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TEXTS AND READINGS IN MATHEMATICS 9

A Course on Topological Groups

K. Chandrasekharan

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A COURSE ON TOPOLOGICAL GROUPS

By K. Chandrasekharan

AUTHOR'S NOTE

This course has for its aim a proof of the Peter-Weyl theorem (1927), that every complex-valued continuous function on a compact topological group is a uniform limit of finite linear combinations of representation functions coming from irreducible representations. The method of proof adopted here is the one expounded by Warren Ambrose in his MIT lectures (1952). It incorporates the ideas originally introduced in this context by John von Neumann, and André Weil, and makes use of the L_2-algebra of the group relative to Haar measure. The topological, analytical, and algebraic groundwork needed for the proof is provided as part of the course. Acknowledgements are due to the following:

F. Peter and H. Weyl, Math. Annalen, 97(1927), 737-755; No. 73 of Weyl's Gesammelte Abhandlungen, Bd. III

A. Haar, Annals of Math. 34 (1933), 147-169

J. von Neumann, Compositio Math. 1(1934), 106-114; No.22 of his Collected Works, Vol. II

J. von Neumann, Trans. American Math. Soc. 36(1934), 445-492; No. 23 of his Collected Works, Vol. II

L.S. Pontrjagin, Comptes Rendus, Paris, 198 (1934), 238-240

L.S. Pontrjagin, Annals of Math. 35 (1934), 361-388

L.S. Pontrjagin, Topological groups, Princeton (1939)

M.H. Stone, Linear transformations in Hilbert space and their applications in analysis, AMS Colloq. Publ. XV (1932)

F.D. Murnaghan, The theory of group representations, Baltimore (1938)

A. Weil, L'integration dans les Groupes topologiques et ses Applications, Paris (1940)

H. Cartan, Comptes Rendus, 211 (1940), 759-762; No. 69 of his Collected Works, Vol. III

C. Chevalley, Lie Groups I, Princeton (1946)

W. Ambrose, Lectures on topological groups (unpublished) Ann Arbor (1946)

W. Ambrose, Trans. American Math. Soc. 65 (1949), 27-48

W. Ambrose, Lectures on topological groups (mimeographed lectures) MIT (Boston), pp. 216, 11 Chrs. (1952)

D. Montgomery and L. Zippin, Topological transformation groups, New York (1955)

It is Ambrose's approach that is the prime influence in this presentation, which was offered as an optional course at the ETH, Zürich, more than once, during the years 1965-88.

Zürich, January 1993

I. Topological Preliminaries

I.1 Topological spaces

A *topology* T in a set X is a class of subsets of X (called *open sets*) satisfying the following axioms:

(1) the union of any number of open sets is open;

(2) the intersection of any two (or finite number of) open sets is open;

(3) X and the empty set ($\emptyset$) are open.

A *topological space* is a set X with a topology T in X. T is called the *trivial topology* if $T = \{X, \emptyset\}$, *the discrete topology* if $T = \{A \mid A \subset X\}$.

Let X be a topological space, and $A \subset X$. We define an *induced topology* in A as follows: the class of open sets in A is the class of sets of the form $U \cap A$, where U runs through *all* open sets in X.

In a topological space X, a *closed set* is any set whose complement is an open set.

The *closure* $\overline{E}$ of any set E is the intersection of all closed sets containing E. By axioms (1) and (3) above, the closure is a closed set.

A *neighbourhood* of $x \in X$ is any open set containing x. Let X_1, X_2 be two topological spaces, and f a *mapping*

$$X_1 \xrightarrow{f} X_2.$$

[A *mapping* is an assignment to each $x \in X_1$ of an element $f(x) \in X_2$.]. Then f is *continuous* if and only if for every open set O_2 in X_2, the set $f^{-1}(O_2)$ is an open set in X_1. [Here $f^{-1}(O_2) = \{x \mid x \in X_1,\ f(x) \in O_2\}$.

Let $x \in X_1$. We say that f is *continuous at* x, if to every neighbourhood U of $f(x)$ there exists a neighbourhood V of x, such that $f(V) \subset U$. We say that f is *continuous*, if f is continuous at every point of X_1. This definition is equivalent to the preceding one.

The *mapping* f is said to be *open*, if for every open set O_1 in X_1, the set $f(O_1)$ is an open set in X_2.

Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two continuous mappings. Then the *composite mapping* $g \circ f : X \longrightarrow Z$ is continuous.

Examples

(i) If X is a discrete topological space, Y is a topological space, then *every* mapping $f : X \longrightarrow Y$ is continuous.

(ii) If X is any topological space, and Y a set with the trivial topology, $f : X \longrightarrow Y$ is continuous.

(iii) $I_X : X \longrightarrow X$ is the *identity mapping.*

The *topological product* of two topological spaces X_1, X_2 is the topological space $X = X_1 \times X_2$, whose set is the cartesian product of X_1 and X_2, namely $\{(x_1, x_2) \mid x_1 \in X_1, x_2 \in X_2\}$, with the *open sets* being *all unions* of sets of the form $O_1 \times O_2$, where O_1 is an open set in X_1, and O_2 an open set in X_2. [Sets of the form $O_1 \times O_2$ form a *basis of open sets* in $X_1 \times X_2$.]

Note that if $p_1 : X_1 \times X_2 \longrightarrow X_1$, and $p_2 : X_1 \times X_2 \longrightarrow X_2$ are the *projections* defined by $p_1(x_1, x_2) = x_1$, and $p_2(x_1, x_2) = x_2$, respectively, then p_1 and p_2 are *continuous* mappings. $(x_1 \in X_1, x_2 \in X_2)$.

Covering. A family $\{V_\alpha\}_{\alpha \in I}$ of subsets of a set X is a *covering* of X, if $\bigcup_{\alpha \in I} V_\alpha = X$. That is to say, each point of X belongs to at least one V_α. If further, X is a topological space, and each V_α is an open subset of X, we say that $\{V_\alpha\}$ is an *open covering* of X.

A covering $\{V_\alpha\}_{\alpha \in I}$ is a *finite covering* if I is a finite set.

Compact sets. A *topological space X is compact,* if every open covering of X contains a finite covering. [That is to say, $X = \bigcup_{\alpha \in I} U_\alpha$, U_α open, implies that there exist $\alpha_2, \alpha_2, \ldots, \alpha_n \in I$ such that $\bigcup_{1 \leq i \leq n} U_{\alpha_i} = X$]. A *subset* $A \subset X$ is compact, if A is compact in the induced topology.

Remarks

(i) $\mathbb{R}$ is not compact. ($\mathbb{R}$ = real numbers)

(ii) Let X be a topological space, $A \subset X$. If A consists of finitely many elements of X, then A is compact.

(iii) Let $a, b \in \mathbb{R}$, $a \leq b$. Then the set $\{x \in \mathbb{R} \mid a \leq x \leq b\}$ is compact.

An *equivalent definition* of compactness is the following: if $\{F_\alpha\}$ is a family of closed sets such that every finite subfamily has a non-empty intersection, then $\bigcap_\alpha F_\alpha \neq \emptyset$. [If $\{F_\alpha\}$ is a family of closed sets with the *finite intersection property*, then the intersection of the whole class is non-empty].

A *closed subset* of a compact set is compact. A *continuous image* of a compact set is compact. [If X_1 is compact, X_2 Hausdorff (see later), and $f : X_1 \longrightarrow X_2$ is continuous, then $f(X_1)$ is closed in X_2].

A product of compact spaces is compact (Tychonoff).

A space X is *locally compact*, if for every $x \in X$, there is a neighbourhood O_1 of x, such that the closure of O_1 is compact.

Every compact space is locally compact, but not vice versa.

Homeomorphisms Let X and Y be topological spaces, and f a mapping, $f : X \longrightarrow Y$. We say that f is a *homeomorphism* if f is one-to-one (i.e. $f(x) = f(y) \Longrightarrow x = y$), onto (i.e. $(f(X) = Y)$, and both f and f^{-1} are continuous.

Two topological spaces X, Y are said to be *homeomorphic*, if there exists a homeomorphism f from X to Y.

Examples

(i) $X = Y = \mathbb{R}$, $f(x) = x^3$, $-x$ for $x \in X$.

(ii) (a) $X = \mathbb{R}$, $Y = \{x \in \mathbb{R} \mid -1 < x < 1\}$. X and Y are homeomorphic under the mapping: $f(x) = \dfrac{x}{1 + |x|}$.

(b) $\mathbb{R}^n$ and the open unit-ball $B = \{x \in \mathbb{R}^n \mid ||x|| < 1\}$.

(iii) If $X = \mathbb{R}$, and $Y = \{x \in \mathbb{R} \mid -1 < x \leq 1\}$, then X and Y are *not* homeomorphic. Note that Y is compact, while $\mathbb{R}$ is not.

(iv) If a mapping is one-to-one, onto, *and* continuous, it does *not* follow that it is a homeomorphism. Let X = a set with more than one element, T = the discrete topology in X, and T' = the trivial topology in X. Then the identity mapping $I_X : (X, T) \longrightarrow (X, T')$ is continuous, but not a homeomorphism.

Separation axioms

Two sets M_1 and M_2 can be ***separated by open sets***, if there exist open sets O_1 and O_2, such that $M_1 \subset O_1$, $M_2 \subset O_2$, and $O_1 \cap O_2 = \emptyset$.

M_1 and M_2 can be ***separated by a real function***, if there exists a continuous real function f on X such that $0 \leq f(x) \leq 1$ for $x \in X$, and $f(x) = 0$ for $x \in M_1$, and $f(x) = 1$ for $x \in M_2$.

T_1: for any two *disjoint* points, there exists a neighbourhood of either point not containing the other. [This implies that the complement of each point is an open set, or that each point is a closed set.]

N.B. ***In this course we assume that all topological spaces*** (are T_1-spaces) ***satisfy the axiom*** T_1.

T_2: Any two distinct points can be separated by open sets. (or, any two distinct points have disjoint neighbourhoods) A ***topological space is Hausdorff*** it it satisfies T_2.

Examples

(i) A set X with the discrete topology.

(ii) If X is a set with more than one element, then X with the trivial topology is not Hausdorff.

(iii) If X is Hausdorff, and $A \subset X$, then A is Hausdorff with the induced topology.

(iv) Let X_1 be compact, X_2 Hausdorff, and $f : X_1 \longrightarrow X_2$ be continuous. Then $f(X_1)$ is closed in X_2.

(v) Let X be Hausdorff, $A \subset X$, with A compact. Then A is closed.

T_3: A closed set F and a point $x \notin F$ can be separated by open sets.

A ***topological space is regular*** if it satisfies T_1 and T_3. A ***completely regular*** topological space is one in which a closed set F and a point $x \notin F$ can be separated by a real function.

T_4: Two disjoint closed sets can be separated by open sets. A topological space is ***normal*** if it satisfies T_1 and T_4.

Urysohn's lemma (Lefschetz, Algebraic Topology, p. 27) In a normal space, every two disjoint closed sets can be separated by a real function.

I.2 Topological groups

A *topological group* is a set G which is both a group and a T_1-space, with the topology and group structure related by the assumption that the functions $(x, y) \longmapsto x \cdot y$ and $x \longmapsto x^{-1}$ are continuous. Here $x, y \in G$, and x^{-1} is the group inverse of x.

Equivalently, the function $(x, y) \longmapsto xy^{-1}$ (from $G \times G$ to G) is continuous.

Examples

(i) The additive group of real numbers is the underlying group of a topological group whose underlying topological space is the usual space of real numbers. More generally, the Euclidean n-space under addition with the usual topology.

(ii) Any group with the discrete topology (every set is its own closure).

(iii) The n-dimensional torus (product of n circles). Here a circle is the topological group $\{z \in \mathbb{C} \mid |z| = 1\}$ under multiplication.

(iv) $GL(n, \mathbb{C})$: *the general linear group*, i.e. the group of all non-singular $n \times n$ matrices with complex coefficients. For the topology use that "induced" by considering the $n \times n$ matrices as a subset of $\mathbb{C}^{n^2}$.

(v) Any product of topological groups.

Trivial properties Let e be the identity in a topological group G. If $E, F \subset G$, then $EF \underset{def}{=} \{xy \mid x \in E, y \in F\}$.

(1) If $z = xy$, O a neighbourhood of z, there exist neighbourhoods P of x and Q of y such that $PQ \subset O$.

Let ϕ denote the mapping $(x, y) \longmapsto xy$, and $\widetilde{O} = \phi^{-1}(O)$. Since ϕ is continuous, and O open, it follows that $\widetilde{O}$ is open. Now $\widetilde{O}$ contains (x, y), hence it contains a set of the form $O_1 \times O_2$, where O_1 is a neighbourhood of x, and O_2 a neighbourhood of y, and $\phi(O_1 \times O_2) = O_1 \cdot O_2 \subset O$. Take $P = O_1$ and $Q = O_2$.

(1′) If $a \in G$, then for every neighbourhood V of a^{-1} there exists a neighbourhood U of a, such that $U^{-1} \subset V$. This follows again from the fact that $a \longmapsto a^{-1}$ is continuous. [If g is a continuous mapping of a topological space R into R', then for every point $a \in R$ and every neighbourhood U' of $a' = g(a) \in R'$ there exists a neighbourhood U of a such that $g(U) \subset U'$]

(2) For each $x \in G$, the mappings $y \longmapsto yx$ and $y \longmapsto xy$ are homeomorphisms (or *topological mappings*)

The mapping $f : y \longmapsto xy$ is one-to-one ($xy = x'y$ implies that $x = x'$). The inverse mapping $f^{-1} : y \longmapsto x^{-1}y$ is of the same form. Hence it suffices to prove that f is continuous. Let O be any neighbourhood of xy. By (1) there exists a neighbourhood O_1 of x, and a neighbourhood O_2 of y, such that $O_1O_2 \subset O$. Hence $xO_2 \subset O$, that is to say $f(O_2) \subset O$. This shows that f is continuous (cf. (1′)).

(3) The mapping $x \longmapsto x^{-1}$ is a homeomorphism. For $x \longmapsto x^{-1}$ is one-to-one, and is its own inverse. It is continuous by definition.

(4) If O is open in G, then O^{-1}, xO, EO, Ox, OE are also open, where $x \in G$, $E \subset G$.

By (2) xO and Ox are open, so is O^{-1} by (3). Now $EO = \bigcup_{x \in E} xO$, hence open; similarly also OE.

(5) If V is any neighbourhood of e, it contains a neighbourhood W of e, such that $W \cdot W^{-1} \subset V$.

Since $ee^{-1} = e$, there exist neighbourhoods V_1 of e, and V_2 of $e^{-1} = e$, such that $V_1V_2 \subset V$ by (1).

Let $V_3 = V_1 \cap V_2^{-1}$. Then V_3 is open, and $e \in V_3$, so that V_3 is a neighbourhood of e.

Next $V_3V_3^{-1} \subset V$. For $x \in V_3$ implies $x \in V_1$, $x \in V_2^{-1}$, and $y^{-1} \in V_3^{-1}$ implies that $y \in V_3$, which in turn implies that $y^{-1} \in V_2$. Hence $xy^{-1} \in V_3V_3^{-1}$ implies that $xy^{-1} \in V_1V_2 \subset V$. Choose $W = V_3$.

(6) A neighbourhood V of e is defined to be *symmetric* if and only if $V = V^{-1}$.

Every neighbourhood W of e contains a symmetric neighbourhood (for example, $W \cap W^{-1}$). By (4) W^{-1} is open. Hence $W \cap W^{-1}$ is a symmetric neighbourhood of e.

(7) Every neighbourhood of $x \in G$ is of the form xV as well as of the form Wx, where V, W are neighbourhoods of the identity.

Let U be any neighbourhood of x. Then $V = x^{-1}U \ni e$, and $x^{-1}U$ is open, by (4). Hence V is a neighbourhood of e. Similarly $W = Ux^{-1}$ is a neighbourhood of e. Hence $U = xV = Wx$.

(8) The continuity of $(x,y) \longmapsto xy^{-1}$ is equivalent to the continuity of $(x,y) \longmapsto xy$ together with the continuity of $x \longmapsto x^{-1}$.

(i) If $(x,y) \longmapsto xy$ and $x \longmapsto x^{-1}$ are continuous, then $(x,y^{-1}) \longmapsto xy^{-1}$ *and* $(x,y) \longmapsto (x,y^{-1})$ are continuous, hence the composite $(x,y) \longmapsto xy^{-1}$ is continuous.

(ii) Conversely, if $(x,y) \longmapsto xy^{-1}$ is continuous, then $(e,y) \longmapsto ey^{-1} = y^{-1}$ is continuous. Further $y \longmapsto (e,y)$ is continuous (obviously). Hence $y \longmapsto y^{-1}$ is continuous. Therefore $(x,y) \longmapsto (x,y^{-1})$ is continuous. But, by hypothesis, $(x,y) \longmapsto xy^{-1}$ is continuous. Hence $(x,y) \longmapsto xy$ is continuous.

Separation properties

Lemma 1 The topological space of a topological group G is Hausdorff.

Proof Let $x, y \in G$, $x \neq y$. Then $y^{-1}x \neq e$. By the T_1-property, we can find a neighbourhood V of e not containing $y^{-1}x$. By (5) there exists a neighbourhood V_1 of e, such that $V_1V_1^{-1} \subset V$. Then xV_1, yV_1 are neighbourhoods of x and of y respectively. And $xV_1 \cap yV_1 = \emptyset$. For if $xV_1 \cap yV_1 \neq \emptyset$ then there exist $v', v'' \in V_1$, such that $xv' = yv''$, so that $y^{-1}x = v''v'^{-1} \in V_1V_1^{-1} \subset V$, contradicting the choice of V. Hence $xV_1 \cap yV_1 = \emptyset$.

Lemma 2 If $E \subset G$, then $\overline{E} = \cap EV = \cap VE$, where V extends over all neighbourhoods of e.

(i) $\overline{E} \supset \cap EV$.

If $x \in EV$ for all V, we shall see that *every* neighbourhood of x intersects E, hence $x \in \overline{E}$.

Let $x \in \cap EV$, and let O be any neighbourhood of x. Then by (7) above, $O = xV$, where V is a neighbourhood of e. By hypothesis, $x \in EV^{-1}$, which implies that $x = ay^{-1}$, where $a \in E$, $y \in V$, or $xy = a$. Hence xV intersects E (i.e. has a non-empty intersection with E). Therefore O intersects E, hence $x \in \overline{E}$.

(ii) $\overline{E} \subset \cap EV$.

If $x \in \overline{E}$, then every neighbourhood of x intersects E. By (7), xV^{-1} is a neighbourhood of x (where V is a neighbourhood of e). Hence xV^{-1} intersects E. This implies that $x \in EV$ (for $\exists\, y \in V$, such that $xy^{-1} \in E$, or $x \in Ey \subset EV$). Hence $x \in \cap EV$.

Remark A topological group is ***homogeneous***. Given any two elements $p, q \in G$, there exists a ***topological mapping*** f of G onto itself, which takes p into q.

Take $a = p^{-1}q$, and take $f(x) = xa$. This implies that $f(p) = q$.

It is sufficient for many purposes therefore to verify *local* properties for a single element only. For example, to show that G is locally compact, it is sufficient to show that its identity e has a neighbourhood U whose closure is compact — so also with regularity.

Lemma 3 The topological space of a topological group G is ***regular*** (Kolmogorov).

Proof We can separate e, and any closed set $F \not\ni e$. Let $O = F^c$. Then O is a neighbourhood of e. Now there exists a neighbourhood V of e, such that $V^2 \subset O$ (because of (5) and (6)). V and $\overline{V}^c$ are disjoint open sets. We shall see that $F \subset \overline{V}^c$.

Since V is a neighbourhood of e, that will prove the lemma. Now

$$\begin{aligned}
\overline{V} &= \cap VW, \quad \text{where } W \text{ is a neighbourhood of } e \text{ (by Lemma 2)},\\
&\subset V^2, \quad \text{(since } V \text{ is a } W\text{, a neighbourhood of } e\text{)},\\
&\subset O, \quad \text{(by choice of } V \text{ and } O\text{)},\\
&= F^c, \quad \text{(by definition of } O\text{)}.
\end{aligned}$$

Hence $\overline{V} \subset F^c$, or $\overline{V}^c \supset F$.

Theorem 1 A topological space which is the underlying space of a topological group G is completely regular.

Proof It is sufficient to show that if F is a *closed* set not containing e, then F and e can be separated by a continuous real function (cf. the remark above).

Let $V = F^c$. Then V is a neighbourhood of e. Choose a sequence $V_1, V_2, \ldots$, of neighbourhoods of e, such that $V_1^2 \subset V$, $V_{k+1}^2 \subset V_k$, $k \geq 1$. [This is possible, see the proof of Lemma 3, in which (5) and (6) were used].

Let α be a finite dyadic real number, with

$$\alpha = 0.\ \alpha_1\alpha_2\cdots\alpha_k000, \qquad \alpha_i = 0, 1.$$

Define

$$O_\alpha = V_1^{\alpha_1} \cdot V_2^{\alpha_2} \cdots V_k^{\alpha_k} \text{ (group product), where } V_i^0 = e.$$

We will show that

$$\alpha < \beta \text{ implies that } \overline{O}_\alpha \subset O_\beta.$$

If $\alpha < \beta$, their first j digits agree for some $j \geq 0$, so that

$$\alpha = 0 \cdot \alpha_1\alpha_2\cdots\alpha_j\ 0\ \alpha_{j+2}\cdots\alpha_k 0\ 0\ 0;$$

$$\beta = 0 \cdot \beta_1\beta_2\cdots\beta_j\ 1\ \beta_{j+2}\cdots\beta_m 0\ 0\ 0,$$

and $\alpha_\ell = \beta_\ell$ for $\ell \leq j$.

Now define

$$\alpha' = 0 \cdot \alpha_1\cdots\alpha_j\ 0\ 1\ 1\ \cdots\ \underset{\alpha_k}{1}\ \ 0\ 0\ 0\cdots$$

$$\beta' = 0 \cdot \alpha_1\cdots\alpha_j\ 1\ 0\ 0\ 0\cdots.$$

Then

$$\alpha \leq \alpha' < \beta' \leq \beta.$$

Clearly $$O_\alpha \subset O_{\alpha'} \subset O_{\beta'} \subset O_\beta, \qquad \text{(since } V_i^0 = e\text{)}.$$

We shall see that

$$\overline{O}_{\alpha'} \subset O_{\beta'}, \text{ (which will imply that } \overline{O}_\alpha \subset O_\beta\text{)}.$$

Let

$$O = V_1^{\alpha_1} \cdots V_j^{\alpha_j}.$$

Then

$$O_{\alpha'} = OV_{j+2} \cdots V_k, \quad \text{and} \quad O_{\beta'} = OV_{j+1}.$$

Hence, by Lemma 2,

$$\begin{aligned} \overline{O}_{\alpha'} \subset O_{\alpha'} V_k &= OV_{j+2} \cdots V_{k-1} V_k^2 \\ &\subset OV_{j+2} \cdots V_{k-2} V_{k-1}^2 \quad (\text{since } V_k^2 \subset V_{k-1}) \\ &\subset OV_{j+2}^2 \subset OV_{j+1} = O_{\beta'} \quad (\text{by definition}). \end{aligned}$$

Hence

$$\overline{O}_{\alpha'} \subset O_{\beta'}, \text{ and therefore } \overline{O}_\alpha \subset O_\beta.$$

Now define

$$f(x) = \begin{cases} 1, & \text{if } x \notin \textit{any } O_\alpha, \\ \inf\{\alpha \mid x \in O_\alpha\}, & \text{if } x \in \text{some } O_\alpha. \end{cases}$$

Then we have $0 \leq f(x) \leq 1$, for all $x \in G$. Further

$$f(x) = \begin{cases} 0, & \text{for } x = e \text{ (since } V_i^0 = e), \\ 1, & \text{for } x \in F, \ (V = F^c, \text{ see above}). \end{cases}$$

We shall see that f is *continuous*. The sets $\{x \mid f(x) > k\}$ and $\{x \mid f(x) < k\}$, $k \in \mathbb{R}$, are *open*. For

$$\begin{aligned} \{x \mid f(x) > k\} &= \left(\bigcap_{\alpha > k} \overline{O}_\alpha\right)^c \Longleftrightarrow \{x \mid f(x) \leq k\} \\ &= \left(\bigcap_{\alpha > k} \overline{O}_\alpha\right), \quad (\text{a closed set}). \end{aligned}$$

Note that $\alpha < \beta \Longrightarrow O_\alpha \subset O_\beta$, hence

$$\{x \mid f(x) \leq k\} \subset \{x \mid x \in \bigcap_{\alpha > k} O_\alpha\};$$

for if x is such that $f(x) \leq k$, and $\beta > k$ such that $x \notin O_\beta$, then $x \notin O_\alpha$ for all $\alpha < \beta$ (since $O_\alpha \subset O_\beta$), hence $f(x) \geq \beta > k$, a contradiction. The opposite inclusion is trivial. Thus

$$\{x \mid f(x) \leq k\} = \left\{x \mid x \in \bigcap_{\alpha > k} O_\alpha\right\}.$$

But $\bigcap_{\alpha>k} \overline{O}_\alpha = \bigcap_{\alpha>k} O_\alpha$, since on the one hand, $O_\alpha \subset \overline{O}_\alpha$, and on the other, $\alpha < \beta \Longrightarrow \overline{O}_\alpha \subset O_\beta$, so that $x \in \bigcap_{\alpha>k} \overline{O}_\alpha$ implies that $x \in \bigcap_{\alpha>k} O_\alpha$, ($\alpha$ is dyadic rational), hence $\{x \mid f(x) > k\}$ is open.

Similarly $\{x \mid f(x) < k\}$ is also open. For if the set contains the point x, it contains a whole neighbourhood of it; for if x is such that $f(x) < k$, then there exists $\alpha < k$, such that $x \in O_\alpha$, which is a neighbourhood of x all of which is contained in $\{x \mid f(x) < k\}$. $[y \in O_\alpha \Longrightarrow f(y) \leq \alpha < k]$.

Lemma 4 If C_1, C_2 are compact subsets of G, then C_1C_2 is compact.

Consider the mapping $G \times G \longrightarrow G$ which takes (x, y) into xy. This is continuous. The product $C_1 \times C_2$ is compact (Tychonoff). The proof follows from the fact that the continuous image of a compact set is compact.

Remark If F_1, F_2 are closed, it does *not* follow that $F_1 \cdot F_2$ is closed. In $\mathbb{R}^1$ let $F_1 = \{n \in \mathbb{Z} \mid n \geq 1\}$, and $F_2 = \{0\} \cup \left\{\frac{1}{n} \mid n \in F_1\right\}$. Then $F_1 \cdot F_2 = Q_{\geq 0} \subset \mathbb{R}^1$. $[Q_{\geq 0} =$ rational numbers $\geq 0]$.

Theorem 2 A locally compact group is normal.

If the group is compact, the proof is easy, since compactness together with regularity (or Hausdorff) implies normality.

Otherwise take a symmetric neighbourhood U of e with compact closure, and consider $G' = \bigcup_{n=1}^{\infty} U^n$. Then G' is an open and closed subgroup of G, and it is sufficient to prove normality for G'. To do that, use the fact that G' is σ-compact, i.e. $G' = \bigcup_{n=1}^{\infty} K_n$, where K_n is compact. ($K_n = \bigcup_{\ell=1}^{n} \overline{U}^\ell$).

[A locally compact (T_1) group is *para-compact*, hence normal] Ref. e.g. Hewitt & Ross: Abstract Harmonic Analysis, I, p. 76, Th. (8.13).

I.3 Subgroups, Quotient groups

Let G be a *topological group*, and H a *subset* of G. Then H is, by definition, a *subgroup of the topological group* G if and only if H is a subgroup of the abstract group G, *and* H is a *closed set* in the topological space G.

Let G be a topological group, and H a subset of it which is a subgroup of G considered as an abstract group. Then H *is* also a topological group with the *induced topology*. In particular, a *subgroup of an abstract group which is a topological group is itself a topological group.*

A subgroup N of the topological group G is defined to be a *normal subgroup* if N is a normal subgroup of the abstract group G.

Let G be a topological group, and let H be a subgroup of the *abstract* group G. Then $\overline{H}$ is a subgroup of the topological group G. If H is a normal subgroup of the abstract group G, then $\overline{H}$ is also normal, i.e. a normal subgroup of the topological group G.

Let us recall that if G is any group, and H a subgroup of G, a *left coset of H* is a subset of G of the form $xH, x \in G$. The *left coset set* is the set of all left cosets of H, denoted by G/H. We have a *natural map* or *projection* $\pi : G \longrightarrow G/H$ $(x \longmapsto xH)$ defined by $\pi(x) =$ the left coset of H which contains x.

If G is a topological group, we shall topologize G/H, *assuming* that H is *closed*. A set $O \subset G/H$ is *open*, if and only if $\pi^{-1}(O)$ is open in G. This means that we *require* π to be a *continuous* map.

Lemma 5 G/H is a T_1-space, and π is open.

Proof Since H is closed, xH is also closed (homeomorphism), so that $(xH)^c$ is open in G. Write $\tilde{x} = xH$, (coset containing x). Now $\pi^{-1}(G/H - \tilde{x}) = (xH)^c$, which is open. Therefore G/H is T_1 (the complement of each point is an open set). We know that π is continuous; we have to show that it is also open. Let O be open, $O \subset G$. Then πO is in the coset space; it is open if and only if $\pi^{-1}(\pi O)$ is open. But $\pi^{-1}(\pi O) = OH$, which is open since $OH = \bigcup_{x \in H} Ox$, where Ox is open. Thus, O open $\Longrightarrow \pi O$ is open.

Lemma 6 G/H is a T_2-space.

Proof Let x_1 and y_1 be two *distinct* points of $G/H = Q$, and $x, y \in G$ such that $\pi(x) = x_1$, $\pi(y) = y_1$. Choose a neighbourhood v of e, such that $Vx \cap yH = \emptyset$. This is possible, because $x \notin yH = \overline{yH}$ (H is closed, so yH is closed. Vx is a neighbourhood of x, use the definition of $\overline{yH}$). It follows that $VxH \cap yH = \emptyset$. For if $VxH \cap yH \neq \emptyset$, then $vxh_1 = yh_2$, say, where $v \in V$, $h_1, h_2 \in H$. Hence $vx = yh_2h_1^{-1} = yh_3$, which contradicts $Vx \cap yH := \emptyset$.

Let V_1 be a neighbourhood of e such that $V_1^{-1}V_1 \subset V$. Then

$V_1^{-1}V_1xH \cap gH = \emptyset$, hence $V_1xH \cap V_1yH = \emptyset$. Therefore

$$\pi(V_1x) \cap \pi(V_1y) = \emptyset.$$

Now $x_1 \in \pi(V_1x)$, since $e \in V_1$; and $y_1 \in \pi(V_1y)$; and $\pi(V_1x)$, $\pi(V_1y)$ are ***open*** (by Lemma 1, V_1x and V_1y are open, and by Lemma 5 π is open). Hence x_1, y_1 are separated by disjoint open sets.

Remark If G is *any* (topological) group, H a *closed* subgroup, we have topologized the coset set G/H in such a way that it is a T_2-space. We may call G/H the *quotient space*, and the given topology the *quotient space topology*. On the other hand, if G is *any* group, and H a *normal* subgroup (i.e. $\forall\ x \in H,\ \forall\ a \in G,\ a\ x\ a^{-1} \in H$ or $a\ H\ a^{-1} \subset H$), then the coset set G/H is, in fact, a group, known as the *quotient group*. The next lemma shows that the quotient group, with the quotient space topology, is a *topological group*, if, to start with, G is a topological group.

Lemma 7 If G is a topological group, and H a normal subgroup, then the quotient group G/H with the quotient space topology is a topological group.

Proof We have only to show that the mapping $\psi_1 : G/H \times G/H \longrightarrow G/H$ given by $\psi_1(x_1, y_1) = x_1y_1^{-1}$, where $x_1, y_1 \in G/H$, is continuous. Let $\psi : G \times G \longrightarrow G$ be given by $(x, y) \overset{\psi}{\longmapsto} xy^{-1}$, $x, y \in G$. We then have

$$\begin{array}{ccc} G \times G & \xrightarrow{\pi \times \pi} & G/H \times G/H \\ \downarrow \psi & & \downarrow \psi_1 \\ G & \xrightarrow{\pi} & G/H. \end{array}$$

Trivially we have $\pi\psi = \psi_1(\pi \times \pi)$. Since π and ψ are continuous, $\pi\psi$ is *continuous*. Hence $\psi_1(\pi \times \pi)$ is continuous.

Let O_1 be an open set in G/H. Then

$$[\psi_1(\pi \times \pi)]^{-1}(O_1)$$

is open in $G \times G$. But $\pi \times \pi$ is open. Hence

$$(\pi \times \pi)(\pi \times \pi)^{-1}\psi_1^{-1}(O_1) \text{ is open in } G/H \times G/H;$$

that is to say, $\psi_1^{-1}(O_1)$ is open, hence ψ_1 is continuous.

Lemma 8 Let G be a topological group, and H a subgroup.

(a) If G is compact, then H and G/H are both compact.

(b) If G is locally compact, then both H and G/H are locally compact.

Proof

(a) H is a closed subset of a compact set, hence compact. G/H is the continuous image of a compact set, hence compact.

(b) H is locally compact since it is a closed subset of a locally compact space. [Let S be a locally compact space, $T \subset S$, $T = \overline{T}$. Then T is locally compact. For, $p \in T \Longrightarrow p \in S \Longrightarrow \exists\, U_p \subset S$ so that $p \in U_p$ and $\overline{U}_p$ is compact. Now $T \cap \overline{U}_p$ is a compact neighbourhood of p].

To prove that G/H is locally compact, let $q \in G/H$, and U_1 a neighbourhood of q (i.e. an open set containing q). Let

$$U = \pi^{-1}(U_1),$$

and let $x \in U$, so that $\pi(x) = q$. Since G is locally compact, and U is open, there exists a neighbourhood O of x, such that $\overline{O} \subset U$, with $\overline{O}$ compact. [Let G be *any* topological group. To every neighbourhood U of e, there exists a neighbourhood V of e, such that $\overline{V} \subset U$. For let V be a *symmetric* neighbourhood of e, such that $V^2 \subset U$. (See (5) and (6) above.) Now $x \in \overline{V} \Longrightarrow (xV) \cap V \neq \emptyset$. Hence $xv_1 = v_2$, where $v_1, v_2 \in V$. Therefore $x = v_2 v_1^{-1} \in V \cdot V^{-1} \subset V^2 \subset U$. Hence $\overline{V} \subset U$.] Then we have

$$\pi(\overline{O}) \subset \pi(U) = U_1.$$

Now $O_1 = \pi(O)$ a neighbourhood of q (O is a neighbourhood of x). And $\pi(\overline{O})$ is compact (since $\overline{O}$ is compact). Since G/H is a T_2-space, $\pi(\overline{O})$ is *closed*.

We have $\quad O_1 \subset \pi(\overline{O}), \quad$ (see above: $O_1 = \pi(O)$)
which implies that

$$\overline{O}_1 \subset \overline{\pi(\overline{O})} = \pi(\overline{O}), \quad \text{(compact)}$$

hence $\overline{O}_1$ is compact (closed subset of a compact set). Thus O_1 is a neighbourhood of q with compact closure. It follows that G/H is locally compact.

I.4 Examples

Let $\mathbb{C}^n$ denote the n-dimensional complex cartesian space. It is a vector space of dimension n over the field $\mathbb{C}$ of complex numbers.

Let $\bar{e}_i = (0,0,0,\dots \underset{i}{1}, 0,0,0)$. Then $\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n$ form a base of $\mathbb{C}^n$ over $\mathbb{C}$.

