Volume 52

Spinors, Twistors, Clifford Algebras and Quantum Deformations

Proceedings of the Second Max Born Symposium held near Wrocław, Poland, September 1992

edited by

Zbigniew Oziewicz
Bernard Jancewicz
and
Andrzej Borowiec
Institute of Theoretical Physics,
University of Wrocław,
Wrocław, Poland

KLUWER ACADEMIC PUBLISHERS
DORDRECHT / BOSTON / LONDON

Library of Congress Cataloging-in-Publication Data

Spinors, twistors, Clifford algebras, and quantum deformations /
edited by Zbigniew Oziewicz, Bernard Jancewicz, and Andrzej
Borowiec.
p. cm. -- (Fundamental theories of physics ; v. 52)
ISBN 0-7923-2251-7 (acid-free paper)
1. Spinor analysis--Congresses. 2. Twistor theory--Congresses.
3. Clifford algebras--Congresses.. 4. Mathematical physics-
-Congresses. 5. Rzewuski, Jan, 1916- . I. Oziewicz, Zbigniew.
II. Jancewicz, Bernard. III. Borowiec, Andrzej, 1950- .
IV. Series.
QC20.7.S65S67 1993
530.1'5257--dc20 93-7413

ISBN 0-7923-2251-7

Published by Kluwer Academic Publishers,
P.O. Box 17, 3300 AA Dordrecht, The Netherlands.

Kluwer Academic Publishers incorporates
the publishing programmes of
D. Reidel, Martinus Nijhoff, Dr W. Junk and MTP Press.

Sold and distributed in the U.S.A. and Canada
by Kluwer Academic Publishers,
101 Philip Drive, Norwell, MA 02061, U.S.A.

In all other countries, sold and distributed
by Kluwer Academic Publishers Group,
P.O. Box 322, 3300 AH Dordrecht, The Netherlands.

Printed on acid-free paper

Printed in the Netherlands

This volume is dedicated to Professor Jan Rzewuski, pioneer and teacher,
on the occasion of his 75-th birthday

TABLE OF CONTENTS

SPINORS

TWISTORS

CLIFFORD ALGEBRAS

QUANTUM DEFORMATIONS

RELATED TOPICS

FOREWORD

ZBIGNIEW OZIEWICZ
University of Wrocław, Poland

December 1992

The First Max Born Symposium in Theoretical and Mathematical Physics, organized by the University of Wrocław, was held in September 1991 with the intent that it would become an annual event. It is the outgrowth of the annual Seminars organized jointly since 1972 with the University of Leipzig. The name of the Symposia was proposed by Professor Jan Łopuszański. Max Born, an outstanding German theoretical physicist, was born in 1883 in Breslau (the German name of Wrocław) and educated here.

The Second Max Born Symposium was held during the four days 24-27 September 1992 in an old Sobótka Castle 30 km west of Wrocław. The Sobótka Castle was built in the eleventh century. The dates engraved on the walls of the Castle are 1024, 1140, and at the last rebuilding, 1885. The castle served as a cloister until the end of the sixteenth century.

The Second Max Born Symposium is dedicated to **Professor Jan Rzewuski**. Professor Rzewuski was born in 1916, earning his doctoral degree in 1948 and his habilitation in 1950 at the University of Warsaw. He was a professor at Copernicus University in Toruń* until 1952 when he was forced by the communist regime to move to the University of Wrocław**. In Wrocław, Professor Rzewuski founded the Institute of Theoretical Physics and became its first director. He was forced to resign from the positions of Dean of Faculty and Director of the Institute after March 1968 when he supported the demands of protesting students. During and after Martial Law in Poland, over the years 1982-1988, Professor Rzewuski supported the fight for independence. ***

In 1958 Rzewuski started to propagate the idea that *the spinor space* is more fundamental than Minkowski space-time and that classical and quantum field theories needed to be formulated in complex spinor space rather than in real Minkowski space. Later, Professor Rzewuski independently discovered the notion of Penrose's twistor and the Penrose transform. In the

* The Stefan Batory University in Vilna, founded in 1570, was relocated in Toruń in 1945.

** Jan Kazimierz University in Lwów was moved in 1945 to Wrocław.

*** When I was in prison in 1982, and again in 1984, Professor Rzewuski bravely worked for my release on his bail. To give bail required an extraordinary amount of courage and a good heart.

1980's, Professor Rzewuski investigated the submanifolds of Hom($\mathbb{C}^n$,$\mathbb{C}^m$) of fixed rank. This approach, presented in Professor Rzewuski's article in the first Chapter of this volume, generalizes the Penrose transform to arbitrary dimension.

The subject of the Second Max Born Symposium *Spinors, Twistors and Clifford Algebras* reflects the domains in which Professor Rzewuski's contributions are notable. The subject has been extended by *Quantum Deformations.*

The lectures in mathematics and theoretical physics attracted 65 participants from 20 countries. During the four days there were 54 lectures with 38 in parallel evening sessions. The lectures at the Symposium were grouped according to subject. In this volume we have again grouped together related contributions.

The Symposium was opened by a welcoming speech and two speeches in honor of Professor Rzewuski. The welcoming speech, not included in this volume, was delivered by the Director of the Institute of Theoretical Physics, Professor Jerzy Lukierski.

The first plenary lecture was given by **Michel Dubois-Violette**. He explored the identification of the Clifford algebra $C\ell_{2\ell}$ of even dimensional euclidean space with the algebra of the fermionic anticommutation relations by means of an isometric complex structure. We learn that neither Cartan's terminology (*simple spinors*) nor Chevalley's (*pure spinors*) are appropriate in this case because these spinors are *Fock states.*

The first Chapter on *SPINORS* contains also papers on spin structures, one by **Andrzej Trautman** who unfortunately was unable to participate because of the tragic death of his son, and the other by **Vladimir Lyakhovsky** who also delivered a second lecture about multiparametric deformations of quantum groups (i.e. Hopf algebras). This second lecture by Lyakhovsky is published elsewhere. A paper related to spin structures is presented also by **Ludwik Dąbrowski**.

Gary Gibbons describes the geometry of the Majorana spinors in terms of the real projective space.

The second chapter, devoted to *TWISTORS*, starts with the plenary lecture by **Dmitri Volkov**, one of the pioneers of graded symmetries. Supersymmetry ($\mathbb{Z}_2$-graded Lie algebras) was introduced by Volkov and Akulov in 1973 and independently by Julius Wess and Bruno Zumino in 1974. I should add that professor Jan Łopuszański "almost" discovered supersymmetry in 1971 and made significant contributions to this domain later on. The organizers were proud to have gathered together at the Sobótka castle the pioneers of supersymmetry, professors **Dmitri Volkov**, **Julius Wess** and **Jan Łopuszański**. Dmitri Volkov showed in his lecture how twistors are related to supersymmetry. Volkov's colaborator, **Aleksandr Zheltukhin** describes superstrings and supermembranes in terms of twistors.

Twistor space is a $U_{2,2}$-module, a four dimensional $\mathbb{C}$-space with a hermitian form and with the automorphism group $U_{2,2}$. The sequence of nested subspaces of the twistor space is called a *flag.* An GL-orbit of a flag is called a *twistor flag space.* Anatol Odzijewicz from Białystok, Poland, has since 1979 been considering twistor flag spaces as phase spaces. **Arkadiusz Jadczyk** in a plenary lecture identified the relativistic conformal phase space with the symmetric homogeneous space of the automorphism group $U_{2,2}$ of twistors. We learn that such a homogeneous space carries *a quantum geometry* with inclusion of Planck's constant, which to me is mysterious. Jadczyk's lecture also contains a discussion of Max Born's scientific works in historical perspective. He showed transparences with citations from Max Born's papers related to Born's reciprocity principle ($q \to -p, p \to q$ invariance). Born's principle is presented by Jadczyk in a new light.

We invited **Professor Albert Crumeyrolle** from Toulouse, France, as the key speaker to the session on *CLIFFORD ALGEBRAS.* With deep sadness we were forced to announce the sudden death of our friend and colleague. Professor Crumeyrolle was one of the major contributors to the theory of Clifford algebras and spinor structures, including ∞-dimensional *symplectic* Clifford algebras and symplectic spinors, invented by him in 1975. Professor Crumeyrolle was born in 1919 and died in June 1992.

Vladimír Souček presented his recent monograph on Clifford analysis written jointly with Delanghe and Sommen.

Professor **Wojciech Królikowski** from Warsaw was unable to participate in person; however, he was kind enough to send us his lecture in which he explains how *a sequence of Clifford algebras* leads to the existence of three and **only** three families of fundamental fermions with a mass spectrum for charged leptons.

David Hestenes and **Garret Sobczyk** are the authors of the monograph, *Clifford Algebra to Geometric Calculus*, published by Kluwer in 1984* soon after Garret Sobczyk was expelled from Poland**. This monograph is a polemic assault against the Cartan's calculus of differential forms.

David Hestenes in a plenary lecture *Reconcilling Clifford and Grassmann*, stressed again that "the modern calculus of differential forms is a step backward" in comparison with *geometric calculus*, i.e. "calculus with the structure of Clifford algebra". The differential forms were known since Pfaff (1815). The calculus of differential forms was completly developed by Ellié Cartan around 1900 and was based on Grassmann's exterior algebra invented in 1844. Hestenes's and Sobczyk's *geometric calculus* is more

* Second printing in 1987.

** Numerous members of our faculty were arrested during Martial Law in Poland. Among others, Garret Sobczyk was forced to live in the underground for two months, was arrested by the SB (Polish KGB) in February 1983 and then brutally expelled from Poland to Helsinki.

known as the (Pseudo)Riemannian Differential Geometry. This differential geometry include the theory of the Dirac operator and utilize completely the Cartan's calculus for riemannian structures. The riemannian differential geometry was developed among other by Dirac, Hodge, Kodaira and de Rham. Hestenes's polemic lecture of 36 pages is published in the Journal of Mathematical Physics (1993) and is not included in the present volume. We include another paper by Hestenes on Hamiltonian mechanics in terms of *orthogonal* Clifford algebra. Hestenes' approach is presented more extensively in his monograph *New Foundations for Classical Mechanics* (fourth printing 1992). Again this approach is controversial because it requires that the *phase space* manifold be endowed with a riemannian structure, whereas no natural riemannian structure seems to exist. Any choice of scalar product*** gives a rather obscure identification of vector fields with differential forms.

Spinor and twistor spaces are minimal one sided *ideals* in Clifford algebras. This approach has a long history. Invented by Elié Cartan (pure spinors), presented by Marcel Riesz at the Mathematical Congress in Stockholm in 1946, reinvented and used by Claude Chevalley in his book in 1954, used by Atiyah, Bott and Shapiro in 1964, and by Penrose in 1967 who utilized Witt's decomposition. Since 1974, Albert Crumeyrolle made these ideas more popular by utilizing Witt's splitting for descriptions of spinors and twistors*.

David Hestenes in 1975 associated *the name* "spinor" with Clifford *subalgebra* of even multivectors. Even subalgebra is *not* an ideal. The trouble is that spinor and twistor spaces are *not* just linear spaces; they are linear spaces *with* structure tensors, namely, bilinear or hermitian forms. SU_2-spinors are members of the two-dimensional *oriented* $\mathbb{C}$-space with a *hermitian* form and with the automorhism group SU_2. Whereas the basis independent identification of spinors as ideals is compatible with these structure forms (inheriting these structures from Clifford algebra), Hestenes' *frame-dependent* "identification" of spinors with even subalgebras seems to be no more than the identification of linear spaces with the same dimension. **Josep Parra** recognized the difference between Dirac spinors and Hestenes's missleading "spinors". Unfortunately most of audience was not able to follows his reasoning**.

*** Hestenes' choice is that the tangent space splits into the direct sum of the two *orthogonal* spaces of equal dimensions in such a way that the directions of the position and momenta are orthogonal.

* Twistors as a minimal ideal in the complexified Dirac-Clifford algebra (or alternatively as a minimal ideal in the *real* anti- De Sitter-Clifford algebra) was explored by Professor Jan Rzewuski in a joint paper with Abłamowicz and myself in 1982.

** The young Cambridge group, Anthony Lasenby, Chris Doran and Steve Gull, presented a paper about how to generalize Hestenes' *basis-dependent* linear isomorphisms from, what they call *2-spinors* ($SL_2(\mathbb{C})$-spinors, which are members of a *symplectic* space), and from twistors to the subalgebra of the even multivectors of *real* Clifford algebra of Minkowski's space-time. These linear isomorphisms are not convincing because they are frame-dependent and need "specific Clifford elements allowed on the right and not allowed on the left".

The next chapter is about one of the central topics of this Symposium: *QUANTUM DEFORMATIONS.* In his plenary lecture **Julius Wess** gave an introduction to quantum groups also known as noncommutative Hopf algebras. He presented a new geometric framework based on the algebra generated by noncommutative spacetime "coordinates". This leads to a discrete spacetime described by eigenvalue equations of operator-valued spacetime "coordinates". **Jerzy Lukierski** in his plenary lecture presented a nonlinear quantum deformation of the Poincaré algebra and pointed out that such deformations lead to the field equations with finite difference time derivatives.

Ursula Carow-Watamura explained the construction of the quantum Lorentz group, the quantum Minkowski space and the q-deformed Dirac γ-matrices.

Differential calculus for noncommutative Hopf algebras has been elaborated by Woronowicz since 1979. A calculus for associative rings, which does not have the Hopf algebra structure, has been considered by Alain Connes, by Michel Dubois-Violette (since 1988), by Julius Wess and Bruno Zumino in 1990 and by many others. **John Madore** applies this noncommutative calculus to electrodynamics and **Satoshi Watamura** applies the bicovariant differential calculus in quantum deformations of gauge theories.

Braided Lie algebras were presented by **Dmitri Gurevich** who invented this generalization.

Shahn Majid delivered two lectures, one of which is included in these Proceedings. Majid considers Hopf algebras with a braided structure on the tensor product. In this way he obtain a generalization of supersymmetry (supergroups and superalgebras).

The last chapter contains several important and interesting lectures which do not fit into any of the previous chapters. One of the most interesting plenary lectures was delivered by **Richard Kerner** on $\mathbb{Z}_3$-graded algebras.

Leopold Halpern was for four years, 1956-1959, an assistant of Erwin Schrödinger at the University of Vienna and for eleven years, 1974-1984, an assistant of Paul Dirac at Florida State University. Halpern claims that every great physical theory contains an equally great *absurdity* "that no reasonable person can believe in it". Halpern has been proposing a way to avoid the absurdity in Einstein's theory of gravity by introducing a Lagrangian nonlinear in curvature. Halpern has proposed also a Kałuża-Klein gauge theory of gravity based on the anti- De Sitter universe $SO(3,2)/SO(3,1)$. In his lecture Halpern considered spin in Einstein's theory of gravitation and explained his philosophy that an absurdity is unavoidable (and *not* obvious) in any "good" physical theory.

Multisymplectic geometry in classical field theory was initiated by De-

decker in 1953 and was developed in Warsaw by Włodzimierz Tulczyjew around 1968, and since 1972 by Jerzy Kijowski, Krzysztof Gawędzki and Wiktor Szczyrba. In this geometrical approach, the presymplectic differential form of classical mechanics is replaced by a symplectic differential form of degree (2+m) for an m-dimensional classical membrane theory, so that the zero-dimensional membrane corresponds to mechanics. A differential form of degree (2+m) is called *symplectic* if

- it is closed, which assures the existence of a variational formulation, and in particular the existence of the local action,
- it is regular, assuring that the maximal integral submanifolds of the ideal generated by the appropriate (1+m)-forms are exactly (1+m)-dimensional.

In the last chapter, **Dan Radu Grigore** gives an overview of the symplectic formulation of the *Lagrangian* formalism, following only his own papers and those by Krupka and Betounes. The paper deals with the Lagrange-Souriau differential form which does not seem to be regular in the above sense. Instead, it vanishes on the bivector fields spanned by the integrable vertical distribution and satisfies another condition. These requirements are needed for the existence of a local Lagrangian density and are just a fixing of the local symplectic potential.

I have limited my remarks mostly to contributions which provoked the lively discussions during the Symposium. The only complaints of participants were about the overcrowded programme.

Organizers and Editors:

The Symposium was organized by Zbigniew Oziewicz (Chairman), Andrzej Borowiec and Bernard Jancewicz with the great help of Professor Jerzy Lukierski.

Acknowledgements

The Editors would like to thank Mrs. Anna Jadczyk and to Dr. Krzysztof Rapcewicz from University of Wrocław for all their help and assistance during the preparation of these Proceedings.

The English of the *Foreword* has been corrected by Dr. Garret Sobczyk.

The Editors are grateful to Ms Margaret Deignan and Ms Anneke Pot from Kluwer Academic Publishers for all of her kind help in the publishing of these Proceedings.

Professor Jan Rzewuski and his wife Alicja at front of their house in Wrocław (1989).

Homage to Professor Jan Rzewuski

by Professor Jan Łopuszański

Most honorable Professor Rzewuski,

Dear Jaś,

It is a pleasant opportunity to celebrate your jubilee in this beautiful scenery: a gathering of the physical community from all over the world in an old castle amid the wooded hills of Sobótka. Outside - a nice Indian Summer. Our Indian Summer, dear Jaś, yours and mine, the Indian Summer of our lives is also quite nice.

Poland has became finally a free country again. The economy, although still ailing, is slowly improving. We both are healthy and able to follow actively the exciting developments in contemporary physics. We have loving partners in life whom we love too. So we may look with confidence to the future.

As mentioned already your scientific activities are still very vital. This is testified by the main issue of this conference: spinors and twistors, as well as by your book "Introduction to Quantum Theory", published recently. The exposition of the subject is clear, straightforward and elegant and the approach is modern.

The theory of spinorial spaces was the main topic of your numerous publications for many, many years. I do not go wrong claiming that you were the founder of this direction in physics. You were the first to emphasize that the spinor space is more fundamental in physics than that of space–time concept.

Dear Jaś, you did also a pioneering work in the theory of non–local field theory, classical and quantum, as well as in the functional approach to quantum field theory, in particular to the theory of the scattering operator, the so called S–Matrix.

Your work was numerous and well received by experts. Your papers were frequently quoted. Your work found also followers among the younger generation of physicists who have been developing your ideas.

This year is also the 40–th anniversary of your arrival in Wrocław. You were the founder of our Institute and your merit is that this Institute was and is still thriving.

It is also the 25-th anniversary of your becoming a member of the Polish Academy of Sciences.

Passing to problems linked closer with daily life, I would like to sketch a picture of this deep thinking scientist as a man. I shall rely on own experience gained during many years of collaboration. I would like to stress the extraordinary moral uprightness and personal charm of Professor Rzewuski. He is a type of man qualified coloquially as a manly type; tall, strongly built and deft, excellent skier and swimmer; his behaviour and conduct is characterized by self–control, an restraint and quickness of decision. In relations with other people he is straightforward, sensitive and of high personal culture, showing a deep wisdom concerning human nature and life. He likes music and is a connoisseur in this field of arts. As a superior he rarely makes use of the prerogatives of power. He is a good organizer who accurately distinguishes among important and insignificient issues. In his work and in every activity he is exact, persistent and careful. He is courageous and firm, if needed. This was testified during the war when he fought as a voluntary soldier in the Warsaw uprising as well as in 1968 when he dared to oppose the totalitarian communist regime as Dean of our Faculty.

Jaś, have a nice time in Sobótka!

Homage to Professor Jan Rzewuski

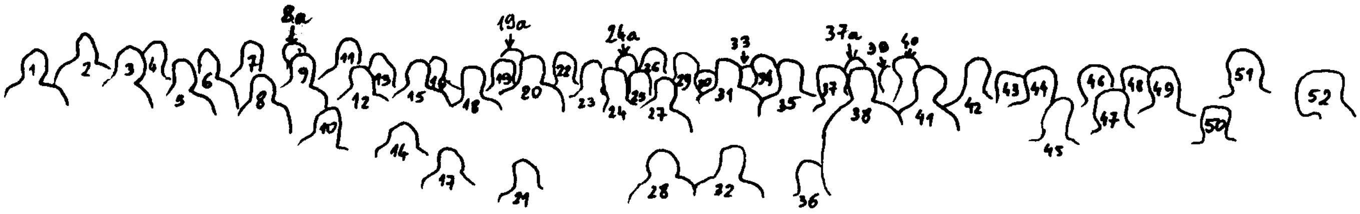
by Professor Jan Mozrzymas

Dear Guests and Participants,

I have known Professor Jan Rzewuski since 1957 which means from the third year of my studies at the University of Wroclaw. I have been, from this time, under the impression of his personality; he has always impressed me as a physicist, a scholarly teacher and a human being. But now, after thirty five years of our mutual acquaintance, I would like to say that the most inspirational and meaningful for me was his behaviour during March 1968. It was the period of the anti-Jewish campaign unleashed by the ruling powers. Professor Rzewuski was at that time the dean of our faculty and, as it turned out, our faculty was the only one that expressed official opposition to this campaign.

In the years 1981-1984, when I was the dean of the faculty and in the years 1984-1987 during which I was the rector of the University of Wroclaw, we lived through Martial Law which was as distressing as March of 1968. Throughout these six years of work first as dean and later as rector, and, especially, in the most difficult situations, I tried to follow the example that Professor Rzewuski provided so many years before. Today on this most solemn occasion, I would like to take the opportunity to thank Professor Rzewuski for all that he has done for me and, in particular, for this example.

Jan Mozrzymas



PARTICIPANTS on the PHOTO

1. Jerzy Cisło, 2. Anthony Lasenby, 3. Helmut Urbantke, 4. Steve Gull, 5. Chris Doran, 6. David Hestenes, 7. Jerzy Hańćkowiak, 8. William Baylis, 8a. Jerzy Różański, 9. Cezary Juszczak. 10. Władysław Marcinek, 11, Przemysław Siemion, 12. Josep Manel Parra, 13. Leopold Halpern, 14. Shahn Majid, 15. Aleksey Isaev, 16. Ireneusz Tobijaszewski, 17. Bernard Jancewicz, 18. Ziemowit Popowicz, 19. Marek Mozrzymas, 19a. Jan Milewski, 20. Mijat Mijatović, 21. Andrzej Borowiec, 22. Jan Sobczyk, 23. Aleksandr Zheltukhin, 24. Zbigniew Oziewicz, 24a. Viktor Abramov, 25. Wojciech Kopczyński, 26. K. Paul Tod, 27. Gary Gibbons, 28. William Pezzaglia, 29. Pertti Lounesto, 30. John Madore, 31. Garret Sobczyk, 32. Jaime Keller, 33. Ralf Grunewald or Uwe Semmelmann, 34. Jerzy Lukierski, 35. Julius Wess, 36. Jakub Rembieliński, 37. Jan Łopuszański, 38. Dan Radu Grigore, 38a. Michel Dubois-Violette, 39. Krzysztof Rapcewicz, 40. Richard Kerner, 41. Dmitrij Volkov, 42. Vladimir Souček, 43. Sorin Marculescu, 44. Valeriy Tolstoy, 45. Ursula Carow-Watamura, 46. Michael Schlieker, 47. Satoshi Watamura, 48. Sergey Merkulov, 49. Vladimir Lyakhovski, 50. Mrs. Susan Dixon, 51. Geoffrey Dixon, 52. Dmitri Gurevich.

SPINORS

STRUCTURE OF MATRIX MANIFOLDS AND A PARTICLE MODEL*)

JAN RZEWUSKI
Institute of Theoretical Physics,
University of Wroclaw, pl. Maxa Borna 9, 50-204 Wroclaw,
Poland

Abstract. The decomposition of matrix manifolds into homogeneous spaces of direct products of certain groups is described. The results are applied to derivation of the internal structure of $SU(2,2) \times SU(m)$ and $P_4 \times SU(m)$ invariant particle models.

1. Introduction

The mathematical description of physical laws is based on the observed symmetries and the underlying geometry. An example is the Poincaré symmetry $P_4 = T_4 \rtimes SO(3,1)$ and the underlying space–time M_4 which is one of the homogeneous spaces of P_4 namely

$$M_4 \cong P_4/SO((3,1). \tag{1.1}$$

This fact inspired some physicists (cf. e.g. [1], [2]) to investigate also other homogeneous spaces of the Poincaré group

$$P_4/H_i \cong \frac{P_4}{SO(3,1)} \cdot \frac{SO(3,1)}{H_i}, \qquad \begin{array}{l} H_i \subset SO(3,1) \\ i = 1,2,\ldots \end{array} \tag{1.2}$$

with respect to their applicability in physics. E.g. the local coordinates on $SO(3,1)/H_i$ can be considered as internal degrees of freedom of a relativistic particle.

In this paper we wish to combine an old idea [3] of describing the particle structure in complex space rather than in Minkowski space with the investigation of homogeneous spaces of the whole physical symmetry group consisting of external as well as internal symmetries [4]. We shall assume, in accordance with experiment, that the physical symmetry is the direct product $SU(2,2) \times SU(m)$ or $P_4 \times SU(m) \subset SU(2,2) \times SU(m)$ of external conformal or Poincaré and internal $SU(m)$ symmetry[1] where P_4 is now to be considered as a subgroup of $SU(2,2)$.

* Supported by Polish Committee of Science, Grant No 2-2419-92-03
[1] We keep m arbitrary to cover such possibilities as $SU(3)$, $SU(3) \times SU(2) \times U(1)$ etc.

The natural representation space for a direct product $GL(n,\mathbb{C}) \times GL(m,\mathbb{C})$ is a complex matrix manifold $\mathbb{C}^{nm}$. In the case of $SU(2,2)\times SU(m)$ it will be $\mathbb{C}^{4m}$. In this space both internal and external symmetries have a common geometrical basis (in contradistinction to space–time where only external symmetries are geometrized).

We shall consider homogeneous manifolds of $SU(2,2) \times SU(m)$ and $P_4 \times SU(m)$ in $\mathbb{C}^{4m}$ and show that there exists one and only one such manifold which admits a unique and consistent projection on the compactified complex Minkowski space. In the case of the smaller $P_4 \times SU(m)$ symmetry we arrive at the homogeneous manifold

$$\frac{P_4 \times SU(m)}{SO(2) \times SU(m-2)} \cong \frac{P_4}{SO(3,1)} \times \frac{SO(3,1) \times SU(m)}{SO(2) \times SU(m-2)}. \tag{1.3}$$

It is seem that the particle structure in this model is described by the m–independent 5–dimensional real manifold $SO(3,1)/SO(2)$ and the manifold $SU(m)/SU(m-2)$ depending on the kind of internal symmetry.

The structure of homogeneous submanifolds of $\mathbb{C}^{nm}$ can be investigated, up to a certain stage, without additional difficulties in the case of arbitrary n and m (Section 2). It provides the theory of n complex m–vectors (or m n–vectors) subject to certain invariant conditions and generalizes in a certain sense the theory of spinors, bispinors, twistors etc. to arbitrary dimensions. In the case when the symmetry is the direct product of more than two groups one has to generalize to matrix manifolds of matrices with more than two indices [5]. One can also consider supermatrices being representation spaces of direct products of supergroups and their decomposition into homogeneous structures [6].

In the case of sets of vector fields the general theory provides a classification of all possible invariant constraints.

In Section 3 we derive the internal structure in the $SU(2,2)\times SU(m)$ and $P_4\times SU(m)$ invariant particle models which follows uniquely from the above mentioned assumptions. This structure is described in terms of homogeneous spaces (cf. e.g. (1.3)) and it remains to describe invariant dynamics and invariant differential operators in these spaces. This task will be the subject of a separate publication.

The present brief report contains only the main features of the theory. Proofs and more details will be published elsewhere (cf. however also [4] and [7]).

2. Matrix Manifolds

Let us consider the set $\mathbb{C}^{nm}$ of all complex $n \times m$ matrices. The elements of this set may be viewed as n complex m–vectors (or m complex n–vectors)

or as homomorphisms $\mathrm{Hom}(\mathbb{C}^m \to \mathbb{C}^n)$ of the vector space $\mathbb{C}^m$ into $\mathbb{C}^n$ (or vice–versa).

The set $\mathbb{C}^{mn} \cong \mathrm{Hom}(\mathbb{C}^m \to \mathbb{C}^n)$ decomposes in a natural way into submanifolds $\mathcal{O}_k^{(n,m)}$ of matrices of equal rank

$$\mathcal{O}_k^{(nm)} := \{M \in \mathbb{C}^{nm} : rank M = k\} \ , \tag{2.1}$$

$$M = \{m_{a\alpha}\} \begin{array}{l} a = 1,\ldots,n \\ \alpha = 1,\ldots,m \end{array} \in \mathbb{C}^{nm} \cong \mathrm{Hom}\,(\mathbb{C}^m \to \mathbb{C}^n) \ ,$$

$$\mathbb{C}^{nm} = \bigcup_{k=0}^{min(n,m)} \mathcal{O}_k^{(n,m)}, \qquad \mathcal{O}_k^{(n,m)} \cap \mathcal{O}_l^{(n,m)} = \delta_{kl}\mathcal{O}_k^{(n,m)} \ . \tag{2.2}$$

A matrix of rank k is characterized by the fact that all determinants

$$m\begin{pmatrix} \alpha_1,\ldots,\alpha_l \\ a_1,\ldots,a_l \end{pmatrix} := det \begin{pmatrix} m_{a_1\alpha_1} & \cdots & m_{a_1\alpha_l} \\ \vdots & & \\ m_{a_l\alpha_1} & \cdots & m_{a_l\alpha_l} \end{pmatrix} \tag{2.3}$$

of order higher than k vanish and that there exists at least one subdeterminant of order k different from zero

$$m\begin{pmatrix} \alpha_1,\ldots,\alpha_k \\ a_1,\ldots,a_k \end{pmatrix} \neq 0 \ . \tag{2.4}$$

Equation (2.4) determines a coordinate neighbourhood for the manifold $\mathcal{O}_k^{(n,m)}$. There are $\binom{n}{k}\binom{m}{k}$ such neighbourhoods according to the $\binom{n}{k}$ possibilities to choose k rows out of n rows out and to the $\binom{m}{k}$ possibilities to choose k columns out of m columns.

Let us choose on $\mathcal{O}_k^{(n,m)}$ a neighborhood corresponding to the square submatrix

$$K = \begin{pmatrix} m_{a_1\alpha_1} & \cdots & m_{a_1\alpha_m} \\ \vdots & & \\ m_{a_k\alpha_1} & \cdots & m_{a_k\alpha_m} \end{pmatrix} , \qquad det\ K \neq 0 \tag{2.5}$$

and denote the complementary matrices by A, B, Y.

$$A = \begin{pmatrix} m_{a_{k+1}\alpha_1} & \dots & m_{a_{k+1}\alpha_k} \\ \vdots & & \\ m_{a_n\alpha_1} & \dots & m_{a_n\alpha_k} \end{pmatrix}, \qquad B = \begin{pmatrix} m_{a_1\alpha_{k+1}} & \dots & m_{a_n\alpha_m} \\ \vdots & & \\ m_{a_n\alpha_{k+1}} & \dots & m_{a_k\alpha_m} \end{pmatrix},$$

$$Y = \begin{pmatrix} m_{a_{k+1}\alpha_{k+1}} & \dots & m_{a_{k+1}\alpha_m} \\ \vdots & & \\ m_{a_n\alpha_{k+1}} & \dots & m_{a_n\alpha_m} \end{pmatrix}^{*)}. \tag{2.6}$$

In the neighborhood

$$m\begin{pmatrix} 1,\dots,k \\ 1,\dots,k \end{pmatrix} \neq 0 \tag{2.7}$$

the picture is particularly simple

$$M = \left(\begin{array}{c|c} K & B \\ \hline A & Y \end{array}\right). \tag{2.8}$$

According to well known relations from linear algebra, we have

$$\begin{aligned} A = aK \ &, \ Y = aB \\ B = Kb \ &, \ Y = Ab \end{aligned} \tag{2.9}$$

where

$$a = \left\{a^{a'}_{a''}\right\} \quad \begin{aligned} a' &= a_1,\dots,a_n \\ a'' &= a_{k+1},\dots,a_n \end{aligned} \qquad \text{and} \qquad b = \left\{b^{\alpha'}_{\alpha''}\right\} \quad \begin{aligned} \alpha' &= \alpha_1,\dots,\alpha_k \\ \alpha'' &= \alpha_{k+1},\dots,\alpha_m \end{aligned} \tag{2.10}$$

are $k \times (n-k)$ and $k \times (m-k)$ matrices resp.

Due to invertibility of K ($det\ K \neq 0$) we obtain from (2.9)

$$Y = aKb = AK^{-1}B \tag{2.11}$$

providing two natural coordinate systems on $\mathcal{O}_k^{(n,m)}$ corresponding to the neighbourhood $det\ K \neq 0$.

---

* The index sets are ordered in the sense that $a_1 < a_2 < \dots < a_i$; $a_{k+1} < a_{k+2} < \dots < a_n$ and similarly for the α's.

The coordinates a and b play a particularly important role because of their invariance properties. The coordinates a are the same for all columns and the coordinates b are the same for all rows. The first (second) are invariant with respect to arbitrary transformations of columns (rows).

It is seen from (2.11) that

$$dim_{\mathbb{C}}\mathcal{O}_k^{(n,m)} = k(n+m-k). \tag{2.12}$$

The space with lowest dimension, for $k = 0$, is the point $m_{a\alpha} = 0, a = 1,\dots,n,\ k = 1,\dots,m$. The next complex dimension is already $n+m-1$. Each space $\mathcal{O}_l^{(n,m)}$ with $l < k$ lies in the boundary of $\mathcal{O}_k^{(n,m)}$ in the sense

$$\mathcal{O}_l^{(n,m)} \subset \bar{\mathcal{O}}_k^{(n,m)}, \qquad l < k \tag{2.13}$$

where the "bar" denotes closure in the topology induced on $\mathcal{O}_k^{(n,m)}$ from the natural topology in $\mathbb{C}^{(n,m)}$.

The manifolds $\mathcal{O}_l^{(n,m)}$ all have elements arbitrary close to $0 := \mathcal{O}_0^{(n,m)}$ and form a flag of manifolds [7] in the sense that $\bar{\mathcal{O}}_k^{(n,m)} \subset \bar{\mathcal{O}}_{k+1}^{(n,m)}$. All closed orbits meet at the point $O = \mathcal{O}_0^{(n,m)}$ and their tangent spaces at this point form a flag of spaces in the usual sense.

The homomorphism $M \in \mathcal{O}_k^{(n,m)}$ admits a canonical decomposition

$$\mathbb{C}^m \xrightarrow{\pi} \mathbb{C}^m/\text{Ker } M \xrightarrow{\iota} Im\ M \xrightarrow{\epsilon} \mathbb{C}^m. \tag{2.14}$$

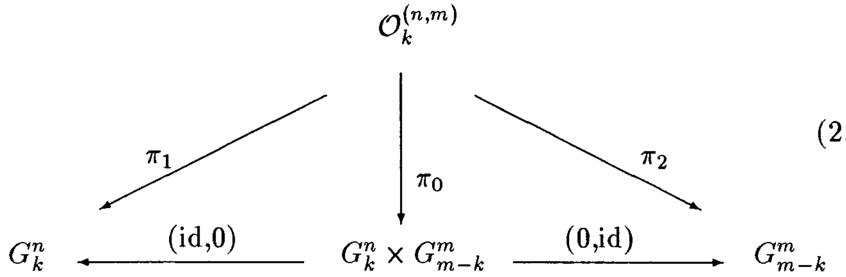
the kernel $Ker\ M$ (the image $Im\ M$) consisting of an $(m-k)$–dimensional (n–dimensional) hyperplane in $\mathbb{C}^m$ ($\mathbb{C}^n$). To this canonical decomposition there corresponds a fibering $\left(\mathcal{O}_k^{(n,m)}, G^m_{m-k} \times G^n_k, \pi_0\right)$ of $\mathcal{O}_k^{(n,m)}$ where

$$\begin{aligned} \pi_0\ :\ \mathcal{O}_k^{(n,m)} &\longrightarrow G^m_{m-k} \times G^n_k \\ M &\longrightarrow Ker\ M \times Im\ M \end{aligned} \tag{2.15}$$

is a projection on the base $G^m_{m-k} \times G^n_k$ and the fibre is homeomorphic with $GL(k,\mathbb{C})$. The G^m_{m-k} and G^n_k are Grassmann manifolds parametrized by the coordinates $a^{a'}_{a''}$ and $b^{\alpha'}_{\alpha''}$ resp. and consisting of all $(m-k)$–dimensional planes in $\mathbb{C}^m$ and all k–dimensional planes in $\mathbb{C}^n$. Two other fiberings are possible

$$\begin{aligned} \pi_1\ :\ \mathcal{O}_k^{(n,m)} &\longrightarrow G^n_k \\ M &\longrightarrow Im\ M, \\ \pi_2\ :\ \mathcal{O}_k^{(n,m)} &\longrightarrow G^m_{m-k} \\ M &\longrightarrow Ker\ M \end{aligned} \tag{2.16}$$

as illustrated on the following graph:

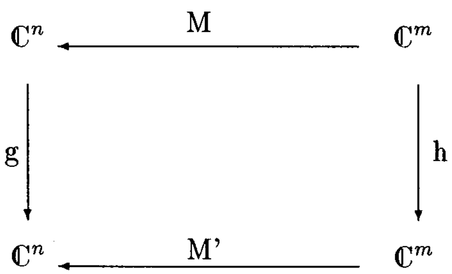


(2.17)

The group theoretical structure of $\mathcal{O}_k^{(n,m)}$ follows from the observation that $\mathbb{C}^{nm}$ is the representation space of the group $GL(n,\mathbb{C}) \times GL(m,\mathbb{C})$:

$$M \to M' = gMh^{-1}, \qquad g \in GL(n,\mathbb{C}), \qquad h \in GL(m,\mathbb{C})$$

according to the commuting diagram



(2.18)

The manifolds $\mathcal{O}_k^{(n,m)}$ are orbits of $GL(n,\mathbb{C}) \times GL(m,\mathbb{C})$ [8]

$$\mathcal{O}_k^{(n,m)} = \frac{GL(n,\mathbb{C}) \times GL(m,\mathbb{C})}{H_k^{(n,m)}}. \tag{2.19}$$

For the point $M_0 = \left(\begin{array}{c|c} \mathbb{1}_k & 0 \\ \hline 0 & 0 \end{array}\right) \in \mathcal{O}_k^{(n,m)}$ the isotropy group is easily calculated to be

$$H_k^{(n,m)} = \left(\begin{array}{c|c} g_1 & g_2 \\ \hline 0 & g_3 \end{array}\right) \times \left(\begin{array}{c|c} g_1^{-1} & 0 \\ \hline h_2 & h_3 \end{array}\right) \tag{2.20}$$

where $g_1 \in GL(k,\mathbb{C})$, $g_2 \in \mathbb{C}^{k(n-k)}$, $g_3 \in GL(n-k,\mathbb{C})$, $h_2 \in \mathbb{C}^{(m-k)k}$, $h_3 \in GL(m-k,\mathbb{C})$.

Another group theoretical description of $\mathcal{O}_k^{(n,m)}$ is obtained if we represent the complex Grassmann manifolds in the base as homogeneous spaces

$$G_k^n \cong \frac{U(n)}{U(k) \times U(n-k)} \cong \frac{GL(n,\mathbb{C})}{H_k^n} \tag{2.21}$$

where

$$H_k^n = \left(\begin{array}{c|c} g_1 & g_2 \\ \hline 0 & g_3 \end{array}\right) \tag{2.22}$$

and $g_1 \in GL(k,\mathbb{C})$, $g_k \in \mathbb{C}^{k(n-k)}$, $g_k \in GL(n-k,\mathbb{C})$.

It is important to note that the Grassmann manifolds G_k^n and G_{m-k}^m of the base in the fibre bundle $\mathcal{O}_k^{(n,m)}$ are invariant with respect to $\mathbb{1}_n \times GL(m,\mathbb{C})$ and $GL(n,\mathbb{C}) \times \mathbb{1}_m$ resp. This follows immediately from the remark after formula (2.11) stating that the coefficients a (b) of the linear combinations $A = aK$ ($B = Kb$) do not depend on the columns (rows) of the matrix M.

Up to now we have considered the general symmetry $GL(n,\mathbb{C}) \times GL(m,\mathbb{C})$. In physical applications we mostly have to do with symmetries restricted by the existence of metric. We shall consider here only the case when the general symmetry is reduced to the direct product $SU(n-p,p) \times SU(m-q,q)$ defined by the invariant hermitean metric tensors F_1 and F_2. In this case there appear real invariants

$$I_n = tr\ r^n, \qquad r = F_2 M^* F_1 M, \tag{2.23}$$

and the manifolds $\mathcal{O}_k^{(n,m)}$ decompose into $SU(n-p) \times SU(m-q)$–invariant submanifolds $\mathcal{O}_k^{(n,m)}$ determined by the invariant equations

$$I_n = \kappa_n\ . \tag{2.24}$$

It can be shown that only the first k invariants are independent (cf. e.g. [4]) so that we have to do with a k–parametric family ($\kappa = \{\kappa_1, \kappa_2, \dots, \kappa_k\}$) of homogeneous spaces of the group $SU(n-p,p) \times SU(m-q,q)$. Analytically these manifolds are obtained by introducing (2.11) into (2.24).

To simplify the notation we extend the matrices $a_{a''}^{b'}$ and $b_{\alpha''}^{\beta'}$ (cf. (2.10)) by the unit matrices $a_{a'}^{b'} = \delta_{a'}^{b'}$, $b_{\alpha'}^{\beta'} = \delta_{\alpha'}^{\beta'}$, so that relation (2.11) can be extended to

$$M = a\ K\ b. \tag{2.25}$$

Introducing now (2.25) into (2.24) we obtain

$$I_n = tr\,(F_2 b^* K^* a^* F_1 a K b)^n = tr\,(f_2 K^* f_1 K)^n \tag{2.26}$$

where

$$f_1 = a^* F_1 a, \qquad f_2 = b F_2 b^* \tag{2.27}$$

are the metrics induced from the metrics F_1 and F_2 on the columns and rows of the $k \times k$ matrix K.

The induced metrics f_1 and f_2 are functions of a and b resp. Their signature is determined by the roots of the secular eqs.

$$det\ (f_i - \lambda \mathbb{1}) = 0 \qquad i = 1, 2\ . \tag{2.28}$$

Not all eigenvalues can appear in the induced metrics. The number of positive (negative) roots can not exceed the number of positive (negative) signs in the original metric. If the original signature in $\mathbb{C}^n$ is $(n-p, p)$ then the admissible signatures on k–dimensional planes in $\mathbb{C}^n$ are $(k - l(k), l(k))$ with the obvious relations

$$k - l(k) \leq n - p, \quad l(k) \leq p, \quad l(k) \leq k, \quad l(k) \geq 0$$

or, jointly,

$$l_{min}(k) = max\{0, k + p - n\} \leq l(k) \leq min\{p, k\} = l_{max}(k). \tag{2.29}$$

In this way the Grassmann manifolds G^n_k and G^m_{m-k} are decomposed into domains corresponding to different induced metrics divided by borders determined by zeros of various multiplicity of the secular equation (2.28). The zeros correspond to degenerate metrics, the number of zeros in the metric equaling the multiplicity of the 0 root. This structure of G^m_{m-k} and G^n_k can be lifted by the inverse of one of the projections π_0, π_1, π_2 to $\mathcal{O}^{(n,m)}_{k,\kappa}$ (cf. (2.17)). Details can be found in [7].

3. The Model

To construct a particle model one has to derive the structure of the space of internal parameters. The derivation is based on two plausible assumptions:

1) The physical symmetry group is represented by the direct product $SU(2,2) \times SU(m)$ or its subgroup $P_4 \times SU(m)$. $SU(2,2)$, the covering group of the conformal group, or its Poincaré subgroup P_4, are supposed to describe the external, $SU(m)$ the internal symmetries in accordance with experimental evidence. External symmetries are represented by $SU(2,2)$ or one of its subgroups in order to have a common geometrical basis ($\mathbb{C}^{4m}$) for both external and internal symmetries. It is not necessary, so far, to specify m. One can think e.g. of $SU(3)$ or $SU(3) \times SU(2) \times U(1) < SU(6)$.

2) The external and internal parameters of the particle are represented by local coordinates of an invariant homogeneous submanifold of the linear representation space $\mathbb{C}^{4m}$ of $SU(2,2) \times SU(m)$. This manifold has to satisfy

the following correspondence principle: It must admit a projection on the Minkowski space–time which is unique and consistent with the symmetry. It is easy to show that there exists one and only one such submanifold of $\mathbb{C}^{4m}$.

To find the manifold satisfying the above conditions we use decomposition (2.2)

$$\mathbb{C}^{4m} = \mathcal{O}^{(4,m)}_0 \cup \mathcal{O}^{(4,m)}_1 \cup \mathcal{O}^{(4,m)}_2 \cup \mathcal{O}^{(4,m)}_3 \cup \mathcal{O}^{(4,m)}_4 \tag{3.1}$$

and the fiberings (2.17)[2]. It is seen immediately that the only submanifold containing G^4_2 is $\mathcal{O}^{(4,m)}_2$ with the local trivialization $G^4_2 \times GL(2,\mathbb{C}) \times G^m_2$. It is well known that G^4_2 is isomorphic with the compactified complex Minkowski space $M^{\mathbb{C}}$, the isomorphism being given by the well known relations

$$\begin{aligned} z_\mu &= \frac{i\lambda}{2}\, Tr\ \sigma_\mu\ a \\ a &= \frac{1}{i\lambda}\, z_\mu\, \tilde{\sigma}^\mu \qquad (\tilde{\sigma}_i = \sigma_i, \tilde{\sigma}_0 = -\sigma_0 = -\mathbb{1}_2) \end{aligned} \tag{3.2}$$

where a is a 2×2 complex matrix its entries being Grassmann coordinates of the two–dimensional hyperplane in $\mathbb{C}^4$. The dimensional parameter λ with dimension of length has to be introduced in a relation connecting the complex vector $z_\mu = x_\mu + i y_\mu$ with the dimensionless ratios $a = AK^{-1}$. According to the remark after formula (2.11) the coordinates $a^{a'}_{a''}$ do not depend on the selection of columns in k which proves uniqueness of the projection π_1. To prove consistency with the group we have to derive the transformation properties of the coordinates z_μ induced by $SU(2,2)$ transformations of the rows in M by the intermediary of the Grassmann coordinates $a = AK^{-1}$.

If d, p_μ, k_μ and $m_{\mu\nu}$ are the generators of dilatations, translations, special conformed transformations and rotations in $\mathbb{C}^4$, then the induced infinitesimal transformations of the z_μ are

$$\begin{aligned} d z_\mu &= -i z_\mu \\ p_\mu z_\lambda &= -i g_{\mu\lambda} \\ k_\mu z_\lambda &= i g_{\mu\lambda} z_\nu z^\nu - 2 i z_\mu z_\lambda \\ m_{\mu\nu} z_\lambda &= -i g_{\mu\lambda} z_\nu + i g_{\nu\lambda} z_\mu \end{aligned} \tag{3.3}$$

The proof of (3.3) can be found in [4] and in the complete version of this report.

[2] The case $m = 2$ corresponds to the Penrose model [9]. In this case $\mathbb{C}^{(4,2)} = \mathcal{O}^{(4,2)}_0 \cup \mathcal{O}^{(4,2)}_1 \cup \mathcal{O}^{(4,2)}_2$ and the internal symmetry is restricted to $SU(2)$.

It is seen from (3.3) that dilatations d and rotations $m_{\mu\nu}$ are linear transformations and, therefore, act in the same way on the real and imaginary parts of the complex vector $z_\mu = x_\mu + iy_\mu$. Special conformal transformation are non–linear and, therefore, they mix x_μ and y_μ according to

$$\begin{aligned} k_\mu x_\lambda &= ig_{\mu\lambda}(x_\nu x^\nu - y_\nu y^\nu) - 2i(x_\mu x_\lambda - y_\mu y_\lambda) \\ k_\mu y_\lambda &= 2ig_{\mu\lambda}x_\nu q^\nu - 2i(x_\mu y_\lambda + y_\mu x_\lambda) \end{aligned} \tag{3.4}$$

Also translations are not linear and it follows from (3.8) that the real part x_μ transforms like a vector $p_\mu x_\lambda = -ig_{\mu\lambda}$ whereas the imaginary part is translationally invariant $p_\mu y_\lambda = 0$.

The transformation properties (3.3) prove that the condition of consistency of the projection (3.2) with the group $SU(2,2)$ is satisfied for the complex vector z_μ.

The real and imaginary parts of $z_\mu = x_\mu + iy_\mu$ transform like vectors with respect to rotations and dilatations. The fact that y_μ is invariant under translations and x_μ transforms like a vector suggests the interpretation of x_μ as the local coordinates of the centre of mass and of y_μ as the relative coordinates with respect to the centre of mass. This interpretation corresponds to Yukawas idea of bilocal theory [10], [11].

Let us go over to the calculation of invariants of the theory. According to (2.26) $I_n = tr(f_2K^*f_1K)^n$, $n = 1,2$ on $\mathcal{O}_2^{(4,m)}$. The metric of the group $SU(m)$ is necessarily $F_2 = \mathbb{1}_m$. The invariant form of the group $SU(2,2)$ must be chosen in accordances with the representation of the generators of $SU(2,2)$ in $\mathbb{C}^4$. It is shown in [4] that we must take

$$F_1 = -\begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}. \tag{3.5}$$

The metric f_1 induced from (3.5) on $\mathbb{C}^2$ is

$$f_1 = -\left(\begin{array}{c|c} a^* + a & 0 \\ \hline 0 & 0 \end{array}\right). \tag{3.6}$$

With the help of the isomorphism (3.2) we can express $a^{a'}_{a''}$ through z_μ and obtain

$$a^* + a = \frac{2}{\lambda}\, y_\mu \tilde{\sigma}^\mu \tag{3.7}$$

and

$$\begin{aligned} I_1 &= \frac{2}{\lambda} y_\mu r^\mu, \\ I_2 &= \frac{4}{\lambda^2}\left\{-\frac{1}{2} y_\mu u^\mu r_\lambda r^\lambda + (y_\mu r^\mu)^2\right\} \end{aligned} \tag{3.8}$$

where

$$r_\mu = -\sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{\alpha=1}^{m} m^*_{\alpha a}(\tilde{\sigma}_\mu)^{\dot{a}b} m_{b\alpha}. \tag{3.9}$$

Details of the derivation are published in [4].

Instead of I_1 and I_2 we can use, equivalently, the invariants $y_\mu r^\mu$ and $y_\mu y^\mu r_\nu r^\nu$ and describe the decomposition of $\mathcal{O}_2^{(4,m)}$ into a two–parameter family of submanifolds by the two $SU(2,2) \times SU(m)$ invariant equations

$$y_\mu r^\mu = -c_{12}, \qquad y_\mu y^\mu r_\nu r^\nu = c_1. \tag{3.10}$$

In the case when we restrict the external symmetry to the Poincaré group P_4 another invariant appears, namely the Poincaré invariant $y_\mu y^\mu$ (cf. (3.3)). A further decomposition of $\mathcal{O}_2^{(4,m)}$ takes place into a three–parameter family of submanifolds $\mathcal{O}_{2,c}^{(4,m)}$ described by the equations

$$y_\mu y^\mu + c_{11} = y_\mu r^\mu + c_{12} = r_\mu r^\mu + c_{22} = 0\,. \tag{3.11}$$

Let us consider now the decomposition of the Grassmann manifold into domains of different induced metrics. According to (2.28) the induced signatures are determined by the roots of the secular equation for the induced metric. In our case (cf. (3.6), (3.7))

$$f_1 = -\frac{2}{\lambda}\, y_\mu \tilde{\sigma}^\mu \tag{3.12}$$

is a 2×2 matrix with the two eigenvalues

$$\begin{aligned} \lambda_{1,2} &= \frac{1}{2} tr\ f_1 \pm \sqrt{\left(\frac{1}{2} tr\ f_1\right)^2 - det\ f_1} \\ det\ f_1 &= -\frac{4}{\lambda^2} y_\mu y^\mu, \qquad tr\ f_1 = -\frac{4}{\lambda}\, y_0. \end{aligned} \tag{3.13}$$

According to the general scheme (2.29), we have the following domains corresponding to the admissible induced metrics: $(++)$, $(+-)$, $(--)$, $(+0)$, $(0-)$,

$(0,0)$. (cf. also [9])

$$
\begin{array}{llll}
(+\,+), & y_0 < 0, & y_\mu y^\mu < 0, & \lambda_1 > 0,\ \lambda_2 > 0 \\
(+\,0), & y_0 < 0, & y_\mu y^\mu = 0, & \lambda_1 = 0,\ \lambda_2 > 0 \\
(+\,-), & y_0 \lessgtr 0, & y_\mu y^\mu > 0, & \lambda_1 < 0,\ \lambda_2 > 0 \\
(0\,-), & y_0 > 0, & y_\mu y^\mu = 0, & \lambda_1 < 0,\ \lambda_2 = 0 \\
(-\,-), & y_0 > 0, & y_\mu y^\mu < 0, & \lambda_1 < 0,\ \lambda_2 < 0 \\
(0\,0), & y_0 = 0, & y_\mu y^\mu = 0, & \lambda_1 = 0,\ \lambda_2 = 0\ .
\end{array}
\tag{3.14}
$$

It is seen that the classification of domains and metrics depends entirely on the character of the fourvector y_μ and does not depend on x_μ.

