who pass the fire exam and the chemical exam
$z$ = people who did not pass the fire exam but did pass the chemical exam
$w$ = people who did not pass either fire exam nor chemical exam

$$x+y+z+w=200$$
$$0.75(x+y)=y$$
$$0.25(z+w)=z$$
$$y+z=120$$

Rewrite the system by solving eq (2) for $x$, eq (3) for $w$, and eq (4) for $z$.

$$x+y+z+w=100$$
$$x=\frac{1}{3}y$$
$$w=3z$$
$$z=-y+120$$

Substitute the values from equations (2), (3), and (4) into equation (1) and solve for $y$.

$$\frac{1}{3}y+y+(-y+120)+3(-y+120)=200$$
$$-\frac{8}{3}y+480=200$$
$$-\frac{8}{3}y=-280$$
$$y=105$$

$z = -105 + 120 = 15$
$w = 3(15) = 45$
$x=\frac{1}{3}(105)=35$

Find the number of people who passed the fire exam ($x + y$)
$x+y=35+105=140$

140 people passed the basic fire science exam.

**59.** $S$ = supply pump
$A$ = outlet pump

$$\begin{cases} \frac{1}{2}S-\frac{1}{2}A=8750 & (1) \\ \frac{3}{4}S-2\left(\frac{3}{4}A\right)=11{,}250 & (2) \end{cases}$$

$$S-A=17{,}500 \quad \text{2 times (1)}$$
$$S-2A=15{,}000 \quad \frac{4}{3}\text{ times (2)}$$
$$A=2500$$
$$S=20{,}000$$

The supply pump can pump 20,000 gal/h.
The outlet pump can pump 2500 gal/h.

**60.** **a.** $x+y=40$

$$V=\pi r^2h+\frac{1}{3}\pi r^2h$$
$$\pi(2)^2y+\frac{1}{3}\pi(2)^2y=477.5$$

The system is $\begin{cases} x+y=40 \\ \pi(2)^2y+\frac{1}{3}\pi(2)^2y=477.5 \end{cases}$

**b.** $y=40-x$

$$4\pi x+\frac{4}{3}\pi(40-x)=477.5$$
$$4\pi x+\frac{4}{3}\pi 40-\frac{4}{3}\pi x=477.5$$
$$x\left(4\pi-\frac{4}{3}\pi\right)=477.5-\frac{160}{3}\pi$$
$$x=\frac{477.5-\frac{160}{3}\pi}{4\pi-\frac{4}{3}\pi}$$
$$x\approx 37.0\text{ ft}$$
$$y=40-x\approx 40-37.0=3.0\text{ ft}$$

## Connecting Concepts

**61.**

$$\begin{aligned}
(3+2i)x+(4-3i)y &= 2-16i\\
3x+2xi+4y-3yi &= 2-16i\\
(3x+4y)+(2x-3y)i &= 2-16i
\end{aligned}$$

$$\begin{cases} 2x+4y=2 & (1)\\ 2x-3y=-16 & (2)\end{cases}$$

$$\begin{aligned}
9x+12y &= 6 && \text{3 times (1)}\\
8x-12y &= -64 && \text{4 times (2)}\\
\hline
17x \quad &= -58\\
x &= -\frac{58}{17}\\
3x+4y &= 2\\
4y &= 2-3\left(-\frac{58}{17}\right)\\
4y &= \frac{208}{17}\\
y &= \frac{52}{17}
\end{aligned}$$

**62.**

$$\begin{aligned}
(4-3i)x+(5+2i)y &= 11+9i\\
4x-3xi+5y+2yi &= 11+9i\\
4x+5y+(-3x+2y)i &= 11+9i
\end{aligned}$$

$$\begin{cases} 4x+5y=11 & (1)\\ -3x+2y=9 & (2)\end{cases}$$

$$\begin{aligned}
12x+15y &= 33 && \text{3 times (1)}\\
-12x+8y &= 36 && \text{4 times (2)}\\
\hline
23y &= 69\\
y &= 3\\
4x+5(3) &= 11\\
4x &= -4\\
x &= -1
\end{aligned}$$

**63.**

$$\begin{aligned}
(2+6i)x+(4-5i)y &= -8-7i\\
2x+6xi+4y-5yi &= -8-7i\\
(2x+4y)+(6x-5y)i &= -8-7i
\end{aligned}$$

$$\begin{cases} 2x+4y=-8 & (1)\\ 6x-5y=-7 & (2)\end{cases}$$

$$\begin{aligned}
-6x-12y &= 24 && \text{−3 times (1)}\\
6x-5y &= -7 && (2)\\
\hline
-17y &= 17\\
y &= -1\\
2x+4(-1) &= -8\\
2x &= -4\\
x &= -2
\end{aligned}$$

## Prepare for Section 9.2

**PS1.**

$$\begin{aligned}
2x-5y &= 15\\
-5y &= -2x+15\\
y &= \frac{2}{5}x-3
\end{aligned}$$

**PS2.**

$$\begin{aligned}
x &= 2c+1\\
y &= -c+3\\
z &= 2x+5y-4\\
z &= 2(2c+1)+5(-c+3)-4\\
&= 4c+2-5c+15-4\\
&= -c+13
\end{aligned}$$

**PS3.** $\begin{cases} 5x-2y=10 \\ 2y=8 \end{cases}$

$y=4$

$$5x-2(4)=10$$
$$5x=18$$
$$x=\frac{18}{5}$$

The solution is $\left(\frac{18}{5}, 4\right)$.

**PS4.** $\begin{cases} 3x-y=11 & (1) \\ 2x+3y=-11 & (2) \end{cases}$

Solve (1) for $y$: $y=3x-11$

$$2x+3(3x-11)=-11$$
$$2x+9x-33=-11$$
$$11x=22$$
$$x=2$$
$$y=3(2)-11$$
$$y=-5$$

The solution is (2, –5).

**PS5.** $\begin{cases} y=3x-4 \\ y=4x-2 \end{cases}$

$$3x-4=4x-2$$
$$x=-2$$

$$y=3(-2)-4$$
$$y=-10$$

The solution is (–2, –10).

**PS6.** $\begin{cases} 4x+y=9 & (1) \\ -8x-2y=-18 & (2) \end{cases}$

Solve (1) for $y$: $y=-4x+9$

$$-8x-2(-4x+9)=-18$$
$$-8x+8x-18=-18$$
$$0=0$$

The system of equations is dependent.
Let $x = c$ and $y = -4c + 9$.
The solutions are $(c, -4c + 9)$.

## Section 9.2

**1.** $\begin{cases} 2x-y+z=8 & (1) \\ 2y-3z=-11 & (2) \\ 3y+2z=3 & (3) \end{cases}$

$$6y-9z=-33 \quad \text{3 times (2)}$$
$$-6y-4z=-6 \quad -2\text{ times (3)}$$
$$-13z=-39$$
$$z=3 \quad (4)$$

$\begin{cases} 2x-y+z=8 \\ 2y-3z=-11 \\ z=3 & (4) \end{cases}$

$$2y-3(3)=-11$$
$$y=-1$$

$$2x-(-1)+3=8$$
$$x=2$$

The solution is $(2, -1, 3)$.

**2.** $\begin{cases} 3x+y+2z=-4 & (1) \\ -3y-2z=-5 & (2) \\ 2y+5z=-4 & (3) \end{cases}$

$$-6y-4z=-10 \quad \text{2 times (2)}$$
$$6y+15z=-12 \quad \text{3 times (3)}$$
$$11z=-22$$
$$z=-2 \quad (4)$$

$\begin{cases} 3x+y+2z=-4 \\ -3y-2z=-5 \\ z=-2 & (4) \end{cases}$

$$-3y-2(-2)=-5$$
$$y=3$$
$$3x+3+2(-2)=-4$$
$$x=-1$$

The solution is $(-1, 3, -2)$.

**3.** $\begin{cases} x+3y-2z=8 & (1) \\ 2x-y+z=1 & (2) \\ 3x+2y-3z=15 & (3) \end{cases}$

$$\begin{array}{rl} -2x-6y+4z=-6 & -2 \text{ times } (1) \\ \underline{2x-\ y+\ z=\ 1} & (2) \\ -7y+5z=-15 & (4) \end{array}$$

$$\begin{array}{rl} -3x-9y+6z=-24 & -3 \text{ times } (1) \\ \underline{3x+2y-3z=15} & (3) \\ -7y+3z=-9 & (5) \end{array}$$

$\begin{cases} x+3y-2z=8 & \\ -7y+5z=-15 & (4) \\ -7y+3z=-9 & (5) \end{cases}$

$$\begin{array}{rl} -7y+5z=-15 & (4) \\ \underline{7y-3z=9} & -1 \text{ times } (5) \\ 2z=-6 & \\ z=-3 & (6) \end{array}$$

$\begin{cases} x+3y-2z=8 & \\ -7y+5z=-15 & \\ z=-3 & (6) \end{cases}$

$$-7y+5(-3)=-15 \qquad x+3(0)-2(3)=8$$
$$y=0 \qquad x=2$$

The solution is $(2,\ 0,\ -3)$.

**4.** $\begin{cases} x-2y+3z=5 & (1) \\ 3x-3y+z=9 & (2) \\ 5x+y-3z=3 & (3) \end{cases}$

$$\begin{array}{rl} -3x+6y-9z=-15 & -3 \text{ times } (1) \\ \underline{3x-3y+\ z=\ 9} & (2) \\ 3y-8z=-6 & (4) \end{array}$$

$$\begin{array}{rl} -5x+10y-15z=-25 & -5 \text{ times } (1) \\ \underline{5x+\ y-\ 3z=\ 3} & (3) \\ 11y-18z=-22 & (5) \end{array}$$

$\begin{cases} x-2y+3z=5 & \\ 3y-8z=-6 & (4) \\ 11y-18z=-22 & (5) \end{cases}$

$$\begin{array}{rl} 33y-88z=-66 & 11 \text{ times } (4) \\ \underline{-33y+54z=66} & -3 \text{ times } (5) \\ -34z=0 & \\ z=0 & (6) \end{array}$$

$\begin{cases} x-2y+3z=5 & \\ 3y-8z=-6 & \\ z=0 & (6) \end{cases}$

$$3y-8(0)=-6 \qquad x-2(-2)+3(0)=5$$
$$y=-2 \qquad x=1$$

The solution is $(1,\ -2,\ 0)$.

**5.** $\begin{cases} 3x+4y-z=-7 & (1) \\ x-5y+2z=19 & (2) \\ 5x+y-2z=5 & (3) \end{cases}$

$$\begin{array}{rl} 3x+4y-z=-7 & (1) \\ \underline{-3x+15y-6z=-57} & -3 \text{ times } (2) \\ 19y-7z=-64 & (4) \end{array}$$

$$\begin{array}{rl} -5x+25y-10z=-95 & -5 \text{ times } (2) \\ \underline{5x+y-2z=5} & (3) \\ 26y-12z=-90 & (5) \end{array}$$

$\begin{cases} 3x+4y-z=-7 \\ 19y-7z=-64 & (4) \\ 26y-12z=-90 & (5) \end{cases}$

$$\begin{array}{rl} 494y-182z=-1664 & 26 \text{ times } (4) \\ \underline{-494y+228z=1710} & -19 \text{ times } (3) \\ 46z=46 & \\ z=1 & (6) \end{array}$$

$\begin{cases} 3x+4y-z=-7 \\ 19y-7z=-64 \\ z=1 & (6) \end{cases}$

$19y-7(1)=-64$
$y=-3$

$3x+4(-3)-1=-7$
$x=2$

The solution is $(2, -3, 1)$.

**6.** $\begin{cases} 2x-3y-2z=12 & (1) \\ x+4y+z=-9 & (2) \\ 4x+2y-3z=6 & (3) \end{cases}$

$$\begin{array}{rl} 2x-3y-2z=12 & (1) \\ \underline{-2x-8y-2z=18} & -2 \text{ times } (2) \\ -11y-4z=30 & (4) \end{array}$$

$$\begin{array}{rl} -4x-16y-4z=36 & -4 \text{ times } (2) \\ \underline{4x+2y-3z=6} & (3) \\ -14y-7z=42 & \\ -2y-z=6 & (5) \end{array}$$

$\begin{cases} 2x-3y-2z=12 \\ -11y-4z=30 & (4) \\ -2y-z=6 & (5) \end{cases}$

$$\begin{array}{rl} -22y-8z=60 & 2 \text{ times } (4) \\ \underline{22y+11z=-66} & -11 \text{ times } (5) \\ 3z=-6 & \\ z=-2 & (6) \end{array}$$

$\begin{cases} 2x-3y-2z=12 \\ -11y-4z=30 \\ z=-2 & (6) \end{cases}$

$-11y-4(-2)=30$
$y=-2$

$2x-3(-2)-2(-2)=12$
$x=1$

The solution is $(1, -2, -2)$.

**7.** $\begin{cases} 2x-5y+3z=-18 & (1) \\ 3x+2y-z=-12 & (2) \\ x-3y-4z=-4 & (3) \end{cases}$

$$\begin{array}{rl} 3x+2y-\quad z=-12 & (2) \\ \underline{-3x+9y+12z=\ \ 12} & -2 \text{ times } (3) \\ 11y+11z=0 & \\ y+z=0 & (4) \end{array}$$

$$\begin{array}{rl} 2x-5y+3z=-18 & (1) \\ \underline{-2x+6y+8z=8} & -2 \text{ times } (3) \\ y+11z=-10 & (2) \end{array}$$

$\begin{cases} 2x-5y+3z=-18 & \\ y+z=0 & (4) \\ y+11z=-10 & (5) \end{cases}$

$$\begin{array}{rl} y+\quad z=0 & (4) \\ \underline{-y-11z=10} & -1 \text{ times } (5) \\ -10z=10 & \\ z=-1 & (6) \end{array}$$

$\begin{cases} 2x-5y+3z=-18 & \\ y+z=0 & \\ z=-1 & (6) \end{cases}$

$$\begin{aligned} y-1&=0 \\ y&=1 \end{aligned}$$

$$\begin{aligned} 2x-5(1)+3(-1)&=-18 \\ x&=-5 \end{aligned}$$

The solution is $(-5,\ 1,\ -1)$.

**8.** $\begin{cases} 4x-y+2z=-1 & (1) \\ 2x+3y-3z=-13 & (2) \\ x+5y+z=7 & (3) \end{cases}$

$$\begin{array}{rl} 4x-y+2z=-1 & (1) \\ \underline{-4x-6y+6z=26} & -2 \text{ times } (2) \\ -7y+8z=25 & (2) \end{array}$$

$$\begin{array}{rl} 2x+3y-3z=-13 & (2) \\ \underline{-2x-10y-2z=-14} & -2 \text{ times } (2) \\ -7y-5z=-27 & (5) \end{array}$$

$\begin{cases} 4x-y+2z=-1 & \\ -7y+8z=25 & (4) \\ -7y-5z=27 & (5) \end{cases}$

$$\begin{array}{rl} -7y+8z=25 & \\ \underline{7y+5z=27} & \\ 13z=52 & \\ z=4 & (6) \end{array}$$

$\begin{cases} 4x-y+2z=-1 & \\ -7y+8z=25 & \\ z=4 & (6) \end{cases}$

$$\begin{aligned} -7y+8(4)&=25 \\ y&=1 \end{aligned}$$

$$\begin{aligned} 4x-1+2(4)&=-1 \\ x&=-2 \end{aligned}$$

The solution is $(-2,\ 1,\ 4)$.

**9.**

$$\begin{cases} x+2y-3z=-7 & (1) \\ 2x-y+4z=11 & (2) \\ 4x+3y-4z=-3 & (3) \end{cases}$$

$$\begin{array}{rl} -2x-4y+6z=14 & -2 \text{ times } (1) \\ \underline{2x-y+4z=11} & (2) \\ -5y+10z=25 & (4) \end{array}$$

$$\begin{array}{rl} -4x-8y+12z=28 & -4 \text{ times } (1) \\ \underline{4x+3y-4z=-3} & (3) \\ -5y+8z=25 & (5) \end{array}$$

$$\begin{cases} x+2y-3z=-7 & \\ -y+2z=5 & (4) \\ -5y+8z=25 & (5) \end{cases}$$

$$\begin{array}{rl} 5y-10z=-25 & -1 \text{ times } (4) \\ \underline{-5y+8z=25} & (5) \\ -2z=0 & \\ z=0 & (6) \end{array}$$

$$\begin{cases} x+2y-3z=-7 & \\ -y+2z=5 & \\ z=0 & (6) \end{cases}$$

$$-y+2(0)=5$$
$$y=-5$$

$$x+2(-5)-3(0)=-7$$
$$x=3$$

The solution is $(3,\ -5,\ 0)$.

**10.**

$$\begin{cases} x-3y+2z=-11 & (1) \\ 3x+y+4z=4 & (2) \\ 5x-5y+8z=-18 & (3) \end{cases}$$

$$\begin{array}{rl} -3x+9y-6z=33 & -3 \text{ times } (1) \\ \underline{3x+y+4z=4} & (2) \\ 10y-2z=37 & (4) \end{array}$$

$$\begin{array}{rl} -5x+15y-10z=55 & -5 \text{ times } (1) \\ \underline{5x-5y+8z=-18} & (3) \\ 10y-2z=37 & (5) \end{array}$$

$$\begin{cases} x-3y+2z=-11 & \\ 10y-2z=37 & (4) \\ 10y-2z=37 & (5) \end{cases}$$

$$\begin{array}{rl} 10y-2z=37 & (4) \\ \underline{-10y+2z=-37} & -1 \text{ times } (5) \\ 0=0 & \end{array}$$

The system is dependent.

Let $z=c$. $10y-2c=37$

$$y=\frac{37+2c}{10}$$

$$x-3\left(\frac{37+2c}{10}\right)+2c=-11$$

$$x-\frac{111}{10}-\frac{6c}{10}=-11-2c$$

$$x=\frac{111}{10}-\frac{110}{10}-\frac{20c}{10}+\frac{6c}{10}$$

$$x=\frac{1-14c}{10}$$

The solution is $\left(\frac{1-14c}{10},\ \frac{37+2c}{10},\ c\right)$.

**11.** $\begin{cases} 2x-5y+2z=-4 & (1) \\ 3x+2y+3z=13 & (2) \\ 5x-3y-4z=-18 & (3) \end{cases}$

$$\begin{array}{rll} 6x-15y+6z=-12 & & \text{3 times (1)} \\ \underline{-6x-\ 4y-6z=-26} & & \text{−2 times (2)} \\ -19y \qquad =-38 & & \\ y \qquad =2 & & (4) \end{array}$$

$$\begin{array}{rll} 10x-25y+10z=-20 & & \text{5 times (1)} \\ \underline{-10x+\ 6y+\ 8z=\ 36} & & \text{− 2 times (3)} \\ -19y+18z=16 & & (5) \end{array}$$

$$\begin{cases} 2x-5y+2z=-4 & \\ y=2 & (4) \\ -19y+18z=16 & (5) \end{cases}$$

$$\begin{array}{rll} 19y \qquad =38 & & \text{19 times (4)} \\ \underline{-19y-18z=16} & & (5) \\ 18z=54 & & \\ z=3 & & (6) \end{array}$$

$$\begin{cases} 2x-5y+2z=-4 & \\ y=2 & \\ z=3 & (6) \end{cases}$$

$$\begin{aligned} 2x-5(2)+2(3)&=-4 \\ x&=0 \end{aligned}$$

The solution is (0, 2, 3).

**12.** $\begin{cases} 3x+2y-5z=6 & (1) \\ 5x-4y+3z=-12 & (2) \\ 4x+5y-2z=15 & (3) \end{cases}$

$$\begin{array}{rll} 15x+10y-25z=30 & & \text{5 times (1)} \\ \underline{-15x+12y-\ 9z=36} & & \text{− 3 times (2)} \\ 22y-34z=66 & & \\ 11y-17z=33 & & (4) \end{array}$$

$$\begin{array}{rll} 12x+\ 8y-20z=\ \ 24 & & \text{4 times (1)} \\ -12x-15y+\ 6z=-45 & & \text{− 3 times (3)} \\ -7y-14z=-21 & & \\ y+2z=3 & & (5) \end{array}$$

$$\begin{cases} 3x+2y-5z=6 & \\ 11y-17z=33 & (4) \\ y+2z=3 & (5) \end{cases}$$

$$\begin{array}{rll} 11y-17z=33 & & (4) \\ \underline{-11y-22z=-33} & & \text{−11 times (5)} \\ -39z=0 & & \\ z=0 & & (6) \end{array}$$

$$\begin{cases} 3x+2y-5z=6 \\ 11y-17z=33 \\ z=0 \end{cases}$$

$$\begin{aligned} 11y-17(0)&=33 \\ y&=3 \end{aligned}$$

$$\begin{aligned} 3x+2(3)-5(0)&=6 \\ x&=0 \end{aligned}$$

The solution is (0, 3, 0).

**13.** $\begin{cases} 2x+y-z=-2 & (1) \\ 3x+2y+3z=21 & (2) \\ 7x+4y+z=17 & (3) \end{cases}$

$$\begin{array}{rl} 6x+3y-3z=-6 & 3 \text{ times } (1) \\ \underline{-6x-4y-6z=-42} & -2 \text{ times } (2) \\ -y-9z=-48 & (4) \end{array}$$

$$\begin{array}{rl} 14x+7y-7z=-14 & 7 \text{ times } (1) \\ \underline{-14x-8y-2z=-34} & -2 \text{ times } (3) \\ -y-9z=-48 & (5) \end{array}$$

$\begin{cases} 2x+y-z=-2 \\ -y-9z=-48 & (4) \\ -y-9z=-48 & (5) \end{cases}$

$$\begin{array}{rl} -y-9z=-48 & (4) \\ \underline{y+9z=48} & -1 \text{ times } (5) \\ 0=0 & (6) \end{array}$$

$\begin{cases} 2x+y-z=-2 \\ -y-9z=-48 \\ 0=0 & (6) \end{cases}$

The system of equations is dependent.

Let $z=c$. $\quad -y-9c=-48$

$y=48-9c$

$2x+(48-9c)-c=-2$

$x=5c-25$

The solution is $(5c-25,\ 48-9c,\ c)$.

**14.** $\begin{cases} 3x+y+2z=2 & (1) \\ 4x-2y+z=-4 & (2) \\ 11x-3y+4z=-6 & (3) \end{cases}$

$$\begin{array}{rl} 12x+4y+8z=8 & 4 \text{ times } (1) \\ \underline{-12x+6y-3z=12} & -3 \text{ times } (2) \\ 10y+5z=20 & \\ 2y+z=4 & (4) \end{array}$$

$$\begin{array}{rl} 33x+11y+22z=22 & 11 \text{ times } (1) \\ \underline{-33x+9y-12z=18} & -3 \text{ times } (3) \\ 20y+10z=40 & \\ 2y+z=4 & (5) \end{array}$$

$\begin{cases} 3x+y+2z=2 \\ 2y+z=4 & (4) \\ 2y+z=4 & (5) \end{cases}$

$$\begin{array}{rl} 2y+z=4 & (4) \\ \underline{-2y-z=-4} & -1 \text{ times } (5) \\ 0=0 & (6) \end{array}$$

$\begin{cases} 3x+y+2z=2 \\ 2y+z=4 \\ 0=0 & (6) \end{cases}$

The system of equations is dependent.

Let $z=c$ $\quad 2y+c=4$

$y=\frac{4-c}{2}$

$3x+\frac{4-c}{2}+2c=2$

$3x=2-2c-\frac{4-c}{2}$

$x=-\frac{c}{2}$

The solution is $\left(-\frac{c}{2},\ \frac{4-c}{2},\ c\right)$.

**15.** $\begin{cases} 3x-2y+3z=11 & (1) \\ 2x+3y+z=3 & (2) \\ 5x+14y-z=1 & (3) \end{cases}$

$$\begin{aligned} 6x-4y+6z&=22 \quad \text{2 times (1)} \\ \underline{-6x-9y-3z}&\underline{=-9} \quad -3\text{ times (2)} \\ 13y+3z&=13 \quad (4) \end{aligned}$$

$$\begin{aligned} 15x-10y+15z&=55 \quad \text{5 times (1)} \\ \underline{-15x-42y+3z}&\underline{=-3} \quad -3\text{ times (3)} \\ -52y+18z&=52 \\ -26y+9z&=26 \quad (5) \end{aligned}$$

$\begin{cases} 3x-2y+3z=11 \\ -13y+3z=13 & (4) \\ -26y+9z=26 & (5) \end{cases}$

$$\begin{aligned} 26y-6z&=-26 \quad -2\text{ times (4)} \\ \underline{-26y+9z}&\underline{=26} \quad (5) \\ z&=0 \quad (6) \end{aligned}$$

$\begin{cases} 3x-2y+3z=11 \\ -13y+3z=13 \\ z=0 & (6) \end{cases}$

$$\begin{aligned} -31y+3(0)&=13 \\ y&=-1 \end{aligned} \qquad \begin{aligned} 3x-2(-1)+3(0)&=11 \\ x&=3 \end{aligned}$$

The solution is $(3,\ -1,\ 0)$.

**16.** $\begin{cases} 2x+3y+2z=14 & (1) \\ x-3y+4z=4 & (2) \\ -x+12y-6z=2 & (3) \end{cases}$

$$\begin{aligned} 2x+3y+2z&=14 \quad (1) \\ \underline{-2x+6y-8z}&\underline{=-8} \quad -2\text{ times (2)} \\ 9y-6z&=6 \\ 3y-2z&=2 \quad (4) \end{aligned}$$

$$\begin{aligned} x-3y+4z&=4 \quad (2) \\ \underline{-x+12y-6z}&\underline{=2} \quad (3) \\ 9y-2z&=6 \quad (5) \end{aligned}$$

$\begin{cases} 2x+3y+2z=14 \\ 3y-2z=2 & (4) \\ 9y-2z=6 & (5) \end{cases}$

$$\begin{aligned} -9y+6z&=-6 \quad -3\text{ times (4)} \\ \underline{9y-2z}&\underline{=6} \quad (5) \\ 4z&=0 \\ z&=0 \quad (6) \end{aligned}$$

$\begin{cases} 2x+3y+2z=14 \\ 3y-2z=2 \\ z=0 \end{cases}$

$$\begin{aligned} 3y-20&=2 \\ y&=\frac{2}{3} \end{aligned} \qquad \begin{aligned} 2x+3\left(\frac{2}{3}\right)+2(0)&=14 \\ x&=6 \end{aligned}$$

The solution is $\left(6,\ \frac{2}{3},\ 0\right)$.

**17.** $\begin{cases} 2x-3y+6z=3 & (1) \\ x+2y-4z=5 & (2) \\ 3x+4y-8z=7 & (3) \end{cases}$

$$\begin{array}{ll} 2x-3y+6z=3 & (1) \\ \underline{-2x-4y+8z=-10} & -2 \text{ times } (2) \\ -7y+14z=-7 & \\ -y+2z=-1 & (4) \end{array}$$

$$\begin{array}{ll} -3x-6y+12z=-15 & -3 \text{ times } (2) \\ \underline{3x+4y-8z=7} & (3) \\ -2y+4z=-8 & \\ -y+2z=-4 & (5) \end{array}$$

$\begin{cases} 2x-3y+6z=3 & \\ -y+2z=-1 & (4) \\ -y+2z=-4 & (5) \end{cases}$

$$\begin{array}{ll} -y+2z=-1 & (4) \\ \underline{y-2z=4} & -1 \text{ times } (5) \\ 0=3 & (6) \end{array}$$

$\begin{cases} 2x-3y+6z=3 & \\ -y+2z=-1 & \\ 0=3 & (6) \end{cases}$

The system of equations is inconsistent and has no solution.

**18.** $\begin{cases} 2x+3y-6z=4 & (1) \\ 3x-2y-9z=-7 & (2) \\ 2x+5y-6z=8 & (3) \end{cases}$

$$\begin{array}{ll} 6x+9y-18z=12 & 3 \text{ times } (1) \\ \underline{-6x+4y+18z=14} & -2 \text{ times } (2) \\ 13y \quad =26 & \\ y=2 & (4) \end{array}$$

$$\begin{array}{ll} 6x-4y-18z=-14 & 3 \text{ times } (2) \\ \underline{-6x-15y+18z=-24} & -3 \text{ times } (3) \\ -19y \quad =-38 & \\ y=2 & (5) \end{array}$$

$\begin{cases} 2x+3y-6z=5 & \\ y \quad =2 & (4) \\ y \quad =2 & (5) \end{cases}$

$$\begin{array}{ll} y=2 & (4) \\ \underline{-y=-2} & -1 \text{ times } (5) \\ 0=0 & (6) \end{array}$$

$\begin{cases} 2x+3y-6z=5 & \\ y \quad =2 & \\ 0=0 & (6) \end{cases}$

The system of equations is dependent.

Let $z=c$. $\quad y=2 \quad 2x+3(2)-6c=4$

$x=3c-1$

The solution is $(3c-1,\ 2,\ c)$.

**19.** $\begin{cases} 2x-3y+5z=14 & (1) \\ x+4y-3z=-2 & (2) \end{cases}$

$$\begin{array}{ll} 2x-3y+5z=14 & (1) \\ \underline{-2x-8y+6z=4} & -2 \text{ times } (2) \\ -11y+11z=18 & (3) \end{array}$$

$\begin{cases} 2x-3y+5z=14 & \\ -11y+11z=18 & (3) \end{cases}$

Let $z=c$. $\quad -11y+11c=18$

$$y=\frac{18-11c}{-11}$$

$$y=\frac{11c-18}{11}$$

$$2x-3\left(\frac{11c-18}{11}\right)+5c=14$$

$$2x=14-5c+\frac{33c-54}{11}$$

$$2x=\frac{154-55c+33c-54}{11}$$

$$x=\frac{50-11c}{11}$$

The solution is $\left(\frac{50-11c}{11},\ \frac{11c-18}{11},\ c\right)$.

**20.** $\begin{cases} x-3y+4z=9 & (1) \\ 3x-8y-2z=4 & (2) \end{cases}$

$$\begin{array}{ll} -3x+9y-12z=-27 & -3 \text{ times } (1) \\ \underline{3x-8y-2z=4} & (2) \\ y-14z=-23 & (3) \end{array}$$

$\begin{cases} x-3y+4z=9 & \\ y-14z=-23 & (3) \end{cases}$

Let $z=c$. $\quad y-14c=-23$

$y=14c-23$

$$x-3(14c-23)+4c=9$$

$$x=9-4c+42c-69$$

$$x=38c-60$$

The solution is $(38c-60,\ 14c-23,\ c)$.

**21.** $\begin{cases} 6x-9y+6z=7 & (1) \\ 4x-6y+4z=9 & (2) \end{cases}$

$$\begin{array}{rl} 24x-36y+24z=28 & 4 \text{ times } (1) \\ -24x+36y-24z=-54 & -6 \text{ times } (2) \\ \hline 0=-26 & \end{array}$$

$\begin{cases} 6x-9y+6z=7 & \\ 0=-26 & (3) \end{cases}$

The system of equations is inconsistent and has no solution.

**22.** $\begin{cases} 4x-2y+6z=5 & (1) \\ 2x-y+3z=2 & (2) \end{cases}$

$$\begin{array}{rl} 4x-2y+6z=5 & (1) \\ -4x+2y-6z=-4 & -2 \text{ times } (2) \\ \hline 0=1 & (3) \end{array}$$

$\begin{cases} 4x-2y+6z=5 & \\ 0=1 & (3) \end{cases}$

The system of equations is inconsistent and has no solution.

**23.** $\begin{cases} 5x+3y+2z=10 & (1) \\ 3x-4y-4z=-5 & (2) \end{cases}$

$$\begin{array}{rl} 15x+9y+6z=30 & 3 \text{ times } (1) \\ -15x+20y+20z=25 & -5 \text{ times } (2) \\ \hline 29y+26z=55 & (3) \end{array}$$

$\begin{cases} 5x+3y+2z=10 & \\ 29y+26z=55 & (3) \end{cases}$

Let $z=c$. $\quad 29y+26c=55$

$$y=\frac{55-26c}{29}$$

$$5x+3\left(\frac{55-26c}{29}\right)+2c=10$$

$$5x=10-2c-\frac{165-78c}{29}$$

$$5x=\frac{290-58c-165+78c}{29}$$

$$x=\frac{25+4c}{29}$$

The solution is $\left(\frac{25+4c}{29}, \frac{55-26c}{29}, c\right)$.

**24.** $\begin{cases} 3x-4y-7z=-5 & (1) \\ 2x+3y-5z=2 & (2) \end{cases}$

$$\begin{array}{rl} 6x-8y-14z=-10 & 2 \text{ times } (1) \\ -6x-9y+15z=-6 & -3 \text{ times } (2) \\ \hline -17y+z=-16 & (3) \end{array}$$

$\begin{cases} 3x-4y-7z=-5 & \\ -17y+z=-16 & (3) \end{cases}$

Let $z=c$ $\quad -17y+c=-16$

$$y=\frac{16+c}{17}$$

$$3x-4y-7z=-5$$

$$3x-4\left(\frac{16+c}{17}\right)-7c=-5$$

$$3x=7c-5+\frac{64+4c}{17}$$

$$3x=\frac{119c-85+64+4c}{17}$$

$$x=\frac{41c-7}{17}$$

The solution is $\left(\frac{41c-7}{17}, \frac{16+c}{17}, c\right)$.

**25.** $\begin{cases} x+3y-4z=0 & (1) \\ 2x+7y+z=0 & (2) \\ 3x-5y-2z=0 & (3) \end{cases}$

$$\begin{array}{rl} -2x-6y+8z=0 & -2 \text{ times } (1) \\ \underline{2x+7y+z=0} & (2) \\ y+9z=0 & (4) \end{array}$$

$$\begin{array}{rl} -3x-9y+12z=0 & -3 \text{ times } (1) \\ \underline{3x-5y-2z=0} & (3) \\ -14y+10z=0 & \\ -7y+5z=0 & (5) \end{array}$$

$\begin{cases} x+3y-4z=0 \\ y+9z=0 & (4) \\ -7y+5z=0 & (5) \end{cases}$

$$\begin{array}{rl} 7y+63z=0 & 7 \text{ times } (4) \\ \underline{-7y+5z=0} & (5) \\ 68z=0 & \\ z=0 & (6) \end{array}$$

$\begin{cases} x+3y-4z=0 \\ y+9z=0 \\ z=0 & (6) \end{cases}$

$$\begin{array}{rr} y+9(0)=0 & x+3(0)-4(0)=0 \\ y=0 & x=0 \end{array}$$

The solution is (0, 0, 0).

**26.** $\begin{cases} x-2y+3z=0 & (1) \\ 3x-7y-4z=0 & (2) \\ 4x-4y+z=0 & (3) \end{cases}$

$$\begin{array}{rl} -3x+6y-9z=0 & -3 \text{ times } (1) \\ \underline{3x-7y-4z=0} & (2) \\ -y-13z=0 & (4) \end{array}$$

$$\begin{array}{rl} -4x+8y-12z=0 & -4 \text{ times } (1) \\ \underline{4x-4y+z=0} & (3) \\ 4y-11z=0 & (5) \end{array}$$

$\begin{cases} x-2y+3z=0 \\ -y-13z=0 & (4) \\ 4y-11z=0 & (5) \end{cases}$

$\begin{cases} x-2y+3z=0 \\ -y-13z=0 \\ z=0 & (6) \end{cases}$

$$\begin{array}{rl} -4y-42z=0 & 4 \text{ times } (4) \\ \underline{4y-11z=0} & (5) \\ -53z=0 & \\ z=0 & (6) \end{array}$$

$z=0$, $y=0$, $x=0$. The solution is (0, 0, 0).

**27.** $\begin{cases} 2x-3y+z=0 & (1) \\ 2x+4y-3z=0 & (2) \\ 6x-2y-z=0 & (3) \end{cases}$

$$\begin{array}{rl} -2x+3y-z=0 & -1 \text{ times } (1) \\ \underline{2x+4y-3z=0} & (2) \\ 7y-4z=0 & (4) \end{array}$$

$$\begin{array}{rl} -6x+9y-3z=0 & -3 \text{ times } (1) \\ \underline{6x-2y-z=0} & (3) \\ 7y-4z=0 & (5) \end{array}$$

$$\begin{cases} 2x-3y+z=0 \\ 7y-4z=0 & (4) \\ 7y-4z=0 & (5) \end{cases}$$

$$\begin{array}{rl} -7y+4z=0 & -1 \text{ times } (4) \\ \underline{7y-4z=0} & (5) \\ 0=0 & \end{array}$$

$$\begin{cases} 2x-3y+z=0 \\ 7y-4z=0 \\ 0=0 \end{cases}$$

Let $z=c$. Then $7y=4c$ or $y=\frac{4}{7}c$. Substitute for $y$ and $z$ in Eq. (1) and solve for $x$.

$$2x-3\left(\frac{4}{7}c\right)+c=0$$
$$2x=\frac{5}{7}c$$
$$x=\frac{5}{14}c$$

The solution is $\left(\frac{5}{14}c,\ \frac{4}{7}c,\ c\right)$.

**28.** $\begin{cases} 5x-4y-3z=0 & (1) \\ 2x+y+2z=0 & (2) \\ x-6y-7z=0 & (3) \end{cases}$

$$\begin{array}{rl} 10x-8y-6z=0 & 2 \text{ times } (1) \\ \underline{-10x-5y-10z=0} & -5 \text{ times } (2) \\ -13y-16z=0 & (4) \end{array}$$

$$\begin{array}{rl} 2x+y+2z=0 & (2) \\ \underline{-2x+12y+14z=0} & -2 \text{ times } (3) \\ 13y+16z=0 & (5) \end{array}$$

$$\begin{cases} 5x-4y-3z=0 \\ -13y-16z=0 & (4) \\ 13y+16z=0 \end{cases}$$

$$\begin{array}{rl} -13y-16z=0 & \\ \underline{13y+16z=0} & \\ 0=0 & (6) \end{array}$$

$$\begin{cases} 5x-4y-3z=0 \\ -13y-16z=0 \\ 0=0 & (6) \end{cases}$$

Let $z=c$. $\quad -13y-16c=0$

$$y=-\frac{16c}{13}$$

$$5x-4\left(-\frac{16c}{13}\right)-3c=0$$
$$5x=3c-\frac{64c}{13}$$
$$5x=-\frac{25c}{13}$$
$$x=-\frac{5}{13}c$$

The solution is $\left(-\frac{5}{13}c,\ -\frac{16}{13}c,\ c\right)$.

**29.** $\begin{cases} 3x-5y+3z=0 & (1) \\ 2x-3y+4z=0 & (2) \\ 7x-11y+11z=0 & (3) \end{cases}$

$$\begin{array}{rl} -6x+10y-6z=0 & -2 \text{ times } (1) \\ \underline{6x-9y+12z=0} & 3 \text{ times } (2) \\ y+6z=0 & (4) \end{array}$$

$$\begin{array}{rl} -21x+35y-21z=0 & -7 \text{ times } (1) \\ \underline{21x-33y+33z=0} & 3 \text{ times } (3) \\ 2y+12z=0 & (5) \end{array}$$

$\begin{cases} 3x-5y+3z=0 & (1) \\ y+6z=0 & (4) \\ 2y+12z=0 & (5) \end{cases}$

$$\begin{array}{rl} -2y-12z=0 & -2 \text{ times } (4) \\ \underline{2y+12z=0} & (5) \\ 0=0 & (6) \end{array}$$

$\begin{cases} 3x-5y+3z=0 & (1) \\ y+6z=0 & (4) \\ 0=0 & (6) \end{cases}$

From Eq. (4), $y=-6z$. Substitute into Eq. (1).

$$\begin{aligned} 3x-5(-6z)+3z&=0 \\ 3x&=-33z \\ x&=-11z \end{aligned}$$

Let $z$ be any real number $c$, then the solutions are $(-11c, -6c, c)$.

**30.** $\begin{cases} 5x-2y-3z=0 & (1) \\ 3x-y-4z=0 & (2) \\ 4x-y-9z=0 & (3) \end{cases}$

$$\begin{array}{rl} 15x-6y-9z=0 & 3 \text{ times } (1) \\ \underline{-15x+5y+20z=0} & -5 \text{ times } (2) \\ -y+11z=0 & (4) \end{array}$$

$$\begin{array}{rl} 12x-4y-16z=0 & 4 \text{ times } (2) \\ \underline{-12x+3y+27z=0} & -3 \text{ times } (3) \\ -y+11z=0 & (5) \end{array}$$

$\begin{cases} 5x-2y-3z=0 & \\ -y+11z=0 & (4) \\ -y+11z=0 & (5) \end{cases}$

$$\begin{array}{rl} -y+11z=0 & (4) \\ \underline{y-11z=0} & -1 \text{ times } (5) \\ 0=0 & (6) \end{array}$$

$\begin{cases} 5x-2y-3z=0 \\ -y+11z=0 \\ 0=0 \end{cases}$

Let $z=c$. $\quad -y+11c=0$

$$y=11c$$

$$\begin{aligned} 5x-2(11c)-3c&=0 \\ 5x&=22c+3c \\ x&=5c \end{aligned}$$

The solution is $(5c, 11c, c)$.

**31.** $\begin{cases} 4x-7y-2z=0 & (1) \\ 2x+4y+3z=0 & (2) \\ 3x-2y-5z=0 & (3) \end{cases}$

$$\begin{array}{rl} 4x-7y-2z=0 & (1) \\ \underline{-4x-8y-6z=0} & -2 \text{ times } (2) \\ -15y-8z=0 & (4) \end{array}$$

$$\begin{array}{rl} 6x+12y+9z=0 & 3 \text{ times } (2) \\ \underline{-6x+4y+10z=0} & -2 \text{ times } (3) \\ 16y+19z=0 & (5) \end{array}$$

$\begin{cases} 4x-7y-2z=0 & \\ -15y-8z=0 & (4) \\ 16y+19z=0 & (5) \end{cases}$

$$\begin{array}{rl} -240y-128z=0 & 16 \text{ times } (4) \\ \underline{240y+285z=0} & 15 \text{ times } (5) \\ 157z=0 & \\ z=0 & (6) \end{array}$$

$\begin{cases} 4x-7y-2z=0 & \\ -15y-8z=0 & \\ z=0 & (6) \end{cases}$

$z=0$, $y=0$, $x=0$. The solution is (0, 0, 0).

**32.** $\begin{cases} 5x+2y+3z=0 & (1) \\ 3x+y-2z=0 & (2) \\ 4x-7y+5z=0 & (3) \end{cases}$

$$\begin{array}{rl} 15x+6y+9z=0 & 3 \text{ times } (1) \\ \underline{-15x-5y+10z=0} & -5 \text{ times } (2) \\ y+19z=0 & (4) \end{array}$$

$$\begin{array}{rl} 12x+4y-8z=0 & 4 \text{ times } (2) \\ -12x+21y-15z=0 & -3 \text{ times } (3) \\ 25y-23z=0 & (5) \end{array}$$

$\begin{cases} 5x+2y+3z=0 & \\ y+19z=0 & (4) \\ 25y-23z=0 & (5) \end{cases}$

$$\begin{array}{rl} -25y-475z=0 & -25 \text{ times } (4) \\ 25y-23z=0 & (5) \\ -498z=0 & \\ z=0 & (6) \end{array}$$

$\begin{cases} 5x+2y+3z=0 & \\ y+19z=0 & \\ z=0 & (6) \end{cases}$

$z=0$, $y=0$, $x=0$. The solution is (0, 0, 0).

**33.**

$$y = ax^2 + bx + c$$
$$3 = a(2)^2 + b(2) + c$$
$$7 = a(-2)^2 + b(-2) + c$$
$$-2 = a(1)^2 + b(1) + c$$

$$\begin{cases} 4a + 2b + c = 3 & (1) \\ 4a - 2b + c = 7 & (2) \\ a + b + c = -2 & (3) \end{cases}$$

$$\begin{array}{rl} 4a + 2b + c = 3 & (1) \\ \underline{-4a + 2b - c = -7} & -1 \text{ times } (2) \\ 4b = -4 & (4) \end{array}$$

$$\begin{array}{rl} 4a + 2b + c = 3 & (1) \\ \underline{-4a - 4b - 4c = 8} & -4 \text{ times } (3) \\ -2b - 3c = 11 & (5) \end{array}$$

$$\begin{cases} 4a + 2b + c = 3 & \\ 4b = -4 & (4) \\ -2b - 3c = 11 & (5) \end{cases}$$

From (4) : $4b = -4$
$b = -1$

From (5) : $-2(-1) - 3c = 11$
$c = -3$

From (1) : $4a + 2(-1) - 3 = 3$
$a = 2$

The equation whose graph passes through the three points is $y = 2x^2 - x - 3$.

**34.**

$$y = ax^2 + bx + c$$
$$-2 = a(1)^2 + b(1) + c$$
$$-4 = a(3)^2 + b(3) + c$$
$$-2 = a(2)^2 + b(2) + c$$

$$\begin{cases} a + b + c = -2 & (1) \\ 9a + 3b + c = -4 & (2) \\ 4a + 2b + c = -2 & (3) \end{cases}$$

Eliminate $c$ from Eq. (2) by multiplying Eq. (1) by –1 and adding to Eq. (2), and eliminate $c$ from Eq. (3) by multiply-ing Eq. (1) by –1 and adding to Eq. (3).

$$\begin{cases} a + b + c = -2 & \\ 8a + 2b = -2 & (4) \\ 3a + b = 0 & (5) \end{cases}$$

Eliminate $b$ from Eq. (5) by multiplying Eq. (4) by $-\frac{1}{2}$ and adding to Eq. (5).

$$\begin{cases} a + b + c = -2 & \\ 8a + 2b = -2 & \\ a = -1 & (6) \end{cases}$$

$a = -1$ $\quad 8(-1) + 2b = -2$ $\quad a + b + c = -2$
$b = 3$ $\quad -1 + 3 + c = -2$
$c = -4$

The equation whose graph passes through the three points is $y = -x^2 + 3x - 4$.

**35.**

$$x^2+y^2+ax+by+c=0$$
$$5^2+3^2+a(5)+b(3)+c=0$$
$$(-1)^2+(-5)^2+a(-1)+b(-5)+c=0$$
$$(-2)^2+2^2+a(-2)+b(2)+c=0$$

$$\begin{cases} 5a+3b+c=-34 & (1) \\ -a-5b+c=-26 & (2) \\ -2a+2b+c=-8 & (3) \end{cases}$$

$$\begin{array}{ll} 5a+3b+c=-34 & (1) \\ \underline{a+5b-c=26} & -1 \text{ times } (2) \\ 6a+8b=-8 & \\ 3a+4b=-4 & (4) \end{array}$$

$$\begin{array}{ll} 5a+3b+c=-34 & (1) \\ \underline{2a-2b-c=8} & -1 \text{ times } (3) \\ 7a+b=-26 & (5) \end{array}$$

$$\begin{cases} 5a+3b+c=-34 & \\ 3a+4b=4 & (4) \\ 7a+b=-26 & (5) \end{cases}$$

$$\begin{array}{ll} 3a+4b=-4 & (4) \\ \underline{-28a-4b=104} & -4 \text{ times } (5) \\ -25a=100 & \\ a=-4 & (6) \end{array}$$

$$\begin{cases} 5a+3b+c=-34 & \\ 3a+4b=-4 & \\ a=-4 & (6) \end{cases}$$

$$3(-4)+4b=-4 \qquad 5(-4)+3(2)+c=-34$$
$$b=2 \qquad c=-20$$

The equation whose graph passes through the three points is $x^2+y^2-4x+2y-20=0$.

**36.**

$$x^2+y^2+ax+by+c=0$$
$$0+36+a(0)+b(6)+c=0$$
$$1+25+a(1)+b(5)+c=0$$
$$49+1+a(-7)+b(-1)+c=0$$

$$\begin{cases} 6b+c=-36 & (1) \\ a+5b+c=26 & (2) \\ -7a-b+c=-50 & (3) \end{cases}$$

$$\begin{array}{ll} 7a+35b+7c=-182 & 7 \text{ times } (2) \\ \underline{-7a-b+c=-50} & (3) \\ 34b+8c=-232 & \\ 17b+4c=-116 & (4) \end{array}$$

$$\begin{cases} 6b+c=-36 & \\ 17b+4c=-116 & (4) \\ -7a-b+c=-50 & \end{cases}$$

$$\begin{array}{ll} -24b-4c=144 & -4 \text{ times } (1) \\ \underline{17b+4c=-116} & (4) \\ -7b=28 & \\ b=-4 & (5) \end{array}$$

$$\begin{cases} b=-4 & (5) \\ 17b+4c=-116 & \\ -7a-b+c=-50 & \end{cases}$$

$$17(-4)+4c=-116 \qquad -7a-(-4)-12=-50$$
$$c=-12 \qquad a=6$$

The equation whose graph passes through the three points is $x^2+y^2+6x-4y-12=0$.

**37.**

$$x^2+y^2+ax+by+c=0$$
$$(-2)^2+10^2+a(-2)+b(10)+c=0$$
$$(-12)^2+(-14)^2+a(-12)+b(-14)+c=0$$
$$5^2+3^2+a(5)+b(3)+c=0$$

$$\begin{cases} -2a+10b+c=-104 & (1)\\ -12a-14b+c=-340 & (2)\\ 5a+3b+c=-34 & (3)\end{cases}$$

$$\begin{array}{ll} -2a+10b+c=-104 & (1)\\ \underline{12a+14b-c=340} & -1\text{ times }(2)\\ 10a+24b=236 & \\ 5a+12b=118 & (4)\end{array}$$

$$\begin{array}{ll} -2a+10b+c=-104 & (1)\\ \underline{-5a-3b-c=34} & -1\text{ times }(3)\\ -7a+7b=-70 & \\ -a+b=-10 & (5)\end{array}$$

$$\begin{cases} -2a+10b+c=-104 & (1)\\ 5a+12b=118 & (4)\\ -a+b=-10 & (5)\end{cases}$$

$$\begin{array}{ll} 5a+12b=118 & (4)\\ \underline{-5a+5b=-50} & 5\text{ times }(5)\\ 17b=68 & \\ b=4 & (6)\end{array}$$

$$\begin{cases} -2a+10b+c=-104 \\ 5a+12b=118 \\ b=4 & (6)\end{cases}$$

$$5a+12(4)=118 \qquad -2(14)+10(4)+c=-104$$
$$a=14 \qquad c=-116$$

The equation whose graph passes through the three points is $x^2+y^2+14x+4y-116=0$.

$$(x^2+14x+49)+(y^2+4y+4)=116+49+4$$
$$(x+7)^2+(y+2)^2=169$$

The center is $(-7,\ -2)$ and radius is 13.

**38.**

$$x^2+y^2+ax+by+c=0$$
$$2^2+5^2+a(2)+b(5)+c=0$$
$$(-4)^2+(-3)^2+a(-4)+b(-3)+c=0$$
$$3^2+4^2+a(3)+b(4)+c=0$$

$$\begin{cases} 2a+5b+c=-29 & (1)\\ -4a-3b+c=-25 & (2)\\ 3a+4b+c=-25 & (3)\end{cases}$$

$$\begin{array}{ll} 2a+5b+c=-29 & (1)\\ \underline{4a+3b-c=25} & -1\text{ times }(2)\\ 6a+8b=-4 & \\ 3a+4b=-2 & (4)\end{array}$$

$$\begin{array}{ll} -4a-3b+c=-25 & (2)\\ \underline{-3a-4b-c=25} & -1\text{ times }(3)\\ -7a-7b=0 & \\ a+b=0 & (5)\end{array}$$

$$\begin{cases} 2a+5b+c=-29 \\ 3a+4b=-2 & (4)\\ a+b=0 & (5)\end{cases}$$

$$\begin{array}{ll} 3a+4b=-2 & (4)\\ \underline{-4a-4b=0} & -4\text{ times }(5)\\ -a=-2 & \\ a=2 & (6)\end{array}$$

$$\begin{cases} 2a+5b+c=-29 \\ 3a+4b=-2 \\ a=2 & (6)\end{cases}$$

$$3(2)+4b=-2 \qquad 2(2)+5(-2)+c=-29$$
$$b=-2 \qquad c=-23$$

The equation whose graph passes through the three points is $x^2+y^2+2x-2y-23=0$.

$$(x^2+2x+1)+(y^2-2y+1)=23+1+1$$
$$(x+1)^2+(y-1)^2=25$$

The center is $(-1,\ 1)$ and radius is 5.

**39.** For intersection $A$, $275+225=x_1+x_2$

$x_1+x_2=500$

For intersection $B$, $x_2+90=x_3+150$

$x_2-x_3=60$

For intersection $C$, $x_1+x_3=240+200$

$x_1+x_3=440$

$$\begin{cases} x_1+x_2=500 & (1) \\ x_2-x_3=60 & (2) \\ x_1+x_3=440 & (3) \end{cases}$$

The equations are dependent.
Solve equation (2) for $x_3$ and substitute the inequality for $x_2$.

$x_3=x_2-60$

Because $150\le x_2\le 250$ , then $90\le x_3\le 190$ .

The flow between $B$ and $C$ is 90 to 190 cars per hour.

**40.** For intersection $A$, $200+x_3=165+x_1$

$x_1-x_3=35$

For intersection $B$, $50+x_1=100+x_2$

$x_1-x_2=50$

For intersection $C$, $200+x_2=185+x_3$

$x_2-x_3=15$

$$\begin{cases} x_1-x_3=35 & (1) \\ x_1-x_2=50 & (2) \\ x_2-x_3=15 & (3) \end{cases}$$

The equations are dependent.
Solve equation (1) for $x_3$ and substitute the inequality for $x_1$.

$x_3=x_1-35$

Solve equation (2) for $x_2$ and substitute the inequality for $x_1$.

$x_2=x_1-50$

Because $60\le x_1\le 80$ , .then $10\le x_2\le 30$ and $25\le x_3\le 45$ .

The flow between $C$ and $A$ is 25 to 45 cars per hour
and the flow between $B$ and $C$ is 10 to 30 cars per hour.

**41.** For intersection $A$, $256+x_4=389+x_1$

$x_1-x_4=-133$

For intersection $B$, $437+x_1=x_2+300$

$x_1-x_2=-137$

For intersection $C$, $298+x_3=249+x_4$

$x_3-x_4=-49$

For intersection $D$, $314+x_2=367+x_3$

$x_2-x_3=53$

$$\begin{cases} x_1-x_4=-133 & (1) \\ x_1-x_2=-137 & (2) \\ x_3-x_4=-49 & (3) \\ x_2-x_3=53 & (4) \end{cases}$$

The equations are dependent. Solving the system gives

$x_1=x_4-133$

$x_2=x_4+4$

$x_3=x_4-49$

Because $125\le x_1\le 175$ , then

$125\le x_4-133\le 175$ and $258\le x_2-4\le 308$ and $258\le x_3+49\le 308$

$258\le x_4\le 308$ $\quad$ $262\le x_2\le 312$ $\quad$ $209\le x_3\le 259$

The flow between $C$ and $A$ is 258 to 308 cars per hour
The flow between $B$ and $D$ is 262 to 312 cars per hour.
The flow between $D$ and $C$ is 209 to 259 cars per hour.

**42.** For intersection $A$, $75+x_4=60+x_1$

$x_1-x_4=-15$

For intersection $B$, $50+x_1=x_2+100$

$x_1-x_2=50$

For intersection $C$, $45+x_2=50+x_3$

$x_2-x_3=5$

For intersection $D$, $80+x_3=40+x_4$

$x_3-x_4=-40$

$$\begin{cases} x_1-x_4=15 & (1) \\ x_1-x_2=50 & (2) \\ x_2-x_3=5 & (3) \\ x_3-x_4=-40 & (4) \end{cases}$$

The equations are dependent. Solving the system gives

$x_1=x_2+50$

$x_3=x_2-5$

$x_4=x_2+35$

Since there cannot be a negative number of cars per hour in an intersection, then $x_3\ge 0$ and therefore $x_2\ge 5$ .

The minimum number of cars traveling between $B$ and $C$ is 5 cars per hour.

**43.** $w_1d_1 + w_2d_2 = w_3d_3$ and $w_1 = 2$, $w_2 = 6$, $w_3 = 9$

$2d_1 + 6d_2 = 9d_3$ (1)

From the words in the exercise,

$d_1 + d_3 = 13$ (2)

$d_2 = \frac{1}{3}d_1$ (3)

$$\begin{array}{ll} 2d_1 + 6d_2 - 9d_3 = 0 & (1) \\ \underline{2d_1 - 6d_2 \qquad = 0} & \text{6 times (3)} \\ 4d_1 \qquad - 9d_3 = 0 & (4) \end{array}$$

$$\begin{array}{ll} 4d_1 - 9d_3 = 0 & (4) \\ \underline{9d_1 + 9d_3 = 0} & \text{9 times (2)} \\ 13d_1 \qquad = 117 & (5) \\ d_1 = 9 & \end{array}$$

Substitute into equation (3) and (2)

$d_2 = \frac{1}{3}(9) = 3$

$9 + d_3 = 13$

$d_3 = 4$

Therefore $d_1 = 9$ in., $d_2 = 3$ in., and $d_3 = 4$ in.

$d_2 + d_3 = 3 + 4 = 7$ in.

$d_1 - d_2 = 9 - 3 = 6$ in.

So the middle chime is 7 in. from the 9 ounce chime and 6 in. from the 2 ounce chime.

**44.**
$$\begin{cases} d_3 + d_4 = 20 & (1) \\ d_1 + d_2 = 10 & (2) \\ d_5 + d_6 = 8 & (3) \\ 6d_1 = 4d_2 & (4) \\ 5d_5 = 3d_6 & (5) \\ (2+6+4)d_3 = (10+5+3)d_4 & (6) \end{cases}$$

$$\begin{array}{ll} 4d_1 + 4d_2 = 40 & \text{4 times (2)} \\ \underline{6d_1 - 4d_2 = 0} & (4) \\ 10d_1 \qquad = 40 & \\ d_1 = 4 & \end{array}$$

$4d_2 = 6(4)$

$d_2 = 6$

$$\begin{array}{ll} 3d_5 + 3d_6 = 24 & \text{3 times (3)} \\ \underline{5d_5 - 3d_6 = 0} & (5) \\ 8d_5 \qquad = 24 & \\ d_5 = 3 & \end{array}$$

$3d_6 = 5(3)$

$d_6 = 5$

$$\begin{array}{ll} 18d_3 + 18d_4 = 360 & \text{18 times (1)} \\ \underline{12d_3 - 18d_4 = 0} & (6) \\ 30d_3 \qquad = 360 & \\ d_3 = 12 & \end{array}$$

$12 + d_4 = 20$

$d_4 = 8$

The lengths are: $d_1 = 4$ in., $d_2 = 6$ in., and $d_3 = 12$ in., $d_4 = 8$ in., $d_5 = 3$ in., and $d_6 = 5$ in.

Connecting Concepts

**45.** $\begin{cases} 2x+y-3z+2w=-1 & (1) \\ 2y-5z-3w=9 & (2) \\ 3y-8z+w=-4 & (3) \\ 2y-2z+3w=-3 & (4) \end{cases}$

$$\begin{array}{rl} 6y-15z-9w=27 & 3\text{ times }(2) \\ \underline{-6y+16z-2w=8} & -2\text{ times }(3) \\ z-11w=35 & (5) \end{array}$$

$$\begin{array}{rl} 2y-5z-3w=9 & (2) \\ \underline{-2y+2z-3w=8} & -1\text{ times }(4) \\ -3z-6w=12 & \\ z+2w=-4 & (6) \end{array}$$

$$\begin{cases} 2x+y-3z+2w=-1 \\ 2y-5z-3w=9 \\ z-11w=35 & (5) \\ z+2w=-4 & (6) \end{cases}$$

$$\begin{array}{rl} z-11w=35 & (5) \\ \underline{-z-2w=4} & -1\text{ times }(6) \\ -13w=39 & \\ w=-3 & (7) \end{array}$$

$$\begin{cases} 2x+y-3z+2w=-1 \\ 2y-5z-3w=9 \\ z-11w=35 \\ w=-3 & (7) \end{cases}$$

$$\begin{aligned} z-11(-3)&=35 \\ z&=2 \end{aligned}$$

$$\begin{aligned} 2y-5(2)-3(-3)&=9 \\ 2y&=10 \\ y&=5 \end{aligned}$$

$$\begin{aligned} 2x+5-3(2)+2(-3)&=-1 \\ 2x&=6 \\ x&=3 \end{aligned}$$

The solution is $(3,\ 5,\ 2,\ -3)$.

**46.** $\begin{cases} 3x-y+2z-3w=5 & (1) \\ 2y-5z+2w=-7 & (2) \\ 4y-9z+w=-19 & (3) \\ 3y+z-2w=-12 & (4) \end{cases}$

$$\begin{array}{rl} -4y+10z-4w=14 & -2\text{ times }(2) \\ \underline{4y-9z+w=-19} & (3) \\ z-3w=-5 & (5) \end{array}$$

$$\begin{array}{rl} 6y-15z+6w=-21 & 3\text{ times }(2) \\ \underline{-6y-2z+4w=24} & -2\text{ times }(4) \\ -17z+10w=3 & (6) \end{array}$$

$$\begin{cases} 3x-y+2z-3w=5 \\ 2y-5z+2w=-7 \\ z-3w=-5 & (5) \\ -17z+10w=3 & (6) \end{cases}$$

$$\begin{array}{rl} 17z-51w=-85 & 17\text{ times }(5) \\ \underline{-17z+10w=3} & (6) \\ -41w=-82 & \\ w=2 & (7) \end{array}$$

$$\begin{cases} 3x-y+2z-3w=5 \\ 2y-5z+2w=-7 \\ z-3w=-5 \\ w=2 & (7) \end{cases}$$

$$\begin{aligned} z-3(2)&=-5 \\ z&=1 \end{aligned}$$

$$\begin{aligned} 2y-5(1)+2(2)&=-7 \\ 2y&=-6 \\ y&=-3 \end{aligned}$$

$$\begin{aligned} 3x-(-3)+2(1)-3(2)&=5 \\ 3x&=6 \\ x&=2 \end{aligned}$$

The solution is $(2,\ -3,\ 1,\ 2)$.

**47.** $\begin{cases} x-3y+2z-w=2 & (1) \\ 2x-5y-3z+2w=21 & (2) \\ 3x-8y-2z-3w=12 & (3) \\ -2x+8y+z+2w=-13 & (4) \end{cases}$

$$\begin{array}{rl} -2x+6y-4z+2w=-4 & -2\text{ times }(1) \\ \underline{2x-5y-3z+2w=21} & (2) \\ y-7z+4w=17 & (5) \end{array}$$

$$\begin{array}{rl} -3x+9y-6z+3w=-6 & -3\text{ times }(1) \\ \underline{3x-8y-2z-3w=12} & (3) \\ y-8z\quad\quad=6 & (6) \end{array}$$

$$\begin{array}{rl} 2x-5y-3z+2w=21 & (2) \\ \underline{-2x+8y+z+2w=-13} & (4) \\ 3y-2z+4w=8 & (7) \end{array}$$

$$\begin{cases} x-3y+2z-w=2 & \\ y-7z+4w=17 & (5) \\ y-8z\quad\quad=6 & (6) \\ 3y-2z+4w=8 & (7) \end{cases}$$

$$\begin{array}{rl} y-7z+4w=17 & (5) \\ \underline{-3y+2z-4w=-8} & -1\text{ times }(7) \\ -2y-5z\quad\quad=9 & (8) \end{array}$$

$$\begin{cases} x-3y+2z-w=2 & \\ y-7z+4w=17 & \\ y-8z\quad\quad=6 & (6) \\ -2y-5z\quad\quad=9 & (8) \end{cases}$$

$$\begin{array}{rl} 2y-16z=12 & 2\text{ times }(6) \\ \underline{-2y-5z=9} & (8) \\ -21z=21 & \\ z=-1 & (9) \end{array}$$

$$\begin{cases} x-3y+2z-w=2 & \\ y-7z+4w=17 & \\ y-8z\quad\quad=6 & \\ z\quad\quad=-1 & (9) \end{cases}$$

$$\begin{aligned} y-8(-1)&=6 \\ y&=-2 \end{aligned}$$

$$\begin{aligned} (-2)-7(-1)+4w&=17 \\ 4w&=12 \\ w&=3 \end{aligned}$$

$$\begin{aligned} x-3(-2)+2(-1)-3&=2 \\ x&=1 \end{aligned}$$

The solution is $(1,\ -2,\ -1,\ 3)$.

**48.** $\begin{cases} x-2y+3z+2w=8 & (1) \\ 3x-7y-2z+3w=18 & (2) \\ 2x-5y+2z-w=19 & (3) \\ 4x-8y+3z+2w=29 & (4) \end{cases}$

$$\begin{array}{rl} -3x+6y-9z-6w=-24 & -3\text{ times }(1) \\ \underline{3x-7y-2z+3w=18} & (2) \\ -y-11z-3w=-6 & (5) \end{array}$$

$$\begin{array}{rl} -2x+4y-6z-4w=-16 & -2\text{ times }(1) \\ \underline{2x-5y+2z-w=19} & (3) \\ -y-4z-5w=3 & (6) \end{array}$$

$$\begin{array}{rl} -4x+8y-12z-8w=-32 & -4\text{ times }(1) \\ \underline{4x-8y+3z+2w=29} & (4) \\ -9z-6w=-3 & \\ 3z+2w=1 & (7) \end{array}$$

$$\begin{cases} x-2y+3z+2w=8 & \\ -y-11z-3w=-6 & (5) \\ -y-4z-5w=3 & (6) \\ 3z+2w=1 & (7) \end{cases}$$

$$\begin{array}{rl} -y-11z-3w=-6 & (5) \\ \underline{y+4z+5w=-3} & -1\text{ times }(6) \\ -7z+2w=-9 & (8) \end{array}$$

$$\begin{cases} x-2y+3z+2w=8 & \\ -y-11z-3w=-6 & \\ -7z+2w=-9 & (8) \\ 3z+2w=1 & (7) \end{cases}$$

$$\begin{array}{rl} -7z+2w=-9 & (8) \\ \underline{-3z-2w=-1} & -1\text{ times }(7) \\ -10z\quad\quad=-10 & \\ z=1 & \end{array}$$

$$\begin{cases} x-2y+3z+2w=8 \\ -y-11z-3w=-6 \\ -7z+2w=-9 \\ z=1 \end{cases}$$

$$\begin{aligned} -7(1)+2w&=-9 \\ 2w&=-2 \\ w&=-1 \end{aligned}$$

$$\begin{aligned} -y-11(1)-3(-1)&=-6 \\ -y&=2 \\ y&=-2 \end{aligned}$$

$$\begin{aligned} x-2(-2)+3(1)+2(-1)&=8 \\ x&=3 \end{aligned}$$

The solution is $(3,\ -2,\ 1,\ -1)$.

**49.** $\begin{cases} x+2y-2z+3w=2 & (1) \\ 2x+5y+2z+4w=9 & (2) \\ 4x+9y-2z+10w=13 & (3) \\ -x-y+8z-5w=3 & (4) \end{cases}$

$$\begin{array}{rl} -2x-4y+4z-6w=-4 & -2 \text{ times} (1) \\ \underline{2x+5y+2z+4w=9} & (2) \\ y+6z-2w=5 & (5) \end{array}$$

$$\begin{array}{rl} -4x-8y+8z-12w=-8 & -4 \text{ times} (1) \\ \underline{4x+9y-2z+10w=13} & (3) \\ y+6z-2w=5 & (6) \end{array}$$

$$\begin{array}{rl} x+2y-2z+3w=2 & (1) \\ \underline{-x-y+8z-5w=3} & (4) \\ y+6z-2w=5 & (7) \end{array}$$

$$\begin{cases} x+2y-2z+3w=2 & \\ y+6z-2w=5 & (5) \\ y+6z-2w=5 & (6) \\ y+6z-2w=5 & (7) \end{cases}$$

$$\begin{cases} x+2y-2z+3w=2 \\ y+6z-2w=5 \\ 0=0 \\ 0=0 \end{cases}$$

Let $z=a, w=b.$

$$y+6a-2b=5$$
$$y=-6a+2b+5$$

$$x+2(-6a+2b+5)-2a+3b=2$$
$$x-12a+4b+10-2a+3b=2$$
$$x=14a-7b-8$$

The solutions are $(14a-7b-8,\ -6a+2b+5,\ a,\ b)$.

**50.** $\begin{cases} x-2y+3z-2w=-1 & (1) \\ 3x-7y-2z-3w=-19 & (2) \\ 2x-5y+2z-w=-11 & (3) \\ -x+3y-2z-w=3 & (4) \end{cases}$

$$\begin{array}{rl} -3x+6y-9z+6w=3 & -3 \text{ times} (1) \\ \underline{3x-7y-2z-3w=-19} & (2) \\ -y-11z+3w=-16 & (5) \end{array}$$

$$\begin{array}{rl} -2x+4y-6z+4w=2 & -2 \text{ times} (1) \\ \underline{2x-5y+2z-w=-11} & (3) \\ -y-4z+3w=-9 & (6) \end{array}$$

$$\begin{array}{rl} x-2y+3z-2w=-1 & (1) \\ \underline{-x+3y-2z-w=3} & (4) \\ y+z-3w=2 & (7) \end{array}$$

$$\begin{cases} x-2y+3z-2w=-1 & \\ -y-11z+3w=-16 & (5) \\ -y-4z+3w=-9 & (6) \\ y+z-3w=2 & (7) \end{cases}$$

$$\begin{array}{rl} y+11z-3w=16 & -1 \text{ times} (5) \\ \underline{-y-4z+3w=-9} & (6) \\ 7z \quad =7 & \\ z=1 & (8) \end{array}$$

$$\begin{array}{rl} -y-4z+3w=-9 & (6) \\ \underline{y+z-3w=2} & (7) \\ -3z \quad =-7 & \\ z=\frac{7}{3} & (9) \end{array}$$

$$\begin{cases} x-2y+3z-2w=-1 & \\ -y-11z+3w=-16 & \\ z \quad =1 & (8) \\ z \quad =\frac{7}{3} & (9) \end{cases}$$

$$\begin{cases} x-2y+3z-2w=-1 \\ -y-11z+3w=-16 \\ z \quad =1 \\ 0=-\frac{4}{3} \end{cases}$$

The system of equations is inconsistent and has no solutions.

**51-52.** $\begin{cases} x-3y-2z=A^2 & (1) \\ 2x-5y+Az=9 & (2) \\ 2x-8y+z=18 & (3) \end{cases}$

Multiply Eq. (1) by –2 and add to Eq. (2). Now multiply Eq. (1) by –2 and add to Eq. (3). The resulting system is

$$\begin{cases} x-3y-2z=A^2 & (4) \\ y+(4+A)z=-2A^2+9 & (5) \\ -2y+5z=-2A^2+18 & (6) \end{cases}$$

Multiply Eq. (5) by 2 and add to Eq. (6). We now have

$$\begin{cases} x-3y-2z=A^2 \\ y+(4+A)z=-2A^2+9 \\ (2A+13)z=-6A^2+36 & (7) \end{cases}$$

For Exercise 51, the system of equations has no solution when $2A+13=0$ or $A=-\frac{13}{2}$.

For Exercise 52, the system of equations has a unique solution when $2A+13\neq 0$ or $A\neq-\frac{13}{2}$.

**53-55.** $\begin{cases} x+2y+z=A^2 & (1) \\ -2x-3y+Az=1 & (2) \\ 7x+12y+A^2z=4A^2-3 & (3) \end{cases}$

Multiply Eq. (1) by 2 and add to Eq. (2). Then multiply Eq. (1) by –7 and add to Eq. (3). The resulting system is

$$\begin{cases} x+2y+z=A^2 & (4) \\ y+(A+2)z=2A^2+1 & (5) \\ -2y+(A^2-7)z=-3A^2-3 & (6) \end{cases}$$

Multiply Eq. (5) by 2 and add to Eq. (6).

$$\begin{cases} x+2y+z=A^2 \\ y+(A+2)z=2A^2+1 \\ (A^2+2A-3)z=A^2-1 & (7) \end{cases}$$

In Exercise 53, the system of equations will have a unique solution when $(A^2+2A-3)\neq 0$ in Eq. (7). That is, $(A+3)(A-1)\neq 0$, or $A\neq -3, A\neq 1$

In Exercise 54, the system will have an infinite number of solutions when $A^2+2A-3=0$ and $A^2-1=0$. This occurs when $A$ = 1.

In Exercise 55, the system of equations will have no solution when $A^2+2A-3=0$ and $A^2-1\neq 0$. This occurs when $A$ = –3.

**56.** $z=ax+by+c$

$$\begin{cases} 2a+b+c=1 & (1) \\ -a+2b+c=12 & (2) \\ 3a+2b+c=0 & (3) \end{cases}$$

$$\begin{array}{rl} 2a+b+c=1 & \\ \underline{-2a+4b+2c=24} & \text{2 times (2)} \\ 5b+3c=25 & (4) \end{array}$$

$$\begin{array}{rl} -3a+6b+3c=36 & \text{3 times (2)} \\ \underline{3a+2b+c=0} & \\ 8b+4c=36 & (5) \end{array}$$

$$\begin{cases} 2a+b+c=1 \\ 5b+3c=25 & (4) \\ 8b+4c=36 & (5) \end{cases}$$

$$\begin{array}{rl} 40b+24c=200 & \text{8 times (4)} \\ \underline{-40b-20c=-180} & -5\text{ times (5)} \\ 4c=20 & \\ c=5 & (6) \end{array}$$

$$\begin{cases} 2a+b+c=1 \\ 5b+3c=25 \\ c=5 & (6) \end{cases}$$

$$\begin{array}{rcl} 5b+3(5)=25 & \quad & 2a+2+5=1 \\ 5b=10 & & 2a=-6 \\ b=2 & & a=-3 \end{array}$$

Thus the equation of the plane is $z=-3x+2y+5$.

**57.** $z=ax+by+c$

$$\begin{cases} a-b+c=5 & (1) \\ 2a-2b+c=9 & (2) \\ -3a-b+c=-1 & (3) \end{cases}$$

$$\begin{array}{rl} 3a-3b+3c=15 & \text{3 times (1)} \\ \underline{-3a-b+c=-1} & (3) \\ -4b+4c=14 & \end{array}$$

$$\begin{array}{rl} -2a+2b-2c=-10 & -2\text{ times (1)} \\ \underline{2a-2b+c=9} & (2) \\ -c=-1 & \\ c=1 & \end{array}$$

$$-4b+4(1)=14$$

$$b=-\frac{5}{2}$$

$$a-\left(-\frac{5}{2}\right)+1=5$$

$$a=\frac{3}{2}$$

Thus the equation of the plane is $z=\frac{3}{2}x-\frac{5}{2}y+1$ or $3x-5y-2z=-2$.

## Prepare for Section 9.3

**PS1.** $x^2+2x-2=0$
$x^2+2x+1=2+1$
$(x+1)^2=3$
$x+1=\pm\sqrt{3}$
$x=-1\pm\sqrt{3}$

**PS2.** $\begin{cases} x+4y=-11 & (1) \\ 3x-2y=9 & (2) \end{cases}$

Solve equation (1) for $x$: $x=-4y-11$

$3(-4y-11)-2y=9$
$-12y-33-2y=9$
$-14y=42$
$y=-3$
$x=-4(-3)-11$
$x=1$

The solution is (1, –3).

**PS3.** parabola

**PS4.** hyperbola

**PS5.** 2

**PS6.** 4

## Section 9.3

**1.** $\begin{cases} y=x^2-x & (1) \\ y=2x-2 & (2) \end{cases}$

Set the expressions for $y$ equal to each other.

$x^2-x=2x-2$
$x^2-3x+2=0$
$(x-2)(x-1)=0$
$x-2=0 \qquad x-1=0$
$x=2 \qquad x=1$

When $x=2$, $y=2^2-2=2$ (From Eq. (1))
When $x=1$, $y=1^2-1=0$

The solutions are (1, 0) and (2, 2).

**2.** $\begin{cases} y=x^2+2x-3 & (1) \\ y=x-1 & (2) \end{cases}$

Set the expressions for $y$ equal to each other.

$x^2+2x-3=x-1$
$x^2+x-2=0$
$(x+2)(x-1)=0$
$x+2=0 \qquad x-1=0$
$x=-2 \qquad x=1$

When $x=-2$, $y=-2-1=-3$ (From Eq. (2))
When $x=1$, $y=1-1=0$

The solutions are (–2, –3) and (1, 0).

**3.** $\begin{cases} y=2x^2-3x-3 & (1) \\ y=x-4 & (2) \end{cases}$

Set the expressions for $y$ equal to each other.

$2x^2-3x-3=x-4$
$2x^2-4x+1=0$
$x=\dfrac{4\pm\sqrt{16-4(2)(1)}}{2\cdot 2}$ (Quadratic Formula)
$=\dfrac{4\pm\sqrt{8}}{4}=\dfrac{4\pm 2\sqrt{2}}{4}=\dfrac{2\pm\sqrt{2}}{2}$

Substitute for $x$ in (1) and solve for $y$.

When $x=\dfrac{2+\sqrt{2}}{2}$, $y=\dfrac{2+\sqrt{2}}{2}-4=\dfrac{-6+\sqrt{2}}{2}$.

When $x=\dfrac{2-\sqrt{2}}{2}$, $y=\dfrac{2-\sqrt{2}}{2}-4=\dfrac{-6-\sqrt{2}}{2}$.

The solutions are

$\left(\dfrac{2+\sqrt{2}}{2}, \dfrac{-6+\sqrt{2}}{2}\right)$ and $\left(\dfrac{2-\sqrt{2}}{2}, \dfrac{-6-\sqrt{2}}{2}\right)$.

**4.** $\begin{cases} y=-x^2+2x-4 \\ y=\dfrac{1}{2}x+1 \end{cases}$

Set the expressions for $y$ equal to each other.

$-x+2x-4=\dfrac{1}{2}x+1$
$-x^2+\dfrac{3}{2}x-5=0$
$2x^2-3x+10=0$
$x=\dfrac{3\pm\sqrt{9-80}}{2\cdot 2}$ (Quadratic Formula)
$x=\dfrac{3\pm\sqrt{-71}}{4}$

Because the solutions are not real numbers, the system of equations has no real number solutions.

**5.** $\begin{cases} y = x^2 - 2x + 3 & (1) \\ y = x^2 - x - 2 & (2) \end{cases}$

Set the expressions for $y$ equal to each other.

$$x^2 - 2x + 3 = x^2 - x - 2$$
$$-x = -5$$
$$x = 5$$

Substitute for $x$ in Eq. (1).

$$y = 5^2 - 2(5) + 3$$
$$y = 18$$

The solution is (5, 18).

**6.** $\begin{cases} y = 2x^2 - x + 1 & (1) \\ y = x^2 + 2x + 5 & (2) \end{cases}$

Set the expressions for $y$ equal to each other.

$$2x^2 - x + 1 = x^2 + 2x + 5$$
$$x^2 - 3x - 4 = 0$$
$$(x-4)(x+1) = 0$$
$$x = 4 \text{ or } x = -1$$

Substitute for $x$ in Eq. (2).

$$y = (4)^2 + 2(4) + 5 \qquad y = (-1)^2 + 2(-1) + 5$$
$$y = 29 \qquad y = 4$$

The solutions are (4, 29) and (–1, 4).

**7.** $\begin{cases} x + y = 10 & (1) \\ xy = 24 & (2) \end{cases}$

Substitute $y$ from Eq. (1) into Eq. (2).

$$x(10 - x) = 24$$
$$10x - x^2 = 24$$
$$0 = x^2 - 10x + 24$$
$$0 = (x-4)(x-6)$$
$$x = 4 \text{ or } x = 6$$

Substitute for $x$ in Eq. (1).

$$4 + y = 10 \qquad 6 + y = 10$$
$$y = 6 \qquad y = 4$$

The solutions are (4, 6) and (6, 4).

**8.** $\begin{cases} x - 2y = 3 & (1) \\ xy = -1 & (2) \end{cases}$

Substitute $x$ from Eq. (1) into Eq. (2).

$$(2y + 3)y = -1$$
$$2y^2 + 3y + 1 = 0$$
$$(2y+1)(y+1) = 0$$
$$y = -\frac{1}{2} \text{ or } y = -1$$

Substitute for $y$ in Eq. (1).

$$x - 2\left(-\frac{1}{2}\right) = 3 \qquad x - 2(-1) = 3$$
$$x = 2 \qquad x = 1$$

The solutions are (2, –1/2) and (1, –1).

**9.** $\begin{cases} 2x - y = 1 & (1) \\ xy = 6 & (2) \end{cases}$

Solve Eq. (1) for $y$.

$$y = 2x - 1 \quad (3)$$

Substitute into Eq. (2).

$$x(2x - 1) = 6$$
$$2x^2 - x = 6$$
$$2x^2 - x - 6 = 0$$
$$(2x+3)(x-2) = 0$$
$$2x + 3 = 0, \text{ or } x - 2 = 0$$
$$x = -\frac{3}{2} \qquad x = 2$$

Substitute for $x$ in Eq. (3).

When $x = -\frac{3}{2}$, $y = 2\left(-\frac{3}{2}\right) - 1 = -4$.

When $x = 2$, $y = 2(2) - 1 = 3$.

The solutions are (–3/2, –4) and (2, 3).

**10.** $\begin{cases} x - 3y = 7 & (1) \\ xy = -4 & (2) \end{cases}$

Solve Eq. (1) for $x$.

$$x = 3y + 7 \quad (3)$$

Substitute into Eq. (2).

$$(3y + 7)y = -4$$
$$3y^2 + 7y = -4$$
$$3y^2 + 7y + 4 = 0$$
$$(3y+4)(y+1) = 0$$
$$3y + 4 = 0, \text{ or } y + 1 = 0$$
$$y = -4/3 \quad y = -1$$

Substitute for $y$ in Eq. (1).

When $y = -4/3$, $x = 3\left(-\frac{4}{3}\right) + 7 = 3$

When $y = -1$, $x = 3(-1) + 7 = 4$

The solutions are (3, –4/3) and (4, –1).

**11.** $\begin{cases} 3x^2 - 2y^2 = 1 & (1) \\ y = 4x - 3 & (2) \end{cases}$

Substitute $y$ from Eq. (2) into Eq. (1).

$$3x^2 - 2(4x-3)^2 = 1$$
$$3x^2 - 32x^2 + 48x - 18 = 1$$
$$29x^2 - 48x + 19 = 0$$
$$(29x - 19)(x-1) = 0$$
$$x = \frac{19}{29} \text{ or } x = 1$$

Substitute for $x$ in Eq. (2).

$$y = 4\left(\frac{19}{29}\right) - 3 \qquad y = 4(1) - 3$$
$$y = \frac{76}{29} - \frac{87}{29} \qquad y = 1$$
$$y = -\frac{11}{29}$$

The solutions are (19/29, –11/29) and (1, 1).

**12.** $\begin{cases} x^2 + 3y^2 = 7 & (1) \\ x + 4y = 6 & (2) \end{cases}$

Substitute $x$ from Eq. (2) into Eq. (1).

$$(6-4y)^2 + 3y^2 = 7$$
$$36 - 48y + 16y^2 + 3y^2 = 7$$
$$19y^2 - 48y + 29 = 0$$
$$(19y - 29)(y-1) = 0$$
$$y = \frac{29}{19} \qquad y = 1$$

Substitute for $y$ in Eq. (2).

$$x + 4\left(\frac{29}{19}\right) = 6 \qquad x + 4(1) = 6$$
$$x = \frac{114}{19} - \frac{116}{19} \qquad x = 2$$
$$x = -\frac{2}{19}$$

The solutions are $\left(-\frac{2}{19}, \frac{29}{19}\right)$ and (2, 1).

**13.** $\begin{cases} y = x^3 + 4x^2 - 3x - 5 & (1) \\ y = 2x^2 - 2x - 3 & (2) \end{cases}$

Set the expressions for $y$ equal to each other.

$$x^3 + 4x^2 - 3x - 5 = 2x^2 - 2x - 3$$
$$x^3 + 2x^2 - x - 2 = 0$$
$$x^2(x+2) - (x+2) = 0$$
$$(x+2)(x^2 - 1) = 0$$
$$(x+2)(x-1)(x+1) = 0$$

$$x = -2,\ x = 1, \text{ or } x = -1$$

Substitute for $x$ in Eq. (2).

$$y = 2(-2)^2 - 2(-2) - 3$$
$$y = 9$$

$$y = 2(1)^2 - 2(1) - 3$$
$$y = -3$$

$$y = 2(-1)^2 - 2(-1) - 3$$
$$y = 1$$

The solutions are (–2, 9), (1, –3) and (–1, 1).

**14.** $\begin{cases} y = x^3 - 2x^2 + 5x + 1 & (1) \\ y = x^2 + 7x - 5 & (2) \end{cases}$

Set the expressions for $y$ equal to each other.

$$x^3 - 2x^2 + 5x + 1 = x^2 + 7x - 5$$
$$x^3 - 3x^2 - 2x + 6 = 0$$
$$x^2(x-3) - 2(x-3) = 0$$
$$(x-3)(x^2 - 2) = 0$$

$$x = 3, x = \sqrt{2}, \text{ or } x = -\sqrt{2}$$

Substitute for $x$ in Eq. (2).

$$y = 3^2 + 7(3) - 5$$
$$y = 25$$

$$y = \left(\sqrt{2}\right)^2 + 7\sqrt{2} - 5$$
$$y = -3 + 7\sqrt{2}$$

$$y = \left(-\sqrt{2}\right)^2 + 7\left(\sqrt{2}\right) - 5$$
$$y = -3 - 7\sqrt{2}$$

The solutions are $(3,\ 25), \left(\sqrt{2},\ -3 + 7\sqrt{2}\right)$ and $(-\sqrt{2},\ -3 - 7\sqrt{2})$.

**15.** $\begin{cases} 2x^2 + y^2 = 9 & (1) \\ x^2 - y^2 = 3 & (2) \end{cases}$

$$\begin{array}{l} 2x^2 + y^2 = 9 \\ \underline{x^2 - y^2 = 3} \\ 3x^2 = 12 \quad \text{Add the equations.} \\ x^2 = 4 \\ x = \pm 2 \end{array}$$

When $x = -2$, $(-2)^2 - y^2 = 3$ From Eq. (2)

$$\begin{aligned} 4 - y^2 &= 3 \\ -y^2 &= -1 \\ y^2 &= 1 \\ y &= \pm 1 \end{aligned}$$

When $x = 2$, $(2)^2 - y^2 = 3$

$$\begin{aligned} 4 - y^2 &= 3 \\ -y^2 &= -1 \\ y^2 &= 1 \\ y &= \pm 1 \end{aligned}$$

The solutions are $(-2, 1)$, $(-2, -1)$, $(2, 1)$ and $(2, -1)$.

**16.** $\begin{cases} 3x^2 - 2y^2 = 19 & (1) \\ x^2 - y^2 = 5 & (2) \end{cases}$

$$\begin{array}{ll} 3x^2 - 2y^2 = 19 & \\ \underline{-3x^2 + 3y^2 = -15} & -3 \text{ times Eq. (2)} \\ y^2 = 4 & \text{Add the equations.} \\ y = \pm 2 & \end{array}$$

When $y = -2$, $x^2 - (-2)^2 = 5$ From Eq. (1)

$$\begin{aligned} x^2 - 4 &= 5 \\ x^2 &= 9 \\ x &= \pm 3 \end{aligned}$$

When $y = 2$, $x^2 - 2^2 = 5$

$$\begin{aligned} x^2 - 4 &= 5 \\ x^2 &= 9 \\ x &= \pm 3 \end{aligned}$$

The solutions are $(3, -2)$, $(3, 2)$, $(-3, 2)$ and $(-3, -2)$.

**17.** $\begin{cases} x^2 - 2y^2 = 8 & (1) \\ x^2 + 3y^2 = 28 & (2) \end{cases}$

Use the elimination method to eliminate $x^2$.

$$\begin{array}{l} x^2 - 2y^2 = 8 \\ \underline{-x^2 - 3y^2 = -28} \\ -5y^2 = -20 \\ y^2 = 4 \\ y = \pm 2 \end{array}$$

Substitute for $y$ in Eq. (1).

$$\begin{array}{ll} x^2 - 2(2)^2 = 8 & x^2 - 2(-2)^2 = 8 \\ x^2 = 16 & x^2 = 16 \\ x = \pm 4 & x = \pm 4 \end{array}$$

The solutions are (4, 2), (–4, 2), (4, –2) and (–4, –2).

**18.** $\begin{cases} 2x^2 + 3y^2 = 5 & (1) \\ x^2 - 3y^2 = 4 & (2) \end{cases}$

Use the elimination method to eliminate $y^2$.

$$\begin{array}{l} 2x^2 + 3y^2 = 5 \quad (1) \\ \underline{x^2 - 3y^2 = 4} \\ 3x^2 = 9 \\ x^2 = 3 \\ x = \pm\sqrt{3} \end{array}$$

Substitute for $y$ in Eq. (2).

$$\begin{aligned} \left(\sqrt{3}\right)^2 - 3y^2 &= 4 \\ y^2 &= -\frac{1}{3} \end{aligned}$$

$y^2 = -1/3$ has no real number solutions. The graphs of the equations do not intersect.

**19.** $\begin{cases} 2x^2 + 4y^2 = 5 & (1) \\ 3x^2 + 8y^2 = 14 & (2) \end{cases}$

Use the elimination method to eliminate $y^2$.

$$\begin{array}{rll} -4x^2 - 8y^2 &= -10 & -2 \text{ times } (1) \\ 3x^2 + 8y^2 &= 14 & \\ \hline -x^2 &= 4 & \\ x^2 &= -4 & \end{array}$$

$x^2 = -4$ has no real number solutions. The graphs of the equations do not intersect.

**20.** $\begin{cases} 2x^2 + 3y^2 = 11 & (1) \\ 3x^2 + 2y^2 = 19 & (2) \end{cases}$

Use the elimination method to eliminate $y^2$.

$$\begin{array}{rll} 4x^2 + 6y^2 &= 22 & 2 \text{ times } (1) \\ -9x^2 - 6y^2 &= -57 & -3 \text{ times}(2) \\ \hline -5x^2 &= -35 & \\ x &= \pm\sqrt{7} & \end{array}$$

Substitute for $x$ in Eq. (1).

$$2(\sqrt{7})^2 + 3y = 11$$
$$3y^2 = -3$$
$$y^2 = -1$$

$y^2 = -1$ has no real number solutions. The graphs of the equations do not intersect.

**21.** $\begin{cases} x^2 - 2x + y^2 = 1 & (1) \\ 2x + y = 5 & (2) \end{cases}$

Substitute $y$ from Eq. (2) into Eq. (1).

$$x^2 - 2x + (5 - 2x)^2 = 1$$
$$x^2 - 2x + 25 - 20x + 4x^2 = 1$$
$$5x^2 - 22x + 24 = 0$$
$$(5x - 12)(x - 2) = 0$$

$x = \frac{12}{5}$ or $x = 2$

Substitute for $x$ in Eq. (2).

$$2\left(\frac{12}{5}\right) + y = 5 \qquad 2(2) + y = 5$$
$$y = \frac{1}{5} \qquad y = 1$$

The solutions are (12/5, 1/5) and (2, 1).

**22.** $\begin{cases} x^2 + y^2 + 3y = 22 & (1) \\ 2x + y = -1 & (2) \end{cases}$

Substitute $y$ from Eq. (2) into Eq. (1).

$$x^2 + (-2x - 1)^2 + 3(-2x - 1) = 22$$
$$x^2 + 4x^2 + 4x + 1 - 6x - 3 = 22$$
$$5x^2 - 2x - 24 = 0$$
$$(5x - 12)(x + 2) = 0$$

$x = \frac{12}{5}$ or $x = -2$

Substitute for $x$ in Eq. (2).

$$2\left(\frac{12}{5}\right) + y = -1 \qquad 2(-2) + y = -1$$
$$y = -\frac{29}{5} \qquad y = 3$$

The solutions are (12/5, –29/5) and (–2, 3).

**23.** $\begin{cases}(x-3)^2+(y+1)^2=5 & (1)\\ x-3y=7 & (2)\\ x=3y+7\end{cases}$

Substitute $x$ from Eq. (2) into Eq. (1).

$$(3y+4)^2+(y+1)^2=5$$
$$9y^2+24y+16+y^2+2y+1=5$$
$$10y^2+26y+12=0$$
$$5y^2+13y+6=0$$
$$(5y+3)(y+2)=0$$
$$y=-\frac{3}{5} \text{ or } y=-2$$

Substitute for $y$ in Eq. (3).

$$x=3\left(-\frac{3}{5}\right)+7 \qquad x=3(-2)+7$$
$$x=\frac{26}{5} \qquad x=1$$

The solutions are (26/5, –3/5) and (1, –2).

**24.** $\begin{cases}(x+2)^2+(y-2)^2=13 & (1)\\ 2x+y=6 & (2)\end{cases}$

Substitute $y$ from Eq. (2) into Eq. (1).

$$(x+2)^2+(4-2x)^2=13$$
$$x^2+4x+4+16-16x+4x^2=13$$
$$5x^2-12x+7=0$$
$$(5x-7)(x-1)=0$$
$$x=\frac{7}{5} \text{ or } x=1$$

Substitute for $x$ in Eq. (2).

$$2\left(\frac{7}{5}\right)+y=6 \qquad 2(1)+y=6$$
$$y=\frac{16}{5} \qquad y=4$$

The solutions are (7/5, 16/5) and (1, 4).

**25.** $\begin{cases}x^2-3x+y^2=4 & (1)\\ 3x+y=11 & (2)\end{cases}$

Substitute $y$ from Eq. (2) into Eq. (1).

$$x^2-3x+(11-3x)^2=4$$
$$x^2-3x+121-66x+9x^2=4$$
$$10x^2-69x+117=0$$
$$(10x-29)(x-3)=0$$
$$x=\frac{39}{10} \text{ or } x=3$$

Substitute for $x$ in Eq. (2).

$$3\left(\frac{39}{10}\right)+y=11 \qquad 3(3)+y=11$$
$$y=-\frac{7}{10} \qquad y=2$$

The solutions are (39/10, –7/10) and (3, 2).

**26.** $\begin{cases}x^2+y^2-4y=4 & (1)\\ 5x-2y=2 & (2)\end{cases}$

Substitute $y$ from Eq. (2) into Eq. (1).

$$x^2+\left(\frac{5}{2}x-1\right)^2-4\left(\frac{5}{2}x-1\right)=4$$
$$x^2+\frac{25}{4}x^2-5x+1-10x+4=4$$
$$\frac{29}{4}x^2-15x+1=0$$
$$29x^2-60x+4=0$$
$$(29x-2)(x-2)=0$$
$$x=\frac{2}{29} \text{ or } x=2$$

Substitute for $x$ in Eq. (2).

$$5\left(\frac{2}{29}\right)-2y=2 \qquad 5(2)-2y=2$$
$$y=-\frac{24}{29} \qquad y=4$$

The solutions are (2/29, –24/29) and (2, 4).

**27.** $\begin{cases}(x-1)^2+(y+2)^2=14 & (1)\\ (x+2)^2+(y-1)^2=2 & (2)\end{cases}$

Expand the binomials and then subtract.

$$\begin{array}{r} x^2-2x+1+y^2+4y+4=14\\ \underline{x^2+4x+4+y^2-2y+1=2}\\ -6x-3 \quad +6y+3=12\\ -6x \quad +6y=12\\ y=x+2\end{array}$$

Substitute for $y$ in Eq. (2).

$$(x+2)^2+(x+2-1)^2=2$$
$$x^2+4x+4+x^2+2x+1=2$$
$$2x^2+6x+3=0$$

$$x=\frac{-6\pm\sqrt{36-4\cdot 2\cdot 3}}{4}=\frac{-6\pm\sqrt{12}}{4}=\frac{-3\pm\sqrt{3}}{2}$$

Substitute for $x$ in $y = x + 2$.

$$y=\frac{-3+\sqrt{3}}{2}+2 \qquad y=\frac{-3-\sqrt{3}}{2}+2$$
$$y=\frac{1+\sqrt{3}}{2} \qquad y=\frac{1-\sqrt{3}}{2}$$

The solutions are
$\left(\frac{-3+\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}\right)$ and $\left(\frac{-3-\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}\right)$.

**28.** $\begin{cases}(x+2)^2+(y-3)^2=10 & (1)\\ (x-3)^2+(y+1)^2=13 & (2)\end{cases}$

Expand the binomials and then subtract.

$$\begin{array}{r} x^2+4x+4+y^2-6y+9=10\\ \underline{x^2-6x+9+y^2+2y+1=13}\\ 10x-5 \quad -8y+8=-3\\ 10x \quad -8y \quad =-6\\ y=\frac{5x+3}{4}\end{array}$$

Substitute for $y$ in Eq. (1): $y-3=\frac{5x-9}{4}$.

$$(x+2)^2+\left(\frac{5x-9}{4}\right)^2=10$$
$$x^2+4x+4+\frac{25x^2-90x+81}{16}=10$$
$$16x^2+64x+64+25x^2-90x+81=160$$
$$41x^2-26x-15=0$$
$$(41x+15)(x-1)=0$$

$x=-\frac{15}{41}$ or $x=1$

Substitute for $x$ in $y=\frac{5x+3}{4}$.

$$y=\frac{5}{4}\left(-\frac{15}{41}\right)+\frac{3}{4} \qquad y=\frac{5(1)+3}{4}$$
$$y=\frac{12}{41} \qquad y=2$$

The solutions are (–15/41, 12/41) and (1, 2).

**29.** $\begin{cases}(x+3)^2+(y-2)^2=20 & (1)\\(x-2)^2+(y-3)^2=2 & (2)\end{cases}$

Expand the binomials and then subtract.

$$\begin{aligned}x^2+6x+9+y^2-4y+4&=20\\ x^2-4x+4+y^2-6y+9&=2\\ \hline 10x+5\qquad +2y-5&=18\\ 10x\qquad +2y&=18\\ y&=-5x+9\end{aligned}$$

Substitute for $y$ in Eq. (2): $y-3=-5x+6$.

$$\begin{aligned}(x-2)^2+(-5x+6)^2&=2\\ x^2-4x+4+25x^2-60x+36&=2\\ 26x^2-64x+38&=0\\ 13x^2-32x+19&=0\\ (13x-19)(x-1)&=0\end{aligned}$$

$x=\dfrac{19}{13}$ or $x=1$

Substitute for $x$ in $y=-5x+9$.

$y=-5\left(\dfrac{19}{13}\right)+9$ $\qquad$ $y=-5(1)+9$

$y=\dfrac{22}{13}$ $\qquad$ $y=4$

The solutions are (19/13, 22/13) and (1, 4).

**30.** $\begin{cases}(x-4)^2+(y-5)^2=8 & (1)\\(x+1)^2+(y+2)^2=34 & (2)\end{cases}$

Expand the binomials and then subtract.

$$\begin{aligned}x^2-8x+16+y^2-10y+25&=8\\ x^2+2x+1+y^2+4y+4&=34\\ \hline -10x+15\qquad -14y+21&=-26\\ -10x\qquad -14y&=-62\\ 5x\qquad +7y&=31\\ y&=\frac{31-5x}{7}\end{aligned}$$

Substitute for $y$ in Eq. (1): $y-5=\dfrac{-5x-4}{7}$.

$$\begin{aligned}(x-4)^2+\left(\frac{-5x-4}{7}\right)^2&=8\\ x^2-8x+16+\frac{25x^2+40x+16}{49}&=8\\ 49x^2-392x+784+25x^2+40x+16&=392\\ 74x^2-352x+408&=0\\ 37x^2-176x+204&=0\\ (37x-102)(x-2)&=0\end{aligned}$$

$x=\dfrac{102}{37}$ or $x=2$

Substitute for $x$ in $y=\dfrac{31-5x}{7}$.

$y=\dfrac{31}{7}-\dfrac{5}{7}\left(\dfrac{102}{37}\right)$ $\qquad$ $y=\dfrac{31-5(2)}{7}$

$y=\dfrac{91}{37}$ $\qquad$ $y=3$

The solutions are (102/37, 91/37) and (2, 3).

**31.** $\begin{cases} (x-1)^2+(y+1)^2=2 & (1) \\ (x+2)^2+(y-3)^2=3 & (2) \end{cases}$

Expand the binomials and then subtract.

$$\begin{array}{l} x^2-2x+1+y^2+2y+1=2 \\ \underline{x^2+4x+4+y^2-6y+9=3} \\ \quad -6x-3 \quad +8y-8=-1 \\ \quad -6x \quad\quad +8y \quad =10 \\ \quad\quad\quad\quad y=\dfrac{3x+5}{4} \end{array}$$

Substitute for $y$ in Eq. (1): $y+1=\dfrac{3x+9}{4}$.

$$(x-1)^2+\left(\frac{3x+9}{4}\right)^2=2$$

$$x^2-2x+1+\frac{9x^2+54x+81}{16}=2$$

$$16x^2-32x+16+9x^2+54x+81=32$$

$$25x^2+22x+65=0$$

$$x=\frac{-22\pm\sqrt{22^2-4(25)(65)}}{2(25)}$$

$$x=\frac{-22\pm\sqrt{-6016}}{50}$$

$x$ is not a real number. There are no real solutions. The curves do not intersect.

**32.** $\begin{cases} (x+1)^2+(y-3)^2=4 & (1) \\ (x-3)^2+(y+2)^2=2 & (2) \end{cases}$

Expand the binomials and then subtract.

$$\begin{array}{l} x^2+2x+1+y^2-6y+9=4 \\ \underline{x^2-6x+9+y^2+4y+4=2} \\ \quad 8x-8 \quad -10y+5=2 \\ \quad 8x \quad\quad -10y \quad =5 \\ \quad\quad\quad\quad y=\dfrac{8x-5}{10} \end{array}$$

Substitute for $y$ in Eq. (1): $y-3=\dfrac{8x-5}{10}$.

$$(x+1)^2+\left(\frac{8x-35}{10}\right)^2=4$$

$$x^2+2x+1+\frac{64x^2-560x+1225}{100}=4$$

$$100x^2+200x+100+64x^2-560x+1225=400$$

$$164x^2-360x+925=0$$

$$x=\frac{360\pm\sqrt{360^2-4(164)(925)}}{2(164)}$$

$$x=\frac{360\pm\sqrt{-477200}}{2(164)}$$

$x$ is not a real number. There are no real solutions. The curves do not intersect.

**33.** $h$ = height
$w$ = weight

$$2h+2w=25$$

$$wh=37.5 \Rightarrow w=\frac{37.5}{h}$$

$$2h+2\left(\frac{37.5}{h}\right)=25$$

$$2h^2+75=25h$$

$$2h^2-25h+75=0$$

$$(2h-15)(h-5)=0$$

$$2h-15=0 \qquad h-5=0$$

$$h=7.5 \qquad h=5$$

Since the height is greater than the width, $h = 7.5$.

$$w=\frac{37.5}{7.5}=5$$

The width is 5 in. and the height is 7.5 in.

**34.** $V=lwh,\ l=w$

$$l^2h=121$$

$$2l+2h=19 \Rightarrow l=\frac{19-2h}{2}$$

$$\left(\frac{19-2h}{2}\right)^2 h=121$$

$$\left(\frac{361-76h+4h^2}{4}\right)h=121$$

$$4h^3-76h^2+361h-484=0$$

$$\begin{array}{r|rrrr} 4 & 4 & -76 & 361 & -484 \\ & & 16 & -240 & 484 \\ \hline & 4 & -60 & 121 & 0 \end{array}$$

$$h=4$$

$$l=\frac{19-2h}{2}=\frac{19-2(4)}{2}=5.5$$

The height is 4 in. and the length and width is 5.5 in.

**35.**

$$x^2 + y^2 = 865$$
$$x^2 - y^2 = 703$$
$$2x^2 = 1568$$
$$x^2 = 784$$
$$x = 28$$

$$28^2 + y^2 = 865$$
$$y^2 = 81$$
$$y = 9$$

The small carpet is 9 ft by 9 ft and the large carpet is 28 ft by 28 ft.

**36.** $h$ = height
$w$ = weight

$$h^2 + w^2 = (25.0)^2$$
$$h = 1.6w$$

$$(1.6w)^2 + w^2 = 625$$
$$3.56w^2 = 625$$
$$w^2 = \frac{625}{3.56}$$
$$w = \sqrt{\frac{625}{3.56}} \approx 13.2$$

$$h = 1.6w = 1.6(13.2) = 21.1$$

The width is 13.2 ft and the height is 21.1 ft

**37.** $r$ = radius of the small globe
$R$ = radius of the large globe

$$V = \frac{4}{3}\pi r^3$$
$$\frac{4}{3}\pi R^3 = 8\left(\frac{4}{3}\pi r^3\right) \Rightarrow R^3 = 8r^3$$
$$\frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3 = 15{,}012.63$$

$$-\frac{4}{3}\pi R^3 + \frac{32}{3}\pi r^3 = 0$$
$$\frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3 = 15{,}012.63$$
$$\frac{28}{3}\pi r^3 = 15{,}012.63$$
$$r^3 = \frac{3(15{,}012.63)}{28\pi}$$
$$r = \sqrt[3]{\frac{3(15{,}012.63)}{28\pi}}$$
$$r \approx 8.0$$

$$R^3 = 8r^3$$
$$R^3 = 8(8.0)^3$$
$$R = 16.0$$

The radius of the large globe is 16.0 in. and the radius of the small globe is 8.0 in.

**38.**

$$\frac{x^2}{47^2} + \frac{(-16)^2}{25^2} = 1$$
$$\frac{x^2}{47^2} = 1 - \left(\frac{16}{25}\right)^2$$
$$x^2 = 47^2\left(1 - \left(\frac{16}{25}\right)^2\right)$$
$$x = \pm\sqrt{47^2\left(1 - \left(\frac{16}{25}\right)^2\right)}$$
$$x \approx -36.1$$

The point $A$ is (–36.1, –16).

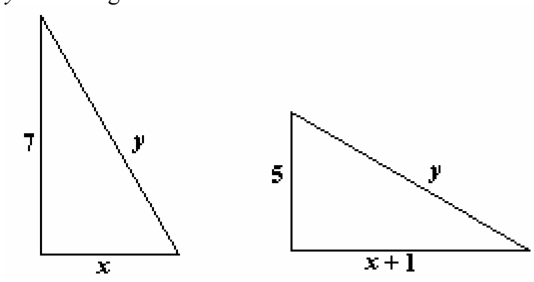
**39.** $\begin{cases} x^2 = y & (1) \\ 18x - 22 = 3y + 5 & (2) \end{cases}$

Substitute for $y$ in Eq. (2).

$$18x - 22 = 3\left(x^2\right) + 5$$
$$0 = 3x^2 - 18x + 27$$
$$0 = 3(x^2 - 6x + 9)$$
$$0 = 3(x-3)^2$$
$$x = 3$$
$$y = 3^2 = 9$$

$$P = x^2 + 3y + 5 + y + 18x - 22$$
$$= 3^2 + 3(9) + 5 + 9 + 18(3) - 22$$
$$= 82 \text{ units}$$

**40.** Let $x$ = the original distance of the ladder from the wall and $y$ = the length of the ladder.




By the Pythagorean Theorem,

$$7^2 + x^2 = y^2 \qquad \text{and} \qquad 5^2 + (x+1)^2 = y^2$$
$$49 + x^2 = y^2 \quad (1) \qquad 25 + x^2 + 2x + 1 = y^2$$
$$x^2 + 2x + 26 = y^2$$

Set the expressions for $y^2$ equal to each other.

$$x^2 + 2x + 26 = x^2 + 49$$
$$2x = 23$$
$$x = 11.5$$

Substitute for $x$ in Eq. (1).

$$y^2 = 49 + (11.5)^2 = 181.25$$
$$y = \sqrt{181.25} \approx 13.46 \text{ meters}$$

**41.** $\begin{cases} x^2 + y^2 = r^2 & (1) \\ y = 2x + 1 & (2) \end{cases}$

Substitute for $y$ in Eq. (1)

$$x^2 + (2x+1)^2 = r^2$$
$$x^2 + 4x^2 + 4x + 1 = r^2$$
$$5x^2 + 4x + 1 = r^2$$

Minimize $r^2$ by completing the square.

$$r^2 = 5x^2 + 4x + 1 = 5\left(x^2 + \frac{4}{5}x + \frac{4}{25}\right) + 1 - \frac{4}{5}$$
$$= 5\left(x + \frac{2}{5}\right)^2 + \frac{1}{5}$$

Thus $\left(-\frac{2}{5}, \frac{1}{5}\right)$ is the point on both $x^2 + y^2 = r^2$ and $y = 2x + 1$ for which $x^2 + y^2 = r^2$ has the smallest radius.

Substitute for $x$ in $r^2 = 5x^2 + 4x + 1$

$$r^2 = 5\left(-\frac{2}{5}\right)^2 + 4\left(-\frac{2}{5}\right) + 1 = \frac{1}{5}.$$
$$r = \sqrt{\frac{1}{5}} \text{ or } \frac{\sqrt{5}}{5} \text{ is the minimum radius.}$$

Therefore $r \geq \frac{\sqrt{5}}{5}$.

**42.** $ab = (a-3)(b+2) = (a+3)(b-1)$

$$ab = (a-3)(b+2)$$
$$ab = ab + 2a - 3b - 6$$
$$6 = 2a - 3b \quad (1)$$

$$ab = (a+3)(b-1)$$
$$ab = ab - a + 3b - 3$$
$$3 = -a + 3b \quad (2)$$

Adding Eq. (1) and Eq. (2) we have $9 = a$.

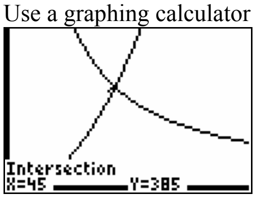
Substitute 9 for $a$ in Eq. (1).

$$6 = 2(9) - 3b$$
$$6 = 18 - 3b$$
$$-12 = -3b$$
$$4 = b$$

Thus, the dimensions of the first rectangle are 9 and 4. Therefore the area is (9)(4) = 36 square units.

**43.** $\begin{cases} x = \dfrac{p^2}{5} - 20 \\ x = \dfrac{17{,}710}{p+1} \end{cases}$

Use a graphing calculator and INTERSECTION.

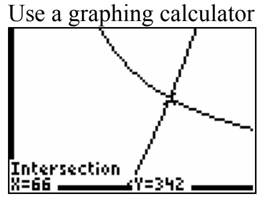



on [0, 100] by [0, 600]
The graphs intersect at (45, 385).

The solution is $45.

**44.** $\begin{cases} x = \dfrac{p^2}{6} - 384 \\ x = \dfrac{22{,}914}{p+1} \end{cases}$

Use a graphing calculator and INTERSECTION.




on [0, 100] by [0, 600]
The graphs intersect at (66, 342).

The solution is $66.

**45.** $\begin{cases} y = 2^x \\ y = x + 1 \end{cases}$

Using a graphing calculator, graph the two equations on the same coordinate grid. Using the ZOOM feature, estimate the coordinates of the points where the graphs intersect. These coordinates are the solutions of the system of equations. For this system of equations, the solutions are (0, 1) and (1, 2).

**46.** $\begin{cases} y = \log_2 x \\ y = x - 3 \end{cases}$

Using a graphing calculator, graph the two equations on the same coordinate grid. Using the ZOOM feature, estimate the coordinates of the points where the graphs intersect. These coordinates are the solutions of the system of equations. For this system of equations, the solutions are approximately (0.1375, –2.8625) and (5.4449, 2.4449).

**47.** $\begin{cases} y = e^{-x} \\ y = x^2 \end{cases}$

Using a graphing calculator, graph the two equations on the same coordinate grid. Using the ZOOM feature, estimate the coordinates of the points where the graphs intersect. These coordinates are the solutions of the system of equations. For this system of equations, the solution is approximately (0.7035, 0.4949).

**48.** $\begin{cases} y = \ln x \\ y = -x + 4 \end{cases}$

Using a graphing calculator, graph the two equations on the same coordinate grid. Using the ZOOM feature, estimate the coordinates of the points where the graphs intersect. These coordinates are the solutions of the system of equations. For this system of equations, the solution is approximately (2.9263, 1.0737).

**49.** $\begin{cases} y = \sqrt{x} \\ y = \dfrac{1}{x-1} \end{cases}$

Using a graphing calculator, graph the two equations on the same coordinate grid. Using the ZOOM feature, estimate the coordinates of the points where the graphs intersect. These coordinates are the solutions of the system of equations. For this system of equations, the solution is approximately (1.7549, 1.3247).

**50.** $\begin{cases} y = \dfrac{6}{x+1} \\ y = \dfrac{x}{x-1} \end{cases}$

Using a graphing calculator, graph the two equations on the same coordinate grid. Using the ZOOM feature, estimate the coordinates of the points where the graphs intersect. These coordinates are the solutions of the system of equations. For this system of equations, the solutions are (2, 2) and (3, 3/2).

## Connecting Concepts

**51.** $\begin{cases} y = x^2 + 4 \\ x = y^2 - 24 \end{cases}$

Solve by substitution.

$x = (x^2 + 4)^2 - 24$

$x = x^4 + 8x^2 + 16 - 24$

$0 = x^4 + 8x^2 - x - 8$

$0 = (x-1)(x^3 + x^2 + 9x + 8)$

$x^3 + x^2 + 9x + 8$ is not factorable over the rational numbers because the Rational Zero Theorem implies the only rational; zeros are $\pm 1, \pm 2, \pm 4, \pm 8$. Thus, the only rational ordered-pair solution is (1, 5).

**52.** $\begin{cases} y = x^2 - 5 \\ x = y^2 - 13 \end{cases}$

Solve by substitution.

$x = (x^2 - 5)^2 - 13$

$x = x^4 - 10x^2 + 25 - 13$

$0 = x^4 - 10x^2 - x + 12$

By the Rational Zero Theorem, the possible rational roots are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$. By Descartes' rule of signs, there are 0 or 2 positive roots and 0 or 2 negative roots.

Using synthetic division, we test possible rational roots.

| | | | | | |
|---|---|---|---|---|---|
| 3 | 1 | 0 | –10 | –1 | 12 |
| | | 3 | 9 | –3 | –12 |
| | 1 | 3 | –1 | –4 | 0 |

Thus, 3 is a root of the equation. The rational ordered-pair solution is (3, 4).

**53.** $x^2 - 3xy + y^2 = 5$

$x^2 - xy - 2y^2 = 0$

Factor the second equation.

$(x - 2y)(x + y) = 0$

Thus $x = 2y$ or $x = -y$. Substituting each expression into the first equation, we have

$$\begin{aligned} (2y)^2 - 3(2y)y + y^2 &= 5 \\ 4y^2 - 6y^2 + y^2 &= 5 \\ -y^2 &= 5 \\ y^2 &= -5 \end{aligned}$$

There are no rational solutions.

$$\begin{aligned} (-y)^2 - 3(-y)y + y^2 &= 5 \\ y^2 + 3y^2 + y^2 &= 5 \\ 5y^2 &= 5 \\ y^2 &= 1 \Rightarrow y = \pm 1 \end{aligned}$$

Substituting into $x = -y$, we have $x = -1$ or $x = 1$. The rational ordered-pair solutions are (–1, 1) and (1, –1).

**54.** $x^2 + 2xy - y^2 = 1$

$x^2 + 3xy + 2y^2 = 0$

Factor the second equation.

$(x - 2y)(x + y) = 0$

Thus $x = -y$ or $x = -2y$. Substitute each expression into the first equation and solve for $y$.

$$\begin{aligned} (-y)^2 + 2(-y)y - y^2 &= 1 \\ y^2 - 2y^2 - y^2 &= 1 \\ -2y^2 &= 1 \\ y^2 &= -\frac{1}{2} \end{aligned}$$

There are no rational solutions.

$$\begin{aligned} (-2y)^2 + 2(-2y)y - y^2 &= 1 \\ 4y^2 - 4y^2 - y^2 &= 1 \\ -y^2 &= 1 \\ y^2 &= -1 \end{aligned}$$

There are no rational solutions.

**55.** $\begin{cases} 2x^2 - 4xy - y^2 = 6 \\ 4x^2 - 3xy - y^2 = 6 \end{cases}$

Subtract the two equations.

$$-2x^2 - xy = 0$$
$$-x(2x + y) = 0$$
$$x = 0 \text{ or } y = -2x$$

Substituting $x = 0$ into the first equation gives $-y^2 = 6$ or $y^2 = -6$. There are no rational solutions.

Substituting $y = -2x$ into the first equation gives

$$\begin{aligned} 2x^2 + 8x^2 - 4x^2 &= 6 \\ 6x^2 &= 6 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

The rational ordered-pair solutions are $(1, -2)$ and $(-1, 2)$.

**56.** $\begin{cases} 3x^2 + 2xy - 5y^2 = 11 \\ x^2 + 3xy + y^2 = 11 \end{cases}$

Subtract the two equations.

$$2x^2 - xy - 6y^2 = 0$$

Factoring, we have $(2x + 3y)(x - 2y) = 0$. Then

$2x = -3y$ or $x = 2y$

$$x = -\frac{3}{2y}$$

Substituting into the first equation, we have

$$\begin{aligned} 3\left(-\frac{3}{2}y\right)^2 + 2\left(-\frac{3}{2}y\right)y - 5y^2 &= 11 \\ \frac{27}{4}y^2 - 3y^2 - 5y^2 &= 11 \\ -\frac{5}{4}y^2 &= 11 \\ y^2 &= -\frac{44}{5} \end{aligned}$$

There are no rational solutions.

$$\begin{aligned} 3(2y)^2 + 2(2y)y - 5y^2 &= 11 \\ 12y^2 + 4y^2 - 5y^2 &= 11 \\ 11y^2 &= 11 \\ y^2 &= 1 \end{aligned}$$

$y = 1$ or $y = -1$

Substituting each of these into the first equation and solving for $x$, we have $(2, 1)$ and $(-2, -1)$ as solutions.

## Prepare for Section 9.4

**PS1.** $x^4 + 14x^2 + 49 = (x^2 + 7)^2$

**PS2.**
$$\begin{aligned} \frac{5}{x-1} + \frac{1}{x+2} &= \frac{x+2}{x+2}\cdot\frac{5}{x-1} + \frac{x-1}{x-1}\cdot\frac{1}{x+2} \\ &= \frac{5x+10}{(x-1)(x+2)} + \frac{x-1}{(x-1)(x+2} \\ &= \frac{6x+9}{(x-1)(x+2)} \end{aligned}$$

**PS3.**
$$\begin{aligned} \frac{7}{x} - \frac{6}{x-1} + \frac{10}{(x-1)^2} &= \frac{(x-1)^2}{(x-1)^2}\cdot\frac{7}{x} - \frac{x(x-1)}{x(x-1)}\cdot\frac{6}{x-1} + \frac{x}{x}\cdot\frac{10}{(x-1)^2} \\ &= \frac{7x^2 - 14x + 7}{x(x-1)^2} - \frac{6x^2 - 6x}{x(x-1)^2} + \frac{10x}{x(x-1)^2} \\ &= \frac{x^2 + 2x + 7}{x(x-1)^2} \end{aligned}$$

**PS4.** $\begin{cases} 1 = A + B & (1) \\ 11 = -5A + 3B & (2) \end{cases}$

Solve equation (1) for $A$ and substitute into equation (2).

$$\begin{aligned} 11 &= -5(1 - B) + 3B \\ 11 &= -5 + 8B \\ 16 &= 8B \\ 2 &= B \end{aligned}$$

$$\begin{aligned} A &= 1 - 2 \\ &= -1 \end{aligned}$$

The solution is $(-1, 2)$.

**PS5.** $\begin{cases} 0=A+B & (1) \\ 3=-2B+C & (2) \\ 16=7A-2C & (3) \end{cases}$

Solve equation (1) for $A$ and substitute into equation (3).
$16 = -7B - 2C$ (4)

Multiply equation (2) by 2 and add to equation (4).

$$\begin{aligned} 6&=-4B+2C \\ 16&=-7B-2C \\ \hline 22&=-11B \\ -2&=B \end{aligned}$$

$A = 2$

$$\begin{aligned} C&=2B+3 \\ &=2(-2)+3 \\ &=-1 \end{aligned}$$

The solution is (2, –2, –1).

**PS6.** $\dfrac{x^3-4x^2-19x-35}{x^2-7x}$

Use long division.

$$\begin{array}{r} x+3 \\ x^2-7x\overline{)\,x^3-4x^2-19x-35} \\ \underline{x^3-7x^2\phantom{-19x-35}} \\ 3x^2-19x\phantom{-35} \\ \underline{3x^2-21x\phantom{-35}} \\ 2x-35 \end{array}$$

$$\frac{x^3-4x^2-19x-35}{x^2-7x}=x+3+\frac{2x-35}{x^2-7x}$$

## Section 9.4

**1.** $\dfrac{x+15}{x(x-5)}=\dfrac{A}{x}+\dfrac{B}{x-5}$

$x+15=A(x-5)+Bx$
$x+15=(A+B)x-5A$

$\begin{cases} 1=A+B \\ 15=-5A \end{cases}$ $\quad A=-3 \quad \begin{aligned} -3+B&=1 \\ B&=4 \end{aligned}$

**2.** $\dfrac{5x-6}{x(x+3)}=\dfrac{A}{x}+\dfrac{B}{x+3}$

$5x-6=A(x+3)+Bx$
$5x-6=(A+B)x+3A$

$\begin{cases} 5=A+B \\ -6=3A \end{cases}$ $\quad A=-2 \quad \begin{aligned} -2+B&=5 \\ B&=7 \end{aligned}$

**3.** $\dfrac{1}{(2x+3)(x-1)}=\dfrac{A}{2x+3}+\dfrac{B}{x-1}$

$$\begin{aligned} 1 &= A(x-1)+B(2x+3) \\ 1 &= Ax-A+2Bx+3B \\ 1 &= (A+2B)x+(-A+3B) \\ 0 &= A+2B \\ 1 &= -A+3B \\ \hline 1 &= \phantom{-A+3}5B \end{aligned}$$

$B=\dfrac{1}{5} \quad 0=A+2\left(\dfrac{1}{5}\right)$

$A=-\dfrac{2}{5}$

**4.** $\dfrac{6x-5}{(x+4)(3x+2)}=\dfrac{A}{x+4}+\dfrac{B}{3x+2}$

$6x-5=A(3x+2)+B(x+4)$
$6x-5=3Ax+2A+Bx+4B$
$6x-5=(3A+B)x+(2A+4B)$

$\begin{cases} 6=3A+B \\ -5=2A+4B \end{cases}$

Solve the system.

$$\begin{aligned} -24 &= -12A-4B \\ -5 &= 2A+4B \\ \hline -29 &= -10A \\ \frac{29}{10} &= A \end{aligned} \qquad \begin{aligned} 6 &= 3\left(\frac{29}{10}\right)+B \\ B &= -\frac{27}{10} \end{aligned}$$

**5.** $\dfrac{x+9}{x(x-3)^2}=\dfrac{A}{x}+\dfrac{B}{x-3}+\dfrac{C}{(x-3)^2}$

$x+9=A(x-3)^2+Bx(x-3)+Cx$

$x+9=Ax^2-6Ax+9A+Bx^2-3Bx+Cx$

$x+9=(A+B)x^2+(-6A-3B+C)x+9A$

$\begin{cases}0=A+B\\1=-6A-3B+C\\9=9A\end{cases}$

$A=1 \qquad A+B=0 \qquad -6A-3B+C=1$

$1+B=0 \qquad -6(1)-3(-1)+C=1$

$B=-1 \qquad C=4$

**6.** $\dfrac{2x-7}{(x+1)(x-2)^2}=\dfrac{A}{x+1}+\dfrac{B}{x-2}+\dfrac{C}{(x-2)^2}$

$2x-7=A(x-2)^2+B(x+1)(x-2)+C(x+1)$

$2x-7=Ax^2-4Ax+4A+Bx^2-Bx-2B+Cx+C$

$2x-7+(A+B)x^2+(-4A-B+C)x+(4A-2B+C)$

$\begin{cases}0=A+B & (1)\\2=-4A-B+C & (2)\\-7=4A-2B+C & (3)\end{cases}$

$2=-4A-B+C \quad (2)$

$\underline{7=-4A+2B-C} \quad -1 \text{ times } (3)$

$9=-8A+B \quad (4)$

$\begin{cases}0=A+B\\2=-4A-B+C\\9=-8A+B & (4)\end{cases}$

$0=A+B \quad (1)$

$\underline{-9=8A-B} \quad -1 \text{ times } (4)$

$9=9A$

$-1=A \quad (5)$

$\begin{cases}-1=A & (5)\\2=-4A-B+C\\9=-8A+B\end{cases}$

$A=-1 \qquad 9=-8(-1)+B \qquad -4(-1)-1+C=2$

$B=1 \qquad C=-1$

**7.** $\dfrac{4x^2+3}{(x-1)(x^2+x+5)}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+x+5}$

$4x^2+3=A(x^2+x+5)+(Bx+C)(x-1)$

$4x^2+3=Ax^2+Ax+5A+Bx^2-Bx+Cx-C$

$4x^2+3=(A+B)x^2+(A-B+C)x+(5A-C)$

$\begin{cases}4=A+B & (1)\\0=A-B+C & (2)\\3=5A-C & (3)\end{cases}$

From (3), $C=5A-3$. From (1), $B=4-A$. Substitute $C$ and $B$ into Eq. (2).

$0=A-(4-A)+5A-3 \qquad C=5(1)-3 \qquad B=4-1$

$0=A-4+A+5A-3 \qquad C=2 \qquad B=3$

$7=7A$

$1=A$

**8.** $\dfrac{x^2+x+3}{(x^2+7)(x-3)}=\dfrac{Ax+B}{x^2+7}+\dfrac{C}{x-3}$

$x^2+x+3=(Ax+B)(x-3)+C(x^2+7)$

$x^2+x+3=Ax^2-3Ax+Bx-3B+Cx^2+7C$

$x^2+x+3=(A+C)x^2+(-3A+B)x+(-3B+7C)$

$\begin{cases}1=A+C & (1)\\1=-3A+B & (2)\\3=-3B+7C & (3)\end{cases}$

$3=3A+3C \quad 3 \text{ times } (1)$

$\underline{1=-3A+B} \quad (2)$

$4=3C+B \quad (4)$

$\begin{cases}1=A+C\\4=B+3C & (4)\\3=-3B+7C\end{cases}$

$12=3B+9C \quad 3 \text{ times } (4)$

$\underline{3=-3B+7C} \quad (3)$

$15=16C \quad (5)$

$\begin{cases}1=A+C\\4=B+3C\\15=16C & (5)\end{cases}$

$C=\dfrac{15}{16} \qquad B+3\left(\dfrac{15}{16}\right)=4 \qquad A+\dfrac{15}{16}=1$

$B=\dfrac{19}{16} \qquad A=\dfrac{1}{16}$

**9.** $\dfrac{x^3+2x}{(x^2+1)^2}=\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{(x^2+1)^2}$

$x^3+2x=(Ax+B)(x^2+1)+(Cx+D)$

$x^3+2x=Ax^3+Ax+Bx^2+B+Cx+D$

$x^3+2x=Ax^3+Bx^2+(A+C)x+(B+D)$

$\begin{cases}1=A\\0=B\\2=A+C\\0=B+D\end{cases}$ $A=1$ $B=0$ $1+C=2$, $C=1$ $0+D=0$, $D=0$

**10.** $\dfrac{3x^3+x^2-x-5}{(x^3+2x+5)^2}=\dfrac{Ax+B}{x^2+2x+5}+\dfrac{Cx+D}{(x^2+2x+5)^2}$

$3x^3+x^2-x-5=(Ax+B)(x^2+2x+5)+Cx+D$

$3x^3+x^2-x-5=Ax^3+2Ax^2+5Ax+Bx^2+2Bx+5B+Cx+D$

$3x^3+x^2-x-5=Ax^3+(2A+B)x^2+(5A+2B+C)x+(5B+D)$

$\begin{cases}3=A\\1=2A+B\\-1=5A+2B+C\\-5=5B+D\end{cases}$ $A=3$ $2(3)+B=1$, $B=-5$ $5(3)+2(-5)+C=-1$, $C=-6$ $5(-5)+D=-5$, $D=20$

**11.** $\dfrac{8x+12}{x(x+4)}=\dfrac{A}{x}+\dfrac{B}{x+4}$

$8x+12=A(x+4)+Bx$

$8x+12=Ax+4A+Bx$

$8x+12=(A+B)x+4A$

$\begin{cases}8=A+B\\12=4A\end{cases}$ $A=3$ $3+B=8$, $B=5$

$\dfrac{8x+12}{x(x+4)}=\dfrac{3}{x}+\dfrac{5}{x+4}$

**12.** $\dfrac{x-14}{x(x-7)}=\dfrac{A}{x}+\dfrac{B}{x-7}$

$x-14=A(x-7)+Bx$

$x-14=Ax-7A+Bx$

$x-14=(A+B)x-7A$

$\begin{cases}1=A+B\\-14=-7A\end{cases}$ $A=2$ $2+B=1$, $B=-1$

$\dfrac{x-14}{x(x-7)}=\dfrac{2}{x}+\dfrac{-1}{x-7}$

**13.** $\dfrac{3x+50}{x^2-7x-18}=\dfrac{3x+50}{(x-9)(x+2)}=\dfrac{A}{x-9}+\dfrac{B}{x+2}$

$3x+50=A(x+2)+B(x-9)$

$3x+50=Ax+2A+Bx-9B$

$3x+50=(A+B)x+(2A-9B)$

$\begin{cases}3=A+B\\50=2A-9B\end{cases}$

$-2A-2B=-6$

$\underline{2A-9B=50}$

$-11B=44$

$B=-4$

$3=A+(-4)$

$7=A$

$\dfrac{3x+50}{x^2-7x-18}=\dfrac{7}{x-9}+\dfrac{-4}{x+2}$

**14.** $\dfrac{7x+44}{x^2+10x+24}=\dfrac{7x+44}{(x+4)(x+6)}=\dfrac{A}{x+4}+\dfrac{B}{x+6}$

$7x+44=A(x+6)+B(x+4)$

$7x+44=Ax+6A+Bx+4B$

$7x+44=(A+B)x+(6A+4B)$

$\begin{cases}7=A+B\\44=6A+4B\end{cases}$

$-4A-4B=-28$

$\underline{6A+4B=44}$

$2A=16$

$A=8$

$8+B=7$

$B=-1$

$\dfrac{7x+44}{x^2+10x+24}=\dfrac{8}{x+4}+\dfrac{-1}{x+6}$

**15.** $$\frac{16x+34}{4x^2+16x+15}=\frac{16x+34}{(2x+3)(2x+5)}=\frac{A}{2x+3}+\frac{B}{2x+5}$$

$$16x+34=A(2x+5)+B(2x+3)$$
$$16x+34=2Ax+5A+2Bx+3B$$
$$16x+34=(2A+2B)x+(5A+3B)$$
$$\begin{cases}16=2A+2B & (1)\\ 34=5A+3B & (2)\end{cases}$$
$$6A+6B=48 \quad \text{3 times (1)}$$
$$\underline{-10A-6B=-68} \quad -2 \text{ times (2)}$$
$$-4A=-20$$
$$A=5$$

$$2(5)+2B=16$$
$$B=3$$
$$\frac{16x+34}{4x^2+16x+15}=\frac{5}{2x+3}+\frac{3}{2x+5}$$

**16.** $$\frac{-15x+37}{9x^2-12x-5}=\frac{-15x+37}{(3x+1)(3x-5)}=\frac{A}{3x+1}+\frac{B}{3x-5}$$

$$-15x+37=A(3x-5)+B(3x+1)$$
$$-15x+37=3Ax-5A+3Bx+b$$
$$-15x+37=(3A+3B)x+(-5A+B)$$
$$\begin{cases}-15=3A+3B & (1)\\ 37=-5A+B & (2)\end{cases}$$

$$-A-B=5 \quad -\frac{1}{3}\text{ times (1)}$$
$$\underline{-5A+B=37} \quad (2)$$
$$-6A=42$$
$$A=-7$$

$$\frac{-15x+37}{9x^2-12x-5}=\frac{-7}{3x+1}+\frac{2}{3x-5}$$

**17.** $$\frac{x-5}{(3x+5)(x-2)}=\frac{A}{3x+5}+\frac{B}{x-2}$$
$$x-5=A(x-2)+B(3x+5)$$
$$x-5=Ax-2A+3Bx+5B$$
$$x-5=(A+3B)x+(-2A+5B)$$
$$\begin{cases}1=A+3B & (1)\\ -5=-2A+5B & (2)\end{cases}$$

$$2A+6B=2 \quad \text{2 times (1)}$$
$$\underline{-2A+5B=-5} \quad (2)$$
$$11B=-3 \qquad A+3\left(-\frac{3}{11}\right)=1$$
$$B=-\frac{3}{11} \qquad A=\frac{20}{11}$$

$$\frac{x-5}{(3x+5)(x-2)}=\frac{20}{11(3x+5)}+\frac{-3}{11(x-2)}$$

**18.** $$\frac{1}{(x+7)(2x-5)}=\frac{A}{x+7}+\frac{B}{2x-5}$$
$$1=A(2x-5)+B(x+7)$$
$$1=2Ax-5A+Bx+7B$$
$$1=(2A+B)x+(-5A+7B)$$
$$\begin{cases}0=2A+B & (1)\\ 1=-5A+7B & (2)\end{cases}$$
$$-14A-7B=0 \quad -7\text{ times (1)}$$
$$\underline{-5A+7B=1} \quad (2)$$
$$-19A=1 \qquad 2\left(-\frac{1}{19}\right)+B=0$$
$$A=-\frac{1}{19} \qquad B=\frac{2}{19}$$

$$\frac{1}{(x+7)(2x-5)}=\frac{-1}{19(x+7)}+\frac{2}{19(2x-5)}$$

**19.**

$$\begin{array}{r}
x+3 \\
x^2-4\,\overline{)\,x^3+3x^2-4x-8} \\
\underline{x^3 \qquad -4x} \\
3x^2 \qquad -8 \\
\underline{3x^2 \qquad -12} \\
4
\end{array}$$

$$\frac{x^3+3x^2-4x-8}{x^2-4}=x+3+\frac{4}{(x-2)(x+2)}$$

$$\frac{4}{(x-2)(x+2)}=\frac{A}{x-2}+\frac{B}{x+2}$$

$4=A(x+20+B(x-2)$

$4=Ax+2A+Bx-2B$

$4=(A+B)x+(2A-2B)$

$\begin{cases}0=A+B & (1)\\ 4=2A-2B & (2)\end{cases}$

$2A+2B=0$ 2 times (1)

$\underline{2A-2B=4}$ (2)

$4A \quad =4$ $\qquad 1+B=0$

$A=1$ $\qquad B=-1$

$$\frac{x^3+3x^2-4x-8}{x^2-4}=x+3+\frac{1}{x-2}+\frac{-1}{x+2}$$

**20.**

$$\begin{array}{r}
x+1 \\
x^2-x-12\,\overline{)\,x^3 \qquad -13x-9} \\
\underline{x^3-x^2-12x} \\
x^2- \quad x-9 \\
\underline{x^2- \quad x-12} \\
3
\end{array}$$

$$\frac{x^3-13x-9}{x^2-x-12}=x+1+\frac{3}{(x-4)(x+3)}$$

$$\frac{3}{(x-4)(x+3)}=\frac{A}{x-4}+\frac{B}{x+3}$$

$3=A(x+3)+B(x-4)$

$3=Ax+3A+Bx-4B$

$3=(A+B)x+(3A-4B)$

$\begin{cases}0=A+B & (1)\\ 3=3A-4B & (2)\end{cases}$

$4A+4B=0$ 4 times (1)

$\underline{3A-4B=3}$ (2)

$7A \quad =3$ $\qquad \frac{3}{7}+B=0$

$A=\frac{3}{7}$ $\qquad B=-\frac{3}{7}$

$$\frac{x^3-13x-9}{x^2-x-12}=x+1+\frac{3}{7(x-4)}+\frac{-3}{7(x+3)}$$

**21.** $\dfrac{3x^2+49}{x(x+7)^2}=\dfrac{A}{x}+\dfrac{B}{x+7}+\dfrac{C}{(x+7)^2}$

$3x^2+49=A(x+7)^2+Bx(x+7)+Cx$

$3x^2+49=Ax^2+14Ax+49A+Bx^2+7Bx+Cx$

$3x^2+49=(A+B)x^2+(14A+7B+C)x+49A$

$\begin{cases}3=A+B\\ 0=14A+7B+C\\ 49=49A\end{cases}$ $\qquad A=1 \qquad 1+B=3 \qquad 14(1)+7(2)+C=0$

$\qquad B=2 \qquad C=-28$

$$\frac{3x^2+49}{x(x+7)^2}=\frac{1}{x}+\frac{2}{x+7}+\frac{-28}{(x+7)^2}$$

**22.** $\dfrac{x-18}{x(x-3)^2}=\dfrac{A}{x}+\dfrac{B}{x-3}+\dfrac{C}{(x-3)^2}$

$x-18=A(x-3)^2+Bx(x-3)+Cx$

$x-18=Ax^2-6Ax+9A+Bx^2-3Bx+Cx$

$x-18=(A+B)x^2+(-6A-3B+C)x+9A$

$\begin{cases}0=A+B\\ 1=-6A-3B+C\\ -18=9A\end{cases}$ $\qquad 9A=-18 \qquad -2+B=0 \qquad -6(-2)-3(2)+C=1$

$\qquad A=-2 \qquad B=2 \qquad C=-5$

$$\frac{x-18}{x(x-3)^2}=\frac{-2}{x}+\frac{2}{x-3}+\frac{-5}{(x-3)^2}$$

**23.** $\dfrac{5x^2-7x+2}{x^3-3x^2+x}=\dfrac{5x^2-7x+2}{x(x^2-3x+1)}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2-3x+1}$

$5x^2-7x+2=A(x^2-3x+1)+(Bx+C)x$
$5x^2-7x+2=Ax^2-3Ax+A+Bx^2+Cx$
$5x^2-7x+2=(A+B)x^2+(-3A+C)x+A$

$\begin{cases} 5=A+B \\ -7=-3A+C \\ 2=A \end{cases}$ $\quad A=2 \quad 2+B=5,\ B=3 \quad -3(2)+C=-7,\ C=-1$

$\dfrac{5x^2-7x+2}{x^3-3x^2+x}=\dfrac{2}{x}+\dfrac{3x-1}{x^2-3x+1}$

**24.** $\dfrac{9x^2-3x+49}{x^3-x^2+10x-10}=\dfrac{9x^2-3x+49}{(x-1)(x^2+10)}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+10}$

$9x^2-3x+49=A(x^2+10)+(Bx+C)(x-1)$
$9x^2-3x+49=Ax^2+10A+Bx^2-Bx+Cx-C$
$9x^2-3x+49=(A+B)x^2+(-B+C)x+(10A-C)$

$\begin{cases} 9=A+B \\ -3=-B+C \\ 49=10A-C \end{cases}$

$-3=-B+C$
$\underline{49=10A-C}$
$46=10A-B$

$10A-B=46$
$\underline{A+B=9}$
$11A\quad=55$
$A=5$

$5+B=9,\ B=4$

$-4+C=-3,\ C=1$

$\dfrac{9x^2-3x+49}{x^3-x^2+10x-10}=\dfrac{5}{x-1}+\dfrac{4x+1}{x^2+10}$

**25.** $\dfrac{2x^3+9x^2+26x+41}{(x+3)^2(x^2+1)}=\dfrac{A}{x+3}+\dfrac{B}{(x+3)^2}+\dfrac{Cx+D}{x^2+1}$

$2x^3+9x^2+26x+41=A(x+3)(x^2+1)+B(x^2+1)+(Cx+D)(x+3)^2$
$2x^3+9x^2+26x+41=Ax^3+Ax+3Ax^2+3A+Bx^2+B+Cx^3+6Cx^2+9Cx+Dx^2+6Dx+9D$
$2x^3+9x^2+26x+41=(A+C)x^3+(3A+B+6C+D)x^2+(A+9C+6D)x+(3A+B+9D)$

$\begin{cases} 2=A+C & (1) \\ 9=3A+B+6C+D & (2) \\ 26=A+9C+6D & (3) \\ 41=3A+B+9D & (4) \end{cases}$

$3A+B+6C+D=9 \quad (3)$
$\underline{-3A-B\quad-9D=-41} \quad -1 \text{ times } (4)$
$6C-8D=-32 \quad (5)$

$A+9C+6D=26 \quad (3)$
$\underline{-A-C\qquad=-2} \quad -1 \text{ times } (1)$
$8C+6D=24 \quad (6)$

$\begin{cases} 2=A+C \\ 9=3A+B+6C+D \\ 24=8C+6D & (6) \\ -32=6C-8D & (5) \end{cases}$

$64C+48D=192 \quad 8 \text{ times } (6)$
$\underline{36C-48D=-192} \quad 6 \text{ times } (5)$
$100C\qquad=0$
$C=0$

$A+0=2,\ A=2 \qquad 8(0)+6D=24,\ D=4 \qquad 3(2)+B+9(4)=41,\ B=-1$

$\dfrac{2x^3+9x^2+26x+41}{(x+3)^2(x^2+1)}=\dfrac{2}{x+3}+\dfrac{-1}{(x+3)^2}+\dfrac{4}{x^2+1}$

**26.** $$\frac{12x^3-37x^2+48x-36}{(x-2)^2(x^2+4)}=\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{Cx+D}{x^2+4}$$

$$12x^3-37x^2+48x-36=A(x-2)(x^2+4)+B(x^2+4)+(Cx+D)(x-2)^2$$

$$12x^3-37x^2+48x-36=Ax^3+4Ax-2Ax^2-8A+Bx^2+4B+Cx^3-4Cx^2+4Cx+Dx^2-4Dx+4D$$

$$12x^3-37x^2+48x-36=(A+C)x^3+(-2A+B-4C+D)x^2+(4A+4C-4D)x+(-8A+4B+4D)$$

$$\begin{cases} 12=A+C & (1)\\ -37=-2A+B-4C+D & (2)\\ 48=4A+4C-4D & (3)\\ -36=-8A+4B+4D & (4)\end{cases}$$

$$\begin{array}{rl} 4A+4C=48 & \text{4 times (1)}\\ \underline{-4A-4C+4D=-48} & -1\text{ times (3)}\\ 4D=0 & \\ D=0 & (5)\end{array}$$

$$\begin{cases} 12=A+C & \\ -37=-2A+B-4C+D & \\ 0=D & (5)\\ -36=-8A+4B+4D & \end{cases}$$

From (1) $C=12-A$

From (4) $4B=8A-36-4D$
$B=2A-9$

Substitute in Eq. (2). $$\begin{aligned} -2A+(2A-9)-4(12-A)+0&=-37\\ -2A+2A-9-48+4A&=-37\\ 4A&=20\\ A&=5\end{aligned}$$

$C=12-5$, $C=7$ $\qquad$ $B=10-9$, $B=1$

$$\frac{12x^3-37x^2+48x-36}{(x-2)^2(x^2+4)}=\frac{5}{x-2}+\frac{1}{(x-2)^2}+\frac{7x}{x^2+4}$$

**27.** $$\frac{3x-7}{(x-4)^2}=\frac{A}{x-4}+\frac{B}{(x-4)^2}$$

$$3x-7=A(x-4)+B$$

$$3x-7=Ax-4A+B$$

$$3x-7=Ax+(-4A+B)$$

$$\begin{cases} 3=A\\ -7=-4A+B\end{cases} \qquad \begin{aligned} B-4(3)&=-7\\ B&=5\end{aligned}$$

$$\frac{3x-7}{(x-4)^2}=\frac{3}{x-4}+\frac{5}{(x-4)^2}$$

**28.** $$\frac{5x-53}{(x-11)^2}=\frac{A}{x-11}+\frac{B}{(x-11)^2}$$

$$5x-53=A(x-11)+B$$

$$5x-53=Ax-11A+B$$

$$5x-53=Ax+(-11A+B)$$

$$\begin{cases} 5=A\\ -53=-11A+B\end{cases} \qquad \begin{aligned} -11(5)+B&=-53\\ B&=2\end{aligned}$$

$$\frac{5x-53}{(x-11)^2}=\frac{5}{x-11}+\frac{2}{(x-11)^2}$$

**29.** $\dfrac{3x^3-x^2+34x-10}{(x^2+10)^2}=\dfrac{Ax+B}{x^2+10}+\dfrac{Cx+D}{(x^2+10)^2}$

$3x^3-x^2+34x-10=(Ax+B)(x^2+10)+Cx+D$

$3x^3-x^2+34x-10=Ax^3+Bx^2+10Ax+10B+Cx+D$

$3x^3-x^2+34x-10=Ax^3+Bx^2+(10A+C)x+(10B+D)$

$\begin{cases}3=A\\-1=B\\34=1-A+C\\-10=10B+D\end{cases}$ $\quad 10(3)+C=34 \quad C=4 \qquad 10(-1)+D=-10 \quad D=0$

$\dfrac{3x^3-x^2+34x-10}{(x^2+10)^2}=\dfrac{3x-1}{x^2+10}+\dfrac{4x}{(x^2+10)^2}$

**30.** $\dfrac{2x^3+9x+1}{x^4+14x^2+49}=\dfrac{2x^3+9x+1}{(x^2+7)^2}=\dfrac{Ax+B}{x^2+7}+\dfrac{Cx+D}{(x^2+7)^2}$

$2x^3+9x+1=(Ax+B)(x^2+7)+Cx+D$

$2x^3+9x+1=Ax^3+Bx^2+7Ax+7B+Cx+D$

$2x^3+9x+1=Ax^3+Bx^2+(7A+C)x+(7B+D)$

$\begin{cases}2=A\\0=B\\9=7A+C\\1=7B+D\end{cases}$ $\quad 7(2)+C=9 \quad C=-5 \qquad 7(0)+D=1 \quad D=1$

$\dfrac{2x^3+9x+1}{x^4+14x^2+49}=\dfrac{2x}{x^2+7}+\dfrac{-5x+1}{(x^2+7)^2}$

**31.** $\dfrac{1}{k^2-x^2}=\dfrac{1}{(k-x)(k+x)}=\dfrac{A}{k-x}+\dfrac{B}{k+x}$

$1=A(k+x)+B(k-x)$

$1=Ak+Ax+Bk-Bx$

$1=(A-B)x+(Ak+Bk)$

$\begin{cases}0=A-B & (1)\\1=Ak+Bk & (2)\end{cases}$

$Ak+Bk=1 \quad (2)$

$\underline{Ak-Bk=0} \quad k \text{ times } (1)$

$2Ak=1$

$A=\frac{1}{2k}$

$\dfrac{1}{2k}-B=0$

$B=\frac{1}{2k}$

$\dfrac{1}{k^2-x^2}=\dfrac{1}{2k(k-x)}+\dfrac{1}{2k(k+x)}$

**32.** $\dfrac{1}{x(k+mx)}=\dfrac{A}{x}+\dfrac{B}{k+mx}$

$1=A(k+mx)+Bx$

$1=Ak+Amx+Bx$

$1=(Am+B)x+Ak$

$\begin{cases}0=Am+B\\1=Ak\end{cases}$

$A=\dfrac{1}{k} \qquad \dfrac{1}{k}m+B=0$

$B=-\dfrac{m}{k}$

$\dfrac{1}{x(k+mx)}=\dfrac{1}{kx}+\dfrac{-m}{k(k+mx)}$

**33.** Long division: $x^3-x^2-x-1$ divided by $x^2-x$ gives quotient $x$; subtract $x^3-x^2$; remainder $-x-1$.

$\dfrac{x^3-x^2-x-1}{x^2-x}=x+\dfrac{-x-1}{x^2-x}$

$\dfrac{-x-1}{x(x-1)}=\dfrac{A}{x}+\dfrac{B}{x-1}$

$-x-1=Ax-A+Bx$

$-x-1=(A+B)x-A$

$\begin{cases}-1=A+B\\-1=-A\end{cases}$ $\quad A=1 \qquad 1+B=-1 \quad B=-2$

$\dfrac{x^3-x^2-x-1}{x^2-x}=x+\dfrac{1}{x}+\dfrac{-2}{x-1}$

**34.** Long division: $2x^3+5x^2+3x-8$ divided by $2x^2+3x-2$ gives quotient $x+1$; subtract $2x^3+3x^2-2x$ to get $2x^2+5x-8$; subtract $2x^2+3x-2$; remainder $2x-6$.