An *endomorphism* α of $\mathbb{C}^n$ is defined when the elements $\alpha\bar{e}_i = \sum_{j=1}^n a_{ji}\bar{e}_j$ are given. To α corresponds the matrix (a_{ij}) of degree n, *and conversely.*

We *use the same letter* α *for the matrix as well as for the endomorphism.*

We define a multiplication $\alpha \circ \beta$ of two endos. α, β with matrices (a_{ij}), (b_{ij}) respectively, by defining the corresponding matrix (c_{ij}) as the product of the matrices, namely

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}. \tag{4.1}$$

Now let $M_n(\mathbb{C})$ denote the set of all matrices of degree n with coefficients in $\mathbb{C}$. If $(a_{ij}) \in M_n(\mathbb{C})$, put $b_{i+(j-1)n} = a_{ij}$. [As j goes from 1 to n, $j-1$ goes from 0 to $n-1$, and $(j-1)n$ from 0 to $n^2 - n$ in steps of n; while i goes from 1 to n; so that $i + (j-1)n$ goes from 1 to n^2].

To (a_{ij}) we associate the point with the coordinates $b_1, b_2, \dots, b_{n^2}$ in $\mathbb{C}^{n^2}$. In this way we get a one-to-one correspondence between $M_n(\mathbb{C})$ and $\mathbb{C}^{n^2}$. Since $\mathbb{C}^{n^2}$ is a topological space, we can define a topology in $M_n(\mathbb{C})$ by *requiring* the correspondence to be a homeomorphism.

Let T be *any* topological space, and let ϕ map T into $M_n(\mathbb{C})$; $\phi : T \longrightarrow M_n(\mathbb{C})$. If $t \in T$, $\phi(t)$ is a matrix with coefficients $a_{ij}(t)$, say. Clearly ϕ is continuous if and only if *each* function $a_{ij}(t)$ is continuous.

$$[\underbrace{T \xrightarrow{\phi} M_n(\mathbb{C}) \xleftrightarrow{\psi} \mathbb{C}^n \xrightarrow{\pi_{ij}} \mathbb{C}}_{\phi_{ij}},\ \phi_{ij}(t) = a_{ij}(t)].$$

[In general, the following situation holds: If $T \xrightarrow{f} \prod_{a\in A} X_a \xrightarrow{\pi_a} X_a$, then "$f$ continuous $\Longleftrightarrow \pi_a \circ f$ continuous for *every* $a \in A$". On the one hand, it is trivial that "f continuous $\Longrightarrow \pi_a \circ f$ continuous". On the other, if $\pi_a \circ f$ is continuous, and U open, with $U \subset X_a$, then $(\pi_a \circ f)^{-1}U$ is open, i.e. $f^{-1}(\pi_a^{-1}(U))$ is open. But $\pi_a^{-1}(U)$ is open, since sets of the form $\pi_a^{-1}(U)$, U open, form a sub-basis[1] of open sets of $\prod_{a\in A} X_a$. So f is continuous].

It follows from this remark, and (4.1), that the product $\sigma\tau$ of two matrices σ and τ is a continuous function of the pair (σ, τ) considered as a point of the space $M_n(\mathbb{C}) \times M_n(\mathbb{C})$.

Notation We denote by ${}^t\alpha$ the *transpose* of the matrix $\alpha = (a_{ij})$; ${}^t\alpha = (a'_{ij})$, $a'_{ij} = a_{ji}$. We denote by $\overline{\alpha}$ the *complex conjugate* of α; $\overline{\alpha} = (\overline{a}_{ij})$.

Clearly $\alpha \longmapsto {}^t\alpha$, and $\alpha \longmapsto \overline{\alpha}$ are homeomorphisms, of order 2, of $M_n(\mathbb{C})$ onto itself.

If α, β are any two matrices, then ${}^t(\alpha\beta) = {}^t\beta \cdot {}^t\alpha$, and $\overline{\alpha\beta} = \overline{\alpha} \cdot \overline{\beta}$.

An $n \times n$ matrix σ is *regular* (or *non-singular)*, if it has an inverse, i.e. if there exists a matrix σ^{-1}, such that $\sigma\sigma^{-1} = \sigma^{-1}\sigma = \varepsilon$, where ε is the *unit matrix* of degree n.

A necessary and sufficient condition for σ to be regular is that its determinant $\det\sigma \neq 0$.

If an endomorphism σ of $\mathbb{C}^n$ maps $\mathbb{C}^n$ onto itself, (and not onto some subspace of lower dimension) the corresponding matrix σ is regular, and σ has a reciprocal endomorphism σ^{-1}.

If σ is a *regular* matrix, we have

$${}^t(\sigma^{-1}) = ({}^t\sigma)^{-1}, \qquad (\overline{\sigma})^{-1} = \overline{(\sigma^{-1})}.$$

If σ and τ are regular matrices, $\sigma\tau$ is also regular, and we have

$$(\sigma\tau)^{-1} = \tau^{-1}\sigma^{-1}.$$

Hence *the regular matrices of degree n form a group with respect to multiplication*, which is called *the general linear group* $GL(n, \mathbb{C})$.

Since the determinant of a matrix is obviously a continuous function of the matrix, $GL(n, \mathbb{C})$ *is an open subset of* $M_n(\mathbb{C})$. [$GL(n, \mathbb{C}) = \{\sigma \mid \det\sigma \neq 0\}$; det is a continuous function.] The elements of $GL(n, \mathbb{C})$ may be considered as points of a topological space which is a subspace of the topological space $M_n(\mathbb{C})$.

If $\sigma = (a_{ij})$ is a regular matrix, the coefficients b_{ij} of σ^{-1} are given by $b_{ij} = A_{ij}(\det\sigma)^{-1}$, where the A_{ij} are polynomials in the coefficients

[1] [subbasis: finite intersections thereof form a basis]

f σ. Hence the mapping

$$\sigma \longmapsto \sigma^{-1}$$

of $GL(n,\mathbb{C})$ onto itself is continuous. Since the mapping coincides with its reciprocal mapping, it is a homeomorphism of $GL(n,\mathbb{C})$ with itself.

The mappings $\sigma \longrightarrow \overline{\sigma}$, and $\sigma \longmapsto {}^t\sigma$ are also homeomorphisms of $GL(n,\mathbb{C})$ with itself. The first is an automorphism of the group, not the second (preserves sums and inverts the order of the products).

$$\text{If } \sigma \in GL(n,\mathbb{C}), \text{ define } \sigma^* = ({}^t\sigma)^{-1}.$$

Then we have: $(\sigma\tau)^* = \sigma^*\tau^*$, $(\sigma^*)^{-1} = (\sigma^{-1})^*$. Hence $\sigma \longmapsto \sigma^*$ is a homeomorphism, and an automorphism of order 2 of $GL(n,\mathbb{C})$.

The subgroups $O(n)$, $O(n,\mathbb{C})$, $U(n)$ of $GL(n,\mathbb{C})$

Let $\sigma \in GL(n,\mathbb{C})$. We say that σ is *orthogonal* if $\sigma = \overline{\sigma} = \sigma^*$. The set of all orthogonal matrices of degree n we denote by $O(n)$. If only $\sigma = \sigma^*$, σ is called *complex orthogonal*, and the set of all such σ we denote by $O(n,\mathbb{C})$. If $\overline{\sigma} = \sigma^*$, σ is called *unitary*, and we denote by $U(n)$ the set of all such σ.

Since $\sigma \longmapsto \overline{\sigma}$ and $\sigma \longmapsto \sigma^*$ are continuous, the sets $O(n)$, $O(n,\mathbb{C})$, $U(n)$ are *closed* subsets of $GL(n,\mathbb{C})$. [Note that $\{x \mid f(x) = c\}$ is closed if f is real and continuous. $\phi_{ij}(\sigma) = \overline{a}_{ij} - a^*_{ij}$ is continuous for all i, j, and $\{\sigma \mid \phi_{ij}(\sigma) = 0\}$ is closed.] Because these mappings are automorphisms, $O(n), O(n,\mathbb{C})$, $U(n)$ are *subgroups* of $GL(n,\mathbb{C})$.

[Note, in parenthesis, that if X is any topological space, and Y a Hausdorff space, and f, g are continuous mappings of X into Y, then the set $E = \{x \mid x \in X,\ f(x) = g(x)\}$ is *closed*. One can see that $F = \{x \mid x \in X,\ f(x) \neq g(x)\}$ is *open* in X. Let $x_0 \in F$. Since $f(x_0) \neq g(x_0)$, and Y is Hausdorff, there exist neighbourhoods U_1, U_2 of $f(x_0)$, $g(x_0)$ respectively, so that $U_1 \cap U_2 = \emptyset$. Since f and g are continuous, there exist neighbourhoods V_1, V_2 of x_0 in X, such that $f(V_1) \subset U_1$, $g(V_2) \subset U_2$. Let $V = V_1 \cdot\cap\cdot V_2$. Then V is a neighbourhood of x_0, and $f(x) \neq g(x)$ for $x \in V$, hence $V \subset F$. It follows that F is open.]

$$\text{Clearly } O(n) = O(n,\mathbb{C}) \cdot\cap\cdot U(n) \qquad \text{(i)}$$

σ is *real*, if its coefficients are real, i.e. if $\sigma = \overline{\sigma}$. The set of all real matrices of degree n we denote by $M_n(\mathbb{R})$, and we define $GL(n,\mathbb{R}) = M_n(\mathbb{R}) \cap GL(n,\mathbb{C})$. Hence $O(n) = GL(n,\mathbb{R}) \cap O(n,\mathbb{C})$.

Since the determinant of the product of two matrices is the product of their determinants, the matrices of determinant 1 form a subgroup of $GL(n,\mathbb{C})$. The group of all matrices with determinant 1 in $GL(n,\mathbb{C})$ is called *the special linear group* $SL(n,\mathbb{C})$.

We set

$$\begin{aligned} SL(n,\mathbb{R}) &= SL(n,\mathbb{C}) \cap GL(n,\mathbb{R}) \\ SU(n) &= SL(n,\mathbb{C}) \cap U(n) \qquad \text{(ii)} \\ SO(n) &= SL(n,\mathbb{C}) \cap O(n). \qquad \text{(iii)} \end{aligned}$$

Clearly $SL(n,\mathbb{C})$, $SL(n,\mathbb{R})$, $SU(n)$, $SO(n)$ are subgroups, and closed subsets of $GL(n,\mathbb{C})$. They may be considered as subspaces of $GL(n,\mathbb{C})$.

Theorem 3 $U(n)$, $O(n)$, $SU(n)$, $SO(n)$ are compact.

Proof We have only to show that $U(n)$ is compact, since $O(n)$, $SU(n)$, $SO(n)$ are closed subsets of $U(n)$. We shall see that $U(n)$ is homeomorphic to a bounded, closed subset of $\mathbb{C}^{n^2}$.

A matrix σ is *unitary* if and only if ${}^t\sigma\overline{\sigma} = \varepsilon$, where ε is the unit matrix. ($\overline{\sigma} = \sigma^* = ({}^t\sigma)^{-1}$).

If $\sigma = (a_{ij})$, then

$$({}^t\sigma)\overline{\sigma} = \varepsilon \Longleftrightarrow \sum_j a_{ji} \cdot \overline{a}_{jk} = \delta_{ik}.$$

(σ is regular, i.e. $\sigma\sigma^{-1} = \varepsilon$).

The left-hand sides of the last equations are continuous functions of σ, $U(n)$ is not only a closed subset of $GL(n,\mathbb{C})$ but also of $M_n(\mathbb{C})$. [For $\{\sigma \mid \sum a_{ji}\overline{a}_{jk} = 0, i \neq k\} \cdot \cap \cdot \{\sigma \mid \sum a_{jk} \cdot \overline{a}_{ji} = 1\}$ is an intersection of closed sets].

Further

$$\sum_j a_{ji}\overline{a}_{ji} = 1 \Longrightarrow |a_{ij}| \leq 1, \quad \text{for} \quad 1 \leq i,j \leq n.$$

Therefore the coefficients of the matrix $\sigma \in U(n)$ are bounded. Since $f : M_n(\mathbb{C}) \longleftrightarrow C^{n^2}$ is a homeomorphism, $f(U(n))$ is closed, and bounded, and a subset of $\mathbb{C}^{n^2}$, hence compact.

II. The Haar measure on a locally compact group

II.1 Regular measures on locally compact spaces

We have used the term 'measure' for any non-negative, additive, set function which vanishes on the empty set [cf. *Course on Integration*].

Given a topological space R which is locally compact, and Hausdorff, let $\mathcal{S}$ denote the σ-ring generated by the compact sets in R. We call $\mathcal{S}$ the *Borel ring* in R.

Remarks.

(i) $E \in \mathcal{S} \implies$ there exist compact sets C_n, $n = 1, 2, \ldots$, such that $E \subset \bigcup_{n=1}^{\infty} C_n$.

(ii) If U is open, C_n compact for $n = 1, 2, \ldots$, and $U \subset \bigcup_{n=1}^{\infty} C_n$, then $U \in \mathcal{S}$.

For if we set $K = \bigcup_{n=1}^{\infty} C_n$, then $K \in \mathcal{S}$, since $C_n \in \mathcal{S}$, and $\mathcal{S}$ is a σ-ring; and $U \subset K$. Since $C_n - U$ is a closed subset of a compact set, it is compact, hence $D = \bigcup_{n=1}^{\infty} C_n - U \in \mathcal{S}$, i.e. $D = K - U \in \mathcal{S}$. Thus $U = K - (K - U) \in \mathcal{S}$.

(iii) The whole set R is a Borel set (i.e. an element of the Borel ring), if and only if there exists a sequence (C_n) of compact sets, such that $R = \bigcup_{n=1}^{\infty} C_n$.

(iv) Every one-point set in R is compact.

(v) If $\mathbb{R}$ is the real line, $\mathcal{S} =$ the σ-ring generated by all open sets U in $\mathbb{R}$.

A measure m is regular on the σ-ring $\mathcal{S}$, if (a) m is countably additive on $\mathcal{S}$; (b) m is finite on compact sets; and (c) for $E \in \mathcal{S}$, there exist open sets $U \in \mathcal{S}$ which contain E such that

$$m(E) = \inf_{U \supset E, U \text{ open}, U \in \mathcal{S}} m(U).$$

Content A content k on R is a real-valued ($\neq \infty$), non-negative, monotone function on the class of all compact sets in R, such that

$$k(C \cup D) \leq k(C) + k(D), \quad \text{for } C, D \text{ compact};$$

$$k(C \cup D) = k(C) + k(D), \quad \text{for } C, D \text{ compact}, C \cap D = \emptyset.$$

[$0 \leq k(C) < \infty$; $C \subset D \Longrightarrow k(C) \leq k(D)$, $k(C \cup D) = k(C) + k(D)$ for $C \cap D = \emptyset$. Note that $\emptyset$ is compact, so that $k(\emptyset) + k(\emptyset) = k(\emptyset)$ with $k(\emptyset) < \infty$, hence $k(\emptyset) = 0$.]

Given a content on a locally compact Hausdorff space, we can construct a regular measure on the Borel ring in the space, as shown in the following

Theorem 1 *Let k be a content on R, which is a locally compact Hausdorff space. For any open set U in R, define*

$$m_0(U) = \sup_{C \subset U, C \text{ compact}} k(C).$$

For any subset $S \subset R$, define

$$m(S) = \inf_{U \supset S,\, U \text{ open}} m_0(U).$$

Then m is an outer measure on R. Every open set in R is m-measurable. The restriction of m to the Borel ring in R is a regular measure. Further m agrees with m_0 on all open sets.

For the proof we require a number of lemmas connecting the measure-theoretic structure of R with the topological.

Lemma 1 For every open set W, we have $m(W) = m_0(W)$.

Proof By definition, we have $m(W) \leq m_0(W)$, (since $W \subset W$). If U is open, and $U \supset W$, then $m_0(U) \geq m_0(W)$, since m_0 is monotone. Hence we have also

$$m_0(W) \leq \inf_{\substack{U \supset W \\ U \text{ open}}} m_0(U) \underset{def}{=} m(W).$$

Lemma 2 m is monotone, i.e., $S \subset T \Longrightarrow m(S) \leq m(T)$.

Proof If W is open, and $W \supset T$, then $W \supset S$, and by definition,

$$m(T) = \inf_{\substack{U \supset T \\ U \text{ open}}} m_0(U) \qquad \text{(i)}$$

$$m(S) = \inf_{\substack{U \supset S \\ U \text{ open}}} m_0(U) \qquad \text{(ii)}$$

and $T \supset S$.

If $m_0(U)$ appears in (i), then it occurs also in (ii). But the infimum (with respect to U) decreases when the class of sets U is enlarged. Hence $m(S) \le m(T)$.

Lemma 3 Let U and V be open sets in R, and E a compact subset of $U \cdot \cup \cdot V$. Then there exist compact sets $C \subset U$ and $D \subset V$, such that $E = C \cup D$.

Proof Let $E \subset U \cdot \cup \cdot V$, E compact. Let $x \in E$. Define a neighbourhood $N(x)$ of x, as follows. If $x \in U$, U open, then $N(x)$ is an *open* neighbourhood of x, with the property $\overline{N(x)}$ is compact *and* $\overline{N(x)} \subset U$. If $x \notin U$, then $x \in V$, V open, in which case choose $N(x)$ as an open neighbourhood of x, with $\overline{N(x)}$ compact, $\overline{N(x)} \subset V$.

[This choice of $N(x)$ is possible: (a) A locally compact Hausdorff space is regular. (b). R regular $\Longleftrightarrow$ ($p \in R$, U_p an open neighbourhood of $p \Longrightarrow$ there exists a neighbourhood V_p of p, such that $\overline{V}_p \subset U_p$). To see this, let U_p be an open neighbourhood of p in R (regular). Then p is closed, and $R - U_p = U_p^c = F$, say, is closed. Since R is regular, there exist open neighbourhoods V_p, $p \in V_p$, and V_F, $F \subset V_F$ such that $V_p \cap V_F = \emptyset$. Hence $V_F \cap \overline{V}_p = \emptyset$. This implies that $\overline{V}_p \cdot \cap \cdot F = \emptyset$, since $F \subset V_F$, hence $\overline{V}_p \subset U_p (= F^c)$. On the other hand, let $p \in R$, and F be any closed set in R, with $p \notin F$. There exists a neighbourhood U_p of p, such that $U_p \cap F = \emptyset$. (Note that F^c is open, $p \in F^c$, hence $\exists\, U_p \subset F^c$). Choose V_p such that $\overline{V}_p \subset U_p$ (by hypothesis). Then V_p (a neighbourhood of p) and $R - \overline{V}_p = \overline{V}_p^c$ (a neighbourhood of F) are disjoint. Hence R is regular (i.e. $T_1 + T_3$). Finally (c). take $p = x$, $U_p = N(x)$, where $\overline{N(x)}$is compact. If $\overline{N(x)} \not\subset U$, take some smaller neighbourhood (than $N(x)$) $N'(x) \subset U$, *such* that $\overline{N'(x)} \subset U$. It exists by regularity. Since $\overline{N'(x)} \subset \overline{N(x)}$, where $\overline{N(x)}$ is compact, it follows that $\overline{N'(x)}$ itself is compact.]

Obviously we have $E \subset \bigcup_{x \in E} N(x)$. Since E is compact, there is a finite open covering $\bigcup_{i=1}^{n} N(x_i)$. Set $C_1 = \bigcup_{i \ni x_i \in U} \overline{N(x_i)}$, $D_1 = \bigcup_{i \ni x_i \notin U} \overline{N(x_i)}$. Then C_1 and D_1 are compact, $E \subset C_1 \cup D_1$, $C_1 \subset U$, $D_1 \subset V$. Set $C = E \cap C_1$, $D = E \cap D_1$. Then $E = C \cup D$.

Lemma 4 If $U_1, U_2, \ldots, U_n$ are open sets in R, then

$$m\left(\bigcup_{i=1}^{n} U_i\right) \leq \sum_{i=1}^{n} m(U_i).$$

Proof First of all, consider the case $n = 2$. We assume that $m(U_1 \cdot \cup \cdot U_2) < \infty$. Let $\varepsilon > 0$, and E a compact set such that $E \subset U_1 \cdot \cup \cdot U_2$, *and*

$$k(E) > m(U_1 \cdot \cup \cdot U_2) - \varepsilon.$$

[Note that k is a content on R]. By Lemma 1, m and m_0 agree on open sets, so that $m(U_1 \cdot \cup \cdot U_2) = m_0(U_1 \cdot \cup \cdot U_2)$. By Lemma 3, $E = C_1 \cup C_2$, where C_1, C_2 are compact, with $C_1 \subset U_1$, $C_2 \subset U_2$. Now

$$\begin{aligned} m(U_1) + m(U_2) &\geq k(C_1) + k(C_2), && \text{since } m(U_1) = m_0(U_1), \\ & && m(U_2) = m_0(U_2), \\ &\geq k(C_1 \cdot \cup \cdot C_2) && \text{and } C_1 \subset U_1,\ C_2 \subset U_2 \\ &= k(E) \\ &> m(U_1 \cdot \cup \cdot U_2) - \varepsilon. \end{aligned}$$

Hence

$$m(U_1) + m(U_2) \geq m(U_1 \cdot \cup \cdot U_2).$$

If $m(U_1 \cup U_2) = \infty$ (which is possible), then for any given real α there exists a compact subset E of $U_1 \cup U_2$ (depending on α), such that $k(E) > \alpha$, and, as before,

$$m(U_1) + m(U_2) \geq k(C_1) + k(C_2) \geq k(C_1 \cdot \cup \cdot C_2) = k(E) > \alpha,$$

hence $m(U_1) + m(U_2) = \infty$.

If $n > 2$, we use induction, and

$$m\left(\bigcup_{i=1}^{n} U_i\right) = m\left[\left(\bigcup_{i<n} U_i \cdot \cup \cdot U_n\right)\right] \leq m\left(\bigcup_{i<n} U_i\right) + m(U_n).$$

Lemma 5 If (U_n), $n = 1, 2, \ldots$, is an infinite sequence of open sets, then we have

$$m\left(\bigcup_{i=1}^{\infty} U_i\right) \leq \sum_{i=1}^{\infty} m(U_i).$$

Proof Let $m(\bigcup_{i=1}^{\infty} U_i) < \infty$, and let $\varepsilon > 0$, and let E be a compact subset of $\bigcup_{i=1}^{\infty} U_i$, such that

$$k(E) > m\left(\bigcup_{i=1}^{\infty} U_i\right) - \varepsilon. \tag{i}$$

Because E is compact, there exists a *finite* indexing set I such that $E \subset \bigcup_{i\in I} U_i$. Hence $k(E) \leq m(\bigcup_{i\in I} U_i)$, since $m = m_0$ on open sets.

By Lemma 4, it follows that $k(E) \leq \sum_{i\in I} m(U_i) \leq \sum_{i=1}^{\infty} m(U_i)$. [Note that $k(\emptyset) = 0$, so $m(\emptyset) = 0$, and by monotonicity $m(U_i) \geq 0$]. Therefore from (i) we get

$$\sum_{i=1}^{\infty} m(U_i) > m\left(\bigcup_{i=1}^{\infty} U_i\right) - \varepsilon.$$

If $m(\bigcup_{i=1}^{\infty} U_i) = \infty$, then for any real α there exists a compact set $E \subset \bigcup_{i=1}^{\infty} U_i$ for which $k(E) > \alpha$. As before we can conclude that $\sum m(U_i) > \alpha$, i.e. $\sum m(U_i) = \infty$, since α is arbitrary.

Lemma 6 For any infinite sequence (S_n), $n = 1, 2, \ldots$ of subsets of R, we have

$$m\left(\bigcup_{i=1}^{\infty} S_i\right) \leq \sum_i m(S_i).$$

Proof Let $m(S_i) < \infty$ for every i; otherwise the result is trivially true. Let $\varepsilon > 0$, and U_i an open set with $U_i \supset S_i$ and

$$m(U_i) < m(S_i) + \frac{\varepsilon}{2^i} \qquad (m = m_0 \text{ on open sets}).$$

Since $\bigcup_i S_i \subset \bigcup_i U_i$, we have

$$\begin{aligned} m\left(\bigcup_i S_i\right) \leq m\left(\bigcup_i U_i\right) \leq \sum_i m(U_i) &< \sum_{i=1}^{\infty}\left(m(S_i) + \frac{\varepsilon}{2^i}\right) \\ \text{(Lemma 2)} \qquad \text{(Lemma 5)} \qquad &< \sum_i m(S_i) + \varepsilon \end{aligned}$$

Lemma 7 m is an outer measure on R.

Proof By Lemma 2, m is monotone. By Lemma 6, m is σ-sub-additive. The empty set is compact, so that $k(\emptyset) < \infty$, and $k(\emptyset) + k(\emptyset) = k(\emptyset)$, so that $k(\emptyset) = 0$, which implies that $m(\emptyset) = 0$, ($\emptyset$ is open!).

Lemma 8 If U and V are open sets in R, and $U \cap V = \emptyset$, then

$$m(U \cdot \cup \cdot V) = m(U) + m(V).$$

Proof This is trivial if $m(U) = \infty$ or $m(V) = \infty$. Let us therefore assume that $m(U) < \infty$, $m(V) < \infty$.

Let $\varepsilon > 0$. Let C be compact, $C \subset U$, D compact, $D \subset V$, such that $k(C) \geq m(U) - \varepsilon$, $k(D) \geq m(V) - \varepsilon$. Then $C \cap D = \emptyset$. Hence $k(C \cup D) = k(C) + k(D)$. Further

$$\begin{aligned} m(U) + m(V) &\leq (k(C) + \varepsilon) + (k(D) + \varepsilon) = k(C \cup D) + 2\varepsilon \\ &\leq m(U \cup V) + 2\varepsilon. \end{aligned}$$

Hence $m(U) + m(V) \leq m(U \cdot \cup \cdot V)$. But by Lemma 5, $m(U \cdot \cup \cdot V) \leq m(U) + m(V)$.

Lemma 9 For any open sets U and V in R, we have

$$m(U) = m(U \cdot \cap \cdot V) + m(U - V).$$

Proof This is trivial if $m(U \cdot \cap \cdot V)$ or $m(U - V)$ is ∞. Now $U = U \cdot \cap \cdot (V \cdot \cup \cdot V^c) = (U \cdot \cap \cdot V) \cdot \cup \cdot (U \cap V^c)$.

Hence

$$m(U) \leq m(U \cap V) + m(U \cdot \cap \cdot V^c) \tag{i}$$

(Lemma 4). On the other hand, let $\varepsilon > 0$, C compact, $C \subset U \cdot \cap \cdot V$, and

$$k(C) > m(U \cdot \cap \cdot V) - \varepsilon. \tag{ii}$$

Let W be an *open* set with the property

$$C \subset W \subset \overline{W} \subset U \cdot \cap \cdot V.$$

Such a set exists because if a space is locally compact and Hausdorff, then it is regular. Now

$$U - V = U - (U \cap V) \subset U - \overline{W}.$$

Hence

$$\begin{aligned} m(U \cap V) + m(U - V) &\le m(U \cap V) + m(U - \overline{W}) \\ &\quad (m \text{ is monotone}) \\ &\le k(C) + \varepsilon + m(U - \overline{W}), \quad \text{by (ii)} \\ &\le m(W) + \varepsilon + m(U - \overline{W}) \\ &\quad (\text{since } C \subset W, \text{ and} \\ &\quad m = m_0 \text{ on open sets}) \\ &= m(W \cdot\cup\cdot (U - \overline{W})) + \varepsilon, \\ &\qquad\qquad \text{by Lemma 8} \\ &\le m(U) + \varepsilon. \qquad\qquad \text{(iii)} \end{aligned}$$

[Note that $W \cap \overline{W}^c = \emptyset$, $W \subset U \cdot\cap\cdot V \subset U$, $U - \overline{W} \subset U$]. The lemma follows from (i) and (iii).

Lemma 10 Every open set in R is m-measurable.

Proof If S is *any* subset of R, and V an open set, we have to show that

$$m(S) \ge m(S \cap V) + m(S - V),$$

for *every* $S \subset R$.

[We recall that $S = S \cdot\cap\cdot (V \cdot\cup\cdot V^c) = (S \cap V) \cdot\cup\cdot (S \cap V^c)$, so that $m(S) \le m(S \cap V) + m(S \cap V^c)$].

We assume that $m(S) < \infty$; the result is otherwise trivial. Let $\varepsilon > 0$. There exists, by definition, an *open* set $V' \supset S$, such that $m(S) + \varepsilon > m(V')$. Now $S \cap V \subset V' \cap V$, and $S - V \subset V' - V$. Hence

$$\begin{aligned} m(S \cap V) + m(S - V) &\le m(V' \cap V) + m(V' - V) \\ &\quad [m \text{ is monotone by Lemma 2}] \\ &= m(V'), \quad \text{by Lemma 9} \\ &< m(S) + \varepsilon \quad (\text{see above}). \end{aligned}$$

Lemma 11 Let $\mathcal{S}$ = the σ-ring generated by the compact sets in a locally compact Hausdorff space R (i.e. the Borel ring in R). Then (1) every element of $\mathcal{S}$ is contained in an open set in $\mathcal{S}$; and (2) every open subset of an open set in $\mathcal{S}$ belongs to $\mathcal{S}$.

Proof (1) Every element of $\mathcal{S}$ is contained in a countable union of compact sets. [For all *such* elements of $\mathcal{S}$ form a σ-ring $\mathcal{S}^*$ say. $\mathcal{S}^*$

contains the compact sets, while $\mathcal{S}$ is the smallest *such* ring. Hence $\mathcal{S}^* \supset \mathcal{S}$].

We will show that every compact set C is contained in an open set $U \in \mathcal{S}$. Let $x \in C$. Since R is locally compact, there exists a neighbourhood $N(x)$ of x, such that $\overline{N(x)}$ is compact. Obviously $C \subset \bigcup_{x \in C} N(x)$, and because of the compactness, there exist x_i, $i = 1, 2, \ldots, n$, such that $C \subset \bigcup_{i=1}^n N(x_i)$, and if $U = \bigcup_{i=1}^n N(x_i)$, then U is open, and $\overline{U} = \overline{\bigcup_{i=1}^n N(x_i)} = \bigcup_{i=1}^n \overline{N(x_i)}$, hence $\overline{U}$ is compact. Thus every compact C is contained in an open set U with $\overline{U}$ compact.

[If we define $F_r X = \overline{X} \cdot \cap \cdot \overline{X^c}$, then (i) $F_r X = F_r X^c$; (ii) $F_r X \subset \overline{X}$, $F_r X^c \subset \overline{X}$; (iii) $F_r X$ is closed. F_r stands for Frontier (Fréchet), called "boundary" by some]

Clearly $U = \overline{U} - F_r U$, where $F_r U$ is compact, since it is a closed subset of $\overline{U}$ which is compact, so that $F_r U \in \mathcal{S}$, and $\overline{U} \in \mathcal{S}$, hence $U \in \mathcal{S}$.

(2) Let $U \in \mathcal{S}$, U open, $V \subset U$, V open; we have then to show that $V \in \mathcal{S}$.

Clearly U is contained in a countable union of compact sets C_n [see Remark (i), p. 19, or beginning of the proof of (1) above]. Now

$$V = U \cdot \cap \cdot V = \bigcup_{i=1}^{\infty} V \cap C_i \qquad \left[V \subset U \subset \bigcup_{i=1}^{\infty} C_i, \text{ and so } V = V \cap \bigcup_{i=1}^{\infty} C_i = \bigcup_{i=1}^{\infty} (V \cap C_i)\right]$$

$$V \cap C_i = C_i - (C_i - V).$$

Since C_i is compact, $C_i \in \mathcal{S}$; and $C_i - V$ is a closed subset of a compact set, hence compact, so $C_i - V \in \mathcal{S}$. Hence $V \cap C_i \in \mathcal{S}$. Since $\mathcal{S}$ is a σ-ring, $V = \bigcup_{i=1}^{\infty} (V \cap C_i) \in \mathcal{S}$.

Remark Let R be a locally compact, Hausdorff space, and $\mathcal{S}$ the σ-ring generated by the compact sets in R, while $\mathcal{S}_0$ is the σ-ring generated by the open sets in $\mathcal{S}$. Then $\mathcal{S} = \mathcal{S}_0$. For obviously $\mathcal{S}_0 \subset \mathcal{S}$, and on the other hand,

C compact $\Longrightarrow$ there exists an open set $U \in \mathcal{S}$, with compact closure, so that $C \subset U \subset \overline{U}$ (cf. proof of Lemma 11).
Now $C = C \cap U = U - (U - C)$, where $U \in \mathcal{S}_0$, $U - C \in \mathcal{S}_0$ (since C is closed, R Hausdorff, and Lemma 11(2)). Hence C compact $\Longrightarrow C \in \mathcal{S}_0$, so that $\mathcal{S} \subset \mathcal{S}_0$.

Proof of Theorem 1 By Lemma 7, m is an outer measure. The m-measurable sets form a σ-algebra $\mathcal{M}$ on which m is σ-additive. By Lemma 10, every open set is m-measurable. By Lemma 11, $\mathcal{S} = \mathcal{S}_0$. [$\mathcal{M}$ is a σ-ring. $\mathcal{M} \supset$ open (Borel) sets by Lemma 10. Hence $\mathcal{M} \supset \mathcal{S}_0 = \mathcal{S}$]. Hence every set in $\mathcal{S}$ (i.e. every Borel set) is m-measurable, and m is σ-additive on $\mathcal{S}$.

If C is compact, we have seen that $C \subset U \subset \overline{U}$, where U is open, and $\overline{U}$ is compact. Hence

$$m(C) \leq m(U) = m_0(U) = \sup_{D \subset U,\ D \text{ compact}} k(D) \leq k(\overline{U}) < \infty,$$

since $\overline{U}$ is compact. It follows that m is finite on compact sets.

By Lemma 1, m and m_0 agree on open sets. Let $S \in \mathcal{S}$. By Lemma 11(i), there exists an open set $U \in \mathcal{S}$, with $U \supset S$. By definition,

$$(*) \qquad m(S) = \inf_{U \supset S,\ U \text{ open}} m_0(U) \leq \inf_{U \supset S,\ U \text{ open},\ U \in \mathcal{S}} m(U).$$

Let $V_1, V_2, \ldots$, be a sequence of *open* sets, containing S, such that $m(V_i) \longrightarrow m(S)$, as $i \longrightarrow \infty$. (Such a sequence exists, since $m(S) = \inf(\ )$).

Let $U \in \mathcal{S}$, U open, and $U \supset S$. Then we have

$$m(V_i) \geq m(V_i \cap U) \geq m(S), \quad \text{since } m \text{ is monotone,}$$

and $V_i \supset S$, $U \supset S$. By Lemma 11, $V_i \cap V \in \mathcal{S}$, since $V_i \cap U$ is an open subset of $U \in \mathcal{S}$. Hence

$$\begin{aligned} \inf_{U \in \mathcal{S},\ U \text{ open},\ U \supset S} m(U) &\leq \inf_i m(V_i \cap U) \\ (\dagger) \qquad &\leq \lim m(V_i \cap U) = m(S). \end{aligned}$$

From $(*)$ and $(\dagger)$ we see that

$$m(S) = \inf_{U \supset S,\ U \text{ open},\ U \in \mathcal{S}} m(U).$$

Therefore m is a regular measure. m is non-trivial if k is non-trivial.

Remark Note that m is not necessarily an extension of k. A content k is called *regular*, if for every compact C,

$$k(C) = \inf\{k(D) \mid C \subset D^0 \subset D\}$$

where D is compact, $D^0 = \text{Int } D$. In this case, $k(C) = m(C)$ for all compact C.