The invariant conditions (3.10), (3.11) contain two translationally invariant Minkowski fourvectors y_μ and r_μ. The variables x_μ do not enter and the variables K and B ($m_{a'\alpha}$, $a' = 1,2$, $\alpha = 1,\dots,m$) enter by the intermediary of the vector r_μ (cf. (3.9)). Solving (3.19) for $\mathbf{y}$ we obtain

$$
\mathbf{y}^2 - \frac{(\mathbf{ry} + c_{12})}{r_0^2} = \frac{c_1}{r_\mu r^\mu} \tag{3.15}
$$

or, after diagonalization,

$$
(y_1')^2 + (y_2')^2 - \frac{r_\mu r^\mu}{r_0^2}\,(y_2')^2 = \frac{c_1 - c_{12}^2}{r_\mu r^\mu} \tag{3.16}
$$

But from (3.9) we easily derive that the vector r_μ is time–like and points towards the future ($r_\mu r^\mu < 0$, $r_0 > 0$). Equation(3.16) represents therefore an ellipsoid which is real when $c_1 - c_{12}^2 < 0$. In the case of conformal symmetry the axes are functions of $r_\mu r^\mu$ and r_0. In the case of Poincaré symmetry $r_\mu r^\mu = -c_{22}$, $c_{22} > 0$, $c_1 = c_{11}c_{22}$ and the ellipsoid (3.16) depends only on the component r_0, the condition for the reality being

$$
det\ c = c_{11}c_{22} - c_{12}^2 < 0\ . \tag{3.17}
$$

It is important to have also a coordinate free description of the spaces determined by equations (3.10) or (3.11). We restrict ourselves here to the (easier) discussion of the Poincaré invariant case (3.11). Consider the point

$\overset{0}{m}_{a'\alpha}$, $\overset{0}{y}_\mu$, $a' = 1,2$, $\alpha = 1,\dots,m$, $\mu = 0,1,2,3$, satisfying conditions

$$
\begin{aligned}
\overset{0}{m}_{1\alpha} &= 0, \qquad \alpha = 2,3,\dots,m, \\
\overset{0}{m}_{2\alpha} &= 0, \qquad \alpha = 3,4,\dots,m, \\
\overset{0}{y}_1 &= \overset{0}{y}_2 = 0.
\end{aligned}
\tag{3.18}
$$

and

$$
\begin{aligned}
\overset{0}{r}_i &= -\sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{\alpha=1}^{m} \overset{0}{m}{}^{*}_{\alpha a}\,(\tilde\sigma_i)^{\dot a b}\, m_{b\alpha} = 0 \\
\overset{0}{r}_0 &= -\sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{\alpha=1}^{m} \overset{0}{m}{}^{*}_{\alpha a}\,(\tilde\sigma_0)^{\dot a b}\, \overset{0}{m}_{b\alpha} = \sqrt{c_{22}} \\
(\overset{0}{y}_3)^2 &- (\overset{0}{y}_0)^2 = -c_{11}, \qquad \overset{0}{y}_0\,\overset{0}{r}_0 = -c_{12}, \qquad det\ c < 0
\end{aligned}
\tag{3.19}
$$

One easily convinces oneselve that this point satisfies conditions (3.11) and that the isotropy group of this point is $SO(2) \times SU(m-2)$.

Moreover, every point satisfying (3.11) can be reached from the point $\overset{0}{m}$, $\overset{0}{y}$ (3.18-19)) by a proper transformation of $SO(3,1) \times SU(m)$. The remaining coordinates x_μ are unrestricted and we have, therefore,

$$
\mathcal{O}_{2,c}^{(4,m)} \cong \frac{P_4}{SO(3,1)} \times \frac{SO(3,1) \times SU(m)}{SO(2) \times SU(m-2)} \tag{3.20}
$$

where $P_4/SO(3,1)$) stands for the real (external) Minkowski space–time parametrized by the coordinates $x_\mu = Re\ z_\mu$ and c denotes the three real parameters c_{ik} satisfying $det\ c < 0$.

The internal space can be considered as the direct product of a five–dimensional outer internal space $SO(3,1)/SO(2)$ parametrized by the coordinates y_μ and r_μ subject to conditions (3.11) and an inner internal space $SU(m)/SU(m-2)$ parametrized by the coordinates $m_{a'\alpha}$, $a' = 1,2$, $\alpha = 1,\dots,m$ subject to conditions $r_\mu = const$, $\mu = 0,1,2,3$.

The domains described by the admissible metrics $(+\,+), (+\,-), (-\,-)$ (of (3,14)) are represented by the inside of the future $(-\,-)$ and past $(+\,+)$ light cones and by the outside $(+\,-)$ of the light cone. The degenerate metrics $(+\,0)$ and $(0\,-)$ are represented by the past and future light cones. The metric $(0,0)$ corresponds to the point $y_\mu = 0$. (cf. also [9]).

The internal space in the model described here is necessarily not compact. It consists of the non–compact outer internal spaces $SO(3,1)/SO(2)$ which is topologically equivalent to the direct product $S^2 \times H^3$ of a two–dimensional sphere and a three–dimensional hyperboloid and of the compact inner internal space $SU(m)/SU(m-2)$ topologically equivalent to $S^{2m-1} \times S^{2m-3}$.

For physical interpretation it remains to find the representations of $SU(2,2)/SU(m)$ and $P_4 \times SU(m)$ in the corresponding homogeneous manifolds. We shall present the results in a separate publication.

References

[1] Finkelstein D., Phys. Rev. **100** (1955), 924.
[2] Bacry H., Kihlberg A., J. Math. Phys. **10** (1969), 2132.
[3] Rzewuski J., Bull. Acad. Polon. Sci. **6** (1958), 261, 335; **7** (1959), 571; **8** (1960), 777, 783; Nuovo Cimento, **5** (1958), 94; Acta Phys. Polon. **17** (1958), 417; **18** (1959), 549.
[4] Rzewuski J., Reports on Math. Phys. **22** (1985), 235.
[5] Rzewuski J., Reports on Math. Phys. **28** (1989), 167.
[6] Rzewuski J., Reports on Math. Phys. **29** (1991), 321.
[7] Kocik J., Rzewuski J., in Symmetries in Science II, Ed. B. Gruber, R. Lenczewski, Plenum Press, 1986.
[8] Crumeyrolle A., Reports on Math. Phys., **28** (1989), 27.
[9] Penrose R., Ann. Phys. **10** (1960), 171; J. Math. Phys. **8** (1967), 345; "Quantum Theory and Structure of Space–Time", München 1975.
[10] Yukawa H., Phys. Rev. **77** (1950), 219; **80** (1950), 1074; **91** (1953), 415.
[11] Rayski J., Nuovo Cimento **2** (1955), 255.

COMPLEX STRUCTURES AND THE ELIE CARTAN APPROACH TO THE THEORY OF SPINORS

MICHEL DUBOIS–VIOLETTE
Laboratoire de Physique Théorique et Hautes Energies,
Bâtiment 211, Université Paris XI,
91405 ORSAY Cedex, France

Abstract. Each isometric complex structure on a 2ℓ-dimensional euclidean space E corresponds to an identification of the Clifford algebra of E with the canonical anticommutation relation algebra for ℓ (fermionic) degrees of freedom. The simple spinors in the terminology of E. Cartan or the pure spinors in the one of C. Chevalley are the associated vacua. The corresponding states are the Fock states (i.e. pure free states), therefore, none of the above terminologies is very good.

1. Introduction

In this lecture, we will discuss complex structures and spinors on euclidean space. This is an extension of the algebraic part of a work [1] describing a sort of generalization of Penrose and Atiyah–Ward transformations in 2ℓ dimension. We shall not describe this work here, refering to [1], but concentrate the lecture upon the notion of simple spinor of E. Cartan [2] (or pure spinor in the terminology of C. Chevalley [3]). Many points of this lecture are well known facts and, in some sense, this may be considered as an introductory review. The notations used here are standard, let us just point out that by an euclidean space we mean a *real* vector space with a positive scalar product and by a Hilbert space we mean a *complex* Hilbert space.

2. Isometric Complex Structures

2.1. Notations

Let E be an oriented 2ℓ-dimensional euclidean space ($E \simeq \mathbb{R}^{2\ell}$) with a scalar product denoted by $(\bullet,\bullet)$. The dual space E^* of E is also, in a canonical way, an euclidean space and we again denote its scalar product by $(\bullet,\bullet)$. On the complexified space $E^*_c = E^* \otimes \mathbb{C}$ of E^* one may extend the scalar product of E^* in two different ways: Either one extends it by bilinearity and the corresponding bilinear form will again be denoted by $(\bullet,\bullet)$ or one extends it by sesquilinearity and the corresponding sesquilinear form will be denoted by $\langle \bullet | \bullet \rangle$. As for any complexified vector space, there is a canonical complex conjugation $\omega \mapsto \bar{\omega}$ on E^*, (an antilinear involution), and the connection between the two scalar products is given by:

$$\langle \omega_1 | \omega_2 \rangle = (\bar{\omega}_1, \omega_2), \qquad \forall \omega_1, \omega_2 \in E^*_c.$$

2.2. Isometric Complex Structures or Hilbertian Structures

Let $\mathcal{H}(E)$ be the set of isometric complex structures on E or, which is the same, the set of orthogonal antisymmetric endormorphisms of E, i.e.

$$\mathcal{H}(E) = \{J \in \mathrm{End}(E) | J \in O(E) \text{ and } J^2 = -\mathbb{1}\} =$$

$$= \{J \in \mathrm{End}(E) | J \in O(E) \text{ and } (X, JY) = -(JX, Y), \quad \forall X, Y \in E\}$$

Let $J \in \mathcal{H}(E)$ and define

$$(x + iy)V = xV + yJV, \qquad \forall (x + iy) \in \mathbb{C}, \qquad \forall V \in E$$

and

$$\langle X | Y \rangle_J = (X, Y) - i(X, JY), \qquad \forall X, Y \in E.$$

Equipped with the above structure, E is a ℓ-dimensional Hilbert space which we denote by E_J. For a basis $(e_1, \dots, e_\ell)$ of the complex vector space E_J, $(e_1, \dots, e_\ell, Je_1, \dots, Je_\ell)$ is a basis of E the orientation of which is independent of $(e_1, \dots, e_\ell)$ but only depends on J. Accordingly, $\mathcal{H}(E)$ splits in two pieces : $\mathcal{H}(E) = \mathcal{H}_+(E) \cup \mathcal{H}_-(E)$. The orthogonal group $O(E)$ acts transitively on $\mathcal{H}(E)$ and the subgroup $SO(E)$ of orientation preserving orthogonal transformations acts transitively on $\mathcal{H}_+(E)$ and on $\mathcal{H}_-(E)$.
Thus one has $\mathcal{H}(E) \simeq O(E)/U(E_J)$ and $\mathcal{H}_+(E) \simeq SO(E)/U(E_J) \simeq \mathcal{H}_-(E)$ where $U(E_J)$ is the unitary group of E_J for a fixed $J \in \mathcal{H}(E)$ (i.e. $U(E_J) \simeq U(\mathbb{C}^\ell)$). We equip $\mathcal{H}(E), \mathcal{H}_+(E)$ and $\mathcal{H}_-(E)$ with the corresponding manifold structure. In particular, $\dim_{\mathbb{R}} \mathcal{H}(E) = \dim_{\mathbb{R}} \mathcal{H}_\pm(E) = \ell(2\ell - 1) - \ell^2 = \ell(\ell - 1)$.

2.3. Identification of Dual Spaces

The dual Hilbert space of E_J can be identified with the Hilbert subspace $\Lambda^{1,0}E^*_J$ of E^*_c defined by

$$\Lambda^{1,0}E^*_J = \{\omega \in E^*_c | \omega \circ J = i\omega\}.$$

One verifies easily that $\Lambda^{1,0}E^*_J$ is maximal isotropic in E^*_c for $(\bullet, \bullet)$ or, which is the same, that $\Lambda^{1,0}E^*_J$ is orthogonal to its conjugate $\overline{\Lambda^{1,0}E^*_J} = \Lambda^{0,1}E^*_J$ in E^*_c for $\langle \bullet | \bullet \rangle$ (i.e. E^*_c is the hilbertian direct sum $\Lambda^{1,0}E^*_J \oplus \Lambda^{0,1}E^*_J$).
Conversely if $F \subset E^*_c$ is a maximal isotropic subspace for $(\bullet, \bullet)$, then there is a unique $J \in \mathcal{H}(E)$ such that $F = \Lambda^{1,0}E^*_J$. It follows that $\mathcal{H}(E)$ identifies with a complex algebraic submanifold of the grassmannian $G_\ell(E^*_c)$ of ℓ-dimensional subspaces of E^*_c, $(G_\ell(E^*_c) \simeq G_{\ell,2\ell}(\mathbb{C}))$. In particular, $\mathcal{H}(E)$ is a compact Kähler manifold of complex dimension $\frac{\ell(\ell-1)}{2}$ and its Kähler metric is given by $ds^2 = \frac{1}{4}\mathrm{tr}((dP^{1,0}_J)^2)$ where $P^{1,0}_J$ is the hermitian projector of E^*_c on $\Lambda^{1,0}E^*_J$. Notice that one has $\overline{P^{1,0}_J} = P^{0,1}_J = \mathbb{1} - P^{1,0}$.
Furthermore $\Lambda^{1,0}E^*_J$ is the fibre at $J \in \mathcal{H}(E)$ of a holomorphic hermitian vector bundle of rank ℓ over $\mathcal{H}(E)$ which we denote by $\Lambda^{1,0}E^*$.
Finally notice that one has the hilbertian sum identifications

$$\Lambda^k E^*_c = \bigoplus_{r+s=k} \Lambda^{r,s}E^*_J, \qquad \forall J \in \mathcal{H}(E)$$

where $\Lambda^{r,s}E^*_J = \Lambda^r(\Lambda^{1,0}E^*_J) \otimes \Lambda^s(\overline{\Lambda^{1,0}E^*_J})$, (here the tensor product is over $\mathbb{C}$).We denote by $P^{r,s}_J$ the corresponding hermitian projectors.

2.4. Examples

One has $\mathcal{H}_+(\mathbb{R}^2) = \{I_+\}, \mathcal{H}_+(\mathbb{R}^4) = \mathbb{C}P^1, \mathcal{H}_+(\mathbb{R}^6) = \mathbb{C}P^3$ and, as will be shown below, $\mathcal{H}_+(\mathbb{R}^{2\ell}) \subset \mathbb{C}P^{2^{\ell-1}-1}$ but the inclusion is strict for $\ell \geq 4$ as it follows by comparison of the dimensions.

2.5. Hodge duality

On ΛE^* there is a linear involution, $*$, defined by $*(\omega^1 \wedge \dots \wedge \omega^p) = \omega^{p+1} \wedge \dots \wedge \omega^{2\ell}$ for any positively oriented orthonormal basis $(\omega^1, \dots, \omega^{2\ell})$. One extends this involution by linearity to ΛE^*_c. One has the following lemma.

Lemma. *Let Ω be an element of $\Lambda^\ell E^*_c$. Then one has $\Omega + i^\ell * \Omega = 0$, (resp. $\Omega - i^\ell * \Omega = 0$), if and only if $P^{0,\ell}_J \Omega = 0, \forall J \in \mathcal{H}_+(E)$, (resp. $\forall J \in \mathcal{H}_-(E)$).*

For $\ell = 2$ (i.e. in dimension 4), this is the basic algebraic lemma for the Penrose–Atiyah–Ward transformation.

3. The Clifford algebra as C.A.R. algebra

3.1. Definition

We define the Clifford algebra $\mathrm{Cliff}(E^*)$ to be the complex associative $*$-algebra with a unit $\mathbb{1}$ generated by the following relations

$$[\gamma(\omega_1), \gamma(\omega_2)]_+ = 2(\omega_1, \omega_2)\mathbb{1} \text{ and } \gamma(\omega)^* = \gamma(\omega) \text{ for } \omega, \omega_i \in E^*.$$

The $\gamma(\omega), \omega \in E^*$, are hermitian generators and $\gamma : E^* \to \mathrm{Cliff}(E^*)$ is an injective $\mathbb{R}$-linear mapping. One extends γ as a $\mathbb{C}$-linear mapping, $\gamma : E^*_c \to \mathrm{Cliff}(E^*)$, by setting $\gamma(\bar{\omega}) = \gamma(\omega)^*$.

3.2. Complex structures and the C.A.R. algebra

Let $J \in \mathcal{H}(E)$ be given. The algebra $\mathrm{Cliff}(E^*)$ is generated by the $\gamma(\omega)$ with $\omega \in \Lambda^{1,0}E^*_J$ and their adjoints $\gamma(\omega)^* = \gamma(\bar{\omega})$. In terms of these generators the relations read

$$[\gamma(\omega_1), \gamma(\omega_2)]_+ = 0 \text{ and } [\gamma(\omega_1)^*, \gamma(\omega_2)]_+ = \langle\omega_1|\omega_2\rangle\mathbb{1}, \quad \forall \omega_i \in \Lambda^{1,0}E^*_J.$$

These are the defining relations of the algebra of canonical anticommutation relations (C.A.R. algebra) for ℓ (fermionic) degrees of freedom. Thus each $J \in \mathcal{H}(E)$ corresponds to an identification of the Clifford algebra with the C.A.R. algebra. Furthermore, the action of the orthogonal group $O(E)$ on $\mathcal{H}(E)$ corresponds to the Bogolioubov transformations. One has, as well known, $\mathrm{Cliff}(E^*) \simeq M_{2^\ell}(\mathbb{C})$.

4. Spinors and Complex Structures

4.1. Definition

We define a space of spinors associated to E to be a Hilbert space S carrying an irreducible $*$-representation of $\mathrm{Cliff}(E^*)$. The spinors are the elements of S. Since $\mathrm{Cliff}(E^*)$ is isomorphic to $M_{2^\ell}(\mathbb{C})$, S is isomorphic to $\mathbb{C}^{2^\ell}$ and the representation is an isomorphism. We shall identify $\mathrm{Cliff}(E^*)$ with the image of this representation.

4.2. The simple spinors of E. Cartan

Let $\psi \in S$ with $\psi \neq 0$ and set $I_\psi = \{\omega \in E^*_c | \gamma(\omega)\psi = 0\}$. If ω_1 and ω_2 are in I_ψ, one has $[\gamma(\omega_1), \gamma(\omega_2)]_+\psi = 2(\omega_1, \omega_2)\psi = 0$, so I_ψ is an isotropic subspace of E^*_c for $(\bullet, \bullet)$.
If I_ψ is maximal isotropic, i.e. if $\dim(I_\psi) = \ell$, then ψ is called a *simple spinor* by E. Cartan [2] or a *pure spinor* by C. Chevalley [3]. We denote by $\mathcal{F}$ the set of these spinors and by $P(\mathcal{F})$ the corresponding algebraic submanifold of $P(S) = \mathbb{C}P^{2^\ell - 1}$, (i.e. $P(\mathcal{F})$ is the set of directions of simple spinors).
For $\psi \in \mathcal{F}, I_\psi = I_{\lambda\psi} \quad \forall \lambda \in \mathbb{C}\backslash\{0\}$, so the maximal isotropic subspace I_ψ of E^*_c does only depend on the direction $[\psi] \in P(\mathcal{F})$ of ψ. On the other hand we know that there is a unique $J \in \mathcal{H}(E)$ such that $I_\psi = \Lambda^{1,0}E^*_J$. It follows that one has a mapping of $P(\mathcal{F})$ in $\mathcal{H}(E)$ which is in fact an isomorphism of complex manifolds. In the following, we shall identify these manifolds, writing $P(\mathcal{F}) = \mathcal{H}(E)$.

4.3. The natural line bundle

The restriction to $P(\mathcal{F}) = \mathcal{H}(E)$ of the tautological bundle of $P(S)$ is a holomorphic hermitian vector bundle of rank one, L, over $\mathcal{H}(E)$. One has $L = \mathcal{F} \cup \{\text{the zero section}\}$.
As holomorphic hermitian line bundles over $\mathcal{H}(E)$, one has the following isomorphisms, see in [1] : $\Lambda^{\ell,0}E^* \simeq L \otimes L$ and $\Lambda^{\frac{\ell(\ell-1)}{2}}T^*\mathcal{H}(E) = L^{\otimes\, 2(\ell-1)}$.

4.4. Semi-spinors and simple spinors

To the action of $SO(E)$ on E^* corresponds a linear representation of its covering $\mathrm{Spin}(E)$ in S. Under this representation, S splits into two irreductible components $S = S_+ \oplus S_-$ with $\dim S_+ = \dim S_- = 2^{\ell-1}$. The elements of S_+ and S_- are called semi-spinors. One the other hand $P(\mathcal{F}) = \mathcal{H}(E)$ splits into two transitive parts under the action of $SO(E)$, $\mathcal{H}(E) = \mathcal{H}_+(E) \cup \mathcal{H}_-(E)$. It follows that $\mathcal{F} = \mathcal{F}_+ \cup \mathcal{F}_-$ with $\mathcal{F}_\pm = \mathcal{F} \cap S_\pm$ and (with an eventual relabelling in the $\pm$) $P(\mathcal{F}_\pm) = \mathcal{H}_\pm(E)$. In other words $\mathcal{F}$ consists of semi-spinors. It turns out that for $\ell \leq 3$ all non vanishing semi-spinors are in $\mathcal{F}$ (i.e. $\mathcal{F}_\pm = S_\pm\backslash\{0\}$) but for $\ell \geq 4$ the inclusions $\mathcal{F}_\pm \subset S_\pm\backslash\{0\}$ are strict inclusions. For $\ell \geq 4$ $\mathcal{H}_+(E)$ is no more a projective space.

5. Fock States and Simple Spinors

5.1. States on algebras

Let $\mathcal{A}$ be an associative complex $*$-algebra with a unit $\mathbb{1}$. We recall that a state on $\mathcal{A}$ is a linear form ϕ on $\mathcal{A}$ such that $\phi(X^*X) \geq 0, \quad \forall X \in \mathcal{A}$ and $\phi(\mathbb{1}) = 1$. The set of all states on $\mathcal{A}$ is a convex subset of the dual space $\mathcal{A}^*$ of $\mathcal{A}$. The extreme points of this convex subset (i.e. which are not convex combinations of two distinct states) are called pure states. To the states on $\mathcal{A}$ correspond cyclic $*$-representations of $\mathcal{A}$ in Hilbert space via the G.N.S. construction; pure states correspond then to irreducible representations.
Coming back to the case $\mathcal{A} = \mathrm{Cliff}(E^*)$, we see that to each spinor $\psi \neq 0$ corresponds a state $X \mapsto \frac{\langle\psi|X\psi\rangle}{\|\psi\|^2}$ (its direction) which is a pure state leading to an irreducible, or simple, representation. This is why the terminology of C. Chevalley or E. Cartan to denote the elements of $\mathcal{F}$ is somehow misleading. What characterizes the elements of $\mathcal{F}$ is that the corresponding states (i.e. elements of $P(\mathcal{F}) = \mathcal{H}(E)$) are Fock states or free states on $\mathrm{Cliff}(E^*)$ (see below); thus the name Fock spinors or free spinors would be better.

5.2. Fock states on the Clifford algebra

First of all it is clear from above that the elements of $\mathcal{F}$ are all possible vacua corresponding to the identifications of $\mathrm{Cliff}(E^*)$ with the C.A.R. algebra. It is well known that given a vacuum, the vacuum expectation values factorize and only depend on the "two-point functions" i.e. on the vacuum expectation values of $\gamma(\omega_1)\gamma(\omega_2)$ for $\omega_i \in E^*$, (this is the very property of the free states). More precisely, a *Fock state*, (see for instance [4]), on $\mathrm{Cliff}(E^*)$ is a pure state ϕ satisfying the following (Q.F.) property:

$$(Q.F.)\begin{cases} \phi(\gamma(\omega_1)\dots\gamma(\omega_{2n+1})) = 0 \\ \phi(\gamma(\omega_1)\dots\gamma(\omega_{2n})) = \sum_{k=2}^{2n}(-1)^k\phi(\gamma(\omega_1)\gamma(\omega_k)).\phi(\gamma(\omega_2).\overset{k}{\vee}.\gamma(\omega_{2n})) \end{cases}$$

for $\omega_i \in E^*$, (where $\overset{k}{\vee}$ means omission of the k$^{\text{th}}$ term). From (Q.F.) one sees that ϕ is determined by the $\phi(\gamma(\omega_1)\gamma(\omega_2)) = h(\omega_1,\omega_2)+i\sigma(\omega_1,\omega_2)$, $\omega_i \in E^*$, where h and σ are real bilinear forms. The defining relations of $\mathrm{Cliff}(E^*)$ implie that $h(\omega_1,\omega_2)+h(\omega_2,\omega_1) = 2(\omega_1,\omega_2)$ and $\sigma(\omega_1,\omega_2)+\sigma(\omega_2,\omega_1) = 0$. The positivity of ϕ is equivalent to $\phi(\gamma(\omega_1+i\omega_2)\gamma(\omega_1-i\omega_2)) \geq 0$ which is equivalent to $h(\omega_1,\omega_2) = (\omega_1,\omega_2)$ and $\sigma(\omega_1,\omega_2) = (A\omega_1,\omega_2) = -(\omega_1,A\omega_2)$ with $\| A \| \leq 1$. By polar decomposition, $A = J|A|$ with $J \in \mathcal{H}(E)$ and $|A| \geq 0$ ($\| |A| \| \geq 1$). Then, ϕ is pure if and only if $|A| = 1$. Therefore, ϕ is a Fock state iff. it satisfies (Q.F.) and $\phi(\gamma(\omega_1)\gamma(\omega_2)) = (\omega_1,\omega_2) + i(J\omega_1,\omega_2)$, $\forall\omega_i \in E^*$, with $J \in \mathcal{H}(E)$. Thus, the Fock states are parametrized by $\mathcal{H}(E) = P(\mathcal{F})$ and, in fact, the set of Fock states is $P(\mathcal{F})$; indeed if $\psi \in \mathcal{F}$ is such that $I_\psi = \Lambda^{1,0}E_J^*$ then one has

$$\frac{\langle\psi|\gamma(\omega_1)\gamma(\omega_2)\psi\rangle}{\| \psi \|^2} = (\omega_1,\omega_2)+i(J\omega_1,\omega_2), \quad \forall\omega_i \in E^*$$

and (Q.F.) is satisfied.

6. Spinors and Fock Space Constructions

The standard construction of the Fock space for the C.A.R. algebra implies that, for each J, S is isomorphic to

$$\bigoplus_n \Lambda^{0,n}E_J^*.$$

However, there is the vacuum, namely an element of L_J, which is hidden here.
In fact, one has an isomorphism Φ of hermitian vector bundles over $\mathcal{H}(E)$ from

$$\bigoplus_n \Lambda^{0,n}E^* \otimes L$$

onto the trivial bundle with fibre equal to S, such that

$$\Phi(\omega\wedge\varphi) = \frac{1}{\sqrt{2}}\gamma(\omega)\Phi(\varphi), \quad \forall\omega \in \Lambda^{0,1}E_J^* \text{ and } \forall\varphi \in \bigoplus_n \Lambda^{0,n}E_J^*\otimes L_J.$$

More precisely one has the following

$$\left.\begin{array}{ll} \Phi_J : \oplus_p \Lambda^{0,2p}E_J^*\otimes L_J \simeq S_+ & (\text{resp. } S_-) \\ \Phi_J : \oplus_p \Lambda^{0,2p+1}E_J^*\otimes L_J \simeq S_- & (\text{resp. } S_+) \end{array}\right\} \forall J \in \mathcal{H}_+(E)\ (\text{resp.} \mathcal{H}_-(E))$$

which gives the identification of semi-spinors.

7. Bundles of Complex Structures

Let M be a 2ℓ-dimensional oriented riemannian manifold. The tangent space $T_x(M)$ at $x \in M$ is an oriented 2ℓ-dimensional euclidean space so one can consider the complex manifold $\mathcal{H}(T_x(M))$ as above. $\mathcal{H}(T_x(M))$ is the fiber at $x \in M$ of a bundle $\mathcal{H}(T(M))$ on M which we call the *bundle of isometric complex structures over* M. This bundle is associated to the orthonormal frame bundle so there is a natural connection on it coming from the Levi–Civita connection of M. On $\mathcal{H}(T(M))$, there is a *natural almost complex structure* defined by the following construction. Let $J_x \in \mathcal{H}(T_x(M))$, then by horizontal lift, J_x defines a complex structure on the tangent horizontal subspace at J_x; on the other hand the tangent vertical subspace at J_x is the tangent space to the complex manifold $\mathcal{H}(T_x(M))$ so it is naturally a complex vector space, so by taking the direct sum one has a complex structure on the tangent space to $\mathcal{H}(T(M))$ at J_x and finally, $\mathcal{H}(T(M))$ becomes an almost complex manifold. It is easy to show that the almost complex manifold $\mathcal{H}(T(M))$ only depends on the conformal structure of M. In particular, the almost complex structure of $\mathcal{H}(T(M))$ is integrable, i.e. $\mathcal{H}(T(M))$ is a complex manifold, whenever M is conformally flat. The Penrose and the Atiyah-Ward transformations are obtained, in the four-dimensional case, by lifting to $\mathcal{H}(T(M))$ various objects living on M (see in [1]).
Let us end this lecture by noticing that the complex manifold $\mathcal{H}(T(S^{2\ell}))$ identifies with the complex manifold $\mathcal{H}(\mathbb{R}^{2\ell+2})$ of isometric complex structures on the euclidean space $\mathbb{R}^{2\ell+2}$, [1]. So, in particular, by restriction to the positively oriented complex structures one has $\mathcal{H}_+(T(S^4)) = \mathcal{H}_+(\mathbb{R}^6) = \mathbb{C}P^3$.

References

[1] DUBOIS–VIOLETTE M. : 1980, "Structures complexes au-dessus des variétés, applications", Séminaire math. E.N.S. in "Mathématique et Physique", L. Boutet de Montvel, A. Douady and J.L. Verdier, eds. Progress in Mathematics **vol. 37**, 1–42.
[2] CARTAN, E. : 1938, Leçons sur la théorie des spineurs I,II, Hermann et Cie.
[3] CHEVALLEY, C. : 1954, "The algebraic theory of spinors", Columbia University Press, Morningside Heights, New York.
[4] MANUCEAU, J. : 1970, "Etude algébrique des états quasi–libres; A. Etats quasi–libres des fermions". In *Cargèse lectures in Physics* **vol. 4**, D. Kastler ed., Gordon and Breach.

SPIN STRUCTURES ON HYPERSURFACES AND THE SPECTRUM OF THE DIRAC OPERATOR ON SPHERES*

ANDRZEJ TRAUTMAN
Institute of Theoretical Physics, Warsaw University,
Hoża 69, 00-681 Warsaw, Poland.

Abstract. Recent results on pin structures on hypersurfaces in spin manifolds are reviewed. A new form of the Dirac operator is used to compute its spectrum on n-dimensional spheres. This constrbution is based on two papers by the author, where details and proofs can be found (Ref.4 and 5).

1. This research has been motivated by, and can be summarized in, the following observations:
(i) In odd dimensions, it is appropriate to use the *twisted* adjoint representation $\rho : \mathrm{Pin}(n) \to \mathrm{O}(n)$ to find a cover of the full orthogonal group $\mathrm{O}(n)$ which extends the standard homomorphism $\mathrm{Spin}(n) \to \mathrm{SO}(n)$. Here ρ is given by $\rho(a)v = \alpha(a)va^{-1}$, where $v \in \mathbf{R}^n$, $a \in \mathrm{Pin}(n) \subset \mathrm{Cl}(n)$ and α is the grading (main) automorphism of the Clifford algebra $\mathrm{Cl}(n)$ [1]. Using the twisted representation leads to modifying the Dirac operator [2].
(ii) The bundles of "Dirac spinors" over even-dimensional spheres are trivial [3]; this observation generalizes to hypersurfaces in $\mathbf{R}^{n+1}$: every such hypersurface, even if it is non-orientable, admits a pin structure with a trivial bundle of Dirac (n even) or Pauli (n odd) spinors [4]
(iii) The spectrum and the eigenfunctions of the Laplace operator Δ on the n-dimensional unit sphere $\mathbf{S}_n$ are easily obtained from the formula

$$\sum_{i=1}^{n+1} \partial^2/\partial x_i^2 = r^{-2}\Delta + r^{-n}\partial/\partial r(r^n \partial/\partial r) \tag{1}$$

This formula generalizes to a foliation of $\mathbf{R}^{n+1}$ by hypersurfaces and extends to the Dirac operator, allowing a simple computation of the Dirac spectrum of n-spheres [5].

* This research was supported in part by the Polish Committee for Scientific Research under grant No.2-0430-9101

2. Consider the vector space $\mathbf{R}^n$ with the standard scalar product $(u \mid v)$ and the associated positive-definite quadratic form $(u \mid u)$, where $u = (u^\mu) \in \mathbf{R}^n$, $\mu = 1, \ldots, n$. The corresponding Clifford algebra $\mathrm{Cl}(n)$ contains $\mathbf{R} \bigoplus \mathbf{R}^n$, one has

$$uv + vu = -2(u \mid v), \quad \text{where} \quad u, v \in \mathbf{R}^n, \tag{2}$$

and uv is the Clifford product of u and v. Let (e_μ) be the canonical frame in $\mathbf{R}^n$ so that $u = u^\mu e_\mu$ for every $u \in \mathbf{R}^n$; similarly, $(e_i), i = 1, \ldots, n+1$, is the canonical frame in $\mathbf{R}^{n+1}$. The group $\mathrm{Pin}(n)$ is defined as the subset of $\mathrm{Cl}(n)$ consisting of products of all finite sequences of unit vectors.

Let $\mathrm{Cl}(n) = \mathrm{Cl}_0(n) \bigoplus \mathrm{Cl}_1(n)$ be the decomposition of $\mathrm{Cl}(n)$ defining its $\mathbf{Z}_2$ grading so that $\mathrm{Spin}(n) = \mathrm{Pin}(n) \bigcap \mathrm{Cl}_0(n)$. Let $a = a_0 + a_1$ be the corresponding decomposition of $a \in \mathrm{Cl}(n)$. The map $h : \mathrm{Cl}(n) \to \mathrm{Cl}_0(n+1)$ given by $a \to a_0 + a_1 e_{n+1}$ is an isomorphism of algebras with units. By restriction, it defines the commutative diagram of group homomorphisms

$$\begin{array}{ccc} \mathrm{Pin}(n) & \xrightarrow{h} & \mathrm{Spin}(n+1) \\ \rho \downarrow & & \downarrow \rho \\ \mathrm{O}(n) & \xrightarrow{H} & \mathrm{SO}(n+1) \end{array} \tag{3}$$

where the horizontal (resp., vertical) arrows are injective (resp., surjective).

For every n, there is a representation γ of $\mathrm{Cl}(n)$ and a representation γ' of $\mathrm{Cl}(n+1)$ in the same complex vector space S. The representation γ' extends γ in the sense that $\gamma = \gamma' \circ h$. One puts

$$\gamma_i = \gamma'(e_i) \quad i = 1, \ldots, n+1. \tag{4}$$

and defines the helicity automorphism $\Gamma = (-1)^{n(n-1)/4} \gamma_1 \gamma_2 \ldots \gamma_n$ so that $\Gamma^2 = 1$. Note that $\gamma(e_\mu) = \gamma_\mu \gamma_{n+1}$ and $\gamma(e_\mu e_\nu) = \gamma_\mu \gamma_\nu$. For $n = 2m$, γ is the Dirac representation in a complex vector space of dimension 2^m and γ' is one of two Pauli representations, characterized, say, by $\gamma_{n+1} = \sqrt{-1}\Gamma$. For $n = 2m - 1$, γ' is the Dirac representation, whereas γ is a faithful representation that decomposes into two irreducible Pauli representations. This terminology generalizes the one used by physicists in dimensions 3 and 4.

3. Consider now an n-dimensional pin manifold M, i.e. a Riemannian manifold with a pin structure

$$Q \xrightarrow{\sigma} P \xrightarrow{\pi} M \tag{5}$$

where P is the $\mathrm{O}(n)$-bundle of all orthonormal frames on M so that $\sigma(q) = (\sigma_\mu(q))$, $q \in Q$, is an orthonormal frame at $\tilde{\pi}(q) = \pi \circ \sigma(q) \in M$ and $\tilde{\pi} : Q \to M$ is a Pin(n)-bundle such that $\sigma \circ \delta(a) = \delta(\rho(a)) \circ \sigma$, where $\delta(a)$ is the (right) translation by $a \in \mathrm{Pin}(n)$ of elements of Q. The Levi-Civita connection on M defines a "spin" connection on the pin-bundle $Q \to M$ which can be described by giving on Q a collection of n horizontal vector fields ∇_μ $(\mu = 1, \ldots, n)$ such that, for every $q \in Q$, one has $T_q \tilde{\pi}(\nabla_\mu(q)) = \sigma_\mu(q)$.

By restriction, one has the representation γ: $\mathrm{Pin}(n) \to \mathrm{GL}(S)$ and one defines a spinor field on M, with its pin structure (5), as a map $\psi : Q \to S$, equivariant with respect to the action on $\mathrm{Pin}(n)$, $\psi \circ \delta(a) = \gamma(a^{-1}) \circ \psi$. Alternatively, and equivalently, a spinor field can be described as a section of the bundle $E \to M$, associated with $Q \to M$ by the representation γ.

The Dirac operator $\nabla = \gamma^\mu \nabla_\mu$ transforms spinor fields into spinor fields.

4. A hypersurface M in an $(n+1)$-dimensional connected Riemannian manifold M' is an n-manifold M with an immersion $f : M \to M'$. The metric tensor on M' induces a Riemannian metric on M. If M' is orientable and P' is its bundle of orthonormal frames of coherent orientation, then the bundle P of all orthonormal frames on M can be identified with the set

$$\{(x, p) \in M \times P' : p = (p_i), i = 1, \ldots, n+1 \text{ where } p \text{ is a frame at } f(x) \text{ such that } p_{n+1} \text{ is orthogonal to } T_x f(T_x M) \subset T_{f(x)} M' \}.$$

The group $\mathrm{O}(n)$ acts in P via H. Assume now that M' has a spin structure $Q' \xrightarrow{\sigma'} P' \to M'$; a spin-structure on M is (5), where $Q \to P$ is the Z_2-bundle induced [6] from $Q' \to P'$ by the map $F : P \to P', F(x, p) = p$, i.e.

$$Q = \{(p, q) \in P \times Q' : F(p) = \sigma'(q)\}.$$

As an example illustrating this construction, one can mention the embedding of real projective spaces, $\mathbf{RP}_n \to \mathbf{RP}_{n+1}$. Since $\mathbf{RP}_{4m+3}$ is a spin manifold, there is a pin structure on $\mathbf{RP}_{4m+2}$ [7].

Immersions of M, which are differentiably homotopic one to another, give rise to equivalent pin structures on M, but otherwise not, in general. For example, the "identity" and the "square" immersions of S_1 in $\mathbf{R}^2$ give rise to the non-trivial and the trivial spin structures on the circle, respectively.

Assume now that the spin structure on M' is trivial, i.e. there exists a map $g : Q' \to \mathrm{Spin}(n+1)$ such that $g(qa) = g(q)a$ for every $q \in Q'$ and $a \in \mathrm{Spin}(n+1)$. *The pin structure on the hypersurface M need not be trivial, but the bundle $E \to M$ of spinors, associated by γ with $Q \to M$, is isomorphic to the direct product $M \times S$.*

Indeed, the bundle E can be identified with the set of equivalence classes of the form $[(p, q, \phi)]$, where $(p, q, \phi) \in P \times Q' \times S$, $F(p) = \sigma'(q)$ and $[(p, q, \phi)] = [(p', q', \phi')]$ iff there is $a \in \mathrm{Pin}(n)$ such that $p' = p\rho(a)$, $q' = qh(a)$ and $\phi = \gamma(a)\phi'$. The map $[(p, q, \phi)] \to (\pi(p), \gamma'(g(q))\phi)$ trivializes E. For example, if M is a hypersurface in $\mathbf{R}^{n+1}$, then its bundle of Dirac or Pauli spinors is trivial. Since $\mathbf{RP}_3 = \mathrm{SO}(3)$ has a trivial spin bundle, the bundle of two-component "Dirac" spinors on $\mathbf{RP}_2$ is also trivial. In general, the bundles of Weyl (half) spinors on even-dimensional hypersurfaces in $\mathbf{R}^{n+1}$ are not trivial (example: even-dimensional spheres).

5. Let $f : M \to M'$ be an embedding (i.e. injective immersion) of the hypersurface M in the manifold M' with a *trivial* spin structure $Q' \to P' \to M'$. The maps $P \to P'$ and $Q \to Q'$ are then also injective and the extension Q" of the Pin(n)-bundle Q to the group Spin(n) is also trivial. A spinor fiels $\psi : Q \to S$ extends to a map $\psi" : Q" \to S$ such that $\psi"(qa) = \gamma'(a^{-1})\psi"(q)$ for every $q \in Q"$ and $a \in$Spin$(n+1)$. Instead of working with ψ, one can now take a global section s of the trivial bundle $Q" \to M$ and the composition $\Psi = \psi" \circ s : M \to S$ as an equivalent way of describing the spinor field. One defines the Dirac operator D acting on Ψ by the formula

$$D\Psi = (\nabla\psi)" \circ s. \tag{6}$$

6. The above considerations are particularly useful and simple when M is an orientable hypersurface embedded in $\mathbf{R}^{n+1}$. This being so, let (X^i) be the unit normal vector field on M and let (x^i) be the Cartesian coordinates in $\mathbf{R}^{n+1}$. Each of the $n(n+1)/2$ vector fields

$$X_{ij} = X_j\partial_i - X_i\partial_j, \quad \text{where} \quad \partial_i = \partial/\partial x_i, \quad 1 \le i < j \le n,$$

is tangent to M. Introducing the notation

$$\sigma_{ij} = (\gamma_i\gamma_j - \gamma_j\gamma_i)/2, \quad \mathbf{X}= X^I\gamma_i, \quad \text{div } X = \partial_i X^i,$$

so that

$$\sigma_{ij} = \delta_{ij} + \gamma_i\gamma_j,$$

one can write (6) as

$$D\Psi = \frac{1}{2}\mathbf{X}(\sigma^{ij}X_{ij} - \text{div}X)\Psi. \tag{7}$$

The right side of (7) is invariant with respect to the replacement of X by $-X$ and one can show that the assumption of orientability of M is irrelevant.

Assume now that $\mathbf{R}^{n+1}$ is foliated by a family of hypersurfaces so that the field X of unit normals is defined over an open subset of $\mathbf{R}^{n+1}$. The identity

$$\gamma^i\partial_i = \mathbf{X}(X^i\partial_i + \tfrac{1}{2}\sigma^{ij}X_{ij}) \tag{8}$$

leads to a decomposition of the Dirac operator $\gamma^i\partial_i$ on $\mathbf{R}^{n+1}$ into parts tangential and transverse to the foliation,

$$\gamma^i\partial_i = D + \mathbf{X}(\partial/\partial r + \tfrac{1}{2}\text{div}X), \tag{9}$$

where $\partial/\partial r = X^i\partial_i$ is the derivative in the "radial" direction, transverse to the foliation. There is an analogous formula for the Laplace operator

[4]. Since the operator D anticommutes with $\mathbf{X}$ and $\mathbf{X}^2 = -1$, if Ψ is an eigenfunction of D, then $(1+\mathbf{X})\Psi$ is an eigenfunction of $\mathbf{X}D$ with the same eigenvalue. Therefore, for M orientable, it is enough to consider the spectrum of the latter operator.

7. As a simple application, consider the spectrum of the Dirac operator on the unit sphere $\mathbf{S}_n$. The space $\mathbf{R}^{n+1}$ with its origin removed is foliated by the spheres $r = \sqrt{(x_1^2 + \ldots + x_{n+1}^2)} =$ const. so that $X^i = x^i/r$,the vector fields X_{ij} are generators of rotations, $\mathbf{div}X = n/r$ and equation (9) gives

$$\mathbf{X}\gamma^i\partial_i = \mathbf{XD} - (\partial/\partial r + n/2r). \tag{10}$$

Let Φ:$\mathbf{R}^{n+1} \to S$ be a spinor-valued harmonic polynomial of degree $l+1$, where $l = 0, 1, \ldots$ The polynomial $\Psi = (\gamma^i\partial_i)\Phi$ is of degree l and is annihilated by the Dirac operator $\gamma^i\partial_i$. Therefore, on the unit sphere $r = 1$, one has

$$\mathbf{XD}\Psi = (l + n/2)\Psi \text{ and } \mathbf{XD}\ \mathbf{X}\Psi = -(l + n/2)\mathbf{X}\Psi. \tag{11}$$

and the spectrum of the Dirac operator on $\mathbf{S}_n$, for $n > 1$, is the set of all numbers of the form $\pm(l + n/2)$, where $l = 0, 1, 2, \ldots$. There is a gap of length n and 0 is never an eigenvalue, this being a simple consequence of the celebrated Lichnerowicz theorem [8]. For $n = 1$, there are two spin structures. The previous formula applies to the non-trivial structure; for the trivial one, the spectrum is $\mathbf{Z}$.

References

[1] M.F. Atiyah, R. Bott and A. Shapiro, *Clifford modules*, Topology, **3** Suppl. 1(1964)3-38

[2] A. Trautman, *The Dirac operator on hypersurfaces in Euclidean space*, Trieste Seminar on Spinors, Letter 13 (10 April 1991).

[3] S. Gutt, *Killing spinors on spheres and projective spaces*, in: "Spinors in Physics and Geometry" (Proc. Conf. Trieste, 11-13 Sept. 1986) ed. by A. Trautman and G. Furlan, World Scientific, Singapore, 1988.

[4] A.Trautman, *Spinors and the Dirac operator on hypersurfaces. I. General Theory*, J.Math.Phys. (in print).

[5] A.Trautman and E. Winkowska, *Spinors and the Dirac operator on hypersurfaces.II. The spheres as an example* (in preparation).

[6] S.Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vol.1, Interscience, New York, 1963.

[7] L. Dabrowski and A.Trautman, *Spinor structures on spheres and projective spaces*, J.Math.Phys. **27**(1986)2022-2028.

[8] A.Lichnerowicz, *Spineurs harmoniques*, C.R. Acad.Sci.Paris A-B **257**(1963)7-9.

ALGEBRAIC CONSTRUCTION OF SPIN STRUCTURES ON HOMOGENEOUS SPACES

VLADIMIR LYAKHOVSKY
Theoretical Department, Institute of Physics, St.Petersburg University, St.Petersburg, 198904, Russia

Abstract. We consider the manyfold B to be the homogeneous space G/H, where G and H are the reductive pair of compact Lie groups. The spin structures on B can be costructed by purely algebraic methods based on the lattice theory.The efficiency of the proposed algorithm is demonstrated on examples.

1. Introduction

The necessary and sufficient condition of existence of spin structures on the oriented Riemanian manifold M is the trivialization of its second Schtiffel-Uitney class (Borel,1959). In practice the problem of evaluating this object is very difficult. It can be totally solved only for the manifolds with quite simple topology or of small dimensions (Avis,1979;Petry,1984)

In multidimensional models for elementary particles interactions it is highly important to know the total classification of spinor fields on the homogeneous spaces of the type $B \approx G/H$ where G and H are the symmetry groups. Thus we have the main fibre bundle (H, G, B) with the structure group H and the base B. Let $(SO(n), E, B)$ be the fibre bundle of the orthonormal frames over the oriented Riemann space B ($n = \dim B$). For the subgroup $K \subset SO(n)$ and K' – the lift of K in Spin(n) the bundle $(SO(n), E, B)$ has the so called reduction $\rho : (K, R, B) \to (SO(n), E, B)$. The spin structure also has the reduction ρ' to (K', R', B). In the corresponding part of the strict homotopic sequence $\pi_1(K) \xrightarrow{\tau} \pi_1(R) \to \pi_1(B)$ the group $\pi_1(K)$ contains the subgroup $\pi_1(K')$.One can prove that for connected space R' the criterium (Baum,1981) must be generalized as follows: the spin structure on B exists iff $\mathrm{Ker}(\tau) \subset \pi_1(K')$ and $N \subset \pi_1(R)$; here the subgroup N is normal in $\pi_1(R)$ and has the properties $\pi_1(R)/N \approx Z_2$, $N \bigcap \tau(\pi_1(K)) = \tau(\pi_1(K'))$. The disconnected R' has the form $R \times Z_2$ and appear only in the case when $K' \approx K \times Z_2$. So the trivial spin structure always exists when K' is disconnected.

In this report we propose the algebraic method of the explicit construction of spin structure on homogeneous spaces $B \approx G/H$ for reductive pair of connected compact Lie groups G and H. It is based on the lattice theory for Lie groups. We shall demonstrate its efficiency on examples that have physical applications.

2. The group structure revised

Consider the fibre bundle of orthonormal frames $(SO(n), E, B)$ on the orientable Riemann manifold $B : SO(n) \to E \to B$, with $n = \dim B$ and the universal covering $\omega : \mathrm{Spin}(n) \to SO(n)$. The existence of the spin structure on B means that the bundle morphism ξ ,

$$\xi : (\mathrm{Spin}(n), E', B) \to (SO(n), E, B)$$

exists and has the following restrictions: $\xi_{\downarrow B} = \mathrm{id}, \xi_{\downarrow \mathrm{Spin}} = \omega$.

In our case $B \approx G/H$ and (G, H) is the reductive pair of the connected compact Lie groups. Let $\mathbf{g}$ and $\mathbf{h}$ be the corresponding Lie algebras and $V_{\mathbf{g}}, V_{\mathbf{h}}$ – their vector spaces. We shall use the direct sum decomposition

$$V_{\mathbf{g}} = V_{\mathbf{h}} \oplus V_{\mathbf{d}}, \tag{1}$$

where $V_{\mathbf{d}}$ is the space of the representation D of the group H induced by the adjoint representation of G,

$$\mathrm{Ad}(G)_{\downarrow H} \approx \mathrm{Ad}(H) \oplus D(H). \tag{2}$$

For its kernel and image the following notations will be used $K \equiv \mathrm{Im} D \subset SO(n)$, $N \equiv \mathrm{Ker} D \subset H$. The important object for us is the reduction ρ of the main fibre bundle

$$\rho : (K, R, B) \longrightarrow (SO(n), E, B),$$

where R is the factorspace G/N. For the morphism ρ the spin structure on B means the existence of such a morphism η and a reduction ρ' that the following diagram is commutative (Dabrovsky,1986)

$$\begin{array}{ccc} (\mathrm{Spin}(n), E', B) & \xrightarrow{\xi} & (SO(n), E, B) \\ \rho' \uparrow & & \rho \uparrow \\ (K', R', B) & \xrightarrow{\eta} & (K, R, B) \end{array} \tag{3}$$

Here K' is the coimage of K for $\omega : K' \equiv \mathrm{Coim}\, \omega(K) \subset \mathrm{Spin}(n)$.

First let us show that N is normal in G and thus R is the factor group (just as $K \approx H/N$). Suppose that N_0 , the connected component of N , is a nontrivial Lie group. Its algebra $\mathbf{n}_0$ commutes with the space $V_{\mathbf{d}}$ and is an ideal in $\mathbf{h}$. So $\mathbf{n}_0$ is an ideal in $\mathbf{g}$ (see the decomposition (1)). For connected group G this means that its subgroup N_0 is normal. Thus in the definition of the factor space B we can use the factor groups G/N_0 and H/N_0.

Having this in mind consider the case of discreat group N. It is central in H because H is connected: $N \subset Z(H)$. Consider the morphisms $\mathrm{Ad}(H)$, $\mathrm{Ad}(G)_{\downarrow H}$, D and their kernels: $\mathrm{Ker}(\mathrm{Ad}(H)) = Z(H)$, $\mathrm{Ker}(\mathrm{Ad}(G)_{\downarrow H}) \subset Z(G)$, $\mathrm{Ker} D = N$. The relation (2) imposes the condition

$$\mathrm{Ker}(\mathrm{Ad}(G)_{\downarrow H}) = Z(H) \cap N = N. \tag{4}$$

The result is that N is the central subgroup of $G : N \subset Z(G)$.