$\dfrac{2x^3+5x^2+3x-8}{2x^2+3x-2}=x+1+\dfrac{2x-6}{2x^2+3x-2}$

$\dfrac{2x-6}{(2x-1)(x+2)}=\dfrac{A}{2x-1}+\dfrac{B}{x+2}$

$2x-6=A(x+2)+B(2x-1)$

$2x-6=Ax+2A+2Bx-B$

$2x-6=(A+2B)x+(2A-B)$

$\begin{cases}2=A+2B\\-6=2A-B\end{cases}$

$2=A+2B$

$\underline{-12=4A-2B}$

$-10=5A$

$-2=A$

$-2+2B=2$

$2B=4$

$B=2$

$\dfrac{2x^3+5x^2+3x-8}{2x^2+3x-2}=x+1+\dfrac{-2}{2x-1}+\dfrac{2}{x+2}$

**35.**

$$\begin{array}{r} 2x-2 \\ x^2-x-1\overline{)2x^3-4x^2 \qquad +5} \\ \underline{2x^3-2x^2-2x} \\ -2x^2+2x+5 \\ \underline{-2x^2+2x+2} \\ 3 \end{array}$$

$$\frac{2x^3-4x^2+5}{x^2-x-1}=2x-2+\frac{3}{x^2-x-1}$$

**36.**

$$\begin{array}{r} x+1 \\ x^3-3x^2\overline{)x^4-2x^3-2x^2-x+3} \\ \underline{x^4-3x^3} \\ x^3-2x^2-x+3 \\ \underline{x^3-3x^2} \\ x^2-x+3 \end{array}$$

$$\frac{x^4-2x^3-2x^2-x+3}{x^2(x-3)}=x+1+\frac{x^2-x+3}{x^2(x-3)}$$

$$x^2-x+3=Ax(x-3)+B(x-3)+Cx^2$$
$$x^2-x+3=Ax^2-3Ax+Bx-3B+Cx^2$$
$$x^2-x+3=(A+C)x^2+(-3A+B)x-3B$$

$$\begin{cases} 1=A+C \\ -1=-3A+B \\ 3=-3B \end{cases} \quad \begin{array}{c} -3B=3 \\ B=-1 \end{array} \quad \begin{array}{c} -3A-1=-1 \\ A=0 \end{array} \quad \begin{array}{c} 0+C=1 \\ C=1 \end{array}$$

$$\frac{x^4-2x^3-2x^2-x+3}{x^2(x-3)}=x+1+\frac{-1}{x^2}+\frac{1}{x-3}$$

## Connecting Concepts

**37.**

$$\frac{x^2-1}{(x-1)(x+2)(x-3)}=\frac{(x-1)(x+1)}{(x-1)(x+2)(x-3)}$$
$$=\frac{x+1}{(x+2)(x-3)}=\frac{A}{x+2}+\frac{B}{x-3}$$

$$x+1=A(x-3)+B(x+2)$$
$$x+1=Ax-3A+Bx+2B$$
$$x+1=(A+B)x+(-3A+2B)$$

$$\begin{cases} A+B=1 \\ -3A+2B=1 \end{cases}$$

$$3A+3B=3$$
$$5B=4$$
$$B=\frac{4}{5} \qquad A=\frac{1}{5}$$

$$\frac{x^2-1}{(x-1)(x+2)(x-3)}=\frac{1}{5(x+2)}+\frac{4}{5(x-3)}$$

**38.**

$$\frac{x^2+x}{x^2(x-4)}=\frac{x(x+1)}{x^2(x-4)}=\frac{x+1}{x(x-4)}$$

Cancel the common term.

$$\frac{x+1}{x(x-4)}=\frac{A}{x}+\frac{B}{x-4}$$

$$x+1=A(x-4)+Bx$$
$$x+1=Ax-4A+Bx$$
$$x+1=(A+B)x-4A$$

$$\begin{cases} 1=A+B \\ 1=-4A \end{cases}$$

$$\begin{array}{c} -4A=1 \\ A=-\frac{1}{4} \end{array} \qquad \begin{array}{c} -\frac{1}{4}+B=1 \\ B=\frac{5}{4} \end{array}$$

$$\frac{x^2+x}{x^2(x-4)}=\frac{-1}{4x}+\frac{5}{4(x-4)}$$

**39.**

$$\frac{-x^4-4x^2+3x-6}{x^4(x-2)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x^4}+\frac{E}{x-2}$$

$$-x^4-4x^2+3x-6=Ax^3(x-2)+Bx^2(x-2)+Cx(x-2)+D(x-2)+Ex^4$$
$$-x^4-4x^2+3x-6=Ax^4-2Ax^3+Bx^3-2Bx^2+Cx^2-2Cx+Dx-2D+Ex^4$$
$$-x^4-4x^2+3x-6=(A+E)x^4+(-2A+B)x^3+(-2B+C)x^2+(-2C+D)x+(-2D)$$

$$\begin{cases} -1=A+E \\ 0=-2A+B \\ -4=-2B+C \\ 3=-2C+D \\ -6=-2D \end{cases} \quad \begin{array}{c} -2D=-6 \\ D=3 \end{array} \quad \begin{array}{c} -2C+3=3 \\ C=0 \end{array} \quad \begin{array}{c} -2B+0=-4 \\ B=2 \end{array} \quad \begin{array}{c} -2A+2=0 \\ A=-2 \end{array} \quad \begin{array}{c} 1+E=-1 \\ E=-2 \end{array}$$

$$\frac{-x^4-4x^2+3x-6}{x^4(x-2)}=\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x^4}+\frac{-2}{x-2}$$

**40.** $\dfrac{3x^2-2x-1}{(x^2-1)^2}=\dfrac{3x^2-2x-1}{[(x-1)(x+1)]^2}=\dfrac{3x^2-2x-1}{(x-1)^2(x+1)^2}=\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{x+1}+\dfrac{D}{(x+1)^2}$

$$3x^2-2x-1=A(x-1)(x+1)^2+B(x+1)^2+C(x+1)(x-1)^2+D(x-1)^2$$
$$3x^2-2x-1=A(x-1)(x^2+2x+1)+B(x^2+2x+1)+C(x+1)(x^2-2x+1)+D(x^2-2x+1)$$
$$3x^2-2x-1=A(x^3+x^2-x-1)+B(x^2+2x+1)+C(x^3-x^2-x+1)+D(x^2-2x+1)$$
$$3x^2-2x-1=Ax^3+Ax^2-Ax-A+Bx^2+2Bx+B+Cx^3-Cx^2-Cx+C+Dx^2-2Dx+D$$
$$3x^2-2x-1=(A+C)x^3+(A+B-C+D)x^2+(-A+2B-C-2D)x+(-A+B+C+D)$$

$$\begin{cases}0=A+C & (1)\\ 3=A+B-C+D & (2)\\ -2=-A+2B-C-2D & (3)\\ -1=-A+B+C+D & (4)\end{cases}$$

$$\begin{array}{lll} A+B-C+D=3 & (2) \\ \underline{A-B-C-D=1} & -1 \text{ times } (4) \\ 2A \quad -2C \quad =4 & \\ A \quad -C \quad =2 & (5)\end{array} \qquad \begin{array}{ll} A+C=0 & (1)\\ \underline{A-C=2} & (5)\\ 2A \quad =2 & \\ A \quad =1 & \end{array} \qquad \begin{array}{l} 1+C=0 \\ C=-1\end{array}$$

$$\begin{array}{ll} -2A+2B+2C+2D=-2 & 2 \text{ times } (4)\\ \underline{A-2B+C+2D=2} & -1 \text{ times } (3)\\ -A \quad +3C+4D=0 & \\ -1 \quad 3(-1)+4D=0 & \\ -4+4D=0 & \\ D=1 & \end{array} \qquad \begin{array}{ll} A+B-C+D=3 & (2)\\ 1+B+1+1=3 & \\ B=0 & \end{array}$$

$$\frac{3x^2-2x-1}{(x^2-1)^2}=\frac{1}{x-1}-\frac{1}{x+1}+\frac{1}{(x+1)^2}$$

**41.** $\dfrac{2x^2+3x-1}{(x^3-1)}=\dfrac{2x^2+3x-1}{(x-1)(x^2+x+1)}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+x+1}$

$$2x^2+3x-1=A(x^2+x+1)+(Bx+C)(x-1)$$
$$2x^2+3x-1=Ax^2+Ax+A+Bx^2-Bx+Cx-C$$
$$2x^2+3x-1=(A+B)x^2+(A-B+C)x+(A-C)$$

$$\begin{cases}2=A+B & (1)\\ 3=A-B+C & (2)\\ -1=A-C & (3)\end{cases}$$

Solve Eq. (1) for $B$ and Eq. (3) for $C$ and substitute into Eq. (2).

$$\begin{array}{l} A+B=2\\ B=2-A\end{array} \qquad \begin{array}{l} A-C=-1\\ C=A+1\end{array}$$

$$\begin{array}{r} A-B+C=3\\ A-(2-A)+(A+1)=3\\ A-2+A+A+1=3\\ 3A=4\\ A=\frac{4}{3}\end{array} \qquad \begin{array}{l} B=2-A\\ B=2-\frac{4}{3}\\ B=\frac{2}{3}\end{array} \qquad \begin{array}{l} C=A+1\\ C=\frac{4}{3}+1\\ C=\frac{7}{3}\end{array}$$

$$\frac{2x^2+3x-1}{x^3-1}=\frac{4}{3(x-1)}+\frac{2x+7}{3(x^2+x+1)}$$

**42.** $\frac{x^3-2x^2+x-2}{x^4-x^3+x-1}=\frac{x^3-2x^2+x-2}{(x-1)(x+1)(x^2-x+1)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2-x+1}$

$$x^3-2x^2+x-2=A(x+1)(x^2-x+1)+B(x-1)(x^2-x+1)+(Cx+D)(x-1)(x+1)$$
$$x^3-2x^2+x-2=Ax^3+A+Bx^3-2Bx^2+2Bx-B+Cx^3-Cx+Dx^2-D$$
$$x^3-2x^2+x-2=(A+B+C)x^3+(-2B+D)x^2+(2B-C)x+(A-B-D)$$

$$\begin{cases} 1=A+B+C & (1)\\ -2=-2B+D & (2)\\ 1=2B-C & (3)\\ -2=A-B-D & (4)\end{cases}$$

$$\begin{array}{rl} -2B+D=-2 & (2)\\ \underline{A-B-D=-2} & (4)\\ A-3B=-4 & (5)\end{array}$$

$$\begin{cases} 1=A+B+C & (1)\\ -2=-2B+D & (2)\\ 1=2B-C & (3)\\ -4=A-3B & (5)\end{cases}$$

Solve Eq. (3) for $C$ and Eq. (5) for $A$ and substitute for $C$ and $A$ in Eq. (1).

$$\begin{array}{l} 2B-C=1\\ C=2B-1\end{array}\qquad \begin{array}{l} A-3B=-4\\ A=3B-4\end{array}\qquad \begin{array}{r} A+B+C=1\\ (3B-4)+B+(2B-1)=1\\ 3B-4+B+2B-1=1\\ 6B=6\\ B=1\end{array}\qquad \begin{array}{l} C=2(1)-1\\ C=1\end{array}\qquad \begin{array}{l} A=3B-4\\ A=-1\end{array}\qquad \begin{array}{r} -2B+D=-2\\ D=0\end{array}$$

$$\frac{x^3-2x^2+x-2}{x^4-x^3+x-1}=\frac{-1}{x-1}+\frac{1}{x+1}+\frac{x}{x^2-x+1}$$

**43.** $\frac{1}{(b-a)(p(x)+a)}+\frac{1}{(a-b)(p(x)+b)}=\frac{(a-b)(p(x)+b)+(b-a)(p(x)+a)}{(b-a)(a-b)(p(x)+a)(p(x)+b)}$

$$=\frac{(a-b)p(x)+(a-b)b+(b-a)p(x)+(b-a)a}{(b-a)(a-b)(p(x)+a)(p(x)+b)}$$
$$=\frac{(a-b)p(x)+(a-b)b-(a-b)p(x)-(a-b)a}{(b-a)(a-b)(p(x)+a)(p(x)+b)}$$
$$=\frac{(a-b)b-(a-b)a}{(b-a)(a-b)(p(x)+a)(p(x)+b)}$$
$$=\frac{(a-b)(b-a)}{(b-a)(a-b)(p(x)+a)(p(x)+b)}$$
$$=\frac{1}{(p(x)+a)(p(x)+b)}$$

**44.** **a.** Let $p(x)=x^2,\ a=4,\ b=1$ (See Exercise 43.)

$$\frac{1}{(x^2+4)(x^2+1)}=\frac{1}{-3(x^2+4)}+\frac{1}{3(x^2+1)}$$

**b.** Let $p(x)=x^2,\ a=1,\ b=9$

$$\frac{1}{(x^2+1)(x^2+9)}=\frac{1}{8(x^2+1)}+\frac{1}{-8(x^2+9)}$$

**c.** Let $p(x)=x^2+x,\ a=1,\ b=2$

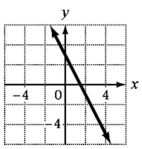
$$\frac{1}{(x^2+x+1)(x^2+x+2)}=\frac{1}{x^2+x+1}-\frac{1}{x^2+x+2}$$

**d.** Let $p(x)=x^2+2x,\ a=4,\ b=9$

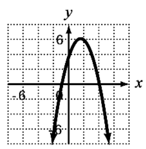
$$\frac{1}{(x^2+2x+4)(x^2+2x+9)}=\frac{1}{5(x^2+2x+4)}-\frac{1}{5(x^2+2x+9)}$$

## Prepare for Section 9.5

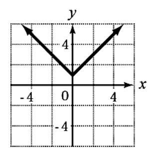
**PS1.**



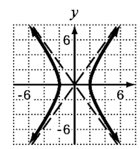
**PS2.**



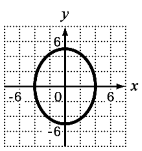
**PS3.**



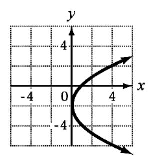
**PS4.**



**PS5.**

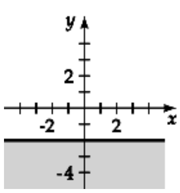


**PS6.**

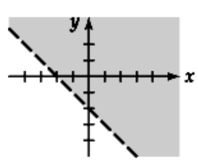


## Section 9.5

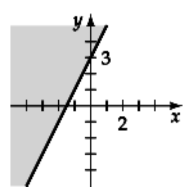
**1.**



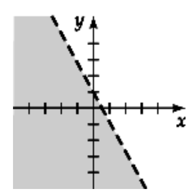
**2.** $y>-x-2$



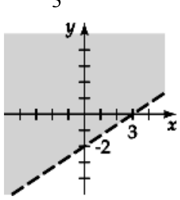
**3.**



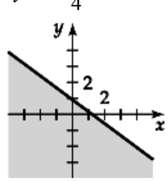
**4.**



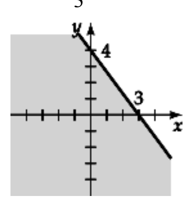
**5.** $y>\frac{2}{3}x-2$



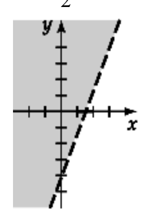
**6.** $y\le-\frac{3}{4}x+1$



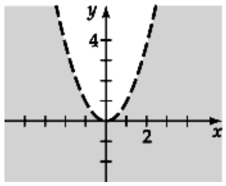
**7.** $y\le-\frac{4}{3}x+4$



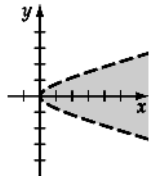
**8.** $y>\frac{5}{2}x-4$



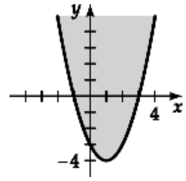
**9.** vertex (0,0)



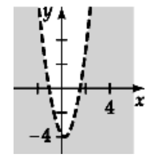

**10.** vertex (0,0)



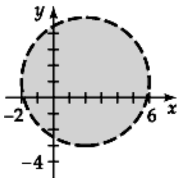

**11.** vertex (1,–4)



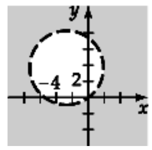

**12.** vertex $\left(\frac{1}{4}, -\frac{25}{8}\right)$



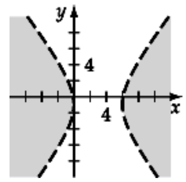

**13.** center (2, 1), $r = 4$



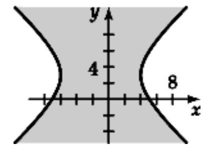

**14.** center (–2, 3), $r = 5$




**15.** center (3, –1), $a = 3$, $b = 4$



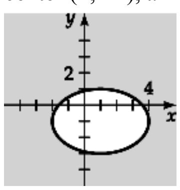

**16.** center (–1, 3), $a = 5$, $b = 4$




**17.**

$$4x^2 + 9y^2 - 8x + 18y \ge 23$$
$$4(x^2 - 2x + 1) + 9(y^2 + 2y + 1) \ge 23 + 4 + 9$$
$$4(x-1)^2 + 9(y+1)^2 \ge 36$$
$$\frac{(x-1)^2}{9} + \frac{(y+1)^2}{4} \ge 1$$

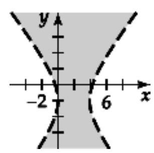
center (1, –1), $a = 3$, $b = 2$



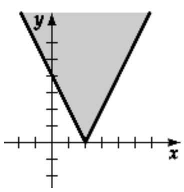

**18.**

$$25x^2 - 16y^2 - 100x - 64y < 64$$
$$25(x^2 - 4x + 4) - 16(y^2 + 4y + 4) < 64 + 100 - 64$$
$$25(x-2)^2 - 16(y+2)^2 < 100$$
$$\frac{(x-2)^2}{4} - \frac{(y+2)^2}{25/4} < 1$$

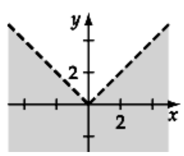
center $(2, -2), a = 2, b = 5/2$



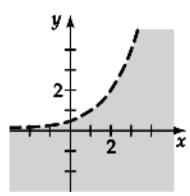

**19.**



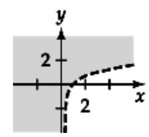

**20.**



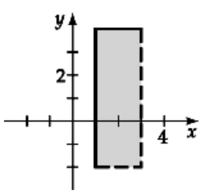

**21.**



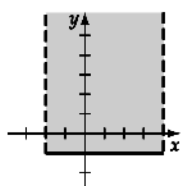

**22.**



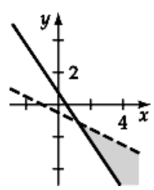

**23.**



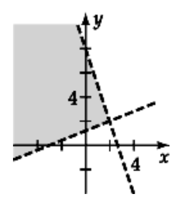

**24.**



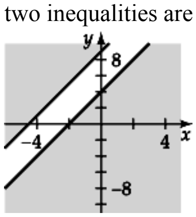

**25.**




**26.**

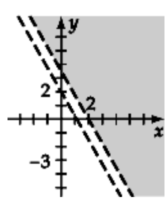



**27.** The graphs of each of the two inequalities are shown.

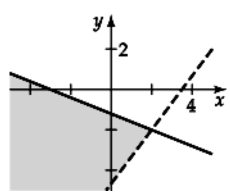



Because the solution sets of the inequalities do not intersect, the system has no solution and cannot be graphed.

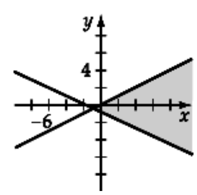
**28.**

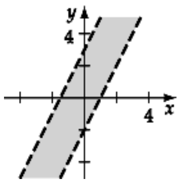



**29.**

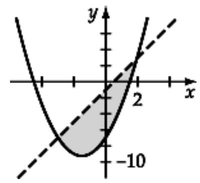


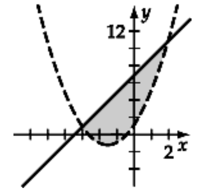

**30.**

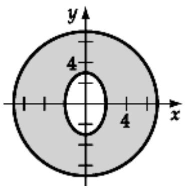


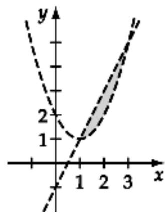
**31.** 

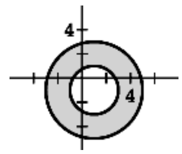

**32.** no solution

**33.** 

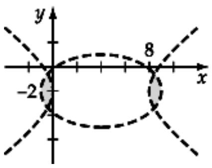

**34.** 

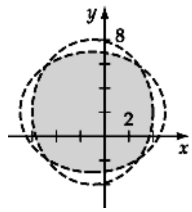

**35.** 

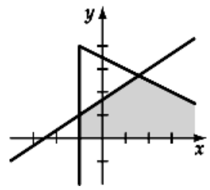

**36.** 

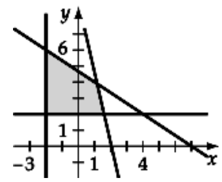

**37.** 

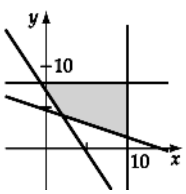

**38.** no solution

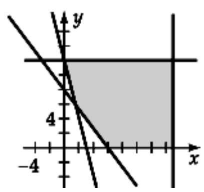
**39.** 


**40.** 


**41.** 


**42.** 


**43.** 


**44.** 


**45.** Substitute Ashley's age, 35, in the first inequality to find the minimum value.

$y \geq 0.55(220 - x)$

$y \geq 0.55(220 - 35)$

$y \geq 0.55(185)$

$y \geq 101.75$

Substitute Ashley's age in the second inequality to find the maximum value.

$y \leq 0.75(220 - x)$

$y \leq 0.75(220 - 35)$

$y \leq 0.75(185)$

$y \leq 138.75$

The minimum is 102 beats per minute and maximum is 139 beats per minute.

**46.** Substitute the sprinter's age, 26, in the first inequality to find the minimum value.

$y \geq 0.80(220 - x)$

$y \geq 0.80(220 - 26)$

$y \geq 0.80(194)$

$y \geq 155.2$

Substitute the sprinter's age in the second inequality to find the maximum value.

$y \leq 0.88(220 - x)$

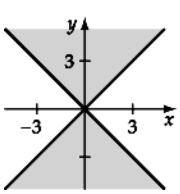
$y \leq 0.88(220 - 26)$

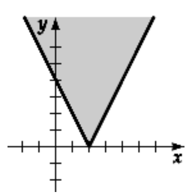
$y \leq 0.88(194)$

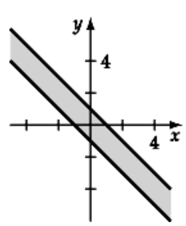
$y \leq 164.9$

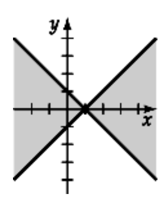
The minimum is 155 beats per minute and maximum is 165 beats per minute.

## Connecting Concepts

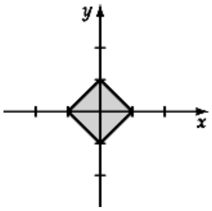
**47.** 


**48.** 

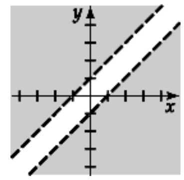

**49.** 


**50.** 

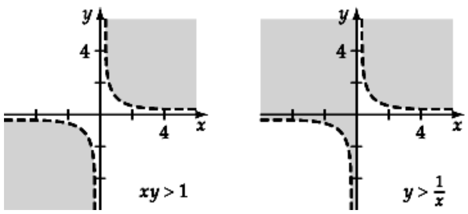
**51.**



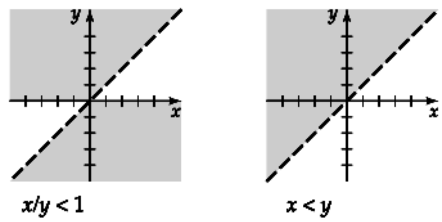

**52.**




**53.** If $x$ is a negative number, then the inequality is reversed when multiplying both sides of the inequality by the negative number $\frac{1}{x}$.

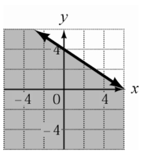



**54.** If $x$ is a negative number, then the inequality is reversed when multiplying both sides of the inequality by a negative number.

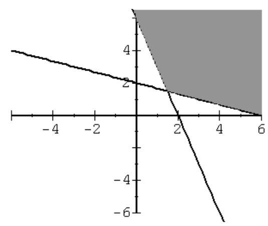



## Prepare for Section 9.6

**PS1.**




**PS2.**




**PS3.**

$$C = 3x + 4y$$
$$C(0,\ 5) = 3(0) + 4(5) = 20$$
$$C(2,\ 3) = 3(2) + 4(3) = 18$$
$$C(6,\ 1) = 3(6) + 4(1) = 22$$
$$C(9,\ 0) = 3(9) + 4(0) = 27$$

**PS4.**

$$C = 6x + 4y + 15$$
$$C(6,\ 0) = 6(0) + 4(20) + 15 = 95$$
$$C(4,\ 18) = 6(4) + 4(18) + 15 = 111$$
$$C(10,\ 10) = 6(10) + 4(10) + 15 = 115$$
$$C(15,\ 0) = 6(15) + 4(0) + 15 = 105$$

**PS5.**

$$\begin{cases} 3x + y = 6 & (1) \\ x + y = 4 & (2) \end{cases}$$

Solve equation (1) for $y$ and substitute into equation (2).

$$x + (-3x + 6) = 4$$
$$-2x + 6 = 4$$
$$-2x = -2$$
$$x = 1$$
$$y = -3(1) + 6$$
$$= 3$$

The solution is (1, 3).

**PS6.**

$$\begin{cases} 300x + 100y = 900 & (1) \\ 400x + 300y = 2200 & (2) \end{cases}$$

$$-900x - 300y = -2700 \quad -3 \text{ times } (1)$$
$$\underline{400x + 300y = 2200} \quad (2)$$
$$-500x \qquad = -500$$
$$x = 1$$

$$300(1) + 100y = 900$$
$$100y = 600$$
$$y = 6$$

The solution is (1, 6).

## Section 9.6

**1.** $C(x, y) = 3x + 4y$

$C(0, 5) = 3(0) + 4(5) = 20$

$C(2, 3) = 3(2) + 4(3) = 18$

$C(6, 1) = 3(6) + 4(1) = 22$

$C(9, 0) = 3(9) + 4(0) = 27$

The minimum of 18 occurs at (2, 3).

**2.** $C(x, y) = 12x + 2y + 48$

$C(0, 24) = 12(0) + 2(24) + 48 = 96$

$C(4, 16) = 12(4) + 2(16) + 48 = 128$

$C(32, 2) = 12(32) + 2(2) + 48 = 436$

$C(40, 0) = 12(40) + 2(0) + 48 = 528$

The minimum of 96 occurs at (0, 24).

**3.** $C(x, y) = 2.5x + 3y + 5$

$C(0, 20) = 2.5(0) + 3(20) + 5 = 65$

$C(5, 19) = 2.5(5) + 3(19) + 5 = 74.5$

$C(20, 4) = 2.5(20) + 3(4) + 5 = 67$

$C(22.5, 0) = 2.5(22.5) + 3(0) + 5 = 61.25$

The maximum of 74.5 occurs at (5, 19).

**4.** $C(x, y) = 6x + 4y + 15$

$C(0, 20) = 6(0) + 4(20) + 15 = 95$

$C(4, 18) = 6(4) + 4(18) + 15 = 111$

$C(10, 10) = 6(10) + 4(10) + 15 = 115$

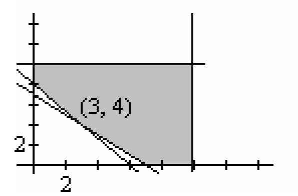
$C(15, 0) = 6(15) + 4(0) + 15 = 105$

The maximum of 115 occurs at (10, 10).

**5.** $C = 4x + 2y$

$$\begin{cases} x + y \ge 7 \\ 4x + 3y \ge 24 \\ x \le 10,\ y \le 10 \\ x \ge 0,\quad y \ge 0 \end{cases}$$

| $C = 4x + 2y$ | | |
|---|---|---|
| (0, 10) | 20 | |
| (0, 8) | 16 | minimum |
| (3, 4) | 20 | |
| (7, 0) | 28 | |
| (10, 0) | 40 | |
| (10, 10) | 60 | |

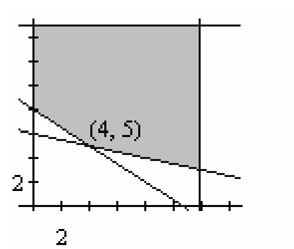



The minimum is 16 at (0, 8).

**6.** $C = 5x + 4y$

$$\begin{cases} 3x + 4y \ge 32 \\ x + 4y \ge 24 \\ x \le 12,\ y \le 15 \\ x \ge 0,\quad y \ge 0 \end{cases}$$

| $C = 5x+4y$ | | |
|---|---|---|
| (0, 15) | 60 | |
| (0, 8) | 32 | minimum |
| (4, 5) | 40 | |
| (12, 3) | 72 | |
| (12, 15) | 120 | |

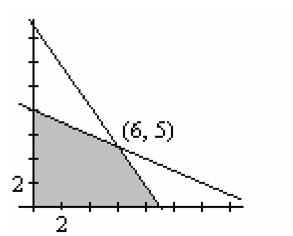



The minimum is 32 at (0, 8).

**7.** $C = 6x + 7y$

$$\begin{cases} x + 2y \le 16 \\ 5x + 3y \le 45 \\ x \ge 0, y \ge 0 \end{cases}$$

| $C = 6x + 7y$ | | |
|---|---|---|
| (0, 8) | 56 | |
| (0, 0) | 0 | |
| (9, 0) | 54 | |
| (6, 5) | 71 | maximum |

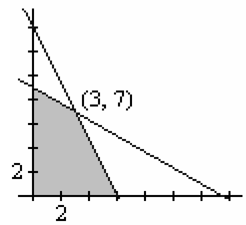



The maximum is 71 at (6, 5).

**8.** $C = 6x + 5y$

$$\begin{cases} 2x + 3y \le 27 \\ 7x + 3y \le 42 \\ x \ge 0, y \ge 0 \end{cases}$$

| $C = 6x + 5y$ | | |
|---|---|---|
| (0, 9) | 45 | |
| (0, 0) | 0 | |
| (6, 0) | 36 | |
| (3, 7) | 53 | maximum |

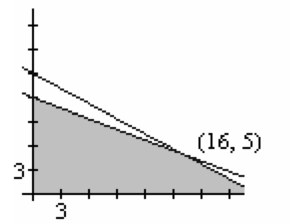



The maximum is 53 at (3, 7).

**9.** $C = x + 6y$

$$\begin{cases} 5x + 8y \le 120 \\ 7x + 16y \le 192 \\ x \ge 0, \ y \ge 0 \end{cases}$$

| $C = x + 6y$ | | |
|---|---|---|
| (0, 12) | 72 | maximum |
| (0, 0) | 0 | |
| (24, 0) | 24 | |
| (16, 5) | 46 | |

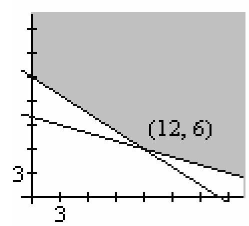



The maximum is 72 at (0, 12).

**10.** $C = 4x + 5y$

$$\begin{cases} x + 3y \ge 30 \\ 3x + 4y \ge 60 \\ x \ge 0, \ y \ge 0 \end{cases}$$

| $C = 4x + 5y$ | | |
|---|---|---|
| (0, 15) | 75 | minimum |
| (12, 6) | 78 | |
| (30, 0) | 120 | |

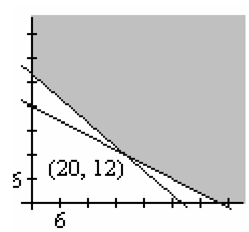



The minimum is 75 at (0, 15).

**11.** $C = 4x + y$

$$\begin{cases} 3x + 5y \ge 120 \\ x + y \ge 32 \\ x \ge 0, \ y \ge 0 \end{cases}$$

| $C = 4x + y$ | | |
|---|---|---|
| (40, 0) | 160 | |
| (0, 32) | 32 | minimum |
| (20, 12) | 92 | |

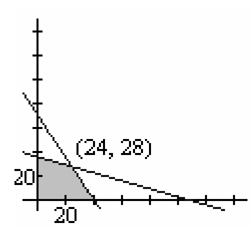



The minimum is 32 at (0, 32).

**12.** $C = 7x + 2y$

$$\begin{cases} x + 3y \le 108 \\ 7x + 4y \le 280 \\ x \ge 0, \ y \ge 0 \end{cases}$$

| $C = 7x + 2y$ | | |
|---|---|---|
| (0, 36) | 72 | |
| (40, 0) | 280 | minimum |
| (24, 28) | 224 | |
| (0, 0) | 0 | |

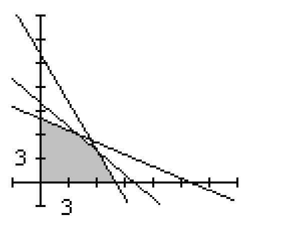



The maximum is 280 at (40, 0).

**13.** $C = 2x + 7y$

$$\begin{cases} x + y \le 10 \\ x + 2y \le 16 \\ 2x + y \le 16 \\ x \ge 0,\ y \ge 0 \end{cases}$$

| $C = 2x + 7y$ | | |
|---|---|---|
| (0, 8) | 56 | maximum |
| (0, 0) | 0 | |
| (8, 0) | 16 | |
| (6, 4) | 40 | |
| (4, 6) | 50 | |

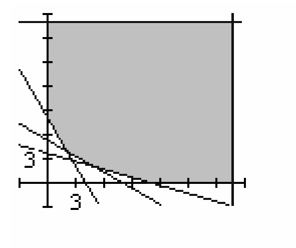



The maximum is 56 at (0, 8).

**14.** $C = 4x + 3y$

$$\begin{cases} 2x + y \ge 8 \\ 2x + 3y \ge 16 \\ x + 3y \ge 11 \\ x \le 20,\ y \le 20 \\ x \ge 0,\ y \ge 0 \end{cases}$$

| $C = 4x + 3y$ | | |
|---|---|---|
| (0, 20) | 60 | |
| (0, 8) | 24 | |
| (2, 4) | 20 | minimum |
| (5, 2) | 26 | |
| (11, 0) | 44 | |
| (20, 0) | 80 | |
| (20, 20) | 140 | |

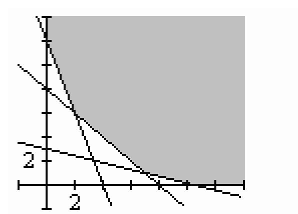



The minimum is 20 at (2, 4).

**15.** $C = 3x + 2y$

$$\begin{cases} 3x + y \ge 12 \\ 2x + 7y \ge 21 \\ x + y \ge 8 \\ x \ge 0,\ y \ge 0 \end{cases}$$

| $C = 3x + 2y$ | | |
|---|---|---|
| (0, 12) | 24 | |
| (2, 6) | 18 | minimum |
| (7, 1) | 23 | |
| (10.5, 0) | 31.5 | |

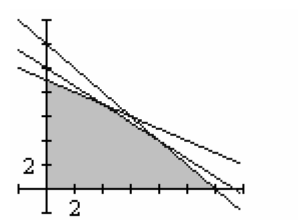



The minimum is 18 at (2, 6).

**16.** $C = 2x + 6y$

$$\begin{cases} x + y \le 12 \\ 3x + 4y \le 40 \\ x + 2y \le 18 \\ x \ge 0,\ y \ge 0 \end{cases}$$

| $C = 2x + 6y$ | | |
|---|---|---|
| (0, 9) | 54 | maximum |
| (4, 7) | 50 | |
| (8, 4) | 40 | |
| (12, 0) | 24 | |
| (0, 0) | 0 | |

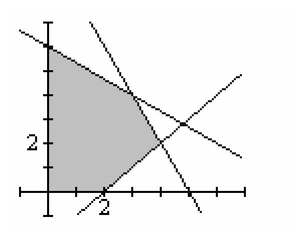



The maximum is 54 at (0, 9).

**17.** $C = 3x + 4y$

$$\begin{cases} 2x + y \le 10 \\ 2x + 3y \le 18 \\ x - y \le 2 \\ x \ge 0,\ y \ge 0 \end{cases}$$

| $C = 3x + 4y$ | | |
|---|---|---|
| (0, 6) | 24 | |
| (3, 4) | 25 | maximum |
| (4, 2) | 20 | |
| (2, 0) | 6 | |
| (0, 0) | 0 | |

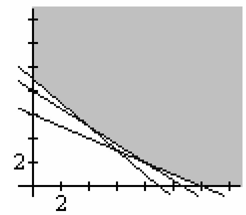



The maximum is 25 at (3, 4).

**18.** $C = 3x + 7y$

$$\begin{cases} x + y \ge 9 \\ 3x + 4y \ge 32 \\ x + 2y \ge 12 \\ x \ge 0,\ y \ge 0 \end{cases}$$

| $C = 3x + 7y$ | | |
|---|---|---|
| (0, 9) | 63 | |
| (4, 5) | 47 | |
| (8, 2) | 38 | |
| (12, 0) | 36 | maximum |

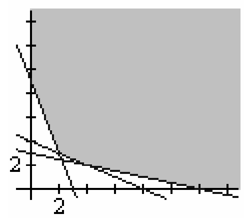



The minimum is 36 at (12, 0).

**19.** $C = 3x + 2y$

$$\begin{cases} x + 2y \ge 8 \\ 3x + y \ge 9 \\ x + 4y \ge 12 \\ x \ge 0,\ y \ge 0 \end{cases}$$

| $C = 3x + 2y$ | | |
|---|---|---|
| (0, 9) | 18 | |
| (2, 3) | 12 | maximum |
| (4, 2) | 16 | |
| (12, 0) | 36 | |

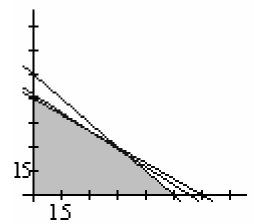



The minimum is 12 at (2, 3).

**20.** $C = 4x + 5y$

$$\begin{cases} 3x + 4y \le 250 \\ x + y \le 75 \\ 2x + 3y \le 180 \\ x \ge 0,\ y \le 0 \end{cases}$$

| $C = 4x + 5y$ | | |
|---|---|---|
| (0, 60) | 300 | |
| (30, 40) | 320 | |
| (50, 25) | 325 | maximum |
| (75, 0) | 300 | |
| (0, 0) | 0 | |

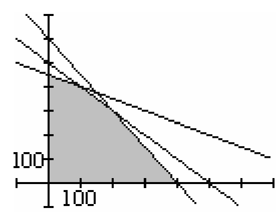



The maximum is 325 at (50, 25).

**21.** $C = 6x + 7y$

$$\begin{cases} x + 2y \le 900 \\ x + y \le 500 \\ 3x + 2y \le 1200 \\ x \ge 0,\ y \ge 0 \end{cases}$$

| $C = 6x + 7y$ | | |
|---|---|---|
| (0, 450) | 3150 | |
| (100, 400) | 3400 | maximum |
| (200, 300) | 3300 | |
| (400, 0) | 2400 | |
| (0, 0) | 0 | |

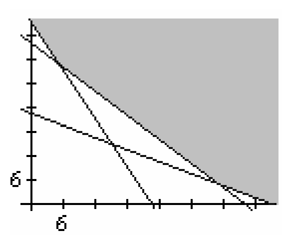



The maximum is 3400 at (100, 400).

**22.** $C = 11x + 16y$

$$\begin{cases} x + 2y \ge 45 \\ x + y \ge 40 \\ 2x + y \ge 45 \\ x \ge 0,\ y \ge 0 \end{cases}$$

| $C = 11x + 16y$ | | |
|---|---|---|
| (0, 45) | 720 | |
| (5, 35) | 615 | |
| (35, 5) | 465 | minimum |
| (45, 0) | 495 | |




The minimum is 465 at (35, 5).

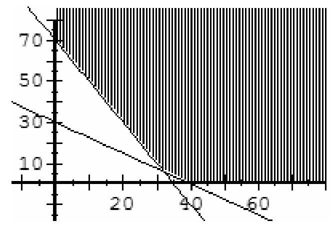
**23.** $x$ = number of cups of Oat Flakes
$y$ = number of cups of Crunchy O's

$C = 0.38x + 0.32y$

Constraints:

$$\begin{cases} 6x + 3y \ge 210 \\ 30x + 40y \ge 1200 \\ x \ge 0 \\ y \ge 0 \end{cases}$$

| $0.38x + 0.32y = C$ | | |
|---|---|---|
| (0, 70) | \$22.40 | |
| (32, 6) | \$14.08 | minimum |
| (40, 0) | \$15.20 | |




The minimum cost of \$14.08 is achieved by mixing 32 cups of Oat Flakes and 6 cups of Crunchy O's.

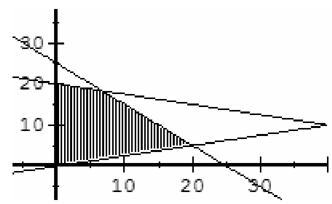
**24.** $x$ = number of two-person tents
$y$ = number of family tents

$P = 3 + x + 49y$

Constraints:

$$\begin{cases} x \le 4y \\ 2x + 2y \le 50 \\ 2x + 4y \le 80 \\ x \ge 0 \\ y \ge 0 \end{cases}$$

| $x + 49y + 3 = P$ | | |
|---|---|---|
| (0, 0) | 0 | |
| (0, 20) | \$980 | |
| (10, 15) | \$1075 | Maximum |
| (20, 5) | \$925 | |




The maximum profit of \$1075 is achieved by making 10 two-person tents and 15 family tents.

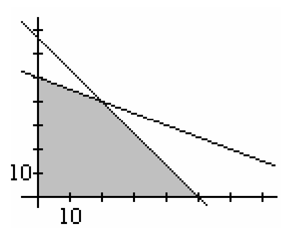
**25.** $W$ = acres of wheat to plant
$B$ = acres of barley to plant

$P = 50W + 70B$

Constraints:

$$\begin{cases} 4W + 3B \le 200 \\ W + 2B \le 100 \\ W \ge 0,\ B \ge 0 \end{cases}$$

| $50W + 70B = P$ | | |
|---|---|---|
| (0, 50) | 3500 | |
| (20, 40) | 3800 | maximum |
| (50, 0) | 2500 | |
| (0, 0) | 0 | |




The maximum profit is achieved by planting 20 acres of wheat and 40 acres of barley.

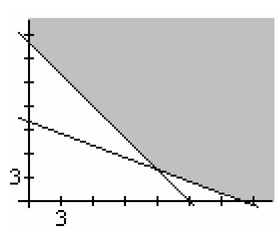
**26.** $x$ = hours of machine 1 use
$y$ = hours of machine 2 use

$\text{Cost} = 28x + 25y$

Constraints:

$$\begin{cases} 4x + 3y \ge 60 \\ 5x + 10y \ge 100 \\ x \ge 0,\ y \ge 0 \end{cases}$$

| $28x + 25y = \text{Cost}$ | | |
|---|---|---|
| (0, 20) | 500 | |
| (12, 4) | 436 | minimum |
| (20, 0) | 560 | |




To achieve the minimum cost, use machine 1 for 12 hours and machine 2 for 4 hours.

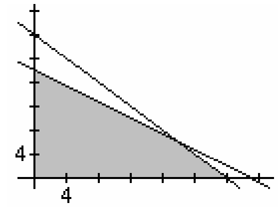
**27.** $S$ = Number of Starter sets
$P$ = Number of Pro sets

Profit $= 35S + 55P$

Constraints:

$$\begin{cases} 4S + 6P \le 108 \\ S + P \le 24 \\ S \ge 0,\ P \ge 0 \end{cases}$$

| Profit $= 35S + 55P$ | | |
|---|---|---|
| (0, 18) | 990 | maximum |
| (18, 6) | 960 | |
| (24, 0) | 840 | |
| (0, 0) | 0 | |




To maximize profit, produce zero starter sets and 18 pro sets.

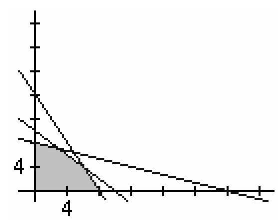
**28.** S = Number of standard model
D = Number of deluxe model

Profit $= 25S + 35D$

Constraints:

$$\begin{cases} S + 3D \le 24 \\ S + D \le 10 \\ 2S + D \le 16 \\ S \ge 0,\ D \ge 0 \end{cases}$$

| Profit $= 25S + 35D$ | | |
|---|---|---|
| (0, 8) | 280 | |
| (3, 7) | 320 | maximum |
| (6, 4) | 290 | |
| (8, 0) | 200 | |
| (0, 0) | 0 | |




To maximize profit, produce 3 standard models and 7 deluxe models.

## Connecting Concepts

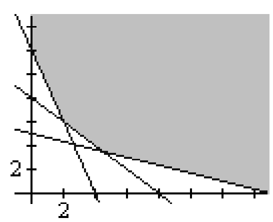
**29.** $A$ = ounces of food group $A$
$B$ = ounces of food group $B$

Cost $= 40A + 10B$

Constraints:

$$\begin{cases} 3A + B \ge 24 \\ A + B \ge 16 \\ A + 3B \ge 30 \\ A \ge 0,\ B \ge 0 \end{cases}$$

| Cost $= 40A + 10B$ | | |
|---|---|---|
| (0, 24) | 240 | minimum |
| (4, 12) | 280 | |
| (9, 7) | 430 | |
| (30, 0) | 1200 | |




To minimize cost, use 24 ounces of food group $B$ and zero ounces of food group $A$. The minimum cost is \$2.40.

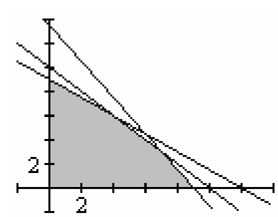
**30.** $A$ = liters of Pymex $A$
$B$ = liters of Pymex $B$

Profit $= 12A + 9B$

Constraints:

$$\begin{cases} A + B \le 10 \\ 3A + 4B \le 36 \\ 3A + 2B \le 27 \\ A \ge 0,\ B \ge 0 \end{cases}$$

| Profit $= 12A + 9B$ | | |
|---|---|---|
| (0, 9) | 81 | |
| (4, 6) | 102 | |
| (7, 3) | 111 | maximum |
| (9, 0) | 108 | |
| (0, 0) | 0 | |




To maximize profit, make 7 liters of Pymex $A$ and 3 liters of Pymex $B$. The maximum profit is \$111.

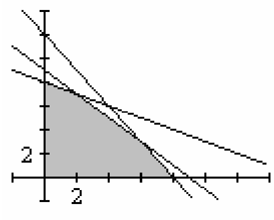
**31.** $x$ = number of 4-cylinder engines
$y$ = number of 6-cylinder engines

Profit $= 150x + 250y$

Constraints:

$$\begin{cases} x + y \le 9 \\ 5x + 10y \le 80 \\ 3x + 2y \le 24 \\ x \ge 0,\ y \ge 0 \end{cases}$$

| $150x + 250y = \text{Profit}$ | | |
|---|---|---|
| (0, 8) | 2000 | |
| (2, 7) | 2050 | maximum |
| (6, 3) | 1650 | |
| (8, 0) | 1200 | |
| (0, 0) | 0 | |




To achieve the maximum profit of $2050, produce two 4-cylinder engines and seven 6-cylinder engines.

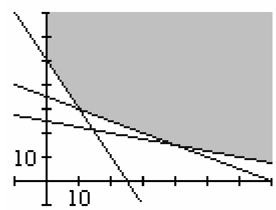
**32.** $x$ = number of pounds of $F_1$
$y$ = number of pounds of $F_2$

Cost $= 450x + 300y$

Constraints:

$$\begin{cases} 200x + 100y \ge 5000 \\ 100x + 200y \ge 7000 \\ 100x + 400y \ge 10{,}000 \\ x \ge 0,\quad y \ge 0 \end{cases}$$

| $450x + 300y = \text{Cost}$ | | |
|---|---|---|
| (0, 50) | 15,000 | |
| (10, 30) | 13,500 | minimum |
| (40, 15) | 22,500 | |
| (100, 0) | 45,000 | |




To achieve the minimum cost, produce 10 pounds of $F_1$ and 30 pounds of $F_2$. The minimum cost is $13,500.

## Exploring Concepts with Technology

**Ill-Conditioned Systems of Equations**

**System 1**
When the coefficients are approximated to:

**a.** the nearest 0.01 the solution is (266.6666667, −1433.333333, 1366,666667).
**b.** the nearest 0.001 the solution is (30.50502049, −209.9900227, 226.6966693).
**c.** the nearest 0.0001 the solution is (27.32317303, −193.65719, 211.5374196).
**d.** the limit of most calculators (nearest 0.00000000001) the solution is (27.00000003, −192.0000002, 210.0000002).

**System 2**
When the coefficients are approximated to:

**a.** the nearest 0.01 the solution is (55.98194131, −295.2595937, 190.9706546, 118.510158).
**b.** the nearest 0.001 the solution is (81.29464132, −708.9158728, 1335.912871, −682.7827636).
**c.** the nearest 0.0001 the solution is (−90.6440169, 1188.443724, −3201.68217, 2258.171289).
**d.** the limit of most calculators (nearest 0.00000000001) the solution is (−64.00000245, 900.0000269, −2520.000064, 1820.000042)

## Assessing Concepts

**1.** True

**2.** False, a graph of a linear equation in three variables is a plane.

**3.** False, a homogeneous system is one where the constant term in each equation is zero.

**4.** True

**5.** True

**6.** Yes

**7.** At a vertex of the set of feasible solutions.

**8.** It is a nonsquare system of equations.

**9.** A line

**10.** (0, 0)

## Chapter Review

**1.** $\begin{cases} 2x-4y=-3 & (1) \\ 3x+8y=-12 & (2) \end{cases}$

$4x-8y=-6$ 2 times (1)

$3x+8y=-12$ (2)

$7x=-18$

$x=-\frac{18}{7}$ $2\left(-\frac{18}{7}\right)-4y=-3$

$y=-\frac{15}{28}$

The solution is $\left(-\frac{18}{7}, -\frac{15}{28}\right)$. [9.1]

**2.** $\begin{cases} 4x-3y=15 \\ 2x=5y=-12 \end{cases}$

$4x-3y=15$ (1)

$-4x-10y=24$ −2 times (2)

$-13y=39$

$y=-3$ $4x-3(-3)=15$

$x=\frac{3}{2}$

The solution is $\left(\frac{3}{2}, -3\right)$.

[9.1]

**3.** $\begin{cases} 3x-4y=-5 & (1) \\ y=\frac{2}{3}x+1 & (2) \end{cases}$

Substitute $y$ from Eq. (2) into Eq. (1).

$3x-4\left(\frac{2}{3}x+1\right)=-5$

$3x-\frac{8}{3}x-4=-5$

$\frac{1}{3}x=-1$

$x=-3$

$y=\frac{2}{3}(-3)+1$

$y=-1$

The solution is $(-3, -1)$. [9.1]

**4.** $\begin{cases} 7x+2y=-14 & (1) \\ y=-\frac{5}{2}x-3 & (2) \end{cases}$

Substitute $y$ from Eq. (2) into Eq. (1).

$7x+2\left(-\frac{5}{2}x-3\right)=-14$

$7x-5x-6=-14$

$2x=-8$

$x=-4$

$y=-\frac{5}{2}(-4)-3$

$y=7$

The solution is (–4, 7). [9.1]

**5.** $\begin{cases} y=2x-5 & (1) \\ x=4y-1 & (2) \end{cases}$

Substitute $x$ from Eq. (2) into Eq. (1).

$y=2(4y-1)-5$

$y=8y-2-5$

$-7y=-7$ $x=4(1)-1$

$y=1$ $x=3$

The solution is $(3, 1)$.

[9.1]

**6.** $\begin{cases} y=3x+4 & (1) \\ x=4y-5 & (2) \end{cases}$

Substitute $x$ from Eq. (2) into Eq. (1).

$y=3(4y-5)+4$ $x=4(1)-5$

$y=12y-15+4$ $x=-1$

$-11y=-11$

$y=1$

The solution is $(-1, 1)$.

[9.1]

**7.** $\begin{cases} 6x+9y=15 & (1) \\ 10x+15y=25 & (2) \end{cases}$

$2x+3y=5$ $\frac{1}{3}$ times (1)

$2x+3y=5$ $\frac{1}{5}$ times (2)

$0=0$

Let $y=c$. $2x+3c=5$

$x=\frac{5-3c}{2}$

The ordered-pair solutions are $\left(\frac{5-3c}{2}, c\right)$. [9.1]

**8.** $4x-8y=9$ (1)

$-4x+8y=-10$ −2 times (2)

$0=-1$

The system of equations is inconsistent.
The system has no solution. [9.1]

**9.** $\begin{cases} 2x-3y+z=-9 & (1) \\ 2x+5y-2z=18 & (2) \\ 4x-y+3z=-4 & (3) \end{cases}$

$$\begin{array}{rl} 2x-3y+z=-9 & (1) \\ \underline{-2x-5y+2z=-18} & -1 \text{ times } (2) \\ -8y+3z=-27 & (4) \end{array}$$

$$\begin{array}{rl} -4x+6y-2z=18 & -2 \text{ times } (1) \\ \underline{4x-y+3z=-4} & (3) \\ 5y+z=14 & (5) \end{array}$$

$\begin{cases} 2x-3y+z=-9 \\ -8y+3z=-27 & (4) \\ 5y+z=14 & (5) \end{cases}$

$$\begin{array}{rl} -8y+3z=-27 & (4) \\ \underline{-15y-3z=-42} & -3 \text{ times } (5) \\ -23y=-69 & \\ y=3 & (6) \end{array}$$

$\begin{cases} 2x-3y+z=-9 \\ -8y+3z=-27 \\ y=3 & (6) \end{cases}$

$$\begin{array}{ll} -8(3)+3z=-27 & 2x-3(3)-1=9 \\ 3z=-3 & 2x=1 \\ z=-1 & x=1 \end{array}$$

The ordered-triple solution is $\left(\frac{1}{2}, 3, -1\right)$. [9.2]

**10.** $\begin{cases} x-3y+5z=1 & (1) \\ 2x+3y-5z=15 & (2) \\ 3x+6y+5z=15 & (3) \end{cases}$

$$\begin{array}{rl} -2x+6y-10z=-2 & -2 \text{ times } (1) \\ \underline{2x+3y-5z=15} & (2) \\ 9y-15z=13 & (4) \end{array}$$

$$\begin{array}{rl} -3x+9y-15z=-3 & -3 \text{ times } (1) \\ \underline{3x+6y+5z=15} & (3) \\ 15y-10z=12 & (5) \end{array}$$

$\begin{cases} x-3y+5z=1 \\ 9y-15z=13 & (4) \\ 15y-10z=12 & (5) \end{cases}$

$$\begin{array}{rl} 18y-30z=26 & 2 \text{ times } (4) \\ \underline{-45y+30z=-36} & -3 \text{ times } (5) \\ -27y=-10 & \\ y=\frac{10}{27} & (6) \end{array}$$

$\begin{cases} x-3y+5z=1 \\ 9y-15z=13 \\ y=\frac{10}{27} & (6) \end{cases}$

$$\begin{array}{ll} -15z=13-9y & x=3y-5z+1 \\ -15z=13-9\left(\frac{10}{27}\right) & x=3\left(\frac{10}{27}\right)-5\left(-\frac{29}{45}\right)+1 \\ -15z=\frac{29}{3} & x=\frac{16}{3} \\ z=-\frac{29}{45} & \end{array}$$

The solution is $\left(\frac{16}{3}, \frac{10}{27}, -\frac{29}{45}\right)$. [9.2]

**11.** $\begin{cases} x+3y-5z=-12 & (1)\\ 3x-2y+z=7 & (2)\\ 5x+4y-9z=-17 & (3)\end{cases}$

$$\begin{array}{rl} -3x-9y+15z=36 & -3 \text{ times } (1)\\ 3x-2y+z=7 & (2)\\ \hline -11y+16z=43 & (4)\end{array}$$

$$\begin{array}{rl} -5x-15y+25z=60 & -5 \text{ times } (1)\\ 5x+4y-9z=-17 & (3)\\ \hline -11y+16z=43 & (5)\end{array}$$

$$\begin{cases} x+3y-5z=-12 & \\ -11y+16z=43 & (4)\\ -11y+16z=43 & (5)\end{cases}$$

$$\begin{cases} x+3y-5z=-12 & \\ -11y+16z=43 & \\ 0=0 & (6)\end{cases}$$

The system of equations is dependent.

Let $z=c$.

$-11y+16c=43$

$$y=\frac{16c-43}{11}$$

$$x+3\left(\frac{16c-43}{11}\right)-5c=-12$$

$$x=\frac{7c-3}{11}$$

The solution is $\left(\frac{7c-3}{11}, \frac{16c-43}{11}, c\right)$. [9.2]

**12.** $\begin{cases} 2x-y+2z=5 & (1)\\ x+3y-3z=2 & (2)\\ 5x-9y+8z=13 & (3)\end{cases}$

$$\begin{array}{rl} 2x-y+2z=5 & (1)\\ -2x-6y+6z=-4 & -2 \text{ times } (2)\\ \hline -7y+8z=1 & (4)\end{array}$$

$$\begin{array}{rl} -5x-15y+15z=-10 & -5 \text{ times } (2)\\ 5x-9y+8z=13 & (3)\\ \hline -24y+23z=3 & (5)\end{array}$$

$$\begin{cases} 2x-y+2z=5 & \\ -7y+8z=1 & (4)\\ -24y+23z=3 & (5)\end{cases}$$

$$\begin{array}{rl} -168y+192z=24 & 24 \text{ times } (4)\\ 168y-161z=-21 & -7 \text{ times } (2)\\ \hline 31z=3 & \\ z=\frac{3}{31} & (6)\end{array}$$

$$\begin{cases} 2x-y+2z=5 & \\ -7y+8z=1 & \\ z=\frac{3}{31} & (6)\end{cases}$$

$$-7y+8\left(\frac{3}{31}\right)=1$$

$$-7y=\frac{31}{31}-\frac{24}{31}$$

$$y=-\frac{1}{31}$$

$$2x-\left(-\frac{1}{31}\right)+2\left(\frac{3}{31}\right)=5$$

$$2x=-\frac{1}{31}-\frac{6}{31}+\frac{155}{31}$$

$$2x=\frac{148}{31}$$

$$x=\frac{74}{31}$$

The solution is $\left(\frac{74}{31}, -\frac{1}{31}, \frac{3}{31}\right)$. [9.2]

**13.** $\begin{cases} 3x+4y-6z=10 & (1) \\ 2x+2y-3z=6 & (2) \\ x-6y+9z=-4 & (3) \end{cases}$

Rearrange the equations so that the equation with the 1 as the coefficient of $x$ is in the first row.

$$\begin{cases} x-6y+9z=-4 & (3) \\ 2x+2y-3z=6 & (2) \\ 3x+4y-6z=10 & (1) \end{cases}$$

$$\begin{array}{ll} -2x+12y-18z=8 & -2 \text{ times } (3) \\ \underline{2x+2y-3z=6} & (2) \\ 14y-21z=14 & \\ 2y-3z=2 & (4) \end{array}$$

$$\begin{array}{ll} -3x+18y-27z=12 & -3 \text{ times } (3) \\ \underline{3x+4y-6z=10} & (1) \\ 22y-33z=22 & \\ 2y-3z=2 & (5) \end{array}$$

$$\begin{array}{ll} x-6y+9z=-4 & (3) \\ 2y-3z=2 & (4) \\ 2y-3z=2 & (5) \end{array}$$

$$\begin{cases} x-6y+9z=-4 \\ 2y-3z=2 \\ 0=0 \end{cases}$$

The system of equations is dependent.

Let $z=c$.

$$2y-3c=2$$
$$y=\frac{3c+2}{2}$$
$$x-6\left(\frac{3c+2}{2}\right)+9c=-4$$
$$x-9c-6+9c=-4$$
$$x=2$$

The solution is $\left(2, \frac{3c+2}{2}, c\right)$. [9.2]

**14.** $\begin{cases} x-6y+4z=6 & (1) \\ 4x+3y-4z=1 & (2) \\ 5x-9y+8z=13 & (3) \end{cases}$

$$\begin{array}{ll} x-6y+4z=6 & (1) \\ \underline{4x+3y-4z=1} & (2) \\ 5x-3y=7 & (4) \end{array}$$

$$\begin{array}{ll} 8x+6y-8z=2 & 2 \text{ times } (2) \\ \underline{5x-9y+8z=13} & (3) \\ 13x-3y=15 & (5) \end{array}$$

$$\begin{cases} x-6y+4z=6 & \\ 5x-3y=7 & (4) \\ 13x-3y=15 & (5) \end{cases}$$

$$\begin{array}{ll} 5x-3y=7 & (4) \\ \underline{-13x+3y=-15} & -1 \text{ times } (5) \\ -8x=-8 & \\ x=1 & (6) \end{array}$$

$$\begin{cases} x-6y+4z=6 & \\ 5x-3y=7 & \\ x=1 & (6) \end{cases}$$

$$5(1)-3y=7 \qquad y=-\frac{2}{3}$$

$$1-6\left(-\frac{2}{3}\right)+4z=6 \qquad 4z=1 \qquad z=\frac{1}{4}$$

The solution is $\left(1, -\frac{2}{3}, \frac{1}{4}\right)$. [9.2]

**15.** $\begin{cases} 2x+3y-2z=0 & (1) \\ 3x-y-4z=0 & (2) \\ 5x+13y-4z=0 & (3) \end{cases}$

$$\begin{array}{rl} 6x+9y-6z=0 & 3 \text{ times } (1) \\ -6x+2y+8z=0 & -2 \text{ times } (2) \\ \hline 11y+2z=0 & (4) \end{array}$$

$$\begin{array}{rl} 15x-5y-20z=0 & 5 \text{ times } (2) \\ -15x-39y+12z=0 & -3 \text{ times } (3) \\ \hline -44y-8z=0 & \\ 11y+2z=0 & (5) \end{array}$$

$$\begin{cases} 2x+3y-2z=0 & (1) \\ 11y+2z=0 & (4) \\ 11y+2z=0 & (5) \end{cases}$$

$$\begin{cases} 2x+3y-2z=0 \\ 11y+2z=0 \\ 0=0 \end{cases}$$

Let $z=c$ $\quad 11y+2c=0$

$$y=-\tfrac{2}{11}c$$

$$2x+3\left(-\tfrac{2c}{11}\right)-2c=0$$

$$2x=\tfrac{28c}{11}$$

$$x=\tfrac{14c}{11}$$

The solution is $\left(\tfrac{14}{11}c,\ -\tfrac{2}{11}c,\ c\right)$. [9.2]

**16.** $\begin{cases} 3x-5y+z=0 & (1) \\ x+4y-3z=0 & (2) \\ 2x+y-2z=0 & (3) \end{cases}$

$$\begin{array}{rl} 3x-5y+z=0 & (1) \\ -3x-12y+9z=0 & -3 \text{ times } (2) \\ \hline -17y+10z=0 & (4) \end{array}$$

$$\begin{array}{rl} -2x-8y+6z=0 & -2 \text{ times } (2) \\ 2x+y-2z=0 & (3) \\ \hline -7y+4z=0 & (5) \end{array}$$

$$\begin{cases} 3x-5y+z=0 & \\ -17y+10z=0 & (4) \\ -7y+4z=0 & (5) \end{cases}$$

$$\begin{array}{rl} -34y+20z=0 & 2 \text{ times } (4) \\ 35y-20z=0 & -5 \text{ times } (5) \\ \hline y=0 & (6) \end{array}$$

$$\begin{cases} 3x-5y+z=0 & \\ -17y+10z=0 & \\ y=0 & (6) \end{cases}$$

$x=0$, $y=0$, $z=0$. The solution is (0, 0, 0). [9.2]

**17.** $\begin{cases} x-2y+z=1 & (1) \\ 3x+2y-3z=1 & (2) \end{cases}$

$$\begin{array}{rl} -3x+6y-3z=-3 & -3 \text{ times } (1) \\ 3x+2y-3z=1 & (2) \\ \hline 8y-6z=-2 & \\ 4y-3z=-1 & (3) \end{array}$$

$$\begin{cases} x-2y+z=1 & \\ 4y-3z=-1 & (3) \end{cases}$$

Let $z=c$. $\quad 4y-3c=-1$

$$y=\frac{3c-1}{4}$$

$$x-2\left(\frac{3c-1}{4}\right)+c=1$$

$$x=\frac{c+1}{2}$$

The solution is $\left(\frac{c+1}{2},\ \frac{3c-1}{4},\ c\right)$. [9.2]

**18.** $\begin{cases} 2x-3y+z=1 & (1) \\ 4x+2y+3z=21 & (2) \end{cases}$

$$\begin{array}{rl} -4x+6y-2z=-2 & -2 \text{ times } (1) \\ 4x+2y+3z=21 & (2) \\ \hline 8y+z=19 & (3) \end{array}$$

$$\begin{cases} 2x-3y+z=1 & \\ 8y+z=19 & (3) \end{cases}$$

Let $z=c$. $\quad 8y=19-c$

$$y=\frac{19-c}{8}$$

$$2x-3\left(\frac{19-c}{8}\right)+c=1$$

$$2x=\frac{65-11c}{8}$$

$$x=\frac{65-11c}{16}$$

The solution is $\left(\frac{65-11c}{16},\ \frac{19-c}{8},\ c\right)$. [9.2]

**19.** $\begin{cases} y = x^2 - 2x - 3 \\ y = 2x - 7 \end{cases}$

$$x^2 - 2x - 3 = 2x - 7$$
$$x^2 - 4x + 4 = 0$$
$$(x-2)(x-2) = 0$$
$$x = 2$$

$$y = 2(2) - 7$$
$$y = -3$$

The solution is (2, –3). [9.3]

**20.** $\begin{cases} y = 2x^2 + x \\ y = 2x + 1 \end{cases}$

$$2x^2 + x = 2x + 1$$
$$2x^2 - x - 1 = 0$$
$$(2x+1)(x-1) = 0$$

$x = -\frac{1}{2}$ $\qquad$ $x = 1$

$y = 2\left(-\frac{1}{2}\right) + 1$ $\qquad$ $y = 2(1) + 1$

$y = 0$ $\qquad$ $y = 3$

The solutions are $\left(-\frac{1}{2}, 0\right)$ and (1, 3). [9.3]

**21.** $\begin{cases} y = 3x^2 - x + 1 \\ y = x^2 + 2x - 1 \end{cases}$

$$3x^2 - x + 1 = x^2 + 2x - 1$$
$$2x^2 - 3x + 2 = 0$$
$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(2)}}{2(2)}$$
$$x = \frac{3 \pm \sqrt{9-16}}{4} = \frac{3 \pm \sqrt{-7}}{4}$$

$x$ has no real number solution. The system of equations is inconsistent. The graphs of the equations do not intersect. No real solution. [9.3]

**22.** $\begin{cases} y = 4x^2 - 2x - 3 \\ y = 2x^2 + 3x - 6 \end{cases}$

$$4x^2 - 2x - 3 = 2x^2 + 3x - 6$$
$$2x^2 - 5x + 3 = 0$$
$$(2x-3)(x-1) = 0$$

$x = \frac{3}{2}$ or $x = 1$

$$y = 2\left(\frac{3}{2}\right)^2 + 3\left(\frac{3}{2}\right) - 6$$
$$y = 3$$

$$y = 2(1)^2 + 3(1) - 6$$
$$y = -1$$

The solutions are $\left(\frac{3}{2}, 3\right)$ and (1, –1). [9.3]

**23.** $(x+1)^2 + (y-2)^2 = 4 \quad (1)$
$2x + y = 4 \quad (2)$

From Eq. (2), $y = -2x + 4$.
Substitute $y$ in Eq. (1).

$$(x+1)^2 + (-2x+4-2)^2 = 4$$
$$(x+1)^2 + (-2x+2)^2 = 4$$
$$x^2 + 2x + 1 + 4x^2 - 8x + 4 = 4$$
$$5x^2 - 6x + 1 = 0$$
$$(5x-1)(x-1) = 0$$

$x = \frac{1}{5}$ or $x = 1$

$y = -2\left(\frac{1}{5}\right) + 4$ $\qquad$ $y = -2(1) + 4$

$y = \frac{18}{5}$ $\qquad$ $y = 2$

The solutions are $\left(\frac{1}{5}, \frac{18}{5}\right)$ and (1, 2). [9.3]

**24.** $\begin{cases} (x-1)^2 + (y+1)^2 = 5 \\ y = 2x - 3 \end{cases}$

$$(x-1)^2 + (2x-3+1)^2 = 5$$
$$(x-1)^2 + (2x-2)^2 = 5$$
$$x^2 - 2x + 1 + 4x^2 - 8x + 4 = 5$$
$$5x^2 - 10x = 0$$
$$5x(x-2) = 0$$

$x = 0$ or $x = 2$

$y = 2(0) - 3$ $\qquad$ $y = 2(2) - 3$

$y = -3$ $\qquad$ $y = 1$

The solutions are $(0, -3)$ and (2, 1). [9.3]

**25.** $\begin{cases}(x-2)^2+(y+2)^2=4 & (1)\\ (x+2)^2+(y+1)^2=17 & (2)\end{cases}$

Expand the binomials. Then subtract.

$$\begin{array}{l} x^2-4x+4+y^2+4y+4=4 \\ \underline{x^2+4x+4+y^2+2y+1=17} \\ -8x+2y+3=-13 \\ -8x+2y=-16 \\ y=4x-8 \end{array}$$

Substitute $y$ into Eq. (1).

$$\begin{aligned}(x-2)^2+(4x-8+2)^2&=4\\ (x-2)^2+(4x-6)^2&=4\\ x^2-4x+4+16x^2-48x+36&=4\\ 17x^2-52x+36&=0\\ (x-2)(17x-18)&=0\end{aligned}$$

$x=2$ or $x=\frac{18}{17}$

$y=4(2)-8$ $\qquad$ $y=4\left(\frac{18}{17}\right)-8$

$y=0$ $\qquad$ $y=-\frac{64}{17}$

The solutions are (2, 0) and $\left(\frac{18}{17}, -\frac{64}{17}\right)$. [9.3]

**26.** $\begin{cases}(x+1)^2+(y-2)^2=1 & (1)\\ (x-2)^2+(y+2)^2=20 & (2)\end{cases}$

Expand the binomials. Then subtract.

$$\begin{array}{l} x^2+2x+1+y^2-4y+4=1 \\ \underline{x^2-4x+4+y^2+4y+4=20} \\ 6x-3-8y=-19 \\ 6x-8y=-16 \\ y=\frac{3x+8}{4} \end{array}$$

$$\begin{aligned}(x+1)^2+\left(\frac{3x+8}{4}-2\right)^2&=1\\ (x+1)^2+\left(\frac{3x}{4}\right)^2&=1\\ x^2+2x+1+\frac{9x^2}{16}&=1\\ 16x^2+32x+16+9x^2&=16\\ 25x^2+32x&=0\\ x(25x+32)&=0\end{aligned}$$

$x=0$ or $x=-\frac{32}{25}$

$y=\frac{3}{4}(0)+2$ $\qquad$ $y=\frac{3}{4}\left(-\frac{32}{25}\right)+2$

$y=2$ $\qquad$ $y=\frac{26}{25}$

The solutions are (0, 2) and $\left(-\frac{32}{25}, \frac{26}{25}\right)$. [9.3]

**27.** $\begin{cases} x^2 - 3xy + \ \ y^2 = -1 & (1) \\ 3x^2 - 5xy - 2y^2 = \ \ 0 & (2) \end{cases}$

Factor Eq. (2).

$(3x + y)(x - 2y) = 0$

$x = \dfrac{-y}{3}$ or $x = 2y$

$x = -\dfrac{y}{3}$ implies $y = -3x.$ Substitute for $y$ in Eq. (1).

$$\begin{aligned} x^2 - 3x(-3x) + (-3x)^2 &= -1 \\ x^2 + 9x^2 + 9x^2 &= -1 \\ 19x^2 &= -1 \\ x^2 &= -\frac{1}{19} \end{aligned}$$

This equation yields no real solutions. Substituting $x = 2y$ in Eq. (1) yields

$$\begin{aligned} (2y)^2 - 3(2y)y + y^2 &= -1 \\ 4y^2 - 6y^2 + y^2 &= -1 \\ -y^2 &= -1 \\ y^2 &= 1 \\ y &= \pm 1 \end{aligned}$$

The solutions are (2, 1) and (−2, −1). [9.3]

**28.** $\begin{cases} 2x^2 + 2xy - y^2 = -1 & (1) \\ 6x^2 \ \ + xy - y^2 = \ \ 0 & (2) \end{cases}$

Factor Eq. (2).

$(3x - y)(2x + y) = 0$

$y = 3x \qquad y = -2x$

$$\begin{aligned} 2x^2 + 2x(3x) - (3x)^2 &= -1 \\ 2x^2 + 6x^2 - 9x^2 &= -1 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

$$\begin{aligned} 2x^2 + 2x(-2x) - (-2x)^2 &= -1 \\ 2x^2 - 4x^2 - 4x^2 &= -1 \\ x^2 &= \frac{1}{6} \\ x &= \pm\frac{\sqrt{6}}{6} \end{aligned}$$

$y = 3(1) \quad y = 3(-1)$

$y = 3 \qquad y = -3$

$y = -2\left(\dfrac{\sqrt{6}}{6}\right) \qquad y = -2\left(-\dfrac{\sqrt{6}}{6}\right)$

$y = -\dfrac{\sqrt{6}}{3} \qquad y = \dfrac{\sqrt{6}}{3}$

The solutions are $(1,\ 3),\ (-1,\ -3),\ \left(\dfrac{\sqrt{6}}{6},\ -\dfrac{\sqrt{6}}{3}\right)$ and $\left(-\dfrac{\sqrt{6}}{6},\ \dfrac{\sqrt{6}}{3}\right)$. [9.3]

**29.** $\begin{cases} 2x^2 - 5xy + 2y^2 = 56 & (1) \\ 14x^2 - 3xy - 2y^2 = 56 & (2) \end{cases}$

Subtract Eq. (1) From Eq. (2)

$12x^2 + 2xy - 4y^2 = 0$

Factor $2(3x + 2y)(2x - y) = 0$. Thus

$3x + 2y = 0$, $y = -\frac{3}{2}$ or $2x - y = 0$, $y = 2x$

Substituting $y = -\frac{3}{2}x$ into Eq. (1), we have

$$2x^2 - 5x\left(-\frac{3}{2}x\right) + 2\left(-\frac{3}{2}x\right)^2 = 56$$
$$2x^2 + \frac{15}{2}x^2 + \frac{9}{2}x^2 = 56$$
$$14x^2 = 56$$
$$x^2 = 4$$
$$x = \pm 2$$

When $x = 2, y = -\frac{3}{2}(2) = -3$;

When $x = -2, y = -\frac{3}{2}(-2) = 3$.

Two solutions are (2, –3) and (–2, 3). Substituting $y = 2x$ into Eq. (1) yields 0 = 56. Thus the only solutions of the system are (2, –3), and (–2, 3). [9.3]

**30.** $\begin{cases} 2x^2 + 7xy + 6y^2 = 1 & (1) \\ 6x^2 + 7xy + 2y^2 = 1 & (2) \end{cases}$

Subtract Eq. (2) from Eq. (1)

$$-4x^2 + 4y^2 = 0$$
$$-x^2 + y^2 = 0$$
$$y = x \quad \text{or} \quad y = -x$$

When $y = x$, we have, from Eq. (1),

$$2x^2 + 7x^2 + 6x^2 = 1$$
$$15x^2 = 1$$
$$x^2 = \frac{1}{15}$$
$$x = \pm\frac{\sqrt{15}}{15}$$

Since $y = x, y = \pm\frac{\sqrt{15}}{15}$. The solutions are

$\left(\frac{\sqrt{15}}{15}, \frac{\sqrt{15}}{15}\right), \left(-\frac{\sqrt{15}}{15}, -\frac{\sqrt{15}}{15}\right)$.

When $y = -x$, we have, from Eq. (1)

$$2x^2 - 7x^2 + 6x^2 = 1$$
$$x^2 = 1$$
$$x = \pm 1$$

Since $y = -x, y = \pm 1$.

The solutions are $(1, -1), (-1, 1)$.

The solutions of the system of equations are

$\left(\frac{\sqrt{15}}{15}, \frac{\sqrt{15}}{15}\right), \left(-\frac{\sqrt{15}}{15}, -\frac{\sqrt{15}}{15}\right), (1, -1), (-1, 1)$. [9.3]

**31.** $\dfrac{7x-5}{x^2-x-2} = \dfrac{7x-5}{(x-2)(x+1)} = \dfrac{A}{x-2} + \dfrac{B}{x+2}$

$$7x - 5 = A(x+1) + B(x-2)$$
$$7x - 5 = Ax + A + Bx - 2b$$
$$7x - 5 = (A+B)x + (A-2B)$$

$\begin{cases} 7 = A + B \\ -5 = A - 2B \end{cases}$

$$7 = A + B$$
$$\underline{5 = -A + 2B}$$
$$12 = 3B \qquad A + 4 = 7$$
$$4 = B \qquad A = 3$$

$\dfrac{7x-5}{x^2-x-2} = \dfrac{3}{x-2} + \dfrac{4}{x+1}$ [9.4]

**32.** $\dfrac{x+1}{(x-1)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2}$

$$x + 1 = A(x-1) + B$$
$$x + 1 = Ax - A + B$$

$$1 = A$$
$$1 = -A + B \qquad -1 + B = 1$$
$$B = 2$$

$\dfrac{x+1}{(x-2)^2} = \dfrac{1}{x-1} + \dfrac{2}{(x-1)^2}$ [9.4]

**33.** $$\frac{2x-2}{(x^2+1)(x+2)}=\frac{Ax+B}{x^2+1}+\frac{C}{x+2}$$

$$2x-2=(Ax+B)(x+2)+C(x^2+1)$$
$$2x-2=Ax^2+2Ax+Bx+2B+Cx^2+C$$
$$2x-2=(A+C)x^2+(2A+B)x+(2B+C)$$

$$\begin{cases} 0=A+C & (1)\\ 2=2A+B & (2)\\ -2=2B+C & (3)\end{cases}$$

$$\begin{cases} 0=A+C & \\ 2=2A+B & \\ 2=A-2B & (4)\end{cases}$$

$$\begin{array}{ll} 0=A \quad +C & (1)\\ \underline{2=\quad -2B-C} & -1\text{ times}(3)\\ 2=A-2B & (4)\end{array}$$

$$\begin{array}{ll} 4=4A+2B & 2\text{ times } (2)\\ \underline{2=\quad A-2B} & (4)\\ 6=5A & (5)\\ \frac{6}{5}=A & \end{array}$$

$$A=\frac{6}{5} \qquad 2\left(\frac{6}{5}\right)+B=2 \qquad \frac{6}{5}+C=0$$
$$B=-\frac{2}{5} \qquad C=-\frac{6}{5}$$

$$\frac{2x-2}{(x^2+1)(x-2)}=\frac{6x-2}{5(x^2+1)}+\frac{-6}{5(x+2)} \quad [9.4]$$

**34.** $$\frac{5x^2-10x+9}{(x-2)^2(x+1)}=\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{x+1}$$

$$5x^2-10x+9=A(x-2)(x+1)+B(x+1)+C(x-2)^2$$
$$5x^2-10x+9=Ax^2-Ax-2A+Bx+B+Cx^2-4Cx+4C$$
$$5x^2-10x+9=(A+C)x^2+(-A+B-4C)x+(-2A+B+4C)$$

$$\begin{cases} 5=A+C & (1)\\ -10=-A+B-4C & (2)\\ 9=-2A+B+4C & (3)\end{cases}$$

$$\begin{array}{ll} -10=-A+B-4C & (2)\\ \underline{-9=2A-B-4C} & -1\text{ times}(3)\\ -19=\ A \quad -8C & (4)\end{array}$$

$$\begin{cases} 5=A+C & \\ -19=A-8C & (4)\\ 9=-2A+B+4C & \end{cases}$$

$$\begin{array}{ll} 5=\quad A+C & (1)\\ \underline{19=-A-8C} & -1\text{ times}(4)\\ 24=\quad 9C & \\ \frac{8}{3}=C & (5)\end{array}$$

$$\begin{cases} 5=A+C & \\ \frac{8}{3}=\quad C & (5)\\ 9=-2A+B+4C & \end{cases}$$

$$A+\frac{8}{3}=5 \qquad -2\left(\frac{7}{3}\right)+B+4\left(\frac{8}{3}\right)=9$$
$$A=\frac{7}{3} \qquad B=9+\frac{14}{3}-\frac{32}{3}$$
$$B=3$$

$$\frac{5x^2-10x+9}{(x-2)^2(x+1)}=\frac{7}{3(x-2)}+\frac{3}{(x-2)^2}+\frac{8}{3(x+1)} \quad [9.4]$$

**35.** $\dfrac{11x^2-x-2}{x^3-x}=\dfrac{11x^2-x-2}{x(x-1)(x+1)}=\dfrac{A}{x}+\dfrac{B}{x-1}+\dfrac{C}{x+1}$

$11x^2-x-2=A(x-1)(x+1)+Bx(x+1)+Cx(x-1)$

$11x^2-x-2=Ax^2-A+Bx^2+Bx+Cx^2-Cx$

$11x^2-x-2=(A+B+C)x^2+(B-C)x+(-A)$

$$\begin{cases} 11=A+B+C & (1) \\ -1=B-C & (2) \\ -2=-A & (3) \end{cases}$$

$$\begin{cases} 11=A+B+C & (1) \\ -1=B-C & (2) \\ 2=-A & (4) \end{cases}$$

$B+C=9$ from (1) with $A=2$

$\underline{B-C=-1}$ (2)

$2B \quad =8$

$B=4$

$4-C=-1$

$C=5$

$\dfrac{11x^2-x-2}{x^3-x}=\dfrac{2}{x}+\dfrac{4}{x-1}+\dfrac{5}{x+1}$ [9.4]

**36.** $\dfrac{x^4+x^3+4x^2+x+3}{(x^2+1)^2}=\dfrac{x^4+x^3+4x^2+x+3}{x^4+2x^2+1}$

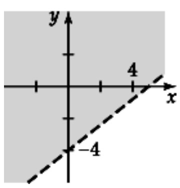
$=1+\dfrac{x^3+2x^2+x+2}{(x^2+1)^2}$

$=1+\dfrac{x^2(x+2)+1(x+2)}{(x^2+1)^2}$

$=1+\dfrac{(x^2+1)(x+2)}{(x^2+1)^2}$

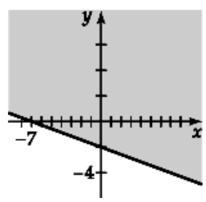
$=1+\dfrac{x+2}{x^2+1}$ [9.4]

**37.**



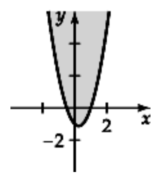

[9.5]

**38.**



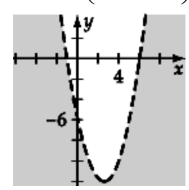

[9.5]

**39.**



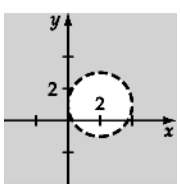

[9.5]

**40.** vertex $\left(\dfrac{5}{2}, -\dfrac{49}{4}\right)$



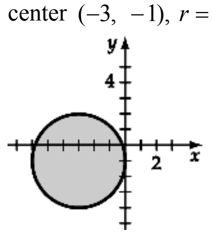

[9.5]

**41.**



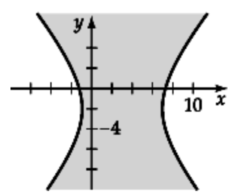

[9.5]

**42.** center $(-3, -1)$, $r=3$



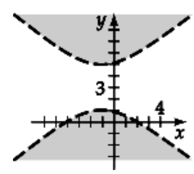

[9.5]

**43.**



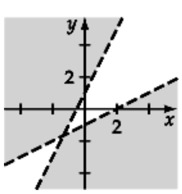

[9.5]

**44.**



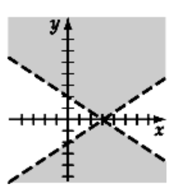

[9.5]

**45.**



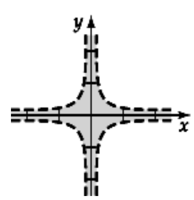

[9.5]

**46.**



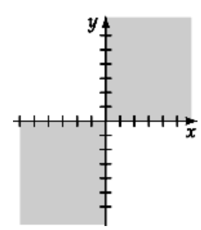

[9.5]

**47.**

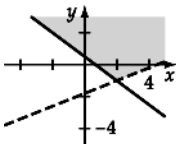



[9.5]

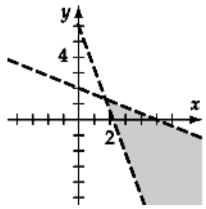
**48.**



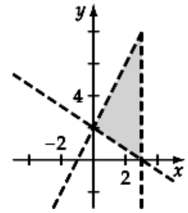

[9.5]

**49.** 

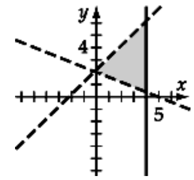
[9.5]

**50.** 

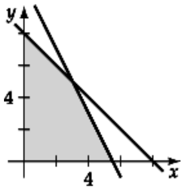
[9.5]

**51.** 

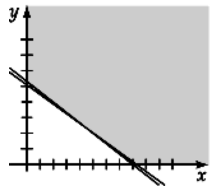
[9.5]

**52.** 

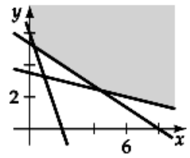
[9.5]

**53.** 

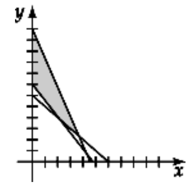
[9.5]

**54.** 

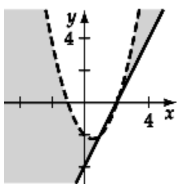
[9.5]

**55.** 

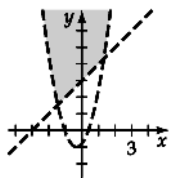
[9.5]

**56.** 

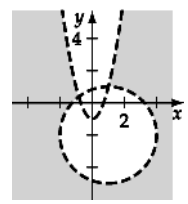
[9.5]

**57.** 

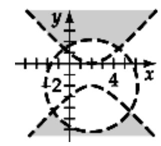
[9.5]

**58.** 

[9.5]

**59.** 

[9.5]

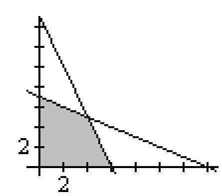
**60.** 

[9.5]

**61.** $P = 2x + 2y$ [9.6]

$$\begin{cases} x + 2y \le 14 \\ 5x + 2y \le 30 \\ x \le 0, y \le 0 \end{cases}$$

| $P = 2x + 2y$ | | |
|---|---|---|
| (0, 7) | 14 | |
| (6, 0) | 12 | |
| (4, 5) | 18 | maximum |
| (0, 0) | 0 | |

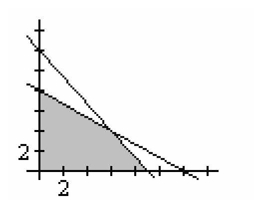



The maximum is 18 at (4, 5).

**62.** $P = 4x + 5y$ [9.6]

$$\begin{cases} 2x + 3y \le 24 \\ 4x + 3y \le 36 \\ x \ge 0, \ y \ge 0 \end{cases}$$

| $4x + 5y = P$ | | |
|---|---|---|
| (0, 8) | 40 | |
| (6, 4) | 44 | maximum |
| (9, 0) | 36 | |
| (0, 0) | 0 | |

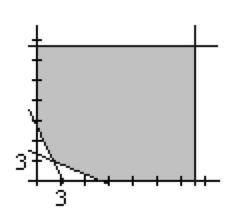



The maximum is 44 at (6, 4).

**63.** $P = 4x + y$ [9.6]

$$\begin{cases} 5x + 2y \ge 16 \\ x + 2y \ge 8 \\ 0 \le x \le 20 \\ 0 \le y \le 20 \end{cases}$$

| $P = 4x + y$ | | |
|---|---|---|
| (0, 8) | 8 | minimum |
| (2, 3) | 11 | |
| (8, 0) | 32 | |
| (20, 0) | 80 | |
| (20, 20) | 100 | |
| (0, 20) | 20 | |

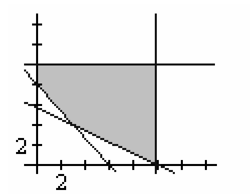



The minimum is 8 at (0, 8).

**64.** $P = 2x + 7y$ [9.6]

$$\begin{cases} 4x + 3y \ge 24 \\ 4x + 7y \ge 40 \\ 0 \le x \le 10 \\ 0 \le y \le 20 \end{cases}$$

| $P = 2x + 7y$ | | |
|---|---|---|
| (0, 8) | 56 | |
| (0, 10) | 70 | |
| (3, 4) | 34 | |
| (10, 0) | 20 | minimun |
| (10, 10) | 90 | |

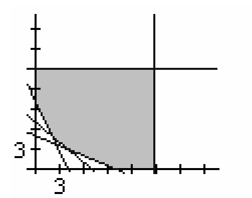



The minimum is 20 at (10, 0).

**65.** $P = 6x + 3y$ [9.6]

$$\begin{cases} 5x + 2y \ge 20 \\ x + \quad y \ge 7 \\ x + 2y \ge 10 \\ 0 \le x \le 15, \quad 0 \le y \le 10 \end{cases}$$

| $6x + 3y = P$ | | |
|---|---|---|
| (0, 10) | 30 | |
| (2, 5) | 27 | minimum |
| (4, 3) | 33 | |
| (10, 0) | 60 | |
| (0, 15) | 45 | |
| (15, 15) | 135 | |
| (15, 0) | 90 | |

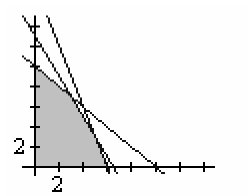



The minimum is 27 at (2, 5).

**66.** $P = 5x + 4y$ [9.6]

$$\begin{cases} x + y \le 10 \\ 2x + y \le 13 \\ 3x + y \le 18 \\ x \ge 0, \quad y \ge 0 \end{cases}$$

| $5x + 4y = P$ | | |
|---|---|---|
| (0, 10) | 40 | |
| (3, 7) | 43 | maximum |
| (5, 3) | 37 | |
| (6, 0) | 30 | |
| (0, 0) | 0 | |




The maximum is 43 at (3, 7).

**67.** $y = ax^2 + bx + c$

$0 = a(1)^2 + b(1) + c$
$5 = a(-1)^2 + b(-1) + c$
$3 = a(2)^2 + b(2) + c$

$$\begin{cases} a+b+c=0 & (1) \\ a-b+c=5 & (2) \\ 4a+2b+c=3 & (3) \end{cases}$$

$$\begin{array}{ll} a+b+c=0 & (1) \\ \underline{-a+b-c=-5} & -1 \text{ times } (2) \\ 2b \quad = -5 & (4) \end{array}$$

$$\begin{array}{ll} a-b+c=5 & (2) \\ \underline{-4a-2b-c=-3} & -1 \text{ times } (3) \\ -3a-3b \quad = 2 & (5) \end{array}$$

$$\begin{cases} a+b+c=0 & \\ 2b=-5 & (4) \\ -3a-3b=2 & (5) \end{cases}$$

$2b = -5$, $b = -\frac{5}{2}$ $\qquad -3a - 3\left(-\frac{5}{2}\right) = 2$, $a = \frac{11}{6}$ $\qquad \frac{11}{6} - \frac{5}{2} + c = 0$, $c = \frac{2}{3}$

The equation of the graph that passes through the three points is

$y = \frac{11}{6}x^2 - \frac{5}{2}x + \frac{2}{3}$. [9.2]

**68.** $x^2 + y^2 + ax + by + c = 0$

$4^2 + 2^2 + 4a + 2b + c = 0$
$0^2 + 1^2 + 0a + 1b + c = 0$
$3^2 + (-1)^2 + 3a + (-1)b + c = 0$

$$\begin{cases} 4a+2b+c=-20 & (1) \\ b+c=-1 & (2) \\ 3a-b+c=-10 & (3) \end{cases}$$

$$\begin{array}{ll} -12a - 6b + 3c = 60 & -3 \text{ times } (1) \\ \underline{12a - 4b + 4c = -40} & 4 \text{ times } (3) \\ -10b + c = 20 & (4) \end{array}$$

$$\begin{cases} 4a+2b+c=-20 & \\ b+c=-1 & \\ -10b+c=20 & (4) \end{cases}$$

$$\begin{array}{ll} b+c=-1 & (2) \\ \underline{10b-c=-20} & -1 \text{ times } (4) \\ 11b \quad =-21 & (5) \\ b \quad = -\frac{21}{11} & \end{array}$$

$$\begin{cases} 4a+2b+c=-20 & \\ b+c=-1 & \\ 11b \quad =-21 & (5) \end{cases}$$

$-\frac{21}{11} + c = -1$, c $= \frac{10}{11}$ $\qquad 4a + 2\left(-\frac{21}{11}\right) + \left(\frac{10}{11}\right) = -20$, $4a = -\frac{188}{11}$, $a = -\frac{47}{11}$

The equation of the circle that passes through the three points is

$x^2 + y^2 - \frac{47}{11}x - \frac{21}{11}y + \frac{10}{11} = 0$. [9.2]

**69.** $x = ax + by + c$

$$2 = 2a + b + c$$
$$0 = 3a + b + c$$
$$-2 = -2a - 3b + c$$

$$\begin{cases} 2a + b + c = 2 & (1) \\ 3a + b + c = 0 & (2) \\ -2a - 3b + c = -2 & (3) \end{cases}$$

$$\begin{array}{rl} 6a + 3b + 3c = 6 & \text{3 times (1)} \\ \underline{-6a - 2b - 2c = 0} & \text{- 2 times (2)} \\ b + c = 6 & (4) \end{array}$$

$$\begin{array}{rl} 2a + b + c = 2 & (1) \\ \underline{-2a - 3b + c = -2} & (3) \\ -2b + 2c = 0 & \\ -b + c = 0 & (5) \end{array}$$

$$\begin{cases} 2a + b + c = 2 & \\ b + c = 6 & (4) \\ -b + c = 0 & (5) \end{cases}$$

$$\begin{array}{rl} b + c = 6 & (4) \\ \underline{-b + c = 0} & (5) \\ 2c = 6 & \\ c = 3 & (6) \end{array}$$

$$\begin{cases} 2a + b + c = 2 & \\ b + c = 6 & \\ c = 3 & (6) \end{cases}$$

$$b + 3 = 6 \qquad 2a + 3 + 3 = 2$$
$$b = 3 \qquad 2a = -4$$
$$a = -2$$

The equation of the graph that passes through the three points is $z = -2x + 3y + 3$. [9.2]

**70.** $x =$ amount of 20% acid

$$0.20x + 0.10(10) = 0.16(x + 10)$$
$$0.20x + 1 = 0.16x + 1.6$$
$$0.04x = 0.6$$
$$x = 15 \text{ liters}$$

[9.1]

**71.** Rate flying with the wind: $r+w$
Rate flying against the wind: $r-w$

$$\begin{cases} 855=(r+w)5 \\ 575=(r-w)5 \end{cases} \qquad \begin{aligned} 171&=r+w \\ \underline{115}&\underline{=r-w} \end{aligned}$$

$$171=143+w \qquad 286=2r$$
$$28=w \qquad 143=r$$

Rate of the wind is 28 mph.
Rate of the plane in calm air is 143 mph.
[9.1]

**72.** $x=$ number of nickels
$y=$ number of dimes
$z=$ number of quarters

$$\begin{cases} x+\ \ y+\ \ z=10 & (1) \\ 5x+10y+25z=125 & (2) \end{cases}$$

$$\begin{cases} x+y+\ \ z=10 \\ \quad y+4z=15 \end{cases}$$

$$\begin{aligned} -5x-\ 5y-\ 5z&=-50 \quad -5 \text{ times } (1) \\ 5x+10y+25z&=125 \\ \hline 5y+20z&=75 \\ y+\ \ 4z&=15 \end{aligned}$$

Solving the last equation for $y$, we have $y=-4z+15$. Subsitute this into Eq. (1) and solve for $x$.

$$x+(-4z+15)+z=10$$
$$x-3z+15=10$$
$$x=3x-5$$

Since $x$, $y$, and $z$ must all be positive integers, from $x=3z-5$, we have $x\ge 2$. For $y=-4x+15$, $x\le 3$. Thus $z=2$ or $z=3$.

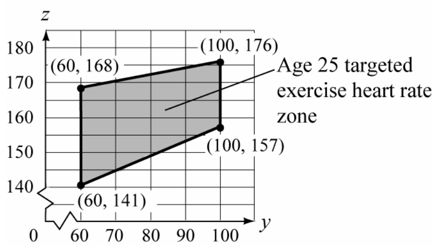
When $z=2$, $x=1$, $y=7$, there could be 1 nickel, 7 dimes and 2 quarters.

When $z=3$, $x=4$, $y=3$, there could be 4 nickels, 3 dimes and 3 quarters. [9.2]

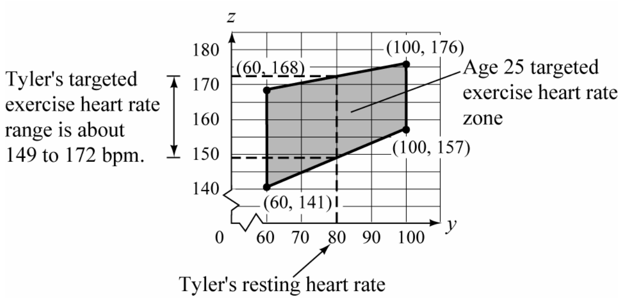
**73.** Given $(a, b, c)$ with $ab=c$, $ac=b$. [9.2]
If $ab=c$, then $b\cdot(ab)=a$ and $b^2=1$, $b=\pm1$, or $a=0$.
If $bc=a$, then $bc\cdot c=b$ and $c^2=1$, $c=\pm1$, or $b=0$
If $ac=b$, then $a\cdot ac=c$ and $a^2=1$, $a\pm1$, or $c=0$
The ordered triples are $(1,\ 1,\ 1)$, $(1,\ -1,\ -1)$, $(-1,\ -1,\ 1)$, $(-1,\ 1,\ -1)$, $(0,\ 0,\ 0)$.

## Quantitative Reasoning

**QR1.**




**QR2.**




**QR3.** Answers will vary.

**QR4.** Answers will vary.

## Chapter Test

**1.** $\begin{cases} 3x+2y=-5 & (1) \\ 2x-5y=-16 & (2) \end{cases}$

$$\begin{aligned} 15x+10y &= -25 \quad 5 \text{ times } (1) \\ 4x-10y &= -32 \quad 2 \text{ times } (2) \\ \hline 19x &= -57 \\ x &= -3 \end{aligned}$$

$$\begin{aligned} 3(-3)+2y &= -5 \\ 2y &= 4 \\ y &= 2 \end{aligned}$$

The solution is (–3, 2). [9.1]

**2.** $\begin{cases} x-\frac{1}{2}y=3 & (1) \\ 2x-\ \ y=6 & (2) \end{cases}$

$$\begin{aligned} -2x+y &= -6 \quad -2 \text{ times } (1) \\ 2x-y &= -6 \\ \hline 0 &= 0 \end{aligned}$$

The system of equations is dependent.

Let $y=c$.

$2x-c=6$

$x=\frac{6+c}{2}$

The solution is $\left(\frac{6+c}{2}, c\right)$. [9.1]

**3.** $\begin{cases} x+3y-\ \ z=8 & (1) \\ 2x-7y+2z=1 & (2) \\ 4x-\ \ y+3z=13 & (3) \end{cases}$

$$\begin{array}{rl} -2x-6y+2z=-16 & -2 \text{ times} (1) \\ 2x-7y+2z=\ \ \ 1 & (2) \\ \hline -13y+4z=-15 & (4) \end{array}$$

$$\begin{array}{rl} -4x-12y+4z=-32 & -4 \text{ times} (1) \\ 4x-\ \ \ \ y+3z=\ \ 13 & (3) \\ \hline -13y+7z=-19 & (5) \end{array}$$

$$\begin{cases} x+3y-\ \ z=\ \ \ \ 8 & \\ -13y+4z=-15 & (4) \\ -13y+7z=\ \ 19 & (5) \end{cases}$$

$$\begin{array}{rl} -13y+4z=-15 & (4) \\ 13y-7z=\ \ 19 & -1 \text{ times} (5) \\ \hline -3z=\ \ 4 & \end{array}$$

$$z=-\frac{4}{3} \qquad (6)$$

$$\begin{cases} x+3y-\ \ z=\ \ \ \ 8 \\ -13y+4z=-15 \\ z=\ \ -\frac{4}{3} \end{cases}$$

$$-13y+4\left(-\frac{4}{3}\right)=-15 \qquad x+3\left(\frac{29}{39}\right)-\left(-\frac{4}{3}\right)=8$$

$$y=\frac{29}{39} \qquad x=\frac{173}{39}$$

The solution is $\left(\frac{173}{39}, \frac{29}{39}, -\frac{4}{3}\right)$. [9.2]

**4.** $\begin{cases} 3x-2y+\ \ z=2 & (1) \\ x+2y-2z=1 & (2) \\ 4x\qquad\ \ -z=3 & (3) \end{cases}$

$$\begin{array}{rl} 3x-2y+\ \ z=\ \ \ 2 & (1) \\ -3x-6y+6z=-3 & -3 \text{ times } (2) \\ \hline -8y+7z=-1 & (5) \end{array}$$

$$\begin{array}{rl} -4x-8y+\ \ 8z=-4 & -4 \text{ times } (2) \\ 4x\qquad -\ \ \ z=\ -3 & (3) \\ \hline -8y+7z=-1 & (5) \end{array}$$

$$\begin{cases} 3x-2y+\ \ z=\ \ \ 2 & \\ -8y+7z=-1 & (4) \\ -8y+7z=-1 & (5) \end{cases}$$

$$\begin{array}{rl} 8y-7z=\ \ \ 1 & -1 \text{ times } (4) \\ -8y+7z=-1 & (5) \\ \hline 0=0 & (6) \end{array}$$

$$\begin{cases} 3x-2y+\ \ z=\ \ \ 2 & \\ -8y+7z=-1 & \\ 0=0 & (6) \end{cases}$$

The system of equations is dependent.

Let $z=c$. $\qquad -8y+7(c)=-1 \qquad 4x-c=3$

$$y=\frac{7c+1}{8} \qquad x=\frac{c+3}{4}$$

The solution is $\left(\frac{c+3}{4}, \frac{7c+1}{8}, c\right)$.

[9.2]

**5.** $\begin{cases} 2x - 3y + z = -1 & (1) \\ x + 5y - 2z = 5 & (2) \end{cases}$

$$\begin{array}{rl} 2x - 3y + z = -1 & (1) \\ -2x - 10y + 4z = -10 & -2 \text{ times } (2) \\ \hline -13y + 5z = -11 & (3) \end{array}$$

$\begin{cases} 2x - 3y + z = -1 \\ -13y + 5z = -11 & (3) \end{cases}$

The system of equations is dependent.

Let $z = c$. $\quad -8y + 7(c) = -1 \quad 4x - c = 3$

$y = \dfrac{7c+1}{8} \qquad x = \dfrac{c+3}{4}$

The solution is $\left(\dfrac{c+10}{13}, \dfrac{5c+11}{13}, c\right)$. [9.2]

**6.** $\begin{cases} 4x + 2y + z = 0 & (1) \\ x - 3y - 2z = 0 & (2) \\ 3x + 5y + 3z = 0 & (3) \end{cases}$

$$\begin{array}{rl} 4x + 2y + z = 0 & (1) \\ -4x + 12y + 8z = 0 & -4 \text{ times } (2) \\ \hline 14y + 9z = 0 & (4) \end{array}$$

$$\begin{array}{rl} -3x + 9y + 6z = 0 & -3 \text{ times } (2) \\ 3x + 5y + 3z = 0 & (3) \\ \hline 14y + 9z = 0 & (5) \end{array}$$

$\begin{cases} 4x + 2y + z = 0 \\ 14y + 9z = 0 & (4) \\ 14y + 9z = 0 & (5) \end{cases}$

$$\begin{array}{rl} 14y + 9z = 0 & \\ -14y - 9z = 0 & -1 \text{ times } (5) \\ \hline 0 = 0 & (6) \end{array}$$

$\begin{cases} 4x + 2y + z = 0 \\ 14y + 9z = 0 \\ 0 = 0 \end{cases}$

The system of equations is dependent.

Let $z = c$. $\quad 14y + 9c = 0 \quad x - 3\left(-\dfrac{9c}{14}\right) - 2c = 0$

$y = -\dfrac{9c}{14} \qquad x = \dfrac{c}{14}$

The solution is $\left(\dfrac{c}{14}, -\dfrac{9c}{14}, c\right)$. [9.2]

**7.** $\begin{cases} y = x + 3 & (1) \\ y = x^2 + x - 1 & (2) \end{cases}$

Set the expressions equal to each other.

$x^2 + x - 1 = x + 3$

$x^2 = 4$

$x = \pm 2$

Substitute for $x$ in Eq. (1).

$y = 2 + 3 \qquad y = -2 + 3$

$y = 5 \qquad y = 1$

The solutions are (2, 5) and (–2, 1). [9.3]

**8.** $\begin{cases} y = x^2 - x - 3 & (1) \\ y = 2x^2 + 2x - 1 & (2) \end{cases}$

Set the expressions equal to each other.

$x^2 + 2x - 1 = x^2 - x - 3$

$x^2 + 3x + 2 = 0$

$(x+2)(x+1) = 0$

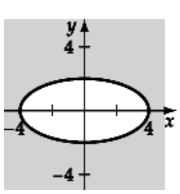
Substitute for $x$ in Eq. (1).

$y = (-2)^2 - (-2) - 3 \qquad y = (-1)^2 - (-1) - 3$

$y = 3 \qquad y = -1$

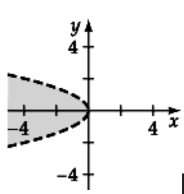
The solutions are (–2, 3) and (–1, –1). [9.3]

**9.**



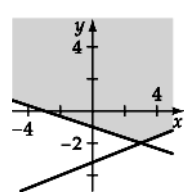

[9.5]

**10.**




[9.5]

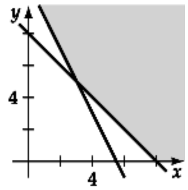
**11.**




[9.5]

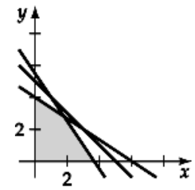
**12.** No solution. The solution set is empty. [9.5]

**13.**




[9.5]

**14.**




[9.5]

**15.**

$$\frac{3x-5}{x^2-3x-4}=\frac{3x-5}{(x-4)(x+1)}=\frac{A}{x-4}+\frac{B}{x+1}$$

$$3x-5=A(x+1)+B(x-4)$$
$$3x-5=Ax+A+Bx-4B$$

$$\begin{cases} 3=A+B \\ -5=A-4B \end{cases}$$

$$3=A+B$$
$$\underline{5=-A+4B}$$

$$8=5B \qquad 3=A+\frac{8}{5}$$

$$\frac{8}{5}=B \qquad \frac{7}{5}=A$$

$$\frac{3x-5}{(x-4)(x+1)}=\frac{7}{5(x-4)}+\frac{8}{5(x+1)}$$ [9.4]

**16.**

$$\frac{2x+1}{x(x^2+1)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$$

$$2x+1=A(x^2+1)+(Bx+C)x$$
$$2x+1=Ax^2+A+Bx^2+Cx$$

$$\begin{cases} 0=A+B \\ 2=C \\ 1=A \end{cases} \qquad \begin{aligned} 0&=1+B \\ -1&=B \end{aligned}$$

$$\frac{2x+1}{x(x^2+1)}=\frac{1}{x}+\frac{-x+2}{x^2+1}$$ [9.4]

**17.**

$$x=20+y$$
$$2x+\pi y=554.16$$

Using substitution,

$$2(20+y)+\pi y=554.16$$
$$40+2y+\pi y=554.16$$
$$y(2+\pi)=514.16$$
$$y=\frac{514.16}{2+\pi}$$
$$y\approx 100$$
$$x=20+y=20+100=120$$

Length is 120 m and width is 100 m. [9.1]

**18.** $x$ = rate of first hour
$y$ = rate of $n$ additional half hour or portion of the half-hour

$$x+6y=14.50$$
$$x+8y=18.00$$

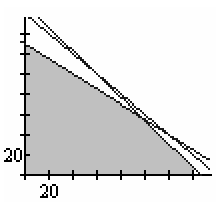
$$-x-6y=-14.50$$
$$\underline{x+8y=18.00}$$
$$14y=3.50$$
$$y=1.75$$
$$x+6(1.75)=14.50$$
$$x=4$$

The fee for the first hour is \$4; the fee for each additional half-hour or portion of the half-hour is \$1.75. [9.1]

**19.** $x$ = Acres of oats
$y$ = Acres of barley
Constraints

$$\begin{cases} x+\ \ y\le 160 \\ 15x+13y\le 2200 \\ 15x+20y\le 2600 \\ x\le 0, \quad y\ge 0 \end{cases}$$

| | maximize $p=120x+150y$ | |
|---|---|---|
| $(0, 130)$ | \$19,500 | |
| $\left(\frac{680}{7}, \frac{400}{7}\right)$ | \$20,228.57 | maximum |
| $\left(146\frac{2}{3}, 0\right)$ | \$17,600 | |
| $(0, 0)$ | 0 | |




To maximize profit, $\frac{680}{7}$ acres of oats and $\frac{400}{7}$ acres of barley must be planted.

[9.6]

**20.**

$$x^2+y^2+ax+by+c=0$$
$$3^2+5^2+3a+5b+c=0$$
$$(-3)^2+(-3)^2-3a-3b+c=0$$
$$4^2+4^2+4a+4b+c=0$$

$$\begin{cases} 3a+5b+c=-34 & (1) \\ -3a-3b+c=-18 & (2) \\ 4a+4b+c=-32 & (3) \end{cases}$$

$$\begin{array}{rl} 3a+5b+c=-34 & (1) \\ \underline{-3a-3b+c=-18} & (2) \\ 2b+2c=-52 & \\ b+c=-26 & (4) \end{array}$$

$$\begin{array}{rl} -12a-12b+4c=-72 & \text{4 times } (2) \\ \underline{12a+12b+3c=-96} & \text{3 times } (3) \\ 7c=-168 & \\ c=-24 & (5) \end{array}$$

$$\begin{cases} 3a+5b+c=-34 & \\ b+c=-26 & (4) \\ c=-24 & (5) \end{cases}$$

$c=-24$ $\quad b+(-24)=-26$ $\quad 3a+5(-2)-24=-34$

$\qquad\qquad b=-2 \qquad\qquad a=0$

The equation of the circle is $x^2+y^2-2y-24=0$. [9.2]

## Cumulative Review

**1.** $f(x)=-x^2+2x-4$ [2.4]

$$=-(x^2-2x)+4$$
$$=-(x^2-2x+1)+4-1$$
$$=-(x-1)^2+3$$

Vertex (1, –3), parabola opens down.
Range: $\{y \mid y \le -3\}$

**2.** $\log_6(x-5)+3\log_6(2x)=\log_6(x-5)+\log_6(2x)^3$ [4.4]

$$=\log_6(x-5)(2x)^3$$
$$=\log[8x^3(x-5)]$$

**3.** Vertex = (4, 2), point (–1, 1), axis of symmetry $x=4$

$$(x-4)^2=4p(y-2)$$
$$(-1-4)^2=4p(1-2)$$
$$25=4p(-1)$$
$$p=-\frac{25}{4}$$
$$(x-4)^2=4\left(-\frac{25}{4}\right)(y-2)$$
$$(x-4)^2=-25(y-2)$$ [8.1]

**4.** even [2.5]

**5.** $\log x - \log(2x-3) = 2$ [4.5]

$$\log \frac{x}{2x-3} = 2$$
$$\frac{x}{2x-3} = 10^2$$
$$x = 100(2x-3)$$
$$x = 200x - 300$$
$$-199x = -300$$
$$x = \frac{300}{199}$$

**6.** vertices (2, 2) and (10, 2), eccentricity = 3 [8.3]

Length of transverse axis = distance between vertices

$2a = |10-2| = 8$

$a = 4$ and $a^2 = 16$

Center (midpoint of transverse axis) is $\left(\frac{2+10}{2}, \frac{2+2}{2}\right) = (6,2)$

Therefore, $h = 6$ and $k = 2$.

Since both vertices lie on the horizontal line $x = 2$, the transverse axis is parallel to the $x$-axis.

Since $e = \frac{c}{a}, c = ae = (4)(3) = 12$

Because $b^2 = c^2 - a^2, b^2 = 144 - 16 = 128$

Substituting $h$, $k$, $a^2$, $b^2$ into the standard equation yields

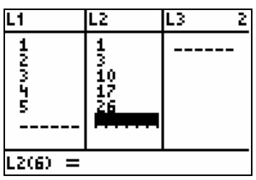
$$\frac{(x-6)^2}{16} - \frac{(y-2)^2}{128} = 1.$$

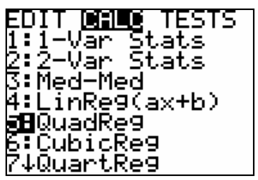
**7.** $g\left(-\frac{1}{2}\right) = \dfrac{-\frac{1}{2} - 2}{-\frac{1}{2}} = \dfrac{-\frac{5}{2}}{-\frac{1}{2}} = 5$ [2.2]

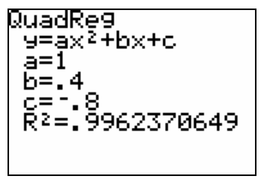
**8.** $f(-2) \cdot g(-2) = [(-2)^2 - 1][(-2)^2 - 4(-2) - 2]$ [2.6]

$$= [4-1][4+8-2]$$
$$= [3][10]$$
$$= 30$$

**9.**




$y = x^2 + 0.4x - 0.8$

**10.** $(x+2)(x-3i)(x+3i) = (x+2)(x^2+9)$ [3.4]

$$= x^3 + 2x^2 + 9x + 18$$

**11.** $Q(r) = \dfrac{2}{1-r}$ [4.1]

$$r = \frac{2}{1-Q}$$
$$r(1-Q) = 2$$
$$r - rQ = 2$$
$$-rQ = 2 - r$$
$$Q = \frac{2-r}{-r}$$
$$Q^{-1}(r) = \frac{r-2}{r}$$

**12.**

$$\begin{array}{r} 2x+1 \\ x^2-x-1 \overline{) 2x^3 - x^2 - 2} \\ \underline{2x^3 - 2x^2} \\ x^2 - 2 \\ \underline{x^2 - x - 1} \\ x+1 \end{array}$$

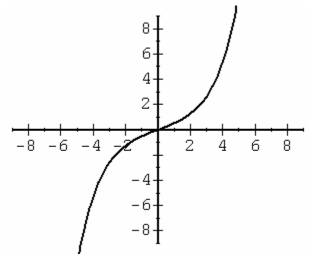
$$H(x) = \frac{2x^3 - x^2 - 2}{x^2 - x - 1} = 2x + 1 + \frac{x+1}{x^2 - x - 1}$$

Slant asymptote: $y = 2x + 1$ [3.5]

**13.** $g[f(1)] = g[2^1]$ [4.2]

$$= g[2]$$
$$= 3^{2(2)}$$
$$= 3^4$$
$$= 81$$

**14.**




[4.2]

**15.** $\cos\left(\frac{11\pi}{6}\right) = \frac{\sqrt{3}}{2}$ [5.4]

**16.** [7.2]

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$B = \cos^{-1}\left(\frac{b^2 - a^2 - c^2}{-2ac}\right)$$
$$= \cos^{-1}\left(\frac{(25)^2 - (30)^2 - (35)^2}{-2(30)(35)}\right)$$
$$\approx 44^\circ$$

**17.** [6.2]

$$\sin 15^\circ = \sin(45^\circ - 30^\circ)$$
$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$
$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$$
$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

**18.** $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j},\ \mathbf{w} = \mathbf{i} + 4\mathbf{j}$ [7.3]

$$\cos\alpha = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{(3\mathbf{i} - 2\mathbf{j})\cdot(\mathbf{i} + 4\mathbf{j})}{\left(\sqrt{(3)^2 + (-2)^2}\right)\left(\sqrt{(1)^2 + (4)^2}\right)}$$
$$= \frac{3 - 8}{\sqrt{13}\sqrt{17}} = \frac{-5}{\sqrt{221}}$$
$$\alpha = \cos^{-1}\left(\frac{-5}{\sqrt{221}}\right) \approx 109.7^\circ$$

**19.** [6.5]

$$\cos^{-1} x + \tan^{-1} x = \frac{\pi}{2}$$
$$\cos\left[\cos^{-1} x + \tan^{-1} x\right] = \cos\left(\frac{\pi}{2}\right)$$
$$\cos(\alpha + \beta) = 0$$
$$\cos\alpha\cos\beta - \sin\alpha\sin\beta = 0$$

$$\cos\alpha = x,\ \sin\alpha = \sqrt{1 + x^2}$$
$$\cos\beta = \sqrt{1 - x^2},\ \sin\beta = \frac{x}{\sqrt{1 + x^2}}$$

$$x\left(\sqrt{1 - x^2}\right) - \left(\sqrt{1 + x^2}\right)\cdot\frac{x}{\sqrt{1 + x^2}} = 0$$
$$x\left(\sqrt{1 - x^2}\right) - x = 0$$
$$x\left(\sqrt{1 - x^2} - 1\right) = 0$$

$$x = 0$$

$$\sqrt{1 - x^2} - 1 = 0$$
$$\sqrt{1 - x^2} = 1$$
$$1 - x^2 = 1$$
$$0 = x^2$$
$$0 = x$$

**20.** $\left(10, \frac{5\pi}{4}\right)$

$$x = 10\cos\frac{5\pi}{4} = -10\frac{\sqrt{2}}{2} = -5\sqrt{2}$$
$$y = 10\sin\frac{5\pi}{4} = -10\frac{\sqrt{2}}{2} = -5\sqrt{2}$$

The rectangular coordinates are $(-5\sqrt{2}, -5\sqrt{2})$. [8.5]

# Chapter 10
# Matrices

## Section 10.1

**1.** $\left[\begin{array}{ccc|c} 2 & -3 & 1 & 1 \\ 3 & -2 & 3 & 0 \\ 1 & 0 & 5 & 4 \end{array}\right], \begin{bmatrix} 2 & -3 & 1 \\ 3 & -2 & 3 \\ 1 & 0 & 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$

**2.** $\left[\begin{array}{ccc|c} 0 & -3 & 2 & 3 \\ 2 & -1 & 0 & -1 \\ 3 & -2 & 3 & 4 \end{array}\right], \begin{bmatrix} 0 & -3 & 2 \\ 2 & -1 & 0 \\ 3 & -2 & 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$

**3.** $\left[\begin{array}{cccc|c} 2 & -3 & -4 & 1 & 2 \\ 0 & 2 & 1 & 0 & 2 \\ 1 & -1 & 2 & 0 & 4 \\ 3 & -3 & -2 & 0 & 1 \end{array}\right], \begin{bmatrix} 2 & -3 & -4 & 1 \\ 0 & 2 & 1 & 0 \\ 1 & -1 & 2 & 0 \\ 3 & -3 & -2 & 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 4 \\ 1 \end{bmatrix}$

**4.** $\left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & -2 \\ 2 & 0 & 1 & -2 & 1 \\ 3 & 0 & 0 & -2 & 3 \\ -1 & 3 & -1 & 0 & 3 \end{array}\right], \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 0 & 1 & -2 \\ 3 & 0 & 0 & -2 \\ -1 & 3 & -1 & 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \\ 3 \end{bmatrix}$

**5.** $\begin{bmatrix} 2 & -1 & 3 & -2 \\ 1 & -1 & 2 & 2 \\ 3 & 2 & -1 & 3 \end{bmatrix} R_1 \longleftrightarrow R_2 \begin{bmatrix} 1 & -1 & 2 & 2 \\ 2 & -1 & 3 & -2 \\ 3 & 2 & -1 & 3 \end{bmatrix} \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & -1 & 2 & 2 \\ 2 & -1 & -1 & -6 \\ 0 & 5 & -7 & -3 \end{bmatrix} \xrightarrow{-5R_2+R_3} \begin{bmatrix} 1 & -1 & 2 & 2 \\ 0 & 1 & -1 & -6 \\ 0 & 0 & -2 & 27 \end{bmatrix}$

$\xrightarrow{(-1/2)R_3} \begin{bmatrix} 1 & -1 & 2 & 2 \\ 0 & 1 & -1 & -6 \\ 0 & 0 & 1 & -27/2 \end{bmatrix}$

**6.** $\begin{bmatrix} 1 & 2 & 4 & 1 \\ 2 & 2 & 7 & 3 \\ 3 & 6 & 8 & -1 \end{bmatrix} \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & -2 & -1 & 1 \\ 0 & 0 & -4 & -4 \end{bmatrix} \xrightarrow{(-1/2)R_2} \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & 1 & 1/2 & -1/2 \\ 0 & 0 & -4 & -4 \end{bmatrix} \xrightarrow{(-1/4)R_3} \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & 1 & 1/2 & -1/2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

**7.** $\begin{bmatrix} 4 & -5 & -1 & 2 \\ 3 & -4 & 1 & -2 \\ 1 & -2 & -1 & 3 \end{bmatrix} R_1 \longleftrightarrow R_3 \begin{bmatrix} 1 & -2 & -1 & 3 \\ 3 & -2 & 1 & -2 \\ 4 & -5 & -1 & 2 \end{bmatrix} \xrightarrow[-4R_1+R_3]{-3R_1+R_2} \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 2 & 4 & -11 \\ 0 & 3 & 3 & -10 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 1 & 2 & -11/2 \\ 0 & 3 & 3 & -10 \end{bmatrix}$

$\xrightarrow{-3R_2+R_3} \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 1 & 2 & -11/2 \\ 0 & 0 & -3 & 13/2 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_3} \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 1 & 2 & -11/2 \\ 0 & 0 & 1 & -13/6 \end{bmatrix}$

**8.** $\begin{bmatrix} -2 & 1 & -1 & 3 \\ 2 & 2 & 4 & 6 \\ 3 & 1 & -1 & 2 \end{bmatrix} R_1 \longleftrightarrow R_2 \begin{bmatrix} 2 & 2 & 4 & 6 \\ -2 & 1 & -1 & 3 \\ 3 & 1 & -1 & 2 \end{bmatrix} \xrightarrow{(1/2)R_1} \begin{bmatrix} 1 & 1 & 2 & 3 \\ -2 & 1 & -1 & 3 \\ 3 & 1 & -1 & 2 \end{bmatrix} \xrightarrow[-3R_1+R_3]{2R_1+R_2} \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 3 & 3 & 9 \\ 0 & -2 & -7 & -7 \end{bmatrix}$

$\xrightarrow{(1/3)R_2} \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & -2 & -7 & -7 \end{bmatrix} \xrightarrow{2R_2+R_3} \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -5 & -1 \end{bmatrix} \xrightarrow{(-1/5)R_3} \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1/5 \end{bmatrix}$

**9.** $\begin{bmatrix}1 & -2 & 3 & -4\\ 3 & -6 & 10 & -14\\ 5 & -8 & 19 & -21\\ 2 & -4 & 7 & -10\end{bmatrix} \xrightarrow[\substack{-5R_1+R_3\\ -2R_1+R_4}]{-3R_1+R_2} \begin{bmatrix}1 & -2 & 3 & -4\\ 0 & 0 & 1 & -2\\ 0 & 2 & 4 & -1\\ 0 & 0 & 1 & -2\end{bmatrix} R_2 \longleftrightarrow R_3 \begin{bmatrix}1 & -2 & 3 & -4\\ 0 & 2 & 4 & -1\\ 0 & 0 & 1 & -2\\ 0 & 0 & 1 & -2\end{bmatrix} \xrightarrow{(1/2)R_2} \begin{bmatrix}1 & -2 & 3 & -4\\ 0 & 1 & 2 & -\frac{1}{2}\\ 0 & 0 & 1 & -2\\ 0 & 0 & 1 & -2\end{bmatrix}$

$\xrightarrow{-R_3+R_4} \begin{bmatrix}1 & -2 & 3 & -4\\ 0 & 1 & 2 & -\frac{1}{2}\\ 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 0\end{bmatrix}$

**10.** $\begin{bmatrix}2 & -1 & 3 & 2\\ 1 & 2 & -1 & 3\\ 3 & 5 & -2 & 2\\ 4 & 3 & 1 & 8\end{bmatrix} R_1 \longleftrightarrow R_2 \begin{bmatrix}1 & 2 & -1 & 3\\ 2 & -1 & 3 & 2\\ 3 & 5 & -2 & 2\\ 4 & 3 & 1 & 8\end{bmatrix} \xrightarrow{\substack{-2R_1+R_2\\ -3R_1+R_3\\ -4R_1+R_4}} \begin{bmatrix}1 & 2 & -1 & 3\\ 0 & -5 & 5 & -4\\ 0 & -1 & 1 & -7\\ 0 & -5 & 5 & -4\end{bmatrix} \xrightarrow{(1/5)R_2} \begin{bmatrix}1 & 2 & -1 & 3\\ 0 & 1 & -1 & \frac{4}{5}\\ 0 & -1 & 1 & -7\\ 0 & -5 & 5 & -4\end{bmatrix}$

$\xrightarrow{\substack{R_2+R_3\\ 5R_2+R_4}} \begin{bmatrix}1 & 2 & -1 & 3\\ 0 & 1 & -1 & \frac{4}{5}\\ 0 & 0 & 0 & -\frac{31}{5}\\ 0 & 0 & 0 & 0\end{bmatrix} \xrightarrow{-\frac{5}{31}R_3} \begin{bmatrix}1 & 2 & -1 & 3\\ 0 & 1 & -1 & \frac{4}{5}\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$

**11.** $\begin{bmatrix}1 & -3 & 4 & 2 & 1\\ 2 & -3 & 5 & -2 & -1\\ -1 & 2 & -3 & 1 & 3\end{bmatrix} \xrightarrow{\substack{-2R_1+R_2\\ R_1+R_3}} \begin{bmatrix}1 & -3 & 4 & 2 & 1\\ 0 & 3 & -3 & -6 & -3\\ 0 & -1 & 1 & 3 & 4\end{bmatrix} \xrightarrow{(1/3)R_2} \begin{bmatrix}1 & -3 & 4 & 2 & 1\\ 0 & 1 & -1 & -2 & -1\\ 0 & -1 & 1 & 3 & 4\end{bmatrix}$

$\xrightarrow{R_2+R_3} \begin{bmatrix}1 & -3 & 4 & 2 & 1\\ 0 & 1 & -1 & -2 & -1\\ 0 & 0 & 0 & 1 & 3\end{bmatrix}$

**12.** $\begin{bmatrix}2 & -1 & 3 & 2 & 2\\ 1 & -2 & 2 & 1 & -1\\ 3 & -5 & -1 & -2 & 3\end{bmatrix} R_1 \longleftrightarrow R_2 \begin{bmatrix}1 & -2 & 2 & 1 & -1\\ 2 & -1 & 3 & 2 & 2\\ 3 & -5 & -1 & -2 & 3\end{bmatrix} \xrightarrow{\substack{-2R_1+R_2\\ -3R_1+R_2}} \begin{bmatrix}1 & -2 & 2 & 1 & -1\\ 0 & 3 & -1 & 0 & 4\\ 0 & 1 & -7 & -5 & 6\end{bmatrix}$

$R_2 \longleftrightarrow R_3 \begin{bmatrix}1 & -2 & 2 & 1 & -1\\ 0 & 1 & -7 & -5 & 6\\ 0 & 3 & -1 & 0 & 4\end{bmatrix} \xrightarrow{-3R_2+R_3} \begin{bmatrix}1 & -2 & 2 & 1 & -1\\ 0 & 1 & -7 & -5 & 6\\ 0 & 0 & 20 & 15 & -14\end{bmatrix} \xrightarrow{(1/20)R_3} \begin{bmatrix}1 & -2 & 2 & 1 & -1\\ 0 & 1 & -7 & -5 & 6\\ 0 & 0 & 1 & \frac{3}{4} & -\frac{7}{10}\end{bmatrix}$

**13.** $\left[\begin{array}{ccc|c}1 & 2 & -2 & -2\\ 5 & 9 & -4 & -3\\ 3 & 4 & -5 & -3\end{array}\right] \xrightarrow{\substack{-5R_1+R_2\\ -3R_1+R_3}} \left[\begin{array}{ccc|c}1 & 2 & -2 & -2\\ 0 & -1 & 6 & 7\\ 0 & -2 & 1 & 3\end{array}\right] \xrightarrow{-1R_2} \left[\begin{array}{ccc|c}1 & 2 & -2 & 2\\ 0 & 1 & -6 & -7\\ 0 & -2 & 1 & 3\end{array}\right] \xrightarrow{2R_2+R_3} \left[\begin{array}{ccc|c}1 & 2 & -2 & -2\\ 0 & 1 & -6 & -7\\ 0 & 0 & -11 & -11\end{array}\right]$

$\xrightarrow{(-1/11)R_3} \left[\begin{array}{ccc|c}1 & 2 & -2 & -2\\ 0 & 1 & -6 & -7\\ 0 & 0 & 1 & 1\end{array}\right]$

$\begin{cases}x+2y-2z=-2\\ \quad y-6z=-7\\ \qquad z=1\end{cases} \quad \begin{array}{l}y-6(1)=-7\\ y=-1\end{array} \quad \begin{array}{l}x+2(-1)-2(1)=-2\\ x=2\end{array}$

The solution is (2, −1, 1).

**14.** $\left[\begin{array}{rrr|r} 1 & -3 & 1 & 8 \\ 2 & -5 & -3 & 2 \\ 1 & 4 & 1 & 1 \end{array}\right] \xrightarrow[-1R_1+R_3]{-2R_1+R_2} \left[\begin{array}{rrr|r} 1 & -3 & 1 & 8 \\ 0 & 1 & -5 & -14 \\ 0 & 7 & 0 & -7 \end{array}\right] \xrightarrow{-7R_2+R_3} \left[\begin{array}{rrr|r} 1 & -3 & 1 & 8 \\ 0 & 1 & -5 & -14 \\ 0 & 0 & 35 & 91 \end{array}\right] \xrightarrow{(1/35)R_3} \left[\begin{array}{rrr|r} 1 & -3 & 1 & 8 \\ 0 & 1 & -5 & -14 \\ 0 & 0 & 1 & 13/5 \end{array}\right]$

$$\begin{cases} x-37+z=8 \\ \quad y-5z=-14 \\ \qquad z=\frac{13}{5} \end{cases}$$

The solution is $\left(\frac{12}{5},\ -1,\ \frac{13}{5}\right)$.

**15.** $\left[\begin{array}{rrr|r} 3 & 7 & -7 & -4 \\ 1 & 2 & -3 & 0 \\ 5 & 6 & 1 & -8 \end{array}\right] R_1 \longleftrightarrow R_2 \left[\begin{array}{rrr|r} 1 & 2 & -3 & 0 \\ 3 & 7 & -7 & -4 \\ 5 & 6 & 1 & -8 \end{array}\right] \xrightarrow[-5R_1+R_3]{-3R_1+R_2} \left[\begin{array}{rrr|r} 1 & 2 & -3 & 0 \\ 0 & 1 & 2 & -4 \\ 0 & -4 & 16 & -8 \end{array}\right] \xrightarrow{4R_2+R_3} \left[\begin{array}{rrr|r} 1 & 2 & -3 & 0 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 24 & -24 \end{array}\right]$

$\xrightarrow{\frac{1}{24}R_3} \left[\begin{array}{rrr|r} 1 & 2 & -3 & 0 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 1 & -1 \end{array}\right]$

$$\begin{cases} x+2y-3z=0 \\ \quad y+2z=-4 \\ \qquad z=-1 \end{cases} \qquad \begin{aligned} y+2(-1)&=-4 \\ y&=-2 \end{aligned} \qquad \begin{aligned} x+2(-2)-3(-1)&=0 \\ x&=1 \end{aligned}$$

The solution is (1, –2, –1).

**16.** $\left[\begin{array}{rrr|r} 2 & -3 & 2 & 13 \\ 3 & -4 & -3 & 1 \\ 3 & 1 & -1 & 2 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{rrr|r} 2 & -3 & 2 & 13 \\ 1 & -1 & -5 & -12 \\ 3 & 1 & -1 & 2 \end{array}\right] R_1 \longleftrightarrow R_2 \left[\begin{array}{rrr|r} 1 & -1 & -5 & -12 \\ 2 & -3 & 2 & 13 \\ 3 & 1 & -1 & 2 \end{array}\right] \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{rrr|r} 1 & -1 & -5 & -12 \\ 0 & -1 & 12 & 37 \\ 0 & 4 & 14 & 38 \end{array}\right]$

$\xrightarrow{-1R_2} \left[\begin{array}{rrr|r} 1 & -1 & -5 & -12 \\ 0 & 1 & -12 & -37 \\ 0 & 4 & 14 & 38 \end{array}\right] \xrightarrow{-4R_2+R_3} \left[\begin{array}{rrr|r} 1 & -1 & -5 & -12 \\ 0 & 1 & -12 & -37 \\ 0 & 0 & 62 & 186 \end{array}\right] \xrightarrow{\frac{1}{62}R_3} \left[\begin{array}{rrr|r} 1 & -1 & -5 & -12 \\ 0 & 1 & -12 & -37 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$$\begin{cases} x-y-5z=-12 \\ \quad y-12z=-37 \\ \qquad z=3 \end{cases} \qquad \begin{aligned} y-12(3)&=-37 \\ y&=-1 \end{aligned} \qquad \begin{aligned} x-(-1)-5(3)&=-12 \\ x&=2 \end{aligned}$$

The solution is (2, −1, 3).