II.2 The Haar measure on a locally compact group

Let G be a locally compact group. Let $\mathcal{S}$ be the Borel ring in G. Let m be a regular measure on $\mathcal{S}$, which is not identically zero. For $a \in G$, $S \in \mathcal{S}$, let $m(aS) = m(S)$. Then m is called a (left-invariant) *Haar measure* on G.

Theorem 2 On every locally compact group there exists a non-trivial Haar measure.

Proof The idea is to construct a content on the group, and then to apply Theorem 1 to obtain a regular measure which has all the properties required of a Haar measure. Let K be a fixed, non-empty, open set in G such that $\overline{K}$ is compact. (Such a K exists, since G is a locally compact group.) Let C be an arbitrary compact set in G. Let N be an open neighbourhood of e, the identity element of G. Then the family $\{aN\}$, $a \in G$, is a covering of G, hence also a covering of C. Since C is compact, there exists a *finite* covering of C. Let $n = n(C, N(e)) \equiv n(C, N)$ be the smallest non-negative integer n, such that $\bigcup_{\nu=1}^{n} a_\nu N(e)$, $a_\nu \in G$, is a covering of C.

Define

$$k_N(C) = \frac{n(C,N)}{n(\overline{K},N)} \qquad (\text{“the relative size of } C \text{ and } \overline{K}\text{”})$$

We shall see that

$$k_N(C) \longrightarrow k(C), \quad \text{as } N(e) \longrightarrow e.$$

Let $\mathcal{C}$ = the class of all compact sets in G. The function $k_N(C)$ has the following properties:

(1) $k_N(C) \leq k_N(D)$, if C, D are compact, $C \subset D$.

(2) $k_N(C \cup D) \leq k_N(C) + k_N(D)$, C, D compact.

(3) If $C \cap D = \emptyset$, there exists a neighbourhood N_0 of e, such that for $e \in N \subset N_0$, (N is a neighbourhood of e)

$$k_N(C \cup D) = k_N(C) + k_N(D).$$

(4) $k_N(aC) = k_N(C)$, $a \in G$ (aC is compact, since $C \longrightarrow aC$ is a homeomorphism).

(5) There exist functionals f and g on $\mathcal{C}$, which are strictly positive, such that

$$\begin{aligned} k_N(C) &\leq f(C) < \infty, \\ k_N(C) &\geq g(C) > 0, \quad \text{if Int } C \neq \emptyset \end{aligned}$$

(where f, g are independent of N).

Properties (1) and (2) are trivial to prove; also (4), for if $\bigcup_{\nu=1}^{j} a_\nu N(e)$ is a covering of C, then $\bigcup_{\nu=1}^{j} a \cdot a_\nu N(e)$ is a covering of aC, and $n(C, N) = n(aC, N)$, $\forall\, a \in G$.

The proof of (3) runs as follows. The set $C \times D$ is *compact* (Tychonoff). The mapping $(x, y) \longmapsto y^{-1}x$ is continuous. Hence $D^{-1}C$ is *compact*, and *closed* (since $D^{-1}C$ is a compact subset of a Hausdorff space). Since $C \cap D = \emptyset$, we have $e \notin D^{-1}C$ ($e \in D^{-1}C \Longrightarrow e = aC$, $a \in D^{-1}$, $c \in C$, which implies $a^{-1} = c$; $a \in D^{-1} \Longrightarrow a^{-1} \in D$; hence $c \in D$, a contradiction). Since G is a *regular* topological space, there exists a neighbourhood $N_1 = N_1(e)$ of e which is *disjoint* with $D^{-1}C$.

For $x, y \in G$, the mapping $(x, y) \longmapsto x^{-1}y$ is continuous; in particular *at* $x = y = e$ ($(e, e) \longmapsto e^{-1}e = e$). Hence there exists a neighbourhood N_0 of e, such that for $(x, y) \in N_0 \times N_0$, we have $x^{-1}y \in N_1$. It follows that if $e \in N \subset N_0$ (N open), and $x, y \in N$, then $x^{-1}y \notin D^{-1}C$.

This implies that no (left) "translation" of N (i.e. aN, $a \in G$) exists which intersects both C and D. For otherwise, let $p \in aN \cap C$, and $q \in aN \cap D$. Then $p = ay$, $y \in N$, and $q = ax$, $x \in N$, and $q^{-1}p \in D^{-1}C$ ($q \in D \Longrightarrow q^{-1} \in D^{-1}$, and $p \in C$). On the other hand, $q^{-1}p = (ax)^{-1}(ay) = x^{-1}y$, so that $x^{-1}y \in D^{-1}C$, which is a contradiction.

Let $n = n(C \cup D, N)$. Then we have $a_1 N, \ldots, a_n N$, such that $\bigcup_{\nu=1}^{n} a_\nu N$, $a_\nu \in G$, is a covering of $C \cup D$. (n is the smallest such integer).

Let $A =$ the set of all those i's (among $i = 1, \ldots, n$) such that $a_i N \cap C \neq \emptyset$;
and $B =$ the set of all those i's for which $a_i N \cap D \neq \emptyset$.

We have noted that $A \cap B = \emptyset$; every i ($i = 1, \ldots, n$) belongs to A or B. Otherwise there would exist an i_0 such that $a_{i_0} N \cap C = a_{i_0} N \cap D = \emptyset$, which implies that $a_{i_0} N \cap (C \cup D) = \emptyset$, i.e. then the translation $a_{i_0} N$ could be dropped from the covering $\bigcup_{\nu=1}^{n} a_\nu N$. This contradicts the assumption that n is the smallest such integer.
Hence $C \cup D \subset \bigcup_{i \in A} a_i N \cdot \cup \cdot \bigcup_{i \in B} a_i N$,
so that

$$(C \cup D) \cdot \cap \cdot C \subset \bigcup_{i \in A} a_i N \cdot \cup \cdot \bigcup_{i \in B} a_i N.$$

Since A and B are disjoint, it follows that $C \subset \bigcup_{i \in A} a_i N$, for $a_i N \cap C = \emptyset$ if $i \in B$. Similarly $D \subset \bigcup_{i \in B} a_i N$. Hence

$$\begin{aligned} n(C, N) &\leq \text{ the number of } i\text{'s in } A, \\ n(D, N) &\leq \text{ the number of } i\text{'s in } B, \end{aligned}$$

therefore

$$n(C, N) + n(D, N) \leq n \underset{def}{=} n(C \cup D, N).$$

Dividing by $n(\overline{K}, N)$, we get

$$k_N(C) + k_N(D) \leq k_N(C \cup D).$$

Property (2), on the other hand, implies the opposite inequality. Thus (3) is proved.

The proof of property (5) runs as follows. Let E be any non-empty *open* set with $\overline{E}$ compact. Then we have

$$(*) \qquad n(C, N) \leq n(C, E) \cdot n(\overline{E}, N).$$

[Here $n(C, E)$ is to be interpreted as follows: If $x \in E$, then $E = xN_0$, where $N_0 = N_0(e)$ is a neighbourhood of the identity. See 'trivial property' (7) on p. 7. And $n(C, E) = n(C, xN_0) = n(C, N_0)$.] We have

$$C \subset \bigcup_i a_i E, \ i = 1, \ldots, n(C, E), \quad a_i \in G,$$

and

$$\overline{E} \subset \bigcup_j b_j N, \quad j = 1, 2, \ldots, n(\overline{E}, N), b_j \in G,$$

so that $C \subset \bigcup_{i,j} a_i b_j N$ (since $E \subset \overline{E}$), there being $n(C, E) \cdot n(\overline{E}, N)$ summands in the last sum, from which $(*)$ follows.

Put $E = K$ in $(*)$. Then we get

$$n(C, N) \leq n(C, K) \cdot n(\overline{K}, N).$$

[Note that K is a fixed, non-empty, open set in G, with $\overline{K}$ compact, with which we began the proof of the theorem]. On dividing by $n(\overline{K}, N) \neq 0$, we get

$$k_N(C) \leq n(C, K).$$

Let $f(C) = n(C, K)$, which is independent of N; $f(C)$ is finite, and

$$(\checkmark) \qquad k_N(C) \leq f(C) > 0$$

which proves *the first part of property* (5). To prove the second part, put $C = \overline{K}$ in $(*)$. Then we get

$$n(\overline{K}, N) \leq n(\overline{K}, E) \cdot n(\overline{E}, N).$$

Dividing by $n(\overline{K}, E) \cdot n(\overline{K}, N)$, we get

$$(\dagger) \qquad \frac{1}{n(\overline{K}, E)} \leq k_N(\overline{E}).$$

If C is compact, with Int $C \neq \emptyset$, then $C \supset \overline{\text{Int } C}$ [Note that Int $X = \overline{X^c}^c$, the largest open set contained in X. Since R is Hausdorff, C is closed, hence $\overline{C} \supset \overline{\text{Int } C}$ i.e. $C \supset \overline{\text{Int } C}$]. Set $E = \text{Int } C$, a non-empty open set. Then we have

$$\begin{aligned} k_N(C) &\geq k_N(\overline{\text{Int } C}), \quad \text{by property (1), p. 28} \\ &\geq \frac{1}{n(\overline{K}, \text{Int } C)}. \end{aligned}$$

(Note that $\overline{\text{Int } C}$ is compact, since it is a closed subset of the compact set C). Let $g(C) = 1/n(\overline{K}, \text{Int } C)$. Then $g(C) > 0$, and $k_N(C) \geq g(C) > 0$, which is the *second part of property* (5).

We shall construct a content k with the help of k_N. Let $\overline{\mathcal{C}} = \{C \mid C$ compact$\}$ p. 28, to each $C \in \mathcal{C}$ make correspond the interval $[0, n(C, K)]$.

The product $\mathcal{F}$ of such intervals, $\mathcal{F} = \prod_{C \in \mathcal{C}}[0, n(C, K)]$ with the product topology, is a compact Hausdorff space. The points are real-valued functions ϕ defined on $\mathcal{C}$, such that for each $C \in \mathcal{C}$ we have

$$0 \leq \phi(C) \leq n(C, K).$$

By Tychonoff's theorem, $\mathcal{F}$ is compact. [$\phi(C)$ is the "Cth co-ordinate" of a point in $\mathcal{F}$].

Let $\mathcal{N}$ = the class of all neighbourhoods of e. For $N \in \mathcal{N}$ let $\mathcal{H}(N)$ = the class of all elements in $\mathcal{F}$ of the form k_M, $N \supset M \in \mathcal{N}, = \{k_M \mid N \supset M \in \mathcal{N},\ k_M \in \mathcal{F}\}$. Then $\mathcal{H}(N) \subset \mathcal{F}$. Since $k_N \in \mathcal{H}(N)$, (see ($\surd$) above), $\mathcal{H}(N) \neq \emptyset$ for every $N \in \mathcal{N}$. $\mathcal{H}(N)$ is an increasing function of N. If $N_1, N_2, \ldots, N_n$ are elements of $\mathcal{N}$ (neighbourhoods of e), then $\bigcap_{i=1}^n N_i$ is also a neighbourhood of e, hence $\bigcap_{i=1}^n N_i \in \mathcal{N}$, and $\bigcap_{i=1}^n N_i \subset N_j$, $j = 1, 2, \ldots, n$. Hence

$$\mathcal{H}\left(\bigcap_{i=1}^n N_i\right) \subset \mathcal{H}(N_j), \quad j = 1, 2, \ldots, n,$$

therefore

$$\mathcal{H}\left(\bigcap_{i=1}^n N_i\right) \subset \bigcap_{i=1}^n \mathcal{H}(N_i),$$

so that

$$\bigcap_{i=1}^n \mathcal{H}(N_i) \neq \emptyset.$$

Hence the class $\{\mathcal{H}(N) \mid N \in \mathcal{N}\}$ has the finite intersection property; therefore also the family $\{\overline{\mathcal{H}}(N) \mid N \in \mathcal{N}\}$.

[Note that $\mathcal{H}(N) \subset \mathcal{F}$, $\mathcal{F}$ is closed, hence $\overline{\mathcal{H}}(N) \subset \mathcal{F}$, and $\overline{\mathcal{H}}(N) \supset \mathcal{H}(N)$.]

Since $\mathcal{F}$ is compact, (Hausdorff) there exists at least one common element k for all $\overline{\mathcal{H}(N)}$, $N \in \mathcal{N}$. That is to say, there exists a k, such that

$$k \in \bigcap_{N \in \mathcal{N}} \overline{\mathcal{H}(N)}.$$

We shall show that k is a content.

By the definition of product topology (T_i = top space, there is a basis in $\prod T_i$ of the form $\prod B_i$, where B_i is an open set in T_i for finitely many i's and $B_i = T_i$ for the rest), given $\varepsilon > 0$, and finitely many compact sets $C_1, C_2, \ldots, C_n$, in G, there exist $N_1 \in \mathcal{N}$, $N_1 \subset N$, $N \in \mathcal{N}$ fixed, such that

$$|k_{N_1}(C_i) - k(C_i)| < \varepsilon, \quad i = 1, 2, \ldots, n.$$

(since every neighbourhood of k intesects $\mathcal{H}(N)$). Now choose $C_1 = C$, $C_2 = aC$, $a \in G$. Then

$$\begin{aligned} |k(C) - k(aC)| &\leq |k(C) - k_{N_1}(C)| + |k_{N_1}(C) - k_{N_1}(aC)| \\ &\quad + |k_{N_1}(aC) - k(aC)| \\ &< 2\varepsilon \quad \text{(the middle term is 0)} \\ &\qquad \text{because of property (3) on p. 29.} \end{aligned}$$

It follows that

$$k(C) = k(aC).$$

The proof that

$$0 \leq k(C) \leq n(C, I) < \infty, \quad \forall\, C \in \mathcal{C}$$

is similar.

If C and D are compact, with $C \subset D$, then $k(C) \leq k(D)$. [For

$$\begin{aligned} |k(C)| &= |k(C) - k_N(C) + k_N(C) - k_N(D) + k_N(D) - k(D) + k(D)| \\ &\leq |k(C) - k_N(C)| + |k_N D) - k(D)| + k(D) \\ &< 2\varepsilon + k(D), \text{since } k_N(C) - k_N(D) \leq 0.] \end{aligned}$$

If C and D are compact, *with* $C \cap D = \emptyset$, then $k(C \cup D) = k(C) + k(D)$. [Choose $N = N_0$ in property (3), p29, so that $k_N(C \cup D) = k_N(C) + k_N(D)$]. We get

$$\begin{aligned} |k(C \cup D) - k(C) - k(D)| &\leq |k(C \cup D) - k_N(C \cup D)| + \\ &\quad |k_N(C) - k(C)| + |k_N(D) - k(D)| \\ &< \varepsilon. \end{aligned}$$

If C and D are arbitrary compct sets (not necessarily disjoint) we use $k_N(C \cup D) \leq k_N(C) + k_N(D)$ and obtain $0 \leq k(C \cup D) \leq 3\varepsilon + k(C) + k(D)$, hence $k(C \cup D) \leq k(C) + k(D)$. From property (5) we can deduce that $k(C) \geq g(C) > 0$ if Iı $C \neq \emptyset$, so that k is not identically zero.

Then *k is a content on G*. By ppealing to Theorem 1, we obtain a regular measure m on the Borel rir in G. Since k is left-invariant, m is also left-invariant, so that m is a har measure.

Remarks

1. Let T be a homeomorphism of G with itself. Let $k'(C) = k(TC)$, for $C \in \mathcal{C}$. By assumption k is a content. It then follows that k' is a content. Let m and m' be the corresponding measures which they generate. Then $m'(E) = m(TE)$ for all Borel sets $E \in \mathcal{S}$.

   To see this we note first of all that

$$\begin{aligned} &\{k'(C) \mid C \in \mathcal{C},\ C \subset U,\ U \text{ open}\} \\ &\quad = \{k(TC) \mid C \in \mathcal{C},\ C \subset U \text{ open}\} \\ &\quad = \{k(D) \mid D = T(C),\ \ C \in \mathcal{C},\ C \subset U \text{ open}\} \\ &\quad = \{k(D) \mid T^{-1}D \subset \mathcal{C},\ T^{-1}D \subset U \text{ open}\} \\ (*) &\quad = \{k(D) \mid D \subset TU, \qquad TU \text{ open},\ D \in \mathcal{C}\} \\ &\qquad\qquad\qquad\qquad (T^{-1}D \in \mathcal{C} \Longrightarrow D \in \mathcal{C}). \end{aligned}$$

   But for U open $m_0(U) = \sup_{C \subset U,\ C \text{ compact}} k(C)$.

   Hence

$$\sup\{k(D) \mid D \subset T(U),\ D \in \mathcal{C}\} = m_0(TU),\ \ TU \text{ open}.$$

   On the other hand,

$$\sup\{k'(C) \mid C \subset U,\ \ U \text{ open},\ \ C \in \mathcal{C}\} = m_0'(U).$$

   Because of $(*)$, we obtain

$$m_0(TU) = m_0'(U).$$

   The measures m and m' are *regular measures* on the Borel ring $\mathcal{S}$. For $E \in \mathcal{S}$, there exist open sets $U \in \mathcal{S}$, which contain E, such that $m(E) = \inf_{U \supset E,\ U \text{ open}} m_0(U)$. Similarly $m'(E) = \inf_{U \supset E,\ U \text{ open}} m_0'(U) = \inf_{TU \supset TE,\ TU \text{ open}} m_0(TU) = m(TE)$.

   It follows that if k left-invariant, so also is m.

2. The existence of a right-invariant Haar measure is similarly proved.

   If G is a given locally compact group, let G' be the *dual* group, with the same elements as G and the same topology but the group operation $\circ$ in G' being defined as: $x \circ y = yx$. Then there exists a left-invariant measure in G' which is right-invariant on G.

3. The Haar measure so obtained is not unique, for if m is one such, then for any constant $c > 0$, cm is likewise such.

II.3 The Riesz-Markoff theorem

Let R be a locally compact Hausdorff space, and $\mathcal{C}_0 = \mathcal{C}_0(R)$ the space of continuous functions vanishing outside a compact set (i.e. each function has a compact support, which may depend on the function).

Let P be a positive, linear functional on $\mathcal{C}_0$ (i.e. P is a real-valued function of functions in $\mathcal{C}_0 : P(\alpha f + \beta g) = \alpha P(f) + \beta P(g)$, for $f, g \in \mathcal{C}_0$, α, β real; $P(f) \geq 0$ for $f \geq 0$, $f \in \mathcal{C}_0$). Then we have the following:

Theorem 3 *There exists a regular measure m on R, such that for $f \in \mathcal{C}_0$, we have*

$$P(f) = \int_R f(x)\, dm(x).$$

The integral refers to the measure space $(R, \mathcal{S}, m)$, where now $\mathcal{S}$ is the Borel ring in R, *with* $R \in \mathcal{S}$.

F. Riesz proved that if R is an interval $[a, b]$ on the real line, then

$$P(f) = \int_a^b f(t)\, dA(t), \qquad -\infty < a < b < +\infty,$$

for some *essentially* monotone function $A(t)$. The general case is due to Markoff. As in the case of proving the existence of a Haar measure, this theorem can again be proved by an application of Theorem 1.

Remarks

(i) The integral in question exists. f is continuous, and if $f \geq 0$, and f has a compact support C, let $\alpha > 0$. Then the set $E_\alpha = \{x \mid f(x) \geq \alpha\}$ is *closed*, and contained in C, hence *compact*. If $\alpha \leq 0$, this set is R, which is *measurable*. Hence f is m-measurable (Borel measurable). If χ is the characteristic function of C, then $f \leq \alpha\chi$, where $\alpha = \sup f$. Since χ is obviously integrable, so is f.

(ii) (a) $\chi_{\cap E_\alpha} = \prod \chi_{E_\alpha}$;

(b) $\chi_{\cup E_\alpha}(x) = \min(1, \sum \chi_{E_\alpha}(x))$;

(c) The E_α's are disjoint if and only if $\sum \chi_{E_\alpha}(x) \leq 1$;

(d) If the E_α's are disjoint, then $\chi_{\cup E_\alpha} = \sum \chi_{E_\alpha}$;

(e) $\chi_{E^c}(x) = 1 - \chi_E(x)$;

(f) $E \subset F \Longrightarrow \chi_E(x) \leq \chi_F(x)$.

We need the following topological result.

Lemma 12 Let R be a locally compact Hausdorff space, $C \subset R$, C compact, U open, $U \supset C$. Then there exists an element $f \in \mathcal{C}_0(R)$, such that

$$f(x) = \begin{cases} 1, & x \in C, \\ 0, & x \in U^c, \end{cases}$$

and $0 \leq f(x) \leq 1$, for $x \in R$.

Proof Every point $x \in C$ has a neighbourhood $N(x)$, such that $\overline{N(x)}$ is compact and $\overline{N(x)} \subset U$. Obviously $C \subset \bigcup_{x \in C} N(x)$. Since C is compact, there exist n points $x_1, \ldots, x_n \in C$ such that $C \subset \bigcup_{i=1}^n N(x_i) = V$, say. Then $C \subset V$, $\overline{V}$ compact, and $\overline{V} \subset U$.

Since $\overline{V}$ is compact and Hausdorff (in the induced topology), $\overline{V}$ is *normal.*

By Urysohn's Lemma, there exists a continuous function ϕ on $\overline{V}$, such that $\phi = 1$ on C (closed), $\phi = 0$ on V^c (closed), and $0 \leq \phi \leq 1$ otherwise.

Define

$$f(x) = \begin{cases} \phi(x), & x \in \overline{V}, \\ 0, & x \in R - \overline{V}. \end{cases}$$

Then we have

$$\begin{aligned} f(x) &= 1, \ \forall\, x \in C, \\ &= 0, \ \forall\, x \in U^c \ \ (U \supset \overline{V} \Longrightarrow U^c \subset \overline{V}^c \subset V^c) \\ \text{and} \quad & 0 \leq f(x) \leq 1, \ \forall\, x \in R. \end{aligned}$$

Proof of Theorem 3 Define for C compact and $f \in \mathcal{C}_0(R)$, $A(C) = \{f \mid f \in \mathcal{C}_0,\ f \geq 0;\ f \geq \chi_C\}$, and $k(C) = \inf_{f \in A(C)} P(f)$. The proof is then divided into five parts;

(i) k is a *content* on R;

(ii) there exists a regular measure m, such that $m(C) = k(C)$, for all $C \in \mathcal{C}$;

(iii) $f \in \mathcal{C}_0$, $f \geq 0 \Longrightarrow P(f) \geq \int f \, dm$;

(iv) for every compact C, and every $\varepsilon > 0$, there exists $g_0 \in A(C)$, $g_0 \leq 1$, such that

$$P(g_0) \leq \int g_0 \, dm + \varepsilon;$$

(v) $f \in \mathcal{C}_0 \Longrightarrow P(f) = \int f \, dm$.

Part (i) Because of the Lemma (just above), $A(C) \neq \emptyset$, thus there exists $f \in A(C)$, and $k(C) \leq P(f) < \infty$.

Let $\varepsilon > 0$, and C, D compact sets. Let $f \in A(C)$ be such that $P(f) < k(C) + \varepsilon$, ($f$ exists by definition of k) and $g \in A(D)$ be such that $P(g) < k(d) + \varepsilon$.

Let $h \in A(C \cup D)$ be such that $h \leq f + g$. [If $h' \in A(C \cup D)$, take $h = \min(h', f+g)$. Then h has the required property. The $\min(\cdot, \cdot) \in \mathcal{C}_0$, and $\chi_{C \cup D} = \min(1, \chi_C + \chi_D)$].

Since the functional P is positive, $P(h) \leq P(f) + P(g)$. Since $h \in A(C \cup D)$, $k(C \cup D) \leq P(h)$. Hence $k(C \cup D) \leq P(f) + P(g) \leq (k(C) + \varepsilon) + (k(D) + \varepsilon) = k(C) + k(D) + 2\varepsilon$.

Hence *k is sub-additive on compact sets. Secondly,* (as we shall see) *k is additive on disjoint compact sets,* i.e. if C and D are disjoint compact sets, then $k(C \cup D) = k(C) + k(D)$. We have only to prove that $k(C \cup D) \geq k(C) + k(D)$, *if* $C \cap D = \emptyset$.

Let $h \in A(C \cup D)$ be such that $P(h) < k(C \cup D) + \varepsilon$. There exist *disjoint, open* sets U, V, with $\overline{U}, \overline{V}$ *compact*, such that $U \supset C$, $V \supset D$. [The space is Hausdorff, C, D compact and disjoint. Hence C, D have disjoint neighbourhoods U, V. Either $\overline{U}$ itself is compact, or (as in Lemma 3) there exists a neighbourhood U' of C with $\overline{U'}$ compact and $\overline{U'} \subset U$.]

Let $f = \min(f', h)$, where $f' \in A(C)$. f' is some element of $A(C)$, so that $f' \geq \chi_C$, $f' \geq 0$, $f' \in \mathcal{C}_0$. For example, $f' = 1$ on C, 0 on U^c; $f' \in \mathcal{C}_0$ since $\overline{U}$ is compact (see the Lemma just above). Then $f \in A(C)$,

$[f, g \in \mathcal{C}_0 \Longrightarrow {}^{\min}_{\max}(f, g) \in \mathcal{C}_0]$ and $f \leq h\chi_U$. $[x \in U \Longrightarrow \chi_U(x) = 1$, so that $f(x) \leq h(x)$. And $x \notin U \Longrightarrow f'(x) = 0$, so that $\min\{f'(x), h(x)\} = 0$, hence $f(x) = 0$ for $x \in U^c$.]

Similarly, there exists $g \in A(D)$, such that $g \leq h\chi_V$. Now

$$\begin{aligned} f + g &\leq h\chi_U + h\chi_V, \\ &\leq h, \quad (\text{since } U \cap V = \emptyset,\ \chi_U + \chi_V \leq 1). \end{aligned}$$

Hence $P(f+g) \leq P(h) < k(C \cup D) + \varepsilon$ (see above). However $P(f) \geq k(C)$, $P(g) \geq k(D)$, since $f \in A(C)$, $g \in A(D)$. Hence

$$k(C) + k(D) \leq P(f) + P(g) = P(f+g) \leq k(C \cup D) + \varepsilon.$$

Thirdly, k is *monotone*. If C and D are compact, and $C \subset D$, then $\chi_C \leq \chi_D$. Now $k(C) = \inf_{f \in A(C)} P(f)$, $k(D) = \inf_{f \in A(D)} P(f)$, and $f \in A(D) \Longrightarrow f \in A(C)$. Hence $k(C) \leq k(D)$.

Fourthly, $k(C) \geq 0$ for every compact C, since $P(f) \geq 0$ for $f \in A(C)$. We have already noted at the beginning that $k(C) < \infty$.

Thus k is a content on R.

Part (ii) By an appeal to Theorem 1 we obtain from k a regular measure m. We shall see that $m(C) = k(C)$ *for any compact* C.

Let U be *open*, $U \supset C$. Then, as in Theorem 1, $m(U) = m_0(U) = \sup_{C \subset U,\ C \text{ compact}} k(C)$. Hence $m(U) \geq k(C)$. But for *any* set S, we have $m(S) = \inf_{U \supset S,\ U \text{ open}} m_0(U)$, which implies that $m(C) \geq k(C)$.

It remains for us therefore only to prove that

$$m(C) \leq k(C).$$

Let $\{f_n\}$, $n = 1, 2, \ldots$, $f_n \in A(C)$, be such that

$$(*) \qquad P(f_n) \longrightarrow k(C). \quad [\text{We have } k(C) = \inf_{f \in A(C)} P(f)]$$

Set $C_n = \left\{ x \mid f_n(x) \geq 1 - \dfrac{1}{n} \right\}$, $n = 2, 3, \ldots$; then C_n is compact [C_n is *closed*, since f_n is continuous]. f_n has compact support, say S_n. Then $C_n \subset S_n$.

Set

$$U_n = \left\{ x \mid f_n(x) > 1 - \frac{1}{n} \right\}, \quad n = 2, 3, \ldots.$$

Then $C_n \supset U_n \supset C$, [since $x \in C \Longrightarrow \chi_C(x) = 1 \Longrightarrow f_n(x) = 1$, since $f_n \geq \chi_C$, $[f_n \in A(C)] \Longrightarrow x \in U_n]$ and

$$\left(1 - \frac{1}{n}\right)^{-1} f_n \in A(C_n).$$

[Note that $\left(1 - \frac{1}{n}\right)^{-1} f_n \in A(C_n) \Longleftrightarrow \left(1 - \frac{1}{n}\right)^{-1} f_n \geq \chi_{C_n}$, since $f_n \in A(C)$. $\left(1 - \frac{1}{n}\right)^{-1} f_n \geq 0$, for $n = \underline{2}, 3, \ldots$ and $\left(1 - \frac{1}{n}\right)^{-1} f_n \in \mathcal{C}_0$. Further $x \in C_n \Longrightarrow \chi_{C_n}(x) = 1$, and $f_n(x) \geq 1 - \frac{1}{n}$, hence $x \in C_n \Longrightarrow \left(1 - \frac{1}{n}\right)^{-1} f_n \geq \chi_{C_n}$]. Hence, by definition of k, we have

$$P\left(\left(1 - \frac{1}{n}\right)^{-1} f_n\right) \geq k(C_n), \quad \text{i.e.} \quad \left(1 - \frac{1}{n}\right)^{-1} P(f_n) \geq k(C_n),$$

which implies that

$$(**) \qquad \liminf_{n \to \infty} k(C_n) \leq k(C). \quad \text{(because of } (*))$$

We next observe that $k(C_n) \geq m(U_n)$. For $C_n \supset U_n$, and $m(U_n) = m_0(U_n) = \sup_{D \subset U_n,\ D \text{ compact}} k(D)$, and every *such* D is contained in C_n, since $U_n \subset C_n$. Because k is monotone, it follows that $k(D) \leq k(C_n)$, which leads to: $m(U_n) \leq k(C_n)$. Using this in $(**)$, we get

$$\liminf_{n \to \infty} m(U_n) \leq k(C).$$

But $U_n \supset C$, hence $m(U_n) \geq m(C)$ (since an outer measure is monotone), and

$$\liminf m(U_n) \geq m(C),$$

thus giving $m(C) \leq k(C)$, which completes part (ii) of the proof.

Part (iii) We shall prove that

$$f \in \mathcal{C}_0, \ \ f \geq 0 \quad \Longrightarrow \quad P(f) \geq \int f \, dm.$$

Since the integral, as well as P, are *linear*, it is sufficient to prove the last inequality for $0 \leq f(x) \leq 1$, $\forall\, x$.

We shall, first of all, effect a decomposition of f as follows: $f(x) = \frac{1}{n}\sum_{i=1}^{n} f_i(x)$, where n is a given positive integer. For $i = 1, 2, \ldots, n$, set

$$f_i(x) = \begin{cases} 0, \text{ if } f(x) < \dfrac{i-1}{n}; \\ nf(x) - (i-1) = \dfrac{f(x) - \dfrac{i-1}{n}}{\dfrac{1}{n}}, \text{ if } \dfrac{i-1}{n} \le f(x) \le \dfrac{i}{n}; \\ 1, \text{ if } \dfrac{i}{n} < f(x). \end{cases}$$

Then we have

$$f_i = \min\{\max\{nf - (i-1), 0\}, 1\} = \max\{\min\{nf - (i-1), 1\}, 0\}.$$

Hence

$$f_i \in C_0, \quad f_i \ge 0, \quad i = 1, 2, \ldots, n.$$

By hypothesis, $0 \le f(x) \le 1$. If x is such that $\frac{j-1}{n} \le f(x) < \frac{j}{n}$, $1 \le j \le n$, then

$$f_i(x) = \begin{cases} 1, \text{ if } 1 \le i \le j-1, & \left(\text{i.e. } \dfrac{i}{n} \le \dfrac{j-1}{n} \le f(x)\right), \\ 0, \text{ if } j+1 \le i \le n, & \left(\text{i.e. } \dfrac{i-1}{n} \ge \dfrac{j}{n} > f(x)\right). \end{cases}$$

Hence

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n} f_i(x) &= \frac{1}{n}\sum_{i=1}^{j-1} f_i(x) + \frac{1}{n} f_j(x) + \frac{1}{n}\sum_{i=j+1}^{n} f_i(x) \\ &= \frac{1}{n}\sum_{i=1}^{j-1} 1 + \frac{1}{n}(nf(x) - (j-1)) + 0 \qquad \text{(cf. defn. } f_j\text{)} \\ &= \frac{j-1}{n} + \frac{1}{n}(nf(x) - (j-1)) = f(x), \quad \textit{for every } x. \end{aligned}$$

For $i = \underline{0}, 1, 2, \ldots, n$ set

$$U_i = \left\{ x \mid f(x) > \frac{i}{n} \right\}.$$

Since f is continuous, with compact support, U_i is *open*, and contained in a compact set. (U_i is measurable). Further we have

$$\chi_{U_i} \leq f_i, \quad i = 1, \ldots, n,$$

for $x \in U_i \Longrightarrow f(x) > \dfrac{i}{n} \Longrightarrow f_i(x) = 1$. *This implies that*

$$m(U_i) \leq P(f_i).$$

For, if C is compact, and $C \subset U_i$, then $\chi_C \leq \chi_{U_i} \leq f_i$, hence $f_i \in A(C)$, and $m(C) = k(C) \leq P(f_i)$, on using Part (ii) and the definition of $k(C)$, so that

$$m(U_i) = m_0(U_i) \underset{def}{=} \sup_{C \subset U_i} k(C) = \sup_{C \subset U_i} m(C) \leq P(f_i).$$

Since $U_0 \supset U_1 \supset U_2 \supset \cdots \supset U_n = \emptyset$, we have

$$P(f) = \frac{1}{n} \sum_{i=1}^{n} P(f_i) \geq \frac{1}{n} \sum_{i=1}^{n} m(U_i) = \sum_{i=1}^{n} \left(\frac{i}{n} - \frac{i-1}{n} \right) m(U_i)$$

$$= \sum_{i=1}^{n-1} \frac{i}{n} (m(U_i) - m(U_{i+1})),$$

since $m(U_n) = m(\emptyset) = 0$,

$$= \sum_{i=1}^{n-1} \frac{i+1}{n} m(U_i - U_{i+1}) - \frac{1}{n} m(U_1)$$

[since $x \in U_i \Longrightarrow f(x) > \dfrac{i}{n}$,

$x \notin U_{i+1} \Longrightarrow f(x) \leq \dfrac{i+1}{n}$, hence

$x \in U_i - U_{i+1} \Longrightarrow \dfrac{i}{n} < f(x) \leq \dfrac{i+1}{n}$

$\Longrightarrow \displaystyle\int_{U_i - U_{i+1}} f(x)\, dm(x) \leq \frac{i+1}{n} m(U_i - U_{i+1})$.]

$$\geq \sum_{i=1}^{n-1} \int_{U_i - U_{i+1}} f\, dm - \frac{1}{n} m(U_1)$$

$$= \int_{U_1} f\, dm - \frac{1}{n} m(U_1)$$

$$\geq \int_{U_0} f\, dm - \frac{1}{n} m(U_0)$$

$$= \int_{R} f\, dm - \frac{1}{n} m(U_0),$$

since $x \in U_0^c \Longrightarrow f(x) = 0$.

[$m(U_0) \geq m(U_1)$

$x \in (U_0 - U_1) \Longrightarrow 0 < f(x) \leq \dfrac{1}{n}$

$m(U_0 - U_1) = m(U_0) - m(U_1)$

$\displaystyle\int_{U_0 - U_1} f\, dm \leq \frac{1}{n}[m(U_0) - m(U_1)]$

$-\dfrac{1}{n} m(U_1) \geq \displaystyle\int_{U_0 - U_1} f\, dm - \frac{1}{n} m(U_0)$.]