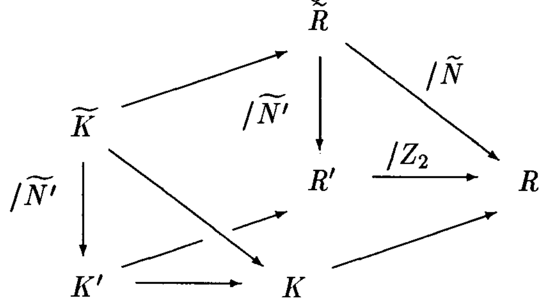
We have proved that in the definition of the factorspace B we can factorize out the kernel of the representation D and reformulate B in terms of the factor groups $R \equiv G/N$ and $K \equiv H/N$,

$$B = G/H \approx R/K. \tag{5}$$

3. Necessary and sufficient conditions

Let the spin structure (η, ρ') exist (see the diagram (3)). The morphism $\eta : R' \longrightarrow R$ is the twofold covering of the group R. So R' is a group. If R' is connected it belongs to the class of locally isomorphic groups with Lie algebra $\mathbf{r}$ and can be realized as a factor of the universal covering group $\widetilde{R}$, $R' \approx \widetilde{R}/\widetilde{N}'$. Consider the coimages $\widetilde{N}$ and $\widetilde{K}$ of the groups e and K in $\widetilde{R}$. Then the following prism is commutative

Fig. 1.



The conclusion is: when the spin structure exists and R' is connected, in the kernel $\widetilde{N}$ of the universal covering $\widetilde{R} \longrightarrow R$ there exists a subgroup $\widetilde{N}' \subset \widetilde{N}$ such that

$$R' \approx \widetilde{R}/\widetilde{N}', \quad K' \approx \widetilde{K}/\widetilde{N}' \text{ and } Z_2 \approx \widetilde{N}/\widetilde{N}'. \tag{6}$$

Note that the covering $\widetilde{K} \longrightarrow K$ is not universal.

When R' is not connected it is equivalent to the direct product $R' \approx R \times Z_2$ and for K' the same is true: $K' \approx K \times Z_2$. So the spin structure $(K \times Z_2, R \times Z_2, B) \xrightarrow{\eta} (K, R, B)$ is trivial.

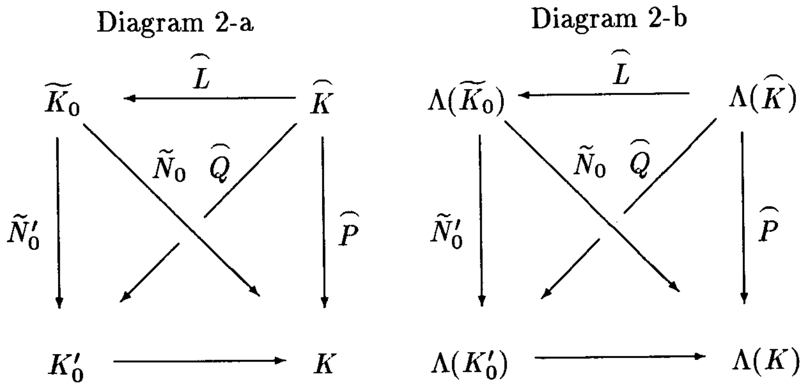
When the existence of the spin structure is in question one can still put the problem whether the bundle (K', R', B) and morphism η exist with K' as a coimage of ω. Now it is clear that to solve this problem one must find the universal covering group $\tilde{R}$ and the corresponding kernel $\tilde{N}$ containing a subgroup $\tilde{N}'$ such that the commutative diagram (Fig. 1) holds. In the case of disconnected $K' \approx K \times Z_2$ the disconnected $R' \approx R \times Z_2$ plays the necessary role and the trivial spin structure on B exists. The number of inequivalent spin structures is just the number of inequivalent subgroups $\tilde{N}'$ with the described properties (7) plus the trivial one in case of disconnected K' .

4. Algebraic construction

The problem is how to construct the subgroups $\widetilde{N'}$. It is sufficient to consider only the front triangle of the diagram (Fig. 1), all the necessary kernels play there. Now we shall expose the algebraic algorithm which solvs the problem. It is based on the lattice theory (Loos,1985;Adams1979).

The group $\widetilde{K}$ must not be connected. Let $\widetilde{K_0}$ be its component of unit and consider the corresponding subgroups of $\tilde{N}$ and $\widetilde{N'}$: $\tilde{N_0} \equiv \tilde{N} \cap \widetilde{K_0}$, $\widetilde{N_0'} \equiv \widetilde{N'} \cap \widetilde{K_0}$. The group K can obviously be written as the factor group $\widetilde{K_0}/\widetilde{N_0}$. Let K_0' be the connected component of K' and $\widehat{K}$ – the universal covering for $\widetilde{K_0}, K_0'$ and K. Then together with the front triangle in Fig.1 we obtain the rectangular commutative diagram, where $\widehat{Q}$ is the kernel of the covering $\widehat{K} \longrightarrow K_0'$.

Fig. 2.



Now the group $\widetilde{N_0}$ is fixed by the factorization

$$\tilde{N}_0 = \widehat{P} \;/\; \widehat{L} \tag{7}$$

The group K is compact and connected so one can look for its maximal torus $T(K)$. Construct all its coimages, contained in the Diagram 2-a . The obtained abelian groups $T(\widetilde{K_0}), T(\widehat{K}), T(K_0')$ together with the initial torus $T(K)$ form the commutative diagram that is just a copy of 2-a. Now take the Lie algebras $\mathbf{t}$ of these groups and consider the corresponding unit lattices $\Lambda \equiv exp^{-1}(l) \subset \mathbf{t}$. Once again one obtaines the diagram induced by the Diagram 2-a .

On the diagram 2-b all the morphisms are injections and the discreat groups indicated on the diagram are the corresponding factors: $\widehat{P} \approx \Lambda(K)/\Lambda(\widehat{K})$, etc. The lattices $\Lambda(K)$ and $\Lambda(\widehat{K})$ are known. It is easy to find $\Lambda(\widetilde{K_0})$. Consider the maximal tori $T(R)$ and $T(\tilde{R})$ and the corresponding unit lattices $\Lambda(R)$ and $\Lambda(\tilde{R})$ The space $V_{\mathbf{t}(K)}$ of algebra $\mathbf{t}(K)$ is the subspace of $V_{\mathbf{t}(\tilde{R})}$. The necessary lattice is obtained by the interseption:

$$\Lambda(\widetilde{K_0}) \approx \Lambda(\tilde{R}) \bigcap V_{\mathbf{t}(K)}. \tag{8}$$

Now the groups $\widehat{L}$ and $\widetilde{N_0}$ are fixed:

$$\begin{aligned} \widehat{L} &\approx \Lambda(\widetilde{K_0})/\Lambda(\widehat{K}), \\ \widetilde{N_0} &\approx \Lambda(K)/\Lambda(\widetilde{K_0}). \end{aligned} \tag{9}$$

So the only object that was not yet defined is the lattice $\Lambda(K_0')$. It can be identified examining the structure of the corresponding sublattice in $\Lambda(\mathrm{Spin}(n))$. For the semisimple groups K and R we propose an easy way to solve this remaining problem.

Consider the complexification $D_{\mathbf{C}}$ of the exact representation $D : K \longrightarrow SO(n)$ and the corresponding complex algebras $\mathbf{k}_{\mathbf{C}}$ and $\mathbf{r}_{\mathbf{C}}$. The decomposition of the type (1) is still valid. $V_{\mathbf{r}_{\mathbf{C}}} \approx V_{\mathbf{k}_{\mathbf{C}}} \oplus V_{\mathbf{d}_{\mathbf{C}}}$. Take the projection $\{\gamma\}$ of the root system $\{\gamma_{\mathbf{r}}\}$ of $\mathbf{r}_{\mathbf{C}}$ of the subspace $\mathbf{k}_{\mathbf{C}}$ and eliminate the subsystem $\{\gamma_{\mathbf{k}}\}$ (the roots of $\mathbf{k}_{\mathbf{C}}$). The obtained set of vectors $\{\beta\} \equiv \{\gamma\} \setminus \{\gamma_{\mathbf{k}}\}$ is the weight diagram of the representation $\mathbf{D}_{\mathbf{C}}$. Let $\{\beta\}^+$ be the subset of positive weights.

When for $b \in T(K)$ the rotation $D(b)$ is lifted to Spin(n) we obtain the product of the operators (Adams,1979) :

$$\exp \pm \pi\beta(b)\widehat{e_i}\widehat{e_k} \tag{10}$$

where $\widehat{e_i}, \widehat{e_k}$ are the Clifford algebra generators corresponding to the plane $V^{\beta} \oplus V^{-\beta}$. Thus $D(b)$ can enter the lattice $\Lambda(K_0')$ only when the sum of $\beta(b)$ for all the positive weights is even.

The lattice $\Lambda(K_0')$ thus constructed,

$$\Lambda(K_0') = \{b \in \Lambda(K) \mid \sum_{\beta \in \{\beta\}^+} \beta(b) = 2\mathbf{Z}\}, \qquad (11)$$

makes it possible to calculate the group $\widehat{Q} = \Lambda(K_0')/\Lambda(\widehat{K})$ and together with $\widehat{L}$ defined earlier (9) one can finally calculate $\widetilde{N_0}' = \widehat{Q} \,/\, \widehat{L}$. Knowing $\widetilde{N_0}'$ and the relations between $\widetilde{R}$ and $\widetilde{K}$ we can easily reestablish $\widetilde{N}'$, obtain the groups K' and R' (see (6)) and thus conclude the construction of the fibre bundle that define the spin structure.

First one must find the group $K_0' = \widehat{K} \,/\, \widehat{Q}$. This almost immediately gives the structure group K' .

$$\begin{aligned} K' &\approx K_0' \times Z_2 \approx K \times Z_2 && \text{for } \Lambda(K_0') = \Lambda(K), \\ K' &\approx K_0' && \text{for } \Lambda(K)/\Lambda(K_0') \approx Z_2. \end{aligned} \qquad (12)$$

Then one must search the subgroups $\widetilde{N}'$ in $\widetilde{N}$ with the properties $\widetilde{N}' \cap \widetilde{K_0} = \widetilde{N_0}'$, $\widetilde{N}/\widetilde{N}' = Z_2$. Every class of equivalent subgroups $\widetilde{N}'$ defines the spin structure (R', K', B) with $R' \approx \widetilde{R}/\widetilde{N}'$. In case of disconnected K' (see (12)) the trivial spin structure with $R' \approx R \times Z_2$ also exists.

5. Examples

5.1 Let us start with the factorspace where it is quite difficult to use the ordinary topological methods. Consider the space $B \approx G/H \approx SO(5)/SO(3)_{\mathrm{sp}}$, where the subgroup $SO(3)_{\mathrm{sp}}$ is the image of the special injection. For this injection the fundamental representations of $SO(5)$ treated as the representations of its $SO(3)_{\mathrm{sp}}$ subgroup remain irreducible (Lyakhovsky,1986): $(5)_{\downarrow SO(3)} = (5)$, $(4)_{\downarrow SO(3)} = (4)$.

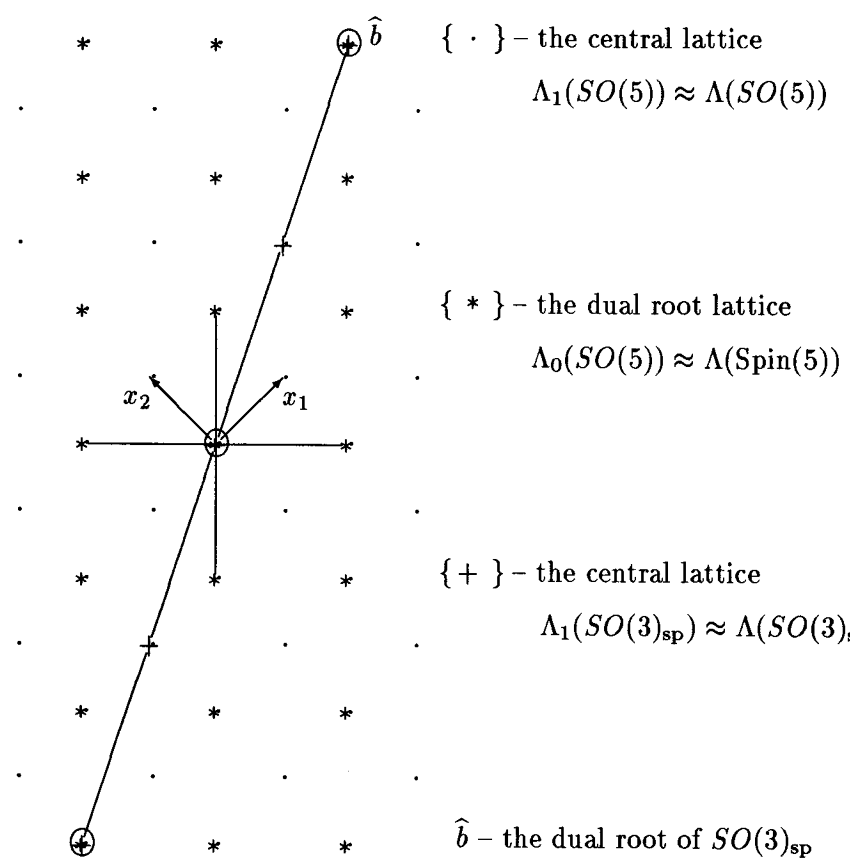
The 7-dimensional space V_{d} in the decomposition (1) is also irreducible. Here the center $Z(G)$ is trivial. So we have $R = SO(5)$, $K = SO(3)_{\mathrm{sp}}$ and, obviously, $\widetilde{R} = \mathrm{Spin}(5)$, $\widehat{K} = SU(2)$. The groups $\widetilde{N}$ and $\widehat{D}$ coinside: $\widetilde{N} \approx \widehat{D} = Z_2$. Now it is necessary to draw the lattices of the groups under consideration in the standard e-frame (see Fig. 3).

Now it is easy to check that the sum $\sum \beta_i(b) = 6\lambda(b)$. is even for every element of $\Lambda(SO(3)_{\mathrm{sp}})$. This means that $\Lambda(K_0') \approx \Lambda(K)$. So the group K' is disconnected: $K' = SO(3) \times \mathbf{Z}_2$, the group $\widehat{Q}$ coinsides with $\widehat{P} : \widehat{Q} \approx \widehat{P} = \mathbf{Z}_2$. So in this case only the trivial spin structure with $R' \approx SO(5) \times \mathbf{Z}_2$ exists.

5.2 For regular injections $H \longrightarrow G$ the proposed algorithm gives the results valid for the whole ansemble of similar factorspaces.

Let $G = SU(n)$ and $H = (SU(p) \times SU(q) \times U(1))/\mathbf{Z}_u$,where $p + q = n$ and u is the minimal proportional to p and q. Here all the necessary lattices

Fig. 3.



are well known (Adams,1979), so we shall expose only the final result. One must consider separately three types of factorspaces.

- There is no spin structure when numbers p and q have different pairity.
- When both p and q are even the only possible spin structure is the trivial one with the total space $R' \approx R \times \mathbf{Z}_2 = (SU(n)/\mathbf{Z}_n) \times \mathbf{Z}_2$.
- For p and q odd the unic nontrivial spin structure exist with

$$\begin{aligned} R' &= SU(n) \,/\, \mathbf{Z}_{n/2}, \\ K' &= (SU(p) \times SU(q) \times U(1)) \,/\, \mathbf{Z}_u \,/\, \mathbf{Z}_{n/2}. \end{aligned}$$

5.3 In multidimensional quantum gauge theories the study of symmetry breaking due to the nontrivial topological configurations leads to the investigation of the model spaces of the type $M^4 \times S^m \,/\, \mathbf{Z}_p$ (Hosotani,1983). In these theories the spinor fields on $S^m \,/\, \mathbf{Z}_p$ can be treated as the $\mathbf{Z}_p$ -invariant spinor fields on S^m. The spinor fields as the global sections of the bundle associated to the pricipal Spin(m)-fibre bundle must retain the initial symmetry. It is known that for $m = 2n+1$ the $SO(2n+2)$ symmetry is broken to $SU(n+1)$ when the factorization by $\mathbf{Z}_p$ $(p \neq 2)$ is produced (Lyakhovsky,1991). So we are forsed to consider S^m as the $SU(n+1)/SU(n)$ factorspace rather then the $SO(2n+2)/SO(2n+1)$.

Let $m = 3$ and take the triple $(e, SU(2), S^3)$ as the initial bundle. Here we obviously have $K = e$ and $R = SU(2) \approx \widetilde{R}$. Thus the group $\widetilde{K_0} = e$ is trivial. The diagram (2) shows that $\widetilde{N_0}'$ and K_0' must also be trivial and $K_0' \approx K_0 = e$. So we have only the trivial spin structure: $K' \approx \mathbf{Z}_2$ and $R' \approx SU(2) \times \mathbf{Z}_2$. The analysis of the injection $R' \longrightarrow SO(4)$ shows that K' is the diagonal subgroup of R'. As a result on the space $(SU(2) \times \mathbf{Z}_2)/\mathbf{Z}_2^{\mathrm{diag}}$ one must study the spinor fields Ψ corresponding to the exact 2-dimensional representation of $\mathbf{Z}_2^{\mathrm{diag}}$. The harmonic expansion for the components Ψ_j of Ψ will contain only those representations of $SU(2) \times \mathbf{Z}_2$ that have the exact 1-dimensional reduction to $\mathbf{Z}_2^{\mathrm{diag}}$. These representations $D^{(l)}(SU(2)) \otimes B(\mathbf{Z}_2^{\mathrm{diag}})$ will contain the exact subrepresentations B for the integer l and the trivial for the halfinteger ones. As a result the basis of the harmonic expansion for Ψ_j is formed by the matrix elements of the full spectrum of representations of $SU(2)$.

Acknowledgements

It is the pleasure for the author to express his gratitude for warm hospitality of the scientists of the Institute of Theoretical Physics in Wroclaw.

References

Borel, A. and Hirzebruch, F.: 1959, *Amer. Journ. of Math.* **81**, 315
Avis, S.J.and Isham, C.J.: 1979, in Quantum Field Theory and Fibre Bundles in General Space-Time, ed(s)., *Recent Developements in Gravitation*, Levy M.,Deser S., Plenum Press:London
Dabrovsky, L. and Trautman, A.: 1986, *Journ. Math. Phys.* **27**, 2022
Petry, H.R.: 1984, *Spin structures on Lorentz manifolds*, ISAS-44/84/EP: Triest
Baum, H.: 1981, *Spin-Structuren und Dirac-Operatoren über Pseudoriemannscher Mannigfaltigkeiten*, Teubner:Leipzig
Loos, O.: 1985, *Symmetric Spaces*, Science:Moscow
Adams, J.: 1979, *Lectures on Lie Groups*, Science:Moscow
Lyakhovsky, V.D. and Vassilevich, D.V.: 1986, *Teor. Matem. Phys.* **66(3)**, 350
Hosotani, Y.: 1983, *Phys. Lett.* **B126**, 309
Lyakhovsky, V.D. and Shtikov, N.N.: 1991, *Yadernaya Physica* **54(8)**, 595

THE KUMMER CONFIGURATION AND THE GEOMETRY OF MAJORANA SPINORS

GARY W. GIBBONS
DAMTP
University of Cambridge
Cambridge CB3 9EW
UK

Abstract. In this article I show how the properties of Majorana spinors in four spacetime dimensions may be understood in terms of the real projective geometry of ordinary three-dimensional space. They may be viewed as points in projective space equipped with a linear line congruence. The discrete group generated by the γ-matrices may be viewed as the automorphism group of Kummer's configuration 16_6. As an application of line geometry which I develop I show how the skies of events of 2 + 1-dimensional Minkowski spacetime correspond to the lines of a linear line complex in projective three space.

1. Introduction

The real Clifford algebra Cliff$(1,3)$ generated by the relations:

$$\gamma_0^2 = -1, \quad \gamma_i^2 = 1, \quad i = 1,2,3 \tag{1.1}$$

is isomorphic to the algebra $R(4)$ of real four by four matrices and therefore admits a representation in which the gamma matrices γ_μ are real four by four matrices and act on a real four dimensional vector space whose elements are called Majorana spinors (see e.g. Dabrowski, 1988). In this representation the charge conjugation matrix C, which satisfies $C\gamma_\mu C^{-1} = -\gamma_\mu t$, γ_5 and γ_0 may be taken to be anti-symmetric and the γ_i to be symmetric.

Viewed projectively one may think of Majorana spinors as points in real projective space $P_3(R)$. This fact allows one to relate the projective geometry of ordinary 3-space and spinor algebra. One aspect of this relationship is that one may identify the 32 element group G_{32} generated by the gamma matrices as the 2-fold cover of the automorphism group G_{16} of Kummer's self-dual configuration 16_6 consisting of 16 points and 16 planes in $P_3(R)$ such that every plane contains 6 points and 6 planes pass through every point (Hudson, 1905). This fact was known to Eddington (Zariski 1932; Eddington,1935,1936) and others in the 30's: it arose in his "Fundamental Theory". The numerological properties of what he called "E-numbers" are in fact just the properties of this Clifford algebra. This talk is intended as an expository account of these matters. Much of the material is extremely ancient if not especially well known nowadays and no particular originality is claimed, indeed, much of what I have to say may be found in the interesting book by Paërls but not in quite the way I shall describe it. The reader may decide for him or herself whether or not the material should remain in the decent obscurity in which it has been left hitherto. My own motivation was not to resurrect Eddington's ghost but rather to develop some geometric intuition for Majorana spinors analogous to that one has for two-component Weyl spinors, considered projectively as points on the complex projective line $P_1(C)$ which has been so successfully exploited by Roger Penrose.

Of course the action of the group G_{16} may be extended complex projective space $P_3(C)$ but something is lost in the way of visualizability and moreover the limitation to real numbers serves to illuminate the differences between the Clifford algebra Cliff(1,3) defined by (1.1) and Cliff(3,1) defined by:

$$\gamma_0^2 = 1, \quad \gamma_i^2 = -1, \quad i = 1,2,3, \tag{1.2}$$

which as a real algebra is isomorphic with the algebra of two by two quaternion valued matrices $H(2)$. Now the group Γ of invertible elements of the Clifford algebra Cliff(1,3) defined by (1.1) is isomorphic to $GL(4,R)$ which acts on $P_3(R)$ as $PS(4,R)$ the natural group associated to the *Projective Geometry* of ordinary three dimensional space. This group is isomorphic to $PSO(3,3)$ the group of linear transformations of six-dimensional space preserving a metric of signature $(3,3)$,[1] and is the basis of Plücker's *Line Geometry* in which lines in ordinary space are associated to null rays in $R^{3,3}$. By contrast the Clifford algebra Cliff(3,1) defined by (1.2) leads to the group $PSO(4,2)$ associated to the *Conformal geometry* of Minkowski spacetime. The analogue of Plücker's construction is Lie's *Sphere Geometry* in which spheres in ordinary space are associated to null rays in $R^{4,2}$. Lie realised that there is no distinction between Line Geometry and Sphere Geometry if one works over the complex numbers and this idea is at the heart of Penrose's Twistor Theory. The passage between these two view points is essentially no more than one of endowing the real four dimensional vector space of Majorana spinors with a complex structure which allows one to identify it with the complex two-dimensional space of Weyl spinors. Explicitly the complex structure is given by the γ-matrix γ_5 whose square is minus one no matter which signature is chosen. Nevertheless for some purposes, supersymmetry and supergravity for example, it is much more convenient to work over the reals and in particular it is often most useful to use Majorana spinors. Indeed it has recently been pointed out by DeWitt and Carlip

[1] A metric of signature (p,p) is sometimes called neutral. I prefer the adjective "Kleinian" by analogy with "Riemannian" for signature (p,0) or (0,p) and "Lorentzian" for signature (p,1) or (1,p). The flat model spaces I like to call Plücker space, Euclidean space and Minkowski space repectively.

that the under certain global conditions the choice of signature may, in a certain sence, actually have have physical consequences.For these reasons I shall restrict myself to real projective geometry.

While working over this material I became aware of various other applications of the projective geometry to be described below to physics. One involves linear line complexes to describe the causal structure of 3-dimensional Minkowski spacetime and will be sketched below. Others relate to the geometry of De-Sitter and Anti-De-Sitter spacetime, the Petrov classification of curvature tensors and even to mechanics.

2. Projective Geometry

The fix notation and terminology it will be useful to recollect some elementary geometrical ideas. Points p in $P_3(R)$ corresponds to rays in in some real four dimensional 4-dimensional vector space V with homogeneous co-ordinates p^α ,$\alpha = 0,1,2,3$. Planes π in $P_3(R)$ correspond to rays in the dual vector space V^* and also have four homogeneous co-ordinates π_α ,$\alpha = 0,1,2,3$. The point p lies in the plane π if and only if:

$$p^\alpha \pi_\alpha = 0. \tag{2.1}$$

Linear maps from V to V are called *collineations*. They induce projective transformations of $P_3(R)$ taking points to points, lines to lines and planes to planes. In homogeneous co-ordinates they have components: $L^\alpha{}_\beta$ Linear maps from V to V^* are called *correlations*. They induce interchanges of points and planes in $P_3(R)$. In homogeneous co-ordinates they have components: $L_{\alpha\beta}$. A correlation is *contact preserving* if it preserves the property of a point lying in a plane. Analytically:

$$p^\alpha \tilde{\pi}_\alpha \propto \tilde{p}^\alpha \pi_\alpha \tag{2.2}$$

For such correlations it follows that $L_{\alpha\beta}$ must either be symmetric or anti-symmetric.

If $L_{\alpha\beta}$ is symmetric the correlation is called a *polarity* or a Legendre transformation and the plane $\pi(p)_\alpha = L_{\alpha\beta}x^\beta$ associated to a point x contains the point x and is tangent to the quadric :

$$L_{\alpha\beta}p^\alpha p^\beta = 0. \tag{2.3}$$

The point p and the plane π are said to be *pole* and *polar* respectively to the quadric. It is clear that there is an equivalence beween quadrics, polarities and real symmetric four by four matrices.

If on the other hand $L_{\alpha\beta}$is skew symmetric the correlation is called a *null correlation*. It then takes a point p to a plane $\pi(p)$ passing through the point. It is clear that there is an equivalence beween null correlations and real four by four skew-symmetric matrices. If the null correlation is non-singular (i.e. invertible) it also defines a symplectic structure on the vector space V which descends to a contact structure on $P_3(R)$. The word "symplectic" was coined by Weyl to replace the confusing use of the word "complex" applied to the group associated to a "linear line complex". This will be defined shortly.

Two points p, q define a unique line l passing through them which corresponds to a simple bi-vector $l^{\alpha\beta}$ in $V \wedge V$

$$l^{\alpha\beta} = p^{[\alpha} q^{\beta]}. \tag{2.4}$$

The space of bi-vectors $V \wedge V$ (i.e. skew-symmetric second rank tensors) is six-dimensional and carries a natural Kleinian metric of signature (3,3) given by the alternating symbol $\epsilon_{\alpha\beta\mu\nu}$. The simple bi-vectors $l^{\alpha\beta}$ satisfy :

$$l^{\alpha\beta} l^{\mu\nu} \epsilon_{\alpha\beta\mu\nu} = 0 \tag{2.5}$$

and therefore constitute the four-dimensional manifold of null rays in $R^{3,3}$ which has topology $(S^2 \times S^2)/\pm 1$. the set of lines inherits a conformal structure because two lines l_1 and l_2 intersect if and only if :

$$\frac{1}{2} l_1^{\alpha\beta} l_2^{\mu\nu} \epsilon_{\alpha\beta\mu\nu} \equiv g_{AB} l_1^A l_2^B = 0, \tag{2.6}$$

where $A = 1,2,3,4,5,6$ and $l^1 = l^{01}$ etc, and g_{AB} has signature (3,3). In what follows, we shall employ the convention that lower indices on bi-vectors have been lower using the alternating symbol. We are now in a position to return to *line complexes* which are 3-dimensional families of lines in $P_3(R)$ ("complex" is here being used to denote a collection of lines which is "plaited together"). *A linear line complex* is the intersection of a hyperplane in $R^{3,3}$ with the Plücker Quadric and thus is of the form:

$$C_{\alpha\beta} l^{\alpha\beta} = 0. \tag{2.7}$$

Clearly associated with every null correlation is a linear line complex and conversely and both are associated to a direction in $R^{3,3}$. A *singular line complex*,sometimes called *special* is associated with a null ray in $R^{3,3}$ and consists of those lines which intersect a fixed line.

A non-singular or non-special linear line complex has the property that every line of the complex passing through a fixed point p in $P_3(R)$ lies in a fixed plane $\pi(p)$ containing p and conversely every line of the complex lying in a given plane π passes through a fixed point $p(\pi)$ in the given plane π. The point p^α and plane π_α are related by

$$\pi_\alpha = C_{\alpha\beta} p^\beta \tag{2.8}$$

The point and plane are sometimes referred to as pole and polar repectively. A more detailed description will be given later. The subgroup of $SO(3,3)$ leaving invariant the 6-vector C_A is the Anti-De-Sitter group $SO(3,2)$ which is double covered by what is usually called, following Weyl's suggestion, the real symplectic group $Sp(4,R)$. We shall return line complexes later when we look at causality in 3-dimensional Minkowski spacetime.

A *quadratic line complex* is the intersection of a quadric in $R^{3,3}$ with the Plücker Quadric and thus is of the form:

$$Q_{\alpha\beta\mu\nu} l^{\alpha\beta} l^{\mu\nu} = 0. \tag{2.9}$$

It has the property that the lines passing through a point p in $P_3(R)$ lie on a quadratic cone. Points at which this cone degenerates to a pair of planes are called *singular points* and they lie on the *singular surface* of the quadratic line complex. These surfaces are called *Kummer surfaces* and turn out to be quartic surfaces. Over the complex numbers they possess 16 double points or *nodes* and 16 singular tangent planes or *tropes*. The nodes and tropes make up a *Kummer configuration*. An example of a real Kummer surface which arises in physics is Fresnel's Wave Surface. The generic quadratic line complex determines a symmetric tensor Q_{AB} on $R^{3,3}$ which may be diagonalized over the reals with respect the Kleinian metric g_{AB} to give a privileged orthonormal sextad for $R^{3,3}$. This will be used later.

Finally we recall the a 2-dimensional family of lines is called a *line congruence* and that a *linear line congruence* is the intersection of the Plücker quadric with 4-dimensional plane in $R^{3,3}$.

3. Spinors in Six Kleinian Dimensions

One may if one wishes pass to the projective space $P_5(R)$ and view the space of lines in $P_3(R)$ as the Plücker quadric in $P_5(R)$ but in what follows it is more useful to remain in $R^{3,3}$. Indeed the key idea (going back at least as far as Cartan (see Paerls)) is to consider the Clifford algebra Cliff(3,3) of $R^{3,3}$. This is isomorphic to the algebra of real eight by eight matrices $R(8)$ and is generated by six real eight by eight matrices Γ_A satisfying:

$$\Gamma_A \Gamma_B + \Gamma_B \Gamma_A = 2 g_{AB} \tag{3.1}$$

which act on an 8-dimensional real vector space S whose elements $\Psi^a$are "Majorana spinors for $SO(3,3)$". Since the $SO(3,3)$-invariant volume element $\Gamma_7 \equiv \Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5$ has square unity, the real vector space of Majorana spinors splits as the direct sum of two real four dimensional vector spaces which may naturally be regarded as the duals of one-another. In fact one vector space summand maybe taken as the original vector space V, rays in which are associated with points in projective space $P_3(R)$ and the other as its dual V^*, rays in which correspond to planes in $P_3(R)$, i.e. :

$$S = V \oplus V^*. \tag{3.2}$$

In other words one may think of a Majorana spinor for $SO(3,3)$, Ψ^a, as a pair (p^α, π_α) consisting of a point p^α and a plane π_α in ordinary (projective) 3-space. Explicitly:

$$\Psi = \begin{pmatrix} p^\alpha \\ \pi_\beta \end{pmatrix}. \tag{3.3}$$

The $SO(3,3)$ invariant Dirac adjoint spinor $\overline{\Psi}_a$ is given by

$$\overline{\Psi} = (\, \pi_\alpha \quad p^\alpha \,). \tag{3.4}$$

So the point lies in the plane if and only if:

$$\overline{\Psi}\Psi = 0. \tag{3.5}$$

With these notational conventions we represent points x^A in $R^{3,3}$, i.e. bivectors $x^{\alpha\beta}$ as elements of the Clifford algebra Cliff$(3,3)$ in terms of the eight by eight dimensional matrices:

$$\not{x} \equiv x^A \Gamma_A = \begin{pmatrix} 0 & x^{\alpha\beta} \\ -\frac{1}{2}\epsilon_{\alpha\beta\mu\nu}x^{\mu\nu} & 0 \end{pmatrix} \tag{3.6}$$

As a check the reader may wish to verify that use of the identity:

$$\frac{1}{2}\epsilon_{\rho\beta\mu\nu}x^{\mu\nu}x^{\beta\sigma} = -\frac{1}{8}\epsilon_{\alpha\beta\mu\nu}x^{\mu\nu}x^{\alpha\beta}\delta^\sigma_\rho \tag{3.7}$$

yield the basic identity:

$$\not{x}\not{x} = g_{AB}x^A x^B \mathbb{1} \tag{3.8}$$

where:

$$\mathbb{1} = \begin{pmatrix} \delta^\alpha{}_\beta & 0 \\ 0 & \delta^\mu{}_\nu \end{pmatrix}. \tag{3.9}$$

Since the $SO(3,3)$ invariant product Γ_7 acts on the space of point-plane pairs as

$$\Gamma_7\Psi = \begin{pmatrix} p^\alpha \\ -\pi_\beta \end{pmatrix}. \tag{3.10}$$

from a geometrical point of view it has no effect. On the other hand $R^{3,3}$ acts on the space S of point-plane pairs as null correlations. One has the following

Proposition. *If*

$$\not{x}\Psi = 0,$$

then the 6-vector x^A must be lightlike, the bi-vector $x^{\alpha\beta}$ must be simple and represents a line in $P_3(R)$, moreover if

$$\Psi = \begin{pmatrix} p^\alpha \\ \pi_\beta \end{pmatrix}.$$

the point p^α lies in the plane π_α and the line $x^{\alpha\beta}$ lies in the plane π_α and passes through the point p^α.

4. Reflection Groups and Their Covers

If we choose an orthonormal basis for $R^{3,3}$, for example that invariantlyassociated to a generic quadratic line complex, we may consider the group of reflections with respect to these basis vectors. Each reflection is an involution and all reflections commute so it is an abelian group G_{32} with 2^6 elements. According to the usual procedure in Clifford algebra we construct a non-abelian double cover G_{64} of this this group by multiplying the six γ-matrices Γ_A. The group G_{64} acts on the eight dimensional spinor space S. The commutator group of G_{64} consists of the elements $\pm\mathbb{1}$ and its quotient by the commutator is of course just the abelian group of reflections G_{32}. Projectively speaking the signs are irrelevant and it is G_{32} which interests us. As mentioned above the six γ-matrices Γ_A act as six null collineations. There are $6 \times 5/2!$ non-trivial ways of multiplying together two γ-matrices. The resulting 15 matrices $\Gamma_{[A}\Gamma_{B]}$ commute with Γ_7 and act on $P_3(R)$ as collineations. There are $6 \times 5 \times 4/3! = 20$ non-trivial products $\Gamma_{[A}\Gamma_B\Gamma_{C]}$ which anticommute with Γ_7 and act as correlations but since the off-diagonal blocks are symmetric rather than anti-symmetric like the off-diagonal blocks of the 6 Γ_A we obtain polarities with associated quadrics. Proceeding in the same way one obtains 15 collineations $\Gamma_{[A}\Gamma_B\Gamma_C\Gamma_{D]}$ and 6 correlations $\Gamma_{[A}\Gamma_B\Gamma_C\Gamma_D\Gamma_{E]}$. Of course there is just one product of the form $\Gamma_{[A}\Gamma_B\Gamma_C\Gamma_D\Gamma_E\Gamma_{F]}$ and this just Γ_7. It would seem therefore that we arrive at $6 + 20 + 6 = 32$ correlations and (including the unit element $\mathit{1}$ and Γ_7) $1 + 15 + 15 + 1 = 32$ collineations. However these will not all be distinct because Ψ and $\Gamma_7\Psi$ correspond to the same geometrical point-plane pair and so projectively speaking we have 16 collineations and 16 correlations, including the identity. The 16 correlations split into 10 polarities and 6 null correlations depending upon whether the off-diagonal blocks are symmetrical or anti-symmetrical. Associated with these are 10 quadrics and 6 linear line complexes. Since one of the symmetrical matrices is the identity matrix, one of these quadrics has no real points.

The 16 correlations form a necessarily invariant subgroup G_{16} of G_{32}, the quotient G_{32}/G_{16} having the effect of interchanging points and planes, i.e. of projective *duality*. The abelian group G_{16} of correlations acting on the space $P_3(R)$ of projective Majorana spinors has a non-abelian 32 element double cover $\tilde{G}_{32}$ which is just that group generated by multiplication of the four by four γ-matrices $\{\gamma_0, \gamma_i\}$ in (1.1). It is this group which so pre-occupied Eddington and which he eventually learnt was the automorphism group of the Kummer configuration which we are now in a position to describe.

Let us start with an arbitrary point p^α and act with the group G_{16} of collineations. A simple argument based on the fact that γ-matrices have square ± 1 allows one to deduce that the group and G_{16} acts effectively on $P_3(R)$ so we shall obtain 15 additional distinct points making up a 16 points orbit of G_{16} in $P_3(R)$. These 16 points are the *points* of Kummer's configuration. Now let us acting with the 6 null correlations associated with the six γ-matrices Γ_A will give 6 planes passing the original point q^α. Acting with the 15 non-identical collineations on one of these planes we obtain 15 other planes making 16 in all. These are the *planes* of Kummer's configuration. By the commutativity and the duality properties of the group that this exhausts the set of planes in the configuration and moreover not only does each point of the configuration lie in 6 of the planes of the configuration but every plane of the configuration contains 6 of the points of the configuration.

5. An Explicit Basis

To make this explicit it is convenient to adopt basis for V. Geometrically a basis for V determines four distinct points in $P_3(R)$ which form as the vertices of a "tetrahedron of reference". The faces of the tetrahedron determine a basis for V^*. The six edges of the tetrahedron determine a real null sextad for $V \wedge V$ falling into two triples $\{l_i^A, n_j^B\}$ such that each vector is lightlike in the Klein metric and the only non-vanishing inner products are between opposite edges: $l_i^A g_{AB} n_i^B = \delta_{ij}$. The tetrahedron of reference is left invariant by the *tetrahedral group* which is the three-dimensional abelian subgroup of correlations corresponding to boosts in the three 2-planes spanned by $\{l_i^A$ and $n_j^B\}$. the orbit in $P_3(R)$ of a one parameter subgroup of the tetrahedral group is called a *W-curve* and the orbit of a two-parmeter subgroup is called a *W-surface*. Interestingly, it has recently been suggested that the shapes of growing buds are well described by W-curves.

With respect to the given tetrahedron of reference a null correlation determines a second tetrahedron of reference each face of which (possibly extended) contains one of the original vertices. Dually each vertex of the second tetrahredron lies on one of the planes of the original tetrahedron. In this way one obtains eight points and eight planes making up the self-dual Moebius configuration 8_4. When we need to we may take the four points of

our tetrahedron of reference to be the origin and the intersection of the three co-ordinate axes with the plane at infinity. The faces of the tetrahedron are then the co-ordinate plane and the plane at infinity.

Algebraically we are now allowd ignore the distinction between co- and contra-variant indices and to decompose the space of bi-vectors $\Lambda^2(V)$ into the orthogonal direct sum of self dual and antiself dual 2-forms: $\Lambda^2(V) = \Lambda^2_+(V) \oplus \Lambda^2_-(V)$. Two mutually commuting bases $\{\rho^i\}$ and $\{\lambda^i\}$, $i = 1,2,3$, for self-dual and anti-selfdual 2-forms respectively may be found generating two copies of the quarternion algebra Cliff(2):

$$\rho^i \rho^j = -\delta^{ij} + \epsilon^{ijk} \rho^k \tag{5.1}$$

and

$$\lambda^i \lambda^j = -\delta^{ij} + \epsilon^{ijk} \lambda^k, \tag{5.2}$$

with

$$\rho^i \lambda - \lambda^j \rho^i = 0. \tag{5.3}$$

One then has:

$$\Gamma^{i+} = \begin{pmatrix} 0 & \rho^i \\ -\rho^i & 0 \end{pmatrix} \tag{5.4}$$

and

$$\Gamma^{i-} = \begin{pmatrix} 0 & \lambda^i \\ \lambda^i & 0 \end{pmatrix}. \tag{5.5}$$

The Γ^{i+} have square plus one and the Γ^{i-} have square minus one.

In this basis the collineations and the correlations consist of $1, \rho^i, \lambda^j$ and $\rho^i \lambda^j$. These are easily seen to generate under multiplication a 32 element subgroup $\tilde{G}_{32}$ of $SO(4)$ isomorphic to $D_4^* \cdot D_4^*$ where D_4^* is the binary dihedral group (which is isomorphic to the multiplicative group of unit quaternions). To see that this group is just the same group that one gets if one multiplies together the usual 4-dimensional γ-matrices it suffices to note that one may take

$$\gamma^i = \lambda^1 \rho^i \tag{5.6}$$

$$\gamma_0 = \lambda^3 . \tag{5.7}$$

It follows that

$$\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \lambda^2, \tag{5.8}$$

and we may take as the charge conjugation matrix:

$$C = \lambda^1 = \gamma_1 \gamma_2 \gamma_3. \tag{5.9}$$

An explicit example is provided by setting:

$$\rho^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \rho^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \rho^3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad (5.10)$$

and

$$\lambda^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \lambda^3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (5.11)$$

Although not the same as those used by Eddington the reader will have no difficulty in verifying that these γ-matrices do indeed satisfy the conditions of his celebrated competition for Caliban's puzzle column in the Christmas 1936 copy of *New Statesman and Nation* (Eddington, 1936,1937) in which three boys and two girls visit a zoo in which the labels on the cages of four pairs, male and female, of animals with known names have unfortunately been lost. For every animal, the Tove for example, John supposes that the animal he supposes to be Mr Tove is the animal he suppose to be Mr Tove while Mary supposes Mr Tove to be the animal she supposes to be Mrs Tove. The same is true for all the boys and all the girls. Moreover the animal which John supposes to be the animal which Mary suposses to be Mr Tove is the animal which John supposes to be Mrs Tove and the same is true for all pairs of children.

The permutations that arise from mistaken identities are represented by the 5 γ-matrices, the boys coresponding to the γ_i and the girls to γ_0 and γ_5. The animal species are associated to points equi-distant from the origin along four orthogonal axes in four dimensional euclidean space. To get the group of Kummer configuration, G_{16} one simply ignores the sex of the animals.

The six null correlations are given by $C, C\gamma_5$ and $C\gamma_5\gamma^\mu$. The ten quadrics are given by $C\gamma^\mu$ and $C\gamma^{[\mu}\gamma^{\nu]}$. The sixteen collineations are of course 1, γ^μ, γ_5, $\gamma^{[\mu}\gamma^{\nu]}$ and $\gamma_5\gamma^\mu$.

To use this representation to construct an example of a Kummer configuration in $P_3(R)$ we start with the plane $(1,1,1,0)$, i.e.

$$x + y + z = 0$$

which passes through the origin and which is perpendicular to one of the four body-diagonals of the cube whose vertices are $(\pm,\pm,\pm,1)$ with all eight combinations of signs and act with ρ^i and λ^j. One obtains six points lying in the plane at the vertices of a regular hexagon:

$$(0,1,-1,1), \quad (-1,0,-1,1), \quad (1,-1,0,1),$$

and

$$(0,-1,1,1), \quad (0,1,0,1), \quad (-1,1,0,1).$$

The vertices of this planar hexagon comprise half of the 12 mid-points of the cube. These twelve points together with four points at lying on intersection of the four body-diagonals of the cube with the plane at infinity comprise the sixteen points of Kummer's configuration. The sixteen planes consist of the four planes passing through the origin perpendicular to the four body-diagonals of the cube, each of which contains six points arranged at the vertices of a regular hexagon, together with twelve other planes. Each of these other twelve planes contains 4 of the mid-points of the cube arranged in a rectangle and is parallel to a pair of body-diagonals. The twelve planes therefore also contain two points at infinity. Thus each plane belonging to the configuration contains six points and it is not difficult to see the truth of the dual proposition that each point lies on six planes of the configuration.

An example of a real Kummer surface in $P_3(R)$ having the maximum complement of 16 nodes and 16 tropes making up the above Kummer configuration is given in affine coordinates by:

$$x^4 + y^4 + z^4 + 1 - y^2z^2 - z^2x^2 - x^2y^2 - x^2 - y^2 - z^2 = 0.$$

6. Causal Structures and Linear line Complexes

The conformal compactification of *any* flat spacetime $R^{p,q}$, $\overline{R^{p,q}}$ is obtained by considering the space of null rays through the origin in $R^{p+1,q+1}$. Two points in $\overline{R^{p,q}}$ are null separated if and only if the rays have vanishing inner product. One recovers $R^{p,q}$ as those null rays which intersect a null hyperplane which does not contain the origin. Four-dimensional Minkowski spacetime is just a special case and we obtain in this way the standard isomorphism between the conformal group and $PSO(4,2)$. If we had started from flat four-dimensional space time with the ultra-hyperbolic signature (2,2) we would have obtained $PSO(3,3)$. The fact that the conformal geomtry of $\overline{R^{2,2}}$ coincides with the projective geometry of $P_3(R)$ is of course the Plücker correspondence.

From the point of view of Clifford algebras it is clear that $p + q + n = 4$ spacetime dimensions is a special case since the dimension of the conformal group $(n+2)(n+1)/2$ is only equal to one less than the dimension of the group invertible elements of the Clifford algebra Cliff(p,q), $2^n - 1$. This is what enabled us to bring in projective geometry. However one can also use projective geometry in lower dimensions and in this section we shall do so in 2+1 spacetime dimensions.

The topologically and metrically $\overline{R^{p,q}} \equiv (S^p \times S^q)/\pm$ with its product metric. In the Lorentzian case $q = 1$ each null geodesic is projects to a great circle on Sp and we can thus identify the space $\overline{N}$ of null geodesics as the bundle of unit tangent vectors of S^p. In the case $p = 2$ the space $\overline{N}$ of null geodesics may thus be identifed with real projective space $P_3(R)$.

In this case the conformal group is the Anti-De-Sitter group $SO(3,2)$ which is double covered by $Sp(4,R)$. The obvious idea now is to exploit the fact that the conformal compactification of $R^{2,1}$ may be obtained as a restriction of the conformal compactification of $R^{2,2}$ and then to use the Plücker correspondence. Thus two points in $\overline{R^{2,1}}$ will be null separated if

(i) two lines in $P_3(R)$ intersect

but

(ii) the lines considered as simple bi-vectors $l^{\alpha\beta}$ are orthogonal to a fixed bi-vector $C_{\alpha\alpha}$, i.e.

$$C_{\alpha\beta}l^{\alpha\beta} = 0. \tag{6.1}$$

In other words the lines in $P_3(R)$ must belong to a non-singular linear line complex.

It follows that the light cones of spacetime points x in $\overline{R^{2,1}}$ considered as the skies, or set of null geodesics sky(x) passing through the event x in may be identified as lines belonging to the linear line complex. Since points of $P_3(R)$ may be identified with null geodesics in $\overline{R^{2,1}}$ one may encode the causal structure of $\overline{R^{2,1}}$ into the projective properties of the linear line complex. For example if two lines $l_1 = \text{sky}(x_1)$ and $l_2 = \text{sky}(x_2)$ belonging to the complex intersect in a point p in $P_3(R)$ then p represents the common null geodesic generator of two light cones of the two spacetime points x_1 and x_2 joining these two events. The plane $\pi(p)$ containing all the lines passing through the point p corresponds to all light cones in $\overline{R^{2,1}}$ sharing p as a generator. Since we wish to investigate the possible relation between linking and causality it will be useful to have a descrption of the lines belonging to the complex. One such description has been provided by Woods (Woods, 1922).

Choosing affine co-ordinates x, y, z such that the contact form:

$$L_{\alpha\beta}x^\alpha dx^\beta = xdy - ydx - dz \tag{6.2}$$

the lines are found to be tangents to helices drawn on cylinders whose axes coincide with the z-axis and whose pitch is $2\pi a^2$ where A is the radius of the cylinder. The lines of the complex passing through points on the z axis (which itself does not belong to the complex) lie in planes perpendicular to the the z-axis. It seems appropriate to recall at this point that the word complex is derived from two Latin words for "plaited together" and the word sym-plectic is derived from two Greek words bearing the same meaning. Clearly each line is linked just once with every other one. Thus linking alone

is not sufficient to determine whether the events in $\overline{R}^{2,1}$ associated to two skies are spacelike or timelike separated. This conclusion is consistent with that of Robert Low, who discuses the case of $R^{2,1}$ (Low, 1990).

To compare it is convenient to abandon the use of affine coordinates and adopt the model of projective space as a solid ball in $R^3 = \{\mathbf{x} = (x, y, z)\}$ with opposite points of the boundary identified. Using *stereograhic* rather than *central* projection one finds that:

$$x^\alpha \propto (\frac{1-r^2}{2}, \mathbf{x}) \tag{6.3}$$

The polar plane through a point $\tilde{\mathbf{x}}$ is:

$$\tilde{x}^0 x^3 + \tilde{x}^1 x^2 - \tilde{x}^2 x^1 - \tilde{x}^3 x^0 = 0. \tag{6.4}$$

This becomes a sphere:

$$(1 - \tilde{r}^2)z + 2(\tilde{x}y - \tilde{y}x) - (1 - r^2)\tilde{z} = 0 \tag{6.5}$$

If the poles lie on the vertical axis $\tilde{\mathbf{x}} = (0, 0, \tilde{z})$ then polar planes are:

$$x^2 + y^2 + (z - \frac{1}{2}(\tilde{z} - \frac{1}{\tilde{z}}))^2 = \frac{1}{4}(\tilde{z} + \frac{1}{\tilde{z}})^2. \tag{6.6}$$

This gives a family of spheres passing through the equator of the unit ball, i.e. $z = 0, x^2 = y^2 = 1$. The lines of the complex belonging to this family of planes are great circles passing through the vertical axis. One may identify the generators of $\mathcal{I}$ as points on the equator of the boundary sphere, i.e. $x^2 + y^2 = 1, z = 0$, and $\mathcal{I}$ itself as the set ofits generators, $\mathcal{I} = \text{sky}(\rangle)$.

7. Majorana Spinors and Linear Line Congruences

We have seen above that endowing $P_3(R)$ with a linear line complex, $C_{\alpha\beta}$ reduces $SO(3,3)$ to $SO(3,2)$. The introduction of a second linear line complex, $C\gamma_5$ which is timelike in the Klein metric will reduce further to the Lorentz group $SO(3,1)$. The lines common to both determine a linear line congruence. One line of the congruence passes through each point p of $P_3(R)$, this line is the intersection of the two polar planes of p determined by the two null correlations C and $C\gamma_5$. The associated Lorentzian 4-plane through the origin in $R^{3,3}$ is left invariant by $SO(2) \times SO(3,2)$, where the $SO(2)$ factor corresponds to *chiral rotations* in the 2-dimensional normal space spanned by C and $C\gamma_5$. The infinitesimal generator of this $SO(2)$ is of course just the involutive collineation γ_5. Acting on $P_3(R)$ the chiral rotations move the points along the lines but leave the lines of the congruence itself invariant. We thus obtain a fibring of $P_3(R)$ by $P_1(R)$ and it is not difficult to see that this is the quotient of the standard *Hopf fibration* of ^{3} by S^1 by the antipodal map. Because as mentioned in the introduction if we pass to complex co-ordinates for V using γ_5 as a complex structure we obtain two-component Weyl spinors and now chiral rotations act by phasing.

References

L. Dąbrowski, *Group Actions on Spinors* Bibliopolis (1988)
A. S. Eddington, *New Pathways in Science* Cambridge University Press p.271 (1935)
A. S. Eddington, *Relativity Theory of Electrons and Protons* Cambridge University Press p.36 (1936)
A. S. Eddington, *The New Statesman and Nation* Dec 19 1936 p 1044, Jan 9 1937 pp. 62-64.
R. W. H. T. Hudson, *Kummer's Quartic Surface* Cambridge University Press 1905reprinted 1990
R. J. Low, *Classical and Quantum Gravity* 7 177-187 (1990)
E. R. Paërls, *Representations of the Lorentz Group and Projective Geometry*
F. S. Woods, *Higher Geometry* Ginn and Company (1922) reprinted by Dover 1961
O. Zariski, *American Journal of Mathematics* 54 466-470 (1932)

PAULI-KOFINK IDENTITIES AND PURE SPINORS

HELMUT URBANTKE
Institut für Theoretische Physik,
Universität Wien
Austria

Abstract. A machinery producing identities between the bilinear covariants of spinors, devised by Pauli and Kofink, is extended to the n-dimensional case and applied to pure spinors.