**17.** $\left[\begin{array}{rrr|r} 1 & 2 & -2 & 3 \\ 5 & 8 & -6 & 14 \\ 3 & 4 & -2 & 8 \end{array}\right] \xrightarrow[-3R_1+R_3]{-5R_1+R_2} \left[\begin{array}{rrr|r} 1 & 2 & -2 & 3 \\ 0 & -2 & 4 & -1 \\ 0 & -2 & 4 & -1 \end{array}\right] \xrightarrow{(1/2)R_2} \left[\begin{array}{rrr|r} 1 & 2 & -2 & 3 \\ 0 & 1 & -2 & 1/2 \\ 0 & -2 & 4 & -1 \end{array}\right] \xrightarrow{2R_2+R_3} \left[\begin{array}{rrr|r} 1 & 2 & -2 & 3 \\ 0 & 1 & -2 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$\begin{cases} x+2y-2z=3 \\ \quad y-2z=\frac{1}{2} \end{cases}$$

$$y=2z+\frac{1}{2}$$

$$x+2\left(2z+\frac{1}{2}\right)-2z=3$$

$$x=2-2z$$

Let $z$ be any real number $c$. The solution is $\left(2-2c,\ 2c+\frac{1}{2},\ c\right)$.

**18.** $\left[\begin{array}{ccc|c} 3 & -5 & 2 & 4 \\ 1 & -3 & 2 & 4 \\ 5 & -11 & 6 & 12 \end{array}\right] R_1 \longleftrightarrow R_2 \left[\begin{array}{ccc|c} 1 & -3 & 2 & 4 \\ 3 & -5 & 2 & 4 \\ 5 & -11 & 6 & 12 \end{array}\right] \xrightarrow[-5R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 4 \\ 0 & 4 & -4 & -8 \\ 0 & 4 & -4 & -8 \end{array}\right] \xrightarrow{\frac{1}{4}R_2} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 4 \\ 0 & 1 & -1 & -2 \\ 0 & 4 & -4 & -8 \end{array}\right]$

$\xrightarrow{-4R_2+R_3} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 4 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\begin{cases} x-3y+2z=4 \\ y-z=-2 \end{cases}$ $\quad y=z-2 \quad$ $x-3(z-2)+2z=4$, $x=z-2$

Let $z$ be any real number $c$. The solution is $(c-2,\ c-2,\ c)$.

**19.** $\left[\begin{array}{ccc|c} 3 & 2 & -1 & 1 \\ 2 & 3 & -1 & 1 \\ 1 & -1 & 2 & 3 \end{array}\right] R_1 \longleftrightarrow R_3 \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 2 & 3 & -1 & 1 \\ 3 & 2 & -1 & 1 \end{array}\right] \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 5 & -5 & -5 \\ 0 & 5 & -7 & -8 \end{array}\right] \xrightarrow{\frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 5 & -7 & -8 \end{array}\right]$

$\xrightarrow{-5R_2+R_3} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -2 & -3 \end{array}\right] \xrightarrow{-\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3/2 \end{array}\right]$

$\begin{cases} x-y+2z=3 \\ y-z=-1 \\ z=\frac{3}{2} \end{cases}$ $\quad y-\frac{3}{2}=-1$, $y=\frac{1}{2} \quad$ $x-\frac{1}{2}+2\left(\frac{3}{2}\right)=3$, $x=\frac{1}{2}$

The solution is $\left(\frac{1}{2},\ \frac{1}{2},\ \frac{3}{2}\right)$.

**20.** $\left[\begin{array}{ccc|c} 2 & 5 & 2 & -1 \\ 1 & 2 & -3 & 5 \\ 5 & 12 & 1 & 10 \end{array}\right] R_1 \longleftrightarrow R_2 \left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 2 & 5 & 2 & -1 \\ 5 & 12 & 1 & 10 \end{array}\right] \xrightarrow[-5R_1+R_3]{-1R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & 1 & 8 & -11 \\ 0 & 2 & 16 & -15 \end{array}\right] \xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 5 \\ 0 & 1 & 8 & -11 \\ 0 & 0 & 0 & 7 \end{array}\right]$

$\begin{cases} x+2y-3z=5 \\ y+8z=-11 \\ 0z=7 \end{cases}$

The system of equations has no solution because the equation $0z=7$ has no solution.

**21.** $\left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 2 & -5 & -2 & 0 \\ 4 & -11 & 2 & 0 \end{array}\right] \xrightarrow[-4R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 1 & -6 & 0 \\ 0 & 1 & -6 & 0 \end{array}\right] \xrightarrow{-1R_2+R_3} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 1 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\begin{cases} x-3y+2z=0 \\ y-6z=0 \end{cases}$ $\quad y=6z \quad$ $x-3(6z)+2z=0$, $x=16z$

Let $z$ be any real number $c$. The solution is $(16c,\ 6c,\ c)$.

**22.** $\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 3 & 4 & -1 & 0 \\ 5 & 6 & -5 & 0 \end{array}\right] \xrightarrow[-5R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 1 & 5 & 0 \end{array}\right] \xrightarrow{-1R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\begin{cases} x+y-2z=0 \\ y+5z=0 \end{cases}$ $\quad y=-5z \quad$ $x+(-5z)-2z=0$, $x=7z$

Let $z$ be any real number $c$. The solution is $(7c,\ -5c,\ c)$**.**

**23.** $\left[\begin{array}{ccc|c} 2 & 1 & -3 & 4 \\ 3 & 2 & 1 & 2 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{ccc|c} 2 & 1 & -3 & 4 \\ 1 & 1 & 4 & -2 \end{array}\right] R_1 \longleftrightarrow R_2 \left[\begin{array}{ccc|c} 1 & 1 & 4 & -2 \\ 2 & 1 & -3 & 4 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 4 & -2 \\ 0 & -1 & -11 & 8 \end{array}\right]$

$\xrightarrow{-1R_2} \left[\begin{array}{ccc|c} 1 & 1 & 4 & -2 \\ 0 & 1 & 11 & -8 \end{array}\right]$

$\begin{cases} x+y+4z=-2 \\ y+11z=-8 \end{cases} \quad y=-11z-8 \quad x+(-11z-8)+4z=-2, \quad x=7z+6$

Let $z$ be any real number $c$. The solution is $(7c+6, -11c-8, c)$.

**24.** $\left[\begin{array}{ccc|c} 3 & -6 & 2 & 2 \\ 2 & 5 & -3 & 2 \end{array}\right] \xrightarrow{-1R_2+R_1} \left[\begin{array}{ccc|c} 1 & -11 & 5 & 0 \\ 2 & 5 & -3 & 2 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & -11 & 5 & 0 \\ 0 & 27 & -13 & 2 \end{array}\right] \xrightarrow{\frac{1}{27}R_2} \left[\begin{array}{ccc|c} 1 & -11 & 5 & 0 \\ 0 & 1 & -13/27 & 2/27 \end{array}\right]$

$\begin{cases} x-11y+5z=0 \\ y-\frac{13}{27}z=\frac{2}{27} \end{cases} \quad y=\frac{13}{27}z+\frac{2}{27} \quad x-11\left(\frac{13}{27}z+\frac{2}{27}\right)+5z=0, \quad x=\frac{8}{27}z+\frac{22}{27}$

Let $z$ be any real number $c$. The solution is $\left(\frac{8}{27}c+\frac{22}{27}, \frac{13}{27}c+\frac{2}{27}, c\right)$.

**25.** $\left[\begin{array}{ccc|c} 2 & 2 & -4 & 4 \\ 2 & 3 & -5 & 4 \\ 4 & 5 & -9 & 8 \end{array}\right] \xrightarrow{(1/2)R_1} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 2 & 3 & -5 & 4 \\ 4 & 5 & -9 & 8 \end{array}\right] \xrightarrow[-4R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{-1R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\begin{cases} x+y-2z=2 \\ y-z=0 \end{cases} \quad y=z \quad x+z-2z=2, \quad x=z+2$

Let $z$ be any real number $c$. The solution is $(c+2, c, c)$.

**26.** $\left[\begin{array}{ccc|c} 3 & -10 & 2 & 34 \\ 1 & -4 & 1 & 13 \\ 5 & -2 & 7 & 31 \end{array}\right] R_1 \longleftrightarrow R_2 \left[\begin{array}{ccc|c} 1 & -4 & 1 & 13 \\ 3 & -10 & 2 & 34 \\ 5 & -2 & 7 & 31 \end{array}\right] \xrightarrow[-5R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|c} 1 & -4 & 1 & 13 \\ 0 & 2 & -1 & -5 \\ 0 & 18 & 2 & -34 \end{array}\right] \xrightarrow{(1/2)R_2} \left[\begin{array}{ccc|c} 1 & -4 & 1 & 13 \\ 0 & 2 & -\frac{1}{2} & -5/2 \\ 0 & 18 & 2 & -34 \end{array}\right]$

$\xrightarrow{-18R_2+R_3} \left[\begin{array}{ccc|c} 1 & -4 & 1 & 13 \\ 0 & 1 & -\frac{1}{2} & -5/2 \\ 0 & 0 & 11 & 11 \end{array}\right] \xrightarrow{(1/11)R_3} \left[\begin{array}{ccc|c} 1 & -4 & 1 & 13 \\ 0 & 1 & -\frac{1}{2} & -5/2 \\ 0 & 0 & 1 & 1 \end{array}\right]$

$\begin{cases} x-4y+z=13 \\ y-\frac{1}{2}z=-\frac{5}{2} \\ z=1 \end{cases} \quad y-\frac{1}{2}(1)=-\frac{5}{2}, \quad y=-2 \quad x-4(-2)+1=13, \quad x=4$

The solution is $(4, -2, 1)$.

**27.** $\left[\begin{array}{ccc|c} 1 & 3 & 4 & 11 \\ 2 & 3 & 2 & 7 \\ 4 & 9 & 10 & 20 \\ 3 & -2 & 1 & 1 \end{array}\right] \xrightarrow[\substack{-4R_1+R_3 \\ -3R_1+R_4}]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 11 \\ 0 & -3 & -6 & -15 \\ 0 & -3 & -6 & -24 \\ 0 & -11 & -11 & -32 \end{array}\right] \xrightarrow{(-1/3)R_2} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 11 \\ 0 & 1 & 2 & 5 \\ 0 & -3 & -6 & -24 \\ 0 & -11 & -11 & -32 \end{array}\right] \xrightarrow{3R_2+R_3} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 11 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & -9 \\ 0 & -11 & -11 & -32 \end{array}\right]$

$R_3 \longleftrightarrow R_4 \left[\begin{array}{ccc|c} 1 & 3 & 4 & 11 \\ 0 & 1 & 2 & 5 \\ 0 & -11 & -11 & -32 \\ 0 & 0 & 0 & -9 \end{array}\right] \xrightarrow{(-1/11)R_3} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 11 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 1 & \frac{32}{11} \\ 0 & 0 & 0 & -9 \end{array}\right] \xrightarrow{-R_2+R_3} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 11 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & -1 & -\frac{23}{11} \\ 0 & 0 & 0 & -9 \end{array}\right] \xrightarrow{-1R_3} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 11 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & \frac{23}{11} \\ 0 & 0 & 0 & -9 \end{array}\right]$

$\begin{cases} x+3y+4z=11 \\ y+2z=5 \\ z=\frac{23}{11} \\ 0z=-9 \end{cases}$

Because $0z=-9$ has no solutions, the system of equations has no solution.

**28.** $$\left[\begin{array}{rrr|r} 1 & -4 & 3 & 4 \\ 3 & -10 & 3 & 4 \\ 5 & -18 & 9 & 10 \\ 2 & 2 & -3 & -11 \end{array}\right] \xrightarrow[-2R_1+R_4]{\substack{-3R_1+R_2\\ -5R_1+R_3}} \left[\begin{array}{rrr|r} 1 & -4 & 3 & 4 \\ 0 & 2 & -6 & -8 \\ 0 & 2 & -6 & -10 \\ 0 & 10 & -9 & -19 \end{array}\right] \xrightarrow{(1/2)R_2} \left[\begin{array}{rrr|r} 1 & -4 & 3 & 4 \\ 0 & 1 & -3 & -4 \\ 0 & 2 & -6 & -10 \\ 0 & 10 & -9 & -19 \end{array}\right] \xrightarrow{-2R_1+R_3} \left[\begin{array}{rrr|r} 1 & -4 & 3 & 4 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 0 & -2 \\ 0 & 10 & -9 & -19 \end{array}\right]$$

$$R_3 \longleftrightarrow R_4 \left[\begin{array}{rrr|r} 1 & -4 & 3 & 4 \\ 0 & 1 & -3 & -4 \\ 0 & 10 & -9 & -19 \\ 0 & 0 & 0 & -2 \end{array}\right] \xrightarrow{-10R_2+R_3} \left[\begin{array}{rrr|r} 1 & -4 & 3 & 4 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 21 & 21 \\ 0 & 0 & 0 & -2 \end{array}\right] \xrightarrow{(1/21)R_3} \left[\begin{array}{rrr|r} 1 & -4 & 3 & 4 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

$$\begin{cases} x-4y+3z=4 \\ \quad y-3z=-4 \\ \qquad z=1 \\ \qquad 0z=-2 \end{cases}$$

Because $0z=-2$ has no solution, the system of equations has no solution.

**29.** $$\left[\begin{array}{rrrr|r} 1 & 2 & -3 & 1 & -7 \\ 3 & 5 & -8 & 5 & -8 \\ 2 & 3 & -7 & 3 & -11 \\ 4 & 8 & -10 & 7 & -10 \end{array}\right] \xrightarrow[-4R_1+R_4]{\substack{-3R_1+R_2\\ -2R_1+R_3}} \left[\begin{array}{rrrr|r} 1 & 2 & -3 & 1 & -7 \\ 0 & -1 & 1 & 2 & 13 \\ 0 & -1 & -1 & 1 & 3 \\ 0 & 0 & 2 & 3 & 18 \end{array}\right] \xrightarrow{-1R_2} \left[\begin{array}{rrrr|r} 1 & 2 & -3 & 1 & -7 \\ 0 & 1 & -1 & -2 & -13 \\ 0 & -1 & -1 & 1 & 3 \\ 0 & 0 & 2 & 3 & 18 \end{array}\right]$$

$$\xrightarrow{R_2+R_3} \left[\begin{array}{rrrr|r} 1 & 2 & -3 & 1 & -7 \\ 0 & 1 & -1 & -2 & -13 \\ 0 & 0 & -2 & -1 & -10 \\ 0 & 0 & 2 & 3 & 18 \end{array}\right] \xrightarrow{(-1/2)R_3} \left[\begin{array}{rrrr|r} 1 & 2 & -3 & 1 & -7 \\ 0 & 1 & -1 & -2 & -13 \\ 0 & 0 & 1 & \frac{1}{2} & 5 \\ 0 & 0 & 2 & 3 & 18 \end{array}\right] \xrightarrow{-2R_3+R_4} \left[\begin{array}{rrrr|r} 1 & 2 & -3 & 1 & -7 \\ 0 & 1 & -1 & -2 & -13 \\ 0 & 0 & 1 & \frac{1}{2} & 5 \\ 0 & 0 & 0 & 2 & 8 \end{array}\right]$$

$$\begin{cases} t+2u-3v+w=-7 \\ \quad u-v-2w=-13 \\ \qquad v+\frac{1}{2}w=5 \\ \qquad 2w=8 \end{cases} \qquad \begin{aligned} v+\tfrac{1}{2}(4)&=5 \\ v&=3 \end{aligned} \qquad \begin{aligned} u-3-2(4)&=-13 \\ u&=-2 \end{aligned} \qquad \begin{aligned} t+2(-2)-3(3)+4&=-7 \\ t&=2 \end{aligned}$$

The solution is $(2, -2, 3, 4)$.

**30.** $$\left[\begin{array}{rrrr|r} 1 & 4 & 2 & -3 & 11 \\ 2 & 10 & 3 & -5 & 17 \\ 4 & 16 & 7 & -9 & 34 \\ 1 & 4 & 1 & -1 & 4 \end{array}\right] \xrightarrow[-1R_1+R_4]{\substack{-2R_1+R_2\\ -4R_1+R_3}} \left[\begin{array}{rrrr|r} 1 & 4 & 2 & -3 & 11 \\ 0 & 2 & -1 & 1 & -5 \\ 0 & 0 & -1 & 3 & -10 \\ 0 & 0 & -1 & 2 & -7 \end{array}\right] \xrightarrow[-1R_3]{(1/2)R_2} \left[\begin{array}{rrrr|r} 1 & 4 & 2 & -3 & 11 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{5}{2} \\ 0 & 0 & 1 & -3 & 10 \\ 0 & 0 & -1 & 2 & -7 \end{array}\right]$$

$$\xrightarrow{R_3+R_4} \left[\begin{array}{rrrr|r} 1 & 4 & 2 & -3 & 11 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{5}{2} \\ 0 & 0 & 1 & -3 & 10 \\ 0 & 0 & 0 & -1 & 3 \end{array}\right]$$

$$\begin{cases} t+4u+2v-3w=11 \\ \quad u-\frac{1}{2}v+\frac{1}{2}w=-\frac{5}{2} \\ \qquad v-3w=10 \\ \qquad -w=3 \end{cases} \qquad \begin{aligned} v-3(3)&=10 \\ v&=1 \end{aligned} \qquad \begin{aligned} u-\tfrac{1}{2}(1)+\tfrac{1}{2}(-3)&=-\tfrac{5}{2} \\ u&=-\tfrac{1}{2} \end{aligned} \qquad \begin{aligned} t+4(-\tfrac{1}{2})+2(1)-3(-3)&=11 \\ t&=2 \end{aligned}$$

The solution is $\left(2, -\frac{1}{2}, 1, -3\right)$.

**31.** $$\left[\begin{array}{cccc|c} 2 & -1 & 3 & 2 & 2 \\ 1 & -1 & 2 & 1 & 2 \\ 3 & 0 & -2 & -3 & 13 \\ 2 & 2 & 0 & -2 & 6 \end{array}\right] \xrightarrow{(1/2)R_1} \left[\begin{array}{cccc|c} 1 & -\frac{1}{2} & \frac{3}{2} & 1 & 1 \\ 1 & -1 & 2 & 1 & 2 \\ 3 & 0 & -2 & -3 & 13 \\ 2 & 2 & 0 & -2 & 6 \end{array}\right] \xrightarrow[\substack{-3R_1+R_3 \\ -2R_1+R_4}]{-1R_1+R_2} \left[\begin{array}{cccc|c} 1 & -\frac{1}{2} & \frac{3}{2} & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 1 \\ 0 & \frac{3}{2} & -\frac{13}{2} & -6 & 10 \\ 0 & 3 & -3 & -4 & 4 \end{array}\right]$$

$$\xrightarrow{-2R_2} \left[\begin{array}{cccc|c} 1 & -\frac{1}{2} & \frac{3}{2} & 1 & 1 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & \frac{3}{2} & -\frac{13}{2} & -6 & 10 \\ 0 & 3 & -3 & -4 & 4 \end{array}\right] \xrightarrow[-3R_2+R_4]{(-3/2)R_2+R_3} \left[\begin{array}{cccc|c} 1 & -\frac{1}{2} & \frac{3}{2} & 1 & 1 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & -5 & -6 & 13 \\ 0 & 0 & 0 & -4 & 10 \end{array}\right]$$

$$\xrightarrow{(1/5)R_3} \left[\begin{array}{cccc|c} 1 & -\frac{1}{2} & \frac{3}{2} & 1 & 1 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 1 & \frac{6}{5} & -\frac{13}{5} \\ 0 & 0 & 0 & -4 & 10 \end{array}\right] \xrightarrow{(-1/4)R_4} \left[\begin{array}{cccc|c} 1 & -\frac{1}{2} & \frac{3}{2} & 1 & 1 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 1 & \frac{6}{5} & -\frac{13}{5} \\ 0 & 0 & 0 & 1 & -\frac{5}{2} \end{array}\right]$$

$$\begin{cases} t-\frac{1}{2}u+\frac{3}{2}v+w=1 \\ u-v=-2 \\ v+\frac{6}{5}w=-\frac{13}{5} \\ w=-\frac{5}{2} \end{cases} \quad \begin{aligned} v+\tfrac{6}{5}(-\tfrac{5}{2}) &= -\tfrac{13}{5} \\ v &= \tfrac{2}{5} \end{aligned} \quad \begin{aligned} u-\tfrac{2}{5} &= -2 \\ u &= -\tfrac{8}{5} \end{aligned} \quad \begin{aligned} t-\tfrac{1}{2}(-\tfrac{8}{5})+\tfrac{3}{2}(\tfrac{2}{5})-\tfrac{5}{2} &= 1 \\ t &= \tfrac{21}{10} \end{aligned}$$

The solution is $\left(\frac{21}{10}, -\frac{8}{5}, \frac{2}{5}, -\frac{5}{2}\right)$.

**32.** $$\left[\begin{array}{cccc|c} 4 & 7 & -10 & 3 & -29 \\ 3 & 5 & -7 & 2 & -20 \\ 1 & 2 & -3 & 1 & -9 \\ 2 & -1 & 2 & -4 & 15 \end{array}\right] \xrightarrow{R_1 \longleftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & -9 \\ 3 & 5 & -7 & 2 & -20 \\ 4 & 7 & -10 & 3 & -29 \\ 2 & -1 & 2 & -4 & 15 \end{array}\right] \xrightarrow[\substack{-4R_1+R_3 \\ -2R_1+R_4}]{-3R_1+R_2} \left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & -9 \\ 0 & -1 & 2 & -1 & 7 \\ 0 & -1 & 2 & -1 & 7 \\ 0 & -5 & 8 & -6 & 33 \end{array}\right]$$

$$\xrightarrow[-R_2+R_3]{-R_2} \left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & -9 \\ 0 & 1 & -2 & 1 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -5 & 8 & -6 & 33 \end{array}\right] \xrightarrow{R_3 \longleftrightarrow R_4} \left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & -9 \\ 0 & 1 & -2 & 1 & -7 \\ 0 & -5 & 8 & -6 & 33 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{5R_2+R_3} \left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & -9 \\ 0 & 1 & -2 & 1 & -7 \\ 0 & 0 & -2 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-(1/2)R_3} \left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & -9 \\ 0 & 1 & -2 & 1 & -7 \\ 0 & 0 & 1 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} t+2u-3v+w=-9 \\ u-2v+w=-7 \\ v+\frac{1}{2}w=1 \end{cases} \quad v=-\tfrac{1}{2}w+1 \quad \begin{aligned} u-2(-\tfrac{1}{2}w+1)+w &= -7 \\ u &= -2w-5 \end{aligned} \quad \begin{aligned} t+2(2w-5)-3(-\tfrac{1}{2}w+1)+w &= -9 \\ t &= \tfrac{3}{2}w+4 \end{aligned}$$

Let $w$ be any real number c. The solution is $\left(\frac{3}{2}c+4, -2c-5, -\frac{1}{2}c+1, c\right)$.

**33.** $\left[\begin{array}{cccc|c} 3 & 10 & 7 & -6 & 7 \\ 2 & 8 & 6 & -5 & 5 \\ 1 & 4 & 2 & -3 & 2 \\ 4 & 14 & 9 & -8 & 8 \end{array}\right] \xrightarrow{R_1 \longleftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 4 & 2 & -3 & 2 \\ 2 & 8 & 6 & -5 & 5 \\ 3 & 10 & 7 & -6 & 7 \\ 4 & 14 & 9 & -8 & 8 \end{array}\right] \xrightarrow[-4R_1+R_4]{\substack{-2R_1+R_2 \\ -3R_1+R_3}} \left[\begin{array}{cccc|c} 1 & 4 & 2 & -3 & 2 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & -2 & 1 & 3 & 1 \\ 0 & -2 & 1 & 4 & 0 \end{array}\right] \xrightarrow{R_2 \longleftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 4 & 2 & -3 & 2 \\ 0 & -2 & 1 & 3 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & -2 & 1 & 4 & 0 \end{array}\right]$

$\xrightarrow{(-1/2)R_2} \left[\begin{array}{cccc|c} 1 & 4 & 2 & -3 & 2 \\ 0 & 1 & -\frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 2 & 1 & 1 \\ 0 & -2 & 1 & 4 & 0 \end{array}\right] \xrightarrow{2R_2+R_4} \left[\begin{array}{cccc|c} 1 & 4 & 2 & -3 & 2 \\ 0 & 1 & -\frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$

$\begin{cases} t+4u+2v-3w=2 \\ u-\frac{1}{2}v-\frac{3}{2}w=-\frac{1}{2} \\ 2v+w=1 \\ w=-1 \end{cases}$ $\quad 2v+(-1)=1$, $v=1$ $\quad u-\frac{1}{2}(1)-\frac{3}{2}(-1)=-\frac{1}{2}$, $u=-\frac{3}{2}$ $\quad t+4(-\frac{3}{2})+2(1)-3(-1)=2$, $t=3$

The solution is $\left(3,\ -\frac{3}{2},\ 1,\ -1\right)$.

**34.** $\left[\begin{array}{cccc|c} 1 & -3 & 2 & 4 & 13 \\ 3 & -8 & 4 & 13 & 35 \\ 2 & -7 & 8 & 5 & 28 \\ 4 & -11 & 6 & 17 & 56 \end{array}\right] \xrightarrow[-4R_1+R_4]{\substack{-3R_1+R_2 \\ -2R_1+R_3}} \left[\begin{array}{cccc|c} 1 & -3 & 2 & 4 & 13 \\ 0 & 1 & -2 & 1 & -4 \\ 0 & -1 & 4 & -3 & 2 \\ 0 & 1 & -2 & 1 & 4 \end{array}\right] \xrightarrow[-1R_2+R_4]{R_2+R_3} \left[\begin{array}{cccc|c} 1 & -3 & 2 & 4 & 13 \\ 0 & 1 & -2 & 1 & -4 \\ 0 & 0 & 2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 8 \end{array}\right]$

$\begin{cases} t-3u+2v+4w=13 \\ u-2v+w=-4 \\ v-w=-1 \\ 0w=8 \end{cases}$

Because equation $0w=8$ has no solution, the system of equations has no solution.

**35.** $\left[\begin{array}{cccc|c} 1 & -1 & 2 & -3 & 9 \\ 4 & 0 & 11 & -10 & 46 \\ 3 & -1 & 8 & -6 & 27 \end{array}\right] \xrightarrow[-3R_1+R_3]{-4R_1+R_2} \left[\begin{array}{cccc|c} 1 & -1 & 2 & -3 & 9 \\ 0 & 4 & 3 & 2 & 10 \\ 0 & 2 & 2 & 3 & 0 \end{array}\right] \xrightarrow{R_2 \longleftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & -1 & 2 & -3 & 9 \\ 0 & 2 & 2 & 3 & 0 \\ 0 & 4 & 3 & 2 & 10 \end{array}\right]$

$\xrightarrow{-2R_2+R_3} \left[\begin{array}{cccc|c} 1 & -1 & 2 & -3 & 9 \\ 0 & 2 & 2 & 3 & 0 \\ 0 & 0 & -1 & -4 & 10 \end{array}\right]$

$\begin{cases} t-u+2v-3w=9 \\ 2u+2v+3w=0 \\ -v-4w=10 \end{cases}$ $\quad v=-4w-10$ $\quad 2u+2(-4w-10)+3w=0$, $u=\frac{5}{2}w+10$ $\quad t-(\frac{5}{2}w+10)+2(-4w-10)-3w=9$, $t=\frac{27}{2}w+39$

Let $w$ be any real number $c$. The solution is $\left(\frac{27}{2}c+39,\ \frac{5}{2}c+10,\ -4c-10,\ c\right)$.

**36.** $\left[\begin{array}{cccc|c} 1 & -1 & 3 & -5 & 10 \\ 2 & -3 & 4 & 1 & 7 \\ 3 & 1 & -2 & -2 & 6 \end{array}\right] \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{cccc|c} 1 & -1 & 3 & -5 & 10 \\ 0 & -1 & -2 & 11 & -13 \\ 0 & 4 & -11 & 13 & -24 \end{array}\right] \xrightarrow{-1R_2} \left[\begin{array}{cccc|c} 1 & -1 & 3 & -5 & 10 \\ 0 & 1 & 2 & -11 & 13 \\ 0 & 4 & -11 & 13 & -24 \end{array}\right]$

$\xrightarrow{-4R_2+R_3} \left[\begin{array}{cccc|c} 1 & -1 & 3 & -5 & 10 \\ 0 & 1 & 2 & -11 & 13 \\ 0 & 0 & -19 & 57 & -76 \end{array}\right] \xrightarrow{(-1/19)R_3} \left[\begin{array}{cccc|c} 1 & -1 & 3 & -5 & 10 \\ 0 & 1 & 2 & -11 & 13 \\ 0 & 0 & 1 & -3 & 4 \end{array}\right]$

$\begin{cases} t-u+3v-5w=10 \\ u+2v-11w=13 \\ v-3w=4 \end{cases}$ $\quad v=3w+4$ $\quad u+2(3w+4)-11w=13$, $u=5w+5$ $\quad t-(5w+5)+3(3w+4)-5w=10$, $t=w+3$

Let $w$ be any real number $c$. The solution is $(c+3,\ 5c+5,\ 3c+4,\ c)$.

**37.** $\left[\begin{array}{cccc|c} 3 & -4 & 1 & 0 & 2 \\ 1 & 1 & -2 & 3 & 1 \end{array}\right] \xrightarrow{(1/3)R_1} \left[\begin{array}{cccc|c} 1 & -\frac{4}{3} & \frac{1}{3} & 0 & \frac{2}{3} \\ 1 & 1 & -2 & 3 & 1 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{cccc|c} 1 & -\frac{4}{3} & \frac{1}{3} & 0 & \frac{2}{3} \\ 1 & \frac{7}{3} & -\frac{7}{3} & 3 & \frac{1}{3} \end{array}\right] \xrightarrow{(3/7)R_2} \left[\begin{array}{cccc|c} 1 & -\frac{4}{3} & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 1 & -1 & \frac{9}{7} & \frac{1}{7} \end{array}\right]$

$\begin{cases} t-\frac{4}{3}u+\frac{1}{3}v = \frac{2}{3} \\ u - v + \frac{9}{7} = \frac{1}{7} \end{cases}$ $\quad u = v-\frac{9}{7}w+\frac{1}{7}$ $\quad t-\frac{4}{3}(v-\frac{9}{7}w+\frac{1}{7})+\frac{1}{3}v=\frac{2}{3}$

$t = v-\frac{12}{7}w+\frac{6}{7}$

Let $v$ be any real number $c_1$ and $w$ be any real number $c_2$. The solution is $\left(c_1-\frac{12}{7}c_2+\frac{6}{7},\ c_1-\frac{9}{7}c_2+\frac{1}{7},\ c_1,\ c_2\right)$.

**38.** $\left[\begin{array}{cccc|c} 2 & 0 & 3 & -4 & 2 \\ 1 & 2 & -4 & 1 & -3 \end{array}\right] \xrightarrow{(1/2)R_1} \left[\begin{array}{cccc|c} 1 & 0 & \frac{3}{2} & -2 & 1 \\ 1 & 2 & -4 & 1 & -3 \end{array}\right] \xrightarrow{1R_1+R_2} \left[\begin{array}{cccc|c} 1 & 0 & \frac{3}{2} & -2 & 1 \\ 0 & 2 & -\frac{11}{2} & 3 & -4 \end{array}\right]$

$\xrightarrow{(1/2)R_2} \left[\begin{array}{cccc|c} 1 & 0 & \frac{3}{2} & -2 & 1 \\ 0 & 1 & -\frac{11}{4} & \frac{3}{2} & -2 \end{array}\right]$

$\begin{cases} t+\frac{3}{2}v-2w = 1 \\ u-\frac{11}{4}v+\frac{3}{2}w = -2 \end{cases}$ $\quad t = -\frac{3}{2}v+2w+1$

$u = \frac{11}{4}v-\frac{3}{2}w-2$

Let $v$ be any real number $c_1$ and $w$ be any real number $c_2$. The solution is $\left(-\frac{3}{2}c_1+2c_2+1,\ \frac{11}{4}c_1-\frac{3}{2}c_2-2,\ c_1,\ c_2\right)$.

**39.** Because there are two points, the degree of the interpolating polynomial is at most 1. The form of the polynomial is $p(x)=a_1x+a_0$
Use this polynomial and the given points to find the system of equations.

$p(-2)=a_1(-2)+a_0=-7$
$p(1)=a_1(1)+a_0=-1$

The system of equations and the associated augmented matrix are $\begin{cases} -2a_1+a_0=-7 \\ a_1+a_0=-1 \end{cases}$ $\quad \left[\begin{array}{cc|c} -2 & 1 & -7 \\ 1 & 1 & -1 \end{array}\right]$

The ref (**r**ow **e**chelon **f**orm) feature of a graphing calculator can be used to rewrite the augmented matrix in echelon form. Consider using the function of your calculator that converts a decimal to a fraction.

The augmented matrix in echelon form and resulting system of equations are $\left[\begin{array}{cc|c} 1 & -1/2 & 7/2 \\ 0 & 1 & -3 \end{array}\right]$ $\quad \begin{cases} a_1-\frac{1}{2}a_0=\frac{7}{2} \\ a_0=-3 \end{cases}$

Solving by back substitution yields $a_0=-3$ and $a_1=2$.
The interpolating polynomial is $p(x) = 2x - 3$.

**40.** Because there are two points, the degree of the interpolating polynomial is at most 1. The form of the polynomial is $p(x)=a_1x+a_0$.
Use this polynomial and the given points to find the system of equations.

$p(-3)=a_1(-3)+a_0=-8$
$p(1)=a_1(1)+a_0=4$

The system of equations and the associated augmented matrix are $\begin{cases} -3a_1+a_0=-8 \\ a_1+a_0=4 \end{cases}$ $\quad \left[\begin{array}{cc|c} -3 & 1 & -8 \\ 1 & 1 & 4 \end{array}\right]$

The ref (**r**ow **e**chelon **f**orm) feature of a graphing calculator can be used to rewrite the augmented matrix in echelon form. Consider using the function of your calculator that converts a decimal to a fraction.

The augmented matrix in echelon form and resulting system of equations are $\left[\begin{array}{cc|c} 1 & -1/3 & 8/3 \\ 0 & 1 & 1 \end{array}\right]$ $\quad \begin{cases} a_1-\frac{1}{3}a_0=\frac{8}{3} \\ a_0=1 \end{cases}$

Solving by back substitution yields $a_0=1$ and $a_1=3$.
The interpolating polynomial is $p(x) = 3x + 1$.

**41.** Because there are three points, the degree of the interpolating polynomial is at most 2.

The form of the polynomial is $p(x) = a_2x^2 + a_1x + a_0$.

Use this polynomial and the given points to find the system of equations.

$$p(-1) = a_2(-1)^2 + a_1(-1) + a_0 = 6$$
$$p(1) = a_2(1)^2 + a_1(1) + a_0 = 2$$
$$p(2) = a_2(2)^2 + a_1(2) + a_0 = 3$$

The system of equations and the associated augmented matrix are $\begin{cases} a_2 - a_1 + a_0 = 6 \\ a_2 + a_1 + a_0 = 2 \\ 4a_2 + 2a_1 + a_0 = 3 \end{cases}$ $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 1 & 1 & 1 & 2 \\ 4 & 2 & 1 & 3 \end{array}\right]$

The ref (**r**ow **e**chelon **f**orm) feature of a graphing calculator can be used to rewrite the augmented matrix in echelon form. Consider using the function of your calculator that converts a decimal to a fraction.

The augmented matrix in echelon form and resulting system of equations are $\left[\begin{array}{ccc|c} 1 & 1/2 & 1/4 & 3/4 \\ 0 & 1 & -1/2 & -7/2 \\ 0 & 0 & 1 & 3 \end{array}\right]$ $\begin{cases} a_2 + \frac{1}{2}a_1 + \frac{1}{4}a_0 = \frac{3}{4} \\ a_1 - \frac{1}{2}a_0 = -\frac{7}{2} \\ a_0 = 3 \end{cases}$

Solving by back substitution yields $a_0 = 3$, $a_1 = -2$, and $a_2 = 1$.

The interpolating polynomial is $p(x) = x^2 - 2x + 3$.

**42.** Because there are three points, the degree of the interpolating polynomial is at most 2.

The form of the polynomial is $p(x) = a_2x^2 + a_1x + a_0$.

Use this polynomial and the given points to find the system of equations.

$$p(-2) = a_2(-2)^2 + a_1(-2) + a_0 = -3$$
$$p(0) = a_2(0)^2 + a_1(0) + a_0 = -1$$
$$p(3) = a_2(3)^2 + a_1(3) + a_0 = 17$$

The system of equations and the associated augmented matrix are $\begin{cases} 4a_2 - 2a_1 + a_0 = -3 \\ a_0 = -1 \\ 9a_2 + 3a_1 + a_0 = 17 \end{cases}$ $\left[\begin{array}{ccc|c} 4 & -2 & 1 & -3 \\ 0 & 0 & 1 & -1 \\ 9 & 3 & 1 & 17 \end{array}\right]$

The ref (**r**ow **e**chelon **f**orm) feature of a graphing calculator can be used to rewrite the augmented matrix in echelon form. Consider using the function of your calculator that converts a decimal to a fraction.

The augmented matrix in echelon form and resulting system of equations are $\left[\begin{array}{ccc|c} 1 & 1/3 & 1/9 & 17/9 \\ 0 & 1 & -1/6 & 19/6 \\ 0 & 0 & 1 & -1 \end{array}\right]$ $\begin{cases} a_2 + \frac{1}{3}a_1 + \frac{1}{9}a_0 = \frac{17}{9} \\ a_1 - \frac{1}{6}a_0 = \frac{19}{6} \\ a_0 = -1 \end{cases}$

Solving by back substitution yields $a_0 = -1$, $a_1 = 3$, and $a_2 = 1$.

The interpolating polynomial is $p(x) = x^2 + 3x - 1$.

**43.** Because there are four points, the degree of the interpolating polynomial is at most 3.

The form of the polynomial is $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ .

Use this polynomial and the given points to find the system of equations.

$$p(-2) = a_3(-2)^3 + a_2(-2)^2 + a_1(-2) + a_0 = -12$$
$$p(0) = a_3(0)^3 + a_2(0)^2 + a_1(0) + a_0 = 2$$
$$p(1) = a_3(1)^3 + a_2(1)^2 + a_1(1) + a_0 = 0$$
$$p(3) = a_3(3)^3 + a_2(3)^2 + a_1(3) + a_0 = 8$$

The system of equations and the associated augmented matrix are $\begin{cases} -8a_3 + 4a_2 - 2a_1 + a_0 = -12 \\ a_0 = 2 \\ a_3 + a_2 + a_1 + a_0 = 0 \\ 27a_3 + 9a_2 + 3a_1 + a_0 = 8 \end{cases}$ $\left[\begin{array}{cccc|c} -8 & 4 & -2 & 1 & -12 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 0 \\ 27 & 9 & 3 & 1 & 8 \end{array}\right]$

The ref (**r**ow **e**chelon **f**orm) feature of a graphing calculator can be used to rewrite the augmented matrix in echelon form.
Consider using the function of your calculator that converts a decimal to a fraction.
The augmented matrix in echelon form and resulting system of equations are

$$\left[\begin{array}{cccc|c} 1 & 1/3 & 1/9 & 1/27 & 8/27 \\ 0 & 1 & -1/6 & 7/36 & -13/9 \\ 0 & 0 & 1 & 5/6 & 2/3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \qquad \begin{cases} a_3 + \frac{1}{3}a_2 + \frac{1}{9}a_1 + \frac{1}{27}a_0 = \frac{8}{27} \\ a_2 - \frac{1}{6}a_1 + \frac{7}{36}a_0 = -\frac{13}{9} \\ a_1 + \frac{5}{6}a_0 = \frac{2}{3} \\ a_0 = 2 \end{cases}$$

Solving by back substitution yields $a_0 = 2,\ a_1 = -1,\ a_2 = -2$ and $a_3 = 1$ .

The interpolating polynomial is $p(x) = x^3 - 2x^2 - x + 2$ .

**44.** Because there are four points, the degree of the interpolating polynomial is at most 3.

The form of the polynomial is $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ .

Use this polynomial and the given points to find the system of equations.

$$p(-1) = a_3(-1)^3 + a_2(-1)^2 + a_1(-1) + a_0 = -5$$
$$p(0) = a_3(0)^3 + a_2(0)^2 + a_1(0) + a_0 = 0$$
$$p(1) = a_3(1)^3 + a_2(1)^2 + a_1(1) + a_0 = 1$$
$$p(2) = a_3(2)^3 + a_2(2)^2 + a_1(2) + a_0 = 4$$

The system of equations and the associated augmented matrix are $\begin{cases} -a_3 + a_2 - a_1 + a_0 = -5 \\ a_0 = 0 \\ a_3 + a_2 + a_1 + a_0 = 1 \\ 8a_3 + 4a_2 + 2a_1 + a_0 = 4 \end{cases}$ $\left[\begin{array}{cccc|c} -1 & 1 & -1 & 1 & -5 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 & 4 \end{array}\right]$

The ref (**r**ow **e**chelon **f**orm) feature of a graphing calculator can be used to rewrite the augmented matrix in echelon form.
Consider using the function of your calculator that converts a decimal to a fraction.
The augmented matrix in echelon form and resulting system of equations are

$$\left[\begin{array}{cccc|c} 1 & 1/2 & 1/4 & 1/8 & 1/2 \\ 0 & 1 & -1/2 & 3/4 & -3 \\ 0 & 0 & 1 & 1/2 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \qquad \begin{cases} a_3 + \frac{1}{2}a_2 + \frac{1}{4}a_1 + \frac{1}{8}a_0 = \frac{1}{2} \\ a_2 - \frac{1}{2}a_1 + \frac{3}{4}a_0 = -3 \\ a_1 + \frac{1}{2}a_0 = 2 \\ a_0 = 0 \end{cases}$$

Solving by back substitution yields $a_0 = 0,\ a_1 = 2,\ a_2 = -2$ and $a_3 = 1$ .

The interpolating polynomial is $p(x) = x^3 - 2x^2 + 2x$ .

**45.** Because there are three points, the degree of the interpolating polynomial is at most 2.

The form of the polynomial is $p(x) = a_2x^2 + a_1x + a_0$.

Use this polynomial and the given points to find the system of equations.

$$p(-1) = a_2(-1)^2 + a_1(-1) + a_0 = 3$$
$$p(1) = a_2(1)^2 + a_1(1) + a_0 = 7$$
$$p(2) = a_2(2)^2 + a_1(2) + a_0 = 9$$

The system of equations and the associated augmented matrix are $\begin{cases} a_2 - a_1 + a_0 = 3 \\ a_2 + a_1 + a_0 = 7 \\ 4a_2 + 2a_1 + a_0 = 9 \end{cases}$ $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 4 & 2 & 1 & 9 \end{array}\right]$

The ref (**r**ow **e**chelon **f**orm) feature of a graphing calculator can be used to rewrite the augmented matrix in echelon form. Consider using the function of your calculator that converts a decimal to a fraction.

The augmented matrix in echelon form and resulting system of equations are $\left[\begin{array}{ccc|c} 1 & 1/2 & 1/4 & 9/4 \\ 0 & 1 & -1/2 & -1/2 \\ 0 & 0 & 1 & 5 \end{array}\right]$ $\begin{cases} a_2 + \frac{1}{2}a_1 + \frac{1}{4}a_0 = \frac{9}{4} \\ a_1 - \frac{1}{2}a_0 = -\frac{1}{2} \\ a_0 = 5 \end{cases}$

Solving by back substitution yields $a_0 = 5$, $a_1 = 2$, and $a_2 = 0$.

The interpolating polynomial is $p(x) = 2x + 5$.

**46.** Because there are three points, the degree of the interpolating polynomial is at most 2.

The form of the polynomial is $p(x) = a_2x^2 + a_1x + a_0$.

Use this polynomial and the given points to find the system of equations.

$$p(-2) = a_2(-2)^2 + a_1(-2) + a_0 = 7$$
$$p(1) = a_2(1)^2 + a_1(1) + a_0 = -2$$
$$p(2) = a_2(2)^2 + a_1(2) + a_0 = -5$$

The system of equations and the associated augmented matrix are $\begin{cases} 4a_2 - 2a_1 + a_0 = 7 \\ a_2 + a_1 + a_0 = -2 \\ 4a_2 + 2a_1 + a_0 = -5 \end{cases}$ $\left[\begin{array}{ccc|c} 4 & -2 & 1 & 7 \\ 1 & 1 & 1 & -2 \\ 4 & 2 & 1 & -5 \end{array}\right]$

The ref (**r**ow **e**chelon **f**orm) feature of a graphing calculator can be used to rewrite the augmented matrix in echelon form. Consider using the function of your calculator that converts a decimal to a fraction.

The augmented matrix in echelon form and resulting system of equations are $\left[\begin{array}{ccc|c} 1 & -1/2 & 1/4 & 7/4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]$ $\begin{cases} a_2 - \frac{1}{2}a_1 + \frac{1}{4}a_0 = \frac{7}{4} \\ a_1 = -3 \\ a_0 = 1 \end{cases}$

Solving by back substitution yields $a_0 = 1$, $a_1 = -3$, and $a_2 = 0$.

The interpolating polynomial is $p(x) = -3x + 1$.

**47.** The form of the polynomial is: $z = ax + by + c$.
Use this polynomial and the given points to find the system of equations.

$$-4 = a(-1) + b(0) + c$$
$$5 = a(2) + b(1) + c$$
$$-1 = a(-1) + b(1) + c$$

The system of equations and the associated augmented matrix are $\begin{cases} -a \quad + c = -4 \\ 2a + b + c = 5 \\ -a + b + c = -1 \end{cases}$ $\left[\begin{array}{ccc|c} -1 & 0 & 1 & -4 \\ 2 & 1 & 1 & 5 \\ -1 & 1 & 1 & -1 \end{array}\right]$

The ref (**r**ow **e**chelon **f**orm) feature of a graphing calculator can be used to rewrite the augmented matrix in echelon form. Consider using the function of your calculator that converts a decimal to a fraction.

The augmented matrix in echelon form and resulting system of equations are $\left[\begin{array}{ccc|c} 1 & 1/2 & 1/2 & 5/2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$ $\begin{cases} a + \frac{1}{2}b + \frac{1}{2}c = \frac{5}{2} \\ b + c = 1 \\ c = -2 \end{cases}$

Solving by back substitution yields $a = 2,\ b = 3,$ and $c = -2$.
The interpolating polynomial is $z = 2x + 3y - 2$.

**48.** The form of the polynomial is: $z = ax + by + c$.
Use this polynomial and the given points to find the system of equations.

$$-3 = a(1) + b(2) + c$$
$$-7 = a(-2) + b(0) + c$$
$$-4 = a(0) + b(1) + c$$

The system of equations and the associated augmented matrix are $\begin{cases} a + 2b + c = -3 \\ -2a \quad + c = -7 \\ b + c = -4 \end{cases}$ $\left[\begin{array}{ccc|c} 1 & 2 & 1 & -3 \\ -2 & 0 & 1 & -7 \\ 0 & 1 & 1 & -4 \end{array}\right]$

The ref (**r**ow **e**chelon **f**orm) feature of a graphing calculator can be used to rewrite the augmented matrix in echelon form. Consider using the function of your calculator that converts a decimal to a fraction.

The augmented matrix in echelon form and resulting system of equations are $\left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 7/2 \\ 0 & 1 & 3/4 & -13/4 \\ 0 & 0 & 1 & -3 \end{array}\right]$ $\begin{cases} a \quad - \frac{1}{2}c = \frac{7}{2} \\ b + \frac{3}{4}c = -\frac{13}{4} \\ c = -2 \end{cases}$

Solving by back substitution yields $a = 2,\ b = -1,$ and $c = -3$.
The interpolating polynomial is $z = 2x - y - 3$.

**49.** The form of the polynomial is: $x^2 + y^2 + ax + by = c$
Use this polynomial and the given points to find the system of equations.

$$(2)^2 + (6)^2 + a(2) + b(6) = c$$
$$(-4)^2 + (-2)^2 + a(-4) + b(-2) = c$$
$$(3)^2 + (-1)^2 + a(3) + b(-1) = c$$

The system of equations and the associated augmented matrix are $\begin{cases} 2a + 6b - c = -40 \\ -4a - 2b - c = -20 \\ 3a - b - c = -10 \end{cases}$ $\left[\begin{array}{ccc|c} 2 & 6 & -1 & -40 \\ -4 & -2 & -1 & -20 \\ 3 & -1 & -1 & -10 \end{array}\right]$

The ref (**r**ow **e**chelon **f**orm) feature of a graphing calculator can be used to rewrite the augmented matrix in echelon form. Consider using the function of your calculator that converts a decimal to a fraction.

The augmented matrix in echelon form and resulting system of equations are $\left[\begin{array}{ccc|c} 1 & 1/2 & 1/4 & 5 \\ 0 & 1 & -3/10 & -10 \\ 0 & 0 & 1 & 20 \end{array}\right]$ $\begin{cases} a - \frac{1}{2}b + \frac{1}{4}c = 5 \\ b - \frac{3}{10}c = -10 \\ c = 20 \end{cases}$

Solving by back substitution yields $a = 2,\ b = -4,$ and $c = 20$.

The interpolating polynomial is $x^2 + y^2 + 2x - 4y = 20$.

**50.** The form of the polynomial is: $x^2+y^2+ax+by=c$

Use this polynomial and the given points to find the system of equations.

$$(2)^2+(1)^2+a(2)+b(1)=c$$
$$(0)^2+(-7)^2+a(0)+b(-7)=c$$
$$(5)^2+(-2)^2+a(5)+b(-2)=c$$

The system of equations and the associated augmented matrix are $\begin{cases} 2a+b-c=-5 \\ -7b-c=-49 \\ 5a-2b-c=-29 \end{cases}$ $\left[\begin{array}{ccc|c} 2 & 1 & -1 & -5 \\ 0 & -7 & -1 & -49 \\ 5 & -2 & -1 & -29 \end{array}\right]$

The ref (**r**ow **e**chelon **f**orm) feature of a graphing calculator can be used to rewrite the augmented matrix in echelon form. Consider using the function of your calculator that converts a decimal to a fraction.

The augmented matrix in echelon form and resulting system of equations are $\left[\begin{array}{ccc|c} 1 & -2/5 & -1/5 & -29/5 \\ 0 & 1 & 1/7 & 7 \\ 0 & 0 & 1 & 7 \end{array}\right]$ $\begin{cases} a-\frac{2}{5}b-\frac{1}{5}c=-\frac{29}{5} \\ b-\frac{1}{7}c=7 \\ c=7 \end{cases}$

Solving by back substitution yields $a=-2$, $b=6$, and $c=7$.
The interpolating polynomial is $x^2+y^2-2x+6y=7$..

**51.** Using a calculator,

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 2 & 3 & 11 \\ 1 & -1 & 2 & -1 & 2 & 0 \\ 2 & 1 & -1 & 2 & -1 & 4 \\ 3 & 2 & -1 & 1 & -2 & 2 \\ 2 & 1 & -1 & -2 & 1 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 2 & -1 & 2 & 3 & 11 \\ 0 & 1 & -1 & 1 & \frac{1}{3} & \frac{11}{3} \\ 0 & 0 & 1 & -\frac{1}{2} & 3 & \frac{7}{2} \\ 0 & 0 & 0 & 1 & \frac{11}{6} & \frac{14}{3} \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{cases} x_1+2x_2-x_3+2x_4+3x_5=11 \\ x_2-x_3+x_4+\frac{1}{3}x_5=\frac{11}{3} \\ x_3-\frac{1}{2}x_4+3x_5=\frac{7}{2} \\ x_4+\frac{11}{6}x_5=\frac{14}{3} \\ x_5=2 \end{cases}$$

The solution is (1, 0, –2, 1, 2).

**52.** Using a calculator,

$$\left[\begin{array}{ccccc|c} 1 & -2 & 2 & -3 & 2 & 5 \\ 1 & -3 & -1 & 2 & -1 & -4 \\ 3 & 1 & -2 & 1 & 3 & 9 \\ 2 & -1 & 3 & -1 & -2 & 2 \\ -1 & 2 & -2 & 3 & -1 & -4 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & -2 & 2 & -3 & 2 & 5 \\ 0 & 1 & 3 & -5 & 3 & 9 \\ 0 & 0 & 1 & -\frac{45}{29} & \frac{24}{29} & \frac{69}{29} \\ 0 & 0 & 0 & 1 & -\frac{3}{2} & -\frac{5}{2} \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{cases} x_1-2x_2+2x_3-3x_4+2x_5=5 \\ x_2+3x_3-5x_4+3x_5=9 \\ x_3-\frac{45}{29}x_4+\frac{24}{29}x_5=\frac{69}{29} \\ x_4-\frac{3}{2}x_5=-\frac{5}{2} \\ x_5=1 \end{cases}$$

The solution is (2, 1, 0, –1, 1).

**53.** Using a calculator,

$$\left[\begin{array}{ccccc|c} 1 & 2 & -3 & -1 & 2 & -10 \\ -1 & -3 & 1 & 1 & -1 & 4 \\ 2 & 3 & -5 & 2 & 3 & -20 \\ 3 & 4 & -7 & 3 & -2 & -16 \\ 2 & 1 & -6 & 4 & -3 & -12 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 2 & -3 & -1 & 2 & -10 \\ 0 & 1 & 2 & 0 & -1 & 6 \\ 0 & 0 & 1 & \frac{4}{3} & -\frac{2}{3} & 2 \\ 0 & 0 & 0 & 1 & 3 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1+2x_2-3x_3-x_4+2x_5=-10 \\ x_2+2x_3-x_5=6 \\ x_3+\frac{4}{3}x_4-\frac{2}{3}x_5=2 \\ x_4+3x_5=-7 \end{cases}$$

$$x_4=-3x_5-7$$
$$x_3+\frac{4}{3}(-3x_5-7)-\frac{2}{3}x_5=2$$
$$x_3=\frac{14}{3}x_5+\frac{34}{3}$$
$$x_2+2\left(\frac{14}{3}x_5+\frac{34}{3}\right)-x_5=6$$
$$x_2=-\frac{25}{3}x_5-\frac{50}{3}$$
$$x_1+2\left(-\frac{25}{3}x_5-\frac{50}{3}\right)-3\left(\frac{14}{3}x_5+\frac{34}{3}\right)-\left(-3x_5-7\right)+2x_5=-10$$
$$x_1=\frac{77}{3}x_5+\frac{151}{3}$$

Let $x_5$ be any real number $c$. The solution is $\left(\frac{77c+151}{3}, \frac{-25c-50}{3}, \frac{14c+34}{3}, -3c-7, c\right)$.

**54.** Using a calculator,

$$\left[\begin{array}{ccccc|c} 1 & -2 & 2 & -3 & 1 & 5 \\ 2 & -3 & 4 & -5 & -1 & 13 \\ 1 & 1 & -2 & 2 & 2 & -11 \\ 3 & -2 & 2 & -2 & -2 & 7 \\ 4 & -4 & 4 & -5 & -1 & 12 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & -2 & 2 & -3 & 1 & 5 \\ 0 & 1 & 0 & 1 & -3 & 3 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{5}{2} & \frac{25}{4} \\ 0 & 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 - 2x_2 + 2x_3 - 3x_4 + x_5 = 5 \\ x_2 + x_4 - 3x_5 = 3 \\ x_3 - \frac{1}{2}x_4 - \frac{5}{2}x_5 = \frac{25}{4} \\ x_4 - 3x_5 = 5 \end{cases}$$

$$x_4 = 3x_5 + 5$$
$$x_3 - \tfrac{1}{2}(3x_5 + 5) - \tfrac{5}{2}x_5 = \tfrac{25}{4}$$
$$x_3 = 4x_5 + \tfrac{35}{4}$$
$$x_2 + (3x_5 + 5) - 3x_5 = 3$$
$$x_2 = -2$$
$$x_1 - 2(-2) + 2\left(4x_5 + \tfrac{35}{4}\right) - 3(3x_5 + 5) + x_5 = 5$$
$$x_1 = -\tfrac{3}{2}$$

Let $x_5$ be any real number $c$. The solution is $\left(-\frac{3}{2}, -2, 4c + \frac{35}{4}, 3c + 5, c\right)$.

## Connecting Concepts

**55.** $$\left[\begin{array}{ccc|c} 1 & 3 & -a^2 & a^2 \\ 3 & 4 & 2 & 3 \\ 2 & 3 & a & 2 \end{array}\right] \xrightarrow[-2R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|c} 1 & 3 & -a^2 & a^2 \\ 0 & -5 & 3a^2+2 & -3a^2+3 \\ 0 & -3 & 2a^2+a & -2a^2+2 \end{array}\right] \xrightarrow[5R_3]{-3R_2} \left[\begin{array}{ccc|c} 1 & 3 & -a^2 & a^2 \\ 0 & 15 & -9a^2-6 & 9a^2-9 \\ 0 & -15 & 10a^2+5a & -10a^2+10 \end{array}\right]$$

$$\xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 1 & 3 & -a^2 & a^2 \\ 0 & 15 & -9a^2-6 & 9a^2-9 \\ 0 & 0 & a^2+5a-6 & -a^2+1 \end{array}\right]$$

$$\begin{cases} x + 3y - a^2 z = a^2 \\ 15y - (9a^2 + 6)z = 9a^2 - 9 \\ (a^2 + 5a - 6)z = -a^2 + 1 \end{cases}$$

For the system of equations to have a unique solution, $a^2 + 5a - 6$ cannot be zero. Thus $a^2 + 5a - 6 \neq 0$, or $a \neq 1$ and $a \neq -6$. The system of equations has a unique solution for all values of $a$ except 1 and –6.

**56.** See the solution to exercise 55. For the system to have an infinite number of solutions, $a^2 + 5a - 6$ and $-a^2 + 1$ both must be zero. Thus

$a^2 + 5a - 6 = 0$ and $-a^2 + 1 = 0$
$(a + 6)(a - 1) = 0$ $a^2 = 1$
$a = -6$ or $a = 1$ $a = 1$ or $a = -1$

Since both equations are zero when $a = 1$, the system of equations has an infinite number of solutions when $a = 1$.

**57.** See the solution to exercise 55. For the system of equations to have no solution, $a^2 + 5a - 6$ must be zero and $-a^2 + 1$ must not equal zero. Thus

$a^2 + 5a - 6 = 0$ and $-a^2 + 1 \neq 0$
$(a + 6)(a - 1) = 0$ $a^2 \neq 1$
$a = -6$ or $a = 1$ $a \neq 1$ or $a \neq -1$

The system of equations will have no solution when $a = -6$.

## Prepare for Section 10.2

**PS1.** 0

**PS2.** $-c$

**PS3.** 1

**PS4.** $\frac{1}{c}$

**PS5.** $3 \times 1$

**PS6.** $\begin{bmatrix} 1 & -3 & 4 \\ 0 & 7 & -5 \\ 0 & -7 & 13 \end{bmatrix}$

## Section 10.2

**1.** **a.** $A+B=\begin{bmatrix} 2 & -1 \\ 3 & 3 \end{bmatrix}+\begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$

**b.** $A-B=\begin{bmatrix} 2 & -1 \\ 3 & 3 \end{bmatrix}-\begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}=\begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}$

**c.** $2B=2\begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}=\begin{bmatrix} -2 & 6 \\ 4 & 2 \end{bmatrix}$

**d.** $2A-3B=2\begin{bmatrix} 2 & -1 \\ 3 & 3 \end{bmatrix}-3\begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}=\begin{bmatrix} 4 & -2 \\ 6 & 6 \end{bmatrix}-\begin{bmatrix} -3 & 9 \\ 6 & 3 \end{bmatrix}=\begin{bmatrix} 7 & -11 \\ 0 & 3 \end{bmatrix}$

**2.** **a.** $A+B=\begin{bmatrix} 0 & -2 \\ 2 & 3 \end{bmatrix}+\begin{bmatrix} 5 & -1 \\ 3 & 0 \end{bmatrix}=\begin{bmatrix} 5 & -3 \\ 5 & 3 \end{bmatrix}$

**b.** $A-B=\begin{bmatrix} 0 & -2 \\ 2 & 3 \end{bmatrix}-\begin{bmatrix} 5 & -1 \\ 3 & 0 \end{bmatrix}=\begin{bmatrix} -5 & -1 \\ -1 & 3 \end{bmatrix}$

**c.** $2B=2\begin{bmatrix} 5 & -1 \\ 3 & 0 \end{bmatrix}=\begin{bmatrix} 10 & -2 \\ 6 & 0 \end{bmatrix}$

**d.** $2A-3B=2\begin{bmatrix} 0 & -2 \\ 2 & 3 \end{bmatrix}-3\begin{bmatrix} 5 & -1 \\ 3 & 0 \end{bmatrix}=\begin{bmatrix} 0 & -4 \\ 4 & 6 \end{bmatrix}-\begin{bmatrix} 15 & -3 \\ 9 & 0 \end{bmatrix}=\begin{bmatrix} -15 & -1 \\ -5 & 6 \end{bmatrix}$

**3.** **a.** $A+B=\begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & -2 \end{bmatrix}+\begin{bmatrix} -3 & 1 & 2 \\ 2 & 5 & -3 \end{bmatrix}=\begin{bmatrix} -3 & 0 & 5 \\ 3 & 5 & -5 \end{bmatrix}$

**b.** $A-B=\begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & -2 \end{bmatrix}-\begin{bmatrix} -3 & 1 & 2 \\ 2 & 5 & -3 \end{bmatrix}=\begin{bmatrix} 3 & -2 & 1 \\ -1 & -5 & 1 \end{bmatrix}$

**c.** $2B=2\begin{bmatrix} -3 & 1 & 2 \\ 2 & 5 & -3 \end{bmatrix}=\begin{bmatrix} -6 & 2 & 4 \\ 4 & 10 & -6 \end{bmatrix}$

**d.** $2A-3B=2\begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & -2 \end{bmatrix}-3\begin{bmatrix} -3 & 1 & 2 \\ 2 & 5 & -3 \end{bmatrix}=\begin{bmatrix} 0 & -2 & 6 \\ 2 & 0 & -4 \end{bmatrix}-\begin{bmatrix} -9 & 3 & 6 \\ 6 & 15 & -9 \end{bmatrix}=\begin{bmatrix} 9 & -5 & 0 \\ -4 & -15 & 5 \end{bmatrix}$

**4.** **a.** $A+B=\begin{bmatrix} 2 & -2 & 4 \\ 0 & -3 & -4 \end{bmatrix}+\begin{bmatrix} 1 & -5 & 6 \\ 4 & -2 & -3 \end{bmatrix}=\begin{bmatrix} 3 & -7 & 10 \\ 4 & -5 & -7 \end{bmatrix}$

**b.** $A-B=\begin{bmatrix} 2 & -2 & 4 \\ 0 & -3 & -4 \end{bmatrix}-\begin{bmatrix} 1 & -5 & 6 \\ 4 & -2 & -3 \end{bmatrix}=\begin{bmatrix} 1 & 3 & -2 \\ -4 & -1 & -1 \end{bmatrix}$

**c.** $2B=2-\begin{bmatrix} 1 & -5 & 6 \\ 4 & -2 & -3 \end{bmatrix}=\begin{bmatrix} 2 & -10 & 12 \\ 8 & -4 & -6 \end{bmatrix}$

**d.** $2A-3B=2\begin{bmatrix} 2 & -2 & 4 \\ 0 & -3 & -4 \end{bmatrix}-3\begin{bmatrix} 1 & -5 & 6 \\ 4 & -2 & -3 \end{bmatrix}=\begin{bmatrix} 4 & -4 & 8 \\ 0 & -6 & -8 \end{bmatrix}-\begin{bmatrix} 3 & -15 & 18 \\ 12 & -6 & -9 \end{bmatrix}=\begin{bmatrix} 1 & 11 & -10 \\ -12 & 0 & 1 \end{bmatrix}$

**5.** **a.** $A+B=\begin{bmatrix}-3 & 4\\ 2 & -3\\ -1 & 0\end{bmatrix}+\begin{bmatrix}4 & 1\\ 1 & -2\\ 3 & -4\end{bmatrix}=\begin{bmatrix}1 & 5\\ 3 & -5\\ 2 & -4\end{bmatrix}$

**b.** $A-B=\begin{bmatrix}-3 & 4\\ 2 & -3\\ -1 & 0\end{bmatrix}-\begin{bmatrix}4 & 1\\ 1 & -2\\ 3 & -4\end{bmatrix}=\begin{bmatrix}-7 & 3\\ 1 & -1\\ -4 & 4\end{bmatrix}$

**c.** $2B=2\begin{bmatrix}4 & 1\\ 1 & -2\\ 3 & -4\end{bmatrix}=\begin{bmatrix}8 & 2\\ 2 & -4\\ 6 & -8\end{bmatrix}$

**d.** $2A-3B=2\begin{bmatrix}-3 & 4\\ 2 & -3\\ -1 & 0\end{bmatrix}-3\begin{bmatrix}4 & 1\\ 1 & -2\\ 3 & -4\end{bmatrix}=\begin{bmatrix}-6 & 8\\ 4 & -6\\ -2 & 0\end{bmatrix}-\begin{bmatrix}12 & 3\\ 3 & -6\\ 9 & -12\end{bmatrix}=\begin{bmatrix}-18 & 5\\ 1 & 0\\ -11 & 12\end{bmatrix}$

**6.** **a.** $A+B=\begin{bmatrix}2 & -2\\ 3 & 4\\ 1 & 0\end{bmatrix}+\begin{bmatrix}-1 & 8\\ 2 & -2\\ -4 & 3\end{bmatrix}=\begin{bmatrix}1 & 6\\ 5 & 2\\ -3 & 3\end{bmatrix}$

**b.** $A-B=\begin{bmatrix}2 & -2\\ 3 & 4\\ 1 & 0\end{bmatrix}-\begin{bmatrix}-1 & 8\\ 2 & -2\\ -4 & 3\end{bmatrix}=\begin{bmatrix}3 & -10\\ 1 & 6\\ 5 & -3\end{bmatrix}$

**c.** $2B=2\begin{bmatrix}-1 & 8\\ 2 & -2\\ -4 & 3\end{bmatrix}=\begin{bmatrix}-2 & 16\\ 4 & -4\\ -8 & 6\end{bmatrix}$

**d.** $2A-3B=2\begin{bmatrix}2 & -2\\ 3 & 4\\ 1 & 0\end{bmatrix}-3\begin{bmatrix}-1 & 8\\ 2 & -2\\ -4 & 3\end{bmatrix}=\begin{bmatrix}4 & -4\\ 6 & 8\\ 2 & 0\end{bmatrix}-\begin{bmatrix}-3 & 24\\ 6 & -6\\ -12 & 9\end{bmatrix}=\begin{bmatrix}7 & -28\\ 0 & 14\\ 14 & -9\end{bmatrix}$

**7.** **a.** $A+B=\begin{bmatrix}-2 & 3 & -1\\ 0 & -1 & 2\\ -4 & 3 & 3\end{bmatrix}+\begin{bmatrix}1 & -2 & 0\\ 2 & 3 & -1\\ 3 & -1 & 2\end{bmatrix}=\begin{bmatrix}-1 & 1 & -1\\ 2 & 2 & 1\\ -1 & 2 & 5\end{bmatrix}$

**b.** $A-B=\begin{bmatrix}-2 & 3 & -1\\ 0 & -1 & 2\\ -4 & 3 & 3\end{bmatrix}-\begin{bmatrix}1 & -2 & 0\\ 2 & 3 & -1\\ 3 & -1 & 2\end{bmatrix}=\begin{bmatrix}-3 & 5 & -1\\ -2 & -4 & 3\\ -7 & 4 & 1\end{bmatrix}$

**c.** $2B=2\begin{bmatrix}1 & -2 & 0\\ 2 & 3 & -1\\ 3 & -1 & 2\end{bmatrix}=\begin{bmatrix}2 & -4 & 0\\ 4 & 6 & -2\\ 6 & -2 & 4\end{bmatrix}$

**d.** $2A-3B=2\begin{bmatrix}-2 & 3 & -1\\ 0 & -1 & 2\\ -4 & 3 & 3\end{bmatrix}-3\begin{bmatrix}1 & -2 & 0\\ 2 & 3 & -1\\ 3 & -1 & 2\end{bmatrix}=\begin{bmatrix}-4 & 6 & -2\\ 0 & -2 & 4\\ -8 & 6 & 6\end{bmatrix}-\begin{bmatrix}3 & -6 & 0\\ 6 & 9 & -3\\ 9 & -3 & 6\end{bmatrix}=\begin{bmatrix}-7 & 12 & -2\\ -6 & -11 & 7\\ -17 & 9 & 0\end{bmatrix}$

**8.** **a.** $A+B=\begin{bmatrix}0 & 2 & 0\\ 1 & -3 & 3\\ 5 & 4 & -2\end{bmatrix}+\begin{bmatrix}-1 & 2 & 4\\ 3 & 3 & -2\\ -4 & 4 & 3\end{bmatrix}=\begin{bmatrix}-1 & 4 & 4\\ 4 & 0 & 1\\ 1 & 8 & 1\end{bmatrix}$

**b.** $A-B=\begin{bmatrix}0 & 2 & 0\\ 1 & -3 & 3\\ 5 & 4 & -2\end{bmatrix}-\begin{bmatrix}-1 & 2 & 4\\ 3 & 3 & -2\\ -4 & 4 & 3\end{bmatrix}=\begin{bmatrix}1 & 0 & -4\\ -2 & -6 & 5\\ 9 & 0 & -5\end{bmatrix}$

**c.** $2B=2\begin{bmatrix}-1 & 2 & 4\\ 3 & 3 & -2\\ -4 & 4 & 3\end{bmatrix}=\begin{bmatrix}-2 & 4 & 8\\ 6 & 6 & -4\\ -8 & 8 & 6\end{bmatrix}$

**d.** $2A-3B=2\begin{bmatrix}0 & 2 & 0\\ 1 & -3 & 3\\ 5 & 4 & -2\end{bmatrix}-3\begin{bmatrix}-1 & 2 & 4\\ 3 & 3 & -2\\ -4 & 4 & 3\end{bmatrix}=\begin{bmatrix}0 & 4 & 0\\ 2 & -6 & 6\\ 10 & 8 & -4\end{bmatrix}-\begin{bmatrix}-3 & 6 & 12\\ 9 & 9 & -6\\ -12 & 12 & 9\end{bmatrix}=\begin{bmatrix}3 & -2 & -12\\ -7 & -15 & 12\\ 22 & -4 & -13\end{bmatrix}$

**9.** $AB=\begin{bmatrix}2&-3\\1&4\end{bmatrix}\begin{bmatrix}-2&4\\2&-3\end{bmatrix}=\begin{bmatrix}(2)(-2)+(-3)(2)&(2)(4)+(-3)(-3)\\(1)(-2)+(4)(2)&(1)(4)+(4)(-3)\end{bmatrix}=\begin{bmatrix}-10&17\\6&-8\end{bmatrix}$

$BA=\begin{bmatrix}-2&4\\2&-3\end{bmatrix}\begin{bmatrix}2&-3\\1&4\end{bmatrix}=\begin{bmatrix}(-2)(2)+(4)(1)&(-2)(-3)+(4)(4)\\(2)(2)+(-3)(1)&(2)(-3)+(-3)(4)\end{bmatrix}=\begin{bmatrix}0&22\\1&-18\end{bmatrix}$

**10.** $AB=\begin{bmatrix}3&-2\\4&1\end{bmatrix}\begin{bmatrix}-1&-1\\0&4\end{bmatrix}=\begin{bmatrix}(3)(-1)+(-2)(0)&(3)(-1)+(-2)(4)\\(4)(-1)+(1)(0)&(4)(-1)+(1)(4)\end{bmatrix}=\begin{bmatrix}-3&-11\\-4&0\end{bmatrix}$

$BA=\begin{bmatrix}-1&-1\\0&4\end{bmatrix}\begin{bmatrix}3&-2\\4&1\end{bmatrix}=\begin{bmatrix}(-1)(3)+(-1)(4)&(-1)(-2)+(-1)(1)\\(0)(3)+(4)(4)&(0)(-2)+(4)(1)\end{bmatrix}=\begin{bmatrix}-7&1\\16&4\end{bmatrix}$

**11.** $AB=\begin{bmatrix}3&-1\\2&3\end{bmatrix}\begin{bmatrix}4&1\\2&-3\end{bmatrix}=\begin{bmatrix}(3)(4)+(-1)(2)&(3)(1)+(-1)(-3)\\(2)(4)+(3)(2)&(2)(1)+(3)(-3)\end{bmatrix}=\begin{bmatrix}10&6\\14&-7\end{bmatrix}$

$BA=\begin{bmatrix}4&1\\2&-3\end{bmatrix}\begin{bmatrix}3&-1\\2&-3\end{bmatrix}=\begin{bmatrix}(4)(3)+(1)(2)&(4)(-1)+(1)(3)\\(2)(3)+(-3)(2)&(2)(-1)+(-3)(3)\end{bmatrix}=\begin{bmatrix}14&-1\\0&-11\end{bmatrix}$

**12.** $AB=\begin{bmatrix}-3&2\\2&-2\end{bmatrix}\begin{bmatrix}0&2\\-2&4\end{bmatrix}=\begin{bmatrix}(-3)(0)+(2)(-2)&(-3)(2)+(2)(4)\\(2)(0)+(-2)(-2)&(2)(2)+(-2)(4)\end{bmatrix}=\begin{bmatrix}-4&2\\4&-4\end{bmatrix}$

$BA=\begin{bmatrix}0&2\\-2&4\end{bmatrix}\begin{bmatrix}-3&2\\2&-2\end{bmatrix}=\begin{bmatrix}(0)(-3)+(2)(2)&(0)(2)+(2)(-2)\\(-2)(-3)+(4)(2)&(-2)(2)+(4)(-2)\end{bmatrix}=\begin{bmatrix}4&-4\\14&-12\end{bmatrix}$

**13.** $AB=\begin{bmatrix}2&-1\\0&3\\1&-2\end{bmatrix}\begin{bmatrix}1&-2&3\\2&0&1\end{bmatrix}=\begin{bmatrix}(2)(1)+(-1)(2)&(2)(-2)+(-1)(0)&(2)(3)+(-1)(1)\\(0)(1)+(3)(2)&(0)(-2)+(3)(0)&(0)(3)+(3)(1)\\(1)(1)+(-2)(2)&(1)(-2)+(-2)(0)&(1)(3)+(-2)(1)\end{bmatrix}=\begin{bmatrix}0&-4&5\\6&0&3\\-3&-2&1\end{bmatrix}$

$BA=\begin{bmatrix}1&-2&3\\2&0&1\end{bmatrix}\begin{bmatrix}2&-1\\0&3\\1&-2\end{bmatrix}=\begin{bmatrix}(1)(2)+(-2)(0)+(3)(1)&(1)(-1)+(-2)(3)+(-2)\\(2)(2)+(0)(0)+(1)(1)&(2)(-1)+(0)(3)+(1)(-2)\end{bmatrix}=\begin{bmatrix}5&-13\\5&-4\end{bmatrix}$

**14.** $AB=\begin{bmatrix}-1&3\\2&1\\-3&-2\end{bmatrix}\begin{bmatrix}0&-1&2\\1&2&-4\end{bmatrix}=\begin{bmatrix}(-1)(0)+(3)(1)&(-1)(-1)+(3)(2)&(-1)(2)+(3)(-4)\\(2)(0)+(1)(1)&(2)(-1)+(1)(2)&(2)(2)+(1)(-4)\\(-3)(0)+(-2)(1)&(-3)(-1)+(-2)(2)&(-3)(2)+(-2)(-4)\end{bmatrix}=\begin{bmatrix}3&7&-14\\1&0&0\\-2&-1&2\end{bmatrix}$

$BA=\begin{bmatrix}0&-1&2\\1&2&-4\end{bmatrix}\begin{bmatrix}-1&3\\2&1\\-3&-2\end{bmatrix}=\begin{bmatrix}(0)(-1)+(-1)(2)+(2)(-3)&(0)(3)+(-1)(1)+(2)(-2)\\(1)(-1)+(2)(2)+(-4)(-3)&(1)(3)+(2)(1)+(-4)(-2)\end{bmatrix}=\begin{bmatrix}-8&-5\\15&13\end{bmatrix}$

**15.** $AB=\begin{bmatrix}2&-1&3\\0&2&-1\\0&0&2\end{bmatrix}\begin{bmatrix}2&0&0\\1&-1&0\\2&-1&-2\end{bmatrix}$

$=\begin{bmatrix}(2)(2)+(-1)(1)+(3)(2)&(2)(0)+(-1)(-1)+(3)(-1)&(2)(0)+(-1)(0)(3)(-2)\\(0)(2)+(2)(1)+(-1)(2)&(0)(0)+(2)(-1)+(-1)(-1)&(0)(0)+(2)(0)+(-1)(-2)\\(0)(2)+(0)(1)+(2)(2)&(0)(0)+(0)(-1)+(2)(-1)&(0)(0)+(0)(0)+(2)(-2)\end{bmatrix}=\begin{bmatrix}9&-2&-6\\0&-1&2\\4&-2&-4\end{bmatrix}$

$BA=\begin{bmatrix}2&0&0\\1&-1&0\\2&-1&-2\end{bmatrix}\begin{bmatrix}2&-1&3\\0&2&-1\\0&0&2\end{bmatrix}$

$=\begin{bmatrix}(2)(2)+(0)(0)+(0)(0)&(2)(-1)+(0)(2)+(0)(0)&(2)(3)+(0)(-1)+(0)(2)\\(1)(2)+(-1)(0)+(0)(0)&(1)(-1)+(-1)(2)(-1)+(0)(0)&(1)(3)+(-1)(-1)+(0)(2)\\(2)(2)+(-1)(0)+(-2)(0)&(2)(-1)+(-1)(2)+(-2)(0)&(2)(3)+(-1)(-1)+(-2)(2)\end{bmatrix}=\begin{bmatrix}4&-2&6\\2&-3&4\\4&-4&3\end{bmatrix}$

**16.** $AB=\begin{bmatrix}-1 & 2 & 0\\ 2 & -1 & 1\\ -2 & 2 & -1\end{bmatrix}\begin{bmatrix}2 & -1 & 0\\ 1 & 5 & -1\\ 0 & -1 & 3\end{bmatrix}$

$$=\begin{bmatrix}(-1)(2)+(2)(1)+(0)(0) & (-1)(-1)+(2)(5)+(0)(-1) & (-1)(0)+(2)(-1)+(0)(3)\\ (2)(2)+(-1)(1)+(1)(0) & (2)(-1)+(-1)(5)+(1)(-1) & (2)(0)+(-1)(-1)+(1)(3)\\ (-2)(2)+(2)(1)+(-1)(0) & (-2)(-1)+(2)(5)+(-1)(-1) & (-2)(0)+(2)(-1)+(-1)(3)\end{bmatrix}=\begin{bmatrix}0 & 11 & -2\\ 3 & -8 & 4\\ -2 & 13 & -5\end{bmatrix}$$

$BA=\begin{bmatrix}2 & -1 & 0\\ 1 & 5 & -1\\ 0 & -1 & 3\end{bmatrix}\begin{bmatrix}-1 & 2 & 0\\ 2 & -1 & 1\\ -2 & 2 & -1\end{bmatrix}$

$$=\begin{bmatrix}(2)(-1)+(-1)(2)+(0)(-2) & (2)(2)+(-1)(-1)+(0)(2) & (2)(0)+(-1)(1)+(0)(-1)\\ (1)(-1)+(5)(2)+(-1)(-2) & (1)(2)+(5)(-1)+(-1)(2) & (1)(0)+(5)(1)+(-1)(-1)\\ (0)(-1)+(-1)(2)+(3)(-2) & (0)(2)+(-1)(-1)+(3)(2) & (0)(0)+(-1)(1)+(3)(-1)\end{bmatrix}=\begin{bmatrix}-4 & 5 & -1\\ 11 & -5 & 6\\ -8 & 7 & -4\end{bmatrix}$$

**17.** $AB=\begin{bmatrix}1 & -2 & 3\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & -1\\ 1 & 2\end{bmatrix}=\begin{bmatrix}(1)(1)+(-2)(2)+(3)(1) & (1)(0)+(-2)(-1)+(3)(2)\end{bmatrix}=\begin{bmatrix}0 & 8\end{bmatrix}$

**18.** $AB=\begin{bmatrix}-2 & 3\\ 1 & -2\\ 0 & 2\end{bmatrix}\begin{bmatrix}3\\ -2\end{bmatrix}=\begin{bmatrix}(-2)(3))+(3)(-2)\\ (1)(3)+(-2)(-2)\\ (0)(3)+(2)(-2)\end{bmatrix}=\begin{bmatrix}-12\\ 7\\ -4\end{bmatrix}$

**19.** The number of columns of the first matrix is not equal to the number of rows of the second matrix. The product is not possible.

**20.** The number of columns of the first matrix is not equal to the number of rows of the second matrix. The product is not possible.

**21.** $AB=\begin{bmatrix}2 & 3\\ -4 & -6\end{bmatrix}\begin{bmatrix}3 & 6\\ -2 & -4\end{bmatrix}=\begin{bmatrix}(2)(3)+(3)(-2) & (2)(6)+(3)(-4)\\ (-4)(3)+(-6)(-2) & (-4)(6)+(-6)(-4)\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

**22.** $AB=\begin{bmatrix}2 & -1 & 3\\ -1 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 3 & 2\\ 2 & -1 & 0\\ 3 & 1 & 2\end{bmatrix}$

$$=\begin{bmatrix}(2)(1)+(-1)(2)+(3)(3) & (2)(3)+(-1)(-1)+(3)(1) & (2)(2)+(-1)(0)+(3)(2)\\ (-1)(1)+(2)(2)+(1)(3) & (-1)(3)+(2)(-1)+(1)(1) & (-1)(2)+(2)(0)+(1)(2)\end{bmatrix}=\begin{bmatrix}9 & 10 & 10\\ 6 & -4 & 0\end{bmatrix}$$

**23.** The number of columns of the first matrix is not equal to the number of rows of the second matrix. The product is not possible.

**24.** $AB=\begin{bmatrix}2 & -2 & 4\\ 1 & 0 & -1\\ 2 & 1 & 3\end{bmatrix}\begin{bmatrix}2 & 1 & -3 & 0\\ 0 & -2 & 1 & -2\\ 1 & -1 & 0 & 2\end{bmatrix}$

$$=\begin{bmatrix}2(2)+(-2)(0)+4(1) & 2(1)+(-2)(-2)+4(1) & 2(-3)+(-2)(1)+4(0) & 2(0)+(-2)(-2)+4(2)\\ 1(2)+0(0)+(-1)(1) & 1(1)+0(-2)+(-1)(-1) & 1(-3)+0(1)+(-1)(0) & 1(0)+0(-2)+(-1)(2)\\ 2(2)+1(0)+3(1) & 2(1)+1(-2)+3(-1) & 2(-3)+1(1)+3(0) & 2(0)+1(-2)+3(2)\end{bmatrix}=\begin{bmatrix}8 & 2 & -8 & 12\\ 1 & 2 & -3 & -2\\ 7 & -3 & -5 & 4\end{bmatrix}$$

**25.**
$$3X + A = B$$
$$3X + \begin{bmatrix} -1 & 3 \\ 2 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & 3 \\ 4 & -3 \end{bmatrix}$$
$$3X = \begin{bmatrix} 0 & -2 \\ 1 & 3 \\ 4 & -3 \end{bmatrix} - \begin{bmatrix} -1 & 3 \\ 2 & -1 \\ 3 & 1 \end{bmatrix}$$
$$3X = \begin{bmatrix} 1 & -5 \\ -1 & 4 \\ 1 & -4 \end{bmatrix}$$
$$X = \begin{bmatrix} \frac{1}{3} & -\frac{5}{3} \\ -\frac{1}{3} & \frac{4}{3} \\ \frac{1}{3} & -\frac{4}{3} \end{bmatrix}$$

**26.**
$$2A - 3X = 5B$$
$$2\begin{bmatrix} -1 & 3 \\ 2 & -1 \\ 3 & 1 \end{bmatrix} - 3X = 5\begin{bmatrix} 0 & -2 \\ 1 & 3 \\ 4 & -3 \end{bmatrix}$$
$$\begin{bmatrix} -2 & 6 \\ 4 & -2 \\ 6 & 2 \end{bmatrix} - 3X = \begin{bmatrix} 0 & -10 \\ 5 & 15 \\ 20 & -15 \end{bmatrix}$$
$$-3X = \begin{bmatrix} 0 & -10 \\ 5 & 15 \\ 20 & -15 \end{bmatrix} - \begin{bmatrix} -2 & 6 \\ 4 & -2 \\ 6 & 2 \end{bmatrix}$$
$$-3X = \begin{bmatrix} 2 & -16 \\ 1 & 17 \\ 14 & -17 \end{bmatrix}$$
$$X = \begin{bmatrix} -2/3 & 16/3 \\ -1/3 & -17/3 \\ -14/3 & 17/3 \end{bmatrix}$$

**27.**
$$2X - A = X + B$$
$$2X - \begin{bmatrix} -1 & 3 \\ 2 & -1 \\ 3 & 1 \end{bmatrix} = X + \begin{bmatrix} 0 & -2 \\ 1 & 3 \\ 4 & -3 \end{bmatrix}$$
$$X - \begin{bmatrix} -1 & 3 \\ 2 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & 3 \\ 4 & -3 \end{bmatrix}$$
$$X = \begin{bmatrix} 0 & -2 \\ 1 & 3 \\ 4 & -3 \end{bmatrix} + \begin{bmatrix} -1 & 3 \\ 2 & -1 \\ 3 & 1 \end{bmatrix}$$
$$X = \begin{bmatrix} -1 & 1 \\ 3 & 2 \\ 7 & -2 \end{bmatrix}$$

**28.**
$$3X + 2B = X - 2A$$
$$3X + 2\begin{bmatrix} 0 & -2 \\ 1 & 3 \\ 4 & -3 \end{bmatrix} = X - 2\begin{bmatrix} -1 & 3 \\ 2 & -1 \\ 3 & 1 \end{bmatrix}$$
$$3X + \begin{bmatrix} 0 & -4 \\ 2 & 6 \\ 8 & -6 \end{bmatrix} = X - \begin{bmatrix} -2 & 6 \\ 4 & -2 \\ 6 & 2 \end{bmatrix}$$
$$2X + \begin{bmatrix} 0 & -4 \\ 2 & 6 \\ 8 & -6 \end{bmatrix} = -\begin{bmatrix} -2 & 6 \\ 4 & -2 \\ 6 & 2 \end{bmatrix}$$
$$2X = -\begin{bmatrix} 0 & -4 \\ 2 & 6 \\ 8 & -6 \end{bmatrix} - \begin{bmatrix} -2 & 6 \\ 4 & -2 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -6 & -4 \\ -14 & 4 \end{bmatrix}$$
$$X = \begin{bmatrix} 1 & -1 \\ -3 & -2 \\ -7 & 2 \end{bmatrix}$$

**29.** $A^2 = A \cdot A = \begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2(2)+(-3)(1) & 2(-3)+(-3)(-1) \\ 1(2)+(-1)(1) & 1(-3)+(-1)(-1) \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix}$

**30.** $A^3 = A \cdot A \cdot A = \begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2(2)+(-3)(1) & 2(-3)+(-3)(-1) \\ 1(2)+(-1)(1) & 1(-3)+(-1)(-1) \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1(2)+(-3)(1) & 1(-3)+(-3)(-1) \\ 1(2)+(-2)(1) & 1(-3)+(-2)(-1) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

**31.** $B^2 = B \cdot B = \begin{bmatrix} 3 & -1 & 0 \\ 2 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ 2 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 3(3)+(-1)(2)+0(1) & 3(-1)+(-1)(-2)+0(0) & 3(0)+(-1)(-1)+0(2) \\ 2(3)+(-2)(2)+(-1)(1) & 2(-1)+(-2)(-2)+(-1)(0) & 2(0)+(-2)(-1)+(-1)(2) \\ 1(3)+0(2)+2(1) & 1(-1)+0(-2)+2(0) & 1(0)+0(-1)+2(2) \end{bmatrix} = \begin{bmatrix} 7 & -1 & 1 \\ 1 & 2 & 0 \\ 5 & -1 & 4 \end{bmatrix}$$

**32.** $$B^3 = B\cdot B\cdot B = \begin{bmatrix} 3 & -1 & 0 \\ 2 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & -1 & 0 \\ 2 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & -1 & 0 \\ 2 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3(3)+(-1)(2)+0(1) & 3(-1)+(-1)(-2)+0(0) & 3(0)+(-1)(-1)+0(2) \\ 2(3)+(-2)(2)+(-1)(1) & 2(-1)+(-2)(-2)+(-1)(0) & 2(0)+(-2)(-1)+(-1)(2) \\ 1(3)+0(2)+2(1) & 1(-1)+0(-2)+2(0) & 1(0)+0(-1)+2(2) \end{bmatrix}\begin{bmatrix} 3 & -1 & 0 \\ 2 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -1 & 1 \\ 1 & 2 & 0 \\ 5 & -1 & 4 \end{bmatrix}\begin{bmatrix} 3 & -1 & 0 \\ 2 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 7(3)+(-1(2)+1(1) & 7(-1)+(-1)(-2)+1(0) & 7(0)+(-1)(-1)+1(2) \\ 1(3)+2(2)+0(1) & 1(-1)+-2(-2)+(0)(0) & 1(0)+2(-1)+0(2) \\ 5(3)+(-1)(2)+4(1) & 5(-1)+(-1)(-2)+4(0) & 5(0)+(-1)(-1)+4(2) \end{bmatrix} = \begin{bmatrix} 20 & -5 & 3 \\ 7 & -5 & -2 \\ 17 & -3 & 9 \end{bmatrix}$$

**33.** $$\begin{bmatrix} 3 & -8 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$$
$$\begin{cases} 3x - 8y = 11 \\ 4x + 3y = 1 \end{cases}$$

**34.** $$\begin{bmatrix} 2 & 7 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 16 \end{bmatrix}$$
$$\begin{cases} 2x + 7y = 1 \\ 3x - 4y = 16 \end{cases}$$

**35.** $$\begin{bmatrix} 1 & -3 & -2 \\ 3 & 1 & 0 \\ 2 & -4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$
$$\begin{cases} x - 3y - 2z = 6 \\ 3x + y = 2 \\ 2x - 4y + 5z = 1 \end{cases}$$

**36.** $$\begin{bmatrix} 2 & 0 & 5 \\ 3 & -5 & 1 \\ 4 & -7 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 14 \end{bmatrix}$$
$$\begin{cases} 2x + 5z = 9 \\ 3x - 5y + z = 7 \\ 4x - 7y + 6z = 14 \end{cases}$$

**37.** $$\begin{bmatrix} 2 & -1 & 0 & 2 \\ 4 & 1 & 2 & -3 \\ 6 & 0 & 1 & -2 \\ 5 & 2 & -1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$
$$\begin{cases} 2x_1 - x_2 + 2x_4 = 5 \\ 4x_1 + x_2 + 2x_3 - 3x_4 = 6 \\ 6x_1 + x_3 - 2x_4 = 10 \\ 5x_1 + 2x_2 - x_3 - 4x_4 = 8 \end{cases}$$

**38.** $$\begin{bmatrix} 5 & -1 & 2 & -3 \\ 4 & 0 & 2 & 0 \\ 2 & -2 & 5 & -4 \\ 3 & 1 & -3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -1 \\ 2 \end{bmatrix}$$
$$\begin{cases} 5x_1 - x_2 + 2x_3 - 3x_4 = -2 \\ 4x_1 + 2x_3 = 2 \\ 2x_1 - 2x_2 + 5x_3 - 4x_4 = -1 \\ 3x_1 + x_2 - 3x_3 + 4x_4 = 2 \end{cases}$$

**39.** **a.** $3 \times 4$. There are three different fish in four different samples.
**b.** Fish A was caught in sample number 4.
**c.** Fish B. There are more 1's in this row than in any other row.

**40.** **a.** $4 \times 4$. There are four animals in this system.
**b.** Rabbits do not prey on hawks.
**c.** The coyote is not preyed on by any other animal in this system.
**d.** The rabbit does not prey on any animal in this system.

**41.** $$0.98\begin{bmatrix} 2.0 & 1.4 & 3.0 & 1.4 \\ 0.8 & 1.1 & 2.0 & 0.9 \\ 3.6 & 1.2 & 4.5 & 1.5 \end{bmatrix} = \begin{bmatrix} 0.98(2.0) & 0.98(1.4) & 0.98(3.0) & 0.98(1.4) \\ 0.98(0.8) & 0.98(1.1) & 0.98(2.0) & 0.98(0.9) \\ 0.98(3.6) & 0.98(1.2) & 0.98(4.5) & 0.98(1.5) \end{bmatrix} = \begin{bmatrix} 1.96 & 1.37 & 2.94 & 1.37 \\ 0.78 & 1.08 & 1.96 & 0.88 \\ 3.53 & 1.18 & 4.41 & 1.47 \end{bmatrix}$$

**42.** $$1.06\begin{bmatrix} 28.0 & 28.9 & 30.0 & 31.5 \\ 29.0 & 30.3 & 32.5 & 34.5 \\ 30.0 & 31.4 & 34.0 & 37.0 \end{bmatrix} = \begin{bmatrix} 1.06(28.0) & 1.06(28.9) & 1.06(30.0) & 1.06(31.5) \\ 1.06(29.0) & 1.06(30.3) & 1.06(32.5) & 1.06(34.5) \\ 1.06(30.0) & 1.06(31.4) & 1.06(34.0) & 1.06(37.0) \end{bmatrix} = \begin{bmatrix} 29.7 & 30.6 & 31.8 & 33.4 \\ 30.7 & 32.1 & 34.5 & 36.6 \\ 31.8 & 33.3 & 36.0 & 39.2 \end{bmatrix}$$

**43.** **a.** $$H + A = \begin{bmatrix} 14 & 3 \\ 14 & 3 \\ 10 & 7 \end{bmatrix} + \begin{bmatrix} 12 & 5 \\ 7 & 10 \\ 8 & 9 \end{bmatrix} = \begin{bmatrix} 26 & 8 \\ 21 & 13 \\ 18 & 16 \end{bmatrix}$$
The matrix represents the total number of wins and losses for each team for the season.

**b.** $$H - A = \begin{bmatrix} 14 & 3 \\ 14 & 3 \\ 10 & 7 \end{bmatrix} - \begin{bmatrix} 12 & 5 \\ 7 & 10 \\ 8 & 9 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 7 & -7 \\ 2 & -2 \end{bmatrix}$$
The matrix represents the difference between performances at home and performances away.

**44.** **a.** $A+B=\begin{bmatrix}23 & 35 & 49\\ 32 & 41 & 24\end{bmatrix}+\begin{bmatrix}19 & 28 & 36\\ 25 & 38 & 26\end{bmatrix}=\begin{bmatrix}42 & 63 & 85\\ 57 & 79 & 50\end{bmatrix}$

**b.** The sum of $A$ and $B$ represents the total number of each of the three television sets available from both stores.

**45.** $A=\begin{bmatrix}530 & 650 & 815\\ 190 & 385 & 715\\ 485 & 600 & 610\\ 150 & 210 & 305\end{bmatrix}$, $B=\begin{bmatrix}480 & 500 & 675\\ 175 & 215 & 345\\ 400 & 350 & 480\\ 70 & 95 & 280\end{bmatrix}$

$A-B=\begin{bmatrix}50 & 150 & 140\\ 15 & 170 & 370\\ 85 & 250 & 130\\ 80 & 115 & 25\end{bmatrix}$

$A-B$ is number sold of each item during the week.

**46.** $A=\begin{bmatrix}315 & 200 & 415\\ 285 & 175 & 300\\ 275 & 195 & 250\end{bmatrix}$, $B=\begin{bmatrix}200 & 175 & 350\\ 150 & 90 & 180\\ 105 & 50 & 175\end{bmatrix}$

$A+B=\begin{bmatrix}515 & 375 & 765\\ 435 & 265 & 480\\ 380 & 245 & 425\end{bmatrix}$

$A+B$ gives the number of employees in each area for both branches of the company.

**47.** $C=\begin{bmatrix}0.04 & 0.06 & 0.05\\ 0.04 & 0.04 & 0.04\\ 0.03 & 0.07 & 0.06\end{bmatrix}$, $S=\begin{bmatrix}500 & 600\\ 250 & 450\\ 600 & 750\end{bmatrix}$

$CS=\begin{bmatrix}0.04 & 0.06 & 0.05\\ 0.04 & 0.04 & 0.04\\ 0.03 & 0.07 & 0.06\end{bmatrix}\begin{bmatrix}500 & 600\\ 250 & 450\\ 600 & 750\end{bmatrix}$

$=\begin{bmatrix}65 & 88.5\\ 54 & 72\\ 68.5 & 94.5\end{bmatrix}$

To minimize commissions costs, customer $S_1$ should use company $T_2$.

**48.** $S=\begin{bmatrix}52 & 50 & 75 & 20\\ 45 & 48 & 80 & 20\\ 62 & 70 & 78 & 25\end{bmatrix}$, $P=\begin{bmatrix}.25 & .50\\ .30 & .75\\ .15 & .45\\ .10 & .50\end{bmatrix}$

Total cost and total revenue can be determined from the product of $S$ and $P$.

$SP=\begin{bmatrix}52 & 50 & 75 & 20\\ 45 & 48 & 80 & 20\\ 62 & 70 & 78 & 25\end{bmatrix}\begin{bmatrix}.25 & .50\\ .30 & .75\\ .15 & .45\\ 10 & .50\end{bmatrix}$

$=\begin{bmatrix}41.25 & 107.25\\ 39.65 & 104.50\\ 50.70 & 131.10\end{bmatrix}$

**49.** $R_x\cdot\begin{bmatrix}2 & -3\\ 5 & 6\\ 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & -3\\ 5 & 6\\ 1 & 1\end{bmatrix}=\begin{bmatrix}2 & -3\\ -5 & -6\\ 1 & 1\end{bmatrix}\Rightarrow \begin{matrix}P'(2,-5)\\ Q'(-3,-6)\end{matrix}$

**50.** $R_y\cdot\begin{bmatrix}-1 & 2\\ 2 & -4\\ 1 & 1\end{bmatrix}=\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-1 & 2\\ 2 & -4\\ 1 & 1\end{bmatrix}=\begin{bmatrix}1 & -2\\ 2 & -4\\ 1 & 1\end{bmatrix}\Rightarrow \begin{matrix}P'(1,2)\\ Q'(-2,-4)\end{matrix}$

**51.** $R_{xy}\cdot\begin{bmatrix}-3 & -5\\ 1 & 3\\ 1 & 1\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-3 & -5\\ 1 & 3\\ 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 3\\ -3 & -5\\ 1 & 1\end{bmatrix}\Rightarrow \begin{matrix}P'(1,-3)\\ Q'(3,-5)\end{matrix}$

**52.** $R_y\cdot R_{90}\cdot\begin{bmatrix}2 & -1\\ 4 & 3\\ 1 & 1\end{bmatrix}=\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & -1\\ 4 & 3\\ 1 & 1\end{bmatrix}=\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-4 & -3\\ 2 & -1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}4 & 3\\ 2 & -1\\ 1 & 1\end{bmatrix}\Rightarrow \begin{matrix}P'(4,2)\\ Q'(3,-1)\end{matrix}$

**53.** $R_{xy}\cdot T_{3,-1}\cdot\begin{bmatrix}-1 & 1 & 3\\ 5 & -2 & 4\\ 1 & 1 & 1\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 3\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-1 & 1 & 3\\ 5 & -2 & 4\\ 1 & 1 & 1\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 4 & 6\\ 4 & -3 & 3\\ 1 & 1 & 1\end{bmatrix}=\begin{bmatrix}4 & -3 & 3\\ 2 & 4 & 6\\ 1 & 1 & 1\end{bmatrix}\Rightarrow \begin{matrix}A'(4,2)\\ B'(-3,4)\\ C'(3,6)\end{matrix}$

**54.** $R_{180}\cdot R_y\cdot\begin{bmatrix}-4 & -2 & 2\\ -1 & 1 & -5\\ 1 & 1 & 1\end{bmatrix}=\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-4 & -2 & 2\\ -1 & 1 & -5\\ 1 & 1 & 1\end{bmatrix}=\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}4 & 2 & -2\\ -1 & 1 & -5\\ 1 & 1 & 1\end{bmatrix}=\begin{bmatrix}-4 & -2 & 2\\ 1 & -1 & 5\\ 1 & 1 & 1\end{bmatrix}\Rightarrow \begin{matrix}A'(-4,1)\\ B'(-2,-1)\\ C'(2,5)\end{matrix}$

**55.** $T_{1,-6}\cdot\begin{bmatrix}-1 & -1 & 4 & 4\\ 2 & 6 & 6 & 2\\ 1 & 1 & 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & -6\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-1 & -1 & 4 & 4\\ 2 & 6 & 6 & 2\\ 1 & 1 & 1 & 1\end{bmatrix}=\begin{bmatrix}0 & 0 & 5 & 5\\ -4 & 0 & 0 & -4\\ 1 & 1 & 1 & 1\end{bmatrix}$

$R_{180}\cdot\begin{bmatrix}0 & 0 & 5 & 5\\ -4 & 0 & 0 & -4\\ 1 & 1 & 1 & 1\end{bmatrix}=\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 5 & 5\\ -4 & 0 & 0 & -4\\ 1 & 1 & 1 & 1\end{bmatrix}=\begin{bmatrix}0 & 0 & -5 & -5\\ 4 & 0 & 0 & 4\\ 1 & 1 & 1 & 1\end{bmatrix}$

$T_{-1,6}\cdot\begin{bmatrix}0 & 0 & -5 & -5\\ 4 & 0 & 0 & 4\\ 1 & 1 & 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 6\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & -5 & -5\\ 4 & 0 & 0 & 4\\ 1 & 1 & 1 & 1\end{bmatrix}=\begin{bmatrix}-1 & -1 & -6 & -6\\ 10 & 6 & 6 & 10\\ 1 & 1 & 1 & 1\end{bmatrix}\Rightarrow$ $A'(-1, 10)$ $C'(-6, 6)$ $B'(-1, 6)$ $D'(-6, 10)$

**56.** $T_{-6,0}\cdot\begin{bmatrix}-3 & -1 & 6 & 4\\ -1 & 4 & 0 & -5\\ 1 & 1 & 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & -6\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-3 & -1 & 6 & 4\\ -1 & 4 & 0 & -5\\ 1 & 1 & 1 & 1\end{bmatrix}=\begin{bmatrix}-9 & -7 & 0 & -2\\ -1 & 4 & 0 & -5\\ 1 & 1 & 1 & 1\end{bmatrix}$

$R_{270}\cdot\begin{bmatrix}-9 & -7 & 0 & -2\\ -1 & 4 & 0 & -5\\ 1 & 1 & 1 & 1\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-9 & -7 & 0 & -2\\ -1 & 4 & 0 & -5\\ 1 & 1 & 1 & 1\end{bmatrix}=\begin{bmatrix}-1 & 4 & 0 & -5\\ 9 & 7 & 0 & 2\\ 1 & 1 & 1 & 1\end{bmatrix}$

$T_{6,0}\cdot\begin{bmatrix}-1 & 4 & 0 & -5\\ 9 & 7 & 0 & 2\\ 1 & 1 & 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 6\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-1 & 4 & 0 & -5\\ 9 & 7 & 0 & 2\\ 1 & 1 & 1 & 1\end{bmatrix}=\begin{bmatrix}5 & 10 & 6 & 1\\ 9 & 7 & 0 & 2\\ 1 & 1 & 1 & 1\end{bmatrix}\Rightarrow$ $A'(5, 9)$ $C'(6, 0)$ $B'(10, 7)$ $D'(1, 2)$

**57.** **a.** For 1 year from now, $n = 2$.

$\begin{bmatrix}0.55 & 0.45\end{bmatrix}\begin{bmatrix}0.989 & 0.011\\ 0.007 & 0.993\end{bmatrix}^2=\begin{bmatrix}0.5443 & 0.4557\end{bmatrix}$

1 year from now 45.6% of the customers will be drinking diet soda.

**b.** For 3 years from now, $n = 6$.

$\begin{bmatrix}0.55 & 0.45\end{bmatrix}\begin{bmatrix}0.989 & 0.011\\ 0.007 & 0.993\end{bmatrix}^6=\begin{bmatrix}0.5334 & 0.4666\end{bmatrix}$

3 years from now 46.7% of the customers will be drinking diet soda.

**58.** When $n = 21$, $d > r$. After 10 years, the number of diet soda drinkers out numbers the number of regular soda drinkers.

**59.** **a.** For 12 months from now, $n = 12$.

$\begin{bmatrix}0.15 & 0.85\end{bmatrix}\begin{bmatrix}0.975 & 0.025\\ 0.014 & 0.986\end{bmatrix}^{12}=\begin{bmatrix}0.2293 & 0.7707\end{bmatrix}$

12 months from now 22.9% of the customers will be renting DVD movies online.

**b.** For 24 months from now, $n = 24$.

$\begin{bmatrix}0.15 & 0.85\end{bmatrix}\begin{bmatrix}0.975 & 0.025\\ 0.014 & 0.986\end{bmatrix}^{24}=\begin{bmatrix}0.2785 & 0.7215\end{bmatrix}$

24 months from now 27.9% of the customers will be renting DVD movies online.

**60.** For 5 months from now, $n = 5$.

$\begin{bmatrix}0.25 & 0.75\end{bmatrix}\begin{bmatrix}0.98 & 0.02\\ 0.05 & 0.95\end{bmatrix}^5=\begin{bmatrix}0.3913 & 0.6087\end{bmatrix}$

5 months from now, 39.1% of the customers will stop at Store *A*.

**61.** When $n = 11$, $a > 0.5053$.

$\begin{bmatrix}0.25 & 0.75\end{bmatrix}\begin{bmatrix}0.98 & 0.02\\ 0.05 & 0.95\end{bmatrix}^{11}=\begin{bmatrix}0.5053 & 0.4947\end{bmatrix}$

After 11 months, Store *A* will have 50% of the town's customers.

**62.** In *n* years,

$\begin{bmatrix}475{,}000 & 375{,}000\end{bmatrix}\begin{bmatrix}0 & 0.65\\ 1.25 & 0\end{bmatrix}^n$

After 4 years,

$\begin{bmatrix}475{,}000 & 375{,}000\end{bmatrix}\begin{bmatrix}0 & 0.65\\ 1.25 & 0\end{bmatrix}^4=\begin{bmatrix}313{,}574 & 247{,}558\end{bmatrix}$

$313{,}574 + 247{,}558 = 561{,}132$

There will be approximately 561,000 plants in the reserve after 4 years.

**63.** Using $A$ and $B$ as given and a calculator,

$$AB=\begin{bmatrix}24 & 21 & -12 & 32 & 0\\ -7 & -8 & 3 & 21 & 20\\ 32 & 10 & -32 & 1 & 5\\ 19 & -15 & -17 & 30 & 20\\ 29 & 9 & -28 & 13 & -6\end{bmatrix}$$

**64.** Using $A$ and $B$ as given and a calculator,

$$BA=\begin{bmatrix}30 & -27 & 3 & 13 & 28\\ 7 & -13 & -3 & -5 & 21\\ 17 & -7 & -3 & 12 & 33\\ 34 & -8 & 23 & 2 & -17\\ 12 & 14 & -7 & 14 & -8\end{bmatrix}$$

**65.** Using $A$ as given and a calculator,

$$A^3=\begin{bmatrix}46 & -100 & 36 & 273 & 93\\ 82 & -93 & 19 & 27 & 97\\ 73 & -10 & -23 & 109 & 83\\ 212 & -189 & 52 & 37 & 156\\ 68 & -22 & 54 & 221 & 58\end{bmatrix}$$

**66.** Using $B$ as given and a calculator,

$$B^3=\begin{bmatrix}55 & -65 & 65 & 291 & -154\\ -60 & -72 & 69 & 87 & -26\\ 98 & -94 & -33 & 128 & -124\\ 149 & 213 & -49 & 114 & -93\\ 44 & -57 & 55 & 63 & -121\end{bmatrix}$$

**67.** Using $A$ and $B$ as given and a calculator,

$$A^2+B^2=\begin{bmatrix}76 & -8 & -25 & 30 & 6\\ 14 & 16 & -10 & 14 & 2\\ 39 & 0 & -45 & 22 & 27\\ 0 & -4 & 23 & 83 & -16\\ 56 & -20 & -22 & 7 & 5\end{bmatrix}$$

**68.** Using $A$ and $B$ as given and a calculator,

$$AB-BA=\begin{bmatrix}-6 & 48 & -15 & 19 & -28\\ -14 & 5 & 6 & 26 & -1\\ 15 & 17 & -29 & -11 & -28\\ -15 & -7 & -40 & 28 & 37\\ 17 & -5 & -21 & -1 & 2\end{bmatrix}$$

## Connecting Concepts

**69.** $$3A=3\begin{bmatrix}2+3i & 1-2i\\ 1+i & 2-i\end{bmatrix}=\begin{bmatrix}6+9i & 3-6i\\ 3+3i & 6-3i\end{bmatrix}$$

**70.** $$-2B=-2\begin{bmatrix}1-i & 2+3i\\ 3+2i & 4-i\end{bmatrix}=\begin{bmatrix}-2+2i & -4-6i\\ -6-4i & -8+2i\end{bmatrix}$$

**71.** $$2iB=2i\begin{bmatrix}1-i & 2+3i\\ 3+2i & 4-i\end{bmatrix}=\begin{bmatrix}2+2i & -6+4i\\ -4+6i & 2+8i\end{bmatrix}$$

**72.** $$3iA=3i\begin{bmatrix}2+3i & 1-2i\\ 1+i & 2-i\end{bmatrix}=\begin{bmatrix}-9+6i & 6+3i\\ -3+3i & 3+6i\end{bmatrix}$$

**73.** $$A+B=\begin{bmatrix}2+3i & 1-2i\\ 1+i & 2-i\end{bmatrix}+\begin{bmatrix}1-i & 2+3i\\ 3+2i & 4-i\end{bmatrix}=\begin{bmatrix}3+2i & 3+i\\ 4+3i & 6-2i\end{bmatrix}$$

**74.** $$A-B=\begin{bmatrix}2+3i & 1-2i\\ 1+i & 2-i\end{bmatrix}-\begin{bmatrix}1-i & 2+3i\\ 3+2i & 4-i\end{bmatrix}=\begin{bmatrix}1+4i & -1-5i\\ -2-i & -2\end{bmatrix}$$

**75.** $$AB=\begin{bmatrix}2+3i & 1-2i\\ 1+i & 2-i\end{bmatrix}\begin{bmatrix}1-i & 2+3i\\ 3+2i & 4-i\end{bmatrix}=\begin{bmatrix}(2+3i)(1-i)+(1-2i)(3+2i) & (2+3i)(2+3i)+(1-2i)(4-i)\\ (1+i)(1-i)+(2-i)(3+2i) & (1+i)(2+3i)+(2-i)(4-i)\end{bmatrix}=\begin{bmatrix}12-3i & -3+3i\\ 10+i & 6-i\end{bmatrix}$$

**76.** $$BA=\begin{bmatrix}1-i & 2+3i\\ 3+2i & 4-i\end{bmatrix}\begin{bmatrix}2+3i & 1-2i\\ 1+i & 2-i\end{bmatrix}=\begin{bmatrix}(1-i)(2+3i)+(2+3i)(1+i) & (1-i)(1-2i)+(2+3i)(2-i)\\ (3+2i)(2+3i)+(4-i)(1+i) & (3+2i)(1-2i)+(4-i)(2-i)\end{bmatrix}=\begin{bmatrix}4+6i & 6+i\\ 5+16i & 14-10i\end{bmatrix}$$

**77.** $$A^2=A\cdot A=\begin{bmatrix}2+3i & 1-2i\\ 1+i & 2-i\end{bmatrix}\begin{bmatrix}2+3i & 1-2i\\ 1+i & 2-i\end{bmatrix}$$
$$=\begin{bmatrix}(2+3i)(2+3i)+(1-2i)(1+i) & (2+3i)(1-2i)+(1-2i)(2-i)\\ (1+i)(2+3i)+(2-i)(1+i) & (1+i)(1-2i)+(2-i)(2-i)\end{bmatrix}$$
$$=\begin{bmatrix}-2+11i & 8-6i\\ 2+6i & 6-5i\end{bmatrix}$$

**78.** $B^2 = B \cdot B = \begin{bmatrix} 1-i & 2+3i \\ 3+2i & 4-i \end{bmatrix} \begin{bmatrix} 1-i & 2+3i \\ 3+2i & 4-i \end{bmatrix}$

$$= \begin{bmatrix} (1-i)(1-i)+(2+3i)(3+2i) & (1-i)(2+3i)+(2+3i)(4-i) \\ (3+2i)(1-i)+(4-i)(3+2i) & (3+2i)(2+3i)+(4-i)(4-i) \end{bmatrix}$$

$$= \begin{bmatrix} +11i & 16+11i \\ 19+4i & 15+5i \end{bmatrix}$$

**79.**

$$(R_{90})^2 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{180}$$

Rotating 90° twice around the origin is the same as rotating 180° once around the origin.

$$(R_{90})^3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{270}$$

Rotating 90° three times around the origin is the same as rotating 270° once around the origin.

**80.**

$$R_x \cdot R_y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_y \cdot R_x = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, $R_x \cdot R_y = R_y \cdot R_x$ .

Reflecting across the $x$-axis and then the $y$-axis is the same as reflecting across the $y$-axis and then the $x$-axis.

## Prepare for Section 10.3

**PS1.** $-\frac{3}{2}$

**PS2.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**PS3.** 1. Interchange any two rows.
2. Multiply all elements in a row by the same nonzero number.
3. Replace a row by the sum of that row and a nonzero multiple of any other row.

**PS4.** $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 4 \\ -3 & 2 & 2 \end{bmatrix} \xrightarrow[3R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -4 & 11 \end{bmatrix}$

**PS5.**

$$AX = B$$
$$A^{-1}AX = A^{-1}B$$
$$X = A^{-1}B$$

**PS6.** $\begin{cases} 2x+3y=9 \\ 4x-5y=7 \end{cases}$

## Section 10.3

**1.** $\left[\begin{array}{rr|rr} 1 & -3 & 1 & 0 \\ -2 & 5 & 0 & 1 \end{array}\right] \xrightarrow{2R_1+R_2} \left[\begin{array}{rr|rr} 1 & -3 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{array}\right] \xrightarrow{-1R_2} \left[\begin{array}{rr|rr} 1 & -3 & 1 & 0 \\ 0 & 1 & -2 & -1 \end{array}\right] \xrightarrow{3R_2+R_1} \left[\begin{array}{rr|rr} 1 & 0 & -5 & -3 \\ 0 & 1 & -2 & -1 \end{array}\right]$

The inverse matrix is $\begin{bmatrix} -5 & -3 \\ -2 & -1 \end{bmatrix}$.

**2.** $$\left[\begin{array}{rr|rr} 1 & 2 & 1 & 0 \\ -2 & -3 & 0 & 1 \end{array}\right] \xrightarrow{2R_1+R_2} \left[\begin{array}{rr|rr} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{rr|rr} 1 & 0 & -3 & -2 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

The inverse matrix is $\begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}$.

**3.** $$\left[\begin{array}{rr|rr} 1 & 4 & 1 & 0 \\ 2 & 10 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{rr|rr} 1 & 4 & 1 & 0 \\ 0 & 2 & -2 & 1 \end{array}\right] \xrightarrow{(1/2)R_2} \left[\begin{array}{rr|rr} 1 & 4 & 1 & 0 \\ 0 & 1 & -1 & \frac{1}{2} \end{array}\right] \xrightarrow{-4R_2+R_1} \left[\begin{array}{rr|rr} 1 & 0 & 5 & -2 \\ 0 & 1 & -1 & \frac{1}{2} \end{array}\right]$$

The inverse matrix is $\begin{bmatrix} 5 & -2 \\ -1 & \frac{1}{2} \end{bmatrix}$.

**4.** $$\left[\begin{array}{rr|rr} -2 & 3 & 1 & 0 \\ -6 & -8 & 0 & 1 \end{array}\right] \xrightarrow{(-1/2)R_1} \left[\begin{array}{rr|rr} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ -6 & -8 & 0 & 1 \end{array}\right] \xrightarrow{6R_1+R_2} \left[\begin{array}{rr|rr} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -17 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{(-1/17)R_2} \left[\begin{array}{rr|rr} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{17} & -\frac{1}{17} \end{array}\right] \xrightarrow{(3/2)R_2+R_1} \left[\begin{array}{rr|rr} 1 & 0 & -\frac{4}{17} & -\frac{3}{34} \\ 0 & 1 & \frac{3}{17} & -\frac{1}{17} \end{array}\right]$$

The inverse matrix is $\begin{bmatrix} -\frac{4}{17} & -\frac{3}{34} \\ \frac{3}{17} & -\frac{1}{17} \end{bmatrix}$.

**5.** $$\left[\begin{array}{rrr|rrr} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 5 & 1 & 0 & 1 & 0 \\ 3 & 6 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{rrr|rrr} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_1} \left[\begin{array}{rrr|rrr} 1 & 0 & -7 & 5 & -2 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right] \xrightarrow[-3R_3+R_2]{7R_3+R_1} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -16 & -2 & 7 \\ 0 & 1 & 0 & 7 & 1 & -3 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$$

The inverse matrix is $\begin{bmatrix} -16 & -2 & 7 \\ 7 & 1 & -3 \\ -3 & 0 & 1 \end{bmatrix}$.

**6.** $$\left[\begin{array}{rrr|rrr} 1 & 3 & -2 & 1 & 0 & 0 \\ -1 & -5 & 6 & 0 & 1 & 0 \\ 2 & 6 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R_1+R_3]{R_1+R_2} \left[\begin{array}{rrr|rrr} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & -2 & 4 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{(-1/2)R_2} \left[\begin{array}{rrr|rrr} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[2R_3+R_1]{2R_3+R_2} \left[\begin{array}{rrr|rrr} 1 & 3 & 0 & -3 & 0 & 2 \\ 0 & 1 & 0 & -\frac{9}{2} & -\frac{1}{2} & 2 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{-3R_2+R_1} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{21}{2} & \frac{3}{2} & -4 \\ 0 & 1 & 0 & -\frac{9}{2} & -\frac{1}{2} & 2 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

The inverse matrix is $\begin{bmatrix} \frac{21}{2} & \frac{3}{2} & -4 \\ -\frac{9}{2} & -\frac{1}{2} & 2 \\ -2 & 0 & 1 \end{bmatrix}$.

**7.** $$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 6 & 1 & 0 & 1 & 0 \\ 3 & 6 & -4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & 3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -3 & 0 & 1 \end{array}\right] \xrightarrow{(1/2)R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 0 & -1 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-1R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 3 & 0 & -1 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 3 & -1 & 0 \\ 0 & 1 & \frac{3}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 3 & 0 & -1 \end{array}\right] \xrightarrow[(-3/2)R_3+R_2]{4R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 15 & -1 & -4 \\ 0 & 1 & 0 & -\frac{11}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & 3 & 0 & -1 \end{array}\right]$$

The inverse matrix is $\begin{bmatrix} 15 & -1 & -4 \\ -\frac{11}{2} & \frac{1}{2} & \frac{3}{2} \\ 3 & 0 & -1 \end{bmatrix}$.

**8.** $$\left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 6 & 4 & -1 & 0 & 1 & 0 \\ 4 & 2 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(1/2)R_1} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 6 & 4 & -1 & 0 & 1 & 0 \\ 4 & 2 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-4R_1+R_3]{-6R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 2 & -3 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-1R_3} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right] \xrightarrow{(-1/2)R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{3}{2} & 2 & -\frac{1}{2} & 0 \\ 0 & 1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right] \xrightarrow[-2R_3+R_2]{(-3/2)R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 1 & 0 & -7 & 1 & 2 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right]$$

The inverse matrix is $\begin{bmatrix} 5 & -\frac{1}{2} & -\frac{3}{2} \\ -7 & 1 & 2 \\ 2 & 0 & -1 \end{bmatrix}$.

**9.** $$\left[\begin{array}{ccc|ccc} 2 & 4 & -4 & 1 & 0 & 0 \\ 1 & 3 & -4 & 0 & 1 & 0 \\ 2 & 4 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(1/2)R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & -2 & \frac{1}{2} & 0 & 0 \\ 1 & 3 & -4 & 0 & 1 & 0 \\ 2 & 4 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R_1+R_3]{-1R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -2 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{3}{2} & -2 & 0 \\ 0 & 1 & -2 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[2R_3+R_2]{-2R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{2} & -2 & -2 \\ 0 & 1 & 0 & -\frac{5}{2} & 1 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

The inverse matrix is $\begin{bmatrix} \frac{7}{2} & -2 & -2 \\ -\frac{5}{2} & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$.

**10.** $$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 & 1 & 0 \\ 3 & -6 & 6 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 1 \end{array}\right]$$

The matrix does not have an inverse.

**11.** $$\left[\begin{array}{rrrr|rrrr} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 2 & -1 & 5 & 1 & 0 & 1 & 0 & 0 \\ 3 & -3 & 7 & 5 & 0 & 0 & 1 & 0 \\ -2 & 3 & -4 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{2R_1+R_2 \\ -3R_1+R_3 \\ 2R_1+R_4}} \left[\begin{array}{rrrr|rrrr} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-1R_2+R_4} \left[\begin{array}{rrrr|rrrr} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & -1 & 0 & 1 \end{array}\right] \xrightarrow{1R_3+R_4} \left[\begin{array}{rrrr|rrrr} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{(1/4)R_4} \left[\begin{array}{rrrr|rrrr} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{rrrr|rrrr} 1 & 0 & 3 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{array}\right]$$

$$\xrightarrow{\substack{-3R_3+R_1 \\ -R_3+R_2}} \left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & -6 & 8 & 1 & -3 & 0 \\ 0 & 1 & 0 & -3 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{array}\right] \xrightarrow{\substack{6R_4+R_1 \\ 3R_4+R_2 \\ -2R_4+R_3}} \left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & \frac{19}{2} & -\frac{1}{2} & -\frac{3}{2} & \frac{3}{2} \\ 0 & 1 & 0 & 0 & \frac{7}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{array}\right]$$

The inverse matrix is $\begin{bmatrix} \frac{19}{2} & -\frac{1}{2} & -\frac{3}{2} & \frac{3}{2} \\ \frac{7}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \\ -\frac{7}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$.

**12.** $$\left[\begin{array}{rrrr|rrrr} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 3 & 2 & -1 & 5 & 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 5 & 0 & 0 & 1 & 0 \\ 4 & 4 & -4 & 7 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{-3R_1+R_2 \\ -2R_1+R_3 \\ -4R_1+R_4}} \left[\begin{array}{rrrr|rrrr} 1 & 1 & -1 & 2 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{-1R_2 \\ -1R_4}} \left[\begin{array}{rrrr|rrrr} 1 & 1 & -1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 3 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 & 0 & -1 \end{array}\right] \xrightarrow{-R_2+R_1} \left[\begin{array}{rrrr|rrrr} 1 & 0 & 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 3 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 & 0 & -1 \end{array}\right]$$

$$\xrightarrow{\substack{-R_3+R_1 \\ 2R_3+R_2}} \left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 3 & -1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 & 0 & -1 \end{array}\right] \xrightarrow{\substack{-3R_4+R_2 \\ -R_4+R_3}} \left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -13 & -1 & 2 & 3 \\ 0 & 0 & 1 & 0 & -6 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 4 & 0 & 0 & -1 \end{array}\right]$$

The inverse matrix is $\begin{bmatrix} 0 & 1 & -1 & 0 \\ -13 & -1 & 2 & 3 \\ -6 & 0 & 1 & 1 \\ 4 & 0 & 0 & -1 \end{bmatrix}$.

**13.** $$\left[\begin{array}{rrrr|rrrr} 1 & -1 & 1 & 3 & 1 & 0 & 0 & 0 \\ 2 & -1 & 4 & 8 & 0 & 1 & 0 & 0 \\ 1 & 1 & 6 & 10 & 0 & 0 & 1 & 0 \\ -1 & 5 & 5 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[1R_1+R_4]{\substack{-2R_1+R_2 \\ -1R_1+R_3}} \left[\begin{array}{rrrr|rrrr} 1 & -1 & 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & -2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 7 & -1 & 0 & 1 & 0 \\ 0 & 4 & 6 & 7 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-4R_2+R_4]{-2R_2+R_3} \left[\begin{array}{rrrr|rrrr} 1 & -1 & 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 3 & -2 & 1 & 0 \\ 0 & 0 & -2 & -1 & 9 & -4 & 0 & 1 \end{array}\right] \xrightarrow{2R_3+R_4} \left[\begin{array}{rrrr|rrrr} 1 & -1 & 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 5 & 15 & -8 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{(1/5)R_4} \left[\begin{array}{rrrr|rrrr} 1 & -1 & 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 & -\frac{8}{5} & \frac{2}{5} & \frac{1}{5} \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{rrrr|rrrr} 1 & 0 & 3 & 5 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 & -\frac{8}{5} & \frac{2}{5} & \frac{1}{5} \end{array}\right]$$

$$\xrightarrow[-2R_3+R_2]{-3R_3+R_1} \left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & -4 & -10 & 7 & -3 & 0 \\ 0 & 1 & 0 & -4 & -8 & 5 & -2 & 0 \\ 0 & 0 & 1 & 3 & 3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 & -\frac{8}{5} & \frac{2}{5} & \frac{1}{5} \end{array}\right] \xrightarrow[-3R_4+R_3]{\substack{4R_4+R_1 \\ 4R_4+R_2}} \left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 2 & \frac{3}{5} & -\frac{7}{5} & \frac{4}{5} \\ 0 & 1 & 0 & 0 & 4 & -\frac{7}{5} & -\frac{2}{5} & \frac{4}{5} \\ 0 & 0 & 1 & 0 & -6 & \frac{14}{5} & -\frac{1}{5} & -\frac{3}{5} \\ 0 & 0 & 0 & 1 & 3 & -\frac{8}{5} & \frac{2}{5} & \frac{1}{5} \end{array}\right]$$

The inverse matrix is $\begin{bmatrix} 2 & \frac{3}{5} & -\frac{7}{5} & \frac{4}{5} \\ 4 & -\frac{7}{5} & -\frac{2}{5} & \frac{4}{5} \\ -6 & \frac{14}{5} & -\frac{1}{5} & -\frac{3}{5} \\ 3 & -\frac{8}{5} & \frac{2}{5} & \frac{1}{5} \end{bmatrix}$.

**14.** $$\left[\begin{array}{rrrr|rrrr} 1 & -1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & -1 & 6 & 6 & 0 & 1 & 0 & 0 \\ 3 & -1 & 12 & 12 & 0 & 0 & 1 & 0 \\ -2 & -1 & -14 & -10 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[2R_1+R_4]{\substack{-2R_1+R_2 \\ -3R_1+R_3}} \left[\begin{array}{rrrr|rrrr} 1 & -1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 2 & -2 & 1 & 0 & 0 \\ 0 & 2 & 9 & 6 & -3 & 0 & 1 & 1 \\ 0 & -3 & -12 & -6 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[3R_2+R_4]{-R_2+R_3} \left[\begin{array}{rrrr|rrrr} 1 & -1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 2 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & -4 & 3 & 0 & 1 \end{array}\right]$$

The matrix does not have an inverse.

**15.** $\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \end{bmatrix}$ (1)

Find the inverse of $\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$.

$$\left[\begin{array}{rr|rr} 1 & 4 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{rr|rr} 1 & 4 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right] \xrightarrow{-1R_2} \left[\begin{array}{rr|rr} 1 & 4 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right] \xrightarrow{-4R_2+R_1} \left[\begin{array}{rr|rr} 1 & 0 & -7 & 4 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

The inverse of $\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$ is $\begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}$. Multiply each side of Eq. (1) by the inverse matrix.

$$\begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The solution is (2, 1).

**16.** $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ (1)

Find the inverse of $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|cc} 1 & \frac{3}{2} & \frac{1}{2} & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{cc|cc} 1 & \frac{3}{2} & \frac{1}{2} & 0 \\ 1 & \frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right] \xrightarrow{2R_2} \left[\begin{array}{cc|cc} 1 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 2 \end{array}\right] \xrightarrow{-\frac{3}{2}R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

The inverse matrix is $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$. Multiply each side of Eq. (1) by the inverse matrix.

$$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

The solution is (–2, 3).

**17.** $\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \end{bmatrix}$ (1)

Find the inverse matrix of $\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] \xrightarrow{-3R_1+R_2} \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 8 & -3 & 1 \end{array}\right] \xrightarrow{(1/8)R_2} \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -\frac{3}{8} & \frac{1}{8} \end{array}\right] \xrightarrow{2R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & -\frac{3}{8} & \frac{1}{8} \end{array}\right]$$

The inverse matrix is $\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{8} & \frac{1}{8} \end{bmatrix}$. Multiply each side of Eq. (1) by the inverse matrix.

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{8} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{8} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 8 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{7}{4} \\ -\frac{25}{8} \end{bmatrix}$$

The solution is $\left(\frac{7}{4}, -\frac{25}{8}\right)$.

**18.** $\begin{bmatrix} 3 & -5 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -18 \\ -11 \end{bmatrix}$ (1)

Find the inverse matrix of $\begin{bmatrix} 3 & -5 \\ 2 & -3 \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 3 & -5 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{array}\right] \xrightarrow[-2R_1+R_2]{(1/3)R_1} \left[\begin{array}{cc|cc} 1 & -\frac{5}{3} & \frac{1}{3} & 0 \\ 1 & \frac{1}{3} & -\frac{2}{3} & 1 \end{array}\right] \xrightarrow{3R_2} \left[\begin{array}{cc|cc} 1 & -\frac{5}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -2 & 3 \end{array}\right] \xrightarrow{(5/3)R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & -3 & 5 \\ 0 & 1 & -2 & 3 \end{array}\right]$$

The inverse matrix is $\begin{bmatrix} -3 & 5 \\ -2 & 3 \end{bmatrix}$. Multiply each side of Eq. (1) by the inverse matrix.

$$\begin{bmatrix} -3 & 5 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -18 \\ -11 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

The solution is (–1, 3).

**19.** $$\begin{bmatrix}1&1&2\\2&3&3\\3&3&7\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}4\\5\\14\end{bmatrix} \qquad (1)$$

Find the inverse matrix of $\begin{bmatrix}1&1&2\\2&3&3\\3&3&7\end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc}1&1&2&1&0&0\\2&3&3&0&1&0\\3&3&7&0&0&1\end{array}\right]\xrightarrow[-3R_1+R_3]{2R_1+R_2}\left[\begin{array}{ccc|ccc}1&1&2&1&0&0\\0&1&-1&-2&1&0\\0&0&1&-3&0&1\end{array}\right]\xrightarrow{-R_2+R_1}\left[\begin{array}{ccc|ccc}1&0&3&3&-1&0\\0&1&-1&-2&1&0\\0&0&1&-3&0&1\end{array}\right]$$

$$\xrightarrow[R_3+R_2]{3R_2+R_1}\left[\begin{array}{ccc|ccc}1&0&0&12&-1&-3\\0&1&0&-5&1&1\\0&0&1&-3&0&1\end{array}\right]$$

The inverse matrix is $\begin{bmatrix}12&-1&-3\\-5&1&1\\-3&0&1\end{bmatrix}$. Multiply each side of Eq. (1) by the inverse matrix.

$$\begin{bmatrix}12&-1&-3\\-5&1&1\\-3&0&1\end{bmatrix}\begin{bmatrix}1&1&2\\2&3&3\\3&3&7\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}12&-1&-3\\-5&1&1\\-3&0&1\end{bmatrix}\begin{bmatrix}4\\5\\14\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\-1\\2\end{bmatrix}$$

The solution is (1, –1, 2).

**20.** $$\begin{bmatrix}1&2&-1\\2&3&-1\\3&6&-2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}5\\8\\14\end{bmatrix} \qquad (1)$$

Find the inverse matrix of $\begin{bmatrix}1&2&-1\\2&3&-1\\3&6&-2\end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc}1&2&-1&1&0&0\\2&3&-1&0&1&0\\3&6&-2&0&0&1\end{array}\right]\xrightarrow[-3R_1+R_3]{-2R_1+R_2}\left[\begin{array}{ccc|ccc}1&2&-1&1&0&0\\0&-1&1&-2&1&0\\0&0&1&-3&0&1\end{array}\right]\xrightarrow{-1R_2}\left[\begin{array}{ccc|ccc}1&2&-1&1&0&0\\0&1&-1&2&-1&0\\0&0&1&-3&0&1\end{array}\right]$$

$$\xrightarrow{-2R_2+R_1}\left[\begin{array}{ccc|ccc}1&0&1&-3&2&0\\0&1&-1&2&-1&0\\0&0&1&-3&0&1\end{array}\right]\xrightarrow[R_3+R_2]{-R_3+R_1}\left[\begin{array}{ccc|ccc}1&0&0&0&2&-1\\0&1&0&-1&-1&1\\0&0&1&-3&0&1\end{array}\right]$$

The inverse matrix is $\begin{bmatrix}0&2&-1\\-1&-1&1\\-3&0&1\end{bmatrix}$. Multiply each side of Eq. (1) by the inverse matrix.

$$\begin{bmatrix}0&2&-1\\-1&-1&1\\-3&0&1\end{bmatrix}\begin{bmatrix}1&2&-1\\2&3&-1\\3&6&-2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0&2&-1\\-1&-1&1\\-3&0&1\end{bmatrix}\begin{bmatrix}5\\8\\14\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\1\\-1\end{bmatrix}$$

The solution is (2, 1, –1).

**21.** $\begin{bmatrix} 1 & 2 & 2 \\ -2 & -5 & -2 \\ 2 & 4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 5 \\ 8 \\ 19 \end{bmatrix}$ (1)

Find the inverse matrix of $\begin{bmatrix} 1 & 2 & 2 \\ -2 & -5 & -2 \\ 2 & 4 & 7 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ -2 & -5 & -2 & 0 & 1 & 0 \\ 2 & 4 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R_1+R_3]{2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 3 & -2 & 0 & 1 \end{array}\right] \xrightarrow{-1R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & -1 & 0 \\ 0 & 0 & 3 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{(1/3)R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & 1 & \frac{1}{3} \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 6 & 5 & 2 & 0 \\ 0 & 1 & -2 & -2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & 0 & \frac{1}{3} \end{array}\right] \xrightarrow[-6R_3+R_1]{2R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & 2 & -2 \\ 0 & 1 & 0 & -\frac{10}{3} & -1 & \frac{2}{3} \\ 0 & 0 & 1 & -\frac{2}{3} & 0 & \frac{1}{3} \end{array}\right]$$

The inverse matrix is $\begin{bmatrix} 9 & 2 & -2 \\ -\frac{10}{3} & -1 & \frac{2}{3} \\ -\frac{2}{3} & 0 & \frac{1}{3} \end{bmatrix}$. Multiply each side of Eq. (1) by the inverse matrix.

$$\begin{bmatrix} 9 & 2 & -2 \\ -\frac{10}{3} & -1 & \frac{2}{3} \\ -\frac{2}{3} & 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ -2 & -5 & -2 \\ 2 & 4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 9 & 2 & -2 \\ -\frac{10}{3} & -1 & \frac{2}{3} \\ -\frac{2}{3} & 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 5 \\ 8 \\ 19 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 23 \\ -12 \\ 3 \end{bmatrix}$$

The solution is (23, –12, 3).

**22.** $\begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & 10 \\ 2 & -2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 5 \\ 16 \\ 9 \end{bmatrix}$ (1)

Find the inverse matrix of $\begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & 10 \\ 2 & -2 & 5 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 3 & -1 & 10 & 0 & 1 & 0 \\ 2 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 1 & -3 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{(1/2)R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{-1R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right]$$

$$\xrightarrow{R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{7}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right] \xrightarrow[(-1/2)R_3+R_2]{(-7/2)R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{15}{2} & \frac{1}{2} & \frac{7}{2} \\ 0 & 1 & 0 & -\frac{5}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right]$$

The inverse matrix is $\begin{bmatrix} -\frac{15}{2} & \frac{1}{2} & \frac{7}{2} \\ -\frac{5}{2} & \frac{1}{2} & \frac{1}{2} \\ 2 & 0 & -1 \end{bmatrix}$. Multiply each side of Eq. (1) by the inverse matrix.

$$\begin{bmatrix} -\frac{15}{2} & \frac{1}{2} & \frac{7}{2} \\ -\frac{5}{2} & \frac{1}{2} & \frac{1}{2} \\ 2 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & 10 \\ 2 & -2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -\frac{15}{2} & \frac{1}{2} & \frac{7}{2} \\ -\frac{5}{2} & \frac{1}{2} & \frac{1}{2} \\ 2 & 0 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 16 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

The solution is (2, 0, 1).

**23.** $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 5 & 1 & 2 \\ 2 & 4 & 1 & 1 \\ 3 & 6 & 0 & 4 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 8 \\ 16 \end{bmatrix} \quad (1)$

Find the inverse matrix of $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 5 & 1 & 2 \\ 2 & 4 & 1 & 1 \\ 3 & 6 & 0 & 4 \end{bmatrix}$.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 2 & 5 & 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 4 & 1 & 1 & 0 & 0 & 1 & 0 \\ 3 & 6 & 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{-2R_1+R_3 \\ -3R_1+R_4}]{-2R_1+R_2} \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & -2 & 1 & 5 & -2 & 0 & 0 \\ 0 & 1 & 1 & -0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-1R_3+R_2]{2R_3+R_1} + \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & -2 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{-R_4+R_2 \\ R_4+R_3}]{R_4+R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -2 & -2 & 2 & 1 \\ 0 & 1 & 0 & 0 & 3 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & -5 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -3 & 0 & 0 & 1 \end{array}\right]$$

The inverse matrix is $\begin{bmatrix} -2 & -2 & 2 & 1 \\ 3 & 1 & -1 & -1 \\ -5 & 0 & 1 & 1 \\ -3 & 0 & 0 & 1 \end{bmatrix}$. Multiply each side of Eq. (1) by the inverse matrix.

$$\begin{bmatrix} -2 & -2 & 2 & 1 \\ -3 & 1 & -1 & -1 \\ -5 & 0 & 1 & 1 \\ -3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 5 & 1 & 2 \\ 2 & 4 & 1 & 1 \\ 3 & 6 & 0 & 4 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & -2 & 2 & 1 \\ 3 & 1 & -1 & -1 \\ -5 & 0 & 1 & 1 \\ -3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 10 \\ 8 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -6 \\ -2 \end{bmatrix}$$

The solution is (0, 4, –6, –2).

**24.** $\begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & -1 & 6 & 2 \\ 3 & -2 & 9 & 4 \\ 1 & -2 & 0 & -1 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 16 \\ 28 \\ 2 \end{bmatrix}$ (1)

Find the inverse matrix of $\begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & -1 & 6 & 2 \\ 3 & -2 & 9 & 4 \\ 1 & -2 & 0 & -1 \end{bmatrix}$.

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 2 & -1 & 6 & 2 & 0 & 1 & 0 & 0 \\ 3 & -2 & 9 & 4 & 0 & 0 & 1 & 0 \\ 1 & -2 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{-2R_1 + R_2 \\ -3R_1 + R_3 \\ -1R_1 + R_4}} \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & -2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 4 & -3 & 0 & 1 & 0 \\ 0 & -1 & -2 & -1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{-1R_2 + R_3 \\ 1R_2 + R_4}} \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 + R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 4 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{-4R_3 + R_1 \\ -2R_3 + R_2}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -6 & 3 & 5 & -4 & 0 \\ 0 & 1 & 0 & -2 & 0 & 3 & -2 & 0 \\ 0 & 0 & 1 & 2 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 & 0 & 1 \end{array}\right] \xrightarrow{\substack{6R_4 + R_1 \\ 2R_4 + R_2 \\ -2R_4 + R_3}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -15 & 11 & -4 & 6 \\ 0 & 1 & 0 & 0 & -6 & 5 & -2 & 2 \\ 0 & 0 & 1 & 0 & 5 & -3 & 1 & -2 \\ 0 & 0 & 0 & 1 & -3 & 1 & 0 & 1 \end{array}\right]$$

The inverse matrix is $\begin{bmatrix} -15 & 11 & -4 & 6 \\ -6 & 5 & -2 & 2 \\ 5 & -3 & 1 & -2 \\ -3 & 1 & 0 & 1 \end{bmatrix}$. Multiply each side of Eq. (1) by the inverse matrix.

$$\begin{bmatrix} -15 & 11 & -4 & 6 \\ -6 & 5 & -2 & 2 \\ 5 & -3 & 1 & -2 \\ -3 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & -1 & 6 & 2 \\ 3 & -2 & 9 & 4 \\ 1 & -2 & 0 & -1 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -15 & 11 & -4 & 6 \\ -6 & 5 & -2 & 2 \\ 5 & -3 & 1 & -2 \\ -3 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 16 \\ 28 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 3 \end{bmatrix}$$

The solution is (1, –2, 1, 3).