Since n is arbitrary, and $m(U_0) < \infty$ (since U_0 is contained in a compact set), we get

$$P(f) \geq \int_R f\, dm, \quad f \in \mathcal{C}_0, f \geq 0.$$

Part (iv) Let $f_0 \in A(C)$, and $P(f_0) \leq k(C) + \varepsilon$. Set $g_0 = \min\{f_0, 1\}$. Then $g_0 \in A(C)$, and

$$P(g_0) \leq P(f_0) \leq k(C) + \varepsilon = m(C) + \varepsilon \leq \int_R g_0\, dm + \varepsilon,$$

since $[g_0 \in A(C) \Longrightarrow g_0 \geq \chi_C]$. Hence for every compact set C, and for every $\varepsilon > 0$, there exists a function $g_0 \in A(C)$, $g_0 \leq 1$, such that $P(g_0) \leq \int_R g_0\, dm + \varepsilon$.

Part (v) If $f \in \mathcal{C}_0$, then $P(f) = \int_R f\, dm$. In order to prove this, let C be a compact set, such that $\{x \mid f(x) \neq 0\} \subset C$, and $\varepsilon > 0$.

After Part (iv), there exists $f_0 \in A(C)$, $f_0 \leq 1$, such that

$$(*) \qquad P(f_0) \leq \int_R f_0\, dm + \varepsilon.$$

Since $\chi_C \leq f_0$, and $f_0 \leq 1$, we have $ff_0 = f$. Let $\alpha > 0$, such that $|f(x)| \leq \alpha$, $\forall\, x$. Then $ff_0 + \alpha f_0 = (f + \alpha)f_0 \in \mathcal{C}_0$, *and* $f + \alpha f_0 \geq 0$. After part (iii) we obtain

$$\begin{aligned} P(f) + \alpha P(f_0) &= P(f + \alpha f_0) \\ &\geq \int (f+\alpha) f_0\, dm \\ &= \int f\, dm + \alpha \int f_0\, dm \quad (\text{since } ff_0 = f). \end{aligned}$$

Hence

$$\begin{aligned} P(f) &\geq \int f\, dm + \alpha \left(\int f_0\, dm - P(f_0) \right), \\ &\geq \int f\, dm - \alpha\varepsilon, \quad (\text{because of } (*)) \end{aligned}$$

which gives $P(f) \geq \int f\, dm$. Replacing f by $-f$, we get $-P(f) \geq -\int f\, dm$, or $P(f) \leq \int f\, dm$, the reverse inequality, thus completing the proof of Theorem 3.

Remark The measure m in the theorem is actually unique. To prove that, it is useful to show that the measure of any Borel set can be approximated from below by compact sets.

Theorem 4 Let m be a regular measure on a locally compact Hausdorff space R. Let E be a Borel set in R (i.e. an element of the Borel ring). Then we have

$$m(E) = \sup_{C \subset E,\ C \text{ compact}} m(C).$$

Proof If $C \subset E$, then $m(C) \leq m(E)$. By definition of the Borel ring, there exists a sequence (C_i'), C_i' compact, with $E \subset \bigcup_{i=1}^{\infty} C_i'$. Set $C_n = \bigcup_{i \leq n} C_i'$. Then C_n is *compact*, and (C_n) is monotonically increasing. Further $E \subset \bigcup_{n=1}^{\infty} C_n$, hence $E = \bigcup_n (E \cap C_n)$. Since m is σ-additive,

$$m(E \cap C_n) \longrightarrow m(E), \quad \text{as} \quad n \longrightarrow \infty.$$

Let $\varepsilon > 0$. Then there exists $n = n(\varepsilon)$, such that

$$m(E \cap C_n) > m(E) - \varepsilon, \quad \text{if} \quad m(E) < \infty,$$

(†) and

$$m(E \cap C_n) > \frac{1}{\varepsilon}, \quad \text{if} \quad m(E) = \infty.$$

The set $C_n - E$ is Borel, since C_n and E are. There exists (because of the regularity of the measure) an open set $U = U(\varepsilon)$, which is also a Borel set, such that $U \supset C_n - E$, and

$$(*) \qquad m(U) \leq m(C_n - E) + \varepsilon.$$

(see Lemma 11, p. 25) [Note that $m(E) = \inf_{U \supset E,\ U \text{ open},\ U \text{ Borel}} m(U)$].

If we define $K = C_n - U$, then K is compact (as it is a closed subset of a compact set). Since $U \supset C_n - E$, we have

$$C_n - U \subset C_n - (C_n - E) = C_n \cap E,$$

hence $K \subset C_n \cap E$. Further

$$(C_n \cap E) - K = (C_n \cap E) - (C_n - U) \subset U - (C_n - E).$$

Hence

$$\begin{aligned} m(C_n \cap E) - m(K) &\leq m(U) - m(C_n - E), \quad [m(K) < \infty] \\ &\leq \varepsilon. \quad \text{(see } (*) \text{ above).} \end{aligned}$$

If $m(E) < \infty$, there exists a compact set $K \subset E$, such that

$$\begin{aligned} m(E) - m(K) &= (m(E) - m(C_n \cap E)) + (m(C_n \cap E) - m(K)) \\ &\leq 2\varepsilon \quad \text{(see (†) above).} \end{aligned}$$

If $m(E) = \infty$, there exists a compact set $K \subset E$, such that $m(K) \geq \frac{1}{\varepsilon} - \varepsilon$, [since $m(K) = m(C_n \cap E) - (m(C_n \cap E) - m(K))$, and (†)], which completes the proof of the theorem.

Theorem 5 The measure m in Theorem 3 is unique.

Proof Let m and m' be two regular measures, such that

$$\int_R f(x)\, dm(x) = \int_R f(x)\, dm'(x), \quad \forall\, f \in \mathcal{C}_0.$$

Then we have to show that $m = m'$ on the ring of Borel sets, $\mathcal{S}$. For that, it suffices to show that $m(C) = m'(C)$ for all compact sets $C \in \mathcal{C}$. If C is compact, and $\varepsilon > 0$, there exist *open*, Borel sets U, U', such that $U \supset C$, $U' \supset C$, $m(U - C) < \varepsilon$, $m'(U' - C) < \varepsilon$. Set $V = U \cap U'$. Then V is open, $V \supset C$, $m(V - C) < \varepsilon$, $m'(V - C) < \varepsilon$. We may assume that $\overline{V}$ is compact (R is locally compact, Hausdorff). Then there exists a *continuous* function f, such that

$$f(x) = \begin{cases} 1, & x \in C \\ 0, & x \notin V, \end{cases} \qquad \text{(Lemma, p. 36)}$$

$$0 \leq f(x) \leq 1, \quad \forall\, x.$$

Hence $f \in \mathcal{C}_0$, and $\chi_C \leq f \leq \chi_V$, which implies that

$$m(C) \leq \int_R f\, dm \leq m(V),$$

and similarly

$$m'(C) \leq \int_R f\, dm' \leq m'(V).$$

By hypothesis, $\int_R f\, dm = \int_R f\, dm' = A$, say. Since we have

$$m(V) - m(C) < \varepsilon, \quad m'(V) - m(C) < \varepsilon,$$

it follows that $A - m(C) \leq m(V) - m(C) < \varepsilon$, and similarly $A - m'(C) < \varepsilon$. Thus $m(C) - m'(C) = m(C) - A + A - m'(C)$, and $|m(C) - m'(C)| \leq 2\varepsilon$. Hence $m = m'$ on $\mathcal{C}$, therefore also on $\mathcal{S}$.

If R is a locally compact Hausdorff space, $\mathcal{R}$ the Borel ring of R, with $R \in \mathcal{R}$ (so that $\mathcal{R}$ is, in fact, a σ-algebra) and r is a regular measure on $\mathcal{R}$, we call $\mathcal{M} = (R, \mathcal{R}, r)$ *a regular measure space*. One can consider *integrable functions* on $\mathcal{M}$, and L_p-spaces. The property of regular measures obtained in Theorem 4 enables us to prove the following

Theorem 6 Let $L_p(\mathcal{M})$, $1 \leq p < \infty$, stand for the class of M-measurable functions f such that $|f|^p$ is integrable. Let $\mathcal{C}_0$ be the set of all (real-valued) continuous functions, with compact support, on R. Then $\mathcal{C}_0$ is "*dense*" in $L_p(\mathcal{M})$.

Proof

(i) Any function $f \in L_p(\mathcal{M})$ can be approximated, arbitrarily closely, by a simple function (which is of the form $\sum_{i=1}^{n} a_i \chi_{E_i}$, where $r(E_i) < \infty$, a_i real). Since $f = f^+ - f^-$, where $f^+, f^- \geq 0$, we may assume, without loss of generality, that $f \geq 0$. Since f^p is integrable by assumption, there exists a monotone increasing sequence (g_n) of non-negative, simple functions, such that $g_n(x) \longrightarrow f^p(x)$, as $n \longrightarrow \infty$, for almost all x. Thus $g_n^{1/p} \longrightarrow f$ pointwise. Now

$$||g_n^{1/p} - f||_p^p = \int_{\mathcal{M}} |g_n^{1/p} - f|^p,$$

while $|g_n^{1/p} - f|^p \leq 2|f|^p$. Hence $||g_n^{1/p} - f|| \longrightarrow 0$ as $n \longrightarrow \infty$.

(ii) Any simple function can be "approximated", arbitrarily closely, by a function in $\mathcal{C}_0$. It is obviously sufficient to consider any characteristic function of a 'chunk', i.e. χ_E, where $E \in \mathcal{R}$, $r(E) < \infty$. Since r is a *regular measure*,

$$r(E) = \sup_{C \subset E,\ C \text{ compact}} r(C). \qquad \text{(Theorem 4)}$$

Given $\varepsilon > 0$, there exists an *open* set $U \supset E$, and a *compact* set $C \subset E$ (depending on ε), such that $r(U - C) < \varepsilon$ *and* $\overline{U}$ is compact. [If $\overline{U}$ itself is not compact, there exists an open V, $\overline{V} \subset U$, $V \supset C$; with $\overline{V}$ compact. cf. Lemma 12, p. 36].

By Lemma 12, there exists a continuous function f on R, such that $f(x) = 1$, $\forall\, x \in C$, $f(x) = 0$, $x \notin U$, and $0 \leq f(x) \leq 1$, for all x. Obviously $f \in \mathcal{C}_0$, and

$$\int |f - \chi_E|^p = \int_{U-C} |f - \chi_E|^p \leq r(U - C) < \varepsilon, \qquad (\text{since } |f - \chi_E| \leq 1)$$

so that $||f - \chi_E||_p < \varepsilon^{1/p}$.

II.4 Baire functions

A *Baire function* on any locally compact, Hausdorff space R, is an element of the smallest class $\mathcal{B}$ of real-valued functions on R, which *contains* $\mathcal{C}_0(R)$, and is *closed for pointwise convergence of sequences* of elements.

Note that $\mathcal{B}$ exists, since the set of all functions on R has the two stated properties, and the intersection of any two such classes is again such a class.

Lemma 13 Let $\{f_n\}$, $n = 1, 2, \ldots$, be a sequence of functions in $\mathcal{B}$, such that $|f_n(x)| < \phi(x)$, a real-valued function (so that $\sup(\{f_n\})$ is real-valued). Then we have $\sup(\{f_n\}) \in \mathcal{B}$.

Proof Since $\sup(\{f_n\}) = \lim_{n\to\infty} \sup(f_1, \ldots, f_n)$, (pointwise limit), it is sufficient to show that $g \in \mathcal{B}$, $h \in \mathcal{B} \Longrightarrow \sup(g, h) \in \mathcal{B}$.

Given $g \in \mathcal{B}$, let $\mathcal{A}(g)$ = the class of all functions h, such that $\sup(g, h) \in \mathcal{B}$. Then

(i) $\mathcal{A}(g)$ is closed for pointwise convergence of sequences;

(ii) $\mathcal{A}(g) \supset \mathcal{C}_0$, *if* $g \in \mathcal{C}_0$. (since $\sup(f, f') \in \mathcal{C}_0$, if $f, f' \in \mathcal{C}_0$).

[Note that

$$g \in \mathcal{B}, \quad \sup(g, h_n) \in \mathcal{B}, \quad \forall\, n, \quad h_n \longrightarrow h \;\text{ imply that }\; \sup(g, h) \in \mathcal{B}.$$

(i) $g(x) < h(x) \Longrightarrow \sup(g(x), h_n(x)) = h_n(x)$ for $n \geq n_0$. But $h_n(x) \longrightarrow h(x) = \sup(g(x), h(x))$, hence $\sup(g(x), h_n(x)) \longrightarrow \sup(g(x), h(x))$, as $n \longrightarrow \infty$. Since $\sup(g(x), h_n(x)) \in \mathcal{B}$, and $\mathcal{B}$ is closed for pointwise convergence of sequences, $\sup(g(x), h(x)) \in \mathcal{B}$.

(ii) $g(x) > h(x) \Longrightarrow \sup(g(x), h_n(x)) = g(x)$ for $n \geq n_0$, and $g(x) = \sup(g(x), h(x))$.

(iii) $g(x) = h(x) \Longrightarrow \sup(g(x), h_n(x)) = \sup(h(x), h_n(x)) \longrightarrow \sup(h(x), h(x)) = h$.]

It follows that

$$\mathcal{A}(g) \supset \mathcal{B}, \quad \textit{if} \quad g \in \mathcal{C}_0. \;\text{(since } \mathcal{B} \text{ is the smallest such class)}$$

Hence $g \in \mathcal{C}_0$, $h \in \mathcal{B} \Longrightarrow \sup(g,h) \in \mathcal{B}$, that is to say, $\mathcal{A}(h) \supset \mathcal{C}_0$. It follows, as before, that $\mathcal{A}(h) \supset \mathcal{B}$ when $h \in \mathcal{B}$. Thus

$$g, h \in \mathcal{B} \Longrightarrow \sup(g,h) \in \mathcal{B}.$$

Theorem 7 Let $\mathcal{M}$ be a regular measure space (as defined on p. 45), and let $\mathcal{B}$ be the class of all Baire functions on $\mathcal{M}$. Then (a) *every Baire function is measurable* (= Borel measurable); and (b) if f is *any* (Borel) *measurable function on* $\mathcal{M}$, *there exists a Baire function which equals* f *almost everywhere.*

Proof (a) The class of all *measurable* Baire functions has the two properties: (i) it contains $\mathcal{C}_0$; (ii) it is closed for pointwise convergence of sequences (f_n measurable, $f_n \longrightarrow g$ imply that g is measurable). However, $\mathcal{B}$ is the smallest such class. Hence every Baire function is measurable.

(b) We may assume that $f \geq 0$. There exists a *monotone increasing* sequence $\{f_n\}$ of *quasi-simple* functions, such that $f_n(x) \longrightarrow f(x)$, as $n \longrightarrow \infty$, almost everywhere. (We recall that a quasi-simple function is a finite linear combination, with real coefficients, of characteristic functions of sets from $\mathcal{R}$, the Borel ring on R).

If the result holds for any quasi-simple function f, then it holds in general. For in that case there exists a sequence $\{g_n\}$ of Baire functions, with $g_n = f_n$ almost everywhere, and since f_n is $\uparrow$, $\sup(g_n)$ is a Baire function which equals f almost everywhere. *It is enough therefore to prove* (b) in the *case of* $f =$ *a quasi-simple function* ($\sum_{i=1}^{n} a_i \chi_{E_i}$, $E_i \in \mathcal{R}$).

Now $S \in \mathcal{R}$ implies that $S \subset \bigcup_{n=1}^{\infty} C_n$, C_n compact, and we may assume that $C_i \subset C_{i+1}$, $i = 1, 2, \ldots$. Then we have

$$\chi_{S \cap C_n} \longrightarrow \chi_S, \quad \text{as } n \longrightarrow \infty. \text{ (pointwise)} \quad \left(S = \bigcup_{n=1}^{\infty} S \cap C_n \right).$$

It suffices therefore to prove (b) *in the case* $f = \chi_E$, *where* E *is measurable, and* $E \subset$ *a compact set, hence* $r(E) < \infty$.

Since r is a regular measure, there exists a sequence $\{C_n\}$ of compact sets, with $C_n \subset E$, such that $r(E - C_n) \longrightarrow 0$. We may assume that $C_n \subset C_{n+1}$. It follows that $\chi_E = \chi_{\bigcup_{n=1}^{\infty} C_n}$ almost everywhere, and $\chi_{\bigcup_{n=1}^{\infty} C_n} = \lim_{n \to \infty} \chi_{C_n}$. *It suffices therefore to prove* (b) *in the case* $f =$

χ_C, *where* C *is compact.*

If C is a given compact set, there exist *open* Borel sets U_n, such that $U_n \supset C$, *and* $r(U_n - C) \longrightarrow 0$, as $n \longrightarrow \infty$ (since $r(E) = \inf_{U \supset E,\ U \text{ open } U \text{ Borel}} r(U)$, where E is a Borel set).

Let

$$f_n(x) = \begin{cases} 1, & x \in C; \\ 0, & x \notin U_n; \end{cases}$$

and $0 \leq f_n(x) \leq 1$, $x \in R$, $f_n \in \mathcal{C}_0$ (by Lemma 12, p. 36).

We *may* assume that U_n is $\downarrow$. Set $f = \inf(\{f_n\})$. By Lemma 13 (and symmetry) f is a Baire function which equals 1 on C and 0 outside U_n, $n = 1, 2, \ldots$. Hence f *equals* 0 *outside* $\bigcap_i U_i$. Since $r(U_n - C) \longrightarrow 0$, we have $r(\bigcap_i U_i - C) = 0$. Hence $f = \chi_C$ almost everywhere, where f is a Baire function.

We shall consider Baire functions on the product of two regular measure spaces, which is not necessarily regular although it can be extended to a regular measure space. One has, however, to be cautious about applying, without qualification, Fubini's theorem to the extension.

We recall (from the course on Integration Theory) the basic facts concerning product measures, and measure spaces.

By a *measure space* we mean a triple $(R, \mathcal{R}, r)$, where R is *any set*, R a σ-algebra of subsets of R, and r a countably additive set function on $\mathcal{R}$. Let $\mathcal{R}_0$ denote the set of all elements of $\mathcal{R}$ on which r is *finite*; they are called *chunks.* The measure space is called *σ-finite* if R is a countable union of chunks. If $(S, \mathcal{S}, s)$ is another measure space, we denote by $\mathcal{S}_0$ the chunks in $\mathcal{S}$. By a *rectangle* we mean the cartesian product $A \times B$, where $A \in \mathcal{R}_0$, $B \in \mathcal{S}_0$. It is known that both $\mathcal{R}_0$ and $\mathcal{S}_0$ are rings. The class of all finite disjoint unions of rectangles is a ring, which is, in fact, the ring $\mathcal{P}_0$ generated by the rectangles. For any $E \in \mathcal{P}_0$, say $E = \dot{\bigcup}_{1 \leq i \leq n} P_i$, where $P_i = A_i \times B_i$, $A_i \in \mathcal{R}_0$, $B_i \in \mathcal{S}_0$, we *define*

$$m_0(E) = \sum_{1 \leq i \leq n} r(A_i) \cdot s(B_i).$$

If m' is *any* measure on $\mathcal{P}_0$, such that $m'(A \times B) = r(A) \cdot s(B)$ for *every* rectangle $A \times B$, then (it can be shown that) $m'(E) = m_0(E)$, for all $E \in \mathcal{P}_0$. *The basic result on product spaces is as follows:*

If $(R, \mathcal{R}, r)$ and $(S, \mathcal{S}, s)$ are two σ-finite measure spaces, then there exists a σ-finite measure space, called *the product measure space*, or simply *the product space*, $(P, \mathcal{P}, m)$, such that $P = R \times S$, $\mathcal{P}$ = the σ-ring generated by $\mathcal{P}_0$, and m a countably additive measure on $\mathcal{P}$, which agrees with m_0 on $\mathcal{P}_0$.

The connection between integrals on the product space and on the individual spaces is established in Fubini's theorem, one form of which is as follows:

Let $f \geq 0$ be $\mathcal{P}$-measurable on the product space $(R \times S, \mathcal{P}, r \times s)$ of two σ-finite measure spaces $(R, \mathcal{R}, r)$ and $(S, \mathcal{S}, s)$. Then for every fixed $y \in S$, $f(x, y)$ is measurable as a function of x, and if we set $g(y) = \int_R f(x, y)\, dr(x)$, then g is measurable, and

$$\int_S g(y)\, ds(y) = \iint_{R \times S} f(x, y) d(r \times s)(x, y).$$

Thus if the integral on the product space on the right-hand side is finite, then the 'repeated integral' on the left-hand side is also finite, and both are equal. Further the two repeated integrals, namely $\int_S g(y)\, ds(y)$, and $\int_R (\int_S f(x, y)\, ds(y))\, dr(x)$, are equal.

Against this general background, we now state the following result on Baire functions on the product of two locally compact Hausdorff spaces.

Theorem 8 A Baire function on the product of two locally compact Hausdorff spaces, say R, R', is measurable relative to the product of *any* two regular measure spaces $\mathcal{M}$, $\mathcal{M}'$ whose underlying topological spaces are the given spaces R and R'.

For the proof we need, and will assume, the following lemma, which is a special case of *the Stone-Weierstrass approximation theorem.*

Lemma 14 If R and R' are locally compact, Hausdorff spaces, and $f \in \mathcal{C}_0(R \times R')$, then there exists a sequence $\{f_n\}$ of finite linear combinations of products of elements from $\mathcal{C}_0(R)$ with elements from $\mathcal{C}_0(R')$ which converges pointwise to f, and the convergence is locally uniform (i.e. uniform on compact sets).

Proof of Theorem 8 Let $h \in \mathcal{C}_0(R)$. Then h is measurable in the product space $\mathcal{M} \times \mathcal{M}'$, as a function on $R \times R'$. For we have

$$\{(x, x') \mid h(x) > \alpha\} = \{x \mid h(x) > \alpha\} \times R';$$

Since every element of $\mathcal{C}_0(R)$ is measurable relative to $\mathcal{M}$, it follows that

$\{x \mid h(x) > \alpha\}$ is a measurable set in R. R' is measurable relative to $\mathcal{M}'$. Hence the product $\{x \mid h(x) > \alpha\} \times R'$ is measurable in the product space $\mathcal{M} \times \mathcal{M}'$. Hence every f_n in Lemma 14 is measurable relative to $\mathcal{M} \times \mathcal{M}'$; therefore f is measurable relative to $\mathcal{M} \times \mathcal{M}'$, where $f \in \mathcal{C}_0(R \times R')$. It follows that every Baire function is likewise measurable.

Corollary *Let m, n be two regular measures on a locally compact group G. Let f, g be Baire functions on G. Then $f(y^{-1}x)g(y)$ is measurable, and vanishes outside a countable union of 'rectangles' in the product space:* $(G, m) \times (G, n)$.

Proof. [If $f_0 \in \mathcal{C}_0(G)$, $g_0 \in \mathcal{C}_0(G)$ with supports A and B respectively, then $f_0(x) \cdot g_0(y)$ is a Baire function (with $A \times B$ compact).]

$f(x) \cdot g(y)$ is a Baire function on $G \times G$ (with the product topology). A homeomorphism $(x, y) \longleftrightarrow (y^{-1}x, y)$, of $G \times G$ with itself, carries Baire functions into Baire functions. Hence $f(y^{-1}x)g(y)$ is again a Baire function on $G \times G$, $(x, y \in G)$, hence *measurable*.

Every Baire function on $R = G \times G$ vanishes outside a countable union of compact sets. For the class of all *such* Baire functions is closed for sequential convergence and contains $\mathcal{C}_0$, and therefore contains $\mathcal{B}$, the class of *all* Baire functions (since it is the smallest such).

Hence $f(y^{-1}x)g(y)$ vanishes outside a countable union of compact sets in $G \times G$.

Now let $C \subset G \times G$, C compact. Then $C \subset \bigcup_{i=1}^{n}(O_i \times O_i')$, where O_i, O_i' are *open* chunks in G. (Sets of the form $O_i \times O_i'$ form a basis for the open sets in $G \times G$). Such products are rectangles. Hence

$$\bigcup_{i=1}^{\infty} C_i \subset \bigcup_{i=1}^{\infty} E_i, \qquad \text{where } C_i \text{ is compact, } E_i \text{ is a rectangle.}$$

Thus $f(y^{-1}x)g(y)$ vanishes outside a countable union of rectangles in $G \times G$.

II.5 Essential uniqueness of the Haar measure

To prove that any two Haar measures on a locally compact group are proportional we need some preparation.

If $\mathcal{M} = (R, \mathcal{R}, r)$, $\mathcal{N} = (S, \mathcal{S}, s)$ are two measure spaces, we call a mapping ϕ of R into S *a measurable transformation* of $\mathcal{M}$ into $\mathcal{N}$, *if* for any $F \in \mathcal{S}$, we have $\phi^{-1}(F) \in \mathcal{R}$, and $r(\phi^{-1}(F)) = s(F)$.

Lemma 15 Let T be a measurable transformation of $\mathcal{M}$ into $\mathcal{N}$, and let f be integrable (or non-negative, measurable) on $\mathcal{N}$. Then

$$\int_{\mathcal{N}} f\, ds = \int_{\mathcal{M}} f \circ T\, dr.$$

Proof It is obviously enough to consider $f \geq 0$. If $f(x) = +\infty$ for $x \in A$, where $s(A) > 0$, then the result is trivial. We may therefore assume that f is *finite almost everywhere.* Since T^{-1} (a null set in $\mathcal{N}$) = a null set in $\mathcal{M}$, we may assume that f is *finite everywhere.* The result is true for $= \chi_E$, where E is a measurable set (by assumption), and by linearity it holds for any *quasi-simple* function f. If $f_n \uparrow f$, f_n quasi-simple, then we have

$$\int_{\mathcal{N}} f_n = \int_{\mathcal{M}} f_n \circ T, \quad n = 1, 2, \ldots,$$

and by the monotone convergence theorem,

$$\int_{\mathcal{N}} f = \int_{\mathcal{M}} f \circ T.$$

Lemma 16 Let $W = \bigcup_{n=1}^{\infty} C_n$, where the C_n are compact sets, and W is an open set, with $\overline{W}$ compact. Then χ_W is a Baire function.

Proof We may assume that $C_n \subset C_{n+1}$. Let $f_n(x) = 1$, $x \in C_n$; $f_n(x) = 0$, $x \notin W$; and $0 \leq f_n(x) \leq 1$, (cf. Lemma 12) f_n continuous. Since $\overline{W}$ is compact, $f_n \in \mathcal{C}_0$. And we have $\lim_{n\to\infty} f_n(x) = \chi_W(x)$, for all x. Hence χ_W is a Baire function.

Theorem 9 *If G is a locally compact group, and m, n are* (possibly) *two Haar measures on G, then $m(E) = \alpha n(E)$ for all Borel sets E in G, where α is a positive real constant.*

Proof Let $f, g \in \mathcal{C}_0(G)$, $0 \leq f \leq 1$, $0 \leq g \leq 1$. Let $C = \{x \mid f(x) = 1\}$. $U = \left\{x \mid g(x) > \frac{1}{2}\right\}$. Then C is compact (as a closed subset of a compact set). χ_C is a Baire function, since it is the pointwise limit of the sequence $f, f^2, \ldots, f^n, \ldots$. Further U is open, since g is continuous, and $\overline{U}$ is compact (since it is contained in the support of g), and $U = \bigcup_{i=1}^{\infty} D_i$, where $D_i = \{x \mid g(x) \geq \frac{1}{2} + \frac{1}{i}\}$, $i = 1, 2, \ldots$, D_i compact. By Lemma 16, χ_U *is a Baire function.*

Now $m(C) = \int_G \chi_C(x)\, dm(x)$, and

$$\begin{aligned} n(U) &= \int_G \chi_U(y)\, dn(y) \\ &= \int_G \chi_U(x^{-1}y)\, dn(y) \quad \text{(left-invariance)} \\ [&= \int_G \chi_{xU}(y)\, dn(y) \quad (x^{-1}y \in U \Longleftrightarrow y \in xU) \\ &= n(xU) = n(U)]. \end{aligned}$$

Hence

$$m(C) \cdot n(U) = \left(\int \chi_C(x)\, dm(x) \right) \cdot \left(\int \chi_U(x^{-1}y)\, dn(y) \right).$$

By the Corollary to Theorem 8, and Fubini's theorem, we have

$$m(C) \cdot n(U) = \iint_{G \times G} \chi_C(x) \chi_U(x^{-1}y)\, dp(x, y).$$

However

$$\chi_C(x)\chi_U(x^{-1}y) \leq \chi_{CU}(y) \cdot \chi_U(x^{-1}y).$$

[The left-hand side is 1, if $x \in C$ *and* $x^{-1}y \in U$ or $y \in xU \subset CU$ so that $\chi_{CU}(y) = 1$]
and χ_{CU} *is a Baire function* [$CU = \bigcup_{n=1}^{\infty} CD_n$, where CU is *open*, and CD_n – as the group-product of compact sets – is *compact*. Further $\overline{CU}$ is compact. C is compact, $\overline{U}$ compact $\Longrightarrow C \times \overline{U}$ compact $\Longrightarrow C \cdot \overline{U}$ compact $\Longrightarrow \overline{C} \cdot \overline{U}$ compact (since $C = \overline{C}$) $\Longrightarrow \overline{C} \cdot \overline{U}$ closed. But $C \cdot U \subset \overline{C} \cdot \overline{U}$, so that $\overline{CU} \subset \overline{C} \cdot \overline{U}$ which is compact. So $\overline{CU}$ is compact. By Lemma 16, χ_{CU} is a Baire function]

By the Corollary to Theorem 8, $\chi_{CU}(y)\chi_U(x^{-1}y)$ is measurable on $G \times G$, and from the double integral above we obtain

$$\begin{aligned} m(C)n(U) &\le \iint\limits_{G\times G} \chi_{CU}(y)\chi_U(x^{-1}y)\,dp(x,y) \\ &= \iint \chi_{CU}(y)\chi_{U^{-1}}(y^{-1}x)\,dp(x,y) \quad \text{(Theorem 8)} \\ &= \iint \chi_{CU}(y)[\chi_{U^{-1}}(y^{-1}x)\,dm(x)]\,dn(y) \\ &= \int \chi_{CU}(y)\left[\int \chi_{U^{-1}}(x)\,dm(x)\right]dn(y) \quad \text{(Lemma 15 +} \\ &= n(CU)m(U^{-1}). \quad \text{Fubini +} \end{aligned}$$

Hence $m(C)n(U) \le n(CU)m(U^{-1})$. left-invariance)

By symmetry we have, for any compact set $D = \{x \mid h(x) = 1\}$, where $h \in \mathcal{C}_0(G)$, $0 \le h \le 1$, with U^{-1} in place of U, and n instead of m, [U^{-1} is related to $g(x^{-1})$ in $\mathcal{C}_0$ just as U is to $g(x)$]

$$n(D)m(U^{-1}) \le m(DU^{-1})n(U).$$

[$U^{-1} = \bigcup_{n=1}^{\infty} D_n^{-1}$, $\overline{U^{-1}} = \overline{U}^{-1}$, $\chi_{U^{-1}}$ is Baire]. It follows that

$$m(C)\cdot n(U)\cdot n(D)\cdot m(U^{-1}) \le n(CU)m(U^{-1})m(DU^{-1})n(U).$$

Now let V be a *non-empty, open* set. Then $m(V) \ne 0$. [Otherwise $m(C) = 0$ for all compact C]. Choose $g \in \mathcal{C}_0(G)$, such that $U \ne \emptyset$, [recall: $U = \left\{x \mid g(x) > \frac{1}{2}\right\}$] hence $n(U) \ne 0$, $m(U^{-1}) \ne 0$. Then we obtain from the above inequality,

$$(*) \qquad m(C)\cdot n(D) \le n(CU)m(DU^{-1}).$$

Since C, D are compact, and m, n are regular measures, given $\varepsilon > 0$, there exist *open, Borel sets* V, W, such that $C \subset V$, $D \subset W$, and

$$(\dagger) \qquad \begin{aligned} n(V) &< n(C)+\varepsilon, \\ m(W) &< m(D)+\varepsilon. \end{aligned}$$

Let U_1, U_2 be *open* sets containing e (the identity element), such that $CU_1 \subset V$, $DU_2 \subset W$, with $\overline{U}_1, \overline{U}_2$ compact.

[There exist such U_1 and U_2. (i) Let C be compact, $C \subset S$, and S open. Then $x \in C \Longrightarrow \exists\ N_x$, N_x open, $x \in N_x$, such that $\overline{N}_x$ is compact. (ii) $x \in C \Longrightarrow \exists\ O_x$, an open neighbourhood of e, such that $xO_x \subset S$. There exists then an open neighbourhood T_x of e, such that

$T_x^2 \subset O_x$ (cf. trivial properties (5) and (6), p. 6).(iii) xT_x is now an open neighbourhood of $x \in C$. Hence $C \subset \bigcup_x xT_x$. But C is compact, so there exist $x_1, \ldots, x_n$ such that $C \subset \bigcup_{i=1}^n x_i T_{x_i}$. Set $T = \bigcap_{j=1}^n T_{x_j}$. Then T is open, T contains e. And

$$CT \subset \bigcup_{i=1}^{n} x_i T_{x_i} T_{x_i} = \bigcup_{i=1}^{n} x_i T_{x_i}^2 \subset \bigcup_{i=1}^{n} x_i O_{x_i} \subset S$$

(since $x_i O_{x_i} \subset S$ for every i). Now take $S = V$, and $T = U_1$. Similarly treat D, W, V_2].

Let $U_3 \subset U_1 \cap U_2$, U_3 *symmetric*, $e \in U_3$; and let U_4 be open, such that $e \in U_4$, *and* $\overline{U}_4 \subset U_3$. [Note that $\overline{U}_4$ is compact, since $\overline{U}_4 \subset U_3 \subset \overline{U}_3 \subset \overline{U_1 \cap U_2} \subset \overline{U}_1 \cap \overline{U}_2$ which is compact].

Let

$$g \in C_0(G), \quad g(x) = \begin{cases} 1, & x \in \overline{U}_4, \\ 0, & x \notin U_3, \end{cases} \quad 0 \le g(x) \le 1,$$

and define U (as above) by means of *this* function g, i.e. $U = \left\{ x \mid g(x) > \dfrac{1}{2} \right\}$. Then obviously U is *open*, non-empty ($\overline{U}_4 \subset U \implies e \in U$), and $\overline{U}$ is compact (since $\overline{U} \subset$ support of g) *and* $CU \subset V$, $DU^{-1} \subset W$.

[Note that U is a set on which $g \ge \dfrac{1}{2}$, while U_3 is a set on which $0 \le g \le 1$. We have $\overline{U}_4 \subset U_3 \subset U_1$, and $U \subset U_3$, so that $CU \subset CU_3 \subset CU_1 \subset V$; and $U^{-1} \subset U_3^{-1} = U_3$, so that $DU^{-1} \subset DU_3 \subset DU_2 \subset W$].

The inequality (*) above then gives

$$m(C)n(D) \le n(V)m(W),$$

which implies, by (†),

$$m(C)n(D) \le (n(C) + \varepsilon)(m(D) + \varepsilon),$$

or

$$m(C)n(D) \le n(C) \cdot m(D).$$

With D in place of C, we get

$$m(D)n(C) \le n(D) \cdot m(C),$$

hence

$$m(C)n(D) = n(C)m(D)$$

Since the measures m and n are *regular*, and non-zero, there exists, by Theorem 4, a compact set C_1, with $m(C_1) \neq 0$. Now define

$$f \in \mathcal{C}_0; \quad f = 1 \text{ on } C_1, \quad m(C_1) \neq 0; \quad 0 \leq f \leq 1.$$

Let $C = \{x \mid f(x) = 1\}$. Then $m(C) \neq 0$, and $n(D) = \alpha m(D)$, where α is a constant for all D, where $D = \{x \mid h(x) = 1\}$, $0 \leq h \leq 1$, h being an arbitrary element of $\mathcal{C}_0$, $\alpha > 0$.