1. Introduction

Space-times of dimension higher than four have found their way into attempts of establishing a unified theory of all interactions, notably through the Kaluza-Klein construction and its generalizations to include non-abelian gauge fields and supersymmetry multiplets. This made it necessary to consider spinors in arbitrary dimensions. Some special dimensions > 4 where spinors are considered in more detail are 6 (twistor theory (Penrose 1986); Calabi-Yau manifolds (Candelas 1985)), 7 (parallelized 7-sphere (Englert 1983)), 8, 10, 11 (extended supergravities – see e.g. Julia (1982)); we also wish to mention the description of classical strings without differential constraints as given by Hughston (1987). In all these works certain identities play a basic role. As probably well-known to the practitioneers, these can be derived according to a scheme first devised by Pauli (1935) and slightly extended by Kofink (1937, 1940); it is probably better known as Fierz rearrangements (Pietschmann 1983). For dimensions ≥ 7, identities of this type are also related to the theory of pure spinors (purity conditions and purity syzygies; see Cartan (1966); Chevalley (1954); Hughston (1987); Budinich (1989)).

In this note we describe the Pauli-Kofink type approach to some of these identities in a unified manner (Sect. 4) after some preparatory material in Sect. 2 and 3. Application to pure spinors are given in Sect. 5. All considerations are restricted to the complex domain, for simplicity.

2. Clifford Algebra and Spinors. Completeness. Semispinors

As a generalization and complexification of ordinary Minkowski vector space, we consider a complex vector space $\mathbf{V}$, $\dim \mathbf{V} = n$ finite, together with a non-degenerate symmetric bilinear form g. The Clifford algebra for $\mathbf{V}, g$ is the complex associative algebra with unity generated by $\mathbf{V}$ in which the relations $xy + yx = g(x,y) \cdot 1$ hold for all $x, y \in \mathbf{V}$. Here iuxtaposition of vectors means algebra multiplication, whereas the scalar product of x, y will always be written $g(x,y)$, 1 is the unit element of the algebra. By spinors we mean the elements of a complex spinor space $\mathbf{S}$ which carries an *irreducible* representation γ of the Clifford algebra by linear operators. The set of these operators is thus generated by the operators $\gamma(e_\mu) = \gamma_\mu$ satisfying

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu} id_{\mathbf{S}}, \tag{2.1}$$

where $\{e_\mu \in \mathbf{V}\}$ is an orthonormal basis, $g(e_\mu, e_\nu) = g_{\mu\nu} = \delta_{\mu\nu}$. The indices appearing in (2.1) can also be considered as "abstract indices". Similarly, we shall have occasion to use indices in spin space as well, either to be thought of as referring to some basis in $\mathbf{S}$, or as abstract indices. For any subset $\Lambda = \{\lambda_1, \ldots, \lambda_{|\Lambda|}\}$ of the set $N = \{1, \ldots, n\}$, where the elements λ_i are ordered according to $1 \leq \lambda_1 < \lambda_2 < \ldots < \lambda_{|\Lambda|} \leq n$, we define

$$\gamma_\Lambda := \gamma_{\lambda_1} \gamma_{\lambda_2} \cdots \gamma_{\lambda_{|\Lambda|}} = \gamma_{[\lambda_1} \cdots \gamma_{\lambda_{|\Lambda|}]}, \qquad \gamma_\emptyset := id_{\mathbf{S}}, \tag{2.2}$$

where $[\ldots]$ indicates antisymmetrization and $|\Lambda|$ is the cardinality of Λ. By (2.1), the γ_Λ linearly span the algebra of operators generated by the γ_λ.

With these notations, we can now state the basic properties of the spin representation of the Clifford algebra; but we have to distinguish the cases n = even and n = odd. (For a fuller treatment, see, e.g. Penrose (1986), Budinich (1988).)

Case n = even $= 2m$. All irreducible representations of the Clifford algebra are equivalent and faithful; $\dim \mathbf{S} = 2^m$; $\operatorname{tr} \gamma_\Lambda = 0$ for all $\Lambda \neq \emptyset$; the γ_Λ are linearly independent and span the whole set End $\mathbf{S}$ of linear operators on $\mathbf{S}$ – in more detail, any $F \in$ End $\mathbf{S}$ may be expanded

$$F \equiv 2^{-m} \sum_{\Lambda \subset N} (\operatorname{tr} \gamma_\Lambda^{-1} F) \gamma_\Lambda. \tag{2.3 even}$$

This formula is the basis of the Fierz rearrangements and for the derivation of Pauli-Kofink type identities (see Sect. 4).

The element γ_N satisfies

$$\gamma_N^2 = (-1)^m id_{\mathbf{S}}. \tag{2.4}$$

The spaces of semispinors (= half spinors = Weyl spinors = chiral spinors) $\mathbf{S}_\pm$ are defined as the projections

$$\mathbf{S}_\pm := \frac{id_{\mathbf{S}} \pm i^m \gamma_N}{2} \mathbf{S}. \tag{2.5}$$

They are invariant under the even part of the Clifford algebra, but get interchanged by the odd part, as we have

$$\gamma_\Lambda \gamma_N = (-1)^{|\Lambda|} \gamma_N \gamma_\Lambda. \tag{2.6}$$

Case n = odd $= 2m+1$. The irreducible representations of the Clifford algebra fall into two equivalence classes, none of them faithful, distinguished by $\gamma_N = \pm i^m id_{\mathbf{S}}$; for both classes the dimension of spin space is again 2^m; $\operatorname{tr} \gamma_\Lambda = 0$ for all $\Lambda \neq \emptyset, N$. The γ_Λ form an overcomplete system, i.e., we have

$$F = 2^{-m-1} \sum_{\Lambda \subset N} (\operatorname{tr} \gamma_\Lambda^{-1} F) \gamma_\Lambda = 2^{-m} \sum_{\Lambda \subset N, |\Lambda| \leq m} (\operatorname{tr} \gamma_\Lambda^{-1} F) \gamma_\Lambda. \tag{2.3 odd}$$

3. The Fundamental Bilinear Form

If $x \mapsto \gamma(x)$ is an irreducible representation of the Clifford algebra on the spin space $\mathbf{S}$, then

$$x \mapsto \gamma'(x) := (-1)^m (\gamma(x))^T \tag{3.1}$$

is, up to a sign, the transpose representation, acting irreducibly on the dual space $\mathbf{S}^*$. Since in the even case $n = 2m$ there is just one equivalence class of irreducible representations, there exists a nonsingular intertwiner $B : \mathbf{S} \to \mathbf{S}^*$, i.e.

$$B \gamma_\lambda = (-1)^m \gamma_\lambda^T B. \tag{3.2}$$

Defining

$$B_\Lambda := B \gamma_\Lambda, \tag{3.3}$$

it can be shown that

$$B_\Lambda^T = (-1)^{(m-|\Lambda|)(m-|\Lambda|+1)/2} B_\Lambda. \tag{3.4}$$

The sign in (3.1) has been chosen in such a way that the same properties for B, including its existence, also hold in the odd case.

In the even case, we can define chiral projections in $\mathbf{S}^*$ by taking the transposes of the chiral projections in $\mathbf{S}$. Then B maps $\mathbf{S}_\pm$ to $\mathbf{S}_\pm^*$ or $\mathbf{S}_\mp^*$ according to m = even ("splitting case") or m = odd ("mixing case"):

$$B_\Lambda \mathbf{S}_\pm = \mathbf{S}^*_{\pm(-1)^{m+|\Lambda|}}. \tag{3.5}$$

In terms of indices, if we use small Latin upper indices for elements of $\mathbf{S}$, lower indices for $\mathbf{S}^*$, B appears as B_{ab}, which is either symmetric or antisymmetric (cf. (3.4)). We can use B_{ab} as a "spinor metric" to move indices, with the usual care for signs in the antisymmetric case. For n even, we can adapt the index notation to the chiral decomposition and will use capital Latin indices, undotted for S_+, S_+^*, dotted for S_-, S_-^*: then B_{ab} defines B_{AB}, $B_{\dot{A}\dot{B}}$ in the splitting case, $B_{A\dot{B}}$, $B_{\dot{A}B}$ in the mixing case. So we have separate semispinor metrics in the splitting case, but an equivalence of semispinors of one chirality and dual semispinors of the opposite chirality in the mixing case. (See Penrose (1986) for details.)

4. Fierz Rearrangements and Pauli-Kofink Type Identities

Pick four spinors $\varphi, \psi, \omega, \chi \in \mathbf{S}$ and two operators M, L on $\mathbf{S}$: we then can form, e.g., the invariants obtained from evaluating $BM\psi \in \mathbf{S}^*$ on φ, $BL\chi \in \mathbf{S}^*$ on ω, written usually as

$$\varphi^a B_{ab} M^b{}_c \psi^c = \varphi^T BM\psi, \qquad \omega^d B_{de} L^e_f \chi^f = \omega^T BL\chi. \tag{4.1}$$

A *Fierz rearrangement* of the product of these,

$$\varphi^a B_{ab} M^b{}_c \underbrace{\psi^c \omega^d B_{de}} L^e_f \chi^f = \varphi^T BM \underbrace{\psi\omega^T B} L\chi \tag{4.2}$$

is an expression that results when the indicated operator $F^c{}_e := \psi^c \omega^d B_{de}$ (of rank one) is expanded according to (2.3), resulting, e.g., in the even n case in

$$(\varphi^T BM\psi)(\omega^T BL\chi) \equiv 2^{-m} \sum_{\Lambda \subset N} (\omega^T B\gamma_\Lambda^{-1}\psi)(\varphi^T BM\gamma_\Lambda L\chi). \tag{4.3}$$

Of course, one could have included the factor M, or L, or both, in the definition of F to arrive at other versions. This procedure has played a role in the discussion of possible interaction terms in weak interactions (Fierz 1937). In concrete examples, it is wise to observe the symmetry properties (3.4) of $B\gamma_\Lambda = B_\Lambda$ if, e.g., $\omega = \psi$ in (4.3), and also the chiral properties of B, M, L, γ_Λ if chiral spinors are involved. One can decompose (4.3) with respect to these two aspects to obtain a group of identities with a considerably smaller number of terms on the right hand side of each.

Generalized Pauli-Kofink type identities are a special case where M and L are replaced by $\gamma_{[\mu_1} \cdots \gamma_{\mu_p]}$, $\gamma_{[\nu_1} \cdots \gamma_{\nu_q]}$, and contractions over some of the indices μ_i, ν_j are taken (Pauli (1935); Kofink (1937, 1940); in this work φ, ω, χ are all assumed to be related to ψ; the scalar identities (i.e. *all* μ_i, ν_j contracted) are also called Fierz identities (Fierz 1937) and have been worked out in n dimensions by Case (1955)). A more recent application with $n = 6$, $M = \gamma^{[\mu}\gamma^{\lambda]}$, $L = \gamma_{[\lambda}\gamma_{\nu]}$, $\psi = \chi =$ (Majorana) spinor, $\varphi = \omega = \gamma_N\psi$, where (4.3) yields $-(\psi^T B\psi)^2\delta^\mu_\lambda$, showing that a (resp. covariantly constant) Majorana spinor field on a 6-dimensional Riemannian manifold defines an almost complex (resp. complex) structure, is given in (Candelas 1985).

The type of identities about which we want to go into more detail here arises if we take $M = \gamma^\mu$, $L = \gamma_\mu$ and $\omega = \psi$. Then only those Λ where $B\gamma_\Lambda = B_\Lambda$ is symmetric will contribute, i.e., according to (3.4), $|\Lambda| = m, m+1; m-3, m+4; \ldots$. If further φ, ψ, ω have a definite chirality, then for (4.3) to be non-trivial φ and χ must have chirality $(-1)^{m+1}$ relative to ψ, cf. (3.5), and we will have only contributions from Λ satisfying $m + |\Lambda| =$ even. These two conditions together then require $m - |\Lambda|$ to be a multiple of four. We also can easily prove that (for even or odd n)

$$\gamma^\mu \gamma_\Lambda \gamma_\mu = (-1)^{|\Lambda|}(n - 2|\Lambda|)\gamma_\Lambda. \tag{4.4}$$

This is the second row of the 'Fierz matrix' and shows, in particular, that for even $n = 2m$ the 'middle' terms $|\Lambda| = m$ in the sum (4.3) drop out. Finally, for odd n quite generally and for even n in the chiral case the sum (4.3) may be restricted to $|\Lambda| \le m$, but must then be taken twice: this is because if Λ, Λ' are (ordered) complements in N, they differ by a factor $\pm\gamma_N$ which is a multiple of $id_{\mathbf{S}}$ in the odd n case and acts as such on chiral spinors in the even case, while the numerical factors (4.4) are the same for Λ, Λ' if n is odd, opposite if n is even but this is then compensated by an additional (-1) that remains from the relative chirality $(-1)^{m+1}$ between ψ and χ.

The identity in question thus becomes, in the even case, with chiral φ, ψ, χ and relative chirality $(-1)^{m+1}$ between ψ and φ, χ:

$$(-2)^{m-4}(\varphi^T B\gamma_\mu\psi)(\psi^T B\gamma^\mu\chi) = \sum_\Lambda c_\Lambda(\psi^T B\gamma_\Lambda^{-1}\psi)(\varphi^T B\gamma_\Lambda\chi)$$

$$|\Lambda| = m-4, m-8, \ldots \quad (\ge 0), \qquad c_\Lambda = s \text{ if } |\Lambda| = m - 4s. \tag{4.5 even}$$

In the odd case, we have

$$2^m(\varphi^T B\gamma_\mu\psi)(\psi^T B\gamma^\mu\chi) = \sum_\Lambda (-1)^{|\Lambda|}(n - 2|\Lambda|)(\psi^T B\gamma_\Lambda^{-1}\psi)(\varphi^T B\gamma_\Lambda\chi)$$

$$|\Lambda| = m, m-3, m-4, m-7, m-8, \ldots \quad (\ge 0). \tag{4.5 odd}$$

Here the coefficient of the highest terms, $|\Lambda| = m$, is $(-1)^m$, and we may get rid of these terms by subtracting the corresponding expansion of the expression $(-1)^m(\varphi^T B\psi)(\psi^T B\chi)$, resulting in

$$(-2)^{m-3}\{(\varphi^T B\gamma_\mu\psi)(\psi^T B\gamma^\mu\chi) - (-1)^m(\varphi^T B\psi)(\psi^T B\chi)\} =$$

$$= \sum_\Lambda c_\Lambda(\psi^T B\gamma_\Lambda^{-1}\psi)(\varphi^T B\gamma_\Lambda\chi), \qquad |\Lambda| = m-3, m-4, m-7, m-8, \ldots \tag{4.6}$$

$$c_\Lambda = \begin{cases} s & \text{for } |\Lambda| = m - 4s + 1 \\ -s & \text{for } |\Lambda| = m - 4s. \end{cases}$$

Before applying (4.5), (4.6) to pure spinors (Sect. 5), we shall discuss them in a few special cases. For $n = 4$ and $n = 6$ the sum on the right is empty; for $n = 4$, one gets a formula well-known from two-component spinor algebra with a well-known geometric content (Penrose (1984); for $n = 6$, using semispinor indices, (3.4) tells us that the quantities $\gamma^\mu{}_{AB}$ are antisymmetric, so that our identity (4.5) says that

$$E_{ABCD} := \gamma_{\mu AB}\gamma^\mu{}_{CD}, \qquad E^{ABCD} := \gamma^{\mu AB}\gamma_\mu{}^{CD} \tag{4.7}$$

are totally antisymmetric. It is well-known that such an object is the basic structure of twistor algebra (Penrose 1986). – $n = 8$ is the first instance in the even case where we have a nonvanishing r.h. side in (4.5); $\dim \mathbf{V} = 2m = 8$ is distinguished among *all* dimensions by the fact that $\mathbf{S}_\pm$ are of the same dimension $\frac{1}{2}\cdot 2^4 = 8$ as $\mathbf{V}$. Further, it here happens for the first time in the even case that B is both symmetric and splitting, thus defining quadratic forms in $\mathbf{S}_+$ and $\mathbf{S}_-$; together with g on $\mathbf{V}$, we have altogether three spaces with quadratic forms and of the same dimension 8. Eqs. (3.4,5) tell us that $B\gamma_\mu$ is symmetric-mixing, so we have $\gamma^\mu{}_{\dot{A}C} = \gamma^\mu{}_{C\dot{A}}$, and the content of our identity (4.5) may be written, after raising one index

$$\gamma_\mu{}^{\dot{A}}{}_B\gamma^\mu{}_{\dot{D}C} + \gamma_\mu{}^{\dot{A}}{}_C\gamma^\mu{}_{\dot{D}B} = 2B_{BC}\delta^{\dot{A}}_{\dot{D}} \tag{4.8}$$

plus an analogous equation with dotted and undotted indices interchanged. If the defining relation (2.1) is also split into its semispinor versions, one recognizes, on comparison with the relations just obtained, that, just as $(\mathbf{S}_+, \mathbf{S}_-)$ are semispinor spaces for $(\mathbf{V}, g)$, $(\mathbf{V}, \mathbf{S}_+)$ are semispinor spaces for $(\mathbf{S}_-, B|_{\mathbf{S}_-})$ and $(\mathbf{S}_-, \mathbf{V})$ are semispinor spaces for $(\mathbf{S}_+, B|_{\mathbf{S}_+})$. This is one version of the 'principle of triality' (cf. Cartan (1966), Chevalley (1954), Penrose (1986)). – For $n = 10$, the right hand side of (4.5) contains only terms with $|\Lambda| = 1$; B is antisymmetric and mixing, $B\gamma_\mu$ is symmetric and splitting. Identity (4.5) then can be rewritten semispinorially

$$\gamma_{\mu\dot{A}(\dot{B}}\gamma^\mu{}_{\dot{C}\dot{D})} = 0 = \gamma_{\mu A(B}\gamma^\mu{}_{CD)}, \tag{4.9}$$

where (...) indicates symmetrization. These relations play a significant role in Hughston (1987), where they are called 'purity syzygies' for reasons to be explained in the next section.

5. Application to Pure Spinors

In this section all spinors are to be chiral if $n = 2m$ is even. For a given nonzero spinor ψ, consider the associated subspaces of $\mathbf{V}$ defined by

$$\mathbf{V}_\psi := \{x = x^\mu e_\mu | x^\mu = \varphi^T B\gamma_\mu\psi, \varphi \in \mathbf{S}\}$$
$$\mathbf{N}_\psi := \mathbf{V}_\psi^\perp = \{x | x^\mu\gamma_\mu\psi = 0\}.$$

If $\mathbf{N}_\psi \neq \{0\}$, it is totally null, as follows from $0 = \gamma(x)^2\psi = g(x,x)\psi$, and then $\mathbf{N}_\psi \subset \mathbf{N}_\psi^\perp = \mathbf{V}_\psi^{\perp\perp} = \mathbf{V}_\psi$; thus $\mathbf{V}_\psi$ is null. (Recall the terminology: a subspace $\mathbf{W} \subset \mathbf{V}$ is called r-fold isotropic or null iff $\dim(\mathbf{W} \cap \mathbf{W}^\perp) = r$, totally null iff $\mathbf{W} \subset \mathbf{W}^\perp$. Recall further that the maximum dimension of totally null subspaces is m and that the null cone of $(\mathbf{V}, g)$ carries two $\frac{1}{2}m(m-1)$ parameter families of maximal totally null subspaces if $n = 2m$ and one $\frac{1}{2}m(m+1)$ parameter family of such if $n = 2m+1$.)

A spinor ψ is called *pure* iff $\mathbf{N}_\psi$ is maximal, i.e., m-dimensional and thus $\mathbf{V}_\psi$ is of dimension m resp. $m+1$ in the even resp. odd case; in the former case it follows that $\mathbf{N}_\psi = \mathbf{V}_\psi$.

Considering first the even case, identity (4.5) shows immediately – because of $\mathbf{V}_\psi = \mathbf{N}_\psi$ – that ψ (chiral!) is pure iff $\psi^T B_\Lambda\psi = 0$ for $|\Lambda| = m-4, m-8, \ldots$. The identity goes, however, somewhat beyond this well-known fact if $m > 4$, as it yields some 'purity syzygies', i.e., identical relations satisfied by the $\psi^T B_\Lambda \psi$ which explain the discrepancy in the correspondence between the number of (linearly independent!) purity conditions just obtained and the dimension of the two families of totally null m-spaces in $\mathbf{V}$: they are obtained by putting $\varphi = \psi$ in (4.5), leaving χ arbitrary. More of these syzygies are obtained if we apply the general Pauli-Kofink machinery mentioned above. It seems difficult, in the general case, to give a detailed count of independent syzygies to make up for the discrepancy mentioned – more knowledge from invariant theory is required, at least.

Considering the odd case, identity (4.5) immediately shows that the purity conditions $\psi^T B_\Lambda\psi = 0$, $|\Lambda| = m-3, m-4, \ldots$ are sufficient to guarantee $\dim \mathbf{N}_\psi = m$, because according to it the spinors φ mapped onto $\mathbf{N}_\psi = \mathbf{V}_\psi^\perp$ by $M_\psi : \varphi \mapsto \varphi^T B_\mu\psi$ then form a subspace $\psi^\perp$ of $\mathbf{S}$ given by $\varphi^T B\psi = 0$, i.e. having codimension 1; thus

$$1 = \dim \mathbf{S}/\psi^\perp = \dim \frac{\mathbf{S}/\ker M_\psi}{\psi^\perp/\ker M_\psi} = \dim \mathbf{V}_\psi/\mathbf{N}_\psi = n - 2\dim \mathbf{N}_\psi.$$

Concerning necessity, we were not able, on the odd case, to get along without resorting to the rank results of Veblen (1955) or the recursive procedure of Budinich (1989); both of these arguments work, however, equally well without the benefit of our identity, whose main advantage thus seems to be to provide a short *sufficiency* proof. (Note that Budinich (1989) aims at providing *simple* arguments in the theory of pure spinors, but refers to Chevalley (1954) for sufficiency – not to Cartan (1966), whose proof contains a point that remains cryptic to the present author.) Again there is a discrepancy in the correspondence between the number of (linearly independent) purity conditions and the dimension of the family of totally null m-spaces in $\mathbf{V}$, if $m > 3$, to be explained in terms of syzygies, and we do not yet have a systematics for that.

Acknowledgements

I am indebted to L.P. Hughston and A. Trautman who know everything that has been said in this paper.

References

Budinich, P. and Trautman, A.: 1988, 'The Spinorial Chessboard', *Springer: Berlin.*
Budinich, P. and Trautman, A.: 1989, *J. Math. Phys.* **30**, 2125.
Candelas, Ph., Horowitz, G., Strominger, A. and Witten, E.: 1985, *Nucl. Phys.* **B258**, 46.
Cartan, E.: 1966, 'The Theory of Spinors', *M.I.T. Press: Cambridge.*
Case, K.M.: 1955, *Phys. Rev.* **97**, 810.
Chevalley, C.: 1954, 'The Algebraic Theory of Spinors', *Columbia U.P.: New York.*
Englert, F., Rooman, M., and Spindel, Ph.:1983, *Phys. Lett.* **130B**, 50.
Fierz, M.: 1937, *Z. Physik* **104**, 553.
Hughston, L.P. and Shaw, W.T.: 1987, *Proc. R. Soc. Lond.* **A414**, 423.
Julia, B.: 1982, 'Marcel Grossmann Meeting on General Relativity', R. Ruffini, editor, *North-Holland: Amsterdam*, 79.
Kofink, W.: 1937, *Ann. Phys. (Leipzig)* **30**, 91.
Kofink, W.: 1940, *Ann. Phys. (Leipzig)*, **38**, 426.
Pauli, W.: 1935, 'Zeeman-Verhandelingen', *M. Nijhoff: s'Gravenhage.*
Penrose, R. and Rindler, W.: 1984, 'Spinors and Space-Time', Vol. 1, *Cambridge U.P.: Cambridge.*
Pietschmann, H.V.R.: 1983, 'Formulae and Results in Weak Interactions and Derivations', *Springer: Wien.*
Veblen, O. and v. Neumann, J.: 1955, 'Geometry of Complex Domains', *The Institute of Advanced Study: Princeton.*

GENERAL COVARIANCE AND SPINORS

LUDWIK DĄBROWSKI
SISSA, Strada Costiera 11,
34014 Trieste, Italy

Spinors play an important role in general relativity. Besides describing the matter fields, they have been employed for example to classify 'types' of Weyl tensor, in the $E > 0$ theorem of Witten and other instances. In this note we shall discuss the issue of general covariance in gravity with spinors. This is usually regarded as a statement that Lie derivative of the Lagrangean L, with respect to arbitrary vector field X, is zero $\partial_X L = 0$. To check this condition, one does not apparently need the Lie derivative of spinors. Namely, if L is a scalar (density) function of spinors, one may substitute the Lie derivative of L with a (fiducial) covariant derivative ∇_X of L and, using the Leibnitz rule, apply ∇_X to spinors, which is a well defined operation. However, similarly to gauge theories, where one insists on the group action of gauge transformations on the gauge potentials, it should be equally useful to have the general covariance as a well defined group action of diffeomorphisms on the configuration space of gravity with spinors. Concerning this point one encounters two quite opposite claims. It is a poular opinion in the literature on gravity and supergravity, that no problem arises as spinors transform as 'scalars' uder coordinate changes and as 'spinors' under (local) vierbein rotations. Instead, differential geometers often claim that there is no satisfactory action of diffeomorphisms on spinors. Our aim is to explain why both these opinions are essentially right. In order to do so, we shall briefly recall:

1. various kinds of spinors
2. 'no-go' theorem for some space-times
3. nonuniquness problem
4. reformulation of the problem in terms of $\overline{GL}(4)$.

We refer to [Dąbrowski, 1986] for more details and the list of relevant references.

1. Let G be a chosen structure group under which tensors transform, e.g. $G = SO_\circ(3,1)$ in the Lorentz space-time. The type of a spinor is specified by a spinorial (projective) representation R of G in a linear space V. For convenience, this is commonly viewed as a true (nonprojective) representation of a double covering of G, i.e. of a group $\overline{G}$ equipped with a 2:1 homomorphism

ρ onto G

$$\begin{array}{ccc} & \mathbf{Z}_2 & \\ & \downarrow & \\ & \overline{G} & \\ \rho & \downarrow & \searrow \overline{R} \\ & G \xrightarrow{R} & L(V)\,. \end{array} \tag{1}$$

If $G = SO_o(3,1)$, $\overline{G} = Spin_o(3,1) \equiv SL_2(\mathbf{C})$, and typically one choses the Weyl (chiral) representation in $V = \mathbf{C}^2$ or the Dirac representation in $V = \mathbf{C}^4$, etc. These standard data are best defined in terms of Clifford algebras and are also well known for other dimensions and signatures. However, one often needs a bigger group G, such as the (Cartesian) product of $SO_o(3,1)$ times one of the following groups: $\mathbf{Z}_2 \times \mathbf{Z}_2$ (reflections), R^* (conformal), $U(1)$ (electric charge), $SU(2)$ (nonabelian gauge), or yet a bigger group. For a chosen G there may be several possibilities for $\overline{G}$ and for ρ, some of them (but not all) defined by means of Clifford algebra (c.f. [Dąbrowski, 1986] for details). Let us mention that $G = GL(4)$ and its (two) nontrivial coverings, though not directly applicable as a structure group (demanding spinorial representations to be finite dimensional), will be employed in the sequel.

In order to pass to manifolds, the above data are usually slightly reinterpreted as follows. Let F_g be the space of orthonormal frames $E = \{E^a\}$ in $\mathbf{R}^{3,1}$, with respect to some chosen metric g. Let $T^{(E)}$ be the components of a tensor T in the frame E. They transfrom as $T^{(E)} = R(g)\ T^{(E')}$, with $E' = Eg \equiv E^b g^a_b$ for $g \in G$. Thus one can regard a tensor as an R-equvariant map, $E \overset{T}{\longmapsto} T^{(E)}$, from F to the space of components. Analogously, for spinors, one makes use of an (abstract) space $\overline{F_g}$ of 'spinor frames', together with a definite 2:1 (equivariant) assignment η between spinor and orthonormal frames ($\overline{G}$ acts freely and transitively on $\overline{F_g}$ and η intertwines this action and the action of G on F_g). Next, spinors are regarded as $\overline{G}$-equivariant functions ψ on $\overline{F_g}$

$$\begin{array}{ccccc} \overline{G} & \hookrightarrow & \overline{F_g} & \xrightarrow{\psi} & V \\ \rho\downarrow & & \downarrow\eta & & \\ G & \hookrightarrow & F_g\,. & & \end{array} \tag{2}$$

2. The content of 1. can be globalized [Haefliger, 1956],[Milnor, 1963] by taking in (2) F_g to be the bundle of (space and time oriented) orthonormal frames defined with respect to a metric tensor g. One can also take a (related) bigger principal G-bundle over the manifold M. Next, $\overline{F_g}$ is some principal $\overline{G}$-bundle over M, and η - an equivariant 2:1 bundle homomorphism. The spinor fields of type $\overline{R}$ are just functions from $\overline{F_g}$ to V, which

are equivariant with respect to the (free and transitive on fibers) action of $\overline{G}$. It is well known, though surprising, that such a *spin structure* $(\overline{F_g}, \eta)$ not always exist. For $G = SO_o(3,1)$, besides the orientability and existence of Lorentzian metric, a topological condition for a manifold is vanishing of the second $\mathbf{Z}_2$ Stiefel-Whitney class. This is clearly a global problem (locally one always have spinors), which can be visualised e.g. by a paradox with a one-parameter family of parallel transports along closed paths in $\mathbf{CP}_1$ [Geroch, 1970]. For another choice of G, the obstruction is generally weaker and may completely disappear.

3. Another known 'complication' is that if it exist, spin structure may be not unique, cf. [Isham, 1978]. Remind that two spin-structures $(\overline{F_g}', \eta')$ and $(\overline{F_g}, \eta)$ are equivalent iff there exist a bundle isomorphism β such that

$$\begin{array}{ccc} \overline{F_g}' & \xrightarrow{\beta} & \overline{F_g} \\ \eta'\downarrow & & \downarrow\eta \\ F_g & \xrightarrow{id} & F_g \end{array} \tag{3}$$

commutes. It can be seen that the number of inequivalent spin structures equals the number of classes in $H^1(M, \mathbf{Z}_2) \simeq HOM(\pi_1, \mathbf{Z}_2)$.

Now, assuming that two inequivalent spin-structures are isomorphic as bundles (which is a typical case), we mention some aspects of the inequivalence between $(\overline{F_g}, \eta')$ and $(\overline{F_g}, \eta)$. The first one concerns the spin connection, defined as a pull back to $\overline{F_g}$ of the Levi-Civita connection on M (composed with the isomorphism of Lie algebras of $SO_o(3,1)$ and $Spin_o(3,1)$). In our case we have two different spin connections $\overline{\Gamma}' \equiv \eta'^*\Gamma \neq \eta^*\Gamma \equiv \overline{\Gamma}$, though locally they are equivalent just by a Lorentz gauge transformation. This yields diffrent covariant derivatives and consequently different Dirac operators. As an alternative, one can perform (locally) a gauge tranformation, i.e. pass to another gauge $(\overline{E}', \eta)$, such that $\eta'(\overline{E}') = \eta(\overline{E})$. Then, the (local) expressions for the covariant derivatives coincide and the same holds for Dirac operators, but the (anti-) posed periodicity conditions (along the loops in π_1) are different. Altogether, it is clear that inequivalence leads, in general, to different spectra of Dirac operator, positive eigenspaces and second quantization.

4. Now we pass to the question of diffeomorphisms. We have seen that the spinor fields are subordinated to metrics; i.e one first needs g, then a spin structure and finally the spinors. Therefore, the configuration space $\mathcal{W}$ of spinor fields coupled to gravity has the structure of (infinite dimensional) vector bundle over the space $\mathcal{M} \times \Sigma$, where $\mathcal{M}$ is the space of metrics and Σ is the (discrete) set of spin-structures. The fiber Γ_g over g, is the space of spinor fields defined as above. Now, to implement the action of a diffeomorphism f, one observes that f transforms metrics by a pull back, $g' = f^*g$, and maps Γ_g to $\Gamma_{g'}$. It may also change a spin structure. In order

to compare spin-structures related to different metrics, and then to define the action of diffeomorphisms on Σ, sort of a canonical isomorphism between the bundles of orthonormal frames for different metrics would be completely sufficient. This may be the case if M has some additional structure (e.g. the complex structure). Related interesting claims were also presented in [Binz and Pferschy, 1983], [Bourguignon and Gauduchon, 1992] and [Hennig and Jadczyk, 1987], but the author is not aware yet of a completely conclusive proof in the case of Lorentzian signature. Our solution uses a reformulation of the definition of spin-structure and spinor fields [Dąbrowski and Percacci, 1986]. First define a (nontrivial) double covering of $GL_+(4)$ by the following commutative diagram

$$\begin{array}{ccc} Spin_o(3,1) & \hookrightarrow & \overline{GL}_+(4) \\ \rho\downarrow & & \downarrow \\ SO_o(3,1) & \hookrightarrow & GL_+(4)\,. \end{array} \qquad (4)$$

Associated with (4) there is a reformulated definition of spin structure, i.e. a double covering of the bundle of all (oriented) linear frames on M.

$$\begin{array}{ccccc} \overline{F_g} & \hookrightarrow & \overline{F} & \stackrel{\psi}{\to} & V \\ \eta_g\downarrow & & \downarrow\eta & & \\ F_g & \hookrightarrow & F\,. & & \end{array} \qquad (5)$$

This is really nothing but a reformulation: the existence conditions are identical (for oriented Lorentzian M) as one easily passes from one definition to the other, with the help of a metric or by the associated bundle method. Also the equivalence of spin structures is preserved. Nevertheless it gives us a possibility to define $\overline{F_g}$ and $\overline{F_{g'}}$ to be equivalent when they originate from the same $\overline{F}$. Note that for a given metric g, spinor fields are functions on $\overline{F}$, which are supported only on $\overline{F_g}$, and which are equivariant with respect to the subgroup $Spin_o(3,1)$ of $\overline{GL}_+(4)$. Note also that $\overline{GL}_+(4)$ is used here merely as a tool and we don't need its representations.

Now, given a diffeomorphism f and some (reformulated) spin-structure $(\overline{F},\eta)$ we ask for a lift $\overline{f}$ of f (and of its derivative Tf)

$$\begin{array}{ccc} \overline{F}' & \stackrel{\overline{f}}{\to} & \overline{F} \\ \eta'\downarrow & & \downarrow\eta \\ F & \stackrel{Tf}{\to} & F \\ \downarrow & & \downarrow \\ M & \stackrel{f}{\to} & M\,. \end{array} \qquad (6)$$

It has been shown that such a lift always exists for precisely one $(\overline{F}',\eta')$. We use this fact to define $(\overline{F}',\eta')$ to be the result of transformation of $(\overline{F},\eta)$ by f (in fact this defines an action of diffeomorphisms on Π, with some nice properties).

Finally, we can define the transformation of ψ by f as

$$\psi' = \psi\circ\overline{f}\,. \qquad (7)$$

It is easily seen that ψ' is a spinor field for the metric f^*g and that (6) defines an action of (a double cover) of diffeomorphisms on spinor fields. The formula (6) has a rather simple meaning. With repect to appropriately chosen local spin frames $\overline{E}'$ in $\overline{F}'$ and $\overline{E}$ in $\overline{F}$, such that $\overline{E} = \overline{f}(\overline{E}')$, the local components of ψ' and ψ are equal; thus we essentially recover the statement "spinors transform as 'scalars' uder coordinate changes", except one important detail that it is not sufficient to refer to the usual linear frames, as is clear from the famous 'sign' ambiguity.

We close with some remarks. The most satisfactory result, the action of diffeomorphisms on the space of spin-structures, was succesfully applied to two-dimensional oriented and nonoriented surfaces with or without boundary [Dąbrowski and Percacci, 1987]. However, some complications arise in the Lorentzian case for bigger structure groups and also for diffeomorphisms, which do not preserve the orientation. The reason is the subtlety that $\overline{GL}(4)$ contains those coverings of $O(3,1)$ (and even of $SO(3,1)$), which are not defined in terms of Clifford algebras. This difficulty should be possible to overcome. A more important issue is that the action of diffeomorphisms on spinor fields is not a representation in a fixed space of spinor fields. As a consequence, given a one-parameter subgroup $\{f\}$ of diffeomorphisms, the notion of a Lie derivative of a spinor fields is still not defined. It is this fact behind the statement that there is no satisfactory action of diffeomorphisms on spinors. A remedy would be a possibility to compare the spaces of spinor fields at least along paths $\{f^*g\}$ in $\mathcal{M}$. For that purpose, a sort of a (natural ?) connection on the infinite dimensional vector bundle (configuration space) $\mathcal{W}$ would be helpful.

References

E. Binz and R. Pferschy, 1983, *C.R.Math.Rep.Acad.Sci. Canada* **V**269
J-P. Bourguignon and P. Gauduchon, 1992, *Commun.Math.Phys.* **144**581
L. Dąbrowski, 1986, *Group Actions on Spinors, Naples Lecture Notes* Bibliopolis, Naples
L. Dąbrowski and R. Percacci, 1986, *Commun.Math.Phys.* **106**691
L. Dąbrowski and R. Percacci, 1987, *J.Math.Phys.* **29**580
R. Geroch, 1970, *J.Math.Phys.* **9**1739
A. Haefliger, 1956, *C.R.Acad.Sci. Paris* **243**558
A. Hennig and A. Jadczyk, 1987, Clausthal T. U. preprint
C.J. Isham, 1978, *Proc.Roy.Soc. London* **362A**383
J. Milnor, 1963, *Enseign.Math.* **9**198

TENSORED DIVISION ALGEBRAS: ORIGIN OF GEOMETRY, SPINORS AND SYMMETRY

GEOFFREY DIXON
Department of Physics
Brandeis University
Waltham, MA 02254
USA

and

Department of Mathematics
University of Massachusetts
Boston, MA 02125
USA

Abstract. A summary of a decade's work on the tensored division algebras is presented, emphasizing the remarkably complete and exact correspondence of this mathematics to the design of our physical reality. The meaning of this correspondence is explored.

1. Playing with Blocks

Clifford Algebras	Lie Groups for Internal Symmetries
$\mathbf{R}_{1,0}$	$\boxed{U(1) \simeq SO(2)}$
$\mathbf{R}_{0,1}$	$SO(3)$
$\mathbf{R}_{2,0}$	$\vdots$
$\vdots$	$\boxed{SU(2)}$
$\mathbf{R}_{2,2}$	$\boxed{SU(3)}$
$\boxed{\mathbf{R}_{1,3}}$	$SU(4)$
$\mathbf{R}_{0,4}$	$\vdots$
$\vdots$	$Sp(2)$
$\boxed{\mathbf{R}_{1,9}?}$	$\vdots$
$\vdots$	

Nature has been too kind to us. Had our mathematics one Clifford algebra corresponding to a single geometry and acting on a solitary spinor space, and did the spinors help describe particles that fell into the only multiplets of a lone internal symmetry, we would have little difficulty explaining our universe. It is the way it is, we would say, because it can not be other than it is.

Instead we are awash in Clifford algebras and their spinors, Lie groups and their multiplets. Out of this surfeit a select few play fundamental roles in the design of physical reality. Why? Larger, encompassing Clifford algebras, groups, or supersymmetries and string theories, offer the hope of combining several unexplained features under a single unexplained banner. But our only hope presently of achieving such a unification is to stumble upon it.

Nature presumably did not stumble upon Her design. It is my belief that what we see of this design is merely the surface, that beneath the surface there is a Truth that functions in the absence of our attempts to perceive it, in the absence of us altogether. Within this Truth are the conditions underlying viable reality and existence, and they are stringent enough to exclude any type of existence other than our own. Flatland is not viable.

With mathematical physics we have gained some understanding, but mathematics and physics are still viewed as different sciences. I believe that to go further we must develop a new instrument to help us see below the surface, a hypermathematics, which would bring mathematics and physics to a common focus. Within this there would be a hypervariational principle with which we might explain the selectness of our physical reality. At the same time, I believe that Truth, and our symbolic representation of it, however encompassing, are distinct.

It is not my intention here to develop this hypermathematics, merely to open a door onto a path that may eventually lead in the right direction.

Division Algebras
R
C
Q
O

As indicated above, there are only four real normed division algebras, the reals, the complexes, the quaternions, and the octonions. Three are already employed in physics. The reals give us the results of measurements, quantum mechanics is done over the complex field, and the Pauli algebra, **P**, without which we could not describe fermions, is isomorphic to $\mathbf{R} \otimes \mathbf{C} \otimes \mathbf{Q}$. On the mathematical end, in Lie group theory, the sequences of orthogonal, unitary, symplectic, and special classical Lie groups arise, respectively, from the algebras **R**, **C**, **Q**, and **O**. In topology the parallelizable spheres, S^1, S^3, and S^7 arise from **C**, **Q**, and **O** (S^0, the null sphere, trivially parallelizable, arises from **R**).

The sequence of division algebras is unique, finite, and generative in both mathematics and physics. Wouldn't it be great if it were in fact an integral part of the design of our universe. I believe that it is. I have demonstrated that it is [1], which tends to confirm me in my belief.

This belief has been shared by many other physicists and mathematicians. Many attempts have been made to make a quantum mechanics based on **Q** or **O** instead of **C**. Because **C** is a subalgebra of **Q**, and **Q** is a subalgebra of **O**, it seems superficially reasonable that in making the jump to **Q** or **O**, we may discard the smaller algebras, which are in any case present as subalgebras. In my opinion this is an example of confusing Truth with the symbolic tools we use to explore Truth. The definition of an algebra did not miraculously appear on a stone tablet; we made it up. And Nature is not obliged to order reality to fit our prejudices.

The sum of all the properties associated with each of the division algebras exceeds those included in the definition of an algebra. Let $\mathbf{K}_L$ and $\mathbf{K}_R$ be the adjoint algebras of left and right actions of the algebra **K** on itself, $\mathbf{K} =$ **R**, **C**, **Q**, or **O**. Then, for example,

$\mathbf{C}_L$ and $\mathbf{C}_R$ are the same algebra, both isomorphic to **C**,
$\mathbf{Q}_L$ and $\mathbf{Q}_R$ are distinct algebras, both isomorphic to **Q**,
$\mathbf{O}_L$ and $\mathbf{O}_R$ are the same algebra, both isomorphic to $\mathbf{R}(8)$,

where $\mathbf{R}(8)$ is the algebra of 8×8 real matrices. (I have found the natural octonion products more useful than those based on defining **O** as an extension of **Q**. In particular, given a basis e_a, a=1,...,7, for the hypercomplex part of **O**, and given that distinct e_a anticommute, that $e_a(e_a e_b) = (e_b e_a)e_a = -e_b$, that $e_a^2 = -1$, then the rest of the multiplication table is determined by the equation $e_a e_{a+1} = e_{a+5}$, indices modulo 7, from 1 to 7. The ease of this multiplication is enhanced by the following property: if $e_a e_b = e_c$ then $e_{(2a)} e_{(2b)} = e_{(2c)}$. So, $e_1 e_2 = e_6$ yields, via this index doubling property, $e_2 e_4 = e_5$.) My point, if it is not already plain, is that the algebraic inclusion property is irrelevant. These algebras stand on their own. I am suggesting that the physical relevance table for division algebras should look like this:

Division Algebras
R
C
Q
O

That is, the algebra

$$\mathbf{H} = \mathbf{R} \otimes \mathbf{C} \otimes \mathbf{Q} \otimes \mathbf{O}$$

must form part of the algebraic design of reality. I assume it underlies the Clifford algebra/spinor, Lie group/multiplet structure of our universe, and with physical reality as a guide (as little as possible), I shall outline the way in which this may be seen to be true.

2. Playing with H

The individual division algebras may be associated with Clifford algebras in the following ways:

$$\begin{aligned} \mathbf{C}_L &\simeq \mathbf{R}_{0,1}, \\ \mathbf{Q}_L &\simeq \mathbf{R}_{0,2}, \\ \mathbf{O}_L &\simeq \mathbf{R}_{0,6}. \end{aligned}$$

(Note that **C**, **Q** and **O** are 2^k-dimensional, k=1,2,3, and their left adjoint algebras are isomorphic to the Clifford algebras of spaces of dimension k!. This may be accidental, but there is a more natural way of extending the sequence than the Cayley-Dickson prescription, and I suggest further work in that direction to resolve this interesting dimensionality question.)

The spinor spaces of these Clifford algebras are just **C**, **Q** and **O**, the object spaces of $\mathbf{C}_L$, $\mathbf{Q}_L$ and $\mathbf{O}_L$. Of the three only $\mathbf{Q}_L$ does not act effectively. $\mathbf{Q}_R$ provides an internal degree of freedom. The group of elements of $\mathbf{Q}_R$ of unit norm is $SU(2)$.

Two physically relevant isomorphisms are

$$\mathbf{P}_L = \mathbf{R}_L \otimes \mathbf{C}_L \otimes \mathbf{Q}_L \simeq \mathbf{C}(2) \simeq \mathbf{R}_{3,0}, \tag{1}$$

and

$$\mathbf{P}_L(2) \simeq \mathbf{C}(4) \simeq \mathbf{C} \otimes \mathbf{R}_{1,3}, \tag{2}$$

this last being the complexification of the Clifford algebra of (1,3)-Minkowski space. This is the Dirac algebra. The spinors of $\mathbf{P}_L(2)$, namely $\mathbf{P}^2$ (the space of 2x1 matrices over **P**) are more complicated than those of $\mathbf{C}(4)$ in having the $\mathbf{Q}_R$ internal freedom. Each $\mathbf{P}_L(2)$ spinor contains a pair of ordinary Dirac spinors.

The isomorphisms (1) are just mathematics, but those in (2) are arrived at only by following the lead of physics, which we know rests on the Dirac algebra. I will follow this lead a bit further with the following isomorphisms:

$$\mathbf{H}_L \simeq \mathbf{C}(16) \simeq \mathbf{R}_{0,9}, \tag{3}$$

and

$$\mathbf{H}_L(2) \simeq \mathbf{C}(32) \simeq \mathbf{C} \otimes \mathbf{R}_{1,9}. \tag{4}$$

There are, of course, indications in other branches of theoretical physics that $\mathbf{R}^{1,9}$ is a physically relevant geometry. Because the complexification of its Clifford algebra is to **H** what the Dirac algebra is to **P**, I believe it necessarily is, although the role of the extra six dimensions is not yet clear.

Step back to just **H** and its adjoints for a moment. **H** is not a division algebra (it is not even alternative), and its identity admits nontrivial resolutions into four orthogonal primitive idempotents. There are at least three (I suspect no more) inequivalent resolutions of this identity, but only one (into elements $\Delta_m, m = 0,1,2,3, \ni \Delta_m\Delta_n = \delta_{mn}\Delta_m$ (orthogonality), and $\Delta_0 + \Delta_1 + \Delta_2 + \Delta_3 = 1$) which satisfies the associativity conditions

$$\begin{aligned} \Delta_m(X\Delta_n) &= (\Delta_m X)\Delta_n, \\ \Delta_m(\Delta_n X) &= (\Delta_m\Delta_n)X, \\ (X\Delta_m)\Delta_n &= X(\Delta_m\Delta_n) \end{aligned} \tag{5}$$

for all $X \in \mathbf{H}$.

These associativity conditions allow us to consistently define the components $\Delta_m X \Delta_n$ of elements $X \in \mathbf{H}$ with respect to the Δ_m. Of particular interest are the diagonal components

$$\Delta_m X \Delta_m = X_m \Delta_m, \tag{6}$$

where $X_m \in \mathbf{C}$. (That is, in a mathematical sense, with respect to the resolution $\{\Delta_m\}$, **H** is a complex algebra. Therefore, if **H** is indeed part of the design of reality, it is not surprising that physics is inherently complex as well.)

It is possible to design a group of **H**-automorphisms $M_m, m = 0,1,2,3, \ni M_m(\Delta_0) = \Delta_m$ (M_0 = identity map). With their help, and the help of Hermitian conjugation ($X \to X^\dagger$), we can define the real part of the trace of $X \in \mathbf{H}$:

$$Re(tr(X)) = \frac{1}{2}\sum_{mn}\{[M_n(\Delta_m X\Delta_m)] + [same]^\dagger\} = \frac{1}{2}\sum_m (X_m + X_m^*)$$

(note: $\Delta_m^\dagger = \Delta_m$). Likewise, if $X, Y \in \mathbf{H}$, we may construct their inner product:

$$< X, Y >= \frac{1}{8} \sum_{mn} \{[M_n((X\Delta_m)^\dagger (Y\Delta_m))] + [same]^\dagger\}. \tag{7}$$

(= real part of $X^\dagger Y$ or $Y^\dagger X$).

The Δ_m in (7) may be replaced by more general Γ_m satisfying

$$\Gamma_m^\dagger \Gamma_n = \delta_{mn} \Delta_n \text{ (no sum)}, \tag{8}$$

and an associativity relation like (5),

$$\Gamma_m^\dagger (X\Gamma_n) = (\Gamma_m^\dagger X)\Gamma_n \tag{9}$$

for all $X \in \mathbf{H}$. The symmetry of (8) and (9) is somewhat complicated, but on each of the Γ_m it boils down to

$$U(2) \times U(3) \tag{10}$$

(in fact, they are each $U(3)$-invariant, but not $U(2)$ invariant, and that $SU(2)$ is chiral, $SU(3)$ nonchiral, is eventually seen to result from this $U(2)/U(3)$ distinction).

I now let physics take the lead and define $\psi \in \mathbf{H}$ which transforms with respect to the symmetry (10) like $\Gamma_0^\dagger$ (any other $\Gamma_m^\dagger$ would have done as well). Since ψ is a general element of $\mathbf{H}$, and $SU(3)$ is a subgroup of the invariance group of $\mathbf{O}$ (ie., G_2), ψ is not $U(3)$ invariant. With respect to $SU(3)$ it transforms as

$$\mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{1}} \oplus \bar{\mathbf{3}}. \tag{11}$$

In fact, with respect to the symmetry (10), ψ transforms exactly like a family plus antifamily, including all correct quantum numbers, of lefthanded leptoquark Weyls spinors (ie., the individual invariant vector components are 2-dimensional over $\mathbf{C}$).

The Δ_m can be expressed as follows:

$$\begin{aligned} \Delta_0 &= \lambda_+ \rho_+, \\ \Delta_1 &= \lambda_- \rho_+, \\ \Delta_2 &= \lambda_+ \rho_-, \\ \Delta_3 &= \lambda_- \rho_-, \end{aligned} \tag{12}$$

where $\lambda_\pm$ are $SU(2)$ eigenvector projectors, and $\rho_\pm$ are $SU(3)$ multiplet projectors (the fact that the Δ_m break ψ down to the vector level with respect to $SU(2)$, and to the multiplet level with respect to $SU(3)$, is eventually seen to be the explanation for why $SU(2)$ breaks and $SU(3)$ is exact). In particular, $\rho_+\psi$ transforms under $SU(3)$ like $\mathbf{1} \oplus \mathbf{3}$, and $\rho_-\psi$ like $\bar{\mathbf{1}} \oplus \bar{\mathbf{3}}$. So the operation of $\rho_\pm$ from the left projects the matter/antimatter half of ψ.

What about righthanded leptoquark Weyl spinors? The symmetry (10) is carried by the M_m from Γ_0 to the other Γ_m. In fact, since the Γ_m are $U(3)$ invariant, only $U(2)$ is nontrivially carried. Therefore each of the elements $\rho_+\psi\Gamma_m$ have the same $U(3)$ charges as $\rho_+\psi\Delta_m$, but altered $U(2)$ charges (in fact, they are $SU(2)$ singlets). They have precisely the charges observed on righthanded leptoquark spinors (ρ_+ from the left assures us that this is matter; the antimatter half is treated similarly).

The particle assignments are

lefthanded	righthanded	assignment
$\rho_+\psi\Delta_0$	$\rho_+\psi\Gamma_0$	neutrino
$\rho_+\psi\Delta_1$	$\rho_+\psi\Gamma_1$	(-1)-lepton
$\rho_+\psi\Delta_2$	$\rho_+\psi\Gamma_2$	(2/3)-quark
$\rho_+\psi\Delta_3$	$\rho_+\psi\Gamma_3$	(-1/3)-quark

(13)

Left- and righthanded assignments can be simultaneously manifested in the larger context of $\mathbf{H}_L(2)$ and its spinor space $\mathbf{H}^2$. The details are developed elsewhere [1].