**25.** The average temperature for the two points,

$$x_1 = \frac{35+50+x_2+60}{4} = \frac{145+x_2}{4} \text{ or } 4x_1 - x_2 = 145$$

$$x_2 = \frac{x_1+50+55+60}{4} = \frac{165+x_1}{4} \text{ or } -x_1 + 4x_2 = 165$$

The system of equations and associated matrix equation are

$$\begin{cases} 4x_1 - x_2 = 145 \\ -x_1 + 4x_2 = 165 \end{cases} \qquad \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 145 \\ 165 \end{bmatrix}$$

Solving the matrix equation by using an inverse matrix gives $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 49.7 \\ 53.7 \end{bmatrix}$

The temperatures are $x_1$ = 49.7°F, $x_2$ = 53.7°F.

**26.** The average temperature for the two points,

$$x_1 = \frac{40+25+x_2+40}{4} = \frac{105+x_2}{4} \text{ or } 4x_1 - x_2 = 105$$

$$x_2 = \frac{x_1+25+60+40}{4} = \frac{125+x_1}{4} \text{ or } -x_1 + 4x_2 = 125$$

The system of equations and associated matrix equation are

$$\begin{cases} 4x_1 - x_2 = 105 \\ -x_1 + 4x_2 = 125 \end{cases} \qquad \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 105 \\ 125 \end{bmatrix}$$

Solving the matrix equation by using an inverse matrix gives $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 36.3 \\ 40.3 \end{bmatrix}$

The temperatures are $x_1$ = 36.3°F, $x_2$ = 40.3°F.

**27.** The average temperature for the two points,

$$x_1 = \frac{50+60+x_2+x_3}{4} = \frac{110+x_2+x_3}{4} \text{ or } 4x_1 - x_2 - x_3 = 110$$

$$x_2 = \frac{x_1+60+60+x_4}{4} = \frac{120+x_1+x_4}{4} \text{ or } -x_1 + 4x_2 - x_4 = 120$$

$$x_3 = \frac{50+x_1+x_4+50}{4} = \frac{100+x_1+x_4}{4} \text{ or } -x_1 + 4x_3 - x_4 = 100$$

$$x_4 = \frac{x_3+x_2+60+50}{4} = \frac{110+x_2+x_3}{4} \text{ or } -x_2 - x_3 + 4x_4 = 110$$

The system of equations and associated matrix equation are

$$\begin{cases} 4x_1 - x_2 - x_3 = 110 \\ -x_1 + 4x_2 - x_4 = 120 \\ -x_1 + 4x_3 - x_4 = 100 \\ -x_2 - x_3 + 4x_4 = 110 \end{cases} \qquad \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 110 \\ 120 \\ 100 \\ 110 \end{bmatrix}$$

Solving the matrix equation by using an inverse matrix gives $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 55 \\ 57.5 \\ 52.5 \\ 55 \end{bmatrix}$

The temperatures are $x_1$ = 55°F, $x_2$ = 57.5°F, $x_3$ = 52.5°F, $x_4$ = 55°F.

**28.** The average temperature for the two points,

$$x_1 = \frac{55+70+x_2+x_3}{4} = \frac{125+x_2+x_3}{4} \text{ or } 4x_1 - x_2 - x_3 = 125$$

$$x_2 = \frac{x_1+70+65+x_4}{4} = \frac{135+x_1+x_4}{4} \text{ or } -x_1 + 4x_2 - x_4 = 135$$

$$x_3 = \frac{55+x_1+x_4+40}{4} = \frac{95+x_1+x_4}{4} \text{ or } -x_1 + 4x_3 - x_4 = 95$$

$$x_4 = \frac{x_3+x_2+65+40}{4} = \frac{105+x_2+x_3}{4} \text{ or } -x_2 - x_3 + 4x_4 = 105$$

The system of equations and associated matrix equation are

$$\begin{cases} 4x_1 - x_2 - x_3 = 125 \\ -x_1 + 4x_2 - x_4 = 135 \\ -x_1 + 4x_3 - x_4 = 95 \\ -x_2 - x_3 + 4x_4 = 105 \end{cases} \qquad \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 125 \\ 135 \\ 95 \\ 105 \end{bmatrix}$$

Solving the matrix equation by using an inverse matrix gives $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 60 \\ 62.5 \\ 52.5 \\ 55 \end{bmatrix}$

The temperatures are $x_1$ = 60°F, $x_2$ = 62.5°F, $x_3$ = 52.5°F, $x_4$ = 55°F.

**29.** $A$ = number of adult tickets
$C$ = number of child tickets

Saturday $A+C=100$

$$20A+15C=1900$$

$$\begin{bmatrix} 1 & 1 \\ 20 & 15 \end{bmatrix}\begin{bmatrix} A \\ C \end{bmatrix}=\begin{bmatrix} 100 \\ 1900 \end{bmatrix}$$

$$\begin{bmatrix} -3 & \frac{1}{5} \\ 4 & -\frac{1}{5} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 20 & 15 \end{bmatrix}\begin{bmatrix} A \\ C \end{bmatrix}=\begin{bmatrix} -3 & \frac{1}{5} \\ 4 & -\frac{1}{5} \end{bmatrix}\begin{bmatrix} 100 \\ 1900 \end{bmatrix}$$

$$\begin{bmatrix} A \\ C \end{bmatrix}=\begin{bmatrix} 80 \\ 20 \end{bmatrix}$$

On Saturday, 80 adults and 20 children took the tour.

Sunday $A+C=120$

$$20A+15C=2275$$

$$\begin{bmatrix} 1 & 1 \\ 20 & 15 \end{bmatrix}\begin{bmatrix} A \\ C \end{bmatrix}=\begin{bmatrix} 120 \\ 2275 \end{bmatrix}$$

$$\begin{bmatrix} -3 & \frac{1}{5} \\ 4 & -\frac{1}{5} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 20 & 15 \end{bmatrix}\begin{bmatrix} A \\ C \end{bmatrix}=\begin{bmatrix} -3 & \frac{1}{5} \\ 4 & -\frac{1}{5} \end{bmatrix}\begin{bmatrix} 120 \\ 2275 \end{bmatrix}$$

$$\begin{bmatrix} A \\ C \end{bmatrix}=\begin{bmatrix} 95 \\ 25 \end{bmatrix}$$

On Sunday, 95 adults and 25 children took the tour.

**30.** $S$ = number of standard models
$D$ = number of deluxe models

January: $S+D=90$

$$45S+60D=4650$$

$$\begin{bmatrix} 1 & 1 \\ 45 & 60 \end{bmatrix}\begin{bmatrix} S \\ D \end{bmatrix}=\begin{bmatrix} 90 \\ 4650 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -\frac{1}{15} \\ -3 & \frac{1}{15} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 45 & 60 \end{bmatrix}\begin{bmatrix} S \\ D \end{bmatrix}=\begin{bmatrix} 4 & -\frac{1}{15} \\ -3 & \frac{1}{15} \end{bmatrix}\begin{bmatrix} 90 \\ 4650 \end{bmatrix}$$

$$\begin{bmatrix} S \\ D \end{bmatrix}=\begin{bmatrix} 50 \\ 40 \end{bmatrix}$$

In January, 50 standard models and 40 deluxe models were manufactured.

February $S+D=100$

$$45S+60D=5250$$

$$\begin{bmatrix} 1 & 1 \\ 45 & 60 \end{bmatrix}\begin{bmatrix} S \\ D \end{bmatrix}=\begin{bmatrix} 100 \\ 5250 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -\frac{1}{15} \\ -3 & \frac{1}{15} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 45 & 60 \end{bmatrix}\begin{bmatrix} S \\ D \end{bmatrix}=\begin{bmatrix} 4 & -\frac{1}{15} \\ -3 & \frac{1}{15} \end{bmatrix}\begin{bmatrix} 100 \\ 5250 \end{bmatrix}$$

$$\begin{bmatrix} S \\ D \end{bmatrix}=\begin{bmatrix} 50 \\ 50 \end{bmatrix}$$

In February, 50 standard models and 50 deluxe models were manufactured.

**31.** $x_1 =$ number of 100-gram portions of additive 1

$x_2 =$ number of 100-gram portions of additive 2

$x_3 =$ number of 100-gram portions of additive 3

Sample 1:

$$\begin{aligned} 30x_1 + 40x_2 + 50x_3 &= 380 \\ 10x_1 + 15x_2 + \phantom{0}5x_3 &= 95 \\ 10x_1 + 10x_2 + \phantom{0}5x_3 &= 85 \end{aligned}$$

$$\begin{bmatrix} 30 & 40 & 50 \\ 10 & 15 & 5 \\ 10 & 10 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 380 \\ 95 \\ 85 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{70} & -\frac{6}{35} & \frac{11}{35} \\ 0 & \frac{1}{5} & -\frac{1}{5} \\ \frac{1}{35} & -\frac{2}{35} & -\frac{1}{35} \end{bmatrix} \begin{bmatrix} 30 & 40 & 50 \\ 10 & 15 & 5 \\ 10 & 10 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{70} & -\frac{6}{35} & \frac{11}{35} \\ 0 & \frac{1}{5} & -\frac{1}{5} \\ \frac{1}{35} & -\frac{2}{35} & -\frac{1}{35} \end{bmatrix} \begin{bmatrix} 380 \\ 95 \\ 85 \end{bmatrix}$$

$$\begin{bmatrix} S \\ D \end{bmatrix} = \begin{bmatrix} 50 \\ 50 \end{bmatrix}$$

For Sample 1, 500 g of additive 1, 200 g of additive 2, and 300 g of additive 3 are required.

Sample 2:

$$\begin{aligned} 30x_1 + 40x_2 + 50x_3 &= 380 \\ 10x_1 + 15x_2 + \phantom{0}5x_3 &= 110 \\ 10x_1 + 10x_2 + \phantom{0}5x_3 &= 90 \end{aligned}$$

$$\begin{bmatrix} 30 & 40 & 50 \\ 10 & 15 & 5 \\ 10 & 10 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 380 \\ 100 \\ 90 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{70} & -\frac{6}{35} & \frac{11}{35} \\ 0 & \frac{1}{5} & -\frac{1}{5} \\ \frac{1}{35} & -\frac{2}{35} & -\frac{1}{35} \end{bmatrix} \begin{bmatrix} 30 & 40 & 50 \\ 10 & 15 & 5 \\ 10 & 10 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{70} & -\frac{6}{35} & \frac{11}{35} \\ 0 & \frac{1}{5} & -\frac{1}{5} \\ \frac{1}{35} & -\frac{2}{35} & -\frac{1}{35} \end{bmatrix} \begin{bmatrix} 380 \\ 110 \\ 90 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix}$$

For Sample 2, 400 g of additive 1, 400 g of additive 2, and 200 g of additive 3 are required.

**32.** $x_1 =$ number of 100-gram portions of Food Type I

$x_2 =$ number of 100-gram portions of Food Type II

$x_3 =$ number of 100-gram portions of Food Type III

First diet:
$$\begin{aligned} 13x_1 + 4x_2 + x_3 &= 23 \\ 10x_1 + 4x_2 \quad &= 18 \\ 13x_1 + 4x_2 + 10x_3 &= 39 \end{aligned}$$

$$\begin{bmatrix} 13 & 4 & 1 \\ 10 & 4 & 0 \\ 13 & 3 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 23 \\ 18 \\ 39 \end{bmatrix}$$

$$\begin{bmatrix} \frac{20}{49} & -\frac{37}{98} & -\frac{2}{49} \\ -\frac{50}{49} & \frac{117}{98} & \frac{5}{49} \\ -\frac{11}{49} & \frac{13}{98} & \frac{6}{49} \end{bmatrix} \begin{bmatrix} 13 & 4 & 1 \\ 10 & 4 & 0 \\ 13 & 3 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{20}{49} & -\frac{37}{98} & -\frac{2}{49} \\ -\frac{50}{49} & \frac{117}{98} & \frac{5}{49} \\ -\frac{11}{49} & \frac{13}{98} & \frac{6}{49} \end{bmatrix} \begin{bmatrix} 23 \\ 18 \\ 39 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

For the first diet, 100g of Food Type I, 200g of Food Type II, and 200g of Food Type III are required.

Second diet:
$$\begin{aligned} 13x_1 + 4x_2 + x_3 &= 35 \\ 10x_1 + 4x_2 \quad &= 28 \\ 13x_1 + 4x_2 + 10x_3 &= 42 \end{aligned}$$

$$\begin{bmatrix} 13 & 4 & 1 \\ 10 & 4 & 0 \\ 13 & 3 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 35 \\ 28 \\ 42 \end{bmatrix}$$

$$\begin{bmatrix} \frac{20}{49} & -\frac{37}{98} & -\frac{2}{49} \\ -\frac{50}{49} & \frac{117}{98} & \frac{5}{49} \\ -\frac{11}{49} & \frac{13}{98} & \frac{6}{49} \end{bmatrix} \begin{bmatrix} 13 & 4 & 1 \\ 10 & 4 & 0 \\ 13 & 3 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{20}{49} & -\frac{37}{98} & -\frac{2}{49} \\ -\frac{50}{49} & \frac{117}{98} & \frac{5}{49} \\ -\frac{11}{49} & \frac{13}{98} & \frac{6}{49} \end{bmatrix} \begin{bmatrix} 35 \\ 28 \\ 42 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

For the second diet, 200g of Food Type I, 200g of Food Type II, and 100g of Food Type III are required.

**33.** Using a calculator,

$$\begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 2 & -2 & 3 \\ 6 & -1 & 2 & 3 \\ 2 & 3 & -1 & 5 \end{bmatrix}^{-1} \approx \begin{bmatrix} -5.667 & -3.667 & 5 & 0.333 \\ -27.667 & -18.667 & 24 & 2.333 \\ -19.333 & -13.333 & 17 & 1.667 \\ 15 & 10 & -13 & -1 \end{bmatrix}$$

**34.** Using a calculator,

$$\begin{bmatrix} 3 & -1 & 0 & 1 \\ 2 & -2 & -3 & 0 \\ -1 & -3 & 5 & 3 \\ 5 & 3 & -2 & 1 \end{bmatrix}^{-1} \approx \begin{bmatrix} 0.1 & 0.143 & -0.057 & 0.071 \\ -0.5 & 0.071 & 0.071 & 0.286 \\ -0.4 & 0.286 & 0.086 & 0.143 \\ 0.2 & -0.357 & 0.243 & 0.071 \end{bmatrix}$$

**35.** Using a calculator,

$$\begin{bmatrix} -\frac{2}{7} & 4 & -\frac{1}{6} \\ -2 & \sqrt{2} & -3 \\ \sqrt{3} & 3 & -\sqrt{5} \end{bmatrix}^{-1} \approx \begin{bmatrix} -0.150 & -0.217 & 0.302 \\ 0.248 & -0.024 & 0.013 \\ 0.217 & -0.200 & -0.195 \end{bmatrix}$$

**36.** Using a calculator,

$$\begin{bmatrix} 6 & \pi & -\frac{4}{7} \\ -5 & \sqrt{7} & 2 \\ \frac{5}{6} & -\sqrt{3} & \sqrt{10} \end{bmatrix}^{-1} \approx \begin{bmatrix} 0.097 & -0.073 & 0.064 \\ 0.143 & 0.159 & -0.075 \\ 0.053 & 0.106 & 0.258 \end{bmatrix}$$

**37.** $X = (I - A)^{-1} D$, where $X$ is consumer demand, $I$ is the identity matrix, $A$ is the input-output matrix, and $D$ is the final demand. Thus

$$X = \left( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.20 & 0.15 & 0.10 \\ 0.10 & 0.30 & 0.25 \\ 0.20 & 0.10 & 0.10 \end{bmatrix} \right)^{-1} \begin{bmatrix} 120 \\ 60 \\ 55 \end{bmatrix}$$

$$= \begin{bmatrix} 0.80 & -0.15 & -0.10 \\ -0.10 & 0.70 & -0.25 \\ -0.20 & -0.10 & 0.90 \end{bmatrix}^{-1} \begin{bmatrix} 120 \\ 60 \\ 55 \end{bmatrix}$$

$$\approx \begin{bmatrix} 194.67 \\ 157.03 \\ 121.82 \end{bmatrix}$$

\$194.67 million worth of manufacturing, \$157.03 million worth of transportation, \$121.82 million worth of services.

**38.** $X = (I - A)^{-1} D$, where $X$ is consumer demand, $I$ the identity matrix, $A$ is the input-output, and $D$ is the final demand. Thus

$$X = \left( \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.10 & 0.05 & 0.20 & 0.15 \\ 0.20 & 0.10 & 0.30 & 0.10 \\ 0.05 & 0.30 & 0.20 & 0.40 \\ 0.10 & 0.20 & 0.15 & 0.20 \end{bmatrix} \right)^{-1} \begin{bmatrix} 80 \\ 100 \\ 50 \\ 80 \end{bmatrix}$$

$$= \begin{bmatrix} 0.90 & -0.05 & -0.20 & -0.15 \\ -0.20 & 0.90 & -0.30 & -0.10 \\ -0.05 & -0.30 & 0.80 & -0.40 \\ -0.10 & -0.20 & -0.15 & 0.80 \end{bmatrix}^{-1} \begin{bmatrix} 80 \\ 100 \\ 50 \\ 80 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1.30 & 0.40 & 0.58 & 0.58 \\ 0.48 & 1.58 & 0.84 & 0.71 \\ 0.44 & 0.92 & 1.92 & 1.16 \\ 0.36 & 0.62 & 0.64 & 1.72 \end{bmatrix}^{-1} \begin{bmatrix} 80 \\ 100 \\ 50 \\ 80 \end{bmatrix}$$ (A computer program was used to calculate the inverse matrix.)

\$219.0 million worth of manufacturing, \$294.3 million worth of agriculture, \$316.7 million worth of service, \$260.3 million worth of transportation.

**39.** The input-output matrix, $A$, is given by

$$A = \begin{bmatrix} 0.05 & 0.20 & 0.15 \\ 0.02 & 0.03 & 0.25 \\ 0.10 & 0.12 & 0.05 \end{bmatrix}$$

Consumer demand is given by

$$X = (I - A)^{-1} D$$

$$X = \left( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.05 & 0.20 & 0.15 \\ 0.02 & 0.03 & 0.25 \\ 0.10 & 0.12 & 0.05 \end{bmatrix} \right)^{-1} \begin{bmatrix} 30 \\ 5 \\ 25 \end{bmatrix}$$

$$= \begin{bmatrix} 0.95 & -0.20 & -0.15 \\ -0.02 & 0.97 & -0.25 \\ -0.10 & -0.12 & 0.95 \end{bmatrix}^{-1} \begin{bmatrix} 30 \\ 5 \\ 25 \end{bmatrix}$$

$$\approx \begin{bmatrix} 39.69 \\ 14.30 \\ 32.30 \end{bmatrix}$$

\$39.69 million worth of coal, \$14.30 million worth of iron, \$32.30 million worth of steel.

**40.** The input-output matrix, $A$, is given by

$$A=\begin{bmatrix}0.01 & 0.08 & 0.20\\ 0.03 & 0.05 & 0.20\\ 0.10 & 0.15 & 0.10\end{bmatrix}$$

Consumer demand is given by

$$X=(I-A)^{-1}D$$

$$X=\left(\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}-\begin{bmatrix}0.01 & 0.08 & 0.20\\ 0.03 & 0.05 & 0.20\\ 0.10 & 0.15 & 0.10\end{bmatrix}\right)^{-1}\begin{bmatrix}100\\ 75\\ 150\end{bmatrix}$$

$$=\begin{bmatrix}0.99 & -0.08 & -0.20\\ -0.03 & 0.95 & -0.20\\ -0.10 & -0.15 & 0.90\end{bmatrix}^{-1}\begin{bmatrix}100\\ 75\\ 150\end{bmatrix}$$

$$=\begin{bmatrix}152.63\\ 126.88\\ 204.77\end{bmatrix}$$

\$152.63 million from the plastics division, \$126.88 million from the semiconductor division, \$204.77 million from the computer division.

## Connecting Concepts

**41.** $$AB=\begin{bmatrix}2 & -3\\ -6 & 9\end{bmatrix}\begin{bmatrix}-3 & 15\\ -2 & 10\end{bmatrix}=\begin{bmatrix}2(-3)+(-3)(-2) & 2(15)+(-3)(10)\\ -6(-3)+9(-2) & -6(15)+9(10)\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=O$$

**42.** Let $A$ be a matrix with an inverse matrix $A^{-1}$.
If $AB=O$, then

$$A^{-1}(AB)=A^{-1}O$$
$$(A^{-1}A)B=O$$
$$IB=O$$
$$B=O$$

**43.** $$AB=\begin{bmatrix}2 & -1\\ -4 & 2\end{bmatrix}\begin{bmatrix}3 & 4\\ 1 & 5\end{bmatrix}$$
$$=\begin{bmatrix}2(3)+(-1)(1) & 2(4)+(-1)(5)\\ -4(3)+2(1) & -4(4)+2(5)\end{bmatrix}=\begin{bmatrix}5 & 3\\ -10 & -6\end{bmatrix}$$
$$AC=\begin{bmatrix}2 & -1\\ -4 & 2\end{bmatrix}\begin{bmatrix}4 & 7\\ 3 & 11\end{bmatrix}$$
$$=\begin{bmatrix}2(4)+(-1)(3) & 2(7)+(-1)(11)\\ -4(4)+2(3) & -4(7)+2(11)\end{bmatrix}=\begin{bmatrix}5 & 3\\ -10 & -6\end{bmatrix}$$

**44.** Let $A$ be a matrix with an inverse matrix $A^{-1}$.
If $AB=AC$, then

$$A^{-1}(AB)=A^{-1}(AC)$$
$$(A^{-1}A)B=(A^{-1}A)C$$
$$IB=IC$$
$$B=C$$

**45.** $$\left[\begin{array}{cc|cc}a & b & 1 & 0\\ c & d & 0 & 1\end{array}\right]\xrightarrow{\frac{1}{a}R_1}\left[\begin{array}{cc|cc}1 & \frac{b}{a} & \frac{1}{a} & 0\\ c & d & 0 & 1\end{array}\right]\xrightarrow{(-c)R_1+R_2}\left[\begin{array}{cc|cc}1 & \frac{b}{a} & \frac{1}{a} & 0\\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1\end{array}\right]\xrightarrow{\frac{a}{ad-bc}R_2}\left[\begin{array}{cc|cc}1 & \frac{b}{a} & \frac{1}{a} & 0\\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc}\end{array}\right]$$

$$\xrightarrow{-\frac{b}{a}R_2+R_1}\left[\begin{array}{cc|cc}1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc}\\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc}\end{array}\right]$$

$$A^{-1}=\begin{bmatrix}\frac{d}{ad-bc} & -\frac{b}{ad-bc}\\ -\frac{c}{ad-bc} & \frac{a}{ad-bc}\end{bmatrix}=\frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$

**46.** $A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$

$A^{-1}$ exists if and only if the denominator $ad - bc \neq 0$.

**47.** **a.** $a = 2, b = -3, c = 4, d = -5$

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{2(-5)-(-3)(4)}\begin{bmatrix} -5 & 3 \\ -4 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -5 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -5/2 & 3/2 \\ -2 & 1 \end{bmatrix}$$

**b.** $a = 5, b = 6, c = 3, d = 4$

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{5(4)-(6)(3)}\begin{bmatrix} 4 & -6 \\ -3 & 5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & -6 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -3/2 & 5/2 \end{bmatrix}$$

**c.** $a = 0, b = -1, c = 4, d = 4$

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{0(4)-(-1)(4)}\begin{bmatrix} 4 & 1 \\ -4 & 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 1 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1/4 \\ -1 & 0 \end{bmatrix}$$

**48.** $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{3(1)-(-2)(1)}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

$B^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{2(3)-(-1)(2)}\begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$

$(AB)^{-1} = \left(\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 2 & 3 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 2 & -9 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{20} & \frac{9}{40} \\ -\frac{1}{10} & \frac{1}{20} \end{bmatrix}$

$B^{-1} \cdot A^{-1} = \begin{bmatrix} 3/8 & 1/8 \\ -1/4 & 1/4 \end{bmatrix}\begin{bmatrix} 1/5 & 2/5 \\ -1/5 & 3/5 \end{bmatrix} = \begin{bmatrix} 1/20 & 9/40 \\ -1/10 & 1/20 \end{bmatrix}$

**49.**
$$\begin{aligned} (AB)(AB)^{-1} &= I \\ A^{-1}(AB)(AB)^{-1} &= A^{-1}I \\ (A^{-1}A)B(AB)^{-1} &= A^{-1} \\ IB(AB)^{-1} &= A^{-1} \\ B(AB)^{-1} &= A^{-1} \\ B^{-1}B(AB)^{-1} &= B^{-1}A^{-1} \\ I(AB)^{-1} &= B^{-1}A^{-1} \\ (AB)^{-1} &= B^{-1}A^{-1} \end{aligned}$$

## Prepare for Section 10.4

**PS1.** 2

**PS2.** $(-1)^{i+j}$

$(-1)^{2+6} = (-1)^8 = 1$

**PS3.**
$$\begin{aligned} &(-1)^{1+1}(-3)+(-1)^{1+2}(-2)+(-1)^{1+3}(5) \\ &=(-1)^2(-3)+(-1)^3(-2)+(-1)^4(5) \\ &=-3+(-1)(-2)+5 \\ &=-3+2+5 \\ &=4 \end{aligned}$$

**PS4.** $a_{23} = 1$

**PS5.** $3\begin{bmatrix} -2 & 1 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 9 & -15 \end{bmatrix}$

**PS6.** $\begin{bmatrix} 1 & 3 & -2 \\ -2 & -1 & 1 \\ 4 & 0 & 1 \end{bmatrix} \xrightarrow[-4R_1+R_3]{2R_1+R_2} \begin{bmatrix} 1 & 3 & -2 \\ 0 & 5 & -3 \\ 0 & -12 & 9 \end{bmatrix}$

## Section 10.4

**1.** $\begin{vmatrix} 2 & -1 \\ 3 & 5 \end{vmatrix} = 2(5)-(-1)(3) = 10-(-3) = 13$

**2.** $\begin{vmatrix} 2 & 9 \\ -6 & 2 \end{vmatrix} = 2(2)-(-6)(9) = 4-(-54) = 58$

**3.** $\begin{vmatrix} 5 & 0 \\ 2 & -3 \end{vmatrix} = 5(-3)-(2)(0) = -15-0 = -15$

**4.** $\begin{vmatrix} 0 & -8 \\ 3 & 4 \end{vmatrix} = 0(4)-(3)(-8) = 0-(-24) = 24$

**5.** $\begin{vmatrix} 4 & 6 \\ 2 & 3 \end{vmatrix} = 4(3)-(2)(6) = 12-12 = 0$

**6.** $\begin{vmatrix} -3 & 6 \\ 4 & -8 \end{vmatrix} = -3(-8)-(4)(6) = 24-24 = 0$

**7.** $\begin{vmatrix} 0 & 9 \\ 0 & -2 \end{vmatrix} = 0(-2)-(0)(9) = 0-0 = 0$

**8.** $\begin{vmatrix} -3 & 9 \\ 0 & 0 \end{vmatrix} = -3(0)-(0)(9) = 0-0 = 0$

**9.** $M_{11} = \begin{vmatrix} 4 & -1 \\ -5 & 6 \end{vmatrix} = 4(6)-(-5)(-1) = 19$

$C_{11} = (-1)^{1+1} M_{11} = M_{11} = 19$

**10.** $M_{21} = \begin{vmatrix} -2 & -3 \\ -5 & 6 \end{vmatrix} = -2(6)-(-5)(-3) = -27$

$C_{21} = (-1)^{2+1} M_{21} = -M_{21} = -(-27) = 27$

**11.** $M_{32} = \begin{vmatrix} 5 & -3 \\ 2 & -1 \end{vmatrix} = 5(-1)-2(-3) = 1$

$C_{32} = (-1)^{3+2} M_{32} = -M_{32} = -1$

**12.** $M_{33} = \begin{vmatrix} 5 & -2 \\ 2 & 4 \end{vmatrix} = 5(4)-2(-2) = 24$

$C_{33} = (-1)^{3+3} M_{33} = M_{33} = 24$

**13.** $M_{22} = \begin{vmatrix} 3 & 3 \\ 6 & 3 \end{vmatrix} = 3(3)-6(3) = -9$

$C_{22} = (-1)^{2+2} M_{22} = M_{22} = -9$

**14.** $M_{13} = \begin{vmatrix} 1 & 3 \\ 6 & -2 \end{vmatrix} = 1(-2)-6(3) = -20$

$C_{13} = (-1)^{1+3} M_{13} = M_{13} = -20$

**15.** $M_{31} = \begin{vmatrix} -2 & 3 \\ 3 & 0 \end{vmatrix} = -2(0)-3(3) = -9$

$C_{31} = (-1)^{3+1} M_{31} = M_{31} = -9$

**16.** $M_{23} = \begin{vmatrix} 3 & -2 \\ 6 & -2 \end{vmatrix} = 3(-2)-6(-2) = 6$

$C_{23} = (-1)^{2+3} M_{23} = -M_{23} = -6$

**17.** $$\begin{vmatrix} 2 & -3 & 1 \\ 2 & 0 & 2 \\ 3 & -2 & 4 \end{vmatrix} = -2\begin{vmatrix} -3 & 1 \\ -2 & 4 \end{vmatrix} + 0\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} - 2\begin{vmatrix} 2 & -3 \\ 3 & -2 \end{vmatrix}$$
$$= -2(-10)+0-2(5)$$
$$= 20-10$$
$$= 10$$

**18.** $$\begin{vmatrix} 3 & 1 & -2 \\ 2 & -5 & 4 \\ 3 & 2 & 1 \end{vmatrix} = 3\begin{vmatrix} -5 & 4 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix} + (-2)\begin{vmatrix} 2 & -5 \\ 3 & 2 \end{vmatrix}$$
$$= 3(-13)-1(-10)-2(19)$$
$$= -39+10-38$$
$$= -67$$

**19.** $$\begin{vmatrix} -2 & 3 & 2 \\ 1 & 2 & -3 \\ -4 & -2 & 1 \end{vmatrix} = -2\begin{vmatrix} 2 & -3 \\ -2 & 1 \end{vmatrix} - 3\begin{vmatrix} 1 & -3 \\ -4 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ -4 & -2 \end{vmatrix}$$
$$= -2(-4)-3(-11)+2(6)$$
$$= 8+33+12$$
$$= 53$$

**20.** $$\begin{vmatrix} 3 & -2 & 0 \\ 2 & -3 & 2 \\ 8 & -2 & 5 \end{vmatrix} = 3\begin{vmatrix} -3 & 2 \\ -2 & 5 \end{vmatrix} - (-2)\begin{vmatrix} 2 & 2 \\ 8 & 5 \end{vmatrix} + 0\begin{vmatrix} 2 & -3 \\ 8 & -2 \end{vmatrix}$$
$$= 3(-11)+2(-6)+0$$
$$= -33-12$$
$$= -45$$

**21.** $$\begin{vmatrix} 2 & -3 & 10 \\ 0 & 2 & -3 \\ 0 & 0 & 5 \end{vmatrix} = 2\begin{vmatrix} 2 & -3 \\ 0 & 5 \end{vmatrix} - 0\begin{vmatrix} -3 & 10 \\ 0 & 5 \end{vmatrix} + 0\begin{vmatrix} -3 & 10 \\ 2 & -3 \end{vmatrix}$$
$$= 2(10)-0+0$$
$$= 20$$

**22.** $$\begin{vmatrix} 6 & 0 & 0 \\ 2 & -3 & 0 \\ 7 & -8 & 2 \end{vmatrix} = 6\begin{vmatrix} -3 & 0 \\ -8 & 2 \end{vmatrix} - 0\begin{vmatrix} 2 & 0 \\ 7 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & -3 \\ 7 & -8 \end{vmatrix}$$
$$= 6(-6)-0+0$$
$$= -36$$

**23.** $\begin{vmatrix} 0 & -2 & 4 \\ 1 & 0 & -7 \\ 5 & -6 & 0 \end{vmatrix} = 0\begin{vmatrix} 0 & -7 \\ -6 & 0 \end{vmatrix} - (-2)\begin{vmatrix} 1 & -7 \\ 5 & 0 \end{vmatrix} + 4\begin{vmatrix} 1 & 0 \\ 5 & -6 \end{vmatrix}$
$= 0 + 2(35) + 4(-6)$
$= 70 - 24$
$= 46$

**24.** $\begin{vmatrix} 5 & -8 & 0 \\ 2 & 0 & -7 \\ 0 & -2 & -1 \end{vmatrix} = 5\begin{vmatrix} 0 & -7 \\ -2 & -1 \end{vmatrix} - (-8)\begin{vmatrix} 2 & -7 \\ 0 & -1 \end{vmatrix} + 0\begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix}$
$= 5(-14) + 8(-2) + 0$
$= -70 - 16$
$= -86$

**25.** $\begin{vmatrix} 4 & -3 & 3 \\ 2 & 1 & -4 \\ 6 & -2 & -1 \end{vmatrix} = 4\begin{vmatrix} 1 & -4 \\ -2 & -1 \end{vmatrix} - (-3)\begin{vmatrix} 2 & -4 \\ 6 & -1 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 6 & -2 \end{vmatrix}$
$= 4(-9) + 3(22) + 3(-10)$
$= -36 + 66 - 30$
$= 0$

**26.** $\begin{vmatrix} -2 & 3 & 9 \\ 4 & -2 & -6 \\ 0 & -8 & -24 \end{vmatrix} = -2\begin{vmatrix} -2 & -6 \\ -8 & -24 \end{vmatrix} - 4\begin{vmatrix} 3 & 9 \\ -8 & -24 \end{vmatrix} + 0\begin{vmatrix} 3 & 9 \\ -2 & -6 \end{vmatrix}$
$= -2(0) - 4(0) + 0$
$= 0$

**27.** Row 2 consists entirely of zeros. Therefore, the determinant is zero.

**28.** Column 3 consists entirely of zeros. Therefore, the determinant is zero.

**29.** 2 was factored from row 2.

**30.** $-3$ was factored from column 2.

**31.** Row 1 was multiplied by $-2$ and added to row 2.

**32.** Row 1 was multiplied by $-1$ and added to row 3.

**33.** 2 was factored from column 1.

**34.** Row 3 is a constant multiple of row 1. $-2R_1 = R_3$.
Therefore, the determinant is zero.

**35.** The matrix is in triangular form.
The product of the elements on the main diagonal is $-12$.
Therefore, the value of the determinant is $-12$.

**36.** The matrix is in triangular form.
The product of the elements on the main diagonal is $-15$.
Therefore, the value of the determinant is $-15$.

**37.** Row 1 and row 3 were interchanged.
Therefore, the sign of the determinant was changed.

**38.** Column 1 and column 2 were interchanged.
Therefore, the sign of the determinant was changed.

**39.** Each row of the first determinant was multiplied by *a* to produce the second determinant.

**40.** Columns 1, 2, and 3 are identical.
Therefore, the determinant is zero.

**41.** $\begin{vmatrix} 2 & 4 & 1 \\ 1 & 2 & -1 \\ 1 & 2 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & -1 \\ 1 & 2 & 2 \end{vmatrix} R_1 \leftrightarrow R_2$

$= -\begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{vmatrix} \begin{matrix} -2R_1 + R_2 \\ -R_1 + R_3 \end{matrix}$

$= -(1)(0)(3) = 0$

**42.** $\begin{vmatrix} 3 & -2 & -1 \\ 1 & 2 & 4 \\ 2 & -2 & 3 \end{vmatrix} = -\begin{vmatrix} -2 & 3 & -1 \\ 2 & 1 & 4 \\ -2 & 2 & 3 \end{vmatrix} C_1 \leftrightarrow C_2$

$= 2\begin{vmatrix} 1 & 3 & -1 \\ -1 & 1 & 4 \\ 1 & 2 & 3 \end{vmatrix}$ Factor 2 from $C_1$

$= 2\begin{vmatrix} 1 & 0 & 0 \\ -1 & 4 & 3 \\ 1 & -1 & 4 \end{vmatrix} \begin{matrix} -3C_1 + C_2 \\ C_1 + C_3 \end{matrix}$

$= 2\begin{vmatrix} 1 & 0 & 0 \\ -1 & 4 & 0 \\ 1 & -1 & \frac{19}{4} \end{vmatrix} -\frac{3}{4}C_2 + C_3$

$= 2(1)(4)\left(\frac{19}{4}\right) = 38$

**43.** $\begin{vmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ 3 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & -2 & 6 \end{vmatrix} \begin{matrix} -2R_1 + R_2 \\ -3R_1 + R_3 \end{matrix}$

$= \begin{vmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{vmatrix} -2R_2 + R_3$

$= (-1)(0) = 0$

**44.** $\begin{vmatrix} 1 & 2 & 5 \\ -1 & 1 & -2 \\ 3 & 1 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 5 \\ 0 & 3 & 3 \\ 0 & -5 & -5 \end{vmatrix} \begin{matrix} R_1 + R_2 \\ -3R_1 + R_3 \end{matrix}$

$= \begin{vmatrix} 1 & 2 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{vmatrix} \frac{5}{3}R_2 + R_3$

$= 1(0)(3) = 0$

**45.** $\begin{vmatrix} 0 & -1 & 1 \\ 1 & 0 & -2 \\ 2 & 2 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & -2 \\ 0 & -1 & 1 \\ 2 & 2 & 0 \end{vmatrix} \quad R_1 \leftrightarrow R_2$

$= -\begin{vmatrix} 1 & 0 & -2 \\ 0 & -1 & 1 \\ 0 & 2 & 4 \end{vmatrix} -2R_1 + R_3$

$= -\begin{vmatrix} 1 & 0 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 6 \end{vmatrix} \quad 2R_2 + R_3$

$= -(1)(-1)(6) = 6$

**46.** $\begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 3 & -4 & 5 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & -4 & 5 \end{vmatrix} \quad R_1 \leftrightarrow R_2$

$= -\begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & -7 & 2 \end{vmatrix} \begin{matrix} -2R_1 + R_2 \\ -3R_1 + R_3 \end{matrix}$

$= -\begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -\frac{1}{3} \end{vmatrix} -\frac{7}{3}R_2 + R_3$

$= -(1)(-3)\left(-\frac{1}{3}\right) = -1$

**47.** $\begin{vmatrix} 1 & 2 & -1 & 2 \\ 1 & -2 & 0 & 3 \\ 3 & 0 & 1 & 5 \\ -2 & -4 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 & 2 \\ 0 & -4 & 1 & 1 \\ 0 & -6 & 4 & -1 \\ 0 & 0 & -1 & 10 \end{vmatrix} \begin{matrix} -1R_1 + R_2 \\ -3R_1 + R_3 \\ 2R_1 + R_4 \end{matrix}$

$= \begin{vmatrix} 1 & 2 & -1 & 2 \\ 0 & -4 & 1 & 1 \\ 0 & 0 & \frac{5}{2} & -\frac{5}{2} \\ 0 & 0 & -1 & 10 \end{vmatrix} -\frac{3}{2}R_2 + R_3$

$= \begin{vmatrix} 1 & 2 & -1 & 2 \\ 0 & -4 & 1 & 1 \\ 0 & 0 & \frac{5}{2} & -\frac{5}{2} \\ 0 & 0 & 0 & 9 \end{vmatrix} \frac{2}{5}R_3 + R_4$

$= 1(-4)\left(\frac{5}{2}\right)(9) = -90$

**48.** $\begin{vmatrix} 1 & -1 & -1 & 2 \\ 0 & 2 & 4 & 6 \\ 1 & 1 & 4 & 12 \\ 1 & -1 & 0 & 8 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 & 2 \\ 0 & 2 & 4 & 6 \\ 0 & 2 & 5 & 10 \\ 0 & 0 & 2 & 6 \end{vmatrix} \begin{matrix} -1R_1 + R_3 \\ -1R_1 + R_4 \end{matrix}$

$= \begin{vmatrix} 1 & -1 & -1 & 2 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 6 \end{vmatrix} -1R_2 + R_3$

$= \begin{vmatrix} 1 & -1 & -1 & 2 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 2 \end{vmatrix} -R_3 + R_4$

$= 1(2)(1)(2) = 4$

**49.** $\begin{vmatrix} 1 & 2 & 3 & -1 \\ 6 & 5 & 9 & 8 \\ 2 & 4 & 12 & -1 \\ 1 & 2 & 6 & -1 \end{vmatrix} = 3\begin{vmatrix} 1 & 2 & 1 & -1 \\ 6 & 5 & 3 & 8 \\ 2 & 4 & 4 & -1 \\ 1 & 2 & 2 & -1 \end{vmatrix}$ Factor 3 from $C_3$

$= 3\begin{vmatrix} 1 & 2 & 1 & -1 \\ 0 & -7 & -3 & 14 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} \begin{matrix} -6R_1 + R_2 \\ -2R_1 + R_3 \\ -1R_1 + R_4 \end{matrix}$

$= 3\begin{vmatrix} 1 & 2 & 1 & -1 \\ 0 & -7 & -3 & 14 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -\frac{1}{2} \end{vmatrix} -\frac{1}{2}R_3 + R_4$

$= 3(1)(-7)(2)\left(-\frac{1}{2}\right) = 21$

**50.** $\begin{vmatrix} 1 & 2 & 0 & -2 \\ -1 & 1 & 3 & 5 \\ 2 & 1 & 4 & 0 \\ -2 & 5 & 2 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 & -2 \\ 0 & 3 & 3 & 3 \\ 0 & -3 & 4 & 4 \\ 0 & 9 & 2 & 2 \end{vmatrix} \begin{matrix} 1R_1 + R_2 \\ -2R_1 + R_3 \\ 2R_1 + R_4 \end{matrix}$

$= \begin{vmatrix} 1 & 2 & 0 & -2 \\ 0 & 3 & 3 & 3 \\ 0 & 0 & 7 & 7 \\ 0 & 0 & -7 & -7 \end{vmatrix} \begin{matrix} R_2 + R_3 \\ -3R_2 + R_4 \end{matrix}$

$= \begin{vmatrix} 1 & 2 & 0 & -2 \\ 0 & 3 & 3 & 3 \\ 0 & 0 & 7 & 7 \\ 0 & 0 & 0 & 0 \end{vmatrix} R_3 + R_4$

$= 1(3)(7)(0) = 0$

**51.** Using a calculator,

$\begin{vmatrix} 2 & -2 & 3 & 1 \\ 5 & 2 & -2 & 3 \\ 6 & -1 & 2 & 3 \\ 2 & 3 & -1 & 5 \end{vmatrix} = 3$

**52.** Using a calculator,

$\begin{vmatrix} 3 & -1 & 0 & 1 \\ 2 & -2 & 3 & 0 \\ -1 & -3 & 5 & 3 \\ 5 & 3 & -2 & 1 \end{vmatrix} = 140$

**53.** Using a calculator,

$\begin{vmatrix} -\frac{2}{7} & 4 & -\frac{1}{6} \\ -2 & \sqrt{2} & -3 \\ \sqrt{3} & 3 & -\sqrt{5} \end{vmatrix} \approx -38.933$

**54.** Using a calculator,

$\begin{vmatrix} 6 & \pi & -\frac{4}{7} \\ -5 & \sqrt{7} & 2 \\ \frac{5}{6} & -\sqrt{3} & \sqrt{10} \end{vmatrix} \approx 122.204$

**Connecting Concepts**

**55.** $$\frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ -1 & 0 & 1 \\ 4 & 8 & 1 \end{vmatrix} = \frac{1}{2}[3C_{12} + 0C_{22} + 8C_{32}]$$
$$= \frac{1}{2}\left[-3\begin{vmatrix} -1 & 1 \\ 4 & 1 \end{vmatrix} + 0 - 8\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix}\right]$$
$$= \frac{1}{2}[-3(5) - 8(3)] = \frac{1}{2}(15 - 24) = \frac{1}{2}(-9) = -\frac{9}{2}$$
$$\left|-\frac{9}{2}\right| = \frac{9}{2}$$

The area of the triangle is $4\frac{1}{2}$ square units.

**56.** $$\frac{1}{2}\begin{vmatrix} -3 & 4 & 1 \\ 1 & 5 & 1 \\ 5 & -2 & 1 \end{vmatrix} = \frac{1}{2}[-3C_{11} + 1C_{21} + 5C_{31}]$$
$$= \frac{1}{2}[-3M_{11} - 1M_{21} + 5M_{31}]$$
$$= \frac{1}{2}\left[-3\begin{vmatrix} 5 & 1 \\ -2 & 1 \end{vmatrix} - 1\begin{vmatrix} 4 & 1 \\ -2 & 1 \end{vmatrix} + 5\begin{vmatrix} 4 & 1 \\ 5 & 1 \end{vmatrix}\right]$$
$$= \frac{1}{2}[-3(7) - 1(6) + 5(-1)]$$
$$= \frac{1}{2}[-21 - 6 - 5] = \frac{1}{2}(-32) = -16$$
$$|-16| = 16$$

The area of the triangle is 16 square units.

**57.** $$\frac{1}{2}\begin{vmatrix} 4 & 9 & 1 \\ 8 & 2 & 1 \\ -3 & -2 & 1 \end{vmatrix} = \frac{1}{2}[4C_{11} + 8C_{21} + (-3)C_{31}]$$
$$= \frac{1}{2}[4M_{11} - 8M_{21} - 3M_{31}]$$
$$= \frac{1}{2}\left[4\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix} - 8\begin{vmatrix} 9 & 1 \\ -2 & 1 \end{vmatrix} - 3\begin{vmatrix} 9 & 1 \\ 2 & 1 \end{vmatrix}\right]$$
$$= \frac{1}{2}[4(4) - 8(11) - 3(7)] = \frac{1}{2}[16 - 88 - 21] = \frac{1}{2}(-93) = -\frac{93}{2}$$
$$\left|-\frac{93}{2}\right| = \frac{93}{2}$$

The area of the triangle is $46\frac{1}{2}$ square units.

**58.** $$\frac{1}{2}\begin{vmatrix} 0 & 4 & 1 \\ -5 & 7 & 1 \\ 2 & 9 & 1 \end{vmatrix} = \frac{1}{2}[(0)C_{11} + (-5)C_{21} + 2C_{31}]$$
$$= \frac{1}{2}[0M_{11} + 5M_{21} + 2M_{31}]$$
$$= \frac{1}{2}\left[0 + 5\begin{vmatrix} 4 & 1 \\ 9 & 1 \end{vmatrix} + 2\begin{vmatrix} 4 & 1 \\ 7 & 1 \end{vmatrix}\right]$$
$$= \frac{1}{2}[5(-5) + 2(-3)] = \frac{1}{2}[-25 - 6] = \frac{1}{2}(-31) = -\frac{31}{2}$$
$$\left|-\frac{31}{2}\right| = \frac{31}{2}$$

The area of the triangle is $15\frac{1}{2}$ square units.

**59.** $$\begin{vmatrix} a & b & c \\ na & nb & nc \\ d & e & f \end{vmatrix} = \begin{vmatrix} a & b & c \\ a & b & c \\ d & e & f \end{vmatrix}$$

Since two rows are identical, the determinant is 0.

**60.** $$\begin{vmatrix} a & d & 0 \\ b & e & 0 \\ c & f & 0 \end{vmatrix} = 0C_{13} + 0C_{23} + 0C_{33} \quad = 0 + 0 + 0 = 0$$

**61.** $$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 0 \end{vmatrix} = xC_{11} + yC_{12} + 1C_{13} = 0$$
$$= xM_{11} - yM_{12} + 1M_{13} = 0$$
$$= x(y_1 - y_2) - y(x_1 - x_2) + (x_1y_2 - x_2y_1) = 0$$

Since $x_1$, $x_2$, $y_1$ and $y_2$ are constants, $x(y_1 - y_2) - y(x_1 - x_2) + (x_1y_2 - x_2y_1) = 0$ is a line in the form $ax + by + c = 0$, and $(x_1, y_1)$ and $(x_2, y_2)$ satisfy this equation.

**62.** $\begin{vmatrix} x & y & 1 \\ 2 & 3 & 1 \\ -1 & 4 & 1 \end{vmatrix} = xC_{11} + yC_{12} + 1C_{13} = 0$

$= xM_{11} - yM_{12} + 1M_{13} = x\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} - y\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix} = 0$

$= x(-1) - y(3) + 1(11) = 0$

$= -x - 3y + 11 = 0$

$x + 3y = 11$ is the equation of the line passing through the points (2,3) and (–1,4).

**63.** $\begin{vmatrix} x & y & 1 \\ -3 & 4 & 1 \\ 2 & -3 & 1 \end{vmatrix} = xC_{11} + yC_{12} + 1C_{13} = 0$

$= xM_{11} - yM_{12} + 1M_{13} = 0$

$= x(7) - y(-5) + 1(1) = 0$

$= 7x + 5y + 1 = 0$

$7x + 5y = -1$ is the equation of the line passing through the points (–3,4) and (2,–3).

**64.** $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$

$\begin{vmatrix} a_1 & b_1 \\ Ka_1 + a_2 & Kb_1 + b_2 \end{vmatrix} = a_1b_1K + a_1b_2 - (a_1b_1K + a_2b_1) = a_1b_2 - a_2b_1$

Adding a multiple of a row to another row does not change the value of the determinant.

**65.** $A = \frac{1}{2}\left( \begin{vmatrix} 8 & 25 \\ -4 & 5 \end{vmatrix} + \begin{vmatrix} 25 & 15 \\ 5 & 9 \end{vmatrix} + \begin{vmatrix} 15 & 17 \\ 9 & 20 \end{vmatrix} + \begin{vmatrix} 17 & 0 \\ 20 & 10 \end{vmatrix} + \begin{vmatrix} 0 & 8 \\ 10 & -4 \end{vmatrix} \right)$

$= \frac{1}{2}[140 + 150 + 147 + 170 + (-80)]$

$= \frac{1}{2}(527) = 263.5$ square units

## Prepare for Section 10.5

**PS1.** $\begin{vmatrix} -5 & 2 \\ 3 & 1 \end{vmatrix} = -5(1) - 3(2) = -5 - 6 = -11$

**PS2.** $\begin{vmatrix} 3 & -1 & 6 \\ 2 & 9 & 0 \\ 1 & -2 & 3 \end{vmatrix} = 3\begin{vmatrix} 9 & 0 \\ -2 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} + 6\begin{vmatrix} 2 & 9 \\ 1 & -2 \end{vmatrix}$

$= 3(27) + 1(6) + 6(-13)$

$= 81 + 6 - 78$

$= 9$

**PS3.** $\begin{bmatrix} 2 & -7 \\ 3 & 5 \end{bmatrix}$

**PS4.** $\begin{vmatrix} 1 & -2 & 1 \\ -1 & 1 & -2 \\ 2 & 3 & -1 \end{vmatrix} = 1\begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} + 2\begin{vmatrix} -1 & -2 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} -1 & 1 \\ 2 & 3 \end{vmatrix}$

$= (5) + 2(5) + (-5)$

$= 5 + 10 - 5$

$= 10$

**PS5.** $\begin{vmatrix} 3 & -1 \\ 2 & -3 \end{vmatrix} = -9 + 2 = -7$ $\quad \begin{vmatrix} 1 & 4 \\ -2 & 5 \end{vmatrix} = 5 + 8 = 13$ $\quad \dfrac{\begin{vmatrix} 3 & -1 \\ 2 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & 4 \\ -2 & 5 \end{vmatrix}} = -\dfrac{7}{13}$

**PS6.** No

## Section 10.5

**1.** $$x_1 = \frac{\begin{vmatrix} 8 & 4 \\ 1 & -5 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 1 & -5 \end{vmatrix}} = \frac{-44}{-31} = \frac{44}{31}$$
$$x_2 = \frac{\begin{vmatrix} 3 & 8 \\ 4 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 4 & -5 \end{vmatrix}} = \frac{-29}{-31} = \frac{29}{31}$$

**2.** $$x_1 = \frac{\begin{vmatrix} 9 & -3 \\ -3 & -4 \end{vmatrix}}{\begin{vmatrix} 1 & -3 \\ 2 & -4 \end{vmatrix}} = \frac{-45}{2} = -\frac{45}{2}$$
$$x_2 = \frac{\begin{vmatrix} 1 & 9 \\ 2 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & -3 \\ 2 & -4 \end{vmatrix}} = \frac{-21}{2} = -\frac{21}{2}$$

**3.** $$x_1 = \frac{\begin{vmatrix} -1 & 4 \\ 5 & -6 \end{vmatrix}}{\begin{vmatrix} 5 & 4 \\ 3 & -6 \end{vmatrix}} = \frac{-14}{-42} = \frac{1}{3}$$
$$x_2 = \frac{\begin{vmatrix} 5 & -1 \\ 3 & 5 \end{vmatrix}}{\begin{vmatrix} 5 & 4 \\ 3 & -6 \end{vmatrix}} = \frac{28}{-42} = -\frac{2}{3}$$

**4.** $$x_1 = \frac{\begin{vmatrix} 9 & 5 \\ 8 & 7 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 5 & 7 \end{vmatrix}} = \frac{23}{-11} = -\frac{23}{11}$$
$$x_2 = \frac{\begin{vmatrix} 2 & 9 \\ 5 & 8 \end{vmatrix}}{\begin{vmatrix} 2 & 5 \\ 5 & 7 \end{vmatrix}} = \frac{-29}{-11} = \frac{29}{11}$$

**5.** $$x_1 = \frac{\begin{vmatrix} 0 & 2 \\ -3 & 1 \end{vmatrix}}{\begin{vmatrix} 7 & 2 \\ 2 & 1 \end{vmatrix}} = \frac{6}{3} = 2$$
$$x_2 = \frac{\begin{vmatrix} 7 & 0 \\ 2 & -3 \end{vmatrix}}{\begin{vmatrix} 7 & 2 \\ 2 & 1 \end{vmatrix}} = \frac{-21}{3} = -7$$

**6.** $$x_1 = \frac{\begin{vmatrix} 1 & -8 \\ -2 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & -8 \\ 4 & 5 \end{vmatrix}} = \frac{-11}{47} = -\frac{11}{47}$$
$$x_2 = \frac{\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & -8 \\ 4 & 5 \end{vmatrix}} = \frac{-10}{47} = -\frac{10}{47}$$

**7.** $$x_1 = \frac{\begin{vmatrix} 0 & -7 \\ 0 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & -7 \\ 2 & 4 \end{vmatrix}} = \frac{0}{26} = 0$$
$$x_2 = \frac{\begin{vmatrix} 3 & 0 \\ 2 & 0 \end{vmatrix}}{\begin{vmatrix} 3 & -7 \\ 2 & 4 \end{vmatrix}} = \frac{0}{26} = 0$$

**8.** $$x_1 = \frac{\begin{vmatrix} -3 & 4 \\ 0 & -1 \end{vmatrix}}{\begin{vmatrix} 5 & 4 \\ 2 & -1 \end{vmatrix}} = \frac{3}{-13} = -\frac{3}{13}$$
$$x_2 = \frac{\begin{vmatrix} 5 & -3 \\ 2 & 0 \end{vmatrix}}{\begin{vmatrix} 5 & 4 \\ 2 & -1 \end{vmatrix}} = \frac{6}{-13} = -\frac{6}{13}$$

**9.** $$x_1 = \frac{\begin{vmatrix} 2.1 & 0.3 \\ -1.6 & -1.4 \end{vmatrix}}{\begin{vmatrix} 1.2 & 0.3 \\ 0.8 & -1.4 \end{vmatrix}} = \frac{-2.46}{-1.92} = 1.28125$$
$$x_2 = \frac{\begin{vmatrix} 1.2 & 2.1 \\ 0.8 & -1.6 \end{vmatrix}}{\begin{vmatrix} 1.2 & 0.3 \\ 0.8 & -1.4 \end{vmatrix}} = \frac{-3.6}{-1.92} = 1.875$$

**10.** $$x_1 = \frac{\begin{vmatrix} 1.1 & -4.2 \\ -3.4 & 3.2 \end{vmatrix}}{\begin{vmatrix} 3.2 & -4.2 \\ 0.7 & 3.2 \end{vmatrix}} = \frac{-10.76}{13.18} \approx -0.82$$
$$x_2 = \frac{\begin{vmatrix} 3.2 & 1.1 \\ 0.7 & -3.4 \end{vmatrix}}{\begin{vmatrix} 3.2 & -4.2 \\ 0.7 & 3.2 \end{vmatrix}} = \frac{-11.65}{13.18} \approx -0.88$$

**11.**
$$D=\begin{vmatrix}3 & -4 & 2\\1 & -1 & 2\\2 & 2 & 3\end{vmatrix}=-17$$
$$D_1=\begin{vmatrix}1 & -4 & 2\\-2 & -1 & 2\\-3 & 2 & 3\end{vmatrix}=-21$$
$$D_2=\begin{vmatrix}3 & 1 & 2\\1 & -2 & 2\\2 & -3 & 3\end{vmatrix}=3$$
$$D_3=\begin{vmatrix}3 & -4 & 1\\1 & -1 & -2\\2 & 2 & -3\end{vmatrix}=29$$
$$x_1=\frac{D_1}{D}=\frac{-21}{-17}=\frac{21}{17}$$
$$x_2=\frac{D_2}{D}=\frac{3}{-17}=-\frac{3}{17}$$
$$x_3=\frac{D_3}{D}=\frac{29}{-17}=-\frac{29}{17}$$

**12.**
$$D=\begin{vmatrix}5 & -2 & 3\\3 & 1 & -2\\1 & -2 & 3\end{vmatrix}=-4$$
$$D_1=\begin{vmatrix}-2 & -2 & 3\\3 & 1 & -2\\-1 & -2 & 3\end{vmatrix}=1$$
$$D_2=\begin{vmatrix}5 & -2 & 3\\3 & 3 & -2\\1 & -1 & 3\end{vmatrix}=39$$
$$D_3=\begin{vmatrix}5 & -2 & -2\\3 & 1 & 3\\1 & -2 & -1\end{vmatrix}=27$$
$$x_1=\frac{D_1}{D}=\frac{1}{-4}=-\frac{1}{4}$$
$$x_2=\frac{D_2}{D}=\frac{39}{-4}=-\frac{39}{4}$$
$$x_3=\frac{D_3}{D}=\frac{27}{-4}=-\frac{27}{4}$$

**13.**
$$D=\begin{vmatrix}1 & 4 & -2\\3 & -2 & 3\\2 & 1 & -3\end{vmatrix}=49$$
$$D_1=\begin{vmatrix}0 & 4 & -2\\4 & -2 & 3\\-1 & 1 & -3\end{vmatrix}=32$$
$$D_2=\begin{vmatrix}1 & 0 & -2\\3 & 4 & 3\\2 & -1 & -3\end{vmatrix}=13$$
$$D_3=\begin{vmatrix}1 & 4 & 0\\3 & -2 & 4\\2 & 1 & -1\end{vmatrix}=42$$
$$x_1=\frac{D_1}{D}=\frac{32}{49}$$
$$x_2=\frac{D_2}{D}=\frac{13}{49}$$
$$x_3=\frac{D_3}{D}=\frac{42}{49}=\frac{6}{7}$$

**14.**
$$D=\begin{vmatrix}4 & -1 & 2\\1 & 3 & -1\\2 & 3 & -2\end{vmatrix}=-18$$
$$D_1=\begin{vmatrix}6 & -1 & 2\\-1 & 3 & -1\\5 & 3 & -2\end{vmatrix}=-47$$
$$D_2=\begin{vmatrix}4 & 6 & 2\\1 & -1 & -1\\2 & 5 & -2\end{vmatrix}=42$$
$$D_3=\begin{vmatrix}4 & -1 & 6\\1 & 3 & -1\\2 & 3 & 5\end{vmatrix}=61$$
$$x_1=\frac{D_1}{D}=\frac{-47}{-18}=\frac{47}{18}$$
$$x_2=\frac{D_2}{D}=\frac{42}{-18}=-\frac{7}{3}$$
$$x_3=\frac{D_3}{D}=\frac{61}{-18}=-\frac{61}{18}$$

**15.**
$$D=\begin{vmatrix}0 & 2 & -3\\3 & -5 & 1\\4 & 0 & 2\end{vmatrix}=-64$$
$$D_1=\begin{vmatrix}1 & 2 & -3\\0 & -5 & 1\\-3 & 0 & 2\end{vmatrix}=29$$
$$D_2=\begin{vmatrix}0 & 1 & -3\\3 & 0 & 1\\4 & -3 & 2\end{vmatrix}=25$$
$$D_3=\begin{vmatrix}0 & 2 & 1\\3 & -5 & 0\\4 & 0 & -3\end{vmatrix}=38$$
$$x_1=\frac{D_1}{D}=\frac{29}{-64}=-\frac{29}{64}$$
$$x_2=\frac{D_2}{D}=\frac{25}{-64}=-\frac{25}{64}$$
$$x_3=\frac{D_3}{D}=\frac{38}{-64}=-\frac{19}{32}$$

**16.**
$$D=\begin{vmatrix}2 & 5 & 0\\1 & 0 & -3\\2 & -1 & 2\end{vmatrix}=-46$$
$$D_1=\begin{vmatrix}1 & 5 & 0\\-2 & 0 & -3\\4 & -1 & 2\end{vmatrix}=-43$$
$$D_2=\begin{vmatrix}2 & 1 & 0\\1 & -2 & -3\\2 & 4 & 2\end{vmatrix}=8$$
$$D_3=\begin{vmatrix}2 & 5 & 1\\1 & 0 & -2\\2 & -1 & 4\end{vmatrix}=-45$$
$$x_1=\frac{D_1}{D}=\frac{-43}{-46}=\frac{43}{46}$$
$$x_2=\frac{D_2}{D}=\frac{8}{-46}=-\frac{4}{23}$$
$$x_3=\frac{D_3}{D}=\frac{-45}{-46}=\frac{45}{46}$$

**17.**
$$D=\begin{vmatrix}4 & -5 & 1\\3 & 1 & 0\\1 & -1 & 3\end{vmatrix}=53$$
$$D_1=\begin{vmatrix}-2 & -5 & 1\\4 & 1 & 0\\0 & -1 & 3\end{vmatrix}=50$$
$$D_2=\begin{vmatrix}4 & -2 & 1\\3 & 4 & 0\\1 & 0 & 3\end{vmatrix}=62$$
$$D_3=\begin{vmatrix}4 & -5 & -2\\3 & 1 & 4\\1 & -1 & 0\end{vmatrix}=4$$
$$x_1=\frac{D_1}{D}=\frac{50}{53}$$
$$x_2=\frac{D_2}{D}=\frac{62}{53}$$
$$x_3=\frac{D_3}{D}=\frac{4}{53}$$

**18.**
$$D=\begin{vmatrix}3 & -1 & 1\\1 & 0 & 3\\2 & 2 & -5\end{vmatrix}=-27$$
$$D_1=\begin{vmatrix}5 & -1 & 1\\-2 & 0 & 3\\0 & 2 & -5\end{vmatrix}=-24$$
$$D_2=\begin{vmatrix}3 & 5 & 1\\1 & -2 & 3\\2 & 0 & -5\end{vmatrix}=89$$
$$D_3=\begin{vmatrix}3 & -1 & 5\\1 & 0 & -2\\2 & 2 & 0\end{vmatrix}=26$$
$$x_1=\frac{D_1}{D}=\frac{-24}{-27}=\frac{8}{9}$$
$$x_2=\frac{D_2}{D}=\frac{89}{-27}=-\frac{89}{27}$$
$$x_3=\frac{D_3}{D}=\frac{26}{-27}=-\frac{26}{27}$$

**19.**
$$D=\begin{vmatrix}2 & 2 & -3\\1 & -3 & 2\\4 & -1 & 3\end{vmatrix}=-37$$
$$D_1=\begin{vmatrix}0 & 2 & -3\\0 & -3 & 2\\0 & -1 & 3\end{vmatrix}=0$$
$$D_2=\begin{vmatrix}2 & 0 & -3\\1 & 0 & 2\\4 & 0 & 3\end{vmatrix}=0$$
$$D_3=\begin{vmatrix}2 & 2 & 0\\1 & -3 & 0\\4 & -1 & 0\end{vmatrix}=0$$
$$x_1=\frac{D_1}{D}=\frac{0}{-37}=0$$
$$x_2=\frac{D_2}{D}=\frac{0}{-37}=0$$
$$x_3=\frac{D_3}{D}=\frac{0}{-37}=0$$

**20.** $D = \begin{vmatrix} 1 & 3 & 0 \\ 2 & -3 & 1 \\ 4 & 5 & -2 \end{vmatrix} = 25$

$D_1 = \begin{vmatrix} -2 & 3 & 0 \\ 1 & -3 & 1 \\ 0 & 5 & -2 \end{vmatrix} = 4$

$D_2 = \begin{vmatrix} 1 & -2 & 0 \\ 2 & 1 & 1 \\ 4 & 0 & -2 \end{vmatrix} = -18$

$D_3 = \begin{vmatrix} 1 & 3 & -2 \\ 2 & -3 & 1 \\ 4 & 5 & 0 \end{vmatrix} = -37$

$x_1 = \frac{D_1}{D} = \frac{4}{25}$

$x_2 = \frac{D_2}{D} = \frac{-18}{25} = -\frac{18}{25}$

$x_3 = \frac{D_3}{D} = \frac{-37}{25} = -\frac{37}{25}$

**21.** $D = \begin{vmatrix} 2 & -3 & 4 & -1 \\ 1 & 2 & 0 & 2 \\ 3 & 1 & 0 & -2 \\ 1 & -3 & 2 & -1 \end{vmatrix} = -38$

$D_2 = \begin{vmatrix} 2 & 1 & 4 & -1 \\ 1 & -1 & 0 & 2 \\ 3 & 2 & 0 & -2 \\ 1 & 3 & 2 & -1 \end{vmatrix} = 70$

$x_2 = \frac{D_2}{D} = \frac{70}{-38} = -\frac{35}{19}$

**22.** $D = \begin{vmatrix} 3 & 1 & -2 & 3 \\ 2 & -3 & 2 & 0 \\ 1 & 1 & -2 & 2 \\ 2 & 0 & 3 & -2 \end{vmatrix} = -3$

$D_4 = \begin{vmatrix} 3 & 1 & -2 & 4 \\ 2 & -3 & 2 & -2 \\ 1 & 1 & -2 & 3 \\ 2 & 0 & 3 & 4 \end{vmatrix} = 51$

$x_4 = \frac{D_4}{D} = \frac{51}{-3} = -17$

**23.** $D = \begin{vmatrix} 1 & -3 & 2 & 4 \\ 3 & 5 & -6 & 2 \\ 2 & -1 & 9 & 8 \\ 1 & 1 & 1 & -8 \end{vmatrix} = -1310$

$D_1 = \begin{vmatrix} 0 & -3 & 2 & 4 \\ -2 & 5 & -6 & 2 \\ 0 & -1 & 9 & 8 \\ -3 & 1 & 1 & -8 \end{vmatrix} = 1210$

$x_1 = \frac{D_1}{D} = \frac{1210}{-1310} = -\frac{121}{131}$

**24.** $D = \begin{vmatrix} 2 & 5 & -5 & -3 \\ 1 & 7 & 8 & -1 \\ 4 & 0 & 1 & 1 \\ 3 & 2 & -1 & 0 \end{vmatrix} = 168$

$D_3 = \begin{vmatrix} 2 & 5 & -3 & -3 \\ 1 & 7 & 4 & -1 \\ 4 & 0 & 3 & 1 \\ 3 & 2 & 0 & 0 \end{vmatrix} = 157$

$x_3 = \frac{D_3}{D} = \frac{157}{168}$

**25.** $D = \begin{vmatrix} 0 & 3 & -1 & 2 \\ 5 & 1 & 3 & -1 \\ 1 & -2 & 0 & 9 \\ 2 & 0 & 2 & 0 \end{vmatrix} = 120$

$D_4 = \begin{vmatrix} 0 & 3 & -1 & 1 \\ 5 & 1 & 3 & -4 \\ 1 & -2 & 0 & 5 \\ 2 & 0 & 2 & 3 \end{vmatrix} = 160$

$x_4 = \frac{D_4}{D} = \frac{160}{120} = \frac{4}{3}$

**26.** $D = \begin{vmatrix} 4 & 1 & 0 & -3 \\ 5 & 2 & -2 & 1 \\ 1 & -3 & 2 & -2 \\ 0 & 0 & 3 & 4 \end{vmatrix} = -254$, $D_1 = \begin{vmatrix} 4 & 1 & 0 & -3 \\ 7 & 2 & -2 & 1 \\ -6 & -3 & 2 & -2 \\ -7 & 0 & 3 & 4 \end{vmatrix} = -77$

$x_1 = \frac{D_1}{D} = \frac{-77}{-254} = \frac{77}{254}$

## Connecting Concepts

**27.** $D = \begin{vmatrix} 2 & -3 & 1 \\ 1 & 1 & -2 \\ 4 & -1 & -3 \end{vmatrix} = 0$

In order for us to use Cramer's Rule, the determinant of the coefficient matrix cannot be zero. The system of equations has infinitely many solutions.

**28.** $a_{11}x_1 + a_{12}x_2 = b_1$

$a_{21}x_1 + a_{22}x_2 = b_2$

$-a_{21}a_{11}x_1 - a_{21}a_{12}x_2 = -a_{21}b_1$

$a_{11}a_{21}x_1 + a_{11}a_{22}x_2 = a_{11}b_2$

$(a_{11}a_{22} - a_{21}a_{12})x_2 = a_{11}b_2 - a_{21}b_1$

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} x_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

**29.** $D = \begin{vmatrix} k & 3 \\ k & -2 \end{vmatrix} = -5k$

For the system of equations to have a unique solution, the determinant of the coefficient matrix cannot be zero.

$-5k = 0$

$k = 0$

The system of equations has a unique solution for all values of $k$ except $k = 0$.

**30.** $D = \begin{vmatrix} k & 4 \\ 9 & -k \end{vmatrix} = -k^2 - 36$

For the system of equations to have a unique solution, the determinant of the coefficient matrix cannot be zero.

$$\begin{aligned} -k^2 - 36 &= 0 \\ k^2 &= -36 \\ k &= \pm\sqrt{-36} \end{aligned}$$

$\sqrt{-36}$ is not a real number. The system of equations has a unique solution for all real values of $k$.

**31.** $D = \begin{vmatrix} 1 & 2 & -3 \\ 2 & k & -4 \\ 1 & -2 & 1 \end{vmatrix} = 4k - 8$

For the system of equations to have a unique solution, the determinant of the coefficient matrix cannot be zero.

$$\begin{aligned} 4k - 8 &= 0 \\ 4(k-2) &= 0 \\ k - 2 &= 0 \\ k &= 2 \end{aligned}$$

The system of equations has a unique solution for all values of $k$ except $k = 2$.

**32.** $D = \begin{vmatrix} k & 1 & 0 \\ 0 & 1 & -4 \\ 1 & 0 & k \end{vmatrix} = k^2 - 4$

For the system of equations to have a unique solution, the determinant of the coefficient matrix cannot be zero.

$$\begin{aligned} k^2 - 4 &= 0 \\ k^2 &= 4 \\ k &= \pm 2 \end{aligned}$$

The system of equations has a unique solution for all values of $k$ except $k = 2$ and $k = -2$.

**33.**
$$\begin{aligned} ru + sv &= w \\ (2+3i)r + (4-2i)s &= -6+15i \\ 2r + 3ri + 4s - 2si &= -6+15i \\ (2r+4s) + (3r-2s)i &= -6+15i \end{aligned}$$

$$2r + 4s = -6$$
$$3r - 2s = 15$$

$$D = \begin{vmatrix} 2 & 4 \\ 3 & -2 \end{vmatrix} = -16$$
$$D_r = \begin{vmatrix} -6 & 4 \\ 15 & -2 \end{vmatrix} = -48$$
$$D_s = \begin{vmatrix} 2 & -6 \\ 3 & 15 \end{vmatrix} = 48$$
$$r = \frac{D_r}{D} = \frac{-48}{-16} = 3$$
$$s = \frac{D_s}{D} = \frac{48}{-16} = -3$$

**34.**
$$\begin{aligned} ru + sv &= w \\ (3-4i)r + (1+2i)s &= 4-22i \\ 3r - 4ri + s + 2si &= 4-22i \\ (3r+s) + (-4r+2s)i &= 4-22i \end{aligned}$$

$$3r + s = 4$$
$$-4r + 2s = -22$$

$$D = \begin{vmatrix} 3 & 1 \\ -4 & 2 \end{vmatrix} = 10$$
$$D_r = \begin{vmatrix} 4 & 1 \\ -22 & 2 \end{vmatrix} = 30$$
$$D_s = \begin{vmatrix} 3 & 4 \\ -4 & -22 \end{vmatrix} = -50$$
$$r = \frac{D_r}{D} = \frac{30}{10} = 3$$
$$s = \frac{D_s}{D} = \frac{-50}{10} = -5$$

## Exploring Concepts with Technology

**Stochastic Matrices**

$$XT = \begin{bmatrix} 0.428 & 0.572 \end{bmatrix}$$
$$XT^2 \approx \begin{bmatrix} 0.45236 & 0.54764 \end{bmatrix}$$
$$XT^3 \approx \begin{bmatrix} 0.47355 & 0.52645 \end{bmatrix}$$

$$XT^{20} \approx \begin{bmatrix} 0.60209 & 0.39791 \end{bmatrix}$$
$$XT^{40} \approx \begin{bmatrix} 0.61456 & 0.38544 \end{bmatrix}$$
$$XT^{60} \approx \begin{bmatrix} 0.61533 & 0.38467 \end{bmatrix}$$
$$XT^{100} \approx \begin{bmatrix} 0.61538 & 0.38462 \end{bmatrix}$$

It appears that as the number of weeks increases, Super A will get slightly more than 61.5% of the neighborhood and Super B will get slightly less than 38.5% of the neighborhood.

Changing the market share *does not* affect the result (to 6 decimal places) after 100 weeks.

**Three Department Stores:** After 100 weeks, Super A will have 23.87% of the market share, Super B will have 33.65% of the market share, and Super C will have 42.48% of the market share.

## Assessing Concepts

**1.** See the Chapter Summary under 10.1.

**2.** See the Chapter Summary under 10.1.

**3.** Yes.

**4.** They must be of the same order.

**5.** The number of columns of the first matrix must equal the number of rows of the second matrix.

**6.** A matrix does not have a multiplicative inverse.

**7.** A square matrix with 1's along the main diagonal and zeros elsewhere.

**8.** No. Nonsquare and singular matrices do not have multiplicative inverses.

**9.** It is the determinant of the matrix obtained by deleting the $i$th row and the $j$th column of $A$.

**10.** No. If the determinant of the coefficient matrix is zero, Cramer's Rule cannot be used.

## Chapter Review

**1.** $3A = 3\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -3 & 9 \\ 9 & 6 & -3 \end{bmatrix}$ [10.2]

**2.** $-2B = -2\begin{bmatrix} 0 & -2 \\ 4 & 2 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -8 & -4 \\ -2 & 6 \end{bmatrix}$ [10.2]

**3.** $-A + D = -\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \end{bmatrix} + \begin{bmatrix} -3 & 4 & 2 \\ 4 & -2 & 5 \end{bmatrix}$ [10.2]

$= \begin{bmatrix} -2 & 1 & -3 \\ -3 & -2 & 1 \end{bmatrix} + \begin{bmatrix} -3 & 4 & 2 \\ 4 & -2 & 5 \end{bmatrix}$

$= \begin{bmatrix} -5 & 5 & -1 \\ 1 & -4 & 6 \end{bmatrix}$

**4.** $2A - 3D = 2\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \end{bmatrix} - 3\begin{bmatrix} -3 & 4 & 2 \\ 4 & -2 & 5 \end{bmatrix}$ [10.2]

$= \begin{bmatrix} 4 & -2 & 6 \\ 6 & 4 & -2 \end{bmatrix} - \begin{bmatrix} -9 & 12 & 6 \\ 12 & -6 & 15 \end{bmatrix}$

$= \begin{bmatrix} 13 & -14 & 0 \\ -6 & 10 & -17 \end{bmatrix}$

**5.** $AB = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 4 & 2 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} -1 & -5 \\ 7 & 1 \end{bmatrix}$ [10.2]

**6.** $DB = \begin{bmatrix} -3 & 4 & 2 \\ 4 & -2 & 5 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 4 & 2 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 18 & 8 \\ -3 & -27 \end{bmatrix}$ [10.2]

**7.** $BA = \begin{bmatrix} 0 & -2 \\ 4 & 2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \end{bmatrix} = \begin{bmatrix} -6 & -4 & 2 \\ 14 & 0 & 10 \\ -7 & -7 & 6 \end{bmatrix}$ [10.2]

**8.** $BD = \begin{bmatrix} 0 & -2 \\ 4 & 2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} -3 & 4 & 2 \\ 4 & -2 & 5 \end{bmatrix} = \begin{bmatrix} -8 & 4 & -10 \\ -4 & 12 & 18 \\ -15 & 10 & -13 \end{bmatrix}$ [10.2]

**9.** $C^2 = C \cdot C = \begin{bmatrix} 2 & 6 & 1 \\ 1 & 2 & -1 \\ 2 & 4 & -1 \end{bmatrix}\begin{bmatrix} 2 & 6 & 1 \\ 1 & 2 & -1 \\ 2 & 4 & -1 \end{bmatrix}$ [10.2]

$= \begin{bmatrix} 12 & 28 & -5 \\ 2 & 6 & 0 \\ 6 & 16 & -1 \end{bmatrix}$

**10.** $C^3 = C \cdot C \cdot C = \begin{bmatrix} 2 & 6 & 1 \\ 1 & 2 & -1 \\ 2 & 4 & -1 \end{bmatrix}\begin{bmatrix} 2 & 6 & 1 \\ 1 & 2 & -1 \\ 2 & 4 & -1 \end{bmatrix}\begin{bmatrix} 2 & 6 & 1 \\ 1 & 2 & -1 \\ 2 & 4 & -1 \end{bmatrix}$ [10.2]

$= \begin{bmatrix} 12 & 28 & -5 \\ 2 & 6 & 0 \\ 6 & 16 & -1 \end{bmatrix}\begin{bmatrix} 2 & 6 & 1 \\ 1 & 2 & -1 \\ 2 & 4 & -1 \end{bmatrix}$

$= \begin{bmatrix} 42 & 108 & -11 \\ 10 & 24 & -4 \\ 26 & 64 & -9 \end{bmatrix}$

**11.** $BAC = \begin{bmatrix} 0 & -2 \\ 4 & 2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 6 & 1 \\ 1 & 2 & -1 \\ 2 & 4 & -1 \end{bmatrix} = \begin{bmatrix} -6 & -4 & 2 \\ 14 & 0 & 10 \\ -7 & -7 & 6 \end{bmatrix}\begin{bmatrix} 2 & 6 & 1 \\ 1 & 2 & -1 \\ 2 & 4 & -1 \end{bmatrix} = \begin{bmatrix} -12 & -36 & -4 \\ 48 & 124 & 4 \\ -9 & -32 & -6 \end{bmatrix}$ [10.2]

**12.** Not possible since $A$ is of order $2 \times 3$ and $D$ is of order $2 \times 3$. [10.2]

**13.** $AB - BA = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 4 & 2 \\ 1 & -3 \end{bmatrix} - \begin{bmatrix} 0 & -2 \\ 4 & 2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -15 \\ 7 & 1 \end{bmatrix} - \begin{bmatrix} -6 & -4 & 2 \\ 14 & 0 & 10 \\ -7 & -7 & 6 \end{bmatrix}$

Not possible since $AB$ is of order $2 \times 2$ and $BA$ is of order $3 \times 3$. [10.2]

**14.** $DB-BD=\begin{bmatrix}-3&4&2\\4&-2&5\end{bmatrix}\begin{bmatrix}0&-2\\4&2\\1&-3\end{bmatrix}-\begin{bmatrix}0&-2\\4&2\\1&-3\end{bmatrix}\begin{bmatrix}-3&4&2\\4&-2&5\end{bmatrix}=\begin{bmatrix}18&8\\-3&-27\end{bmatrix}-\begin{bmatrix}-8&4&-10\\-4&12&18\\-15&10&-13\end{bmatrix}$

Not possible since *DB* is of order 2 × 2 and *BD* is of order 3 × 3. [10.2]

**15.** $(A-D)C=\left(\begin{bmatrix}2&-1&3\\3&2&-1\end{bmatrix}-\begin{bmatrix}-3&4&2\\4&-2&5\end{bmatrix}\right)\begin{bmatrix}2&6&1\\1&2&-1\\2&4&-1\end{bmatrix}=\begin{bmatrix}5&-5&1\\-1&4&-6\end{bmatrix}\begin{bmatrix}2&6&1\\1&2&-1\\2&4&-1\end{bmatrix}=\begin{bmatrix}7&24&9\\-10&-22&1\end{bmatrix}$ [10.2]

**16.** $AC-DC=\begin{bmatrix}2&-1&3\\3&2&-1\end{bmatrix}\begin{bmatrix}2&6&1\\1&2&-1\\2&4&-1\end{bmatrix}-\begin{bmatrix}-3&4&2\\4&-2&5\end{bmatrix}\begin{bmatrix}2&6&1\\1&2&-1\\2&4&-1\end{bmatrix}=\begin{bmatrix}9&22&0\\6&18&2\end{bmatrix}\begin{bmatrix}2&-2&-9\\16&40&1\end{bmatrix}=\begin{bmatrix}7&24&9\\-10&-22&1\end{bmatrix}$ [10.2]

**17.** $\left[\begin{array}{ccc|ccc}2&6&1&1&0&0\\1&2&-1&0&1&0\\2&4&-1&0&0&1\end{array}\right]\xrightarrow{\frac{1}{2}R_1}\left[\begin{array}{ccc|ccc}1&3&\frac{1}{2}&\frac{1}{2}&0&0\\1&2&-1&0&1&0\\2&4&-1&0&0&1\end{array}\right]\xrightarrow[-2R_1+R_3]{-1R_1+R_2}\left[\begin{array}{ccc|ccc}1&3&\frac{1}{2}&\frac{1}{2}&0&0\\0&-1&-\frac{3}{2}&-\frac{1}{2}&1&0\\0&-2&-2&-1&0&1\end{array}\right]$

$\xrightarrow[\substack{-3R_2+R_1\\2R_2+R_3}]{-1R_2}\left[\begin{array}{ccc|ccc}1&0&-4&-1&3&0\\0&1&\frac{3}{2}&\frac{1}{2}&-1&0\\0&0&1&0&-2&1\end{array}\right]\xrightarrow[(-3/2)R_3+R_2]{4R_3+R_1}\left[\begin{array}{ccc|ccc}1&0&0&-1&-5&4\\0&1&0&\frac{1}{2}&2&-\frac{3}{2}\\0&0&1&0&-2&1\end{array}\right]$

$C^{-1}=\begin{bmatrix}-1&-5&4\\\frac{1}{2}&2&-\frac{3}{2}\\0&-2&1\end{bmatrix}$ [10.3]

**18.** Determinant of $C=\begin{bmatrix}2&6&1\\1&2&-1\\2&4&-1\end{bmatrix}=2C_{11}+1C_{21}+2C_{31}$ [10.4]

$=2M_{11}-1M_{21}+2M_{31}$

$=2\begin{vmatrix}2&-1\\4&-1\end{vmatrix}-1\begin{vmatrix}6&1\\4&-1\end{vmatrix}+2\begin{vmatrix}6&1\\2&-1\end{vmatrix}=2(2)-1(-10)+2(-8)=4+10-16=-2$

**19.** $\left[\begin{array}{cc|c}2&-3&7\\3&-4&10\end{array}\right]\xrightarrow{-1R_1+R_2}\left[\begin{array}{cc|c}2&-3&7\\1&-1&3\end{array}\right]\xleftrightarrow{R_1+R_2}\left[\begin{array}{cc|c}1&-1&3\\2&-3&7\end{array}\right]\xrightarrow{-2R_1+R_2}\left[\begin{array}{cc|c}1&-1&3\\0&-1&1\end{array}\right]\xrightarrow{-1R_2}\left[\begin{array}{cc|c}1&-1&3\\0&1&-1\end{array}\right]$

$\begin{cases}x-y=3\\y=-1\end{cases}\quad x-(-1)=3,\quad x=2$

The solution is (2, −1). [10.1]

**20.** $\left[\begin{array}{cc|c}3&4&-9\\2&3&-7\end{array}\right]\xrightarrow{-R_2+R_1}\left[\begin{array}{cc|c}1&1&-2\\2&3&-7\end{array}\right]\xrightarrow{-2R_1+R_2}\left[\begin{array}{cc|c}1&1&-2\\0&1&-3\end{array}\right]$

$\begin{cases}x+y=-2\\y=-3\end{cases}\quad x+(-3)=-2,\quad x=1$

The solution is (1, −3). [10.1]

**21.** $\left[\begin{array}{cc|c} 4 & -5 & 12 \\ 3 & 1 & 9 \end{array}\right] \xrightarrow{-1R_2+R_1} \left[\begin{array}{cc|c} 1 & -6 & 3 \\ 3 & 1 & 9 \end{array}\right] \xrightarrow{-3R_1+R_2} \left[\begin{array}{cc|c} 1 & -6 & 3 \\ 0 & 19 & 0 \end{array}\right] \xrightarrow{\frac{1}{19}R_2} \left[\begin{array}{cc|c} 1 & -6 & 3 \\ 0 & 1 & 0 \end{array}\right]$

$\begin{cases} x-6y=3 \\ y=0 \end{cases} \quad \begin{aligned} x-6(0)&=3 \\ x&=3 \end{aligned}$

The solution is (3, 0). [10.1]

**22.** $\left[\begin{array}{cc|c} 2 & -5 & 10 \\ 5 & 2 & 4 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{cc|c} 2 & -5 & 10 \\ 1 & 12 & -16 \end{array}\right] R_1 \longleftrightarrow R_2 \left[\begin{array}{cc|c} 1 & 12 & -16 \\ 2 & -5 & 10 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{cc|c} 1 & 12 & -16 \\ 0 & -29 & 42 \end{array}\right]$

$\xrightarrow{-\frac{1}{29}R_2} \left[\begin{array}{cc|c} 1 & 12 & -16 \\ 0 & 1 & -\frac{42}{29} \end{array}\right]$

$\begin{cases} x+12y=-16 \\ y=-\frac{42}{29} \end{cases} \quad \begin{aligned} x+12\left(-\frac{42}{29}\right)&=-16 \\ x-\frac{504}{29}&=-16 \\ x&=\frac{40}{29} \end{aligned}$

The solution is $\left(\frac{40}{29}, -\frac{42}{29}\right)$. [10.1]

**23.** $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 3 & 8 & 11 & 17 \\ 2 & 6 & 7 & 12 \end{array}\right] \xrightarrow[-2R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 1 & 2 \end{array}\right] \xrightarrow{(1/2)R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 2 \end{array}\right] \xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right] \xrightarrow{-1R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\begin{cases} x+2y+3z=5 \\ y+z=1 \\ z=0 \end{cases} \quad \begin{aligned} y+0&=1 \\ y&=1 \end{aligned} \quad \begin{aligned} x+2(1)+3(0)&=5 \\ x&=3 \end{aligned}$

The solution is (3, 1, 0). [10.1]

**24.** $\left[\begin{array}{ccc|c} 1 & -1 & 3 & 10 \\ 2 & -1 & 7 & 24 \\ 3 & -6 & 7 & 21 \end{array}\right] \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 10 \\ 0 & 1 & 1 & 4 \\ 0 & -3 & -2 & -9 \end{array}\right] \xrightarrow{3R_2+R_3} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 10 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$\begin{cases} x-y+3z=10 \\ y+z=4 \\ z=3 \end{cases} \quad \begin{aligned} y+3&=4 \\ y&=1 \end{aligned} \quad \begin{aligned} x-1+3(3)&=10 \\ x&=2 \end{aligned}$

The solution is (2, 1, 3). [10.1]

**25.** $\left[\begin{array}{ccc|c} 2 & -1 & -1 & 4 \\ 1 & -2 & -2 & 5 \\ 3 & -3 & -8 & 19 \end{array}\right] R_1 \longleftrightarrow R_2 \left[\begin{array}{ccc|c} 1 & -2 & -2 & 5 \\ 2 & -1 & -1 & 4 \\ 3 & -3 & -8 & 19 \end{array}\right] \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & -2 & -2 & 5 \\ 0 & 3 & 3 & -6 \\ 0 & 3 & -2 & 4 \end{array}\right] \xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & -2 & -2 & 5 \\ 0 & 1 & 1 & -2 \\ 0 & 3 & -2 & 4 \end{array}\right]$

$\xrightarrow{-3R_2+R_3} \left[\begin{array}{ccc|c} 1 & -2 & -2 & 5 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & -5 & 10 \end{array}\right] \xrightarrow{-\frac{1}{5}R_3} \left[\begin{array}{ccc|c} 1 & -2 & -2 & 5 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$

$\begin{cases} x-2y-2z=5 \\ y+z=-2 \\ z=-2 \end{cases} \quad \begin{aligned} y+(-2)&=-2 \\ y&=0 \end{aligned} \quad \begin{aligned} x-2(0)-2(-2)&=5 \\ x&=1 \end{aligned}$

The solution is (1, 0, –2). [10.1]

**26.** $\left[\begin{array}{ccc|c} 3 & -7 & 8 & 10 \\ 1 & -3 & 2 & 0 \\ 2 & -8 & 7 & 5 \end{array}\right] R_1 \longleftrightarrow R_2 \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 3 & -7 & 8 & 10 \\ 2 & -8 & 7 & 5 \end{array}\right] \xrightarrow[-2R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 2 & 2 & 10 \\ 0 & -2 & 3 & 5 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 1 & 1 & 5 \\ 0 & -2 & 3 & 5 \end{array}\right]$

$\xrightarrow{2R_2+R_3} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 5 & 15 \end{array}\right] \xrightarrow{\frac{1}{5}R_3} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$\begin{cases} x-3y+2z=0 \\ y+\ \ z=5 \\ z=3 \end{cases} \quad \begin{aligned} y+3&=5 \\ y&=2 \end{aligned} \quad \begin{aligned} x-3(2)+2(3)&=0 \\ x&=0 \end{aligned}$

The solution is (0, 2, 3). [10.1]

**27.** $\left[\begin{array}{ccc|c} 4 & -9 & 6 & 54 \\ 3 & -8 & 8 & 49 \\ 1 & -3 & 2 & 17 \end{array}\right] R_1 \longleftrightarrow R_3 \left[\begin{array}{ccc|c} 1 & -3 & 2 & 17 \\ 3 & -8 & 8 & 49 \\ 4 & -9 & 6 & 54 \end{array}\right] \xrightarrow[-4R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 17 \\ 0 & 1 & 2 & -2 \\ 0 & 3 & -2 & -14 \end{array}\right] \xrightarrow{-3R_2+R_3} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 17 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & -8 & -8 \end{array}\right]$

$\xrightarrow{(-1/8)R_3} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 17 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$

$\begin{cases} x-3y+2z=17 \\ y+2z=-2 \\ z=\ 1 \end{cases} \quad \begin{aligned} y+2(1)&=-2 \\ y&=-4 \end{aligned} \quad \begin{aligned} x-3(-4)+2(1)&=17 \\ x&=3 \end{aligned}$

The solution is (3, –4, 1). [10.1]

**28.** $\left[\begin{array}{ccc|c} 3 & 8 & -5 & 6 \\ 2 & 9 & -1 & -8 \\ 1 & -4 & -2 & 16 \end{array}\right] R_1 \longleftrightarrow R_3 \left[\begin{array}{ccc|c} 1 & -4 & -2 & 16 \\ 2 & 9 & -1 & -8 \\ 3 & 8 & -5 & 6 \end{array}\right] \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & -4 & -2 & 16 \\ 0 & 17 & 3 & -40 \\ 0 & 20 & 1 & -42 \end{array}\right] \xrightarrow{\frac{1}{17}R_2} \left[\begin{array}{ccc|c} 1 & -4 & -2 & 16 \\ 0 & 1 & \frac{3}{17} & -\frac{40}{17} \\ 0 & 20 & 1 & -42 \end{array}\right]$

$\xrightarrow{-20R_2+R_3} \left[\begin{array}{ccc|c} 1 & -4 & -2 & 16 \\ 0 & 1 & \frac{3}{17} & -\frac{40}{17} \\ 0 & 0 & -\frac{43}{17} & \frac{86}{17} \end{array}\right] \xrightarrow{-\frac{17}{43}R_3} \left[\begin{array}{ccc|c} 1 & -4 & -2 & 16 \\ 0 & 1 & \frac{3}{17} & -\frac{40}{17} \\ 0 & 0 & 1 & -2 \end{array}\right]$

$\begin{cases} x-4y-\ 2z=16 \\ y+\frac{3}{17}z=-\frac{40}{17} \\ z=-2 \end{cases} \quad \begin{aligned} y+\tfrac{3}{17}(-2)&=-\tfrac{40}{17} \\ y&=-2 \end{aligned} \quad \begin{aligned} x-4(-2)-2(-2)&=16 \\ x&=4 \end{aligned}$

The solution is (4, –2, –2). [10.1]

**29.** $\left[\begin{array}{ccc|c} 1 & 1 & 2 & -5 \\ 2 & 3 & 5 & -13 \\ 2 & 5 & 7 & -19 \end{array}\right] \xrightarrow[-2R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -5 \\ 0 & 1 & 1 & -3 \\ 0 & 3 & 3 & -9 \end{array}\right] \xrightarrow{-3R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\begin{cases} x+y+2z=-5 \\ y+\ \ z=-3 \end{cases} \quad y=-z-3 \quad \begin{aligned} x+(-z-3)+2z&=-5 \\ x&=-z-2 \end{aligned}$

Let $z$ be any real number $c$.
The solution is $(-c-2,\ -c-3,\ c)$. [10.1]

**30.** $\left[\begin{array}{rrr|r} 1 & -2 & 3 & 9 \\ 3 & -5 & 8 & 25 \\ 1 & 0 & -1 & 5 \end{array}\right] \xrightarrow{\substack{-3R_1+R_2 \\ -1R_1+R_3}} \left[\begin{array}{rrr|r} 1 & -2 & 3 & 9 \\ 0 & 1 & -1 & -2 \\ 0 & 2 & -4 & -4 \end{array}\right] \xrightarrow{-2R_2+R_3} \left[\begin{array}{rrr|r} 1 & -2 & 3 & 9 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & -2 & 0 \end{array}\right] \xrightarrow{(-1/2)R_3} \left[\begin{array}{rrr|r} 1 & -2 & 3 & 9 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\begin{cases} x-2y+3z = 9 \\ y - z = -2 \\ z = 0 \end{cases} \quad \begin{aligned} y-0 &= -2 \\ y &= -2 \end{aligned} \quad \begin{aligned} x-2(-2)+3(0) &= 9 \\ x &= 5 \end{aligned}$

The solution is (5, –2, 0). [10.1]

**31.** $\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 2 & 1 \\ 3 & 8 & 1 & 4 & 1 \\ 2 & 7 & 3 & 2 & 0 \\ 1 & 3 & -2 & 5 & 6 \end{array}\right] \xrightarrow{\substack{-3R_1+R_2 \\ -2R_1+R_3 \\ -1R_1+R_4}} \left[\begin{array}{rrrr|r} 1 & -2 & -1 & 2 & 1 \\ 0 & 2 & 4 & -2 & -2 \\ 0 & 3 & 5 & -2 & -2 \\ 0 & 1 & -1 & 3 & 5 \end{array}\right] \xrightarrow{(-1/2)R_2} \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 2 & 1 \\ 0 & 1 & 2 & -1 & -1 \\ 0 & 3 & 5 & -2 & -2 \\ 0 & 1 & -1 & 3 & 5 \end{array}\right]$

$\xrightarrow{\substack{-3R_2+R_3 \\ -1R_2+R_4}} \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 2 & 1 \\ 0 & 1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & -3 & 4 & 6 \end{array}\right] \xrightarrow{-1R_3} \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 2 & 1 \\ 0 & 1 & 2 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & -3 & 4 & 6 \end{array}\right] \xrightarrow{-3R_3+R_4} \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 2 & 1 \\ 0 & 1 & 2 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$

$\begin{cases} w+2x-y+2z = 1 \\ x+2y-z = -1 \\ y-z = -1 \\ z = 3 \end{cases} \quad \begin{aligned} y-3 &= -1 \\ y &= 2 \end{aligned} \quad \begin{aligned} x+2(2)-3 &= -1 \\ x &= -2 \end{aligned} \quad \begin{aligned} w+2(-2)-2+2(3) &= 1 \\ w &= 1 \end{aligned}$

The solution is (1, –2, 2, 3). [10.1]

**32.** $\left[\begin{array}{rrrr|r} 1 & -3 & -2 & 1 & -1 \\ 2 & -5 & 0 & 3 & 1 \\ 3 & -7 & 3 & 0 & -18 \\ 2 & -3 & -5 & -2 & -8 \end{array}\right] \xrightarrow{\substack{-2R_1+R_2 \\ -3R_1+R_3 \\ -2R_1+R_4}} \left[\begin{array}{rrrr|r} 1 & -3 & -2 & 1 & -1 \\ 0 & 1 & 4 & 1 & 3 \\ 0 & 2 & 9 & -3 & -15 \\ 0 & 3 & -1 & -4 & -6 \end{array}\right] \xrightarrow{\substack{-2R_2+R_3 \\ -3R_2+R_4}} \left[\begin{array}{rrrr|r} 1 & -3 & -2 & 1 & -1 \\ 0 & 1 & 4 & 1 & 3 \\ 0 & 0 & 1 & -5 & -21 \\ 0 & 0 & -13 & -7 & -15 \end{array}\right]$

$\xrightarrow{13R_3+R_4} \left[\begin{array}{rrrr|r} 1 & -3 & -2 & 1 & -1 \\ 0 & 1 & 4 & 1 & 3 \\ 0 & 0 & 1 & -5 & -21 \\ 0 & 0 & 0 & -72 & -288 \end{array}\right] \xrightarrow{(-1/72)R_4} \left[\begin{array}{rrrr|r} 1 & -3 & -2 & 1 & -1 \\ 0 & 1 & 4 & 1 & 3 \\ 0 & 0 & 1 & -5 & -21 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$

$\begin{cases} w-3x-2y+z = -1 \\ x+4y+z = 3 \\ y-5z = -21 \\ z = 4 \end{cases} \quad \begin{aligned} y-5(4) &= -21 \\ y &= -1 \end{aligned} \quad \begin{aligned} x+4(-1)+4 &= 3 \\ x &= 3 \end{aligned} \quad \begin{aligned} w-3(3)-2(-1)+4 &= -1 \\ w &= 2 \end{aligned}$

The solution is (2, 3, –1, 4). [10.1]

**33.** $\left[\begin{array}{cccc|c} 1 & 3 & 1 & -4 & 3 \\ 1 & 4 & 3 & -6 & 5 \\ 2 & 8 & 7 & -5 & 11 \\ 2 & 5 & 0 & -6 & 4 \end{array}\right] \xrightarrow[\substack{-2R_1+R_3 \\ -2R_1+R_4}]{-1R_1+R_2} \left[\begin{array}{cccc|c} 1 & 3 & 1 & -4 & 3 \\ 0 & 1 & 2 & -2 & 2 \\ 0 & 2 & 5 & 3 & 5 \\ 0 & -1 & -2 & 2 & -2 \end{array}\right] \xrightarrow[1R_2+R_4]{-2R_2+R_3} \left[\begin{array}{cccc|c} 1 & 3 & 1 & -4 & 3 \\ 0 & 1 & 2 & -2 & 2 \\ 0 & 0 & 1 & 7 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$\begin{cases} w+3x+y-4z=3 \\ x+2y-2z=2 \\ y+7z=1 \end{cases}$ $y=-7z+1$ $\quad x+2(-7z+1)-2z=2$, $x=16z$ $\quad w+3(16z)+(-7z+1)-4z=3$, $w=-37z+2$

Let $z$ be any real number $c$.

The solution is $(-37c+2,\ 16c,\ -7c+1,\ c)$. [10.1]

**34.** $\left[\begin{array}{cccc|c} 1 & 4 & -2 & 3 & 6 \\ 2 & 9 & -1 & 5 & 13 \\ 1 & 7 & 6 & 5 & 9 \\ 3 & 14 & 0 & 7 & 20 \end{array}\right] \xrightarrow[\substack{-1R_1+R_3 \\ -3R_1+R_4}]{-2R_1+R_2} \left[\begin{array}{cccc|c} 1 & 4 & -2 & 3 & 6 \\ 0 & 1 & 3 & -1 & 1 \\ 0 & 3 & 8 & 2 & 3 \\ 0 & 2 & 6 & -2 & 2 \end{array}\right] \xrightarrow[-2R_2+R_4]{-3R_2+R_3} \left[\begin{array}{cccc|c} 1 & 4 & -2 & 3 & 6 \\ 0 & 1 & 3 & -1 & 1 \\ 0 & 0 & -1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-1R_3} \left[\begin{array}{cccc|c} 1 & 4 & -2 & 3 & 6 \\ 0 & 1 & 3 & -1 & 1 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$\begin{cases} w+4x-2y+3z=6 \\ x+3y-z=1 \\ y-5z=0 \end{cases}$ $y=5z$ $\quad x+3(5z)-z=1$, $x=-14z+1$ $\quad w+4(-14z+1)-2(5z)+3z=6$, $w=63z+2$

Let $z$ be any real number $c$.
The solution is $(63c+2,\ -14c+1,\ 5c,\ c)$. [10.1]

**35.** Because there are three points, the degree of the interpolating polynomial is at most 2.
The form of the polynomial is $p(x)=a_2x^2+a_1x+a_0$.
Use this polynomial and the given points to find the system of equations.

$$p(-1)=a_2(-1)^2+a_1(-1)+a_0=-4$$
$$p(2)=a_2(2)^2+a_1(2)+a_0=8$$
$$p(3)=a_2(3)^2+a_1(3)+a_0=16$$

The system of equations and the associated augmented matrix are $\begin{cases} a_2-a_1+a_0=-4 \\ 4a_2+2a_1+a_0=8 \\ 9a_2+3a_1+a_0=16 \end{cases}$ $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -4 \\ 4 & 2 & 1 & 8 \\ 9 & 3 & 1 & 16 \end{array}\right]$

The ref (**r**ow **e**chelon **f**orm) feature of a graphing calculator can be used to rewrite the augmented matrix in echelon form. Consider using the function of your calculator that converts a decimal to a fraction.

The augmented matrix in echelon form and resulting system of equations are $\left[\begin{array}{ccc|c} 1 & 1/3 & 1/9 & 16/9 \\ 0 & 1 & -2/3 & 13/3 \\ 0 & 0 & 1 & -2 \end{array}\right]$ $\begin{cases} a_2+\frac{1}{3}a_1+\frac{1}{9}a_0=\frac{16}{9} \\ a_1-\frac{2}{3}a_0=\frac{13}{2} \\ a_0=-2 \end{cases}$

Solving by back substitution yields $a_0=-2$, $a_1=3$, and $a_2=1$.

The interpolating polynomial is $p(x)=x^2+3x-2$. [10.1]

**36.** Because there are three points, the degree of the interpolating polynomial is at most 2.

The form of the polynomial is $p(x) = a_2x^2 + a_1x + a_0$.

Use this polynomial and the given points to find the system of equations.

$$p(-1) = a_2(-1)^2 + a_1(-1) + a_0 = 4$$
$$p(1) = a_2(1)^2 + a_1(1) + a_0 = 0$$
$$p(2) = a_2(2)^2 + a_1(2) + a_0 = -5$$

The system of equations and the associated augmented matrix are $\begin{cases} a_2 - a_1 + a_0 = 4 \\ a_2 + a_1 + a_0 = 0 \\ 4a_2 + 2a_1 + a_0 = -5 \end{cases}$ $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 1 & 1 & 1 & 0 \\ 4 & 2 & 1 & -5 \end{array}\right]$

The ref (**r**ow **e**chelon **f**orm) feature of a graphing calculator can be used to rewrite the augmented matrix in echelon form. Consider using the function of your calculator that converts a decimal to a fraction.

The augmented matrix in echelon form and resulting system of equations are $\left[\begin{array}{ccc|c} 1 & 1/2 & 1/4 & -5/4 \\ 0 & 1 & -1/2 & -7/2 \\ 0 & 0 & 1 & 3 \end{array}\right]$ $\begin{cases} a_2 + \frac{1}{2}a_1 + \frac{1}{4}a_0 = -\frac{5}{4} \\ a_1 - \frac{1}{2}a_0 = -\frac{7}{2} \\ a_0 = 3 \end{cases}$

Solving by back substitution yields $a_0 = 3$, $a_1 = -2$, and $a_2 = -1$.

The interpolating polynomial is $p(x) = -x^2 - 2x + 3$. [10.1]

**37.** $\left[\begin{array}{cc|cc} 2 & -2 & 1 & 0 \\ 3 & -2 & 0 & 1 \end{array}\right] \xrightarrow{(1/2)R_1} \left[\begin{array}{cc|cc} 1 & -1 & \frac{1}{2} & 0 \\ 3 & -2 & 0 & 1 \end{array}\right] \xrightarrow{-3R_1 + R_2} \left[\begin{array}{cc|cc} 1 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & 1 \end{array}\right] \xrightarrow{R_2 + R_1} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & -\frac{3}{2} & 1 \end{array}\right]$

The inverse matrix is $\begin{bmatrix} -1 & 1 \\ -\frac{3}{2} & 1 \end{bmatrix}$. [10.3]

**38.** $\left[\begin{array}{cc|cc} 3 & 4 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{cc|cc} 1 & \frac{4}{3} & \frac{1}{3} & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1 + R_2} \left[\begin{array}{cc|cc} 1 & \frac{4}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & -\frac{2}{3} & 1 \end{array}\right] \xrightarrow{3R_2} \left[\begin{array}{cc|cc} 1 & \frac{4}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -2 & 3 \end{array}\right] \xrightarrow{-\frac{4}{3}R_2 + R_1} \left[\begin{array}{cc|cc} 1 & 0 & 3 & -4 \\ 0 & 1 & -2 & 3 \end{array}\right]$

The inverse matrix is $\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$. [10.3]

**39.** $\left[\begin{array}{cc|cc} -2 & 3 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] \xrightarrow{-\frac{1}{2}R_1} \left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1 + R_2} \left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 7 & 1 & 1 \end{array}\right] \xrightarrow{\frac{1}{7}R_2} \left[\begin{array}{cc|cc} 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{7} & \frac{1}{7} \end{array}\right]$

$\xrightarrow{\frac{3}{2}R_2 + R_1} \left[\begin{array}{cc|cc} 1 & 0 & -\frac{2}{7} & \frac{3}{14} \\ 0 & 1 & \frac{1}{7} & \frac{1}{7} \end{array}\right]$

The inverse matrix is $\begin{bmatrix} -\frac{2}{7} & \frac{3}{14} \\ \frac{1}{7} & \frac{1}{7} \end{bmatrix}$. [10.3]

**40.** $\left[\begin{array}{cc|cc} 5 & -4 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] \xrightarrow{(1/5)R_1} \left[\begin{array}{cc|cc} 1 & -\frac{4}{5} & \frac{1}{5} & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] \xrightarrow{-3R_1 + R_2} \left[\begin{array}{cc|cc} 1 & -\frac{4}{5} & \frac{1}{5} & 0 \\ 0 & \frac{22}{5} & -\frac{3}{5} & 1 \end{array}\right] \xrightarrow{(5/22)R_2} \left[\begin{array}{cc|cc} 1 & -\frac{4}{5} & \frac{1}{5} & 0 \\ 0 & 1 & -\frac{3}{22} & \frac{5}{22} \end{array}\right]$

$\xrightarrow{(4/5)R_2 + R_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{11} & \frac{2}{11} \\ 0 & 1 & -\frac{3}{22} & \frac{5}{22} \end{array}\right]$

The inverse matrix is $\begin{bmatrix} \frac{1}{11} & \frac{2}{11} \\ -\frac{3}{22} & \frac{5}{22} \end{bmatrix}$. [10.3]

**41.** $\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 6 & 4 & 0 & 1 & 0 \\ 3 & 8 & 6 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & -2 & 1 & 0 \\ 0 & 2 & 3 & -3 & 0 & 1 \end{array}\right] \xrightarrow{(1/2)R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & \frac{1}{2} & 0 \\ 0 & 2 & 3 & -3 & 0 & 1 \end{array}\right]$

$\xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 1 & 1 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \xrightarrow[-R_3+R_2]{R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -2 & 1 \\ 0 & 1 & 0 & 0 & \frac{3}{2} & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$

The inverse matrix is $\begin{bmatrix} 2 & -2 & 1 \\ 0 & \frac{3}{2} & -1 \\ -1 & -1 & 1 \end{bmatrix}$. [10.3]

**42.** $\left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ 3 & -8 & 7 & 0 & 1 & 0 \\ 2 & -3 & 6 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -3 & 1 & 0 \\ 0 & 3 & 2 & -2 & 0 & 1 \end{array}\right] \xrightarrow{-3R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -3 & 1 & 0 \\ 0 & 0 & -1 & 7 & -3 & 1 \end{array}\right]$

$\xrightarrow{-1R_3} \left[\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -3 & 1 & 0 \\ 0 & 0 & 1 & -7 & 3 & -1 \end{array}\right] \xrightarrow{3R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & -8 & 3 & 0 \\ 0 & 1 & 1 & -3 & 1 & 0 \\ 0 & 0 & 1 & -7 & 3 & -1 \end{array}\right] \xrightarrow[-R_3+R_2]{-5R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 27 & -12 & 5 \\ 0 & 1 & 0 & 4 & -2 & 1 \\ 0 & 0 & 1 & -7 & 3 & -1 \end{array}\right]$

The inverse matrix is $\begin{bmatrix} 27 & -12 & 5 \\ 4 & -2 & 1 \\ -7 & 3 & -1 \end{bmatrix}$. [10.3]

**43.** $\left[\begin{array}{ccc|ccc} 3 & -2 & 7 & 1 & 0 & 0 \\ 2 & -1 & 5 & 0 & 1 & 0 \\ 3 & 0 & 10 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(1/3)R_1} \left[\begin{array}{ccc|ccc} 1 & -\frac{2}{3} & \frac{7}{3} & \frac{1}{3} & 0 & 0 \\ 2 & -1 & 5 & 0 & 1 & 0 \\ 3 & 0 & 10 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & -\frac{2}{3} & \frac{7}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & 1 & 0 \\ 0 & 2 & 3 & -1 & 0 & 1 \end{array}\right]$

$\xrightarrow{3R_2} \left[\begin{array}{ccc|ccc} 1 & -\frac{2}{3} & \frac{7}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & -2 & 3 & 0 \\ 0 & 2 & 3 & -1 & 0 & 1 \end{array}\right] \xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & -\frac{2}{3} & \frac{7}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & -2 & 3 & 0 \\ 0 & 0 & 1 & 3 & -6 & 1 \end{array}\right] \xrightarrow{(2/3)R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -1 & 2 & 0 \\ 0 & 1 & 1 & -2 & 3 & 0 \\ 0 & 0 & 1 & 3 & -6 & 1 \end{array}\right]$

$\xrightarrow[-1R_3+R_2]{-3R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -10 & 20 & -3 \\ 0 & 1 & 0 & -5 & 9 & -1 \\ 0 & 0 & 1 & 3 & -6 & 1 \end{array}\right]$

The inverse matrix is $\begin{bmatrix} -10 & 20 & -3 \\ -5 & 9 & -1 \\ 3 & -6 & 1 \end{bmatrix}$. [10.3]

**44.** $\left[\begin{array}{ccc|ccc} 4 & 9 & -11 & 1 & 0 & 0 \\ 3 & 7 & -8 & 0 & 1 & 0 \\ 2 & 6 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(1/4)R_1} \left[\begin{array}{ccc|ccc} 1 & \frac{9}{4} & -\frac{11}{4} & \frac{1}{4} & 0 & 0 \\ 3 & 7 & -8 & 0 & 1 & 0 \\ 2 & 6 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & \frac{9}{4} & -\frac{11}{4} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & -\frac{3}{4} & 1 & 0 \\ 0 & \frac{3}{2} & \frac{5}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right]$

$\xrightarrow{4R_2} \left[\begin{array}{ccc|ccc} 1 & \frac{9}{4} & -\frac{11}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 1 & 1 & -3 & 4 & 0 \\ 0 & \frac{3}{2} & \frac{5}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right] \xrightarrow{(-3/2)R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & \frac{9}{4} & -\frac{11}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 1 & 1 & -3 & 4 & 0 \\ 0 & 0 & 1 & 4 & -6 & 1 \end{array}\right]$

$\xrightarrow{(-9/4)R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 7 & -9 & 0 \\ 0 & 1 & 1 & -3 & 4 & 0 \\ 0 & 0 & 1 & 4 & -6 & 1 \end{array}\right] \xrightarrow[-1R_3+R_2]{5R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 27 & -39 & 5 \\ 0 & 1 & 0 & -7 & 10 & -1 \\ 0 & 0 & 1 & 4 & -6 & 1 \end{array}\right]$

The inverse matrix is $\begin{bmatrix} 27 & -39 & 5 \\ -7 & 10 & -1 \\ 4 & -6 & 1 \end{bmatrix}$. [10.3]

**45.** $$\left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 2 & -1 & 6 & 5 & 0 & 1 & 0 & 0 \\ 3 & -1 & 9 & 6 & 0 & 0 & 1 & 0 \\ 2 & -2 & 4 & 7 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{-2R_1+R_2 \\ -3R_1+R_3 \\ -2R_1+R_4}} \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 & -2 & 1 & 0 & 0 \\ 0 & 2 & 3 & -3 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_2+R_3} \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-1R_3} \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 4 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{-4R_3+R_1 \\ -2R_3+R_2}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -2 & 3 & -7 & 4 & 0 \\ 0 & 1 & 0 & -3 & 0 & -3 & 2 & 0 \\ 0 & 0 & 1 & 1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{2R_4+R_1 \\ 3R_4+R_2 \\ -1R_4+R_3}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & -7 & 4 & 2 \\ 0 & 1 & 0 & 0 & -6 & -3 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right]$$

The inverse matrix is $\begin{bmatrix} -1 & -7 & 4 & 2 \\ -6 & -3 & 2 & 3 \\ 1 & 2 & -1 & -1 \\ -2 & 0 & 0 & 1 \end{bmatrix}$. [10.3]

**46.** $$\left[\begin{array}{cccc|cccc} 1 & 2 & -2 & 1 & 1 & 0 & 0 & 0 \\ 3 & 7 & -3 & 1 & 0 & 1 & 0 & 0 \\ 2 & 7 & 4 & 3 & 0 & 0 & 1 & 0 \\ 1 & 4 & 2 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{-3R_1+R_2 \\ -2R_1+R_3 \\ -1R_1+R_4}} \left[\begin{array}{cccc|cccc} 1 & 2 & -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -2 & -3 & 1 & 0 & 0 \\ 0 & 3 & 8 & 1 & -2 & 0 & 1 & 0 \\ 0 & 2 & 4 & 3 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{-3R_2+R_3 \\ -2R_2+R_4}} \left[\begin{array}{cccc|cccc} 1 & 2 & -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -2 & -3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 7 & 7 & -3 & 1 & 0 \\ 0 & 0 & -2 & 7 & 5 & -2 & 0 & 1 \end{array}\right] \xrightarrow{-1R_3} \left[\begin{array}{cccc|cccc} 1 & 2 & -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -2 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -7 & -7 & 3 & -1 & 0 \\ 0 & 0 & -2 & 7 & 5 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{2R_3+R_4} \left[\begin{array}{cccc|cccc} 1 & 2 & -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -2 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -7 & -7 & 3 & -1 & 0 \\ 0 & 0 & 0 & -7 & -9 & 4 & -2 & 1 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & -8 & 5 & 7 & -2 & 0 & 0 \\ 0 & 1 & 3 & -2 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -7 & -7 & 3 & -1 & 0 \\ 0 & 0 & 0 & -7 & -9 & 4 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{8R_3+R_1 \\ -3R_3+R_2}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -51 & -49 & 22 & -8 & 0 \\ 0 & 1 & 0 & 19 & 18 & -8 & 3 & 0 \\ 0 & 0 & 1 & -7 & -7 & 3 & -1 & 0 \\ 0 & 0 & 0 & -7 & -9 & 4 & -2 & 1 \end{array}\right] \xrightarrow{(-1/7)R_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -51 & -49 & 22 & -8 & 0 \\ 0 & 1 & 0 & 19 & 18 & -8 & 3 & 0 \\ 0 & 0 & 1 & -7 & -7 & 3 & -1 & 0 \\ 0 & 0 & 0 & 1 & \frac{9}{7} & -\frac{4}{7} & \frac{2}{7} & -\frac{1}{7} \end{array}\right]$$

$$\xrightarrow{\substack{7R_4+R_3 \\ -19R_4R_2 \\ 51R_4+R_1}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{116}{7} & -\frac{50}{7} & \frac{46}{7} & -\frac{51}{7} \\ 0 & 1 & 0 & 0 & -\frac{45}{7} & \frac{20}{7} & -\frac{17}{7} & \frac{19}{7} \\ 0 & 0 & 1 & 0 & 2 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & \frac{9}{7} & -\frac{4}{7} & \frac{2}{7} & -\frac{1}{7} \end{array}\right]$$

The inverse matrix is $\begin{bmatrix} \frac{116}{7} & -\frac{50}{7} & \frac{46}{7} & -\frac{51}{7} \\ -\frac{45}{7} & \frac{20}{7} & -\frac{17}{7} & \frac{19}{7} \\ 2 & -1 & 1 & -1 \\ \frac{9}{7} & -\frac{4}{7} & \frac{2}{7} & -\frac{1}{7} \end{bmatrix}$. [10.3]

**47.**

$$\left[\begin{array}{cccc|cccc} 3 & 7 & -1 & 8 & 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 5 & 0 & 1 & 0 & 0 \\ 3 & 6 & -4 & 8 & 0 & 0 & 1 & 0 \\ 2 & 4 & -4 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(1/3)R_1} \left[\begin{array}{cccc|cccc} 1 & \frac{7}{3} & -\frac{1}{3} & \frac{8}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 2 & 5 & 0 & 5 & 0 & 1 & 0 & 0 \\ 3 & 6 & -4 & 8 & 0 & 0 & 1 & 0 \\ 2 & 4 & -4 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{-2R_1+R_2 \\ -3R_1+R_3 \\ -2R_1+R_4}} \left[\begin{array}{cccc|cccc} 1 & \frac{7}{3} & -\frac{1}{3} & \frac{8}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} & 1 & 0 & 0 \\ 0 & -1 & -3 & 0 & -1 & 0 & 1 & 0 \\ 0 & -\frac{2}{3} & -\frac{10}{3} & -\frac{4}{3} & -\frac{2}{3} & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{3R_2} \left[\begin{array}{cccc|cccc} 1 & \frac{7}{3} & -\frac{1}{3} & \frac{8}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 & -2 & 3 & 0 & 0 \\ 0 & -1 & -3 & 0 & -1 & 0 & 1 & 0 \\ 0 & -\frac{2}{3} & -\frac{10}{3} & -\frac{4}{3} & -\frac{2}{3} & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{1R_2+R_3 \\ (2/3)R_2+R_4}} \left[\begin{array}{cccc|cccc} 1 & \frac{7}{3} & -\frac{1}{3} & \frac{8}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 & -2 & 3 & 0 & 0 \\ 0 & 0 & -1 & -1 & -3 & 3 & 1 & 0 \\ 0 & 0 & -2 & -2 & -2 & 2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-1R_3} \left[\begin{array}{cccc|cccc} 1 & \frac{7}{3} & -\frac{1}{3} & \frac{8}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 & -2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & -3 & -1 & 0 \\ 0 & 0 & -2 & -2 & -2 & 2 & 0 & 1 \end{array}\right] \xrightarrow{2R_3+R_4} \left[\begin{array}{cccc|cccc} 1 & \frac{7}{3} & -\frac{1}{3} & \frac{8}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 & -2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 4 & -4 & -2 & 1 \end{array}\right]$$

The matrix does not have an inverse. [10.3]

**48.**

$$\left[\begin{array}{cccc|cccc} 3 & 1 & 5 & -5 & 1 & 0 & 0 & 0 \\ 2 & 1 & 4 & -3 & 0 & 1 & 0 & 0 \\ 3 & 0 & 4 & -3 & 0 & 0 & 1 & 0 \\ 4 & 1 & 8 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(1/3)R_1} \left[\begin{array}{cccc|cccc} 1 & \frac{1}{3} & \frac{5}{3} & -\frac{5}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 2 & 1 & 4 & -3 & 0 & 1 & 0 & 0 \\ 3 & 0 & 4 & -3 & 0 & 0 & 1 & 0 \\ 4 & 1 & 8 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{-2R_1+R_2 \\ -3R_1+R_3 \\ -4R_1+R_4}} \left[\begin{array}{cccc|cccc} 1 & \frac{1}{3} & \frac{5}{3} & -\frac{5}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} & 1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -\frac{1}{3} & \frac{4}{3} & \frac{23}{3} & -\frac{4}{3} & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{3R_2 \\ 1R_2+R_3 \\ (1/3)R_2+R_4}} \left[\begin{array}{cccc|cccc} 1 & \frac{1}{3} & \frac{5}{3} & -\frac{5}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & -2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 & -3 & 3 & 1 & 0 \\ 0 & 0 & 2 & 8 & -2 & 1 & 0 & 1 \end{array}\right] \xrightarrow{-2R_3+R_4} \left[\begin{array}{cccc|cccc} 1 & \frac{1}{3} & \frac{5}{3} & -\frac{5}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & -2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 & -3 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 & 4 & -5 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{(1/2)R_4} \left[\begin{array}{cccc|cccc} 1 & \frac{1}{3} & \frac{5}{3} & -\frac{5}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & -2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 & -3 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & -\frac{5}{2} & -1 & \frac{1}{2} \end{array}\right] \xrightarrow{(-1/3)R_2+R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & -2 & 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 1 & -2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 & -3 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & -\frac{5}{2} & -1 & \frac{1}{2} \end{array}\right]$$

$$\xrightarrow{\substack{-2R_3+R_2 \\ -1R_3+R_1}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -5 & 4 & -4 & -1 & 0 \\ 0 & 1 & 0 & -5 & 4 & -3 & -2 & 0 \\ 0 & 0 & 1 & 3 & -3 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & -\frac{5}{2} & -1 & \frac{1}{2} \end{array}\right] \xrightarrow{\substack{5R_4+R_1 \\ 5R_4+R_2 \\ -3R_4+R_3}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 14 & -\frac{33}{2} & -6 & \frac{5}{2} \\ 0 & 1 & 0 & 0 & 14 & -\frac{31}{2} & -7 & \frac{5}{2} \\ 0 & 0 & 1 & 0 & -9 & \frac{21}{2} & 4 & -\frac{3}{2} \\ 0 & 0 & 0 & 1 & 2 & -\frac{5}{2} & -1 & \frac{1}{2} \end{array}\right]$$

The inverse matrix is $\begin{bmatrix} 14 & -\frac{33}{2} & -6 & \frac{5}{2} \\ 14 & -\frac{31}{2} & -7 & \frac{5}{2} \\ -9 & \frac{21}{2} & 4 & -\frac{3}{2} \\ 2 & -\frac{5}{2} & -1 & \frac{1}{2} \end{bmatrix}$. [10.3]

**49.** **a.**

$$\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 18 \\ -13 \end{bmatrix}$$

The solution is (18, –13).

**b.**

$$\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -22 \\ 16 \end{bmatrix}$$

The solution is (–22, 16). [10.3]

**50.** **a.**

$$\begin{bmatrix} 2 & -5 \\ 3 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} -7 & 5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ 3 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -7 & 5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 41 \\ 17 \end{bmatrix}$$

The solution is (41, 17).

**b.**

$$\begin{bmatrix} 2 & -5 \\ 3 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} -7 & 5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ 3 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -7 & 5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -39 \\ -16 \end{bmatrix}$$

The solution is (–39, –16). [10.3]

**51.** **a.**

$$\begin{bmatrix} 2 & -5 \\ 3 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{14} & -\frac{5}{14} \\ -1 & \frac{2}{7} & \frac{3}{7} \\ 0 & \frac{1}{7} & -\frac{2}{7} \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ 4 & 4 & 1 \\ 2 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 & -\frac{1}{14} & -\frac{5}{14} \\ -1 & \frac{2}{7} & \frac{3}{7} \\ 0 & \frac{1}{7} & -\frac{2}{7} \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -\frac{18}{7} \\ \frac{23}{7} \\ -\frac{6}{7} \end{bmatrix}$$

The solution is $\left(-\frac{18}{7}, \frac{23}{7}, -\frac{6}{7}\right)$.

**b.**

$$\begin{bmatrix} 2 & 1 & -1 \\ 4 & 4 & 1 \\ 2 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{14} & -\frac{5}{14} \\ -1 & \frac{2}{7} & \frac{3}{7} \\ 0 & \frac{1}{7} & -\frac{2}{7} \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ 4 & 4 & 1 \\ 2 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 & -\frac{1}{14} & -\frac{5}{14} \\ -1 & \frac{2}{7} & \frac{3}{7} \\ 0 & \frac{1}{7} & -\frac{2}{7} \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -\frac{31}{14} \\ \frac{20}{7} \\ \frac{3}{7} \end{bmatrix}$$

The solution is $\left(-\frac{31}{14}, \frac{20}{7}, \frac{3}{7}\right)$. [10.3]

**52.** **a.**

$$\begin{bmatrix} 3 & -2 & 1 \\ 3 & -1 & 3 \\ 6 & -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{11}{3} & \frac{2}{3} & \frac{5}{3} \\ -5 & 1 & 2 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 1 \\ 3 & -1 & 3 \\ 6 & -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{11}{3} & \frac{2}{3} & \frac{5}{3} \\ -5 & 1 & 2 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} \\ -1 \\ 2 \end{bmatrix}$$

The solution is $\left(-\frac{4}{3}, -1, 2\right)$.

**b.**

$$\begin{bmatrix} 3 & -2 & 1 \\ 3 & -1 & 3 \\ 6 & -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{13} & \frac{2}{3} & \frac{5}{3} \\ -5 & 1 & 2 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 1 \\ 3 & -1 & 3 \\ 6 & -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{13} & \frac{2}{3} & \frac{5}{3} \\ -5 & 1 & 2 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -9 \\ -11 \\ 6 \end{bmatrix}$$

The solution is $(-9, -11, 6)$. [10.3]

**53.** $$T_{3,-1} \cdot \begin{bmatrix} -5 & 4 \\ 3 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 & 4 \\ 3 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ 2 & -3 \\ 1 & 1 \end{bmatrix}$$

$$R_{xy} \cdot \begin{bmatrix} -2 & 7 \\ 2 & -3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 7 \\ 2 & -3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -2 & 7 \\ 1 & 1 \end{bmatrix} \Rightarrow \begin{matrix} A'(2, -2) \\ B'(-3, 7) \end{matrix}$$ [10.2]

**54.** $$T_{-2,-3} \cdot \begin{bmatrix} -3 & 1 & 0 \\ 4 & -1 & 5 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 & 0 \\ 4 & -1 & 5 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -1 & -2 \\ 1 & -4 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$R_{90} \cdot \begin{bmatrix} -5 & -1 & -2 \\ 1 & -4 & 2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 & -1 & -2 \\ 1 & -4 & 2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 4 & -2 \\ -5 & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$T_{2,3} \cdot \begin{bmatrix} -1 & 4 & -2 \\ -5 & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 & -2 \\ -5 & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 0 \\ -2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{matrix} A'(1, -2) \\ B'(6, 2) \\ C'(0, 1) \end{matrix}$$ [10.2]

**55.** $\begin{vmatrix} 2 & 6 & 4 \\ 1 & 2 & 1 \\ 3 & 8 & 6 \end{vmatrix} \xrightarrow{\text{Factor 2 from row 1}} 2\begin{vmatrix} 1 & 3 & 2 \\ 1 & 2 & 1 \\ 3 & 8 & 6 \end{vmatrix}$

$\xrightarrow[-3R_1+R_3]{-1R_1+R_2} 2\begin{vmatrix} 1 & 3 & 2 \\ 0 & -1 & -1 \\ 0 & -1 & 0 \end{vmatrix}$

$\xrightarrow{-R_2+R_3} 2\begin{vmatrix} 1 & 3 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{vmatrix}$

$= 2(1)(-1)(1) = -2$ [10.4]

**56.** $\begin{vmatrix} 3 & 0 & 10 \\ 3 & -2 & 7 \\ 2 & -1 & 5 \end{vmatrix} \xrightarrow{(-10/3)C_1+C_3} \begin{vmatrix} 3 & 0 & 0 \\ 3 & -2 & -3 \\ 2 & -1 & -\frac{5}{3} \end{vmatrix}$

$\xrightarrow{(-3/2)C_2+C_3} \begin{vmatrix} 3 & 0 & 0 \\ 3 & -2 & 0 \\ 2 & -1 & -\frac{1}{6} \end{vmatrix}$

$=3(-2)\left(-\frac{1}{6}\right)=1$ [10.4]

**57.** $\begin{vmatrix} 3 & -8 & 7 \\ 2 & -3 & 6 \\ 1 & -3 & 2 \end{vmatrix} \xrightarrow[(-1/3)R_1+R_3]{(-2/3)R_1+R_2} \begin{vmatrix} 3 & -8 & 7 \\ 0 & \frac{7}{3} & \frac{4}{3} \\ 0 & -\frac{1}{3} & -\frac{1}{3} \end{vmatrix}$

$\xrightarrow{(1/7)R_2+R_3} \begin{vmatrix} 3 & -8 & 7 \\ 0 & \frac{7}{3} & \frac{4}{3} \\ 0 & 0 & -\frac{1}{7} \end{vmatrix}$

$= 3\frac{7}{3}\left(-\frac{1}{7}\right) = -1$ [10.4]

**58.** $\begin{vmatrix} 4 & 9 & -11 \\ 2 & 6 & -3 \\ 3 & 7 & -8 \end{vmatrix} \xrightarrow[(-3/4)R_1+R_3]{(-1/2)R_1+R_2} \begin{vmatrix} 4 & 9 & -11 \\ 0 & \frac{3}{2} & \frac{5}{2} \\ 0 & \frac{1}{4} & \frac{1}{4} \end{vmatrix}$

$\xrightarrow{(-1/6)R_2+R_3} \begin{vmatrix} 4 & 9 & -11 \\ 0 & \frac{3}{2} & \frac{5}{2} \\ 0 & 0 & -\frac{1}{6} \end{vmatrix}$

$= 4\left(\frac{3}{2}\right)\left(-\frac{1}{6}\right) = -1$ [10.4]

**59.** $\begin{vmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & 6 & 3 \\ 3 & -1 & 8 & 7 \\ 3 & 0 & 9 & 9 \end{vmatrix} \xrightarrow{\text{Factor 3 from row 4}} 3\begin{vmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & 6 & 3 \\ 3 & -1 & 8 & 7 \\ 1 & 0 & 3 & 3 \end{vmatrix}$

$\xrightarrow[\substack{-3R_1+R_3 \\ -1R_1+R_4}]{-R_1+R_2} 3\begin{vmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & 1 & 2 \end{vmatrix}$

$\xrightarrow[-1R_2+R_4]{-2R_2+R_3} 3\begin{vmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -1 & 1 \end{vmatrix}$

$\xrightarrow[R_3+R_4]{\text{Factor } -2 \text{ from row 3}} -6\begin{vmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{vmatrix}$

$= -6(1)(1)(1)(0) = 0$ [10.4]

**60.** $$\begin{vmatrix}1&2&-2&3\\3&7&-3&11\\2&3&-5&11\\2&6&1&8\end{vmatrix}\xrightarrow[-2R_1+R_4]{\substack{-3R_1+R_2\\-2R_1+R_3}}\begin{vmatrix}1&2&-2&3\\0&1&3&2\\0&-1&-1&5\\0&2&5&2\end{vmatrix}$$

$$\xrightarrow[-2R_2+R_4]{R_2+R_3}\begin{vmatrix}1&2&-2&3\\0&1&3&2\\0&0&2&7\\0&0&-1&-2\end{vmatrix}$$

$$\xrightarrow{(1/2)R_3+R_4}\begin{vmatrix}1&2&-2&3\\0&1&3&2\\0&0&2&7\\0&0&0&\frac{3}{2}\end{vmatrix}$$

$$=1(1)(2)\left(\frac{3}{2}\right)=3\ \ [10.4]$$

**61.** $$\begin{vmatrix}1&2&-2&1\\2&5&-3&1\\2&0&-10&1\\3&8&-4&1\end{vmatrix}\xrightarrow[-R_4+R_3]{\substack{-R_4+R_1\\-R_4+R_2}}\begin{vmatrix}-2&-6&2&0\\-1&-3&1&0\\-1&-8&-6&0\\3&8&-4&1\end{vmatrix}$$

$$\xrightarrow[(1/3)R_3+R_1]{(1/6)R_3+R_2}\begin{vmatrix}-\frac{7}{3}&-\frac{26}{3}&0&0\\-\frac{7}{6}&-\frac{13}{3}&0&0\\-1&-8&-6&0\\3&8&-4&1\end{vmatrix}$$

$$\xrightarrow{-2R_2+R_1}\begin{vmatrix}0&0&0&0\\-\frac{7}{6}&-\frac{13}{3}&0&0\\-1&-8&-6&0\\3&8&-4&1\end{vmatrix}$$

$$=0\left(-\frac{13}{3}\right)(-6)(-1)=0\ \ [10.4]$$

**62.** $$\begin{vmatrix}1&3&-2&0\\3&11&-4&4\\2&9&-8&2\\3&12&-10&2\end{vmatrix}\xrightarrow[\text{Factor 2 from column 4}]{\text{Factor }-2\text{ from column 3}}-4\begin{vmatrix}1&3&1&0\\3&11&2&2\\2&9&4&1\\3&12&5&1\end{vmatrix}$$

$$\xrightarrow[-1R_4+R_3]{-2R_4+R_2}-4\begin{vmatrix}1&3&1&0\\-3&-13&-8&0\\-1&-3&-1&0\\3&12&5&1\end{vmatrix}$$

$$\xrightarrow[-8R_3+R_2]{R_3+R_1}-4\begin{vmatrix}0&0&0&0\\5&11&0&0\\-1&-3&-1&0\\3&12&5&1\end{vmatrix}$$

$$=-4(0)(11)(-1)(1)=0\ \ [10.4]$$

**63.** $$x_1=\frac{\begin{vmatrix}2&-3\\2&5\end{vmatrix}}{\begin{vmatrix}2&-3\\3&5\end{vmatrix}}=\frac{16}{19}$$

$$x_2=\frac{\begin{vmatrix}2&2\\3&2\end{vmatrix}}{\begin{vmatrix}2&-3\\3&5\end{vmatrix}}=\frac{-2}{19}=-\frac{2}{19}\ \ [10.5]$$

**64.** $$x_1=\frac{\begin{vmatrix}-3&4\\2&-2\end{vmatrix}}{\begin{vmatrix}3&4\\5&-2\end{vmatrix}}=\frac{-2}{-26}=\frac{1}{13}$$

$$x_2=\frac{\begin{vmatrix}3&-3\\5&2\end{vmatrix}}{\begin{vmatrix}3&4\\5&-2\end{vmatrix}}=\frac{21}{-26}=-\frac{21}{26}\ \ [10.5]$$

**65.**
$$D=\begin{vmatrix}2 & 1 & -3\\3 & 2 & 1\\1 & -3 & 4\end{vmatrix}=44\ [10.5]$$
$$D_1=\begin{vmatrix}2 & 1 & -3\\1 & 2 & 1\\-2 & -3 & 4\end{vmatrix}=13$$
$$D_2=\begin{vmatrix}2 & 2 & -3\\3 & 1 & 1\\1 & -2 & 4\end{vmatrix}=13$$
$$D_3=\begin{vmatrix}2 & 1 & 2\\3 & 2 & 1\\1 & -3 & 4\end{vmatrix}=-17$$
$$x_1=\frac{D_1}{D}=\frac{13}{44}$$
$$x_2=\frac{D_2}{D}=\frac{11}{44}=\frac{1}{4}$$
$$x_3=\frac{D_3}{D}=\frac{-17}{44}=-\frac{17}{44}$$

**66.**
$$D=\begin{vmatrix}3 & 2 & -1\\1 & 3 & -2\\4 & -1 & -5\end{vmatrix}=-44\ [10.5]$$
$$D_1=\begin{vmatrix}0 & 2 & -1\\3 & 3 & -2\\-1 & -1 & -5\end{vmatrix}=34$$
$$D_2=\begin{vmatrix}3 & 0 & -1\\1 & 3 & -2\\4 & -1 & -5\end{vmatrix}=-38$$
$$D_3=\begin{vmatrix}3 & 2 & 0\\1 & 3 & 3\\4 & -1 & -1\end{vmatrix}=26$$
$$x_1=\frac{D_1}{D}=\frac{34}{-44}=-\frac{17}{22}$$
$$x_2=\frac{D_2}{D}=\frac{-38}{-44}=\frac{19}{22}$$
$$x_3=\frac{D_3}{D}=\frac{26}{-44}=-\frac{13}{22}$$

**67.**
$$D=\begin{vmatrix}0 & 2 & 5\\2 & -5 & 1\\4 & 3 & 0\end{vmatrix}=138\ [10.5]$$
$$D_1=\begin{vmatrix}2 & 2 & 5\\4 & -5 & 1\\2 & 3 & 0\end{vmatrix}=108$$
$$D_2=\begin{vmatrix}0 & 2 & 5\\2 & 4 & 1\\4 & 2 & 0\end{vmatrix}=-52$$
$$D_3=\begin{vmatrix}0 & 2 & 2\\2 & -5 & 4\\4 & 3 & 2\end{vmatrix}=76$$
$$x_1=\frac{D_1}{D}=\frac{108}{138}=\frac{54}{69}=\frac{18}{23}$$
$$x_2=\frac{D_2}{D}=\frac{-52}{138}=-\frac{26}{69}$$
$$x_3=\frac{D_3}{D}=\frac{76}{138}=\frac{38}{69}$$

**68.**
$$D=\begin{vmatrix}2 & -3 & -4\\1 & -2 & 2\\2 & 7 & -1\end{vmatrix}=-83\ [10.5]$$
$$D_1=\begin{vmatrix}2 & -3 & -4\\-1 & -2 & 2\\2 & 7 & -1\end{vmatrix}=-21$$
$$D_2=\begin{vmatrix}2 & 2 & -4\\1 & -1 & 2\\2 & 2 & -1\end{vmatrix}=-12$$
$$D_3=\begin{vmatrix}2 & -3 & 2\\1 & -2 & -1\\2 & 7 & 2\end{vmatrix}=40$$
$$x_1=\frac{D_1}{D}=\frac{-21}{-83}=\frac{21}{83}$$
$$x_2=\frac{D_2}{D}=\frac{-12}{-83}=\frac{12}{83}$$
$$x_3=\frac{D_3}{D}=\frac{40}{-83}=-\frac{40}{83}$$

**69.**
$$D=\begin{vmatrix}1 & -3 & 1 & 2\\2 & 7 & -3 & 1\\-1 & 4 & 2 & -3\\3 & 1 & -1 & -2\end{vmatrix}=-252\ [10.5]$$
$$D_3=\begin{vmatrix}1 & -3 & 3 & 2\\2 & 7 & 2 & 1\\-1 & 4 & -1 & -3\\3 & 1 & 0 & -2\end{vmatrix}=-230$$
$$x_3=\frac{D_3}{D}=\frac{-230}{-252}=\frac{115}{126}$$

**70.**
$$D=\begin{vmatrix}2 & 3 & -2 & 1\\1 & -1 & -3 & 2\\3 & 3 & -4 & -1\\5 & -5 & -1 & 2\end{vmatrix}=-230\ [10.5]$$
$$D_2=\begin{vmatrix}2 & -2 & -2 & 1\\1 & 2 & -3 & 2\\3 & 4 & -4 & -1\\5 & 7 & -1 & 2\end{vmatrix}=289$$
$$x_2=\frac{D_2}{D}=\frac{289}{-230}=-\frac{289}{230}$$

**71.** The input-output matrix $A$ is given by

$$A=\begin{bmatrix}0.05 & 0.06 & 0.08\\ 0.02 & 0.04 & 0.04\\ 0.03 & 0.03 & 0.05\end{bmatrix}$$

Consumer demand $X$ is given by $X=(I-A)^{-1}D$.

$$X=\left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}0.05 & 0.06 & 0.08\\ 0.02 & 0.04 & 0.04\\ 0.03 & 0.03 & 0.05\end{bmatrix}\right)^{-1}\begin{bmatrix}30\\12\\21\end{bmatrix}$$
$$=\begin{bmatrix}0.95 & -0.06 & -0.08\\ -0.02 & 0.96 & -0.04\\ -0.03 & -0.03 & 0.95\end{bmatrix}^{-1}\begin{bmatrix}30\\12\\21\end{bmatrix}\approx\begin{bmatrix}34.47\\14.20\\23.64\end{bmatrix}$$

\$34.47 million computer division, \$14.20 million monitor division, \$23.64 million disk drive division. [10.3]

**72.** The input-output matrix $A$ is given by

$$A=\begin{bmatrix}0.07 & 0.04 & 0.07\\ 0.03 & 0.07 & 0.04\\ 0.03 & 0.03 & 0.02\end{bmatrix}$$

The consumer demand $X$ is given by $X=(I-A)^{-1}D$.

$$X=\left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}0.07 & 0.04 & 0.07\\ 0.03 & 0.07 & 0.04\\ 0.03 & 0.03 & 0.02\end{bmatrix}\right)^{-1}\begin{bmatrix}27\\18\\10\end{bmatrix}$$
$$=\begin{bmatrix}0.93 & -0.04 & -0.07\\ -0.03 & 0.93 & -0.04\\ -0.03 & -0.03 & 0.98\end{bmatrix}^{-1}\begin{bmatrix}27\\18\\10\end{bmatrix}=\begin{bmatrix}30.82\\20.86\\11.79\end{bmatrix}$$

\$30.82 million lumber division, \$20.86 million paper division, \$11.79 million prefabricated wall division. [10.3]

## Quantitative Reasoning

**QR1. a.** There are 4 vertices, so the adjacency matrix $A$ is a $4\times 4$ matrix.

$$A=\begin{bmatrix}0&1&0&1\\1&0&2&1\\0&2&0&0\\1&1&0&0\end{bmatrix}$$

**b.** For a walk of length 3, find $A^3$

$$A^3=\begin{bmatrix}0&1&0&1\\1&0&2&1\\0&2&0&0\\1&1&0&0\end{bmatrix}^3=\begin{bmatrix}2&7&2&3\\7&2&12&7\\2&12&0&2\\3&7&2&2\end{bmatrix}$$

## Chapter Test

**1.** $\left[\begin{array}{ccc|c}2&3&-3&4\\3&0&2&-1\\4&-4&2&3\end{array}\right],\ \begin{bmatrix}2&3&-3\\3&0&2\\4&-4&2\end{bmatrix},\ \begin{bmatrix}4\\-1\\3\end{bmatrix}$ [10.1]

**2.** $\begin{cases}3x-2y+5z-w=9\\2x+3y-z+4w=8\\x+3z+2w=-1\end{cases}$ [10.1]

**3.**
$$\left[\begin{array}{ccc|c}1&-2&3&10\\2&-3&8&23\\-1&3&-2&-9\end{array}\right]\xrightarrow[R_1+R_3]{-2R_1+R_2}\left[\begin{array}{ccc|c}1&-2&3&10\\0&1&2&3\\0&1&1&1\end{array}\right]\xrightarrow{-2R_2+R_3}\left[\begin{array}{ccc|c}1&-2&3&10\\0&1&2&3\\0&0&-1&-2\end{array}\right]\xrightarrow{-R_3}\left[\begin{array}{ccc|c}1&-2&3&10\\0&1&2&3\\0&0&1&2\end{array}\right]$$

$\begin{cases}x-2y+3z=10\\y+2z=3\\z=2\end{cases}$ $\quad y+2(2)=3$, $y=-1$ $\quad x-2(-1)+3(2)=10$, $x=2$

The solution is (2, –1, 2). [10.1]

**4.**
$$\left[\begin{array}{ccc|c}2&6&-1&1\\1&3&-1&1\\3&10&-2&1\end{array}\right]R_1\leftrightarrow R_2\left[\begin{array}{ccc|c}1&3&-1&1\\2&6&-1&1\\3&10&-2&1\end{array}\right]\xrightarrow[-3R_1+R_3]{-2R_1+R_2}\left[\begin{array}{ccc|c}1&3&-1&1\\0&0&1&-1\\0&1&1&-2\end{array}\right]R_2\leftrightarrow R_3\left[\begin{array}{ccc|c}1&3&-1&1\\0&1&1&-2\\0&0&1&-1\end{array}\right]$$

$\begin{cases}x+3y-z=1\\y+z=-2\\z=-1\end{cases}$ $\quad y+(-1)=-2$, $y=-1$ $\quad x+3(-1)-(-1)=1$, $x=3$

The solution is (3, –1, –1). [10.1]

**5.** $\left[\begin{array}{rrrr|r} 1 & 2 & -3 & 2 & 11 \\ 2 & 5 & -8 & 5 & 28 \\ -2 & -4 & 7 & -1 & -18 \end{array}\right] \xrightarrow[2R_1+R_3]{-2R_1+R_2} \left[\begin{array}{rrrr|r} 1 & 2 & -3 & 2 & 11 \\ 0 & 1 & -2 & 1 & 6 \\ 0 & 0 & 1 & 3 & 4 \end{array}\right]$

$\begin{cases} w+2x-3y+2z=11 \\ \quad x-2y+\ z=6 \\ \qquad y+3z=4 \end{cases}$ $\quad y=4-3z \quad$ $x-2(4-3z)+z=6$, $x=-7z+14$ $\quad$ $w+2(-7z+14)-3(4-3z)+2z=11$, $w=3z-5$

Let $z$ be any real number $c$.