If E is an ***arbitrary compact set***, there exist sequences $\{Y_i\}$, $\{Z_i\}$ of *open*, ***Borel sets***, such that $Y_i \supset E$, $m(Y_i - E) \longrightarrow 0$ and $Z_i \supset E$, $n(Z_i - E) \longrightarrow 0$, as $i \longrightarrow \infty$.

Set $W_i = Y_i \cdot \cap \cdot Z_i$. Then (W_i) is a sequence of *open*, ***Borel sets***, such that $W_i \supset E$, and $m(W_i - E) \longrightarrow 0$, $n(W_i - E) \longrightarrow 0$, as $i \longrightarrow \infty$.

Now define $f_i \in \mathcal{C}_0$, $0 \leq f_i \leq 1$, $f_i = \begin{cases} 1 \text{ on } E, \\ 0 \text{ on } W_i^c, \end{cases}$

and let $D_i = \{x \mid f_i(x) = 1\}$. Then, as above, we have $m(D_i) = \alpha \cdot m(D_i)$. Since $W_i \supset D_i \supset E$, we have $m(W_i) \geq m(D_i) \geq m(E)$, and $n(W_i) \geq n(D_i) \geq n(E)$. $[D_i \supset E$, since $f_i = 1$ on E; $x \in W_i \Longrightarrow 0 \leq f(x) \leq 1$; $x \in D_i \Longrightarrow f_i(x) = 1]$.

It follows that $m(D_i) \longrightarrow m(E)$, $n(D_i) \longrightarrow m(E)$, as $i \longrightarrow \infty$, hence $n(E) = \alpha m(E)$, for ***any compact*** E. This holds for all Borel sets E, by Theorem 4.

Examples of Haar integrals

1. $G = (\mathbb{R}^n, +)$, $\mu(f) = \int_{R^n} f(x)\,dx$ (n-dimensional Lebesgue integral).

2. $G = S^1 = \mathbb{R}/Z$, $f : S^1 \longrightarrow \mathbb{R}$, equivalent to $f^* : \mathbb{R} \longrightarrow \mathbb{R}$, of period 1, $\mu(f^*) = \int_0^1 f^*(x)\,dx$.

3. $S^1 = \{e^{i\theta}, \ 0 \leq \theta \leq 2\pi\}$, $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(e^{i\theta})\,d\theta$.

4. $G =$ finite discrete group of order n, $L^1(G) =$ all functions $f : G \longrightarrow \mathbb{R}$, and $\dfrac{1}{n}\displaystyle\sum_{x \in G} f(x)$.

An ***automorphism*** of a topological group G is a homeomorphism of G with itself such that $A(ab) = A(a) \cdot A(b)$, where $a, b \in G$, and $\cdot$ denotes the group operation.

Corollary 1 Let $\mathcal{A}$ be the class of all automorphisms of the locally compact group G. Then there exists a real-valued, non-zero, function σ on $\mathcal{A}$, such that

$$m(A(E)) = \sigma(A) \cdot m(E)$$

where E is a Borel set, and m a Haar measure on G. Further σ has the property: $\sigma(AA') = \sigma(A) \cdot \sigma(A')$ for $A, A' \in \mathcal{A}$. [If A and A' are automorphisms of G, the product AA' of A and A' is the automorphism defined by the requirement: $(AA')(x) = A(x)A'(x)$, $x \in G$. It can be verified that AA' *is* an automorphism].

Proof $m(A(E))$, as a *function of* E, is again a Haar measure. Theorem 9 implies that there exists a constant $\sigma(A)$, depending only on A, such that

$$m(A(E)) = \sigma(A) \cdot \sigma(E).$$

Since all Haar measures on G are proportional, $\sigma(A)$ is a constant *for all measures* m.

Now

$$m((AA')E) = \sigma(AA')m(E).$$

On the other hand,

$$m((AA')E) = m(A(A'(E))) = \sigma(A) \cdot m(A'(E)) = \sigma(A) \cdot \sigma(A') \cdot m(E).$$

Since $m(E) \neq 0, \infty$ for at least one E, we obtain $\sigma(AA') = \sigma(A) \cdot \sigma(A')$.

Corollary 2 Let m be a left-invariant Haar measure on G. There exists a real-valued, *continuous* function ρ on G, such that $m(Ex) = \rho(x) \cdot m(E)$, where E is a *Borel set* and $x \in G$; ρ has the property $\rho(xy) = \rho(x) \cdot \rho(y)$, $x, y \in G$.

Proof If E is Borel, $x \in G$, then Ex is Borel, also E^{-1}. [If $\mathcal{K} = \{E$ Borel $\mid Ex$ Borel$\}$, then $\mathcal{K}$ is a σ-ring, and $\mathcal{K}$ contains all compact sets].

Consider the inner automorphism $\phi_x : a \longmapsto x^{-1}ax$, $a, x \in G$; and apply Corollary 1. Then we have $m(x^{-1}Ex) = \rho(x)m(E)$, where E is Borel, and $\rho(x) = \sigma(\phi_x)$. Since $\phi_y \cdot \phi_x = \phi_{x \cdot y}$, it follows that $\rho(xy) = \rho(x) \cdot \rho(y)$ for $x, y \in G$. Further $0 < \rho(x) < \infty$. Because of the left-invariance, $m(Ex) = \rho(x)m(E)$.

To show that ρ is *continuous*, let $f \in C_0(G)$. Then

$$(*) \qquad \int f(a)\, dm_x(a) = \rho(x) \int f(a)\, dm(a),$$

where $m_x(E) = m(Ex)$, hence $m(E) = m_x(Ex^{-1})$. By Lemma 15 [$a \overset{T}{\longmapsto} ax^{-1}$, $\int_{\mathcal{N}} f\, ds = \int_{\mathcal{M}} f \circ T\, dr$; $s(F) = r(T^{-1}(F))$, $F \in \mathcal{S}$, $T^{-1}(F) \in \mathcal{R}$, $T^{-1}(b) = bx$].

$$\int f(a)\, dm_x(a) = \int f(ax^{-1})\, dm(a).$$

If C is a compact set, such that f vanishes outside C, then

$$\int f(a)\, dm(a) = \int_C f(a)\, dm(a). \quad \text{(some number)}$$

We shall see that

$$\int f(ay)\, dm(a),$$

as a function of y, is continuous at $y = e$.

Let N be a *compact, symmetric neighbourhood of* e, then we have, for $y \in N$,

$$\begin{aligned} \left| \int f(ay)\, dm(a) - \int f(a)\, dm(a) \right| &\leq \int |f(ay) - f(a)|\, dm(a) \\ &= \int_{CN} |f(ay) - f(a)|\, dm(a). \end{aligned}$$

[$C \subset CN$; $a \notin CN \Longrightarrow a \notin C \Longrightarrow f(a) = 0$. And $ay \notin C \Longrightarrow f(ay) = 0$].

It can be shown (see Lemma 17, Remarks, which come below) that

$$|f(ay) - f(a)| \longrightarrow 0, \quad \text{uniformly as } y \longrightarrow e.$$

Hence $\int |f(ay) - f(a)|\, dm(a) \longrightarrow 0$, as $y \longrightarrow e$.(CN is compact, so that $m(CN) < \infty$). Hence $\int f(ax^{-1})\, dm(a)$ is a continuous function of x at $x = e$.

Let $f \in \mathcal{C}_0$, such that $\int f(a)\, dm(a) \neq 0$. Such an f exists since m is not identically zero. It then follows from $(*)$ above that $\rho(x)$ is continuous at $x = e$. Since $\rho(x) = \rho(xx_0^{-1})\rho(x_0)$, ρ is continuous at $x = x_0$, for any $x_0 \in G$.

Unimodular groups A locally compact group G is called unimodular if $\rho = 1$ in Corollary 2, i.e., "m is left-invariant" implies that "m is right-invariant".

Corollary 3 Every compact or abelian group is unimodular. If m is a Haar measure on a unimodular group G, then

$$m(E^{-1}) = m(E), \quad \text{for any Borel set } E.$$

Proof If G is compact, $m(Gx) = \rho(x)m(G)$, where m is a Haar measure, and $Gx = G$. Hence $\rho(x) = 1$ (since $m(G) < \infty$).

Every abelian group is obviously unimodular. If G is unimodular, then *define* $n(E) = m(E^{-1})$, where E is a Borel set. n is a Haar measure, [Note that $n(Ex) = m((Ex)^{-1})) = m(x^{-1}E^{-1}) = m(E^{-1}) = n(E)$, because m is left-invariant. And $n(xE) = m((xE)^{-1}) = m(E^{-1}x^{-1}) = m(E^{-1})$ by right-invariance] as can be verified.

Since all Haar measures on G are proportional, we have

$$m(E^{-1}) = \alpha \cdot m(E), \qquad \alpha > 0;$$

that is, $m(E) = m((E^{-1})^{-1}) = \alpha m(E^{-1}) = \alpha^2 m(E)$, so that $\alpha^2 = 1$, or $\alpha = 1$ (since $\alpha > 0$).

II.6 The L^1-algebra of a locally compact group

Our aim is to show that the integrable functions on a locally compact group, relative to a Haar measure, form an associative algebra relative to the usual addition and scalar multiplication and the operation of 'convolution' as multiplication.

Lemma 17 Let $f \in C_0(G)$. Then, given $\varepsilon > 0$, there exists a neighbourhood U of e, such that

$$|f(x) - f(y)| < \varepsilon, \quad \text{for } x^{-1}y \in U.$$

Proof Set $y = xs$, so that $x^{-1}y = s$. We seek a neighbourhood U of e, such that $s \in U \implies |f(x) - f(xs)| < \varepsilon$, $\forall\, x \in G$.

Define $V = \{s \in G \mid |f(x) - f(xs)| < \varepsilon, \ \forall\, x \in G\}$. Then $e \in V$. We shall see that V contains an open set U containing e.

On $G \times G$ define h as follows: $h(x, s) = |f(x) - f(xs)|$, $x, s \in G$. Then h is continuous at (x, e) with $h(x, e) = 0$, $\forall\, x \in G$, and $V = \{s \in G \mid h(x, s) < \varepsilon, \ \forall\, x \in G\}$. By hypothesis there exists a *compact* set C, such that $f = 0$ on $G - C$.

Let V be a *symmetric* neighbourhood of e such that $\overline{V}$ is *compact* and *symmetric* (for example, $\overline{V} = N \cap N^{-1}$, N being a compact neighbourhood of e). Define

$$D = C\overline{V} = \{xy \mid x \in C,\ y \in \overline{V}\}.$$

Then D is compact ($C \times \overline{V}$ is compact, and $(x, y) \longmapsto xy$ is continuous). Obviously $C \subset D$ (since $e \in V$).

Given $y \in D$, there exists a neighbourhood N_y of (y, e) such that $h(x, s) < \varepsilon$, for $(x, s) \in N_y$ (since h is continuous at (y, e), with $h(y, e) = 0$). We may assume that N_y is of the form $A_y \times U_y$, where A_y is a neighbourhood of y, and U_y the corresponding neighbourhood of e.

Since D is compact, there exist $y_1, \dots, y_n$ such that $D \subset \bigcup_{i=1}^{n} A_{y_i}$. Now define $U = \bigcap_{i=1}^{n} U_{y_i} \cdot \cap \cdot V$, the U_{y_i} "corresponds" to A_{y_i}. Then U is open and contains e.

Now $x \in G \Longrightarrow x \in D$ or $x \notin D$.

(i) $x \in D \Longrightarrow x \in A_{y_i}$ for an i, $1 \leq i \leq n$. And $s \in U \Longrightarrow s \in U_{y_i}$ (the same i).
Hence $x \in D$, $s \in U$ $\Longrightarrow (x, s) \in N_{y_i}$ (since $N_{y_i} = A_{y_i} \times U_{y_i}$).
$\Longrightarrow h(x, s) < \varepsilon$.

(ii) $x \notin D \Longrightarrow x \notin C$ (since $C \subset D$) $\Longrightarrow f(x) = 0$, and $x \notin D$, $s \in U \Longrightarrow xs \notin C \Longrightarrow f(x, s) = 0$ (since $s \in U \Longrightarrow s \in V \Longrightarrow s^{-1} \in V$, and $xs \in C \Longrightarrow x \in Cs^{-1} \subset CV \subset D$). Hence $h(x, s) = 0$, $\forall\, x \in G$, $s \in U$.

Remarks By Lemma 17, if $f \in C_0(G)$, there exists a neighbourhood U of e such that $|f(y) - f(x)| < \varepsilon$ for $x^{-1}y \in U$. That is to say, $|f(xt) - f(x)| < \varepsilon$, for $t \in U$ and $\forall\, x \in G$.

By considering $x \longrightarrow f(x^{-1})$, we obtain similarly $|f(x) - f(y)| < \varepsilon$, for $xy^{-1} \in U$, hence $|f(t^{-1}x) - f(x)| < \varepsilon$, for $t \in U = U(e)$, $\forall\, x \in G$.

Theorem 10 Let G be a locally compact group, and m a Haar measure on G. Let $f \in L^p(G)$, $1 \leq p < \infty$ (relative to that measure). Let $f_a(x) = f(a^{-1}x)$, for $x, a \in G$. Then the mapping $a \longmapsto f_a$ from G to $L^p(G)$ is continuous, that is to say

$$\|f_a - f_{a_0}\|_p \longrightarrow 0, \quad \text{as } a \longrightarrow a_0. \quad (a, a_0 \in G)$$

Proof Let $\varepsilon > 0$, and let $g \in \mathcal{C}_0(G)$ be such that $||f - g||_p < \varepsilon$, after Theorem 6. Because of the left-invariance of the measure m, we have

$$(*) \qquad ||f_a - g_a||_p = ||f - g||_p, \quad \text{for} \quad a \in G.$$

Let C be a compact set such that $g(x) = 0$ for $x \notin C$. Since $f_a - f = (f_a - g_a) + (g_a - g) + (g - f)$, we have

$$||f_a - f||_p \leq ||f_a - g_a||_p + ||g_a - g||_p + ||g - f||_p.$$

Since $g_a(x) - g(x) = 0$ for $x \notin aC \cdot \cup \cdot C$, and

$$|g_a(x) - g(x)| < \varepsilon, \text{ for } x \in aC \cdot \cup \cdot C, \ a \longrightarrow e, \text{ (by Lemma 17)}$$

we have

$$||g_a - g||_p < \left(\int_{aC \cup C} \varepsilon^p \, dm(x) \right)^{1/p},$$

which implies that

$$||f_a - f||_p < 2\varepsilon + 2\varepsilon(m(C))^{1/p} = \varepsilon(2 + 2(m(C))^{1/p}). \qquad [m(C) = m(aC)]$$

Hence $f_a \longrightarrow f$ in L^p-norm as $a \longrightarrow e$. However,

$$(f_a)_b(x) = f(b^{-1}a^{-1}x) = f_{ab}(x), \quad \text{hence} \quad f_a = (f_{aa_0^{-1}})_{a_0}.$$

As $a \longrightarrow a_0$, $aa_0^{-1} \longrightarrow e$, hence $||f_{aa_0^{-1}} - f||_p \longrightarrow 0$. It follows that

$$||f_{(aa_0^{-1})a_0} - f_{a_0}||_p \longrightarrow 0, \quad \text{i.e.} \quad ||f_a - f_{a_0}||_p \longrightarrow 0 \qquad \text{(cf. } (*) \text{ above)}.$$

The convolution Let G be a locally compact group, and m denote a Haar measure on G. The convolution of two functions f and g on G, denoted $f * g$, is said *to exist at a point* $x \in G$ if $f(y)g(y^{-1}x)$ is y-integrable, and

$$f * g(x) \underset{def}{=} \int f(y)g(y^{-1}x)\, dy,$$

where "dy" stands for the measure m (in Theorem 1). The convolution $f * g$ exists if it exists *almost everywhere on* G.

Remark If f and g are measurable, and vanish outside Borel sets, then $f(y)g(y^{-1}x)$ is measurable (as a function of y) for fixed x. For there exist Baire functions $f'(y)$, $(g'(y)$ which are almost everywhere equal to $f(y)$ and $g(y)$ respectively, so that $f(y)g(y^{-1}x) = f'(y)g'(y^{-1}x)$ for almost every y. Now $g'(y^{-1}x)$, as a function of y, is Baire, so that $f'(y)g'(y^{-1}x)$ is Baire, hence measurable.

Theorem 11 If $f \in L^1(G)$, $g \in L^p(G)$, $1 \le p \le \infty$, then $f * g$ exists, $f * g \in L^p(G)$, and $||f * g||_p \le ||f||_1 \cdot ||g||_p$. ($G$ is a locally compact group).

Proof This is trivial in case $p = \infty$, for $||f||_\infty = \inf_N \sup_{x \notin N} |f(x)|$, where N denotes *any* null set. We may therefore assume that $p < \infty$.

Let f be *m-integrable* on G, m being a Haar measure. Then f vanishes outside a countable union of "chunks", i.e. sets E_n with $m(E_n) < \infty$. [Here measurable $\Longleftrightarrow$ Borel]. For every such E_n there exists a compact set E'_n, such that $m(E_n - E'_n) < \varepsilon$, for any given $\varepsilon > 0$. [Theorem 4] Hence *f vanishes outside the union of a Borel set and a null set*, $[f(x) = 0, x \notin \bigcup E_n;\ E_n = E_n - E'_n \cup E'_n\!:\ \bigcup E_n = \bigcup E'_n \cdot\cup\cdot \bigcup(E_n - E'_n)]$ Hence [Theorem 7(b)] f is equal to a Baire function almost everywhere. The same is true of g.

Set
$$h(x) = \int |f(y)g(y^{-1}x)|\, dm(y).$$
By Fubini's theorem [$|f(y)g(y^{-1}x)| \ge 0$, and measurable on the product space], h is m-measurable, and
$$h(x) = \int |(f(y)^{1/p}g(y^{-1}x)| \cdot |f(y)^{1/q}|\, dm(y), \qquad \frac{1}{p} + \frac{1}{q} = 1.$$
By Hölder's inequality,
$$h(x) \le \left(\int |f(y) \cdot g^p(y^{-1}x)|\, dy\right)^{1/p} \left(\int |f(y)|\, dy\right)^{1/q},$$
and
$$(h(x))^p \le \int |f(y)| \cdot |g(y^{-1}x)|^p\, dy \cdot ||f||_1^{p/q}.$$
By Fubini's theorem again,
$$\begin{aligned}\int (h(x))^p\, dx &= ||f||_1^{p/q} \cdot \int \left(\int |f(y)| \cdot |g(y^{-1}x)|^p\, dx\right) dy \\ &= ||f||_1^{p/q} \cdot \int |f(y)| \cdot ||g||_p^p\, dy \quad \text{(by left-invariance)} \\ &= ||f||_1^{1+p/q} \cdot ||g||_p^p.\end{aligned}$$

Hence $h(x)$ is finite almost everywhere. It follows that $f*g$ exists. Since $|f*g(x)| \le h(x)$, we have

$$\begin{aligned} ||f*g||_p &\le ||h||_p \\ &\le (||f||_1^{1+p/q} \cdot ||g||_p^p)^{1/p} \\ &= ||f||_1 \cdot ||g||_p. \end{aligned}$$

[Note that if $f \ge 0$, $g \ge 0$, then $h = f*g \ge 0$, and $||f*g||_1 = ||f||_1 \cdot ||g||_1$.]

Theorem 12 Let G be a unimodular group, and let $f \in L^p(G)$, $g \in L^q(G)$, $1 \le p \le \infty$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then $f*g$ exists and is continuous, with

$$|f*g| \le ||f||_p \cdot ||g||_q.$$

Proof Consider $g(y^{-1}x)$ as a function of y. It belongs to $L^q(G)$, with the norm $||g||_q$, since $\int |g(y^{-1}x)|^q\, dy = \int |g(yx)|^q\, dy^{-1}$ [by Lemma 15; dy^{-1} denotes the measure $m'(E) = m(E^{-1})$, where m is a *right* Haar measure. By Corollary 3 to Theorem 9, $m(E^{-1}) = m(E)$.], and $\int |g(yx)|^q\, dy = \int |g(y)|^q\, dy$, because of the right-invariance. Since the product of an element in $L^p(G)$ with an element in $L^q(G)$ is *integrable*, the convolution *$f*g$ exists everywhere.* $[|(f*g)(x)| \le (\int |f(y)|^p\, dy)^{1/p} (\int |g(y^{-1}x)|^q\, dy)^{1/q} = ||f||_p \cdot ||g||_q]$.

To prove the continuity, we note that

$$(f*g)(x) - (f*g)(x') = \int f(y)\{g(y^{-1}x) - g(y^{-1}x')\}\, dy,$$

hence

$$|(f*g)(x) - (f*g)(x')|$$

$$\le \left(\int |f(y)|^p\, dy\right)^{1/p} \left(\int |g(y^{-1}x) - g(y^{-1}x')|^q\, dy\right)^{1/q}$$

Writing $h(x) = g(x^{-1})$, and as before $h_a(x) = h(a^{-1}x)$, we have

$$\begin{aligned} \int |g(y^{-1}x) - g(y^{-1}x')|^q\, dy &= \int |h(x^{-1}y) - h(\underbrace{x'^{-1} \cdot x} \cdot x^{-1}y)|^q\, dy \\ &= ||h_{x^{-1}x'} - h||_q^q, \end{aligned}$$

where $h \in L^q(G)$, [since $\int |h(x)|^q\, dx = \int |g(x^{-1})|^q\, dx = \int |g(y)|^q\, dy^{-1} = \int |g(y)|^q\, dy$ by hypothesis], so that, by Theorem 10, $||h_{x^{-1}x'} - h||_q \longrightarrow 0$ as $x' \longrightarrow x$. It follows that

$$(f * g)(x') \longrightarrow (f * g)(x), \quad \text{as } x' \longrightarrow x.$$

Theorem 13 *If G is a locally compact group, with a Haar measure m, the m-integrable functions on G form an (associative) algebra relative to ordinary addition and scalar multiplication and relative to convolution as multiplication.*

Remarks

(i) The associativity is a consequence of Fubini's theorem for Baire functions.

If f, g are *integrable* Baire functions on G, then $f(y)g(y^{-1}x)$ is an *integrable* Baire function on $G \times G$. Take $p = 1$ in Theorem 11. Then $||f * g||_1 \leq ||f||_1||g||_1 < \infty$. Hence $f * g$ is integrable with respect to x. That is to say, $\int f(y)g(y^{-1}x)\, dy$ is not only measurable, but integrable with respect to x almost everywhere. Hence the iterated integral exists, therefore (strong form!) the double integral exists and is equal.

(ii) The L^1-*algebra* of G is also called the '*group algebra*' of G. In the general case of *locally compact* groups, the L^p-spaces, for $p > 1$, do not form algebras; in the case of *compact* groups they do.

III. Hilbert spaces and the spectral theorem

III.1 Banach spaces

A ***Banach space*** over the complex numbers $\mathbb{C}$, or the real numbers $\mathbb{R}$, is a ***linear space*** (over $\mathbb{C}$ or $\mathbb{R}$), with a ***norm*** '$\|\ \|$', such that the space is ***complete*** with respect to the "metric" $d(x,y) = \|x-y\|$ defined by the norm. [A norm is a function '$\|\ \|$", which is ***non-negative***, and ***real-valued***, with the properties: (i) $\|ax\| = |a|\cdot\|x\|$, $a \in \mathbb{C}$; (ii) $\|x+y\| \leq \|x\|+\|y\|$; (iii) $\|x\| = 0 \iff x = 0$.]

Example $1 \leq p < \infty$, $L^p(M)$, $f \in L^p(M)$, $\|f\| = (\int |f|^p\, dr)^{1/p}$, where M is a ***measure space***: $M = (R, \mathcal{R}, r)$, R a set, $\mathcal{R}$ a σ-algebra of subsets of R, r a countably additive measure on $\mathcal{R}$.

A set A is said to be ***partially ordered***, if and only if a relation "$\leq$" is defined in A, such that for $\alpha, \beta, \gamma \in A$ we have (i) $\alpha \leq \alpha$, (ii) $\alpha \leq \beta$, $\beta \leq \gamma \Longrightarrow \alpha \leq \gamma$, and (iii) $\alpha \leq \beta$, $\beta \leq \alpha \Longrightarrow \alpha = \beta$.

A partially ordered set A is called a ***directed set*** if and only if for $\alpha, \beta \in A$ there exists $\gamma \in A$, such that $\alpha \leq \gamma$, $\beta \leq \gamma$.

By a ***sequence*** is meant a family of elements $\{x_\alpha\}$ where the indices α belong to a ***directed set***. (i.e. not necessarily integers)

If $x_\alpha \in X$, where X is a ***topological space***, x_α is said to ***converge*** to x ($x \in X$) if and only if, for every neighbourhood O of x, there exists an index α_0, such that $x_\alpha \in O$, for all $\alpha \geq \alpha_0$. (The ***Moore-Smith convergence***) [We do not assume any countability axioms in the space considered: so the notion of convergence needs to be generalized so as to allow subscripts other than integers].

Given a family of elements $\{x_i\}$ in an additively written abelian group - so, in particular, the x_i's may be from a Banach space - we define the ***infinite*** sum $\sum_i x_i$ as follows: consider the set A of all ***finite*** subsets of the indices i, made into a directed set by the relation of set-inclusion.

For every $\alpha \in A$, $\alpha = \{i_1, \ldots, i_k\}$, define the "partial sum" $S_\alpha = \sum_{j=1}^{k} x_{i_j}$. ***Define*** $\sum_i x_i = x$, if and only if the sequence (S_α) ***converges*** to x in the sense just defined.

Note that if x_i's are real numbers, with $\mathbb{N}$ as the index set, infinite sums converge in *this* sense, if and only if they are *absolutely* convergent in the ordinary sense, since no linear order is imposed on the indices.

If X is a *metric space*, $\{x_\alpha\}$ is said to be a *Cauchy sequence* if and only if given $\varepsilon > 0$, there exists an α_0, such that $d(x_\alpha, x_\beta) < \varepsilon$, $\forall\ \alpha, \beta \geq \alpha_0$, where d denotes the metric in X. The space X is said to be *complete* if every Cauchy sequence is "convergent".

Proposition If X is a complete metric space in the ordinary sense, then every Cauchy sequence in the generalized sense is convergent in the generalized sense.

Proof Let (x_α) be a Cauchy sequence in the generalized sense. Given a positive integer n, choose an index α_n such that $d(x_\alpha, x_{\alpha_n}) < \dfrac{1}{n}$, $\forall\ \alpha \geq \alpha_n$, *and* such that $\alpha_m \geq \alpha_n$ for $m \geq n$. This is possible since A (the set of all finite subsets of indices i) is a directed set. Then (x_{α_n}) is a Cauchy sequence in the ordinary sense. Since X is complete by assumption, it converges to $x \in X$. Because of the choice of α_n, it follows that x_n converges to X in the generalized sense.

III.2 Hilbert spaces

A *Hilbert space* H is a Banach space over $\mathbb{C}$, in which the norm is given by an *inner product.*

An *inner product* is a complex-valued function '(,)' defined on $H \times H$, such that for $x, y, z \in H$, and $a, b \in \mathbb{C}$, we have

(i) $(ax + by, z) = a(x, z) + b(y, z)$,

(ii) $(x, y) = \overline{(y, x)}$,

(iii) $(x, x) \geq 0$ and $(x, x) = 0 \Longleftrightarrow x = 0$,

(iv) $(x, x)^{1/2} = ||x||$.

Example $L_2(\mathbb{R})$ with $(f, g) = \int f(x)\overline{g(x)}\, dx$, $f = g \Longleftrightarrow f(x) = g(x)$

almost everywhere.

$$\ell_2: \quad x = (x_n),\ y = (y_n), \quad \text{with} \quad \sum |x_n|^2 < \infty,\ \sum |y_n|^2 < \infty,$$
$$(x, y) = \sum x_n \overline{y}_n \qquad\qquad x_n, y_n \in \mathbb{C}$$

Remarks Let $\lambda \in \mathbb{C}$.

(i) $(x, \lambda y) = \overline{(\lambda y, x)} = \overline{\lambda}\ \overline{(y, x)} = \overline{\lambda}(x, y), \quad x, y \in H,\ \lambda \in \mathbb{C}$.

(ii) $(x, y+z) = \overline{(y+z, x)} = \overline{(y, x)} + \overline{(z, x)} = (x, y) + (x, z)$, $\forall\ x, y, z \in H$.

(iii) $(x, 0) = (x, 0 \cdot x) = 0 \cdot (x, x) = 0, \ \ \forall\, x \in H$. In particular, $(0, 0) = 0$.

Lemma (Schwarz). $|(x, y)| \leq ||x|| \cdot ||y||$.

Proof If $(x, y) = 0$, this is evident; otherwise, for $a \in \mathbb{C}$, we have $0 \leq (x - ay, x - ay) = (x, x) - a(y, x) - \overline{a}(x, y) + a\overline{a}(y, y)$. Choose $a = \dfrac{(x, y)}{(y, y)}$. [Then

$$(x, x) - \frac{(x, y)(y, x)}{(y, y)} - \frac{(y, x)(x, y)}{(y, y)} + \frac{(x, y)(y, x)}{(y, y)} \geq 0$$

$\Longrightarrow (x, x)(y, y) \geq (x, y)(y, x) = |(x, y)|^2]$.

Remarks (i) If (,) is an inner product with the properties (i), (ii), (iii) as above, and the function '$||\ ||$' is defined by (iv) as above, then '$||\ ||$' is automatically a norm. This follows from Lemma 1.

(ii) The function (x, y) is '*conjugate-linear*' in y, i.e.

$$(z, ax + by) = \overline{a}(z, x) + \overline{b}(z, y), \quad a, b \in \mathbb{C}, \quad x, y, z \in H.$$

Lemma 2 If $x, y \in H$, we have

$$||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2$$

(for $||x + y||^2 + ||x - y||^2 = (x + y, x + y) + (x - y, x - y) = (x, x) + (x, y) + (y, x) + (y, y) + (x, x) - (x, y) - (y, x) + (y, y) = 2||x||^2 + 2||y||^2$)

Definition Let F be a non-empty subset of H.

1. F is *linear* $\Longleftrightarrow (x, y \in F \Longrightarrow ax + by \in F,\ \forall\ a, b \in \mathbb{C})$.

2. F is *convex* $\Longleftrightarrow$ $(x, y \in F \Longleftrightarrow ax + by \in F, \forall\, a, b \in \mathbb{R},\ a \geq 0,$ $b \geq 0,\ a + b = 1)$.

Lemma 3 If F is a closed, convex subset of H, it has a point at minimal distance from the origin, i.e. there exists $x \in F$, such that $||x|| \leq ||y||$ for all $y \in F$.

Proof Let $d = \inf_{y \in F} ||y||$, and let (y_n) be a sequence in F, such that $||y_n|| \longrightarrow d$. Then (y_n) is a Cauchy sequence (in F). For

$$\begin{aligned} \left\|\frac{y_n - y_m}{2}\right\|^2 &= \frac{1}{2}||y_n||^2 + \frac{1}{2}||y_m||^2 - \left\|\underbrace{\frac{y_m}{2} + \frac{y_n}{2}}_{\in F \text{ (convex)}}\right\|^2 \qquad \text{(Lemma 2)} \\ &= \frac{1}{2}\left(||y_n||^2 + ||y_m||^2 - 2\left\|\frac{y_m}{2} + \frac{y_n}{2}\right\|^2\right) \\ &\leq \frac{1}{2}(||y_n||^2 + ||y_m||^2 - 2d^2) \\ &\longrightarrow 0, \text{ as } m, n \longrightarrow \infty. \end{aligned}$$

Hence (y_n) has a limit $x \in H$ (completeness); but then $x \in F$, since F is closed, and $||x|| = d$.

Definition Let $E \subset H$. E is a *subspace* of H (a Hilbert space) if (a) E is linear, (i.e. $E \neq \emptyset$, $x, y \in E \Longrightarrow \alpha x + \beta y \in E$, $\alpha, \beta \in \mathbb{C}$), and (b) E is closed in the topology induced by the norm.

Definition Let $x, y \in H$, *and* $H_0 \subset H$. Define $x \perp y \Longleftrightarrow (x, y) = 0$, $x \perp H_0 \Longleftrightarrow (x, y) = 0$ for every $y \in H_0$, $H_0^\perp = \{y \mid y \perp H_0\}$. $H_0^\perp$ is the *orthogonal complement* of H_0.

Note that (i) $x \perp y \Longrightarrow ||x + y||^2 = ||x||^2 + ||y||^2$

(ii) $H_0^\perp$ is linear, if H_0 is linear. For let $x_1, x_2 \in H_0^\perp$; $\alpha, \beta \in \mathbb{C}$. Then $\forall\, y \in H_0$, we have $(\alpha x_1 + \beta x_2, y) = \underset{=0}{\alpha(x, y)} + \underset{=0}{\beta(x_2, y)} = 0$. Hence $\alpha x_1 + \beta x_2 \in H_0^\perp$.

(iii) $H_0^\perp$ is closed, if H_0 is linear. Let $x \in \overline{H_0^\perp}$. Then there exists a sequence $(x_n) \subset H_0^\perp$, such that (x_n) converges to x.

Now for all $y \in H_0$ we have $(x, y) = \lim_{n \to \infty} (x_n, y)$ [(*) since the inner product is a continuous function of each of the variables] $= 0$, since

$x_n \in H_0^\perp$ and $y \in H_0$. Hence $x \in H_0^\perp$.

[((*)). If $x_n \longrightarrow x$, $y_n \longrightarrow y$, then $(x_n, y_n) \longrightarrow (x, y)$, as $n \longrightarrow \infty$, for

$$\begin{aligned} |(x_n,y_n)-(x,y)| &= |(x_n,y_n)-(x,y_n)+(x,y_n)-(x,y)| \\ &= |(x_n-x,y_n)+(x,y_n-y)| \\ &\le |(x_n-x,y_n)|+|(x,y_n-y)| \\ &\le ||x_n-x||\cdot||y_n||+||x||\cdot||y_n-y|| \\ &\to 0.] \end{aligned}$$

Hence $H_0^\perp$ is a *subspace.*

Lemma 4 If H_0 is a proper subspace of H, then $H_0^\perp$ contains an element which is non-zero.

Proof Choose $y \in H$, $y \notin H_0$. Then $y + H_0$ is a closed, convex subset of H. [For if $a \ge 0$, $b \ge 0$, with $a + b = 1$, and $p \in y + H_0$, $q \in y + H_0$, then $ap + bq \in y + H_0$, since $p = y + h_1$, $q = y + h_2$, with $h_1, h_2 \in H_0$, hence $ab + bq = \underbrace{(a+b)}_{1}y + \underbrace{(h_1+h_2)}_{\in H_0} \in y + H_0$.]. By Lemma 3 there exists a point $x \in (y + H_0)$ at *minimal* distance d from the origin. Now $x \neq 0$, since $y \notin H_0$. [$x \in y + H_0$, $x = 0 \Longrightarrow -y \in H_0 \Longrightarrow y \in H_0$]. If $c \in \mathbb{C}$, $x_0 \in H_0$, then $x + cx_0 \in y + H_0$. Hence

$$\begin{aligned} 0 \le ||x+cx_0||^2 - ||x||^2 &= (x+cx_0, x+cx_0)-(x,x) \\ &= c(x_0,x)+\bar{c}(x,x_0)+|c|^2||x_0||^2. \end{aligned}$$

If $(x, x_0) \neq 0$ for some x_0, set $c_1 = \dfrac{c}{(x, x_0)}$. Consider all those c_1's which are real. For *such* c_1 we have

$$\begin{aligned} 0 &\le c_1|(x,x_0)|^2+c_1|(x,x_0)|^2+c_1^2\cdot||x_0||^2\cdot|(x,x_0)|^2 \\ &= 2c_1|(x,x_0)|^2+c_1^2\cdot||x_0||^2\cdot|(x,x_0)|^2. \end{aligned}$$

But this is false for c_1 such that $\dfrac{1}{c_1} < -\dfrac{||x_0||^2}{2}$. [For $\dfrac{c_1^2}{c_1} < -c_1^2\dfrac{||x_0||^2}{2}$, hence $2c_1 + c_1^2||x_0||^2 < 0$.] It follows that $(x, x_0) = 0$, i.e. $x \in H_0^\perp$.