There is more one can derive from the mathematics, including a natural weak mixing and spontaneous symmetry breaking, but I want to conclude this section with a look beyond the Standard Model. Most of the early development was based on $\mathbf{H}^2$-fields Ψ considered functions of (1,3)-Minkowski space. I recently looked at what would happen if this were expanded to a functional dependence on (1,9)-space. The extra six dimensions carry $SU(3)$ charges. I have demonstrated that inorder for the (1,9)-Dirac operator to reduce to the (1,3)-Dirac operator, the lepton fields above must be independent of the extra six dimensions, and the quark field may depend only upon parameters carrying $SU(3)$ charges parallel to those carried by the fields themselves. The reason this is essential is phenomenological. With respect to (1,9)-Minkowski space, matter and antimatter are indistinguishable, and a Lagrangian of the form

$$\begin{aligned} \mathcal{L}_{1,9} &=< \Psi, \partial\!\!\!/_{1,9}\Psi > \\ &=< \rho_+\Psi, \partial\!\!\!/_{1,3}\rho_+\Psi > + < \rho_-\Psi, \partial\!\!\!/_{1,3}\rho_-\Psi > \\ &\quad + < \rho_+\Psi, \partial\!\!\!/_{0,6}\rho_-\Psi > + < \rho_-\Psi, \partial\!\!\!/_{0,6}\rho_+\Psi >, \end{aligned} \tag{14}$$

where the (1,9)-Dirac operator $\partial\!\!\!/_{1,9}$ is constructed on $\mathbf{H}_L(2)$, gives rise to unmediated (or space mediated) matter into antimatter transitions via the last two terms above. Such transitions have not been observed. They disappear from the mathematics if the lepto-quark functional dependencies on the extra six dimensions are arranged as outlined above (ie., the last two terms in (14) are identically zero in this case).

3. If the Shoe Fits, Wear It

Division Algebras

R | **C** | **Q** | **O**

Clifford Algebras: $\mathbf{R}_{1,3}$, $\mathbf{R}_{1,9}$

Internal Symmetries: $U(2)$, $U(3)$

Multiplets, Quantum Numbers, and Families
Parity Nonconservation and Chirality
⋮

For some reason Nature sprouts only certain geometries and symmetries and multiplets and no others. A finite number are chosen, infinitely many rejected. There must exist a kernel - a seed - which guides Nature in its very select development. **H** does this, simply, elegantly, and at a mathematical level below geometry and symmetry, using algebraic objects of great and generative importance to many branches of our mathematics. As I see it, while it is necessary now to use physics to guide the researcher in the development of this mathematical application, ultimately other generative mathematical ideas would be included, and physics could then be derived from pure mathematics, a mathematics in many respects unlike any we have yet seen. This would take us down a little deeper, a little closer to whatever ultimate Truths generate reality. Quite frankly I do not imagine these truths are totally accessible to us. But no matter, this just means there'll always be records to break. In the meantime, whether it be based on strings or twistors or ought else, no theory not based on **H** has a chance of succeeding. Nature is the way it is, I would say, because it can not be other than it is.

References

[1] In a series of papers -
Geoffrey Dixon: *Il Nuovo Cim* **105B** 349(1990).
Geoffrey Dixon: *Phys. Lett.* **152B** 343(1985).
Geoffrey Dixon: *Phys. Rev.* D **29** 1276(1984).
Geoffrey Dixon: *Phys. Rev.* D **28** 833(1983).
- I developed most of the features outlined in this paper. However, as these ideas evolved some notions supplanted others (in particular, there is no mention of adjoint algebras in the earlier papers). I am presently at work on a book in which a decade of mathematical and physical research will be presented as a unified and coherent whole.

G – STRUCTURE FOR HYPERMANIFOLD

ANDRZEJ BOROWIEC
Institute of Theoretical Physics
University of Wrocław
pl. Maxa Borna 9, 50–204 Wrocław,
Poland, e-mail: borowiec@plwruw11.bitnet

Abstract. This paper extends the earlier version (Borowiec 1989). It points out the fact that Riemannian geometry plays a very exceptional role among geometries represented by a symmetric tensor $g_{\mu_1 \ldots \mu_n}$ of degree n. In particular, we question the claim made by Holm (1986) that there exist torsion free Christoffel coefficients on a hyperspin manifold. we develope a G–structure formalism for hypermanifolds.

Key words: G–structure, tensor concomitant, connection, chronometric, hypermanifold

1. Introduction

Recently David Finkelstein (1986) proposed a new idea in physics – a hyperspin theory. While Riemannian geometry is based on a second–rank symmetric metric tensor $g_{\mu\nu}$, he suggested replacing it by an n- symmetric form $g_{\mu_1 \ldots \mu_n}$ called a chronometric. A manifold M equipped with such 0–deformable tensor will be called a hypermanifold. The name hyperspin manifold was reserved for a particular kind of hypermanifold. C. Holm (1986) discussed the geometry of hyperspin manifolds in a complete analogy with the metric geometry. Our purpose is to explain why some important concepts of Riemannian geometry work differently in the chronometric case. In particular, we will criticize some oversimplified statements formulated by C. Holm.

We will confine ourselves to the case of $n = 3$ (i.e. a cubic metric). A generalization from 3 to n is evident. A flat space with a cubic metric and the construction of the classical and quantum mechanics of a particle moving in it has been recently considered by Yamaleev (1989).

Let $\eta = (\eta_{ijk})$ be a symmetric 3–rd rank covariant tensor on a linear space $V = I\!R^N$, i.e. $\eta \in V^* \odot V^* \odot V^*$ where $\odot$ denotes a symmetric tensor product. Assume that η is nondegenerate in the following sense: the mapping (also denoted by η)

$$V \ni (x^i) \stackrel{\eta}{\longrightarrow} (v_{kl} = x^i \eta_{ikl}) \in V^* \odot V^* \qquad (1)$$

* Dedicated to Prof. Jan Rzewuski
** Supported by KBN under Grant PB # 2 2419 03

is injective (we use the Einstein summation convention). The Riemannian case is an exception because (1) becomes an isomorphism of V onto V^*. Decomposing $V^* \odot V^* = L'' \oplus L'$ into a direct sum, where $L'' = Im\ \eta$, one can define (depending of the choice of L') the linear mapping called an inverse or dual chronometric $\tilde{\eta} : V^* \odot V^* \to V$, by

$$\tilde{\eta}|_{L'} = 0 \qquad \tilde{\eta}|_{L'} = \eta^{-1} \tag{2}$$

By definition $\tilde{\eta} = \left(\eta^{ijk}\right)$ is any solution of the eq.

$$\eta^{kij}\ \eta_{ijm} = \delta^k_m \tag{3}$$

Notice that in general η^{kij} will be symmetric only in the last two indices. Even if there exists a completely symmetric solution of (3) it is nonunique. In the following we will fix some dual tensor $\tilde{\eta}$.

It is convenient to introduce the tensor $\lambda^{ij}_{kl} = \eta^{mij}\eta_{mkl}$. Then (3) guarantees the following properties of $\lambda : V^* \odot V^* \longrightarrow V^* \odot V^*$

$$\begin{gathered}\lambda^{ij}_{kl}\,\lambda^{kl}_{mn} = \lambda^{ij}_{mn} \\ \lambda^{ij}_{kl}\eta^{mkl} = \eta^{mij}, \qquad \lambda^{ij}_{kl}\eta_{mij} = \eta_{mkl}\end{gathered} \tag{4}$$

This means that λ is a projection operator on a subspace L'' along the direction L". Immediately one obtains that the conditions

$$\lambda^{ij}_{kl}v_{ij} = v_{kl} \tag{5}$$

as sufficient and necessary for an element $(v_{ij}) \in V^* \odot V^*$ to be a value of some vector $(v^m = \eta^{mij}v_{ij}) \in V$ under the action of (1).

2. Tensorial G–structures

Let LM denote the frame bundle over an N–dimensional paracompact manifold M. LM is a principal $GL(\mathbb{R}, N)$–bundle. Consider a tensor field $t \in \Gamma\left(T^p_q M\right)$ of type (p, q) on M. Let $\overset{\vee}{t} :\ LM \to T^p_q\left(\mathbb{R}^N\right)$ denote the corresponding equivariant mapping (see e.g. Gancarzewicz 1987). A tensor t is called 0–deformable if $GL(\mathbb{R}, N)$ acts transitively on $\overset{\vee}{t}(LM) \subset T^p_q(\mathbb{R}^N)$. Let $\tau \in \overset{\vee}{t}(LM)$ and $G(\tau) \subset GL(\mathbb{R}, N)$ be the isotropy subgroup of τ. 0–deformability of t is equivalent to the existence of a subbundle $P(\tau)$ of LM on which t is constant and equal to τ. τ is called a canonical form of t. $P(\tau)$ is a $G(\tau)$–structure on M. If τ' is an another element in $\overset{\vee}{t}\,(LM)$ then $P(\tau)$ and $P(\tau')$ are isomorphic. Notice that there is no canonical isomorphism between $P(\tau)$ and $P(\tau')$.

A tensor field $s \in \Gamma\left(T^{p'}_{q'}M\right)$ is called a concomitant of $t \in \Gamma\left(T^p_q M\right)$ if there exists a $GL(\mathbb{R}, N)$–equivariant map $f : \overset{\vee}{t}\,(LM) \longrightarrow \overset{\vee}{s}(LM)$ such that $\overset{\vee}{s} = f \circ \overset{\vee}{t}$. By definition each equivariant map produces some concomitant. If t is 0–deformable and s is a concomitant of t then $\overset{\vee}{s}$ is constant on $P(\tau)$ and $G(\tau) \subset G(f(\tau))$ (Zajtz 1985).

A given linear connection Γ on M is said to be t–connection if $\nabla t = 0$. The existence of a t–connection is equivalent to the 0–deformability of t.

A $G(t)$–structure $P(\tau)$ is said to be integrable if every point $x \in M$ has a coordinate chart (x^μ) such that the (local) cross section $\left(\partial/\partial x^1, \ldots, \partial/\partial x^N\right)$ of LM is a (local) cross section of $P(\tau)$. An integrable G–structure is locally flat and it admits a torsion free connection (Kobayashi 1972).

Kobayashi and Nagano (1965) have found all subgroup G of $GL(\mathbb{R}, N)$ which satisfy the following condition: every G–structure P on a manifold M admits a torsionfree connection. It should be noticed that the group $G(\eta)$ does not belong to this class, where η is taken from the previous Section.

Example.

Each (pseudo-)–Riemannian metric on manifold M is 0–deformable. The metric $g = (g_{\mu\nu})$ and its inverse $\tilde{g} = (g^{\mu\nu})$ are mutually concomitant to each other. The Riemannian connection is a unique torsionfree g–connection on M.

3. Hypermanifolds

Let M be an N–dimensional manifold with given $G(\eta)$–structure, where $\eta = V^* \odot V^* \odot V^*$. Let $g = (g_{\mu\nu\lambda})$ denote the corresponding tensor field on M. According to the hyperspin philosophy one has to assume that there exists a tensor concomitant $\tilde{g} = \left(g^{\mu\nu\lambda}\right)$ of g such that

$$g_{\mu\nu\alpha}g^{\mu\nu\beta} = \delta^\beta_\alpha \tag{6}$$

The resulting structure will be called a hypermanifold. It is also convenient to introduce a tensor concomitant h by

$$h^{\mu\nu}_{\alpha\beta} = g^{\mu\nu\rho}g_{\rho\alpha\beta} \tag{7}$$

The algebraic relations (3), (4) and (5) remain valid for tensor fields g, $\tilde{g}$, and h.

Let $\Gamma^{\ \lambda}_{\mu\nu}$ be the connection coefficients for some g–connection on M. From $\nabla g = 0$ one gets

$$\partial_\mu g_{\nu\lambda\rho} = \Gamma_{\mu\nu\lambda\rho} + \Gamma_{\mu\lambda\rho\nu} + \Gamma_{\mu\rho\nu\lambda} \tag{8}$$

where $\Gamma_{\mu\nu\lambda\rho} = \Gamma^{\ \alpha}_{\mu\nu} g_{\alpha\lambda\rho}$. Holm (1986) has found a "unique torsionfree" solution of (8) of the form

$$\Omega_{\mu\nu\lambda\rho} = \frac{1}{3}\left(\partial_\mu g_{\nu\lambda\rho} + \partial_\nu g_{\mu\lambda\rho}\right) - \frac{1}{6}\left(\partial_\rho g_{\lambda\mu\nu} + \partial_\lambda g_{\rho\mu\nu}\right) \tag{9}$$

Indeed, (9) is an algebraic solution of (8) and is symmetric in the first pair of indices. Unfortunately the $\Omega_{\mu\nu\lambda\rho}$ do not transform into themselves under the gauge transformations, so they depend on the choice of coordinates. In particular (9) does not yield a global object. The next remark is that the solutions (9) are not connection coefficients at all. To see this let us observe that $\Omega_{\mu\nu\lambda\rho}$ does not satisfy the constraints (5), i.e.

$$h^{\lambda\rho}_{\alpha\beta}\Omega_{\mu\nu\lambda\rho} \neq \Omega_{\mu\nu\alpha\beta} \tag{10}$$

and hence there do not exist coefficients $\Gamma^{\ \lambda}_{\mu\nu}$ such that $\Omega_{\mu\nu\alpha\beta} = \Gamma^{\ \lambda}_{\mu\nu} g_{\lambda\alpha\beta}$.

For a paracompact M, a g–connection always exists, but the problem of finding of a canonical g–connection of M remains open. If it even exists it will be in general with torsion.

4. Particle Trajectories

It follows from the above considerations that one cannot use of the geodesic principle of general relativity in order to obtain particle trajectories in a space–time M with a geometry represented by the tensor field g.

Let $\delta g = g' - g$ b e a symmetric tensor of type (0 for matter regularly concentrated on a curve $K \subset M$ can be written in the form

$$\int_K \left(\sigma^{\mu\nu\lambda}\delta g_{\mu\nu\lambda}\right) ds = 0 \tag{11}$$

where the x^μ's are a coordinate system on M and K respectively, and $\sigma^{\mu\nu\lambda}$ denotes a density on K with values in symmetric tensors of type $(0,3)$. $\delta g_{\mu\nu\lambda}$ is of the form $L_Y g_{\mu\nu\lambda}$ with Y being any vector field on M. Then after some long calculations and using the methods developed by Jadczyk (1983), one finds

$$\frac{d}{ds}\left(v^\mu\ v^\nu g_{\mu\nu\lambda}\right) = -\frac{1}{3} v^\mu v^\nu \partial_\lambda g_{\mu\nu\rho} = 0 \tag{12}$$

where $v^\mu = \frac{dx^\mu}{ds}$ are velocity components. It is remarkable that the same equations can be derived from the variational principle for the functional $\int_a^b \sqrt[3]{g_{\mu\nu\lambda} v^\mu v^\nu v^\lambda}\ ds$.

As an accidental result we should noticed that (12) turns out to be "geodesic motion" for Holm's connection.

Acknowledgment

I am grateful to A. Jadczyk for valuable suggestions.

Note added

After completing this work, the author has become aware of an interesting paper by Urbantke (1989) in which the historical remarks on the "space problem of Weyl" are also contained.

References

Borowiec, A., (1989) Int. Jour. Theor. Phys. **28** (10) 1229
Finkelstein, D., (1986), Phys. Rev. Lett. **56** (15) 1532
Finkelstein, D., Finkelstein, S.R., Holm, C. (1986), Int. Journ. Theor. Phys. **25** (4) 441
Holm, C. (1986) Int. Journ. Theor. Phys. **25** (11) 1209
Jadczyk, A. (1983) Annales de L'Institut Henri Poincare - Section A, **XXXVIII** (2) 99
Gancarzewicz, J. (1987) "Differential Geometry" PWN, Warsaw (in Polish)
Kobayashi, S. (1972), "Transformation Groups in Differential Geometry", Springer – Verlag, Berlin
Kobayashi, S., Nagano, T., (1965) J. Math. Soc. Japan, **17** (1) 84
Souriau, J.M. (1974), Annales de l'Institute Henri Poincare - Section A, vol. XX, 967
Yamaleev, R.S. (1989), "On Geometric Form in Three Dimensional Space with Cubic Metric", Communications, JINR P5 - 89 - 269
Yamaleev, R.S. (1989), JINR Rapid Communications, **1** (34) 50
Urbantke, H., (1989), Int. Journ. Theor. Phys. **28** (10) 1233
Zajtz, A. (1985), Colloquium Mathematicum XLIX (2) 232

TOWARDS A UNIFICATION OF "EVERYTHING" WITH GRAVITY

JERZY RAYSKI
Institute of Physics
Jagellonian University, Kraków, Poland

Abstract. Combining the ideas of gauge interactions with a global supersymmetry a unified model in six dimensions is built up step by step starting with a single generation of leptons and ending with three generations of leptons and coloured quarks forming a supermultiplet characterized by a most general extension $N = 8$. The puzzle of supersymmetric partners like gravitino, photino, s-leptons and s-quarks is seen in a new light.

Let us begin with a simple model of one generation of leptons interacting electro-weakly and gravitionally. Consider a set of fields with the same number of fermionic and bosonic degrees of freedom consisting of one tensor field, two Rarita-Schwinger fields, four vector fields, six Weyl fields and six scalars. All these fields are at first massless two-component fields except for one-component scalars. This set may be split either into ($N = 1$)- supermultiplets or into extended ($N = 2$)-supermultiplets (see the tables 1 and 2 where the rows denote fields with spins $2, 3/2, 1, 1/2$ and 0).

table 1

$$\begin{bmatrix}1\\2\\4\\6\\6\end{bmatrix} = \begin{bmatrix}1\\1\\-\\-\\-\end{bmatrix} + \begin{bmatrix}-\\1\\1\\-\\-\end{bmatrix} + 3\times\begin{bmatrix}-\\-\\1\\1\\-\end{bmatrix} + 3\times\begin{bmatrix}-\\-\\-\\1\\2\end{bmatrix}$$

table 2

$$\begin{bmatrix}1\\2\\4\\6\\6\end{bmatrix} = \begin{bmatrix}1\\2\\1\\-\\-\end{bmatrix} + 3\times\begin{bmatrix}-\\-\\1\\2\\2\end{bmatrix}$$

The first column to the right of the table 1 is interpretable as graviton and gravitino. The second represents photon and photino whose spin is unexpectedly not smaller but higher by 1/2 from unity. The triplet appearing in the third column denotes three vector fields $W^{\pm}$ and Z^0 intermediating weak interactions and is combined with a triplet of their supersymmetric partners with spins 1/2 to be called W-ino and Z-ino. A triplet of Weyl fields appearing in the last column may be interpreted as a triplet of weakly interacting leptons, possibly e_R, e_L and ν_L^e. Two of them *viz.* e_R, e_L may be fused into a 4-component Dirac field but the third Weyl field has no partner with an oppposite helicity and, consequently, exhibits a chiral character of the weak interactions. Also the number of Z-ino is unity, *i.e.* is an odd number and, together with Z^0 must violate parity conservation of weak coupling.

The next problem is that of spontaneous symmetry breaking. Three of six scalars appearing in the tables 1 and 2 should be swallowed up by the triplet of vector fields appearing in the third row of table 1 endowing them with big masses in agreement with experimental evidence. It confirms our previous guess that these fields are $W^{\pm}$ and Z^0. Similarly two of the Weyl fields, *viz.* W- ino's have also to be swallowed up by the two spin 3/2 fields endowing them with very big masses which fact explains why corresponding particles could not be observed yet. The table 3 shows the results of spontaneous symmetry breakdown.

table 3

$$\begin{bmatrix} 1 \\ 2 \\ 4 \\ 6 \\ 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 \\ 2' \\ 1+3' \\ 1+3 \\ 1+2 \end{bmatrix}$$

where "prime" denotes "heavy". It is seen that besides a triplet of leptons: electron and left-handed neutrino one more Weyl field has been playing a twofold role: it is a Z-ino which at the same time may be regarded as a right-handed neutrino. It is excluded from weak interactions but it is involved into a supersymmetric interaction within a local super-doublet with two scalars. According to the presence of a triplet and a singlet of vector fields, the symmetry of gauge interactions is $G = U(1) \times SU(2)$. The above sketched model of a lepton generation appears much simpler and intelligible if looked upon from a six-dimensional viewpoint. Let us assume that spacetime is six-dimensional with topology of $M_4 \times S_2$ or $AdS \times S_2$ where S_2 is a two-dimensional spherical surface. Its radius is assumed to be extremely small.

The metric field is

$$(g_{MN}) = \left(\begin{array}{c|c} g_{\mu\nu} & g_{\mu\eta} \\ \hline g_{\xi\nu} & g_{\xi\eta} \end{array}\right) \tag{1}$$

with $M, N = 0, 1, ..., 5$; $\mu, \nu = 0, ..., 3$; and $\xi, \eta = 4, 5$. The mixed metric tensor components $g_{\mu\xi}$ appearing as if components of two fourvectors assume the form [4]

$$g_{\mu\xi} = \sum_{a|1}^{3} A_\mu^a K_\xi^a \tag{2}$$

where K_ξ^a are the Killing vectors of a sphere. The fields A_μ^a are to be identified just with $W_\mu^{\pm}$ and Z_μ^0 exhibiting the symmetry of a sphere being a fundamental representation of $SU(2)$. From the point of view of Minkowskian observer the components $g_{\xi\eta}$ look like scalars.

Three fourvectors A_μ^a have been incorporated intrinsically into the six-dimensional metric field g_{MN} in agreement with Kaluza's original assumption however the electromagnetic field is not interpretable as a constituent of generalized metric but denotes the first four components of a six-vector:

$$V_M = \{V_\mu, V_\xi\} \tag{3}$$

It follows that the common view that all vector fields have a metrical origin may be regarded as a prejudice. The extra components of the six-vector V_ξ form something like a "tail" and look like scalars for macroscopic observers. They may be identified with two apparent scalars appearing in the last row of the table 3 so that the number of genuine scalars in our scheme reduces to a singlet a Goldstone scalar.

Inasmuch as the number of independent components of the generalized metric field is eleven (two $g_{\mu\nu}$, six A_μ^a and three $g_{\xi\eta}$) while that of massles Rarita-Schwinger field components in $D = 6$ is twelve whereas the numbers of components of massles vector fields as well as of Weyl spinors in $D = 6$ is four the table 1 may be rewritten and simplified enormously, see the table 4

table 4

$$\begin{bmatrix} 1 \\ 2 \\ 4 \\ 6 \\ 6 \end{bmatrix}_{D=4} \longrightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}_{D=6}$$

which justifies *ex post* our initial choice of the multiplet appearing to the left hand sides of the tables 1,...,4.

Three generations of leptons.

In order to perform a transition to a triplet of leptonic generations we proceed as follows: Assume a (reducible) supermultiplet involving 1 tensor field, 4 Rarita-Schwinger spinors, 12 vector fields, 24 Weyl spinors and 30 scalars. This supermultiplet consists of 56 fermionic and 56 bosonic degrees of freedom and splits into irreducible supermultiplets characterized by the following indices of extension: once $N = 4$, six times $N = 2$ and eight times $N = 1$ as seen from the table 5

table 5

$$\begin{bmatrix} 1 \\ 4 \\ 12 \\ 24 \\ 30 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 6 \\ 4 \\ 2 \end{bmatrix} + 6 \times \begin{bmatrix} - \\ - \\ 1 \\ 2 \\ 2 \end{bmatrix} + 8 \times \begin{bmatrix} - \\ - \\ - \\ 1 \\ 2 \end{bmatrix} .$$

Applying Higgs mechanism for spontaneous symmetry breaking eleven from twelve vector fields acquire considerable masses by swallowing up eleven scalar fields. Similarly four Rarita-Schwinger fields become very heavy too by swallowing up four of the Weyl spinors as is to be seen from the table 6

table 6

$$\begin{bmatrix} 1 \\ 4 \\ 12 \\ 24 \\ 30 \end{bmatrix} \overrightarrow{ssb} \begin{bmatrix} 1 \\ 4' \\ 1+3'+8' \\ 12+8 \\ 18+1 \end{bmatrix}_{D=4} = \begin{bmatrix} 1 \\ 2' \\ 1+8' \\ 6 \\ 1 \end{bmatrix}_{D=6} .$$

According to the appearance of a set of $1 + 3' + 8'$ vector fields in the third row of the middle column of the table 6 it is seen that the gauge group is

$$G_l = U(1) \times SU(2) \times SU(3) \tag{4}$$

due to the fact that the octet of heavy vector fields means a fundamental representation of the $SU(3)$ group. The particles forming this octet may be called para-gluons. They must possess a considerable mass value in order to prevent a quick decay of higher into lower generations.

The adequacy of the $SU(3)$ symmetry group is confirmed also by a discussion of the set of fields appearing in the fourth column of the table 6. Their number 24 splits naturally into $12 + 8 + 4$ wherefrom four have to be swallowed up by the Rarita-Schwinger fields, further eight are also related to Rarita-Schwinger fields inasmuch as they form their "tails" if going over to a six-dimensional description so that finally we are left with only 12 two-component (or six fourcomponent) Weyl spinors in four (or six) dimensions. These numbers are factorizable by the factor 3, *viz.* $12 = 2 \times 2 \times 3$ which

means that the corresponding spinor fields form triplets. Thus the number 12 of Weyl spinors denotes nothing else but three generations of leptons. They include also right-handed neutrina although the latter do not participate in weak interactions.

The 18 scalars appearing in the last row of the middle column of the table 6 mean "tails", *i.e.* additional components of the $8 + 1$ six-vectors so that finally we are left with only one single genuine scalar a Goldstone boson if regarding and interpreting the multiplet from a 6-dimensional viewpoint.

Three generations of coloured quarks

Let us consider now a supermultiplet consisting of 96 bosonic and 96 fermionic degrees of freedom, *viz.* 1 tensor, 6 Rarita- Schwinger spinors (spin 3/2), 20 vectors, 42 Weyl spinors and 54 scalars. It forms a (reducible) supermultiplet splitting into the following irreducible constituents (see table 7)

table 7

$$\begin{bmatrix} 1 \\ 6 \\ 20 \\ 42 \\ 54 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 16 \\ 26 \\ 30 \end{bmatrix} + 4 \times \begin{bmatrix} - \\ - \\ 1 \\ 4 \\ 6 \end{bmatrix} .$$

The irreducible constituents are: a single $(N = 6)$-extension and a quartet of $(N = 4)$-extensions with the highest spin values 2 and 1 respectively.

The number of vector fields splits as follows: $20 = 1 + 3 + 2 \times 8$ and is compatible with a symmetry group of gauge interactions

$$G_{qg} = U(1) \times SU(2)_L \times SU(3)_g \times SU(3)_c \tag{5}$$

where one of the two octets is related to the symmetry group of three generations, the other with the group of colour. In order to prevent a quick decay of higher into lower generations the para-gluons must be massive which may be achieved by a (generalized) Higgs mechanism. Three vector fields representing the group $SU(2)_L$ together with eight representing the group $SU(3)_g$ have to swallow up eleven scalars while the six spin 3/2 fermions must swallow up six Weyl spinors endowing the respective particles with high masses (see the table 8) table 8

$$\begin{bmatrix} 1 \\ 6 \\ 20 \\ 42 \\ 54 \end{bmatrix} \xrightarrow{ssb} \begin{bmatrix} 1 \\ 6' \\ 1+3'+8+8' \\ 36 \\ 34+9 \end{bmatrix}_{D=4} = \begin{bmatrix} 1 \\ 3' \\ 1+8+8' \\ 12 \\ 8+1 \end{bmatrix}_{D=6} .$$

From the fourth row of the middle column it is seen that the number of Weyl fields is 36 interpretable as three generations of coloured quarks according to the splitting $36 = 2 \times 2 \times 3 \times 3$ where 2×2 denotes a doublet of helicities times a doublet of charm or flavour whereas 3×3 denotes a triplet of generations times a triplet of colour. It should be stressed that the octet of lepton and quark generations must be different one from the other in order to prevent decay of quarks and hadrons into leptons. In order to account for the masses of leptons the $SU(3)$ symmetries of generations must be (slightly) broken but the mechanisms of these breakdowns are not yet clear.

The problem of a $N = 8$ extension

It is suggestive to assume that the set of all fundamental fields and particle types existing in Nature should form the most general, irreducible supermultiplet characterized by the index of extension $N = 8$. This supermultiplet includes 1 tensor, 8 Rarita-Schwinger fields, 28 vectors, 56 Weyl fields and 70 scalars. In view of the splitting $28 = 1 + 3 + 8 \times 3$ it could be supposed that the symmetry group of gauge interactions were

$$G = U(1) \times SU(2) \times SU(3) \times SU(3) \times SU(3) \tag{6}$$

of rank 8. However, it seems impossible to "put quarks and leptons into one basket". Inasmuch as strong gauge interactions are not universal and quarks interact with leptons only via universal gravitational and electro-weak couplings we assume that the Lagrangian splits into a leptonic and quarkonic parts interacting only via subgroups (4) and (5) of the most general possible group (6) of gauge interactions of ranks 4 and 6 respectively. The generality of the scheme will be preserved only in so far that **all** fields appearing within the ($N = 8$)-multiplet partake either in the leptonic or quarkonic parts of the Lagrangian

$$\mathcal{L} = \mathcal{L}_l + \mathcal{L}_q + \mathcal{L}_b. \tag{7}$$

The lepton and quark parts $\mathcal{L}_l$ and $\mathcal{L}_q$ involve the interaction terms with bosons whereas the bosonic part $\mathcal{L}_b$ denotes a sum of interaction-free bosonic fields. The gauge groups of interactions with leptons and quarks in $\mathcal{L}_l$ and $\mathcal{L}_q$ are the groups (4) and (5) respectively. If all interactions are of a gauge type and gravitational ones and of Yukawa-Higgs type then writing down a Lagrangian (7) is rather a matter of standard techniques.

Assume that the triplet $W^{\pm} Z^0$ as well as both para-gluonic octets become massive as visualized by the table 9

table 9

$$\begin{bmatrix} 1 \\ 8 \\ 28 \\ 56 \\ 70 \end{bmatrix} = \begin{bmatrix} 1 \\ 8' \\ 1+3'+8+8'+8' \\ 48 \\ 1+50 \end{bmatrix}$$

The large masses of the two octets of para-gluons prevent a possibility of a quick decay of higher into lower generations. The appearance of two different octets of para-gluons for leptons and quarks assures a lack of decay of hadrons into leptons.

The reduction of the number of Weyl spinors from 56 to 48 is just sufficient and necessary to interpret the remainig 48 as three generations of leptons and quarks. To see this let us perform the following splitting

$$48 = 12 + 36 = 2 \times 2 \times 3 + 2 \times 2 \times 3 \times 3$$

where 12 denotes the number of leptons and 36 that of quarks. As before 2×2 means two helicities times a doublet of charm (or flavour). One of the triplets accounts for the three generations of leptons or quarks while the second triplet accounts for the colour of quarks. Nevertheless, the multiplet $N = 8$ is not simply a sum of formerly discussed lepton and quark multiplets because of different roles of universal electro-weakly-gravitational and specific $SU(3)$ interactions. Adding simply the schemes would mean doubling the gravitational field and the number of fields representing the $W^{\pm}$, Z^0 bosons. Instead, we may perform a decomposition according to the table 10 where the first column to the right hand side of this table

table 10

$$\begin{bmatrix} 1 \\ 8 \\ 28 \\ 56 \\ 70 \end{bmatrix} = \begin{bmatrix} - \\ 2 \\ 8 \\ 12+2 \\ 16 \end{bmatrix} + \begin{bmatrix} 1 \\ - \\ 4 \\ - \\ 22 \end{bmatrix} + \begin{bmatrix} - \\ 6 \\ 16 \\ 36+6 \\ 32 \end{bmatrix}$$

denotes the leptonic sector, the third describes the quarkonic sector while the middle column denotes their common part, *i.e.* the terms joining leptonic with quarkonic worlds via universal gravitational, electro-weak and Yukawa-like interactions.

The additional numbers 2 and 6 of Weyl spinors from the fourth rows of the table 10 are to be swallowed up by the corresponding spin-3/2 fields endowing them with masses by means of a mechanism of spontaneous symmetry breaking. The numbers of scalars are explicable if reinterpreting the supermultiplet from a six-dimensional viewpoint. The 16 and 32 scalars appearing in the first and third column are the "tails" of the corresponding six-vectors whereas the number of $22 = 1 + 3 + 2 + 16$ scalars from the middle row denotes respectively a Goldstone boson, a triplet of metric tensor components $g_{\xi\eta}$, the "tail" of the six-vector of electromagnetic potentials, and a set of further 16 apparent scalars which will be swallowed up by the two octets of para-gluons endowing them with masses.

If reinterpreted from a four-dimensional to a six-dimensional viewpoint the $(N = 8)$-dimensional extended supermultiplet assumes the following form

table 11

$$\begin{bmatrix} 1 \\ 8 \\ 28 \\ 56 \\ 70 \end{bmatrix}_{D=4} \longrightarrow \begin{bmatrix} 1 \\ 1+3 \\ 25 \\ 5+15 \\ 1+16 \end{bmatrix}_{D=6}$$

(still prior to the spontaneous symmetry breakdown).

Let us notice the following remarkable circumstance: the numbers of Weyl spinors viewed from a six-dimensional perspective are odd, *viz.* 5 and 15 in the fourth row of the table 11 if decomposed into leptonic and quarkonic sectors, *i.e.* into singlets and triplets. This explains why parity conservation must be violated by weak interactions. Weyl spinors cannot be fused into Dirac spinors in $D = 6$.

Concluding remarks

Our tables may be regarded as analogues of the Mendelejev table of chemical elements, but this time applied to elementary particles. Similarly

as the original table of Mendelejev exhibited some signs of periodicity (and therefore was called "periodic table of elements") also here we notice signs of periodicity since the rows denote bosonic and fermionic fields alternately. Moreoveer, similarly as in the case of Mendelejev tables there appear also here empty places to be filled up by some expected but not yet discovered elements (particle types).

The table 11 reveals also a possibility of another gauge symmetry *viz.* $U(1)\times SU(2)\times SU(5)$. This possibility should be worked out also. The fact that our scheme fits so well and smoothly into a six-dimensional framework shows decisively the inadequacy of the concept of superstrings, the latter requiring at least a 10-dimensional space.

In spite of the fact that now the problem of writing down explicitly a Lagrangian for unified theory is only a matter of standard techniques, the above sketched assumptions cannot be regarded yet as a full unification because they do not predict all possible relations between coupling constants, provided they are indeed constants if viewed upon from a point of view of a cosmological time scale.

A possible objection that the above models may turn out to be mathematically inconsistent (because of non-renormalizable couplings, *etc.*) is to be refuted since obviously it does not apply only to such or similar endavours of unification but equally well to any contemporary quantum formalism of gravitation. Probably such objections cannot be avoided unless a profound modification of the concept of general relativity will be achieved.

References

[1] J.Rayski, *Acta Phys. Polonica* **27**, 89 (1965)
[2] J.Rayski, in *Unified Field Theories in more than 4 dimensions*, ed. by V. De Sabbata and E.Schmutzer, World Sc Singapore, 1982
[3] J. Rayski and J. M. Rayski jr, *Nuovo Cim* **103 A**, 1729 (1990)
[4] E. Witten, *Nuclear Physics* **B 186**, 412 (1981)

GENERALIZED FIERZ IDENTITIES AND THE SUPERSELECTION RULE FOR GEOMETRIC MULTISPINORS

WILLIAM M. PEZZAGLIA JR.
Department of Physics and Astronomy
San Francisco State University
1600 Holloway Avenue
San Francisco, California 94132
U.S.A.

Abstract. The inverse problem, to reconstruct the general multivector wave function from the observable quadratic densities, is solved for 3D geometric algebra. It is found that operators which are applied to the right side of the wave function must be considered, and the standard Fierz identities do not necessarily hold except in restricted situations, corresponding to the spin-isospin superselection rule. The Greider idempotent and Hestenes quaterionic spinors are included as extreme cases of a single superselection parameter.

Key words: fierz – multivector – superselection – spinors

1. Introduction

In a recent paper Crawford[1] explored the inverse problem of Dirac bispinor algebra, to reconstruct the wave function from the observable quadratic densities. Other authors[2,3] have presented parallel developments for multivector quantum mechanics in which column spinors are replaced by Clifford algebra aggregates. However, these expositions have only considered restricted cases (e.g. minimal ideals) for which the multivector analogies of the observable bispinor densities obey the standard Fierz[1,2,4] identities.

Previously we have proposed[5,6] a more general multivector wave function in which all the geometric degrees of freedom are used. To obtain the complete set of observable multispinor densities one needs to augment the standard *sinistral*[6] operators (applied to the left side of the wave function) with new *dextral*[6] (right-side applied) operators, and also *bilateral*[6] operators (multivectors applied on both sides of the wave function). It is found that if and only if the multivector wave function is restricted will the multispinor densities obey the standard Fierz identities. In this paper we propose to solve the inverse problem for the general unrestricted multivector wave function of the 8-element 3D geometric algebra $\mathbf{C}(2)$, i.e. the Pauli algebra.

2. The Algebra of Standard Pauli Spinor Densities

In non-relativistic quantum mechanics, the electron is represented by a two-component Pauli spinor. The endomorphism algebra (module structure on spinors) is $\mathbf{C}(2)$, i.e. two by two complex matrices. This Clifford algebra has as its basis the 8 element group generated by 3 mutually anticommuting basis vectors, $\{\sigma_j, \sigma_k\} = 2\delta_{jk}$ for (j,k = 1,2,3), and where $i = \sigma_1\sigma_2\sigma_3$. As operators, their "bilinear expectation values" yield real densities which are interpreted to be the projection of the spin along the j-th spatial axis, $S_j = <\psi|\sigma_j|\psi>$.

The 4 densities $\{\rho, S_i\}$, where $\rho = <\psi|\psi>$, are invariant with respect to the phase parameter of the spinor. Hence they satisfy a single constraint equation which can be derived by substituting the projection operator into the square of the normalization. The magnitude of the spin is found to be constrained by the Fierz[1,2,4] identity,

$$|\mathbf{S}|^2 = S^k S_k = \rho^2. \tag{1}$$

The spinor can be reconstructed in terms of a $U(2)$ unitary rotation matrix,

$$U(\lambda,\theta,\phi,\alpha) = \exp(i\alpha/2)\exp(i\sigma_3\phi/2)\exp(i\sigma_2\theta/2)\exp(i\sigma_3\lambda/2), \tag{2}$$

where (θ,ϕ) are the orientation angles of the spin and (λ,α) do not contribute in the bilinear form $\sigma^j S_j(\theta,\phi) = \rho U\sigma_3 U^\dagger$. Choosing a starting spinor to be the "plus" eigenstate of σ_3, the wave function can be expressed, $\psi(\rho,\theta,\phi,\beta) = \rho U|\eta>$, where the net unobservable phase is $\beta = \lambda + \alpha$.

3. The Algebra of Geometric Multispinors

We consider an unrestricted multivector wave function[5] which has the same *8 degrees of freedom* as the Clifford group,

$$\psi = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = (a + b\sigma_1)\frac{(1+\sigma_3)}{2} + (c + d\sigma_1)\frac{(1+\sigma_3)}{2}\sigma_1, \tag{3}$$

where $\{a,b,c,d\}$ are complex coefficients. Note each *column* of the matrix is a *minimal left ideal* of the algebra and will hence behave like a column spinor[8] for all *sinistral* [6] (left-sided) operations. Each *row* of the matrix is a *minimal right ideal* of the algebra and will behave like a row *isospinor* for all dextral[6] (right-sided) operations. Hence the complete solution can be interpreted[6] as an isospin doublet (of spinors) coupled by now-allowable dextral application of the Pauli operators.

3.1. Multivector Densities

A complete set of 16 generalized quadratic forms are defined in terms of the matrix trace (i.e. half the scalar part of the Clifford multivector)[9],

$$\rho = Tr(\psi^\dagger\psi) = Tr(\psi\psi^\dagger) = |a|^2 + |b|^2 + |c|^2 + |d|^2, \tag{4a}$$

$$S_j = Tr(\psi^\dagger\sigma_j\psi) = Tr(\psi\psi^\dagger\sigma_j), \quad (j = 1,2,3), \tag{4b}$$

$$T_j = Tr(\psi\sigma_j\psi^\dagger) = Tr(\psi^\dagger\psi\sigma_j), \quad (j = 1,2,3), \tag{4c}$$

$$R_{jk} = Tr(\psi^\dagger\sigma_j\psi\sigma_k) = Tr(\psi\sigma_k\psi^\dagger\sigma_j), \quad (j,k = 1,2,3). \tag{4d}$$

They are interpreted to be the probability, spin, isospin and *bilateral* densities respectively. From these we can construct the multivector densities,

$$\psi\psi^\dagger = (\rho + \sigma_k S^k)/2, \tag{5a}$$

$$\psi^\dagger\psi = (\rho + \sigma_k T^k)/2, \tag{5b}$$

$$\psi^\dagger\sigma_j\psi = (S_j + R_{jk}\sigma^k)/2, \tag{5c}$$

$$\psi\sigma_k\psi^\dagger = (T_k + \sigma^j R_{jk})/2. \tag{5d}$$

3.2. Generalized Fierz Identities

The 16 densities are all independent of the phase parameter, hence must satisfy 9 constraint equations. In general these identities have the form,

$$Tr[(\psi^\dagger\sigma_\alpha\psi)\sigma_\beta(\psi^\dagger\sigma_\gamma\psi)\sigma_\delta] = Tr[(\psi\sigma_\beta\psi^\dagger)\sigma_\gamma(\psi\sigma_\delta\psi^\dagger)\sigma_\alpha] \tag{6a}$$

where the indices can take on values 0 through 3, and $\sigma_0 = 1$. The parenthesis indicate where one inserts eqs. (5abcd). It can be shown from these relations that the bilateral density eq. (4d) contains all the other densities. Further, we find that the magnitudes of the spin and isospin are equal, but that eq. (1) is no longer valid,

$$|\mathbf{S}|^2 = |\mathbf{T}|^2 \le \rho^2. \tag{6b}$$

3.3. Interpretation of the Bilateral Density

Counting degrees of freedom,we see that there is one free internal "hidden variable" contained in R_{jk} which does not affect the other densities. To gain some insight as to the nature of this parameter we consider the class of unitary (hence ρ invariant) transformations that will leave the densities $\{S^j, T^j\}$ invariant, but modify the bilateral density.

The special case sinistral operator, $U(\lambda) = \exp[i\sigma_j S^j/(2|\mathbf{S}|)]$, will leave the spin invariant (as well as the isospin) as it corresponds to a rotation about the spin axis by angle λ. The bilateral density will be modified by this transformation, hence we should be able to parametrize R_{jk} in terms of the densities $\{\rho, S^j, T^j\}$ and a bilateral phase angle λ.

4. Inverse Theorem and Superselection Rule

We assert that the projection operator for the multivector wave function has the bilateral form, $\psi = (\rho\psi + S_k\sigma^k\psi + \psi\sigma_k T^k + \sigma^j\psi\sigma^k R_{jk})/(4\rho)$.

4.1. Inverse Theorem

The multivector wave function can be reconstructed from the observable densities by a applying the projection operator to an arbitrary starting solution η, and renormalizing. Hence we assert,

$$\Psi(\alpha, S^k, T^k, R^{jk}) = e^{i\alpha}(\rho\eta + S^k\sigma_k\eta + \eta\sigma_k T^k + \sigma^j\eta\sigma^k R_{jk})/N, \qquad (7a)$$

where α is a phase factor and η is an arbitrary starting multivector subject only to the normalized trace constraint $Tr(\eta^\dagger\eta) = 1$.

The normalization factor is most directly determined by requiring the reconstructed wave function to reproduce the probability density eq. (4a),

$$N^2 = 4[\rho + Tr(\eta^\dagger\sigma^j\eta\sigma^k R_{jk})] + 2[Tr(\eta^\dagger\eta\sigma^k T_k) + Tr(\eta\eta^\dagger\sigma^k S_k)], \qquad (7b)$$

where identities have been used to reduce the quadradic terms to linear ones in terms of the observable densities. This construction will fail if eq. (7b) yields zero, in which case a different starting solution should be used.

4.2. Special Classes of Solutions

In order to insure a scalar norm, Hestenes[3,9] proposed a unitary or *quaternionic* solution which has 5 parameters,

$$\psi(\alpha,\rho,\lambda,\theta,\phi) = \sqrt{\frac{\rho}{2}}\, U(\lambda,\theta,\phi,\alpha) = \sqrt{\frac{\rho}{2}}\, e^{i\alpha/2}\, [r + i\sigma^j B_j], \qquad (8)$$

where unitary matrix $U(\lambda,\theta,\phi,\alpha)$ is given by eq. (2). The alternate quaternionic *Cayley-Klein* components $\{r, B_j\}$ are all real numbers, subject to constraint $r^2 + B^2 = 1$. Only 4 parameters are however needed to describe an electron, hence Hestenes (arbitrarily?) sets the parameter α to zero.

This unitary class of solutions is synonymous with *zero* magnitude spin and isospin as defined by eqs. (4bc). The bilateral density eq. (4d) is proportional (by a factor of ρ) to the $O(3)$ rotation matrix $R(\lambda,\theta,\phi)$ associated with the $U(2)$ matrix $U(\lambda,\theta,\phi,\alpha)$. This allows Hestenes to make an alternate definition of a "spin" vector in terms of the bilateral density, $S'_j = R_{j3} = Tr(\psi^\dagger\sigma_j\psi\sigma_3) = \frac{1}{2}Tr(U\sigma_3U^\dagger\sigma_j)\rho$. It is easily verified that R_{jk} is invariant with respect to the λ parameter of the unitary matrix, allowing Hestenes to reinterpret it as quantum phase, and dextrally applied $i\sigma^3$ as the quantum phase generator (replacing the usual commuting i).

In contrast, Greider[7] proposed an idempotent spinor which has the algebraic form of the projection operator,

$$\psi = e^{i\alpha}\frac{(\rho + S_k\sigma^k)}{2\sqrt{\rho}} = \sqrt{\rho}\; U(0,\theta,\phi,\alpha)\frac{(1+\sigma^3)}{2}U^\dagger(0,\theta,\phi,-\alpha), \qquad (9)$$

where the magnitude of the spin is subject to the standard Fierz constraint of eq. (1). This makes the determinant zero, hence the wave function is of the "singular class" distinctly different from the "unitary class" discussed above. There are only 4 degrees of freedom, exactly that needed to describe a single Pauli spinor (i.e. isospin is everywhere parallel to spin).

Isospin degrees of freedom can be re-introduced by applying a dextrad rotation operator to eq. (10). Equivalently, consider the following factorized idempotent form,

$$\psi = \sqrt{\rho}\; U(\lambda,\theta_S,\phi_S,\alpha)\;\frac{(1+\sigma^3)}{2}\; U^\dagger(\lambda,\theta_T,\phi_T,-\alpha), \qquad (10a)$$

$$= e^{i\alpha}\;(\rho + S_k\sigma^k)(\rho + T_j\sigma^j)\;[4\rho(\rho^2 + S_kT^k)]^{-\frac{1}{2}}, \qquad (10b)$$

where the singularity constraint eq. (1) still holds. The angles $\{\theta_S, \phi_S\}$ give the orientation of the spin, while $\{\theta_T, \phi_T\}$ that of the isospin. Our solution has 6 degrees of freedom, exactly that which is needed to describe an isospin doublet of Pauli spinors (i.e. two particle generations in the family). The net phase $\beta = \lambda + \alpha$ shows λ is indistinguishable from parameter α, hence $R_{jk} = \rho S_j T_k/|\mathbf{S}|^2$, has no λ dependence.

4.3. Superselection Parameter

Our multispinor solution is subject to a *spin-isospin superselection rule*[10]. While certain linear combinations are allowable, the superposition of "spin & isospin up" with "spin & isospin down" would yield a "forbidden" unitary class solution with zero spin and isospin. Equivalently such a state is inaccessible by any spin/isospin rotation from a "spin & isospin up" state. Mathematically this constraint manifests as requiring the determinant of our wave function to be zero.

Consider a new superselection parameter δ, defined: $|\mathbf{S}| = \rho\cos\delta$,

$$\Psi(\alpha,\rho,\lambda,\delta,\theta_S,\phi_S) = \sqrt{\rho}\; e^{i\alpha}\exp(i\sigma^k n_k\lambda/2)\;(1 + e^{i\delta}n_k\sigma^k)/2, \qquad (11)$$

where $n_k(\theta_S,\phi_S) = S_k/|\mathbf{S}|$. For $\delta = 0$ the wave function becomes a Greider[7] idempotent with zero determinant, and when the spin vanishes in the limit of $\delta = \pi/2$, the solution is of the Hestenes[3,9] quaternionic form. Note the bilateral phase λ is independent of the ordinary imaginary phase α for $\delta > 0$.

The remaining two isospin degrees of freedom can be reintroduced as before by a dextrad rotation operator. A complete parameterization of the general 8 degree of freedom solution can be expressed in polar form,

$$\Psi(\rho,\alpha,\lambda,\delta,\theta_S,\phi_S,\theta_T,\phi_T) = \sqrt{\rho}\ U(\lambda,\theta_S,\phi_S,\alpha)\frac{(1+e^{i\delta}\sigma_3)}{2}U^{\dagger}(\lambda,\theta_T,\phi_T,-\alpha). \quad (12)$$

5. Summary

We have solved the inverse problem for the completely general eight degree of freedom wave function of 3D geometric space. Our results are more general than other treatments in that a more complete set of quadratic multispinor densities is introduced which includes sinistral, dextral and bilateral operations. The 16 densities satisfy generalized Fierz-type identities. The new *bilateral density* is found to contain one new independent "hidden" variable which does not affect the more familiar probability, spin and isospin densities. It is an open question as to whether this quantity can be physically measured, or is unobservable like the overall quantum phase parameter.

The standard Fierz identities (for column spinors) are found not to hold except for a restricted singular class of wave functions. This appears to be a manifestation of the spin-isospin superselection rule, and may be the critical constraint which classifies the solution as being a fermionic particle. A continuous superselection parameter is introduced for which the singular class of solutions (which includes the Greider idempotent form) is one extreme case; the Hestenes quaternionic spinor form is at the other extreme.

Extending the work to 4D Minkowski space with a 16 degree of freedom wave function we will find 136 quadratic forms, which obey 121 generalized identities. One or more new "hidden" variables will be found, and the standard Fierz identities will not be valid except for a restricted wavefunction, corresponding to the charge superselection rule of bispinors.

References

[1] Crawford, J.P.: 1985, *J. Math. Phys.* **26**, 1439-41.
[2] Lounesto, P.: 1993, 'Clifford algebras and Hestenes spinors', *Found. Phys.*to appear in May 1993 issue, special edition to honor D. Hestenes 60th birthday (P. Lounesto, ed.).
[3] Hestenes, D.: 1979, *Am. J. Phys.* **47**, 399.
[4] Fierz, M.: Z. Phys., *1937* **104**, 553.
[5] Pezzaglia, W.: 1992, *Found. Phys. Lett.* **5**, 57.
[6] Pezzaglia, W.:1993, 'Dextral and Bilateral Multivector Gauge Field Description of Light-Unflavored Mesonic Interactions',*Found. Phys.*to appear in May 1993 issue.
[7] Greider, K.:1984, *Found. Phys.***14**,467; :1980,*Phys. Rev. Lett.***44**, 1718.
[8] Hestenes, D.: 1965, *Spacetime Algebra*, Gordon and Breach Publ., 38; : 1971, *Am. J. Phys.* **39**, 1013-27.
[9] Hestenes, D. and Gurtler, R.: 1971, *Am. J. Phys.* **39**, 1028.
[10] Mohapatra, R.N.: 1987, *Phys. Rev. Lett* **59**, 1510.

ELECTRONS, PHOTONS, AND SPINORS IN THE PAULI ALGEBRA

WILLIAM E. BAYLIS
Department of Physics, University of Windsor
Windsor, Ontario, Canada N9B 3P4

Abstract. The multivectors ("cliffors") of three-dimensional Euclidean space form a complex four-dimensional vector space with the Minkowski metric. In fact all elements of the real Clifford algebra of Minkowski space (the 'Dirac' or 'spacetime' algebra) can be mapped (in two mappings) onto the Pauli algebra. The Pauli algebra is used here to provide a covariant description of elementary charges and electromagnetic radiation fields in terms of 'spinors' which represent Lorentz transformations describing their motion.

Key words: electrons – photons – spinors – Pauli algebra – Dirac equation

1. Introduction

One of the simplest Clifford algebras with widespread applications to the physical world is the real geometric algebra of three-dimensional Euclidean space (Hestenes 1966, Baylis 1992). It is a convenient starting point for mathematicians in their abstract study of more general Clifford algebras since, in spite of its simplicity, it is sufficiently complex to include the complex field (as its centre) and the quaternions (as its even subalgebra), and these together with its nonabelian product and zero-divisors produce a mathematical structure of considerable richness. Its standard representation in terms of Pauli spin matrices is also familiar to physicists, especially in quantum theory, although its real power and beauty is better revealed in the representation-free algebraic form. Here for simplicity, the *Pauli algebra* $\mathcal{P}$ is used to refer to the geometric algebra of three-dimensional Euclidean space; no specific matrix representation is implied. Similarly, the *Dirac algebra* $\mathcal{D}$ refers to the real Clifford algebra of Minkowski spacetime, and the real quaternion algebra is labeled by $\mathcal{H}$.