The solution is $(3c-5,\ -7c+14,\ 4-3c,\ c)$ [10.2]

**6.** $-3A=-3\begin{bmatrix} -1 & 3 & 2 \\ 1 & 4 & -1 \end{bmatrix}=\begin{bmatrix} 3 & -9 & -6 \\ -3 & -12 & 3 \end{bmatrix}$ [10.2]

**7.** $A+B=\begin{bmatrix} -1 & 3 & 2 \\ 1 & 4 & -1 \end{bmatrix}+\begin{bmatrix} 2 & -1 & 3 \\ 4 & -2 & -1 \\ 3 & 2 & 2 \end{bmatrix}$

$A + B$ is not defined because the matrices do not have the same order. [10.2]

**8.** $3B-2C=3\begin{bmatrix} 2 & -1 & 3 \\ 4 & -2 & -1 \\ 3 & 2 & 2 \end{bmatrix}-2\begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & 8 \\ -1 & 3 & -2 \end{bmatrix}=\begin{bmatrix} 6 & -3 & 9 \\ 12 & -6 & -3 \\ 9 & 6 & 6 \end{bmatrix}-\begin{bmatrix} 2 & -4 & 6 \\ 4 & -6 & 16 \\ -2 & 6 & -4 \end{bmatrix}=\begin{bmatrix} 4 & 1 & 3 \\ 8 & 0 & -19 \\ 11 & 0 & 10 \end{bmatrix}$ [10.2]

**9.** $AB=\begin{bmatrix} -1 & 3 & 2 \\ 1 & 4 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 \\ 4 & -2 & -1 \\ 3 & 2 & 2 \end{bmatrix}=\begin{bmatrix} (-1)(2)+3(4)+2(3) & (-1)(-1)+3(-2)+2(2) & (-1)3+3(-1)+2(2) \\ 1(2)+4(4)+(-1)(3) & 1(-1)+4(-2)+(-1)(2) & 1(3)+4(-1)+(-1)(2) \end{bmatrix}=\begin{bmatrix} 16 & -1 & -2 \\ 15 & -11 & -3 \end{bmatrix}$ [10.2]

**10.** Use $AB$ from Problem 9. $AB-A=\begin{bmatrix} 16 & -1 & -2 \\ 15 & -11 & -3 \end{bmatrix}-\begin{bmatrix} -1 & 3 & 2 \\ 1 & 4 & -1 \end{bmatrix}=\begin{bmatrix} 17 & -4 & -4 \\ 14 & -15 & -2 \end{bmatrix}$ [10.2]

**11.** $CA=\begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & 8 \\ -1 & 3 & -2 \end{bmatrix}\begin{bmatrix} -1 & 3 & 2 \\ 1 & 4 & -1 \end{bmatrix}$ [10.2]

The number of columns of the first matrix is not equal to the number of rows of the second matrix. The product is not possible.

**12.** $BC-CB=\begin{bmatrix} 2 & -1 & 3 \\ 4 & -2 & -1 \\ 3 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & 8 \\ -1 & 3 & -2 \end{bmatrix}-\begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & 8 \\ -1 & 3 & -2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 \\ 4 & -2 & -1 \\ 3 & 2 & 2 \end{bmatrix}$

$=\begin{bmatrix} -3 & 8 & -8 \\ 1 & -5 & -2 \\ 5 & -6 & 21 \end{bmatrix}-\begin{bmatrix} 3 & 9 & 11 \\ 16 & 20 & 25 \\ 4 & -9 & -10 \end{bmatrix}$

$=\begin{bmatrix} -6 & -1 & -19 \\ -15 & -25 & -27 \\ 1 & 3 & 31 \end{bmatrix}$ [10.2]

**13.** $A^2=\begin{bmatrix} -1 & 3 & 2 \\ 1 & 4 & -1 \end{bmatrix}\begin{bmatrix} -1 & 3 & 2 \\ 1 & 4 & -1 \end{bmatrix}$

The number of columns of the first matrix is not equal to the number of rows of the second matrix. The product is not possible. [10.2]

**14.** $B^2=\begin{bmatrix} 2 & -1 & 3 \\ 4 & -2 & -1 \\ 3 & 2 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 \\ 4 & -2 & -1 \\ 3 & 2 & 2 \end{bmatrix}=\begin{bmatrix} 9 & 6 & 13 \\ -3 & -2 & 12 \\ 20 & -3 & 11 \end{bmatrix}$ [10.2]

**15.** $\left[\begin{array}{rrr|rrr} 1 & -2 & 3 & 1 & 0 & 0 \\ 2 & -3 & 8 & 0 & 1 & 0 \\ -1 & 3 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1+R_3]{-2R_1+R_2} \left[\begin{array}{rrr|rrr} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{-2R_2+R_3} \left[\begin{array}{rrr|rrr} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -1 & 3 & -1 & 1 \end{array}\right]$

$\xrightarrow{2R_2+R_1} \left[\begin{array}{rrr|rrr} 1 & 0 & 7 & -3 & 2 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -1 & 3 & -1 & 1 \end{array}\right] \xrightarrow{\substack{7R_3+R_1 \\ 2R_3+R_2 \\ -R_3}} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 18 & -5 & 7 \\ 0 & 1 & 0 & 4 & -1 & 2 \\ 0 & 0 & 1 & -3 & 1 & -1 \end{array}\right]$

The inverse matrix is $\begin{bmatrix} 18 & -5 & 7 \\ 4 & -1 & 2 \\ -3 & 1 & -1 \end{bmatrix}$. [10.3]

**16.** $M_{21} = \begin{vmatrix} -1 & 3 \\ 2 & 2 \end{vmatrix} = -2 - 6 = -8$ [10.4]

$C_{21} = (-1)^{2+1} M_{21} = -(-8) = 8$

**17.** $|B| = \begin{vmatrix} 2 & -1 & 3 \\ 4 & -2 & -1 \\ 3 & 2 & 2 \end{vmatrix} = 3C_{31} + 2C_{32} + 2C_{33}$ [10.4]

$= 3(-1)^{3+1}\begin{vmatrix} -1 & 3 \\ -2 & -1 \end{vmatrix} + 2(-1)^{3+2}\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix} + 2(-1)^{3+3}\begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix}$

$= 3(1+6) - 2(-2-12) + 2(-4+4) = 21 + 28 + 0 = 49$

**18.** $\begin{vmatrix} 1 & -2 & 3 \\ 2 & -3 & 8 \\ -1 & 3 & -2 \end{vmatrix} = \begin{matrix} -2R_1 + R_2 \\ R_1 + R_3 \end{matrix} \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = -R_2 + R_3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{vmatrix} = (1)(1)(-1) = -1$ [10.4]

**19.** $D = \begin{vmatrix} 3 & 2 & -1 \\ 2 & -3 & 2 \\ 5 & 6 & 3 \end{vmatrix} = -82$ $D_3 = \begin{vmatrix} 3 & 2 & 12 \\ 2 & -3 & -1 \\ 5 & 6 & 4 \end{vmatrix} = 280$ $z = \frac{D_3}{D} = \frac{280}{-82} = -\frac{140}{41}$ [10.5]

**20.** $X = (I - A)^{-1} D$ [10.3]

$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.15 & 0.23 & 0.11 \\ 0.08 & 0.10 & 0.05 \\ 0.16 & 0.11 & 0.07 \end{bmatrix}\right)^{-1} \begin{bmatrix} 50 \\ 32 \\ 8 \end{bmatrix}$

## Cumulative Review

**1.** $y - 5 = -\frac{1}{2}(x - (-4))$ [2.3]

$y - 5 = -\frac{1}{2}x - 2$

$y = -\frac{1}{2}x + 3$

**2.** [3.1]

$$\begin{array}{r} 2x^2 + x - 10 \\ x - 3\overline{)2x^3 - 5x^2 - 13x + 30} \\ \underline{2x^3 - 6x^2} \qquad\qquad \\ x^2 - 13x \qquad \\ \underline{x^2 - 3x} \qquad \\ -10x + 30 \\ \underline{-10x + 30} \\ 0 \end{array}$$

**3.** $h[k(0)] = h[3^0]$ [4.2]

$= h[1]$

$= e^{-1}$

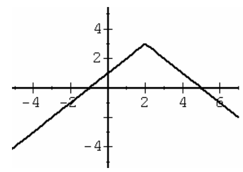
$\approx 0.3679$

**4.** $$2x^2-4x+3y-1=0 \quad [8.1]$$
$$2x^2-4x=-3y+1$$
$$2(x^2-2x)=-3y+1$$
$$2(x-1)^2=-3y+3$$
$$2(x-1)^2=-3(y-1)$$
Vertex = (1, 1)

**5.** $$\begin{cases} 3x-4y=4 & (1) \\ 2x-3y=1 & (2) \end{cases}$$
Solve (2) for $x$ and substitute into (1).
$$2x-3y=1$$
$$2x=3y+1$$
$$x=\frac{3y+1}{2}$$
$$3\left(\frac{3y+1}{2}\right)-4y=4$$
$$9y+3-8y=8$$
$$y=5$$
$$x=\frac{3(5)+1}{2}=8$$
The solution is (8, 5). [9.1]

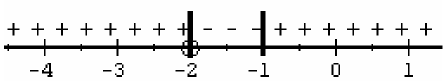
**6.** Domain: $\{x \mid -3 \le x \le 3\}$ [2.2]

**7.** $x^2+4x-5=(x+5)(x-1)$ [3.5]
$x+5=0 \qquad x-1=0$
$x=-5 \qquad x=1$
Vertical asymptotes: $x=-5$, $x=1$

**8.**


[2.5]

**9.** $$\frac{x+2}{x+1}>0$$
The quotient $\frac{x+2}{x+1}$ is positive.
The critical values are –2 and –1.

$\frac{x+2}{x+1}$



$(-\infty,-2)\cup(-1,\infty)$ [1.5]

**10.** $$2a=12$$
$$a=6$$
$$a^2=b^2+c^2$$
$$(6)^2=b^2+(4)^2$$
$$36-16=b^2$$
$$20=b^2$$
$$\frac{(x-3)^2}{36}+\frac{(y+4)^2}{20}=1 \quad [8.2]$$

**11.** $$\frac{f(x+h)-f(x)}{h}=\frac{(x+h)^2-3(x+h)+2-(x^2-3x+2)}{h} \quad [2.6]$$
$$=\frac{x^2+2xh+h^2-3x-3h+2-x^2+3x-2}{h}$$
$$=\frac{h^2+2xh-3h}{h}$$
$$=2x-3+h$$

**12.** $$125^x=\frac{1}{25} \quad [4.5]$$
$$5^{3x}=5^{-2}$$
$$3x=-2$$
$$x=-\frac{2}{3}$$

**13.** $\dfrac{x-2}{x^2-5x-6}=\dfrac{x-2}{(x+1)(x-6)}$

$$\frac{x-2}{(x+1)(x-6)}=\frac{A}{x+1}+\frac{B}{x-6}$$
$$x-2=A(x-6)+B(x+1)$$
$$x-2=Ax-6A+Bx+B$$
$$x-2=(A+B)x+(-6A+B)$$
$$\begin{cases} 1=A+B & (1) \\ -2=-6A+B & (2) \end{cases}$$

$-1=-A-B$    $-1$ times (1)
$\underline{-2=-6A+B}$    (2)
$-3=-7A$

$$\frac{3}{7}=A$$

$$1=\frac{3}{7}+B$$
$$\frac{4}{7}=B$$

$$\frac{x-2}{x^2-5x-6}=\frac{4}{7(x+1)}+\frac{3}{7(x-6)} \quad [9.4]$$

**14.** $10^x-10^{-x}=2$    [4.5]

$$10^x(10^x-10^{-x})=2(10^x)$$
$$(10^x)^2-2(10^x)-1=0$$

Let $u=10^x$

$$u^2-2u-1=0$$
$$u=\frac{2\pm\sqrt{4-4(1)(-1)}}{2}$$
$$=\frac{2\pm\sqrt{8}}{2}$$
$$=1\pm\sqrt{2}$$
$$10^x=1+\sqrt{2}$$
$$\log 10^x=\log(1+\sqrt{2})$$
$$x=\log(1+\sqrt{2})\approx 0.3828$$

**15.** $\cos 30^\circ \sin 30^\circ + \sec 45^\circ \tan 60^\circ$    [5.2]

$$=\cos 30^\circ \sin 30^\circ+\frac{1}{\sin 45^\circ}\tan 60^\circ$$
$$=\frac{\sqrt{3}}{2}\cdot\frac{1}{2}+\frac{2}{\sqrt{2}}\cdot\sqrt{3}$$
$$=\frac{\sqrt{3}}{4}+\frac{2\sqrt{6}}{2}$$
$$=\frac{\sqrt{3}+4\sqrt{6}}{4}$$

**16.** $\sin 292\frac{1}{2}^\circ=\sin\frac{585^\circ}{2}$

$$=-\sqrt{\frac{1-\cos 585^\circ}{2}}$$
$$=-\sqrt{\frac{1-\cos 225^\circ}{2}}$$
$$=-\sqrt{\frac{1-\left(-\frac{\sqrt{2}}{2}\right)}{2}}$$
$$=-\sqrt{\frac{2+\sqrt{2}}{4}}$$
$$=-\frac{1}{2}\sqrt{2+\sqrt{2}} \quad [6.3]$$

**17.** $K=\frac{1}{2}bc\sin A$    [7.2]

$$=\frac{1}{2}(11)(4)\sin 65^\circ$$
$$=22\sin 65^\circ$$
$$\approx 20\text{ m}^2$$

**18.** $\tan\theta+\dfrac{\cos\theta}{1-\sin\theta}=\tan\theta+\dfrac{\cos\theta(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}$

$$=\tan\theta+\frac{\cos\theta(1-\sin\theta)}{1-\sin^2\theta}$$
$$=\tan\theta+\frac{\cos\theta(1-\sin\theta)}{\cos^2\theta}$$
$$=\frac{\sin\theta}{\cos\theta}+\frac{1-\sin\theta}{\cos\theta}$$
$$=\frac{\sin\theta+1-\sin\theta}{\cos\theta}$$
$$=\frac{1}{\cos\theta}$$
$$=\sec\theta \quad [5.4]$$

**19.** $z = 2 - 2i\sqrt{3}$ [7.4]

$r = \sqrt{2^2 + (-2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4$

$\alpha = \tan^{-1}\left|\frac{b}{a}\right| = \tan^{-1}\left|\frac{-2\sqrt{3}}{2}\right| = \tan^{-1}\left|-\sqrt{3}\right| = 60^\circ$

$z$ is in the fourth quadrant $270^\circ < \theta < 360^\circ$,

$\theta = 360^\circ - 60^\circ = 300^\circ$

$z = r\,cis\theta$

$= 4(\cos 300^\circ + i\sin 300^\circ)$

**20.** $\sin\frac{x}{2} + \cos x = 1$ [6.6]

$\sqrt{\frac{1-\cos x}{2}} + \cos x = 1$

$\sqrt{\frac{1-\cos x}{2}} = 1 - \cos x$

$\frac{1-\cos x}{2} = 1 - 2\cos x + \cos^2 x$

$1 - \cos x = 2 - 4\cos x + 2\cos^2 x$

$0 = 2\cos^2 x - 3\cos x + 1$

$0 = (2\cos x - 1)(\cos x - 1)$

$2\cos x - 1 = 0$ $\qquad \cos x = 1$

$\cos x = \frac{1}{2}$ $\qquad x = 0$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

# Chapter 11
# Sequences, Series, and Probability

## Section 11.1

**1.** $a_1 = 1(1-1) = 1\cdot 0 = 0$
$a_2 = 2(2-1) = 2\cdot 1 = 2$
$a_3 = 3(3-1) = 3\cdot 2 = 6$
$a_8 = 8(8-1) = 8\cdot 7 = 56$

**2.** $a_1 = 2\cdot 1 = 2$
$a_2 = 2\cdot 2 = 4$
$a_3 = 2\cdot 3 = 6$
$a_8 = 2\cdot 8 = 16$

**3.** $a_1 = 1 - \frac{1}{1} = 0$
$a_2 = 1 - \frac{1}{2} = \frac{1}{2}$
$a_3 = 1 - \frac{1}{3} = \frac{2}{3}$
$a_8 = 1 - \frac{1}{8} = \frac{7}{8}$

**4.** $a_1 = \frac{1+1}{1} = \frac{2}{1} = 2$
$a_2 = \frac{2+1}{2} = \frac{3}{2}$
$a_3 = \frac{3+1}{3} = \frac{4}{3}$
$a_8 = \frac{8+1}{8} = \frac{9}{8}$

**5.** $a_1 = \frac{(-1)^{1+1}}{1^2} = \frac{(-1)^2}{1} = 1$
$a_2 = \frac{(-1)^{2+1}}{2^2} = \frac{(-1)^3}{4} = -\frac{1}{4}$
$a_3 = \frac{(-1)^{3+1}}{3^2} = \frac{(-1)^4}{9} = \frac{1}{9}$
$a_8 = \frac{(-1)^{8+1}}{8^2} = \frac{(-1)^9}{64} = -\frac{1}{64}$

**6.** $a_1 = \frac{(-1)^{1+1}}{1(1+1)} = \frac{(-1)^2}{1\cdot 2} = \frac{1}{2}$
$a_2 = \frac{(-1)^{2+1}}{2(2+1)} = \frac{(-1)^3}{2\cdot 3} = -\frac{1}{6}$
$a_3 = \frac{(-1)^{3+1}}{3(3+1)} = \frac{(-1)^4}{3\cdot 4} = \frac{1}{12}$
$a_8 = \frac{(-1)^{8+1}}{8(8+1)} = \frac{(-1)^9}{8\cdot 9} = -\frac{1}{72}$

**7.** $a_1 = \frac{(-1)^{2\cdot 1-1}}{3\cdot 1} = \frac{(-1)^{2-1}}{3} = -\frac{1}{3}$
$a_2 = \frac{(-1)^{2\cdot 2-1}}{3\cdot 2} = \frac{(-1)^{4-1}}{6} = -\frac{1}{6}$
$a_3 = \frac{(-1)^{2\cdot 3-1}}{3\cdot 3} = \frac{(-1)^{6-1}}{9} = -\frac{1}{9}$
$a_8 = \frac{(-1)^{2\cdot 8-1}}{3\cdot 8} = \frac{(-1)^{16-1}}{24} = -\frac{1}{24}$

**8.** $a_1 = \frac{(-1)^1}{2\cdot 1-1} = \frac{-1}{2-1} = -1$
$a_2 = \frac{(-1)^2}{2\cdot 2-1} = \frac{1}{3}$
$a_3 = \frac{(-1)^3}{2\cdot 3-1} = -\frac{1}{5}$
$a_8 = \frac{(-1)^8}{2\cdot 8-1} = \frac{1}{15}$

**9.** $a_1 = \left(\frac{2}{3}\right)^1 = \frac{2}{3}$
$a_2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$
$a_3 = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$
$a_8 = \left(\frac{2}{3}\right)^8 = \frac{256}{6561}$

**10.** $a_1 = \left(\frac{-1}{2}\right)^1 = -\frac{1}{2}$
$a_2 = \left(\frac{-1}{2}\right)^2 = \frac{1}{4}$
$a_3 = \left(\frac{-1}{2}\right)^3 = -\frac{1}{8}$
$a_8 = \left(\frac{-1}{2}\right)^8 = \frac{1}{256}$

**11.** $a_1 = 1 + (-1)^1 = 1 + (-1) = 0$
$a_2 = 1 + (-1)^2 = 1 + 1 = 2$
$a_3 = 1 + (-1)^3 = 1 + (-1) = 0$
$a_8 = 1 + (-1)^8 = 1 + 1 = 2$

**12.** $a_1 = 1 + (-0.1)^1 = 1 + (-0.1) = 0.9$
$a_2 = 1 + (-0.1)^2 = 1 + 0.01 = 1.01$
$a_3 = 1 + (-0.1)^3 = 1 + (-0.001) = 0.999$
$a_8 = 1 + (-0.1)^8 = 1 + 0.00000001 = 1.00000001$

**13.** $a_1 = (1.1)^1 = 1.1$
$a_2 = (1.1)^2 = 1.21$
$a_3 = (1.1)^3 = 1.331$
$a_8 = (1.1)^8 = 2.14358881$

**14.** $a_1 = \dfrac{1}{1^2+1} = \dfrac{1}{1+1} = \dfrac{1}{2}$

$a_2 = \dfrac{2}{2^2+1} = \dfrac{2}{4+1} = \dfrac{2}{5}$

$a_3 = \dfrac{3}{3^2+1} = \dfrac{3}{9+1} = \dfrac{3}{10}$

$a_8 = \dfrac{8}{8^2+1} = \dfrac{8}{64+1} = \dfrac{8}{65}$

**15.** $a_1 = \dfrac{(-1)^{1+1}}{\sqrt{1}} = \dfrac{(-1)^2}{1} = 1$

$a_2 = \dfrac{(-1)^{2+1}}{\sqrt{2}} = \dfrac{(-1)^3}{\sqrt{2}} = -\dfrac{1}{\sqrt{2}} = -\dfrac{\sqrt{2}}{2}$

$a_3 = \dfrac{(-1)^{3+1}}{\sqrt{3}} = \dfrac{(-1)^4}{\sqrt{3}} = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

$a_8 = \dfrac{(-1)^{8+1}}{\sqrt{8}} = \dfrac{(-1)^9}{2\sqrt{2}} = -\dfrac{1}{2\sqrt{2}} = -\dfrac{\sqrt{2}}{4}$

**16.** $a_1 = \dfrac{3^{1-1}}{2^1} = \dfrac{3^0}{2} = \dfrac{1}{2}$

$a_2 = \dfrac{3^{2-1}}{2^2} = \dfrac{3^1}{4} = \dfrac{3}{4}$

$a_3 = \dfrac{3^{3-1}}{2^3} = \dfrac{3^2}{8} = \dfrac{9}{8}$

$a_8 = \dfrac{3^{8-1}}{2^8} = \dfrac{3^7}{256} = \dfrac{2187}{256}$

**17.** $a_1 = 1! = 1$

$a_2 = 2! = 2 \cdot 1 = 2$

$a_3 = 3! = 3 \cdot 2 \cdot 1 = 6$

$a_8 = 8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40{,}320$

**18.** $a_1 = \dfrac{1!}{(1-1)!} = \dfrac{1}{0!} = \dfrac{1}{1} = 1$

$a_2 = \dfrac{2!}{(2-1)!} = \dfrac{2 \cdot 1}{1!} = \dfrac{2}{1} = 2$

$a_3 = \dfrac{3!}{(3-1)!} = \dfrac{3 \cdot 2 \cdot 1}{2!} = \dfrac{3 \cdot 2 \cdot 1}{2 \cdot 1} = 3$

$a_8 = \dfrac{8!}{(8-1)!}$

$= \dfrac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{7!}$

$= \dfrac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 8$

**19.** $a_1 = \log 1 = 0$

$a_2 = \log 2 \approx 0.3010$

$a_3 = \log 3 \approx 0.4771$

$a_8 = \log 8 \approx 0.9031$

**20.** $a_1 = \ln 1 = 0$

$a_2 = \ln 2 \approx 0.6931$

$a_3 = \ln 3 \approx 1.0986$

$a_8 = \ln 8 \approx 2.0794$

**21.** $a_1 = 1$ $\quad \dfrac{1}{7} = 0.\overline{142857}$

$a_2 = 4$

$a_3 = 2$

$a_8 = 4$

**22.** $a_1 = 0$ $\quad \dfrac{1}{13} = 0.\overline{076923}$

$a_2 = 7$

$a_3 = 6$

$a_8 = 7$

**23.** $a_1 = 3$

$a_2 = 3$

$a_3 = 3$

$a_8 = 3$

**24.** $a_1 = -2$

$a_2 = -2$

$a_3 = -2$

$a_8 = -2$

**25.** $a_1 = 5$

$a_2 = 2 \cdot a_1 = 2 \cdot 5 = 10$

$a_3 = 2 \cdot a_2 = 2 \cdot 10 = 20$

**26.** $a_1 = 2$

$a_2 = 3 \cdot a_1 = 3 \cdot 2 = 6$

$a_3 = 3 \cdot a_2 = 3 \cdot 6 = 18$

**27.** $a_1 = 2$

$a_2 = 2 \cdot a_1 = 2 \cdot 2 = 4$

$a_3 = 3 \cdot a_2 = 3 \cdot 4 = 12$

**28.** $a_1 = 1$

$a_2 = 2^2 \cdot a_1 = 4 \cdot 1 = 4$

$a_3 = 3^2 \cdot a_2 = 9 \cdot 4 = 36$

**29.** $a_1 = 2$

$a_2 = (a_1)^2 = (2)^2 = 4$

$a_3 = (a_2)^2 = (4)^2 = 16$

**30.** $a_1 = 4$

$a_2 = \dfrac{1}{a_1} = \dfrac{1}{4}$

$a_3 = \dfrac{1}{a_2} = \dfrac{1}{\frac{1}{4}} = 4$

**31.** $a_1 = 2$

$a_2 = 2 \cdot 2 \cdot a_1 = 4 \cdot 2 = 8$

$a_3 = 2 \cdot 3 \cdot a_2 = 6 \cdot 8 = 48$

**32.** $a_1 = 2$
$a_2 = (-3)\cdot 2\cdot a_1 = -6\cdot 2 = -12$
$a_3 = (-3)\cdot 3\cdot a_2 = -9\cdot(-12) = 108$

**33.** $a_1 = 3$
$a_2 = (a_1)^{1/2} = (3)^{1/2} = \sqrt{3}$
$a_3 = (a_2)^{1/3} = \left(3^{1/2}\right)^{1/3} = 3^{1/6} = \sqrt[6]{3}$

**34.** $a_1 = 2$
$a_2 = (a_1)^2 = (2)^2 = 4$
$a_3 = (a_2)^3 = (4)^3 = 64$

**35.** $a_1 = 1$
$a_2 = 3$
$a_3 = \frac{1}{2}(a_2 + a_1) = \frac{1}{2}(3+1) = \frac{1}{2}(4) = 2$
$a_4 = \frac{1}{2}(a_3 + a_2) = \frac{1}{2}(2+3) = \frac{1}{2}(5) = \frac{5}{2}$
$a_5 = \frac{1}{2}(a_4 + a_3) = \frac{1}{2}\left(\frac{5}{2}+2\right) = \frac{1}{2}\left(\frac{9}{2}\right) = \frac{9}{4}$

**36.** $a_1 = 1$
$a_2 = 4$
$a_3 = a_2\cdot a_1 = 4\cdot 1 = 4$
$a_4 = a_3\cdot a_2 = 4\cdot 4 = 16$
$a_5 = a_4\cdot a_3 = 16\cdot 4 = 64$

**37.** $a_n = a_{n-1} + a_{n-2}$
$a_3 = 3+1 = 4$
$a_4 = 4+3 = 7$
$a_5 = 7+4 = 11$

**38.**

$$\begin{array}{r} 668 \\ +\ 866 \\ \hline 1534 \end{array}$$

Sorting 1534 gives
1345

$$\begin{array}{r} 1345 \\ +\ 5431 \\ \hline 6776 \end{array}$$

Sorting 6776 gives
6677

**39.** $7! - 6! = 7\cdot 6! - 6! = 6!(7-1) = 6!\cdot 6 = 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1\cdot 6 = 4320$

**40.** $(4!)^2 = (4\cdot 3\cdot 2\cdot 1)^2 = (24)^2 = 576$

**41.** $\frac{9!}{7!} = \frac{9\cdot 8\cdot 7!}{7!} = 72$

**42.** $\frac{10!}{5!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5!}{5!} = 30,240$

**43.** $\frac{8!}{3!5!} = \frac{8\cdot 7\cdot 6\cdot 5!}{3\cdot 2\cdot 1\cdot 5!} = 56$

**44.** $\frac{12!}{4!8!} = \frac{12\cdot 11\cdot 10\cdot 9\cdot 8!}{4\cdot 3\cdot 2\cdot 1\cdot 8!} = 495$

**45.** $\frac{100!}{99!} = \frac{100\cdot 99!}{99!} = 100$

**46.** $\frac{100!}{98!2!} = \frac{100\cdot 99\cdot 98!}{98!\cdot 2\cdot 1} = 4950$

**47.** $\sum_{i=1}^{5} i = 1+2+3+4+5 = 15$

**48.** $\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 1+4+9+16 = 30$

**49.** $\sum_{i=1}^{5} i(i-1) = 1(1-1) + 2(2-1) + 3(3-1) + 4(4-1) + 5(5-1)$
$= 1\cdot 0 + 2\cdot 1 + 3\cdot 2 + 4\cdot 3 + 5\cdot 4$
$= 0+2+6+12+20 = 40$

**50.** $\sum_{i=1}^{7} i(2i+1) = (2\cdot 1+1) + (2\cdot 2+1) + 2(2\cdot 3+1) + (2\cdot 4+11) + (2\cdot 5+1) + (2\cdot 6+1) + (2\cdot 7+1)$
$= (2+1) + (4+1) + (6+1) + (8-1) + (10+1) + (12+1) + (14+1)$
$= 3+5+7+9+11+13+15 = 63$

**51.** $$\sum_{k=1}^{4}\frac{1}{k}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{12}{12}+\frac{6}{12}+\frac{4}{12}+\frac{3}{12}=\frac{25}{12}$$

**52.** $$\sum_{k=1}^{6}\frac{1}{k(k+1)}=\frac{1}{1(1+1)}+\frac{1}{2(2+1)}+\frac{1}{3(3+1)}+\frac{1}{4(4+1)}+\frac{1}{5(5+1)}+\frac{1}{6(6+1)}$$
$$=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\frac{1}{6\cdot 7}$$
$$=\frac{210}{420}+\frac{70}{420}+\frac{35}{420}+\frac{21}{420}+\frac{14}{420}+\frac{10}{420}=\frac{360}{420}=\frac{6}{7}$$

**53.** $$\sum_{j=1}^{8}2j=2\sum_{j=1}^{8}j=2(1+2+3+4+5+6+7+8)=2(36)=72$$

**54.** $$\sum_{i=1}^{6}(2i+1)(2i-1)=\sum_{i=1}^{6}\left(4i^2-1\right)=4\sum_{i=1}^{6}i^2-\sum_{i=1}^{6}(1)=4\left(1^2+2^2+3^2+4^2+5^2+6^2\right)-(1+1+1+1+1+1)$$
$$=4(1+4+9+16+25+36)-6$$
$$=4(91)-6=358$$

**55.** $$\sum_{i=3}^{5}(-1)^i 2^i=(-1)^3 2^3+(-1)^4 2^4+(-1)^5 2^5=(-1)8+(1)16+(-1)32=-8+16-32=-24$$

**56.** $$\sum_{i=3}^{5}\frac{(-1)^i}{2^i}=\frac{(-1)^3}{2^3}+\frac{(-1)^4}{2^4}+\frac{(-1)^5}{2^5}=\frac{-1}{8}+\frac{1}{16}+\frac{-1}{32}=\frac{-4}{32}+\frac{2}{32}+\frac{-1}{32}=-\frac{3}{32}$$

**57.** $$\sum_{n=1}^{7}\log\left(\frac{n+1}{n}\right)=\log\left(\frac{1+1}{1}\right)+\log\left(\frac{2+1}{2}\right)+\log\left(\frac{3+1}{3}\right)+\log\left(\frac{4+1}{4}\right)+\log\left(\frac{5+1}{5}\right)+\log\left(\frac{6+1}{6}\right)+\log\left(\frac{7+1}{7}\right)$$
$$=\log\left(2\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdot\frac{6}{5}\cdot\frac{7}{6}\cdot\frac{8}{7}\right)=\log 8=3\log 2\approx 0.9031$$

**58.** $$\sum_{n=2}^{8}\ln\frac{n}{n+1}=\ln\frac{2}{3}+\ln\frac{3}{4}+\ln\frac{5}{6}+\ln\frac{6}{7}+\ln\frac{7}{8}+\ln\frac{8}{9}=\ln\left(\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}\cdot\frac{6}{7}\cdot\frac{7}{8}\cdot\frac{8}{9}\right)=\ln\frac{2}{9}\approx-1.5041$$

**59.** $$\sum_{k=0}^{8}\frac{8!}{k!(8-k)!}=\frac{8!}{0!(8-0)!}+\frac{8!}{1!(8-1)!}+\frac{8!}{2!(8-2)!}+\frac{8!}{3!(8-3)!}+\frac{8!}{4!(8-4)!}+\frac{8!}{5!(8-5)!}+\frac{8!}{6!(8-6)!}+\frac{8!}{7!(8-7)!}+\frac{8!}{8!(8-8)!}$$
$$=\frac{8!}{0!8!}+\frac{8!}{1!7!}+\frac{8!}{2!6!}+\frac{8!}{3!5!}+\frac{8!}{4!4!}+\frac{8!}{5!3!}+\frac{8!}{6!2!}+\frac{8!}{7!1!}+\frac{8!}{8!0!}$$
$$=1+8+\frac{8\cdot 7}{2}+\frac{8\cdot 7\cdot 6}{3\cdot 2}+\frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2}+\frac{8\cdot 7\cdot 6}{3\cdot 2\cdot 1}+\frac{8\cdot 7}{2}+8+1$$
$$=1+8+28+56+70+56+28+8+1=256$$

**60.** $\sum_{k=0}^{7} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040}$

$= \frac{5040}{5040} + \frac{5040}{5040} + \frac{2520}{5040} + \frac{840}{5040} + \frac{210}{5040} + \frac{42}{5040} + \frac{7}{5040} + \frac{1}{5040}$

$= \frac{13700}{5040} = \frac{1370}{504} = \frac{685}{252}$

**61.** $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} = \sum_{i=1}^{6} \frac{1}{i^2}$

**62.** $2 + 4 + 6 + 8 + 10 + 12 + 14 = 2(1) + 2(2) + 2(3) + 2(4) + 2(5) + 2(6) + 2(7) = \sum_{i=1}^{7} 2i$

**63.** $2-4+8-16+32-64+128 = 2^1(-1)^{1+1} + 2^2(-1)^{2+1} + 2^3(-1)^{3+1} + 2^4(-1)^{4+1} + 2^5(-1)^{5+1} + 2^6(-1)^{6+1} + 2^7(-1)^{7+1} = \sum_{i=1}^{7} 2^i(-1)^{i+1}$

**64.** $1 - 8 + 27 - 64 + 125 = 1^3 - 2^3 + 3^3 - 4^3 + 5^3 = 1^3(-1)^{1+1} + 2^3(-1)^{2+1} + 3^3(-1)^{3+1} + 4^3(-1)^{4+1} \;\; + 5^3(-1)^{5+1} = \sum_{i=1}^{5} i^3(-1)^{i+1}$

**65.** $7 + 10 + 13 + 16 + 19 = 7 + (7+3) + (7 + 3\cdot 2) + (7 + 3\cdot 3) + (7 + 3\cdot 4) = \sum_{i=0}^{4} (7 + 3i)$

**66.** $30 + 26 + 22 + 18 + 14 + 10 = 30 + (30-4) + (30 - 4\cdot 2) + (30 - 4\cdot 3) + (30 - 4\cdot 4) + (30 - 4\cdot 5) = \sum_{i=0}^{5} (30 - 4i)$

**67.** $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} = \sum_{i=1}^{4} \frac{1}{2^i}$

**68.** $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \frac{16}{81} - \frac{32}{243} = 1 - \frac{2}{3} + \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 - \left(\frac{2}{3}\right)^5 = \sum_{i=0}^{5} \left(\frac{2}{3}\right)^i (-1)^i = \sum_{i=0}^{5} \left(-\frac{2}{3}\right)^i$

## Connecting Concepts

**69.** Let $N = 7$.

$a_1 = \frac{7}{2} = 3.5$

$a_2 = \frac{1}{2}\left(3.5 + \frac{7}{3.5}\right) = 2.75$

$a_3 = \frac{1}{2}\left(2.75 + \frac{7}{2.75}\right) \approx 2.6477273$

$a_4 \approx \frac{1}{2}\left(2.6477273 + \frac{7}{2.6477273}\right) \approx 2.6457520$

**70.** Let $N = 10$.

$a_1 = \frac{10}{2} = 5$

$a_2 = \frac{1}{2}\left(5 + \frac{10}{5}\right) = 3.5$

$a_3 = \frac{1}{2}\left(3.5 + \frac{10}{3.5}\right) \approx 3.1785714$

$a_4 = \frac{1}{2}\left(3.1785714 + \frac{10}{3.1785714}\right) \approx 3.1623194$

$a_5 = \frac{1}{2}\left(3.1623194 + \frac{10}{3.1623194}\right) \approx 3.1622777$

**71.** $a_{20} \approx 1.0000037$

$a_{100} \approx 1$

**72.** $a_1 = i,\ a_2 = -1,\ a_3 = -i,\ a_4 = 1$

$a_5 = i,\ a_6 = -1,\ a_7 = -i,\ a_8 = 1$

Notice that the sequence repeats itself in groups of 4. To find $a_{237}$ divide 237 by 4. $a_{237} = i^r$ where $r$ is the remainder after the division. Thus $a_{237} = i^1 = i$.

**73.** $1+1=2$

$1+1+2=4$

$1+1+2+3=7$

The sum of the first $n$ terms of a Fibonacci sequence equals the $(n + 2)$ term – 1.
Therefore the sum of the first ten terms is (10+2) term – 1, i.e. 12th term – 1.
144  1 = 143

**74.** 81 = 5 + 21 + 55

**75.**
$$F_{10} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{10} - \left(\frac{1-\sqrt{5}}{2}\right)^{10}}{\sqrt{5}} = 55$$
$$F_{15} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{15} - \left(\frac{1-\sqrt{5}}{2}\right)^{15}}{\sqrt{5}} = 610$$

**76.** $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

When $n = 10$ we have $\sqrt{2\pi(10)}\left(\frac{10}{e}\right)^{10} \approx 3.5986956 \times 10^6$.

When $n = 20$ we have $\sqrt{2\pi(20)}\left(\frac{20}{e}\right)^{20} \approx 2.4227869 \times 10^{18}$.

When $n = 30$ we have $\sqrt{2\pi(30)}\left(\frac{30}{e}\right)^{30} \approx 2.6451710 \times 10^{32}$.

**77.**
$$\sum_{i=1}^{n} ca_i = ca_1 + ca_2 + ca_3 + \cdots + ca_n$$
$$= c(a_1 + a_2 + a_3 + \cdots + a_n)$$
$$= c\sum_{i=1}^{n} a_i$$

## Prepare for Section 11.2

**PS1.**
$$-3 = 25 + (15-1)d$$
$$-3 = 25 + 14d$$
$$-28 = 14d$$
$$-2 = d$$

**PS2.**
$$13 = 3 + (5-1)d$$
$$13 = 3 + 4d$$
$$10 = 4d$$
$$\frac{5}{2} = d$$

**PS3.**
$$S = \frac{50\left[2(2) + (50-1)\frac{5}{4}\right]}{2} = \frac{50\left[4 + \frac{245}{4}\right]}{2}$$
$$= \frac{50\left[\frac{261}{4}\right]}{2} = \frac{6525}{4}$$

**PS4.** $a_5 = 5 + (5-1)4 = 21$

**PS5.** $a_{20} = 52 + (20-1)(-3) = -5$

**PS6.** 5 – 2 = 3
8 – 5 = 3
Yes

## Section 11.2

**1.** $d = 10 - 6 = 4$

$a_n = 6 + (n-1)4 = 6 + 4n - 4 = 4n + 2$
$a_9 = 4 \cdot 9 + 2 = 36 + 2 = 38$
$a_{24} = 4 \cdot 24 + 2 = 98$

**2.** $d = 12 - 7 = 5$

$a_n = 7 + (n-1)5 = 5n + 2$
$a_9 = 5 \cdot 9 + 2 = 45 + 2 = 47$
$a_{24} = 5 \cdot 24 + 2 = 120 + 2 = 122$

**3.** $d = 4 - 6 = -2$

$a_n = 6 + (n-1)(-2) = 6 - 2n + 2 = 8 - 2n$
$a_9 = 8 - 2 \cdot 9 = 8 - 18 = -10$
$a_{24} = 8 - 2 \cdot 24 = 8 - 48 = -40$

**4.** $d = 4 - 11 = -7$

$a_n = 11 + (n-1)(-7) = 11 - 7n + 7$
$= 18 - 7n$
$a_9 = 18 - 7 \cdot 9 = 18 - 63 = -45$
$a_{24} = 18 - 7 \cdot 24 = 18 - 168 = -150$

**5.** $d = -5 - (-8) = 3$
$a_n = -8 + (n-1)3 = -8 + 3n - 3$
$= 3n - 11$
$a_9 = 3 \cdot 9 - 11 = 27 - 11 = 16$
$a_{24} = 3 \cdot 24 - 11 = 72 - 11 = 61$

**6.** $d = -9 - (-15) = 6$
$a_n = -15 + (n-1)6 = -15 + 6n - 6$
$= 6n - 21$
$a_9 = 6 \cdot 9 - 21 = 33$
$a_{24} = 6 \cdot 24 - 21 = 123$

**7.** $d = 4 - 1 = 3$
$a_n = 1 + (n-1)3 = 1 + 3n - 3 = 3n - 2$
$a_9 = 3 \cdot 9 - 2 = 27 - 2 = 25$
$a_{24} = 3 \cdot 24 - 2 = 72 - 2 = 70$

**8.** $d = 1 - (-4) = 5$
$a_n = -4 + (n-1)5 = -4 + 5n - 5 = 5n - 9$
$a_9 = 5 \cdot 9 - 9 = 45 - 9 = 36$
$a_{24} = 5 \cdot 24 - 9 = 120 - 9 = 111$

**9.** $d = (a+2) - a = 2$
$a_n = a + (n-1)2 = a + 2n - 2$
$a_9 = a + 2 \cdot 9 - 2 = a + 18 - 2 = a + 16$
$a_{24} = a + 2 \cdot 24 - 2 = a + 48 - 2 = a + 46$

**10.** $d = (a+1) - (a-3) = a + 1 - a + 3 = 4$
$a_n = a - 3 + (n-1)4 = a - 3 + 4n - 4$
$= a + 4n - 7$
$a_9 = a + 4 \cdot 9 - 7 = a + 36 - 7 = a + 29$
$a_{24} = a + 4 \cdot 24 - 7 = a + 96 - 7 = a + 89$

**11.** $d = \log 14 - \log 7 = \log \frac{14}{7} = \log 2$
$a_n = \log 7 + (n-1)\log 2$
$a_9 = \log 7 + 8\log 2$
$a_{24} = \log 7 + 23\log 2$

**12.** $d = \ln 16 - \ln 4 = \ln \frac{16}{4} = \ln 4$
$a_n = \ln 4 + (n-1)\ln 4 = \ln 4 + \ln 4^{(n-1)}$
$= \ln\left(4 \cdot 4^{(n-1)}\right) = \ln 4^n = n \ln 4$
$a_9 = 9\ln 4$
$a_{24} = 24\ln 4$

**13.** $d = \log a^2 - \log a = \log \frac{a^2}{a} = \log a$
$a_n = \log a + (n-1)\log a$
$= (1 + n - 1)\log a = n\log a$
$a_9 = 9\log a$
$a_{24} = 24\log a$

**14.** $d = \log_2 5a - \log_2 5 = \log_2 \frac{5a}{5} = \log_2 a$
$a_n = \log_2 5 + (n-1)\log_2 a$
$a_9 = \log_2 5 + 8\log_2 a$
$a_{24} = \log_2 5 + 23\log_2 a$

**15.** $d = 15 - 13 = 2$

$$a_4 = a_1 + (4-1)2 = 13$$
$$a_1 + 6 = 13$$
$$a_1 = 7$$
$$a_{20} = 7 + (20-1)2 = 7 + (19)2 = 7 + 38 = 45$$

**16.** $a_6 = -14 = a_1 + (6-1)d$

$$-14 = a_1 + 5d$$
$$-14 - 5d = a_1$$

$a_8 = a_1 + (8-1)d = -20$

$$a_1 + 7d = -20$$
$$14 - 5d + 7d = -20$$
$$-14 + 2d = -20$$
$$2d = -6$$
$$d = -3$$
$$a_1 = -14 - 5(-3) = -14 + 15 = 1$$
$$a_{15} = 1 + (15-1)(-3) = 1 + 14(-3) = 1 - 42 = -41$$

**17.** $a_5 = -19 = a_1 + (5-1)d$

$$-19 = a_1 + 4d$$
$$-19 - 4d = a_1$$

$a_7 = -29 = a_1 + (7-1)d$

$$-29 = a_1 + 6d$$
$$-29 = (-19 - 4d) + 6d$$
$$-29 = -19 + 2d$$
$$-10 = 2d$$
$$d = -5$$
$$a_1 = -4(-5) - 19 = 1$$
$$a_{17} = 1 + (17-1)(-5) = 1 + 16(-5) = 1 - 80 = -79$$

**18.** $a_4 = 22 = a_1 + (4-1)d$

$a_7 = 34 = a_1 + (7-1)d$

$$34 = a_1 + 6d$$
$$34 = 22 - 3d + 6d$$
$$34 = 22 + 3d$$
$$3d = 12$$
$$d = 4$$
$$22 = a_1 + 3d$$
$$a_1 = 22 - 3d$$
$$a_1 = 22 - 3 \cdot 4 = 10$$
$$a_{23} = 10 + (23-1)4 = 10 + (22)4 = 10 + 88 = 98$$

**19.** $a_1 = 3(1) + 2 = 5$

$$a_{10} = 3(10) + 2 = 32$$
$$S_{10} = \tfrac{10}{2}(a_1 + a_{10}) = 5(5+32) = 185$$

**20.** $a_1 = 4(1) - 3 = 1$

$$a_{12} = 4(12) - 3 = 45$$
$$S_{12} = \tfrac{12}{2}(a_1 + a_{12}) = 6(1+45) = 276$$

**21.** $a_1 = 3 - 5(1) = -2$

$$a_{15} = 3 - 5(15) = -72$$
$$S_{15} = \frac{15}{2}(a_1 + a_{15}) = \frac{15}{2}(-2 + (-72)) = \frac{15}{2}(-74) = -555$$

**22.** $a_1 = 1 - 2(1) = -1$

$$a_{20} = 1 - 2(20) = -39$$
$$S_{20} = \frac{20}{2}(a_1 + a_{20}) = 10(-1 + (-39)) = 10(-40) = -400$$

**23.** $a_1 = 6(1) = 6$

$$a_{12} = 6(12) = 72$$
$$S_{12} = \frac{12}{2}(a_1 + a_{12}) = 6(6 + 72) = 6(78) = 468$$

**24.** $a_1 = 7(1) = 7$

$$a_{14} = 7(14) = 98$$
$$S_{14} = \frac{14}{2}(a_1 + a_{14}) = 7(7 + 98) = 735$$

**25.** $a_1 = 1 + 8 = 9$

$$a_{25} = 25 + 8 = 33$$
$$S_{25} = \frac{25}{2}(a_1 + a_{25}) = \frac{25}{2}(9 + 33) = \frac{25}{2}(42) = 525$$

**26.** $a_1 = 1 - 4 = -3$

$$a_{25} = 25 - 4 = 21$$
$$S_{25} = \frac{25}{2}(a_1 + a_{25}) = \frac{25}{2}(-3 + 21) = \frac{25}{2}(18) = 225$$

**27.** $a_1 = -1$

$$a_{30} = -30$$
$$S_{30} = \frac{30}{2}(a_1 + a_{30}) = 15(-1 + (-30)) = 15(-31) = -465$$

**28.** $a_1 = 4 - 1 = 3$

$$a_{40} = 4 - 40 = -36$$
$$S_{40} = \frac{40}{2}(a_1 + a_{40}) = 20(3 + (-36)) = 20(-33) = -660$$

**29.** $a_1 = 1 + x$

$$a_{12} = 12 + x$$
$$S_{12} = \frac{12}{2}(a_1 + a_{12}) = 6(1 + x + 12 + x) = 78 + 12x$$

**30.** $a_1 = 2(1) - x = 2 - x$

$$a_{15} = 2(15) - x = 30 - x$$
$$S_{15} = \frac{15}{2}(a_1 + a_{15}) = \frac{15}{2}(2 - x + 30 - x) = \frac{15}{2}(32 - 2x)$$
$$= 240 - 15x$$

**31.** $a_1 = (1)x = x$
$a_{20} = (20)x = 20x$

$S_{20} = \frac{20}{2}(a_1 + a_{20}) = 10(x + 20x) = 210x$

**32.** $a_1 = -(1)x = -x$
$a_{14} = -(14)x = -14x$

$S_{14} = \frac{14}{2}(a_1 + a_{14}) = 7[-x + (-14x)] = 7(-15x) = -105x$

**33.** $-1, c_2, c_3, c_4, c_5, c_6, 23$
$a_1 = -1$
$a_7 = a_1 + (n-1)d$
$23 = -1 + (7-1)d$
$23 = -1 + 6d$
$24 = 6d$
$d = 4$
$c_2 = -1 + (2-1)4 = -1 + 4 = 3$
$c_3 = -1 + (3-1)4 = -1 + (2)4 = 7$
$c_4 = -1 + (4-1)4 = -1 + (3)4 = 11$
$c_5 = -1 + (5-1)4 = -1 + (4)4 = 15$
$c_6 = -1 + (6-1)4 = -1 + (5)4 = 19$

**34.** $7, c_2, c_3, c_4, c_5, c_6, 19$
$a_1 = 7$
$a_7 = a_1 + (n-1)d$
$19 = 7 + (7-1)d$
$12 = 6d$
$d = 2$
$c_2 = 7 + (2-1)2 = 7 + 2 = 9$
$c_3 = 7 + (3-1)2 = 7 + (2)2 = 11$
$c_4 = 7 + (4-1)2 = 7 + (3)2 = 13$
$c_5 = 7 + (5-1)2 = 7 + (4)2 = 15$
$c_6 = 7 + (6-1)2 = 7 + (5)2 = 17$

**35.** $3, c_2, c_3, c_4, c_5, \frac{1}{2}$
$a_1 = 3$
$a_6 = a_1 + (n-1)d$
$\frac{1}{2} = 3 + (6-1)d$
$-\frac{5}{2} = 5d$
$d = -\frac{1}{2}$
$c_2 = 3 + (2-1)\left(-\frac{1}{2}\right) = 3 - \frac{1}{2} = \frac{5}{2}$
$c_3 = 3 + (3-1)\left(-\frac{1}{2}\right) = 3 - 1 = 2$
$c_4 = 3 + (4-1)\left(-\frac{1}{2}\right) = 3 - \frac{3}{2} = \frac{3}{2}$
$c_5 = 3 + (5-1)\left(-\frac{1}{2}\right) = 3 - 2 = 1$

**36.** $\frac{11}{3}, c_2, c_3, c_4, c_5, 6$
$a_1 = \frac{11}{3}$
$a_6 = a_1 + (n-1)d$
$6 = \frac{11}{3} + (6-1)d$
$\frac{7}{3} = 5d$
$d = \frac{7}{15}$
$c_2 = \frac{11}{3} + (2-1)\frac{7}{15} = \frac{11}{3} + \frac{7}{15} = \frac{62}{15}$
$c_3 = \frac{11}{3} + (3-1)\frac{7}{15} = \frac{11}{3} + \frac{14}{15} = \frac{69}{15}$
$c_4 = \frac{11}{3} + (4-1)\frac{7}{15} = \frac{11}{3} + \frac{21}{15} = \frac{76}{15}$
$c_5 = \frac{11}{3} + (5-1)\frac{7}{15} = \frac{11}{3} + \frac{28}{15} = \frac{83}{15}$

**37.** $a_1 = 1, d = 2$
$S_n = \frac{n[2(1) + (n-1)2]}{2} = \frac{n}{2}[2n] = n^2$

**38.** $a_1 = 2$
$a_n = 2n$
$S_n = \frac{n}{2}(a_n + a_1) = \frac{n}{2}[2n + 2] = n^2 + n$

**39.** $a_1 = 25, d = -1$
$a_6 = 25 + (6-1)(-1) = 25 - 5 = 20$
$S_6 = \frac{6}{2}(25 + 20) = 3(45) = 135$
20 logs stacked on sixth row,
135 logs in the six rows

**40.** $a_1 = 27, d = 2$
$a_{10} = 27 + (10-1)(2) = 27 + 18 = 45$
$S_{10} = \frac{10}{2}(27 + 45) = 5(72) = 360$
45 seats in the tenth row,
360 seats in the ten rows

**41.** $a_1 = 5000, d = -250$

$a_{15} = 5000 + (15-1)(-250) = 5000 - 3500 = 1500$

The fifteenth prize is \$1500.

$S_{15} = \frac{15}{2}(5000 + 1500) = \frac{15}{2}(6500) = 48,750$

The total amount of money distributed is \$48,750.

**42.** $a_1 = 15$, $d = 5$

$a_n = a_1 + (n-1)d$

$60 = 15 + (n-1)5$

$60 = 15 + 5n - 5$

$50 = 5n$

$n = 10$

In 10 weeks, a person will be walking 60 minutes a day.

**43.** $a_1 = 16, d = 32$

$S_7 = \frac{7}{2}\left[2(16) + (7-1)32\right] = \frac{7}{2}[32 + 192] = \frac{7}{2}[224] = 784$

The total distance the object falls is 784 ft.

**44.** $a_n = 2n - 1$

$a_{10} = 2(10) - 1 = 20 - 1 = 19$

$a_1 = 2(1) - 1 = 1$

$S_{10} = \frac{10}{2}(1 + 19) = 100$

The distance the ball rolls in the tenth second is 19 ft.
The total distance is 100 ft.

**45.** $a_n = a_1 + (n-1)d$

$46.9 = 3.5 + (n-1)1.4$

$43.4 = (n-1)1.4$

$31 = n - 1$

$32 = n$

**46.** $S_n = \frac{n}{2}(a_1 + a_n)$

$S_{32} = \frac{32}{2}(3.5 + 46.9)$

$S_{32} = 806.4$ mm

## Connecting Concepts

**47.** If $f(x)$ is a linear function, then $f(x) = mx + b$. To show that $f(n)$, where $n$ is a positive integer, is an arithmetic sequence, we must show that $f(n+1) - f(n)$ is a constant. We have

$$\begin{aligned} f(n+1) - f(n) &= (m(n+1) + b) - (m(n) + b) \\ &= mn + m + b - mn - b \\ &= m \end{aligned}$$

Thus, the difference between any two successive terms is $m$, the slope of the linear function.

**48.** $a_1 = 3$, $a_n = a_{n-1} + 5$

Rewriting $a_n = a_{n-1} + 5$ as $a_n - a_{n-1} = 5$, the difference between successive terms is the same constant 5. Thus the sequence is an arithmetic sequence with $a_1 = 3$ and $d = 5$.

$a_n = a_1 + (n-1)d$

Substituting,

$a_n + 3 + (n-1)5 = 5n - 2$

**49.** $a_1 = 4$, $a_n = a_{n-1} - 3$

Rewriting $a_n = a_{n-1} - 3$ as $a_n - a_{n-1} = -3$, we find that the difference between successive terms is the same constant –3.
Thus the sequence is an arithmetic sequence with $a_1 = 4$ and $d = -3$.

$a_n = a_1 + (n-1)d$

Substituting,

$a_n = 4 + (n-1)(-3) = 4 - 3n + 3 = 7 - 3n$

**50.** $a_1 = 4, a_n = b_{n-1} + 5; b_1 = 2, b_n = a_{n-1} + 1$

To show that $a_n$ is an arithmetic sequence, we must show that the difference between successive terms is some constant *d*.
We begin by finding a relationship between $a_n$ and $a_{n-2}$.

$$\begin{aligned} a_n &= b_{n-1} + 5 \\ &= a_{n-1-1} + 1 + 5 \\ &= a_{n-2} + 6 \end{aligned}$$

$$\begin{aligned} a_1 &= 4 \\ a_2 &= b_1 + 5 = 2 + 5 = 7 \\ a_3 &= a_1 + 6 \\ a_4 &= a_2 + 6 \\ a_5 &= a_3 + 6 = (a_1 + 6) + 6 = a_1 + 2(6) \\ a_6 &= a_4 + 6 = (a_2 + 6) + 6 = a_2 + 2(6) \\ a_7 &= a_5 + 6 = (a_3 + 6) + 6 = a_1 + 3(6) \\ a_8 &= a_6 + 6 = (a_4 + 6) + 6 = a_2 + 3(6) \end{aligned}$$

Thus we have $a_n = a_{n-2} + 6$. This establishes a relationship between *alternate* successive terms. We now examine the terms of $a_n$.

Now consider two cases. First, $n$ is an even integer, $n = 2k$. From the list of terms, we have

$a_{2k} = a_2 + (k-1)6 \quad k \geq 2$

Now consider the case when $n$ is an odd integer, $n = 2k - 1$.

$a_{2k-1} = a_1 + (k-1)6 \quad k \geq 2$

Thus $a_{2k} - a_{2k-1} = (a_2 + (k-1)6) - (a_1 + (k-1)6) = a_2 - a_1 = 7 - 4 = 3$

Therefore, the difference between successive terms is the constant 3. To find $a_{100}$, use $a_n = a_1 + (n-1)d$.

$a_{100} = 4 + (100-1)(3) = 4 + (99)(3) = 301$

To show that $b_n$ is also an arithmetic sequence, we have

$b_n - b_{n-1} = (a_{n-1} + 1) - (a_{n-2} + 1) = a_{n-1} - a_{n-2}$

Because $a_n$ is an arithmetic sequence, $a_{n-1} - a_{n-2}$ is a constant. Thus $b_n$ is an arithmetic sequence.

**51.** $a_1 = 1, a_n = b_{n-1} + 7; b_1 = -2, b_n = a_{n-1} + 1$

To show that $a_n$ is an arithmetic sequence, we must show that $a_n - a_{n-1} = d$, where $d$ is a constant. We begin by finding a relationship between $a_n$ and $a_{n-2}$.

$a_n = b_{n-1} + 7 = a_{n-2} + 1 + 7$

$a_n = a_{n-2} + 8$

This establishes a relationship between *alternate* successive terms. We now examine some terms of $a_n$.

$a_1 = 1$

$a_2 = b_1 + 7 = -2 + 7 = 5$

$a_3 = a_1 + 8 \qquad (a_n = a_{n-2} + 8)$

$a_4 = a_2 + 8$

$a_5 = a_3 + 8 = (a_1 + 8) + 8 = a_1 + 2(8)$

$a_6 = a_4 + 8 = (a_2 + 8) + 8 = a_2 + 2(8)$

$a_7 = a_5 + 8 = (a_1 + 2(8)) + 8 = a_1 + 3(8)$

$a_8 = a_6 + 8 = (a_2 + 2(8)) = a_2 + 3(8)$

Now consider two cases. First, $n$ is an even integer, $n = 2k$.

$a_{2k} = a_2 + (k-1)8 \qquad k \ge 2$

When $n$ is an odd integer, $n = 2k - 1$.

$a_{2k-1} = a_1 + (k-1)8 \qquad k \ge 2$

Thus $a_{2k} - a_{2k-1} = (a_2 + (k-1)8) - (a_1 + (k-1)8) = a_2 - a_1 = 5 - 1 = 4$

Therefore, the difference between successive terms is the constant. To find $a_{50}$, use $a_n = a_1 + (n-1)d$.

$a_{50} = 1 + (49)(4) = 197$

To show that $b_n$ is an arithmetic sequence, we have

$b_n - b_{n-1} = (a_{n-1} + 1) - (a_{n-2} + 1) = a_{n-1} - a_{n-2}$

Because $a_n$ is an arithmetic sequence, $a_{n-1} - a_{n-2}$ is a constant. Thus $b_n$ is an arithmetic sequence.

## Prepare for Section 11.3

**PS1.** $\frac{4}{2} = 2, \quad \frac{8}{4} = 2$

The ratio is 2.

**PS2.** $\sum_{n=1}^{4} \frac{1}{2^{n-1}} = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} = \frac{15}{8}$

**PS3.** $S = \frac{3(1-(-2)^5)}{1-(-2)} = 33$

**PS4.**

$S - rS = a - ar^2$

$S(1-r) = a(1-r^2)$

$S = \frac{a(1+r)(1-r)}{(1-r)}$

$S = a(1+r)$

**PS5.**

$a_1 = 3\left(-\frac{1}{2}\right)^1 = -\frac{3}{2}$

$a_2 = 3\left(-\frac{1}{2}\right)^2 = \frac{3}{4}$

$a_3 = 3\left(-\frac{1}{2}\right)^3 = -\frac{3}{8}$

**PS6.**

$S_1 = 2$

$S_2 = 2 + 4 = 6$

$S_3 = 2 + 4 + 8 = 14$

## Section 11.3

**1.** $\frac{4^{i+1}}{4^i} = 4$, geometric, $r = 4$

**2.** $\frac{6^{i+1}}{6^i} = 6$, geometric, $r = 6$

**3.** $\frac{\frac{1}{i+1}}{\frac{1}{i}} = \frac{i}{i+1}$, not geometric

**4.** $\frac{\frac{1}{2^{i+1}}}{\frac{1}{2^i}} = \frac{2^i}{2^{i+1}} = \frac{1}{2}$, geometric, $r = \frac{1}{2}$

**5.** $\frac{2^{(i+1)x}}{2^{ix}} = \frac{2^{ix+x}}{2^{ix}} = 2^x$, geometric, $r = 2^x$

**6.** $\frac{(-1)^{(i+1)-1}e^{(i+1)x}}{(-1)^{i-1}e^{ix}} = \frac{(-1)^i e^{ix+x}}{(-1)^{i-1}e^{ix}} = (-1)e^x$

geometric, $r = (-1)e^x$

**7.** $\frac{3(2^{(i+1)-1})}{3(2^{i-1})} = \frac{2^i}{2^{i-1}} = 2$, geometric, $r = 2$

**8.** $\frac{5(-2)^{(i+1)-1}}{5(-2)^{i-1}} = \frac{(-2)^i}{(-2)^{i-1}} = -2$, geometric, $r = -2$

**9.** $\frac{x^{2(i+1)}}{x^{2i}} = \frac{x^{2i+2}}{x^{2i}} = x^2$, geometric, $r = x^2$

**10.** $\frac{3(i+x)x^{i+1}}{3ix^i} = \frac{(i+1)x}{i}$, not geometric

**11.** $\frac{\ln 5(i+1)}{\ln 5i} = \left(\frac{\ln 5}{\ln 5i}\right)\left(\frac{\ln(i+1)}{\ln 5i}\right)$, not geometric

**12.** $\frac{\log x^{2(i+1)-1}}{\log x^{2i-1}} = \frac{2^i \log x}{2^{i-1}\log x} = \frac{2^i}{2^i 2^{-1}} = 2$, geometric, $r = 2$

**13.** $r = \frac{8}{2} = 4$,

$a_n = 2 \cdot 4^{n-1} = 2 \cdot 2^{2(n-1)} = 2^{2n-1}$

**14.** $r = \frac{5}{1} = 5, a_n = 1(5)^{n-1} = 5^{n-1}$

**15.** $r = \frac{12}{-4} = -3, a_n = -4(-3)^{n-1}$

**16.** $r = \frac{6}{-3} = -2, a_n = -3(-2)^{n-1}$

**17.** $r = \frac{4}{6} = \frac{2}{3}, a_n = 6\left(\frac{2}{3}\right)^{n-1}$

**18.** $r = \frac{6}{8} = \frac{3}{4}, a_n = 8\left(\frac{3}{4}\right)^{n-1}$

**19.** $r = -\frac{5}{6}, a_n = -6\left(-\frac{5}{6}\right)^{n-1}$

**20.** $r = -\frac{\frac{4}{3}}{-2} = -\frac{4}{6} = -\frac{2}{3}, a_n = -2\left(-\frac{2}{3}\right)^{n-1}$

**21.** $r = \frac{-3}{9} = -\frac{1}{3}, a_n = 9\left(-\frac{1}{3}\right)^{n-1} = \left(-\frac{1}{3}\right)^{n-3}$

**22.** $r = \frac{-\frac{4}{3}}{8} = \frac{-1}{6}, a_n = 8\left(-\frac{1}{6}\right)^{n-1}$

**23.** $r = \frac{-x}{1} = -x, a_n = 1(-x)^{n-1} = (-x)^{n-1}$

**24.** $r = \frac{2a}{2} = a, a_n = 2(a)^{n-1} = 2a^{n-1}$

**25.** $r = \frac{c^5}{c^2} = c^3, a_n = c^2\left(c^3\right)^{n-1} = c^2c^{3n-3} = c^{3n-1}$

**26.** $r = \frac{x^4}{-x^2} = -x^2, a_n = -x^2\left(-x^2\right)^{n-1} = -1 \cdot x^2\left[(-1)^{n-1} \cdot x^{2n-2}\right] = (-1)^n x^{2n}$

**27.** $r = \frac{\frac{3}{10,000}}{\frac{3}{100}} = \frac{1}{100}, a_n = \frac{3}{100}\left(\frac{1}{100}\right)^{n-1} = 3\left(\frac{1}{100}\right)^n$

**28.** $r = \frac{\frac{7}{10,000}}{\frac{7}{10}} = \frac{1}{1000}, a_n = \frac{7}{100}\left(\frac{1}{1000}\right)^{n-1} = 7\left(\frac{1}{10}\right)^{3n-2}$

**29.** $r = \frac{0.05}{0.5} = 0.1, a_n = 0.5(0.1)^{n-1} = 5(0.1)^n$

**30.** $r = \frac{0.004}{0.4} = 0.01, a_n = 0.4(0.01)^{n-1} = 4(0.1)^{2n-1}$

**31.** $r = \frac{0.0045}{0.45} = 0.01,\ a_n = 0.45(0.01)^{n-1} = 45(0.01)^n$

**32.** $r = \frac{0.000234}{0.234} = 0.001,\ a_n = 0.234(0.001)^{n-1} = 234(0.001)^n$

**33.** $a_1 = 2, \quad a_5 = 162, \quad a_n = a_1 r^{n-1}$

$162 = 2r^{5-1}$

$r^4 = 81$

$r = 3$

$a_3 = 2(3)^{3-1} = 2 \cdot 9 = 18$

**34.** $a_3 = 1, \quad a_8 = \frac{1}{32}$

$\frac{1}{\frac{1}{32}} = \frac{a_1 r^2}{a_1 r^7}$

$32 = \frac{1}{r^5}$

$r^5 = \frac{1}{32}$

$r = \frac{1}{2}$

$1 = a_1\left(\frac{1}{2}\right)^2$

$a_1 = 4$

$a_4 = 4\left(\frac{1}{2}\right)^{4-1} = 4\left(\frac{1}{8}\right) = \frac{1}{2}$

**35.** $a_3 = \frac{4}{3}, \quad a_6 = -\frac{32}{81}$

$\frac{\frac{4}{3}}{-\frac{32}{81}} = \frac{a_1(r)^{3-1}}{a_1(r)^{6-1}}$

$\frac{-27}{8} = \frac{1}{r^3}$

$r^3 = -\frac{8}{27}$

$r = \frac{-2}{3}$

$\frac{4}{3} = a_1\left(-\frac{2}{3}\right)^{3-1}$

$a_1 = \frac{4}{3}\left(\frac{9}{4}\right) = 3$

$a_2 = 3\left(\frac{-2}{3}\right) = -2$

**36.** $a_4 = \frac{8}{9}, \quad a_7 = \frac{64}{243}$

$\frac{\frac{8}{9}}{\frac{64}{243}} = \frac{a_1 r^{4-1}}{a_1 r^{7-1}}$

$\frac{27}{8} = \frac{1}{r^3}$

$r^3 = \frac{8}{27}$

$r = \frac{2}{3}$

$\frac{8}{9} = a_1\left(\frac{2}{3}\right)^{4-1}$

$a_1 = \frac{8}{9} \cdot \frac{27}{8} = 3$

$a_5 = 3\left(\frac{2}{3}\right)^{5-1} = \frac{16}{27}$

**37.** $a_1 = 3, \quad a_2 = 9, \quad r = \frac{9}{3} = 3$

$S_5 = \frac{3\left(1-3^5\right)}{1-3} = \frac{3(-242)}{-2} = 363$

**38.** $a_1 = 2, \quad a_2 = 4, \quad r = \frac{4}{2} = 2$

$S_7 = \frac{2\left(1-2^7\right)}{1-2} = \frac{2(-127)}{-1} = 254$

**39.** $a_1 = \frac{2}{3}, \quad a_2 = \frac{4}{9}, \quad r = \frac{\frac{4}{9}}{\frac{2}{3}} = \frac{2}{3}$

$$S_6 = \frac{\frac{2}{3}\left[1-\left(\frac{2}{3}\right)^6\right]}{1-\frac{2}{3}} = \frac{\frac{2}{3}\left(\frac{665}{729}\right)}{\frac{1}{3}} = \frac{1330}{729}$$

**40.** $a_1 = \frac{4}{3},\ a_2 = \frac{16}{9},\ r = \frac{\frac{16}{9}}{\frac{4}{3}} = \frac{4}{3}$

$$S_{14} = \frac{\frac{4}{3}\left[1-\left(\frac{4}{3}\right)^{14}\right]}{1-\frac{4}{3}} = \frac{\frac{4}{3}\left(\frac{-263,652,487}{4,782,969}\right)}{-\frac{1}{3}} = \frac{1,054,609,948}{4,782,969} \approx 220.49$$

**41.** $a_1 = 1, a_2 = -\frac{2}{5}, r = -\frac{2}{5}$

$$S_9 = \frac{1\left[1-\left(-\frac{2}{5}\right)^9\right]}{1-\left(-\frac{2}{5}\right)} = \frac{\frac{1,953,637}{1,953,125}}{\frac{7}{5}} = \frac{279,091}{390,625}$$

**42.** $a_1 = 1, a_2 = -\frac{1}{3}, r = -\frac{1}{3}$

$$S_8 = \frac{1\left[1-\left(-\frac{1}{3}\right)^8\right]}{1-\left(-\frac{1}{3}\right)} = \frac{\frac{6560}{6561}}{\frac{4}{3}} = \frac{1640}{2187}$$

**43.** $a_1 = 1, a_2 = -2, r = -2$

$$S_7 = \frac{1[1-(-2)^{10}]}{1-(-2)} = -341$$

**44.** $a_1 = 2, a_2 = 10, r = 5$

$$S_8 = \frac{2[1-5^8]}{1-5} = 195,312$$

**45.** $a_1 = 5, a_2 = 15, r = 3$

$$S_{10} = \frac{5[1-3^{10}]}{1-3} = 147,620$$

**46.** $a_1 = 2, a_2 = -8, r = -4$

$$S_{11} = \frac{2[1-(-4)^{11}]}{1-(-4)} = 1,677,722$$

**47.** $a_1 = \frac{1}{3}, a_2 = \frac{1}{9}, r = \frac{1}{3}$

$$S = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

**48.** $a_1 = \frac{3}{4}, a_2 = \frac{9}{16}, r = \frac{3}{4}$

$$S = \frac{\frac{3}{4}}{1-\left(\frac{3}{4}\right)} = \frac{\frac{3}{4}}{\frac{1}{4}} = 3$$

**49.** $a_1 = -\frac{2}{3}, r = -\frac{2}{3}$

$$S = \frac{-\frac{2}{3}}{1-\left(-\frac{2}{3}\right)} = -\frac{\frac{2}{3}}{\frac{5}{3}} = -\frac{2}{5}$$

**50.** $a_1 = -\frac{3}{5}, r = -\frac{3}{5}$

$$S = \frac{-\frac{3}{5}}{1-\left(-\frac{3}{5}\right)} = \frac{-\frac{3}{5}}{\frac{8}{5}} = -\frac{3}{8}$$

**51.** $a_1 = \frac{9}{100}, r = \frac{9}{100}$

$$S = \frac{\frac{9}{100}}{1-\frac{9}{100}} = \frac{\frac{9}{100}}{\frac{91}{100}} = \frac{9}{91}$$

**52.** $a_1 = \frac{7}{10}, r = \frac{7}{10}$

$$S = \frac{\frac{7}{10}}{1-\frac{7}{10}} = \frac{\frac{7}{10}}{\frac{3}{10}} = \frac{7}{3}$$

**53.** $a_1 = 0.1, r = 0.1$

$$S = \frac{0.1}{1-0.1} = \frac{0.1}{0.9} = \frac{1}{9}$$

**54.** $a_1 = 0.4, r = 0.5$

$$S = \frac{0.5}{1-0.5} = \frac{0.5}{0.5} = 1$$

**55.** $a_1 = 1,\ r = -0.4$

$$S = \frac{1}{1-(-0.4)} = \frac{1}{1.4} = \frac{5}{7}$$

**56.** $a_1 = 1,\ r = -0.8$

$$S = \frac{1}{1-(-0.8)} = \frac{1}{1.8} = \frac{5}{9}$$

**57.** $0.\overline{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots$

$$a_1 = \frac{3}{10}, r = \frac{\frac{3}{100}}{\frac{3}{10}} = \frac{1}{10}$$

$$0.\overline{3} = \frac{\frac{3}{10}}{1-\frac{1}{10}} = \frac{\frac{3}{10}}{\frac{9}{10}} = \frac{1}{3}$$

**58.** $0.\overline{5} = \frac{5}{10} + \frac{5}{100} + \frac{5}{1000} + \cdots$

$$a_1 = \frac{5}{10}, r = \frac{\frac{5}{100}}{\frac{5}{10}} = \frac{1}{10}$$

$$0.\overline{5} = \frac{\frac{5}{10}}{1-\frac{1}{10}} = \frac{\frac{5}{10}}{\frac{9}{10}} = \frac{5}{9}$$

**59.** $0.\overline{45} = \frac{45}{100} + \frac{45}{10,000} + \frac{45}{1,000,000} + \cdots$

$$a_1 = \frac{45}{100}, r = \frac{\frac{45}{10,000}}{\frac{45}{100}} = \frac{1}{100}$$

$$0.\overline{45} = \frac{\frac{45}{100}}{1-\frac{1}{100}} = \frac{\frac{45}{100}}{\frac{99}{100}} = \frac{5}{11}$$

**60.** $0.\overline{63} = \frac{63}{100} + \frac{63}{10,000} + \frac{63}{1,000,000} + \cdots$

$$a_1 = \frac{63}{100}, r = \frac{\frac{63}{10,000}}{\frac{63}{100}} = \frac{1}{100}$$

$$0.\overline{63} = \frac{\frac{63}{100}}{1-\frac{1}{100}} = \frac{63}{99} = \frac{7}{11}$$

**61.** $0.\overline{123} = \frac{123}{100} + \frac{123}{1,000,000} + \frac{123}{1,000,000,000} + \cdots$

$$a_1 = \frac{123}{1000}, r = \frac{1}{1000}$$

$$0.\overline{123} = \frac{\frac{123}{1000}}{1-\frac{1}{1000}} = \frac{123}{999} = \frac{41}{333}$$

**62.** $0.3\overline{95} = \frac{3}{10} + \left[\frac{95}{1000} + \frac{95}{100,000} + \frac{95}{10,000,000} + \cdots\right]$

$$a_1 = \frac{95}{1000}, r = \frac{1}{100}$$

$$0.3\overline{95} = \frac{3}{10} + \frac{\frac{95}{1000}}{1-\frac{1}{100}} = \frac{3}{10} + \frac{95}{990} = \frac{392}{990} = \frac{196}{495}$$

**63.** $0.\overline{422} = \frac{422}{1000} + \frac{422}{1,000,000} + \frac{422}{1,000,000,000} + \cdots$

$a_1 = \frac{422}{1000}, r = \frac{1}{1000}$

$0.\overline{422} = \frac{\frac{422}{1000}}{1-\frac{1}{1000}} = \frac{422}{999}$

**64.** $0.\overline{355} = \frac{355}{1000} + \frac{355}{1,000,000} + \frac{355}{1,000,000,000} + \cdots$

$a_1 = \frac{355}{1000}, r = \frac{1}{1000}$

$0.\overline{355} = \frac{\frac{355}{1000}}{1-\frac{1}{1000}} = \frac{355}{999}$

**65.** $0.25\overline{4} = \frac{25}{100} + \left[\frac{4}{1,000} + \frac{4}{10,000} + \frac{4}{100,000} + \cdots\right]$

$a_1 = \frac{4}{1000}, r = \frac{1}{10}$

$0.25\overline{4} = \frac{25}{100} + \frac{\frac{4}{1000}}{1-\frac{1}{10}} = \frac{25}{100} + \frac{4}{900} = \frac{229}{900}$

**66.** $0.37\overline{2} = \frac{37}{100} + \left[\frac{2}{1,000} + \frac{2}{10,000} + \frac{2}{100,000} \cdots\right]$

$a_1 = \frac{2}{1000}, r = \frac{1}{10}$

$0.37\overline{2} = \frac{37}{100} + \frac{\frac{2}{1000}}{1-\frac{1}{10}} = \frac{37}{100} + \frac{2}{900} = \frac{335}{900} = \frac{67}{180}$

**67.** $1.20\overline{84} = 1 + \frac{2}{10} + \left[\frac{84}{10,000} + \frac{84}{1,000,000} + \frac{84}{100,000,000} + \cdots\right]$

$a_1 = \frac{84}{10,000}, r = \frac{1}{100}$

$1.20\overline{84} = 1 + \frac{2}{10} + \frac{\frac{84}{10,000}}{1-\frac{1}{100}} = \frac{12}{10} + \frac{7}{825} = \frac{1994}{1650} = \frac{997}{825}$

**68.** $02.25\overline{90} = \frac{225}{100} + \left[\frac{90}{10,000} + \frac{90}{1,000,000} + \frac{90}{100,000,000} \cdots\right]$

$a_1 = \frac{90}{10,000}, r = \frac{1}{100}$

$2.25\overline{90} = \frac{225}{100} + \frac{\frac{90}{10,000}}{1-\frac{1}{100}} = \frac{9}{4} + \frac{1}{110} = \frac{497}{220}$

**69.** $A = 100, i = 0.09, n = 2, r = 8, r = \frac{i}{n} = \frac{0.09}{2} = 0.049, m = nt = 2 \cdot 8 = 16$

$P = 100\frac{\left[(1+0.045)^{16} - 1\right]}{0.045} \approx 2271.93367$

The future value is \$2271.93.

**70.** $A = 250, i = 0.08, n = 12, t = 4, r = \frac{8}{12} = \frac{2}{3}, m = 12 \cdot 4 = 48$

$P = 250\frac{\left[\left(1+\frac{0.08}{12}\right)^{48} - 1\right]}{\frac{0.08}{12}} \approx \$14,087.48$

**71.** When a name was removed from the top of the list, the letter had been sent to

$5(5^5) = 15,625$ people

who sent 10 cents each for a total of

$0.10(15,625) = \$1562.50$ for each recipient**.**

**72.** $S_{11} = \frac{5(1-5^{11})}{1-5} = 61,035,155$

$S_{12} = \frac{5(1-5^{12})}{1-5} = 305,175,780$

For a population of 127,000,000, the entire population would receive the letter on the 12th level.

**73.** $A = 0.5,\ n = 4,\ k = -0.876,\ t = 4$

$A + Ae^{kt} + Ae^{2kt}Ae^{3kt}$

$= 0.5 + 0.5e^{-0.867(4)} + 0.5e^{2(-0.867)(4)} + 0.5e^{3(-0.867)(4)}$

$\approx 0.52$ mg

Or $A = 0.5,\ r = e^{kt},\ n = 4,\ k = -0.876,\ t = 4$

$S_4 = \frac{0.5(1-e^{-.867(4)(3)})}{1-e^{-.867(4)}} \approx 0.52$ mg

**74.** $A + Ae^{kt} + \cdots + Ae^{(n-1)kt} + \cdots = 2$

$A(1 + e^{kt} + \cdots + e^{(n-1)kt} + \cdots) = 2$

$\frac{A}{1-e^{kt}} = 2$

$A = 2(1-e^{kt})$

$A = 2(1-e^{-0.25(12)})$

$A \approx 1.90$ mg

**75.** Stock value $= \dfrac{D(1+g)}{i-g}$

$= \dfrac{1.87(1+0.15)}{0.20-0.15}$ $\quad D = 1.87,\ g = 0.15,\ i = 0.20$

$= \$43.01$

**76.** Stock value $= \dfrac{D(1+g)}{i-g}$

$67 = \dfrac{1.32(1+g)}{0.20-g}$ $\quad$ Solve for $g$.

$67(0.20-g) = 1.32(1+g)$

$13.4 - 67g = 1.32 + 1.32g$

$12.08 = 68.32g$

$0.1768 = g$

The dividend growth rather is 17.68%.

**77.** Stock value (no dividend growth) $= \dfrac{D}{i} = \dfrac{2.94}{0.15} = \$19.60$

**78.** Stock value (no dividend growth) $= \dfrac{D}{i}$

$16 = \dfrac{3.24}{i}$

$i = \dfrac{3.24}{16}$

$i = 0.2025$

The expected rate of return is 20.25%.

**79.** If $g$ was **not** less than $i$ in the Gordon model of stock valuation, the common ratio of the geometric sequence would be greater than 1 and the sum of the infinite geometric series would not be defined.

**80.** Using the multiplier effect,

$\dfrac{50}{1-0.90} = 500$

The net effect of \$50 million is \$500 million.

**81.** Using the multiplier effect,

$\dfrac{25}{1-0.75} = 100$

The net effect of \$25 million is \$100 million.

**82.** Using the multiplier effect,

$\dfrac{500{,}000}{1-0.40} \approx \$833{,}000$

About \$833,000 is used before it is removed.

## Connecting Concepts

**83.** Let $a_n$ be a geometric sequence. Thus

$a_n = a_1 r^{n-1}, a_1 \neq 0, \quad r \neq 0$

and $\log a_n = \log a_1 r^{n-1} = \log a_1 + \log r^{n-1}$

$= \log a_1 + (n-1)\log r$

Since $r$ is a constant, $\log r$ is a constant. Conjecture: The sequence $\log a_n$ is an arithmetic sequence. To prove this conjecture, we must show that $\log a_n - \log a_{n-1}$ is a constant.

$\log a_n - \log a_{n-1} = (\log a_1 + (n-1)\log r) - (\log a_1 + (n-2)\log r)$

$= \log r$

Since $\log r$ is a constant, the sequence $\log a_n$ is an arithmetic sequence.

**84.** Let $a_n$ be a geometric sequence. Thus

$a_n = a_1 + (n-1)d$ where $d$ is a constant.

$2^{a_n} = 2^{a_1+(n-1)d} = 2^{a_1} \cdot 2^{(n-1)d} = 2^{a_1} \cdot \left(2^d\right)^{n-1}$

Conjecture : The sequence $2^{a_n}$ is a geometric sequence. To prove this conjecture let $b_n = 2^{a_n}$ and show that

$\frac{b_n + 1}{b_n} = r$ where $r$ is a constant.

$$\frac{b_{n+1}}{b_n} = \frac{2^{a_1} \cdot \left(2^d\right)^n}{2^{a_1} \cdot \left(2^d\right)^{n-1}} = 2^d.$$

Because $d$ is a constant (see above), $2^d$ is a constant. Thus $2^{a_n}$ is a geometric sequence.

**85.** Yes. Because $x \neq 0$, the first term is 1 and the common ratio is $x$. If $|x| < 1$, the geometric series converges to $\frac{1}{1-x}$. If $|x| \geq 1$, the geometric series does not converge.

**86.** To find the area of the $n^{\text{th}}$ inscribed square, begin by finding the area of the first few squares. The area of the first square is $1^2 = 1$. Thus

$a_1 = 1$

The length of a side of the second square can be found by using the Pythagorean Theorem.

$$S^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Notice here that $S^2$ is the area of the second square. Thus

$a_2 = \frac{1}{2}$.

The length of the side of the third square is

$$S^2 = \left(\frac{1}{2\sqrt{2}}\right)^2 + \left(\frac{1}{2\sqrt{2}}\right)^2 = \frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{1}{4}$$

Again $S^2$ is the area of the square. Thus

$a_3 = \frac{1}{4}$.

From here we conjecture that the area of the $n^{\text{th}}$ square is

$a_n = \frac{1}{2^{n-1}}$.

**87.** If $a_n$ is a geometric sequence, then $a_n = a_1 r^{n-1}$

Since $a_n = ar^{n-1}$, then $a_1 = a$.

$$\begin{aligned} P_n &= a_1 \cdot a_2 \cdot a_3 \cdot \ \cdots \ \cdot a_n \\ &= a \cdot a_2 \cdot a_3 \cdot \ \cdots \ \cdot a_n \\ &= a \cdot ar \cdot ar^2 \cdot \ \cdots \ \cdot ar^{n-1} \\ &= a^n r^{[(n-1)n]/2} \end{aligned}$$

The exponent on $r$ is found by using the sum of an arithmetic series formula. Note that

$a \cdot ar \cdot ar^2 \cdot ar^3 \cdot \ \cdots \ \cdot ar^{n-1} = a^n \cdot r^{1+2+3+\cdots+n-1}$ and $1 + 2 + 3 + \ \cdots \ + n - 1 = \frac{(n-1)n}{2}$.

**88.** Let $a_n = f(n) = ab^n$. Thus

$$\frac{a_{n+1}}{a_n} = \frac{ab^{n+1}}{ab^n} = b$$

Since $b$ is a constant, $a_n$ is a geometric sequence.

**89.** The distance the ball travels each bounce is given by

$a_1 = 5$
$a_2 = 2(0.8)5$ Multiply by 2 for the distance up and down.
$a_3 = 2(0.8)(0.8)(5) = 2(0.8)^2 5$
$a_4 = 2(0.8)(0.8)^2 5 = 2(0.8)^3 5$
$\vdots$
$a_n = 2(0.8)^{n-1} \cdot 5$

This is a geometric sequence (after $a_1$). The sum of this sequence is the total distance travelled by the ball.

$$\frac{a_{n+1}}{a_n} = \frac{2(0.8)^n(5)}{2(0.8)^{n+1}(5)} = 0.8$$

Because $0.8 < 1$, the geometric series converges.

$$S = 5 + \frac{8}{1-0.8} = 5 + \frac{8}{0.2} = 5 + 40 = 45$$

The distance traveled is 45 feet. Note from our calculation that the geometric series begins with $a_2$. The first term $(a_1 = 5)$ is added to the series.

**90.** The distance of each swing of the bob is a term of a sequence

$a_1 = 30$
$a_2 = (0.9)(30)$
$a_3 = (0.9)(0.9)(30) = (0.9)^2(30)$
$\vdots$
$a_n = (0.9)^{n-1}(30)$

Because $\frac{a_{n+1}}{a_n} = \frac{(0.9)^n(30)}{(0.9)^{n-1}(30)} = 0.9$, a constant, the sequence is a geometric sequence. Since $0.9 < 1$, the infinite geometric series converges. The sum of this series is the total distance traveled by the bob.

$$S = \frac{30}{1-0.9} = \frac{30}{0.1} = 300$$

The bob traveled 300 inches.

**91.** The $n^{\text{th}}$ generation has $a_n = 2^n$ grandparents. Since this is a geometric sequence, the sum can be found by a formula.

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_{10} = \frac{2(1-2^{10})}{1-2} = \frac{2(1-1024)}{-1} = 2046$$

When $n = 1, a_n = 2$ and there are no grandparents. Therefore there are $2046 - 2 = 2044$ grandparents by the $10^{\text{th}}$ generation.

## Prepare for Section 11.4

**PS1.** $$\sum_{i=1}^{4} \frac{1}{i(i+1)} = \frac{1}{1(1+1)} + \frac{1}{2(2+1)} + \frac{1}{3(3+1)} + \frac{1}{4(4+1)}$$
$$= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{4}{5} = \frac{4}{4+1}$$

**PS2.** $$k(k+1)(2k+1) + 6(k+1)^2 = (k+1)[k(2k+1) + 6(k+1)]$$
$$= (k+1)[2k^2 + k + 6k + 6]$$
$$= (k+1)[2k^2 + 7k + 6]$$
$$= (k+1)(k+2)(2k+3)$$

**PS3.** $$\frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k+2}{k+2} \cdot \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$
$$= \frac{k^2 + 2k + 1}{(k+1)(k+2)} = \frac{(k+1)(k+1)}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

**PS4.** $1^2 \not> 2(1) + 1 = 3$
$2^2 \not> 2(2) + 1 = 5$
$3^2 > 2(3) + 1 = 7$
3

**PS5.** $$S_n + a_{n+1} = \frac{n(n+1)}{2} + n + 1$$
$$= \frac{n^2 + n}{2} + \frac{2n+2}{2} = \frac{n^2 + 3n + 2}{2}$$
$$= \frac{(n+1)(n+2)}{2}$$

**PS6.** $$S_n + a_{n+1} = 2^{n+1} - 2 + 2^{n+1}$$
$$= 2^{n+2} - 2$$
$$= 2(2^{n+1} - 1)$$

# Section 11.4

**1.** 1. Let $n = 1$. $S_1 = 3 \cdot \; - 2 = 1 = \frac{1(3 \cdot 1 - 1)}{2}$

2. Assume the statement is true for some positive integer $k$.

$S_k = 1 + 4 + 7 + \cdots + 3k - 2 = \frac{k(3k-1)}{2}$ (Induction Hypothesis)

Verify that the statement is true when $n = k + 1$

$S_{k+1} = \frac{(k+1)(3k+2)}{2}$.

$a_k = 3k - 2, a_{k+1} = 3k + 1$

$S_{k+1} = S_k + a_{k+1}$

$= \frac{k(3k-1)}{2} + 3k + 1 = \frac{3k^2 - k}{2} + \frac{6k+2}{2}$

$= \frac{3k^2 + 5k + 2}{2} = \frac{(k+1)(3k+2)}{2}$

By the Principle of Mathematical Induction, the statement is true for all positive integers.

**2.** 1. Let $n = 1$. $S_1 = 2 \cdot 1 = 2 = 1(1+1)$

2. Assume $S_k = 2 + 4 + 6 + \cdots + 2k = k(k+1)$ is true for some positive integer $k$ (Induction Hypothesis).

Verify that $S_{k+1} = (k+1)(k+2)$ is true when $n = k + 1$

$a_k = 2k, \quad a_{k+1} = 2(k+1)$

$S_{k+1} = S_k + a_{k+1} = k(k+1) + 2(k+1) = (k+1)(k+2)$

By the Principle of Mathematical Induction, the statement is true for all positive integers.

**3.** 1. Let $n = 1$. $S_1 = 1^3 = 1 = \frac{1^2(1+1)^2}{4}$

2. Assume $S_k = 1 + 8 + 27 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}$ is true for some positive integer $k$.(Induction Hypothesis).

Verify that $S_{k+1} = \frac{(k+1)^2(k+2)^2}{4}$.

$a_k = k^3, \; a_{k+1} = (k+1)^3$

$S_{k+1} = S_k + a_{k+1} = \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$

$= \frac{(k+1)^2(k^2 + 4k + 4)}{4} = \frac{(k+1)^2(k+2)^2}{4}$

By the Principle of Mathematical Induction, the statement is true for all positive integers.

**4.** 1. Let $n = 1$ $S_1 = 2^1 = 2 = 2(2^1 - 1)$

2. Assume that $S_k = 2 + 4 + 8 + \cdots + 2^k = 2(2k - 1)$ is true for some positive integer $k$ (Induction Hypothesis).

Verify that $S_{k+1} = 2\left(2^{k+1} - 1\right)$.

$a_k = 2^k, \quad a_{k+1} = 2^{k+1}$

$S_{k+1} = S_k + a_{k+1} = 2\left(2^k - 1\right) + 2^{k+1} = 2^{k+1} - 2 + 2^{k+1}$

$2 \cdot 2^{k+1} - 2 = 2\left(2^{k+1} - 1\right)$

By the Principle of Mathematical Induction, the statement is true for all positive integers.

**5.** 1. Let $n = 1$. $S_1 = 4 \cdot 1 - 1 = 3 = 1(2 \cdot 1 + 1)$

2. Assume that $S_k = 3 + 7 + 11 + \cdots + 4k - 1 = k(2k + 1)$ is true for some positive integer $k$ (Induction Hypothesis).

Verify that $S_{k+1} + (k+1)(2k+3)$ .

$a_k = 4k - 1, \quad a_{k+1} = 4k + 3$

$$S_{k+1} = S_k + a_{k+1}$$
$$= k(2k+1) + 4k + 3$$
$$= 2k^2 + 5k + 3 = (k+1)(2k+3)$$

By the Principle of Mathematical Induction, the statement is true for all positive integers.

**6.** 1. Let $n = 1$. $S_1 = 3^1 = \dfrac{3(3^1 - 1)}{2} = 3$

2. Assume that $S_k = 3 + 9 + 27 + \cdots + 3^k = \dfrac{3(3^k - 1)}{2}$ is true for some positive integer $k$ (Induction Hypothesis).

Verify that $S_{k+1} = \dfrac{3(3^{k+1} - 1)}{2}$ .

$a_k = 3^k, \quad a_{k+1} = 3^{k+1}$

$$S_{k+1} = S_k + a_{k+1} = \frac{3(3^k - 1)}{2} + 3^{k+1}$$
$$= \frac{3^{k+1} - 3 + 2 \cdot 3^{k+1}}{2} = \frac{3 \cdot 3^{k+1} - 3}{2} = \frac{3(3^{k+1} - 1)}{2}$$

By the Principle of Mathematical Induction, the statement is true for all positive integers.

**7.** 1. Let $n = 1$. $S_1 = (2 \cdot 1 - 1)^3 = 1 = 1^2(2 \cdot 1^2 - 1)$

2. Assume that $S_k = 1 + 27 + 125 + \cdots + (2k-1)^3 = k^2(2k^2 - 1)$ is true for some positive integer $k$ (Induction Hypothesis).

Verify that $S_{k+1} = (k+1)^2(2k^2 + 4k + 1)$ .

$a_k = (2k-1)^3, \quad a_{k+1} = (2k+1)^3$

$$S_{k+1} = S_k + a_{k+1} = k^2(2k^2 - 1) + (2k+1)^3 = 2k^4 - k^2 + 8k^3 + 12k^2 + 6k + 1$$
$$= 2k^4 + 8k^3 + 11k^2 + 6k + 1 = (k+1)(2k^3 + 6k^2 + 5k + 1)$$
$$= (k+1)^2(2k^2 + 4k + 1)$$

By the Principle of Mathematical Induction, the statement is true for all positive integers.

**8.** 1. Let $n = 1$. $S_1 = 1(1+1) = 2 = \dfrac{1(1+1)(1+2)}{3}$

2. Assume that $S_k = 2 + 6 + 12 + \cdots + k(k+1) = \dfrac{k(k+1)(k+2)}{3}$ is true for some positive integer $k$ (Induction Hypothesis).

Verify that $S_{k+1} = \dfrac{(k+1)(k+2)(k+3)}{3}$ .

$a_k = k(k+1), \quad a_{k+1} = (k+1)(k+2)$

$$S_{k+1} = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) = \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}$$
$$= \frac{(k+1)(k+2)(k+3)}{3}$$

By the Principle of Mathematical Induction, the statement is true for all positive integers.

**9.** 1. Let $n = 1$. $S_1 = \frac{1}{(2\cdot 1-1)(1\cdot 1+1)} = \frac{1}{3} = \frac{1}{2\cdot 1+1}$

2. Assume that $S_k = \frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$ for some positive integer $k$ (Induction Hypothesis).

Verify that $S_{k+1} = \frac{k+1}{2k+3}$.

$a_k = \frac{1}{(2k-1)(2k+1)}, \quad a_{k+1} = \frac{1}{(2k+1)(2k+3)}$

$S_{k+1} = \frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)} = \frac{2k^2+3k+1}{(2k+1)(2k+3)} = \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$

By the Principle of Mathematical Induction, the statement is true for all positive integers.

**10.** 1. Let $n = 1$. $S_1 = \frac{1}{2\cdot 1(2\cdot 1+2)} = \frac{1}{8} = \frac{1}{4(1+1)}$

2. Assume that $S_k = \frac{1}{2\cdot 4}+\frac{1}{4\cdot 6}+\frac{1}{6\cdot 7}+\cdots+\frac{1}{2k(2k+2)} = \frac{k}{4(k+1)}$ for some positive integer $k$ (Induction Hypothesis).

Verify that $S_{k+1} = \frac{k+1}{4(k+2)}$.

$a_k = \frac{1}{2k(2k+2)}, \quad a_{k+1} = \frac{1}{(2k+2)(2k+4)}$

$S_{k+1} = \frac{k}{4(k+1)}+\frac{1}{(2k+2)(2k+4)} = \frac{k}{4(k+1)}+\frac{1}{4(k+1)(k+2)} = \frac{k(k+2)+1}{4(k+1)(k+2)} = \frac{k^2+2k+1}{4(k+1)(k+2)}$

$= \frac{(k+1)^2}{4(k+1)(k+2)} = \frac{k+1}{4(k+2)}$

By the Principle of Mathematical Induction, the statement is true for all positive integers.

**11.** 1. Let $n = 1$. $S_1 = 1^4 = 1 = \frac{1(1+1)(2\cdot 1+1)(3\cdot 1^2+3\cdot 1-1)}{30} = \frac{2\cdot 3\cdot 5}{30} = 1$

2. Assume that $S_k = 1+16+81+\cdots+k^4 = \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30}$ for some positive integer $k$ (Induction Hypothesis).

Verify that $S_{k+1} = \frac{(k+1)(k+2)(2k+3)(3k^2+9k+5)}{30}$.

$a_k = k^4, \quad a_{k+1} = (k+1^4)$

$S_{k+1} = \frac{k(k+1)(2k+1)(3k^2+3k-1)}{30}+(k+1)^4$

$= \frac{(k+1)[k(2k+1)(3k^2+3k-1)+30(k+1)^3]}{30}$

$= \frac{(k+1)[6k^4+39k^3+91k^2+89k+30]}{30} = \frac{(k+1)(k+2)(6k^3+27k^2+37k+15)}{30}$

$= \frac{(k+1)(k+2)(2k+3)(3k^2+9k+5)}{30}$

By the Principle of Mathematical Induction, the statement is true for all positive integers.

**12.** 1. Let $n = 1$. $P_1 = \left(1 - \frac{1}{1+1}\right) = \frac{1}{2} = \frac{1}{1+1}$

2. Assume $P_k = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{k+1}\right) = \frac{1}{k+1}$ is true for some positive integer $k$ (Induction Hypothesis).

Verify $P_{k+1} = \frac{1}{k+2}$.

$$\begin{aligned} P_{k+1} &= \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{k+1}\right)\left(1-\frac{1}{k+2}\right) \\ &= P_k\left(1-\frac{1}{k+2}\right) \\ &= \frac{1}{k+1}\left(1-\frac{1}{k+2}\right) \\ &= \frac{1}{k+1} - \frac{1}{(k+1)(k+1)} = \frac{k+2-1}{(k+1)(k+2)} = \frac{k+1}{(k+1)(k+2)} = \frac{1}{k+2} \end{aligned}$$

By the Principle of mathematical Induction, the statement is true for all positive integers.

**13.** 1. Let $n = 4$. Then $\left(\frac{3}{2}\right)^4 = \frac{81}{16} = 5\frac{1}{16}; 4+1=5$

Thus, $\left(\frac{3}{2}\right)^n > n+1$ for $n = 4$.

2. Assume $\left(\frac{3}{2}\right)^k > k+1$ is true for some positive integer $k \geq 4$ (Induction Hypothesis).

Verify that $\left(\frac{3}{2}\right)^{k+1} > k+2$.

$$\left(\frac{3}{2}\right)^{k+1} = \left(\frac{3}{2}\right)^k\left(\frac{3}{2}\right) > (\mathrm{k}+1)\left(\frac{3}{2}\right) = \frac{1}{2}(3k+3) = \frac{1}{2}(2k+k+3) > \frac{1}{2}(2k+1+3) = k+2$$

Thus $\left(\frac{3}{2}\right)^{k+1} > k+2$. By the Principle of Mathematical Induction, $\left(\frac{3}{2}\right)^n > n+1$ for all $n \geq 4$.

**14.** 1. Let $n = 7$. $\left(\frac{4}{3}\right)^7 = \frac{16384}{2187} = 7\frac{1075}{2187} > 7$

Thus, $\left(\frac{4}{3}\right)^n > n$ for $n = 7$.

2. Assume $\left(\frac{4}{3}\right)^k > k$ is true for some positive integer $k \geq 7$ (Induction Hypothesis).

Verify that $\left(\frac{4}{3}\right)^{k+1} > k+1$.

$$\left(\frac{4}{3}\right)^{k+1} = \left(\frac{4}{3}\right)^k\left(\frac{4}{3}\right) > k\left(\frac{4}{3}\right) = \frac{1}{3}(4k) = \frac{1}{3}(3k+k) > \frac{1}{3}(3k+3) = k+1$$

Thus $\left(\frac{4}{3}\right)^{k+1} > k+1$. By the Principle of Mathematical Induction, $\left(\frac{4}{3}\right)^n > n$ for all $n \geq 7$.

**15.** 1. Let $n = 1$.

$0 < a < 1$

$0 < a \cdot a < a \cdot 1$

$a^{1+1} = a^2 < 1$

Thus, if $0 < a < 1$, then $a^{1+1} < a^n$ for $n = 1$.

2. Assume $a^{k+1} < a^k$ is true for some positive integer $k$, if $0 < a < 1$ (Induction Hypothesis).

Verify $a^{k+2} < a^{k+1}$.

$0 < a < 1$

$0 < a \cdot a^{k+1} < 1 \cdot a^{k+1}$

$a^{k+2} < a^{k+1}$

By the Principle of Mathematical Induction, if $0 < a < 1$, then $a^{n+1} < a^n$ for all positive integers.

**16.** 1. Let $n = 1$.

$a > 1$

$a \cdot a^1 > 1 \cdot a^1$

$a^{1+1} > a^1$

Thus, if $a > 1$, then $a^{n+1} > a^n$ for $n = 1$.

2. Assume that if $a > 1$, then $a^{k+1} > a^k$ for some positive integer $k$ (Induction Hypothesis).

Verify $a^{k+2} > a^{k+1}$.

$a > 1$

$a \cdot a^{k+1} > 1 \cdot a^{k+1}$

$a^{k+2} > a^{k+1}$

By the Principle of Mathematical Induction, if $a > 1$, then $a^{n+2} > a^{n+1}$ for all positive integers.

**17.** 1. Let $n = 4$. $1 \cdot 2 \cdot 3 \cdot 4 = 24$, $2^4 = 16$

Thus, $1 \cdot 2 \cdot 3 \cdot 4 > 2^n$ for $n = 4$.

2. Assume $1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot k > 2^k$ is true for some positive integer $k \ge 4$ (Induction Hypothesis).

Verify $1 \cdot 2 \cdot 3 \cdot \cdots \cdot k \cdot (k+1) > 2^{k+1}$.

$1 \cdot 2 \cdot 3 \cdot \cdots \cdot k \cdot (k+1) > 2^k (k+1) > 2^k \cdot 2 = 2^{k+1}$

Thus, $1 \cdot 2 \cdot 3 \cdot \cdots \cdot k \cdot (k+1) > 2^{k+1}$. By the Principle of Mathematical Induction, $1 \cdot 2 \cdot 3 \cdot \cdots \cdot n > 2^n$ for all $n \ge 4$.

**18.** 1. Let $n = 1$. $\frac{1}{\sqrt{1}} = 1 = \sqrt{1}$

Thus $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \ge \sqrt{n}$ for $n = 1$.

Assume $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} \ge \sqrt{k}$ is true for some positive integer $k$ (Induction Hypothesis).

Verify $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \ge \frac{1}{\sqrt{k+1}} > \sqrt{\text{k}+1}.$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \ge \sqrt{\text{k}} + \frac{1}{\sqrt{\text{k}+1}}$$

$$= \frac{\sqrt{k}\sqrt{k+1}}{\sqrt{k+1}} + \frac{1}{\sqrt{k+1}} > \frac{\sqrt{k}\sqrt{k}}{\sqrt{k+1}} + \frac{1}{\sqrt{k+1}}$$

$$= \frac{k+1}{\sqrt{k+1}} = \sqrt{k+1}$$

Thus, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$. By the Principle of Mathematical Induction, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \ge \sqrt{n}$ for all positive integers $n$.

**19.** 1. Let $n = 1$ and a > 0. $(1+a)^1 = 1+a = 1+1\cdot a$

Thus $(1+a)^n \geq 1+na$ for $n = 1$.

2. Assume $(1+a)^k \geq 1+ka$ is true for some positive integer $k$ (Induction Hypotheses).

Verify $(1+a)^{k+1} > 1+(k+1)a$.

$(1+a)^{k+1} = (1+a)^k(1+a) \geq (1+ka)(1+a) = 1+(k+1)a+ka^2 > 1+(k+1)a$

Thus $(1+a)^{k+1} > 1+(k+1)a$. By the Principle of Mathematical Induction, $(1+a)^n > 1+na$ for all positive integers $n$.

**20.** 1. Let $n = 1$. $\log_{10} 1 = 0 < 1$

Thus $\log_{10} n < n$ for $n = 1$.

2. Assume $\log_{10} k < k$ is true for some positive integer $k$ (Induction Hypothesis).

Verify $\log_{10}(k+1) < k+1$.

$\log_{10}(k+1) \leq \log_{10}(k+k) = \log_{10} 2k = \log_{10} 2 + \log_{10} k < 1+k$ because $\log_{10} 2 < 1$ and $\log_{10} k < k$.

Thus $\log_{10}(k+1) < k+1$. By the Principle of Mathematical Induction, $\log_{10} n < n$ for all positive integers $n$.

**21.** 1. Let $n = 1$. $1^2 + 1 = 2$, $2 = 2\cdot 1$

Thus 2 is a factor of $n^2 + n$ for $n = 1$.

2. Assume 2 is a factor of $k^2 + k$ for some positive integer $k$ (Induction Hypothesis).

Verify 2 is a factor of $(k+1)^2 + k + 1$.

$(k+1)^2 + k + 1 = (k+1)(k+1+1) = (k+1)(k+2)$

Since $k^2 + k = k(k+1)$, 2 is a factor of $k$ or $k + 1$.
If 2 is a factor of $k + 1$, then 2 is a factor of $(k+1)(k+2)$.
If 2 is a factor of $k$, then 2 is a factor of $k + 2$.

Thus, 2 is a factor of $(k+1)^2 + k + 1$. By the Principle of Mathematical Induction, 2 is a factor of $n^2 + n$ for all positive integers.

**22.** 1. Let $n = 1$. $1^3 - 1 = 0$, $0 = 0\cdot 3$

Thus, 3 is a factor of $n^3 - n$ for $n = 1$.

2. Assume 3 is a factor of $k^3 - k$ for some positive integer $k$ (Induction Hypothesis).

Verify 3 is a factor of $(k+1)^3 - (k+1)$ when $n = k + 1$.

$$(k+1)^3 - (k+1) = (k+1)[(k+1)^2 - 1] = (k+1)(k^2 + 2k)$$
$$= (k+1)(k+2)\ k = k(k+1)(k+2)$$

Since $k^3 - k = k(k+1)(k-1)$, then 3 is a factor of $k$, $k + 1$, or $k - 1$.
If 3 is a factor of $k$, then 3 is a factor of $k(k+1)(k+2)$.
If 3 is a factor of $k + 1$, then 3 is a factor of $k(k+1)(k+2)$.
If 3 is a factor of $k - 1$, then 3 is a factor of $k + 2$. Since $k + 2 = k - 1 + 3$, the sum of two multiples of 3 is a multiple of 3, so 3 is a factor of $k(k+1)(k+2)$.

Thus, 3 is a factor of $(k+1)^3 - (k+1)$. By the Principle of Mathematical Induction, 3 is a factor of $n^3 - n$ for all positive integers.

**23.** 1. Let $n = 1$. $5^1 - 1 = 4$, $4 = 4\cdot 1$

Thus, 4 is a factor of $5^n - 1$ for $n = 1$.

2. Assume 4 is a factor of $5^k - 1$ for some positive integer $k$ (Induction Hypothesis).

Verify 4 is a factor of $5^{k+1} - 1$.

Now $5^{k+1} - 1 = 5\cdot 5^k - 5 + 4 = 5\,(5^k - 1) + 4$ which is the sum of two multiples of 4.

Thus, 4 is a factor of $5^{k+1} - 1$. By the Principle of Mathematical Induction, 4 is a factor of $5^n - 1$ for all positive integers.

**24.** 1. Let $n = 1$. $6^1 - 1 = 5$, $5 = 5 \cdot 1$

Thus, 5 is a factor of $6^n - 1$ for $n = 1$.

2. Assume 5 is a factor of $6^k - 1$ for some positive integer $k$ (Induction Hypothesis).

Verify 5 is a factor of $6^{k+1} - 1$.

Now $6^{k+1} - 1 = 6 \cdot 6^k - 6 + 5 = 6(6^k - 1) + 5$ which is the sum of two multiples of 5.

Thus, 5 is a factor of $6^{k+1} - 1$. By the Principle of Mathematical Induction, 5 is a factor of $6^n - 1$ for all positive integers.

**25.** 1. Let $n = 1$. $(xy)^1 = xy = x^1 y^1$

Thus, $(xy)^n = x^n y^n$ for $n = 1$.

2. Assume $(xy)^k = x^k y^k$ is true for some positive integer $k$ (Induction Hypothesis).

Verify $(xy)^{k+1} = x^{k+1} y^{k+1}$.

$(xy)^{k+1} = (xy)^k (xy)^1 = x^k y^k \cdot xy = x^{k+1} y^{k+1}$

Thus $(xy)^{k+1} = x^{k+1} y^{k+1}$. By the Principle of Mathematical Induction, $(xy)^n = x^n y^n$ for all positive integers.

**26.** 1. Let $n = 1$. $\left(\frac{x}{y}\right)^1 = \frac{x}{y} = \frac{x^1}{y^1}$.

Thus, $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$ for $n = 1$.

2. Assume $\left(\frac{x}{y}\right)^k = \frac{x^k}{y^k}$ is true for some positive integer $k$ (Induction Hypothesis).

Verify $\left(\frac{x}{y}\right)^{k+1} = \frac{x^{k+1}}{y^{k+1}}$.

$\left(\frac{x}{y}\right)^{k+1} = \left(\frac{x}{y}\right)^k \left(\frac{x}{y}\right)^1 = \frac{x^k}{y^k} \cdot \frac{x}{y} = \frac{x^{k+1}}{y^{k+1}}$

Thus, $\left(\frac{x}{y}\right)^{k+1} = \frac{x^{k+1}}{y^{k+1}}$. By the Principle of Mathematical Induction, $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$ for all positive integers.

**27.** 1. Let $n = 1$. $a^1 - b^1 = a - b$

Thus $a - b$ is a factor of $a^n - b^n$ for $n = 1$.

2. Assume $a - b$ is a factor of $a^k - b^k$ for some positive integer $k$ (Induction Hypothesis).

Verify $a - b$ is a factor of $a^{k+1} - b^{k+1}$.

$a^{k+1} - b^{k+1} = (a \cdot a^k - ab^k) + (ab^k - b \cdot b^k) = a(a^k - b^k) + b^k(a - b)$

The sum of two multiples of $a - b$ is a multiple of $a - b$. Thus, $a - b$ is a factor of $a^{k+1} - b^{k+1}$. By the Principle of Mathematical Induction, $a - b$ is a factor of $a^n - b^n$ for all positive integers.

**28.** 1. Let $n = 1$. $a^{2\cdot 1+1} + b^{2\cdot 1+1} = a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

Thus, $a + b$ is a factor of $a^{2n+1} + b^{2n+1}$ for $n = 1$.

2. Assume $a + b$ is a factor of $a^{2k+1} + b^{2k+1}$ for some positive integer $k$ (Induction Hypothesis).

Verify $a + b$ is a factor of $a^{2k+3} + b^{2k+3}$.

$a^{2k+3} + b^{2k+3} = (a^{2k+2} + b^{2k+2})(a+b) - ab(a^{2k+1} + b^{2k+1})$

The sum of two multiples of $a + b$ is a multiple of $a + b$. Thus, $a + b$ is a factor of $a^{2k+3} + b^{2k+3}$. By the Principle of Mathematical Induction, $a + b$ is a factor of $a^{2n+1} + b^{2n+1}$ for all positive integers.

**29.** 1. Let $n = 1$. $ar^{1-1} = a \cdot 1 = a = \dfrac{a(1-r^1)}{1-r}$

Thus, the statement is true for $n = 1$.

2. Assume $\sum_{k=1}^{j} ar^{k-1} = \dfrac{a(1-r^j)}{1-r}$ is true for some positive integer $j$.

Verify $\sum_{k=1}^{j+1} ar^{k-1} = \dfrac{a(1-r^{j+1})}{1-r}$ is true for $n = j+1$.

$$\sum_{k=1}^{j+1} ar^{k-1} = \sum_{k=1}^{j}(ar^{k+1}-1) = \frac{a(1-r^j)}{1-r} + ar^j$$
$$= \frac{a(1-r^j) + ar^j(1-r)}{1-r} = \frac{a[1-r^j+r^j-r^{j+1}]}{1-r}$$
$$= \frac{a(1-r^{j+1})}{1-r}$$

By the Principle of Mathematical Induction, $\sum_{k=1}^{n} ar^{k-1} = \dfrac{a(1-r^n)}{1-r}$

**30.** 1. Let $n = 1$. $\sum_{k=1}^{1}(ak+b) = a+b$ and $\dfrac{1[(1+1)a+2b]}{2} = a+b$

Therefore, the statement is true for $n = 1$.

2. Assume the statement is true for $n = i$.

$\sum_{k=1}^{i}(ak+b)$ and $\dfrac{i[(i+1)a+2b]}{2}$

Prove the statement is true for $n = i+1$. That is, prove $\sum_{k=1}^{i+1}(ak+b) = \dfrac{(i+1)[(i+2)a+2b]}{2}$

$$\sum_{n=1}^{i+1}(ak+b) = \sum_{k=1}^{i}(ak+b) + a(i+1) + b$$
$$= \frac{i[(i+1)a+2b]}{2} + a(i+1) + b = \frac{i[(i+1)a+2b]+[2a(i+1)+2b]}{2}$$
$$= \frac{i(i+1)a+2bi+2a(i+1)+2b}{2} = \frac{i(i+1)a+2b(i+1)+2a(i+1)}{2}$$
$$= \frac{(i+1)[ai+2a+2b]}{2} = \frac{(i+1)[a(i+2)+2b]}{2}$$

Therefore, the statement is true for all positive integers $n$.

## Connecting Concepts

**31.** 1. If $N = 25$, then $\log 25! \approx 25.19 > 25$.

2. Assume $\log k! > k$ for $k > 25$ (Induction Hypothesis).

Prove $\log(k+1)! > k+1$.

$\log(k+1)! = \log[(k+1)k!] = \log(k+1) + \log k! > \log(k+1) + k$

Because $k > 25$, $\log(k+1) > 1$. Thus, $\log(k+1)! > k+1$.

Therefore, $\log n! > n$ for all $n > 25$.

**32.** 1. Let $\frac{a_{n+1}}{a_n} < r$ for $n \geq N$. We are to prove that $a_{N+k} < a_N r^k$ for each positive integer $k$.

When $k = 1$, we have $\frac{a_{N+1}}{a_N} < r$. Thus $a_{N+1} < a_N r$ which is true by the condition of the sequence.

Thus, the statement is true for $k$ = 1.

2. Assume that for some integer $k$, $a_{N+k} < a_N r^k$ (Induction Hypothesis).

Prove $a_{N+k+1} < a_N r^{k+1}$.

Since $\frac{a_{N+k+1}}{a_{N=k}} < r$ by the Induction Hypothesis, we have

$a_{N+k+1} < ra_{N+k} < r(a_N r^k) = a_N r^{k+1}$

Thus, the statement is true for all positive integers $n$.

**33.** 1. When $n$ = 1, we have $(x^m)^1 = x^m$ and $x^{m \cdot 1} = x^m$.

Therefore, the statement is true for $n$ = 1.

2. Assume the statement is true for $n = k$. That is, assume $(x^m)^k = x^{mk}$ (Induction Hypothesis).

Prove the statement is true for $n = k + 1$.

$x^{m(k+1)} = x^{mk+m} = x^{mk} \cdot x^m = (x^m)^k \cdot x^m = (x^m)^{k+1}$

Thus, the statement is true for all positive integers $n$ and $m$.

**34.** 1. When $n$ = 1,

$$\sum_{i=0}^{1} \frac{1}{i\,!} = \frac{1}{0\,!} + \frac{1}{1!} = 1 + 1 = 2$$

$$3 - \frac{1}{1} = 3 - 1 = 2$$

Thus, the statement is true for $n$ = 1.

2. Assume the statement is true for $n = k$. That is, assume $\sum_{i=0}^{k} \frac{1}{i\,!} \leq 3 - \frac{1}{k}$ (Induction Hypothesis).

Now prove the statement is true for $n = k + 1$.

$$\sum_{i=0}^{k+1} \frac{1}{i!} = \sum_{i=0}^{k} \frac{1}{i\,!} + \frac{1}{(k+1)!} \leq 3 - \frac{1}{k} + \frac{1}{(k+1)!}$$

Because $\frac{1}{(k+1)!} \leq \frac{1}{k(k+1)}, 3 - \frac{1}{k} + \frac{1}{(k+1)!} \leq 3 - \frac{1}{k} + \frac{1}{k(k+1)} = 3 - \frac{1}{(k+1)}$.

Thus, $\sum_{i=0}^{k+1} \frac{1}{i\,!} \leq 3 - \frac{1}{k+1}$.

The statement is true for all positive integers $n$.

**35.** 1. When $n = 3$, we have $\left(\frac{3+1}{3}\right)^3 = \left(\frac{4}{3}\right)^3 = \frac{64}{27} < 3.$

Thus the statement is true for $n = 3$.

2. Assume the statement is true for $n = k$. That is, assume $\left(\frac{k+1}{k}\right)^k < k$ (Induction Hypothesis).

Prove the statement is true for $n = k + 1$. That is, prove $\left(\frac{k+2}{k+1}\right)^{k+1} < k+1.$

We begin by noting that $\left(\frac{k+2}{k+1}\right) < \frac{k+1}{k}$. Therefore

$$\left(\frac{k+2}{k+1}\right)^{k+1} < \left(\frac{k+1}{k}\right)^{k+1} = \left(\frac{k+1}{k}\right)^k \left(\frac{k+1}{k}\right)$$

By the Induction Hypothesis, $\left(\frac{k+1}{k}\right)^k < k$; thus

$$\left(\frac{k+1}{k}\right)^k \left(\frac{k+1}{k}\right) < k\left(\frac{k+1}{k}\right) = k+1$$

We now have $\left(\frac{k+2}{k+1}\right)^{k+1} < k+1$. The induction is complete.

Thus $\left(\frac{n+1}{n}\right)^n < n$ is true for all $n \ge 3$.

## Prepare for Section 11.5

**PS1.** $(a+b)^3 = (a+b)(a+b)(a+b)$
$= a^3 + 3a^2b + 3ab^2 + b^3$

**PS2.** $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$

**PS3.** $0! = 1$

**PS4.** $\frac{6!}{2!(6-2)!} = \frac{720}{2(24)} = 15$

**PS5.** $\frac{7!}{3!(7-3)!} = \frac{5040}{6(24)} = 35$

**PS6.** $\frac{10!}{10!(10-10)!} = \frac{3,628,800}{3,628,000(1)} = 1$

## Section 11.5

**1.** $\binom{7}{4} = \frac{7!}{4!(7-4)!} = \frac{7\cdot 6\cdot 5\cdot 4!}{4!(3\cdot 2\cdot 1)} = 35$

**2.** $\binom{8}{6} = \frac{8!}{6!(8-6)!} = \frac{8\cdot 7\cdot 6!}{6!\cdot 2\cdot 1} = 28$

**3.** $\binom{9}{2} = \frac{9!}{2!(9-2)!} = \frac{9\cdot 8\cdot 7!}{2\cdot 1\cdot 7!} = 36$

**4.** $\binom{10}{5} = \frac{10!}{5!(10-5)!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5!}{5!\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = 252$

**5.** $\binom{12}{9} = \frac{12!}{9!(12-9)!} = \frac{12\cdot 11\cdot 10\cdot 9!}{9!\cdot 3\cdot 2\cdot 1} = 220$

**6.** $\binom{6}{5} = \frac{6!}{5!(6-5)!} = \frac{6\cdot 5!}{5!\cdot 1} = 6$

**7.** $\binom{11}{0} = \frac{11!}{0!(11-0)!} = \frac{11!}{1\cdot 11!} = 1$

**8.** $\binom{14}{14} = \frac{14!}{14!(14-14)!} = \frac{14!}{14!\cdot 1} = 1$

**9.** $(x-y)^6 = \binom{6}{0}x^6 + \binom{6}{1}x^5(-y) + \binom{6}{2}x^4(-y)^2 + \binom{6}{3}x^3(-y)^3 + \binom{6}{4}x^2(-y)^4 + \binom{6}{5}x(-y)^5 + \binom{6}{6}(-y)^6$
$= x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6$

**10.** $(a-b)^5=\binom{5}{0}a^5+\binom{5}{1}a^4(-b)+\binom{5}{2}a^3(-b)^2+\binom{5}{3}a^2(-b)^3+\binom{5}{4}a(-b)^4+\binom{5}{5}(-b)^5$
$=a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$

**11.** $(x+3)^5=\binom{5}{0}x^5+\binom{5}{1}x^4\cdot 3+\binom{5}{2}x^3\cdot 3^2+\binom{5}{3}x^2\cdot 3^3+\binom{5}{4}x\cdot 3^4+\binom{5}{5}3^5$
$=x^5+15x^4+90x^3+270x^2+405x+243$

**12.** $(x-5)^4=\binom{4}{0}x^4+\binom{4}{1}x^3(-5)+\binom{4}{2}x^2(-5)^2+\binom{4}{3}x(-5)^3+\binom{4}{4}(-5)^4$
$=x^4-20x^3+150x^2-500x+625$

**13.** $(2x-1)^7=\binom{7}{0}(2x)^7+\binom{7}{1}(2x)^6(-1)+\binom{7}{2}(2x)^5(-1)^2+(-1)^2+\binom{7}{3}(2x)^4(-1)^3+\binom{7}{4}(2x)^3(-1)^4+\binom{7}{5}(2x)^2(-1)^5$
$+\binom{7}{6}(2x)(-1)^6+\binom{7}{7}(-1)^7$
$=128x^7-448x^6+672x^5-560x^4+280x^3-84x^2+14x-1$

**14.** $(2x+y)^6=\binom{6}{0}(2x)^6+\binom{6}{1}(2x)^5y+\binom{6}{2}(2x)^4y^2+\binom{6}{3}(2x)^3y^3+\binom{6}{4}(2x)^2y^4+\binom{6}{5}(2x)y^5+\binom{6}{6}y^6$
$=64x^6+192x^5y+240x^4y^2+160x^3y^3+60x^2y^4+12xy^5+y^6$

**15.** $(x+3y)^6=\binom{6}{0}x^6+\binom{6}{1}x^5(3y)+\binom{6}{2}x^4(3y)^2+\binom{6}{3}x^3(3y)^3+\binom{6}{4}x^2(3y)^4+\binom{6}{5}x(3y)^5+\binom{6}{6}(3y)^6$
$=x^6+18x^5y+135x^4y^2+540x^3y^3+1215x^2y^4+1458xy^5+729y^6$

**16.** $(x-4y)^5=\binom{5}{0}x^5+\binom{5}{1}x^4(-4y)+\binom{5}{2}x^3(-4y)^2+\binom{5}{3}x^2(-4y)^3+\binom{5}{4}x(-4y)^4+\binom{5}{5}(-4y)^5$
$=x^5-20x^4y+160x^3y^2-640x^2y^3+1280xy^4-1024y^5$

**17.** $(2x-5y)^4=\binom{4}{0}(2x)^4+\binom{4}{1}(2x)^3(-5y)+\binom{4}{2}(2x)^2(-5y)^2+\binom{4}{3}(2x)(-5y)^3+\binom{4}{4}(-5y)^4$
$=16x^4-160x^3y+600x^2y^2-1000xy^3+625y^4$

**18.** $(3x+2y)^4=\binom{4}{0}(3x)^4+\binom{4}{1}(3x)^2(2y)+\binom{4}{2}(3x)^2(2y)^2+\binom{4}{3}(2x)(2y)^3+\binom{4}{4}(2y)^4$
$=81x^4+216x^3y+216x^2y^2+96xy^3+16y^4$

**19.** $\left(x+\frac{1}{x}\right)^6=\binom{6}{0}x^6+\binom{6}{1}x^5\left(\frac{1}{x}\right)+\binom{6}{2}x^4\left(\frac{1}{x}\right)^2+\binom{6}{3}x^3\left(\frac{1}{x}\right)^3+\binom{6}{4}x^2\left(\frac{1}{x}\right)^4+\binom{6}{5}x\left(\frac{1}{x}\right)^5+\binom{6}{6}\left(\frac{1}{x}\right)^6$
$=x^6+6x^4+15x^2+20+\frac{15}{x^2}+\frac{6}{x^4}+\frac{1}{x^6}$

**20.** $\left(2x-\sqrt{y}\right)^7=\binom{7}{0}(2x)^7+\binom{7}{1}(2x)^6\left(-\sqrt{y}\right)+\binom{7}{2}(2x)^5\left(-\sqrt{y}\right)^2+\binom{7}{3}(2x)^4\left(-\sqrt{y}\right)^3+\binom{7}{4}(2x)^3\left(-\sqrt{y}\right)^4+\binom{7}{5}(2x)^2\left(-\sqrt{y}\right)^5$
$+\binom{7}{6}(2x)\left(-\sqrt{y}\right)^6+\binom{7}{7}\left(-\sqrt{y}\right)^7$
$=128x^7-448x^6\sqrt{y}+672x^5y-560x^4y\sqrt{y}+280x^3y^2-84x^2y^2\sqrt{y}+14xy^3-y^3\sqrt{y}$

**21.** $\left(x^2-4\right)^7=\binom{7}{0}\left(x^2\right)^7+\binom{7}{1}(2x)^6(-4)+\binom{7}{2}\left(x^2\right)^5(-4)^2+\binom{7}{3}\left(x^2\right)^4(-4)^3+\binom{7}{4}\left(x^2\right)^3(-4)^4+\binom{7}{5}\left(x^2\right)^2(-4)^5$
$+\binom{7}{6}\left(x^2\right)(-4)^6+\binom{7}{7}(-4)^7$
$=x^{14}-28x^{12}+336x^{10}-2240x^8+8960x^6-21504x^4+28672x^2-16384$

**22.** $\left(x-y^3\right)^6=\binom{6}{0}x^6+\binom{6}{1}x^5\left(-y^3\right)+\binom{6}{2}x^4\left(-y^3\right)^2+\binom{6}{3}x^3\left(-y^3\right)^3+\binom{6}{4}x^2\left(-y^3\right)^4+\binom{6}{5}x\left(-y^3\right)^5+\binom{6}{6}\left(-y^3\right)^6$

$=x^6-6x^5y^3+15x^4y^6-20x^3y^9+15x^2y^{12}-6xy^{15}+y^{18}$

**23.** $\left(2x^2+y^3\right)^5=\binom{5}{0}\left(2x^2\right)^5+\binom{5}{1}\left(2x^2\right)^4\left(y^3\right)+\binom{5}{2}\left(2x^2\right)^3\left(y^3\right)^2+\binom{5}{3}\left(2x^2\right)^2\left(y^3\right)^3+\binom{5}{4}\left(2x^2\right)\left(y^3\right)^4+\binom{5}{5}\left(y^3\right)^5$

$=32x^{10}+80x^8y^3+80x^6y^6+40x^4y^9+10x^2y^{12}+y^{15}$

**24.** $\left(2x-y^3\right)^6=\binom{6}{0}(2x)^6+\binom{6}{1}(2x)^5(-y^3)+\binom{6}{2}(2x)^4(-y^3)^2+\binom{6}{3}(2x)^3(-y^3)^3+\binom{6}{4}(2x)^2(-y^3)^4+\binom{6}{5}(2x)(-y^3)^5+\binom{6}{6}(-y^3)^6$

$=64x^6-192x^5y^3+240x^4y^6-160x^3y^9+60x^2y^{12}-12xy^{15}+y^{18}$

**25.** $\left(\frac{2}{x}-\frac{x}{2}\right)^4=\binom{4}{0}\left(\frac{2}{x}\right)^4+\binom{4}{1}\left(\frac{2}{x}\right)^3\left(-\frac{x}{2}\right)+\binom{4}{2}\left(\frac{2}{x}\right)^2\left(-\frac{x}{2}\right)^2+\binom{4}{3}\left(\frac{2}{x}\right)\left(-\frac{x}{2}\right)^3+\binom{4}{4}\left(-\frac{x}{2}\right)^4=\frac{16}{x^4}-\frac{16}{x^2}+6-x^2+\frac{x^4}{16}$

**26.** $\left(\frac{a}{b}+\frac{b}{a}\right)^3=\binom{3}{0}\left(\frac{a}{b}\right)^3+\binom{3}{1}\left(\frac{a}{b}\right)^2\left(\frac{b}{a}\right)+\binom{3}{2}\left(\frac{a}{b}\right)\left(\frac{b}{a}\right)^2+\binom{3}{3}\left(\frac{b}{a}\right)^3=\frac{a^3}{b^3}+\frac{3a}{b}+\frac{3b}{a}+\frac{b^3}{a^3}$

**27.** $\left(s^{-2}+s^2\right)^6=\binom{6}{0}\left(s^{-2}\right)^6+\binom{6}{1}\left(s^{-2}\right)^5\left(s^2\right)+\binom{6}{2}\left(s^{-2}\right)^4\left(s^2\right)^2+\binom{6}{3}\left(s^{-2}\right)^3\left(s^2\right)^3+\binom{6}{4}\left(s^{-2}\right)^2\left(s^2\right)^4+\binom{6}{5}\left(s^{-2}\right)\left(s^2\right)^5+\binom{6}{6}\left(s^2\right)^6$

$=s^{-12}+6s^{-8}+15s^{-4}+20+15s^4+6s^8+s^{12}$

**28.** $\left(2r^{-1}+s^{-1}\right)^5=\binom{5}{0}\left(2r^{-1}\right)^5+\binom{5}{1}\left(2r^{-1}\right)^4\left(s^{-1}\right)+\binom{5}{2}\left(2r^{-1}\right)^3\left(s^{-1}\right)^2+\binom{5}{3}\left(2r^{-1}\right)^2\left(s^{-1}\right)^3+\binom{5}{4}\left(2r^{-1}\right)\left(s^{-1}\right)^4+\binom{5}{5}\left(s^{-1}\right)^5$

$=32r^{-5}+80r^{-4}s^{-1}+80r^{-3}s^{-2}+40r^{-2}s^{-3}+10r^{-1}s^{-4}+s^{-5}$

**29.** eighth term is $\binom{10}{7}(3x)^3(-y)^7=-3240x^3y^7$

**30.** fourth term is $\binom{12}{3}x^9(2y)^3=1760x^9y^3$

**31.** third term is $\binom{12}{2}x^{10}(4y)^2=1056x^{10}y^2$

**32.** thirteenth term is $\binom{14}{12}(2x)^2(-1)^{12}=364x^2$

**33.** fifth term is $\binom{9}{4}\left(\sqrt{x}\right)^5\left(-\sqrt{y}\right)^4=126x^2y^2\sqrt{x}$

**34.** sixth term is $\binom{10}{5}\left(x^{-1/2}\right)^5\left(x^{1/2}\right)^5=252$

**35.** ninth term is $\binom{11}{8}\left(\frac{a}{b}\right)^3\left(\frac{b}{a}\right)^8=\frac{165b^5}{a^5}$

**36.** seventh term is $\binom{13}{6}\left(\frac{3}{x}\right)^7\left(-\frac{x}{3}\right)^6=\frac{5148}{x}$

**37.** $\binom{n}{i-1}a^{n-(i-1)}b^{i-1}$, if $b^{i-1}=b^8$, then $i=9$.

ninth term is $\binom{10}{8}(2a)^2(-b)^8=180a^2b^8$

**38.** eighth term is $\binom{9}{7}(3r)^2(2s)^7=41{,}472r^2s^7$

**39.** $\left(y^2\right)^{i-1}=y^8$, if $2i-2=8$, then $2i=10$ or $i=5$.

fifth term is $\binom{6}{4}(2x)^2\left(y^2\right)^4=60x^2y^8$

**40.** $\left(b^3\right)^{i-1}=b^9$, if $3i-3=9$, then $3i=12$ or $i=4$

fourth term is $\binom{8}{3}a^5\left(-b^3\right)^3=-56a^5b^9$

**41.** sixth term is $\binom{10}{5}(3a)^5(-b)^5 = -61{,}236a^5b^5$

**42.** fifth term is $\binom{8}{4}(a)^4\left(b^2\right)^4 = 70a^4b^8$

**43.** fifth term is $\binom{9}{4}\left(s^{-1}\right)^5(s)^4 = 126s^{-1}$

sixth term is $\binom{9}{5}\left(s^{-1}\right)^4(s)^5 = 126s$

**44.** fourth term is $\binom{7}{3}\left(x^{1/2}\right)^4\left(-y^{1/2}\right)^3 = -35x^2y^{3/2}$

fifth term is $\binom{7}{4}\left(x^{1/2}\right)^3\left(-y^{1/2}\right)^4 = 35x^{3/2}y^2$

**45.** $$(2-i)^4 = \binom{4}{0}\left(2^4\right) + \binom{4}{1}(2)^3(-i)^1 + \binom{4}{2}(2)^2(-i)^2 + \binom{4}{3}2(-i)^3 + \binom{4}{4}(-i)^4$$
$$= 16 + 32(-i) + 24(-1) + 8(-i)^3 + 1$$
$$= 16 - 32i - 24 + 8i + 1$$
$$= -7 - 24i$$

**46.** $$(3+2i)^3 = \binom{3}{0}(3)^3 + \binom{3}{1}(3)^2(2i)^1 + \binom{3}{2}(3)(2i)^2 + \binom{3}{3}(2i)^3$$
$$= 27 + 54i - 36 - 8i$$
$$= -9 + 46i$$

**47.** $$(1+2i)^5 = \binom{5}{0}(1)^5 + \binom{5}{1}(1)^4(2i)^1 + \binom{5}{2}(1)^2(2i)^2 + \binom{5}{3}(1)^2(2i)^3 + \binom{5}{4}(1)(2i)^4 + \binom{5}{5}(2i)^5$$
$$= 1 + 10i - 40 - 80i + 80 + 32i$$
$$= 41 - 38i$$

**48.** $$(1-3i)^5 = \binom{5}{0}(1)^5 + \binom{5}{1}(1)^4(-3i)^1 + \binom{5}{2}(1)^3(-3i)^2 + \binom{5}{3}(1)^2(-3i)^3 + \binom{5}{4}1(-3i)^4 + \binom{5}{5}(-3i)^5$$
$$= 1 - 15i - 90 + 270i + 405 - 243i = 316 + 12i$$

**49.** $$\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^8 = \left(\frac{\sqrt{2}}{2}\right)^8(1+i)^8 + \frac{1}{16}(1 + 8i - 28 - 56i + 70 + 56i - 28 - 8i + 1) = \frac{1}{16}(16) = 1$$

**50.** $$\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^6 = \left(\frac{1}{2}\right)^6\left(1 + i\sqrt{3}\right)^6 = \frac{1}{64}\left(1 + 6i\sqrt{3} - 45 - 60i\sqrt{3} + 135 + 54i\sqrt{3} - 27\right) = \frac{1}{64}(64) = 1$$

## Connecting Concepts

**51.** $$\frac{(x+h)^n - x^n}{h} = \frac{x^n + nx^{n-1}h + \dfrac{n(n-1)x^{n-2}h^2}{2} + \dfrac{n(n-1)(n-2)x^{n-3}h^3}{6} + \cdots + h^n - x^n}{h}$$
$$= nx^{x-1} + \frac{(n)(n-1)x^{n-2}h}{2} + \frac{n(n-1)(n-2)x^{n-3}h^2}{6} + \cdots + h^{n-1}$$

**52.** $$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad \binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k!}$$

Thus $\binom{n}{k} = \binom{n}{n-k}$

**53.** $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$. With $x = 1$ and $y = 1$, we have $(1+1)^n = \sum_{k=0}^{n}\binom{n}{k}(1)^{n-k}(1)^k$

$$2^n = \sum_{k=0}^{n}\binom{n}{k}$$

**54.** $\binom{n}{k}+\binom{n}{k+1}=\frac{n!}{k!(n-k)!}+\frac{n!}{(k+1)!(n-(k+1))!}$
$=\frac{n!(k+1)+n!(n-k)}{(k+1)!(n-k)!}$
$=\frac{n!k+n!+n!n-n!k}{(k+1)!(n-k)!}$
$=\frac{n!(n+1)}{(k+1)!(n-k)!}$
$=\frac{(n+1)!}{(k+1)!(n+1-(k+1))!}=\binom{n+1}{k+1}$

**55.** Let $x = 1$ and $y = -1$ in the expansion of $(x+y)^n$.

$$(x+y)^n=\sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^i$$

$$(1+(-1))^n=0^n=0=\sum_{i=0}^{n}\binom{n}{i}1^{n-i}(-1)^i=\sum_{i=0}^{n}(-1)^i\binom{n}{i}$$

Thus $\sum_{i=0}^{n}(-1)^i\binom{n}{i}=0$

**56.** $(0.98)^8=(1-0.02)^8$

The sum of the first three terms is $\binom{8}{0}1^8+\binom{8}{1}1^7(-0.02)^1+\binom{8}{2}1^6(-0.02)^2=1^8+8(-0.02)+28(-0.02)^2$

$(0.98)^8\approx 1-0.16+0.0112=0.8512$

**57.** $(1.02)^8=(1+0.02)^8=\sum_{8-i}^{8}\binom{8}{i}(1)^{8-1}(0.02)^i$

The sum of the first three terms is

$\binom{8}{0}(0.02)^0+\binom{8}{1}(0.02)^1+\binom{8}{2}(0.02)^2=1+0.16+0.0112$

$(1.02)^8\approx 1.1712$

**58.** $\frac{10!}{2!3!5!}=2520$

**59.** $\frac{9!}{5!2!2!}=756$

**60.** $\frac{9!}{4!5!0!}=126$

**61.** $\frac{8!}{3!5!0!}=56$

## Prepare for Section 11.6

**PS1.** $7!=7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1=5040$

**PS2.** $(7-3)!=4!=4\cdot3\cdot2\cdot1=24$

**PS3.** $\binom{7}{1}=\frac{7!}{1!(7-1)!}=\frac{7\cdot6!}{1(6!)}=7$

**PS4.** $\binom{8}{5}=\frac{8!}{5!(8-5)!}=\frac{8\cdot7\cdot6\cdot5!}{5!(3\cdot2\cdot1)}=56$

**PS5.** $\frac{10!}{(10-2)!}=\frac{10\cdot9\cdot8!}{8!}=90$

**PS6.** $\frac{6!}{(6-6)!}=\frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{0!}=720$

## Section 11.6

**1.** $P(6,2)=\frac{6!}{(6-2)!}=\frac{6\cdot5\cdot4!}{4!}=30$

**2.** $P(8,7)=\frac{8!}{(8-7)!}=\frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{1}=40{,}320$

**3.** $C(8,4)=\frac{8!}{4!(8-4)!}=\frac{8\cdot7\cdot6\cdot5\cdot4!}{4!\cdot4\cdot3\cdot2\cdot1}=70$

**4.** $C(9,2)=\frac{9!}{2!(9-2)!}=\frac{9\cdot8\cdot7!}{2\cdot1\cdot7!}=36$

**5.** $P(8,0)=\frac{8!}{(8-0)!}=\frac{8!}{8!}=1$

**6.** $P(9,9)=\frac{9!}{(9-9)!}=\frac{9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{1}=362{,}880$

**7.** $C(7,7)=\frac{7!}{7!(7-7)!}=\frac{7!}{7!\cdot1}=1$

**8.** $C(6,0)=\frac{6!}{0!(6-0)!}=\frac{6!}{1\cdot6!}=1$

**9.** $C(10,4)=\frac{10!}{4!(10-4)!}=\frac{10\cdot9\cdot8\cdot7\cdot6!}{4\cdot3\cdot2\cdot1\cdot6!}=210$

**10.** $P(10,4)=\frac{10!}{(10-4)!}=\frac{10\cdot9\cdot8\cdot7\cdot6!}{6!}=5040$

**11.** Use the counting principle.
$3 \cdot 2 \cdot 2 = 12$
There are 12 different possible computer systems.

**12.** Use the counting principle.
$4 \cdot 4 \cdot 4 \cdot 4 = 256$
256 possible colors could be formed, assuming that each palate must be used each time.