Theorem 1 Let H_0 be a subspace of H. Then $H = H_0 \oplus H_0^\perp$, i.e. $x \in H \Longrightarrow x = x_0 + x_1$, $x_0 \in H_0$, $x_1 \in H_0^\perp$, and this expression is unique.

Proof (a) the uniqueness. If $x = x_0 + x_1 = y_0 + y_1$, $\begin{cases} x_0, y_0 \in H_0, \\ x_1, y_1 \in H_0^{\perp}, \end{cases}$ then $x_0 - y_0 = y_1 - x_1 \in H_0 \cap H_0^{\perp} = 0$

(b) the existence. Note that $H_0 + H_0^{\perp}$ is a *subspace*. For, if $x^n \in H_0 + H_0^{\perp}$, and $x^n \longrightarrow x$, i.e. $x^n = x_0^n + x_1^n \longrightarrow x$, then $||x^n - x^m||^2 \equiv ||x_0^n - x_0^m + x_1^n - x_1^m||^2 = ||x_0^n - x_0^m||^2 + ||x_1^n - x_1^m||^2 \longrightarrow 0$, since $x \perp y \Longrightarrow (||x+y||^2 = ||x||^2 + ||y||^2)$ and $x^n \longrightarrow x$. Hence (x_0^n), (x_1^n) are Cauchy sequences and have limits x_0 and x_1, where $x_0 \in H_0$, $x_1 \in H_0^{\perp}$, since H_0 is closed by assumption, and $H_0^{\perp}$ is closed. Then trivially $(x^n) \longrightarrow x_0 + x_1 = x \in H_0 + H_0^{\perp}$. Hence $H_0 + H_0^{\perp}$ is a *subspace*. If $H_0 + H_0^{\perp} \neq H$, then (by Lemma 4) there exists $y \neq 0$, such that $y \perp (H_0 + H_0^{\perp})$, in particular $y \perp H_0$ which implies that $y \in H_0^{\perp}$, a contradiction.

Remark If H_0 is a *subspace* of H, then $(H_0^{\perp})^{\perp} = H_0$. For, $x \in H_0 \Longrightarrow (x, y) = 0$, $\forall\, y \in H_0^{\perp} \Longrightarrow x \in (H_0^{\perp})^{\perp}$, hence $H_0 \subset (H_0^{\perp})^{\perp}$. On the other hand, if $x \in (H_0^{\perp})^{\perp}$, then $x \in H$, so that $x = y + z$, $y \in H_0$, $z \in H_0^{\perp}$ (Theorem 1). Hence $(x, z) = (y, z) + (z, z)$. But $(x, z) = 0$, since $x \in (H_0^{\perp})^{\perp}$ and $z \in H_0^{\perp}$. Further $(y, z) = 0$, since $y \in H_0$, $z \in H_0^{\perp}$. Hence $||z||^2 = 0$; or $z = 0$, i.e. $x = y \in H_0$.

Definition Let E be a *subset* of H. E is said to be *linearly independent*, if and only if no finite linear combination of elements of E is 0 except the linear combination with all the coefficients equal to 0.

E is a *generating set* if and only if every element of H is a finite linear combination of elements of E.

E is a *base* if and only if it is a linearly independent generating set.

It can be proved that every Hilbert space H has a base (in fact, every vector space) and all the bases have the same power. ("number" of elements). We do not here distinguish between different infinities.

The *dimension* of H = the power of a base.

Definition Let $(x_\alpha) \subset H$. Then (x_α) is said to be a *complete orthonormal set*, if (i) $||x_\alpha|| = 1$; (ii) $(x_\alpha, x_\beta) = 0$ if $\alpha \neq \beta$; (iii) (x_α) is not contained in a larger set satisfying (i) and (ii). Such sets always exist in H.

Examples $L_2[0,2\pi]$; $t \longmapsto \dfrac{1}{\sqrt{2\pi}}, \dfrac{\cos t}{\sqrt{\pi}}, \dfrac{\sin t}{\sqrt{\pi}}, \dfrac{\cos 2t}{\sqrt{\pi}}, \dfrac{\sin 2t}{\sqrt{\pi}}, \ldots; \dfrac{e^{inx}}{\sqrt{2\pi}}$, $n \in \mathbb{Z}$.

Lemma 5 The dimension of H = the power of a complete orthonormal set.

Proof If $\dim H = n < \infty$, and (x_i) a complete orthonormal set, then (x_i) contains at most n elements, since it is linearly independent. If it contained less than n elements, it could be expanded to a base for H, and the additional elements, when normalized, could be added to (x_i), contrary to completeness.

If $\dim H = \infty$, and (x_i) is a complete orthonormal set, then (x_i) is infinite. For, supposing (x_i) is finite, the closed linear subspace H_0 which it generates is a proper subset of H, so that (by Lemma 4) there exists an element $x \notin H_0$, such that $x_0 \perp H_0$, and this element (normalized) could be added to (x_i) contrary to completeness.

Lemma 6 $\dim H < \infty \iff H$ is locally compact. For if $\dim H = n < \infty$, then H has the topology of $\mathbb{R}^{2n}$. If $\dim H = \infty$, there exists an infinite orthonormal set (x_α). An ε-sphere around 0 contains $\left(\frac{\varepsilon}{2}x_\alpha\right)$, which have no limit point since $\left\|\frac{\varepsilon}{2}x_\alpha - \frac{\varepsilon}{2}x_\beta\right\| = \frac{\varepsilon}{2}\cdot\sqrt{2}$.

Definition A bounded linear functional on H is a linear function $f : H \longrightarrow \mathbb{C}$ with the property that there exists a real number M, such that

$$|f(x)| \leq M\|x\|, \quad \forall\, x \in H.$$

Remarks

(i) Note that f is *continuous*, for by linearity,

$$|f(x) - f(y)| = |f(x-y)| \leq M\|x-y\|.$$

(ii) Let B be a Banach space. The *dual space* of B is the space of bounded linear functionals on B.

(iii) Note that linearity together with continuity imply boundedness.

(iv) Let $a \in H$. Then $f : x \longmapsto (x,a)$ is a bounded linear functional. For $f(\alpha x_1+\beta x_2) = (\alpha x_1+\beta x_2, a) = \alpha(x_1,a)+\beta(x_2,a) = \alpha f(x_1)+\beta f(x_2)$, and $|f(x)| = |(x,a)| \leq \|a\|\cdot\|x\|$, $\forall\, x \in H$ (Schwarz).

(v) **Definition** If $H \neq 0$, $||f|| = \sup_{x \neq 0} \frac{|f(x)|}{||x||}$.

$f(x) = 0$, $\forall\, x \Longrightarrow ||f|| = 0$, and conversely.

Lemma 7 If f is a bounded linear functional on H, then there exits a unique element $x^* \in H$, such that $f(x) = (x, x^*)$, $\forall\, x \in H$.

Proof Uniqueness. If $(x, x^*) = (x, y^*)$ for all $x \in H$, then we take $x = x^* - y^*$, so that $(x^* - y^*, x^*) = (x^* - y^*, y^*)$, hence $(x^* - y^*, x^* - y^*) = ||x^* - y^*||^2 = 0$, and $x^* = y^*$.

Existence. If $f = 0$, take $x^* = 0$. If $f \neq 0$, then $H_0 \underset{def}{=} \{y \mid f(y) = 0\}$ is a *proper* (closed, linear) *subspace* of H. By Lemma 4 there exists an $x_1 \in H_0^{\perp}$ (with $x_1 \neq 0$), with $||x_1|| = 1$. Take $x^* = \overline{f(x_1)} \cdot x_1$.

(a) If $x = cx^*$, then

$$\begin{aligned} f(x) &= cf(x^*) = cf(\overline{f(x_1)}, x_1) = c\overline{f(x_1)} \cdot f(x_1) \\ &= (c\overline{f(x_1)} \cdot f(x_1) \cdot (x_1, x_1) = (c\overline{f(x_1)}x_1, \overline{f(x_1)}x_1) \\ &= (cx^*, x^*) = (x, x^*) \quad \text{(since } x = cx^*\text{)} \end{aligned}$$

(b) *If* $x \in H_0$, then $f(x) = 0$, *and* $(x, x^*) = 0$, since $x_1 \in H_0^{\perp}$. Hence $f(x) = (x, x^*) = 0$.

(c) *Every element* x of H *is of the form* $cx^* + x_0$ with $x_0 \in H_0$. For if we take $c = \frac{f(x)}{f(x^*)}$, then $f(x - cx^*) = 0$, by linearity. Hence $x - cx^* \in H_0$, by the definition of H_0; and $x = cx^* + (x - cx^*)$, where $x - cx^* \in H_0$. Because of (a) and (b) the lemma is proved.

III.3 Bounded operators

Definition An *operator* (*or bounded operator*) is a linear transformation $T : H \longrightarrow H$, which satisfies the following condition of boundedness: there exists a real number M, such that

$$||Tx|| \leq M \cdot ||x||, \quad \forall\, x \in H.$$

Define, for $H \neq 0$, $||T|| = \sup\limits_{||x|| \neq 0} \dfrac{||Tx||}{||x||}$. If $Tx = 0$, $\forall\, x \in H$, then $||T|| = 0$. An operator is *continuous*. For, if $x_n \longrightarrow x$, then $||Tx_n - Tx|| = ||T(x_n - x)|| \leq M||x_n - x|| \longrightarrow 0$. Conversely, linearity together with continuity implies boundedness.

Lemma 8 If T is an operator on H, $H \neq 0$, and M a non-negative real number, then the following four conditions are equivalent.

(a) $||Tx|| \leq M||x||, \forall\, x \in H$.

(b) $|(Tx, y)| \leq M \cdot ||x|| \cdot ||y||, \quad \forall\, x, y \in H$

(c) $||Tx|| \leq M, \forall\, x \in H$, with $||x|| = 1$.

(d) $|(Tx, y)| \leq M, \forall\, x, y \in H$, such that $||x|| = ||y|| = 1$.

Proof (a) $\Longleftrightarrow$ (c), (b) $\Longleftrightarrow$ (d), and (a) $\Longleftrightarrow$ (b).

Obviously we have: (a) $\Longrightarrow$ (c) and (b) $\Longrightarrow$ (d)

If $x \neq 0$, (c) $\Longrightarrow$ (a), for $||Tx|| = ||x|| \cdot \left\| T\left(\dfrac{x}{||x||}\right)\right\| \leq M||x||$.

If $x = 0$, "(c) $\Longrightarrow$ (a)" is obvious.

Similarly, if $x \neq 0$, $y \neq 0$, (d) $\Longrightarrow$ (b), since

$$|(Tx, y)| = ||x|| \cdot ||y|| \left(T\left(\frac{x}{||x||}, \frac{y}{||y||}\right)\right) \leq ||x|| \cdot ||y|| \cdot M.$$

If $x = 0$, or $y = 0$, "(d) $\Longrightarrow$ (b)" is obvious. Hence we have (a) $\Longleftrightarrow$ (c) and (b) $\Longleftrightarrow$ (d).

By Schwarz's inequality, we have

$$|(Tx, y)| \leq ||Tx|| \cdot ||y|| \underset{(a)}{\leq} M||x|| \cdot ||y||,$$

hence (a) $\Longrightarrow$ (b). On the other hand, take $y = Tx$ in (b). Then

$$||Tx||^2 \leq M \cdot ||x|| \cdot ||Tx||.$$

If $Tx = 0$, this is trivial. Otherwise, divide by $||Tx||$. Hence (b) $\Longrightarrow$ (a)

Remarks on Lemma 7 Note that $||f|| = ||x^*||$. We may restrict ourselves to the case $f \neq 0$, so that $x^* \neq 0$ (since $(x, 0) = 0$). Then we have

$$||f|| = \sup_{x \neq 0} \frac{|f(x)|}{||x||} = \sup_{x \neq 0} \frac{|(x, x^*)|}{||x||} \leq \sup_{x \neq 0} \frac{||x|| \cdot ||x^*||}{||x||} = ||x^*||,$$

so that $||f|| \le ||x^*||$; on the other hand, if $H \neq 0$,

$$||f|| = \sup_{x \neq 0} \frac{|f(x)|}{||x||} = \sup_{x \neq 0} \frac{|(x, x^*)|}{||x||} \ge \frac{(x^*, x^*)}{||x^*||} = ||x^*||, \text{ (one term with } x = x^*)$$

so that $||f|| \ge ||x^*||$.

Definition Let T be an operator on H. The operator T^* is said to be the *adjoint operator* of T, *if* for all $x, y \in H$, we have

$$(Tx, y) = (x, T^*y).$$

Lemma 9 To each operator T on H, there exists a unique adjoint operator T^*, and $||T^*| \le ||T||$.

Proof (1) For a fixed $y \in H$, $f : x \longmapsto (Tx, y)$, $x \in H$, is a bounded linear functional.

For, let $x_1, x_2 \in H$; $\alpha, \beta \in \mathbb{C}$. Then $f(\alpha x_1 + \beta x_2) = (T(\alpha x_1 + \beta x_2), y) = (\alpha T x_1 + \beta T x_2, y) = \alpha(Tx_1, y) + \beta(Tx_2, y) = \alpha f(x_1) + \beta f(x_2)$. And for every $x \in H$, we have

$$|f(x)| = \underset{\text{(Schwarz)}}{|(Tx, y)| \le ||Tx|| \cdot ||y||} \le ||T|| \cdot ||y|| \cdot ||x||,$$

where $||T|| \cdot ||y||$ is a number independent of x.

(2) By Lemma 7 there exists a unique element $y^* \in H$, such that $f(x) \underset{def}{=} (Tx, y) = (x, y^*)$, for *all* $x \in H$. This implies that there exists a unique mapping $T^* : y \longmapsto T^*y = y^*$, of H into H, such that $(Tx, y) = (x, T^*y)$, for all $x, y \in H$.

(3) T^* is *linear*. Let $y_1, y_2 \in H$; $\alpha, \beta \in \mathbb{C}$. Then we have for all $x \in H$,

$$\begin{aligned}(x, T^*(\alpha y_1 + \beta y_2)) \underset{def}{=} (Tx, \alpha y_1 + \beta y_2) &= \overline{\alpha}(Tx, y_1) + \overline{\beta}(Tx, y_2) \\ &= \overline{\alpha}(x, T^*y_1) + \overline{\beta}(x, T^*y_2) \\ &= (x, \alpha T^*y_1 + \beta T^*y_2).\end{aligned}$$

Hence $T^*(\alpha y_1 + \beta y_2) = \alpha T^* y_1 + \beta T^* y_2$.

[Note that $(x, 0) = \overline{(x, 0 \cdot x)} = \overline{0 \cdot (x, x)} = 0$, and $(0, 0) = 0$, and $(x, y+z) = \overline{(y+z, x)} = \overline{(y, x)} + \overline{(z, x)} = (x, y) + (x, z)$.]

(4) T^* is *bounded*. Let $x, y \in H$. Then $(Tx, y) = (x, T^*y)$. Set $x = T^*y$. Then we have $(TT^*y, y) = (T^*y, T^*y) = ||T^*y||^2$, while

$(TT^*y, y) \leq ||T(T^*y)|| \cdot ||y|| \leq ||T|| \cdot ||T^*y|| \cdot ||y||$. If $T^*y \neq 0$, then this implies that $||T^*y|| \leq ||T|| \cdot ||y||$; which holds trivially if $T^*y = 0$.

Hence T^* is a *bounded* operator, i.e. an operator and $||T^*|| \leq ||T||$.

Remarks

1. $T^{**} = T$, since

$$(x, (T^*)^*y) = (T^*x, y) = \overline{(y, T^*x)} = \overline{(Ty, x)} = (x, Ty), \quad \text{for } x, y \in H.$$

2. $||T|| \leq ||T^*||$, for, by Lemma 9, $||T^*|| \leq ||T||$, hence $||(T^*)^*|| \leq ||T^*||$, or $||T|| \leq ||T^*||$ by the above remark. *Thus we have* $||T^*|| = ||T||$.

3. $(T_1+T_2)^* = T_1^*+T_2^*$; $(\lambda T)^* = \overline{\lambda}T^*$, $\lambda \in \mathbb{C}$; $(T_1T_2)^* = T_2^*T_1^*$, where the 'product' T_1T_2 is defined as follows: $T_1T_2 : x \longmapsto T_1(T_2x)$, the 'range' of T_2 being contained in the 'domain' of T_1, and the domain of T_1T_2 is the same as the domain of T_2.

Lemma 10 Let H_1 and H_2 be Hilbert spaces, and $f(x, y)$ a bounded, semilinear functional on $H_1 \times H_2$ (i.e. linear in the variable "x", and conjugate linear in the variable "y" with $|f(x, y)| \leq M \cdot ||x|| \cdot ||y||$ for a real number M). Then there exists a unique linear transformation $T : H_1 \longrightarrow H_2$, such that

$$f(x, y) = (Tx, y).$$

Proof For every $x \in H_1$, we see that $\overline{f(x, \cdot)}$ is a bounded, linear functional on H_2. By Lemma 7, therefore, there exists a *unique* element $z_x \in H_2$, such that $\overline{f(x, y)} = (y, z_x)$, $\forall\ y \in H_2$, or $f(x, y) = (z_x, y)$. Then $T : x \longmapsto z_x$ is the sought transformation.

Lemma 11 Let T be an operator on H. Then

$$(Tx, x) = 0, \quad \forall\ x \Longleftrightarrow T = 0.$$

Proof Let $a, b \in \mathbb{C}$; $x, y \in H$. We have the identity

$$a\bar{b}(Tx, y) + \bar{a}b(Ty, x) = (T(ax + by), ax + by) - |a|^2(Tx, x) - |b|^2(Ty, y).$$

[If $(Tx, x) = 0$, $\forall\ x$, then $(T(ax + by), ax + by) = 0$]. Set $a = b = 1$. Then we have:

$$(Tx, y) + (Ty, x) = 0;$$

set $a = i$, $b = 1$. Then we have $i(Tx, y) - i(Ty, x) = 0$. Hence

$$2i(Tx, y) = 0.$$

The converse is trivial.

Lemma 12 Let T be an operator on H. Then we have:

$$T = T^* \iff (Tx, x) \text{ real, for all } x \in H.$$

Proof If $T = T^*$, then $(Tx, x) = (x, T^*x) = (x, Tx) = \overline{(Tx, x)}$. On the other hand, if (Tx, x) is real, then

$$(Tx, x) = \overline{(Tx, x)} = \overline{(x, T^*x)} = (T^*x, x),$$

hence

$$((T - T^*)x, x) = 0, \quad \forall\ x, \text{ i.e. } T = T^*.$$

Definition An operator T on H is called:

1. *real*, or *self-adjoint*, or *Hermitian*, or *symmetric*, if and only if $T = T^*$. (i.e. $(Tx, y) = (x, Ty)$, $\forall\ x, y \in H$).
2. *positive*, if and only if $(Tx, x) \geq 0$, $\forall\ x \in H$.
3. *unitary*, if and only if T maps H *onto* H, and $||Tx|| = ||x||$. [In the finite dimensional case, the "isometry" of T implies that it is *onto*.]
4. *a projector*, if and only if T is *real* and *idempotent*, i.e. $T^2 = T = T^*$. [The zero operator $O : x \longmapsto 0$, and the identity operator $I : x \longmapsto x$ are counted as projectors].

Remarks Let H_0 be a subspace of H (i.e. a closed, linear subset). If $x \in H$, then, by Theorem 1, we have $x = x_0 + x_1$, $x_0 \in H_0$, $x_1 \in H_0^\perp$, and the decomposition is unique.

If we *define* $Px = x_0$, P is called *the projection of* H *on* H_0. It is actually a *projector* as defined in (4) above.

For $y \in H_0 \Longrightarrow P(y) = y$ (since $y = y + 0$), so that $PP(x) = P(x_0) = x_0 = P(x)$, and P is *idempotent*. Further, P is *symmetric*. For $(Px_1, x_2) = (x_1, Px_2)$, $\forall\ x_1, x_2 \in H$, which can be seen as follows: $x_1 = y_1 + z$, say, and $x_2 = y_2 + z_2$, where $y_1, y_2 \in H_0$, and $z_1, z_2 \in H_0^{\perp}$. Hence

$$\begin{aligned}(Px_1, x_2) = (y_1, y_2 + z_2) &= (y_1, y_2) + \underbrace{(y_1, z_2)}_{=0} \\ &= (y_1, y_2) \\ &= (y_1, y_2) + \underbrace{(z_1, y_2)}_{=0} \\ &= (y_1 + z_1, y_2) = (x_1, Px_2).\end{aligned}$$

Conversely let P be a *projector* on H. Then $W_p = \{P(x) \mid x \in H\}$ is *linear*, since P is linear. Hence $\overline{W}_p$ is a *subspace*, say H_0. Then P is *the projection of H on H_0*. For we have $x = Px + x - Px$, $\forall\ x \in H$; and we have only to show that $Px \in H_0$, $x - Px \in H_0^{\perp}$. Now (i) $Px \in W_p \subset H_0$, $\forall\ x$; and (ii) if $y \in H$, then $(x - Px, Py) = (P(x - Px), y) = (Px - PPx, y)$ [since P is *real* and linear] $= (Px - Px, y) = (0, y) = 0$ [since P is idempotent]. Hence $(x - Px) \perp W_p$, $(Py \in W_p)$, and therefore $(x - Px) \perp \overline{W}_p (= H_0)$. [Note that if $y^0 \in H_0$, then $\exists\ (y_n)$, $y_n \in W_p \ni y_n \longrightarrow y^0$. Now $(x - Px, y) = 0$, $\forall\ y \in W_p$. Because of the continuity of $(\ ,\ )$, it follows that '$(x - Px, y_n) = 0$, $\forall\ n$' $\Longrightarrow$ '$(x - Px, y^0) = 0$'].

Lemma 13

(1) $(T_1T_2)^* = T_2^*T_1^*$.

(2) Let P be the projection of H on the subspace H_0. Then we have: "$TP = PT$" $\Longleftrightarrow$ "$TH_0 \subset H_0$ *and* $TH_0^{\perp} \subset H_0^{\perp}$".

(3) Let P and H_0 be as in (2). Then we have

$$TP = PTP \Longleftrightarrow TH_0 \subset H_0.$$

(4) Let T be unitary, $TH_0 \subset H_0$, $T^{-1}H_0 \subset H_0$ (or T real). Then $TH_0^{\perp} \subset H_0^{\perp}$.

Proof (1) We have $(T_1T_2x, y) = (T_2x, T_1^*y) = (x, T_2^*T_1^*y)$, and T^* is unique for any T.

(3) If $TP = PTP$, and $x_0 \in H_0$, then we have $Tx_0 = TPx_0 = PTPx_0 = P(Tx_0)$, hence $Tx_0 \in H_0$, which implies that $TH_0 \subset H_0$. Conversely, if $TH_0 \subset H_0$, then $T\underbrace{Px}_{\in H_0} \in H_0$, $\forall\, x \in H$. Hence $P(TPx) = TPx$, $\forall\, x \in H$, or $PTP = TP$.

(2) Let $TP = PT$. Then we have $TP^2 = PTP$, or $TP = PTP$, and by (3) we get: $TH_0 \subset H_0$.

Further $PTP = P^2T = PT$, so that $TP = PTP = PT$, and by taking the adjoints: $(TP)^* = (PTP)^* = (PT)^*$, that is to say $P^*T^* = (TP)^*P^* = P^*T^*P^* = PT^*P$, ($P$ real), $= T^*P^*$. Hence $P^*T^* = PT^*P = T^*P^*$, or $PT^* = \underbrace{PT^*P = T^*P}$ ($P^* = P$), which by (3) implies that $T^*H_0 \subset H_0$. (they are, in fact, equivalent)

Now $TH_0 \subset H_0 \iff T^*H_0^\perp \subset H_0^\perp$. Since $T^{**} = T$ and $H_0^{\perp\perp} = H_0$, to see this it suffices to prove that

$$(\dagger) \qquad TH_0 \subset H_0 \Longrightarrow T^*H_0^\perp \subset H_0^\perp.$$

Let $TH_0 \subset H_0$, *and* $x \in H_0$, $y \in H_0^\perp$. Then we have $(x, T^*y) = (Tx, y) = 0$, $\forall\, x \in H_0$ (since $Tx \in H_0$ by hypothesis). Hence $T^*y \in H_0^\perp$, and this holds for all $y \in H_0^\perp$, so that $T^*H_0^\perp \subset H_0^\perp$. We have already seen that $T^*H_0 \subset H_0$. Hence (†)

$$T^*H_0 \subset H_0 \iff T^{**}H_0^\perp \subset H_0^\perp \quad (T^{**} = T)$$

(4) 'T *unitary*' $\Longrightarrow$ '$||Tx|| = ||x||$' $\iff$ '$(Tx, Ty) = (x, y)$'.

[$||Tx|| = ||x|| \Longrightarrow (Tx, Tx) = (x, x)$. Hence

$$\left(T\frac{x+y}{2}, T\frac{x+y}{2}\right) - \left(T\frac{x-y}{2}, T\frac{x-y}{2}\right)$$
$$= \left(\frac{x+y}{2}, \frac{x+y}{2}\right) - \left(\frac{x-y}{2}, \frac{x-y}{2}\right).$$

Hence $\text{Re}(Tx, Ty) = \text{Re}(x, y)$. Similarly ($y \longrightarrow iy$), $\text{Im}(Tx, Ty) = \text{Im}(x, y)$.]

However, $(Tx, Ty) = (x, T^*Ty) = (x, y)$, so that $(x, T^*Ty - y) = 0$, $\forall\, x, y \in H$. It follows that $T^*Ty = y$, or $T^*T = 1$. Similarly $TT^* = 1$. Hence $T^*T = TT^* = 1$, i.e. $T^* = T^{-1}$. Thus

$$\text{“}TH_0 \subset H_0,\ T^{-1}H_0 \subset H_0\text{”} \iff \text{“}TH_0 \subset H_0,\ T^*H_0 \subset H_0\text{”}$$
$$\iff \text{“}TH_0 \subset H_0,\ TH_0^\perp \subset H_0^\perp\text{”}$$

(See above (3)).

If T is real, then $T^* = T$, and the result follows from (†) above.

Theorem 2 If T is a real operator on H, $H \neq 0$, then

$$||T|| = \sup_{||x||=1} |(Tx, x)|.$$

Proof Let $M = \sup_{||x||=1} |(Tx, x)|$. Then $M \leq ||T||$, for

$$|(Tx, x)| \leq ||Tx|| \cdot ||x|| \leq ||T|| \cdot ||x||^2, \quad \text{hence} \quad \sup_{||x||=1} |(Tx, x)| \leq ||T||.$$

We have therefore only to show that $||T|| \leq M$. By Lemma 8, it suffices to show that $|(Tx, y)| \leq M$, $\forall\ x, y \in H$ such that $||x|| = ||y|| = 1$. If $(Tx, y) = 0$, this is trivial. If $(Tx, y) \neq 0$, set

$$z = \frac{|(Tx, y)|}{(Tx, y)} \cdot x.$$

Then we have

$$(Tz, y) = \overline{(y, Tz)} = \overline{\left(y, \frac{|(Tx, y)|}{(Tx, y)} \cdot Tx\right)} = \frac{|(Tx, y)|}{(Tx, y)} \cdot \overline{(y, Tx)} = |(Tx, y)|,$$

and, since $T = T^*$,

$$(Ty, z) = (y, Tz) = \overline{(Tz, y)} = |(Tx, y)|.$$

Further, if $z + y \neq 0$,

$$(T(z + y), z + y) = ||z + y||^2 \left(T\left(\frac{z + y}{||z + y||}\right), \frac{z + y}{||z + y||}\right)$$

or, $|(T(z+y), z+y)| \leq M \cdot ||z+y||^2$, (by the definition of M). Similarly

$$|(T(z - y), z - y)| \leq M \cdot ||z - y||^2.$$

Now

$$(T(z + y), z + y) = (Tz, z) + (Tz, y) + (Ty, z) + (Ty, y), \quad \text{and}$$
$$(T(z - y), z - y) = (Tz, z) - (Tz, y) - (Ty, z) + (Ty, y),$$

hence

$$\begin{aligned}(T(z + y), z + y) - (T(z - y), z - y) &= 2(Tz, y) + 2(Ty, z) \\ &= 4|(Tx, y)| \quad \text{(see above)}.\end{aligned}$$

Thus

$$4|(Tx,y)| \leq M(||z+y||^2 + ||z-y||^2) = 2M(||z||^2 + ||y||^2) \text{ (Lemma 2)}$$

or

$$|(Tx,y)| \leq M\left(\frac{||z||^2}{2} + \frac{||y||^2}{2}\right) = M, \text{ for } ||x|| = ||y|| = 1.$$

Thus $||T|| \leq M$.

Definition Let T be an operator on H. An *eigenvalue* of T is a complex number λ, such that there exists a non-zero $x \in H$ with $Tx = \lambda x$. x is then called an *eigenvector of eigenvalue* λ. The set of all eigenvectors of eigenvalue λ is a (closed, linear) subspace of H, called the *eigenspace corresponding to* λ, say H_λ.

[H_λ is linear: $x_1, x_2 \in H_\lambda$; $\alpha, \beta \in \mathbb{C} \Longrightarrow \alpha x_1 + \beta x_2 \in H_\lambda$, for $T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2 = \alpha\lambda x_1 + \beta\lambda x_2 = \lambda(\alpha x_1 + \beta x_2)$, hence $\alpha x_1 + \beta x_2 \in H_\lambda$.

H_λ is closed: $\overline{H}_\lambda = H_\lambda$. Let $x \in \overline{H}_\lambda$. Then $\exists$ $(x_n) \subset H_\lambda$, with $x_n \longrightarrow x$. Hence $||Tx_n - Tx|| = ||T(x_n - x)|| \leq M||x_n - x|| \longrightarrow 0$. But $Tx_n = \lambda x_n$, hence $Tx = \lambda x$, i.e. $x \in H_\lambda$].

Remarks If T is a *real* operator, then (a) all eigenvalues of T are real, and (b) eigenvectors corresponding to distinct eigenvalues are "perpendicular". For $\overline{\lambda}(x,x) = (x, \lambda x) = (x, Tx) = (Tx, x) = (\lambda x, x) = \lambda(x,x)$, implying (a). (b) means that if $Tx = \lambda x$, $Ty = \lambda y$, $\lambda \neq \mu$, then $(x,y) = 0$. This follows from: $\lambda(x,y) = (\lambda x, y) = (Tx, y) = (x, Ty) = (x, \mu y) = \mu(x,y)$ (μ is real by (a)) and $\lambda \neq \mu$, so that $(x,y) = 0$.

Definition Let (H_α) be a family of (closed, linear) subspaces of H. Then H is the *direct sum* of the H_α's, written $H = \bigoplus_\alpha H_\alpha$, if and only if each $x \in H$ is uniquely expressible as $x = \sum x_\alpha$, with $x_\alpha \in H_\alpha$. If $H = H_1 \oplus H_2$, H_1 and H_2 are said to be complementary, in Hilbert space sense.

Definition An operator T on H is *completely continuous* (or *compact*) if it carries the unit sphere (or, equivalently, any bounded subset of H) into a *relatively compact set* (i.e. a set whose closure is compact).

[The image of any bounded sequence contains a convergent subsequence].

Remarks

(1) *All* operators on E^n are compact. [A bounded set is carried into a bounded set, and in E^n all bounded sets are relatively compact]. $E = \mathbb{C}$.

(2) If A is an unbounded, linear operator, then there exists a sequence (x_n) with $||x_n|| = 1$, and $||Ax_n|| > n$, $n = 1, 2, 3, \ldots$. The set $(x_1, x_2, \ldots)$ is bounded, and the image $(Ax_1, Ax_2, \ldots)$ is not relatively compact, so that A is *not* compact.

(3) Not all bounded linear operators are compact. If H is an *infinite* dimensional Hilbert space, then the identity operator is *not compact*. [There exist sets which are bounded but not relatively compact, e.g. the unit sphere. There exist infinite sequences of orthonormal functions which contain no convergent subsequences $\frac{e^{inx}}{\sqrt{2\pi}}$, $n = 1, 2, \ldots$].

(4) If H is *finite* dimensional, then every bounded operator is compact, since every bounded set is relatively compact.

(5) If T_1 is compact, and T_2 bounded, then T_1T_2 and T_2T_1 are compact.

(6) If T is compact, T^* is compact.

(7) If $x(t) \in L_2[a, b]$, $-\infty \leq a < b \leq +\infty$; $k(s, t) \in L_2\{(s, t) \mid a \leq s, t \leq b\}$;

$$T : x(t) \longmapsto \int_a^b k(s, t)x(s)\, ds;$$

then T is (bounded, linear) *compact*

III.4 The spectral theorem

Theorem 3 (Spectral Theorem) If T is a real, completely continuous operator on H and (λ_n) is the set of all eigenvalues of T, then

1. for each $\varepsilon > 0$, there exist only finitely many λ_n with $|\lambda_n| > \varepsilon$, so the set (λ_n) is countable, and λ_n converges to 0;

2. for each $\lambda_n \neq 0$, the corresponding eigenspace is finite dimensional;

3. if $H_{\lambda_n} \equiv H_n$ is the eigenspace which corresponds to λ_n, then $H = \sum H_n$;

4. If R is the closure of the image of T, then $R = \sum H_n$ over the n such that $\lambda_n \neq 0$;

5. an operator S on H commutes with T if and only if it leaves each eigenspace of T invariant: $S(H_n) \subset H_n$ for all n.

Proof

1. If λ and ν are eigenvalues, with $\lambda \neq \mu$, and $|\lambda|$, $|\mu| > \varepsilon > 0$, and x and y are the corresponding eigenvectors (*normalized*, so that $||x|| = ||y|| = 1$), then
$$||Tx - Ty||^2 = ||\lambda x - \mu y||^2 = |\lambda|^2 + |\mu|^2 > 2\varepsilon^2,$$
since T is real, and hence (by remarks (a) and (b) on p. 80) $(x, y) = (y, x) = 0$. If there were infinitely many such eigenvalues, the corresponding eigenvectors would map into an infinite set of points in the image of the unit sphere having no limit point, contradicting the assumption that T be compact.

2. Let $\lambda_n \neq 0$, and let H_n be the corresponding eigenspace. The set $\{H_n \cap \{x \mid ||x|| = 1\}\}$ is bounded. If $x \in H_n$, then $Tx = \lambda_n x$, and $||x|| = 1$ implies that $||\lambda_n x|| = |\lambda_n| \cdot ||x|| = |\lambda_n|$. Hence the above set maps into $H_n \cap \{x \mid ||x|| = |\lambda_n|\}$ which is relatively compact only if H_n is finite dimensional (cf. Remarks (3) and (4) on p. 81).

3. Let T be real and compact, and $T \neq 0$. Then the proof is in four steps.

   3(a). T has an eigenvalue $\lambda \neq 0$.