The aims of the present contribution are to summarize applications of $\mathcal{P}$ to the study of 'elementary' particles and radiation fields (with much more emphasis on the former), to show how naturally the algebra models the structure of spacetime, and to see how elements of $\mathcal{P}$ provide covariant descriptions of particles and fields and yield a simple interpretation of the Dirac equation. The following section reviews the structure of the Pauli algebra and its relation to Minkowski space. While the mathematics is familiar to anyone with an interest in Clifford algebras, the physical content is so beautiful in its simplicity and so magnificent in its power that it deserves close attention. The following formulation also serves to introduce the idiosyncrasies of my notation. Although it is given for convenience in terms of flat spacetime, it can be extended to curved manifolds (Rastall 1964).

2. From Euclid to Minkowski

Recall that the Pauli algebra $\mathcal{P}$ is generated by an associative product of real three-dimensional vectors which satisfies the condition $\mathbf{a}^2 = \mathbf{a} \cdot \mathbf{a}$ for any vector $\mathbf{a}$ in $\mathcal{P}$. From three orthonormal, anticommuting basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of the underlying Euclidean space, the resulting fundamental relation

$$\mathbf{e}_j\mathbf{e}_k + \mathbf{e}_k\mathbf{e}_j = 2\delta_{jk} \tag{1}$$

allows eight independent real basis forms of $\mathcal{P}$ to be constructed:

$$\{1; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3; \mathbf{e}_1\mathbf{e}_2, \mathbf{e}_2\mathbf{e}_3, \mathbf{e}_3\mathbf{e}_1; \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\}. \tag{2}$$

As in $\mathcal{D}$ and $\mathcal{H}$, the canonical element squares to -1, but in contrast $\mathcal{D}$ and $\mathcal{H}$, the canonical element $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ of $\mathcal{P}$, because of the odd number of dimensions of the ground space, also commutes with all other elements. It may therefore be identified with the imaginary i:

$$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = i. \tag{3}$$

The product of i with a vector gives the bivector dual to it. In particular

$$i\mathbf{e}_1 = \mathbf{e}_2\mathbf{e}_3,\ i\mathbf{e}_2 = \mathbf{e}_3\mathbf{e}_1,\ i\mathbf{e}_3 = \mathbf{e}_1\mathbf{e}_2. \tag{4}$$

The identification of the canonical element of $\mathcal{P}$ with the imaginary i allows one to consider $\mathcal{P}$ to be spanned not only by the eight basis forms (2) over the reals, but also by the four forms $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ over the complex field. That is, a general element $p \in \mathcal{P}$ is the sum $p_0 + \mathbf{p}$ of a scalar p_0 and a three-vector $\mathbf{p} = p^k\mathbf{e}_k$, both of which may be complex. Thus, the products of real three-dimensional vectors generate in $\mathcal{P}$ both complex numbers and a four-dimensional space, and the imaginary i has geometric content: i times a scalar is a pseudoscalar (trivector) and represents a volume; i times a vector is a pseudovector (bivector) and represents a plane.

There are two fundamental involutions on $\mathcal{P}$. Reversion $p \to p^\dagger$ plays the role of complex conjugation and changes the sign of the imaginary part of a element whereas spatial reversal $p \to \bar{p}$ changes the sign on the three-vector part. Thus if

$$p = p_0 + \mathbf{p} = p^\mu\mathbf{e}_\mu \tag{5}$$

where $\mathbf{e}_0 = 1$, then

$$p^\dagger = p^{\mu *}\mathbf{e}_\mu \tag{6}$$

$$\bar{p} = p_0 - \mathbf{p}. \tag{7}$$

The two may be combined into the automorphism $p \to \bar{p}^\dagger = \bar{p^\dagger}$. If $p = p^\dagger$, it is *real*; if $p = \bar{p}$, it is a (possibly complex) scalar. The scalar part of a product pr is indicated as a dot product:

$$p \cdot r = (pr + \overline{pr})/2 = p_0r_0 + \mathbf{p} \cdot \mathbf{q}. \tag{8}$$

Note that $p \cdot q = 1 \cdot (pr) = 1 \cdot (rp)$.

An element p is *invertible* if there exists another element, say q, whose product with p is a nonzero scalar: $pq = \overline{pq} = \bar{q}\bar{p}$. It is straightforward to see that q is proportional to $\bar{p}$ and that the inverse of p is

$$p^{-1} = \bar{p}/(p\bar{p}). \tag{9}$$

If the scalar *norm* $p\bar{p}$ vanishes, the element p is a *zero-divisor* and is not invertible. (Such elements add to the structure of the algebra by allowing proper ideals.) A metric for the space is naturally defined for real elements by the bilinear scalar norm

$$p\bar{p} = p_0^2 - \mathbf{p}^2. \tag{10}$$

It is seen to be exactly the Minkowski metric, with the scalar part of p playing the role of the 'time' component.

A statement made above can now be strengthened: *the products of real three-dimensional vectors generate in $\mathcal{P}$ both complex numbers and a four-dimensional space with the Minkowski metric.*

3. Lorentz Transformations and Covariance

Physical four-vectors form a real subspace $\mathcal{M}$ of $\mathcal{P}$. They include (in units with $c = 1$) the four-velocity $u = \gamma + \mathbf{u}$, the (four-)momentum $p = E + \mathbf{p}$, the (four-)current density $j = \rho + \mathbf{j}$, and the four-potential $A = \phi + \mathbf{A}$. Transformations which leave the norm of any four-vector invariant are called (homogeneous) *Lorentz transformations.* It follows that such transformations also leave invariant scalar products of the form $p \cdot \bar{A}$. The restricted (*i.e.*, proper orthochronous) subset of these transformations, when operating on any four-vector p, can be written (Baylis and Jones 1989a)

$$p \to LpL^\dagger \tag{11}$$

where L is any unimodular element of $\mathcal{P}$: $L\bar{L} = 1$. (The proper nonorthochronous Lorentz transformations take the form $p \to -LpL^\dagger$ whereas the improper ones are $p \to \pm L\bar{p}L^\dagger$.) The elements L which give the Lorentz transformations form the subalgebra $sl(2, C)$ of $\mathcal{P}$ and can always be written as the product

$$L = \mathcal{BR} = \exp(\mathbf{w}/2)\exp(-i\boldsymbol{\theta}/2) \tag{12}$$

of a *boost* by rapidity $\mathbf{w}$ and a *rotation* by θ in the plane $i\boldsymbol{\theta}$. The rotation elements $\mathcal{R}$ constitute the Lie algebra $su(2) \subset sl(2, C)$.

The separation of any four-vector in $\mathcal{P}$ into scalar and vector parts corresponds to the physical partition into 'time' and space components. This corresponds to an obvious differentiation which every conscious observer makes, and it is desirable that any algebra modeling the physical world contain an analogous natural partitioning. However, the separation is obviously *frame dependent* since boosts generally scramble time and space components. How can $\mathcal{P}$ provide a *covariant* description of nature, *i.e.*, one in which the basic physical equations take the same form in all inertial frames?

The obvious answer is simply to avoid splitting elements of $\mathcal{P}$ into scalar and three-vector parts. Of course it is the essence of Clifford algebras that they are most useful when elements are not expanded in components or expressed as the sum of homogenous (scalar, vector, etc.) parts. Just as the power of complex numbers is largely lost when every such number is written as the sum of a real number and an imaginary one, so is the efficiency of $\mathcal{P}$ degraded by expressing four-vectors as sums of scalars and three-vectors.

When four-vectors are expressed as single elements of $\mathcal{P}$, relations among them can be covariant, for example $p = mu$. Products in $\mathcal{P}$ involving four-vectors also appear frequently in covariant relations. They transform simply under Lorentz transformations if four-vectors $p \in \mathcal{M}$ are alternated with barred four-vectors $\bar{q} \in \bar{\mathcal{M}}$:

$$p\bar{q} \to (LpL^\dagger)(\overline{LqL^\dagger}) = L(p\bar{q})\bar{L}. \tag{13}$$

If $p\bar{q}$ is expanded in scalar and vector parts:

$$p\bar{q} = p \cdot \bar{q} + p \wedge \bar{q} \tag{14}$$

where the scalar part is as above $p \cdot \bar{q} = 1 \cdot (p\bar{q}) = (p\bar{q} + q\bar{p})/2$ and the vector part is

$$p \wedge \bar{q} := (p\bar{q} - q\bar{p})/2 = p\bar{q} - p \cdot \bar{q} = -q \wedge \bar{p}, \tag{15}$$

then the scalar part, as seen above, is invariant whereas the vector part transforms as $p \wedge \bar{q} \to Lp \wedge \bar{q}\bar{L}$. The vector part of the product $p\bar{q}$ of $p \in \mathcal{M}$ and $q \in \bar{\mathcal{M}}$ is called a *six-vector* because it generally has three real-vector and three pseudovector (imaginary-vector or bivector) components.

An important six-vector is the electromagnetic field

$$F = \partial \wedge \bar{A} = \partial\bar{A} - \partial \cdot \bar{A} = \mathbf{E} + i\mathbf{B} \tag{16}$$

where ∂ is the four-vector operator $\partial_t - \nabla$. The Lorentz-force equation in $\mathcal{P}$ for a particle of momentum p takes the covariant form

$$dp/d\tau = e\Re(Fu) \tag{17}$$

where τ is the proper time, and Maxwell's equations are just the scalar, pseudoscalar, vector and pseudovector parts of

$$\bar{\partial}F = 4\pi K\bar{j} \tag{18}$$

where K is a constant depending on units: $K = 1$ in Gaussian units, $(4\pi)^{-1}$ in Heaviside-Lorentz units, and $(4\pi\epsilon_0)^{-1}$ in SI units. Note that just as the Pauli product $p\bar{q} = p \cdot \bar{q} + p \wedge \bar{q}$ contains both inner and outer parts, so does differentiation with ∂. In terms of differential forms, $\partial = \delta + d$.

Thus the Pauli algebra, while allowing a physically intuitive separation of four-vectors into 'time' and space parts, also generates a Minkowski space and provides a naturally covariant formalism for problems in relativity. For problems in physics, it is usually unnecessary and often needlessly inefficient to use the Dirac algebra $\mathcal{D}$. To underscore this point, it is worthwhile to observe that any element of $\mathcal{D}$ can be mapped onto $\mathcal{P}$ (Baylis and Jones 1988). The mapping from even elements of $\mathcal{D}$ onto $\mathcal{P}$ is a well-known isomorphism. Odd elements of $\mathcal{D}$ are changed into even elements through multiplication by a basis vector of the ground space $\mathcal{M}$ of $\mathcal{D}$; traditionally one uses the time-like vector γ^0. These elements can then also be mapped onto $\mathcal{P}$ by the same isomorphism. Of course the result is a two-to-one mapping, which means that a given type of element in $\mathcal{P}$ can represent two types in $\mathcal{D}$. In practice, however, this causes no problems. Thus a scalar in $\mathcal{P}$ can be either a Lorentz scalar or the 'time' component of a four-vector, and a three-vector in $\mathcal{P}$ can be either part of a four-vector or part of a six-vector (a Dirac bivector), but no one is likely to confuse the possiblilites. Indeed a covariant algebraic notation keeps the identities quite distinct.

4. Electrons and Neutrinos

In this section, the Pauli algebra is applied to descriptions of 'elementary' particles (Baylis 1992). The spin and translational motion of an 'elementary' particle is described by a characteristic Lorentz transformation Λ: the transformation of the particle from its rest frame to the observer's frame (Gürsey 1957; Rastall 1964, 1988). Of course a compound particle like a nucleon may require several Lorentz transformations to fully describe its motion, one, say, for each quark, but I want to assume that there exist particles which are 'elementary' in the sense that they require only one Lorentz transformation for their full description. However, it is *not* necessary to assume that the 'elementary' particle is a point, only that its motion is described by a single Lorentz transformation. The particle must be structureless but may have a finite extent. In order that a single Lorentz transformation describe a non-point particle, the time-development of the transformation must be governed by a *linear* equation of motion, and this together with the Lorentz-force equation, can be shown to constrain the g-factor of the particle to be 2 (Baylis 1992). The idea of having a single such transformation for a particle is essentially classical; the transition to a quantum picture involves replacing the single Lorentz transformation by a field.

By using Λ, one can transform any property of the rest frame to the lab frame of the observer. For example, any one of the orthonormal unit vectors $e_\mu, \mu = 0,1,2,3$ in the rest frame is transformed as a four-vector to the corresponding *Frenet* vector

$$u_\mu = \Lambda e_\mu \Lambda^\dagger \tag{19}$$

in the lab frame. In $\mathcal{P}$ the timelike unit vector is simply $e_0 = 1$ and the corresponding Frenet four-vector is the four-velocity

$$u \equiv u_0 = \Lambda\Lambda^\dagger. \tag{20}$$

The six-vectors constructed from the Frenet four-vectors are Minkowski-space bivectors:

$$u^\mu \bar{u}^\nu = \Lambda e^\mu \bar{e}^\nu \bar{\Lambda}. \tag{21}$$

The characteristic Lorentz transformation Λ is usually different for different observers. If the frame of the one observer is transformed to that of another by a transformation element L, then Λ is transformed according to

$$\Lambda \to L\Lambda. \tag{22}$$

The transformation (22) is just what is needed for the Frenet four-vectors and six-vectors to be covariant: they are the *bilinear covariants* of the classical theory. However, the form (22) shows that Λ itself can not be a four-vector or any product constructed from four-vectors; its transformation behaviour is distinct. Transformation (22) is appropriate for *spinors*, and Λ may be seen to be a vector in the representation space of the group $SL(2,C)$ of restricted Lorentz transformations. Although the space is reducible, it is the smallest space to give a faithful representation of $SL(2,C)$. Λ is called the *eigenspinor* of the particle. For restricted transformations, it is unimodular: $\Lambda\bar{\Lambda} = 1$.

The eigenspinor of an accelerating particle is a function of the proper time τ of the particle: $\Lambda = \Lambda(\tau)$. Eigenspinors at different times are related by a Lorentz transformation $L(\tau_2, \tau_1)$ which serves as the *time evolution operator* of the particle:

$$\Lambda(\tau_2) = L(\tau_2, \tau_1)\Lambda(\tau_1). \tag{23}$$

The four-momentum p of a classical particle is its mass times its four-velocity u:

$$p = mu = m\Lambda\Lambda^\dagger. \tag{24}$$

Because the eigenspinor Λ for a restricted transformation is unimodular, the trivial identification (24) can be put in a form which is linear in the space containing both Λ and $\bar{\Lambda}^\dagger$:

$$p\bar{\Lambda}^\dagger = m\Lambda. \tag{25}$$

This is the classical Dirac equation. It is put in its traditional quantum form $\gamma_\mu p^\mu \psi = m\psi$ by defining a four-component column spinor

$$\psi = \begin{pmatrix} \bar{\eta}^\dagger \\ \xi \end{pmatrix} \tag{26}$$

where the Weyl two-spinors enter as columns of $\Lambda = (\eta, \xi)$ and $\bar{\Lambda}^\dagger = (-\bar{\xi}^\dagger, \bar{\eta}^\dagger)$. Defining the rest-frame two-spinors

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \bar{\beta}^\dagger,\ \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \bar{\alpha}^\dagger, \tag{27}$$

one can write $\bar{\eta}^\dagger$ and ξ as transformations from the rest frame:

$$\bar{\eta}^\dagger = \bar{\Lambda}^\dagger \bar{\alpha}^\dagger,\ \xi = \Lambda\beta. \tag{28}$$

(Note that in terms of spinors with abstract indices, the bar lowers the index, and the dagger dots it.)

The correspondence between the classical eigenspinor Λ and the quantum four-spinor ψ is further strengthened by calculating the bilinear covariants and the CPT transformations in terms the Weyl-spinor components (Baylis 1992). Comparisons with the quantum forms shows that the quantum amplitude ψ must, within a normalization constant and an arbitrary initial rotation of the rest frame, represent the Lorentz transformation of the particle from its rest frame to the lab frame. An association of ψ with a Lorentz transformation of the particle is not new. It was made by Gürsey (1957), Rastall (1964, 1988), and Hestenes (1975, 1990).

As a sample calculation of Λ in $\mathcal{P}$, consider the eigenspinor at a given proper time, say $\tau = 0$. Like any other Lorentz transformation, it can be written as the product of a boost and a rotation:

$$\Lambda(0) = \mathcal{B}(0)\mathcal{R}(0). \tag{29}$$

From $u = \Lambda\Lambda^\dagger = \mathcal{B}^2$ we find $\mathcal{B} = u^{1/2}$. It is readily verified that the solution can be either timelike ($\mathcal{B}_1$) or spacelike ($\mathcal{B}_2$):

$$\mathcal{B}_1 = \pm\frac{p+m}{\sqrt{2m(E+m)}},\ \mathcal{B}_2 = \mathcal{B}_1\hat{\mathbf{p}}. \tag{30}$$

The solution $\mathcal{B}_1$ is unimodular and therefore a proper boost, whereas $\mathcal{B}_2$ is anti-unimodular and therefore an improper boost. If $\Lambda = \mathcal{B}_1\mathcal{R}$ is a solution of the Dirac equation (25), it must represent a positive-energy particle:

$$1 \cdot p = m\Lambda \cdot \Lambda^\dagger > 0 \tag{31}$$

whereas $\Lambda = \mathcal{B}_2\mathcal{R}$, as an anti-unimodular solution of (25), represents a negative-energy one:

$$1 \cdot p = -m\Lambda \cdot \Lambda^\dagger = -m\mathcal{B}_2 \cdot \mathcal{B}_2 < 0. \tag{32}$$

Unfortunately, this conflicts with Eq. (24) which $\mathcal{B}_2$ was supposed to satisfy: the sign of p is different! Either one changes the sign of p in $\mathcal{B}_2$ so that it represents the momentum of the negative- energy particle or one leaves the sign and reinterprets p in $\mathcal{B}_2$ as the momentum of the /em antiparticle.

Since rotations are unitary as well as unimodular, $\mathcal{R} = \bar{\mathcal{R}}^\dagger$. Consequently the matrix representation can be expressed in terms of a single 2-spinor χ: $\mathcal{R} = (-\bar{\chi}^\dagger, \chi)$. When this is combined with the standard matrix representations of the boost parts $\mathcal{B}_1$ and $\mathcal{B}_2$, the usual expressions (within a constant normalization factor) are obtained for the momentum eigenstates $\psi_p(0)$ of quantum theory. Now the full momentum eigenstates are plane waves with the dependence $\psi_p(x) = \psi_p(0)\exp(-ip \cdot \bar{r}/\hbar)$. Since the Lorentz scalar $p \cdot \bar{r} = m\tau$, the relation of ψ to the eigenspinor Λ means that

$$\Lambda(\tau) = \Lambda(0)\exp(i\mathbf{e}_3 m\tau/\hbar). \tag{33}$$

In words: the particle spins about its rest-frame direction $-\mathbf{e}_3$ at the *Zitterbewegung* (proper) frequency $2m/\hbar = 2mc^2/\hbar$.

The physical interpretation of the quantum phase as a rotation or spin is a great success of the Clifford-algebra approach to the Dirac theory (Hestenes 1975, 1990). Many other useful insights also surface. For example, at low velocity, the rotational two-spinor χ gives the rotation $\mathcal{R}$ of the spin from $-\mathbf{e}_3$ to its direction in the lab frame, and the intrinsic parity of fermions and antifermions is given immediately. The theory in $\mathcal{P}$ has so far been essentially classical, but the transition to the full quantum theory is surprisingly easy. The first step is to demand local gauge invariance which now corresponds to invariance under a rotation about the spin axis. This invariance demands the introduction of a gauge field and the replacement of the rotation factor $\exp(i\mathbf{e}_3 m\tau) = \exp(i\mathbf{e}_3 p \cdot \bar{r})$ by $\exp[i\mathbf{e}_3 \int (p + eA) \cdot d\bar{r}]$. However, this factor is now path-dependent. The second and crucial step is to sum over the Λ from all contributing paths. The result is a path integral over contributions which all satisfy the differential relation

$$\bar{\partial}\Lambda(i\mathbf{e}_3) = (p + eA)\Lambda. \tag{34}$$

The combination of this differential relation with the classical Dirac equation (25) gives the full Dirac theory in a form ripe with physical significance and often very convenient for computations.

The concept of a rest frame becomes meaningless for massless particles like neutrinos, but the theory developed above for 'elementary' charged particles works well in the limits $e \to 0$ and $m \to 0$. Of course the wavefunction must be normalized to correspond, say, to $p^{1/2}$ rather than to $u^{1/2}$, but then it is finite and its 'plane-wave' solutions vary as $m^{1/2}\Lambda(r) = m^{1/2}\Lambda(0)\exp(i\mathbf{e}_3 p \cdot \bar{r}/\hbar)$. As with electrons, any spin state can be written as a linear combination of the two helicity states, but in the limit $m \to 0$ one of the helicity states vanishes.

5. Electromagnetic Radiation

The vector potential for circularly polarized electromagnetic plane waves can be written in the same form as the neutrino eigenspinors:

$$A(r) = \mathbf{A}(r) = \mathbf{A}(0)\exp(i\kappa\hat{\mathbf{k}}k \cdot \bar{r}) \tag{35}$$

where for simplicity the transverse ('radiation') gauge has been adopted: $\mathbf{k} \cdot \mathbf{A} = \phi = 0$ and $\kappa = \pm 1$ is the helicity. The Lorentz-gauge condition is also satisfied: $\partial \cdot \bar{A} = 0$. The vector potential (35) is a real vector which rotates about the propagation direction. This is more obvious if (35) is written

$$A(r) = \mathcal{R}A(0)\mathcal{R}^\dagger, \tag{36}$$

where $\mathcal{R} = \exp(-i\kappa\hat{\mathbf{k}}k \cdot \bar{r}/2)$ is the *rotational eigenspinor* of the wave. Of course a boost can also be applied.

Maxwell's equation (18) for A for source-free space requires the propagation four-vector to be null, that is to be a zero divisor: $k\bar{k} = 0$. As a consequence, the associated electromagnetic field is

$$\mathbf{F} = \mathbf{E} + i\mathbf{B} = \partial \wedge \bar{A} = i\kappa k\mathbf{A}(r) = i\kappa k\mathbf{A}(0)\exp(-i\kappa k \cdot \bar{r}). \tag{37}$$

The unit vector $\hat{\mathbf{k}}$ could be dropped because $k\hat{\mathbf{k}} = k$.

6. Conclusions

The three real basis vectors of physical space generate in the Pauli algebra $\mathcal{P}$ a four-dimensional space with the Minkowski metric and complex numbers. It provides a covariant framework for problems in relativity without appealing to higher-order Clifford algebras. Application of the Pauli algebra to radiation fields gives simple real expressions for the fields in terms of a rotating vector. Application to 'elementary' particles turns the innocent kinematical relation $p = mu$ into the classical Dirac equation, with the quantum probability amplitude identified with the Lorentz transformation (the 'eigenspinor') of the particle. The plane waves of quantum theory imply a rotation about the spin axis at the *Zitterbewegung* frequency. Local gauge invariance requires a gauge field A which gives path-dependent eigenspinors, and a linear superposition of contributions from different paths leads to path integrals and the differential operator for the canonical momentum. The full Dirac theory results, but in a form that clearly displays physical features which are well hidden in traditional treatments. In the nonrelativistic limit, the energy eigenvalue equation from the Dirac theory gives the Schrödinger equation with the Pauli Hamiltonian, and even there all appearances of the imaginary i in the differential operators and in commutation relations are seen to originate with the spin of the particle (Hestenes 1971, 1975, 1990; Baylis et al. 1992).

The above assertion is liable to engender disbelief: "Surely you don't mean that all quantum particles must have spin?" No, just the 'elementary' ones. Of course, if a scalar Higgs boson is discovered, there may be more work to do.

Satisfying progress can be reported in applications of the Pauli algebra to fundamental problems of physics, including research not reported here on the electroweak theory, radiation reaction, and general relativity. However, much work remains to be done. Especially pressing are problems in many-body interactions and second-quantization, and there are potential applications in quantum gravity which will probably occupy a generation of physicists.

7. Acknowledgements

This work has been supported by the Natural Sciences and Engineering Research Council of Canada.

References

Baylis. W.E.: 1992, 'Classical eigenspinors and the Dirac equation', *Phys. Rev. A* **45**, 4293-4302

Baylis, W.E. and Huschilt, J. and Wei, J.: 1992, 'Why i?', *Am. J. Phys.* **60**, 788-797

Baylis, W.E. and Jones, G.: 1988, 'Special relativity with Clifford algebras and 2 × 2 matrices, and the exact product of two boosts', *J. Math. Phys.* **29**, 57-62

Baylis, W.E. and Jones, G.: 1989a, 'The Pauli-algebra approach to special relativity', *J. Phys. A* **22**, 1-16

Baylis, W.E. and Jones, G.: 1989b, 'Relativistic dynamics of charges in external fields: the Pauli-algebra approach', *J. Phys. A* **22**, 17-29

Gürsey, F.: 1957, 'Relativistic kinematics of a classical point particle in spinor form', *Nuovo Cimento* **5**, 784-809

Hestenes, D.: 1966, *Spacetime Algebra*, Gordon and Breach: New York

Hestenes, D.: 1975, 'Observables, operators, and complex numbers in the Dirac theory', *J. Math. Phys.* **16**, 556-572

Hestenes, D.: 1990, 'The Zitterbewegung interpretation of quantum mechanics', *Found. Phys.* **20**, 1213-1232

Rastall, P.: 1964, 'Quaternions in relativity', *Rev. Mod. Phys.* **36**, 820-832

Rastall, P.: 1988, 'New forms of the Dirac equation', *Nuovo Cimento* **101 B**, 479-494

TWISTORS

TWISTORS AND SUPERSYMMETRY

DMITRIJ VOLKOV
Kharkov Institute of Physics and Technology,
Kharkov 310108, the Ukraine

In modern theoretical physics there are two branches of research which are closely related to fundamental problems of space-time. These are twistors and supersymmetry. Recently a growing tendency to intertwining these two branches has been appearing, and it is now widely believed that twistors may play an essential part in the foundation of supersymmetric theories. The main point where twistors and supersymmetry are in touch is the integrability conditions for N-extended and $D = 10$ super Yang-Mills theories and supergravity which, in its turn, is connected with the fact that the trajectory of a massless superparticle in superspace is not a world line but a supersurface with one bosonic and a number of fermionic directions.

The way of reasoning leading from supersymmetry to twistors is rather complicated and grounded more on guesswork than on a solid constructive basis. A more direct way which incorporates twistors into supersymmetric theories as a primary ingredient has been proposed recently (Volkov 1988, 1989, 1990, Sorokin 1989a,b, 1990) and developed in a number of works[1].

The proposed "twistor like" reformulation of the superparticle and superstring actions contains two kinds of off-shell supersymmetry: the global supersymmetry of the super Poincaré group in the target superspace and a local supersymmetry on the world line (sheet) of the superparticle (superstring). The latter holds off shell and in a particular gauge transforms into the Siegel symmetry.

In respect to the local supertransformations on the world line the Grassmann θ coordinates of the target superspace and the components of twistor connected with the momentum of the superparticle are superpartners. This fact is of great importance and reflects a fundamental role which twistors play in the proposed formulation.

Here (after reviewing some aspects of supersymmetry and twistor theory) we discuss motivation, underlying ideas and an outcome of the proposed reformulation. The $D = 3, 4, 6$ and 10 superparticle action will be considered.

[1] The references are given in the following text.

1. Twistors

Twistors has been introduced by Penrose (1967) as an alternative to the coordinate description of space- time and simultaneously as a bridge between space-time and quantum properties of matter. The latter appeared hopeful because of the complex structure of both the twistor space and the quantum wave functions. In "the twistor programme" (Penrose 1977) a relation of twistors to the theory of elementary particles has been argued and unified twistor description of fundamental interactions had been preconceived. The unification of twistor theory with supersymmetry may contribute to revival of some ideas of "the twistor programme".

In the case of the $D = 4$ space-time a twistor is defined by four complex numbers which are usually represented as a pair of two-component complex spinors $\lambda_\alpha, \mu^\alpha$. Their relation to the space time coordinates is given by the basic equation

$$\mu^{\dot{\alpha}} = ix^{\dot{\alpha}\alpha}\lambda_\alpha \tag{1.1}$$

which for constant μ^α and λ_α satisfying

$$S = \frac{1}{2}(\lambda^\alpha\mu_\alpha + \overline{\lambda}^{\dot{\alpha}}\overline{\mu}_{\dot{\alpha}}) = 0 \tag{1.2}$$

describes a light-like line in Minkowski space. (1.1) together with the Cartan relation for the momentum of a massless relativistic particle

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha\overline{\lambda}_{\dot{\alpha}} \tag{1.3}$$

allows one to transform the world line action

$$S_1 = \int d\tau(p_m\dot{x}^m - e\frac{p^2}{2}) \tag{1.4}$$

to the twistor world line action

$$S_2 = i\int d\tau(\lambda^\alpha\dot{\mu}_\alpha + \dot{\overline{\lambda}}^{\dot{\alpha}}\overline{\mu}_{\dot{\alpha}}) \tag{1.5}$$

or to the third one

$$S_3 = -\int d\tau p_{\alpha\dot{\alpha}}(\dot{x}^{\alpha\dot{\alpha}} - \lambda^\alpha\overline{\lambda}^{\dot{\alpha}}) \tag{1.6}$$

which is intermediate between (1.4) and (1.5) as it contains the coordinates and the twistor components simultaneously. In performing the transformation it is important to note that $p^2 = 0$ is the constraint equation which follows from (1.4). The Cartan relation (1.3) solves this constraint explicitly. As a result the actions (1.5) and (1.6) are invariant under reparametrization of τ without the presence of the one-dimensional metric $e(\tau)$. This is very

essential for the subsequent supersymmetric generalization. The action (1.5) will not be used below. It is written down to stress that the approach under consideration is tightly connected with the standard twistor theory. It is evident that the supersymmetrized actions also admit a number of transformations similar to (1.4-6), which give a way to formulating different variants of supertwistor theory.[2] We will not touch this interesting question here.

The action (1.6) and its generalization to the string provide us with a tool to incorporate twistors into the superparticle and superstring theory. Now we proceed to those problems of the superstring and superparticle theory which require their reformulation.

2. Supersymmetry

While up to now twistor theory, providing a rather powerful technique for the investigating modern field theories, has not noticeably influenced the basic concepts of space-time, supersymmetry has lead to their revision and pretends now to be a physical theory. In the first papers on supersymmetry (Golfand 1971, Volkov 1972, Wess 1974) the Poincaré group has been generalized to the super-Poincaré group. The next years contributed to the development of supersymmetric field theories, supergravity and superstring theory.

Here we consider flat superspace with even coordinates x^m and odd Grassmann spinor coordinates θ^α their transformation law being defined as

$$\begin{aligned} \theta' &= \theta + \epsilon \\ x'^m &= x^m + i\overline{\theta}\Gamma^m\epsilon, \end{aligned} \tag{2.1}$$

which together with the Poincaré transformations is the super Poincaré group.[3]

In this section we use the notation being conventional for the description of spaces of dimensions $D = 3, 4, 6$ and 10; i.e. spinors are Majorana, Γ^m are symmetric, the Fiertz identity is fulfilled.

A remarkable peculiarity of the supersymmetry theories is that to a great extent their content depends on the symmetry properties of free objects propagating in superspace, which, in turn, can be defined classically. The proper starting point of any geometrical theory of objects propagating in a target space should include postulating that the inner space of objects is a subspace of the target space. Postulates of such kind were widely applied by

[2] See, for example, (Berkovits 1991)

[3] This formulation of superspace as well as the technique of using the invariant differential forms (2.2) for constructing supersymmetric actions was first proposed in (Volkov 1972) and in the widely unknown paper (Volkov 1974) where to describe the fermion Goldstone particles superspace (2.1) was used as a target space and Minkovski space was used as an inner space.

E. Cartan to his classical geometrical constructions and the method of external differential forms created by him gave nice mathematical formulation to them.

The objects (superparticles, superstrings, supermembranes) propagating in superspace sweep some supersurfaces, which locally in the small neighborhood are described by the differential forms

$$\begin{aligned} \Pi^m(d) &= dx^m - i\bar{\theta}\Gamma^m d\theta \\ \Pi(d) &= d\theta \end{aligned} \qquad (2.2)$$

being invariant under supertransformations (2.1). With the use of the differential forms (2.2) manifestly super Poincaré actions are easily constructed.

The Brink-Schwarz superparticle action (Brink 1981) is

$$S_{B-S} = \frac{1}{2}\int d\tau e^{-1}(\dot{x} - i\bar{\theta}\Gamma\theta)^2 \qquad (2.3)$$

where e is a one-dimensional metric. The Green-Schwarz superstring action consists of two terms (Green 1981)

$$S_{G-S} = S_1 + S_2 \qquad (2.4)$$

$$S_1 = -\frac{1}{2\pi}\int d^2\sigma\sqrt{h}h^{\alpha\beta}\Pi_\alpha\Pi_\beta \qquad (2.5)$$

where $\Pi^m_\alpha = \partial_\alpha x^m - i\bar{\theta}^A\Gamma^m\partial_\alpha\theta^A$ and $A = 1,2$. $h^{\alpha\beta}$ is the two-dimensional metric and A=1,2, which is the direct supersymmetrization of the bosonic string action and

$$\begin{aligned} S_2 = \tfrac{1}{\pi}\textstyle\int d^2\sigma \Big\{ &-i\epsilon^{\alpha\beta}\partial_\alpha x^m(\bar{\theta}^1\Gamma_m\partial_\beta\theta^1 - \bar{\theta}^2\Gamma_m\partial_\beta\theta^2) + \\ &\epsilon^{\alpha\beta}\bar{\theta}^1\Gamma^m\partial_\alpha\theta^1\bar{\theta}^2\Gamma_m\partial_\beta\theta^2\Big\} \end{aligned} \qquad (2.6)$$

which is a Wess-Zumino term, corresponding to the differential 3-form

$$W_3 = i\Pi^m(d)(d\bar{\theta}^1\Gamma^m d\theta^1 - d\bar{\theta}^2\Gamma^m d\theta^2) \qquad (2.7)$$

S_{B-S}, S_1 and S_2 are manifestly invariant under reparametrization. The actions (2.3) and (2.4) also have a remarkable fermionic symmetry under the local transformations

$$\delta\theta = 2i(\Gamma^m\Pi_m)\kappa, \quad \delta x^m = i\bar{\theta}\Gamma^m\delta\theta \qquad (2.8)$$

where $\Pi^m = i\bar{\theta}\Gamma^m\dot{\theta}$ and $\delta e = 0$ for the superparticle case and $\Pi^m_\alpha = \partial_\alpha x^m - i\bar{\theta}\Gamma^m\partial_\alpha\theta$, $\delta(\sqrt{h}h^{\alpha\beta}) = 0$ and $\kappa \to \kappa_\alpha$ for the superstring case. Spinor as well as extended symmetry indices are omitted.[4]

[4] The thorough discussion of the actions (2.3) and (2.4) and of the Siegel symmetry is given in the monograph (Green 1987) so we only make same general remarks.

Afterwards Green and Schwarz succeeded in writing down their celebrated superstring action (2.4) which is invariant under (2.8) only if the both terms S_1 and S_2 are present.

The discovery of the Siegel symmetry solved, in principle, the problem of unwanted fermionic degrees of freedom appearing due to disbalance in the number of the vector and spinor coordinates of superspace. Nevertheless, because of the presence of singularity in (2.8) no Lorentz covariant procedure for solving it practically has been proposed up to now, which is a serious drawback of the GS superstring theory. So, despite all the attractiveness of the GS superstring action the above mentioned drawback is an unsurmounted barrier to further development of the theory.

Therefore it may be instructive to go back to another formulation of superstring which was proposed earlier by Ramond, Neveu and Schwarz (Ramond 1971, Neveu 1971) and reanalyze its virtues and shortcomings. As it is well known, this formulation nicely deals with the unphysical fermionic components since a local superconformal invariance of the superstring world sheet has been introduced into the theory from the beginning. On the other hand, the existing procedure for checking the invariance of the RNS superstring under the super Poincaré transformations in the target superspace is highly artificial and tedious. So, we see that there is striking duality of virtues and shortcomings of GS and RNS superstrings with respect to their target space and world sheet supersymmetries.

Just as the GS superstring has a simplified version which is the BS superparticle, and which is a useful training ground for learning the symmetry properties of the theory, there similarly is a simplified version of the RNS superstring. This is the so called spinning particle. The latter has attracted less attention than the BS superparticle. But as we will see later the theory of the spinning particle may be looked at from a rather unexpected point of view as a consequence, twistors appear on the scene and a new approach to the theory of superparticles and superstrings arises.

3. The spinning particle

A covariant action for a spinning particle has been proposed in (Brink, 1976). The authors aimed to demonstrate the presence of supersymmetry in the Dirac equation and to give a simple one-dimensional model for interacting matter and supergravity. In the massless case the proposed action is following

$$S = \int d\tau(p_m\dot{x}^m - \frac{1}{2}ep^2 - \frac{1}{2}\psi^m\dot{\psi}_m - i\xi\psi^m p_m) \qquad (3.1)$$

(3.1) is invariant under τ reparametrizations and the local supersymmetry transformations

$$\delta\psi^m = \alpha(\tau)p^m,\ \delta x^m = i\alpha(\tau)\psi^m,\ \delta p^m = 0\ \delta e = i\alpha(\tau)\xi,\ \delta\xi = 2\dot{\alpha} \quad (3.2)$$

where ψ^m is a fermionic vector superpartner of x^m, e is the (one-dimensional) vielbein field and ξ its fermionic superpartner. (3.1) represents the limit of the RNS superstring when the string tension goes to infinity. The RNS superstring action contains the same functions x^m, p_m,and ψ^m depending on the world-sheet coordinates and two- dimensional supergravity multiplet. The absence in (3.1) of any spinor variable makes it difficult to formulate supersymmetry properties in the target space. Varying (8) which respect to e and ξ gives the constraints

$$p^2 = 0 \quad (3.3)$$
$$p\psi = 0 \quad (3.4)$$

which are the conditions for the particle to have zero mass and to obey the Dirac equation. We begin considering the twistor representation of the action (3.1) with the $D=3$ case, as it does not contain complications peculiar to the $D = 4,6$ and 10 cases. In section 1 the equation (3.3) has been solved by taking into account the Cartan relation:

$$p_{\alpha\beta} = \lambda_\alpha\lambda_\beta. \quad (3.5)$$

As (3.4) is a superpartner of (3.3) it is naturally to try to solve it by a similar substitution. We try

$$\psi_{\alpha\beta} = \lambda_\alpha\theta_\beta + \lambda_\beta\theta_\alpha \quad (3.6)$$

where θ_α is a real Grassmann spinor. Under local supersymmetry λ and θ transform as $\delta\lambda = 0$, $\delta\theta = \alpha(\tau)\lambda$ in accord with (3.2). In terms of spinor variables λ and θ the action of spinning particle (3.1) reads

$$S_{\frac{1}{2}} = \int d\tau(\lambda^\alpha\lambda^\beta\dot{x}_{\alpha\beta} - \frac{i}{2}(\lambda^\alpha\theta^\beta + \lambda^\beta\theta^\alpha)\frac{d}{d\tau}(\lambda_\alpha\theta_\beta + \lambda_\beta\theta_\alpha). \quad (3.7)$$

The equations of motion for $x_{\alpha\beta}$ generated by the action (3.7) have the form

$$\dot{\lambda}_\alpha = 0 \quad (3.8)$$

Taking into account Eqs. (3.5) and (3.8), we may rewrite the (3.7) in the form

$$S_{\frac{1}{2}} = \int d\tau\lambda^\alpha\lambda^\beta\{\dot{x}_{\alpha\beta} - \frac{i}{2}(\dot{\theta}_\alpha\theta_\beta - \theta_\alpha\dot{\theta}_\beta)\} \quad (3.9)$$

It is wonderful that the representation (3.9) is invariant under the super Poincaré transformations

$$\delta\theta_\alpha = \epsilon_\alpha,\ \delta\lambda_\alpha = 0,\ \delta x_{\alpha\beta} = \frac{1}{2}(\theta_\alpha\epsilon_\beta + \theta_\beta\epsilon_\alpha) \quad (3.10)$$

and coincides with the Brink-Schwarz superparticle action

$$S_{B-S} = \int d\tau\{p_{\alpha\beta}(\dot{x}^{\alpha\beta} - i\theta^\alpha\dot{\theta}^\beta) - \frac{e}{2}p^2\} \quad (3.11)$$

So the action (3.1) and (3.11) are classically equivalent on the mass shell. Upon quantization in the $D=4$ case (3.1) leads to the Dirac equation, and (3.7) and (3.11) lead to the Dirac equation for a Majorana spinor and the Klein-Gordon equation for a complex scalar field. So in the both cases there is a fourfold degeneration of the states, and the quantum systems are also equivalent. Nonequivalence arises either when an interaction is included or as a result of second quantization when the difference in statistics comes into play.

Consider now the Siegel transformations. For the action (3.11) they are

$$\delta\theta_\alpha = p_{\alpha\beta}\kappa^\beta, \delta x_{\alpha\beta} = i(\theta_\alpha\delta\theta_\beta + \theta_\beta\delta\theta\alpha), \delta p_\alpha = 0, \delta e = 4\theta^\alpha\delta\theta_\alpha \quad (3.12)$$

Due to the Cartan relations (3.12) transforms into

$$\delta\theta_\alpha = \alpha(\tau)\lambda_\alpha, \delta\lambda_\alpha = 0, \delta x_{\alpha\beta} = i\alpha(\tau)(\lambda_\alpha\theta_\beta + \lambda_\beta\theta_\alpha), \quad (3.13)$$

where $\alpha = \lambda_\alpha\kappa^\alpha$. Comparing (3.13) with (3.2) we see that they coincide. Since the transformations (3.2) are an off-shell symmetry of (3.9), we get an off-shell formulation of the Siegel symmetry. Note that the relation between $\delta\theta$ and δx in (3.12) and (3.13) has the opposite sign from that for the super Poincaré group. No explanation to this fact has been proposed. Now, because of the above relation between the Siegel and local superconformal transformations, this can be explained as usual difference of the signs of the left and right Cartan differential forms on a group manifold. The above consideration can be generalized up to the $D = 4,6,10$ for the $D = 4$ case, for example, eqs. (3.12) and (3.13) are the same, excepting that κ, θ, and λ are complex and

$$\delta x_{\alpha\dot{\beta}} = i\alpha(\tau)(\lambda_\alpha\overline{\theta}_{\dot{\beta}} + \overline{\lambda}_{\dot{\beta}}\theta_\alpha) \quad (3.14)$$

retaining to be real. It is natural to generalize (3.14) to complex α, so that, for example, (3.14) gets the form

$$\delta x_{\alpha\dot{\beta}} = i\alpha(\tau)(\lambda_\alpha\overline{\theta}_{\dot{\beta}} + \overline{\lambda}_{\dot{\beta}}\theta_\alpha) \quad (3.15)$$

One can easily show that (3.9) is also invariant under the complex transformations, but only on shell for an imaginary $\alpha(\tau)$. To get the off shell local supersymmetry it is necessary to consider its $n=2$ generalization. For $D = 6$ and 10,$n = 4$ and 8 generalizations are needed. Their consideration is more convenient in superfield formulation.

4. Twistor like action for superparticle

In the preceding section we have proved the equivalence of actions of the spinning particle and the superparticle, and we have also got an important conclusion that the θ-coordinates and the λ-components of twistor are superpartners under the local world line superconformal group. Using the mass shell equations has been an essential drawback of our reasoning. Here we give a twistor like reformulation of the $D = 3, 4, 6$ and 10 superparticle action with the off-shell $n = 1$ local superconformal invariance, which will be our primary principle. To do this we use the superfields

$$P_m = p_m + i\eta\rho_m, X_m = x_m + i\eta\psi_m,\ \Theta_\alpha = \theta_\alpha + \eta\lambda_\alpha \tag{4.1}$$

where ρ_m and ψ_m are Grassmann superpartners of p_m and x_m , respectively. Now with the use of (4.1) we supersymmetrize the action (1.6) and get

$$S = -i\int d\tau d\eta P_m(DX^m - i\overline{\Theta}\Gamma^m D\Theta), \tag{4.2}$$

where $D = \frac{\partial}{\partial\eta} + i\eta\frac{\partial}{\partial\tau}$. Note that, due to the structure of the Θ-superfield which contains λ as a multiplier of η , the second term in (4.2) contains derivatives of Θ . Besides, the sum of the first and the second term being supersymmetrized independently is super Poincaré invariant. Integrating over η gives the component form of (4.2)

$$S = \int d\tau\{p_m(\dot{x}^m - i\overline{\theta}\Gamma^m\dot{\theta} + \overline{\lambda}\Gamma^m\lambda) + i\rho_m(\psi^m + \overline{\theta}\Gamma^m\lambda)\}, \tag{4.3}$$

Excluding some variables from (4.3) one can get different forms of the action, including that of the BS superparticle and of the spinning superparticle.

Since the action (1.6) is off-shell reparametrization invariant the action (4.2-3) is off-shell invariant under superconformal transformations of τ and η independently from the fact that the derivative D does not contain vielbein variables and has the form corresponding to flat superspace. It is also invariant under $D - 2$ Siegel transformations one of which coincides with superconformal one.[5] To get the full equivalence of the Siegel and local superconformal transformations for superspaces with $D = 4, 6$ and 10 it is necessary to consider the $n = 2, 4$ and 8 extensions of the local superconformal group. This has been done in a number of papers (Sorokin 1989 a,b; Berkovits 1991; Howe 1991, Ganntlet 1991, Galperin 1992; Pashnev 1992; Chikalov 1992) which have elaborated a route from $D = 3$ to $D = 10$ dimensions. Because of the lack of place for reviewing all of them, here we briefly present only some of the results received recently. In (Galperin 1992) the general form of $D = 3, 4, 6$ and 10 superparticle action has been proposed. It looks as follows

$$S = \int d\tau d^n\eta P_{am}(D_a X^m - iD_a\Theta\gamma^m\Theta) \tag{4.4}$$

and is the most straight-forward generalization of the action (4.2). In (4.4) $D_a = \frac{\partial}{\partial\eta^a} + i\eta_a\frac{\partial}{\partial\tau}, a = 1, 2, ..., n$ and $n = 1, 2, 4$ and 8 depending on the dimension of superspace. (4.4) is gauge invariant under local superconformal transformations, which in the superfunction form is[6]

$$\delta\tau = \Lambda - \frac{1}{2}\eta^a D_a\Lambda\ \delta\eta_a = -\frac{i}{2}D_a\Lambda, \tag{4.5}$$

where Λ is an unconstrained world-line superfunction. Secondly, there is a large abelian gauge invariance

$$\delta P_a^m = D_b(\zeta_{abc}\Gamma^m D_c\Theta) \tag{4.6}$$

where the spinor parameter ζ_{abc} is totally symmetric with respect to its indices.

5. Twistor like actions for strings

To generalize the twistor-like approach to strings it is convenient to find a twistor realization of the constraints

$$T_{++} = \partial_+ x\partial_+ x = T-- = \partial--x\partial_{--}x = 0 \tag{5.1}$$

(which is analogous to constraint for the bosonic particle in the form of the Cartan relation (1.3). It can be easily shown that such a realization exists and has the form

$$\partial_\pm X^m = T^2\overline{\lambda}_\pm\Gamma^m\lambda_\pm \tag{5.2}$$

Using (5.2), the following form of bosonic string action can be written (Soroka 1990, Pashnev 1992)

$$S = \int d\tau d\sigma det(e^a_\nu)\overline{\lambda}\Gamma_m\rho^a e^\mu_a\lambda(\partial_\mu x^m - \frac{T^2}{2}\overline{\lambda}\Gamma^m\rho_b e^b_\mu\lambda). \tag{5.3}$$

Generalized to two spinors (A=1,2) the analogous to (5.3) form of the action for the type II GS superstring has been recently received (Chikalov 1992), which is invariant under diffeomorphism transformations on the world-sheet superspace, the latter is achieved by using a new ingeniously constructed

[5] Formally this is related to the fact that the expression under the integral sign in (4.2) is the 1-differential form on superspace in which even *commuting* differential $d\tilde{\eta}$ is substituted by the product of two anticommuting differentials $d\tau d\eta$. The substitution transforms the 1-differential form on superspace into Berezin integral. The detailed discussion of (4.2) as Chern—Simons action is given in (Howe 1991)

[6] It has been recently shown that this gauge invariance can be extended to the whole superdiffeomorphism group (Chikalov 1992) .

representation of the diffeomorphism group, in which all the coordinates of superspace and spinor superfields representing the left and right moving modes transform simultaneously and nonlinearly.

Much attention has been paid to twistor like reformulation of the heterotic string action (Berkovits 1989, 1991; Tonin 1991, Aoyama 1992; Delduc 1992). Since the heterotic superstring is supersymmetric only for right moving modes it is similar, to a certain extent, to the superparticle. This considerably simplifies consideration. $n = (1,0), (2,0), (4,0)$ and $(8,0)$ variants of world-sheet conformal supersymmetry has been proposed.

References

De Azcarraga, J.A. and Lukierski (1982) Supersymmetric particles with internal symmetries and central charges. *Phys. Lett.* **B113**, 170.

Aoyama, S. Pasti, P. Tonin, M. (1992) The GS and NRS heterotic strings from twistor string models. Preprint DFPD/92/TN/9.

Aoyama, S. Kowalski-Glikman, J. van Holten, J.W. Lukierski, J. (1987) The spinning superparticle. *Phys. Lett.* **201B**, 487. Kowalski-Glikman, J., Lukierski, J. (1989) *Mod. Phys. Lett.* **A4**, 2437

Berkovits, N. (1989) A covariant action for the heterotic string with manifest space time supersymmetry and world sheet conformal invariance. *Phys. Lett.* **B 232**, 184.

Berkovits, N. (1991a) A supertwistor description of the massless superparticle in ten-dimensional superspace. *Nucl. Phys.* **B350**, 193.

Berkovits, N. (1991b) The heterotic Green-Schwarz superstring on an $N = (2,0)$ super-world sheet. Preprint ITP- SB-91-63.

Brink, L. and Schwarz, J.H. (1981) Supersymmetric dual string theory. *Nucl. Phys.* **B181**, 502.

Chikalov, V. Pashnev, A. (1992) Twistor - like type II superstring and bosonic string. Preprint JINR-E2/92-337.

Delduc, F., Ivanov, E., Sokatchev, E. (1992) Twistor - like superstring with $D = 3, 4$ and 6 target superspace and $N = (1,0), (2,0), (4,0)$ world-sheet supersymmetry. ENSLAPP-L- 371/92 BONN-HE-92-11.

Galperin, A. and Sokatchev, E. (1992) A twistor - like $D = 10$ superparticle action with manifest $N = 8$ world-line supersymmetry. Preprint Jhu-TIPAC-920010 BONN-HE 92-07.

Gauntlet, J.P. (1991) A κ - symmetry calculus for superparticle EFI 91-41.

Gol'fand, Y.A. and Lichtman, E.P. (1971) Extension of the algebra of Poincaré group generators and violation of P invariance. JETR. 13, 323.

Green, M.B. Schwarz, J.H, Witten E. (1987) Superstring theory, v.1.

Howe, P.S. and Towndsend (1991) The massless superparticle as Chern-Simons mechanics. *Phys. Lett.* **259**, 285.

Ivanov, E.A. and Kapustnikov, A.A. (1991) Towards a tensor calculus for κ - supersymmetry. *Phys. Lett.* **267**, 175.

Neveu, A. Schwarz, J.H. (1971) Factorizable dual model of pions. *Nucl. Phys.* **B31**, 86.

Pashnev, A. and Sorokin, D. (1992) Note on superfield formulation of $D = 2, 3, 4, 6$ and 10 superparticle. Preprint JINR E2-92-27.

Penrose, R. (1967) Twistor algebra *J. Math. Phys.* **8**, 345.

Penrose, R. (1977) The twistor programme. *Rep. Math. Phys.* **12**, 65.

Ramond, P. (1971), Dual theory for free fermions. *Phys. Rev.* **D3**, 2415.

Siegel, W. (1983) Hidden local supersymmetry in the supersymmetric particle action. *Phys. Lett.* **128B**, 397.

Sorokin, D.P. (1990) Double supersymmetric particle theories *Fortshr. der Phys.* **38**, 923.

Sorokin, D.P., Tkach, V.I. and Volkov, D.V. (1989a) Superparticles, twistors and Siegel symmetry. *Mod. Phys. Lett.* **A4**, 901.