**13.** Use the counting principle.
$2 \cdot 2 \cdot 2 \cdot 2 = 16$
There are 16 possible light switch configurations.

**14.** Use the counting principle.
$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7 = 128$
There are 128 different positive integers that can be stored.

**15.** $P(6,6) = 6! = 720$

**16.** $P(12,3) = 1320$

**17.** $5 \cdot 5 \cdot 5 = 125$ ways

**18.** $C(9,3) = 84$

**19.** Use the combination formula with $n = 25$, $r = 5$.
$C(25,5) = \frac{25!}{5!(20)!} = 53{,}130$

**20.** Use the combination formula with $n = 26$, $r = 2$.
$C(26,2) = \frac{26!}{2!23!} = 325$

**21.** There are 676 ways to arrange 26 letters taken two at a time ($26 \cdot 26 = 676$). Now if there are more than 676 employees, then at least two employees have the same first and last initials.

**22.** $C(4,2) \cdot P(3,3) \cdot P(3,3) = 216$

**23.** $C(6,3) \cdot C(8,3) = 1120$

**24.** **a.** $3! \cdot 3! = 6 \cdot 6 = 36$
**b.** $6 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 72$

**25.** $2^{10} = 1024$

**26.** $4^{20}$

**27.** $C(40,6) = 3{,}838{,}380$

**28.** $C(10,8) = 45$

**29.** **a.** $C(7,5) = 21$
**b.** $C(7,4) \cdot C(3,1) = 35 \cdot 3 = 105$
**c.** $C(7,2) \cdot C(3,3) = 21 \cdot 1 = 21$

**30.** **a.** The number of ways 10 finalists can be selected from 15 semifinalists is the combination of 15 students selected 10 at a time.
$C(15,10) = 3003$
There are 3003 ways the finalists can be chosen.

**b.** The number of ways the 10 finalists can contain three seniors is the product of the combination of 7 seniors selected 3 at a time and the combination of 8 remaining students selected 7 at a time.
$C(7,3)C(8,7) = 35 \cdot 8 = 280$
There are 280 ways the finalists can contain 3 seniors.

**c.** At least five seniors means 5 or 6 or 7 seniors are finalists (there are only 7 seniors). Since the events are related by “or,” sum the number of ways each event can occur.
$C(7,3) \cdot C(8,5) + C(7,6) \cdot C(8,3) = 21 \cdot 56 + 7 \cdot 70 + 56 = 1176 + 490 + 56 = 1722$
There are 1722 ways the finalists can contain at least 5 seniors.

**31.** $3 \cdot 12 \cdot 5 \cdot 10^7 = 1.8 \times 10^9$

**32.** **a.** $C(13,5) = 1287$
**b.** $4 \cdot C(13,5) = 5148$
**c.** $C(48,2) \cdot C(4,3) = 4512$
**d.** $C(4,4) \cdot C(48,1) + C(4,3) \cdot C(48,2) + C(4,2) \cdot C(48,3) = 108{,}336$

**33.** $C(10,3) - C(8,1) = 120 - 8 = 112$

**34.** **a.** $C(12,5) = 792$
**b.** $C(10,3) = 120$

**35.** $P(5,5) = 120$

**36.** $7 \cdot 6 = 42$

**37.** $C(7,2) = 21$

**38.** $C(12,2) = 66$

**39.** $\frac{16 \cdot 14}{2} = 112$

**40.** $C(12,3) = 220$

**41.** $C(20,10) = 184{,}756$

**42.** $C(15,8)\cdot P(8,3)=6{,}435\cdot 336=2{,}162{,}160$

**43.** $C(20,12)\cdot C(12,4)=125{,}970\cdot 495=62{,}355{,}150$

**44.** A total of 14 moves are required, 7 to the right and 7 down.

$C(14,\ 7)=\dfrac{14!}{7!7!}=3432$

**45.** A triple-decker cone could have all one flavor ice cream, or two different flavors with one scoop of the first flavor and two scoops of the second flavor (such as one scoop of vanilla and two scoops of chocolate), or two different flavors with two scoops of the first flavor and one scoop of the second flavor (such as two scoops of vanilla and one scoop of chocolate), or three different flavors.

$$\begin{aligned} C(31,\ 1)+C(31,\ 2)+C(31,\ 2)+C(31,\ 3) &= \frac{31!}{1!30!}+\frac{31!}{2!29!}+\frac{31!}{2!29!}+\frac{31!}{3!28!} \\ &= 31 + 465 + 465 + 4495 \\ &= 5456 \end{aligned}$$

**46.** $$\begin{aligned}\left(2^{16}\right)^{1024\cdot 768} &= 2^{16\cdot 1024\cdot 768} \\ &= 2^{12{,}582{,}912}\end{aligned}$$

**47.** 19!

## Connecting Concepts

**48.** There are $n$ ways to choose the first point and $n-1$ ways to choose the second point. Thus there are $n(n-1)$ ways to choose both points. Since the direction of the line is not important, the number of lines is

$\dfrac{n(n-1)}{2}$ or $\dbinom{n}{2}$

**49.** **a.** $12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6=3{,}991{,}680$

**b.** $12^7-12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6=31{,}840{,}128$

**50.** $C(4,\ 1)+C(4,\ 2)+C(4,\ 3)+C(4,\ 4)=15$ different sums

**51.** Let $a_1,\ a_2,...,\ a_5$ be the long pieces and $b_1,\ b_2,...,\ b_5$ be the short pieces. The pairs must have one $a$ with one $b$. For $a_1$ there are 5 $b$'s, for $a_2$ there are 4 $b$'s, … . Thus there are $5\cdot 4\cdot 3\cdot 2\cdot 1=120$ pairs consisting of a long piece and a short piece.

**52.** We use the following: (Number of numbers with no repetitions) plus (Number of numbers with two or more repetitions) equals $10^4$. Number of numbers with no repetitions $=10\cdot 9\cdot 8\cdot 7$. Thus the number of numbers with two or more repetitions $=10^4-10\cdot 9\cdot 8\cdot 7=4960$.

**53.** To return to the original spot, the tourist must toss an equal number of heads and tails. This is $C(10, 5) = 252$. There are 252 different toss combinations that return the tourist to the origin.

## Prepare for Section 11.7

**PS1.** See 11.6.

**PS2.** Use the counting principle.
$4\cdot 3=12$
There are 12 different possible two-digit numbers.

**PS3.** $P(7,2)=\dfrac{7!}{(7-2)!}=\dfrac{7\cdot 6\cdot 5!}{5!}=42$

**PS4.** $C(7,2)=\dfrac{7!}{2!(7-2)!}=\dfrac{7\cdot 6\cdot 5!}{2!\cdot 5!}=21$

**PS5.** $\dbinom{8}{5}\left(\dfrac{1}{4}\right)^5\left(\dfrac{3}{4}\right)^{8-5}=\dfrac{8!}{5!(8-5)!}\left(\dfrac{1}{4}\right)^5\left(\dfrac{3}{4}\right)^3=\dfrac{189}{8192}$

**PS6.** Use the counting principle.
$2\cdot 2\cdot 2\cdot 2=16$
There are 16 possible light switch configurations.

# Section 11.7

**1.** Label senators $S_1$, $S_2$ and representatives $R_1$, $R_2$, $R_3$.
The sample space is $S = \{S_1R_1,\ S_1R_2,\ S_1R_3,\ S_2R_1,\ S_2R_2,\ S_2R_3,\ R_1R_2,\ R_1R_3,\ R_2R_3,\ S_1S_2\}$

**2.** $S = \{\text{T, e, n, s}\}$

**3.** Label coin $H$, $T$ and integers 1, 2, 3, 4.
$S = \{H1,\ H2,\ H3,\ H4,\ T1,\ T2,\ T3,\ T4\}$

**4.** $S = \{HHHH, THHH, HTHH, HHHT, TTHH, THHT, HTTH, HHTT, THTH, HTHT, HHTH, HTTT, TTTH, THTT, TTHT, TTTT\}$

**5.** Let the three cans be represented by $A$, $B$, and $C$ and $(x, y)$ represent the cans that balls 1 and 2 are placed in; e.g., $(A, B)$ means ball 1 is in can $A$ and ball 2 is in can $B$.

$S = \{(A,A),\ (A,B),\ (A,C),\ (B,A),\ (B,B),\ (B,C),\ (C,A),\ (C,B),\ (C,C)\}$

**6.** $S = \{RD, RI, DI\}$

**7.** $S = \{HSC, HSD, HCD, SCD\}$

**8.** Let $(L_1, L_2, L_3)$ denote letter $L_1$ in envelope $A$, letter $L_2$ in envelope $B$, letter $L_3$ in envelope $C$. Then the sample space is
$\{(ABC),(ACB),(BAC),(BCA),(CAB),(CBA)\}$

**9.** $S = \{\text{ae,ai,ao,au,ei,eo,eu,io,iu,ou}\}$

**10.** $S = \{DN_1N_2, DN_1N_3, DN_2N_3, N_1N_2N_3\}$

**11.** $E = \{HHHH\}$

**12.** $E = \{TTHH, THHT, HTTH, HHTT, THTH, HTHT\}$

**13.** $E = \{TTTT, HTTT, THTT, TTHT, TTTH, TTHH, THTH, HTHT, THHT, HTTH, HHTT\}$

**14.** $E = \{HHHH, HHHT, HHTH, HTHH, THHH\}$

**15.** $E = \varnothing$

The sample space $S$ for the events in Exercises 16—20 is
$S = \{(1,1),\ (1,2),\ (1,3),\ (1,4),\ (1,5),\ (1,6),\ (2,1),\ (2,2),\ (2,3),\ (2,4),\ (2,5),\ (2,6),\ (3,1),\ (3,2),\ (3,3),\ (3,4),\ (3,5),\ (3,6),$
$(4,1),\ (4,2),\ (4,3),\ (4,4),\ (4,5),\ (4,6),\ (5,1),\ (5,2),\ (5,3),\ (5,4),\ (5,5),\ (5,6),\ (6,1),\ (6,2),\ (6,3),\ (6,4),\ (6,5),\ (6,6)\}$

**16.** $E = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

**17.** $E = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$

**18.** $E = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3),(5,1),(5,2),(5,3),(5,4),(6,1),(6,2),(6,3),(6,4),(6,5)\}$

**19.** $E = \{(1,4),(2,4),(3,4),(4,4),(5,4),(6,4)\}$

**20.** $E = S = \{(1,1),\ (1,2),\ (1,3),\ (1,4),\ (1,5),\ (1,6),\ (2,1),\ (2,2),\ (2,3),\ (2,4),\ (2,5),\ (2,6),\ (3,1),\ (3,2),\ (3,3),\ (3,4),\ (3,5),\ (3,6),$
$(4,1),\ (4,2),\ (4,3),\ (4,4),\ (4,5),\ (4,6),\ (5,1),\ (5,2),\ (5,3),\ (5,4),\ (5,5),\ (5,6),\ (6,1),\ (6,2),\ (6,3),\ (6,4),\ (6,5),\ (6,6)\}$

**21.** **a.** $P(\text{king}) = \dfrac{4}{52} = \dfrac{1}{13}$

**b.** $P(\text{spade}) = \dfrac{13}{52} = \dfrac{1}{4}$

**22.** $P(\text{even}) + P(\div 3) - P(\text{even and} \div 3) = \dfrac{3}{6} + \dfrac{2}{6} - \dfrac{1}{6} = \dfrac{5}{6} - \dfrac{1}{6} = \dfrac{2}{3}$

**23.** $P(\text{increase GNP}) + P(\text{increase inflation}) - P(\text{increase GNP and inflation}) = 0.64 + 0.55 - 0.22 = 0.97$

**24.** $P(\text{greater than 3000}) = \dfrac{2 \cdot 4 \cdot 4 \cdot 4}{4^4} = \dfrac{128}{156} = \dfrac{1}{2}$

**25.** $P(\text{1st}) + P(\text{2nd}) - P(\text{1st and 2nd}) = \dfrac{1}{2} + \dfrac{1}{5} - \dfrac{1}{10} = \dfrac{6}{10} = \dfrac{3}{5}$

**26.** **a.** There are $C(10, 2) = 45$ ways in which 2 calculators can be randomly chosen from 10 calculators. There is only one way in which 2 defective calculators can be chosen from 2 defective calculators. Therefore, the probability is $\frac{1}{45}$.

**b.** At least 1 defective calculator means 1 defective calculator or 2 defective calculators. We first calculate the probability of 1 defective calculator in a sample of 2 defective calculators. The number of ways of choosing 1 defective calculator from 2 is $C(2, 1) = 2$. Because there are two calculators in the sample and 1 is defective, the other calculator is not defective. There are $C(8, 1)$ ways of selecting 1 calculator that is not defective. The probability of choosing 1 defective calculator is

$$\frac{C(2,1)C(8,1)}{C(10,2)} = \frac{2 \cdot 8}{45} = \frac{16}{45}$$

From part (a), the probability of 2 defective calculators is $\frac{1}{45}$. Therefore, the probability of at least 1 defective calculator is $\frac{16}{45} + \frac{1}{45} = \frac{17}{45}$

**27.** Because sampling is with replacement, the events are independent. On one trial, the probability of not selecting a 0 is $\frac{9}{10}$. Therefore, the probability of not selecting a 0 on five trials is

$$\left(\frac{9}{10}\right)^5 = 0.59.$$

**28.** The number of ways in which 6 people can be arranged in a line is $P(6, 6) = 720$. To determine how many ways 2 people $A$ and $B$ can be together, consider the number of arrangements of ($AB$), $C$, $D$, $E$, $F$. (We have grouped $A$ and $B$ together and considered arranging 5 items.) Since there is the same number of arrangements with $BA$ (instead of $AB$), the number of arrangements is $2 \cdot P(5,5) = 240$. Therefore, the probability of any 2 people being together is

$$\frac{240}{720} = \frac{1}{3}.$$

**29.** To receive at least \$50, an envelope with \$50 in cash or \$100 in cash must be selected. The probability is

$$\frac{75}{500} + \frac{50}{500} = \frac{125}{500} = \frac{1}{4} = 0.25.$$

**30.** The number of possible juries that can be chosen is $C(30,12) = 86,493,225$. It is possible to choose 6 women from 15 in $C(15,6) = 5005$ ways. It is possible to choose 6 men from 15 in $C(15, 6) = 5005$ ways. Therefore, the probability of 6 men and 6 women is

$$\frac{C(15,6)C(15,6)}{C(30,12)} = \frac{5005 \cdot 5005}{86493225} \approx 0.2896.$$

**31.** There are $P(6, 6) = 720$ seating arrangements for the 6 children. There are $2 \cdot 3! \cdot 3!$ ways to have boys and girls alternate. Therefore the probability of boys and girls alternating is

$$\frac{2 \cdot 3! \cdot 3!}{720} = \frac{72}{720} = \frac{1}{10} = 0.1.$$

**32.** Four committee members can be selected from 8 people in $C(8, 4) = 70$ ways. There are $C(3, 2) = 3$ ways to choose 2 accountants from 3 and $C(5, 2) = 10$ ways to choose 2 actuaries from 5. The probability of 2 accountants and 2 actuaries is $\frac{C(3,2)C(5,2)}{C(8,4)} = \frac{3 \cdot 10}{70} = \frac{3}{7}$

**33.** The subject can select $C(5, 2) = 10$ different sets of 2 cards. The magician must name the set the subject has drawn. Therefore, the probability that a magician can guess the answers is $\frac{1}{10}$ or 0.1.

**34.** Yes. The probability of drawing an ace from a regular deck of playing cards is $\frac{4}{52} = \frac{1}{13}$. Because the card is replaced before the second draw, the probability of drawing an ace on the second draw is also $\frac{1}{13}$. Therefore, the probability of drawing two aces with replacement is $\frac{1}{13} \cdot \frac{1}{13} = \frac{1}{169}$.

**35.** The probability of choosing Monday is $\frac{1}{5}$. The probability of choosing 8:00 A.M. is $\frac{1}{8}$. Therefore the probability of choosing Monday at 8:00 A.M. is

$$\frac{1}{5} \cdot \frac{1}{8} = \frac{1}{40} \text{ or } 0.025.$$

**36.** Calculate the probability that the first radar system detects the missile but the second radar system does not plus the probability that the second but not the first detects the missile plus the probability that both detect the missile.
Probability $= 0.95(0.05) + 0.05(0.95) + 0.95(0.95)$
$= 0.9975$

**37.** The probability of at least one unprofitable
$= 1 -$ probability of all profitable
$= 1 - [(0.10)(0.10)(0.10)(0.10)]$
$= 1 - 0.0001 = 0.9999$

**38.** The probability that a single CD-ROM is not defective is $\frac{999}{1000} = 1 - \frac{1}{1000}$. The probability that three are not defective is $\left(\frac{999}{1000}\right)^3 \approx 0.997$.

**39.** Assuming there is no preference, then the probability of choosing program $A$ is $\frac{1}{2}$. The probability of all four companies choosing program $A$ is $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$.

**40.** This is a binomial probability with $n = 8$, $k = 3.$, $p = \frac{1}{4}$, and $q = \frac{3}{4}$.
Probability $= \binom{8}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^5 \approx 0.2076$

**41.** The probability of at least one defective equals 1 minus the probability of no defectives. The probability of no defectives is $(0.95)^5$. Therefore the probability of at least one defective is $1-(0.95)^5 \approx 0.2262$

**42.** The probability of winning a prize is 1 minus the probability of not winning a prize. The probability of not winning a prize is
$\frac{C(998,10)}{C(1000,10)} \approx \frac{2.58\times10^{23}}{2.63\times10^{23}} \approx 0.98$.
Therefore the probability of winning a prize is 1 – 0.98, or 0.02.

**43.** This is a binomial probability with $p = \frac{3}{4}$, $q = \frac{1}{4}$, $n = 25$, and $k = 21, 22, 23, 24$, and $25$.

$$P = \binom{25}{21}\left(\frac{3}{4}\right)^{21}\left(\frac{1}{4}\right)^4 + \binom{25}{22}\left(\frac{3}{4}\right)^{22}\left(\frac{1}{4}\right)^3 + \binom{25}{23}\left(\frac{3}{4}\right)^{23}\left(\frac{1}{4}\right)^2 + \binom{25}{24}\left(\frac{3}{4}\right)^{24}\left(\frac{1}{4}\right)^1 + \binom{25}{25}\left(\frac{3}{4}\right)^{25}$$

The probability is approximately 0.2137.

**44.** In a two-engine plane, safe flight occurs when one or two engines operate. The probability of this event is equal to
1 – probability that both engines fail $= 1 - [(0.03)(0.03)] = 1 - 0.0009 = 0.9991$.
For a four-engine plane, safe flight occurs when two, three, or four engines operate. The probability of this event is equal to

$$\binom{4}{2}(0.03)^2(0.97)^2 + \binom{4}{1}(0.03)(0.97)^3 + \binom{4}{0}(0.97)^4 \approx 0.9999$$

This probability is higher than that of two engines. Therefore, a plane that has four engines is safer.

## Connecting Concepts

**45.** Let the originator tell a member of the club. (Rumor told once.) This person repeats it to any of 7 remaining members with probability $\frac{7}{8}$. (Rumor told twice.) That person repeats it to any of 7 members with probability $\frac{7}{8}$. Probability of both events is $\left(\frac{7}{8}\right)^2$.

**46.** After the cards are shuffled, number the cards from 1 to 10, Let $\{c_1, c_2, c_3, c_4, c_5\}$ be the numbers of the cards the subject said were white. For example, {1, 5, 7, 9, 10} means that the subject named those cards white. The sample space is all possible five-element subsets. Thus, $N(S) = C(10, 5)$. The event $E$ is that the subject named exactly 8 correctly. Then $N(E) = C(5,4)\cdot C(5,1)$.
$P(E) = \frac{5\cdot5}{252} = \frac{25}{252}$

**47.** Let $a$, $b$, c, $d$ be the first number chosen. Then event $E$ is choosing a second number such that the first digit is not $a$, the second digit is not $b$, the third digit is not $c$, and the fourth digit matches the third digit. Since there are 9 digits available (digits 1 through 9), the probability that the first digit is not $a$ is $\frac{8}{9}$, the probability that the second digit is not $b$ is $\frac{8}{9}$, the probability that the third digit is not $c$ is $\frac{8}{9}$, and the probability that the fourth digit matches the third is $\frac{1}{9}$.

Thus, $P(E) = \frac{8}{9}\cdot\frac{8}{9}\cdot\frac{8}{9}\cdot\frac{1}{9} = \frac{512}{6561} \approx 0.078$.

**48.** Arrangements of Tennessee $= \frac{9!}{1!4!2!2!} = 3780$

Arrangements in which $T$ occurs first $= \frac{8!}{4!2!2!} = 430$

Probability $T$ occurs first $= \frac{420}{3780} = \frac{1}{9}$

**49.** Recall from Section 11.3 the Sum of a Geometric Series

$$S = \frac{a_1}{1-r}$$

$$S = \frac{p^2}{1-2p(1-p)} \quad a_1 = p^2,\ r = 2p(1-p)$$

$$S = \frac{(0.55)^2}{1-2(0.55)(1-0.45)}$$

$$S \approx 0.599$$

The probability is 0.599.

**50.** Recall from Section 11.3 the Sum of a Geometric Series

$$S = \frac{a_1}{1-r}$$

$$S = \frac{\frac{5}{36}}{1-\frac{11}{36}} \quad a_1 = \frac{5}{36},\ r = \frac{11}{36}$$

$$S = 0.2$$

The probability is 0.2.

## Exploring Concepts with Technology

**Mathematical Expectation**

Casino 1

| Mark | Catch | Win | Expectation |
|---|---|---|---|
| 6 | 4 | \$8 | \$.23 |
| 6 | 5 | \$176 | \$.54 |
| 6 | 6 | \$2960 | \$.38 |
| | | | \$1.15 |

Casino 2

| Mark | Catch | Win | Expectation |
|---|---|---|---|
| 6 | 4 | \$6 | \$.17 |
| 6 | 5 | \$160 | \$.50 |
| 6 | 6 | \$3900 | \$.50 |
| | | | \$1.17 |

Casino 3

| Mark | Catch | Win | Expectation |
|---|---|---|---|
| 6 | 4 | \$8 | \$.23 |
| 6 | 5 | \$180 | \$.56 |
| 6 | 6 | \$3000 | \$.39 |
| | | | \$1.18 |

Casino 4

| Mark | Catch | Win | Expectation |
|---|---|---|---|
| 6 | 4 | \$6 | \$.17 |
| 6 | 5 | \$176 | \$.54 |
| 6 | 6 | \$3000 | \$.39 |
| | | | \$1.10 |

Each mathematical expectation was determined using the formula

$$\text{Mathematical expectation} = \frac{C(20,\ c)\cdot C(60,\ 6-c)}{C(80,\ 6)}W$$

where $W$ is the number of dollars you win for $c$ catches.

Thus, casino 3 offers the greatest mathematical expectation. For each \$2 bet at casino 3, the gambler has a mathematical expectation of winning \$1.18.

## Assessing Concepts

1. $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$

2. An arithmetic sequence is one in which the difference between any two successive terms is the same constant.

3. A geometric sequence is one in which the ratio of any two successive terms is the same constant.

4. A sequence is an ordered list of numbers. A series is the sum of the terms of a sequence.

5. The $n$th partial sum of a sequence is the sum of the first $n$ terms of the sequence.

6. A permutation takes into consideration the order of the elements of a list. A combination does not.

7. No

8. See the Chapter 11 Summary under Section 11.4.

9. The Binomial Theorem for Positive Integers is used to expand $(a+b)^n$, where $n$ is a positive integer.

10. 0

11. $0 \le P \le 1$

12. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

## Chapter Review

1. $a_n = n^2$ [11.1]
$a_3 = 3^2 = 9$
$a_7 = 7^2 = 49$

2. $a_n = n!$ [11.1]
$a_3 = 3! = 6$
$a_7 = 7! = 5040$

3. $a_n = 3n + 2$ [11.1]
$a_3 = 11$
$a_7 = 23$

4. $a_n = 1 - 2n$ [11.1]
$a_3 = 1 - 2(3) = 1 - 6 = -5$
$a_7 = 1 - 2(7) = 1 - 14 = -13$

5. $a_n = 2^{-n}$ [11.1]
$a_3 = 2^{-3} = \frac{1}{8}$
$a_7 = 2^{-7} = \frac{1}{128}$

6. $a_n = 3n$ [11.1]
$a_3 = 3^3 = 27$
$a_7 = 3^7 = 2187$

7. $a_n = \frac{1}{n!}$ [11.1]
$a_3 = \frac{1}{3!} = \frac{1}{6}$
$a_7 = \frac{1}{7!} = \frac{1}{5040}$

8. $a_n = \frac{1}{n}$ [11.1]
$a_3 = \frac{1}{3}$
$a_7 = \frac{1}{7}$

9. $a_n = \left(\frac{2}{3}\right)^n$ [11.1]
$a_3 = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$
$a_7 = \left(\frac{2}{3}\right)^7 = \frac{128}{2187}$

10. $a_n = \left(-\frac{4}{3}\right)^n$ [11.1]
$a_3 = \left(-\frac{4}{3}\right)^3 = -\frac{64}{27}$
$a_7 = \left(-\frac{4}{3}\right)^7 = -\frac{16,384}{2187}$

11. $a_1 = 2, a_n = 3a_{n-1}$ [11.1]
$a_2 = 3a_1 = 3 \cdot 2 = 6$
$a_3 = 3a_2 = 3 \cdot 6 = 18$ •
$a_4 = 3a_3 = 3 \cdot 18 = 54$
$a_5 = 3a_4 = 3 \cdot 54 = 162$
$a_6 = 3a_5 = 3 \cdot 162 = 486$
$a_7 = 3a_6 = 3 \cdot 486 = 1458$ •

12. $a_1 = -1, a_n = 2a_{n-1}$ [11.1]
$a_2 = 2a_1 = 2(-1) = -2$
$a_3 = 2a_2 = 2(-2) = -4$ •
$a_4 = 2a_3 = 2(-4) = -8$
$a_5 = 2a_4 = 2(-8) = -16$
$a_6 = 2a_5 = 2(-16) = -32$
$a_7 = 2a_6 = 2(-32) = -64$ •

**13.** $a_1 = 1, a_n = -na_{n-1}$ [11.1]

$a_2 = -2a_1 = -2 \cdot 1 = -2$

$a_3 = -3a_2 = -3 \cdot (-2) = 6$ •

$a_4 = -4a_3 = -4 \cdot 6 = -24$

$a_5 = -5a_4 = -5 \cdot (-24) = 120$

$a_6 = -6a_5 = -6 \cdot 120 = -720$

$a_7 = -7a_6 = -7 \cdot (-720) = 5040$ •

**14.** $a_1 = 2, a_n = n^2 a_{n-1}$ [11.1]

$a_2 = 2^2 a_1 = 2^2 \cdot 2 = 8$

$a_3 = 3^2 a_2 = 9 \cdot 8 = 72$ •

$a_4 = 4^2 a_3 = 16 \cdot 72 = 1152$

$a_5 = 5^2 a_4 = 25 \cdot 1152 = 28{,}800$

$a_6 = 6^2 a_5 = 36 \cdot 28{,}800 = 1{,}036{,}800$

$a_7 = 7^2 a_6 = 49 \cdot 1{,}036{,}800 = 50{,}803{,}200$ •

**15.** $a_1 = 4, a_n = a_{n-1} + 2$ [11.1]

$a_2 = a_1 + 2 = 4 + 2 = 6$

$a_3 = a_2 + 2 = 6 + 2 = 8$ •

$a_4 = a_3 + 2 = 8 + 2 = 10$

$a_5 = a_4 + 2 = 10 + 2 = 12$

$a_6 = a_5 + 2 = 12 + 2 = 14$

$a_7 = a_6 + 2 = 14 + 2 = 16$ •

**16.** $a_1 = 3, a_n = a_{n-1} - 3$ [11.1]

$a_2 = a_1 - 3 = 3 - 3 = 0$

$a_3 = a_2 - 3 = 0 - 3 = -3$ •

$a_4 = a_3 - 3 = -3 - 3 = -6$

$a_5 = a_4 - 3 = -6 - 3 = -9$

$a_6 = a_5 - 3 = -9 - 3 = -12$

$a_7 = a_6 - 3 = -12 - 3 = -15$ •

**17.** $a_1 = 1, a_2 = 2, a_n = a_{n-1}a_{n-2}$ [11.1]

$a_3 = a_2 \cdot a_1 = 2 \cdot 1 = 2$ •

$a_4 = a_3 \cdot a_2 = 2 \cdot 2 = 4$

$a_5 = a_4 \cdot a_3 = 4 \cdot 2 = 8$

$a_6 = a_5 \cdot a_4 = 8 \cdot 4 = 32$

$a_7 = a_6 \cdot a_5 = 32 \cdot 8 = 256$ •

**18.** $a_1 = 1, a_2 = 2, a_n = \dfrac{a_n - 1}{a_n - 2}$ [11.1]

$a_3 = \dfrac{a_2}{a_1} = \dfrac{2}{1} = 2$ •

$a_4 = \dfrac{a_3}{a_2} = \dfrac{2}{2} = 1$

$a_5 = \dfrac{a_4}{a_3} = \dfrac{1}{2}$

$a_6 = \dfrac{a_5}{a_4} = \dfrac{1/2}{1} = \dfrac{1}{2}$

$a_7 = \dfrac{a_6}{a_5} = \dfrac{1/2}{1/2} = 1$ •

**19.** $a_1 = -1, a_n = 3_n a_{n-1}$ [11.1]

$a_2 = 3(2)a_1 = 3(2)(-1) = -6$

$a_3 = 3(3)a_2 = 3(3)(-6) = -54$ •

$a_4 = 3(4)a_3 = 3(4)(-54) = -648$

$a_5 = 3(5)a_4 = 3(5)(-648) = -9720$

$a_6 = 3(6)a_5 = 3(6)(-9720) = -174{,}960$

$a_7 = 3(7)a_6 = 3(7)(-174{,}960) = -3{,}674{,}160$ •

**20.** $a_1 = 2, a_n = -2na_{n-1}$ [11.1]

$a_2 = -2(2)a_1 = -2(2)(2) = -8$

$a_3 = -2(3)a_2 = -2(3)(-8) = 48$ •

$a_4 = -2(4)a_3 = -2(4)(48) = -384$

$a_5 = -2(5)a_4 = -2(5)(-384) = 3840$

$a_6 = -2(6)a_5 = -2(6)(3840) = -46{,}080$

$a_7 = -2(7)a_6 = -2(7)(-46{,}080) = 645{,}120$ •

**21.** $a_{n+1} - a_n = (n+1)^2 - n^2 = 2n + 1 \neq$ constant.

Thus not an arithmetic sequence.

$\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)^2}{n^2} = \left(1 + \dfrac{1}{n}\right)^2 \neq$ constant.

Thus not a geometric sequence.

Neither an arithmetic nor a geometric sequence. [11.1]

**22.** $a_{n+1} - a_n = (n+1)! - n!$

$= (n+1)n! - n! = n \cdot n! \neq$ constant.

Thus not an arithmetic sequence.

$\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)!}{n!} = \dfrac{(n-1)n!}{n!} = n + 1 \neq$ constant.

Thus not a geometric sequence.

Neither an arithmetic nor a geometric sequence. [11.1]

**23.** $a_{n+1} - a_n = 3(n+1) + 2 - (3n + 2) = 3 =$ constant. [11.2]

Arithmetic sequence.

**24.** $a_{n+1} - a_n = (1 - 2(n+1)) - (1 - 2n) = -2 =$ constant. [11.2]

Arithmetic sequence.

**25.** $\dfrac{a_{n+1}}{a_n} = \dfrac{2^{-(n+1)}}{2^{-n}} = \dfrac{2^{-n} \cdot 2^{-1}}{2^{-n}} = 2^{-1} = \dfrac{1}{2} =$ constant. [11.3]

Geometric sequence.

**26.** $\dfrac{a_{n+1}}{a_n} = \dfrac{3^{n+1}}{3^n} = \dfrac{3^n \cdot 3}{3^n} = 3 =$ constant. [11.3]

Geometric sequence.

**27.** $a_{n+1} - a_n = \frac{1}{(n+1)!} - \frac{1}{n!}$
$= \frac{1}{(n+1)n!} - \frac{1}{n!} = \frac{1}{n!}\left(\frac{1}{n+1} - 1\right) \neq \text{constant}$

$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{n!}{(n+1)!} = \frac{n!}{(n+1)n!} = \frac{1}{n+1} \neq \text{constant}$

Neither an arithmetic nor a geometric sequence. [11.1]

**28.** $a_{n+1} - a_n = \frac{1}{n+1} - \frac{1}{n} = -\frac{1}{n(n+1)} \neq \text{constant}$

$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} \neq \text{constant}$

Neither an arithmetic nor a geometric sequence. [11.1]

**29.** $\frac{a_{n+1}}{a_n} = \frac{\left(\frac{2}{3}\right)^{n+1}}{\left(\frac{2}{3}\right)^n} = \frac{2}{3} = \text{constant.}$

Geometric sequence. [11.3]

**30.** $\frac{a_{n+1}}{a_n} = \frac{\left(-\frac{4}{3}\right)^{n+1}}{\left(-\frac{4}{3}\right)^n} = -\frac{4}{3} = \text{constant.}$

Geometric sequence [11.3]

**31.** Since each successive term is 3 times the previous term, the sequence has a common ratio of 3.
Geometric sequence. [11.1]

**32.** Since each successive term is 2 times the previous term, the sequence has a common ratio of 2.
Geometric sequence. [11.1]

**33.** Examining the terms of the sequence reveals that there is no common difference and no common ratio.
Neither an arithmetic nor a geometric sequence. [11.1]

**34.** Examining the terms of the sequence reveals that there is no common difference and no common ratio.
Neither an arithmetic nor a geometric sequence. [11.1]

**35.** Since each successive term is 2 more than the preceding term, the sequence has a common difference of 2.
Arithmetic sequence. [11.2]

**36.** Since each successive term is 3 less than the preceding term, the sequence has a common difference of $-3$.
Arithmetic sequence. [11.2]

**37.** Examining the terms of the sequence reveals that there is no common difference and no common ratio.
Neither an arithmetic nor a geometric sequence. [11.1]

**38.** Examining the terms of the sequence reveals that there is no common difference and no common ratio.
Neither an arithmetic nor a geometric sequence. [11.1]

**39.** Examining the terms of the sequence reveals that there is no common difference and no common ratio.
Neither an arithmetic nor a geometric sequence. [11.1]

**40.** Examining the terms of the sequence reveals that there is no common difference and no common ratio.
Neither an arithmetic nor a geometric sequence. [11.1]

**41.** $\sum_{n=1}^{9}(2n-3)$ is an arithmetic series with common difference 2. $a_1 = -1, a_9 = 15, n = 9$ [11.2]

$S_n = \frac{n}{2}(a_1 + a_n)$

$S_9 = \frac{9}{2}(-1+15) = \frac{9}{2}(14) = 63$

**42.** $\sum_{i=1}^{11}(1-3i)$ is an arithmetic series with common difference $-3$. $a_1=-2, a_{11}=-32, n=11$ [11.2]

$$S_n=\frac{n}{2}\left(a_1+a_n\right)$$
$$S_{11}=\frac{11}{2}\left(-2+(-32)\right)=\frac{11}{2}(-34)=-187$$

**43.** $\sum_{k=1}^{8}(4k+1)$ is an arithmetic series with common difference 4. $a_1=5, a_8=33, n=8$ [11.2]

$$S_n=\frac{n}{2}\left(a_1+a_n\right)$$
$$S_8=\frac{8}{2}(5+33)=4(38)=152$$

**44.** $\sum_{i=1}^{10}(i^2+3)$ is neither an arithmetic nor a geometric series. [11.1]

$$\sum_{i=1}^{10}(i^2+3)=(1+3)+(4+3)+(9+3)+(16+3)+(25+3)+(36+3)+(49+3)+(64+3)+(81+3)+(100+3)=415$$

**45.** $\sum_{n=1}^{6}3\cdot 2^n$ is a geometric series with common ratio 2. $a_1=6,\ n=6,\ r=2$ [11.3]

$$S_n=\frac{a_1(1-r^n)}{1-r}$$
$$S_6=\frac{6(1-2^6)}{1-2}=\frac{6(-63)}{-1}=378$$

**46.** $\sum_{i=1}^{5}2\cdot 4^{i-1}$ is a geometric series with common ratio 4. $a_1=2,\ n=5,\ r=4$ [11.3]

$$S_n=\frac{a_1(1-r^n)}{1-r}$$
$$S_5=\frac{2(1-4^5)}{1-4}=\frac{2(-1023)}{-3}=682$$

**47.** $\sum_{k=1}^{9}(-1)^k(3^k)$ is a geometric series with common ratio $-3$. $a_1=-3,\ n=9,\ r=-3$ [11.3]

$$S_n=\frac{a_1(1-r^n)}{1-r}$$
$$S_9=\frac{-3(1-(-3)^9)}{1-(-3)}=\frac{-3(19,684)}{4}=-14,763$$

**48.** $\sum_{i=1}^{8}(-1)^{i+1}2^i$ is a geometric series with common ratio $-2$. $a_1 = 2,\ n = 8,\ r = -2$ [11.3]

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_8 = \frac{2(1-(-2)^8)}{1-(-2)} = \frac{2(-255)}{3} = -170$$

**49.** $\sum_{i=1}^{10}\left(\frac{2}{3}\right)^i$ is a geometric series with common ratio $\frac{2}{3}$. $a_1 = \frac{2}{3},\ n = 10,\ r = \frac{2}{3}$ [11.3]

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_{10} = \frac{\frac{2}{3}\left(1-\left(\frac{2}{3}\right)^{10}\right)}{1-\frac{2}{3}} = \frac{\frac{2}{3}\left(1-\left(\frac{2}{3}\right)^{10}\right)}{\frac{1}{3}} = 2\left(1-\left(\frac{2}{3}\right)^{10}\right) = 2\left(1-\frac{1{,}024}{59{,}049}\right) = 2\left(\frac{58{,}025}{59{,}049}\right) = \frac{116{,}050}{59{,}049} \approx 1.9653$$

**50.** $\sum_{i=1}^{11}\left(\frac{3}{2}\right)^i$ is a geometric series with common ratio $\frac{3}{2}$. $a_1 = \frac{3}{2},\ n = 11,\ r = \frac{3}{2}$ [11.3]

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_{11} = \frac{\frac{3}{2}\left(1-\left(\frac{3}{2}\right)^{11}\right)}{1-\frac{3}{2}} \approx \frac{\frac{3}{2}(-85.4976)}{-\frac{1}{2}} \approx 256.4927$$

**51.** $\sum_{n=1}^{9}\frac{(-1)^{n+1}}{n^2}$ is neither a geometric nor an arithmetic series. [11.1]

$$\sum_{n=1}^{9}\frac{(-1)^{n+1}}{n^2} = 1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{26}+\frac{1}{49}-\frac{1}{64}+\frac{1}{81} \approx 0.8280$$

**52.** $\sum_{k=1}^{5}\frac{(-1)^{k+1}}{k!}$ is neither an arithmetic nor a geometric series. [11.1]

$$\sum_{k=1}^{5}\frac{(-1)^{k+1}}{k!} = 1-\frac{1}{2}+\frac{1}{6}-\frac{1}{24}+\frac{1}{120} = \frac{19}{30} \approx 0.6\overline{3}$$

**53.** $\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n$ is an infinite geometric series with common ratio $\frac{1}{4}$. $a_1 = \frac{1}{4},\ r = \frac{1}{4}$ [11.3]

$$S = \frac{a_1}{1-r}$$

$$S = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

**54.** $\sum_{i=1}^{\infty}\left(-\frac{5}{6}\right)^i$ is an infinite geometric series with common ratio $-\frac{5}{6}$. $a_1=-\frac{5}{6}$, $r=-\frac{5}{6}$ [11.3]

$$S=\frac{a_1}{1-r}$$

$$S=\frac{-\frac{5}{6}}{1-\left(-\frac{5}{6}\right)}=\frac{-\frac{5}{6}}{\frac{11}{6}}=-\frac{5}{11}$$

**55.** $\sum_{k=1}^{\infty}\left(-\frac{4}{5}\right)^k$ is an infinite geometric series with common ratio $-\frac{4}{5}$. $a_1=-\frac{4}{5}$, $r=-\frac{4}{5}$ [11.3]

$$S=\frac{a_1}{1-r}$$

$$S=\frac{-\frac{4}{5}}{1-\left(-\frac{4}{5}\right)}=\frac{-\frac{4}{5}}{\frac{9}{5}}=-\frac{4}{9}$$

**56.** $\sum_{j=0}^{\infty}\left(\frac{1}{5}\right)^j$ is an infinite geometric series with common ratio $\frac{1}{5}$. $a_1=1$, $r=\frac{1}{5}$ [11.3]

$$S=\frac{a_1}{1-r}$$
$$S=\frac{1}{1-\frac{1}{5}}=\frac{1}{\frac{4}{5}}=\frac{5}{4}$$

**57.** $\sum_{i=1}^{n}(5i+1)=\frac{n(5n+7)}{2}$ [11.4]

1. For $n=1$, we have $\sum_{i=1}^{1}(5i+1)=6$ and $\frac{1(5+7)}{2}=6$. Therefore that statement is true for $n=1$.
2. Assume the statement is true for $n=k$.

$$\sum_{i=1}^{k}(5i+1)=\frac{k(5k+7)}{2} \quad \text{Induction Hypothesis}$$

Prove the statement is true for $n=k+1$. That is, prove

$$\sum_{i=1}^{k+1}(5i+1)=\frac{(k+1)(5k+12)}{2}$$

$$\sum_{i=1}^{k+1}(5i+1)=\sum_{i=1}^{k}(5i+1)+5(k+1)+1=\sum_{i=1}^{k}(5i+1)+5k+6$$

$$=\frac{k(5k+7)}{2}+5k+6 \quad \text{Using the Induction Hypothesis}$$

$$=\frac{k(5k+7)+10k+12}{2}=\frac{5k^2+7k+10k+12}{2}$$

$$=\frac{5k^2+17k+12}{2}=\frac{(k+1)(5k+12)}{2}$$

Thus the statement is true for $n=k+1$. By the Induction Axiom, the statement is true for all positive integers.

**58.** $\sum_{i=1}^{n}(3-4i)=n(1-2n)$ [11.4]

1. For $n=1,$ we have $\sum_{i=1}^{1}(3-4i)=-1$ and $1[1-2(1)]=-1.$

   Therefore the statement is true for $n=1$.

2. Assume the statement is true for $n=k$.

   $\sum_{i=1}^{k}(3-4i)=k(1-2k)$ Induction Hypothesis

   Prove the statement is true for $n=k+1$. That is, prove

   $\sum_{i=1}^{k+1}(3-4i)=(k+1)(-1-2k)=-(k+1)(2k+1)$

   $$\begin{aligned}\sum_{i=1}^{k+1}(3-4i)&=\sum_{i=1}^{k}(3-4i)+3-4(k+1)=\sum_{i=1}^{k}(3-4i)-1-4k\\&=k(1-2k)-1-4k \quad \text{Using the Induction Hypothesis}\\&=k-2k^2-1-4k=-2k^2-3k-1\\&=-(2k^2+3k+1)=-(k+1)(2k+1)\end{aligned}$$

Thus the statement is true for $n=k+1$. By the Induction Axiom, the statement is true for all positive integers.

**59.** $\sum_{i=0}^{n}\left(-\frac{1}{2}\right)^i=\frac{2\left(1-\left(-\frac{1}{2}\right)^{n+1}\right)}{3}$ [11.4]

1. This induction begins with $n=0$.

   $\sum_{i=0}^{0}\left(-\frac{1}{2}\right)^i=\left(-\frac{1}{2}\right)^0=1$ and $\frac{2\left(1-\left(-\frac{1}{2}\right)^{1}\right)}{3}=1$

   Thus the statement is true when $n=0$.

2. Assume the statement is true for $n=k$.

   $\sum_{i=0}^{k}\left(-\frac{1}{2}\right)^i=\frac{2\left(1-\left(-\frac{1}{2}\right)^{k+1}\right)}{3}$ Induction Hypothesis

   Prove the statement is true for $n=k+1$. That is, prove

   $$\sum_{i=0}^{k+1}\left(-\frac{1}{2}\right)^i=\frac{2\left(1-\left(-\frac{1}{2}\right)^{k+2}\right)}{3}=\frac{2-2\left(-\frac{1}{2}\right)^{k+2}}{3}=\frac{2-2\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)^{k+1}}{3}=\frac{2+\left(-\frac{1}{2}\right)^{k+1}}{3}$$

   $$\begin{aligned}\sum_{i=0}^{k+1}\left(-\frac{1}{2}\right)^i&=\sum_{i=0}^{k}\left(-\frac{1}{2}\right)^i+\left(-\frac{1}{2}\right)^{k+1}\\&=\frac{2\left(1-\left(-\frac{1}{2}\right)^{k+1}\right)}{3}+\left(-\frac{1}{2}\right)^{k+1}\\&=\frac{2\left(1-\left(-\frac{1}{2}\right)^{k+1}\right)+3\left(-\frac{1}{2}\right)^{k+1}}{3}=\frac{2-2\left(-\frac{1}{2}\right)^{k+1}+3\left(-\frac{1}{2}\right)^{k+1}}{3}\\&=\frac{2+\left(-\frac{1}{2}\right)^{k+1}}{3}\end{aligned}$$

   Thus the statement is true for all integers $n\ge 0$.

**60.** $\sum_{i=0}^{n}(-1)^i = \frac{1-(-1)^{n+1}}{2}$ [11.4]

1. This induction begins with $n = 0$.

   $\sum_{i=0}^{0}(-1)^i = (-1)^0 = 1$ and $\frac{1-(-1)}{2} = \frac{1+1}{2} = 1$

   Thus the statement is true for $n = 0$.

2. Assume the statement is true for $n = k$.

   $\sum_{i=0}^{k}(-1)^i = \frac{1-(-1)^{k+1}}{2}$

   Prove the statement is true for $n = k+1$. That is, prove

   $\sum_{i=0}^{k+1}(-1)^i = \frac{1-(-1)^{k+2}}{2} = \frac{1-(-1)(-1)^{k+1}}{2} = \frac{1+(-1)^{k+1}}{2}$

   $\sum_{i=0}^{k+1}(-1)^i = \sum_{i=0}^{k}(-1)^i + (-1)^{k+1}$

   $= \frac{1-(-1)^{k+1}}{2} + (-1)^{k+1}$ Using the Induction Hypothesis

   $= \frac{1-(-1)^{k+1}+2(-1)^{k+1}}{2}$

   $= \frac{1+(-1)^{k+1}}{2}$

   Thus the statement is true for all integers $n \ge 0$.

**61.** $n^n \ge$ n! [11.4]

1. When $n = 1$, $1^1 = 1$ and $n! = 1$. The statement is true for $n = 1$.
2. Assume the statement is true for $n = k$.

   $k^k \ge k!$ Induction Hypothesis

   Prove the statement is true for $n = k+1$. That is, prove

   $(k+1)^{k+1} \ge (k+1)!$

   $(k+1)^{k+1} = (k+1)(k+1)^k > (k+1)k^k$

   By Induction Hypothesis $k^k \ge k!$. We have

   $(k+1)k^k \ge (k+1)k! = (k+1)!$

   Therefore $(k+1)^{k+1} \ge (k+1)!$

   Therefore the statement is true for all integers $n \ge 1$.

**62.**

1. For $n = 9$, $9! = 362{,}880$, $4^9 = 262{,}144$ [11.4]

   Since $362{,}880 > 262{,}144$, the statement is true for $n = 9$.

2. Assume the statement is true for $n = k$.

   $k! > 4^k$ Induction Hypothesis

   Prove the statement is true for $n = k+1$. That is, prove

   $(k+1)! > 4^{k+1}$

   $(k+1)! = (k+1)k! > (k+1)4^k$ By Induction Hypothesis

   Since $k \ge 9$, $k+1 \ge 4$. Thus

   $(k+1)4^k > 4 \cdot 4^k = 4^{k+1}$

   Thus $(k+1)! > 4^{k+1}$

   The statement is true for all integers $n \ge 9$.

**63.** 1. When $n=1$, we have $1^3+2(1)=3$. Since 3 is a factor of 3, the statement is true for $n=1$. [11.4]

2. Assume the statement is true for $n=k$.

3 is a factor of $k^3+2k$ Induction Hypothesis

Prove the statement is true for $n=k+1$. That is, prove

3 is a factor of $(k+1)^3+2(k+1)$.

$$\begin{aligned}(k+1)^3+2(k+1)&=k^3+3k^2+3k+1+2k+2\\&=(k^3+2k)+3(k^2+k+1)\end{aligned}$$

By Induction Hypothesis, 3 is a factor of $k^3+2k$. Three is also a factor of $3(k^2+k+1)$. Thus 3 is a factor of $(k+1)^3+2(k+1)$. The statement is true for all positive integers $n$.

**64.** 1. When $n=1$, $a_1=\sqrt{2}<2$. The statement is true for $n=1$. [11.4]

2. Assume the statement is true for some integer $k$.

$a_k<2$ Induction Hypothesis

Prove the statement is true for $n=k+1$. That is, prove

$a_{k+1}<2$.

By the Induction Hypothesis, $a_k<2$. Thus

$(\sqrt{2})^{a_k}<(\sqrt{2})^2=2$

But $(\sqrt{2})^{a_k}=a_{k+1}$. Thus

$a_{k+1}<2$

The statement is true for all positive integers $n$.

**65.** $$(4a-b)^5=\sum_{i=0}^{5}\binom{5}{i}(4a)^{5-i}(-b)^i \qquad [11.5]$$

$$\begin{aligned}&=\binom{5}{0}(4a)^5+\binom{5}{1}(4a)^4(-b)^1+\binom{5}{2}(4a)^3(-b)^2+\binom{5}{3}(4a)^2(-b)^3+\binom{5}{4}(4a)(-b)^4+\binom{5}{5}(-b)^5\\&=1(1024a^5)+5(-256a^4b)+10(64a^3b^2)+10(-16a^2b^3)+5(4ab^4)+(-b^5)\\&=1024a^5-1280a^4b+640a^3b^2-160a^2b^3+20ab^4-b^5\end{aligned}$$

**66.** $$(x+3y)^6=\sum_{i=0}^{6}\binom{6}{i}x^{6-i}(3y)^i \qquad [11.5]$$

$$\begin{aligned}&=\binom{6}{0}x^6+\binom{6}{1}x^5(3y)+\binom{6}{2}x^4(3y)^2+\binom{6}{3}x^3(3y)^3+\binom{6}{4}x^2(3y)^4+\binom{6}{5}x(3y)^5+\binom{6}{6}(3y)^6\\&=x^6+6x^5(3y)+15x^4(9y^2)+20x^3(27y^3)+15x^2(81y^4)+6x(243y^5)+1(729y^6)\\&=x^6+18x^5y+135x^4y^2+540x^3y^3+1215x^2y^4+1458xy^5+729y^6\end{aligned}$$

**67.** $$\left(\sqrt{a}+2\sqrt{b}\right)^8=\sum_{i=0}^{8}\binom{8}{i}\left(\sqrt{a}\right)^{8-i}\left(2\sqrt{b}\right)^i \quad [11.5]$$

$$\begin{aligned}&=\binom{8}{0}(\sqrt{a})^8+\binom{8}{1}(\sqrt{a})^7(2\sqrt{b})+\binom{8}{2}(\sqrt{a})^6(2\sqrt{b})^2+\binom{8}{3}(\sqrt{a})^5(2\sqrt{b})^3+\binom{8}{4}(\sqrt{a})^4(2\sqrt{b})^4+\binom{8}{5}(\sqrt{a})^3(2\sqrt{b})^5\\&\quad+\binom{8}{6}(\sqrt{a})^2(2\sqrt{b})^6+\binom{8}{7}(\sqrt{a})(2\sqrt{b})^7+\binom{8}{8}(2\sqrt{b})^8\\&=1(a^4)+8a^{7/2}(2b^{1/2})+28a^3(4b)+56a^{5/2}(8b^{3/2})+70a^2(16b^2)+56a^{3/2}(32b^{5/2})+28a(64b^3)\\&\quad+8a^{1/2}(128b^{7/2})+1(256b^4)\\&=a^4+16a^{7/2}b^{1/2}+112a^3b+448a^{5/2}b^{3/2}+1120a^2b^2+1792a^{3/2}b^{5/2}+1792ab^3+1024a^{1/2}b^{7/2}+256b^4\end{aligned}$$

**68.** $\left(2x-\dfrac{1}{2x}\right)^7=\displaystyle\sum_{i=0}^{7}\binom{7}{i}(2x)^{7-i}\left(-\dfrac{1}{2x}\right)^i$ [11.5]

$$=\binom{7}{0}(2x)^7+\binom{7}{1}(2x)^6\left(-\frac{1}{2x}\right)+\binom{7}{2}(2x)^5\left(-\frac{1}{2x}\right)^2+\binom{7}{3}(2x)^4\left(-\frac{1}{2x}\right)^3+\binom{7}{4}(2x)^3\left(-\frac{1}{2x}\right)^4+\binom{7}{5}(2x)^2\left(-\frac{1}{2x}\right)^5$$
$$+\binom{7}{6}(2x)\left(-\frac{1}{2x}\right)^6+\binom{7}{7}\left(-\frac{1}{2x}\right)^7$$
$$=1(128x^7)-7(64x^6)\left(\frac{1}{2x}\right)+21(32x^5)\left(\frac{1}{4x^2}\right)-35(16x^4)\left(\frac{1}{8x^3}\right)+35(8x^3)\left(\frac{1}{16x^4}\right)-21(4x^2)\left(\frac{1}{32x^5}\right)$$
$$+7(2x)\left(\frac{1}{64x^6}\right)-1\left(\frac{1}{128x^7}\right)$$
$$=128x^7-224x^5+168x^3-70x+\frac{35}{2x}-\frac{21}{8x^3}+\frac{7}{32x^5}-\frac{1}{128x^7}$$

**69.** The fifth term of $(3x-4y)^7$ is $\binom{7}{4}(3x)^3(-4y)^4=35(27x^3)(256y^4)=241{,}920x^3y^4$. [11.5]

**70.** The eighth term of $(1-3x)^9$ is $\binom{9}{7}(1)^2(-3x)^7=36\cdot 1\cdot(-2187x^7)=-78{,}732x^7$. [11.5]

**71.** There are 26 choices for each letter. By the Fundamental Counting Principle, there are $26^8$ possible passwords. [11.6]

**72.** Using the Fundamental Counting Principle, we have $10^6\cdot 26$ possible serial numbers. [11.6]

**73.** This is a permutation with $n=15$ and $r=3$. [11.6]

$$P(15,3)=\frac{15!}{(15-3)!}=\frac{15!}{12!}=2730$$

**74.** There are $\binom{4}{1}$ ways to choose a supervisor and $\binom{12}{3}$ ways to choose 3 regular employees. Thus, there are $\binom{4}{1}\binom{12}{3}$ ways to do both.

$\binom{4}{1}\binom{12}{3}=4\cdot 220=880$ shifts have 1 supervisor. [11.6]

**75.** This problem is solved in stages. First, there are $\binom{10}{5}$ ways to choose a committee excluding both people who refuse to serve together. Second, there are $\binom{10}{4}\binom{2}{1}$ ways to choose a committee that includes one person but not the other.

Altogether there are $\binom{10}{5}+\binom{10}{4}\binom{2}{1}$ ways to choose the committee.

$\binom{10}{5}+\binom{10}{4}\binom{2}{1}=252+210(2)=672$ possible committees [11.6]

**76.** There are $\binom{10}{4}=210$ ways of choosing 4 calculators from 10. If the inspector is to choose 1 defective calculator, then 3 nondefective calculators must also be chosen. There are $\binom{2}{1}\binom{8}{3}=2(56)=112$ ways to accomplish that. Therefore, the probability of the event is

$\frac{112}{210}=\frac{8}{15}$. [11.7]

**77.** The probability is $\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{8}$.

The probability of one tail and therefore two heads is $3\left(\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\right)=\frac{3}{8}$. [11.4]

**78.** There are $\binom{10}{4}$ ways to draw 4 cards from 10. There are $\binom{5}{2}\binom{5}{2}$ ways to draw 2 red and 2 black cards The probability of drawing 2 red and 2 black cards is $\frac{\binom{5}{2}\binom{5}{2}}{\binom{10}{4}} = \frac{10\cdot 10}{210} = \frac{10}{21}$. [11.7]

**79.** We look at the possibility for each case.

| | | |
|---|---|---|
| If the middle digit is zero, there are | 0 | numbers |
| If the middle digit is one, there is | 1 | number |
| If the middle digit is two, there are | 4 | numbers |
| If the middle digit is three, there are | 9 | numbers |
| If the middle digit is four, there are | 16 | numbers |
| If the middle digit is five, there are | 25 | numbers |
| If the middle digit is six, there are | 36 | numbers |
| If the middle digit is seven, there are | 49 | numbers |
| If the middle digit is eight, there are | 64 | numbers |
| If the middle digit is nine, there are | 81 | numbers |
| Total | 285 | numbers |

The probability is $\frac{285}{1000} = 0.285$. [11.7]

**80.** The probability that the sum of two numbers is 9 when the numbers are selected with replacement from 1, 2, 3, 4, 5, 6 is $\frac{4}{36} = \frac{1}{9}$.

The probability that the sum is 7 is $\frac{6}{36} = \frac{1}{6}$. Therefore the probability that it is not 7 and not 9 is

$1 - \left(\frac{1}{9} + \frac{1}{6}\right) = 1 - \frac{10}{36} = \frac{13}{18}$.

First selection, sum is 9.

Second selection, probability of 9 is $\frac{1}{9}$.

Third selection, probability is $\frac{13}{18} \cdot \frac{1}{9}$.

Fourth selection, probability is $\frac{13}{18} \cdot \frac{13}{18} \cdot \frac{1}{9}$.

$\vdots$

The total probability is the sum of this infinite process.

$\frac{1}{9} + \frac{13}{18} \cdot \frac{1}{9} + \left(\frac{13}{18}\right)^2 \left(\frac{1}{9}\right) + \cdots$

This is a geometric series with $a_1 = \frac{1}{9}, r = \frac{13}{18}$.

$S = \frac{\frac{1}{9}}{1 - \frac{13}{18}} = \frac{\frac{1}{9}}{\frac{5}{18}} = \frac{2}{5}$

The probability is $\frac{2}{5}$. [11.7]

**81.** The probability of drawing an ace and a 10 card from one regular deck of playing cards is

$$\frac{\binom{4}{1}\binom{16}{1}}{\binom{52}{2}} = \frac{4\cdot 16}{\frac{52\cdot 51}{2}} \approx 0.0483.$$

The probability of drawing an ace and a 10 card from two regular decks of playing cards is

$$\frac{\binom{8}{1}\binom{32}{1}}{\binom{104}{2}} = \frac{8\cdot 32}{\frac{104\cdot 103}{2}} \approx 0.0478.$$

Drawing an ace and a 10 card from *one* deck has the greater probability. [11.7]

**82.** Probability = (probability of 2)(probability of 1) [11.7]
+ (probability of 3)(probability of 1 or 2)
+ (probability of 4)(probability of 1 or 2 or 3)
+ (probability of 5)

$$= \frac{1}{5}\cdot\frac{1}{4} + \frac{1}{5}\cdot\frac{2}{4} + \frac{1}{5}\cdot\frac{3}{4} + \frac{1}{5}\cdot\frac{4}{4}$$
$$= \frac{1}{20} + \frac{2}{20} + \frac{3}{20} + \frac{4}{20} = \frac{10}{20} = \frac{1}{2}$$

**83.** There are $\binom{12}{3}$ ways of choosing 3 people from 12. There are $\binom{11}{2}\cdot 1$ ways of choosing 2 people and the person with [11.7] badge number 6.

$$\text{Probability} = \frac{\binom{11}{2}\cdot 1}{\binom{12}{3}} = \frac{\frac{11\cdot 10}{2}}{\frac{12\cdot 11\cdot 10}{3\cdot 2}} = \frac{1}{4}$$

**84.** $\text{Stock value} = \frac{D(1+g)}{i-g} = \frac{1.27(1+0.03)}{0.12-0.03} \approx \$14.53$ [11.3]

**85.** Using the multiplier effect, [11.3]

$$\frac{15}{1-0.80} = 75$$

The net effect of \$15 million is \$75 million.

## Quantitative Reasoning

**QR1.** $PMT = L\left[\frac{i}{1-(1+i)^{-nt}}\right],\ i = \frac{r}{n}$

$$= 12{,}000\left[\frac{\frac{0.09}{12}}{1-\left(1+\frac{0.09}{12}\right)^{-12(5)}}\right]$$
$$\approx \$249.10$$

**QR2.** $PMT = L\left[\frac{i}{1-(1+i)^{-nt}}\right],\ i = \frac{r}{n}$

$$= 18{,}000\left[\frac{\frac{0.06}{12}}{1-\left(1+\frac{0.06}{12}\right)^{-12(4)}}\right]$$
$$\approx \$422.73$$

**QR3.** $PMT = L\left[\frac{i}{1-(1+i)^{-nt}}\right],\ i = \frac{r}{n}$

$$= 15{,}000\left[\frac{\frac{0.085}{12}}{1-\left(1+\frac{0.085}{12}\right)^{-12(5)}}\right]$$
$$\approx \$307.75$$

Total payments made over 12 years:
$307.75(12)(5) = 18{,}465$
Total interest paid
$18{,}465 - 15{,}000 = \$3465$.

## Chapter Test

**1.** $a_3 = \frac{2^3}{3!} = \frac{8}{6} = \frac{4}{3}$ [11.1]

$a_5 = \frac{2^5}{5!} = \frac{32}{120} = \frac{4}{15}$

**2.** $a_2 = 2\cdot a_1 = 2\cdot 3 = 6$ [11.1]

$a_3 = 2\cdot a_2 = 12$ •

$a_4 = 2\cdot a_3 = 24$

$a_5 = 2\cdot a_4 = 48$ •

**3.** $a_{n+1} - a_n = [-2(n+1)+3] - (-2n+3)$ [11.3]
$= -2n - 2 + 3 + 2n - 3$
$= -2 =$ constant

arithmetic

**4.** $a_{n+1} - a_n = 2(n+1)^2 - 2n^2$ [11.1]
$= 4n + 2 \neq$ constant

$$\frac{a_{n+1}}{a_n} = \frac{2(n+1)^2}{2n^2}$$
$$= 1 + \frac{2}{n} + \frac{1}{n^2} \neq \text{ constant}$$

neither

**5.** $$\frac{a_{n+1}}{a_n} = \frac{\frac{(-1)^{n+1-1}}{3^{n+1}}}{\frac{(-1)^{n-1}}{3^n}} = \frac{-1}{3} = \text{constant} \quad [11.3]$$

geometric

**6.** $$\sum_{i=1}^{6} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{120}{120} + \frac{60}{120} + \frac{40}{120} + \frac{30}{120} + \frac{24}{120} + \frac{20}{120} \quad [11.1]$$
$$= \frac{294}{120} = \frac{49}{20}$$

**7.** $$\sum_{j=1}^{10} \frac{1}{2j} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \frac{1}{1024} = \frac{\frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^{10}\right)}{1 - \frac{1}{2}} = 1 - \left(\frac{1}{2}\right)^{10} \quad [11.3]$$
$$= 1 - \frac{1}{1024} = \frac{1023}{1024}$$

**8.** $$\sum_{k=1}^{20} (3k-2) = 1 + 4 + 7 + 10 + \cdots + 58 = \frac{20}{2}(1+58) = 10(59) \quad [11.2]$$
$$= 590$$

**9.** $a_3 = a_1 + (3-1)d = 7,$
$a_8 = a_1 + (8-1)d = 22$

$$\begin{aligned} a_1 + 2d &= 7 \\ \underline{a_1 + 7d} &\underline{= 22} \\ -5d &= -15 \\ d &= 3 \end{aligned}$$

$$\begin{aligned} a_1 &= a_3 - 2(3) \\ &= 7 - 6 \\ &= 1 \end{aligned}$$

$$\begin{aligned} a_{20} &= a_1 + (20-1)d \quad [11.2] \\ &= 1 + (19)(3) \\ &= 58 \end{aligned}$$

**10.** $$\sum_{k=1}^{\infty} \left(\frac{3}{8}\right)^k = \frac{a_1}{1-r} = \frac{\frac{3}{8}}{1 - \left(\frac{3}{8}\right)} = \frac{\frac{3}{8}}{\frac{5}{8}} = \frac{3}{5} \quad [11.3]$$

**11.** $$0.\overline{15} = 0.15 + 0.0015 + 0.000015 + \ldots = \frac{0.15}{1 - 0.01} = \frac{0.15}{0.99} = \frac{15}{99} = \frac{5}{33} \quad [11.3]$$

**12.** 1. Let $n = 1$. $2 - 3(1) = -1$ $\quad \frac{1(1-3(1))}{2} = -1$ [11.4]

Thus the statement is true for $n = 1$.

2. Assume $\sum_{i=1}^{k}(2-3i) = \frac{k(1-3k)}{2}$ is true for some positive number $k$.

Verify $\sum_{i=1}^{k+1}(2-3i) = \frac{(k+1)[1-3(k+1)]}{2} = \frac{(k+1)(1-3k-3)}{2} = \frac{(k+1)(-3k-2)}{2} = -\frac{(k+1)(3k+2)}{2}$

$$\begin{aligned}\frac{k(1-3k)}{2} + [2-3(k+1)] &= \frac{k(1-3k)}{2} + (-3k-1)\\ &= \frac{k-3k^2-6k-2}{2}\\ &= -\frac{(3k^2+5k+2)}{2}\\ &= -\frac{(k+1)(3k+2)}{2}\end{aligned}$$

Thus the formula has been established by the extended principle of mathematical induction.

**13.** 1. Let $n = 7$ [11.4]

$7! = 50{,}407$ $\quad 3^7 = 2187$

Thus $n! > 3^n$ for $n = 7$.

2. Assume $k! > 3^k$

Verify $(k+1)! > 3^{k+1}$

$$\begin{aligned} k! &> 3^k\\ k+1 &> 3\\ (k+1)k! &> 3 \cdot 3^k\\ (k+1)! &> 3^{k+1}\end{aligned}$$

Thus the formula has been established by the extended principle of mathematical induction.

**14.** $$\begin{aligned}(x-2y)^5 &= x^5 - 5(x)^4(2y) + 10(x)^3(2y)^2 - 10(x)^2(2y)^3 + 5(x)(2y)^4 - (2y)^5\\ &= x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5\end{aligned}$$ [11.5]

**15.** $(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$ [11.5]

$$\begin{aligned}\left(x+\frac{1}{x}\right)^6 &= x^6 + 6(x)^5\left(\frac{1}{x}\right) + 15(x)^4\left(\frac{1}{x}\right)^2 + 20(x)^3\left(\frac{1}{x}\right)^3 + 15(x)^2\left(\frac{1}{x}\right)^4 + 6x\left(\frac{1}{x}\right)^5 + \left(\frac{1}{x}\right)^6\\ &= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}\end{aligned}$$

**16.** $$\begin{aligned}\text{6th term of } (3x+2y)^8 &= \binom{8}{6-1}(3x)^3(2y)^{6-1} \quad [11.5]\\ &= \binom{8}{5}(3x)^3(2y)^5\\ &= 56 \cdot 27x^3 \cdot 32y^5\\ &= 48{,}384x^3y^5\end{aligned}$$

**17.** $52 \cdot 51 \cdot 50 = 132{,}600$ [11.6]

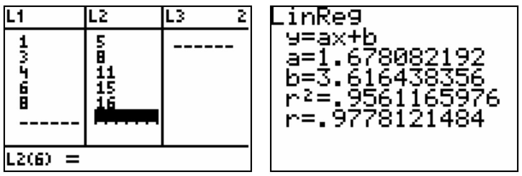
**18.** $26 \cdot 25 \cdot 24 \cdot 9 \cdot 8 \cdot 23 \cdot 22 = 568{,}339{,}200$ [11.6]

**19.** $\frac{C(8,\ 3)C(10,\ 2)}{C(18,\ 5)} = \frac{56 \cdot 45}{8568} = \frac{5}{17} \approx 0.294118$ [11.7]

**20.** Stock value $= \dfrac{D(1+g)}{i-g} = \dfrac{0.86(1+0.06)}{0.15-0.06} \approx \$10.13$ [11.3]

## Cumulative Review

**1.**




$y = 1.7x + 3.6$ [2.7]

**2.**

$$5 + \frac{x}{3} = -3 \quad [2.3]$$
$$\frac{x}{3} = -8$$
$$x = -24$$

**3.**

$$2x^2 - 3x = 4 \quad [1.3]$$
$$2x^2 - 3x - 4 = 0$$
$$x = \frac{3 \pm \sqrt{(-3)^2 - 4(2)(-4)}}{2(2)} = \frac{3 \pm \sqrt{9+32}}{4}$$
$$= \frac{3 \pm \sqrt{41}}{4}$$

**4.**

$$\log_b\left(\frac{xy^2}{z^3}\right) = \log_b x + \log_b y^2 - \log_b z^3 \quad [4.4]$$
$$= \log_b x + 2\log_b y - 3\log_b z$$

**5.**

$$16x^2 + 25y^2 - 96x + 100y - 156 = 0$$
$$16(x^2 - 6x) + 25(y^2 + 4y) = 156$$
$$16(x-3)^2 + 25(y+2)^2 = 156 + 144 + 100$$
$$16(x-3)^2 + 25(y+2)^2 = 400$$
$$\frac{16(x-3)^2}{400} + \frac{25(y+2)^2}{400} = 1$$
$$\frac{(x-3)^2}{25} + \frac{(y+2)^2}{16} = 1$$
$$a^2 = 25, b^2 = 16$$
$$c^2 = a^2 - b^2 = 25 - 16 = 9$$
$$c = 3$$
$$e = \frac{c}{a} = \frac{3}{5} \quad [5.2]$$

**6.**

$$\begin{cases} 2x - 3y = 8 & (1) \\ x + 4y = -7 & (2) \end{cases}$$

Solve (2) for $x$ and substitute into (1).

$$x = -4y - 7$$
$$2(-4y - 7) - 3y = 8$$
$$-8y - 14 - 3y = 8$$
$$-11y = 22$$
$$y = -2$$
$$x = -4(-2) - 7 = 1$$

The solution is (1, –2). [9.1]

**7.**

$$3A - 2B = 3\begin{bmatrix} -1 & 2 \\ 5 & 3 \\ 0 & 3 \end{bmatrix} - 2\begin{bmatrix} 7 & -3 \\ 6 & 5 \\ 1 & -2 \end{bmatrix} \quad [10.2]$$
$$= \begin{bmatrix} -3 & 6 \\ 15 & 9 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 14 & -6 \\ 12 & 10 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} -17 & 12 \\ 3 & -1 \\ -2 & 13 \end{bmatrix}$$

**8.**

$$\left(\frac{h}{g}\right)(-3) = \frac{h(-3)}{g(-3)} = \frac{-3-2}{(-3)^2 - (-3) + 4} = \frac{-5}{16} \quad [2.6]$$

**9.** $y = 0$ [3.5]

**10.** $\log_{1/2} 64 = -6$ [4.3]

**11.** [4.5]

$$\begin{aligned} 4^{2x+1} &= 3^{x-2} \\ \ln 4^{2x+1} &= \ln 3^{x-2} \\ (2x+1)\ln 4 &= (x-2)\ln 3 \\ 2x\ln 4 + \ln 4 &= x\ln 3 - 2\ln 3 \\ x\ln 4^2 - x\ln 3 &= -2\ln 3 - \ln 4 \\ x(\ln 16 - \ln 3) &= -2\ln 3 - \ln 4 \\ x &= \frac{-2\ln 3 - \ln 4}{\ln 16 - \ln 3} \approx -2.1 \end{aligned}$$

**12.**

$$\begin{cases} x^2 + y^2 + xy = 10 & (1) \\ x - y = 1 & (2) \end{cases}$$

Solve (2) for $x$ and substitute into (1).

$x = y + 1$

$$\begin{aligned} (y+1)^2 + y^2 + (y+1)y &= 10 \\ y^2 + 2y + 1 + y^2 + y^2 + y &= 10 \\ 3y^2 + 3y - 9 &= 0 \\ 3(y^2 + y - 3) &= 0 \end{aligned}$$

$$y = \frac{-1 \pm \sqrt{1^2 - 4(1)(-3)}}{2(1)} = \frac{-1 \pm \sqrt{1+12}}{2} = \frac{-1 \pm \sqrt{13}}{2}$$

$$x = \frac{-1 \pm \sqrt{13}}{2} + 1 = \frac{1 \pm \sqrt{13}}{2}$$

The solutions are $\left(\frac{1+\sqrt{13}}{2}, \frac{-1+\sqrt{13}}{2}\right)$ and $\left(\frac{1-\sqrt{13}}{2}, \frac{-1-\sqrt{13}}{2}\right)$. [9.3]

**13.**

$$\begin{bmatrix} 3 & 2 \\ -2 & 1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 & 1 \\ -2 & 0 & 4 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 9 & 11 & -3 \\ -6 & -6 & 2 & -5 \\ 10 & 3 & -15 & 13 \end{bmatrix}$$ [10.2]

**14.** Let $t = 5$, [4.5]

$$\begin{aligned} 5 &= -\frac{175}{32}\ln\left(1 - \frac{v}{175}\right) \\ 5\left(-\frac{32}{175}\right) &= \ln\left(1 - \frac{v}{175}\right) \\ e^{-32/35} &= 1 - \frac{v}{175} \\ -175\left(e^{-32/35} - 1\right) &= v \\ v &\approx 105 \text{ mph} \end{aligned}$$

**15.** [5.3]

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{-1}{2}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{2\sqrt{3}}{3} = \frac{\text{hyp}}{\text{adj}} = \frac{2}{\sqrt{3}}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\text{adj}}{\text{opp}} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

**16.** [6.1]

$$\begin{aligned} \frac{\sin x}{1+\cos x} + \frac{1+\cos x}{\sin x} &= \frac{\sin x(1-\cos x)}{1-\cos^2 x} + \frac{1+\cos x}{\sin x} \\ &= \frac{\sin x(1-\cos x)}{\sin^2 x} + \frac{1+\cos x}{\sin x} \\ &= \frac{1-\cos x}{\sin x} + \frac{1}{\sin x} + \frac{\cos x}{\sin x} \\ &= \frac{1}{\sin x} + \frac{1}{\sin x} \\ &= 2\csc x \end{aligned}$$

**17.** $C = 180^\circ - 40^\circ - 65^\circ = 75^\circ$ [7.1]

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$a = \frac{20\sin 40^\circ}{\sin 75^\circ} \approx 13 \text{ cm}$$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$b = \frac{20\sin 65^\circ}{\sin 75^\circ} \approx 19 \text{ cm}$$

**18.** $A = 9, B = 4, C = 6$ [8.4]

$$\cot 2\alpha = \frac{A-C}{B} = \frac{9-6}{4} = \frac{3}{4}$$

$$2\alpha \approx 54^\circ$$

$$\alpha \approx 27^\circ$$

**19.** $\sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right)$ [6.5]

Let $\theta = \cos^{-1}\frac{4}{5}$

$$\cos\theta = \frac{4}{5}$$

$$\sin\frac{1}{2}\theta = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\frac{4}{5}}{2}}$$
$$= \sqrt{\frac{5-4}{10}} = \frac{1}{\sqrt{10}}$$
$$= \frac{\sqrt{10}}{10}$$

**20.** $2\mathbf{v} - 3\mathbf{w} = 2(2\mathbf{i} + 5\mathbf{j}) - 3(3\mathbf{i} - 6\mathbf{j})$ [7.3]

$$= 4\mathbf{i} + 10\mathbf{j} - 9\mathbf{i} + 18\mathbf{j}$$
$$= -5\mathbf{i} + 28\mathbf{j}$$

# Responses to Projects in the Text

**Preliminary Concepts**
CHAPTER P of *College Algebra and Trigonometry*

---

## P.1 The Real Number System

1. **Number Puzzle** $n$ is 1 less than a multiple of 6 and also 1 less than a multiple of 5 and also 1 less than a multiple of 4 and also 1 less than a multiple of 3 and also 1 less than a multiple of 2. The smallest value of $n$ is 1 less than the least common multiple of 6, 5, 4, 3, and 2. Thus $n$ is 1 less than 60, or $n = 59$.
2. **Operations on Intervals**
   - **a.** $(-4)^2 = 16,\ 0^2 = 0,\ \ 2^2 = 4$. The square of every number in the interval (–4, 2) is $(-4,\ 2)^2 = [0,\ 16)$.
   - **b.** |4| = 4, |0| = 0, |5| = 5. The absolute value of every number in the interval (–4, 5) is *ABS*(–4, 5) = [0, 5).
   - **c.** $\sqrt{0} = 0, \sqrt{9} = 3$. The square root of every number in the interval (0, 9) is $\sqrt{(0,9)} = (0,3)$
   - **d.** The reciprocal of every number in the interval (0, 1) is $(1,\infty)$.
3. **Factors of a Number** Whenever you square a natural number, there is a repeated factor, so in listing the pairs of factors, that one would repeat, therefore the number of factors is an odd number. For example, squaring 6 gives us 36. The factors are 1 × 36, 2 × 18, 3 × 12,4 × 9, and 6 × 6. We do not count the repeated 6 twice, so there are an odd number of factors.

## P.2 Integer and Rational Number Exponents

1. **Relativity Theory** % error $= \dfrac{\left| mc^2\left[\dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}-1\right]-\dfrac{1}{2}mv^2\right|}{mc^2\left[\dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}-1\right]}\times 100 = \dfrac{\left| c^2\left[\dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}-1\right]-\dfrac{1}{2}v^2\right|}{c^2\left[\dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}-1\right]}\times 100$
   - **a.** For $v = 30$ meters per second, % error = $0.000303176$
   - **b.** For $v = 240$ meters per second, % error = $1.75483\times 10^{-6}$
   - **c.** For $v = 3\times 10^7$ meters per second, % error = $0.750628$
   - **d.** For $v = 1.5\times 10^8$ meters per second, % error = $19.1987$
   - **e.** For $v = 2.7\times 10^8$ meters per second, % error = $68.0755$
   - **f.** The percent errors is very small for everyday speeds.
   - **g.** % change $= \dfrac{|m_0 - m|}{m_0}\times 100$ where $m = \dfrac{m_0}{\sqrt{\frac{1-v^2}{c^2}}}$ and $v = 0.99c$. % change is approximately 609%.
   - **h.** As the speed of the object approaches the speed of light, the denominator of kinetic energy equation approaches 0, which implies that the kinetic energy is approaching infinity. Thus it would require an infinite amount of energy to move a particle at the speed of light.

## P.3 Polynomials

1. **Odd Numbers**

   An even number is a number that is a multiple of 2. Let $m$ and $n$ be natural numbers. Then $2m$ and $2n$ are even natural numbers. The product $(2m)(2n) = 4mn = 2(2mn)$, which is an even number (it is a multiple of 2). Therefore, an even number times an even number is an even number.

   An odd number is a number that is not a multiple of 2. Let $m$ and $n$ be natural numbers. Then $2m+1$ and $2n+1$ are odd numbers because they are not multiples of 2. The product $(2m+1)(2n+1) = 4mn + 2(m+n)+1$ is not a multiple of 2 (2 is not a common factor) and is therefore an odd number. Therefore, the product of two odd numbers is an odd number.

   Consider the product $2m(2n+1) = 4mn + 2m = 2(2mn+m)$. Because the product is a multiple of 2, it is an even number. Thus the product of an odd number and an even number is an even number.

**2. Prime Numbers**

**a.** The contrapositive of "If $A$, then $B$" is "If not $B$, then not $A$." The contrapositive of "If two triangles are congruent, then they are similar" is "If two triangles are not similar, then they are not congruent."

**b.** Fermat's Little Theorem does not say anything about $n$ when $n$ is not a prime number.

**c.** The converse of "If $A$, then $B$" is "If $B$, then $A$." The converse of Fermat's Little Theorem is "If $a$ is any natural number and $a^n - a$ is divisible by $n$, then $n$ is a prime number."

**d.** No. In fact, 561 is not a prime number. The converse of Fermat's Little Theorem is false.

**e.** No. See part (d).

**f.** A Carmichael number $n$ is a number for which $a^n - a$ is divisible by every natural number $a$ and $n$ is not prime. The first three Carmichael numbers are $561$, $1105$, and $1729$. Any Carmichael number can be used to show that the converse of Fermat's Little Theorem is false. Carmichael numbers are sometimes called pseudoprimes.

**P.4 Factoring**

**1. Geometry**

**a.** I + II + III

**b.** II + III + V

**c.** Because the area of I is the same as the area of V, the sum of the areas of region I, II, and III equals the sum of the areas of regions II, III, and V.

**2. Geometry**

$$(x+y)^2 = x^2 + 2xy + y^2$$

**3. Geometry**

$$\begin{aligned} x^3 - y^3 &= x(x-y)^2 + xy(x-y) + xy(x-y) + y^2(x-y) \\ &= (x-y)\left(x(x-y) + 2xy + y^2\right) \\ &= (x-y)\left(x^2 + xy + y^2\right) \end{aligned}$$

**P.5 Rational Expressions**

**1. Continued Fractions**

**a.** $C_2 = \dfrac{1}{1+\frac{1}{1+1}} = \dfrac{1}{1+\frac{1}{2}} = \dfrac{1}{1.5} = 0.\overline{6}$

**b.** $C_3 = \dfrac{1}{1+\dfrac{1}{1+\frac{1}{1+1}}} = \dfrac{1}{1+\dfrac{1}{1+\frac{1}{2}}} = \dfrac{1}{1+\frac{2}{3}} = \dfrac{1}{\frac{5}{3}} = \dfrac{3}{5} = 0.6$

**c.** $C_5 = \dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+1}}}}} = \dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\frac{1}{2}}}}} = \dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\frac{2}{3}}}} = \dfrac{1}{1+\dfrac{1}{1+\frac{3}{5}}} = \dfrac{1}{1+\frac{5}{8}} = \dfrac{8}{13}$

**d.** $\dfrac{-1+\sqrt{5}}{2} \approx 0.618$, $C_5 = \dfrac{8}{13} \approx 0.615$

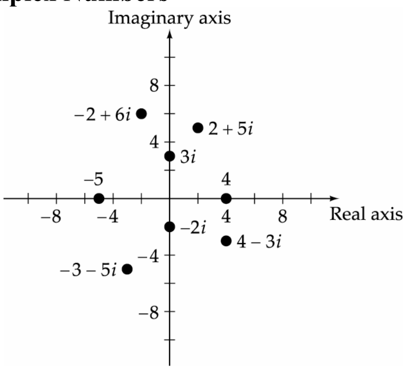
**2. Representation of $\pi$**

An excellent source for $\pi$ and its history can be found in an article by Dario Castellanos in *Mathematics Magazine*, vol. 61, no. 2 (April 1988). Here are two results (both attributable to Euler) from that article.

$$\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \frac{5^2}{6+\cdots}}} \quad \text{and} \quad \pi = 2\left(1 + \cfrac{2}{3 + \cfrac{1\cdot 3}{4 + \cfrac{3\cdot 5}{4 + \frac{5\cdot 7}{4+\cdots}}}}\right)$$

## P.6 Complex Numbers

**1. – 8.**




**9.** $|2+5i| = \sqrt{2^2+5^2}$

$= \sqrt{4+25}$

$= \sqrt{29}$

**10.** $|4-3i| = \sqrt{4^2+(-3)^2}$

$= \sqrt{16+9}$

$= \sqrt{25}$

$= 5$

**11.** $|-2+6i| = \sqrt{(-2)^2+6^2}$

$= \sqrt{4+36}$

$= \sqrt{40}$

$= \sqrt{4\cdot 10}$

$= 2\sqrt{10}$

**12.** $|-3-5i| = \sqrt{(-3)^2+(-5)^2} = \sqrt{9+25} = \sqrt{34}$

**13.** $|a+bi| = \sqrt{(a)^2+(b)^2} = \sqrt{a^2+b^2}$

$|-a-bi| = \sqrt{(-a)^2+(-b)^2} = \sqrt{a^2+b^2}$

**14.** A complex number and its additive inverse are the same distance from the origin in the complex plane. The real parts of a complex number and its additive inverse are the same distance from the imaginary axis but on opposite sides of the imaginary axis. The imaginary parts of a complex number and its additive inverse are the same distance from the real axis but on opposite sides of the real axis.

## Equations and Inequalities

CHAPTER 1 of *College Algebra and Trigonometry*

### 1.1 Linear Equations

**1. Perfect Games**

**a.** The probability that one of the teams will get a perfect game is $0.7^{27}$. Because every game involves exactly two teams, the probability of a perfect game in any one game is $2\left(0.7^{27}\right)$. Therefore, the number of perfect games we can expect in *x* games is $p = 2\left(0.7^{27}\right)x.$

**b.** A baseball almanac shows that during the years 1962 to 2002, there were 10 perfect games. It is difficult to determine the exact number of games played in this period, but it is about 71,000.

**c.** From part (a), we should expect the number of perfect games during the period 1962 to 2002 to be $p = 2\left(0.7^{27}\right)71{,}000 \approx 9.3$. This result is very close to the actual result found in part (b).

### 1.2 Formulas and Applications

**1. A Work Problem and Its Extensions**

**a.** If a pump can fill a pool in *A* hours, then the part of the pool it fills every hour is $\frac{1}{A}$.

If a pump can fill a pool in *B* hours, then the part of the pool it fills every hour is $\frac{1}{B}$.

Let *T* be the total time it takes the pumps to fill the pool when they both work together. $T\left(\frac{1}{A}\right)$ = the part of the pool filled by pump *A*. $T\left(\frac{1}{B}\right)$ = the part of the pool filled by pump *B*. Because we fill exactly 1 pool, we have

$$T\left(\frac{1}{A}\right)+T\left(\frac{1}{B}\right)=1$$
$$T\left(\frac{1}{A}\right)AB+T\left(\frac{1}{B}\right)AB=1\cdot AB$$
$$TB+TA=AB$$
$$T\left(B+A\right)=AB$$
$$T=\frac{AB}{A+B}$$

**b.** Using the procedure in part (a) yields $T=\dfrac{ABC}{AB+AC+BC}$

Observe that *T* is the product of the individual times divided by the sum of the products of the times taken two at a time.

**c.** $$T=\frac{A_1A_2A_3\cdots A_n}{\left(A_2A_3A_4\cdots A_n\right)+\left(A_1A_3A_4\cdots A_n\right)+\left(A_1A_2A_4\cdots A_n\right)\cdots+\left(A_1A_2A_3\cdots A_{n-1}\right)}$$

That is, *T* is given by the product of the *A*'s divided by the sum of products of the *A*'s taken ($n$ – 1) at a time.

**d.** One method is as follows: Use the alignment chart to determine that together the pump that can fill it in 6 hours working with the pump that can fill it in 12 hours would take a total of 4 hours. Think of these two pumps as one 4-hour pump. Now use the alignment chart again, using 4 hours and 6 hours as the individual times for two pumps to produce the final answer of 2.7 hours (nearest 0.1 hour).

**2. Resistance of Parallel Circuits**

In electronics it can be shown that if two resistors (one with resistance $R_1$ ohms and the other with resistance $R_2$ ohms) are placed in parallel, the total resistance *R* provided by the two resistors is

$$R=\frac{R_1R_2}{R_1+R_2}$$

This formula has the same form as the formula give in part 1(a). Although the setting is different, the mathematics needed to solve a parallel resistance problem is exactly the same as that used to solve a combined work problem. The parallel resistance problem can also be extended to consider more than two resistors, in a manner analogous to the work problems in 1(b) and 1(c).

**1.3 Quadratic Equations**

**1. The Sum and Product of the Roots Theorem**

The quadratic equation $ax^2+bx+c=0$ has roots

$$r_1=\frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad r_2=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

The sum of the roots is

$$r_1+r_2=\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2z}=\frac{-2b}{2a}=-\frac{b}{a}$$

The product of the roots is

$$r_1r_2=\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)=\frac{b^2-\left(\sqrt{b^2-4ac}\right)^2}{4a^2}=\frac{b^2-b^2+4ac}{4a^2}=\frac{c}{a}$$

**2. Visual Insight**

The reasons for the steps in President Garfield's proof of the Pythagorean Theorem are as follows:

**i)** The area of the large triangle plus the area of the two small triangles is equal to the total area of the region, which is a trapezoid.

**ii)** The height of the trapezoid is $a+c$. The sum of the bases of the trapezoid is also $a+b$.

**iii)** Expand the right side and subtract $ab$ from each side of the equation.

**iv)** Multiply each side of the equation by 2.

**1.4 Other Types of Equations**

**1. The Reduced Cubic**

**a.** Given the reduced cubic $x^3+mx+n=0$. Let $x=\frac{m}{3z}-z$.

$$\left(\frac{m}{3z}-z\right)^3+m\left(\frac{m}{3z}-z\right)+n=0$$

$$\frac{m^3-9m^2z^2+27mz^4-27z^6}{27z^3}+\frac{m^2-3mz^2}{3z}+n=0$$

$$\frac{m^3-9m^2z^2+27mz^4-27z^6+9m^2z^2-27mz^4+27nz^3}{27z^3}=0$$

$$m^3-9m^2z^2+27mz^4-27z^6+9m^2z^2-27mz^4+27nz^3=0$$

$$-27z^6+27z^3n+m^3=0$$

$$z^6-nz^3-\frac{m^3}{27}=0$$

**b.** Let $u=z^3$ and $u^2=z^6$. Then the last equation in part (a) can be written as $u^2-nu-\frac{m^3}{27}=0$.

At this point Francois Vieta knew he could solve the original reduced cubic in part (a), because he knew he could use the quadratic formula, which had been around for centuries, to solve the foregoing quadratic.

The work follows in part (c).

**c.** $$z^3=u=\frac{n\pm\sqrt{n^2+\frac{4m^3}{27}}}{2}=\frac{n}{2}\pm\frac{1}{2}\sqrt{n^2+\frac{4m^3}{27}}=\frac{n}{2}\pm\sqrt{\frac{n^2}{4}+\frac{m^3}{27}}$$

If we use the positive root in this equation, we can show that $z=\sqrt[3]{\frac{n}{2}+\sqrt{\frac{n^2}{4}+\frac{m^3}{27}}}$.

**d.** Rewrite the reduced cubic equation $x^3 + 3x = 14$ as $x^3 + 3x - 14 = 0$. Thus we have a reduced cubic equation with $m = 3$ and $n = -14$. Substituting for $z$ in the equation from part (c) gives

$$z = \sqrt[3]{\frac{-14}{2} + \sqrt{\frac{(-14)^2}{4} + \frac{3^3}{27}}} = \sqrt[3]{-7 + \sqrt{49+1}} = \sqrt[3]{-7 + 5\sqrt{2}} = \sqrt{2} - 1$$

It is easy to verify that $\sqrt[3]{-7 + 5\sqrt{2}} = \sqrt{2} - 1$. Just show that $\left(\sqrt{2} - 1\right)^3 = -7 + 5\sqrt{2}$. It is somewhat more difficult to determine that $\sqrt[3]{-7 + 5\sqrt{2}} = \sqrt{2} - 1$. The motivating idea is first to suspect that $\sqrt[3]{-7 + 5\sqrt{2}}$ can be expressed as $\sqrt{2} + k$ for some real constant $k$ and then solve $\left(\sqrt{2} + k\right)^3 = -7 + 5\sqrt{2}$ for $k$. Hence the real solution of $x^3 + 3x = 14$ is

$$x = \frac{m}{3z} - z = \frac{3}{3\left(\sqrt{2} - 1\right)} - \left(\sqrt{2} - 1\right) = \frac{1}{\sqrt{2} - 1} - \left(\sqrt{2} - 1\right) = 2 .$$

**2. Fermat's Last Theorem**

**The History of Fermat's Last Theorem**

The conjecture that $x^n + y^n = z^n$ is impossible for all integers $n > 2$ is known as *Fermat's Last Theorem*. The actual statement of the conjecture was given by Pierre de Fermat in 1637. Fermat wrote the conjecture as a paragraph in the margin of the text *The Arithmetic of Diophantus*.

> It is impossible to write a cubic as the sum of two cubes, a fourth power as the sum of two fourth powers, and in general any power beyond the second as the sum of two similar powers. For this I have discovered a truly wonderful proof but the margin of this book is too small to contain it.

Many mathematicians have tried to find either a proof of Fermat's Last Theorem or a counterexample to disprove the result. In 1780 Leonhard Euler proved the theorem for $n = 3$. Other mathematicians proved it true for $n = 5$, $n = 7$, and $n = 13$. Before June of 1993, the theorem had been established for $2 < n < 25{,}000$ .

**The Relationship Between Fermat's Last Theorem and the Pythagorean Theorem**

If we remove the restriction that $n$ must be larger than 2 and consider the equation $x^2 + y^2 = z^2$, then we can find an infinite number of solutions. For example, we know that $x = 3$, $y = 4$, and $z = 5$ are solutions of this equation, as are $3k$, $4k$, and $5k$ for any positive integer $k$. Because $x^2 + y^2 = z^2$ is in the form of the Pythagorean Theorem, positive-integer solutions of this equation are called Pythagorean triples.

**Dr. Andrew Wile's Proof of Fermat's Last Theorem**

In June of 1993, Andrew Wiles of Princeton University announced that he had produced a proof of Fermat's Last Theorem. At first it appeared that he had, in fact, written a proof of Fermat's Last Theorem. However, an error was soon discovered. At this time Andrew Wiles was extremely disappointed. He had spent over 7 years working on his proof. He had even given a presentation at Cambridge University in which he outlined his proof to his peers. At first Wiles could not find a way to repair his proof, but eventually, after an additional year of work, and with the assistance of the mathematician Richard Taylor, a valid proof of Fermat's Last Theorem was achieved. Additional information about Wiles' proof of Fermat's Last Theorem can be found on the NOVA ONLINE internet site *Solving Fermat: Andrew Wiles* at:

http://www.pbs.org/wgbh/nova/proof/wiles.html

Andrew Wiles' proof consists of a 150-page document. It makes use of 20th century mathematics that was not available to Fermat. Thus we are certain that Wiles' proof is not the same as the proof that Fermat indicated he had produced. It is interesting to note that on the NOVA ONLINE internet site mentioned above, Andrew Wiles states "I don't believe that Fermat had a proof."

## 1.5 Inequalities

**1. Triangles**

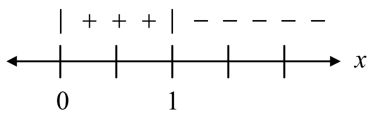
**a.** $x + x + 5 > x + 9, \Rightarrow x > 4$

**b.** $x + x^2 + x > 2x^2 + x$

$-x^2 + x > 0$

$-x(x-1) > 0$

$-x(x-1)$




$x$ is between 0 and 1.

**c.**

$$\frac{1}{x+2} + \frac{1}{x+1} > \frac{1}{x}$$

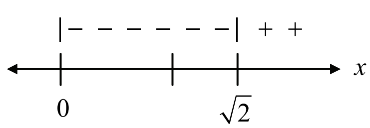
$$\frac{1}{x+2} + \frac{1}{x+1} - \frac{1}{x} > 0$$

$$\frac{x(x+1) + x(x+2) - (x+2)(x+1)}{x(x+1)(x+2)} > 0$$

$$\frac{x^2 + x + x^2 + 2x - x^2 - 3x - 2}{x(x+1)(x+2)} > 0$$

$$\frac{x^2 - 2}{x(x+1)(x+2)} > 0$$

$$\frac{x^2 - 2}{x(x+1)(x+2)} > 0 \text{ and } x > 0$$




The solution of the above inequalities is given by $x > \sqrt{2}$ .

**2. Fair Coins**

**a.** $\left|\frac{t - 500}{15.81}\right| \le 2.33$ if and only if

$$-2.33 \le \frac{t-500}{15.81} \le 2.33$$

$$-36.8373 \le t - 500 \le 36.8373$$

$$463.1627 \le t \le 536.8373$$

Because $t$ must be a nonnegative integer, we have $464 \le t \le 536$.

Therefore, according to this definition, a coin will be considered a fair coin if, in 1000 flips of the coin, the number of tails is greater than or equal to 464, but less than or equal to 536.

**b.** Answers will vary.

## 1.6 Variation and Applications

### 1. A Direct Variation Formula

Given $f(x) = k\,x$, prove $f(x_2) = f(x_1)\dfrac{x_1}{x_2}$.

Proof: $f(x_1) = k\,x_1$ and $f(x_2) = k\,x_2$, so

$$\frac{f(x_2)}{f(x_1)} = \frac{k\,x_2}{k\,x_1}$$
$$\frac{f(x_2)}{f(x_1)} = \frac{x_2}{x_1}$$
$$f(x_2) = f(x_1)\frac{x_2}{x_1}$$

Let $f_1 = 17$ kg, $f_2 = 22$ kg, and $d(f_1) = 8.5$ centimeters. Then

$$d(f_2) = d(f_1)\frac{f_2}{f_1}$$
$$d(22) = 8.5 \cdot \frac{22}{17}$$
$$d(22) = 11 \text{ centimeters}$$

### 2. An Inverse Variation Formula

Given $f(x) = \dfrac{k}{x}$, prove $f(x_2) = f(x_1)\dfrac{x_1}{x_2}$.

Proof: $f(x) = \dfrac{k}{x}$, so $f(x_2) = \dfrac{k}{x_2}$ and $f(x_1) = \dfrac{k}{x_1}$.

The ratio $\dfrac{f(x_2)}{f(x_1)}$ is given by

$$\frac{f(x_2)}{f(x_1)} = \frac{\frac{k}{x_2}}{\frac{k}{x_1}}$$
$$f(x_2) = f(x_1)\frac{\frac{k}{x_2}}{\frac{k}{x_1}} = f(x_1)\frac{k}{x_2}\cdot\frac{x_1}{k}$$

Thus $f(x_2) = f(x_1)\dfrac{x_1}{x_2}$.

Let $x_1 = 280$, $x_2 = 330$, and $V(280) = 2.4$.

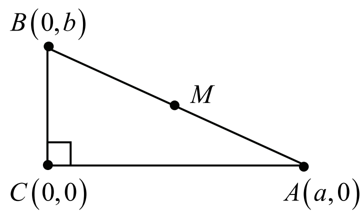
$$V(x_2) = V(x_1)\frac{x_1}{x_2}$$
$$V(330) = V(280)\frac{280}{330} = (2.4)\frac{280}{330} \approx 2.0 \text{ liters}$$

**Functions and Graphs**
CHAPTER 2 of *College Algebra and Trigonometry*

## 2.1 A Two-Dimensional Coordinate System and Graphs

**1. Verify a Geometric Theorem**

Consider any right triangle *ABC*. Place the triangle in a coordinate system with the vertex point of its right angle at the origin and its legs on the *x*-axis and the *y*-axis as shown on page 177 of the text.




By the midpoint formula, the coordinates of point *M* are

$$\left(\frac{a+0}{2},\frac{0+b}{2}\right)=\left(\frac{a}{2},\frac{b}{2}\right)$$

$$d(A,M)=\sqrt{\left(a-\frac{a}{2}\right)^2+\left(0-\frac{b}{2}\right)^2}=\sqrt{\left(\frac{a}{2}\right)^2+\left(-\frac{b}{2}\right)^2}=\frac{\sqrt{a^2+b^2}}{2}$$

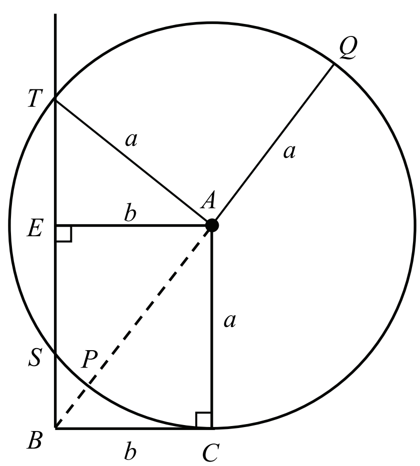
$$d(B,M)=\sqrt{\left(0-\frac{a}{2}\right)^2+\left(b-\frac{b}{2}\right)^2}=\sqrt{\left(-\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2}=\frac{\sqrt{a^2+b^2}}{2}$$

$$d(C,M)=\sqrt{\left(0-\frac{a}{2}\right)^2+\left(0-\frac{b}{2}\right)^2}=\sqrt{\left(-\frac{a}{2}\right)^2+\left(-\frac{b}{2}\right)^2}=\frac{\sqrt{a^2+b^2}}{2}$$

Thus the midpoint of the hypotenuse of any right triangle is equidistant from each of the vertices of the triangle.

**2. Solve a Quadratic Equation Geometrically**

**a.** See the following figure.




**b.** Using the figure from part (a), we see that

$$d(Q,B)=a+d(A,B)=a+\sqrt{a^2+b^2}$$

Let $x=a+\sqrt{a^2+b^2}$ in the equation $x^2=2ax+b^2$ to show that both sides equal the expression $2a^2+2a\sqrt{a^2+b^2}+b^2$. Thus $d(Q,B)$ is a solution of $x^2=2ax+b^2$.

**c.** Using the figure from part (a), we see that

$$d(P,B)=d(A,B)-d(A,P)=\sqrt{a^2+b^2}-a$$

Let $x=\sqrt{a^2+b^2}-a$ in the equation $x^2=-2ax+b^2$ to show that both sides equal the expression $2a^2-2a\sqrt{a^2+b^2}+b^2$. Thus $d(P,B)$ is a solution of $x^2=-2ax+b^2$.

**d.** Because $d(T,E)=\sqrt{a^2-b^2}$ and $d(E,B)=a,$ we know that $d(T,B)=a+\sqrt{a^2-b^2}$. Let $x=a+\sqrt{a^2-b^2}$ in $x^2=2ax-b^2$ to show that each side simplifies to $2a^2+2a\sqrt{a^2-b^2}-b^2$. Thus $d(T,B)$ is a solution of $x^2=2ax-b^2$.

Also note that $d(S,B)=a-d(E,T)=a-\sqrt{a^2-b^2}$. Let $x=a-\sqrt{a^2-b^2}$ in $x^2=2ax-b^2$ to show that each side simplifies to $2a^2-2a\sqrt{a^2-b^2}-b^2$. Thus $d(S,B)$ is a solution of $x^2=2ax-b^2$.

**2.2 Introduction to Functions**

**1. Day of Week**

**a.** Let $m=10, d=7, c=19,$ and $y=41$. Then

$$\begin{aligned} z &= \left\lfloor\frac{13m-1}{5}\right\rfloor+\left\lfloor\frac{y}{4}\right\rfloor+\left\lfloor\frac{c}{4}\right\rfloor+d+y-2c \\ &= \left\lfloor\frac{13\cdot 10-1}{5}\right\rfloor+\left\lfloor\frac{41}{4}\right\rfloor+\left\lfloor\frac{19}{4}\right\rfloor+7+41-2\cdot 19 \\ &= 25+10+4+7+41-38 \\ &= 49 \end{aligned}$$

The remainder of 49 divided by 7 is 0. Thus December 7, 1941, was a Sunday.

**b.** This one is tricky. Because we are finding a date in the month of January, we must use 11 for the month and we must use the previous year, which is 2009. Thus we let $m=11,\ d=1,\ c=19,$ and $y=109$. Then

$$\begin{aligned} z &= \left\lfloor\frac{13m-1}{5}\right\rfloor+\left\lfloor\frac{y}{4}\right\rfloor+\left\lfloor\frac{c}{4}\right\rfloor+d+y-2c \\ &= \left\lfloor\frac{13\cdot 11-1}{5}\right\rfloor+\left\lfloor\frac{109}{4}\right\rfloor+\left\lfloor\frac{19}{4}\right\rfloor+1+109-2\cdot 19 \\ &= 28+27+4+1+109-38 \\ &= 131 \end{aligned}$$

The remainder of 131 divided by 7 is 5. Thus January 1, 2010, will be a Friday.

**c.** Let $m=5,\ d=4,\ c=17,$ and $y=76$. Then

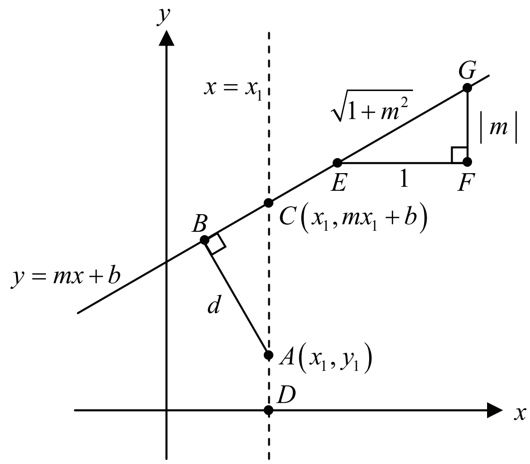
$$\begin{aligned} z &= \left\lfloor\frac{13m-1}{5}\right\rfloor+\left\lfloor\frac{y}{4}\right\rfloor+\left\lfloor\frac{c}{4}\right\rfloor+d+y-2c \\ &= \left\lfloor\frac{13\cdot 5-1}{5}\right\rfloor+\left\lfloor\frac{76}{4}\right\rfloor+\left\lfloor\frac{17}{4}\right\rfloor+4+76-2\cdot 17 \\ &= 12+19+4+4+76-34 \\ &= 81 \end{aligned}$$

The remainder of 81 divided by 7 is 4. Thus July 4, 1776 was a Thursday.

**d.** Answers will vary.

## 2.3 Linear Functions

### 1. Visual Insight




The distance $d$ between the point $A(x_1, y_1)$ and the line $y = mx + b$ is $d = \dfrac{|mx_1 + b - y_1|}{\sqrt{1+m^2}}$.

Triangle $ABC$ is similar to triangle $EFG$. Thus

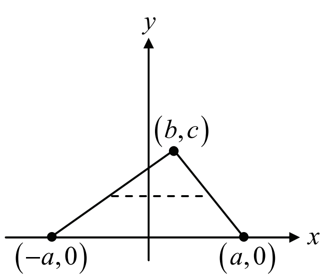
$$\frac{d(A,B)}{d(A,C)} = \frac{d(E,F)}{d(E,G)}$$
- Corresponding sides of similar triangles are proportional.

$$\frac{d}{|mx_1 + b - y_1|} = \frac{1}{\sqrt{1+m^2}}$$
- Substitute for $d(A,B)$, $d(A,C)$, and $d(E,G)$.

$$d = \frac{|mx_1 + b - y_1|}{\sqrt{1+m^2}}$$
- Solve for $d$, which is the distance from point $A$ to the line.

### 2. Verify Geometric Theorems

Place an arbitrary triangle in a coordinate system and label its vertices as shown.




The coordinates of the endpoints of the line segment that connects the midpoints of two of the sides of the triangle are given by

$$\left(\frac{-a+b}{2}, \frac{0+c}{2}\right) \quad \text{and} \quad \left(\frac{a+b}{2}, \frac{0+c}{2}\right)$$

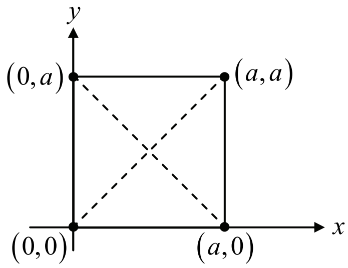
The slope of the line segment that connects the midpoints of two sides of the triangle is

$$\frac{\frac{c}{2} - \frac{c}{2}}{\frac{a+b}{2} - \frac{-a+b}{2}} = \frac{0}{\frac{2a}{2}} = \frac{0}{a} = 0$$

The slope of the third side of the triangle is also 0. Thus the two line segments are parallel.

**3. Verify Geometric Theorems**

Place an arbitrary square in a coordinate system and label its vertices as shown.




The slope of the diagonal through the origin is $a/a = 1$. The slope of the other diagonal is $a/(-a) = -1$. Applying the Parallel and Perpendicular Lines Theorem from Section 2.3 enables us to state that the diagonals are perpendicular.

The midpoint of the diagonal through the origin is the point $\left(\frac{a+0}{2}, \frac{0+a}{2}\right) = \left(\frac{a}{2}, \frac{a}{2}\right)$. The midpoint of the other diagonal is the point $\left(\frac{0+a}{2}, \frac{a+0}{2}\right) = \left(\frac{a}{2}, \frac{a}{2}\right)$. Thus the midpoint of each diagonal is $M = \left(\frac{a}{2}, \frac{a}{2}\right)$. Use the distance formula to determine that the distance from the midpoint *M* to each of the vertices is

$$\sqrt{\left(\frac{a}{2}-0\right)^2 + \left(\frac{a}{2}-a\right)^2} = \sqrt{\left(\frac{a}{2}\right)^2 + \left(-\frac{a}{2}\right)^2} = \sqrt{\frac{a^2}{a}} = \frac{a}{\sqrt{2}} = \frac{\sqrt{2}}{2}a$$

Thus the diagonals of any square are perpendicular bisectors of each other.

## 2.4 Quadratic Functions

**1. The Cubic Formula**

In the fifteenth century, the Italian mathematician Luca Pacioli (ca. 1445-1509) expressed his belief that cubic equations of the form $x^3 + bx^2 + cx + d = 0$ could not be solved, in general, by algebraic procedures involving radicals. His assertion challenged mathematicians to search for a "solution." The mathematician Scipione del Ferro (1465-1526) was not able to find such a solution, however, he did discover a formula that solved "depressed cubic" equations of the form $x^3 + mx = n$. At that time del Ferro decided to keep his solution a secret. The reason for this decision was based on the fact that mathematicians of that time period occasionally faced challenges from other mathematicians. If someone challenged him with a set of problems to be solved, then he could challenge the person to solve a set of reduced cubic equations. Even if he did poorly on his problems, he was confident that the challenger would not be able to solve any of the depressed cubic equations. In such a situation, del Ferro would be considered the winner of the challenge. This is in sharp contrast to the present time where a mathematician's reputation is based on his or her published works.

Just prior to his death, del Ferro shared his solution with his student Antonio Fior (ca. 1506-?). In 1535 Fior challenged the scholar Niccolo Fontana (1499-1557), (also known as Tartaglia – the Stammerer) to solve a set of depressed cubic equations. The challenge problems that Fior received from Tartaglia concerned several different mathematical topics. Thus Tartaglia was in a difficult situation. If he could find the solution to depressed cubic equations, then he would be able to solve all of the challenge problems, but if he could not discover the solution to depressed cubics, then he would probably not be able to solve any of the challenge problems.

Tartaglia began a desperate assault on the problem of finding the solution for depressed cubics. Many days passed and Tartaglia had not found the solution. However, Tartaglia was a talented mathematician and with continued effort he did discover the solution. With the solution in hand, it was easy for Tartaglia to solve all of the challenge problems. Fior was unable to solve all of the challenge problems he had received from Tartaglia. Thus Tartaglia was the undisputed winner of the challenge.

Tartaglia had discovered the solution for the depressed cubic equation, but the solution of the general cubic equations $x^3 + bx^2 + cx + d = 0$ was still unknown. It was at this point that the Italian mathematician Gerolamo Cardano (1501-1576) contacted Tartaglia to learn about his wonderful solution for depressed cubics. Tartaglia was not eager to share his solution with Cardano, but Cardano was persistent and after a period of about 4 years, Tartaglia met with Cardano. Before Tartaglia divulged the solution, he made Cardano take an oath to never publish the solution.

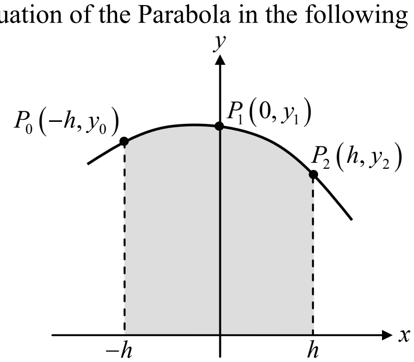
For many years Cardano kept Tartaglia's solution a secret. Cardano even determined how to solve the general cubic equation $x^3 + bx^2 + cx + d = 0$, but the procedure consisted of using a substitution to write the general cubic equation in the form of a depressed cubic equation. Cardano was stymied. He could not publish his wondrous new solution because of the promise he had made to Tartaglia. Ludovico Ferrari, a protégé of Cardano's even discovered how to solve general fourth degree polynomial equations, but his solution also depended upon writing the polynomial in the form of the depressed cubic equation and then applying Tartaglia's solution.

Then in 1543, Cardano and Ferrari happened to examine the mathematical papers of Scipione del Ferro. To their surprise, they found that del Ferro had in fact been the first to discover the solution of the depressed cubic equation and that he had left a written copy of his discovery. Cardano felt that he no longer needed to keep his oath to Tartaglia, because the original solution had been done by del Ferro and only rediscovered by Tartaglia. In 1545 Cardano published a mathematical manuscript that included the solution to the general cubic equation $x^3 + bx^2 + cx + d = 0$.

Note that the above account is only a brief discussion of the events surrounding the development of the solution of the general cubic equation. The mathematical techniques used in solving a cubic equation are given in the response to Project 1, of Section 3.4.

**2. Simpson's Rule**

The equation of the Parabola in the following figure is $y = Ax^2 + Bx + C$.




$$y_0 = A(-h)^2 + B(-h) + C = Ah^2 - Bh + C$$
$$y_1 = A(0)^2 + B(0) + C = C$$
$$y_2 = A(h)^2 + B(h) + C = Ah^2 + Bh + C$$

Thus $y_0 + 4y_1 + y_2 = Ah^2 - Bh + C + 4C + Ah^2 + Bh + C = 2Ah^2 + 6C$.

## 2.5 Properties of Graphs

**1. Dirichlet Function**

**a.** The domain of the Dirichlet function is the set of all real numbers.

**b.** The range of the Dirichlet function is $\{0,1\}$.

**c.** The Dirichlet function has an *x*-intercept at every point $(a, 0)$ where *a* is a rational number.

**d.** The Dirichlet function has a *y*-intercept of $(0, 0)$.

**e.** The Dirichlet function is an even function.

**f.** Graphing calculators use only rational numbers. Thus a graphing calculator will fail to show any of the points $(b, 1)$ where *b* is an irrational number.

**g.** The Dirichlet function is said to be discontinuous at every point because between any two rational numbers we can find an irrational number, and between any two irrational numbers we can find a rational number. The graph of the Dirichlet function looks like to horizontal lines (one at $y = 0$ and one at $y = 1$), except that the lines are "full of holes." Some mathematicians have called the Dirichlet function a shotgun function because the graph can be thought of as two horizontal lines in which we use a shotgun to blast holes. The holes on the line $y = 1$ occur at $(a, 1)$ for all rational numbers *a*. The holes on the line $y = 0$ occur at $(b, 0)$ for all irrational numbers *b*.

2. **Isolated Point**

   The point (1,1) is a solution of $y=\sqrt{(x-1)^2(x-2)}+1$ because $1=\sqrt{((1)-1)^2((1)-2)}+1$.

   Graphing utilities use only a finite number of domain values in the construction of a graph. Your graphing utility may have graphed the function for some values of $x<1$ and for some values of $x>1$, but not for $x=1$.

3. **A Line with a Hole**

   Graphing utilities use only a finite number of domain values in the construction of a graph. Thus your graphing utility may fail to show the proper result exactly at $x=2$.

4. **Finding a Complete Graph**

   To produce the part of the graph to the left of the *y*-axis, it may be necessary to enter the function as

   $$y=-3|x|^{5/3}-6|x|^{4/3}+2 \quad \text{for} \quad x\le 0$$

   This procedure may be necessary because some graphing utilities do not evaluate fractional powers of negative numbers. You need $y_1$ as shown in text and $y_2$ as above with a condition at the end:

   $$y_2=\left(-3|x|^{5/3}-6|x|^{4/3}+2\right)(x\le 0) \quad \text{(using TI-83)}$$

## 2.6 The Algebra of Functions

1. **A Graphing Utility Project**

   **a.** Let $x=5$ and $y=9$. Then $\text{Maximum}(5,9)=\frac{5+9}{2}+\frac{|5-9|}{2}=\frac{14}{2}+\frac{4}{2}=7+2=9$.

   **b.** Let $x=201$ and $y=80$.

   Then $\text{Maximum}(201,80)=\frac{201+80}{2}+\frac{|201-80|}{2}=\frac{281}{2}+\frac{121}{2}=140.5+60.5=201$.

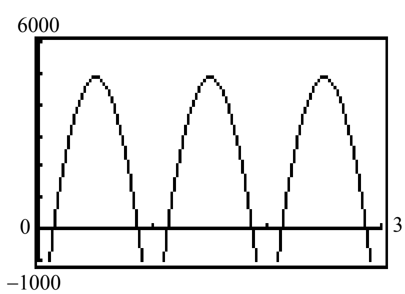
   **c.** For each real number $x\ge 0$, the graph of $y_3$ is a graph whose range value is the maximum of $y_1(x)$ and $y_2(x)$. Thus for each $x\ge 0$, $y_3$ can be graphed by plotting points from the graph of $y_1$ if its graph is higher than the graph of $y_2$, and by plotting points from the graph of $y_2$ if it is the higher graph.

   **d.** The domain of $y_1:\{x|x\in\Re\}$ (that is, all real numbers). The domain of $y_2:\{x|x\ge 0\}$. The domain of $y_3:\{x|x>0\}$. The domain of $y_3$ is the intersection of the domain of $y_1$ and the domain of $y_2$.

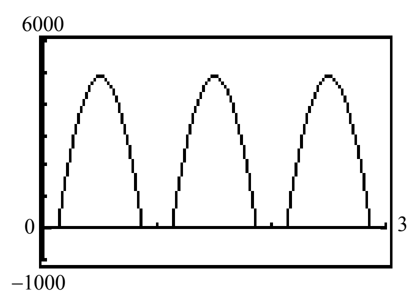
   **e.** $\text{Minimum}(f(x),g(x))=\frac{f(x)+g(x)}{2}-\frac{|f(x)-g(x)|}{2}$

2. **The Never-Negative Function**

   **a.**




   **b.**




   **a.** The graph of $M_+$ is the graph of *M* provided that $M\ge 0$. For each *x* such that $M<0$, the graph of $M_+$ is the point (*x*, 0).

   **b.** The maximum mosquito population is 4850 mosquitoes per acre. This maximum occurs at $t=\frac{1}{2}$, $t=\frac{3}{2}$ and $t=\frac{5}{2}$ (the middle of each month).

**c.** Assume the months have 30 days. The best time to fish if you wish to minimize your exposure to mosquitoes is during the first 3 or the last 3 days of each month. During this period, the mosquito population will be zero. During the period from the 4th day to the 26th day of the month, you will be exposed to mosquitoes.

## 2.7 Modeling Data Using Regression
## The Median-Median Line

**1.** $y \approx 24.8558x - 41.9712$

**2.** $y \approx 3.5429x + 58.0952$

**3.** **a.** $y = 2.0000x + 1.0000$

**b.** $y = 2.0000x + 1.0000$

**c.** If a least-squares regression line provides an exact fit for a set of data, then the median-median line for the data will be exactly the same as the least-squares regression line.

**4.** **a.** median-median line: $y \approx 1.3571x - 0.3333$; regression line: $y \approx 1.2636x + 0.5182$

**b.** median-median line: $y \approx 1.2143x + 0.3333$; regression line: $y \approx 0.6727x + 3.4727$

**c.** A median for a set of data is generally not changed by increasing one of the larger data values or by decreasing one of the smaller data values. For instance, consider the data 3, 5, 7, 11, 20, which has a median of 7. Now change the 20 to a larger number, say 30. This new set of data, 3, 5, 7, 11, 30, still has a median of 7. In general, a median-median line is less sensitive to a single change in one value than is a least-squares regression line. In fact, any single change in a set of data will cause a change in the least-squares regression line for the data, but often this same change will not change the median-median line for the data. Some mathematical texts refer to the median-median line of a set of data as the “resistant line” for the data.

**Polynomial and Rational Functions**
**CHAPTER 3 of *College Algebra and Trigonometry***

**3.1 The Remainder Theorem and the Factor Theorem**

**1. Horner's Polynomial Form**

**a.** $P(x) = 3x^5 - 4x^4 + 5x^3 - 2x^2 + 3x - 8$

$P(6) = 3(6)^5 - 4(6)^4 + 5(6)^3 - 2(6)^2 + 3(6) - 8 = 3(7776) - 4(1296) + 5(216) - 2(36) + 3(6) - 8$
$= 23328 - 5184 + 1080 - 72 + 18 - 8 = 18144 + 1080 - 72 + 18 - 8$
$= 19224 - 72 + 18 - 8 = 19152 + 18 - 8 = 19170 - 8 = 19162$

**b.** $P(x) = 3x^5 - 4x^4 + 5x^3 - 2x^2 + 3x - 8$
$= (3x^4 - 4x^3 + 5x^2 - 2x + 3)x - 8 = ((3x^3 - 4x^2 + 5x - 2)x + 3)x - 8$
$= (((3x^2 - 4x + 5)x - 2)x + 3)x - 8 = ((((3x - 4)x + 5)x - 2)x + 3)x - 8$

$P(6) = ((((3 \cdot 6 - 4)(6) + 5)(6) - 2)(6) + 3)(6) - 8 = (((18 - 4)(6) + 5)(6) - 2)(6) + 3)(6) - 8$
$= (((14)(6) + 5)(6) - 2)(6) + 3)(6) - 8 = (((84 + 5)(6) - 2)(6) + 3)(6) - 8$
$= (((89)(6) - 2)(6) + 3)(6) - 8 = ((534 - 2)(6) + 3)(6) - 8 = ((532)(6) + 3)(6) - 8$
$= (3192 + 3)(6) - 8 = (3195)(6) - 8 = 19170 - 8 = 19162$

The evaluation in part **b.** involves easier calculations.

**3.2 Polynomial Functions of Higher Degree**

**1.** The student is correct. The polynomial $P(n) = n^3 - n$ can be written in factored form as $P(n) = n(n-1)(n+1)$. In this form it is easy to see that $P(n)$ is the product of three consecutive natural numbers, one of which must be an even number and one of which must be a multiple of three. Thus $P(n)$ must be a multiple of 6 for any natural number $n$.

**3.3 Zeros of Polynomial Functions**

**1. Relationships Between Zeros and Coefficients**

**a.** $r_1 = 1,\ r_2 = 2,\ r_3 = 3.$
$P(x) = x^3 + C_1x^2 + C_2x + C_3 = x^3 - 6x^2 + 11x - 6 \Rightarrow C_1 = -6,\ C_2 = 11,\ C_3 = -6$
$1 + 2 + 3 = 6 = -(-6) \Rightarrow r_1 + r_2 + r_3 = -C_1$
$1(2) + 1(3) + 2(3) = 2 + 3 + 6 = 11 \Rightarrow r_1r_2 + r_1r_3 + r_2r_3 = C_2$
$1(2)(3) = 6 = -(-6) \Rightarrow r_1r_2r_3 = -C_3$
$1(2)(3) = 6 = (-1)^3(-6) \Rightarrow r_1r_2r_3 = (-1)^n C_n$

**b.** Responses will vary.

**3.4 The Fundamental Theorem of Algebra**

**1. Investigate the Roots of Cubic Equation**

**a.** Given $x^3 + bx^2 + cx + d = 0$. Let $x = y - \frac{b}{3}$ and substitute into the equation. Then simplify.

$$x^3 + bx^2 + cx + d = 0$$
$$\left(y - \frac{b}{3}\right)^3 + b\left(y - \frac{b}{3}\right)2 + c\left(y - \frac{b}{3}\right) + d = 0$$
$$y^3 - by^2 + \frac{yb^2}{3} - \frac{b^3}{27} + b\left(y^2 - \frac{2by}{3} + \frac{b^2}{9}\right) + c\left(y - \frac{b}{3}\right) + d = 0$$
$$y^3 + \left(c - \frac{b^2}{3}\right)y + \left(\frac{2b^2}{27} - \frac{bc}{3} + d\right) = 0$$

Now let $m = c - \frac{b^2}{3}$ and $n = -\left(\frac{2b^3}{27} - \frac{bc}{3} + d\right)$. Then $y^3 + my = n$.

**b.** To solve the equation $x^3 - 6x^2 + 20x - 33 = 0$, let $b = -6, c = 20$ and $d = -33$. Then

$$m = 20 - \frac{(-6)^2}{3} = 8 \text{ and } n = -\left(\frac{2(-6)^3}{27} - \frac{(-6)(20)}{3} - 33\right) = 9$$

A solution is given by

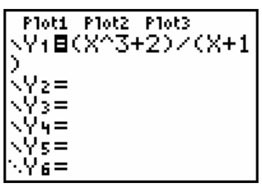
$$\begin{aligned} y &= \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} - \sqrt[3]{-\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} \\ &= \sqrt[3]{\frac{9}{2} + \sqrt{\frac{9^2}{4} + \frac{8^3}{27}}} - \sqrt[3]{-\frac{9}{2} + \sqrt{\frac{9^2}{4} + \frac{8^3}{27}}} \\ &= 1 \end{aligned}$$

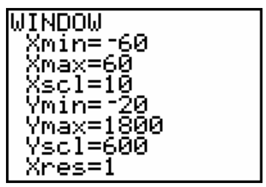
Substituting $y$ and $b$ into $x = y - \frac{b}{3}$ gives $x = 3$ is a solution of $x^3 - 6x^2 + 20x - 33 = 0$. To find the remaining solutions, use synthetic division to determine the reduced equation, which is $x^2 - 3x + 11 = 0$. The solutions of this equation are $\frac{3 + i\sqrt{35}}{2}$ and $\frac{3 - i\sqrt{35}}{2}$.

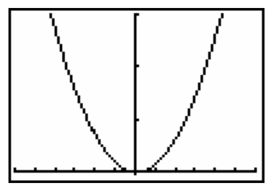
## 3.5 Graphs of Rational Functions and Their Applications

**1. Parabolic Asymptotes**

**a.**





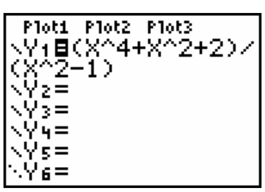


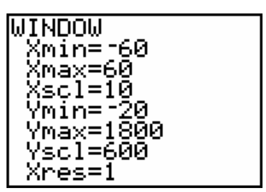
Yes. As $x \to \infty$ and $x \to -\infty$, the graph of $F$ approaches the graph of the parabola.

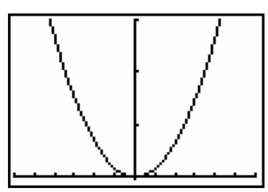
**b.** Divide $R(x)$ by $S(x)$ to find the quotient $Q(x)$. The equation $y = Q(x)$ is the equation of the parabolic asymptote.

**c.**

$$\begin{array}{r} x^2 + 2 \\ x^2 - 1\overline{\big)\,x^4 + x^2 + 2} \\ \underline{x^4 - x^2} \\ 2x^2 + 2 \\ \underline{2x^2 - 2} \\ 4 \end{array}$$







Yes

**d.** Answers will vary.

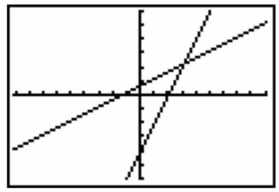
**Exponential and Logarithmic Functions**
CHAPTER 4 of *College Algebra and Trigonometry*

## 4.1 Inverse Functions

### 1. Intersection Points for the Graphs of $f$ and $f^{-1}$

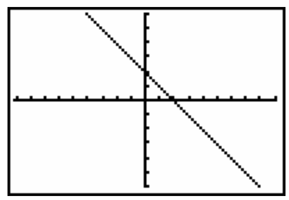
**i.**

$$f(x) = 2x - 4$$
$$x = 2y - 4$$
$$x + 4 = 2y$$
$$\tfrac{1}{2}x + 2 = y$$
$$f^{-1}(x) = \tfrac{1}{2}x + 2$$



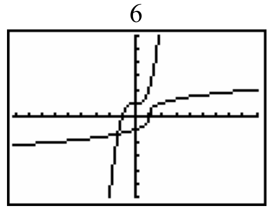

**ii.**

$$f(x) = -x + 2$$
$$x = -y + 2$$
$$y = -x + 2$$
$$f^{-1}(x) = -x + 2$$



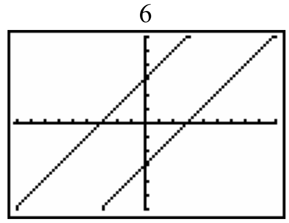

**iii.**

$$f(x) = x^3 + 1$$
$$x = y^3 + 1$$
$$x - 1 = y^3$$
$$\sqrt[3]{x-1} = y$$
$$f^{-1}(x) = \sqrt[3]{x-1}$$



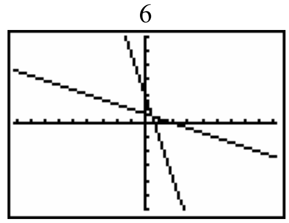

**iv.**

$$f(x) = x - 3$$
$$x = y - 3$$
$$x + 3 = y$$
$$f^{-1}(x) = x + 3$$



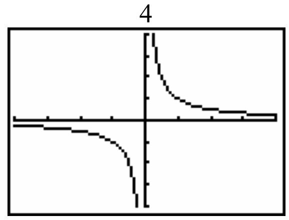

**v.**

$$f(x) = -3x + 2$$
$$x = -3y + 2$$
$$3y = -x + 2$$
$$f^{-1}(x) = -\tfrac{1}{3}x + \tfrac{2}{3}$$




**vi.**

$$f(x) = \tfrac{1}{x}$$
$$x = \tfrac{1}{y}$$
$$y = \tfrac{1}{x}$$
$$f^{-1}(x) = \tfrac{1}{x}$$

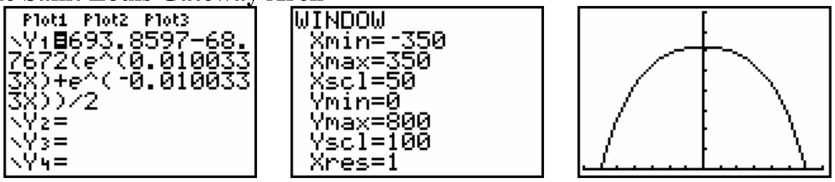



**a.** No **b.** They are equal. **c.** Yes. Consider the function $f(x) = x$.

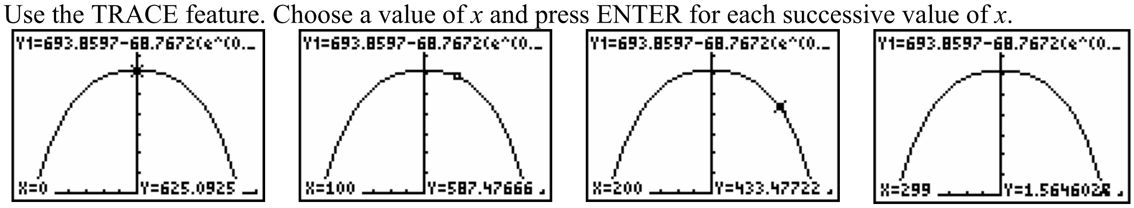
## 4.2 Exponential Functions and Their Applications

### 1. The Saint Louis Gateway Arch

**a.**

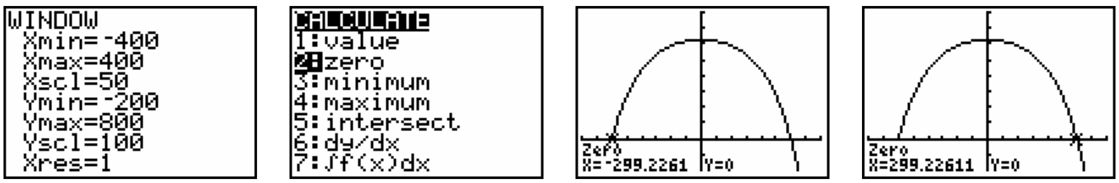


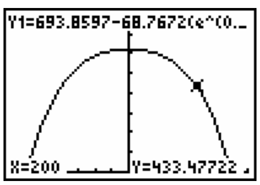

**b.** Use the TRACE feature. Choose a value of $x$ and press ENTER for each successive value of $x$.




$h(0) \approx 625.1$ feet  $h(100) \approx 587.5$ feet  $h(200) \approx 433.5$ feet  $h(299) \approx 1.6$ feet

**c.**




Width of the catenary at ground level $=|-299.2261 - 299.22611| = |-598.45221| \approx 598.5$ feet

Maximum height of the catenary $= h(0) \approx 625.1$ feet

**d.** $625.1 - 598.5 = 26.6$ feet

### 2. An Exponential Reward

**a.** From the chart, create a pattern from the total number of grains of wheat on squares 1 to $n$

For square $n = 1$, $1 = 2^1 - 1$
For square $n = 2$, $3 = 2^2 - 1$
For square $n = 3$, $7 = 2^3 - 1$
For square $n = 4$, $15 = 2^4 - 1$
For square $n = 5$, $31 = 2^5 - 1$
For square $n = 6$, $63 = 2^6 - 1$

So for square $n = 64$, $2^{64} - 1 = 1.8446744 \times 10^{19}$ grains is the total.

**b.** To find the total weight, multiply the number of grains by the weight for each one

$$(1.8446744 \times 10^{19})(0.000008) = 1.4757395 \times 10^{14} \text{ kilograms}$$

**c.** To find how long it would take, convert the weight to metric tons and divide by the amount per year

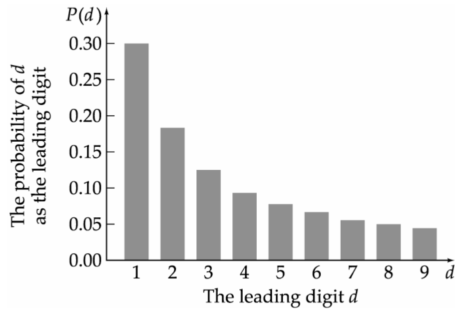
$$1.4757395 \times 10^{14}\,\text{kg} \cdot \frac{1 \text{ metric ton}}{1000 \text{ kg}} \cdot \frac{1\text{year}}{6.5 \times 10^{8} \text{ metric tons}} \approx 227 \text{ years}$$

838 

## 4.3 Logarithmic Functions and Their Applications

**1. a.**

| $d$ | $P(d)=\log\left(1+\frac{1}{d}\right)$ |
|---|---|
| 1 | 0.301 |
| 2 | 0.176 |
| 3 | 0.125 |
| 4 | 0.097 |
| 5 | 0.079 |
| 6 | 0.067 |
| 7 | 0.058 |
| 8 | 0.051 |
| 9 | 0.046 |




**b.** $P(6)=\log\left(1+\frac{1}{6}\right)=\log\left(\frac{7}{6}\right)\approx 0.067$ or 6.7%

**c.** $P(1)=\log\left(1+\frac{1}{1}\right)=\log(2)\approx 0.301$ or 30.1%

$P(9)=\log\left(1+\frac{1}{9}\right)=\log\left(\frac{10}{9}\right)\approx 0.046$ or 4.6%

$\frac{P(1)}{P(9)}=\frac{0.301}{0.046}\approx 6.54$ times as many

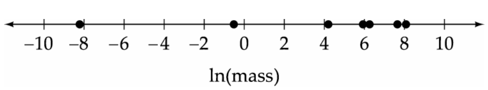
**d.** Most high school students are teenagers.