   3(b). $Tx = \sum_k \lambda_k P_k x$, $\forall\, x \in H$, where (λ_k) are eigenvalues of T, and (P_k) the Projections on the corresponding eigenspaces (H_k). i.e. $||Tx - \sum_{k=1}^{n} \lambda_k P_k x|| \longrightarrow 0$, for $n \longrightarrow \infty$.

   3(c). 0 is an eigenvalue if and only if there exists $x \neq 0$, such that $x \perp H_k$, $\lambda_k \neq 0$, $k = 1, 2, \ldots$.

3(d). It follows from 3(c) that the eigenspace H_0 corresponding to the eigenvalue 0 is given by $H_0 = \{x \mid x \perp H_k, \quad k = 1, 2, 3, \ldots\}$, and is the orthogonal complement of the subspace spanned as Hilbert space by $H_1, H_2, \ldots$, written as $H_0^{\perp} = \overline{\sum_{\lambda_k \neq 0} H_k}$.

Proof of (3) Introduce in H_k, $k = 1, 2, \ldots$, an orthonormal basis:

$$\begin{array}{ccccccc} & H_1 & & & H_2 & & \\ e_{11}, e_{12}, & \ldots, & e_{1n_1}, e_{21}, e_{22}, & \ldots, & e_{2n_2}, & \ldots . \end{array}$$

Rewrite this in a new notation as follows

$$e_1, e_2, \ldots, e_{n_1}, e_{n_1+1}, e_{n_1+2}, \ldots e_{n_1+n_2}, \ldots .$$

Since the spaces H_k, $k = 1, 2, \ldots$ are mutually orthogonal (see Remark (b), on p. 80, after Theorem 2),

$$(e_1, e_2, \ldots)$$

is an orthonormal set. It is complete if and only if 0 is not an eigenvalue of T. Hence, *if 0 is not an eigenvalue of T*, we have the Fourier expansion

$$x = \sum_{n=1}^{\infty} (x, e_n) e_n, \qquad x \in H$$

which is unique. Here $(x, e_n)e_n \in H_{x_n}$, say. This means $H = \sum_{k=1}^{\infty} H_k$.

If 0 *is* an eigenvalue, then $H = H_0 + H_0^{\perp}$, where $H_0^{\perp} = \bigvee_{\lambda_k \neq 0} H_k$ is a Hilbert space by 3(d). The restriction of T to this space has all its eigenvalues *different* from zero. Hence, as above, $\bigvee_{\lambda_k \neq 0} H_k = \sum_{k=1}^{\infty} H_k$. That is to say, $H = H_0 + \bigvee_{\lambda_k \neq 0} H_k$, hence

$$x = \sum_{k=0}^{\infty} x_k, \quad \forall\, x \in H, \quad x_k \in H_k,$$

as claimed in (3).

Now we have to prove 3(a), (b), (c) and (d).

Proof of 3(a) (Smithies) $T \neq 0 \Longrightarrow ||T|| \neq 0$. Since T is real, by Theorem 2,

$$\sup_{||x||=1} |(Tx, x)| = ||T||.$$

It follows that either for $M = ||T||$ or for $M = -||T||$, or both, there exists a sequence $\{x_n\}$ of unit vectors (i.e. $x_n \in H$, $||x_n|| = 1$) with the property

$$\lim_{n \to \infty} (Tx_n, x_n) = M.$$

Since T is completely continuous, there exists a convergent subsequence, which we can also denote by (x_n), such that $Tx_n \longrightarrow y$. Since $M \neq 0$, it follows that $y \neq 0$.
We shall see that

$$Ty = My.$$

Now

$$\begin{aligned}
||Tx_n - Mx_n||^2 &= (Tx_n - Mx_n, Tx_n - Mx_n) \\
&= (Tx_n, Tx_n - Mx_n) - (Mx_n, Tx_n - Mx_n) \\
&= (Tx_n, Tx_n) - (Tx_n, Mx_n) - (Mx_n, Tx_n) \\
&\qquad + (Mx_n, Mx_n) \\
&= ||Tx_n||^2 - M(Tx_n, x_n) - M(x_n, Tx_n) \\
&\qquad + M^2||x_n||^2 \\
&\leq 2M^2 - 2M(Tx_n, x_n) \quad \text{(Lemma 12)} \\
&= 2M(M - (Tx_n, x_n)) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned}$$

Hence

$$||Ty - My|| \leq ||Ty - TTx_n|| + ||TTx_n - TMx_n|| + ||TMx_n - My||.$$

Now

$||Ty - TTx_n|| \longrightarrow 0$, as $n \longrightarrow \infty$, since T is continuous, and

$Tx_n \longrightarrow y$, and $||TTx_n - TMx_n|| \leq ||T|| \cdot ||Tx_n - Mx_n|| \longrightarrow 0$, since $||Tx_n - Mx_n|| \longrightarrow 0$ as just seen, and $||T|| \neq 0$, and

$$||TMx_n - My|| = M||Tx_n - y|| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

It follows that $Ty = My$, i.e. M is an eigenvalue of T, $M \neq 0$.

Proof of 3(b) Define the operators:

$$B_n \underset{def}{=} T - \sum_{k=1}^{n} \lambda_k P_k, \quad \text{for} \quad n = 1, 2, \ldots,$$

(*) with real coefficients. B_n is *real*; (finite linear combination of *real* operators with *real* coefficients) B_n is *compact*; and B_n has the eigenvalues $\lambda_{n+1}, \lambda_{n+2}, \ldots$, and none others which are different from zero.

For, let $\lambda \neq 0$, λ an eigenvalue of B_n. Then there exists a vector $x \neq 0$, such that $B_n x = \lambda x$, $x \in H$. (†). Now we have

$$x = y_1 + y_2 + \cdots + y_n + z,$$

where $y_1 \in H_1$, $y_2 \in H_2$, …, $y_n \in H_n$, and $z \perp H_k$ $(k = 1, 2, \ldots, n)$. And

$$\begin{aligned} B_n x &= \left(T - \sum_{k=1}^{n} \lambda_k P_k\right)\left(z + \sum_{k=1}^{n} y_k\right) \\ &= Tz + \sum_{k=1}^{n} T y_k - \sum_{k=1}^{n} \lambda_k P_k z - \sum_{k=1}^{n} \lambda_k P_k y_k = Tz, \end{aligned}$$

since $TY_k = \lambda_k y_k$, $\lambda_k P_k z = 0$, $P_k y_k = y_k$. Hence, for $k = 1, 2, \ldots, n$, we have

$$\begin{aligned} \lambda(y_k, y_k) &= \lambda(y_1 + y_2 + \cdots + y_n + z, y_k) \\ &\quad \text{(since } (y_k, z) = 0, \text{ and } (y_k, y_j) = 0, \text{ for } k \neq j) \\ &= \lambda(x, y_k) = (\lambda x, y_k) \\ &= (B_n x, y_k) \quad \text{(by assumption)} \\ &= (Tz, y_k) \quad \text{(see above)} \\ &= (z, T y_k) \quad \text{(since } T = T^*) \\ &= (z, \lambda_k y_k) = \lambda_k (z, y_k) = 0, \quad \text{since } \lambda_k \text{ is real.} \end{aligned}$$

However, $\lambda \neq 0$. It follows therefore that $y_k = 0$, for $k = 1, 2, \ldots, n$, hence $x = z$ $(z \neq 0$, since $x \neq 0)$, where $z \perp H_k$, $k = 1, 2, \ldots, n$ $(P_k z = 0, k = 1, 2, \ldots, n)$.

Thus we have

$$Tz = B_n x = B_n z = \lambda z, \qquad \text{(see (†) above)}$$

where $z \neq 0$, and $z \perp H_k$, $k = 1, 2, \ldots, n$. It follows that λ is an eigenvalue of T, and $\lambda \neq \lambda_k$, $k = 1, 2, \ldots, n$, since $z \perp H_k$ for $k = 1, 2, \ldots, n$.

On the other hand one sees that $\lambda_{n+1}, \lambda_{n+2}, \ldots$ *are* eigenvalues of B_n. [λ_{n+1} is an eigenvalue of T, hence there exist $x_{n+1} \neq 0$, such that $Tx_{n+1} = \lambda_{n+1}x_{n+1}$, and $P_k x_{n+1} = 0$, for $k = 1, 2, \ldots, n$. Therefore $B_n x_{n+1} = \lambda_{n+1}x_{n+1}$]. Thus the statement marked $(*)$ above is proved.

Now (Theorem 2 + Schwarz's inequality),

$$||B_n|| \leq \max(|\lambda_{n+1}|, |\lambda_{n+2}|, \ldots) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

[If T has only finitely many eigenvalues, then $||B_n|| = 0$ for sufficiently large n]. Thus

$$\left\| T - \sum_{k=1}^{n} \lambda_k P_k \right\| \longrightarrow 0 \text{ as } n \longrightarrow \infty,$$

which proves 3(b).

Proof of 3(c) and (d) We have $Tx = 0 \Longleftrightarrow \sum_k \lambda_k P_k(x) = 0$ [by 3(b)]. But $\sum \lambda_k P_k x = 0 \Longrightarrow P_j \sum_k \lambda_k P_k x = 0 \Longleftrightarrow \sum_k \lambda_k P_j P_k x = 0$, and $P_j P_k x = 0$ for $j \neq k$, and $P_j^2 = P_j$ (Remark (b), p. 80, after Theorem 2). Thus $\sum \lambda_k P_j P_k x = 0 \Longleftrightarrow \lambda_j P_j(x) = 0$, and this holds for all j. But $\lambda_j \neq 0$, hence $P_j(x) = 0$, $\forall\, j$. Thus

$$\begin{aligned} \sum_k \lambda_k P_j P_k x = 0 \quad &\Longleftrightarrow \quad P_k x = 0 \text{ for } k = 1, 2, \ldots \\ &\Longleftrightarrow \quad x \perp H_k, \text{ for } k = 1, 2, \ldots \end{aligned}$$

which settles 3(c), from which 3(d) follows.

4. If $x = Ty$, $y \in H$, then, by 3(b), we have $x = Ty = \sum_k \lambda_k P_k(y)$, obviously with $\lambda_k \neq 0$. Now $P_k(y)$ is an element in H_k (finite dimensional) with the basis vectors (e_{kj}). Hence $P_k y = \sum_j (y, e_{kj}) e_{kj}$

(finite sum). Thus

$$
\begin{aligned}
x = Ty = \sum_k \lambda_k P_k y &= \sum_{k,j} \lambda_k (y, e_{kj}) e_{kj} \\
&= \sum_{k,j} (y, \lambda_k e_{kj}) e_{kj}, \quad \text{since } \lambda_k \text{ is real} \\
&= \sum_{k,j} (y, T e_{kj}) e_{kj} \\
&= \sum_{k,j} (Ty, e_{kj}) e_{kj}, \quad \text{since } T \text{ is real} \\
&= \sum_{k,j} (x, e_{kj}) e_{kj}, \quad \text{since } x = Ty \text{ by assumption} \\
&= \sum_n (x, e_n) e_n \quad ((x, e_n) e_n \in H_{k_n}, \text{ say}).
\end{aligned}
$$

Hence $R = \sum_{\lambda_k \neq 0} H_k$.

5. Let T be real and compact. Then, by the proof of 3(b), $T = \sum \lambda_k P_k$. If $x_k \in H_k$, then $Tx_k = \sum_j \lambda_j P_j x_k = \lambda_k x_k$, and

(i) $$STx_k = \lambda_k S x_k.$$

On the other hand,

(ii) $$TS(x_k) = \sum_j \lambda_j P_j (s x_k),$$

and

(iii) $$P_k(Sx_k) = Sx_k \iff Sx_k \in H_k; \quad \forall\, k.$$

[For, $P_k(Sx_k) = Sx_k \Longrightarrow Sx_k \in H_k$, otherwise $Sx_k \in H_i$, $i \neq k$, so that $P_k(Sx_k) = 0$, since $H_i \perp H_j$ for $i \neq j$. Conversely, $Sx_k \in H_k \Longrightarrow P_k(Sx_k) = Sx_k$, by definition of P_k].

[Note: $Tx = 0 \iff \sum \lambda_k P_k x = 0$ (see Proof of 3(c) and (d))
$\iff P_k x = 0 \iff x \perp H_k, \quad k = 1, 2, \ldots$].

Hence

$$
\begin{aligned}
SH_k \subset H_k,\ \forall\, k \geq 1 &\Longrightarrow TS = ST \text{ on } H_k,\ \forall\, k \geq 1 \\
&\Longrightarrow TS = ST \text{ on } H = \sum_{k=1}^{\infty} H_k.
\end{aligned}
$$

While

$$ST = TS \text{ on } H_k, \forall k \geq 1 \Longrightarrow P_k(Sx_k) = Sx_k, \forall k \geq 1, \text{ by (i) and (ii)}$$
$$\Longleftrightarrow Sx_k \in H_k, \forall k \geq 1, \text{ where } x_k \in H_k.$$

This proves 5.

Definition An operator T on H is said to be

(i) *positive* if and only if $(Tx, x) \geq 0, \forall x \in H$,

(ii) *strictly positive* if and only if $(Tx, x) > 0, \forall x \neq 0$.

Corollaries

1. A completely continuous, real operator T is positive (strictly positive) if and only if its eigenvalues are ≥ 0 (or > 0).

   For, if $x \in H_\lambda$, the eigenspace corresponding to λ, then $(Tx, x) = (\lambda x, x) = \lambda(x, x)$, and $(x, x) \geq 0$ and, by Theorem 3, this can be extended to H.

2. A completely continuous, real operator T which is positive (strictly positive) has a positive (strictly positive) square root $\sqrt{T}$ which is unique and commutes with it.

   For if we define $\sqrt{T}$ on the eigenspace H_n corresponding to the eigenvalue λ_n by: $\sqrt{T}x = \sqrt{\lambda_n}x$, and extend linearly, then $(\sqrt{T})^2 x = Tx$ on $\sum H_n = H$.

3. If H is finite dimensional, and T is strictly positive, then T has an inverse.

   For if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of T, and $H_1, \ldots, H_n$ the corresponding eigenspaces, we can define $T^{-1}x = \dfrac{1}{\lambda_n}x$ on H_n, and extend linearly.

4. The spectral theorem for completely continuous real operators on a *finite* dimensional H amounts to the standard theorem that if A is a Hermitian matrix, then there exists a unitary matrix U, such that UAU^{-1} is diagonal.

IV. Compact groups and their representations

IV.1 Equivalence of every finite-dimensional representation to a unitary representation

Our aim is to prove the following results on finite-dimensional representations of compact groups. We wish to show (1) that every representation is equivalent to a unitary representation (2) that every representation is completely reducible, (3) the orthogonality relations, and (4) the Peter-Weyl theorem.

G will denote a topological group. If it is compact, it will be explicitly so stated, in which case we normalize the Haar measure by taking $\int_G 1\, dx = 1$.

$GL(n, \mathbb{C})$ *will denote* (as in Ch. I, p. 5) *the general linear group*, with the topology given by considering its elements as coordinates in $\mathbb{R}^{2n^2}$. $TL(V, n, \mathbb{C}) = TL(n, \mathbb{C})$ *will denote the group of non-singular linear transformations on an n-dimensional vector space* V (over $\mathbb{C}$), and $U(H, n, \mathbb{C}) = U(n, \mathbb{C})$ *the group of unitary transformations on a Hilbert space* H which contains a complete orthonormal set of power n (n is invariant, as can be proved). Here n is any cardinal number.

The topology of $TL(n, \mathbb{C})$. Let V be the vector space on which the transformations operate, and V^* its dual. Define for each $v \in V$ and $f \in V^*$ a complex-valued function g on $TL(n, \mathbb{C})$ as follows:

$$g(T) = f(Tv), \qquad T \in TL(n, \mathbb{C}).$$

The topology of $TL(n, \mathbb{C})$ is that *generated* by all such g's, i.e. for each open set O in the complex plane, and each such g, we get a set E, where

$$E = \{T \mid g(T) \in O)\},$$

and we define the open sets of $TL(n, \mathbb{C})$ to be all unions of finite intersections of such E's. This is the "coarsest" topology for which all the g's are continuous.

To know that representations by linear transformations and by matrices are essentially the same, one has to know that $TL(n,\mathbb{C})$ and $GL(n,\mathbb{C})$ are topologically isomorphic, i.e. there is an algebraic isomorphism of $TL(n,\mathbb{C})$ onto $GL(n,\mathbb{C})$, which is also a homeomorphism. [Let $v_1, v_2, \ldots, v_n$ any basis for V. If T is any linear transformation, associate with it a matrix $F(T) = (t_{ij}(T))$ defined by $Tv_j = \sum_i t_{ij}(T)v_i$. Then F is an isomorphism onto. To see that it is a homeomorphism, consider the functions $t_{ij}(T)$. They are of the type used above in defining the topology of $TL(n,\mathbb{C})$. For if $f_1, f_2, \ldots, f_n$ is a dual basis to $v_1, v_2, \ldots, v_n$, then

$$f_i(Tv_j) = f_i\left(\sum_k t_{kj}(T)v_k\right) = \sum_k t_{kj}(T)\delta_{ki} = t_{ij}(T).$$

Every g of the form $g(T) = f(Tv)$, $f \in V^*$, is a linear combination of the n^2 t_{ij}'s since the f_i's form a dual basis. It follows that the topology of $TL(n,\mathbb{C})$ is *generated* by the t_{ij}'s, which is the topology of $GL(n,\mathbb{C})$.]

A matrix M, and a linear transformation T, are said to correspond if $M = (t_{ij}(T))$ for some basis in V.

It can be proved that $TL(n,\mathbb{C})$ is a *topological* group, which is topologically isomorphic to $GL(n,\mathbb{C})$.

The topology of $U(n,\mathbb{C})$ is the topology generated by all functions of the form

$$f(U) = (Ux, y), \quad \text{for } x, y \in H,\ U \in U(n,\mathbb{C}).$$

If n is finite, then $U(n,\mathbb{C})$ is a subgroup of $TL(n,\mathbb{C})$, where $U(n,\mathbb{C})$ and $TL(n,\mathbb{C})$ are transformations of the same space, and the topology of $U(n,\mathbb{C})$ is that induced from $TL(n,\mathbb{C})$, using the fact that every linear functional on H is given by an inner product. (cf. p. 72).

Definition *A finite dimensional representation* of G by linear transformations (or matrices) is a continuous homomorphism of G into $TL(n,\mathbb{C})$ (or $GL(n,\mathbb{C})$). The vector space on which $TL(n,\mathbb{C})$ operates is the *representation space*; and n *the degree*, of the representation.

Examples

1. $G =$ real numbers, $\phi(r) = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$

2. $G =$ real numbers, $\phi(r) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ r & 1 & 0 & 0 \\ r & 0 & 1 & 0 \\ r^2 & r & r & 1 \end{pmatrix}$

3. $G = GL(n, \mathbb{C})$, $\phi(x) = \begin{pmatrix} 1 & 0 \\ \log|\det x| & 1 \end{pmatrix}$

4. $G =$ group of matrices of the form $x = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ * & & \mu \end{pmatrix}$, $\phi(x) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$.

Definition Let ϕ and ψ be two finite dimensional representations of G by linear transformations. Then ϕ and ψ are said to be *equivalent*, written $\phi \equiv \psi$, if and only if there exists a linear isomorphism T of the representation space V of ϕ onto the representation space W of ψ, such that

$$\phi(x) = T^{-1}\psi(x)T, \quad \text{for all } x \in G.$$

Two finite dimensional representations ϕ and ψ of G by matrices are equivalent if and only if they have the same degree and there exists a non-singular matrix T of that degree such that

$$\phi(x) = T^{-1}\psi(x)T, \quad \text{for all } x \in G.$$

Lemma 1 Let H be a finite dimensional Hilbert space. (A finite dimensional vector space over $\mathbb{C}$ can be made into a Hilbert space). Let ϕ be a representation of G by linear transformations on H. If $k(x) = (\phi(x)v, \phi(x)w)$, for any $v, w \in H$, $x \in G$, then k is a continuous function on G.

Proof Note that $T \longrightarrow T^*$ is a continuous $\underline{\text{automorphism}}$ of $TL(n, \mathbb{C})$, since it is open $(\{T^* \mid (Tv, w) \in O\} = \{T^* \mid \overline{(T^*w, v)} \in O\}$, O open), and is its own inverse $(T^{**} = T)$.

Further every step in

$$T \longmapsto (T, T) \longmapsto (T^*, T) \longmapsto T^* \cdot T \longmapsto (T^* \cdot Tv, w)$$

is continuous, and $(T^*Tv, w) = (Tv, Tw)$. Hence $T \longmapsto (Tv, Tw)$ is continuous. k is the composite of the continuous maps $x \longmapsto \phi(x)$, and $T \longmapsto (Tv, Tw)$.

Theorem 1 Every finite dimensional representation of a compact group G by linear transformations is equivalent to a unitary representation. (i.e. a representation by unitary transformations).

Proof Let ϕ be the given representation, and V the representation space of ϕ (considered as a Hilbert space by choosing any inner product). We shall find an equivalent unitary representation ψ with the same representation space.

Define a semi-bilinear functional on V by:

$$f(v, w) = \int_G (\phi(x)v, \phi(x)w)\, dx,$$

where $v, w \in V$, $x \in G$, $dx =$ the Haar measure on G, $\phi(x)$ a linear transformation.

By Lemma 1, $(\phi(x)v, \phi(x)w)$ is a continuous function on G, which is compact. By Lemma 10 of Chapter III on semibilinear functionals, there exists a linear transformation T_1 of $V \longrightarrow V$, such that

$$(T_1 v, w) = f(v, w) = \int_G (\phi(x)v, \phi(x)w)\, dx,$$

and T_1 is strictly positive, since

$$(T_1 v, v) = \int ||\phi(x)v||^2\, dx > 0, \quad \text{if} \quad v \neq 0,$$

since $\phi(x)$ has an inverse (non-singular).

Now $\dim V = n < \infty$. Hence T_1 has a strictly positive square root T, and T has an inverse (Corollary 3 to Theorem 3, Chapter III, p. 88).

Define $\psi(x) = T\phi(x)T^{-1}$. Then ψ is a representation, and $\psi \equiv \phi$.

To show that $\psi(y)$ is *unitary* $\forall\, y \in G$, note, first of all, that T is real [T strictly positive $\Longrightarrow (Tx, x) > 0$, $\forall\, x \neq 0$, which implies that T is real, cf. Lemma 12, Chapter III, p. 76]. Hence

$$\begin{aligned}
(Tv, Tw) &= (T^2 v, w) \quad [(Tx, y) = (x, T^* y) = (x, Ty)] \\
& \qquad\qquad\qquad\qquad x = Tv, \quad y = w \\
&= (T_1 v, w) \\
(*) \qquad &= \int (\phi(x)v, \phi(x)w)\, dx
\end{aligned}$$

and

$$\begin{aligned}
(\psi(y)v, \psi(y)w) &= (T\phi(y)T^{-1}v,\ T\phi(y)T^{-1}w), \text{ by definition of } \psi; \\
&= \int (\phi(x)\phi(y)T^{-1}v, \phi(x)\phi(y)T^{-1}w)\, dx, \quad \text{by } (*) \\
&= \int (\phi(xy)T^{-1}v, \phi(xy)T^{-1}w)\, dx \\
&= \int (\phi(x)T^{-1}v, \phi(x)T^{-1}w)\, dx \\
& \quad (G \text{ compact} \Longrightarrow dx \text{ is right-invariant}) \\
&= (TT^{-1}v, TT^{-1}w), \text{ by } (*); \\
(\dagger) \qquad &= (v, w).
\end{aligned}$$

Remarks Every compact group of linear transformations on a finite dimensional vector space leaves a positive-definite semi-bilinear form invariant. The inner product [,] defined by $[x, y] = (Tx, Ty)$ is such a form and the invariance is given by (†).

IV.2 Complete reducibility

Definition *A set* $\mathcal{T}$ *of linear transformations* of a finite dimensional vector space V is said to be (i) *irreducible* if and only if there exists no proper linear subspace of V which is invariant under all $T \in \mathcal{T}$; (ii) *completely reducible* if and only if for each linear subspace V_1 of V invariant under $\mathcal{T}$, there exists a complementary invariant subspace V_2.

A representation of G is irreducible, or completely reducible, if and only if its image is. These properties are *invariant under equivalence* of representations.

The corresponding formulation for matrices runs: *a set of matrices* $\mathcal{M}$ is said to be

(i) *irreducible* if and only if it is ***not*** possible to put them simultaneously (i.e. by taking each T into PTP^{-1} with P fixed) into the form

$$\begin{bmatrix} * & 0 \\ * & * \end{bmatrix};$$

(ii) *completely reducible* if and only if it is possible to put them simultaneously into the form

$$\begin{bmatrix} * & & & 0 \\ & * & & \\ & & \ddots & \\ 0 & & & * \end{bmatrix}$$

where each block is irreducible.

Theorem 2 Every finite dimensional representation of a compact group G by linear transformations is completely reducible.

Proof By Theorem 1 it is sufficient to prove this for *unitary* representations. If a unitary transformation on a *finite* dimensional vector space leaves a linear subspace invariant, then it also leaves the orthogonal complement invariant. [Note that if the dimension is finite, isometry implies onto, so that $TH_0 \subset H_0 \Longrightarrow TH_0 = H_0 \Longrightarrow TH_0^{\perp} = H_0^{\perp}$. See Lemma 13, Ch. III, p. 77.]

Definition If V_1 is an invariant subspace of V for the representation ϕ of G, and ϕ_1 the restriction of ϕ to V_1, we call ϕ_1 *the representation induced by ϕ on V_1*. A representation ϕ is said to be the *direct sum* of the representations $\phi_1, \phi_2, \ldots, \phi_n$ on subspaces $V_1, \ldots, V_n$ of the representation space V, if and only if (i) ϕ_i is the representation induced by ϕ on V_i (so that, in particular, V_i is invariant under ϕ), and (ii) $V = V_1 \oplus \cdots \oplus V_n$ (a direct sum).

Corollary to Theorem 2 Every finite dimensional representation of a compact G by linear transformations is a direct sum of irreducible representations.

Proof If ϕ is not irreducible, let V_1 be an invariant subspace, and V_2 its invariant complementary subspace. Then ϕ is the direct sum of the representations it induces on V_1 and V_2. If *these* are not irreducible, they can be decomposed again. Since V is finite dimensional, this process must end.

IV.3 Orthogonality relations

Definition If ϕ is a finite dimensional representation of G by linear transformations on V, we define a family of functions on G, called the *representation functions* coming from ϕ. They are functions F of the form

$$F(x) = f(\phi(x)v), \quad \text{where} \quad x \in G,\ v \in V,\ f \in V^* \quad \text{(the dual of } V\text{)}.$$

Note that

1. Representation functions are *continuous*, for F is the composite of $x \longmapsto \phi(x)$ and $T \longmapsto f(Tv)$. (where f is, by definition, continuous)

2. If ϕ is a finite dimensional representation of G by linear transformations, and ϕ' the corresponding representation by matrices (see pp. 89-90), then the family of representation functions coming from ϕ is the family of linear combinations of matrix coefficients of ϕ'.

3. If ϕ and ψ are equivalent representations of G by linear transformations, they have the same family of representation functions.

[$f(\phi(x)v) = f(T^{-1}\psi(x)Tv) = f'(\psi(x)w)$, where $w = Tv$, and $f'(w) = f(T^{-1}w)$.] $f \in V^*$, $f' \in W^*$, $v \in V$, $w \in W$.

Lemma 2 (Schur's lemma) Let V and V' be finite dimensional vector spaces, J and J' irreducible sets of linear transformations on V and V', and S a linear transformation of V into V', such that

$$SJ = J'S.$$

Then either $S = 0$, or V and V' have the same dimension, and S is an isomorphism onto.

Proof Let k be the *kernel* of S and I the *image* of S. Then *I is invariant under J'*. For, if $x' \in I$, $T' \in J'$, then there exist $x \in V$, and $T \in J$, such that $x' = Sx$, and $ST = T'S$, hence $T'x' = T'Sx = STx = S(Tx) \in I$. (since $Tx \in V$). That is to say, $x' \in I \Longrightarrow T'x' \in I$.

Secondly *K is invariant under J*. For, if $x \in K$, and $T \in J$, there exists $T' \in J'$, such that $ST = T'S$, hence $S(Tx) = T'(Sx) = T'(0) = 0$ (since $Sx = 0$ by definition of K). That is to say, $x \in K \Longrightarrow S(Tx) = 0$ for $T \in J \Longrightarrow Tx \in K$.

But J and J' are *irreducible*, hence $I = \{0\}$ or V', and $K = \{0\}$ or V. If $S \neq 0$, then $I \neq \{0\}$, so that $K \neq V$ (for otherwise $K = V$, which implies that $I = \{0\}$), hence *$K = \{0\}$ and $I = V'$*.

Corollary to Lemma 2 If J is an irreducible family of linear transformations on a finite dimensional vector space V over an algebraically closed field F, and S a linear transformation on V such that $SJ = JS$, then $S = \lambda I$ for some $\lambda \in F$. [Here I is the identity transformation].

Proof It is known that there exists a $\lambda \in F$ such that $S - \lambda I$ has a non-zero kernel. [If M is a matrix associated with S, then λ is an eigenvalue of M.] And $J(S - \lambda I) = (S - \lambda I)J$. Hence Lemma 2 gives: $S - \lambda I = 0$ (since the kernel is V, i.e. $(S - \lambda I)v = 0, \forall\, v \in V$).

Theorem 3 (First half of the orthogonality relations) If ϕ and ψ are finite dimensional, inequivalent, irreducible, unitary representations of a compact group G, then all representation functions of ϕ are orthogonal to all representation functions of ψ.

Proof The representation spaces V and W are Hilbert spaces. Then for any representation functions F_1 of ϕ and F_2 of ψ, there exist $v, v' \in V$ and $w, w' \in W$ for which

$$F_1(x) = (\phi(x)v, v'), \quad F_2(x) = (\psi(x)w, w').$$

[$F(x) \underset{def}{=} f(\phi(x)v)$, $\quad x \in G, \quad v \in V, \quad f \in V^*$. F is a bounded linear functional. By Lemma 7 of Chapter III, $F(x) = (\phi(x)v, v')$ for a $v' \in V$].

We shall show that the semi-bilinear function f on $V \times W$ defined, for fixed v', w', by

$$f(v, w) = \int (\phi(x)v, v')\overline{(\psi(x)w, w')}\, dx,$$

vanishes. By Lemma 10, Chapter III, there exists a (unique) linear transformation T of V into W, such that

$$f(v, w) = (Tv, w).$$

However, for $y \in G$, we have

$$f(v, w) = f(\phi(y)v, \psi(y)w) = (T\phi(y)v, \psi(y)w).$$

[For

$$\begin{aligned} \int (\phi(x)\phi(y)v, v')\overline{(\phi(x)\phi(y)w, w')}\, dx &= \int (\phi(xy)v, v')\overline{(\psi(xy)w, w')}\, dx \\ &= \int (\phi(t)v, v')\overline{(\psi(t)w, w')}\, dt, \end{aligned}$$

because of the right-invariance of dt].

Hence we obtain:

$$(T\phi(y)v, \psi(y)w) = (Tv, w).$$

With $\psi(y^{-1})w$ in place of w, we get

$$(T\phi(y)v, w) = (Tv, \psi(y^{-1})w) = (\psi(y)Tv, w),$$

since ψ is unitary (over $\mathbb{C}$). It follows that

$$T\phi(y) = \psi(y)T, \quad \forall\, y \in G;$$

however ϕ and ψ are *not* equivalent; hence by Schur's lemma $T = 0$, which implies that $(Tv, w) = 0 = f(v, w)$.

Theorem 4 (Second half of the orthogonality relations) If ϕ is an irreducible, unitary representation, of degree $n < \infty$, of a compact group G, and F_1, F_2 are representation functions of ϕ defined by

$$F_1(x) = (\phi(x)v, v'), \quad F_2(x) = (\phi(x)w, w'),$$

then we have

$$\int F_1(x)\overline{F_2(x)}\, dx = \frac{1}{n}(v, w)\overline{(v', w')}.$$

Remark The corresponding relations for matrix coefficients can be obtained as follows: choose a complete orthonormal set $v_1, v_2, \ldots, v_n$

in V, and define $a_{ij}(x) = (\phi(x)v_i, v_j)$. Then by Lemma 1, $a_{ij}(x)$ is continuous. Choose $v = v_i$, $v' = v_j$, $w = v_{k'}$, $w' = v_l$. Then

$$\int a_{ij}(x)\overline{a_{kl}(x)}\, dx = \frac{1}{n}(v_i, v_k)\overline{(v_j, v_l)} = \delta_{ik}\delta_{jl}\frac{1}{n}.$$

Proof of Theorem 4 Define f as in Theorem 3, with ϕ in place of ψ,

$$(*) \qquad f(v,w) = \int (\phi(x)v, v')\overline{(\phi(x)w, w')}\, dx.$$

As before we get $f(v,w) = (Tv, w)$ and

$$(\underbrace{T\phi(y)}\, v, w) = (\underbrace{\phi(y)T}\, v, w), \quad \text{hence} \quad T\phi(y) = \phi(y)T.$$

By the Corollary to Schur's Lemma 2, $T = \lambda I$ for some constant λ, hence

$$(\dagger) \qquad f(v,w) = \lambda(v,w).$$

Now let $v_1, \ldots, v_n$ be an orthonormal basis in V. Then we have $v_i(x) =_{def} \phi$ also as an orthonormal basis [$\phi(x)$ is unitary $\Longrightarrow$ $(\phi(x)v_i, \phi(x)v_j) = (v_i \cdot v_j)$, and $||\phi(x)v_i|| = ||v_i||$]. Now choose $v = w = v_i$ and sum from $i = 1$ to $i = n$. Then we get, from $(*)$ and $(\dagger)$,

$$(**) \qquad n\lambda = \int_G \sum_{i=1}^{n} (v_i(x), v')(w', v_i(x))\, dx.$$

Now $w' = \sum_{i=1}^{n}(w', v_i(x)), v_i(x)$, and $v' = \sum_{j=1}^{n}(v', v_j(x)), v_j(x)$, so that

$$(w', v') = \left(\sum_{i=1}^{n}(w', v_i(x))v_i(x), \sum_{j=1}^{n}(v', v_j(x))v_j(x) \right).$$

However $v_i(x) \perp v_j(x)$ for $i \neq j$. Hence

$$(w', v') = \sum_{i=1}^{n}(w', v_i(x)) \cdot |\underbrace{|v_i(x)|}_{=1}|^2 \cdot \overline{(v', v_i(x))},$$

and $(**)$ gives

$$n\lambda = \int_G (w', v')\, dx = \overline{(v', w')}, \text{ because of the normalization: } \int_G dx = 1;$$

or

$$\lambda = \overline{(v', w')}/n,$$

which together with $(\dagger)$ proves the theorem.

Characters Let ϕ be a finite dimensional representation of G by linear transformations. Let $v_1, v_2, \dots, v_n$ be a basis for the representation space V. Let $(t_{ij}(x))$ be the matrix associated with $\phi(x)$ by this basis. We define the *character* of ϕ to be the function F on G given by

$$F(x) = \operatorname{tr}(t_{ij}(x)) = \sum_{i=1}^{n} t_{ii}(x), \quad (\text{tr} = \text{trace}).$$

This definition is independent of the particular basis chosen.

Remarks

(1) Let ϕ, ψ be *in*equivalent, irreducible unitary representations of a compact group G. Then they have *different* characters.