Sorokin, D.P., Tkach, V.I., Volkov, D.V. and Zheltukhin, A.A. (1989b). From the superparticle Siegel symmetry to the spinning particle proper - time supersymmetry. *Phys. Lett.* **B216**, 302.

Tonin, M. (1991) κ κ -symmetry as world sheet supersymmetry in $D = 10$ heterotic superstring DFPD/91/TN/29.

Volkov, D.V. and Akulov, V.P. (1972) On a possible universal interaction of neutrino. JETR Lett. 16 621.

Volkov, D.V. and Akulov, V.P. (1974) Goldstone fields of 1/2 spin. *Theor. Math. Phys.* **18**, 39.

Volkov, D.V. and Zheltukhin, A.A. (1988) Extension of the Penrose representation and its use to describe supersymmetric models. JETP Lett. 48 61.

Volkov, D.V. and Zheltukhin, A.A. (1989) On the equivalence of the massless Dirac and supersymmetric particles *Lett. Math. Phys.* **17**, 141.

Volkov, D.V. and Zheltukhin, A.A. (1990) Lagrangian for massless particles and strings with local and global supersymmetry. *Nucl. Phys.* **B335**, 723.

Wess, J. and Zumino, B. (1974) Supergauge transformations in four dimensions. *Nucl. Phys.* **B70**, 39.

A TWISTOR–LIKE DESCRIPTION OF $D = 10$ SUPERSTRINGS AND $D = 11$ SUPERMEMBRANES

IGOR BANDOS and ALEKSANDR ZHELTUKHIN
Kharkov Institute of Physics and Technology
Kharkov, 310108, Ukraine

1. Introduction

The covariant description of the supersymmetric theories of particles and strings is hampered by the problem of the local κ–symmetry covariant description [1,2]. Recently the new approach [3-13] for solving this problem has been suggested, where twistor–like variables have been introduced in addition to the world coordinates.

The introduction of twistor–like fields gives the simple solution of the problem of covariant division for the primary Grassmannian spinor constraints for superparticles and superstrings. After this covariant division of the constraints a new way is opened for the solving problem of the covariant BRST–BFV quantization along the line considered in [9a,b,d,e] for null superstrings and supermembranes in $D = 4$.

Here we give the general prescription for the constructing of the superstring and super p–brane actions in the extended space of world coordinates and twistor–like (or spinor harmonic) variables. This prescription is reduced to constructing the realization of the Cartan moving repere $\mathbf{n}_m(\tau, \sigma^n) \equiv n_m^{(e)}(\tau, \sigma^n)$ in D–dimensional space–time in terms of the generalized Newman – Penrose "dyades" or Lorentz harmonics $v_\alpha^a(\tau, \sigma^n) \in Spin(1, D-1)$ [14, 7, 9b,e,f, 11,12]. This realization has the form

$$n_m^{(e)}(\tau, \sigma^n) = 2^{-\left[\frac{D}{2}\right]} v_\alpha^a (C\Gamma_m)^{\alpha\beta} v_\beta^b \left(\Gamma^{(e)} C^{-1}\right)_{ab} \qquad (1)$$

For the most interesting case $D = 10$ superstring we show the mechanism of the covariant division for the Grassmannian constraints and find the covariant irreducible representation for the k–symmetry generators. The harmonicity (or generalized pure spinors) conditions for $D = 11$ supermembrane are proposed.

2. Super p–brane in twistor–like harmonic formulation

The suggested formulation for the super p–brane action in D–dimensional space–time is (see also [15])

$$S_{D,N,P} \; = \; \int d^{p+1}\zeta \; e \left[-\frac{1}{\sqrt{\alpha'}} \; e^{\mu}_{f} \left(\mathbf{n}^{[f]} \; \mathbf{w}_{\mu} \right) + c \right] + S^{WZ}_{D,N,P} \; , \qquad (2)$$

where $e^{f}_{\mu}(f; \mu = 0, \ldots, p)$ is the world–hypersheet vielbein, $w^{m}_{\mu} \equiv \partial_{\mu} x^{m} - i\partial_{\mu}\theta^{I} C\Gamma^{m}\theta^{I}$ $(I = 1, \ldots, N)$, c is a dimensionless ("cosmological") constant, $S^{WZ}_{D,N,P}$ denotes the Wess–Zumino term [16]. The symbols $\mathbf{n}^{[f]}$ denote a set of (p+1) tangent to world hypersheet vectors of the Cartan moving repere $\mathbf{n}^{(e)} \equiv \left(\mathbf{n}^{[f]}(\zeta), \mathbf{n}^{(i)}(\zeta) \right)$

$$\left(\mathbf{n}^{(e)} \cdot \mathbf{n}^{(k)} \right) \equiv n^{(e)}_{m} \; n^{(k)m} = \eta^{(e)(k)} = diag(1, -1, \ldots, -1) \; . \qquad (3)$$

The orthogonality conditions (3) are automatically satisfied if the twistor–like representation (1) for $\mathbf{n}^{(e)}$ is used and the harmonicity conditions for the majorana spinor $2^{[D/2]} \times 2^{[D/2]}$ matrix $v^{a}_{\alpha}(\zeta)$ are taken into account $(a, \alpha = 1, \ldots, 2^{[D/2]})$.

$$\Xi_{\mathcal{M}} \equiv Sp \left(v\Gamma^{(n)} C^{-1} v^{T} \Gamma_{m_1 \ldots m_k} \right) = 0, \qquad k = 2, \ldots, [D/2] \qquad (4)$$

Eqs. (4) have been considered for the cases $D = 4$ in [7], $D = 3, 6, 10$ [12,11] and $D = 11$ in [9c,f] and may be treated as the generalized "pure spinors"–type defining conditions [17].

After the substitution of Eqs. (1) into (2a) we get the twistor–like representation for $S_{D,N,P}$

$$S_{D,N,P} \; =$$

$$= \int d^{p+1}\zeta \; e \left[-\frac{1}{\sqrt{\alpha'} 2^{[D/2]}} \; e^{\mu}_{f} \; \mathbf{w}_{\mu} \; Sp \left(v^{T} C \; \boldsymbol{\Gamma} \; v \; \Gamma^{[f]} \; C^{-1} \right) + c \right] + S^{WZ}_{D,N,P}, \qquad (5)$$

Let prove the classical equivalence of the representation (2) to the standard Dirac–Nambu one.

The motion equations $\delta S_{D,N,P} \big/ \delta e^{\mu}_{f} \; = \; 0$ give

$$e^{f}_{\mu}(\zeta) = \frac{p}{c\sqrt{\alpha'}} \; \left(\mathbf{w}_{\mu} \cdot \mathbf{n}^{[f]} \right) \Rightarrow \frac{-1}{\sqrt{\alpha'}} \; e^{\mu}_{f} \; \left(\mathbf{w}_{\mu} \cdot \mathbf{n}^{[f]} \right) = \frac{p+1}{p} \; c \qquad (6)$$

The variations of $S_{D,N,P}$ (2) with respect to the additional variables $\mathbf{n}^{(e)}(\zeta)$ or $v^{a}_{\alpha}(\zeta)$ must take into account the restrictions (3) or (4). Then the motion Eqs. $\delta S_{D,N,P} \big/ \delta v^{a}_{\alpha} \Big|_{\Xi_{\mathcal{M}}=0}$ may be presented in the form

$$\left(\mathbf{w}_{\mu} \cdot \mathbf{n}^{(i)} \right) = 0 \Rightarrow \mathbf{n}^{[f]}(\zeta) = \frac{p}{c\sqrt{\alpha'}} \; e^{\mu f} \; \mathbf{w}_{\mu} \qquad (7)$$

Using Eqs. (6) and (7) we find that

$$g_{\mu\nu}(\zeta) = e^{f}_{\mu} \; e_{f\nu} = \left(\frac{p}{c\sqrt{\alpha'}} \right)^{2} (\mathbf{w}_{\mu} \cdot \mathbf{w}_{\nu})$$
$$e(\zeta) = det \; e^{f}_{\mu} = \left(\frac{p}{c\sqrt{\alpha'}} \right)^{p+1} \sqrt{(-1)^{p} \; det \, (\mathbf{w}_{\mu} \cdot \mathbf{w}_{\nu})} \qquad (8)$$

and after use of Eqs. (6–8) we get the Dirac–Nambu representation for $S_{D,N,P}$ (2), i.e.

$$S_{D,N,P} = -\left(\frac{p}{c} \right)^{P} \left(\frac{1}{\sqrt{\alpha'}} \right)^{p+1} \int d^{p+1}\zeta \sqrt{(-1)^{p} \; det \, (\mathbf{w}_{\mu} \cdot \mathbf{w}_{\nu})} + S^{W,Z}_{D,N,P}$$

3. $D = 10$ Superstring in twistor–like approach

In this case $(D, N, P) = (10, 2, 1)$ and $v^{a}_{\alpha}(\zeta)$ is a majorana spinor 16×16 matrix. Since the local plane tangent to the superstring world–sheet is two–dimensional $(p = 1)$ its local fixation is defined by the choice $\mathbf{n}^{[0]} \equiv \mathbf{n}^{(0)}$ and $\mathbf{n}^{[1]} = \mathbf{n}^{(9)}$ as the vector tangent to the world–sheet. This choice reduces the local Lorentz group $SO(1,9)$ to it's subgroup $SO(1,1) \times SO(8)$ and index a is splitted $a = (A^{+}, \dot{A}^{-})$. Here $A = 1, \ldots, 8$, $\dot{A} = 1, \ldots, 8$ are the indices of $8(s)$ and $8(c)$ spinor representations of $SO(8)$ group, and $+, -$ belong to unit weight's spinor indices of $SO(1,1)$ local tangent group.

Now the harmonicity conditions (4) for $v^{a}_{\alpha}(\zeta) \equiv \left(v^{+}_{\alpha A}, v^{-}_{\alpha \dot{A}} \right)$, are presented as [11]

$$\Xi^{(n)}_{m_1, \ldots, m_5} \equiv \frac{1}{16} \; v^{a}_{\alpha} \; \tilde{\sigma}^{\alpha\beta}_{m_1 \ldots m_5} v^{b}_{\beta} \; \sigma^{(n)}_{ab} = 0, \qquad (9a)$$

$$\Xi \equiv \frac{1}{128} \left(v_{\alpha \dot{A}}^{\;-} \; \tilde{\sigma}^{\alpha\beta}_{n} \; v_{\beta \dot{A}}^{\;-} \right) \left(v_{\rho A}^{\;+} \; \tilde{\sigma}^{n\rho\lambda} \; v_{\lambda A}^{\;+} \right) - 1 = 0 \qquad (9b)$$

and the tangent light–cone vectors $\mathbf{n}^{[\pm 2]} \equiv \mathbf{n}^{(0)} \pm \mathbf{n}^{(1)}$ together with the 8 vectors $\mathbf{n}^{(c)}$ orthogonal to $\mathbf{n}^{[\pm 2]}$ are parametrized as [9b,f]

$$n^{[+2]}_{m}(\zeta) = \frac{1}{8} v_{\alpha A}^{\;+} \; \tilde{\sigma}^{\alpha\beta}_{n} \; v_{\beta A}^{\;+}, \qquad n^{[-2]}_{m}(\zeta) = \frac{1}{8} v_{\alpha \dot{A}}^{\;-} \; \tilde{\sigma}^{\alpha\beta}_{m} \; v_{\beta \dot{A}}^{\;-},$$
$$n^{[i]}_{m}(\zeta) = \frac{1}{8} v_{\alpha A}^{\;+} \; \tilde{\sigma}^{\alpha\beta}_{m} \; \gamma^{(i)}_{A\dot{B}} \; v_{\beta \dot{B}}^{\;-}, \qquad (9)$$

The orthonormality conditions (3) are automatically satisfied due to (9) and the identity $\tilde{\sigma}_n^{(\alpha\beta}\tilde{\sigma}^{\gamma)\delta m} = 0$. Among 1261 harmonicity conditions only 210+1 = 211 ones are independent. However, due to invariance of Eqs. (9) under the transformations from $SO(1,1) \times SO(8)$ gauge group, killing 28+1=29 components of $v_\alpha^a(\zeta)$, only 16=256-211-29 of them are independent degrees of freedom. Therefore the harmonics v_α^a may be considered as the coordinates of the coset space $SO(1,9)/SO(1,1) \times SO(8)$. Together with the two non pure gauge degrees of freedom

$$\rho^{[+2]\tau} \equiv \frac{e}{\sqrt{\alpha'}}(e_0^\tau - e_1^\tau) \quad \text{and} \quad \rho^{[-2]\tau} = \frac{e}{\sqrt{\alpha'}}(e_0^\tau + e_1^\tau) \quad \text{of} \quad e_\mu^f(\zeta)$$

the 16 independent components of $v_\alpha^a(\zeta)$ parametrize the 18 independent components of the two light–like combinations $\mathbf{K}^I (I = 1,2)$ [9b,f]

$$K_n^I \equiv \mathcal{P}_n - \frac{(-1)^I}{c\,\alpha'}\left(\partial_\sigma x_n - 2i\,\partial_\sigma\,\theta^I\,\sigma_n\,\theta + \partial_\sigma \mathcal{A}^I\right) \tag{10a}$$

of the Virasoro reparametrization constraints

$$\mathcal{K}^{1m}\mathcal{K}_m^1 \approx 0, \qquad \mathcal{K}^{2m}\mathcal{K}_m^2 \approx 0, \tag{10b}$$

Therefore we prove that the introduction of the harmonic variables does not introduce any additional degrees of freedom.

The action (2) for $D = 10$ $N = II\ B$ Green–Schwarz superstring is rewritten in the form [9b,f]

$$S_{10,2,1} = \frac{1}{2}\int d\tau d\sigma \left\{\left[\left(\rho^{[+2]\mu}\mathbf{n}^{[-2]} - \frac{i}{c\,\alpha'}\epsilon^{\mu\nu}\partial_\nu\theta^1\boldsymbol{\sigma}\theta^1\right) + \left(\phi^{[-2]\mu}\mathbf{n}^{[+2]} + \right.\right.\right.$$
$$\left.\left. + \frac{i}{c\alpha'}\epsilon^{\mu\nu}\partial_\nu\theta^2\boldsymbol{\sigma}\theta^2\right)\right]\mathbf{w}_\mu + \epsilon^{\mu\nu}\left[\frac{1}{2}c\alpha'\left(\rho_\mu^{[+2]}\mathbf{n}^{[-2]}\right)\left(\rho_\nu^{[+2]}\mathbf{n}^{[+2]}\right) + \right.$$
$$\left.\left. + \frac{1}{c\alpha'}\left(\partial_\mu\theta^1\boldsymbol{\sigma}\theta^1\right)\left(\partial_\nu\theta^2\boldsymbol{\sigma}\theta^2\right)\right]\right\} \tag{11}$$

Note that $S_{10,2,1}$ (11) may be presented in the Chern–Simons–like form after the redefinition of the combinations $\rho^{[+2]\mu}\mathbf{n}^{[-2]}$ and $\rho^{[-2]\mu}\mathbf{n}^{[+2]}$ (containing the world–sheet metric e_μ^f) into new momentum–like variables. The action (11) is invariant under the k–symmetry transformations [9f]

$$\begin{aligned}
&\delta\theta^{\alpha 1} = v^{\alpha-}{}_A\epsilon_A^+, \qquad \delta\rho^{[+2]\mu} = \frac{4i}{c\alpha'}\,\epsilon_A^+\epsilon^{\mu\nu}v_{\alpha A}{}^+\partial_\nu\theta^{\alpha 1},\\
&\delta\theta^{\alpha 2} = v^{\alpha-}{}_{\dot A}{}^+\epsilon_{\dot A}^-, \qquad \delta\rho^{[-2]\mu} = -\frac{4i}{c\alpha'}\,\epsilon_{\dot A}^-v_{\alpha\dot A}{}^+\epsilon^{\mu\nu}\partial_\nu\theta^{\alpha 2},\\
&\delta x^n = i\left(v_A^{\alpha-}\epsilon_A^+\theta^{\beta 1} + v_{\dot A}^{\alpha+}\epsilon_{\dot A}^-\theta^{\beta 2}\right)\sigma_{\alpha\beta}^n\,,\\
&\delta v_{\alpha A}^+ = \frac{-2i}{c\alpha'}\,\zeta^{i[+2]}\mathcal{D}^{i[-2]}v_{\alpha A}{}^+, \qquad \delta v_{\alpha\dot A}^- = \frac{2i}{c\alpha'}\,\zeta^{i[-2]}\mathcal{D}^{i[+2]}v_{\alpha\dot A}{}^-,
\end{aligned} \tag{12}$$

where $\epsilon_A^+, \epsilon_{\dot A}^-$ are the k–symmetry Grassmannian parameters, $D^{i[\pm 2]}$–the covariant derivatives generating the Lorentz boosts and $\zeta^{i[\pm 2]}$ are

$$\zeta^{i[\pm 2]} = \epsilon_A^+\gamma_{A\dot A}^i v_{\alpha\dot A}^-\partial^{[\pm 2]}\theta^{\alpha 1} + \epsilon_{\dot A}^-\tilde{\gamma}_{\dot A A}^i v_{\alpha A}{}^+\partial^{[\pm 2]}\theta^{\alpha 2}.$$

The primary Grassmannian spinor constraints $\tilde{\mathcal{D}}_\alpha^I$ corresponding to the k–symmetry (12)

$$\begin{aligned}
\tilde{\mathcal{D}}_\alpha^I &\equiv -\pi_\alpha^I + i\left[\mathcal{P}_m + (-1)^I/\alpha'\left(\partial_\sigma x_m - i\partial_\sigma\theta^I\sigma_m\theta^I\right)\right]\sigma_{\alpha\beta}^m\theta^{\beta I} -\\
&-2i(-1)^I/c\alpha'\left(\partial_\sigma\theta^I\sigma^m\right)_\alpha \mathcal{A}_m^I \approx 0
\end{aligned} \tag{13}$$

are the mixture of the first and second class constraints. The covariant division of these constraints may be done with the help of $v_A^{\alpha-}(\zeta)$ and $v_{\dot A}^{\alpha+}(\zeta)$ harmonics.

The irreducible first class constraints D_a^{1-} and $D_{\dot A}^{2+}$ are the 16 generators of the k–symmetry

$$\mathcal{D}_A^{1-} \equiv v_A^{\alpha-}\,\tilde{\mathcal{D}}_\alpha^1 \approx 0, \qquad \mathcal{D}_{\dot A}^{2+} \equiv v_{\dot A}^{\alpha+}\,\tilde{\mathcal{D}}_\alpha^2 \approx 0, \tag{14a}$$

The irreducible second class constraints $\mathcal{D}_A^{2-}$ and $\mathcal{D}_{\dot A}^{1+}$ are presented as the following products

$$\mathcal{D}_A^{2-} \equiv v_A^{\alpha-}\,\tilde{\mathcal{D}}_\alpha^2 \approx 0, \qquad \mathcal{D}_{\dot A}^{1+} \equiv v_{\dot A}^{\alpha+}\,\tilde{\mathcal{D}}_\alpha^1 \approx 0, \tag{14b}$$

Due to the limited volume of the report we have no possibility to reproduce all constraints characterizing the twistor–like representation (11).

Since all these constraints [9] are covariant and irreducible, the BRST-BFV formalism may be employed for the covariant quantization of $D = 10$ IIB superstring, as it has been done for the null super p–brane ($p = 0, 1, 2$) in $D = 4$ [9a,d,e].

Concluding the twistor–like description of $D = 10$ superstring list its motion equations

$$v_{\alpha A}^{+}\, e^{\mu[-2]}\, \partial_\mu\, \theta^{\alpha 2} = 0, \qquad v_{\alpha \dot{A}}^{-}\, e^{\mu[+2]}\, \partial_\mu\, \theta^{\alpha 1} = 0,$$

$$\partial_\mu \left(e e^{\mu[\pm 2]}\, \mathbf{n}^{[\mp 2]} \right) - \frac{1}{c\sqrt{\alpha'}}\, \epsilon^{\mu\nu} \left(\partial_\mu \theta^1\, \boldsymbol{\sigma}\, \partial_\nu\, \theta^1 - (1 \to 2) \right) = 0 \qquad (15)$$

which is to be completed by Eqs. (6) and (7).

4. $D = 11$ supermembrane in twistor–like approach

The local space tangent to the supermembrane ($(D, N, P) = (11, 1, 2)$) world–volume is 3–dimensional and its basis is built from the repere vectors $\mathbf{n}^{[f]} \equiv \left(\mathbf{n}^{(0)}, \mathbf{n}^{(9)}, \mathbf{n}^{(10)} \right)$. Therefore the local repere group $SO(1, 10)$ is reduced to $SO(1,2) \times SO(8)$ one. Then the majorana spinor 32×32 matrix $v_\alpha^a(\zeta)$ is presented as $v_\alpha^a(\zeta) \equiv \left(v_{\alpha, A}^{\tilde{a}}, v_{\alpha, \dot{A}\tilde{a}} \right)$, where $\tilde{a} = (1, 2)$ is spinor index of $SO(1, 2,)$ group. The harmonicity conditions for $D = 11$ have the form [9c,f].

$$\Xi \equiv v_\alpha^a\, C^{\alpha\beta}\, v_\beta^b - C^{ab} = 0, \qquad \Xi^{(n)}_{m_1 m_2} \equiv v_\alpha^a \left(\Gamma_{m_1 m_2} \right)^{\alpha\beta}\, v_\beta^b \left(\Gamma^{(n)}\, C^{-1} \right)_{ab} = 0,$$

$$\Xi^{(n)}_{m_1 \ldots m_5} \equiv v_\alpha^a \left(C \Gamma_{m_1 \ldots m_5} \right)^{\alpha\beta}\, v_\beta^b \left(\Gamma^{(n)} C^{-1} \right)_{ab} = 0 \qquad (16)$$

and reduce the number of independent variables among v_α^a to 55=1024-496-11-462. Due to the invariance of Eqs. (16) under $SO(1,2) \times SO(8)$ gauge group the resulting number of independent components of v_α^a equals to 24 and coincides with the dimension of the coset $SO(1, 10) \big/ SO(1, 2) \times SO(9)$.

$SO(1,2) \times SO(8)$ invariant representations for C^{ab} and the $\Gamma^{(m)} \equiv \left(\Gamma^{[f]}, \Gamma^{(i)} \right)$ in $D = 11$ are

$$C^{ab} = -C^{ba} = diag \left(\epsilon^{\tilde{a}\, \tilde{b}}\, \delta_{AB},\, -\epsilon_{\tilde{a}\, \tilde{b}}\, \delta_{\dot{A}\dot{B}} \right),$$

$$\Gamma^{[f]} = diag \left(\gamma_{\tilde{a}}^{[f]\tilde{b}}\, \delta_{AB},\, \gamma_{\tilde{b}}^{[f]\tilde{a}}\, \delta_{\dot{A}\dot{B}} \right), \qquad (17)$$

$$\Gamma^i = \begin{pmatrix} 0 & \epsilon_{\tilde{a}\tilde{b}} \gamma^{(i)}{}_{A\dot{B}} \\ -\epsilon^{\tilde{a}\tilde{b}} \tilde{\gamma}^{(i)}{}_{\dot{A}B} & 0 \end{pmatrix}$$

Then the twistor–like expression (2) for $D = 11$ $N = 1$ supermembrane is

presented in the form [9c,f].

$$S_{11,1,1} = \int d^3\, \zeta\, e \Big[c - \frac{1}{\sqrt{\alpha'} 5!}\, e_f^\mu\, \mathbf{w}_\mu \left(v_{\alpha A}^{\tilde{a}}\, v_{\beta A}^{\tilde{c}}\, \epsilon_{\tilde{b}\, \tilde{c}} - v_{\alpha \dot{A} \tilde{c}}\, v_{\beta \dot{A} \tilde{b}}\, \epsilon^{\tilde{a}\, \tilde{c}} \right) \gamma_{\tilde{a}}^{[f]\tilde{b}}\, (C\boldsymbol{\Gamma})^{\alpha\beta} \Big] + S^{WX}_{11,1,2}\,, \qquad (18)$$

where $\mathbf{w}_\mu \equiv \partial_\mu \mathbf{x} - i \partial_\mu \theta^\alpha \left(\boldsymbol{\Gamma} C^{-1} \right)_{\alpha\beta}\, \theta^\beta$. The motion equations generated by $S_{11,1,2}$ (18) are

$$v_{\alpha \dot{A} \tilde{b}}\, \gamma_{\tilde{a}}^{[f]\tilde{b}}\, e_f^\mu\, \partial_\mu\, \theta^\alpha = 0,$$

$$\partial_\mu \left[e\, e_f^\mu\, n_m^{[f]} \right] - \frac{4}{\alpha' c^2}\, \epsilon^{\mu\nu\rho}\, w_\mu{}^n \left(\Gamma_{mn} C^{-1} \right) \partial_\rho\, \theta = 0,$$

and Eqs. (6) and (7). The presence of $v_\alpha^a(\zeta)$ in $S_{11,1,2}$ (18) provides the Grassmannian constraints covariant division into irreducible constraints of the first and second class and carrying out the covariant quantization along the line [9a,b,d,e].

References

[1] De Azcarraga, I.A. and Lukierski, J., (1982) *Phys. Lett.* **B113**, 170; Siegel, W., ibid. **B128**, (1983) 397

[2] Green, M.B., Schwarz, J.H. and Witten, E., Superstring Theory, V.1. Cambridge Univ. Press 1987

[3] Volkov, D.V. and Zheltukhin, A.A. ((1988), Pis'ma Zh. Eksp. Teor. Fiz. **48**, 61; *Nucl. Phys.* **B335** (1990) 723

[4] Sorokin, D.P., Tkach, V.I. and Volkov, D.V., (1989) *Mod. Phys. Lett.* **A4**, 901 Sorokin, D.P., Tkach, V.I., Volkov, D.V. and Zheltukhin, A.A., (1989) *Phys. Lett.* **B216**, 302

[5] Nissimov, E., Pacheva, S. and Solomon, S., *Nucl. Phys.* **B296** (1988), 469; **B297** (1988) 349; **B317** (1989) 344

[6] Berkovits, N., *Phys. Lett.* **B232** (1989) 184; **B241** (1990) 497

[7] Bandos, I.A., *Yad. Fiz.* **51** (1990) 1429

[8] Bengtsson, I. and Cederwall, M., *Nucl. Phys.* **B302** (1988) 104

[9] Bandos, I.A. and Zheltukhin, A.A., *Pis'ma Zh. Eksp.Teor. FIz.* **51** (1990) 547[a]; **54** (1991)421[b]; **55** (1992) 81[c]; *Phys. Lett.* **B261** (1991) 245[d]; *Teor. Mat. Fiz.* **88** (1991) 358[e]; *Yader. Fiz.* **55** (1992) 3082[f]

[10] Ivanov, E.A. and Kapustnikov, A.A., *Phys. Lett.* **B267** (1991) 175

[11] Galperin, A.S., Howe, P.S. and Stelle, K.S., *Nucl. Phys.* **B308** (1992) 248

[12] Delduc, F., Galperin, A. and Sokatchev, E., *Nucl. Phys.* **B368** (1992) 143

[13] Howe, P.S. and Townsend, P.K., *Phys. Lett.* **B259** (1991) 285

[14] Penrose, R. and Rindler, W., *Spinor and Space-time* v. 1,2 Cambridge Univ. Press, 1986

[15] Volkov, D.V. and Zheltukhin, A.A., *Ukrain. Fiz. Zhurnal,* **30** (1985) 809; Zheltukhin, A.A., *Teor. Mat. Fiz.* **77** (1988) 377

[16] Bergshoeff, E., Sezgin, E. and Townsend, P.K., *Phys. Lett.* **B189** (1987) 75

[17] Howe, P., *Phys. Lett.* **B258** (1991) 141

BORN'S RECIPROCITY IN THE CONFORMAL DOMAIN

ARKADIUSZ JADCZYK
Institute of Theoretical Physics, University of Wrocław, pl. Maksa Borna 9, PL-50-204 Wrocław, Poland

Abstract. Max Born's reciprocity principle is revisited and complex four dimensional Kähler manifold $D_4 \approx SU(2,2)/S(U(2) \times U(2))$ is proposed as a replacement for space-time on the micro scale. It is suggested that the geodesic distance in D_4 plays a role of a quark binding super-Hamiltonian.

1. Introduction

Some 55 years ago, in the Scottish city of Edinburgh, Max Born wrote '*A suggestion for unifying quantum theory and relativity*'[Born, 1938], the paper that introduced his 'principle of reciprocity'. He started there with these words:

> 'There seems to be a general conviction that the difficulties of our present theory of ultimate particles and nuclear phenomena (the infinite values of the self energy, the zero energy and other quantities) are connected with the problem of merging quantum theory and relativity into a consistent unit. Eddington's book, "Relativity of the Proton and the Electron", is an expression of this tendency; but his attempt to link the properties of the smallest particles to those of the whole universe contradicts strongly my physical intuition. Therefore I have considered the question whether there may exist (other possibilities of unifying quantum theory and the principle of general invariance, which seems to me the essential thing, as gravitation by its order of magnitude is a molar effect and applies only to masses in bulk, not to the ultimate particles. I present here an idea which seems to be attractive by its simplicity and may lead to a satisfactory theory. '

Born then went on to introduce the *principle of reciprocity* - a primary symmetry between coordinates and momenta. He explained that

> 'The word *reciprocity* is chosen because it is already generally used in the lattice theory of crystals where the motion of the particle is described in the p-space with help of the *reciprocal lattice*.'

A year later, in a paper "*Reciprocity and the Number 137. Part I*", [Born, 1939] he makes an attempt to derive from his new principle the numerical value of the fine structure constant. [1] The most recent and clear exposition of the principle of reciprocity appears in his paper '*Reciprocity Theory of Elementary Particles* ', published in 1949 in honor of 70th birthday of Albert Einstein [Born, 1949]. The following extensive quotation from the Introduction to this paper brings us closer to Born's original motivations.

'The theory of elementary particles which I propose in the following pages is based on the current concepts of quantum mechanics and differs widely from the ideas which Einstein himself has developed in regard to this problem.(...) Relativity postulates that all laws of nature are invariant with respect to such linear transformations of space time $x^k = (\mathbf{x}, t)$ for which the quadratic form $R = x^k x_k = t^2 - \mathbf{x}^2$ is invariant (...). The underlying physical assumption is that the 4-dimensional distance $r = R^{\frac{1}{2}}$ has an absolute significance and can be measured. This is natural and plausible assumption as long as one has to do with macroscopic dimensions when measuring rods and clocks can be applied. But is it still plausible in the domain of atomic phenomena? (...) I think that the assumptions of the observability of the 4-dimensional distance of two events inside atomic dimensions is an extrapolation which can only be justified by its consequences; and I am inclined to interpret the difficulties which quantum mechanics encounters in describing elementary particles and their interactions as indicating the failure of this assumption.

The well-known limits of observability set by Heisenberg's uncertainty rules have little to do with this question; they refer to the measurements and momenta of a particle by an instrument which defines a macroscopic frame of reference, and they can be intuitively understood by taking into account that even macroscopic instruments must react according to quantum laws if they are of any use for measuring atomic phenomena. Bohr has illustrated this by many instructive examples. The determination of the distance $R^{\frac{1}{2}}$ of two events needs two neighboring space-time measurements; how could they be made with macroscopic instruments if the distance is of atomic size?

If one looks at this question from the standpoint of momenta, one encounters another paradoxical situation. There is of course a quantity analogous to R, namely $P = p^2 = p_k p^k = E^2 - \mathbf{p}^2$, where $p_k = (\mathbf{p}, E)$ represents momentum and energy. But this is not a continuous variable as it represents the square of the rest mass. A determination of P means therefore not a real measurement but a choice between a number of values corresponding to the particles with which one has possibly to do. (...) It looks therefore, as if the distance P in momentum space is capable of an infinite number of discrete values which can be roughly determined while the distance R in coordinate space is not an observable quantity at all.

This lack of symmetry seems to me very strange and rather improbable. There is strong formal evidence for the hypothesis , which I have called *the principle of reciprocity*, that the laws of nature are symmetrical with regard to space-time and momentum-energy, or more precisely, that they are invariant

[1] He failed, but many years later Armand Wyler [Wyler, 1968,1969,1971] obtained a reasonable value by playing, as we shall see, with a similar geometrical idea. Wyler failed however in another respect: he was unable to formulate all the principles that are necessary to justify his derivation . His work was criticized (cf. [Robertson, 1971; Gilmore, 1971; Vigier, 1976]), his ideas not understood, his name disappeared from the lists of publishing scientists.

under the transformation

$$x_k \to p_k, \qquad p_k \to -x_k. \tag{I.1}$$

The most obvious indications are these. The canonical equations of classical mechanics

$$\dot{x}^k = \partial H/\partial p_k, \qquad \dot{p}_k = -\partial H/\partial x^k \tag{I.2}$$

are indeed invariant under the transformation (I.1), if only the first 3 components of the 4–vectors x^k and p_k are considered. These equations hold also in the matrix or operator form of quantum mechanics. The commutation rules

$$x^k p_l - p_l x^k = i\hbar\delta_l^k, \tag{I.3}$$

and the components of the angular momentum,

$$m_{kl} = x_k p_l - x_l p_k, \tag{I.4}$$

show the same invariance, for all 4 components. These examples are, in my opinion, strongly suggestive, and I have tried for years to reformulate the fundamental laws of physics in such a way that the reciprocity transformation (I.1) is valid (...). I found very little resonance in this endeavor; apart from my collaborators, K. Fuchs and K. Sarginson, the only physicist who took it seriously and tried to help us was A. Landé (...). But our efforts led to no practical results; there is no obvious symmetry between coordinate and momentum space, and one had to wait until new experimental discoveries and their theoretical interpretation would provide a clue. (...) There must be a general principle to determine all possible field equations, in particular all possible rest masses.(...) I shall show that the principle of reciprocity provides a solution to this new problem – whether it is the correct solution remains to be seen by working out all consequences. But the simple results which we have obtained so far are definitely encouraging (...).'

[2] The very problem of a serious contradiction between quantum theory and relativity was addressed again, in 1957, by E.P. Wigner in a remarkable paper '*Relativistic Invariance and Quantum Phenomena*', [Wigner, 1957]. Wigner starts with the assertion that 'there is hardly any common ground between the general theory of relativity and quantum mechanics'. He then goes on to analyze the limits imposed on space-time localization of events by quantum theory to conclude that:

'The events of the general relativity are coincidences, that is collisions between particles. The founder of the theory, when he created this concept, had evidently macroscopic bodies in mind. Coincidences, that is, collisions between such bodies, are immediately observable. This is not

[2] It must be said that later on, in his autobiographical book '*My life and my views*', [Born, 1968], Born hardly devoted more than a few lines to the principle of reciprocity. Apparently he was discouraged by its lack of success in predicting new experimental facts.

the case for elementary particles; a collision between these is something much more evanescent. In fact, the point of a collision between two elementary particles can be closely localized in space-time only in case of high–energy collisions.'

3

Wigner analyzes the quantum limitations on the accuracy of clocks, and he finds that "a clock, with a running time of a day and an accuracy of 10^{-8} second, must weigh almost a gram–for reasons stemming solely from uncertainty principles and similar considerations". [4]

2. Reciprocity the Twistor Way

Max Born's original idea of reciprocity was clear but imprecise. We will try to interpret it using more modern concepts. The interpretation below is ours. And so are its faults.

2.1. Interpretation

We will interpret the reciprocity symmetry (I.1) as a tangent space symmetry rather than as a global one. So, we assume that the fundamental arena D in which relativistic quantum processes take place is an 8-dimensional manifold with local coordinates (x^μ, p^μ). The symmetry (I.1) should hold in each tangent space. Since the square of the operation (I.1) is $-I$, we interpret (I.1) as the requirement that D should be equipped with a *complex structure*, which is respected by the fundamental equations. It is clear from Born's papers that D should be also endowed with a metric tensor. The simplest complex Riemannian manifolds are those that are Kählerian symmetric domains. I choose the Cartan domain $D_4 \approx SU(2,2)/S(U(2) \times U(2)) \approx SO(4,2)/S(O(2) \times O(2))$ as *the* candidate. It has many nice properties - some of them will be discussed later. There are also many possible objections against such a choice. Let me try to anticipate some of them.

– $D4$ has *positive-definite* metric - it cannot contain Minkowski space
  = True, indeed. On the other hand one can argue that according to Born's original idea, and according to the analysis by Wigner, Minkowski space-time of events is only an *approximation*. High–energy or high–mass approximation. Thus it is reassuring that the *Shilov boundary* of D_4 (the important concept that will be discussed later) is naturally isomorphic to the (compactified) Minkowski space endowed with its *indefinite conformal structure*. Let us interpret the points of $D4$ as elementary micro *event–processes*, that is micro-events accompanied by energy transfers. A coordinate of such an event is $z^k = x^k + \hbar p^k/p^2$, with $p^2 = E^2 - \mathbf{p}^2 > 0$ (see Sec. ???). In the limit of large energy transfers $\hbar^2/p^2 \to 0$ the positive definite metric blows up. What remains is the Minkowskian conformal metric for $z^k = x^k + 0$ - the finite part of the Shilov boundary. The positive definiteness of the Riemannian metric on D_4 can be thus viewed as an advantage rather than as a fault.

– D_4 is not invariant under time inversion.
  = Indeed, time inversion is not a symmetry of D_4 – it would change the complex structure into the opposite one. We will see that when realizing D_4 as a part of the Grassmannian in $\mathbf{C}^4$ one gets automatically two copies of the domain. Then time inversion can be thought of as the transposition of these two copies. We consider D_4 as useful for the modeling of physical processes on a micro–scale (say, inside mesons and hadrons). We know that on this scale time–inversion need not be a symmetry. On the other hand such a primary arrow-of-time on a micro-scale may well be connected with the observed macroscopic irreversibility as dealt with in thermodynamics. Thus breaking of the time-inversion symmetry can also viewed as an advantage rather than as a fault.

– D_4 has *constant* curvature and it is hard to imagine how models based on D_4 can be constructed that include gravity and/or gauge fields.
  = True, one of the original reasons for introducing the principle of reciprocity was unification of gravity and quantum theory. On the other hand, let us recall that, according to Born, gravitation 'is a molar effect and applies only to masses in bulk, not to the ultimate particles.'If so, and according to our interpretation above, there is no place for gravitation (and for other gauge fields as well) inside a meson or a hadron. Of, course, one could object that then there is also no place for space,time,energy and for momentum. It is of course an extrapolation, perhaps unjustified, that these concepts apply to such a micro-scale. However, extrapolating Einstein's scheme of general relativity into this domain would be unjustified even more. Therefore the idea that the primary arena of elementary event-processes is *homogenous* under a sufficient "zoom" may be rather attractive than appalling. [5]

– There is nothing new in the idea. Everything has been already said.

[3] I will return to this conclusion when interpreting space-time as the Shilov boundary of the conformal domain D_4.

[4] In 1986 Károlyházy et al. in the paper 'On the possible role of gravity in the reduction of the wave function', [Károlyházy, 1986], presented another analysis of the imprecision in space–time structure imposed by the quantum phenomena. They proposed 'to put the *proper amount of haziness* into the space–time structure'. Their ideas, as well as the ideas of a "stochastic space-time" most notably represented by E. Prugovečky (cf. [Prugovečki, 1991]) and references therein) all point in a similar direction.

[5] I heard this idea from Rudolph Haag.

= This objection is a serious one. There are extensive papers dealing with the domain D_4, mainly by Roger Penrose and his group (cf. [Penrose and Rindler, 1986] and references there), but also by Odzijewicz and collaborators (see [Odzijewicz, 1976; Karpio et al., 1986; Odzijewicz, 1988] and references there), and by Unterberger [Unterberger, 1987]. Many of these papers are too difficult for me to understand all their conclusions. Therefore there is a chance that the ideas presented here are simplistic and naive, mainly owing to my inadequate knowledge. If so, I will beg your pardon, and I will do my best to (at least) present those ideas that, I believe, deserve propagation. [6]

3. Algebraic description of the conformal domain D_4

There are many ways of describing the same domain D_4. I choose the algebraic description because it is simple. On the other hand it so happens that many years ago I studied its geometry, by algebraic means, without being fully aware of the full impact of the study [Jadczyk, 1971].

Let V be a complex vector space of complex dimension $n = p+q$, equipped with a Hermitean scalar product $\langle\ ,\ \rangle$ of signature (p,q). The domain D_n^+ is then defined as the manifold of p dimensional, *positive* linear subspaces of V[7]. In the following we will write D_n to denote D_n^+. Let $L(V)$ denote the algebra of linear operators on V. For each subspace $W \in D_n$ let E_W denote the orthogonal projection on W, and let $S_W \equiv 2E_W - I$. Then $S_W = S_W^\star$, $S_W^2 = I$, and $(v,w)_{S_W} \doteq (v, S_W w)$ is a *positive definite* scalar product on V. The last statement follows from the fact that S_W reverses the sign on $W^\perp$. Conversely, if $S \in L(V)$ satisfies the three conditions above, then the subspace $W \doteq \{v : Sv = v\}$ is in D_n and $S = S_W$. Geometrically, S_W plays the role of a *geodesic reflection symmetry* with respect to the point $W \in D_n$. The parametrization of the points of D_n through their symmetries is in many respects the most convenient one - the fact that is little known! Whenever we speak about a point of D_n, we have in mind one of its representing objects: subspace W, projection E, or symmetry operator S. We will use the '$\star$'symbol to denote the Hermitean conjugate with respect to the indefinite scalar product on V. Given $S \in D_n$, the Hermitean conjugate of $Y \in L(V)$ with respect to the positive-definite scalar product $(u,v)_S$ will be denoted by Y^S. Notice that $Y^S = SY^\star S,\ \ Y^\star = SY^S$.

It is evident from the very definition that the unitary group $U(V)$ of $(V, \langle\ ,\ \rangle)$, which is isomorphic to $U(p,q)$, acts transitively on D_n with the

[6] A review with a different emphasis can also be found in [Coquereaux and Jadczyk, 1990]

[7] The orthocomplements of the subspaces from D_n^+ are q dimensional *negative* subspaces. They form D_n^-. For $p = q$ this is the second copy of D_n^+ - as mentioned in the discussion of time inversion above.

stability group $U(p) \times U(q)$. The same is true about $SU(p,q)$, which acts effectively on D_n, so that

$$D_n \simeq SU(p,q)/S(U(p) \times U(q))$$

. By differentiating the defining equations

$$S = S^\star, \qquad S^2 = I, \tag{1}$$

of D_n we find that the tangent space T_S at S can be identified with the set of operators $X \in L(V)$ such that

$$X = X^\star, \quad \text{and} \quad XS + SX = 0. \tag{2}$$

Suppose now that $p = q$, thus $n = 2p$ (the most symmetric case). Call a basis $\{e_i\}$ in V isotropic if the scalar product of V in this basis reads $\langle v, w\rangle = v^\dagger G w$, where G is the block matrix

$$G = \begin{pmatrix} 0_p & iI_p \\ -iI_p & 0_p \end{pmatrix}. \tag{3}$$

Fix an isotropic basis, then D_n is isomorphic to the space of all $p\times p$ complex matrices T such that

$$i(T^\star - T) > 0, \tag{4}$$

the isomorphism $W \Longleftrightarrow T$ being given by

$$W = \{\begin{pmatrix} Tu \\ u \end{pmatrix} : u \in \mathbf{C}^p\}. \tag{5}$$

This parametrization defines complex structure on D_n. In terms of the operators X of Eq.(2) the complex structure J_S of the tangent space T_S at S is given by the map $J_S : X \to iXS$. Notice that (in the chosen isotropic basis) the orthogonal subspace to W_T is

$$W_T{}^\perp = \{\begin{pmatrix} T^\dagger u \\ u \end{pmatrix} : u \in \mathbf{C}^p\} = W_{T^\dagger}. \tag{6}$$

D_n is naturally equipped with an $U(V)$ – invariant positive definite Riemannian metric:

$$g(X,Y)_S \doteq -Tr(XY), \qquad X, Y \in T_S. \tag{7}$$

That g is positive definite follows from $X = X^\star = -X^S$, thus

$$g(X,X)_S = -Tr(XX) = Tr(X^S X) > 0 \quad \text{for} \quad X \neq 0. \tag{8}$$

D_n carries also an $U(V)$ - invariant symplectic structure ω: [8]

$$\omega(X,Y)_S \doteq g(X, J_S Y)_S = iTr(SXY). \tag{9}$$

D_n is a homogeneous Kählerian manifold. For $p = q = 2$ its interpretation as a conformal–relativistic phase space comes from the T–parametrization: [9] with T as in Eq. (5), we write

$$T = t^\mu \sigma_\mu = (x^\mu + \frac{q^\mu}{q^2})\sigma_\mu, \tag{10}$$

where $q^\mu = i\hbar p^\mu$, and $\sigma_\mu = \{I_2, \sigma\}$ are the Pauli matrices. The condition (4) reads now $p^2 = (p^0)^2 - \mathbf{p}^2 > 0$. Thus topologically, and also with respect to the action of the Poincaré group, D_4 is nothing but the future tube of the Minkowski space, endowed with a nontrivial Riemannian metric. It is to be stressed that special conformal transformations act on the variables p^μ not in the way one would normally expect. Thus (x^μ, p^μ) refer to some *extended process* rather than to a point event. Till now no interpretation of the points of D_4 in terms of space-time concepts, i.e. an interpretation that would explain their transformation properties, has been given.

The second important representation of D_n is as a bounded domain in $\mathbf{C}^{p^2}$. This representation can be obtained via the Cayley transform from the T-representation:

$$Z = i\frac{T-i}{T+i}, \qquad iT = \frac{Z+i}{Z-i}. \tag{11}$$

Geometrically, Z can be thought of as an orthogonal graph of the subspace W_T with respect to a fixed subspace $W_0 = W_{\{T=i\}}$. The condition (4) reads now $ZZ^\dagger < I$. The topological boundary ∂D_n is $(p^2 - 1)$ dimensional. The *Shilov* boundary $\check{\partial} D_n$ is defined as consisting of those points of ∂D_n at which functions analytic on the domain reach their maxima. $\check{\partial} D_n$ is isomorphic to the set of $p \times p$ unitary matrices; thus, for $p = 2$, to the compactified Minkowski space. $\check{\partial} D_n$ carries a unique $U(p,p)$–invariant conformal structure of signature (p,p). For $p = 2$ – the one induced by a flat Minkowski metric. The Cayley transform maps Minkowski space $t^\mu = x^\mu + 0$ onto the finite (affine) part of $\check{\partial} D_4$. We see from Eq. (10) that Minkowski space can interpreted as the *very–high–mass*, or *very–high–energy–momentum–transfer* limit of D_4. Elementary micro-processes that are characterized by very high energy-momentum transfers can be described as pure space-time events. It is only for such processes that the standard concepts of space, time and causality are applicable. For generic micro-processes there is no distinction between space and time, no distinction between space-time and energy-momentum. This would be an extreme manifestation of the Born reciprocity idea! *Thus, we propose to consider D_4 as the replacement for space-time on the micro scale.* In an analogy to the harmonic oscillator, the (square of) geodesic distance in D_4 may play a role of the quark binding super-Hamiltonian. One obtains in this way, again in the spirit of Born's reciprocity, an interesting and non-trivial version of the relativistic harmonic oscillator. Here we can only sketch the idea. [10]

[8] Although it is clear that ω is a non–degenerate, $U(V)$–invariant two-form, to prove that it is *closed* needs a computation.

[9] A justification for such a parametrization can be found in [Odzijewicz, 1976], [Coquereaux and Jadczyk, 1990]

Given two points S, S' in D_n, the fundamental two-point object is the unitary operator $t(S', S) \doteq (S'S)^{\frac{1}{2}}$. Many of the algebraic properties of these operators (including the case of $n = \infty$) have been studied in [Jadczyk, 1971]. Notice that $t(S', S)$ is unitary w.r.t the indefinite scalar product of V, but positive w.r.t both p.d. scalar products $(u,v)_S$, $(u,v)_{S'}$. In the next paragraph we will show that the map

$$X \longmapsto t(S'S)Xt(S'S)^\star$$

is the geodesic transport from the tangent space at S to the tangent space at S'.

3.1. Reductive decomposition of $U(V)$

For the Lie algebra of $U(V)$ we have:

$$Lie(U(V)) = \{Y \in L(V) : Y = -Y^\star\}, \tag{12}$$

while $L(V)$ coincides with the complexified $Lie(U(V))$. The Killing form $B(X,Y)$ is then given by

$$B(X,Y) \approx Tr(XY). \tag{13}$$

Given $S \in D_n$, the isotropy subalgebra K_S at S is

$$K_S = \{X \in L(V) : X^\star = -X, [X,S] = 0\}. \tag{14}$$

Every $X \in L(V)$ can be uniquely decomposed as

$$X = X_S^+ + iX_S^-,$$

where

$$[X_S^\pm, S]_\mp = 0.$$

The decomposition is given by

$$X_S^+ = \frac{1}{2}(SXS + X),$$

[10] More can be found in the forthcoming Thesis of W. Mulak (cf. also [Mulak, 1992] for an SU(1,1) version)

$$X_S^- = \frac{i}{2}(SXS - X).$$

We have $(X^\star)_S^\pm = (X_S^\pm)^\star$, and also

$$Tr(X_S^+ Y_S^-) = 0, \qquad \forall X, Y \in L(V).$$

Therefore the orthogonal complement of K_S w.r.t. the Killing form $B(X,Y)$ is the subspace $M_S \subset Lie(U(V))$ given by

$$M_S = \{X \in L(V) : X^\star = -X, XS + SX = 0\}.$$

3.2. $t(S', S)$ AS THE GEODESIC PARALLEL TRANSPORT

We will show that $t(S', S)$ implements parallel transport from the tangent space at S to that at S', and also how it can be used for computing of the geodesic distance between the two points. First notice that each geodesic through S is generated by a unique element $X \in M_S$ as follows (cf. [Kobayashi, 1969], p.192):

$$t \longmapsto S(t) = e^{tX} S e^{-tX} = e^{2tX} S, \tag{15}$$

the last equality follows from $XS + SX = 0$. If $Y(t)$ is a parallel vector field along $S(t)$, then (because D_n is a symmetric space; see [Chavel, 1972], p.64)

$$Y(t) = S(t/2)S(0)Y(0)S(0)S(t/2), \tag{16}$$

which by (15) gives

$$Y(t) = e^{tX} Y(0) e^{-tX}. \tag{17}$$

On the other hand

$$t(S(t), S(0)) = (S(t)S)^{\frac{1}{2}} = (e^{2tX})^{\frac{1}{2}} = e^{tX}, \tag{18}$$

and so

$$Y(t) = t(S(t), S)Y(0)t(S(t), S)^{-1}, \tag{19}$$

which proves that $t(S(t)S)$ is the parallel transport operator. To find the geodesic distance formula, notice that $e^{2tX}S$ is a geodesic through S with the tangent vector field $\dot{S} = 2Xe^{2tX}S$ of length $-Tr(\dot{S}^2) = 4Tr(X^2)$. For $Tr(X^2) = \frac{1}{4}$, $S(t)$ is parametrized by its length. But, from Eq.(18), we have that $tX = \ln t(S(t)S)$, $t^2X^2 = \ln^2 t(S(t)S)$, thus

$$dist(S, S(t)) = t = 4Tr(\ln^2 t(S(t)S))), \tag{20}$$

or

$$dist(S, S') = Tr(\ln^2(SS')). \tag{21}$$

4. Conclusions: quantum conformal oscillator

The relativistic quark model based on the Lorentz-covariant harmonic oscillator has been considered by many authors (cf. [Kim and Noz, 1991], and references there). Extending Max Born's reciprocity principle we propose to investigate a similar model, but based on the geometry of D_4.

For simplicity let us consider the *spinless* two-body problem in D_4. Quantum states of the two-body system will be described by analytic functions[11] $\Psi(S, S')$ on $D_4 \times D_4$, integrable with respect to an appropriate invariant measure. We take for super–Hamiltonian H of the system the Toeplitz projection of $dist(S, S')^2$. One can prove that by introducing the 'center of mass 'coordinates, the problem reduces to a one body problem. The spectrum of H can be computed in terms of the coherent states on D_4 (cf. [Mulak, 1992]). Such a model is nonrealistic, as it does not take into account spin. To consider spinning quarks we have to take for a model Hilbert space the space of sections of an appropriate vector bundle. The most natural one is the holomorphic tautological bundle Q^+ that associates to each $S \in D_4$ the subspace $W_S = \{u \in V : Su = u\}$. This bundle is endowed with a natural Hermitean connection. The operators $t(S, S')$ provide a natural parallel transport also in this bundle. Using its natural connection a Dirac–like operator can be constructed on Q^+. Much work must still have been done in order to see if models constructed along these lines have anything to do with reality.