## 4.4 Logarithms and Logarithmic Scales

**1. Logarithmic Scales**

**a.**

| Animal | Mass (g) | ln(Mass) |
|---|---|---|
| Rotifer | 0.000000006 | –8.22 |
| Dwarf Goby | 0.30 | –0.52 |
| Lobster | 15,900 | 4.20 |
| Leatherback Turtle | 851,000 | 5.93 |
| Giant Squid | 1,820,000 | 6.26 |
| Whale Shark | 44,700,000 | 7.65 |
| Blue Whale | 120,000,000 | 8.08 |



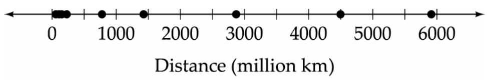

**b.** The logarithmic number line is more helpful when comparing different animals.

**c.** $10^1=10$ times heavier

$10^2=100$ times heavier

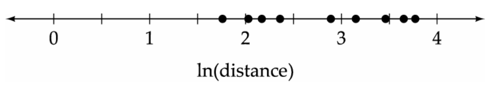
**2. Logarithmic Scales**

**a.**




**b.**

| Planet | Distance (million km) | ln(Distance) |
|---|---|---|
| Mercury | 58 | 1.76 |
| Venus | 108 | 2.03 |
| Earth | 150 | 2.18 |
| Mars | 228 | 2.36 |
| Jupiter | 778 | 2.89 |
| Saturn | 1427 | 3.15 |
| Uranus | 2871 | 3.46 |
| Neptune | 4497 | 3.65 |
| Pluto | 5913 | 3.77 |




**c.** Answers will vary.

**d.** $10^3 = 1000$. One distance is 1000 times greater than the other.

**3. Biological Diversity**

**a.** $D = -\left(\frac{1}{5}\log_2 \frac{1}{5} + \frac{1}{5}\log_2 \frac{1}{5} + \frac{1}{5}\log_2 \frac{1}{5} + \frac{1}{5}\log_2 \frac{1}{5} + \frac{1}{5}\log_2 \frac{1}{5}\right) = -\log_2 \frac{1}{5} = -\frac{\log \frac{1}{5}}{\log 2} \approx 2.322$

**b.** $D = -\left(\frac{1}{8}\log_2 \frac{1}{8} + \frac{3}{8}\log_2 \frac{3}{8} + \frac{1}{16}\log_2 \frac{1}{16} + \frac{1}{8}\log_2 \frac{1}{8} + \frac{5}{16}\log_2 \frac{5}{16}\right)$

$= -\left(\frac{\frac{1}{8}\log \frac{1}{8} + \frac{3}{8}\log \frac{3}{8} + \frac{1}{16}\log \frac{1}{16} + \frac{1}{8}\log \frac{1}{8} + \frac{5}{16}\log \frac{5}{16}}{\log 2}\right) \approx 2.055$

This system has less diversity than the one given in Table 1.

**c.** $D = -\left(\frac{1}{4}\log_2 \frac{1}{4} + \frac{3}{4}\log_2 \frac{3}{4}\right)$

$= -\left(\frac{0.25\log 0.25 + 0.75\log 0.75}{\log 2}\right) \approx 0.811$

This system has less diversity than the one given in Table 2.

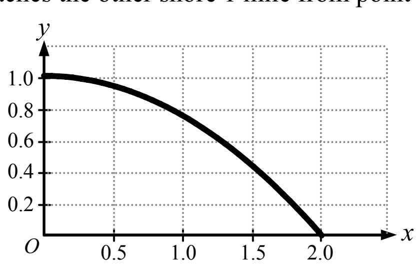
**d.** $D = -(1\ \log 1) = 0$

This value means that the system has no variety of species.

**4.5 Exponential and Logarithmic Equations**

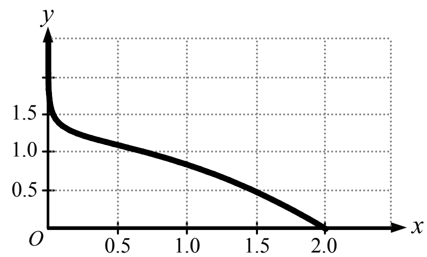
**1. Navigating**

**a.** If $v = w$, then the equation becomes $y = 1 - \left(\frac{x}{2}\right)^2$. The graph of this equation is shown below. The boat reaches the other shore 1 mile from point *O*.

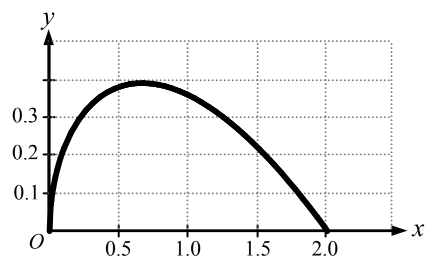


**b.** Now suppose $w > v$. For instance, let $w = 1.1v$. Then the equation becomes $y = \left(\frac{x}{2}\right)^{-0.1} - \left(\frac{x}{2}\right)^{2.1}$. The graph of this equation is shown below. In this case, the *y*-axis is a vertical asymptote and the boat never reaches the shore.




**c.** Now suppose $w < v$. For instance, let $w = 0.5v$. Then the equation becomes $y = \left(\frac{x}{2}\right)^{0.5} - \left(\frac{x}{2}\right)^{1.5}$. The graph of this equation is shown below. In this case, the boat reaches the shore at point *O*.




## 4.6 Exponential Growth and Decay

**1. A Declining Fish Population**

**a.** $P_0 = \dfrac{1000}{1 + (-0.3333)} = \dfrac{1000}{0.6667} = 1500$ fish

**b.** As $t \to \infty$, $P(t) \to 1000$ fish.

**2. A Declining Deer Population**

**a.** $P_2 = \dfrac{1800}{1 + (-0.25)(0.869)} = \dfrac{1800}{0.782} = 2300$ deer

**b.** As $t \to \infty$, $P(t) \to 1800$ deer.

**3. Modeling World Record Times in the Men's Mile Race**

**a.** $WR(107) = \dfrac{199.13}{1 + (-0.21726)(0.42536)} = 219.41 \text{ s} = 3 \text{ min}, 39.41 \text{ s}$

$WR(137) = \dfrac{199.13}{1 + (-0.21726)(0.33471)} = 214.75 \text{ s} = 3 \text{ min}, 34.75 \text{ s}$

**b.** As $t \to \infty$, $WR(t) \to 199.13 \text{ s} = 3 \text{ min}, 19.13 \text{ s}$.

## 4.7 Modeling Data with Exponential and Logarithmic Functions

**1. A Modeling Project**

**a.** Answers will vary
**b.** Answers will vary
**c.** Answers will vary
**d.** Answers will vary
**e.** Answers will vary
**f.** Answers will vary

**Trigonometric Functions**
CHAPTER 5 of *College Algebra and Trigonometry*

## 5.1 Angles and Arcs

**1. Conversion of Units**

**a.** $14 \text{ tollars} = \dfrac{14 \text{ tollars}}{1}\left(\dfrac{4 \text{ nollars}}{7 \text{ tollars}}\right)\left(\dfrac{3 \text{ mollars}}{5 \text{ nollars}}\right) = \dfrac{14\cdot 4\cdot 3}{7\cdot 5} \text{ mollars} = \dfrac{24}{5} \text{ mollars} = 4\dfrac{4}{5} \text{ mollars}$

**b.** $4\dfrac{4}{5} \text{ mollars} = 4 \text{ mollars} + \dfrac{4}{5} \text{ mollars} = 4 \text{ mollars} + \dfrac{4 \text{ mollars}}{5}\left(\dfrac{5 \text{ lollars}}{1 \text{ mollar}}\right) = 4 \text{ mollars} + 4 \text{ lollars}$

**c.** Start with the given quantity. Multiply by unit fractions that eliminate the given units and yield results in terms of the desired units. We knew we wanted to eliminate the tollars unit in the numerator. Thus we need to multiply by the unit fraction: $\left(\dfrac{4 \text{ nollars}}{7 \text{ tollars}}\right)$

**d.** $\left(\dfrac{\pi}{180°}\right)$

**2. Space Shuttle** Let $d$ be the portion of a revolution that the Galápagos Islands rotates as the shuttle revolves $1+d$ revolutions. The time required for the shuttle will be

$$t_1 = \left(\frac{(1+d) \text{ revolutions}}{1}\right)\left(\frac{2.231 \text{ hours}}{1 \text{ revolution}}\right) = (1+\text{d})(2.231) \text{ hours}$$

The time required for the earth to rotate $d$ revolutions is

$$t_2 = \left(\frac{d \text{ revolutions}}{1}\right)\left(\frac{23.933 \text{ hours}}{1 \text{ revolution}}\right) = 23.933d \text{ hours}$$

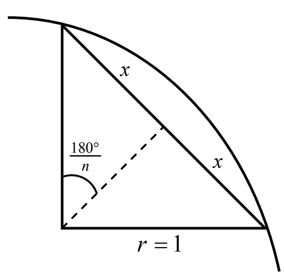
Setting $t_1 = t_2$ yields $2.231 + 2.231d = 23.933d$. Solving this equation gives $d \approx 0.1028$ revolution. The time required for the shuttle to complete $1+d$ revolutions is

$$(1+0.1028) \text{ revolutions}\left(\frac{2.231 \text{ hours}}{1 \text{ revolution}}\right) = 2.460 \text{ hours (to the nearest 0.001 hour)}$$

## 5.2 Trigonometric Functions of Acute Angles

**1. Perimeter of a Regular *n*-gon**

**a.**




$$P = 2xn$$

$$\sin\frac{180°}{n} = \frac{x}{r}$$

$$x = r\sin\frac{180°}{n} = \sin\frac{180°}{n}$$

$$P = 2n\sin\frac{180°}{n}$$

**b.** $P_{10} = 2(10)\sin\dfrac{180°}{10} \approx 6.18034$

$P_{50} = 2(50)\sin\dfrac{180°}{50} \approx 6.27905$

$P_{100} = 2(100)\sin\dfrac{180°}{100} \approx 6.282152$

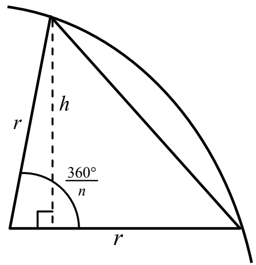
$P_{1000} = 2(1000)\sin\dfrac{180°}{1000} \approx 6.283175$

$P_{10,000} = 2(10,000)\sin\dfrac{180°}{10,000} \approx 6.283185$

$P_n$ approaches $2\pi$ as $n$ increases because the perimeter approaches the circumference of the circle.

**2. Area of a Regular *n*-gon**

**a.**




$$A = \frac{1}{2}hr \cdot n = \frac{n}{2} r \cdot r \sin\frac{360°}{n} = \frac{n}{2}\sin\frac{360°}{n}$$

$$\sin\frac{360°}{n} = \frac{h}{r}$$
$$h = r\sin\frac{360°}{n}$$
$$r = 1$$

**b.**
$$A_{20} = \frac{10}{2}\sin\frac{360°}{10} \approx 2.938926261$$
$$A_{50} = \frac{50}{2}\sin\frac{360°}{50} \approx 3.133330839$$
$$A_{100} = \frac{100}{2}\sin\frac{360°}{100} \approx 3.139525976$$
$$A_{1000} = \frac{1000}{2}\sin\frac{360°}{1000} \approx 3.141571983$$
$$A_{10,000} = \frac{10,000}{2}\sin\frac{360°}{10,000} \approx 3.141592447$$

$A_n$ approaches $\pi$ as $n$ increases because the area of the polygon approaches the area of the circle, which is $\pi \cdot 1^2 = \pi$.

## 5.3 Trigonometric Functions of Any Angle

**1. Find Sums or Products**

**a.** 0 The sum is 0 because $\cos n° = -\cos(180° - n°)$ for all integers $n$ such that $0° \le n \le 90°$.

**b.** 0 The sum is 0 because $\sin n° = -\sin(360° - n°)$ for all integers $n$ such that $0° \le n \le 180°$.

**c.** 0 The sum is 0 because $\cot n° = -\cot(180° - n°)$ for all integers $n$ such that $1° \le n \le 89°$.

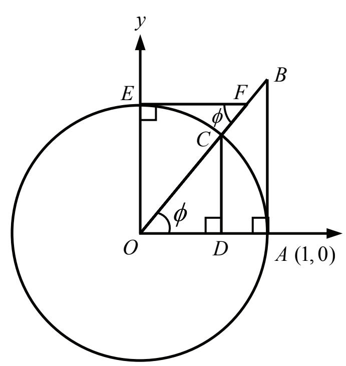
**d.** 0 The product is 0 because $\cos 90° = 0$ and the product of 0 and any real number is 0.

**e.** 179 This is the sum of 359 numbers. However, the sum can be regrouped so that it consists of 179 pairs of the form $(\cos x)^2 + (\cos 90° - x)^2 = (\cos x)^2 + (\sin x)^2 = 1$. The term that is not paired up with another number in the list is $\cos 90° = 0$. Thus the sum is 179.

## 5.4 Trigonometric Functions of Real Numbers

**1. Visual Insight**

Unit circle
$OD = \cos\phi$
$DC = \sin\phi$




**a.** Consider the triangle $ABC$. By definition,

$$\tan\phi = \frac{\text{opp}}{\text{adj}} = \frac{d(A,B)}{d(O,A)} = \frac{d(A,B)}{1} = \text{the length of line segment } AB.$$

**b.** Triangle *ODC* is similar to triangle *FEO*. Therefore,

$$\cot\phi = \frac{d(O,D)}{d(D,C)} = \frac{d(F,E)}{d(E,O)} = \frac{d(F,E)}{1} = \text{the length of line segment } EF.$$

**c.** Consider triangle *AOB*. By definition,

$$\sec\gamma = \frac{\text{hyp}}{\text{adj}} = \frac{d(O,B)}{d(O,A)} = \frac{d(O,B)}{1} = \text{the length of line segment } OB.$$

**d.** Triangle *OEF* is similar to triangle *CDO*. Therefore,

$$\csc\gamma = \frac{d(O,C)}{d(C,D)} = \frac{d(F,O)}{d(O,E)} = \frac{d(F,O)}{1} = \text{the length of line segment } OF.$$

**2. Functions Defined by a Square**

**a.** $\text{ssin } 3.2 = 0.8$ because traveling 3.2 units counterclockwise around the square from $(1,0)$ places you at the point $(-1,0.8)$. The *y*-value of this ordered pair is 0.8.

**b.** $\text{scos } 4.4 = -1$ because traveling 4.4 units counterclockwise around the square from $(1,0)$ places you at the point $(-1,-0.4)$. The *x*-value of this ordered pair is –1.

**c.** $\text{ssin } 5.5 = -1$ and $\text{scos } 5.5 = -\frac{1}{2}$, thus, $\text{stan } 5.5 = \frac{-1}{\left(-\frac{1}{2}\right)} = 2$.

**d.** $\text{ssin } 11.2 = 0.8$ (Note that the function $y = \text{ssin } x$ is periodic with a period of 8. Thus $\text{ssin } 11.2 =$ ssin 3.2 which equals 0.8 from part (a).)

**e.** $\text{scos} - 5.2 = -0.8$.

**f.** $\text{stan} - 6.5 = \frac{\text{ssin}(-6.5)}{\text{scos}(-6.5)} = \frac{1}{0.5} = 2$

## 5.5 Graphs of the Sine and Cosine Functions

**1. Cepheid Variable Stars and the Period-Luminosity Relationship**

**a.** A Cepheid is usually a population I giant yellow star, pulsing regularly by expanding and contracting, resulting in a regular oscillation of its luminosity. Named for the prototype of this class found in the constellation Cepheus, classical Cepheids have periods from about 1.5 days to over 50 days and are Population I stars. The longer the period of such a star, the greater its natural brightness; this relationship was discovered in 1912 by the American astronomer Henrietta Leavitt (b. 1868-d. 1921). The relationship between a Cepheid variable's luminosity and variability period is quite precise, and has been used as a standard candle for almost a century.

**b.** Because of this correlation (discovered by Henrietta Leavitt in 1912), a Cepheid variable can be used as a standard candle to determine the distance to its host cluster or galaxy. Since the period-luminosity relation can be calibrated with great precision using the nearest Cepheid stars, the distances found with this method are among the most accurate available.

## 5.6 Graphs of the Other Trigonometric Functions

**1. A Technology Question**

The function $f(x) = \tan x$ is undefined at $x = \frac{3}{2}\pi$, which is between the domain values 4.7123 and 4.7124. Because $y = \tan x$ approaches $\infty$ as *x* approaches $\frac{3}{2}\pi$ from the left and $y = \tan x$ approaches $-\infty$ as *x* approaches $\frac{3}{2}\pi$ from the right, it is possible to produce large changes in your range values with small changes in your domain values as they change from slightly less than $\frac{3}{2}\pi$ to slightly greater than $\frac{3}{2}\pi$.

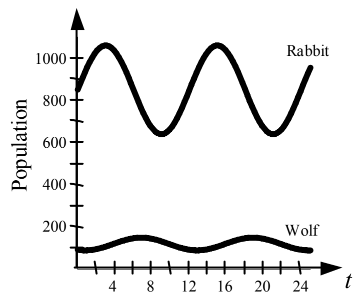
**2. Solutions of a Trigonometric Equation**

The equation $\tan\left(\frac{1}{x}\right) = 0$ has an infinite number of solutions on the interval $-1 \le x \le 1$. This can be determined by observing that as $x$ varies from –1 to 1, the fraction $1/x$ takes on all values less than or equal to –1 and all values greater than or equal to 1. Because the tangent function is periodic with a period of $\pi$, the expression $\tan\left(\frac{1}{x}\right)$ will equal 0 whenever $1/x$ is a multiple of $\pi$ for – for instance, when $x = \frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \ldots$. Thus there are an infinite number of solutions.

## 5.7 Graphing Techniques

**1. Predator-Prey Relationships**

The graphs are shown below. Because the assumption is that the wolves prey on the rabbits, as the rabbit population increases, there is more food for the wolves, which in turn allows the wolf population to increase. However, as the wolf population increases, the demand for rabbits increases, and the rabbit population starts to decline. This effects a decline in the wolf population. But as the wolf population declines, there is less danger to the rabbits, and their population starts to rise. The process then repeats itself.

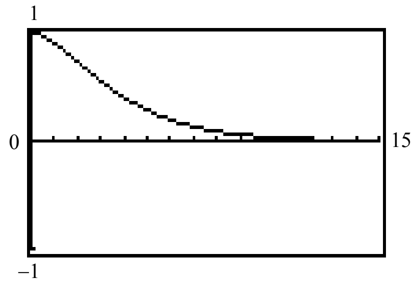



A possible equation for the graph given in the text of the population model for rabbits is $r(t) = \frac{50}{3}t + 200\sin\left(\frac{\pi t}{6}\right) + 400$. This equation is based on the assumption that there is a linear increase in the sine function. Other answers may be given.

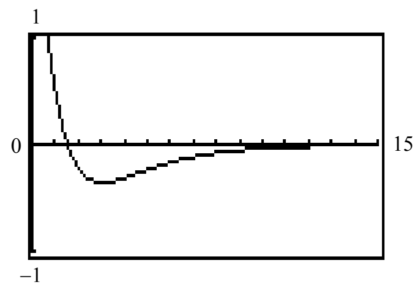
**5.8 Harmonic Motion—An Application of the Sine and Cosine Functions**

**1. Three Types of Damped Harmonic Motion**

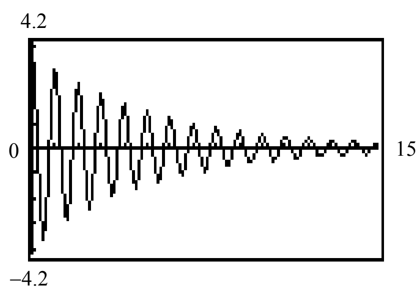
The displacement of $f(t)$ starts at its maximum when $t = 0$ and then descends toward its equilibrium point, getting closer and closer as $t$ increases without bound. See the following figure.




The displacement of $g(t)$ starts at its maximum when $t = 0$. It next falls to a minimum displacement below the equilibrium position and then rises to approach equilibrium as $t$ increases without bound. See the following figure.




The displacement of $h(t)$ starts at its maximum displacement when $t = 0$. As $t$ increases, it oscillates about its equilibrium point. As it oscillates, the maximum displacement of each cycle tends to 0. See the following figure.




**2. Logarithmic Decrement**

**a.** $\gamma \approx 3.51$

**b.** $\Delta \approx 1.26$; $\ln \gamma = \Delta$

**Trigonometric Equations and Identities**
CHAPTER 6 of *College Algebra and Trigonometry*

## 6.1 Verification of Trigonometric Identities

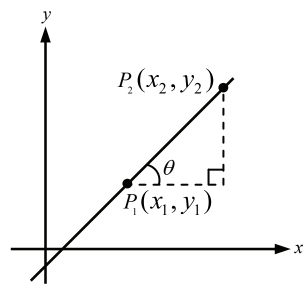
### 1. Grading a Quiz

1,2, and 4 are correct.

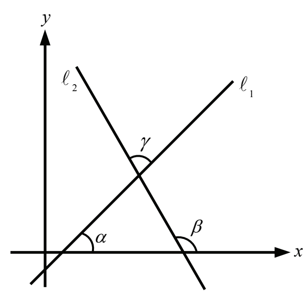
## 6.2 Sum, Difference, and Cofunction Identities

### 1. Intersecting Lines

**a.** In the following figure, we see that if a line intersects the $x$-axis at an angle of $\theta$, then $\tan\theta = \dfrac{y_2 - y_1}{x_2 - x_1} = m$, which is the slope of the line.




Let $\gamma$ be the smallest positive angle from $\ell_1$ to $\ell_2$, as shown in the following figure.




It is possible to show that $\beta = \alpha + \gamma$. (Use the exterior angle theorem and the theorem on vertical angles.) Thus $\gamma = \beta - \alpha$, and $\tan\gamma = \tan(\beta - \alpha) = \dfrac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha} = \dfrac{m_2 - m_1}{1 + m_1 m_2}$.

**b.** The line $\ell_1$ given by $y = x + 5$ has a slope of $m_1 = 1$. The slope of the line $\ell_2$ given by $y = 3x - 4$ has a slope of $m_2 = 3$. From part (a), we know that the tangent of the angle $\gamma$ (the acute angle between the lines) is

$$\tan\gamma = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{3 - 1}{1 + (1)(3)} = \frac{2}{4} = \frac{1}{2}$$

A graph of $y = \tan\gamma$ and $y = \dfrac{1}{2}$ (on the interval $0 < \gamma < 90°$) shows that $\gamma \approx 26.6°$.

**c.** Let $m_2$ be the slope of the second line. From part (a),

$$\tan 60° = \frac{m_2 - 0.5}{1+(0.5)(m_2)}$$

$$\sqrt{3} = \frac{m_2 - 0.5}{1+(0.5)(m_2)}$$

$$\sqrt{3} + 0.5\sqrt{3}m_2 = m_2 - 0.5$$

$$\sqrt{3} + 0.5 = m_2 - 0.5\sqrt{3}m_2$$

$$\sqrt{3} + 0.5 = m_2\left(1 - 0.5\sqrt{3}\right)$$

$$\frac{\sqrt{3}+0.5}{1-0.5\sqrt{3}} = m_2$$

$$m_2 \approx 16.66$$

The second line passes through the point (1,5) with a slope of about 16.66, so

$$y - 5 \approx 16.66(x-1)$$

$y \approx 16.66x - 11.66$ • The equation of line $\ell_2$ in slope-intercept form

## 6.3 Double- and Half-Angle Identities

**1. Visual Insight**

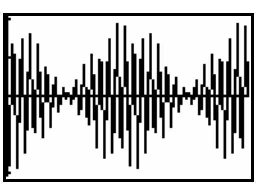
The measure of a central angle is equal to the measure of its intercepted arc. The measure of an inscribed angle is one-half the measure of its intercepted arc. Therefore, the measure of the small marked angle at the right must be half the measure of the central angle $\theta$. Thus the measure of the small marked angle is $\theta/2$. The side opposite $\theta$ in the small triangle is $\sin\theta$. The side adjacent to $\theta$ in the small triangle is $\cos\theta$.

By definition, $\tan\frac{\theta}{2} = \frac{\text{opp}}{\text{adj}} = \frac{\sin\theta}{1+\cos\theta}$.

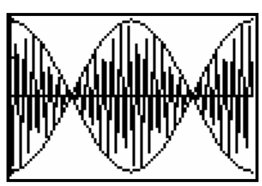
## 6.4 Identities Involving the Sum of Trigonometric Functions

**1. Beats**

**a.**



Y3



Y3, Y4, Y5

The graph of Y3 is bounded by the graphs of Y4 and Y5.

**b.** $Y3 = 2\sin 2\pi\left(\frac{442+440}{2}x\right)\cdot\cos 2\pi\left(\frac{442-440}{2}x\right)$

$= 2\sin(2\pi\cdot 441x)\cos(2\pi x)$

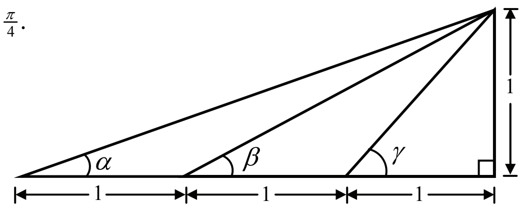
**c.** $568 - 564 = 4$ beats per second

**d.** 2 beats per second

## 6.5 Inverse Trigonometric Functions

**1. Visual Insight**

**a.** Verify the identity $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}=\frac{\pi}{4}$.




Let $\alpha=\tan^{-1}\frac{1}{3}$ and $\beta=\tan^{-1}\frac{1}{2}$, as in the figure. Thus $\tan\alpha=\frac{1}{3}$ and $\tan\beta=\frac{1}{2}$.

$$\begin{aligned}\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}&=\alpha+\beta\\&=\tan^{-1}\left[\tan(\alpha+\beta)\right]\\&=\tan^{-1}\left(\frac{\tan\alpha+\tan\beta}{1-\tan\beta\tan\beta}\right)\\&=\tan^{-1}\left[\frac{1/3+1/2}{1-(1/3)(1/2)}\right]\\&=\tan^{-1}1=\frac{\pi}{4}\end{aligned}$$

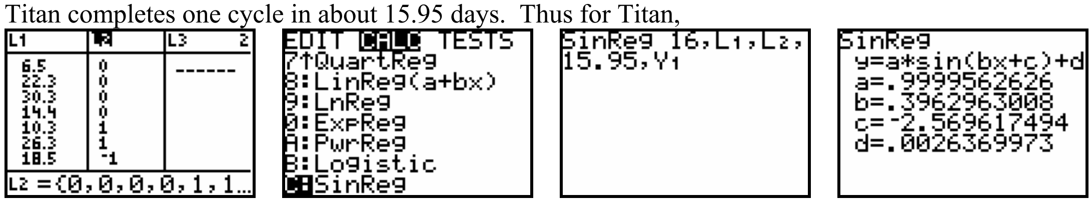
Hence $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}=\frac{\pi}{4}$.

**b.** The figure in part (a) shows that $\alpha=\tan^{-1}\frac{1}{3}$, $\beta=\tan^{-1}\frac{1}{2}$, and $\gamma=\frac{\pi}{4}$. Substituting in the result from part (a) produces $\alpha+\beta=\gamma$.

## 6.6 Trigonometric Equations

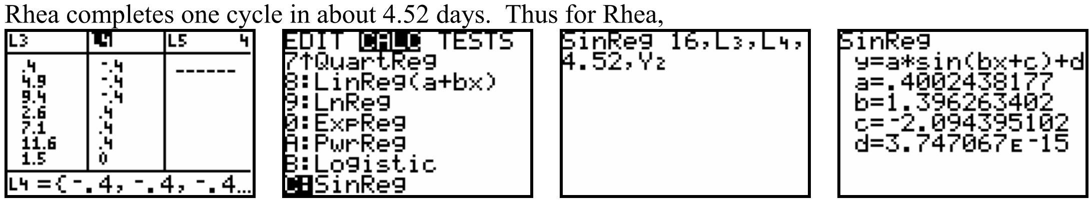
**1. The Moons of Saturn**

**a.** Titan completes one cycle in about 15.95 days. Thus for Titan,




$y_1\approx1.0000\sin(0.3963x-2.5696)+0.0026$

**b.** Rhea completes one cycle in about 4.52 days. Thus for Rhea,

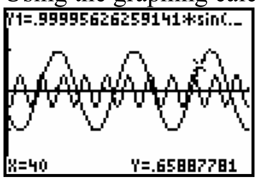


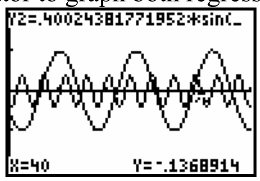

$y_2\approx0.4002\sin(-1.3963x-2.0944)-0.0000$

**c.** This solution can be found by using the regression formula directly, or using the graphing calculator graph at value $x = 40$ as shown below.
At 11:59PM on May 10, 2000, $x = 40$,

Using the regression formulas from parts **a** and **b**,
$y_1(40) \approx 1.0000\sin(0.3963(40) - 2.5696) + 0.0026 \approx 0.65888$
$y_2(40) \approx 0.4002\sin(-1.3963(40) - 2.0944) - 0.0000 \approx -0.13689$

Using the graphing calculator to graph both regression formulas.



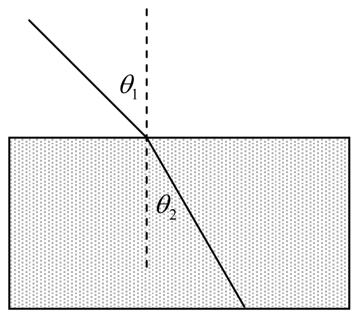

They were on opposite sides of Saturn.

**Applications of Trigonometry**
CHAPTER 7 of *College Algebra and Trigonometry*

**7.1 The Law of Sines**

1. **Fermat's Principle and Snell's Law**
   Fermat's Principle: Light traveling from one point to another will follow a path such that, compared with nearby paths, the time required is either a minimum or maximum or will remain unchanged. Snell's Law, which follows, is derived by using Fermat's Principle.

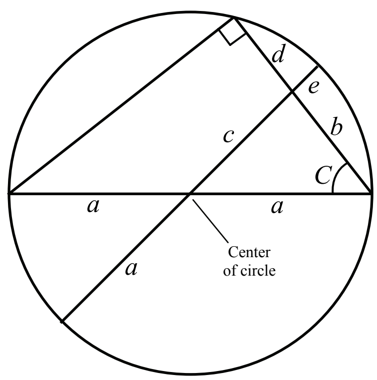
   Snell's Law: $\frac{\sin\theta_1}{\sin\theta_2} = n_{21}$, where $n_{21}$ is a constant called the index of refraction of medium 2 with respect to medium 1. See the diagram below. (*Note*: The index of refraction depends on the wavelength. As the wavelength increases, the index of refraction decreases.)




Because a diamond has a higher index of refraction than glass, light entering the diamond is refracted at a greater angle than the same light entering a piece of glass. The result is a narrower "rainbow" as the light leaves the diamond than as it leaves the glass.

**7.2 The Law of Cosines and Area**

1. **Visual Insight**




$bd = (a+c)e$ • If two chords of a circle intersect, then the product of the lengths of the segments on one chord equals the product of the lengths of the segments on the other chord.

$bd = (a+c)(a-c)$ • $e = a-c$

$b(2a\cos C - b) = (a+c)(a-c)$ • In the right triangle $\cos C = (d+b)/(2a)$. Solving for $d$ gives us $d = 2a\cos C - b$.

$2ab\cos C - b^2 = a^2 - c^2$ • Simplify.

$c^2 = a^2 + b^2 - 2ab\cos C$ • Solve for $c^2$.

## 7.3 Vectors

**1. Same Direction or Opposite Directions** If $c > 0$ and $\mathbf{v} = c\mathbf{w}$, then **v** and **w** have the same direction, and the angle between the vectors is $\theta = 0°$. Thus $\dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|} = \cos 0° = 1$.

If $c < 0$ and $\mathbf{v} = c\mathbf{w}$, then **v** and **w** are vectors that have opposite directions, and the angle between the vectors is $\theta = 180°$. Thus $\dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|} = \cos 180° = -1$.

**2. The Law of Cosines and Vectors**

$$\begin{aligned}\|\mathbf{v}-\mathbf{w}\|^2 &= (\mathbf{v}-\mathbf{w})(\mathbf{v}-\mathbf{w})\\ &= \mathbf{v}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{w} - \mathbf{w}\cdot\mathbf{v} + \mathbf{w}\cdot\mathbf{w}\\ &= \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\mathbf{v}\cdot\mathbf{w}\end{aligned}$$

- Because $\mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\,\|\mathbf{w}\|\cos\alpha$, we see that this equation is a restatement of the Law of Cosines in vector form.

**3. Projection Relationships** Let **v** and **w** be two nonzero vectors. Let $\alpha$ be the measure of the angle between the vectors. By definition, $\text{proj}_{\mathbf{w}}\mathbf{v} = \|\mathbf{v}\|\cos\alpha$.

**a.** If $\text{proj}_{\mathbf{w}}\mathbf{v} = 0$, then $\|\mathbf{v}\|\cos\alpha = 0$, and $\cos\alpha = 0$, which implies $\alpha = 90°$. Hence **v** and **w** are perpendicular (orthogonal).

**b.** If $\text{proj}_{\mathbf{w}}\mathbf{v} = \|\mathbf{v}\|$, then $\|\mathbf{v}\|\cos\alpha = \|\mathbf{v}\|$, and $\cos\alpha = 1$. Thus, $\alpha = 0°$. Hence **v** and **w** have the same direction.

## 7.4 Trigonometric Form of Complex Numbers

**1. A Geometrical Interpretation**

Multiplication of a real number *a* by *i* produces the product $ai$. Note that in an Argand diagram, the numbers *a* and $ai$ are both placed the same distance from the origin. Also note that the position of $ai$ can be determined by rotating the position of the real umber $a$ $90°$ counterclockwise about the origin.

Multiplication of a complex number $a+bi$ by $i$ also produces a number that is located in an Argand diagram the same distance from the origin as the original number $a+bi$. Once again, the position of the product $-b+ai$ can be determined by rotating the position of the original number $a+bi$ $90°$ counterclockwise about the origin.

It is also worth noting that multiplicity a number by $-1 = i^2$ can be thought of geometrically as a $180°$ counterclockwise rotation of the original number about the origin.

## 7.5 De Moivre's Theorem

**1. Verify Identities**

$$z = (\cos\theta + i\sin\theta)$$

$$z^2 = \cos 2\theta + i\sin 2\theta \text{ by De Moivre's Theorem.}$$

$$z^2 = \cos^2\theta + 2i\sin\theta\cos\theta + i^2\sin^2\theta \text{ by squaring both sides.}$$

Because $z^2 = z^2$, we have

$$(\cos 2\theta + i\sin 2\theta) = (\cos^2\theta + 2i\sin\theta\cos\theta + i^2\sin^2\theta)$$

$$(\cos 2\theta + i\sin 2\theta) = (\cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta)$$

Equating the real parts, we have

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

Equating the imaginary parts, we have

$$\sin 2\theta = 2\sin\theta\cos\theta$$

**2. Discover Identities**

$x = (\cos\theta + i\sin\theta)$

$z^4 = \cos 4\theta + i\sin 4\theta$ by De Moivre's Theorem.

$z^4 = \cos^4\theta + 4i\cos^3\theta\sin\theta + 6i^2\cos^2\theta\sin^2\theta + 4i^3\cos\theta\sin^3\theta + i^4\sin\theta$ by taking $z$ to the fourth power.

Because $z^4 = z^4$, we have

$$\begin{aligned}(\cos 4\theta + i\sin 4\theta) &= (\cos^4\theta + 4i\cos^3\theta\sin\theta + 6i^2\cos^2\theta\sin^2\theta + 4i^3\cos\theta\sin^3\theta + i^4\sin\theta)\\ &= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta\end{aligned}$$

Equating the real parts, we have

$$\begin{aligned}\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta\\ &= \cos^4\theta - 2\cos^2\theta\sin^2\theta + \sin^4\theta - 4\cos^2\theta\sin^2\theta\\ &= \left(\cos^2\theta - \sin^2\theta\right)^2 - 4\cos^2\theta\sin^2\theta\\ &= \cos^2 2\theta - 2\sin^2 2\theta\end{aligned}$$

Equating the imaginary parts, we have

$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$$

**Topics in Analytic Geometry**
CHAPTER 8 of *College Algebra and Trigonometry*

## 8.1 Parabolas

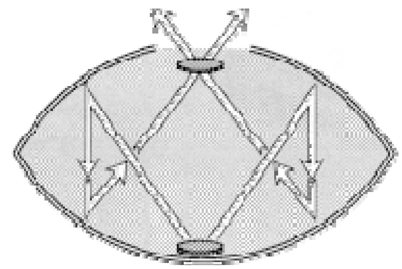
1. **3-D Optical Illusion**

Responses will vary. For example, http://www.exploratorium.edu/snacks/parabolas.html gives the following explanation.

You are seeing an image formed by two concave mirrors facing one another. The object is placed at the center of the bottom mirror. The curvature of the mirrors is such that the object is at the focal point of the top mirror.

When light from a point on the object hits the top mirror, it reflects in parallel rays. These parallel rays hit the bottom mirror and reflect so that they reassemble to form a point located at one focal length from the bottom mirror. The mirrors are placed so that the focal point of the bottom mirror is located at the hole in the top of the device. The end result is that light from every point on the object is assembled into an image in the hole.

The ray diagram may help explain this effect.



The image produced by this apparatus is known as a *real image*, because the light that forms it actually passes through the location of the image. However, if you place a piece of wax paper or onionskin paper at the location of the real image, the image will not appear on the paper. The outside regions of the mirrors that do not reflect light to your eyes do reflect light to the paper. The edges of the mirrors have large aberrations and create an image so blurred that it cannot be seen.

## 8.2 Ellipses

1. **Kepler's Laws**

Kepler was born Dec. 17, 1571, and died Nov. 15, 1630. He was among the first strong supporters of the heliocentric theory. In 1596 Kepler published *Cosmographic Mystery*, in which he defended the Copernican theory.

Tycho Brahe, mathematician at the court of Emperor Rudolph II, was impressed with the work of Kepler and invited him to Prague as his assistant. When Brahe died the following year (1601), Kepler was appointed to the position held by Brahe. Between 1609 and 1619, Kepler published his three laws of motion:

**1.** Each planet moves about the sun in an orbit that is an ellipse, with the sun at one focus of the ellipse.

**2.** The straight line joining a planet and the sun sweeps out equal areas in space in equal intervals of time.

**3.** The squares of the sidereal periods of the planets are in direct proportion to the cubes of the semimajor axes of their orbits. That is, $P^2 = ka^3$. The value of $k$ depends on the units of measurement. If astronomical units are used, then $k = 1$.

Kepler also made contributions to optics and telescope lenses and gave a physical explanation of nova. His text *Introduction to Copernican Astronomy* was one of the most widely read treatises on astronomy.

**a.** A planet's velocity is greatest when it is at perihelion. This follows from Kepler's second law.

**b.** The period of Mars is 1.87 years. This follows from the third law.

2. **Neptune**

Neptune was discovered as a result of mathematical prediction. The perturbative effects of Jupiter and Saturn on Uranus alone did not allow for observed discrepancies in Uranus's orbit. Using Newtonian gravitational theory, John Couch Adams produced mathematical evidence that an unknown planet could account for the irregularities in Uranus's orbit.

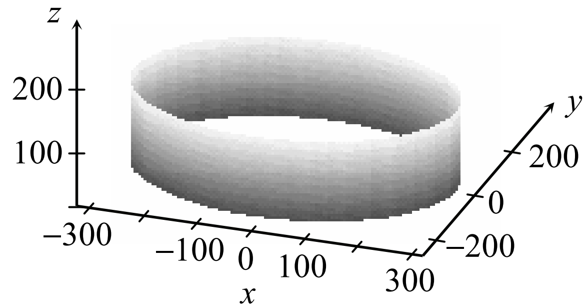
Adams sent his calculations, along with the region of the sky in which to search for the new planet, to Sir George Airy, the Royal Astronomer, and asked him to begin a search for the planet. But Airy had no faith in Adams' calculations and did not look for Neptune.

In June of 1846, Leverrier, a French mathematician, independently reproduced the work of Adams. His calculations also were sent to Airy. This time the Royal Astronomer suggested that Cambridge University begin a search for the new planet.

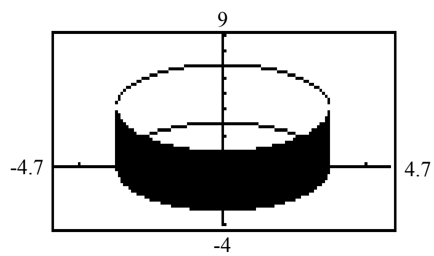
The work of Challis, director of Cambridge Observatory, was sloppy and negligent. Consequently, he did not find the planet. In September 1846, Leverrier suggested that Galle of the Berlin Observatory search the *Aquarius* region of the sky for the planet. Galle found the planet during his first observation. The discovery of Neptune was a great achievement for the time and was a major triumph of gravitational theory.

**3. Graph the Colosseum**

**a.** A graph of the Colosseum produced by using the *Maple* commands given in the text.




**b.** The following image appears to be constructed with ellipses because the window is not a "square"

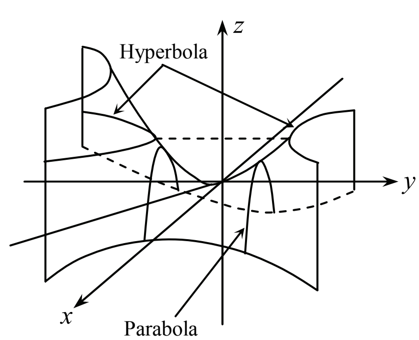



window. That is, one unit on the $x$-axis is not the same length as one unit on the $y$-axis. The graphs of the equations on a "square" window would appear as semicircles, but on this non-square window they are distorted and appear to be elliptical.

**8.3 Hyperbolas**

**a. A Hyperbolic Paraboloid**

The general equation is $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}$. The graph for $a = 3$, $b = 4$, and $c = 1$ is shown below. The graph is symmetric with respect to the planes $x = 0$ and $y = 0$.




The section in the plane $x = 0$ is $y^2 = \frac{b^2}{c} z$, which is a parabola that opens upward and has its vertex at the origin.

The section in the plane $y = 0$ is $x^2 = -\frac{a^2}{c} z$, which is a parabola that opens downward and has its vertex at the origin.

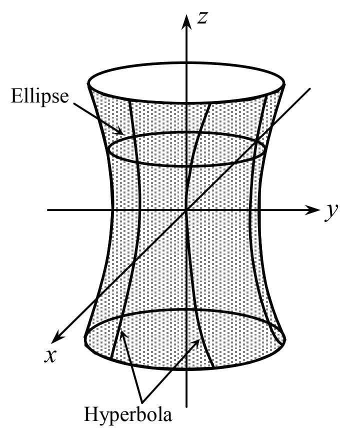
 

If the surface is cut by the plane $z = k > 0$, then the section is the hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{k}{c}$, whose focal axis is parallel to the *y*-axis and whose vertices are on the parabola $y^2 = \frac{b^2}{c} z$.

If $k < 0$, then the section is the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{|k|}{c}$, whose focal axis is parallel to the *x*-axis and whose vertices lie on the parabola $x^2 = -\dfrac{a^2}{c} z$.

**b. A Hyperboloid of One Sheet**

The general equation is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. The graph for $a = 3$, $b = 4$, and $c = 5$ is shown below. The graph is symmetric with respect to each of the coordinate planes.




The sections cut by the coordinate planes are

$$x = 0:\ \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad y = 0:\ \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 \qquad z = 0:\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The plane $z = k$ cuts the surface in an ellipse with equation $\frac{x^2}{a^2} = \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2}$. The center of the ellipse is on the *z*-axis, and its vertices fall on the hyperbolas $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ and $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$.

Cooling towers are built in the shape of a hyperboloid of one sheet, because such structures are strong, they have a large carrying capacity, and they can be constructed of many straight, narrow boards.

## 8.4 Rotation of Axes

**1. Use the Invariant Theorems**

We are given $10x^2 + 24xy + 17y^2 - 26 = 0$. Thus $A = 10,\ B = 24,\ C = 17,\ D = 0,\ E = 0,$ and $F = 26$. We seek the equation of the form $A'(x')^2 + C'(y')^2 - F = 0$.

The invariant theorems in Exercises 41 and 41 of Section 8.4 indicate that

$$A' + C' = A + C \quad \text{and} \quad (B')^2 - 4A'C' = B^2 - 4AC.$$

$A' + C' = A + C$ implies $A' + C' = 10 + 17 = 27$. Hence $C' = 27 - A'$.

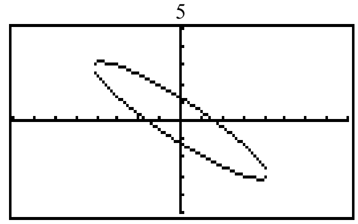
$(B')^2 - 4A'C' = B^2 - 4AC = 24^2 - 4(10)(17) = -104$. In the $x'y'$-coordinate system $B' = 0$, so we have

$$-4A'C' = -104$$
$$-4A'(27 - A') = -104 \quad \bullet \text{ Substitute } 27 - A' \text{ for } C'.$$
$$4(A')^2 - 108A' + 104 = 0$$
$$(A')^2 - 27A' + 26 = 0$$
$$(A' - 1)(A' - 26) = 0$$

Hence $A' = 1$ or $A' = 26$. If $A' = 1$, then $C' = 26$. If $A' = 26$, then $C' = 1$. Thus we conclude that the conic given by $10x^2 + 24xy + 17y^2 - 26 = 0$ is also represented by

$$26(x')^2 + 1(y')^2 - 26 = 0 \text{ or } 1(x')^2 + 26(y') - 26 = 0$$

in the $x'y'$-coordinate system. An examination of the following graph shows that we should pick $(x')^2 + 26(y')^2 - 26 = 0$ if we plan to obtain the $x'$-axis by a $53°$ counterclockwise rotation of the *x*-axis and that we should pick $26(x')^2 + (y')^2 - 26 = 0$ if we plan to obtain the $x'$-axis by a $143°$ counterclockwise rotation of the *x*-axis.




## 8.5 Introduction to Polar Coordinates

**1. A Polar Distance Formula**

**a.** Let $P_1$, the origin, and $P_2$ form a triangle $\Delta P_1OP_2$.

If we let *a* = the distance from $P_1$ to the origin, *b* = the distance from the origin to $P_2$, and *c* = the distance from $P_1$ to $P_1$, the Law of Cosines defines $c^2 = a^2 + b^2 - 2ab\cos C$, where *C* is the angle formed at the origin.

Substituting $r_1$ for the distance to $P_1$ from the origin $(r_1 = a)$ and $r_2$ for the distance from the origin to $P_2\,(r_2 = b)$, we find that the distance squared between the two points $P_1$ and $P_2$ is now $d[P_1, P_2]^2 = r_1^2 + r_2^2 - 2r_1r_2\cos C$.

Because the angle *C* is simply the difference between the angles formed by the two points and the polar axis, we can substitute $(\theta_2 - \theta_1)$ for *C* and arrive at the desired result:

$$d(P_1, P_2) = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}.$$

**b.** $d((3, 60°),\ (5, 170°)) = \sqrt{3^2 + 5^2 - 2(3)(5)\cos(170° - 60°)} \approx 6.65$

**c.** Because $(\theta_2 - \theta_1) = -(\theta_1 - \theta_2)$ and because $\cos x = \cos(-x)$, this distance formula can also be written as $d(P_1, P_2) = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$.

**2. Another Polar Form for a Circle**

**a.**

$$
\begin{aligned}
r &= a\sin\theta + b\cos\theta \\
r^2 &= ar\sin\theta + br\cos\theta \\
x^2 + y^2 &= ay + bx \\
x^2 - bx + y^2 - ay &= 0 \\
x^2 - bx + \frac{b^2}{4} + y^2 - ay + \frac{a^2}{4} &= \frac{a^2}{4} + \frac{b^2}{4} \\
\left(x - \frac{b}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 &= \frac{a^2 + b^2}{4} \\
\left(x - \frac{b}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 &= \left(\frac{\sqrt{a^2 + b^2}}{2}\right)^2
\end{aligned}
$$

Thus the center of the circle is $\left(\frac{b}{2}, \frac{a}{2}\right)$ and the radius is $r = \frac{\sqrt{a^2+b^2}}{2}$.

## 8.6 Polar Equations of the Conics

**1. Polar Equation of a Line**

From the figure $k = d\sin\theta_p$, $h = d\cos\theta_p$ and the slope of the line segment perpendicular to the line is $m = \tan\theta_p$. Therefore, the equation of the line is given by

$$y - d\sin\theta_p = \left(-\frac{1}{\tan\theta_p}\right)\left(x - d\cos\theta_p\right)$$

Switching to polar, we have

$$
\begin{aligned}
r\sin\theta - d\sin\theta_p &= \left(-\frac{1}{\tan\theta_p}\right)\left(r\cos\theta - d\cos\theta_p\right) \\
-r\sin\theta\tan\theta_p + d\sin\theta_p\tan\theta_p &= r\cos\theta - d\cos\theta_p \\
-r\sin\theta\frac{\sin\theta_p}{\cos\theta_p} + d\sin\theta_p\frac{\sin\theta_p}{\cos\theta_p} &= r\cos\theta - d\cos\theta_p \\
-r\sin\theta\sin\theta_p + d\sin^2\theta_p &= r\cos\theta\cos\theta_p - d\cos^2\theta_p \\
d\sin^2\theta_p + d\cos^2\theta_p &= r\cos\theta\cos\theta_p + r\sin\theta\sin\theta_p \\
d\left(\sin^2\theta_p + \cos^2\theta_p\right) &= r\left(\cos\theta\cos\theta_p + \sin\theta\sin\theta_p\right) \\
d &= r\cos\left(\theta - \theta_p\right) \\
r &= \frac{d}{\cos\left(\theta - \theta_p\right)}
\end{aligned}
$$

**3. Polar Equation of a Circle That Passes Through the Pole**

From the figure, we have the center of the circle $(h,\ k)$ with $h = a\cos\theta_C$, $k = a\sin\theta_C$, and radius $a$.

$$(x-h)^2 + (y-k)^2 = a^2$$
$$(x - a\cos\theta_C)^2 + (y - a\sin\theta_C)^2 = a^2$$

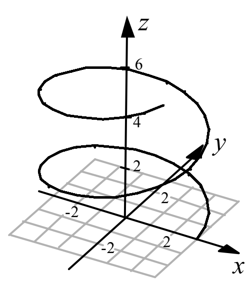
Changing to polar yields

$$(r\cos\theta - a\cos\theta_C)^2 + (r\sin\theta - a\sin\theta_C)^2 = a^2$$
$$\left(r^2\cos^2\theta - 2ra\cos\theta\cos\theta_C + a^2\cos^2\theta_C\right) + \left(r^2\sin^2\theta - 2ra\sin\theta\sin\theta_C + a^2\sin^2\theta_{C1}\right) = a^2$$
$$r^2\left(\cos^2\theta + \sin^2\theta\right) - 2ra\left(\cos\theta\cos\theta_C + \sin\theta\sin\theta_C\right) + a^2\left(\cos^2\theta_C + \sin^2\theta_C\right) = a^2$$
$$r^2 - 2ra\left(\cos\theta\cos\theta_C + \sin\theta\sin\theta_C\right) + a^2 = a^2$$
$$r^2 - 2ra\left[\cos(\theta - \theta_C)\right] = 0$$
$$r - 2a\left[\cos(\theta - \theta_C)\right] = 0$$
$$r = 2a\left[\cos(\theta - \theta_C)\right]$$

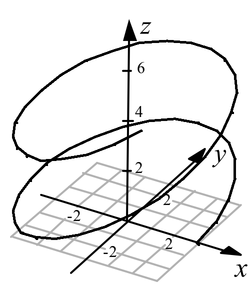
## 8.7 Parametric Equations

**1. Parametric Equations in an *xyz*-Coordinate System**

**a.** Graph of $x = 3\cos t,\ y = 3\sin t,\ z = 0.5t,$ for $0 \le t \le 12$.




**b.** Graph of $x = 3\cos t,\ y = 6\sin t,\ z = 0.5t,$ for $0 \le t \le 12$.




**c.** The graph in part (a) is "circular," whereas the graph in part (b) is "elliptical."

**d.** Each of the curves is a helix.

## Systems of Equations

CHAPTER 9 of *College Algebra and Trigonometry*

### 9.1 Systems of Linear Equations in Two Variables

**1. Independent and Dependent Conditions**

**a.** Independent. The system of equations is $\begin{cases} x+y=30 \\ x-y=10 \end{cases}$.

The solution of this system of equations is $(20,\ 10)$.

**b.** Dependent. The equation is $20x+30y=1000$. Because three are two variables and only one equation, there are an infinite number of possible solutions. The ordered pairs $(50,\ 0)$ and $(35,\ 10)$ are two possible solutions.

**c.** The system of equations is $\begin{cases} x+y=20 \\ 2x=10-2y \end{cases}$. The system of equations has no solution. This means it is impossible to satisfy the two conditions of the problem at the same time.

**d.** Answers will vary. The answer should contain a word problem with two independent conditions.

**e.** Answers will vary. The answer should contain a word problem with two dependent conditions.

### 9.2 Nonlinear Systems of Equations

**1. Concept of Dimension**

In *Flatland*, people are two-dimensional polygons. A person's station in life is determined by the number of sides of the polygon. People walk by sliding along the plane.

The Flatlanders visit the people from the world of one dimension. These people live on a line. When the Flatlanders tell the ruler of the one-dimensional people that they can change position with their neighbors, the one-dimensional people cannot comprehend how such a movement would take place.

The Flatlanders are dutifully smug about their ability to move in the plane and think they are superior to the one-dimensional people.

Then strange phenomena begin when a three-dimensional person enters the world of the Flatlanders. For example, the Flatlanders cannot understand how the three-dimensional people enter their homes even though all the doors and windows are latched.

**2. Abilities of a Four-Dimensional Human**

Some of the best accounts have appeared in *Scientific American*. Descriptions of the capabilities of a four-dimensional person range from turning a tennis ball inside out without tearing it, to reaching into a closed safe and removing the contents.

### 9.3 Nonlinear Systems of Equations

**1. Proving a Geometry Theorem**

**a.** Substitute $y=mx$ into $(x-a)^2+y^2=a^2$ and solve for $x$.

$$\begin{aligned} (x-a)^2+m^2x^2 &= a^2 \\ x^2-2ax+a^2+m^2x^2 &= a^2 \\ (1+m^2)x^2-2ax &= 0 \\ x\left((1+m^2)x-2a\right) &= 0 \end{aligned}$$

Thus $x=0$ or $(1+m^2)x-2a=0$.

$$x=\frac{2a}{1+m^2}$$

If $x=0$, then $y=m(0)=0$. One intersection is $(0,0)$.

If $x=\frac{2a}{1+m^2}$, then $y=m\left(\frac{2a}{1+m^2}\right)=\frac{2am}{1+m^2}$. A second intersection is $\left(\frac{2a}{1+m^2},\frac{2am}{1+m^2}\right)$.

**b.** The slope of the line through $P$ and $(2a, 0)$ is: Slope $= \dfrac{\frac{2am}{1+m^2} - 0}{\frac{2a}{1+m^2} - 2a} = \dfrac{\frac{2am}{1+m^2}}{\frac{-2am^2}{1+m^2}} = -\dfrac{1}{m}$.

**c.** The line segment $\overline{OP}$ is on the line given by $y = mx$. Thus the slope of $\overline{OP}$ is $m$.

**d.** $\overline{OP}$ is perpendicular to $\overline{PQ}$ because their slopes are negative reciprocals of each other.

**e.** Since $\overline{OP}$ is perpendicular to $\overline{PQ}$, angle $OPQ$ is a right angle and triangle $OPQ$ is a right triangle.

**2. Finding Zeros of a Polynomial**

Let $a$, $b$, and $c$ be zeros of $P(x) = x^3 + 2x^2 + Cx - 6$ such that $a = b + c$. Because these are the zeros of $P$, the polynomial must factor (by the Factor Theorem) as $(x - a)(x - b)(x - c)$. Now multiply this out and equate the coefficients to those of $P$. The results enables us to form the system of equations

$a + b + c = -2$ • Equating coefficients of $x$.

$abc = -6$ • Equating the constant term.

$a = b + c$ • This is a condition of the problem.

Solving this system yields $a = -1$, $b = \frac{-1+i\sqrt{23}}{2}$, and $c = \frac{-1-i\sqrt{23}}{2}$.

To find $C$, solve $P(-1) = 0 = (-1)^3 + 2(-1) + C(-1) - 6$, or $C = -9$.

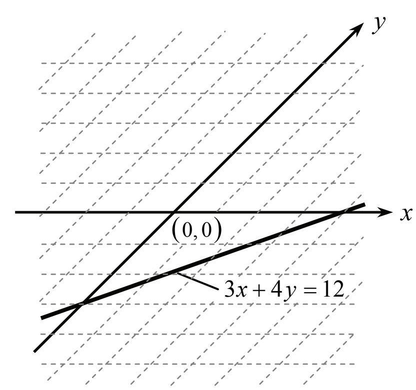
## 9.4 Partial Fractions

**1. Using a Computer Algebra System**

The answer to this question will depend on the rational functions for which the student attempted a partial fraction decomposition using some type of computer algebra system.

## 9.5 Inequalities in Two Variables and Systems of Inequalities

**1. A Parallelogram Coordinate System**




Coordinate lines are drawn parallel to the coordinate axes and form a parallelogram. See the accompanying illustration. A point is located in much the same manner as in a rectangular coordinate system. However, displacement is along the edge of a parallelogram rather than of a rectangle.

The graph of $3x + 4y = 12$ is a straight line. In fact, all linear equations in two variables have a graph that is a straight line.

Here are some other observations a student may include:

Transformation equations between rectangular coordinates and parallelogram coordinates are given by $x' = x$ and $y' = y - x$, where $x'$ and $y'$ are the coordinates in the parallelogram system.

One way to define the distance $d$ between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is $d = |x_1 - x_2| + |y_1 - y_2|$.

## 9.6 Linear Programming

1. **History of Linear Programming**

   Linear programming is a *program*, or method, of allocating resources. It is used in planning food distribution, building rockets, supplying military units with essential equipment, and farming, as well as in many other applications.

   George Dantzig developed the "simplex method" of solving linear programming problems in the late 1940s. The simplex method is basically a matrix method, analogous to row reduction. The result is the best method of allocating resources.

   In the early 1980s, Narendra Karmarkar of AT&T suggested an improvement on the simplex method that greatly reduced the time required for a computer to determine the optimal solution of a linear programming problem.

   In the mid-1980s, L. G. Khachian introduced a method that was supposed to revolutionize the technique of solving linear programming problems. Although his method has some theoretical importance, its practical applications have not been demonstrated.

**Matrices**
CHAPTER 10 of *College Algebra and Trigonometry*

**10.1 Gaussian Elimination Method**

1. **Echelon Form by Using a Graphing Calculator**

**a.** $\begin{bmatrix} 2 & -3 & 1 & 4 \\ 1 & 2 & -2 & -2 \\ 3 & 1 & -3 & 4 \end{bmatrix}$

**b.** $R_1 \leftrightarrow R_2$ $\begin{bmatrix} 1 & 2 & -2 & -2 \\ 2 & -3 & 1 & 4 \\ 3 & 1 & -3 & 4 \end{bmatrix}$ $-2R_1 + R_2 \to R_2$ $\begin{bmatrix} 1 & 2 & -2 & -2 \\ 0 & -7 & 5 & 8 \\ 3 & 1 & -3 & 4 \end{bmatrix}$

$-3R_1 + R_3 \to R_3$ $\begin{bmatrix} 1 & 2 & -2 & -2 \\ 0 & -7 & 5 & 8 \\ 0 & -5 & 3 & 10 \end{bmatrix}$ $\left(-\frac{1}{7}\right)R_2$ $\begin{bmatrix} 1 & 2 & -2 & -2 \\ 0 & 1 & -\frac{5}{7} & -\frac{8}{7} \\ 0 & -5 & 3 & 10 \end{bmatrix}$

$5R_2 + R_3 \to R_3$ $\begin{bmatrix} 1 & 2 & -2 & -2 \\ 0 & 1 & -\frac{5}{7} & -\frac{8}{7} \\ 0 & 0 & -\frac{4}{7} & -\frac{30}{7} \end{bmatrix}$ $\left(-\frac{7}{4}\right)R_3$ $\begin{bmatrix} 1 & 2 & -2 & -2 \\ 0 & 1 & -\frac{5}{7} & -\frac{8}{7} \\ 0 & 0 & 1 & -\frac{15}{2} \end{bmatrix}$

**10.2 The Algebra of Matrices**

1. **Transformations**

**a.** $R_{90} \cdot \begin{bmatrix} t \\ t+2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ t+2 \\ 1 \end{bmatrix} = \begin{bmatrix} -t-2 \\ t \\ 1 \end{bmatrix} \Rightarrow (-t-2,\ t) \Rightarrow x = -t-2,\ y = t$

Using substitution, $x = -y - 2$

$y = -x - 2$

**b.** $R_y \cdot \begin{bmatrix} t \\ 3t-1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ 3t-1 \\ 1 \end{bmatrix} = \begin{bmatrix} -t \\ 3t-1 \\ 1 \end{bmatrix} \Rightarrow (-t,\ 3t-1) \Rightarrow x = -t,\ y = 3t-1$

$t = -x$

Using substitution, $y = 3(-x) - 1$

$y = -3x - 1$

**c.** $T_{-1,-1} \cdot \begin{bmatrix} t \\ \frac{1}{t} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ \frac{1}{t} \\ 1 \end{bmatrix} = \begin{bmatrix} t-1 \\ \frac{1}{t}-1 \\ 1 \end{bmatrix}$

$R_{180} \cdot \begin{bmatrix} t-1 \\ \frac{1}{t}-1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t-1 \\ \frac{1}{t}-1 \\ 1 \end{bmatrix} = \begin{bmatrix} -t+1 \\ -\frac{1}{t}+1 \\ 1 \end{bmatrix}$

$T_{1,1} \cdot \begin{bmatrix} -t+1 \\ -\frac{1}{t}+1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -t+1 \\ -\frac{1}{t}+1 \\ 1 \end{bmatrix} = \begin{bmatrix} -t+2 \\ -\frac{1}{t}+2 \\ 1 \end{bmatrix} \Rightarrow (-t+2,\ -\frac{1}{t}+2) \Rightarrow x = -t+2,\ y = -\frac{1}{t}+2$

$t = -x + 2$

Using substitution, $y = -\dfrac{1}{-x+2} + 2 = \dfrac{-1}{-x+2} + \dfrac{-2x+4}{-x+2} = \dfrac{-2x+3}{-x+2} = \dfrac{2x-3}{x-2}$

**d.** $T_{2,-1}\cdot\begin{bmatrix} t \\ t^2 \\ 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} t \\ t^2 \\ 1\end{bmatrix}=\begin{bmatrix} t+2 \\ t^2-1 \\ 1\end{bmatrix} \Rightarrow (t+2,\ t^2-1) \Rightarrow x=t+2,\ y=t^2-1$

$t=x-2$

Using substitution,

$y=(x-2)^2-1$

$y=x^2-4x+3$

**e.** $R_{270}\cdot\begin{bmatrix} t \\ t^2 \\ 1\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} t \\ t^2 \\ 1\end{bmatrix}=\begin{bmatrix} t^2 \\ t \\ 1\end{bmatrix} \Rightarrow (t^2,\ t) \Rightarrow x=t^2,\ y=t$

Using substitution,

$x=y^2$

**f.** $R_{90}\cdot\begin{bmatrix} t \\ t^2 \\ 1\end{bmatrix}=\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} t \\ t^2 \\ 1\end{bmatrix}=\begin{bmatrix} -t^2 \\ t \\ 1\end{bmatrix}$

$T_{-2,\,1}\cdot\begin{bmatrix} -t^2 \\ t \\ 1\end{bmatrix}=\begin{bmatrix}1 & 0 & -2\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} -t^2 \\ t \\ 1\end{bmatrix}=\begin{bmatrix} -t^2-2 \\ t-1 \\ 1\end{bmatrix} \Rightarrow (-t^2-2,\ t-1) \Rightarrow x=-t^2-2,\ y=t-1$

$t=y+1$

Using substitution,

$x=-(y+1)^2-2$

$x=-y^2-2y-3$

2. **Translations**
This project appears on our Internet site at college.hmco.com/info/aufmannCAT.

**10.3 The Inverse of a Matrix**

1. **Multiple Regression Models**

**a.** $A=\begin{bmatrix}20 & 16 & 19 & 13 & 13\\ 6 & 5 & 12 & 4 & 7\\ 1 & 1 & 1 & 1 & 1\end{bmatrix}$ **b.** $B=\begin{bmatrix}90{,}500\\ 73{,}750\\ 117{,}200\\ 59{,}500\\ 74{,}800\end{bmatrix}$ **c.** $A^T=\begin{bmatrix}20 & 6 & 1\\ 16 & 5 & 1\\ 19 & 12 & 1\\ 13 & 4 & 1\\ 13 & 7 & 1\end{bmatrix}$

**d.** $\begin{bmatrix} L \\ M \\ N\end{bmatrix}=\left(\begin{bmatrix}20 & 16 & 19 & 13 & 13\\ 6 & 5 & 12 & 4 & 7\\ 1 & 1 & 1 & 1 & 1\end{bmatrix}\cdot\begin{bmatrix}20 & 6 & 1\\ 16 & 5 & 1\\ 19 & 12 & 1\\ 13 & 4 & 1\\ 13 & 7 & 1\end{bmatrix}\right)^{-1}\cdot\left(\begin{bmatrix}20 & 16 & 19 & 13 & 13\\ 6 & 5 & 12 & 4 & 7\\ 1 & 1 & 1 & 1 & 1\end{bmatrix}\cdot\begin{bmatrix}90{,}500\\ 73{,}750\\ 117{,}200\\ 59{,}500\\ 74{,}800\end{bmatrix}\right)$

$=\begin{bmatrix}1355 & 571 & 81\\ 571 & 270 & 34\\ 81 & 34 & 5\end{bmatrix}^{-1}\cdot\begin{bmatrix}6{,}962{,}700\\ 3{,}079{,}750\\ 415{,}750\end{bmatrix}\approx\begin{bmatrix}0.03098 & -0.01613 & -0.39214\\ -0.01613 & 0.03417 & 0.02890\\ -0.39214 & 0.02890 & 6.35622\end{bmatrix}\cdot\begin{bmatrix}6{,}962{,}700\\ 3{,}079{,}750\\ 415{,}750\end{bmatrix}$

$\approx\begin{bmatrix}2974.14\\ 4963.20\\ 1219.11\end{bmatrix}$

**e.** $S = 2974.14x + 4963.20y + 1219.11$

**f.** $S(10,\ 8) = 2974.14(10) + 4963.20(8) + 1219.11$
$= \$70,666.10$

**2. Cryptography**

**a.** The ASCII (American Standard Code for Information Interchange) is a method by which each letter, punctuation mark, and numeral is given a two-number code. This system is used by all computer systems to exchange information.

**b.** Answers will vary. The students should have an $m \times n$ matrix, *W*, in which *m* represents the length of a code packet and *n* represents the number of characters in the message.

**c.** Answers will vary. The student should construct an $m \times m$ matrix that has an inverse. We will call this matrix *E*.

**d.** Answers will vary depending on parts (b) and (c) above. However, the student should exhibit the product $EW = M$ .

**e.** Answers will vary. The student should show that $E^{-1}M = W$ . That is, the product of the inverse of *E* and the encoded message should be the original message.

**10.4 Determinants**

**1. Determinants, Matrices, and Area**

**a.** $AM = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 & 4 & 10 & 10 \\ 4 & 8 & 8 & 4 \end{bmatrix} = \begin{bmatrix} 12 & 16 & 28 & 24 \\ 20 & 28 & 46 & 38 \end{bmatrix}$

The new figure is a parallelogram. To find the area of the parallelogram, we will use the distance formula to find the length of the base and the formula for the distance between a point and a line to find the height.

$$d = \sqrt{(12-24)^2 + (20-38)^2} = \sqrt{468} = 6\sqrt{13}$$

To find the height, first determine the equation of the line through (12, 20) and (24, 38).

$$m = \frac{38-20}{24-12} = \frac{3}{2}$$

$$y - 20 = \frac{3}{2}(x-12)$$

$$y = \frac{3}{2}x + 2$$

$$h = \frac{\left|\frac{3}{2}(16) + 2 - 28\right|}{\sqrt{\left(\frac{3}{2}\right)^2 + 1}} = \frac{|-2|}{\sqrt{\frac{13}{4}}} = \frac{4\sqrt{13}}{13}$$

Area of parallelogram $= \left(6\sqrt{13}\right)\left(\frac{4\sqrt{13}}{13}\right) = 24$

Area of rectangle $= (6)(4) = 24$

The areas are the same, and $det(A) = 4 - 3 = 1$.

**b.** $AM = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 & 10 & 10 \\ 4 & 8 & 8 & 4 \end{bmatrix} = \begin{bmatrix} 16 & 20 & 38 & 34 \\ 8 & 12 & 18 & 14 \end{bmatrix}$

The new figure is a parallelogram. To find the area of the parallelogram, we will use the distance formula to find the length of the base and the formula for the distance between a point and a line to find the height.

$$d = \sqrt{(16-34)^2 + (8-14)^2} = \sqrt{360} = 6\sqrt{10}$$

To find the height, first determine the equation of the line through (12, 20) and (24, 38).

$$m = \frac{14-8}{34-16} = \frac{1}{3}$$

$$y - 14 = \frac{1}{3}(x - 34)$$

$$y = \frac{1}{3}x + \frac{8}{3}$$

$$h = \frac{\left|\left(\frac{1}{3}\right)20 + \frac{8}{3} - 12\right|}{\sqrt{\left(-\frac{1}{3}\right)^2 + 1}} = \frac{\left|-\frac{8}{3}\right|}{\sqrt{\frac{10}{9}}} = \frac{4\sqrt{10}}{5}$$

Area of parallelogram $= \left(6\sqrt{10}\right)\left(\frac{4\sqrt{10}}{5}\right) = 48$

Area of rectangle $= (6)(4) = 24$

The area of the new figure is twice that of the rectangle, and $det(A) = 3 - 1 = 2$.

**c.** $AM = \begin{bmatrix} 1 & 2 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 & 10 & 10 \\ 4 & 8 & 8 & 4 \end{bmatrix} = \begin{bmatrix} 12 & 20 & 26 & 18 \\ 6 & 10 & 13 & 9 \end{bmatrix}$

The new figure is a line segment and therefore has no area. The student should verify this by showing that all the points lie on the same line.

The area is 0; $det(A) = 1 - 1 = 0$.

**d.** The absolute value of the determinant of *A* times the original area equals the area of the figure produced by *AM*.

**e.** Some students may not have realized that the absolute value of the determinant is required for their conjecture (in part d).

## 10.5 Cramer's Rule

**1. Cramer's Rule**

Prove Cramer's Rule for a 3 by 3 system of linear equations. Consider the following system, where *x*, *y*, and *z* are variables.

$$\begin{aligned} ax+by+cz&=d \qquad \text{(I)} \\ ex+fy+gz&=h \qquad \text{(II)} \\ jx+ky+mz&=n \qquad \text{(III)} \end{aligned}$$

Multiply Equation (I) by *g*, and Equation (II) by (–*c*), and add.

$$\begin{aligned} agx+bgy+cgz&=dg \\ -cex-cfy-cgz&=-ch \\ \hline (ag-ce)x+(bg-cf)y&=dg-ch \qquad \text{(IV)} \end{aligned}$$

Multiply Equation (I) by *m*, and Equation (III) by (–*c*), and add.

$$\begin{aligned} amx+bmy+cmz&=dm \\ -cjx-cky-cmz&=-cn \\ \hline (am-cj)x+(bm-ck)y&=dm-cn \qquad \text{(V)} \end{aligned}$$

Multiply Equation (IV) by $(bm-ck)$, and Equation (V) by $(-bg+cf)$, and add.

$$\begin{aligned} (bm-ck)(ag-ce)x+(bm-ck)(bg-cf)y&=(bm-ck)(dg-ch) \\ (-bg+cf)(am-cj)x+(-bg+cf)(bm-ck)y&=(-bg+cf)(dm-cn) \\ \hline \left[(bm-ck)(ag-ce)+(-bg+cf)(am-cj)\right]x&=(bm-ck)(dg-ch)+(-bg+cf)(dm-cn) \end{aligned}$$

Thus

$$\begin{aligned} x&=\frac{(bm-ck)(dg-ch)+(-bh+cf)(dm-cn)}{(bm-ck)(ag-ce)+(-bg+cf)(am-ej)} \\ &=\frac{\cancel{bdgm}-bchm-cdgk+c^2hk-\cancel{bdgm}+bcgn+cdfm-c^2fn}{\cancel{abgm}-bcem-acgk+c^2ek-\cancel{abgm}+begj+acfm-cefj} \\ &=\frac{c\{(dfm+bgn+chk)+(-cfn-bhm-dgk)\}}{c\{(afm+bgj+cek)+(-efj-bem-akg)\}} \\ &=\frac{\begin{vmatrix} d & b & c \\ h & f & g \\ n & k & m \end{vmatrix}}{\begin{vmatrix} a & b & c \\ e & f & g \\ j & k & m \end{vmatrix}} \end{aligned}$$

The results $y=\dfrac{\begin{vmatrix} a & d & c \\ e & h & g \\ j & n & m \end{vmatrix}}{\begin{vmatrix} a & b & c \\ e & f & g \\ j & k & m \end{vmatrix}}$, $z=\dfrac{\begin{vmatrix} a & b & d \\ e & f & h \\ j & k & n \end{vmatrix}}{\begin{vmatrix} a & b & c \\ e & f & g \\ j & k & m \end{vmatrix}}$ can by established by using a similar approach.

**Sequences, Series, and Probability**
CHAPTER 11 of *College Algebra and Trigonometry*

## 11.1 Infinite Sequences and Summation Notation

1. **Formulas for Infinite Sequences**
   **a.** The idea is to write a formula that gives $2n$ for $n = 1, 2, 3, 4$ and 43 when $n = 5$. The formula contains two parts. The first part is zero for $n = 1, 2, 3, 4$. The second part is $2n$.
   $$a_n = (43-2n)\left[\frac{n(n-1)(n-2)(n-3)(n-4)}{n!}\right]+2n$$
   The first five terms of this sequence are 2, 4, 6, 8, 43.
   **b.** The idea is to write a formula that gives $2n$ for $n = 1, 2, 3, 4$ and $x$ when $n = 5$. The formula contains two parts. The first part is zero for $n = 1, 2, 3, 4$. The second part is $2n$.
   $$a_n = (x-2n)\left[\frac{n(n-1)(n-2)(n-3)(n-4)}{n!}\right]+2n$$
   The first five terms of this sequence are 2, 4, 6, 8, $x$.

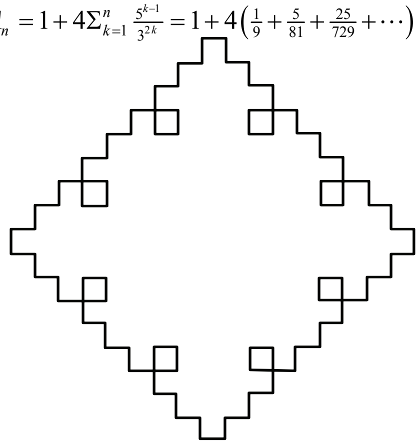
## 11.2 Arithmetic Sequences and Series

1. **Angles of a Triangle**
   **a.** 360°
   **b.** 540°
   **c.** 720°
   **d.** $(n-3)180°$

2. **Prove a Formula**
   We know that $S_n = \frac{n}{2}(a_1 + a_n)$ and that $a_n = a_1 + (n-1)d$. Substitute for $a_n$ in the formula for $S_n$ and simplify.
   $$S_n = \frac{n}{2}\left[a_1 + a_1 + (n-1)d\right] = \frac{n}{2}\left[2a_1 + (n-1)d\right]$$

## 11.3 Geometric Sequences and Series

1. **Fractals**
   **a.** The perimeter after completing the process $n$ times is $P_n = 4\left(\frac{5}{3}\right)^{n-1}$.
   **b.** As $n \to \infty$, $P_n = 4\left(\frac{5}{3}\right)^{n-1}$ approaches infinity. Therefore, the perimeter is infinite.
   **c.** $A_n = 1 + 4\Sigma_{k=1}^{n} \frac{5^{k-1}}{3^{2k}}$
   **d.** $A_n = 1 + 4\Sigma_{k=1}^{n} \frac{5^{k-1}}{3^{2k}} = 1 + 4\left(\frac{1}{9} + \frac{5}{81} + \frac{25}{729} + \cdots\right)$



Beginning with $\frac{5}{81}$, the series is an infinite geometric series with $a = \frac{5}{81}$ and $r = \frac{5}{9}$.

$$\begin{aligned} 1+4\left(\frac{1}{9}+\frac{5}{81}+\frac{25}{729}+\cdots\right) &= 1+4\left(\frac{1}{9}+\frac{5/81}{1-5/9}\right) \\ &= 1+4\left(\frac{1}{9}+\frac{9}{36}\right) \\ &= 1+4\left(\frac{1}{4}\right) = 2 \end{aligned}$$

The area is 2 square units.

### 11.4 Mathematical Induction

**1. Steps in a Mathematical Induction Proof**

**a.** Assume $2+4+8+\cdots+2^k = 2^{k+1}+1$. Prove the formula is true for $n=k+1$. That is, prove that $2+4+8+\cdots+2^k+2^{k+1} = 2^{k+2}+1$.

$$\begin{aligned} S_{k+1} &= 2+4+8+\cdots+2^k+2^{k+1} \\ &= S_k + 2^{k+1} \\ &= 2^{k+1}+1+2^{k+1} \\ &= 2\cdot 2^{k+1}+1 = 2^{k+2}+1 \end{aligned}$$

Thus the statement is true for $n=k+1$.

**b.** Let $n=1$. $2^1 \neq 2^2+1=5$. This statement is not true for $n=1$.

**c.** $2+4+8+\cdots+2^k = 2\left(1+2+4+8+\cdots+2^{k-1}\right)$

Let $N = 1+2+4+8+\cdots+2^{k-1}$. Thus

$$2+4+8+\cdots+2^k = 2(N) = \text{even number}$$

$$2^{k+1}+1 = 2\cdot 2^k+1 = \text{even number} +1 = \text{odd number}$$

Therefore, the left side is always an even number and the right side is always an odd number. Thus the two values can never be equal.

**d.** The Principle of Mathematical Induction requires that we first establish that there is at least one element in the set *S*. We did not do that, and consequently, we apparently "proved" a statement that is always false.

**2. The Tower of Hanoi**

**a.** The proof is by induction. If there is one disk $(n=1)$, then $2^1-1=2-1=1$ and the game is completed in one move.

Assume that for *k* disks, the game can be completed in $2^k-1$ moves. Prove that for $n=k+1$ disks, the game is completed in $2^{k+1}-1$ moves.

Consider one peg in which $k+1$ disks are arranged. By the Induction Hypothesis, $2^k-1$ moves are required to move the first *k* disks to another peg. Now move the $k+1$ disk to the unoccupied peg. The total number of moves is now $2^k-1+1$. Now move the *k* disks back to the disk containing the $k+1$ disk. This requires $2^k-1$ moves (Induction Hypothesis). The total number of moves is

$$2^k-1+1+2^k-1 = 2\cdot 2^k-1 = 2^{k+1}-1$$

Thus the statement is true for $n=k+1$, and the formula is established.

**b.** From Exercise 2(a), it will take $2^{64}-1$ seconds to complete the transfer.

$$2^{64}-1 \text{ seconds} \approx 5.85\times 10^{11} \text{ years} = 585 \text{ billion years}$$

Thus the legend predicts that the universe will exist for approximately 580 billion more years.

**11.5 The Binomial Theorem**

1. **Pascal's Triangle**
   Two references a student can check are *The History of Mathematics: An Introduction* by David M. Burton, Dubuque, Iowa: William C. Brown, publishers, 1988 and *The History of Mathematics: An Introduction* by Victor J. Katz, New York: Harper Collins, 1993.

2. **Some Other Factorial Functions**
   **a.** For integers, Pochammer $(m,n)=\frac{(m+n-2)!}{(n-1)!}$.
   **b.** $n!!=n(n-2)(n-4)\ldots 2$ when $n$ is an even integer and $n!!=n(n-2)(n-4)\ldots 1$ when $n$ is an odd integer.

**8.6 Permutations and Combinations**

1. **Explain Permutations and Combinations**
   The student should prepare a lesson to explain permutations and combinations to a classmate. The lesson should contain at least five examples of permutations and five examples of combinations.

2. **Application of Counting**
   **a.** $n=5, k=3$

$$\binom{3}{0}3^5-\binom{3}{1}(3-1)^5+\binom{3}{2}(3-2)^5=3^5-3\left(2^5\right)+3=150$$

   **b.** $n=10, k=4$

$$\binom{4}{0}4^{10}-\binom{4}{1}(4-1)^{10}+\binom{4}{2}(4-2)^{10}-\binom{4}{3}(4-3)^{10}=4^{10}-4\left(3^{10}\right)+6\left(2^{10}\right)-4=818,520$$

**11.7 Introduction to Probability**

1. **Monte Hall Problem**
   If the contestant stays with his or her first choice, then the probability that the contestant will win the grand prize is 1/3. The only other possibility is to go with one of the other curtains when given the opportunity to switch to a different curtain. This event of switching to another curtain is the compliment of staying with the first choice. Thus the probability of winning the grand prize by switching to a different curtain is $1-\frac{1}{3}=\frac{2}{3}$. This analysis shows that a contestant will double their chance of winning the grand prize by switching rather than staying with their first choice. Many Internet sites discuss this famous problem. Here are two recommended sites:

   **http://www.cut-the-knot.com/hall.html**
   **http://www.math.rice.edu/~ddonavan/montyurl.html**

2. **Probability and Automatic Garage Door Openers**
   The probability of at least two having the same garage door opener sequence is 1 – (the probability of none having the same sequence).

$$P=1-\binom{500}{0}\left(\frac{1}{64}\right)^0\left(\frac{63}{64}\right)^{500}\approx 1-0.00038=0.99962$$

   There is more than a 99.9% chance that at least two people will choose the same code sequence.

3. **Overbooking by Airlines**
   This project appears on our Internet site at **http://www.college.hmco.com**.

# Additional
## *College Trigonometry*
## Solutions

# Additional *College Trigonometry*, 6*e* Solutions

## Section 1.1

**1.**
$$\begin{aligned}2x+10&=40\\2x&=40-10\\2x&=30\\x&=15\end{aligned}$$

**2.**
$$\begin{aligned}-3y+20&=2\\-3y&=2-20\\-3y&=-18\\y&=6\end{aligned}$$

**3.**
$$\begin{aligned}5x+2&=2x-10\\5x-2x&=-10-2\\3x&=-12\\x&=-4\end{aligned}$$

**4.**
$$\begin{aligned}4x-11&=7x+20\\4x-7x&=20+11\\-3x&=31\\x&=-\frac{31}{3}\end{aligned}$$

**5.**
$$\begin{aligned}2(x-3)-5&=4(x-5)\\2x-6-5&=4x-20\\2x-11&=4x-20\\2x-4x&=-20+11\\-2x&=-9\\x&=\frac{9}{2}\end{aligned}$$

**6.**
$$\begin{aligned}6(5s-11)-12(2s+5)&=0\\30s-66-24s-60&=0\\6s-126&=0\\6s&=126\\s&=21\end{aligned}$$

**7.**
$$\begin{aligned}\frac{3}{4}x+\frac{1}{2}&=\frac{2}{3}\\12\left(\frac{3}{4}x+\frac{1}{2}\right)&=12\left(\frac{2}{3}\right)\\9x+6&=8\\9x&=8-6\\9x&=2\\x&=\frac{2}{9}\end{aligned}$$

**8.**
$$\begin{aligned}\frac{x}{4}-5&=\frac{1}{2}\\4\left(\frac{x}{4}-5\right)&=4\left(\frac{1}{2}\right)\\x-20&=2\\x&=2+20\\x&=22\end{aligned}$$

**9.**
$$\begin{aligned}\frac{2}{3}x-5&=\frac{1}{2}x-3\\6\left(\frac{2}{3}x-5\right)&=6\left(\frac{1}{2}x-3\right)\\4x-30&=3x-18\\4x-3x&=-18+30\\x&=12\end{aligned}$$

**10.**
$$\begin{aligned}\frac{1}{2}x+7-\frac{1}{4}x&=\frac{19}{2}\\4\left(\frac{1}{2}x+7-\frac{1}{4}x\right)&=4\left(\frac{19}{2}\right)\\2x+28-x&=38\\x&=38-28\\x&=10\end{aligned}$$

**11.**
$$\begin{aligned}0.2x+0.4&=3.6\\0.2x&=3.2\\x&=16\end{aligned}$$

**12.**
$$\begin{aligned}0.04x-0.2&=0.07\\0.04x&=0.27\\x&=6.75\end{aligned}$$

**13.**
$$\begin{aligned}\frac{3}{5}(n+5)-\frac{3}{4}(n-11)&=0\\20\left[\frac{3}{5}(n+5)-\frac{3}{4}(n-11)\right]&=20-0\\12(n+5)-15(n-11)&=0\\12n+60-15n+165&=0\\-3n&=-225\\n&=75\end{aligned}$$

**14.**
$$\begin{aligned}-\frac{5}{7}(p+11)+\frac{2}{5}(2p-5)&=0\\35\left[-\frac{5}{7}(p+11)+\frac{2}{5}(2p-5)\right]&=35-0\\-25(p+11)+14(2p-5)&=0\\-25p-275+28p-70&=0\\3p&=345\\p&=115\end{aligned}$$

**15.**
$$\begin{aligned}3(x+5)(x-1)&=(3x+4)(x-2)\\3x^2+12x-15&=3x^2-2x-8\\14x&=7\\x&=\frac{1}{2}\end{aligned}$$

**16.**
$$\begin{aligned}5(x+4)(x-4)&=(x-3)(5x+4)\\5x^2-80&=5x^2-11x-12\\11x&=68\\x&=\frac{68}{11}\end{aligned}$$

**17.**
$$\begin{aligned}0.08x+0.12(4000-x)&=432\\0.08x+480-0.12x&=432\\-0.04x&=-48\\x&=1200\end{aligned}$$

**18.**
$$\begin{aligned}0.075y+0.06(10{,}000-y)&=727.50\\0.075y+600-0.06y&=727.50\\0.015y&=127.5\\y&=8500\end{aligned}$$

**19.**
$$\begin{aligned}x+2y&=8\\2y&=-x+8\\y&=-\frac{1}{2}x+4\end{aligned}$$

**20.**
$$\begin{aligned}3x-5y&=15\\-5y&=-3x+15\\y&=\frac{3}{5}x-3\end{aligned}$$

**21.** $2x+5y=10$
$$2x=-5y+10$$
$$x=-\frac{5}{2}y+5$$

**22.** $5x-4y=10$
$$5x=4y+10$$
$$x=\frac{4}{5}y+2$$

**23.** $ay-by=c$
$$y(a-b)=c$$
$$y=\frac{c}{a-b}$$

**24.** $ax+by=c$
$$by=-ax+c$$
$$y=\frac{-ax+c}{b}$$

**25.** $x=\frac{y}{1-y}$
$$x(1-y)=y$$
$$x-xy=y$$
$$x=xy+y$$
$$x=y(x+1)$$
$$\frac{x}{x+1}=y$$

**26.** $x=\frac{2y-3}{y-1}$
$$x(y-1)=2y-3$$
$$xy-x=2y-3$$
$$xy-2y=x-3$$
$$y(x-2)=x-3$$
$$y=\frac{x-3}{x-2}$$

**27.** $x^2-2x-15=0$
$a=1 \quad b=-2 \quad c=-15$
$$x=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(-15)}}{2(1)}$$
$$x=\frac{2\pm\sqrt{4+60}}{2}$$
$$x=\frac{2\pm\sqrt{64}}{2}=\frac{2\pm 8}{2}$$
$$x=\frac{2+8}{2}=\frac{10}{2}=5 \quad \text{or}$$
$$x=\frac{2-8}{2}=\frac{-6}{2}=-3$$

**28.** $x^2-5x-24=0$
$a=1 \quad b=-5 \quad c=-24$
$$x=\frac{-(-5)\pm\sqrt{(-5)^2-4(1)(-24)}}{2(1)}$$
$$x=\frac{5\pm\sqrt{25+96}}{2}$$
$$x=\frac{5\pm\sqrt{121}}{2}=\frac{5\pm 11}{2}$$
$$x=\frac{5+11}{2}=\frac{16}{2}=8 \quad \text{or}$$
$$x=\frac{5-11}{2}=\frac{-6}{2}=-3$$

**29.** $x^2+x-1=0$
$a=1 \quad b=1 \quad c=-1$
$$x=\frac{-1\pm\sqrt{1^2-4(1)(-1)}}{2}$$
$$x=\frac{-1\pm\sqrt{1+4}}{2}$$
$$x=\frac{-1\pm\sqrt{5}}{2}$$

**30.** $x^2+x-2=0$
$a=1 \quad b=1 \quad c=-2$
$$x=\frac{-1\pm\sqrt{1^2-4(1)(-2)}}{2(1)}$$
$$x=\frac{-1\pm\sqrt{1+8}}{2}=\frac{1\pm\sqrt{9}}{2}=\frac{-1\pm 3}{2}$$
$$x=\frac{-1+3}{2}=\frac{2}{2}=1 \quad \text{or}$$
$$x=\frac{-1-3}{2}=\frac{-4}{2}=-2$$

**31.** $2x^2+4x+1=0$
$a=2 \quad b=4 \quad c=1$
$$x=\frac{-4\pm\sqrt{4^2-4(2)(1)}}{2(2)}$$
$$x=\frac{-4\pm\sqrt{16-8}}{4}$$
$$x=\frac{-4\pm\sqrt{8}}{4}=\frac{-4\pm 2\sqrt{2}}{4}$$
$$x=\frac{-2\pm\sqrt{2}}{2}$$

**32.** $2x^2+4x-1=0$
$a=2 \quad b=4 \quad c=-1$
$$x=\frac{-4\pm\sqrt{4^2-4(2)(-1)}}{2(2)}$$
$$x=\frac{-4\pm\sqrt{16+8}}{4}$$
$$x=\frac{-4\pm\sqrt{24}}{4}=\frac{-4\pm 2\sqrt{6}}{4}$$
$$x=\frac{-2\pm\sqrt{6}}{2}$$

**33.** $3x^2-5x-3=0$
$a=3 \quad b=-5 \quad c=-3$
$$x=\frac{-(-5)\pm\sqrt{(-5)^2-4(3)(-3)}}{2(3)}$$
$$x=\frac{5\pm\sqrt{25+36}}{6}$$
$$x=\frac{5\pm\sqrt{61}}{6}$$

**34.** $3x^2-5x-4=0$
$a=3 \quad b=-5 \quad c=-4$
$$x=\frac{-(-5)\pm\sqrt{(-5)^2-4(3)(-4)}}{2(3)}$$
$$x=\frac{5\pm\sqrt{25+48}}{6}$$
$$x=\frac{5\pm\sqrt{73}}{6}$$

**35.** $\frac{1}{2}x^2 + \frac{3}{4}x - 1 = 0$

$a = \frac{1}{2} \quad b = \frac{3}{4} \quad c = -1$

$x = \dfrac{-\frac{3}{4} \pm \sqrt{\left(\frac{3}{4}\right)^2 - 4\left(\frac{1}{2}\right)(-1)}}{2\left(\frac{1}{2}\right)}$

$x = \dfrac{-\frac{3}{4} \pm \sqrt{\frac{9}{16} + 2}}{1}$

$x = -\frac{3}{4} \pm \sqrt{\frac{41}{16}}$

$x = -\frac{3}{4} \pm \frac{\sqrt{41}}{4}$

$x = \dfrac{-3 \pm \sqrt{41}}{4}$

**36.** $\frac{2}{3}x^2 - 5x + \frac{1}{2} = 0$

$a = \frac{2}{3} \quad b = -5 \quad c = \frac{1}{2}$

$x = \dfrac{-(-5) \pm \sqrt{(-5)^2 - 4\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)}}{2\left(\frac{2}{3}\right)}$

$x = \dfrac{5 \pm \sqrt{25 - \frac{4}{3}}}{\frac{4}{3}}$

$x = \dfrac{5 \pm \sqrt{\frac{71}{3}}}{\frac{4}{3}}$

$x = \dfrac{\left(5 \pm \frac{\sqrt{213}}{3}\right)}{\frac{4}{3}} \left(\frac{3}{3}\right)$

$x = \dfrac{15 \pm \sqrt{213}}{4}$

**37.** $\sqrt{2}x^2 + 3x + \sqrt{2} = 0$

$a = \sqrt{2} \quad b = 3 \quad c = \sqrt{2}$

$x = \dfrac{-3 \pm \sqrt{3^2 - 4 \cdot \sqrt{2} \cdot \sqrt{2}}}{2\sqrt{2}}$

$x = \dfrac{-3 \pm \sqrt{9 - 8}}{2\sqrt{2}}$

$x = \dfrac{-3 + 1}{2\sqrt{2}} = \dfrac{-2}{2\sqrt{2}} = -\dfrac{\sqrt{2}}{2}$ or

$x = \dfrac{-3 - 1}{2\sqrt{2}} = \dfrac{-4}{2\sqrt{2}} = -\sqrt{2}$

**38.** $2x^2 + \sqrt{5}x - 3 = 0$

$a = 2 \quad b = \sqrt{5} \quad c = -3$

$x = \dfrac{-\sqrt{5} \pm \sqrt{(\sqrt{5})^2 - 4(2)(-3)}}{2 \cdot 2}$

$x = \dfrac{-\sqrt{5} \pm \sqrt{5 + 24}}{4}$

$x = \dfrac{-5 \pm \sqrt{29}}{4}$

**39.** $x^2 = 3x + 5$

$x^2 - 3x - 5 = 0$

$a = 1 \quad b = -3 \quad c = -5$

$x = \dfrac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(5)}}{2(1)}$

$x = \dfrac{3 \pm \sqrt{9 + 20}}{2}$

$x = \dfrac{3 \pm \sqrt{29}}{2}$

**40.** $-x^2 = 7x - 1$

$-x^2 - 7x + 1 = 0$

$a = -1 \quad b = -7 \quad c = 1$

$x = \dfrac{-(-7) \pm \sqrt{(-7)^2 - 4(-1)(1)}}{2(-1)}$

$x = \dfrac{7 \pm \sqrt{49 + 4}}{-2}$

$x = \dfrac{7 \pm \sqrt{53}}{-2}$

$x = \dfrac{-7 \pm \sqrt{53}}{2}$

**41.** $x^2 - 2x - 15 = 0$

$(x + 3)(x - 5) = 0$

$x + 3 = 0$ or $x - 5 = 0$

$x = -3 \qquad x = 5$

**42.** $y^2 + 3y - 10 = 0$

$(y + 5)(y - 2) = 0$

$y + 5 = 0$ or $y - 2 = 0$

$y = -5 \qquad y = 2$

**43.** $8y^2 + 189y - 72 = 0$

$(8y - 3)(y + 24) = 0$

$8y - 3 = 0$ or $y + 24 = 0$

$y = \frac{3}{8} \qquad y = -24$

**44.** $12w^2 - 41w + 24 = 0$

$(4w - 3)(3w - 8) = 0$

$4w - 3 = 0$ or $3w - 8 = 0$

$w = \frac{3}{4} \qquad w = \frac{8}{3}$

**45.** $3x^2 - 7x = 0$

$x(3x - 7) = 0$

$x = 0$ or $3x - 7 = 0$

$x = \frac{7}{3}$

**46.** $5x^2 = -8x$

$5x^2 + 8x = 0$

$x(5x + 8) = 0$

$x = 0$ or $5x + 8 = 0$

$x = -\frac{8}{5}$

**47.** $(x - 5)^2 - 9 = 0$

$[(x - 5) - 3][(x - 5) + 3] = 0$

$x - 8 = 0$ or $x - 2 = 0$

$x = 8 \qquad x = 2$

**48.** $(3x + 4)^2 - 16 = 0$

$[(3x + 4) - 4)][(3x + 4) + 4] = 0$

$3x = 0$ or $3x + 8 = 0$

$x = 0 \qquad x = -\frac{8}{3}$

**49.** $2x + 3 < 11$

$2x < 8$

$x < 4$

**50.** $3x-5>16$
$3x>21$
$x>7$

**51.** $x+4>3x+16$
$-2x>12$
$x<-6$

**52.** $5x+6<2x+1$
$3x<-5$
$x<-\frac{5}{3}$

**53.** $-6x+1\geq 19$
$-6x\geq 18$
$x\leq -3$

**54.** $-5x+2\leq 37$
$-5x\leq 35$
$x\geq -7$

**55.** $-3(x+2)\leq 5x+7$
$-3x-6\leq 5x+7$
$-8\leq 13$
$x\geq -\frac{13}{8}$

**56.** $-4(x-5)\geq 2x+15$
$-4x+20\geq 2x+15$
$-6x\geq -5$
$x\leq \frac{5}{6}$

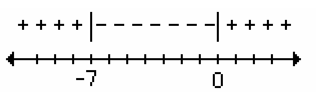
**57.** $-4(3x-5)>2(x-4)$
$-12x+20>2x-8$
$-14x>-28$
$x<2$

**58.** $3(x+7)\leq 5(2x-8)$
$3x+21\leq 10x-40$
$-7x\leq -61$
$x\geq \frac{61}{7}$

**59.** $x^2+7x>0$
$x(x+7)>0$

The product is positive.
The critical values are 0 and –7.

$x(x-7)$

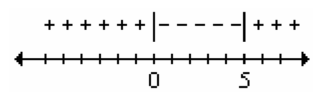



$(-\infty,-7)\cup(0,\infty)$

**60.** $x^2-5x\leq 0$
$x(x-5)\leq 0$

The product is negative or zero.
The critical values are 0 and 5.

$x(x-5)$

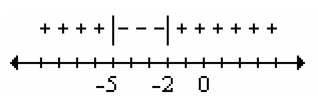



[0, 5]

**61.** $x^2+7x+10<0$
$(x+5)(x+2)<0$

The product is negative.
The critical values are –5 and –2.

$(x+5)(x+2)$

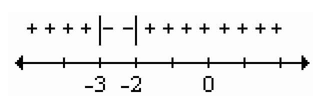



$(-5,-2)$

**62.** $x^2+5x+6<0$
$(x+3)(x+2)<0$

The product is negative.
The critical values are –3 and –2.

$(x+3)(x+2)$

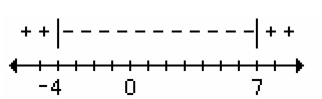



$(-3,-2)$

**63.** $x^2-3x\geq 28$
$x^2-3x-28\geq 0$
$(x+4)(x-7)\geq 0$

The product is positive or zero.
The critical values are –4 and 7.

$(x+4)(x-7)$

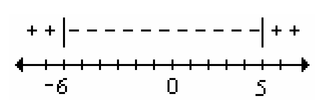



$(-\infty,\ -4]\cup[7,\ \infty)$

**64.** $x^2<-x+30$
$x^2+x-30<0$
$(x-5)(x+6)<0$

The product is negative.
The critical values are 5 and –6.

$(x-5)(x+6)$




$(-6,5)$

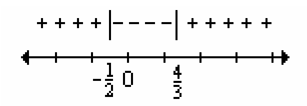
**65.** $6x^2 - 4 \le 5x$

$6x^2 - 5x - 4 \le 0$

$(3x-4)(2x+1) \le 0$

The product is negative or zero.
The critical values are $\frac{4}{3}$ and $-\frac{1}{2}$.

$(3x-4)(2x+1)$




$\left[-\frac{1}{2}, \frac{4}{3}\right]$

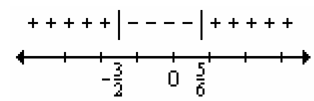
**66.** $12x^2 + 8x \ge 15$

$12x^2 + 8x - 15 \ge 0$

$(6x-5)(2x+3) \ge 0$

The product is positive or zero.
The critical values are $\frac{5}{6}$ and $-\frac{3}{2}$.

$(6x-5)(2x+3)$




$\left(-\infty, -\frac{3}{2}\right] \cup \left[\frac{5}{6}, \infty\right)$

**67.** $|x| < 4$

$-4 < x < 4$

$(-4,\ 4)$

**68.** $|x| > 2$

$x < -2$ or $x > 2$

$(-\infty,\ -2) \cup (2,\ \infty)$

**69.** $|x-1| < 9$

$-9 < x - 1 < 9$

$-8 < x < 10$

$(-8,\ 10)$

**70.** $|x-3| < 10$

$-10 < x - 3 < 10$

$-7 < x < 13$

$(-7,\ 13)$

**71.** $|x+3| > 30$

$x + 3 < -30$ or $x + 3 > 30$

$x < -33 \qquad x > 27$

$(-\infty,\ -33) \cup (27,\ \infty)$

**72.** $|x+4| < 2$

$-2 < x + 4 < 2$

$-6 < x < -2$

$(-6, -2)$

**73.** $|2x-1| > 4$

$2x - 1 < -4$ or $2x - 1 > 4$

$2x < -3 \qquad 2x > 5$

$x < -\frac{3}{2} \qquad x > \frac{5}{2}$

$\left(-\infty,\ -\frac{3}{2}\right) \cup \left(\frac{5}{2},\ \infty\right)$

**74.** $|2x-9| < 7$

$-7 < 2x - 9 < 7$

$2 < 2x < 16$

$1 < x < 8$

$(1,\ 8)$

**75.** $|x+3| \ge 5$

$x + 3 \le -5$ or $x + 3 \ge 5$

$x \le -8 \qquad x \ge 2$

$(-\infty, -8] \cup [2, \infty)$

**76.** $|x-10| \ge 2$

$x - 10 \le -2$ or $x - 10 \ge 2$

$x \le 8 \qquad x \ge 12$

$(-\infty,\ 8) \cup (12, \infty)$

**77.** $|3x-10| \le 14$

$-14 \le 3x - 10 \le 14$

$-4 \le 3x \le 24$

$-\frac{4}{3} \le x \le 8$

$\left[-\frac{4}{3},\ 8\right]$

**78.** $|2x-5| \ge 1$

$2x - 5 \le -1$ or $2x - 5 \ge 1$

$2x \le 4 \qquad 2x \ge 6$

$x \le 2$ or $x \ge 3$

$(-\infty, 2] \cup [3, \infty)$

**79.** $|4-5x| \ge 24$

$4 - 5x \le 24$ or $4 - 5x \ge 24$

$-5x \le -28 \qquad -5x \ge 20$

$x \ge \frac{28}{5} \qquad x \le -4$

$(-\infty,\ -4] \cup \left[\frac{28}{5},\ \infty\right)$

**80.** $|3-2x| \le 5$

$-5 \le 3 - 2x \le 5$

$-8 \le -2x \le 2$

$4 \ge x \ge -1$

$[-1,\ 4]$

**81.** $|x-5| \ge 0$

Because an absolute value is always nonnegative, the inequality is always true. The solution set consists of all real numbers.

$(-\infty, \infty)$

**82.** $|x-7| \geq 0$

Because an absolute value is always nonnegative, the inequality is always true. The solution set consists of all real numbers.

$(-\infty, \infty)$

**83.** $|x-4| \leq 0$

Because an absolute value is always nonnegative, the inequality $|x-4| < 0$ has no solution. Thus the only solution of the inequality $|x-4| \leq 0$ is the solution of the equation $x - 4 = 0$.

$x = 4$

**84.** $|2x+7| \leq 0$

$$2x+7=0$$
$$2x=-7$$
$$x=-\frac{7}{2}$$

**85.**

$$A = 35$$
$$A = LW$$
$$LW = 35$$
$$L = \frac{35}{W}$$

$$P = 27$$
$$P = 2L + 2W$$
$$2L + 2W = 27$$
$$2\left(\frac{35}{W}\right) + 2W = 27$$
$$70 + 2W^2 = 27W$$
$$2W^2 - 27W + 70 = 0$$
$$(2W-7)(W-10) = 0$$

$W = \frac{7}{2}$ or $W = 10$

$$35 = \frac{7}{2}L \qquad 35 = LW$$
$$70 = 7L \qquad 35 = 10L$$
$$10 = L \qquad 3.5 = L$$

The rectangle measures 3.5 cm by 10 cm.

**86.**

$$A = 60$$
$$A = LW$$
$$LW = 60$$
$$W = \frac{60}{L}$$

$$P = 34$$
$$P = 2L + 2W$$
$$2L + 2W = 34$$
$$L + W = 17$$
$$L + \frac{60}{L} = 17$$
$$L^2 - 17L + 60 = 0$$
$$(L-12)(L-5) = 0$$

$L = 12$, $W = \frac{60}{12} = 5$ $\qquad$ $L = 5$, $W = \frac{60}{5} = 12$

The rectangle measures 5 ft by 12 ft.

**87.**

$$A = 1500 = lw$$
$$P = 600 - 2l + 3w$$
$$l = \frac{15000}{w}$$

$$2l + 3w = 600$$
$$2\left(\frac{15000}{w}\right) + 3w = 600$$
$$30{,}000 + 3w^2 = 600w$$
$$3w^2 - 600w + 30{,}000 = 0$$
$$3(w^2 - 200w + 10{,}000) = 0$$
$$3(w-100)(w-100) = 0$$

$w = 100$ ft

$l = \frac{15000}{100} = 150$ ft

The dimensions are 100 feet by 150 feet.

**88.**

$$P = 4w + 2l = 400$$
$$2w + l = 200$$
$$A = 4800 = lw$$
$$l = \frac{4800}{w}$$

$$2w + \frac{4800}{w} = 200$$
$$2w^2 + 4800 = 200w$$
$$w^2 - 100w + 2400 = 0$$
$$(w-60)(w-40) = 0$$

$w = 60$, $l = \frac{4800}{60} = 80$ $\qquad$ $w = 40$, $l = \frac{4800}{40} = 120$

There are two solutions: 60 yd × 80 yd or 40 yd × 120 yd.

**89.** Plan A: $5 + 0.01x$

Plan B: $1 + 0.08x$

$$
\begin{aligned}
5 + 0.01x &< 1 + 0.08x \\
4 &< .07x \\
57.1 &< x
\end{aligned}
$$

Plan A is less expensive if you use at least 58 checks.

**90.** Company A: $19 + 0.12M$

Company B: $12 + 0.21M$

$$
\begin{aligned}
19 + 0.12M &< 12 + 0.21M \\
7 &< 0.09M \\
77.7 &< M
\end{aligned}
$$

Company A is less expensive if you drive at least 78 miles.

**91.** Plan A: $100 + 8x$

Plan B: $250 + 3.5x$

$$
\begin{aligned}
100 + 8x &> 250 + 3.5x \\
4.5x &> 150 \\
x &> 33.3
\end{aligned}
$$

Plan A pays better if at least 34 sales are made.

**92.** Plan A: $15 + 1.49x$

Plan B: $1.99x$

$$
\begin{aligned}
1.99x &< 15 + 1.49x \\
0.50x &< 15 \\
x &< 30
\end{aligned}
$$

If fewer than 30 videos are rented, Plan B is less expensive.

**93.**

$$
\begin{aligned}
68 &\le F \le 104 \\
68 &\le \frac{9}{5}C + 32 \le 104 \\
36 &\le \frac{9}{5}C \le 72 \\
20 &\le C \le 40
\end{aligned}
$$

## Connecting Concepts

**94.** **a.**

$$\frac{l}{w} = \frac{w}{l - w}$$

$$
\begin{aligned}
l(l - w) &= w^2 \\
l^2 - lw &= w^2 \\
0 &= w^2 + lw - l^2 \\
w &= \frac{-l \pm \sqrt{l^2 - 4(-l^2)}}{2}
\end{aligned}
$$

Since $w > 0$, $w = \dfrac{-l + l\sqrt{5}}{2} = l\left(\dfrac{-1 + \sqrt{5}}{2}\right)$

**b.** $w = 101\left(\dfrac{-l + \sqrt{5}}{2}\right) \approx 62.4$ ft

**95.**

$$
\begin{aligned}
253 &= \frac{1}{2}n(n + 1) \\
n^2 + n - 506 &= 0 \\
(n + 23)(n - 22) &= 0 \\
n &= 22
\end{aligned}
$$

So $1 + 2 + 3 + \cdots + 21 + 22 = 253$.

**96.** **a.**

$$
\begin{aligned}
464 &= \frac{n}{2}(n - 3) \\
n^2 - 3n - 928 &= 0 \\
(n - 32)(n + 29) &= 0 \\
n &= 32
\end{aligned}
$$

The polygon has 32 sides.

**b.**

$$
\begin{aligned}
12 &= \frac{n}{2}(n - 3) \\
24 &= n^2 - 3 \\
0 &= n^2 - 3 - 24
\end{aligned}
$$

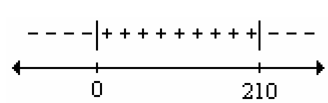
$n^2 - 3n - 24$ is not factorable over the integers. Thus, the polygon in **a.** cannot have 12 diagonals.

**97.** $R = 420x - 2x^2$

$$
\begin{aligned}
420x - 2x^2 &> 0 \\
2x(210 - x) &> 0
\end{aligned}
$$

The product is positive.
The critical values are 0 and 210.

$2x(210 - x)$




(0, 210)

**98.** **a.** $|x-3|<8$

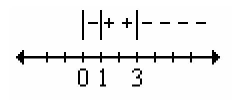
**b.** $|x-j|<k$

**99.** $s=-16t^2+v_0t+s_0 \quad s>48, \quad v_0=64, \quad s_0=0$

$$-16t+64t>48$$
$$-16t^2+64t-48>0$$
$$-16(t^2-4t+3)>0$$
$$-16(t-3)(t-1)>0$$

The product is positive.
The critical values are $t=3$ and $t=1$.

$-16(t-3)(t-1)$




1 second $< t <$ 3 seconds

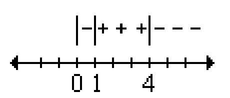
The ball is higher than 48 ft between 1 and 3 seconds.

**100.** $v_0=80, \quad s_0=32$

$$-16t^2+80t+32>96$$
$$-16t^2+80t-64>0$$
$$-16(t^2-5t+4)>0$$
$$-16(t-4)(t-1)>0$$

The product is positive.
The critical values are $t=4$ and $t=1$.

$-16(t-4)(t-1)$




1 second $< t <$ 4 seconds

The ball is higher than 96 ft between 1 and 4 seconds.

**101.** **a.** $|s-4.25|\le 0.01$

**b.** $s-4.25=0.01$, or $s-4.25=-0.01$
$s=4.26$ $\quad s=4.24$ critical values

$4.24 \le s \le 4.26$

## Prepare for Section 1.2

**PS1.** $\dfrac{4+(-7)}{2}=\dfrac{-3}{2}$

**PS2.** $\sqrt{50}=\sqrt{25}\sqrt{2}=5\sqrt{2}$

**PS3.** $y=3x-2$
$5\overset{?}{=}3(-1)-2$
$5\ne-5$ No, the equation is not true.

**PS4.** $y=(-3)^2-3(-3)-2$
$y=9+9-2$
$y=16$

**PS5.** $|-3-(-1)|=|-3+1|=|-2|=2$

**PS6.** $\sqrt{(-3)^2-4(-2)(2)}=\sqrt{9+16}=\sqrt{25}=5$

## Section 1.2

### Prepare for Section 1.3

**PS1.**
$$y = 3x + 12$$
$$0 = 3x + 12$$
$$-12 = 3x$$
$$-4 = x$$

**PS2.**
$$y = x^2 - 4x + 3$$
$$0 = x^2 - 4x + 3$$
$$0 = (x-1)(x-3)$$
$$x - 1 = 0 \quad x - 3 = 0$$
$$x = 1 \quad x = 3$$
1 and 3

**PS3.**
$$y^2 = x \qquad y^2 = x$$
$$a^2 = 9 \qquad b^2 = 9$$
$a = -3$ or 3 $\qquad b = -3$ or 3

**PS4.** $d = \sqrt{(3-(-4))^2 + (-2-1)^2} = \sqrt{49+9} = \sqrt{58}$

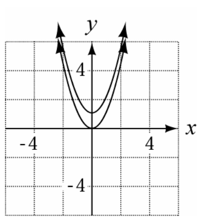
**PS5.** –4

**PS6.** Decrease

## Section 1.3

### Prepare for Section 1.4

**PS1.**




The graph of *g* is one unit above the graph of *f*.

**PS2.**
$$f(3) = \frac{3(3)^4}{(3)^2 + 1} = \frac{243}{10} = 24.3$$
$$f(-3) = \frac{3(-3)^4}{(-3)^2 + 1} = \frac{243}{10} = 24.3$$
$$f(3) = f(-3)$$

**PS3.**
$$f(-2) = 2(-2)^3 - 5(-2) = -16 + 10 = -6$$
$$-f(2) = -[2(2)^3 - 5(2)] = -[16 - 10] = -6$$
$$f(-2) = -f(2)$$

**PS4.**
$$f(-2) - g(-2) = (-2)^2 - [-2 + 3] = 4 - 1 = 3$$
$$f(-1) - g(-1) = (-1)^2 - [-1 + 3] = 1 - 2 = -1$$
$$f(0) - g(0) = (0)^2 - [0 + 3] = 0 - 3 = -3$$
$$f(1) - g(1) = (1)^2 - [1 + 3] = 1 - 4 = -3$$
$$f(2) - g(2) = (2)^2 - [2 + 3] = 4 - 5 = -1$$

**PS5.** $\frac{-a+a}{2} = 0, \quad \frac{b+b}{2} = b$

midpoint is (0, *b*)

**PS6.** $\frac{-a+a}{2} = 0, \quad \frac{-b+b}{2} = 0$

midpoint is (0, 0)

## Section 1.4

### Prepare for Section 1.5

**PS1.**
$$f(3) - g(3) = \left(3^2 + 3(3) + 1\right) - \left(4(3) + 5\right)$$
$$= 19 - 17$$
$$= 2$$

**PS2.**
$$f(-2) \cdot g(-2) = \left(3(-2)^2 - (-2) - 4\right) \cdot \left(2(-2) - 5\right)$$
$$= (12 + 2 - 4) \cdot (-9)$$
$$= 10 \cdot (-9)$$
$$= -90$$

**PS3.** $f(3a) = 2(3a)^2 - 5(3a) + 2$
$= 18a^2 - 15a + 2$

**PS4.** $f(2+h) = 2(2+h)^2 - 5(2+h) + 2$
$= 2h^2 + 8h + 8 - 5h - 10 + 2$
$= 2h^2 + 3h$

**PS5.** Domain: all real numbers except $x = 1$

**PS6.** $2x - 8 = 0$
$x = 4$
Domain: $x \geq 4$ or $[4, \infty)$

## Section 1.5

### Prepare for Section 1.6

**PS1.** $2x + 5y = 15$
$5y = -2x + 15$
$y = -\frac{2}{5}x + 3$

**PS2.** $x = \frac{y+1}{y}$
$xy = y + 1$
$xy - y = 1$
$y(x-1) = 1$
$y = \frac{1}{x-1}$

**PS3.** $f(-1) = \frac{2(-1)^2}{(-1)-1} = \frac{2}{-2} = -1$

**PS4.** (3, 7)

**PS5.** All real numbers.

**PS6.** $x + 2 \geq 0$
$x \geq -2$
$\{x \mid x \geq -2\}$

## Section 1.6

### Prepare for Section 1.7

See *CAT Prepare for Section* 2.7 solutions on page 156.

### Chapter 1 Assessing Concepts

**1.** a, c, d, e

**2.** $f[g(x)] = 3(2x+4) + 8$
$= 6x + 12 + 8$
$= 6x + 20$

$g[f(x)] = 2(3x+8) + 4$
$= 6x + 16 + 4$
$= 6x + 20$

Thus $f[g(x)] = 6x + 20 = g[f(x)]$.

To be inverse functions,
$f[g(x)] = x = g[f(x)]$.
No. They are not inverse functions.

**3.** $f(2) = 3$
$f(x) = f(x+4)$
$f(2) = f(2+4) = f(6)$
$f(6) = f(6+4) = f(10)$
$f(10) = f(10+4) = f(14)$
$f(14) = f(14+4) = f(18)$
Thus, $f(18) = f(2) = 3$.

**4.** $|x+2| < 3$

**5.** It is the slope of the line between $(a, f(a))$ and $(b, f(b))$.

**6.** (3, –2)

**7.** (7, 3)

**8.** (–3, 6)

**9.** (3, 4)

**10.** Yes. The slope of the regression line is negative.

## Chapter 1 Chapter Review

**1.**
$$3-4z=12$$
$$-4z=9$$
$$z=-\frac{9}{4}$$

**2.**
$$4y-3=6y+5$$
$$-2y=8$$
$$y=-4$$

**3.**
$$2x-3(2-3x)=14x$$
$$2x-6+9x=14x$$
$$-6=3x$$
$$-2=x$$

**4.**
$$5-2(3m+2)=3(1-m)$$
$$5-6m-4=3-3m$$
$$1-6m=3-3m$$
$$-3m=2$$
$$m=-\frac{2}{3}$$

**5.**
$$y^2-3y-18=0$$
$$(y-6)(y+3)=0$$
$$y-6=0 \quad \text{or} \quad y+3=0$$
$$y=6 \quad \text{or} \quad y=-3$$

**6.**
$$2z^2-9z+4=0$$
$$(2z-1)(z-4)=0$$
$$2z-1=0 \quad \text{or} \quad z-4=0$$
$$2z=1 \qquad z=4$$
$$z=\frac{1}{2}$$

**7.**
$$3v^2+v=1$$
$$3v^2+v-1=0$$
$$v=\frac{-1\pm\sqrt{1^2-4(3)(-1)}}{2(3)}$$
$$v=\frac{-1\pm\sqrt{13}}{6}$$

**8.**
$$3s=4-2s^2$$
$$2s^2+3s-4=0$$
$$s=\frac{-3\pm\sqrt{3^2-4(2)(-4)}}{2(2)}$$
$$s=\frac{-3\pm\sqrt{9+32}}{4}$$
$$s=\frac{-3\pm\sqrt{41}}{4}$$

**9.**
$$3c-5\le 5c+7$$
$$-2c\le 12$$
$$c\ge -6$$

**10.**
$$7a>5-2(3a-4)$$
$$7a>5-6a+8$$
$$13a>13$$
$$a>1$$

**11.**
$$x^2-x-12\ge 0$$
$$(x-4)(x+3)\ge 0$$

Critical values are 4 and $-3$.

$$(-\infty,-3]\cup[4,\infty)$$

**12.**
$$2x^2-x<1$$
$$2x^2-x-1<0$$
$$(2x+1)(x-1)<0$$

Critical values are $-\frac{1}{2}$ and 1.

$$-\frac{1}{2}<x<1$$

**13.**
$$|2x-5|>3$$
$$2x-5>3 \quad \text{or} \quad 2x-5<-3$$
$$2x>8 \qquad 2x<2$$
$$x>4 \qquad x<1$$
$$(-\infty,1)\cup(4,\infty)$$

**14.**
$$|1-3x|\le 4$$
$$-4\le 1-3x\le 4$$
$$-5\le -3x\le 3$$
$$\frac{5}{3}\ge x\ge -1$$

**15.**
$$c^2=a^2+b^2$$
$$c^2=6^2+8^2$$
$$c=\sqrt{36+64}$$
$$c=\sqrt{100}=10$$

**16.**
$$c^2=a^2+b^2$$
$$20^2=11^2+b^2$$
$$b=\sqrt{400-121}$$
$$b=\sqrt{279}$$
$$b\approx 16.7$$

**17.**
$$c^2=a^2+b^2$$
$$13^2=a^2+12^2$$
$$a=\sqrt{169-144}$$
$$a=\sqrt{25}$$
$$a=5$$

**18.**
$$c^2=a^2+b^2$$
$$c^2=7^2+14^2$$
$$c=\sqrt{49+196}$$
$$c=\sqrt{245}$$
$$c\approx 15.7$$

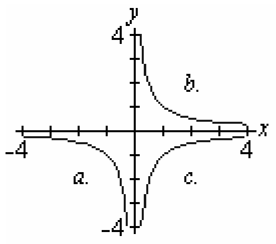
**19.** $d = \sqrt{(7-(-3))^2 + (11-2)^2}$
$= \sqrt{100+81}$
$= \sqrt{181}$

**20.** $d = \sqrt{(-3-5)^2 + (-8-(-4))^2}$
$= \sqrt{64+16}$
$= \sqrt{80} = 4\sqrt{5}$

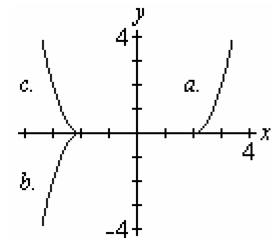
**21.** $\left(\frac{2+(-3)}{2}, \frac{8+12}{2}\right) = \left(-\frac{1}{2}, 10\right)$

**22.** $\left(\frac{-4+8}{2}, \frac{7+(-11)}{2}\right) = (2, -2)$

**23.**




**24.**




**25.** $y = x^2 - 7$

Replace $y$ with $-y$.

$-y = x^2 - 7$
$y = -x^2 + 7$

Thus, $y$ is not symmetric with respect to the $x$-axis.

Replace $x$ with $-x$.

$y = (-x)^2 - 7$
$y = x^2 - 7$

Thus, $y$ is symmetric with respect to the $y$-axis.

Replace $x$ with $-x$ and replace $y$ with $-y$.

$-y = (-x)^2 - 7$
$y = -x^2 + 7$

Thus, $y$ is not symmetric with respect to the origin.

Therefore, the graph of $y = x^2 - 7$ is symmetric with respect to the $y$-axis.

**26.** $x = y^2 + 3$
$y^2 = x - 3$
$y = \pm\sqrt{x-3}$

Replace $y$ with $-y$.

$x = (-y)^2 + 3$
$x = y^2 + 3$
$y^2 = x - 3$
$y = \pm\sqrt{x-3}$

Thus, $y$ is symmetric with respect to the $x$-axis.

Replace $x$ with $-x$.

$-x = y^2 + 3$
$y^2 = -x - 3$
$y = \pm\sqrt{-x-3}$

Thus, $y$ is not symmetric with respect to the $y$-axis.

Replace $x$ with $-x$ and replace $y$ with $-y$.

$-x = (-y)^2 + 3$
$-x = y^2 + 3$
$y^2 = -x - 3$
$y = \pm\sqrt{-x-3}$

Thus, $y$ is not symmetric with respect to the origin.

Therefore, the graph of $x = y^2 + 3$ is symmetric with respect to the $x$-axis.

**27.** $y = x^3 - 4x$

Replace $y$ with $-y$.

$$-y = x^3 - 4x$$
$$y = -x^3 + 4x$$

Thus, $y$ is not symmetric with respect to the $x$-axis.

Replace $x$ with $-x$.

$$y = (-x)^3 - 4(-x)$$
$$y = -x^3 + 4x$$

Thus, $y$ is not symmetric with respect to the $y$-axis.

Replace $x$ with $-x$ and replace $y$ with $-y$.

$$-y = (-x)^3 - 4(-x)$$
$$-y = -x^3 + 4x$$
$$y = x^3 - 4x$$

Thus, $y$ is symmetric with respect to the origin.

Therefore, the graph of $y = x^3 - 4x$ is symmetric with respect to the origin.

**28.** $y^2 = x^2 + 4$

$$y = \pm\sqrt{x^2 + 4}$$

Replace $y$ with $-y$.

$$(-y)^2 = x^2 + 4$$
$$y^2 = x^2 + 4$$
$$y = \pm\sqrt{x^2 + 4}$$

$y$ is symmetric with respect to the $x$-axis.

Replace $x$ with $-x$.

$$y^2 = (-x)^2 + 4$$
$$y^2 = x^2 + 4$$
$$y = \pm\sqrt{x^2 + 4}$$

Thus, $y$ is symmetric with respect to the $y$-axis.

Replace $x$ with $-x$ and replace $y$ with $-y$.

$$(-y)^2 = (-x)^2 + 4$$
$$y^2 = x^2 + 4$$
$$y = \pm\sqrt{x^2 + 4}$$

Thus, $y$ is symmetric with respect to the origin.

Therefore, the graph of $y^2 = x^2 + 4$ is symmetric with respect to the $x$-axis, the $y$-axis, and the origin.

**29.** $\dfrac{x^2}{3^2} + \dfrac{y^2}{4^2} = 1$

Replace $y$ with $-y$.

$$\frac{x^2}{3^2} + \frac{(-y)^2}{4^2} = 1 \Rightarrow \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$$

Thus, $y$ is symmetric with respect to the $x$-axis.

Replace $x$ with $-x$.

$$\frac{(-x)^2}{3^2} + \frac{y^2}{4^2} = 1 \Rightarrow \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$$

Thus, $y$ is symmetric with respect to the $y$-axis.

Replace $x$ with $-x$ and replace $y$ with $-y$.

$$\frac{(-x)^2}{3^2} + \frac{(-y)^2}{4^2} = 1 \Rightarrow \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$$

Thus, $y$ is symmetric with respect to the origin.

Therefore, the graph of $\dfrac{x^2}{3^2} + \dfrac{y^2}{4^2} = 1$ is symmetric with respect to the $x$-axis, the $y$-axis, and the origin.

**30.** $xy = 8$

$$y = \frac{8}{x}$$

Replace $y$ with $-y$.

$$x(-y) = 8$$
$$y = -\frac{8}{x}$$

Thus, $y$ is not symmetric with respect to the $x$-axis.

Replace $x$ with $-x$.

$$-xy = 8$$
$$y = -\frac{8}{x}$$

Thus, $y$ is not symmetric with respect to the $y$-axis.

Replace $x$ with $-x$ and replace $y$ with $-y$.

$$-x(-y) = 8$$
$$y = \frac{8}{x}$$

Thus, $y$ is symmetric with respect to the origin.

Therefore, the graph of $xy = 8$ is symmetric with respect to the origin.

**31.** $|y| = |x|$

Replace $y$ with $-y$.

$|-y| = |x| \Rightarrow |y| = |x|$

Thus, $y$ is symmetric with respect to the $x$-axis.

Replace $x$ with $-x$.

$|y| = |-x| \Rightarrow |y| = |x|$

Thus, $y$ is symmetric with respect to the $y$-axis.

Replace $x$ with $-x$ and replace $y$ with $-y$.

$|-y| = |-x| \Rightarrow |y| = |x|$

Thus, $y$ is symmetric with respect to the origin.

Therefore, the graph of $|y| = |x|$ is symmetric with respect to the $x$-axis, the $y$-axis, and the origin.

**32.** $|x + y| = 4$

Replace $y$ with $-y$.

$|x + (-y)| = 4 \Rightarrow |x - y| = 4$

Thus, $y$ is not symmetric with respect to the $x$-axis.

Replace $x$ with $-x$.

$|(-x) + y| = 4 \Rightarrow |y - x| = 4$

Thus, $y$ is not symmetric with respect to the $y$-axis.

Replace $x$ with $-x$ and replace $y$ with $-y$.

$|(-x) + (-y)| = 4 \Rightarrow |-(x + y)| = 4 \Rightarrow |x + y| = 4$

Thus, $y$ is symmetric with respect to the origin.

Therefore, the graph of $|x + y| = 4$ is symmetric with respect to the origin.

**33.** center $(3, -4)$, radius 9

**34.**

$$\begin{aligned} x^2 + 10x \quad + y^2 + 4y \quad &= -20 \\ x^2 + 10x + 25 \quad + y^2 + 4y + 4 &= 25 + 4 - 20 \\ (x+5)^2 \quad + \quad (y+2)^2 \quad &= 9 \end{aligned}$$

center $(-5, -2)$, radius 3

**35.** $(x-2)^2 + (y+3)^2 = 5^2$

**36.** $(x+5)^2 + (y-1)^2 = 8^2$, radius $= |-5-(3)| = 8$

**37.**

**a.** $f(1) = 3(1)^2 + 4(1) - 5$
$= 2$

**b.** $f(-3) = 27 - 12 - 5$
$= 10$

**c.** $f(t) = 3t^2 + 4t - 5$

**d.** $f(x+h) = 3(x+h)^2 + 4(x+h) - 5$
$= 3x^2 + 6xh + 3h^2 + 4x + 4h - 5$

**e.** $3f(t) = 9t^2 + 12t - 15$

**f.** $f(3t) = 3(3t)^2 + 4(3t) - 5$
$= 27t^2 + 12t - 5$

**38.**

**a.** $g(3) = \sqrt{64 - 3^2}$
$= \sqrt{55}$

**b.** $g(-5) = \sqrt{64 - (-5)^2}$
$= \sqrt{64 - 25}$
$= \sqrt{39}$

**c.** $g(8) = \sqrt{64 - 8^2}$
$= 0$

**d.** $g(-x) = \sqrt{64 - (-x)^2}$
$= \sqrt{64 - x^2}$

**e.** $2g(t) = 2\sqrt{64 - t^2}$

**f.** $g(2t) = \sqrt{64 - (2t)^2}$
$= \sqrt{64 - 4t^2}$
$= 2\sqrt{16 - t^2}$

**39.** **a.** $(f \circ g)(3) = f[g(3)]$
$= f[3-8]$
$= f[-5]$
$= (-5)^2 + 4(-5)$
$= 5$

**b.** $(g \circ f)(-3) = g[f(-3)]$
$= g[(-3)^2 + 4(-3)]$
$= g[-3]$
$= [-3-8]$
$= -11$

**c.** $(f \circ g)(x) = f[g(x)]$
$= f[x-8]$
$= (x-8)^2 + 4(x-8)$
$= x^2 - 16x + 64 + 4x - 32$
$= x^2 - 12x + 32$

**d.** $(g \circ f)(x) = g[f(x)]$
$= g[x^2 + 4x]$
$= [x^2 + 4x] - 8$
$= x^2 + 4x - 8$

**40.** **a.** $(f \circ g)(-5) = f[g(-5)]$
$= f\left[\,|(-5)-1|\,\right]$
$= f[6]$
$= 2(6)^2 + 7$
$= 72 + 7$
$= 79$

**b.** $(g \circ f)(-5) = g[f(-5)]$
$= g[2(-5)^2 + 7]$
$= g[57]$
$= |57-1|$
$= 56$

**c.** $(f \circ g)(x) = f[g(x)]$
$= f\left[\,|x-1|\,\right]$
$= 2\left(|x-1|\right)^2 + 7$
$= 2(x-1)^2 + 7$
$= 2(x^2 - 2x + 1) + 7$
$= 2x^2 - 4x + 2 + 7$
$= 2x^2 - 4x + 9$

**d.** $(g \circ f)(x) = g[f(x)]$
$= g[2x^2 + 7]$
$= \left|2x^2 + 7 - 1\right|$
$= \left|2x^2 + 6\right|$
$= 2x^2 + 6$

**41.** $$\frac{f(x+h)-f(x)}{h} = \frac{4(x+h)^2 - 3(x+h) - 1 - (4x^2 - 3x - 1)}{h}$$
$$= \frac{4x^2 + 8xh + 4h^2 - 3x - 3h - 1 - 4x^2 + 3x + 1}{h}$$
$$= \frac{8xh + 4h^2 - 3h}{h}$$
$$= 8x + 4h - 3$$

**42.** $$\frac{g(x+h)-g(x)}{h} = \frac{(x+h)^3 - (x+h) - (x^3 - x)}{h}$$
$$= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h}$$
$$= \frac{3x^2h + 3xh^2 + h^3 - h}{h}$$
$$= 3x^2 + 3xh + h^2 - 1$$

**43.** $f(x) = -2x^2 + 3$

Domain: All real numbers

**44.** $f(x)=\sqrt{6-x}$

$6-x\ge 0$

$-x\ge -6$

$x\le 6$

Domain: $\{x|x\le 6\}$

**45.** $f(x)=\sqrt{25-x^2}$

$25-x^2\ge 0$

$(5-x)(5+x)\ge 0$

Critical values $-5$ and 5.

Domain: $\{x|-5\le x\le 5\}$

**46.** $f(x)=\dfrac{3}{x^2-2x-15}$

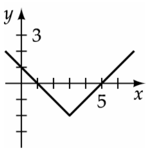
$x^2-2x-15=0$

$(x+3)(x-5)=0$

$x=-3 \quad x=5$

Domain: $\{x|x\ne -3 \text{ and } x\ne 5\}$

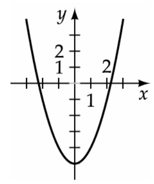
**47.**




$f$ is increasing on $[3,\ \infty)$

$f$ is decreasing on $(-\infty,\ 3]$

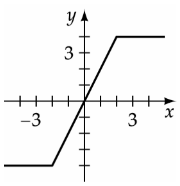
**48.**




$f$ is increasing on $[0,\ \infty)$

$f$ is decreasing on $(-\infty,\ 0]$

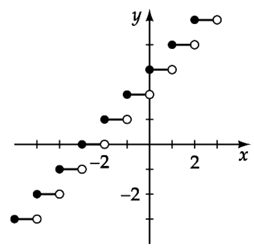
**49.**




$f$ is increasing on $[-2,\ 2]$

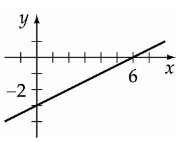
$f$ is constant on $(-\infty,\ -2]\cup[2,\ \infty)$

**50.**



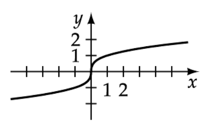

$f$ is constant on $\ldots,[-6,\ -5),\ [-5,\ -4),\ [-4,\ -3),$
$[-3,\ -2),\ [-2,\ -1),\ [-1,\ 0),\ [0,\ 1),\ldots$

**51.**



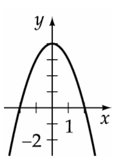

$f$ is increasing on $(-\infty,\infty)$

**52.**




$f$ is increasing on $(-\infty,\infty)$

**53.**

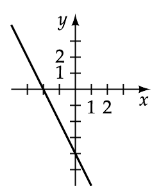



**a.** Domain $\{x|x \text{ is a real number}\}$

Range $\{y|y\le 4\}$

**b.** $g$ is an even function

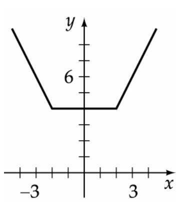
**54.**




**a.** Domain all real numbers
Range all real numbers

**b.** $g$ is neither even nor odd

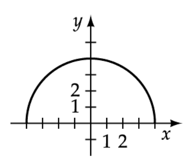
**55.**




**a.** Domain all real numbers
Range $\{y|y\ge 4\}$

**b.** $g$ is an even function

**56.**

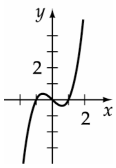



**a.** Domain $\{x|-4\le x\le 4\}$

Range $\{y|0\le y\le 4\}$

**b.** $g$ is an even function

**57.**

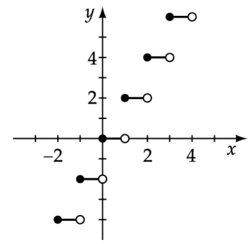



**a.** Domain $\{x|x \text{ is a real number}\}$

Range $\{y|y \text{ is a real number}\}$

**b.** $g$ is an odd function

**58.**




**a.** Domain $\{x|x \text{ is a real number}\}$

Range $\{y|y \text{ is an even integer}\}$

**b.** $g$ is neither even or odd.

**59.** $(f+g)(x) = x^2 - 9 + x + 3$
$= x^2 + x - 6$

Domain of $(f+g)(x)$ is $\{x|x \text{ is a real number}\}$.

$(f-g)(x) = x^2 - 9 - (x+3)$
$= x^2 - x - 12$

Domain of $(f-g)(x)$ is $\{x|x \text{ is a real number}\}$.

$(fg)(x) = (x^2 - 9)(x+3)$
$= x^3 + 3x^2 - 9x - 27$

Domain of $(fg)(x)$ is $\{x|x \text{ is a real number}\}$.

$\left(\frac{f}{g}\right)(x) = \frac{x^2 - 9}{x+3}$
$= x - 3$

Domain of $\left(\frac{f}{g}\right)(x)$ is $\{x|x \neq -3\}$.

**60.** $(f+g)(x) = x^3 + 8 + x^2 - 2x + 4$
$= x^3 + x^2 - 2x + 12$

Domain of $(f+g)(x)$ is the set of all real numbers.

$(f-g)(x) = x^3 + 8 - (x^2 - 2x + 4)$
$= x^3 + 8 - x^2 + 2x - 4$
$= x^3 - x^2 + 2x + 4$

Domain of $(f-g)(x)$ is the set of all real numbers.

$(fg)(x) = (x^3 + 8)(x^2 - 2x + 4)$
$= x^5 - 2x^4 + 4x^3 + 8x^2 - 16x + 32$

Domain of $(fg)(x)$ is the set of all real numbers.

$\left(\frac{f}{g}\right)(x) = \frac{x^3 + 8}{x^2 - 2x + 4}$
$= \frac{(x+2)(x^2 - 2x + 4)}{x^2 - 2x + 4}$
$= x + 2$

Domain of $\left(\frac{f}{g}\right)(x)$ is the set of all real numbers.

**61.** $F[G(x)] = 2\left(\frac{x+5}{2}\right) - 5$ and $G[F(x)] = \frac{(2x-5)+5}{2}$

$F[G(x)] = x + 5 - 5 = x$

$G[F(x)] = \frac{2x}{2} = x$

Because $F[G(x)] = x$ and $G[F(x)] = x$ for all real numbers $x$, $F$ and $G$ are inverses.

**62.** $h[k(x)] = \sqrt{x^2} = x$ Since $x \ge 0$

$k[h(x)] = \left(\sqrt{x}\right)^2 = x$

Because $h[k(x)] = x$ for all $x$ in the domain of $k$ and $k[h(x)] = x$ for all $x$ in the domain of $h$, we have shown that $h$ and $k$ are inverses.

**63.** $l[m(x)] = \dfrac{\frac{3}{x-1}+3}{\frac{3}{x-1}}$

$= \dfrac{3+3x-3}{x-1}, \dfrac{x-1}{3}$

$= \dfrac{3x}{3}$

$= x$

$m[l(x)] = m\left[\dfrac{x+3}{x}\right]$

$= \dfrac{3}{\frac{x+3}{x}-1}$

$= \dfrac{3}{\frac{x+3-x}{x}}$

$= 3 \cdot \dfrac{x}{3} = x$

Thus, $l$ and $m$ are inverse functions.

**64.** $p[q(x)] = \dfrac{\frac{2x}{x-5}-5}{2\left(\frac{2x}{x-5}\right)}$

$= \dfrac{\frac{2x-5x+25}{x-5}}{\frac{4x}{x-5}}$

$= \dfrac{-3x+25}{x-5} \cdot \dfrac{x-5}{4x}$

$= \dfrac{-3x+25}{4x}$

Thus, $p$ and $q$ are not inverse functions.

**65.** $f(x) = 3x-4$

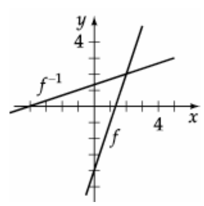
$y = 3x-4$

$x = 3y-4$

$x+4 = 3y$

$\dfrac{x+4}{3} = y$

Thus $f^{-1}(x) = \dfrac{x+4}{3} = \dfrac{1}{3}x + \dfrac{4}{3}$.




**66.** $g(x) = -2x+3$

$y = -2x+3$

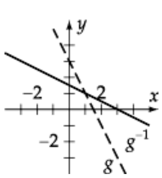
$x = -2y+3$

$x-3 = -2y$

$\dfrac{x-3}{-2} = y$

$\dfrac{3-x}{2} = y$

Thus, $g^{-1}(x) = \dfrac{3-x}{2} = -\dfrac{1}{2}x + \dfrac{3}{2}$.




**67.** $h(x) = -\dfrac{1}{2}x - 2$

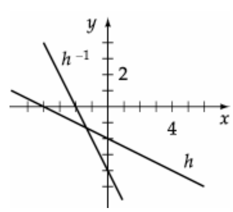
$y = -\dfrac{1}{2}x - 2$

$x = -\dfrac{1}{2}y - 2$

$x+2 = -\dfrac{1}{2}y$

$-2x-4 = y$

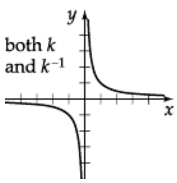
Thus, $h^{-1}(x) = -2x-4$.



**68.** $k(x) = \dfrac{1}{x}$

$$y = \frac{1}{x}$$
$$x = \frac{1}{y}$$
$$y = \frac{1}{x}$$

Thus $k^{-1}(x) = k(x) = \dfrac{1}{x}$.