If F_ϕ, F_ψ are the characters of ϕ, ψ, and $(a_{ij}), (b_{ij})$ the associated matrices, with respect to an orthonormal basis, then

$$\begin{aligned} \int_G F_\phi(x)\overline{F_\phi(x)}\,dx &= \int_G \sum_{i,j} a_{ii}(x)\overline{a_{jj}(x)}\,dx \quad \text{(by defn.)} \\ &= \int_G \sum_{i} a_{ii}(x)\overline{a_{ii}(x)}\,dx \quad \text{(orthogonality Th. 4)} \\ &= n \cdot \frac{1}{n} = 1, \end{aligned}$$

and

$$\int F_\phi(x)\overline{F_\psi(x)}\,dx \underset{\text{(Th. 3)}}{= 0 =} \int \sum_{i,j} a_{ii}(x)\overline{b_{jj}(x)}\,dx,$$

so that $F_\phi \neq F_\psi$.

[For Theorems 3 and 4 one needs the assumption that the representation is unitary. But Theorem 1 gives an *equivalent* representation. If $\phi_1 \equiv \phi_2$, then they have the same character]

(2) ϕ is irreducible $\Longleftrightarrow \int F_\phi(x) \cdot \overline{F_\phi(x)}\,dx = 1$.

The implication from left to right follows from (1). On the other hand *if* ϕ is *not* irreducible, then $\phi = \sum_{i=1}^{n} \phi_i$, where $n > 1$, and

ϕ_i is irreducible. Hence $F_\phi = \sum_{i=1}^n F_i$, and

$$\begin{aligned}\int_G F_\phi(x)\overline{F_\phi(x)}\,dx &= \sum_{i,j}\int_G F_{\phi_i}(x)\overline{F_{\phi_j(x)}}\,dx \\ &= n + \sum_{i\neq j}\int F_{\phi_i}(x)\overline{F_{\phi_j}(x)}\,dx \\ &\geq n > 1.\end{aligned}$$

IV.4 The Peter-Weyl theorem

Our aim is to prove *the Peter-Weyl theorem* in the following form. *If G is compact, then every complex-valued continuous function on G is a uniform limit of finite linear combinations of representation functions from irreducible representations.* For the proof we use the L_2-algebra of G.

By Theorem 12 of Chapter II we know that if G is compact, and $f, g \in L_2(G)$, then the convolution of f with g, written $f * g$, and defined by

$$f * g(x) = \int_G f(y)g(y^{-1}x)\,dy,$$

exists everywhere, and is continuous, with

$$|f * g(x)| \leq ||f||_2 \cdot ||g||_2.$$

Since we have normalized the Haar measure "dy" by taking $\int_G dy = 1$, we also have

$$||f * g||_2 \leq ||f||_2 \cdot ||g||_2.$$

We know that $L_2(G)$ is a Hilbert space; because G is compact, $L_2(G)$ is an associative algebra with the convolution as multiplication. But it does not contain G and does not have an identity (as compared with the group algebra of a finite group). But it does have "approximate" identities.

Definition Let $\varepsilon > 0$, $f \in L_2(G)$. An element $\delta \in L_2(G)$ is an *approximate identity for f with respect to ε*, if and only if

$$||f * \delta - f|| < \varepsilon, \quad \text{and} \quad ||\delta * f - f|| < \varepsilon. \quad (\text{Here } ||\cdot|| = ||\cdot||_2).$$

Theorem 5 If G is compact, $\varepsilon > 0$, and $f_1, f_2, \ldots, f_n \in L_2(G)$, then there exists a right approximate identity of all the f_i with respect to ε.

That is to say, there exists $\delta \in \mathcal{C}_0(G)$, such that

$$||f_i * \delta - f_i|| < \varepsilon, \quad i = 1, 2, \ldots, n.$$

Further, if $f_i \in \mathcal{C}_0(G)$, δ can be so chosen that

$$||f_i * \delta - f_i||_\infty = \sup_{p \geq 1} ||f_i * \delta - f_i||_p < \varepsilon, \quad i = 1, \ldots, n.$$

Proof Suppose, to begin with, that $f_i \in \mathcal{C}_0(G)$. Let v_i be an "ε-uniformity neighbourhood" of e. [By Lemma 17, Remark 1, of Chapter II, p. 60, there exists a neighbourhood V_i of e, such that $|f_i(xt) - f_i(x)| < \varepsilon$ for $t \in V_i$ and $\forall\ x \in G$] Choose $\delta \in \mathcal{C}_0(G)$, such that

$$\text{(i)}\ \ \delta \geq 0; \quad \text{(ii)}\ \ \int_G \delta\, dx = 1; \quad \text{(iii)}\ \ S(\delta) \subset \bigcap_{i=1}^{n} V_i,$$

where $S(\delta)$ is the support of δ, and V_i is symmetric for all $i = 1, 2, \ldots, n$. [δ does not depend on i]. We then have

$$
\begin{aligned}
(*) \quad ||f_i * \delta - f_i||_p^p &= \int \left| \int f_i(xy^{-1})\delta(y)\, dy - f_i(x) \right|^p dx \\
&= \int \left| \int f_i(xy^{-1})\delta(y)\, dy - \int f_i(x)\delta(y)\, dy \right|^p dx \\
&\qquad\qquad \int \delta(y)\, dy = 1 \\
&= \int \left| \int (f_i(xy^{-1}) - f_i(x))\delta(y)\, dy \right|^p dx \\
&\leq \int \left[\int |f_i(xy^{-1}) - f_i(x)| \cdot \delta(y)\, dy \right]^p dx \\
&\leq \int \left[\varepsilon \int \delta(y)\, dy \right]^p dx = \varepsilon^p.
\end{aligned}
$$

Next, we suppose that $f \in L_2(G)$. Let (f_n) be a sequence in $\mathcal{C}_0(G)$ such that $f_n \longrightarrow f$ (in L_2) [Theorem 6, Chapter II, p. 46], and (δ_n) a sequence in $\mathcal{C}_0(G)$, such that $||f_n * \delta_n - f_n|| < \frac{1}{n}$, $||\delta_n|| = 1$ (see (*) above). Then

$$\begin{aligned} ||f * \delta_n - f||_2 &\leq ||(f - f_n) * \delta_n||_2 + ||f_n * \delta_n - f_n||_2 + ||f_n - f||_2 \\ &\leq ||f - f_n||_2 \cdot ||\delta_n||_1 + \frac{1}{n} + \varepsilon_n \\ & \qquad\qquad [\text{Theorem 11, Chapter II, p. 62} \\ &= ||f - f_n||_2 + \frac{1}{n} + \varepsilon_n \qquad f \in L_2,\ g \in L_1 \Longrightarrow \\ &\longrightarrow 0, \text{ as } n \longrightarrow \infty \qquad ||f * g||_2 \leq ||f||_2 \cdot ||g||_1]. \end{aligned}$$

This can also be done for *finitely many* f's.

Lemma 3 Let G be compact, and $f, g \in L_2(G)$. Then $f*g$ is continuous, and $|f*g(x)| \leq ||f||_2 \cdot ||g||_2$ (by Th. 12, Ch. II, p. 63). Hence, if $f_n \longrightarrow f$ in L_2, and $g_n \longrightarrow g$ in L_2, then $f_n * g_n \rightrightarrows f * g$. ($\rightrightarrows$ indicates uniform convergence pointwise).

For

$$\begin{aligned} |f * g(x) - f_n * g_n(x)| &\leq |(f - f_n) * g(x)| + |f_n * (g - g_n)(x)| \\ &\leq ||f - f_n|| \cdot ||g|| + ||f_n|| \cdot ||g - g_n||, \end{aligned}$$

by the first part.

Definition Let G be compact. For each $f \in L_2(G)$, we define the operator $\mathcal{L}_f$ (of "*left multiplication by f*") by:

$$\mathcal{L}_f(g) = f * g, \quad g \in L_2(G).$$

[$\mathcal{L}_f$ exists by Th. 12, Ch. II, p. 63].

Theorem 6 $\mathcal{L}_f$ is bounded, and completely continuous (or, compact) for each $f \in L_2(G)$ (where G is compact).

Proof That $\mathcal{L}_f$ is bounded follows from the fact that

$$||\mathcal{L}_f(g)|| \leq ||f||_2 \cdot ||g||_2 \text{ (Th. 12, Chapter II, p. 63) and } m(G) = 1.$$

For the proof that $\mathcal{L}_f$ is completely continuous, we need the following *two properties*:

(A) If f is fixed, and (g_α) is any bounded set in $L_2(G)$, then the set $(f * g_\alpha)$ is *equicontinuous*. [If $\mathcal{C}(G)$ denotes the set of all continuous functions on a compact group G, and E is a subset of $\mathcal{C}(G)$, then E is said to be *equicontinuous* if and only if for every $\varepsilon > 0$ there exists a neighbourhood V of e, such that $|f(x) - f(y)| < \varepsilon$, for $xy^{-1} \in V$, and $\forall\, f \in E$.]

(B) Every infinite, uniformly bounded, equicontinuous family E of complex-valued functions on a compact Hausdorff space contains a uniformly convergent sequence.

If (A) and (B) hold, it follows that the image, under $\mathcal{L}_f$, of any bounded sequence (g_α) in $L_2(G)$ always contains a convergent subsequence, [the image is $(f * g_\alpha)$], hence $\mathcal{L}_f$ is completely continuous (cf. Ch. III, p. 80).

Proof of Proposition (A) To begin with, let $f, g \in \mathcal{C}_0(G)$. Then

$$(\dagger)\ |f * g(x) - f * g(y)| \le \int |f(xz^{-1}) - f(yz^{-1})| \cdot |g(z)|\, dz$$

$$\le \left(\int |f(xz^{-1}) - f(yz^{-1})|^2 dz \cdot \int |g(z)|^2 dz\right)^{1/2}$$

$$\le \left(\int |f(xz) - f(yz)|^2\, dz\right)^{1/2} \cdot ||g||$$

(G compact)

$$= ||f_x - f_y|| \cdot ||g||, \quad \text{where} \quad f_x(z) = f(xz).$$

Remark [In Chapter III we defined $f * g(x) = \int f(y) g(y^{-1}x)\, dy$. But $\int f(xy^{-1}) g(y)\, dy = \int f(z) g(z^{-1}x)\, d(z^{-1}x) = \int f(z) g(z^{-1}x)\, dz^{-1} = \int f(z) g(z^{-1}x)\, dz$, since G is uni-modular].

Now let $f, g \in L_2(G)$. Then there exist sequences $(f_n), (g_n) \subset \mathcal{C}_0(G)$ such that $f_n \longrightarrow f$ in L_2, and $g_n \longrightarrow g$ in L_2, and

$$|f_n * g_n(x) - f_n * g_n(y)| \le ||(f_n)_x - (f_n)_y|| \cdot ||g_n||, \text{ by } (\dagger) \text{ above.}$$

Since $(f_n)_x \longrightarrow f_x$ as $n \longrightarrow \infty$, we have

$$|f * g(x) - f * g(y)| \le ||f_x - f_y|| \cdot ||g||.$$

[By Lemma 3, $f_n * g_n \rightrightarrows f * g$]. If $x \longrightarrow y$, then $xy^{-1} \longrightarrow e$, and we know from Theorem 10 of Chapter II that $||f_x - f_y|| \longrightarrow 0$. Hence

$$|f * g_\alpha(x) - f * g_\alpha(y)| \le ||f_x - f_y|| \cdot ||g_\alpha|| \le M ||f_x - f_y|| < \varepsilon,$$

for $xy^{-1} \in V(e)$, and $\forall\ \alpha$.

Proof of Proposition (B) [Ref. Chevalley's Lie Groups I, pp. 204-205; Pontrjagin, Th. 19] A different proof from theirs is as follows.

Let M be a uniform bound for all $f \in E$. Let P be the product space of a set of closed balls S_x of radius M indexed by the points of G. Every function $f \in E$ defines a point in P, namely the point whose xth coordinate is $f(x)$, and since P is compact, by Tychonoff's theorem, there is a limit point f_0 in P of f's in E. f_0 is a complex-valued function on G with $|f_0(x)| \leq M$. We shall see that f_0 is *continuous, and is the uniform limit of a sequence in E.*

The topology of P for which Tychonoff's theorem holds is given as follows: if f is a point in P, i.e. $f(x) \in S_x$, $x_1, x_2, \ldots, x_n$ are indices, and $N_1, N_2, \ldots, N_n$ are neighbourhoods of $f(x_1), \ldots, f(x_n)$ in $S_{x_1}, S_{x_2}, \ldots, S_{x_n}$ respectively, then the set of points $g \in P$ such that $g(x_i) \in N_i$, $i = 1, \ldots, n$, is a neighbourhood of f in P. In this proof we take N_i to be a sphere of radius $\varepsilon_i > 0$.

To prove that f_0 is uniformly continuous, for any $\varepsilon > 0$, we take a neighbourhood V of e, such that

(i) $$|f(x) - f(y)| < \tfrac{\varepsilon}{3}, \quad \text{for } x, y \text{ with } xy^{-1} \in V,$$

and for *all* $f \in E$ (by the assumption of equicontinuity). We shall show that this is an ε-uniformity neighbourhood of e for f_0.

Let x and y be such that $xy^{-1} \in V$, and let N be the neighbourhood of f_0 in P defined by (the set of g's such that)

(ii) $$|g(x) - f_0(x)| < \frac{\varepsilon}{3}, \qquad (x \text{ instead of } x_1, \text{ and}$$

(iii) $$|g(y) - f_0(y)| < \frac{\varepsilon}{3}, \qquad y \text{ instead of } x_2)$$

and let f be a function in $N \cap E$ (such an f exists since f_0 is a limit point). Then we have, by (i), (ii) and (iii),

(iv) $|f_0(x) - f_0(y)| \leq |f_0(x) - f(x)| + |f(x) - f(y)| + |f(y) - f(y_0)| < \varepsilon$, for $xy^{-1} \in V$.

Hence f_0 is uniformly continuous.

To show that f_0 is the uniform limit of a sequence in E, let V be a neighbourhood of e, such that if $xy^{-1} \in V$, then

(i) $$|f(x) - f(y)| < \frac{\varepsilon}{3}, \quad \text{for } all \ f \in E$$

and

(iv) $$|f_0(x) - f_0(y)| < \frac{\varepsilon}{3}.$$

From the set of *all* V_x for $x \in G$, choose a finite subset $Vx_1, \ldots, Vx_n$ which covers (the compact) G, and a neighbourhood N^0 of f_0 in P such that

(v) $$|g(x_i) - f_0(x_i)| < \frac{\varepsilon}{3}, \quad \text{for } i = 1, 2, \ldots, n \text{ and for all } g \in N^0.$$

There exists an $f \in N^0 \cap E$ (since f_0 is a limit point), i.e. f is such a "g"; and

$$|f_0(x) - f(x)| \leq |f_0(x) - f_0(x_i)| + |f_0(x_i) - f(x_i)| + |f(x_i) - f(x)|.$$

Given $x \in G$, there exists a Vx_i such that $x \in Vx_i \Longleftrightarrow xx_i^{-1} \in V$. Hence by (iv), with $y = x_i$, we have $|f_0(x) - f_0(x_i)| < \frac{\varepsilon}{3}$. By (v) with $g = f$ we have $|f(x_i) - f_0(x_i) < \frac{\varepsilon}{3}$. By (i) with $y = x_i$, we have $|f(x_i) - f(x)| < \frac{\varepsilon}{3}$. Hence

$$|f_0(x) - f(x)| < \varepsilon, \qquad x \in Vx_i.$$

Given a sequence (ε_n), $\varepsilon_n > 0$, $\varepsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$, there exists a sequence $f^{(n)}(x)$ (in E) such that

$$|f_0(x) - f^{(n)}(x)| < \varepsilon_n \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Uniformity of convergence follows since there are only finitely many Vx_i's which cover G.

Adjoint function If G is compact, $f \in C_0(G)$, the *adjoint function of* f is the function in $C_0(G)$ defined by

$$f^*(x) = \overline{f(x^{-1})}, \quad x \in G.$$

(Since G is compact, we have $||f|| = ||f^*||$)

If $f \in L_2(G)$, and $f_n \in \mathcal{C}_0(G)$ such that $f_n \longrightarrow f$ in L_2-norm, then *define*

$$f^* = \lim_{\text{in } L_2} f_n^*.$$

This definition is independent of the sequence (f_n). For

(i) $||f_n^* - f_m^*|| = ||f_n - f_m||$, and since $f_n \longrightarrow f$ in L_2 norm, it follows that (f_n^*) is a Cauchy sequence in L_2, hence f^* exists; and

(ii) $||f^*|| \underset{def}{=} \lim ||f_n^*|| = \lim ||f_n|| = ||f||$, and

(iii) $||f_n^* - g_n^*|| = ||f_n - g_n|| \longrightarrow 0$, *if* $g_n \longrightarrow f$ in L_2, $g_n \in \mathcal{C}_0(G)$, hence $||f^* - g_n^*|| \le ||f^* - f_n^*|| + ||f_n^* - g_n^*|| \longrightarrow 0$, so that $f^* = \lim_{(\text{in } L_2)} g_n^*$.

Definition $f \in L_2(G)$ is called *self-adjoint* if $f = f^*$.

Lemma 4 We have

$$\mathcal{L}_{f^*} = (\mathcal{L}_f)^*; (\textit{assuming that } G \textit{ is compact})$$

hence, if $f = f^*$, then $\mathcal{L}_f$ is *a real operator*.

Proof Since G is compact, we have

$$\mathcal{L}_{f^*}(g)(y) = (f * g)(y) = \int_G f^*(yt^{-1})g(t)\, dt.$$

By the definition of $\mathcal{L}_f^*$, we have

(i) $$(\mathcal{L}_f g, h) = (g, \mathcal{L}_f^* h), \quad (f, g, h \in L_2(G))$$

where

$$\begin{aligned}(\mathcal{L}_f g, h) &= \int \overline{h}(x) \left(\int f(xy^{-1}) g(y)\, dy \right) dx \\ &= \int g(y) \left(\int f(xy^{-1}) \overline{h(x)}\, dx \right) dy \\ &= (g, H), \quad \text{say,}\end{aligned}$$

(ii) $$\text{where } \overline{H(y)} = \int f(xy^{-1}) \overline{h(x)}\, dx$$

$$\text{or } H(y) = \int \overline{f(xy^{-1})} h(x)\, dx$$

But $f^*(t) = \overline{f(t^{-1})}$, hence

$$\begin{aligned} H(y) &= \int f^*(yx^{-1})h(x)\,dx \\ &= \mathcal{L}_f^*(h)(y), \quad \text{because of (i) and (ii).} \end{aligned}$$

But, by definition, $\mathcal{L}_{f^*}(h)(y) = \int f^*(yx^{-1})h(x)\,dx$. It follows that

$$\mathcal{L}_{f^*}(h) = \mathcal{L}_f^*(h).$$

Remark If T is any operator on a Hilbert space H, we can write

$$T = T_1 + iT_2 = \frac{T+T^*}{2} + i\left(\frac{1}{2i}(T-T^*)\right),$$

where T_1 and T_2 are *self-adjoint.*

Similarly we can write a *function* $f \in L_2(G)$ as

$$f = f_1 + if_2 = \frac{f+f^*}{2} + i\left(\frac{1}{2i}(f-f^*)\right)$$

where f_1 and f_2 are *self-adjoint.*

Definition *The right regular representation* is the representation R, with the representation space $L_2(G)$, and $R(x)f = f^x$, $x \in G$, that is to say

$$(R(x)f)(y) = f^x(y) = f(yx).$$

The *representation functions* F : $F(x) = (f^x, g)$, $f, g \in L_2(G)$, $x \in G$, *of* R *are continuous*, since $x \longrightarrow f^x$ is continuous, and $f \longmapsto (f, g)$ is continuous. [Theorem 10, Chapter II; p. 69, Chapter III, see (*)]

[The topology of $U(n, \mathbb{C})$ is generated by the functions of the form $f(U) = (Ux, y)$, $x, y \in H$ (a Hilbert space), $U \in U(n, \mathbb{C})$, $\dim H = n$ see p. 90.]

If E is an *open* set in $U(n, C)$, we may take it to be of the form

$$E = \{T \mid (Tf, g) \in O\}, \quad \text{where } O \text{ is an open set in } C, \text{ and } f, g \in L_2(G).$$

Then we have

$$R^{-1}(E) = F^{-1}(O).$$

Hence the continuity of F implies the *continuity of* R.

[Note that $Tf = f^x$, $(Tf, g) = (f^x, g) = F(x)$; and $R(x)$ is a *unitary operator*, since $||R(x)f(\underline{y})|| = ||f(\underline{yx})|| = ||f(\underline{\underline{y}})||$ and $R(x)^{-1} = R(x^{-1})$].

We shall now consider ***finite dimensional representations*** obtained by restricting R to finite dimensional ***invariant subspaces.*** We shall see that the eigenspaces of the operator $\mathcal{L}_f$, for *self-adjoint* $f \in L_2(G)$, are invariant under R, and linear combinations of the corresponding representation functions approximate f, leading to the proof of the Peter-Weyl theorem.

Lemma 5 If G is compact, and R is the right regular representation of G, then $R(x)$ commutes with $\mathcal{L}_f$ for all $x \in G$, $f \in L_2(G)$.

Proof Let $f, g \in \mathcal{C}_0(G)$. Then

$$\mathcal{L}_f R(x) g(z) = f^* R(x) g(z) = \int f(zy^{-1}) g(yx)\, dy,$$

while

$$\begin{aligned} R(x)\mathcal{L}_f(g)(z) &= \mathcal{L}_f g(zx) \\ &= (f * g)(zx) \\ &= \int f(zxy^{-1}) g(y)\, dy \quad (y \longrightarrow yx) \\ &= \int f(zy^{-1}) g(yx)\, dy. \end{aligned}$$

Hence $\mathcal{L}_f R(x) = R(x)\mathcal{L}_f$ on $\mathcal{C}_0(G)$, which implies that
⊛ $\mathcal{L}_f R(x) g(z) = R(x)\mathcal{L}_f g(z)$ for all $g \in L_2(G)$, *provided that* $f \in \mathcal{C}_0(G)$. (because R as well as $\mathcal{L}_f$ are continuous).

Finally, if $f \in L_2(G)$, there exists $(f_n) \subset \mathcal{C}_0(G)$, such that $f_n \longrightarrow f$ in L_2-norm, hence $\mathcal{L}_{f_n}(g) \longrightarrow \mathcal{L}_f(g)$ for $g \in L_2$, *and*

$$\begin{aligned} \mathcal{L}_f R(x) g &= \lim_{n\to\infty} \mathcal{L}_{f_n} R(x) g \\ &= \lim_{n\to\infty} R(x)\mathcal{L}_{f_n}(g), \quad [\text{since } f_n \in \mathcal{C}_0(G) \\ &= R(x)\mathcal{L}_f(g) \qquad\qquad \text{and ⊛ above}] \end{aligned}$$

since R is continuous.

Theorem 7 (Peter-Weyl, 1927) If G is a compact, topological group, R its right regular representation, f any self-adjoint element of $L_2(G)$, (λ_i) the non-zero eigenvalues of $\mathcal{L}_f$, and H_i the corresponding eigenspaces, then

(1) Each H_i is a finite dimensional invariant subspace of $R = (R(x)$ for all $x \in G)$, and is also invariant under right-multiplication $*$ by any $g \in L_2(G)$ [i.e. H_i is a right ideal in the L_2-algebra].

(2) If R^i is the representation of G induced on H_i by R, then each eigenfunction of eigenvalue λ_i is a representation function of R^i.

(3) f is an L_2-limit of finite linear combinations of representations functions from the R^i.

(4) If $f \in C_0(G)$ then f is a uniform limit of finite linear combinations of representation functions from the R^i.

Proof

(1) By Theorem 3 of Chapter III, p. 81, parts 2 and 5, H_i is finite dimensional, and H_i is invariant for all $R(x)$. [Note that $T = \mathcal{L}_f$ in Theorem 3, Ch. III, since $\mathcal{L}_f$ is real and compact and $\mathcal{L}_f R(x) = R(x)\mathcal{L}_f$ by Lemma 5 just proved.].

Next, let $h \in H_i$. Then $h * g \in H_i$, for all $g \in L_2(G)$. For $\mathcal{L}_f h = \lambda_i h \Longrightarrow (f * h) = \lambda_i h \Longrightarrow (f * h) * g = \lambda_i(h * g)$. However, $(f * h) * g = f * (h * g) = \mathcal{L}_f(h * g)$. Hence $\mathcal{L}_f(h * g) = \lambda_i(h * g)$. Thus $h * g \in H_i$.

(2) The representation functions of R^i are all the functions of the form

$$\phi(x) = (R(x)h, k), \quad \text{for} \quad h, k \in H_i.$$

In order to show that *each* $h \in H_i$ is of this form, choose a complete orthonormal set $(k_1, \ldots, k_n)$ in H_i. Then we have

$$R(x)h = \sum (R(x)h, k_i)k_i. \quad \text{(finite Fourier expansion)}$$

In particular,

$$\begin{aligned} R(x)h(e) &= \sum (R(x)h, k_i)k_i(e) \\ &= \left(R(x)h, \sum \overline{k_i(e)} k_i\right). \end{aligned}$$

But $R(x)h(e) = h(ex) = h(x)$, hence

$$h(x) = \left(R(x)h, \sum \overline{k_i(e)} k_i\right),$$

so that $h(x)$ is a representation function of R^i.

(3) By part 3 of Theorem 3, Chapter III, every element in the image of $\mathcal{L}_f$, i.e. every $f * g$ for $g \in L_2(G)$, is an L_2-limit of finite linear combinations of eigenfunctions of $\mathcal{L}_f$, hence (by (2) just proved) of representation functions of the R^i's.

Let $\delta_n \in \mathcal{C}_0(G)$ be a right approximate identity of f with respect to $\frac{1}{n}$, and $\sum_{i=1}^N c_i^n h_i^n$ a finite linear combination of eigen-functions, (N depends on n) such that

$$\left\|f * \delta_n - \sum c_i^n h_i^n\right\| < \frac{1}{n}.$$

Then

$$\begin{aligned}\left\|f - \sum c_i^n h_i^n\right\| &\leq \|f - f * \delta_n\| + \left\|f * \delta_n - \sum c_i^n h_i^n\right\| \\ &< \frac{2}{n} \longrightarrow 0 \text{ as } n \longrightarrow \infty\end{aligned}$$

(4) Let $f \in \mathcal{C}_0(G)$, and (f_n) a sequence of representation functions such that $f_n \longrightarrow f$ in L_2 (by (3) just proved), and $(\delta_n) \subset \mathcal{C}_0(G)$ a sequence such that

$$\|f * \delta_n - f\|_\infty < \frac{1}{n}. \qquad \text{(Theorem 5)}$$

Then $f_n * \delta_n$ is a representation function, for $f_n = \sum_{i=1}^N c_i^n h_i^n$, $c_i^n \in \mathbb{C}$, $h_i^n \in H_i$. By (1) above, $h_i^n \in H_i \Longrightarrow h_i^n * \delta_n \in H_i$. Hence $f_n * \delta_n$ is a finite linear combination of representation functions of the R^i.

Secondly $f_n * \delta_n \rightrightarrows f$ (uniform convergence pointwise). For $f_n * \delta_n - f = (f_n - f) * \delta_n + (f * \delta_n) - f$. By Theorem 5 we have $\|f * \delta_n - f\|_\infty < \frac{1}{n}$. Given (f_n) such that $\|f_n - f\| \longrightarrow 0$, choose a subsequence f_{n_k}, such that $\|f - f_{n_k}\| < \frac{\varepsilon}{\|\delta_n\|} = \frac{\varepsilon}{M_n}$, say. ($\| \ \|_2$). Then $f_{n_k} * \delta_n \rightrightarrows f$, for

$$\left|\int f_{n_k}(xy^{-1})\delta_n(y)\,dy - \int f(x)\delta_n(y)\,dy\right|$$
$$= \left|\int \{f_{n_k}(xy^{-1}) - f(x)\}\delta_n(y)\,dy\right|$$
$$\le \left(\int |f_{n_k}(xy^{-1}) - f(x)|^2\,dy\right)^{1/2}\left(\int \delta_n^2(y)\,dy\right)^{1/2}$$

A Corollary to Theorem 7 is the following:

Theorem 8 A compact Lie group G has at least one faithful representation.

For the proof we assume known the following (Chevalley, p. 193).

Proposition If G is a compact Lie group, there exists a representation ϕ such that the kernel of ϕ is contained in the kernel of every representation.

Proof of Th. 8 Let $f \in C_0(G)$, with $f(x) \neq f(e)$, $x \neq e$. [Such an f exists by Urysohn's lemma]. By Theorem 7, f can be uniformly approximated by (linear combinations of) representation functions. Hence there exists a representation function F, such that $F(x) \neq F(e)$.

By the Proposition, if $\phi(x) = \phi(e) = I$, the identity transformation, then $\psi(x) = \psi(e)$ for *every* representation ψ, which contradicts $F(x) \neq F(e)$. Hence $\phi(x) \neq \phi(e)$ if $x \neq e$, and ϕ is faithful.

IV.5 Harald Bohr's Almost Periodic Functions

Let f be a complex-valued, continuous function on $\mathbb{R}_1$. Given $\varepsilon > 0$, the real number $\tau = \tau(\varepsilon) = \tau_f(\varepsilon)$ is called a *translation number of f corresponding to* ε, if we have

$$|f(x+\tau) - f(x)| \le \varepsilon$$

for all $x \in \mathbb{R}_1$. The function f is said to be *almost periodic* (in the sense of Bohr) if given $\varepsilon > 0$ there exists a length L, such that each interval (= open interval) of length L contains at least one translation number $\tau(\varepsilon)$.

It is a basic theorem of Bohr that if f is almost periodic, then $f(x)$ can be uniformly approximated by exponential sums of the form

$\sum_{j=1}^{n} c_j e^{i\lambda_j x}$, where c_j is complex, λ_j is real, and $-\infty < x < \infty$. It can be shown that Bohr's theorem is a consequence of the Peter-Weyl theorem. For that purpose it is convenient to use an alternative definition of an almost periodic function, due to S. Bochner, which results from his theorem that a function f is almost periodic if and only *if the set (of its translates)* $H = \{f(t+a)\}$, $-\infty < t < \infty$, a real, is *relatively compact*; that is to say, every sequence in H contains a uniformly convergent subsequence.

For let G be the set of uniform limits of functions in H (i.e. limits of uniformly convergent sequences from H). Then $G = \overline{H}$, which is to say that H is *dense* in G. Now G is compact (as a topological space, with the topology of uniform convergence), and G has a countable base. We can make G a group relative to an operation of addition $\oplus$ defined as follows:

$$f(t+a') \oplus f(t+a'') \underset{def}{=} f(t+a'+a'') \qquad (\in H).$$

This definition can be extended uniquely to all of G by continuity. Thus G is compact, and abelian, and has a countable base.

If Δ is an orthogonal set of continuous functions on a compact group with a *countable base*, then Δ is *countable.*

Let Δ' be the set of characters of all inequivalent irreducible representations of G. Since Δ' is an orthonormal set of functions on G, it follows that Δ' is countable. [Equivalent representations have the same character].

Hence G has countably many representations: $\phi_1, \phi_2, \ldots, \phi_n, \ldots$.

Since G is abelian, all irreducible representations are of degree 1. (The corresponding matrices are commutative, therefore of the form λI, where λ is complex, and I is the unit matrix. But the matrices are irreducible, hence the degree is 1, and each matrix is a complex number). If ϕ is a representation, and $x \in G$, then $\phi(x) = \chi(x)$ (character). If we consider equivalence-classes of representations, we may assume that they are unitary, hence $|\chi(x)| = 1$.

It follows that ϕ_n is a continuous homomorphism of G into the multiplicative group of complex numbers z with $|z| = 1$.

[$x \in H \Longrightarrow x = f(t+a) \underset{def}{=} x(a) \in H$. Now $x(a)$ is a continuous function of a, and $x(a')+x(a'') = x(a'+a'')$ by the definition given above.

Hence $\phi_n(x(a'+a'')) = \phi_n(x(a')) + \phi_n(x(a''))$, and $\phi_n(x(a)) = e^{i\lambda_n a}$, λ_n real.].

For $x(a) \in H$, define $f'(x) \equiv f'(x(a)) = f(a)$, and extend the definition, by continuity, to G. Then f' is continuous on G (with the strong topology). [$x \longrightarrow x_0$ implies that $\lim_{x\to x_0} \sup_{t\in\mathbb{R}_1} |f(t+x) - f(t+x_0)| = 0$].

By the theorem of Peter-Weyl, we can uniformly approximate f' by linear forms of representation functions.

On H we therefore have an approximation of $f(a)$ by linear forms of $\phi_n(x(a)) = e^{i\lambda_n a}$, $n = 1, 2, 3, \ldots$. [In the old notation: $(R(x)h, k)$, $h \in H^i$, $k \in H^i$, with $R(x) = \alpha(x)$, where $|\alpha(x)| = 1$, leads to $(R(x)h, k) = \alpha(x)(h, k) = ce^{i\lambda_n x}$].

IV.6 Statement of Hilbert's Fifth Problem

Let E_n denote the n-dimensional Euclidean space with n *real* coordinates.

After L.E.J. Brouwer, one knows that an open set in E_m cannot be homeomorphic to an open set in E_n for $m \neq n$. (n is a topological invariant, the dimension).

Let T be a topological space. Then T is *locally Euclidean* if and only if $p \in T$ implies that there exists a neighbourhood U_p of p which is homeomorphic to an open set in E_n; the neighbourhood is called a *coordinate neighbourhood*. For example, open subsets of E_n are locally Euclidean.

A locally Euclidean, Hausdorff space, which is connected, is called a *manifold*. [If $M \subset T$, then M is connected if and only if ($M = A \cup B$, A and B relatively open, $A \neq \emptyset$, $B \neq \emptyset$, imply that $A \cap B \neq \emptyset$). This is equivalent to saying that ($A \subset M$, A is relatively open, A is relatively closed, imply that $A = M$ or $A = \emptyset$).

Let T be a manifold. It is called a real *analytic manifold* (or has a real analytic structure) if there exists a covering of T by coordinate neighbourhoods such that for any two overlapping neighbourhoods the coordinate transformation, in both directions, is given by n real *analytic functions* (i.e. power series)

[$p \in T$; $\exists\ \phi(p) \in E_n$, $\phi(p) = (f_1(p), f_2(p), \ldots, f_n(p))$. Let $\mathfrak{A}(p) =$ the class of all real-valued functions defined on the neighbourhoods of p, with the property that they are real analytic functions of $f_1, f_2, \ldots, f_n$ in a neighbourhood of p. Each function in $\mathfrak{A}(P)$ is defined in a neighbourhood of p.]

One defines analogously a *complex analytic manifold*, and a *complex analytic structure*. Every complex analytic manifold is automatically real analytic but not conversely.

The union of connected sets is again connected provided that every two of them have points in common. To any $x \in T$ (a topological space) there exists a *maximal, connected subset* M of T which contains x. The maximal connected subsets of T are called *components*. [The closure of a component is connected, therefore the component is closed].

Let G be a topological group, and e the identity element. Let G_0 be the component of e. Then G_0 is a *closed*, *invariant*, *subgroup*. [$gG_0 = G_0g$, $g \in G$. Ref. Montgomery and Zippin].

A topological group G is called a (real or complex) *Lie group* if the component of the identity is *open*, and has an (real or complex) analytic structure, such that the coordinates of $z = xy$ are (real or complex) analytic functions of the coordinates of x and y, and the coordinates of z^{-1} are (real or complex) analytic functions of the coordinates of z.

If G is a compact Lie group, there exists a representation ϕ of G, such that the kernel of ϕ is contained in the kernel of every representation. (Chevalley, Vol. I).

Every locally Euclidean group is *isomorphic* (group isomorphism and homeomorphism of the space) to a Lie group (Hilbert's Fifth Problem, 1900). [Proved by D. Montgomery, L. Zippin and A. Gleason, 1952-53.).

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