**69.** Let $x$ = one of the numbers and $50 - x$ = the other number. Their product $y$ is given by

$$y = x(50 - x) = 50x - x^2 = x^2 + 50x$$

Now $y$ takes on its maximum value when

$$x = \frac{-b}{2a} = \frac{-50}{2(-1)} = 25$$

Thus, the two numbers are 25 and $(50 - 25) = 25$. That is, both numbers are 25.

**70.** Let $x$ = the smaller number. Let $x + 10$ = the larger number. The sum of their squares $y$ is given by

$$y = x^2 + (x+10)^2$$
$$= x^2 + x^2 + 20x + 100$$
$$= 2x^2 + 20x + 100$$

Now $y$ takes on its minimum value when

$$x = \frac{-b}{2a} = \frac{-20}{2(2)} = -5$$

Thus, the numbers are $-5$ and $(-5 + 10) = 5$.

**71.** $h(t) = -16t^2 + 220$ and $h(t) = 0$ when

$$-16t^2 + 220 = 0$$
$$220 = 16t^2$$
$$\frac{220}{16} = t^2$$
$$\sqrt{\frac{220}{16}} = t$$
$$\frac{2\sqrt{55}}{4} = t$$

Thus, $t \approx 3.7$ seconds.

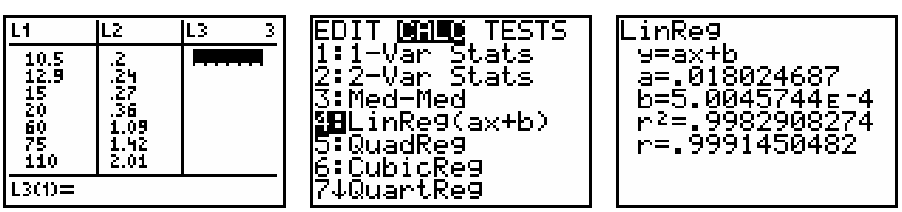
**72.** **a.** $h(1050) = \dfrac{1}{8820}(1050)^2 + 25$

$$= \frac{1{,}102{,}500}{8{,}820} + 25$$
$$= 125 + 25$$
$$= 150 \text{ feet}$$

**b.** $h(2100) = \dfrac{1}{8820}(2100)^2 + 25$

$$= \frac{4{,}410{,}000}{8{,}820} + 25$$
$$= 500 + 25$$
$$= 525 \text{ feet}$$

**73.** **a.** Enter the data on your calculator. The technique for a TI-83 is illustrated here.



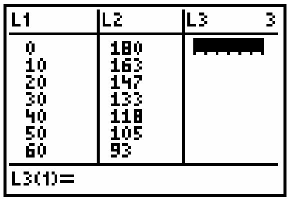

$y = 0.018024687x + 0.00050045744$

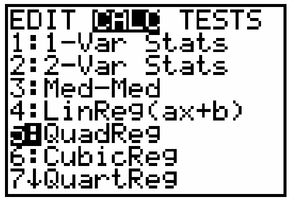
**b.** Yes. $r \approx 0.999$, which is very close to 1.

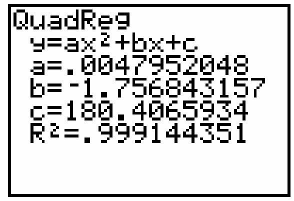
**c.** $y = 0.018024687(100) + 0.00050045744$

$$= 1.8024687 + 0.00050045744$$
$$\approx 1.8 \text{ seconds}$$

**74.** **a.** Enter the data on your calculator. The technique for a TI-83 is illustrated here.




$h = 0.0047952048t^2 - 1.756843157t + 180.4065934$

**b.** If the can is empty, then height = 0.

$$0 = 0.0047952048t^2 - 1.756843157t + 180.4065934$$

$$t = \frac{-(-1.756843157) \pm \sqrt{(-1.756843157)^2 - 4(0.0047952048)(180.4065934)}}{2(0.0047952048)}$$

$$= \frac{1.756843157 \pm \sqrt{3.086497878 - 3.46034625}}{0.0095904096}$$

$$= \frac{1.756843157 \pm \sqrt{-0.373848372}}{0.0095904096}$$

This does not represent a real number. Thus, no, according to the model, the can will never empty.

**c.** The regression line is a model of the data and is not based on physical principles.

## Quantitative Reasoning

**QR1.** **a.** $15 \div 4 = 3$ remainder 3
$15 \bmod 4 \equiv 3$

**b.** $37 \div 5 = 7$ remainder 2
$37 \bmod 5 \equiv 2$

**c.** $52 \div 321 = 0$ remainder 52
$52 \bmod 321 \equiv 52$

**QR2.** Factor the modulus, *M*.

**QR3.** Answers will vary.

## Chapter 1 Chapter Test

**1.**
$$\begin{aligned} 4x - 2(2 - x) &= 5 - 3(2x + 1) \\ 4x - 4 + 2x &= 5 - 6x - 3 \\ 6x - 4 &= 2 - 6x \\ 12x &= 6 \\ x &= \tfrac{1}{2} \end{aligned}$$

**2.**
$$\begin{aligned} 6 - 3x &\ge 3 - 4(2 - 2x) \\ 6 - 3x &\ge 3 - 8 + 8x \\ -11x &\ge -11 \\ x &\le 1 \end{aligned}$$

**3.**
$$\begin{aligned} 2x^2 - 3x &= 2 \\ 2x^2 - 3x - 2 &= 0 \\ (2x + 1)(x - 2) &= 0 \\ x = -\tfrac{1}{2} &\text{ or } x = 2 \end{aligned}$$

**4.**
$$\begin{aligned} 3x^2 - x &= 2 \\ 3x^2 - x - 2 &= 0 \\ (3x + 2)(x - 1) &= 0 \\ x = -\tfrac{2}{3} &\text{ or } x = 1 \end{aligned}$$

**5.**
$$|4 - 5x| > 6$$
$$\begin{aligned} 4 - 5x &< -6 \quad \text{or} \quad & 4 - 5x &> 6 \\ -5x &< -10 & -5x &> 2 \\ x &> 2 & x &< -\tfrac{2}{5} \end{aligned}$$

$$\left(-\infty, -\tfrac{2}{5}\right) \cup (2, \infty)$$

**6.**
$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$d=\sqrt{[4-(-2)]^2+(-2-5)^2}$$
$$d=\sqrt{36+49}$$
$$d=\sqrt{85}$$

**7.**
$$x_m=\frac{4-2}{2}=1$$
$$y_m=\frac{-1+3}{2}=1$$

midpoint = (1, 1)

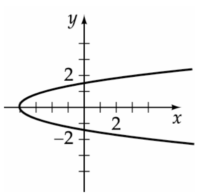
$$\text{length}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$=\sqrt{[4-(-2)]^2+(-1-3)^2}$$
$$=\sqrt{6^2+(-4)^2}$$
$$=\sqrt{36+16}=\sqrt{52}$$
$$=2\sqrt{13}$$

**8.** $x=2y^2-4$

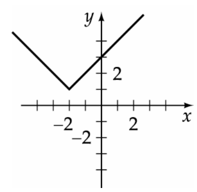
$x=0$ $\quad 2y^2-4=0$
$$2y^2=4$$
$$y^2=2$$
$$y=\pm\sqrt{2}$$

If $y=0$, $\quad x=-4$

intercepts $(0,\ -\sqrt{2}),\ (0,\ \sqrt{2}),\ (-4,\ 0)$




**9.**




**10.**
$$x^2-4x+y^2+2y-4=0$$
$$(x^2-4x+4)+(y^2+2y+1)=4+4+1$$
$$(x-2)^2+(y+1)^2=9$$

center: $(2,-1)$ radius: 3

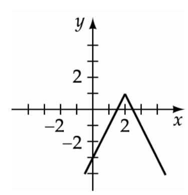
**11.**
$$f(x)=-\sqrt{25-x^2}$$
$$f(-3)=-\sqrt{25-(-3)^2}$$
$$f(-3)=-\sqrt{16}$$
$$f(-3)=-4$$

**12.**
$$x^2-16\ge 0$$
$$(x-4)(x+4)\ge 0$$

Critical values 4 and –4.

Domain $\{x \mid x\ge 4 \text{ or } x\le -4\}$

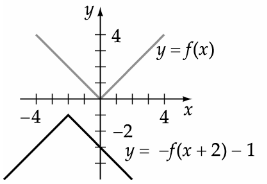
**13.**




*f* is increasing on $(-\infty, 2]$

*f* is decreasing on $[2, \infty)$

**14.** First shift the graph of $f(x)$ horizontally 2 units to the left. Next, reflect the graph across the $x$-axis. Finally, shift the graph vertically down 1 unit.




**15.** **a.** $f(-x) = (-x)^4 - (-x)^2$
$= x^4 - x^2$
$= f(x)$

The function of $f(x) = x^4 - x^2$ is an even function.

**b.** $f(-x) = (-x)^3 - (-x)$
$= -x^3 + x$
$= -(x^3 - x)$
$= -f(x)$

The function $f(x) = x^3 - x$ is an odd function.

**c.** $f(-x) = -x - 1$

The function $f(x) = x - 1$ is neither an even nor an odd function.

Thus, only **b** defines an odd function.

**16.** $(f+g)(x) = (x^2 - 1) + (x - 2)$
$= x^2 + x - 3$

$\left(\frac{f}{g}\right)(x) = \frac{x^2-1}{x-2}, \quad x \neq 2$

**17.** $\frac{f(x+h)-f(x)}{h} = \frac{\left[(x+h)^2+1\right]-(x^2+1)}{h}$
$= \frac{x^2+2xh+h^2+1-x^2-1}{h}$
$= \frac{2xh+h^2}{h}$
$= 2x + h$

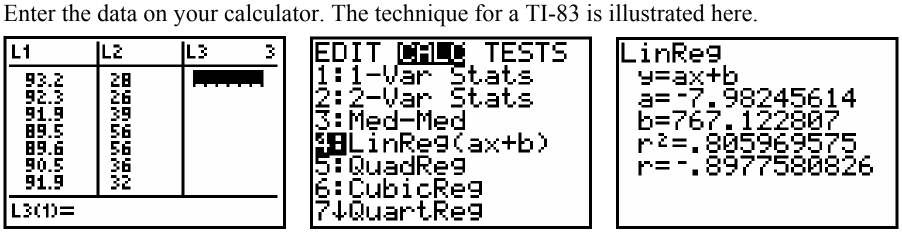
**18.** $(f \circ g)(x) = (\sqrt{x-2})^2 - 2\sqrt{x-2} + 1$
$= x - 2 - 2\sqrt{x-2} + 1$
$= x - 2\sqrt{x-2} - 1$

**19.** $y = \frac{x}{x+1}$

Interchange $x$ and $y$. Then solve for $y$.

$x = \frac{y}{y+1}$
$x(y+1) = y$
$xy + x = y$
$xy - y = -x$
$y(x-1) = -x$
$y = \frac{-x}{x-1} = \frac{x}{1-x}$
$f^{-1}(x) = \frac{x}{1-x}$

**20.** **a.** Enter the data on your calculator. The technique for a TI-83 is illustrated here.




$y = -7.98245614x + 767.122807$

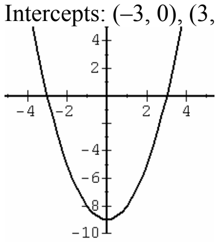
**b.** $y = -7.98245614(89) + 767.122807$
$\approx 57$ Calories

## Chapter 2 Cumulative Review

**1.** $d=\sqrt{(4-(-3))^2+(1-2)^2}=\sqrt{49+1}=\sqrt{50}$ [1.1]

**2.**
$$c^2=a^2+b^2 \quad [1.1]$$
$$1^2=\left(\frac{1}{2}\right)^2+b^2$$
$$\frac{3}{4}=b^2$$
$$\frac{\sqrt{3}}{2}=b$$

**3.** Intercepts: (–3, 0), (3, 0), (–9, 0) [1.2]




**4.** $f(-x)=\dfrac{-x}{(-x)^2+1}=\dfrac{-x}{x^2+1}=-f(x)$ [1.4]

Odd function

**5.**
$$f(x)=\frac{x}{2x-3} \quad [1.5]$$
$$x=\frac{y}{2y-3}$$
$$x(2y-3)=2xy-3x=y$$
$$2xy-y=y(2x-1)=3x$$
$$y=\frac{3x}{2x-1}$$
$$f^{-1}(x)=\frac{3x}{2x-1}$$

**6.** Domain: $(-\infty, 4)\cup(4,\infty)$ [1.3]

**7.**
$$x^2+x-6=0$$
$$(x+3)(x-2)=0$$
$$x+3=0 \qquad x-2=0$$
$$x=-3 \qquad x=2$$

The solutions are –3 and 2. [1.1]

**8.** Shift the graph of $y=f(x)$ horizontally 3 units to the right. [1.4]

**9.** Reflect the graph of $y=f(x)$ across the $y$-axis. [1.4[

**10.** $300^\circ=300^\circ\left(\dfrac{\pi}{180^\circ}\right)=\dfrac{5\pi}{3}$ [2.1]

**11.** $\dfrac{5\pi}{4}=\dfrac{5\pi}{4}\left(\dfrac{180^\circ}{\pi}\right)=225^\circ$ [2.1]

**12.** $f\left(\dfrac{\pi}{3}\right)=\sin\left(\dfrac{\pi}{3}+\dfrac{\pi}{6}\right)=\sin\left(\dfrac{\pi}{2}\right)=1$ [2.3]

**13.** $f\left(\dfrac{\pi}{3}\right)=\sin\left(\dfrac{\pi}{3}\right)+\sin\left(\dfrac{\pi}{6}\right)=\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}=\dfrac{\sqrt{3}+1}{2}$ [2.2]

**14.** $\cos^2 45^\circ+\sin^2 60^\circ+=\left(\dfrac{\sqrt{2}}{2}\right)^2+\left(\dfrac{\sqrt{3}}{2}\right)^2=\dfrac{2}{4}+\dfrac{3}{4}=\dfrac{5}{4}$ [2.2]

**15.** negative [2.3]

**16.**
$\theta=210^\circ$ [2.3]

Since $180^\circ<\theta<270^\circ$,

$$\theta'+180^\circ=\theta$$
$$\theta'=30^\circ$$

**17.** $\theta = \frac{2\pi}{3}$ [2.3]

Since $\frac{\pi}{2} < \theta < \pi$,

$\theta + \theta' = \pi$

$\theta' = \frac{\pi}{3}$

**18.** Domain: $(-\infty, \infty)$ [2.4]

**19.** Range: [–1, 1] [2.4]

**20.** $\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{3}{4}$

$$\begin{aligned}\text{hypotenuse} &= \sqrt{3^2 + 4^2}\\ &= \sqrt{9+16}\\ &= \sqrt{25}\\ &= 5\end{aligned}$$

$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{3}{5}$ [2.2]

## Chapter 3 Cumulative Review

**1.**
$$\begin{aligned}-2x + 1 &< 7\\ -2x &< 6\\ x &> -3\end{aligned}$$

**2.** Shift the graph of $y = f(x)$ horizontally 1 unit to the left and up 2 units.

**3.** Reflect the graph of $y = f(x)$ across the $x$-axis.

**4.**
$$\begin{aligned}f(-x) &= -x - \sin(-x)\\ &= -x + \sin x\\ &= -(x - \sin x)\\ &= -f(x)\end{aligned}$$

odd function

**5.**
$$\begin{aligned}f(x) &= \frac{5x}{x-1}\\ y &= \frac{5x}{x-1}\\ x &= \frac{5y}{y-1}\\ x(y-1) &= 5y\\ xy - 5y &= x\\ y(x-5) &= x\\ y &= \frac{x}{x-5}\\ f^{-1}(x) &= \frac{x}{x-5}\end{aligned}$$

**6.** $240^\circ = 240^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{4\pi}{3}$ [2.1]

**7.** $\frac{5\pi}{3} = \frac{5\pi}{3}\left(\frac{180^\circ}{\pi}\right) = 300^\circ$ [2.1]

**8.** $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

**9.** $\csc 60^\circ = \frac{1}{\sin 60^\circ} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

**10.** $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{2}{3}$

$$\begin{aligned}\text{adjacent side} &= \sqrt{3^2 - 2^2}\\ &= \sqrt{9-4}\\ &= \sqrt{5}\end{aligned}$$

$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$

**11.** $\cot\theta > 0$ in quadrant III
Positive [2.3]

**12.** $\theta = 310^\circ$ [2.3]
Since $270^\circ < \theta < 360^\circ$,
$\theta = \theta' = 360^\circ$
$\theta' = 50^\circ$

**13.** $\theta = \frac{5\pi}{3}$ [2.3]
Since $\frac{3\pi}{2} < \theta < 2\pi$,
$\theta = \theta' = 2\pi$
$\theta' = \frac{\pi}{3}$

**14.** $t = \frac{\pi}{3}$ [2.4]
$y = \sin t = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$
$x = \cos t = \cos\frac{\pi}{3} = \frac{1}{2}$
The point on the unit circle corresponding to $t = \frac{\pi}{3}$ is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

**15.** $y = 0.43\cos\left(2x - \frac{\pi}{6}\right)$ [2.7]
amplitude: 0.43
$0 \le 2x - \frac{\pi}{6} \le 2\pi$
$\frac{\pi}{6} \le 2x \le \frac{13\pi}{6}$
$\frac{\pi}{12} \le x \le \frac{13\pi}{12}$
period $= \pi$, phase shift $= \frac{\pi}{12}$

**16.** $y = \sin^{-1}\frac{1}{2}$ [3.5]
$\sin y = \frac{1}{2}$ $\quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \frac{\pi}{6}$

**17.** $\cos^{-1}(-0.8) = 2.498$ [3.5]

**18.** Domain: [–1, 1]. [3.5]

**19.** Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ [3.5]

**20.**
$$2\cos^2 x - 1 = -\sin x$$
$$1 - 2\sin^2 x = -\sin x$$
$$0 = 2\sin^2 x - \sin x - 1$$
$$0 = (2\sin x + 1)(\sin x - 1)$$

$2\sin x + 1 = 0$ $\qquad$ $\sin x - 1 = 0$
$\sin x = -\frac{1}{2}$ $\qquad$ $\sin x = 1$
$x = \frac{7\pi}{6}, \frac{11\pi}{6}$ $\qquad$ $x = \frac{\pi}{2}$

The solutions are $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

## Chapter 4 Assessing Concepts

**1.** An oblique triangle that does not contain a right angle.

**2.** The Law of Cosines

**3.** SSA

**4.** The variable *s* represents the semiperimeter of the triangle.

**5.** A scalar

**6.** A scalar

**7.** True

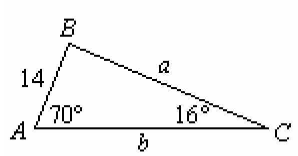
**8.** False

**9.** True

**10.** True

## Chapter 4 Test

**1.**




$B = 180° - 70° - 16°$

$B = 94°$

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$a = \frac{14 \sin 70°}{\sin 16°}$$

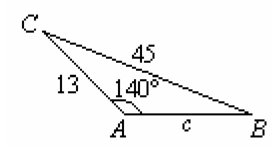
$$a \approx 48$$

$$\frac{c}{\sin C} = \frac{b}{\sin B} \quad [4.1]$$

$$b = \frac{14 \sin 94°}{\sin 16°}$$

$$b \approx 51$$

**2.**



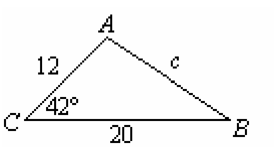

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

$$B = \sin^{-1}\left(\frac{b \sin A}{a}\right) \quad [4.1]$$

$$B = \sin^{-1}\left(\frac{13 \sin 140°}{45}\right)$$

$$B \approx 11°$$

**3.**

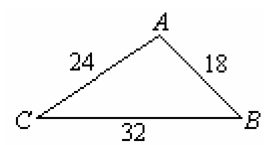



$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = 20^2 + 12^2 - 2(20)(12)\cos 42° \quad [4.2]$$

$$c \approx 14$$

**4.**

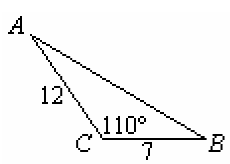



$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$B = \cos^{-1}\left(\frac{32^2 + 18^2 - 24^2}{2(32)(18)}\right) \quad [4.2]$$

$$B \approx 48°$$

**5.**



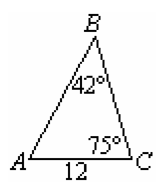

$$K = \frac{1}{2}ab\sin C$$

$$K = \frac{1}{2}(7)(12)(\sin 110°)$$

$K \approx 39$ square units

[4.2]

**6.**




$A = 180° - 42° - 75°$

$A = 63°$

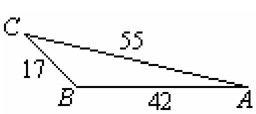
$$K = \frac{b^2 \sin A \sin C}{2\sin B}$$

$$K = \frac{12^2 \sin 63° \sin 75°}{2\sin 42°}$$

$K \approx 93$ square units

[4.2]

**7.**




$$s = \frac{1}{2}(a + b + c)$$

$$s = \frac{1}{2}(17 + 55 + 42) = 57$$

$$K = \sqrt{s(s-a)(s-b)(s-c)}$$

$$K = \sqrt{57(57-17)(57-55)(57-42)}$$

$K \approx 260$ square units [4.2]

**8.** $|\mathbf{v}| = \sqrt{(-2)^2 + (3)^2} = \sqrt{13}$ [4.3]

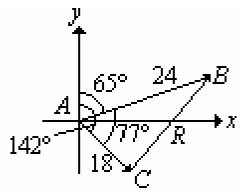
**9.** $a_1 = 12\cos 220° \approx -9.2$ [4.3]
$a_2 = 12\sin 220° \approx -7.7$
$\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j}$
$\mathbf{v} = -9.2\mathbf{i} - 7.7\mathbf{j}$

**10.** $3\mathbf{u} - 5\mathbf{v} = 3(2\mathbf{i} - 3\mathbf{j}) - 5(5\mathbf{i} + 4\mathbf{j})$ [4.3]
$= (6\mathbf{i} - 9\mathbf{j}) - (25\mathbf{i} + 20\mathbf{j})$
$= (6-25)\mathbf{i} + (-9-20)\mathbf{j}$
$= -19\mathbf{i} - 29\mathbf{j}$

**11.** $\mathbf{u}\cdot\mathbf{v} = (-2\mathbf{i} + 3\mathbf{j})\cdot(5\mathbf{i} + 3\mathbf{j})$ [4.3]
$= (-2\cdot 5) + (3\cdot 3)$
$= -10 + 9$
$= -1$

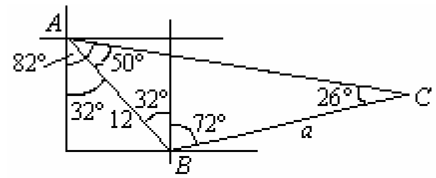
**12.** $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\ \|\mathbf{v}\|} = \dfrac{\langle 3,5\rangle\cdot\langle -6,2\rangle}{\sqrt{3^2+5^2}\sqrt{(-6)^2+2^2}}$ [4.3]
$\cos\theta = \dfrac{-18+10}{\sqrt{34}\sqrt{40}} = \dfrac{-8}{\sqrt{34}\sqrt{40}}$
$\theta \approx 103°$

**13.**



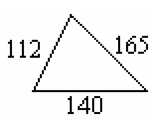

$A = 142° - 65° = 77°$
$R^2 = 24^2 + 18^2 - 2(24)(18)\cos 77°$
$R \approx 27$ miles [4.3]

**14.**




$A = 82° - 32° = 50°$
$C = 180° - 50° - 32° - 72° = 26°$
$\dfrac{a}{\sin 50°} = \dfrac{12}{\sin 26°}$
$a = \dfrac{12\sin 50°}{\sin 26°}$
$a \approx 21$ miles [4.3]

**15.**




$S = \dfrac{1}{2}(112+165+140) = 208.5$
$K = \sqrt{208.5(208.5-112)(208.5-165)(208.5-140)}$
$K \approx 7743$
cost $\approx 8.50(7743)$
cost $\approx \$66,000$ [4.2]

## Chapter 4 Cumulative Review

**1.** $d = \sqrt{(-3-4)^2 + (4-(-1))^2} = \sqrt{49+25} = \sqrt{74}$ [1.2]

**2.** $f(x) + g(x) = \sin x + \cos x$ [1.5]

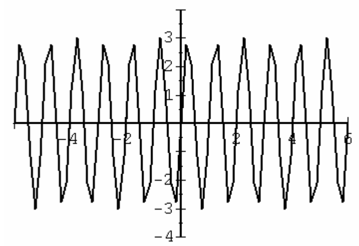
**3.** $(f \circ g)(x) = f[g(x)]$ [1.5]
$= f[\cos x]$
$= \sec(\cos x))$

**4.** $f(x) = \dfrac{1}{2}x - 3$ [1.6]
$y = \dfrac{1}{2}x - 3$
$x = \dfrac{1}{2}y - 3$
$2(x+3) = y$
$f^{-1}(x) = 2x + 6$

**5.** Shifted 2 units to the right and 3 units up. [1.4]

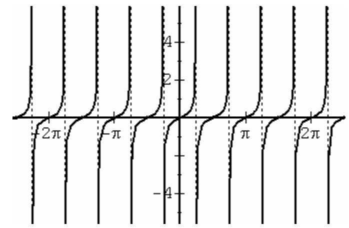
**6.** $\sin 27° = \dfrac{15}{a}$ [2.2]
$a = \dfrac{15}{\sin 27°} \approx 33$ cm

**7.** $y = 3\sin(2\pi x)$



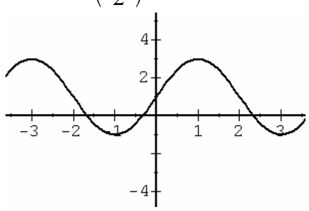

[2.5]

**8.** $y = \frac{1}{4}\tan(2x)$




[2.6]

**9.** $y = 2\sin\left(\frac{\pi x}{2}\right) + 1$




[2.7]

**10.** $y = 3\sin\left(\frac{1}{3}x - \frac{\pi}{2}\right)$ [2.7]

$0 \le \frac{1}{3}x - \frac{\pi}{2} \le 2\pi$

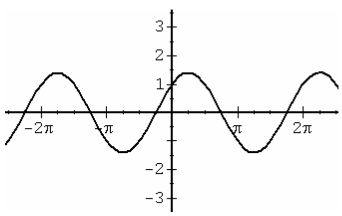
$\frac{\pi}{2} \le \frac{1}{3}x \le \frac{5\pi}{2}$

$\frac{3\pi}{2} \le x \le \frac{15\pi}{2}$

amplitude = 3, period $= 6\pi$, phase shift $= \frac{3\pi}{2}$

**11.** $y = \sin x + \cos x$ [2.7]

Amplitude: $\sqrt{2}$, period: $2\pi$, phase shift: $\frac{\pi}{4}$




**12.** $c^2 = a^2 + b^2 - 2ab\cos C$ [4.2]

$c^2 = (10)^2 + (12)^2 - 2(10)(12)\cos 50^\circ$

$= 244 - 240\cos 50^\circ$

$\approx 9.473$

$c \approx 9.5$ cm

**13.** $\frac{1}{\cos x} - \cos x = \frac{1 - \cos^2 x}{\cos x}$ [3.1]

$= \frac{\sin^2 x}{\cos x}$

$= \sin x \tan x$

**14.** $\sin^{-1}\sin\left(\frac{2\pi}{3}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ [3.5]

**15.** $\tan\left(\cos^{-1}\left(\frac{12}{13}\right)\right)$ [3.5]

Let $\theta = \cos^{-1}\frac{12}{13}$ and find $y = \tan\theta$.

Then $\cos\theta = \frac{12}{13}$, and $0 \le \theta \le \pi$.

$\sqrt{13^2 - 12^2} = \sqrt{25} = 5$

Thus $\tan\theta = \frac{5}{12}$.

$y = \frac{5}{12}$

**16.** $\sin x \tan x - \frac{1}{2}\tan x = 0$ [3.6]

$\tan x\left(\sin x - \frac{1}{2}\right) = 0$

$\tan x = 0$

$x = 0, \pi$

$\sin x - \frac{1}{2} = 0$

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$

**17.** $\|\mathbf{v}\| = \sqrt{4^2 + (-3)^2}$
$\|\mathbf{v}\| = \sqrt{16+9}$
$\|\mathbf{v}\| = 5$

$\alpha = \tan^{-1}\left|\frac{-3}{4}\right| = \tan^{-1}\frac{3}{4}$ [4.3]
$\alpha \approx 36.9°$
$\theta = 360° - \alpha$
$\theta \approx 360° - 36.9°$
$\theta \approx 323.1°$

**18.** $\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|}$ [4.3]

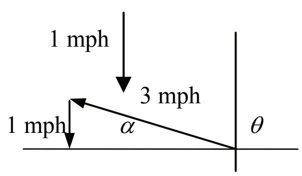
$\cos\theta = \frac{\langle 1,\ 2\rangle \cdot \langle -2,\ 3\rangle}{\sqrt{1^2+(2)^2}\sqrt{(-2)^2+3^2}}$

$\cos\theta = \frac{1(-2)+(2)(3)}{\sqrt{5}\sqrt{13}}$

$\cos\theta = \frac{4}{\sqrt{5}\sqrt{13}} \approx 0.49613$

$\theta = 60.3°$

**19.**




$\alpha = \sin^{-1}\left|\frac{1}{3}\right| = \sin^{-1}\frac{1}{3}$
$\alpha \approx 19.5°$

heading $= \theta = 270° + \alpha$ [4.3]
$\theta \approx 270° + 19.5°$
$\theta \approx 289.5°$

**20.** $\mathbf{AB} = 515(\cos 36\mathbf{i} + \sin 36\mathbf{j}) \approx 416.6\mathbf{i} + 302.7\mathbf{j}$ [4.3]
$\mathbf{AD} = 150[\cos(-30°)\mathbf{i} + \sin(-30°)\mathbf{j}] \approx 129.9\mathbf{i} - 75\mathbf{j}$
$\mathbf{AC} = \mathbf{AB} + \mathbf{AD}$
$\mathbf{AC} = 416.6\mathbf{i} + 302.7\mathbf{j} + 129.9\mathbf{i} - 75\mathbf{j}$
$\mathbf{AC} \approx 546.5\mathbf{i} + 227.7\mathbf{j}$
$\|\mathbf{AC}\| = \sqrt{546.5^2 + (227.7)^2}$
$\|\mathbf{AC}\| \approx 592$ mph

$\alpha = 90° - \theta = 90° - \tan^{-1}\left(\frac{227.7}{546.5}\right) \approx 67.4°$

## Section 5.1

See *CAT Section* P.6 solutions on page 26 for exercises 1 – 62.

**63.** $z_1 = (-2+i) + 4 + 3i = 2 + 4i$
$z_2 = (2+4i) + 4 + 3i = 6 + 7i$
$z_3 = (6+7i) + 4 + 3i = 10 + 10i$
$z_4 = (10+10i) + 4 + 3i = 14 + 13i$
$z_5 = (14+13i) + 4 + 3i = 18 + 16i$

**64.** $z_1 = 2i(1+3i) = 2i + 6i^2 = -6 + 2i$
$z_2 = 2i(-6+2i) = -12i + 4i^2 = -4 - 12i$
$z_3 = 2i(-4-12i) = -8i - 24i^2 = 24 - 8i$
$z_4 = 2i(24-8i) = 48i - 16i^2 = 16 + 48i$
$z_5 = 2i(16+48i) = 32i + 96i^2 = -96 + 32i$

**65.** $z_1 = i(1-i) = i - i^2 = 1 + i$
$z_2 = i(1+i) = i + i^2 = -1 + i$
$z_3 = i(-1+i) = -i + i^2 = -1 - i$
$z_4 = i(-1-i) = -i - i^2 = 1 - i$
$z_5 = z_1$
$z_6 = z_2$
$z_7 = z_3$
$z_8 = z_4$

**66.** $z_1 = (0.5i)^2 = 0.25i^2 = -0.25$
$z_2 = (-0.25)^2 = 0.0625$
$z_3 = (0.0625)^2 = 0.00390625$

**67.** Use $a = 3$, $b = -3$, $c = 3$.

$$\frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{-(-3)+\sqrt{(-3)^2-4(3)(3)}}{2(3)} = \frac{3+\sqrt{9-36}}{6} = \frac{3+\sqrt{-27}}{6} = \frac{3+i\sqrt{27}}{6} = \frac{3+3i\sqrt{3}}{6} = \frac{3}{6}+\frac{3\sqrt{3}}{6}i = \frac{1}{2}+\frac{\sqrt{3}}{2}i$$

**68.** Use $a = 2$, $b = 4$, $c = 4$.

$$\frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{-(4)+\sqrt{(4)^2-4(2)(4)}}{2(2)} = \frac{-4+\sqrt{16-32}}{4} = \frac{-4+\sqrt{-16}}{4} = \frac{-4+i\sqrt{16}}{4} = \frac{-4+4i}{4} = \frac{-4}{4}+\frac{4i}{4} = -1+i$$

**69.** Use $a = 2$ $b = 6$, $c = 6$.

$$\frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{-(6)+\sqrt{(6)^2-4(2)(6)}}{2(2)} = \frac{-6+\sqrt{36-48}}{4} = \frac{-6+\sqrt{-12}}{4} = \frac{-6+i\sqrt{12}}{4} = \frac{-6+2i\sqrt{3}}{4} = \frac{-6}{4}+\frac{2i\sqrt{3}}{4} = -\frac{3}{2}+\frac{\sqrt{3}}{2}i$$

**70.** Use $a =2$, $b = 1$, $c = 3$.

$$\frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{-(1)+\sqrt{(1)^2-4(2)(3)}}{2(2)} = \frac{-1+\sqrt{1-24}}{4} = \frac{-1+\sqrt{-23}}{4} = \frac{-1+i\sqrt{23}}{4} = -\frac{1}{4}+\frac{\sqrt{23}}{4}i$$

**71.** Use $a = 4$, $b = -4$, $c = 2$.

$$\frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{-(-4)+\sqrt{(-4)^2-4(4)(2)}}{2(4)} = \frac{4+\sqrt{16-32}}{8} = \frac{4+\sqrt{-16}}{8} = \frac{4+i\sqrt{16}}{8} = \frac{4+4i}{8} = \frac{4}{8}+\frac{4i}{8} = \frac{1}{2}+\frac{1}{2}i$$

**72.** Use $a = 3$, $b = -2$, $c = 4$.

$$\frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{-(-2)+\sqrt{(-2)^2-4(3)(4)}}{2(3)} = \frac{2+\sqrt{4-48}}{6} = \frac{2+\sqrt{-44}}{6} = \frac{2+i\sqrt{44}}{6} = \frac{2+2i\sqrt{11}}{6} = \frac{2}{6}+\frac{2i\sqrt{11}}{6} = \frac{1}{3}+\frac{\sqrt{11}}{3}i$$

## Connecting Concepts

**73.** $x^2+16 = x^2+4^2 = (x+4i)(x-4i)$

**74.** $x^2+9 = x^2+3^2 = (x+3i)(x-3i)$

**75.** $z^2+25 = z^2+5^2 = (z+5i)(z-5i)$

**76.** $z^2+64 = z^2+8^2 = (z+8i)(z-8i)$

**77.** $4x^2+81 = (2x)^2+9^2 = (2x+9i)(2x-9i)$

**78.** $9x^2+1 = (3x)^2+1^2 = (3x+i)(3x-i)$

**79.** If $x = 1 + 2i$, then $x^2-2x+5 = (1+2i)^2-2(1+2i)+5 = 1+4i+4i^2-2-4i+5 = 1+4i+4(-1)-2-4i+5$
$= 1+4i-4-2-4i+5 = (1-4-2+5)+(4i-4i) = 0$

**80.** If $x = 1 - 2i$, then $x^2-2x+5 = (1-2i)^2-2(1-2i)+5 = 1-4i+4i^2-2+4i+5 = 1-4i+4(-1)-2+4i+5$
$= 1-4i-4-2+4i+5 = (1-4-2+5)+(-4i+4i) = 0$

**81.** Verify that $(-1+i\sqrt{3})^3 = 8$.

$$\begin{aligned}(-1+i\sqrt{3})^3 &= (-1+i\sqrt{3})(-1+i\sqrt{3})^2 = (-1+i\sqrt{3})[(-1)^2 + 2(-1)(i\sqrt{3}) + (i\sqrt{3})^2] \\ &= (-1+i\sqrt{3})[1-2i\sqrt{3}+3i^2] = (-1+i\sqrt{3})[1-2i\sqrt{3}+3(-1)] = (-1+i\sqrt{3})[1-2i\sqrt{3}-3] \\ &= (-1+i\sqrt{3})(-2-2i\sqrt{3}) = -1(-2-2i\sqrt{3}) + i\sqrt{3}(-2-2i\sqrt{3}) = 2+2i\sqrt{3}-2i\sqrt{3}-2i^2(\sqrt{3})^2 \\ &= 2+2i\sqrt{3}-2i\sqrt{3}-2(-1)(3) = 2+2i\sqrt{3}-2i\sqrt{3}+6 = (2+6)+(2i\sqrt{3}-2i\sqrt{3}) \\ &= 8\end{aligned}$$

Verify that $(-1-i\sqrt{3})^3 = 8$.

$$\begin{aligned}(-1-i\sqrt{3})^3 &= (-1-i\sqrt{3})(-1-i\sqrt{3})^2 = (-1-i\sqrt{3})[(-1)^2 + 2(-1)(-i\sqrt{3}) + (-i\sqrt{3})^2] \\ &= (-1-i\sqrt{3})[1+2i\sqrt{3}+3i^2] = (-1-i\sqrt{3})[1+2i\sqrt{3}+3(-1)] = (-1-i\sqrt{3})[1+2i\sqrt{3}-3] \\ &= (-1-i\sqrt{3})(-2+2i\sqrt{3}) = -1(-2+2i\sqrt{3}) - i\sqrt{3}(-2+2i\sqrt{3}) = 2-2i\sqrt{3}+2i\sqrt{3}-2i^2(\sqrt{3})^2 \\ &= 2-2i\sqrt{3}+2i\sqrt{3}-2(-1)(3) = 2-2i\sqrt{3}+2i\sqrt{3}+6 = (2+6)+(-2i\sqrt{3}+2i\sqrt{3}) \\ &= 8\end{aligned}$$

**82.** Verify that $\left[\frac{\sqrt{2}}{2}(1+i)\right]^2 = i$.

$$\left[\frac{\sqrt{2}}{2}(1+i)\right]^2 = \frac{\sqrt{2}^2}{2^2}(1+i)^2 = \frac{2}{4}(1+2i+i^2) = \frac{1}{2}[1+2i+(-1)] = \frac{1}{2}(1+2i-1) = \frac{1}{2}(2i) = i$$

**83.** $i + i^2 + i^3 + i^4 + \ldots + i^{28} = 7(i+i^2+i^3+i^4) = 7(i+(-1)+(-i)+1) = 7(0) = 0$

**84.** $i + i^2 + i^3 + i^4 + \ldots + i^{100} = 25(i+i^2+i^3+i^4) = 25(i+(-1)+(-i)+1) = 25(0) = 0$

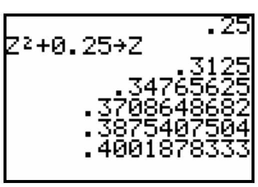
## Prepare for Section 5.2

See *CAT Prepare for Section* 7.4 solutions on page 462.

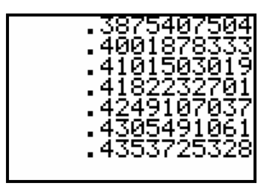
## Exploring Concepts with Technology

**The Mandelbrot Iteration Procedure**

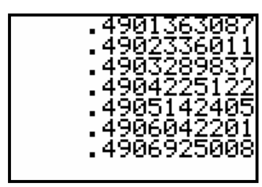
**1.**



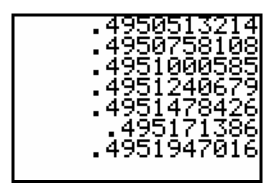

$z_5 \approx 0.4001878333$




$z_{10} \approx 0.4353725328$




$z_{100} \approx 0.4906925008$




$z_{200} \approx 0.4951947016$

Answers will vary.

**2.** $z_0 = i$
$z_1 = -1+i$
$z_2 = -1$
$z_3 = -1+i$
$z_4 = -i$

The iterates continue to cycles back and forth between $-1 + i$ and $-i$.

## Chapter 5 Assessing Concepts

**1.** True

**2.** True

**3.** True

**4.** False

**5.** The four roots are equally spaced around a circle with center (0, 0) and radius 1.

**6.** 1

**7.** 5

**8.** No

**9.** $-3 + 5i$

**10.** 2

## Chapter 5 Review

**1.** $3-\sqrt{-64}=3-8i$ [5.1]
$3+8i$

**2.** $\sqrt{-4}+6=6+2i$ [5.1]
$6-2i$

**3.** $-2+\sqrt{-5}=-2+i\sqrt{5}$ [5.1]
$-2-i\sqrt{5}$

**4.** $-5+\sqrt{-27}=-5+3i\sqrt{3}$ [5.1]
$-5-3i\sqrt{3}$

**5.** $\left(\sqrt{-4}\right)\left(\sqrt{-4}\right)=(2i)(2i)=4i^2=-4$
[5.1]

**6.** $\left(-\sqrt{-27}\right)\left(\sqrt{-3}\right)=\left(-3i\sqrt{3}\right)\left(i\sqrt{3}\right)$ [5.1]
$=-9i^2=9$

**7.** $(3+7i)+(2-5i)=5+2i$ [5.1]

**8.** $(3-4i)+(-6+8i)=-3+4i$ [5.1]

**9.** $(6-8i)-(9-11i)=-3+3i$ [5.1]

**10.** $(-3-5i)-(2+10i)=-5-15i$ [5.1]

**11.** $(5+3i)(2-5i)=10-19i-15i^2$ [5.1]
$=25-19i$

**12.** $(-2-3i)(-4+7i)=8-2i-21i^2$ [5.1]
$=29-2i$

**13.** $\dfrac{-2i}{3-4i}\cdot\dfrac{3+4i}{3+4i}=\dfrac{-6i-8i^2}{9+16}=\dfrac{8}{25}-\dfrac{6}{25}i$ [5.1]

**14.** $\dfrac{4+i}{7-2i}\cdot\dfrac{7+2i}{7+2i}=\dfrac{28+15i+2i^2}{49+4}=\dfrac{26}{53}+\dfrac{15}{53}i$ [5.1]

**15.** $i(2i)-(1+i)^2=2i^2-1-2i-i^2$ [5.1]
$=-2-2i$

**16.** $(2-i)^3=(4-4i+i^2)(2-i)$ [5.1]
$=(3-4i)(2-i)$
$=6-11i+4i^2$
$=2-11i$

**17.** $(3+\sqrt{-4})-(-3-\sqrt{-16})=3+2i+3+4i=6+6i$ [5.1]

**18.** $(-2+\sqrt{-9})+(-3-\sqrt{-81})=-2+3i-3-9i=-5-6i$ [5.1]

**19.** $(2-\sqrt{-3})(2+\sqrt{-3})=4-(-3)=7$ [5.1]

**20.** $(3-\sqrt{-5})(2+\sqrt{-5})=6+i\sqrt{5}-(-5)$ [5.1]
$=11+i\sqrt{5}$

**21.** $i^{27}=i^3=-i$ [5.1]

**22.** $i^{105}=i$ [5.1]

**23.** $\dfrac{i}{i^{17}}=\dfrac{1}{i^{16}}=\dfrac{1}{1}=1$ [5.1]

**24.** $i^{62}=i^2=-1$ [5.1]

**25.** $|-8i|=\sqrt{(0)^2+(-8)^2}$ [5.2]
$=8$

**26.** $|2-3i|=\sqrt{(2)^2+(-3)^2}$ [5.2]
$=\sqrt{4+9}=\sqrt{13}$

**27.** $|-4+5i|=\sqrt{(-4)^2+(5)^2}$ [5.2]
$=\sqrt{16+25}=\sqrt{41}$

**28.** $|-1-i|=\sqrt{(-1)^2+(-1)^2}$ [5.2]
$=\sqrt{1+1}=\sqrt{2}$

**29.** $r=\sqrt{(2)^2+(-2)^2}$ [5.2]
$r=2\sqrt{2}$

$\alpha=\tan^{-1}\left|\dfrac{-2}{2}\right|$
$=\tan^{-1}1=45^\circ$
$\alpha=360^\circ-45^\circ=315^\circ$

$z=2\sqrt{2}\text{ cis }315^\circ$

**30.** $r=\sqrt{\left(-\sqrt{3}\right)^2+(1)^2}$ [5.2]
$r=2$

$\alpha=\tan^{-1}\left|\dfrac{1}{\sqrt{3}}\right|$
$=\tan^{-1}\dfrac{1}{\sqrt{3}}=30^\circ$
$\alpha=180^\circ-30^\circ=150^\circ$

$z=2\text{ cis }150^\circ$

**31.** $r=\sqrt{(-3)^2+(2)^2}$ [5.2]
$r=\sqrt{13}$

$\alpha=\tan^{-1}\left|\frac{2}{-3}\right|$

$=\tan^{-1}\frac{2}{3}=33.7^\circ$

$\alpha=180^\circ-33.7^\circ=146.3^\circ$

$z=\sqrt{13}\text{ cis }146.3^\circ$

**32.** $r=\sqrt{(4)^2+(-1)^2}$ [5.2]
$r=\sqrt{17}$

$\alpha=\tan^{-1}\left|\frac{-1}{4}\right|$

$=\tan^{-1}\frac{1}{4}=14.04^\circ$

$\alpha=360^\circ-14.04^\circ=345.96^\circ$

$z=\sqrt{17}\text{ cis }345.96^\circ$

**33.** $z=5(\cos 315^\circ+i\sin 315^\circ)$ [5.2]
$z=5\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)$
$z=\frac{5\sqrt{2}}{2}-\frac{5\sqrt{2}}{2}i$
$\approx 3.536-3.536i$

**34.** $z=6\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$ [5.2]
$z=6\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$
$z=-3-3\sqrt{3}i$

**35.** $z=2(\cos 2+i\sin 2)$ [5.2]
$z\approx 2(-0.4161+0.9093i)$
$z\approx -0.832+1.819i$

**36.** $z=3(\cos 115^\circ+i\sin 115^\circ)$ [5.2]
$z\approx 3(-0.4226+0.9063i)$
$z\approx -1.27+2.72i$

**37.** $z_1z_2=3(\cos 225°+i\sin 225°)\cdot 10(\cos 45°+i\sin 45°)$ [5.2]
$z_1z_2=30[\cos(225°+45°)+i\sin(225°+45°)]$
$z_1z_2=30(\cos 270°+i\sin 270°)$
$z_1z_2=30(0-1i)$
$z_1z_2=-30i$

**38.** $z_1z_2=5(\cos 162°+i\sin 162°)\cdot 2(\cos 63°+i\sin 63°)$ [5.2]
$z_1z_2=10[\cos(162°+63°)+i\sin(162°+63°)]$
$z_1z_2=10(\cos 225°+i\sin 225°)$
$z_1z_2=10\left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)$
$z_1z_2=-5\sqrt{2}-5i\sqrt{2}$

**39.** $z_1z_2=3(\cos 12°+i\sin 12°)\cdot 4(\cos 126°+i\sin 126°)$ [5.2]
$z_1z_2=12[\cos(12°+126°)+i\sin(12°+126°)]$
$z_1z_2=12(\cos 138°+i\sin 138°)$
$z_1z_2\approx 12(-0.74314+0.66913i)$
$z_1z_2\approx -8.918+8.030i$

**40.** $z_1z_2=(\cos 23°+i\sin 23°)\cdot 4(\cos 233°+i\sin 233°)$ [5.2]
$z_1z_2=4[\cos(23°+233°)+i\sin(23°+233°)]$
$z_1z_2=4(\cos 256°+i\sin 256°)$
$z_1z_2\approx 4(-0.24192-0.97030i)$
$z_1z_2\approx -0.968-3.881i$

**41.** $z_1z_2=3(\cos 1.8+i\sin 1.8)\cdot 5(\cos 2.5+i\sin 2.5)$ [5.2]
$z_1z_2=15[\cos(1.8+2.5)+i\sin(1.8+2.5)]$
$z_1z_2=15(\cos 4.3+i\sin 4.3)$
$z_1z_2\approx 15(-0.4008-0.9162i)$
$z_1z_2\approx -6.012-13.743i$

**42.** $z_1z_2=6(\cos 3.1+i\sin 1.8)\cdot 5(\cos 4.3+i\sin 4.3)$ [5.2]
$z_1z_2=30[\cos(3.1+4.3)+i\sin(3.1+4.3)]$
$z_1z_2=30(\cos 7.4+i\sin 7.4)$
$z_1z_2\approx 30(0.439+0.899i)$
$z_1z_2\approx 13.2+27.0i$

**43.** $\frac{z_1}{z_2}=\frac{6(\cos 50^\circ+i\sin 50^\circ)}{2(\cos 150^\circ+i\sin 150^\circ)}$ [5.2]

$\frac{z_1}{z_2}=3\,[\cos(50^\circ-150^\circ)+i\sin(50^\circ-150^\circ)]$

$\frac{z_1}{z_2}=3(\cos -100^\circ+i\sin -100^\circ)=3\text{cis }(-100^\circ)$

**44.** $\frac{z_1}{z_2}=\frac{30(\cos 165^\circ+i\sin 165^\circ)}{10(\cos 55^\circ+i\sin 55^\circ)}$ [5.2]

$\frac{z_1}{z_2}=3\,[\cos(165^\circ-55^\circ)+i\sin(165^\circ-55^\circ)]$

$\frac{z_1}{z_2}=3(\cos 110^\circ+i\sin 110^\circ)=3\text{cis }(110^\circ)$

**45.** $\dfrac{z_1}{z_2} = \dfrac{40(\cos 66^\circ + i\sin 66^\circ)}{8(\cos 125^\circ + i\sin 125^\circ)}$ [5.2]

$\dfrac{z_1}{z_2} = 5\,[\cos(66^\circ - 125^\circ) + i\sin(66^\circ - 125^\circ)]$

$\dfrac{z_1}{z_2} = 5(\cos -59^\circ + i\sin -59^\circ) = 5\text{cis}\,(-59^\circ)$

**46.** $\dfrac{z_1}{z_2} = \dfrac{2(\cos 150^\circ + i\sin 150^\circ)}{\sqrt{2}(\cos 200^\circ + i\sin 200^\circ)}$ [5.2]

$\dfrac{z_1}{z_2} = \sqrt{2}\,[\cos(150^\circ - 200^\circ) + i\sin(150^\circ - 200^\circ)]$

$\dfrac{z_1}{z_2} = \sqrt{2}(\cos -50^\circ + i\sin -50^\circ) = \sqrt{2}\text{cis}\,(-50^\circ)$

**47.** $\dfrac{z_1}{z_2} = \dfrac{10(\cos 3.7 + i\sin 3.7)}{6(\cos 1.8 + i\sin 1.8)}$ [5.2]

$\dfrac{z_1}{z_2} = \dfrac{5}{3}\,[\cos(3.7 - 1.8) + i\sin(3.7 - 1.8)]$

$\dfrac{z_1}{z_2} = \dfrac{5}{3}(\cos 1.9 + i\sin 1.9) = \dfrac{5}{3}\text{cis}\,(1.9)$

**48.** $\dfrac{z_1}{z_2} = \dfrac{4(\cos 1.2 + i\sin 1.2)}{8(\cos 5.2 + i\sin 5.2)}$ [5.2]

$\dfrac{z_1}{z_2} = \dfrac{1}{2}\,[\cos(1.2 - 5.2) + i\sin(1.2 - 5.2)]$

$\dfrac{z_1}{z_2} = \dfrac{1}{2}(\cos -4 + i\sin -4) = \dfrac{1}{2}\text{cis}\,(-4)$

**49.** $[3(\cos 45^\circ + i\sin 45^\circ)]^5 = 3^5[\cos(5\cdot 45^\circ) + i\sin(5\cdot 45^\circ)]$ [5.3]

$= 243[\cos 225^\circ + i\sin 225^\circ]$

$= 243\left(-\dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2}i\right)$

$= -\dfrac{243\sqrt{2}}{2} - \dfrac{243\sqrt{2}}{2}i$

$\approx -171.827 - 171.827i$

**50.** $\left[\cos\dfrac{11\pi}{8} + i\sin\dfrac{11\pi}{8}\right]^8 = \cos\left(8\cdot\dfrac{11\pi}{8}\right) + i\sin\left(8\cdot\dfrac{11\pi}{8}\right)$ [5.3]

$= \cos(11\pi) + i\sin(11\pi)$

$= \cos\pi + i\sin\pi$

$= -1 + 0i$

$= -1$

**51.** $z = 1 - i\sqrt{3}$ [5.3]

$r = \sqrt{1^2 + (-\sqrt{3})^2}$ $\qquad \alpha = \tan^{-1}\left|\dfrac{-\sqrt{3}}{1}\right| = 60^\circ$

$r = 2$ $\qquad \theta = 300^\circ$

$z = 2(\cos 300^\circ + i\sin 300^\circ)$

$(1 - i\sqrt{3})^7 = [2(\cos 300^\circ + i\sin 300^\circ)]^7$

$= 128[\cos(7\cdot 300^\circ) + i\sin(7\cdot 300^\circ)]$

$= 128(\cos 2100^\circ + i\sin 2100^\circ)$

$= 128(\cos 300^\circ + i\sin 300^\circ)$

$= 64 - 64i\sqrt{3}$

$\approx 64 - 110.851i$

**52.** $z = -2 - 2i$ [5.3]

$r = \sqrt{(-2)^2 + (-2)^2}$ $\qquad \alpha = \tan^{-1}\left|\dfrac{-2}{-2}\right| = 45^\circ$

$r = 2\sqrt{2}$ $\qquad \theta = 225^\circ$

$z = 2\sqrt{2}(\cos 225^\circ + i\sin 225^\circ)$

$(-2 - 2i)^{10} = [2\sqrt{2}(\cos 225^\circ + i\sin 225^\circ)]^{10}$

$= 32,768[\cos(10\cdot 225^\circ) + i\sin(10\cdot 225^\circ)]$

$= 32,768(\cos 2250^\circ + i\sin 2250^\circ)$

$= 32,768(\cos 90^\circ + i\sin 90^\circ)$

$= 0 + 32,768i$

**53.** $z = \sqrt{2} - i\sqrt{2}$ [5.3]

$r = \sqrt{(\sqrt{2})^2 + (-\sqrt{2})^2}$ $\qquad \alpha = \tan^{-1}\left|\dfrac{-\sqrt{2}}{\sqrt{2}}\right| = 45^\circ$

$r = 2$ $\qquad \theta = 315^\circ$

$z = 2(\cos 315^\circ + i\sin 315^\circ)$

$(\sqrt{2} - i\sqrt{2})^5 = [2(\cos 315^\circ + i\sin 315^\circ)]^5$

$= 32[\cos(5\cdot 315^\circ) + i\sin(5\cdot 315^\circ)]$

$= 32(\cos 1575^\circ + i\sin 1575^\circ)$

$= 32(\cos 135^\circ + i\sin 135^\circ)$

$= -16\sqrt{2} + 16i\sqrt{2}$

$\approx -22.627 + 22.627i$

**54.** $z = 3 - 4i$ [5.3]

$r = \sqrt{3^2 + (-4)^2}$ $\qquad \alpha = \tan^{-1}\left|\dfrac{-4}{3}\right| = 53.13^\circ$

$r = 5$ $\qquad \theta = 306.87^\circ$

$z = 5(\cos 306.87^\circ + i\sin 306.87^\circ)$

$(3 - 4i)^5 = [5(\cos 306.87^\circ + i\sin 306.87^\circ)]^5$

$= 3125[\cos(5\cdot 306.87^\circ) + i\sin(5\cdot 306.87^\circ)]$

$= 3125(\cos 1534.35^\circ + i\sin 1534.35^\circ)$

$= 3125(\cos 94.35^\circ + i\sin 94.35^\circ)$

$= -237 + 3116i$

**55.** $27i = 27\left(\cos 90^\circ + i\sin 90^\circ\right)$ [5.3]

$$w_k = 27^{1/3}\left(\cos\frac{90^\circ + 360^\circ k}{3} + i\sin\frac{90^\circ + 360^\circ k}{3}\right) \qquad k = 0, 1, 2$$

$$w_0 = 3\left(\cos\frac{90^\circ}{3} + i\sin\frac{90^\circ}{3}\right)$$
$$w_0 = 3(\cos 30^\circ + i\sin 30^\circ)$$
$$w_0 = 3 \text{ cis } 30^\circ$$

$$w_1 = 3\left(\cos\frac{90^\circ + 360^\circ}{3} + i\sin\frac{90^\circ + 360^\circ}{3}\right)$$
$$w_1 = 3(\cos 150^\circ + i\sin 150^\circ)$$
$$w_1 = 3 \text{ cis } 150^\circ$$

$$w_2 = 3\left(\cos\frac{90^\circ + 360^\circ \cdot 2}{3} + i\sin\frac{90^\circ + 360^\circ \cdot 2}{3}\right)$$
$$w_2 = 3(\cos 270^\circ + i\sin 270^\circ)$$
$$w_2 = 3 \text{ cis } 270^\circ$$

**56.** $8i = 8\left(\cos 90^\circ + i\sin 90^\circ\right)$ [5.3]

$$w_k = 8^{1/4}\left(\cos\frac{90^\circ + 360^\circ k}{4} + i\sin\frac{9^\circ + 360^\circ k}{4}\right) \qquad k = 0, 1, 2, 3$$

$$w_0 = \sqrt[4]{8}\left(\cos\frac{90^\circ}{4} + i\sin\frac{90^\circ}{4}\right)$$
$$w_0 = \sqrt[4]{8}(\cos 22.5^\circ + i\sin 22.5^\circ)$$
$$w_0 = \sqrt[4]{8} \text{ cis } 22.5^\circ$$

$$w_1 = \sqrt[4]{8}\left(\cos\frac{90^\circ + 360^\circ}{4} + i\sin\frac{90^\circ + 360^\circ}{4}\right)$$
$$w_1 = \sqrt[4]{8}(\cos 112.5^\circ + i\sin 112.5^\circ)$$
$$w_1 = \sqrt[4]{8} \text{ cis } 112.5^\circ$$

$$w_2 = \sqrt[4]{8}\left(\cos\frac{90^\circ + 360^\circ \cdot 2}{4} + i\sin\frac{90^\circ + 360^\circ \cdot 2}{4}\right)$$
$$w_2 = \sqrt[4]{8}(\cos 202.5^\circ + i\sin 202.5^\circ)$$
$$w_2 = \sqrt[4]{8} \text{ cis } 202.5^\circ$$

$$w_3 = \sqrt[4]{8}\left(\cos\frac{90^\circ + 360^\circ \cdot 3}{4} + i\sin\frac{90^\circ + 360^\circ \cdot 3}{4}\right)$$
$$w_3 = \sqrt[4]{8}(\cos 292.5^\circ + i\sin 292.5^\circ)$$
$$w_3 = \sqrt[4]{8} \text{ cis } 292.5^\circ$$

**57.** $256 = 256\left(\cos 0^\circ + i\sin 0^\circ\right)$ [5.3]

$$w_k = 81^{1/4}\left(\cos\frac{0^\circ + 360^\circ k}{4} + i\sin\frac{0^\circ + 360^\circ k}{4}\right) \qquad k = 0, 1, 2, 3$$

$$w_0 = 3(\cos 0^\circ + i\sin 0^\circ)$$
$$w_0 = 3 \text{ cis } 0^\circ$$

$$w_1 = 3\left(\cos\frac{0^\circ + 360^\circ}{4} + i\sin\frac{0^\circ + 360^\circ}{4}\right)$$
$$w_1 = 3(\cos 90^\circ + i\sin 90^\circ)$$
$$w_1 = 3 \text{ cis } 90^\circ$$

$$w_2 = 3\left(\cos\frac{0^\circ + 360^\circ \cdot 2}{4} + i\sin\frac{0^\circ + 360^\circ \cdot 2}{4}\right)$$
$$w_2 = 3(\cos 180^\circ + i\sin 180^\circ)$$
$$w_2 = 3 \text{ cis } 180^\circ$$

$$w_3 = 3\left(\cos\frac{0^\circ + 360^\circ \cdot 3}{4} + i\sin\frac{0^\circ + 360^\circ \cdot 3}{4}\right)$$
$$w_3 = 3(\cos 270^\circ + i\sin 270^\circ)$$
$$w_3 = 3 \text{ cis } 270^\circ$$

**58.** $-16\sqrt{2}-16i\sqrt{2}=32(\cos 225°+i\sin 225°)$ [5.3]

$$w_k=32^{1/5}\left(\cos\frac{225°+360°k}{5}+i\sin\frac{225°+360°k}{5}\right)\quad k=0,1,2,3,4$$

$w_0=2\left(\cos\frac{225°}{5}+i\sin\frac{225°}{5}\right)$
$w_0=2(\cos 45°+i\sin 45°)$
$w_0=2\text{ cis }45°$

$w_1=2\left(\cos\frac{225°+360°}{5}+i\sin\frac{225°+360°}{5}\right)$
$w_1=2(\cos 117°+i\sin 117°)$
$w_1=2\text{ cis }117°$

$w_2=2\left(\cos\frac{225°+360°\cdot 2}{5}+i\sin\frac{225°+360°\cdot 2}{5}\right)$
$w_2=2(\cos 189°+i\sin 189°)$
$w_2=2\text{ cis }189°$

$w_3=2\left(\cos\frac{225°+360°\cdot 3}{5}+i\sin\frac{225°+360°\cdot 3}{5}\right)$
$w_3=2(\cos 261°+i\sin 261°)$
$w_3=2\text{ cis }261°$

$w_4=2\left(\cos\frac{225°+360°\cdot 4}{5}+i\sin\frac{225°+360°\cdot 4}{5}\right)$
$w_4=2(\cos 333°+i\sin 333°)$
$w_4=2\text{ cis }333°$

**59.** $81=81\left(\cos 0°+i\sin 0°\right)$ [5.3]

$$w_k=256^{1/4}\left(\cos\frac{0°+360°k}{4}+i\sin\frac{0°+360°k}{4}\right)\quad k=0,1,2,3$$

$w_0=4(\cos 0°+i\sin 0°)$
$w_0=4\text{ cis }0°$

$w_1=4\left(\cos\frac{0°+360°}{4}+i\sin\frac{0°+360°}{4}\right)$
$w_1=4(\cos 90°+i\sin 90°)$
$w_1=4\text{ cis }90°$

$w_2=4\left(\cos\frac{0°+360°\cdot 2}{4}+i\sin\frac{0°+360°\cdot 2}{4}\right)$
$w_2=4(\cos 180°+i\sin 180°)$
$w_2=4\text{ cis }180°$

$w_3=4\left(\cos\frac{0°+360°\cdot 3}{4}+i\sin\frac{0°+360°\cdot 3}{4}\right)$
$w_3=4(\cos 270°+i\sin 270°)$
$w_3=4\text{ cis }270°$

**60.** $-125=125\left(\cos 180°+i\sin 180°\right)$ [5.3]

$$w_k=125^{1/3}\left(\cos\frac{180°+360°k}{3}+i\sin\frac{180°+360°k}{3}\right)\quad k=0,1,2$$

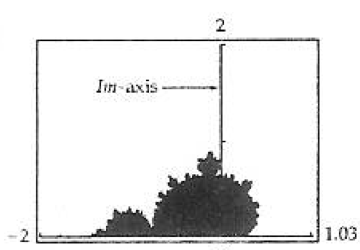
$w_0=5\left(\cos\frac{180°}{3}+i\sin\frac{180°}{3}\right)$
$w_0=5(\cos 60°+i\sin 60°)$
$w_0=5\text{ cis }60°$

$w_1=5\left(\cos\frac{180°+360°}{3}+i\sin\frac{180°+360°}{3}\right)$
$w_1=5(\cos 180°+i\sin 180°)$
$w_1=5\text{ cis }180°=-5$

$w_2=5\left(\cos\frac{180°+360°\cdot 2}{3}+i\sin\frac{180°+360°\cdot 2}{3}\right)$
$w_2=5(\cos 300°+i\sin 300°)$
$w_2=5\text{ cis }300°$

## Chapter 5 Quantitative Reasoning

**QR1.**




**QR2.** $-0.25+0.25i$, $-1+0.1i$, and $0.1+0.2i$

**QR3.** –2 is an element of the Mendelbrot set because all of its iterates equal 2.

**QR4.** The first iterate of $2i$ is $z_1 = -4+2i$ . $2i$ is not an element of the Mandelbrot set because $|-4+2i| > 3$ .

**QR5.** $(Z^2 + S) \to Z$

**QR6.** Answers will vary.

## Chapter 5 Test

**1.** $6+\sqrt{-9} = 6+3i$ [5.1]

**2.** $\sqrt{-18} = 3i\sqrt{2}$ [5.1]

**3.** $(3+\sqrt{-4})+(7-\sqrt{-9}) = 3+2i+7-3i$
$= 10-i$
[5.1]

**4.** $(-1+\sqrt{-25})-(8-\sqrt{-16}) = -1+5i-8+4i$ [5.1]
$= -9+9i$

**5.** $(\sqrt{-12})(\sqrt{-3}) = (2i\sqrt{3})(i\sqrt{3})$ [5.1]
$= 6i^2 = -6$

**6.** $i^{263} = i^3 = -i$ [5.1]

**7.** $(3+7i)-(-2-9i) = 5+16i$ [5.1]

**8.** $(-6-9i)(4+3i) = -24-54i-27i^2$
$= 3-54i$
[5.1]

**9.** $(3-5i)(-3+5i) = -9+30i-25i^2$ [5.1]
$= 16+30i$

**10.** $\dfrac{4-5i}{i}\cdot\dfrac{-i}{-i} = \dfrac{-4i+5i^2}{-i^2} = -5-4i$ [5.1]

**11.** $\dfrac{2-7i}{4+3i}\cdot\dfrac{4-3i}{4-3i} = \dfrac{8-34i+21i^2}{16+9} = -\dfrac{13}{25}-\dfrac{34}{25}i$ [5.1]

**12.** $\dfrac{6+2i}{1-i}\cdot\dfrac{1+i}{1+i} = \dfrac{6+8i+2i^2}{1+1} = \dfrac{4}{2}+\dfrac{8}{2}i = 2+4i$ [5.1]

**13.** $|z| = \sqrt{(3)^2+(-5)^2}$ [5.2]
$= \sqrt{9+25} = \sqrt{34}$

**14.** $r = \sqrt{(3)^2+(-3)^2}$ [5.2]
$r = 3\sqrt{2}$

$\alpha = \tan^{-1}\left|\dfrac{-3}{3}\right|$

$= \tan^{-1} 1 = 45^\circ$

$\alpha = 360^\circ - 45^\circ = 315^\circ$

$z = 3\sqrt{2}\text{ cis }315^\circ$

**15.** $r = \sqrt{(0)^2+(-6)^2}$ [5.2]
$r = 6$

$\alpha = \tan^{-1}\left|\dfrac{-6}{0}\right| = 90^\circ$

$\alpha = 360^\circ - 90^\circ = 270^\circ$

$z = 6\text{ cis }270^\circ$

**16.** $z = 4(\cos 120^\circ + i\sin 120^\circ)$ [5.2]
$z = 4\left(-\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i\right)$
$z = -2+2i\sqrt{3}$

**17.** $z = 5(\cos 225^\circ + i\sin 225^\circ)$ [5.2]
$z = 5\left(-\dfrac{\sqrt{2}}{2}-\dfrac{\sqrt{2}}{2}i\right)$
$z = -\dfrac{5\sqrt{2}}{2}-\dfrac{5\sqrt{2}}{2}i$

**18.** $z_1z_2 = 3(\cos 28° + i\sin 28°)\cdot 4(\cos 17° + i\sin 17°)$ [5.2]
$z_1z_2 = 12[\cos(28°+17°) + i\sin(28°+17°)]$
$z_1z_2 = 12(\cos 45° + i\sin 45°)$
$z_1z_2 = 12\left(\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}i\right)$
$z_1z_2 = 6\sqrt{2}+6i\sqrt{2}$

**19.** $z_1z_2 = 5(\cos 115° + i\sin 115°)\cdot 4(\cos 10° + i\sin 10°)$ [5.2]
$z_1z_2 = 20[\cos(115°+10°) + i\sin(115°+10°)]$
$z_1z_2 = 20(\cos 125° + i\sin 125°)$
$z_1z_2 \approx 20(-0.5736+0.81915i)$
$z_1z_2 \approx -11.472+16.383i$

**20.** $\frac{z_1}{z_2} = \frac{24(\cos 258^\circ + i \sin 258^\circ)}{6(\cos 78^\circ + i \sin 78^\circ)}$ [5.2]

$\frac{z_1}{z_2} = 4\,[\cos(258^\circ - 78^\circ) + i \sin(258^\circ - 78^\circ)]$

$\frac{z_1}{z_2} = 4(\cos 180^\circ + i \sin 180^\circ) = -4 + 0i = -4$

**21.** $\frac{z_1}{z_2} = \frac{18(\cos 50^\circ + i \sin 50^\circ)}{3(\cos 140^\circ + i \sin 140^\circ)}$ [5.2]

$\frac{z_1}{z_2} = 6\,[\cos(50^\circ - 140^\circ) + i \sin(50^\circ - 140^\circ)]$

$\frac{z_1}{z_2} = 6(\cos -90^\circ + i \sin -90^\circ) = 0 - 6i = -6i$

**22.** $z = 2 - 2i\sqrt{3}$ [5.3]

$r = \sqrt{2^2 + (-2\sqrt{3})^2}$ $\quad \alpha = \tan^{-1}\left|\frac{-2\sqrt{3}}{2}\right| = 60^\circ$

$r = 4$ $\quad \theta = 300^\circ$

$$\begin{aligned} z &= 4(\cos 300^\circ + i \sin 300^\circ) \\ (2 - 3i\sqrt{3})^{12} &= [4(\cos 300^\circ + i \sin 300^\circ)]^{12} \\ &= 16,777,216[\cos(12 \cdot 300^\circ) + i \sin(10 \cdot 300^\circ)] \\ &= 16,777,216(\cos 3600^\circ + i \sin 3600^\circ) \\ &= 16,777,216(\cos 0^\circ + i \sin 0^\circ) \\ &= 16,777,216 + 0i \end{aligned}$$

**23.** $64 = 64(\cos 0^\circ + i \sin 0^\circ)$ [5.3]

$w_k = 64^{1/6}\left(\cos\frac{0^\circ + 360^\circ k}{6} + i \sin\frac{0^\circ + 360^\circ k}{6}\right) \quad k = 0, 1, 2, 3, 4, 5$

$w_0 = 2(\cos 0^\circ + i \sin 0^\circ)$

$w_0 = 2 \text{ cis } 0^\circ$

$w_1 = 2\left(\cos\frac{0 + 360^\circ}{6} + i \sin\frac{0 + 360^\circ}{6}\right)$

$w_1 = 2(\cos 60^\circ + i \sin 60^\circ)$

$w_1 = 2 \text{ cis } 60^\circ$

$w_2 = 2\left(\cos\frac{0^\circ + 360^\circ \cdot 2}{6} + i \sin\frac{0^\circ + 360^\circ \cdot 2}{6}\right)$

$w_2 = 2(\cos 120^\circ + i \sin 120^\circ)$

$w_2 = 2 \text{ cis } 120^\circ$

$w_3 = 2\left(\cos\frac{0^\circ + 360^\circ \cdot 3}{6} + i \sin\frac{0^\circ + 360^\circ \cdot 3}{6}\right)$

$w_3 = 2(\cos 180^\circ + i \sin 180^\circ)$

$w_3 = 2 \text{ cis } 180^\circ$

$w_4 = 2\left(\cos\frac{0^\circ + 360^\circ \cdot 4}{6} + i \sin\frac{0^\circ + 360^\circ \cdot 4}{6}\right)$

$w_4 = 2(\cos 240^\circ + i \sin 240^\circ)$

$w_4 = 2 \text{ cis } 240^\circ$

$w_5 = 2\left(\cos\frac{0^\circ + 360^\circ \cdot 5}{6} + i \sin\frac{0^\circ + 360^\circ \cdot 5}{6}\right)$

$w_5 = 2(\cos 300^\circ + i \sin 300^\circ)$

$w_5 = 2 \text{ cis } 300^\circ$

**24.** $-1 + \sqrt{3}i = 2(\cos 120^\circ + i \sin 120^\circ)$ [5.3]

$w_k = 2^{1/3}\left(\cos\frac{120^\circ + 360^\circ k}{3} + i \sin\frac{120^\circ + 360^\circ k}{3}\right) \quad k = 0, 1, 2$

$w_0 = \sqrt[3]{2}\left(\cos\frac{120^\circ}{3} + i \sin\frac{120^\circ}{3}\right)$

$w_0 = \sqrt[3]{2}(\cos 40^\circ + i \sin 40^\circ)$

$w_0 = \sqrt[3]{2} \text{ cis } 40^\circ$

$w_1 = \sqrt[3]{2}\left(\cos\frac{120^\circ + 360^\circ}{3} + i \sin\frac{120^\circ + 360^\circ}{3}\right)$

$w_1 = \sqrt[3]{2}(\cos 160^\circ + i \sin 160^\circ)$

$w_1 = \sqrt[3]{2} \text{ cis } 160^\circ$

$w_2 = \sqrt[3]{2}\left(\cos\frac{120^\circ + 360^\circ \cdot 2}{3} + i \sin\frac{120^\circ + 360^\circ \cdot 2}{3}\right)$

$w_2 = \sqrt[3]{2}(\cos 280^\circ + i \sin 280^\circ)$

$w_2 = \sqrt[3]{2} \text{ cis } 280^\circ$

**25.** $x^5 + 32 = 0$ [5.3]

$x^5 = -32$

Find the five fifth roots of –32.

$-32 = 32(\cos 180° + i \sin 180°)$

$w_k = 32^{1/5}\left(\cos\frac{180° + 360° k}{5} + i \sin\frac{180° + 360° k}{5}\right) \quad k = 0, 1, 2, 3, 4$

$w_0 = 2 \text{ cis } \frac{180°}{5}$

$w_0 = 2 \text{ cis } 36°$

$w_1 = 2 \text{ cis } \frac{180° + 360°}{5}$

$w_1 = 2 \text{ cis } 108°$

$w_2 = 2 \text{ cis } \frac{180° + 360° \cdot 2}{5}$

$w_2 = 2 \text{ cis } 180°$

$w_3 = 2 \text{ cis } \frac{180° + 360° \cdot 3}{5}$

$w_3 = 2 \text{ cis } 252°$

$w_4 = 2 \text{ cis } \frac{180° + 360° \cdot 4}{5}$

$w_4 = 2 \text{ cis } 324°$

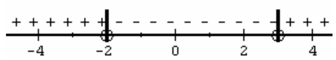
## Chapter 5 Cumulative Review

**1.** $x^2 - x - 6 \le 0$ [1.1

$(x-3)(x+2) \le 0$

The product is negative or zero.
The critical values are –2 and 3.

$(x-3)(x+2)$




[–2, 3]

**2.** $x^2 - 4 = 0$

$(x-2)(x+2) = 0$

$x = -2, 2$

Domain: all real numbers except –2 and 2.

**3.** $f(c) = 2$ [1.3]

$2 = \frac{c}{c+1}$

$2(c+1) = c$

$2c + 2 = c$

$c = -2$

**4.** $(f \circ g)(x) = f[g(x)]$ [1.5]

$= f\left[\frac{x^2-1}{3}\right]$

$= \sin\left(3 \cdot \frac{x^2-1}{3}\right)$

$= \sin(x^2 - 1)$

**5.** $f(x) = \frac{x}{x-1}$ [1.6]

$y = \frac{x}{x-1}$

$x = \frac{y}{y-1}$

$x(y-1) = y$

$xy - x = y$

$xy - y = x$

$y(x-1) = x$

$y = \frac{x}{x-1}$

$f^{-1}(x) = \frac{x}{x-1}$

$f^{-1}(3) = \frac{3}{3-1} = \frac{3}{2}$

**6.** $\frac{3\pi}{2}\left(\frac{180°}{\pi}\right) = 270°$ [2.1]

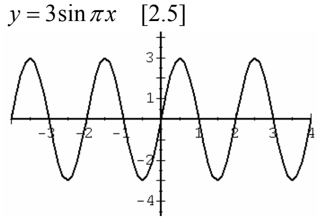
**7.** $\cos 38° = \frac{20}{c}$ [2.2]

$c = \frac{20}{\cos 38°} \approx 25.4 \text{ cm}$

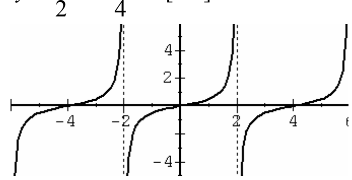
**8.** $-1 \le \sin t \le 1$ [2.4]

$a = -1, b = 1$

**9.** $y = 3 \sin \pi x$ [2.5]




**10.** $y = \frac{1}{2} \tan \frac{\pi x}{4}$ [2.6]



**11.** $\dfrac{\cos x}{1+\sin x}=\dfrac{\cos x}{1+\sin x}\cdot\dfrac{1-\sin x}{1-\sin x}$ [3.1]

$$=\frac{\cos x-\sin x\cos x}{1-\sin^2 x}$$
$$=\frac{\cos x-\sin x\cos x}{\cos^2 x}$$
$$=\frac{1}{\cos x}-\frac{\sin x}{\cos x}$$
$$=\sec x-\tan x$$

**12.** $\sin 2x\cos 3x-\cos 2x\sin 3x=\sin(2x-3x)$ [3.2]

$$=\sin(-x) \text{ or } -\sin x$$

**13.** $\sin\alpha=\dfrac{4}{5},\ \cos\alpha=\dfrac{3}{5}$ [3.2]

$$\cos\beta=\frac{12}{13},\ \sin\beta=-\frac{5}{13}$$
$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$=\left(\frac{3}{5}\right)\left(\frac{12}{13}\right)-\left(\frac{4}{5}\right)\left(-\frac{5}{13}\right)$$
$$=-\frac{48}{65}+\frac{15}{65}=\frac{56}{65}$$

**14.** $y=\sin\left[\sin^{-1}\left(\dfrac{3}{5}\right)+\cos^{-1}\left(-\dfrac{5}{13}\right)\right]$ [3.5]

Let $\alpha=\sin^{-1}\dfrac{3}{5},\ \sin\alpha=\dfrac{3}{5},\ \cos\alpha=\sqrt{1-\left(\dfrac{3}{5}\right)^2}=\dfrac{4}{5}.$

$\beta=\cos^{-1}\left(-\dfrac{5}{13}\right),\ \cos\beta=-\dfrac{5}{13},\ \sin\beta=\sqrt{1-\left(-\dfrac{5}{13}\right)^2}=\dfrac{12}{13}.$

$$y=\sin(\alpha+\beta)$$
$$=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$=\frac{3}{5}\left(-\frac{5}{13}\right)+\frac{4}{5}\left(\frac{12}{13}\right)$$
$$=-\frac{15}{65}+\frac{48}{65}=\frac{33}{65}$$

**15.** $\sin 2x=\sqrt{3}\sin x$ [3.6]

$$2\sin x\cos x-\sqrt{3}\sin x=0$$
$$\sin x(2\cos x-\sqrt{3})=0$$

$\sin x=0$ $\qquad$ $2\cos x-\sqrt{3}=0$

$x=0,\ \pi$ $\qquad$ $\cos x=\dfrac{\sqrt{3}}{2}$

$x=\dfrac{\pi}{6},\ \dfrac{11\pi}{6}$

The solutions are $0,\ \dfrac{\pi}{6},\ \pi,\ \dfrac{11\pi}{6}$ .

**16.** $c^2=a^2+b^2-2ab\cos C$ [4.2]

$$=(140)^2+(130)^2-2(140)(130)\cos 78^\circ$$
$$=36{,}500-36{,}400\cos 78^\circ$$
$$\approx 28{,}932$$
$$c\approx 170 \text{ cm}$$

**17.** $\cos\theta=\dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\ \|\mathbf{w}\|}$ [4.3]

$$\cos\theta=\frac{(3\mathbf{i}+2\mathbf{j})\cdot(5\mathbf{i}-3\mathbf{j})}{\sqrt{3^2+2^2}\sqrt{5^2+(-3)^2}}$$
$$\cos\theta=\frac{3(5)+(2)(-3)}{\sqrt{13}\sqrt{34}}$$
$$\cos\theta=\frac{9}{\sqrt{442}}=0$$
$$\theta=64.7^\circ$$

**18.** $W=\mathbf{F}\cdot\mathbf{s}$ [4.3]

$$W=\|\mathbf{F}\|\ \|\mathbf{s}\|\cos\alpha$$
$$W=(100)(15)(\cos 15^\circ)$$
$$W\approx 1449 \text{ foot-pounds}$$

**19.** $r=\sqrt{(2)^2+(2)^2}$ [5.2]

$$r=2\sqrt{2}$$
$$\alpha=\tan^{-1}\left|\frac{2}{2}\right|$$
$$=\tan^{-1}1=45^\circ$$
$$z=2\sqrt{2}\ \text{cis}\ 45^\circ$$

**20.** $-27=27\left(\cos 180^\circ+i\sin 180^\circ\right)$ [5.3]

$$w_k=27^{1/3}\left(\cos\frac{180^\circ+360^\circ k}{3}+i\sin\frac{180^\circ+360^\circ k}{3}\right)\qquad k=0,1,2$$

$$w_0=3\left(\cos\frac{180^\circ}{3}+i\sin\frac{180^\circ}{3}\right)$$
$$w_0=3(\cos 60^\circ+i\sin 60^\circ)$$
$$w_0=3\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=\frac{3}{2}+\frac{3\sqrt{3}}{2}$$

$$w_1=3\left(\cos\frac{180^\circ+360^\circ}{3}+i\sin\frac{180^\circ+360^\circ}{3}\right)$$
$$w_1=3(\cos 180^\circ+i\sin 180^\circ)$$
$$w_1=3(-1+0i)=-3$$

$$w_2=3\left(\cos\frac{180^\circ+360^\circ\cdot 2}{3}+i\sin\frac{180^\circ+360^\circ\cdot 2}{3}\right)$$
$$w_2=3(\cos 300^\circ+i\sin 300^\circ)$$
$$w_2=3\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=\frac{3}{2}-\frac{3\sqrt{3}}{2}i$$

## Section 6.7

See *CAT Section* 8.7 solutions on page 577 for exercises 1 – 18.

**19.**

$x = 2 + 3\cos t \qquad y = 3 + 2\sin t$

$x - 2 = 3\cos t \qquad y - 3 = 2\sin t$

$\frac{x-2}{3} = \cos t \qquad \frac{y-3}{2} = \sin t$

$$\left(\frac{x-2}{3}\right)^2 + \left(\frac{y-3}{2}\right)^2 = \cos^2 t + \sin^2 t$$

$$\left(\frac{x-2}{3}\right)^2 + \left(\frac{y-3}{2}\right)^2 = 1$$

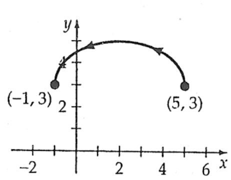
At $t = 0$,

$x = 2 + 3\cos 0 = 5 \qquad y = 3 + 2\sin 0 = 3$

At $t = \pi$,

$x = 2 + 3\cos\pi = -1 \qquad y = 3 + 2\sin\pi = 3$

The point traces the top half of the ellipse $\left(\frac{x-2}{3}\right)^2 + \left(\frac{y-3}{2}\right)^2 = 1$, as shown in the figure. The point starts at (5, 3) and moves counterclockwise along the ellipse until it reaches the point (–1, 3) at time $t = \pi$.




**20.** $x = \sin t \qquad y = -\cos t$

$$x^2 + y^2 = \sin^2 t + \cos^2 t$$

$$x^2 + y^2 = 1$$

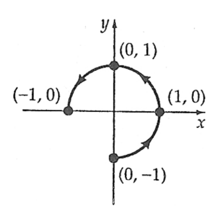
At $t = 0$,

$x = \sin 0 = 0 \qquad y = -\cos 0 = -1$

At $t = \frac{3\pi}{2}$,

$x = \sin\frac{3\pi}{2} = -1 \qquad y = -\cos\frac{3\pi}{2} = 0$

The point traces a portion of the circle $x^2 + y^2 = 1$, as shown in the figure. The point starts at (0, –1) and moves counter clockwise along the circle until it reaches the point (–1, 0) at time $t = \frac{3\pi}{2}$.



**21.** $y = t + 1 \Rightarrow t = y - 1$

$x = 2t - 1$

$x = 2(y - 1) - 1$

$x = 2y - 3$

$x + 3 = 2y$

$y = \frac{1}{2}x + 3$

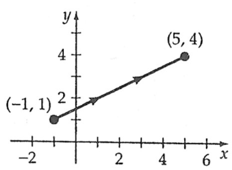
At $t = 0$,

$x = 2(0) - 1 = -1 \qquad y = 0 + 1 = 1$

At $t = 3$,

$x = 2(3) - 1 = 5 \qquad y = 3 + 1 = 4$

The point traces a line segment, as shown in the figure. The point starts at (–1, 1) and moves along the line segment until it reaches the point (5, 4) at time $t = 3$.




**22.** $x = t + 1 \Rightarrow t = x - 1$

$y = \sqrt{t}$

$y = \sqrt{x - 1}$ or $y^2 = x - 1$

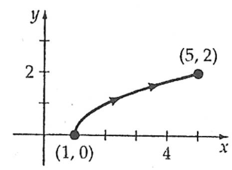
At $t = 0$,

$x = 0 + 1 = 1 \qquad y = \sqrt{0} = 0$

At $t = 4$,

$x = 4 + 1 = 5 \qquad y = \sqrt{4} = 2$

The point traces a portion of the parabola $y^2 = x - 1$, as shown in the figure. The point starts at (1, 0) and moves along the parabola until it reaches the point (5, 2) at time $t = 4$.




**23.** $x = \tan\left(\frac{\pi}{4} - t\right) \qquad y = \sec\left(\frac{\pi}{4} - t\right)$

$y^2 - x^2 = \sec^2\left(\frac{\pi}{4} - t\right) - \tan^2\left(\frac{\pi}{4} - t\right)$

$y^2 - x^2 = 1 \quad$ Since $1 + \tan^2\theta = \sec^2\theta$

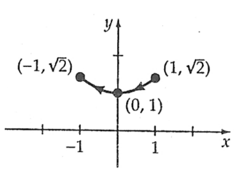
At $t = 0$,

$x = \tan\left(\frac{\pi}{4} - 0\right) = 1 \qquad y = \sec\left(\frac{\pi}{4} - 0\right) = \sqrt{2}$

At $t = \frac{\pi}{2}$,

$x = \tan\left(\frac{\pi}{4} - \frac{\pi}{2}\right) = -1 \qquad y = \sec\left(\frac{\pi}{4} - \frac{\pi}{2}\right) = \sqrt{2}$

The point traces a portion of the top branch of the hyperbola $y^2 - x^2 = 1$, as shown in the figure. The point starts at $(1, \sqrt{2})$ and moves along the hyperbola until it reaches the point $(-1, \sqrt{2})$ at time $t = \frac{\pi}{2}$.




**24.** $x = 1 - t \Rightarrow t = 1 - x$

$y = t^2$

$y = (1 - x)^2$ or $y = (x - 1)^2$

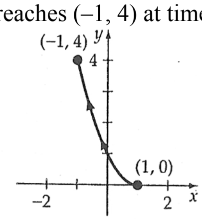
At $t = 0$,

$x = 1 - 0 = 1 \qquad y = (0)^2 = 0$

At $t = 2$,

$x = 1 - 2 = -1 \qquad y = (2)^2 = 4$

The point traces a portion of the parabola given by $y = (x - 1)^2$, as shown in the figure. The point starts at (1, 0) and moves along the parabola until it reaches (–1, 4) at time $t = 2$.



**25.** $C_1: x = 2 + t^2$

$y = 1 - 2t^2$

$x = 2 + t^2 \rightarrow t^2 = x - 2$

$y = 1 - 2(x - 2)$

$y = -2x + 5 \quad x \geq 2, y \leq 1$

$C_2: x = 2 + t$

$y = 1 - 2t$

$x = 2 + t \rightarrow t = x - 2$

$y = 1 - 2(x - 2)$

$y = -2x + 5 \quad x \in R, y \in R$

The graph of $C_1$ is a ray beginning at (2, 1) with slope –2.

The graph of $C_2$ is a line passing through (2, 1) with slope –2.

**26.** $C_1: x = \sec^2 t$

$y = \tan^2 t$

$\tan^2 t + 1 = \sec^2 t$

$y + 1 = x$

$y = x - 1$

Because $0 \leq t < \frac{\pi}{2}$,

$1 \leq \sec^2 t < \infty$ and $0 \leq \tan^2 t < \infty$

$1 \leq \quad x \quad < \infty$ and $0 \leq \quad y \quad < \infty$

Thus, $x \geq 1$ and $y \geq 0$.

$C_2: x = 1 + t^2$

$y = t^2$

$x = 1 + y$

$y = x - 1$

Because $0 < t < \frac{\pi}{2}$,

$0 \leq t^2 < \frac{\pi^2}{4}$ and $1 \leq t^2 + 1 < 1 + \frac{\pi^2}{4}$

Thus, $1 \leq x < 1 + \frac{\pi^2}{4}$ and $0 \leq y < \frac{\pi^2}{4}$.

$C_1$ is a ray from (1, 0) in the direction of $\left(1 + \frac{\pi^2}{4}, \frac{\pi^2}{4}\right)$.

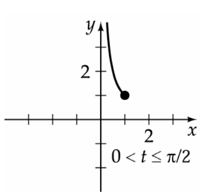
$C_2$ is the points on the line $y = x - 1$ between (1, 0) and $\left(1 + \frac{\pi^2}{4}, \frac{\pi^2}{4}\right)$. $C_2$ includes (1, 0) but not $\left(1 + \frac{\pi^2}{4}, \frac{\pi^2}{4}\right)$.

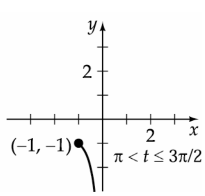
**27.** $x = \sin t$

$y = \csc t$

$\csc t = \frac{1}{\sin t}$

$y = \frac{1}{x}$




Range for graph 1: $0 < t \leq \frac{\pi}{2}$

$0 < x \leq 1$

$y \geq 1$

Range for graph 2: $\pi \leq t \leq \frac{3\pi}{2}$

$-1 \leq x \leq 0$

$y \leq -1$

**28.** $C_1: x = \cos t \qquad 0 \leq t \leq \pi, \ -1 \leq x \leq 1$

$y = \cos^2 t \qquad 0 \leq y \leq 1$

$(\cos t)^2 = \cos^2 t$

$(x)^2 = y$

$y = x^2$ for $-1 \leq x \leq 1$ and $0 \leq y \leq 1$

$C_2: x = \sin t \qquad 0 \leq t \leq \pi, 0 \leq x \leq 1$

$y = \sin^2 t \qquad 0 \leq y \leq 1$

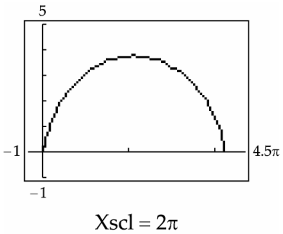
$(\sin t)^2 = \sin^2 t$

$(x)^2 = y$

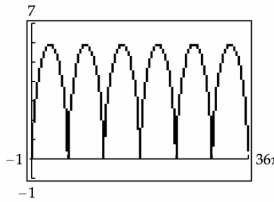
$y = x^2$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$

$C_1$ is the graph of the parabola $y = x^2$ for $-1 \leq x \leq 1$, while $C_2$ is the graph of the parabola $y = x^2$ for $0 \leq x \leq 1$.

**29.**




**30.**




**31.** **a.** For the Hummer,

$x = 6$

$y = 60t$ for $t \geq 0$

**b.** Using the graphing calculator in SIMUL and PAR mode, the Hummer is the first to reach the intersection.

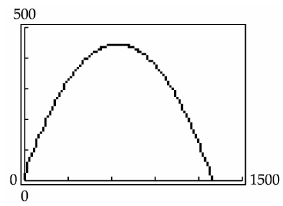
**32.** **a.** For the Learjet,

$x = 300 - 420t$

$y = 200$ for $t \geq 0$

**b.** Using the graphing calculator in SIMUL and PAR mode, the Piper Seneca is the first to reach the intersection point.

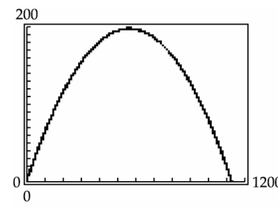
**33.**




Maximum height (to the nearest foot) of 462 feet is attained when $t \approx 5.38$ seconds.

The projectile has a range (to the nearest foot) of 1295 feet and hits the ground in about 10.75 seconds.

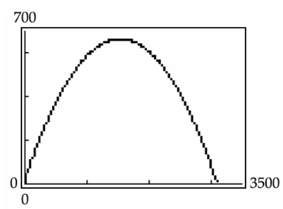
**34.**




Maximum height (to the nearest foot) of 195 feet is attained when $t \approx 3.50$ seconds.

The projectile has a range (to the nearest foot) of 1117 feet and hits the ground in about 6.99 seconds.

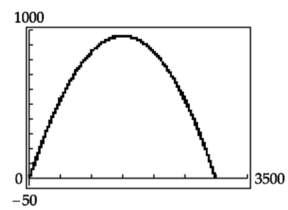
**35.**




Maximum height (to the nearest foot) of 694 feet is attained when $t \approx 6.59$ seconds.

The projectile has a range (to the nearest foot) of 3084 feet and hits the ground in about 13.17 seconds.

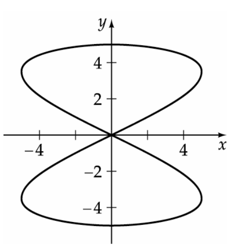
**36.**



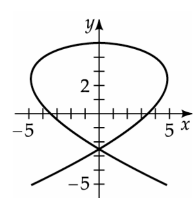

Maximum height: 963 ft at time $t \approx 7.76$ sec

Range: 3009 ft at time $t \approx 15.51$ sec

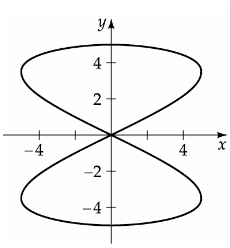
**37.**



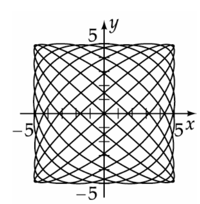

**38.**




**39.**




**40.**



## Connecting Concepts

**41.** Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two distinct points on a line.

If $P(x, y)$ is any other point on the line, then

$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$ (Slope is constant along entire line.)This equation can be rewritten as

$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$ Let this value equal $t$.

Thus, $\frac{x-x_1}{x_2-x_1} = t$ and $\frac{y-y_1}{y_2-y_1} = t$.

Solving for $x$ and $y$, respectively, we have

$x = x_1 + t(x_2 - x_1)$ and $y = y_1 + t(y_2 - y_1)$

**42.** $x = h + a\sin t \quad a > 0, x \in R,\ 0 \le t < 2\pi$

$y = k + b\cos t \quad b > 0, y \in R,\ 0 \le t < 2\pi$

$x = h + a\sin t \to \sin t = \frac{x-h}{a}$

$y = k + b\cos t \to \cos t = \frac{y-k}{b}$

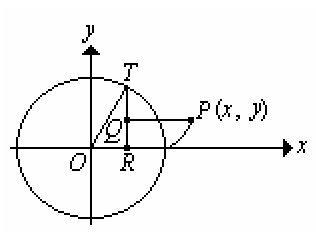
$\sin^2 t + \cos^2 t = 1$

$\left(\frac{x-h}{a}\right)^2 + \left(\frac{y-k}{b}\right)^2 = 1$

$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, which is the standard equation for an ellipse at $(h, k)$.

**43.** radius $= a$, $\theta = \angle TOR$

The $x$-coordinate of $P(x, y)$ is given by $x = OR + QP$.
The $y$-coordinate is given by $y = TR - TQ$.




From the figure,

$OR = a\cos\theta$ and $QP = a\theta\sin\theta$. Thus,

$x = a\cos\theta + a\theta\sin\theta$

$TR = a\sin\theta$ and $TQ = a\theta\cos\theta$. Thus,

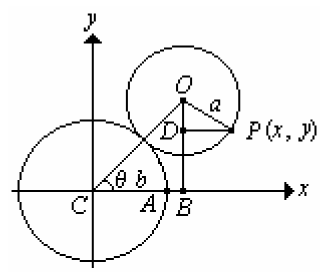
$y = a\sin\theta - a\theta\cos\theta$

The parametric equations are

$x = a\cos\theta + a\theta\sin\theta$

$y = a\sin\theta - a\theta\cos\theta$

**44.** Let $\alpha = \angle COP$. Because the smaller circle does not slip, $b\theta = a\alpha$ or $\alpha = \frac{b}{a}\theta$.




The coordinates of $P(x, y)$ are given by

$x = BC + DP$

$y = OB - OD$

Thus, $x = (a+b)\cos\theta + a\sin\left(\frac{a+b}{a}\theta - \frac{\pi}{2}\right)$

$= (a+b)\cos\theta - a\cos\left(\frac{a+b}{a}\theta\right)$

$y = (a+b)\sin\theta - a\cos\left(\frac{a+b}{a}\theta - \frac{\pi}{2}\right)$

$= (a+b)\sin\theta - a\cos\left(\frac{a+b}{a}\theta\right)$

The parametric equations are

$x = (a+b)\cos\theta - a\cos\left(\frac{a+b}{a}\theta\right)$

$y = (a+b)\sin\theta - a\sin\left(\frac{a+b}{a}\theta\right)$

**45.** Because the circle moves without slipping, $b\theta = a\alpha$.

Therefore, $\alpha = \dfrac{b\theta}{a}$. Let $P(x, y)$ be the coordinates of the moving point.

Angle $\phi = \dfrac{\pi}{2} - \left(\dfrac{b-a}{a}\right)\theta$

Thus, $x = (b-a)\cos\theta + a\sin\left[\dfrac{\pi}{2} - \left(\dfrac{b-a}{a}\right)\theta\right]$

$y = (b-a)\sin\theta - a\cos\left[\dfrac{\pi}{2} - \left(\dfrac{b-a}{a}\right)\theta\right]$

Simplifying, we have

$x = (b-a)\cos\theta + a\cos\left(\dfrac{b-a}{a}\theta\right)$

$y = (b-a)\sin\theta - a\sin\left(\dfrac{b-a}{a}\theta\right)$

## Chapter 6 Chapter Review

**1.** $\log_5 25 = x$ [7.2]

$5^x = 25$

$5^x = 5^2$

$x = 2$

**2.** $\log_3 81 = x$ [7.2]

$3^x = 81$

$3^x = 3^4$

$x = 4$

**3.** $\ln e^3 = x$ [7.2]

$e^x = e^3$

$x = 3$

**4.** $\ln e^{\pi} = x$ [7.2]

$e^x = e^{\pi}$

$x = \pi$

**5.** $3^{2x+7} = 27$ [7.4]

$3^{2x+7} = 3^3$

$2x + 7 = 3$

$2x = -4$

$x = -2$

**6.** $5^{x-4} = 625$ [7.4]

$5^{x-4} = 5^4$

$x - 4 = 4$

$x = 8$

**7.** $2^x = \dfrac{1}{8}$ [7.4]

$2^x = 2^{-3}$

$x = -3$

**8.** $27(3^x) = 3^{-1}$ [7.4]

$27(3^x) = \dfrac{1}{3}$

$3^x = \dfrac{1}{81}$

$3^x = 3^{-4}$

$x = -4$

**9.** $\log x^2 = 6$ [7.4]

$10^6 = x^2$

$1{,}000{,}000 = x^2$

$\pm\sqrt{1{,}000{,}000} = x$

$\pm 1000 = x$

**10.** $\dfrac{1}{2}\log|x| = 5$ [7.4]

$\log|x| = 10$

$10^{10} = |x|$

$x = \pm 10^{10}$

**11.** $10^{\log 2x} = 14$ [7.4]

$2x = 14$

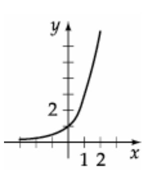
$x = 7$

**12.** $e^{\ln x^2} = 64$ [7.4]

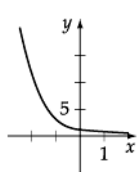
$x^2 = 64$

$x = \pm 8$

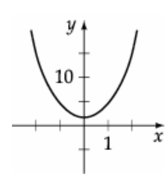
**13.**

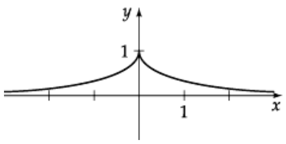


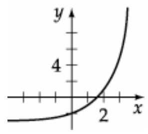
**14.**

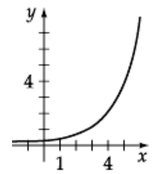


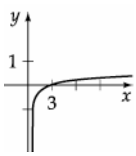
**15.**

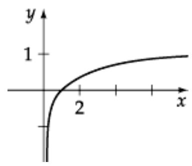


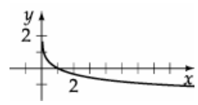
**16.** 

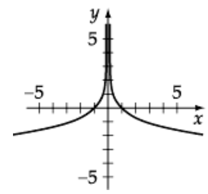

**17.** 

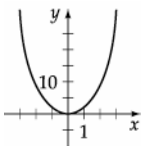

**18.** 

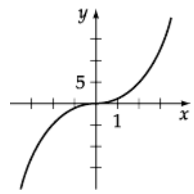

**19.** 


**20.** 


**21.** 


**22.** 


**23.** 


**24.** 


**25.** $\log_4 64 = 3$ [7.2]
$4^3 = 64$

**26.** $\log_{1/2} 8 = -3$ [7.2]
$\left(\frac{1}{2}\right)^{-3} = 8$

**27.** $\log_{\sqrt{2}} 4 = 4$ [7.2]
$\left(\sqrt{2}\right)^4 = 4$

**28.** $\ln 1 = 0$ [7.2]
$e^0 = 1$

**29.** $5^3 = 125$ [7.2]
$\log_5 125 = 3$

**30.** $2^{10} = 1024$ [7.2]
$\log_2 1024 = 10$

**31.** $10^0 = 1$ [7.2]
$\log_{10} 1 = 0$

**32.** $8^{1/2} = 2\sqrt{2}$ [7.2]
$\log_8 2\sqrt{2} = \frac{1}{2}$

**33.** $\log_b \frac{x^2 y^3}{z} = 2\log_b x + 3\log_b y - \log_b z$ [7.3]

**34.** $\log_b \frac{\sqrt{x}}{y^2 z} = \frac{1}{2}\log_b x - \left(2\log_b y + \log_b z\right)$ [7.3]
$= \frac{1}{2}\log_b x - 2\log_b y - \log_b z$

**35.** $\ln xy^3 = \ln x + 3\ln y$ [7.3]

**36.** $\ln \frac{\sqrt{xy}}{z^4} = \frac{1}{2}\left(\ln x + \ln y\right) - 4\ln z$ [7.3]
$= \frac{1}{2}\ln x + \frac{1}{2}\ln y - 4\ln z$

**37.** $2\log x + \frac{1}{3}\log(x+1) = \log\left(x^2 \sqrt[3]{x+1}\right)$ [7.3]

**38.** $5\log x - 2\log(x+5) = \log \frac{x^5}{(x+5)^2}$ [7.3]

**39.** $\frac{1}{2}\ln 2xy - 3\ln z = \ln \frac{\sqrt{2xy}}{z^3}$ [7.3]

**40.** $\ln x - (\ln y - \ln z) = \ln \frac{x}{y/z} = \ln \frac{xz}{y}$ [7.3]

**41.** $\log_5 101 = \frac{\log 101}{\log 5} \approx 2.86754$ [7.3]

**42.** $\log_3 40 = \frac{\log 40}{\log 3} \approx 3.35776$ [7.3]

**43.** $\log_4 0.85 = \frac{\log 0.85}{\log 4} \approx -0.117233$ [7.3]

**44.** $\log_8 0.3 = \frac{\log 0.3}{\log 8} \approx -0.578989$ [7.3]

**45.** $4^x = 30$ [7.4]

$$\log 4^x = \log 30$$
$$x \log 4 = \log 30$$
$$x = \frac{\log 30}{\log 4}$$

**46.** $5^{x+1} = 41$ [7.4]

$$(x+1)\log 5 = \log 41$$
$$x + 1 = \frac{\log 41}{\log 5}$$
$$x = \frac{\log 41}{\log 5} - 1$$

**47.** $\ln(3x) - \ln(x-1) = \ln 4$ [7.4]

$$\ln \frac{3x}{x-1} = \ln 4$$
$$\frac{3x}{x-1} = 4$$
$$3x = 4(x-1)$$
$$3x = 4x - 4$$
$$4 = x$$

**48.** $\ln(3x) + \ln 2 = \ln 1$ [7.4]

$$\ln(3x \cdot 2) = 1$$
$$\ln(6x) = 1$$
$$e^1 = 6x$$
$$\frac{e}{6} = x$$

**49.** $e^{\ln(x+2)} = 6$ [7.4]

$$(x+2) = 6$$
$$x + 2 = 6$$
$$x = 4$$

**50.** $10^{\log(2x+1)} = 31$ [7.4]

$$2x + 1 = 31$$
$$2x = 30$$
$$x = 15$$

**51.**

$$\frac{4^x + 4^{-x}}{4^x - 4^{-x}} = 2$$
$$4^x\left(4^x + 4^{-x}\right) = 2\left(4^x - 4^{-x}\right)4^x$$
$$4^{2x} + 1 = 2\left(4^{2x} - 1\right)$$
$$4^{2x} + 1 = 2\left(4^{2x} - 1\right)$$
$$4^{2x} - 2 \cdot 4^{2x} + 3 = 0$$
$$4^{2x} = 3$$
$$2^x \ln 4 = \ln 3$$
$$x = \frac{\ln 3}{2 \ln 4} \quad [7.4]$$

**52.**

$$\frac{5^x + 5^{-x}}{2} = 8$$
$$5^x\left(5^x + 5^{-x}\right) = 16\left(5^x\right)$$
$$5^{2x} + 1 = 16\left(5^x\right)$$
$$5^{2x} - 16\left(5^x\right) + 1 = 0$$

Let $5^x = u$

$$u^2 - 16u + 1 = 0$$
$$u = \frac{16 \pm \sqrt{16^2 - 4(1)(1)}}{2}$$
$$u = \frac{16 \pm \sqrt{252}}{2}$$
$$u = \frac{16 \pm 6\sqrt{7}}{2}$$
$$u = 8 \pm 3\sqrt{7}$$
$$5^x = 8 \pm 3\sqrt{7}$$
$$x = \frac{\ln\left(8 \pm 3\sqrt{7}\right)}{\ln 5} \quad [7.4]$$

**53.** $\log(\log x) = 3$ [7.4]

$$10^3 = \log x$$
$$10^{(10^3)} = x$$
$$10^{1000} = x$$

**54.** $\ln(\ln x) = 2$ [7.4]

$$e^2 = \ln x$$
$$e^{(e^2)} = x$$

**55.** $\log\sqrt{x-5} = 3$ [7.4]

$$10^3 = \sqrt{x-5}$$
$$10^6 = x - 5$$
$$10^6 + 5 = x$$
$$x = 1{,}000{,}005$$

**56.**
$$\log x + \log(x-15) = 1$$
$$\log x(x-15) = 1$$
$$10 = x^2 - 15x$$
$$0 = x^2 - 15x - 10$$
$$x = \frac{15 \pm \sqrt{15^2 - 4(1)(-10)}}{2}$$
$$x = \frac{15 \pm \sqrt{265}}{2}$$
$$x = \frac{15 + \sqrt{265}}{2} \quad [7.4]$$

**57.**
$$\log_4(\log_3 x) = 1$$
$$4 = \log_3 x$$
$$3^4 = x$$
$$81 = x \quad [7.4]$$

**58.**
$$\log_7(\log_5 x^2) = 0$$
$$7^0 = \log_5 x^2$$
$$1 = \log_5 x^2$$
$$5 = x^2$$
$$\pm\sqrt{5} = x \quad [7.4]$$

**59.**
$$\log_5 x^3 = \log_5 16x \quad [7.4]$$
$$x^3 = 16x$$
$$x^2 = 16$$
$$x = 4$$

**60.**
$$25 = 16^{\log_4 x} \quad [7.4]$$
$$25 = 4^{2\log_4 x}$$
$$25 = 4^{\log_4 x^2}$$
$$25 = x^2$$
$$\pm 5 = x$$
$$5 = x$$

**61.**
$$m = \log\left(\frac{I}{I_0}\right) \quad [7.3]$$
$$= \log\left(\frac{51,782,000 I_0}{I_0}\right)$$
$$= \log 51,782,000$$
$$\approx 7.7$$

**62.**
$$M = \log A + 3\log 8t - 2.92 \quad [7.3]$$
$$= \log 18 + 3\log 8(21) - 2.92$$
$$= \log 18 + 3\log 168 - 2.92$$
$$\approx 5.0$$

**63.**
$$\log\left(\frac{I_1}{I_0}\right) = 7.2 \quad \text{and} \quad \log\left(\frac{I_2}{I_0}\right) = 3.7 \quad [7.3]$$
$$\frac{I_1}{I_0} = 10^{7.2} \qquad \frac{I_2}{I_0} = 10^{3.7}$$
$$I_1 = 10^{7.2} I_0 \qquad I_2 = 10^{3.7} I_0$$
$$\frac{I_1}{I_2} = \frac{10^{7.2} I_0}{10^{3.7} I_0} = \frac{10^{3.5}}{1} \approx \frac{3162}{1}$$
3162 to 1

**64.**
$$\frac{I_1}{I_2} = 600 = 10^x \quad [7.3]$$
$$\log 600 = \log 10^x$$
$$\log 600 = x$$
$$2.8 \approx x$$

**65.**
$$\text{pH} = -\log\left[H_3O^+\right] \quad [7.3]$$
$$= -\log\left[6.28 \times 10^{-5}\right]$$
$$\approx 4.2$$

**66.**
$$5.4 = -\log\left[H_3O^+\right] \quad [7.3]$$
$$-5.4 = \log\left[H_3O^+\right]$$
$$10^{-5.4} = H_3O^+$$
$$0.00000398 \approx H_3O^+$$
$$H_3O^+ \approx 3.98 \times 10^{-6}$$

**67.** $P = 16{,}000, r = 0.08, t = 3$ [7.5]

**a.** $B = 16{,}000\left(1 + \frac{0.08}{12}\right)^{36} \approx \$20{,}323.79$

**b.** $B = 16{,}000e^{0.08(3)}$

$B = 16{,}000e^{0.24} \approx \$20{,}339.99$

**68.** $P = 19{,}000, r = 0.06, t = 5$ [7.5]

**a.** $B = 19{,}000\left(1 + \frac{0.06}{365}\right)^{1825} \approx \$25{,}646.69$

**b.** $B = 19{,}000e^{0.3} \approx \$25{,}647.32$

**69.** $S(n) = P(1-r)^n$, $P = 12{,}400, r = 0.29, t = 3$ [7.5]

$S(n) = 12{,}400(1-0.29)^3 \approx \$4438.10$

**70.** **a.**

$$N(t) = N_0 e^{-0.12t} \quad [7.5]$$
$$N(10) = N_0 e^{-0.12(10)}$$
$$\frac{N(10)}{N_0} = e^{-1.2}$$
$$= .301$$
$$\frac{N(10)}{N_0} = 30.1\% \text{ healed}$$
$$100\% - 30.1\% = 69.9\% \text{ healed}$$

**b.**

$$\frac{N(t)}{N_0} = 0.5$$
$$0.5 = e^{-0.12t}$$
$$\ln 0.5 = -0.12t$$
$$\frac{\ln 0.5}{-0.12} = t$$
$$t \approx 6 \text{ days}$$

**c.**

$$\frac{N(t)}{N_0} = 0.1$$
$$0.1 = e^{-0.12t}$$
$$\ln 0.1 = -0.12t$$
$$\frac{\ln 0.1}{-0.12} = t$$
$$t \approx 19 \text{ days}$$

**71.** $N(0) = 1$

$$1 = N_0 e^{k(0)}$$
$$1 = N_0$$

$N(2) = 5$

$$5 = e^{2k}$$
$$\ln 5 = 2k$$
$$k = \frac{\ln 5}{2} \approx 0.8047$$

Thus $N(t) = e^{0.8047t}$ [7.5]

**72.** $N(0) = N_0 = 2$ and $N(3) = N_0 e^{3k} = 2e^{3k} = 11$

$$e^{3k} = \frac{11}{2}$$
$$e^{3k} = \frac{11}{2}$$
$$3k = \ln\left(\frac{11}{2}\right)$$
$$k = \frac{1}{3}\ln\left(\frac{11}{2}\right)$$
$$\approx 0.5682$$

Thus $N(t) = 2e^{0.5682t}$ [7.5]

**73.** $4 = N(1) = N_0 e^k$ and thus $\dfrac{4}{N_0} = e^k$. Now, we also have $N(5) = 5 = N_0 e^{5k} = N_0\left(\dfrac{4}{N_0}\right)^5 = \dfrac{1024}{N_0^4}$.

$$N_0 = \sqrt[4]{\frac{1024}{5}} \approx 3.783$$

Thus $4 = 3.783e^k$.

$$\ln\left(\frac{4}{3.783}\right) = k$$
$$k \approx 0.0558$$

Thus $N_0 = 3.783e^{0.0558t}$. [7.5]

**74.** $1 = N(0) = N_0$ and $2 = N(-1) = N_0 e^{-k}$.

Since $N_0 = 1$, we have $2 = 1 \cdot e^{-k}$.

$$\ln 2 = -k$$
$$k \approx -0.6931$$

Thus $N(t) = e^{-0.6931t}$. [7.5]

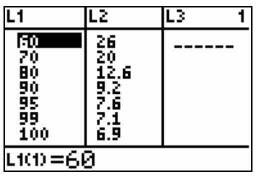
**75.** **a.**
$$N(1) = 25{,}200e^{k(1)} = 26{,}800 \quad [7.5]$$
$$e^k = \frac{26{,}800}{25{,}200}$$
$$\ln e^k = \ln\left(\frac{26{,}800}{25{,}200}\right)$$
$$k \approx 0.061557893$$
$$N(t) = 25{,}200e^{0.061557893\,t}$$

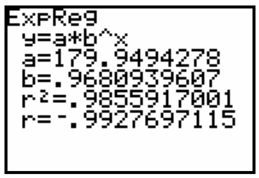
**b.**
$$N(7) = 25{,}200e^{0.061557893(7)}$$
$$= 25{,}200e^{0.430905251}$$
$$\approx 38{,}800$$

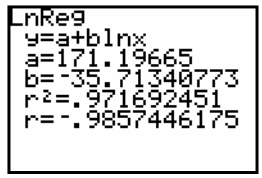
**76.**
$$P(t) = 0.5^{\,t/5730} = 0.96 \quad [7.5]$$
$$\log\left(0.5^{\,t/5730}\right) = \log 0.96$$
$$\frac{t}{5730}\log 0.5 = \log 0.96$$
$$\frac{t}{5730} = \frac{\log 0.96}{\log 0.5}$$
$$t = 5730\left(\frac{\log 0.96}{\log 0.5}\right) \approx 340 \text{ years}$$

**77.** Answers will vary.

**78.** **a.**




exponential: $R \approx 179.949\left(0.968094^t\right)$, $r \approx -0.99277$

logarithmic: $R \approx 171.19665 - 35.71341\ln t$, $r \approx -0.98574$

**b.** The exponential equation provides a better fit to the data.

**c.** $R \approx 179.949\left(0.968094^{108}\right) \approx 5.4$ per 1000 live births [7.6]

**79.** **a.**
$$P(t) = \frac{mP_0}{P_0 + (m - P_0)e^{-kt}}$$
$$P(3) = 360 = \frac{1400(210)}{210 + (1400 - 210)e^{-k(3)}}$$
$$360 = \frac{294000}{210 + 1190e^{-3k}}$$
$$360\left(210 + 1190e^{-3k}\right) = 294000$$
$$210 + 1190e^{-3k} = \frac{294000}{360}$$
$$1190e^{-3k} = \frac{29400}{36} - 210$$
$$e^{-3k} = \frac{29400/36 - 210}{1190}$$
$$\ln e^{-3k} = \ln\left(\frac{29400/36 - 210}{1190}\right)$$
$$-3k = \ln\left(\frac{29400/36 - 210}{1190}\right)$$
$$k = -\frac{1}{3}\ln\left(\frac{29400/36 - 210}{1190}\right)$$
$$k \approx 0.2245763649$$
$$P(t) = \frac{294000}{210 + 1190e^{-0.22458t}} = \frac{1400}{1 + \frac{17}{3}e^{-0.22458t}}$$

**b.**
$$P(13) = \frac{294000}{210 + 1190e^{-0.22458(13)}}$$
$$= \frac{294000}{210 + 1190e^{-2.919492744}}$$
$$\approx 1070 \text{ coyotes} \quad [7.5]$$

**80.** **a.**
$$P(0) = \frac{128}{1 + 5e^{-0.27(0)}} = \frac{128}{1 + 5e^0} = \frac{128}{1+5} = \frac{128}{6} = 21\frac{1}{3}$$

**b.** As $t \to \infty$, $e^{-0.27t} \to 0$.
$$P(t) \to \frac{128}{1 + 5(0)} = \frac{128}{1} = 128 \quad [7.5]$$

## Chapter 6 Chapter Test

**1.** **a.** $\log_b(5x-3)=c$ [7.2]

$b^c = 5x-3$

**b.** $3^{x/2} = y$

$\log_3 y = \frac{x}{2}$

**2.** $\log_b \frac{z^2}{y^3\sqrt{x}} = \log_b z^2 - \log_b y^3 - \log_b x^{1/2}$ [7.3]

$= 2\log_b z - 3\log_b y - \frac{1}{2}\log_b x$

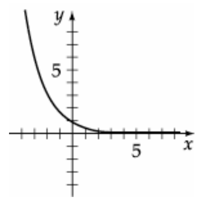
**3.** $\log_{10}(2x+3) - 3\log_{10}(x-2) = \log_{10}(2x+3) - \log_{10}(x-2)^3$

$= \log_{10}\frac{2x+3}{(x-2)^3}$ [7.3]

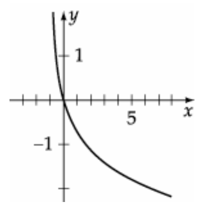
**4.** $\log_4 12 = \frac{\log 12}{\log 4}$ [7.3]

$\approx 1.7925$

**5.**




**6.**




**7.** $5^x = 22$ [7.4]

$x\log 5 = \log 22$

$x = \frac{\log 22}{\log 5}$

$x \approx 1.9206$

**8.** $4^{5-x} = 7^x$ [7.4]

$\ln 4^{5-x} = \ln 7^x$

$(5-x)\ln 4 = x\ln 7$

$5\ln 4 - x\ln 4 = x\ln 7$

$5\ln 4 = x\ln 7 + x\ln 4$

$5\ln 4 = x(\ln 7 + \ln 4)$

$\frac{5\ln 4}{\ln 28} = x$

**9.** $\log(x+99) - \log(3x-2) = 2$ [7.4]

$\log\frac{x+99}{3x-2} = 2$

$\frac{x+99}{3x-2} = 10^2$

$x+99 = 100(3x-2)$

$x+99 = 300x-200$

$-299x = -299$

$x = 1$

**10.** $\ln(2-x) + \ln(5-x) = \ln(37-x)$

$\ln(2-x)(5-x) = \ln(37-x)$

$(2-x)(5-x) = (37-x)$

$10 - 7x + x^2 = 37 - x$

$x^2 - 6x - 27 = 0$

$(x-9)(x+3) = 0$

$x = 9$ (not in domain) or $x = -3$

$x = -3$ [7.4]

**11.** **a.** $A = P\left(1+\frac{r}{n}\right)^{nt}$ [7.5]

$= 20{,}000\left(1+\frac{0.078}{12}\right)^{12(5)}$

$= 20{,}000(1.0065)^{60}$

$= \$29{,}502.36$

**b.** $A = Pe^{rt}$

$= 20{,}000e^{0.078(5)}$

$= 20{,}000e^{0.39}$

$= \$29{,}539.62$

**12.** $A = P\left(1+\frac{r}{n}\right)^{nt}$ [7.5]

$2P = P\left(1+\frac{0.04}{12}\right)^{12t}$

$2 = \left(1+\frac{0.04}{12}\right)^{12t}$

$\ln 2 = \ln\left(1+\frac{0.04}{12}\right)^{12t}$

$\ln 2 = 12t\ln\left(1+\frac{0.04}{12}\right)$

$12t = \frac{\ln 2}{\ln\left(1+\frac{0.04}{12}\right)}$

$t = \frac{1}{12}\cdot\frac{\ln 2}{\ln\left(1+\frac{0.04}{12}\right)}$

$t \approx 17.36$ years

**13.** **a.**
$$M = \log\left(\frac{I}{I_0}\right) \qquad [7.3]$$
$$= \log\left(\frac{42{,}304{,}000 I_0}{I_0}\right)$$
$$= \log 42{,}304{,}000$$
$$\approx 7.6$$

**b.**
$$\log\left(\frac{I_1}{I_0}\right) = 6.3 \qquad \text{and} \qquad \log\left(\frac{I_2}{I_0}\right) = 4.5$$
$$\frac{I_1}{I_0} = 10^{6.3} \qquad \frac{I_2}{I_0} = 10^{4.5}$$
$$I_1 = 10^{6.3} I_0 \qquad I_2 = 10^{4.5} I_0$$
$$\frac{I_1}{I_2} = \frac{10^{6.3} I_0}{10^{4.5} I_0} = \frac{10^{1.8}}{1} \approx \frac{63}{1}$$

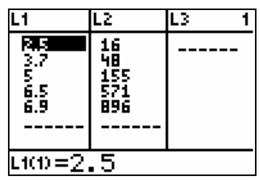
Therefore the ratio is 63 to 1.

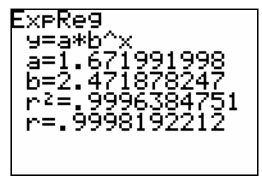
**14.** **a.**
$$N(3) = 34600e^{k(3)} = 39800$$
$$34600e^{3k} = 39800$$
$$e^{3k} = \frac{39800}{34600}$$
$$\ln e^{3k} = \ln\left(\frac{398}{346}\right)$$
$$3k = \ln\left(\frac{398}{346}\right)$$
$$k = \frac{1}{3}\ln\left(\frac{398}{346}\right)$$
$$k \approx 0.0466710767$$
$$N(t) = 34600e^{0.0466710767\, t} \quad [7.5]$$

**b.**
$$N(10) = 34600e^{0.0466710767(10)}$$
$$= 34600e^{0.466710767}$$
$$\approx 55{,}000$$

**15.**
$$P(t) = 0.5^{\,t/5730} = 0.92 \quad [7.5]$$
$$\log 0.5^{\,t/5730} = \log 0.92$$
$$\frac{t}{5730}\log 0.5 = \log 0.92$$
$$\frac{t}{5730} = \frac{\log 0.92}{\log 0.5}$$
$$t = 5730\left(\frac{\log 0.92}{\log 0.5}\right)$$
$$t \approx 690 \text{ years}$$

**16.**

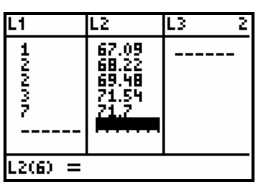


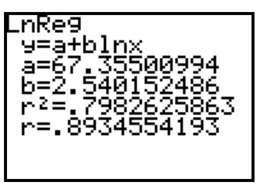



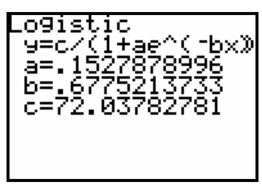

**a.** $y = 1.671991998(2.471878247)^{x}$

**b.** $y = 1.671991998(2.471878247)^{7.8}$ [7.6]
$\approx 1945$

**17.**




**a.** Logarithmic: $d \approx 67.35501 + 2.54015\ln t$ ; Logistic: $d \approx \dfrac{72.03783}{1 + 0.15279e^{-0.67752\, t}}$

**b.** Logarithmic: $d \approx 67.35501 + 2.54015\ln(12) \approx 73.67$ m ;

Logistic: $d \approx \dfrac{72.03783}{1 + 0.15279e^{-0.67752\,(12)}} \approx 72.03$ m [7.6]

**18.** **a.** $a = \dfrac{c - P_0}{P_0} = \dfrac{1100 - 160}{160} = 5.875$

$$P(t) = \frac{c}{1 + ae^{-bt}}$$
$$P(1) = \frac{1100}{1 + 5.875e^{-b(1)}}$$
$$190 = \frac{1100}{1 + 5.875e^{-b}}$$
$$190(1 + 5.875e^{-b}) = 1100$$
$$190 + 1116.25e^{-b} = 1100$$
$$1116.25e^{-b} = 910$$
$$e^{-b} = \frac{910}{1116.25}$$
$$\ln e^{-b} = \ln\frac{910}{1116.25}$$
$$-b = \ln\frac{910}{1116.25}$$
$$b = -\ln\frac{910}{1116.25}$$
$$b \approx 0.20429$$
$$P(t) = \frac{1100}{1 + 5.875e^{-0.20429t}} \quad [7.5]$$

**b.** $P(t) = \dfrac{1100}{1 + 5.875e^{-0.20429(7)}} \approx 457$ raccoons

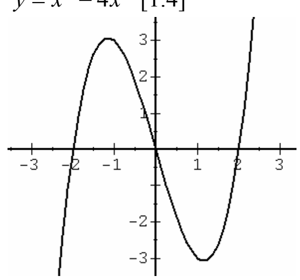
## Chapter 6 Cumulative Review

**1.** $x^2 + 4x - 6 = 0$ [1.1]

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-6)}}{2(1)} = \frac{-4 \pm \sqrt{40}}{2}$$
$$= \frac{-4 \pm 2\sqrt{10}}{2} = -2 \pm \sqrt{10}$$

The solutions are 2 $\pm\sqrt{10}$.

**2.** $y = x^3 - 4x$ [1.4]




symmetric with respect to the origin

**3.** $(g \circ f)(x) = g[f(x)]$ [1.5]
$$= g[\sin x]$$
$$= 3\sin x - 2$$

**4.** $240^\circ = 240^\circ\left(\dfrac{\pi}{180^\circ}\right) = \dfrac{4\pi}{3}$ [2.1]

**5.** $v = \omega r$ [2.1]
$$= \frac{3 \cdot 2\pi \cdot 60 \cdot 60 \cdot 10}{12 \cdot 5280}$$
$$\approx 11 \text{ mph}$$

**6.** $\sin t = -\dfrac{\sqrt{3}}{2} = \dfrac{\text{opp}}{\text{hyp}}$ [2.4]
$$\text{adj} = \sqrt{2^2 - \left(\sqrt{-3}\right)^2} = \sqrt{1} = 1$$
$$\tan t = \frac{\text{opp}}{\text{adj}} = \frac{-\sqrt{3}}{1} = -\sqrt{3}$$

**7.** $\tan 43^\circ = \dfrac{a}{20}$ [2.4]

$a = 20\tan 43^\circ$

$a = 19$ cm

**8.** $y = \dfrac{1}{2}\cos\left(\dfrac{\pi x}{3}\right)$ [2.5]

period: 6, amplitude: $\dfrac{1}{2}$

**9.** $y = 2\tan\left(\dfrac{\pi x}{3}\right)$ [2.6]

period: 3

**10.** [3.1]

$$\begin{aligned}\frac{\sin x}{1-\cos x} &= \frac{\sin x}{1-\cos x}\cdot\frac{1+\cos x}{1+\cos x}\\ &= \frac{\sin x(1+\cos x)}{1-\cos^2 x}\\ &= \frac{\sin x(1+\cos x)}{\sin^2 x}\\ &= \frac{1}{\sin x}+\frac{\cos x}{\sin x}\\ &= \csc x + \cot x\end{aligned}$$

**11.** $\sin\alpha = \dfrac{3}{5},\ \cos\alpha = -\dfrac{4}{5}$ [3.2]

$\cos\beta = -\dfrac{5}{13},\ \sin\beta = \dfrac{12}{13}$

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\\ &= \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right)+\left(-\frac{4}{5}\right)\left(\frac{12}{13}\right)\\ &= -\frac{15}{65}-\frac{48}{65} = -\frac{63}{65}\end{aligned}$$

**12.** $y = \sin\left[\cos^{-1}\left(\dfrac{1}{5}\right)\right]$ [3.5]

Let $\alpha = \cos^{-1}\dfrac{1}{5},\ \cos\alpha = \dfrac{1}{5} = \dfrac{\text{adj}}{\text{hyp}}$

$\sin\alpha = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{\sqrt{5^2-(1)^2}}{5} = \dfrac{2\sqrt{6}}{5}$

$y = \sin\left[\cos^{-1}\left(\dfrac{1}{5}\right)\right] = \dfrac{2\sqrt{6}}{5}$

**13.** $\cos^{-1} x = \sin^{-1}\left(\dfrac{12}{13}\right)$ [3.6]

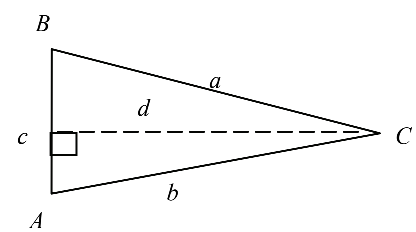
$x = \cos\left[\sin^{-1}\left(\dfrac{12}{13}\right)\right]$

Let $\alpha = \sin^{-1}\dfrac{12}{13},\ \sin\alpha = \dfrac{12}{13} = \dfrac{\text{opp}}{\text{hyp}}$

$\cos\alpha = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{\sqrt{13^2-12^2}}{13} = \dfrac{5}{13}$

$x = \cos\left[\sin^{-1}\left(\dfrac{12}{13}\right)\right] = \dfrac{5}{13}$

**14.**




$$\begin{aligned}C &= 180^\circ - A - B\\ &= 180^\circ - 71^\circ - 80^\circ\\ &= 29^\circ\end{aligned}$$

$\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$b = \dfrac{155\sin 80^\circ}{\sin 29^\circ} \approx 314.9$ ft

$\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$

$a = \dfrac{155\sin 71^\circ}{\sin 29^\circ} \approx 302.3$ ft

$\sin 71^\circ = \dfrac{d}{314.9}$

$d = 314.9\sin 71^\circ$

$\approx 298$ ft [4.1]

**15.** $b^2 = a^2 + c^2 - 2ac\cos B$ [4.2]

$\cos B = \dfrac{a^2+c^2-b^2}{2ac} = \dfrac{4^2+3.6^2-2.5^2}{2(4)(3.6)} = \dfrac{22.7}{28.8}$

$B \approx 38^\circ$

**16.** $a_1 = 30\cos 145^\circ \approx -24.6$ [4.3]

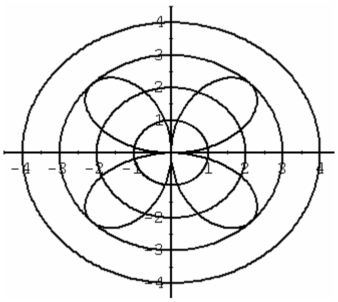
$a_2 = 30\sin 145^\circ \approx 17.2$

$\mathbf{v} = -24.6\mathbf{i} + 17.2\mathbf{j}$

**17.** $\mathbf{v}\cdot\mathbf{w} = (3\mathbf{i}+2\mathbf{j})\cdot(5\mathbf{i}-7\mathbf{j})$ [4.3]
$= 3(5)+(2)(-7)$
$= 1 \neq 0$
No. the vectors are not orthogonal.

**18.** $z = -2 + 2i\sqrt{3}$ [5.2]
$r = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{16} = 4$
$\alpha = \tan^{-1}\left|\frac{2\sqrt{3}}{-2}\right| = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$
$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$
$z = 4 \text{ cis}\left(\frac{2\pi}{3}\right)$

**19.** $r = 3\sin(2\theta)$ [6.5]



**20.** $x = 2t - 1$ [6.7]
$x + 1 = 2t$
$t = \frac{x+1}{2}$
$y = 4t^2 + 1$
$y = 4\left(\frac{x+1}{2}\right)^2 + 1 = 4\left(\frac{x^2+2x+1}{4}\right) + 1$
$y = x^2 + 2x + 2$

## Chapter 7 Cumulative Review

**1.** $(f \circ g)(x) = f[g(x)]$ [1.5]
$= f(x^2+1)$
$= \cos(x^2+1)$

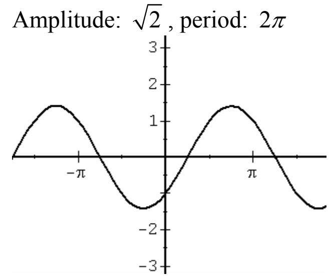
**2.** $f(x) = 2x + 8$ [1.6]
$y = 2x + 8$
$x = 2y + 8$
$x - 8 = 2y$
$\frac{x-8}{2} = y$
$f^{-1}(x) = \frac{1}{2}x - 4$

**3.** $c = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$ [2.2]
$\sin\theta = \frac{3}{5},\ \cos\theta = \frac{4}{5},\ \tan\theta = \frac{3}{4}$

**4.** $\cos 26^\circ = \frac{15}{a}$ [2.2]
$a = \frac{15}{\cos 26^\circ} \approx 16.7$ cm

**5.** $y = 3\sin\left(\frac{1}{3}x - \frac{\pi}{2}\right)$ [2.5]
$0 \le 2x - \frac{\pi}{2} \le 2\pi$
$\frac{\pi}{2} \le 2x \le \frac{5\pi}{2}$
$\frac{\pi}{4} \le x \le \frac{5\pi}{4}$
amplitude = 4, period $= \pi$,
phase shift $= \frac{\pi}{4}$

**6.** $y = \sin x + \cos x$ [3.4]
Amplitude: $\sqrt{2}$, period: $2\pi$



**7.** $f(-x) = \sin(-x) = -\sin x = -f(x)$
odd function [2.4]

**8.** $\frac{1}{\sin x} - \sin x = \frac{1-\sin^2 x}{\sin x}$ [3.1]

$= \frac{\cos^2 x}{\sin x}$

$= \cos x \frac{\cos x}{\sin x}$

$= \cos x \cot x$

**9.** $\tan\left(\sin^{-1}\left(\frac{12}{13}\right)\right)$ [3.5]

Let $\alpha = \sin^{-1}\frac{12}{13}$, $\sin\alpha = \frac{12}{13} = \frac{\text{opp}}{\text{hyp}}$

$\tan\alpha = \frac{\text{opp}}{\text{adj}} = \frac{12}{\sqrt{13^2 - 12^2}} = \frac{12}{5}$

$\tan\left[\sin^{-1}\left(\frac{12}{13}\right)\right] = \frac{12}{5}$

**10.** $2\cos^2 x + \sin x - 1 = 0$ [3.6]

$2(1-\sin^2 x) + \sin x - 1 = 0$

$2\sin^2 x - \sin x - 1 = 0$

$(2\sin x + 1)(\sin x - 1) = 0$

$2\sin x + 1 = 0$ $\quad$ $\sin x - 1 = 0$

$\sin x = -\frac{1}{2}$ $\quad$ $\sin x = 1$

$x = \frac{7\pi}{6}, \frac{11\pi}{6}$ $\quad$ $x = \frac{\pi}{2}$

**11.** $\|\mathbf{v}\| = \sqrt{(-3)^2 + 4^2}$

$\|\mathbf{v}\| = \sqrt{9+16}$

$\|\mathbf{v}\| = 5$

$\alpha = \tan^{-1}\left|\frac{4}{-3}\right| = \tan^{-1}\frac{4}{3}$ [4.3]

$\alpha \approx 53.1°$

$\theta = 180° - \alpha$

$\theta \approx 180° - 53.1°$

$\theta \approx 126.9°$

**12.** $\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|}$ [4.3]

$\cos\theta = \frac{(2\mathbf{i} - 3\mathbf{j})\cdot(-3\mathbf{i} + 4\mathbf{j})}{\sqrt{2^2 + (-3)^2}\sqrt{(-3)^2 + 4^2}}$

$\cos\theta = \frac{2(-3) + (-3)(4)}{\sqrt{13}\sqrt{25}}$

$\cos\theta = \frac{-18}{\sqrt{325}} = -0.9985$

$\theta = 176.8^\circ$

**13** $\mathbf{AB} = 400(\cos 42\mathbf{i} + \sin 42\mathbf{j}) \approx 297.3\mathbf{i} + 267.7\mathbf{j}$ [4.3]

$\mathbf{AD} = 55[\cos(-25°)\mathbf{i} + \sin(-25°)\mathbf{j}] \approx 49.8\mathbf{i} - 23.2\mathbf{j}$

$\mathbf{AC} = \mathbf{AB} + \mathbf{AD}$

$\mathbf{AC} = 297.3\mathbf{i} + 267.7\mathbf{j} + 49.8\mathbf{i} - 23.2\mathbf{j}$

$\mathbf{AC} \approx 347.1\mathbf{i} + 244.5\mathbf{j}$

$\|\mathbf{AC}\| = \sqrt{347.1^2 + (244.5)^2}$

$\|\mathbf{AC}\| \approx 425$ mph

$\alpha = 90^\circ - \theta = 90^\circ - \tan^{-1}\left(\frac{244.5}{347.1}\right) \approx 55^\circ$

**14.** $\frac{a}{\sin A} = \frac{b}{\sin B}$ [4.1]

$\sin A = \frac{a\sin B}{b} = \frac{42\sin 32^\circ}{50} \approx 0.445$

$A \approx 26^\circ$

**15.** $z = 1 - i$ [5.3]

$r = \sqrt{1^2 + (-1)^2}$

$r = \sqrt{2}$

$\alpha = \tan^{-1}\left|\frac{-1}{1}\right| = 45^\circ$

$\theta = 315^\circ$

$z = \sqrt{2}(\cos 315° + i\sin 315°)$

$(1-i)^8 = [\sqrt{2}(\cos 315° + i\sin 315°)]^8$

$= 16[\cos(8\cdot 315°) + i\sin(8\cdot 315°)]$

$= 16(\cos 2520° + i\sin 2520°)$

$= 16(\cos 0° + i\sin 0°)$

$= 16 + 0i = 16$

**16.** $i = 1(\cos 90^\circ + i\sin 90^\circ)$ [5.3]

$w_k = 1^{1/2}\left(\cos\frac{90° + 360°k}{2} + i\sin\frac{90° + 360°k}{2}\right)$ $\quad k = 0, 1$

$w_0 = \cos\frac{90°}{2} + i\sin\frac{90°}{2}$

$w_0 = \cos 45° + i\sin 45°$

$w_0 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

$w_1 = \cos\frac{90° + 360°}{2} + i\sin\frac{90° + 360°}{2}$

$w_1 = \cos 225° + i\sin 225°$

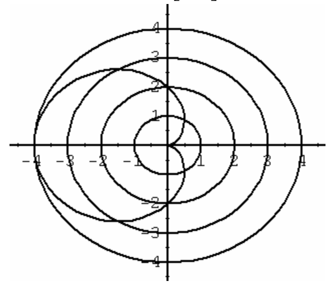
$w_1 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

**17.** $r = \sqrt{x^2 + y^2}$
$= \sqrt{1^2 + 1^2}$
$= \sqrt{2}$

$\theta = \tan^{-1}\frac{y}{x}$ [6.5]
$= \tan^{-1}\left(\frac{1}{1}\right)$
$= 45^\circ$

The polar coordinates of the point are $(\sqrt{2},\ 45^\circ)$.

**18.** $r = 2 - 2\cos\theta$ [6.5]




**19.** $5^x = 10$ [7.4]
$\log 5^x = \log 10$
$x \log 5 = 1$
$x = \frac{1}{\log 5} \approx 1.43$

**20.** $N(t) = N_0 e^{kt}$
$N(138) = N_0 e^{138k}$
$0.5N_0 = N_0 e^{138k}$
$0.5 = e^{138k}$
$\ln 0.5 = \ln e^{138k}$
$\ln 0.5 = 138k \ln e$
$\ln 0.5 = 138k$
$\frac{\ln 0.5}{138} = k$
$-0.005023 \approx k$
$N(t) = N0(0.5)^{t/138} \approx N_0 e^{-0.005023t}$
$N(100) = 3(0.5)^{100/138} \approx 1.8$ mg [7.5]