References

Born, M., 'A suggestion for unifying quantum theory and relativity', *Proc. Roy.Soc.*, **A165**:291–303, 1938

Born, M., 'Reciprocity and the Number 137. Part I', *Proc. Roy. Soc. Edinburgh*, **59**:219–223, 1939

Born, M., 'Reciprocity Theory of Elementary Particles', *Rev. Mod. Phys.*, **21**:463–473, 1949

Born, M., 'My Life and my Views', Charles Scribner's Sons, New York, 1968

Chavel, I., 'Riemannian Symmetric Spaces of Rank One' Marcel Dekker, Inc. New York 1972

Coquereaux, R. and A. Jadczyk, 'Conformal theories, curved phase spaces, relativistic wavelets and the geometry of complex domains', *Rev. Math. Phys.*, **2**:1–44, 1990

Gilmore, R., 'Scaling of Wyler's Expression for α', *Phys. Rev. Lett.*, **28**:462–464

Jadczyk, A., 'Geometry of indefinite metric spaces', *Rep. Math. Phys*, 1:263–276, 1971.

Károlyházy, F., A. Frenkel, and B. Lukács, 'On the possible role of gravity in the reduction of the wave function', in *Quantum Concepts in Space and Time*, Ed. by R. Penrose and C. J. Isham, Clanderon Press, Oxford 1986

Kim, Y.S. and M.E. Noz, *Phase Space Picture of Quantum Mechanics* World Scientific, Singapore 1991

Kobayashi, S., and K. Nomizu, 'Foundations of Differential Geometry, Volume II' Interscience Publ., New York-London-Sydney 1969

[11] More precisely: by holomorphic densities

Karpio, A., A. Kryszeń and A. Odzijewicz, 'Two-twistor conformal Hamiltonian spaces', *Rep. Math. Phys.* , **24**:65–80, 1986
Mulak, W., 'Quantum SU(1,1) Harmonic Oscillator'. in Proc. XXV Symp. Math. Phys., Torun 1992, Preprint IFT UWr, Wroclaw 1993 - to appear
Odzijewicz, A. , 'A Model of Conformal Kinematics', *Int. J. Theor. Phys.* ,**15**: 576–593, 1976
Odzijewicz, A. , 'On reproducing kernels and quantization of states', *Commun. math. Phys.* , **114**:577–597, 1988.
Penrose, R., and W. Rindler, 'Spinors and Space-Time. Volume 2: Spinor and Twistor Methods in Space-Time Geometry', Cambridge University Press, 1986
Prugovečki, E., 'Geometro–Stochastic Locality in Quantum Spacetime and Quantum Diffusions', *Found. Phys.*, **21**:93–124
Robertson,B., 'Wyler's Expression for the Fine-Structure Constant α', *Phys. Rev. Lett.*, **27**:1545–1547
Unterberger,A., 'Analyse Harmonique et Analyse Pseudo–differentielle du Cône de la Lumière', *Astérisque*, **156**, 1987
Vigier,J-P., 'On the geometrical quantization of the electric charge in five dimensions'and its numerical determination as a consequence of asymptotic SO(5,2) group invariance, in *Coll. Int. CNRS*, **237**:411–418, 1976
Wigner,E.P., 'Relativistic Invariance and Quantum Phenomena', *Rev. Mod. Phys.*, **29**:255–269, 1957
Wigner,E.P., 'Geometry of Light Paths between Two Material Bodies', *J. Math. Phys.*,**2**:207–211, 1961
Wyler,A., 'On the Conformal Groups in the Theory of Relativity and their Unitary Representations', *Arch. Rat. Mech. and Anal.*,**31**:35–50, 1968
Wyler,A., 'L'espace symétrique du groupe des équations de Maxwell' *C. R. Acad. Sc. Paris*,**269**:743–745
Wyler,A., 'Les groupes des potentiels de Coulomb et de Yukava', *C. R. Acad. Sc. Paris*,**271**:186–188

SELF-DUAL EINSTEIN SUPERMANIFOLDS AND SUPERTWISTOR THEORY

SERGEY A. MERKULOV*
Department of Mathematics and Computer Science, Odense University, Campusvej 55, DK-5230 Odense M, Denmark

Abstract. A supersymmetric generalization of Penrose's non-linear graviton construction is presented.

Key words: Supertwistor – Supermanifold – Supergravity

1. Self-dual Einstein supermanifolds. Let M be a complex $(4|4)$-dimensional supermanifold equipped with a superconformal structure (Manin 1984) which means a pair of integrable rank-$0|2$ distributions T_lM and T_rM satisfying the conditions:

– the sum of T_lM and T_rM in TM is direct;
– the Frobenius form

$$\begin{aligned}\Phi : T_lM \otimes T_rM &\longrightarrow T_0M = TM/(T_lM + T_rM)\\ X \otimes Y &\longrightarrow [X,Y] \bmod (T_lM + T_rM)\end{aligned}$$

is an isomorphism.

The rank-$(2|0)$ holomorphic vector bundles, $S = \Pi T_lM$ and $\tilde{S} = \Pi T_rM$, Π denoting the parity change functor, are called spinor bundles on M. The tangent bundle to the manifold $M_{\rm red}$ underlying a superconformal supermanifold M factors as a tensor product, $TM_{\rm red} = S_{\rm red} \otimes \tilde{S}_{\rm red}$, of two rank-$(2|0)$ vector bundles, and hence $M_{\rm red}$ comes equipped with an induced conformal structure. It is also clear that the second Stiefel-Whitney cohomology class of $M_{\rm red}$ vanishes. In fact *any* such a conformal 4-manifold is a reduction of some (non-unique) conformal $(4|4)$-supermanifold.

PROPOSITION 1. *There is a covariant functor from the category of 4-dimensional conformal manifolds with vanishing second Stiefel-Whitney class to the category of* $(4|4)$*-dimensional conformal supermanifolds.*

Any 4-dimensional conformal manifold has a distinguished family of curves called null geodesics and defined usually as solutions of some second-order differential equation. An analogous concept in supergeometry can be introduced without referring to any connection and associated differential equations.

* On leave from Theoretical Problems Department, Russian Academy of Sciences, Moscow, Russia

DEFINITION 2 (Merkulov 1991*a*). *A null supergeodesic L in a conformal supermanifold M is a $(1|2)$-dimensional subsupermanifold equipped with a pair of $(0|1)$-dimensional distributions $T_lL \subset T_lM|_L$ and $T_rL \subset T_rM|_L$ such that their sum in TL is direct and the Frobenius form*

$$\begin{aligned} \Phi : T_lL \otimes T_rL &\longrightarrow T_0L = TL/(T_lL + T_rL) \\ X \otimes Y &\longrightarrow [X,Y] \bmod (T_lL + T_rL) \end{aligned}$$

is non-degenerate.

Thus a holomorphic null supergeodesic in M has an induced structure of N=2 SUSY curve (Cohn 1987). Since distributions T_lM and T_rM are integrable, they define a pair of $(4|2)$-dimensional supermanifolds, M_l and M_r, whose structure sheaves, $\mathcal{O}_{M_l}$ and $\mathcal{O}_{M_r}$, are those subsheaves of $\mathcal{O}_M$ which are annihilated by vector fields from T_rM and T_lM respectively. The embeddings $\mathcal{O}_{M_{l,r}} \subset \mathcal{O}_M$ define canonical projections

$$M_l \xleftarrow{\pi_l} M \xrightarrow{\pi_r} M_r$$

A supergeometry analogue of the notion of metric is the notion of *scale.*

DEFINITION 3 (Manin 1984). *A scale on a conformal supermanifold M is a choice of particular non-vanishing volume forms on supermanifolds M_l and M_r.*

A choice of scale on M induces both a volume form on M (thus giving a well-defined integration theory of functions) and non-degenerate symplectic forms on spinor bundles S and $\tilde{S}$(thus generating a metric on the underlying 4-manifold $M_{\rm red}$). A scale on M determines also a unique Levi-Civita superconnection (Ogievetsky and Sokatchev 1981) with torsion and curvature tensors being expressed in terms of algebraically independent Weyl superfields $\mathcal{W}_{\alpha\beta\gamma}$ and $\tilde{\mathcal{W}}_{\dot\alpha\dot\beta\dot\gamma}$, Einstein's superfield $\mathcal{G}_{\alpha\dot\alpha}$, and Ricci scalars $\mathcal{R}$ and $\tilde{\mathcal{R}}$ (here and throughout the paper undotted and dotted Greek indices take values 1,2 and refer to some trivializations of spinor bundles S and $\tilde{S}$ respectively). Now we have a series of definitions mimicking the standard terminology of classical Riemannian geometry:

- A conformal supermanifold is called conformally self-dual if $\tilde{W} = 0$.
- A scaled conformal supermanifold is called Einstein's if $\mathcal{G} = 0$.
- A scaled conformal supermanifold is called self-dual Einstein's if $\tilde{\mathcal{W}} = \mathcal{G} = 0$.
- A self-dual Einstein's supermanifold is called self-dual Ricci's if $\tilde{\mathcal{R}} = 0$.

An Einstein supermanifold is a solution of Einstein's N=1 supergravity dynamical equations with cosmological constant. **2. Integrable conical structure on a conformally self-dual supermanifold.** Let F be a relative projective line bundle, $P_M(\tilde{S})$, on a conformal supermanifold M. The exact sequence

$$0 \longrightarrow \Pi S + \Pi\tilde{S} \longrightarrow TM \longrightarrow S \otimes \tilde{S} \longrightarrow 0$$

produces a canonical $(2|3)$-dimensional conical structure (Manin 1984) on M, i. e. an embedding $F \hookrightarrow G_M(2|3; TM)$ which is given in a local structure frame (Ogievetsky and Sokatchev 1981, Manin 1984) as a map

$$\begin{aligned} F &\longrightarrow G_M(2|3;TM) \\ [\pi^{\dot\alpha}] &\longrightarrow \operatorname{span}(\pi^{\dot\alpha}\nabla_{\alpha\dot\alpha}, \nabla_\alpha, \nabla_{\dot\alpha}) \end{aligned}$$

where $\pi^{\dot\alpha}$ are homogeneous coordinates in a fibre of the bundle $\nu : F \to M$. A Levi-Civita superconnection associated with some scale on M induces a $(2|3)$-dimensional conical superconnection on F which actually does not depend on a choice of scale used in the construction. The integrability condition for this conical superconnection is the equation $\tilde{\mathcal{W}} = 0$ (Merkulov 1991*a*,1992*a*). Supposing that M is conformally self-dual and sufficiently "small", we obtain a double fibration

$$Z \xleftarrow{\mu} F \xrightarrow{\nu} M$$

with leaves of the integrable $(2|3)$-conical connection as fibres of μ. The resulting $(3|1)$-dimensional superspace Z is called a *twistor superspace* associated with a conformally self-dual supermanifold M. In the flat case $M = F(2|0,2|1;C^{4|1})$, $F = F(1|0,2|0,2|1;C^{4|1})$ and $Z = G(1|0;C^{4|1})$. **3. Deformations of a standardly embedded rational curve.** The projection $\mu : F \longrightarrow Z$ embeds fibres of ν into Z with one and the same normal bundle N which fits into an exact sequence

$$0 \longrightarrow \Pi\mathcal{O}(1) \longrightarrow N \longrightarrow C^2 \otimes \mathcal{O}(1) \longrightarrow 0.$$

Here $\mathcal{O}(1)$ is the hyperplane section bundle on $CP^{1|0}$. Thus Z comes equipped with a $(4|2)$-dimensional family of standardly embedded rational curves. In fact this family encodes full information about the original $(4|4)$-dimensional conformally self-dual supermanifold M.

THEOREM 4. *Given any embedding of a rational curve P into a $(3|1)$-dimensional complex supermanifold Z with normal bundle N which fits into an exact sequence,*

$$0 \longrightarrow N_1 \longrightarrow N \longrightarrow N_0 \longrightarrow 0$$

with $N_1 \cong \Pi\mathcal{O}(1)$ and $N_2 \cong C^2 \otimes \mathcal{O}(1)$, then there is an associated conformally self-dual $(4|4)$-supermanifold M.

Proof. Let $\mathcal{O}_Z$ be the structure sheaf of the supermanifold Z and $J \subset \mathcal{O}_Z$ an ideal of functions vanishing on the subsupermanifold $P \cong CP^{1|0}$,

$$0 \longrightarrow J \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_P \longrightarrow 0.$$

Then,

$$0 \longrightarrow J/J^2 \longrightarrow \mathcal{O}_Z/J^2 \longrightarrow \mathcal{O}_P \longrightarrow 0,$$

and, by definition of the conormal bundle,

$$0 \longrightarrow N_0^* \longrightarrow J/J^2 \longrightarrow N_1^* \longrightarrow 0.$$

Hence we can define a sheaf $\mathcal{O}_{P'}$ on P by an exact sequence

$$0 \longrightarrow N_0^* \longrightarrow \mathcal{O}_Z/J^2 \longrightarrow \mathcal{O}_{P'} \longrightarrow 0$$

which in turn fits into an exact sequence

$$0 \longrightarrow N_1^* \longrightarrow \mathcal{O}_{P'} \longrightarrow \mathcal{O}_P \longrightarrow 0.$$

Therefore the pair $P' = (P, \mathcal{O}_{P'})$ is a complex supermanifold of dimension $1|1$ which contains P as a subsupermanifold. Since $N_1^* = \Pi\mathcal{O}(-1)$, P' is biholomorphic to the projective superspace $CP^{1|1}$. Thus we conclude that, for any rational curve $CP^{1|0}$ standardly embedded into a complex $(3|1)$-supermanifold Z, there is a projective superspace $CP^{1|1}$ which contains this curve as a subsupermanifold and is embedded into Z with normal bundle $C^2 \otimes \mathcal{O}_{CP^{1|1}}(1)$ (cf. Merkulov 1991*a*). Now consider the projectivized cotangent bundle $P_Z(\Omega^1 Z)$ of Z. Following ideas of LeBrun (1986) we define a subsupermanifold $P'' \subset P_Z(\Omega^1 Z)$ as consisting of those 1-forms on Z which vanish when restricted on TP'. Since the normal bundle of the embedding $P' \hookrightarrow Z$ is isomorphic to $C^2 \otimes \mathcal{O}_{CP^{1|1}}(1)$, $P'' \longrightarrow P'$ is a trivial $CP^{1|0}$-bundle over P' and hence over P. One may check (cf. LeBrun 1986, 1991) that the quadric $Q = P''\mid_P$ is embedded into $P_Z(\Omega^1 Z)$ with normal bundle $N \cong \Pi\mathcal{O}(1,0) + \Pi\mathcal{O}(0,1) + J^1\mathcal{O}(1,1)$. Relative deformations of such quadrics have been investigated by McHugh (1991) who proved that the locally complete $(4|4)$-dimensional parameter family, M, of deformations of Q in $P_Z(\Omega^1 Z)$ comes equipped with a conformal structure,

$$M_l \overset{\pi_l}{\longleftarrow} M \overset{\pi_r}{\longrightarrow} M_r.$$

We recognize M_l as a $(4|2)$-supermanifold parameterizing relative deformations of P, while M_r as a $(4|2)$-supermanifold parameterizing relative deformations of P' in Z. It is also clear that M comes equipped with an integrable $(3|2)$-conical superconnection. Then the theorem follows from the fact (cf. Merkulov 1991*b*) that such a superconnection always admits a lift to a Levi-Civita superconnection. **4. Twistor transform of self-dual Einstein and Ricci supermanifolds.** Let $\mathcal{O}_F(-1)$ be the tautological sheaf on F. Consider the composition

$$\nabla_{F/Z} : \mathcal{O}_F \overset{\nu^*(\nabla)}{\longrightarrow} \mathcal{O}_F(-1) \otimes \nu^* \left(\Pi(\mathrm{Ber} M_l)^{-1/6} \otimes \Omega^1 M\right) \overset{\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{res}}{\longrightarrow}$$

$$\overset{\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{res}}{\longrightarrow} \mathcal{O}_F(-1) \otimes \nu^*\Pi(\mathrm{Ber} M_l)^{-1/6} \otimes \Omega^1 F/Z,$$

where ∇ is a Levi-Civita superconnection and res denotes restriction of 1-forms on F on μ-vertical vector fields, and define an invertible holomorphic sheaf, $L = \mu_*(\ker\nabla_{F/Z})$, on Z.

THEOREM 5. *There is a one-to-one correspondence between scales on a conformally self-dual supermanifold M satisfying Einstein's equations $\mathcal{G} = 0$ and nowhere vanishing sections of $\Omega^1 Z \otimes (L^*)^2$ on Z.*

This theorem (proved in Merkulov (1991*a*)) is a supersymmetry extension of the result due to Ward (1980). The family of rational curves in Z which lie in the kernel of a global section of $\Omega^1 Z \otimes (L^*)^2$ have a special meaning (cf. LeBrun 1982) — they generate a $(3|2)$-dimensional conformal supermanifold, Y, as defined in Merkulov (1992*b*). There exists an inverse construction (cf. LeBrun 1982).

THEOREM 6. *Let Y be a $(3|2)$-dimensional conformal supermanifold. There is an associated $(4|4)$-dimensional self-dual Einstein supermanifold M.*

Proof. If Y is $(3|2)$-dimensional conformal supermanifold, there is an associated $(3|1)$-dimensional "ambitwistor" superspace Z parameterizing null supergeodesics of Y (Merkulov 1992*b*). The supermanifold Z has a family of rational curves embedded with normal bundle $N \cong \Pi\mathcal{O}(1) + C^2 \otimes \mathcal{O}(1)$. Hence Theorem 4 can be used to generate a conformally self-dual $(4|4)$-supermanifold M from this structure. Moreover Z comes equipped with a contact structure which gives a nowhere vanishing global section of $\Omega^1 Z \otimes (L^*)^2$. Hence, by Theorem 5, the supermanifold M satisfies Einstein's equations.

REMARK 7. Since a general $(3|2)$-dimensional conformal supermanifold Y is specified locally by metric and gravitino fields on Y_{red} defined up to general coordinate and SUSY-transformations, we infer from Theorem 6 that *a solution of self-dual Einstein equations for $N = 1, D = 4$ supergravity can be constructed for each choice of 3 even and 2 odd analytic functions of 3 variables.*

Let τ be a section of $\Omega^1 Z \otimes (L^*)^2$ associated with a self-dual Einstein supermanifold M and $\nu^*(\nabla)$ a lift of an associated Levi-Civita connection to F.

LEMMA 8. *A (4|4)-dimensional distribution on F which is annihilated by the lift $\mu^*(\tau)$ of the section τ coincides precisely with the horizontal distribution $\nu^*(\nabla)$.*

This lemma implies

THEOREM 9. *A self-dual Einstein supermanifold is a self-dual Ricci supermanifold if and only if the (2|1)-dimensional distribution on Z annihilated by τ is integrable.*

The latter statement provides a supersymmetry generalization of Penrose's (1976) non-linear graviton construction.

References

Cohn, J. D. : 1987, 'N=2 super Riemann surfaces', *Nucl. Phys.* **B284**, 349-364.
LeBrun, C. R. : 1982 'H-space with a cosmological constant', *Proc. R. Soc. Lond. Ser.* **A380**, 171-185.
LeBrun, C. R. : 1986, 'Thickenings and gauge fields', *Class. Quantum Grav.* **3**, 1039-1059.
LeBrun, C. R. : 1991, 'Thickenings and conformal gravity', *Commun. Math. Phys.* **139**, 1-43.
Manin, Yu. I. : 1984 *Gauge Field theory and Complex Geometry.* Nauka: Moscow [English transl.: Berlin: Springer 1988].
McHugh, A. P. : 1991 'The space for super light rays for complex conformal spacetimes', Ph. D. Thesis, State University of New York at Stony Brook.
Merkulov, S. A. : 1991*a* 'Geometry of simple supergravity and supertwistor theory', In *Classical Field Theory and Gravitation Theory,* VINITI: Moscow, **3**, pp.59-107 [In Russian].
Merkulov, S. A. : 1991*b* 'Twistor transform of Einstein-Weyl superspaces', *Class. Quantum Grav.* **8**, 2149-2162.
Merkulov, S. A. : 1992*a* 'Supersymmetric non-linear graviton' , *Funct. Anal. and Its Appl.* **26**, *no.* 1 .
Merkulov, S. A. : 1992*b* 'Superconformal geometry in three dimensions', *J. Math. Phys.* **33**, 735-757.
Penrose, R. : 1976 'Non-linear gravitons and curved twistor theory', Gen. Rel. Grav. **7**, 31-52.
Ward, R. S. : 1980 Self-dual space-times with cosmological constant, *Commun. Math. Phys.* **78**, 1-17.

AN APPROACH TO THE CONSTRUCTION OF COHERENT STATES FOR MASSLESS PARTICLES

ANDRZEJ KARPIO
Institute of Physics, Warsaw University Branch, 15-441 Białystok, ul. Lipowa 41, Poland

1. INTRODUCTION

My intention is to present some results related to the construction of coherent states [Perelomov, 1987] in a Hilbert space whose elements are cohomology classes. Such spaces appear in twistor theory and they have very important physical aplications as the quantum spaces of massless particles. This work is a small part of my doctoral thesis which will be published soon. It contains details and proofs of all facts I mention below.

I will restrict my considerations to the manifold of positive, projective twistors PT^+ [Wells, 1979]. From a physical point of view, it is the phase space of massless particles with helicity greater then zero [Hughston, 1979] [Tod, 1977] [Karpio, 1986]. The quantization procedure leads to the first cohomology group $H^1(PT^+, \mathcal{O}(-n-2))$ [Penrose, 1977].

In order to introduce the structure of the Hilbert space one can use the scalar product given in [Eastwood, 1981] [Ginsberg, 1983]. It was formulated for Czech's cohomologies and I mean just this realisation when speaking about cohomology groups. It turns out that this Hilbert space is the first cohomology group of covering PT^+ by two open subsets. The orthonormal basis is formed by "elementary states" [MacCallum, 1972] which are cocycles with representatives chosen in the following way:

$$\mathrm{B}^{(n)}_{l\,k_1 k_2}(Z) = (-1)^{l+1} \imath^{n+1} \sqrt{\frac{k_1!(n+l-k_1)!}{k_2!(l-k_2)!}} \times$$

$$\times \frac{\left(\frac{< A^*, Z >}{\sqrt{-< A^*, A^* >}}\right)^{k_2} \left(\frac{< B^*, Z >}{\sqrt{-< B^*, B^* >}}\right)^{l-k_2}}{\left(\frac{< A, Z >}{\sqrt{< A, A >}}\right)^{k_1+1} \left(\frac{< B, Z >}{\sqrt{< B, B >}}\right)^{n+l-k_1+1}} \qquad (1)$$

where

$k_1 = 0, ..., n+l;\ k_2 = 0, ..., l;\ l = 0, ..., \infty;$

$< , >$ is a twistor form ;

A, B, A^*, B^* form an orthogonal basis for C^4 ; A, B are positive twistors with respect to the twistor form but A^*, B^* are negative ones.

These sections are defined on the intersection of two open subsets in PT^+ : $U_A \cap U_B$ where

$$U_A = \{Z \in PT^+ : < A, Z > \neq 0\},\ U_B = \{Z \in PT^+ : < B, Z > \neq 0\}$$

The above Hilbert space I will denote by $H^1_{AB}(PT^+, \mathcal{O}(-n-2))$. It depends on the choice of twistors A, B but for another choice we obtain a space which is isomorphic with the previous one. More details relating to the construction of $H^1_{AB}(PT^+, \mathcal{O}(-n-2))$ can be found, for instance, in [Penrose, 1979] and in my work which will appear soon.

2. COHERENT STATES IN $H^1_{AB}(PT^+, \mathcal{O}(-n-2))$

The basic concept of the further construction is the Reproduction Kernel for Hilbert space in the sense of Bergman. In order to find it for the space $H^1_{AB}(PT^+, \mathcal{O}(-n-2))$ we can use our orthonormal basis. By definition we have to calculate the sum of the series:

$$\sum_{l=0}^{\infty} \sum_{k_1=0}^{n+l} \sum_{k_2=0}^{l} \mathrm{B}^{*(n)}_{l\,k_1k_2} \otimes \mathrm{B}^{(n)}_{l\,k_1k_2} \tag{2}$$

where $*$ is conjugation defined by the natural duality $PT^+ \simeq PT^{*+}$ and the sections $\mathrm{B}^{(n)}_{l\,k_1k_2}$ are identified with their pullbacks in the double fibering $PT^{*+} \times PT^+$ over PT^{*+} and PT^+. The result of the calculations belongs to the cohomology group $H^2(PT^{*+} \times PT^+, \mathcal{O}(-n-2, -n-2))$ and its representative is given by the formula:

$$\Phi^{(++)}_n(\bar{W}, Z) = \frac{d^{n+1}}{d\lambda^{n+1}} \frac{1}{\lambda} \ln \frac{(x-\lambda)(y-\lambda)}{xy} \tag{3}$$

where

$\lambda = < W, Z >$; $x = \frac{<W,A><A,Z>}{<A,A>}$; $y = \frac{<W,B><B,Z>}{<B,B>}$;

$\bar{W}$ means conjugation with respect to the twistor form.

I will call it the Reproduction Kernel for the Hilbert space $H^1_{AB}(PT^+, \mathcal{O}(-n-2))$ which describes quantum states of massless particles with positive helicity. This terminology is justified by the reproduction property understood as below:

$$(\Phi^{(++)}_n(\bar{Z}, \cdot), \mathrm{f}) = \mathrm{f}(Z) \tag{4}$$

where f represents the cocycle from $H^1_{AB}(PT^+, \mathcal{O}(-n-2))$, the variable Z is fixed and (,) denotes the scalar product for the cohomology classes mentioned in the introduction.

The details of these considerations as well as calculations will be published later. Having the Reproduction Kernel we define coherent states as the elements of the Hilbert space obtained by the evaluation of the kernel in the one of its variables [Perelomov, 1987]. This procedure is obvious if one considers sections of some bundle but in this place we are dealing with cocycles, and therefore with more complicated objects. My proposition is to perform the Penrose Transform to $\Phi^{(++)}_n$ in order to obtain the elements of $H^1_{AB}(PT^+, \mathcal{O}(-n-2))$ which come from our Reproduction Kernel. It is the most natural operation we can do in this case. The use of the Penrose Transform requires the choosing of an element from M^{++} (the manifold of 2-dimensional linear subspaces in C^4, positive with respect to twistor form) which is determined by the pair of positive twistors C, D and some section of the universal bundle over PT^+ given by the constant spinor field $\eta^{A'}$. Calculations are not difficult but need some patience; their result can be presented in the following statement:

STATEMENT 1. *Coherent states in the Hilbert space $H^1_{AB}(PT^+, \mathcal{O}(-n-2))$ of massless particles with non-vanishing helicity $s = \frac{n}{2}$ obtained by evaluating the Reproduction Kernel (3) are cocycles with the following representatives:*

$$\Phi^{(++)}_n([\bar{C}, \bar{D}], \eta)(Z) = 2^{n+1} \imath^n n! \sum_{p=0}^{n} \frac{(\bar{\eta}^0)^{n-p}(\bar{\eta}^1)^p}{< C, Z >^{p+1} < D, Z >^{n-p+1}} \tag{5}$$

for $n > 0$

$$\Phi^{(++)}_0([\bar{C}, \bar{D}])(Z) = \frac{2}{< C, Z > < D, Z >} \tag{6}$$

for $n = 0$
were $[\bar{C}, \bar{D}]$ is an element from M^{++} spanned by the twistors $\bar{C}, \bar{D}$. They realize the embedding of the flag manifold $F^{+,++}$ into $H^1_{AB}(PT^+, \mathcal{O}(-n-2))$.

This statement needs some remarks. First of all I have considered the case n=0 which does not complicate our considerations but it is very natural not only in this place but also in the physical applications of the above results. By $F^{+,++}$ I mean flag manifold which elements are the pairs: 1-dimensional positive subspace in C^4 contained in 2-dimensional positive subspace in C^4. Where "positive " means positive with respect to the twistor form. All results presented in this chapter can be extended to the case of the negative twistors and negative helicities.

3. CONCLUSIONS

The states appearing in the statement were known earlier and were the subject of the considerations of many works, see for example [Hughston, 1979] [Eastwood, 1979]. They are the linear combinations of the simplest elements belonging to the Hilbert space $H^1_{AB}(PT^+, \mathcal{O}(-n-2))$. Moreover, they realize the embedding of $F^{+,++}$ into $H^1_{AB}(PT^+, \mathcal{O}(-n-2))$ which is very important from physical point of view and results in the quantization of classical objects which are the special congruences of null geodesics, the so called Robinson congruences [Ward, 1979] . On the other hand, the Penrose Transform of the coherent states gives us the elements obtained from the reproduction kernel for the Hilbert space of holomorphic spinor fields on the future tube M^{++} [Jacobsen,] . I think that the facts I have just mentioned justify using the name "coherent" for the states with so great importance for physics.

More details, physical interpretation and much more will be published soon.

References

M. G. Eastwood, M. L. Ginsberg - Duality in twistor theory - Duke Math. J. vol. 48, No 1, 1981.

M. G. Eastwood, L.P. Hughston - Massless field based on a line - in Advances in twistor theory, ed. L. P. Hughston, R.S. Ward, 1979.

M. L. Ginsberg - Scattering theory and the geometry of multi-twistor spaces - Trans. Am. Math. Soc. vol. 276, No 2, 1983.

L. P. Hughston - The twistor cohomology of local Hertz potential - in Advances in twistor theory, ed. L. P. Hughston, R.S. Ward, 1979.

H. P. Jacobsen, M. Verne - Wave and Dirac operators and representation of the conformal group - preprint

A. Karpio, A. Kryszen, A. Odzijewicz - 2-twistor conformal, hamiltonian spaces - Rep. Math. Phys. vol. 24, 1986.

M. A. H. MacCallum, R. Penrose - Twistor theory: an approach to the quantisation of fields and space-time - Phys. Rep. sec C, vol. 6, No 4, 1972.

R. Penrose - The twistor programme - Rep. Math. Phys. vol. 12, 1977.

R. Penrose - Twistor function and sheaf cohomology - in Advances in twistor theory, ed. L. P. Hughston, R.S. Ward, 1979.

A. M. Perelomov - Obobshchenyje kogierentnyje sostojania i ih primenenia. Moskva "Nauka" 1987.

P. Tod - Rep. Math. Phys. vol. 11, 1977.

R.S. Ward - Massless fields and sheaf cohomology - in Advances in twistor theory, ed. L. P. Hughston, R.S. Ward, 1979.

R. O. Wells - Complex manifolds and mathematical physics. - Bull. Am. Math. Soc. vol. 1, No 2, 1979.

CLIFFORD ALGEBRAS

WHAT IS A BIVECTOR?

PERTTI LOUNESTO
Institute of Mathematics
Helsinki University of Technology
SF-02150 ESPOO, Finland

May 25, 1993

Abstract. Bivectors do not exist in Clifford algebras over arbitrary fields, especially they do not exist in a canonical way in char 2. However, there is a natural way to introduce bivectors in all other char $\neq 2$, whilst the polarization formula gives a one to one correspondence between quadratic forms and *symmetric* bilinear forms. This paper reviews Chevalley's construction for a quadratic form Q, and arbitrary, *not necessarily symmetric*, bilinear forms such that $B(\mathbf{x},\mathbf{x}) = Q(\mathbf{x})$. The exterior product is obtained from the Clifford product by Riesz's formula $\mathbf{x} \wedge u = \frac{1}{2}(\mathbf{x}u + (-1)^k u\mathbf{x})$, where $\mathbf{x} \in V$ and $u \in \bigwedge^k V$.

Key words: Exterior algebra – contraction – bivectors – Clifford algebra

1. Chevalley's Identification of $C\ell(Q) \subset \mathrm{End}(\bigwedge V)$

Chevalley 1954, pp. 38-42, introduced a linear operator $\gamma_{\mathbf{x}} \in \mathrm{End}(\bigwedge V)$ such that (cf. Oziewicz 1986, page 252 line 3 of formula (23))

$$\gamma_{\mathbf{x}}(u) = \mathbf{x} \wedge u + \mathbf{x} \rfloor u \quad \text{for} \quad \mathbf{x} \in V,\ u \in \bigwedge V.$$

From the derivation rule $\mathbf{x} \rfloor (u \wedge v) = (\mathbf{x} \rfloor u) \wedge v + \hat{u} \wedge (\mathbf{x} \rfloor v)$ and $\mathbf{x} \wedge \mathbf{x} \wedge u = 0$, $\mathbf{x} \rfloor (\mathbf{x} \rfloor u) = 0$ one can conclude the identity $(\gamma_{\mathbf{x}})^2 = Q(\mathbf{x})$. Chevalley's inclusion map $V \to \mathrm{End}(\bigwedge V)$, $\mathbf{x} \to \gamma_{\mathbf{x}}$ was a Clifford map and could be extended to an algebra homomorphism $\psi : C\ell(Q) \to \mathrm{End}(\bigwedge V)$, whose image evaluated at $1 \in \bigwedge V$ yielded the map $\phi : \mathrm{End}(\bigwedge V) \to \bigwedge V$. The composite linear map $\theta = \phi \circ \psi$ was the right inverse of the natural map $\bigwedge V \to C\ell(Q)$ and

$$\bigwedge V \to C\ell(Q) \xrightarrow{\psi} \mathrm{End}(\bigwedge V) \xrightarrow{\phi} \bigwedge V$$

was the identity mapping on $\bigwedge V$. The faithful representation ψ sent $C\ell(Q)$ onto an isomorphic subalgebra of $\mathrm{End}(\bigwedge V)$.

Chevalley's identification works fine with a contraction defined by an arbitrary, **not necessarily symmetric**, bilinear form B such that $B(\mathbf{x},\mathbf{x}) = Q(\mathbf{x})$. The following properties uniquely determine the contraction also for an arbitrary, **not necessarily non-degenerate**, Q:

$$\mathbf{x} \rfloor \mathbf{y} = B(\mathbf{x},\mathbf{y}) \quad \text{for} \quad \mathbf{x},\mathbf{y} \in V \qquad \text{(a)}$$

$$\mathbf{x} \rfloor (u \wedge v) = (\mathbf{x} \rfloor u) \wedge v + \hat{u} \wedge (\mathbf{x} \rfloor v) \qquad \text{(b)}$$

$$(u \wedge v) \rfloor w = u \rfloor (v \rfloor w) \quad \text{for} \quad u,v,w \in \bigwedge V \qquad \text{(c)}$$

(see Helmstetter 1982). The identity (a) fixes the dependence of the contraction on the symmetric bilinear form on V. The identity (b) means that $\mathbf{x} \in V$ operates like a **derivation** (cf. Greub 1978 p. 118 and Crumeyrolle 1990 p. 35). The identity (c) introduces a scalar multiplication on $\bigwedge V$ making it a left module over $\bigwedge V$. The identity (b) allows elements of higher degree on the right hand side (cf. Oziewicz 1986, p. 249 (13)) and the identity (c) allows elements of higher degree on the left hand side (cf. Oziewicz 1986, p. 248 (12)). Evidently, $\mathbf{x} \rfloor \mathbf{a} \in \bigwedge^{k-1} V$ for $\mathbf{a} \in \bigwedge^k V$ and

$$\mathbf{x} \rfloor (\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \ldots \wedge \mathbf{x}_k) = \sum_{i=1}^{k} (-1)^{i-1} B(\mathbf{x},\mathbf{x}_i)\, \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \ldots \wedge \mathbf{x}_{i-1} \wedge \mathbf{x}_{i+1} \wedge \ldots \wedge \mathbf{x}_k.$$

The faithful representation ψ sends the Clifford algebra $C\ell(Q)$ onto an isomorphic subalgebra of $\mathrm{End}(\bigwedge V)$ which as a subspace depends on B.

Remark. Chevalley introduced his identification $C\ell(Q) \subset \mathrm{End}(\bigwedge V)$ in order to be able to include the exceptional case of characteristic 2. In characteristic $\neq 2$ the theory of quadratic forms is the same as the theory of symmetric bilinear forms and Chevalley's identification gives the Clifford algebra of the **symmetric** bilinear form $<\mathbf{x},\mathbf{y}> = \frac{1}{2}(B(\mathbf{x},\mathbf{y}) + B(\mathbf{y},\mathbf{x}))$ satisfying $\mathbf{xy} + \mathbf{yx} = 2<\mathbf{x},\mathbf{y}>$. ∎

Remark. We could also define the right contraction $v \lfloor u$ of $v \in \bigwedge V$ by $u \in \bigwedge V$. The right and the left contractions are related by the formulas $v \lfloor u = u_0 \rfloor v_0 + u_0 \rfloor v_1 - v_1 \rfloor u_0 + v_1 \rfloor u_1$ and $u \rfloor v = v_0 \lfloor u_0 - v_0 \lfloor u_1 + v_1 \lfloor u_0 + v_1 \lfloor u_1$. The notation $\mathbf{a} \cdot \mathbf{b}$ may be used for the contraction when it is clear from the context which factor is contracted and which is the contractor. This **dot product** $\mathbf{a} \cdot \mathbf{b}$ can be used when at least one of the factors is homogeneous. If both factors are homogeneous, then we agree that the one with lower (or not higher) degree is the contractor ($\mathbf{a} \in \bigwedge^i V$, $\mathbf{b} \in \bigwedge^j V$)

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \rfloor \mathbf{b} \quad \text{for} \quad i \leq j \quad \text{and} \quad \mathbf{a} \cdot \mathbf{b} = \mathbf{a} \lfloor \mathbf{b} \quad \text{for} \quad i \geq j.$$

When precisely one factor is known to be homogeneous we agree that it is the contractor ($\mathbf{a} \in \bigwedge^i V$, $u \in \bigwedge V$)

$$\mathbf{a} \cdot u = \mathbf{a} \rfloor u \quad \text{and} \quad u \cdot \mathbf{a} = u \lfloor \mathbf{a}.$$

Note that the contraction obeys the rules $1 \rfloor u = u$, $u \in \bigwedge V$, and $\mathbf{x} \rfloor 1 = 0$, $\mathbf{x} \in V$, but for the dot product $1 \cdot u = u \cdot 1 = u$. Note also that the dot product **does not** act like a scalar multiplication on the left $\bigwedge V$-module $\bigwedge V$, that is, $(\mathbf{a} \wedge \mathbf{b}) \cdot u \neq \mathbf{a} \cdot (\mathbf{b} \cdot u)$. As an exercise the reader may verify that for $\mathbf{a} = \mathbf{e}_1$, $\mathbf{b} = \mathbf{e}_1 + \mathbf{e}_2$ and $u = \mathbf{e}_1 + \mathbf{e}_1 \wedge \mathbf{e}_2$ in $\bigwedge \mathbb{R}^2$ all the expressions $\mathbf{a} \rfloor (\mathbf{b} \rfloor u)$, $\mathbf{a} \rfloor (\mathbf{b} \lfloor u)$, $\mathbf{a} \lfloor (\mathbf{b} \rfloor u)$, $\mathbf{a} \lfloor (\mathbf{b} \lfloor u)$ and $(\mathbf{a} \wedge \mathbf{b}) \rfloor u$, $(\mathbf{a} \wedge \mathbf{b}) \lfloor u$ are unequal with the exception of $(\mathbf{a} \wedge \mathbf{b}) \rfloor u = \mathbf{a} \rfloor (\mathbf{b} \rfloor u) =$?

— The lack of $\bigwedge V$-linearity renders less useful any extension of $u \cdot v$ for arbitrary $u, v \in \bigwedge V$. Such an extension was introduced under the name of 'inner product' by Hestenes&Sobczyk 1984 p. 6 who display only formulas with at least one homogeneous factor. The non-$\bigwedge V$-linear 'inner product' is not consistent with the contraction (in the sense that the 'inner product' is not a special case of the contraction), because $u \cdot v$ might differ simultaneously both from $u \rfloor v$ and $u \lfloor v$. Boudet 1992 p. 345 mentioned a formalization of the 'inner product' but his rules are not sufficient to permit the evaluation of $(\mathbf{x} \wedge \mathbf{y}) \cdot u$ when $u \in \bigwedge V$, $u \notin V$ (though they do permit a construction of the 'inner product' with an additional rule $(\mathbf{x} \wedge \mathbf{y}) \cdot u = \mathbf{x} \cdot (\mathbf{y} \cdot u)$ where $u \in \bigwedge^k V$, $k \geq 2$). ∎

The above remark shows how the asymmetric contraction solves a problem of Hestenes&Sobczyk 1984, who postulate the 'inner product' to be 0 [p. 6, r. 12 formula (1.21b)] if one of the factors is a scalar, and run into difficulties on p. 20 rows 8-18 formula (2.9). However, as the following example shows the problem is deeper than that since the 'inner product' is not equal to the contraction even if scalars were excluded.

Example. Let $\mathbf{e}_1, \mathbf{e}_2$ be an orthonormal basis for $\mathbb{R}^2 = \mathbb{R}^{2,0}$. Compute

$$(\mathbf{e}_1 - \mathbf{e}_1 \wedge \mathbf{e}_2) \cdot (\mathbf{e}_2 + \mathbf{e}_1 \wedge \mathbf{e}_2) = 1 - \mathbf{e}_1 + \mathbf{e}_2$$

in the sense of Hestenes&Sobczyk. The same elements have the contractions

$$(\mathbf{e}_1 - \mathbf{e}_1 \wedge \mathbf{e}_2) \rfloor (\mathbf{e}_2 + \mathbf{e}_1 \wedge \mathbf{e}_2) = 1 + \mathbf{e}_2$$

$$(\mathbf{e}_1 - \mathbf{e}_1 \wedge \mathbf{e}_2) \lfloor (\mathbf{e}_2 + \mathbf{e}_1 \wedge \mathbf{e}_2) = 1 - \mathbf{e}_1.$$

This shows that neither the left contraction nor the right contraction coincides with the 'inner product' of Hestenes&Sobczyk. ∎

In other words, the 'inner product' is not dual/adjoint to the exterior product. To summarize: the **inner product** of Hestenes&Sobczyk is not the same as the **contraction** or the **interior product** of Cartan.

In char $\neq 2$ we may re-obtain the dot product in terms of the Clifford product as follows $\mathbf{a} \cdot \mathbf{b} = \langle \mathbf{ab} \rangle_{|i-j|}$ for $\mathbf{a} \in \bigwedge^i V$ and $\mathbf{b} \in \bigwedge^j V$, where $\langle u \rangle_k$ is the k-vector part of $u \in \bigwedge V \simeq C\ell(Q)$.

For arbitrary Q but char $K \neq 2$ there is the natural choice of the unique symmetric bilinear form B such that $B(\mathbf{x},\mathbf{x}) = Q(\mathbf{x})$ giving rise to the canonical/privileged linear isomorphism $C\ell(Q) \to \bigwedge V$. The case char $K = 2$ is quite different. In general, there are no symmetric bilinear forms such that $B(\mathbf{x},\mathbf{x}) = Q(\mathbf{x})$ and in case that there is such a symmetric bilinear form, it is not unique since any alternating bilinear form is also symmetric and could be added to the symmetric bilinear form without changing Q. [Recall that antisymmetric means $B(\mathbf{x},\mathbf{y}) = -B(\mathbf{y},\mathbf{x})$ and alternating $B(\mathbf{x},\mathbf{x}) = 0$; alternating is always antisymmetric, though in characteristic 2 antisymmetric is not necessarily alternating.] Thereby the contraction is not unique, and there is an ambiguity in γ_u.

In characteristic 2 the theory of quadratic forms is not the same as the theory of symmetric bilinear forms.

In the next example we need the matrix of $v \to uv$, $u = u_0 + u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_{12}\mathbf{e}_1\wedge\mathbf{e}_2$ with respect to the basis $1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1\wedge\mathbf{e}_2$ for $\bigwedge V$, where $\dim_K V = 2$, $B(\mathbf{x},\mathbf{y}) = ax_1y_1 + bx_1y_2 + cx_2y_1 + dx_2y_2$ and $Q(\mathbf{x}) = B(\mathbf{x},\mathbf{x})$:

$$u \simeq \gamma_u = \begin{pmatrix} u_0 & au_1 + cu_2 & bu_1 + du_2 & -(ad-bc)u_{12} \\ u_1 & u_0 + cu_{12} & du_{12} & -(bu_1 + du_2) \\ u_2 & -au_{12} & u_0 - bu_{12} & au_1 + cu_2 \\ u_{12} & -u_2 & u_1 & u_0 + (-b+c)u_{12} \end{pmatrix}.$$

The commutation relations are $\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1 = b + c$ and $\mathbf{e}_1^2 = a$, $\mathbf{e}_2^2 = d$, and we have the following multiplication table

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_1\wedge\mathbf{e}_2$
$\mathbf{e}_1$	a	$\mathbf{e}_1\wedge\mathbf{e}_2 + b$	$-b\mathbf{e}_1 + a\mathbf{e}_2$
$\mathbf{e}_2$	$-\mathbf{e}_1\wedge\mathbf{e}_2 + c$	d	$-d\mathbf{e}_1 + c\mathbf{e}_2$
$\mathbf{e}_1\wedge\mathbf{e}_2$	$c\mathbf{e}_1 - a\mathbf{e}_2$	$d\mathbf{e}_1 - b\mathbf{e}_2$	$-ad + bc + (-b+c)\mathbf{e}_1\wedge\mathbf{e}_2$

In characteristic $\neq 2$ we find $\frac{1}{2}(\mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_2\mathbf{e}_1) = \mathbf{e}_1\wedge\mathbf{e}_2 + \frac{1}{2}(b-c)$ and more generally for $\frac{1}{2}(\mathbf{xy}-\mathbf{yx}) = \mathbf{x}\wedge\mathbf{y} + A(\mathbf{x},\mathbf{y})$ with an alternating scalar valued form $A(\mathbf{x},\mathbf{y}) = \frac{1}{2}(B(\mathbf{x},\mathbf{y}) - B(\mathbf{y},\mathbf{x}))$ (cf. the last equation in Oziewicz 1986 p. 252). The symmetric bilinear form associated with $Q(\mathbf{x})$ is

$$\mathbf{x}\cdot\mathbf{y} = \frac{1}{2}(B(\mathbf{x},\mathbf{y}) + B(\mathbf{y},\mathbf{x})) = ax_1y_1 + \frac{1}{2}(b+c)(x_1y_2 + x_2y_1) + dx_2y_2$$

and we have $\mathbf{xy} + \mathbf{yx} = 2\mathbf{x}\cdot\mathbf{y}$ for $\mathbf{x},\mathbf{y} \in V \subset C\ell(Q)$.

It is convenient to regard $\bigwedge V$ as the subalgebra of $\mathrm{End}(\bigwedge V)$ with the canonical choice of the symmetric $B = 0$. We may also regard $C\ell(Q)$ as a subalgebra of $\mathrm{End}(\bigwedge V)$ obtained with some B such that $B(\mathbf{x},\mathbf{x}) = Q(\mathbf{x})$ and choose the symmetric B in char $\neq 2$.

Example. Let $K = \{0,1\}$, $\dim_K V = 2$ and $Q(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = x_1x_2$. There are only two bilinear forms B_i such that $B_i(\mathbf{x},\mathbf{x}) = Q(\mathbf{x})$, namely $B_1(\mathbf{x},\mathbf{y}) = x_1y_2$ and $B_2(\mathbf{x},\mathbf{y}) = x_2y_1$, and neither is symmetric. The difference $A = B_1 - B_2$, $A(\mathbf{x},\mathbf{y}) = x_1y_2 - x_2y_1$ $(= x_1y_2 + x_2y_1)$ is alternating (and thereby symmetric). Therefore, there are only two representations of $C\ell(Q)$ in $\mathrm{End}(\bigwedge V)$

$$\text{for } B_1: \qquad u \simeq \begin{pmatrix} u_0 & 0 & u_1 & 0 \\ u_1 & u_0 & 0 & -u_1 \\ u_2 & 0 & u_0 - u_{12} & 0 \\ u_{12} & -u_2 & u_1 & u_0 - u_{12} \end{pmatrix}$$

$$\text{for } B_2: \qquad u \simeq \begin{pmatrix} u_0 & u_2 & 0 & 0 \\ u_1 & u_0 + u_{12} & 0 & 0 \\ u_2 & 0 & u_0 & u_2 \\ u_{12} & -u_2 & u_1 & u_0 + u_{12} \end{pmatrix}$$

These representations have the following multiplication tables

B_1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_1\wedge\mathbf{e}_2$
$\mathbf{e}_1$	0	$1 + \mathbf{e}_1\wedge\mathbf{e}_2$	$-\mathbf{e}_1$
$\mathbf{e}_2$	$-\mathbf{e}_1\wedge\mathbf{e}_2$	0	0
$\mathbf{e}_1\wedge\mathbf{e}_2$	0	$-\mathbf{e}_2$	$-\mathbf{e}_1\wedge\mathbf{e}_2$

B_2	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_1\wedge\mathbf{e}_2$
$\mathbf{e}_1$	0	$\mathbf{e}_1\wedge\mathbf{e}_2$	0
$\mathbf{e}_2$	$1 - \mathbf{e}_1\wedge\mathbf{e}_2$	0	$\mathbf{e}_2$
$\mathbf{e}_1\wedge\mathbf{e}_2$	$\mathbf{e}_1$	0	$\mathbf{e}_1\wedge\mathbf{e}_2$

with respect to the basis $1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1\wedge\mathbf{e}_2$ for $\bigwedge V$. In this case there are only two linear isomorphisms $\bigwedge V \to C\ell(Q)$ which are identity mappings when restricted to $K + V$ and which preserve parity. It is easy to verify that the above tables describe actually the only representations of $C\ell(Q)$ in $\bigwedge V$. In this case **there is no canonical linear isomorphism** $\bigwedge V \to C\ell(Q)$, i.e., neither of the above multiplication tables can be preferred over the other. In particular, $\bigwedge^2 V$ cannot be canonically embedded in $C\ell(Q)$, and **there are no bivectors in characteristic 2.** ■

In the next section we try to answer the question: Are there bivectors in characteristics other than 2?

2. Riesz's Introduction of an Exterior Product in $C\ell(Q)$

We start from an object, which we suppose well-known here, the Clifford algebra $C\ell(Q)$ over K, char $K \neq 2$, and introduce another product in $C\ell(Q)$. The isometry $\mathbf{x} \to -\mathbf{x}$ of V when extended to an automorphism of $C\ell(Q)$ is called the **grade involution** $u \to \hat{u}$. Define the **exterior product** of $\mathbf{x} \in V$ and $u \in C\ell(Q)$ by (see Riesz 1958 p. 61-67)

$$\mathbf{x}\wedge u = \frac{1}{2}(\mathbf{x}u + \hat{u}\mathbf{x}), \qquad u\wedge\mathbf{x} = \frac{1}{2}(u\mathbf{x} + \mathbf{x}\hat{u})$$

and extend it by linearity to all of $C\ell(Q)$ which then becomes isomorphic as an associative algebra to $\bigwedge V$. The exterior products of two vectors $\mathbf{x}\wedge\mathbf{y} = \frac{1}{2}(\mathbf{xy} - \mathbf{yx})$ are simple **bivectors** and they span $\bigwedge^2 V$. The exterior product of a vector and a bivector $\mathbf{x}\wedge\mathbf{B} = \frac{1}{2}(\mathbf{xB}+\mathbf{Bx})$ is a 3-vector in $\bigwedge^3 V$. The subspace of k-vectors is constructed recursively by

$$\mathbf{x}\wedge\mathbf{a} = \frac{1}{2}(\mathbf{xa}+(-1)^{k-1}\mathbf{ax}) \in \bigwedge^{k} V \quad \text{for } \mathbf{a}\in \bigwedge^{k-1} V.$$

Riesz's construction shows that **bivectors do exist in all characteristics $\neq 2$.**

Introduce the **contraction** of $u \in C\ell(Q)$ by $\mathbf{x}\in V$ so that (see Riesz 1958 p. 61-67)

$$\mathbf{x} \rfloor u = \frac{1}{2}(\mathbf{x}u - \hat{u}\mathbf{x})$$

and show that this contraction is a derivation of $C\ell(Q)$ while

$$\mathbf{x}\rfloor(uv) = \frac{1}{2}(\mathbf{x}uv - \widehat{uv}\mathbf{x}) = \frac{1}{2}(\mathbf{x}uv - \hat{u}\hat{v}\mathbf{x})$$
$$= \frac{1}{2}(\mathbf{x}uv - \hat{u}\mathbf{x}v + \hat{u}\mathbf{x}v - \hat{u}\hat{v}\mathbf{x}) = (\mathbf{x}\rfloor u)v + \hat{u}(\mathbf{x}\rfloor v).$$

Thus one and the same contraction is indeed a derivation for both the exterior product and the Clifford product. Kähler 1962 (p. 435 (4.4) and p. 456 (10.3)) was aware of the equations $\mathbf{x}\rfloor(u\wedge v) = (\mathbf{x}\rfloor u)\wedge v + \hat{u}\wedge(\mathbf{x}\rfloor v)$ and $\mathbf{x}\rfloor(uv) = (\mathbf{x}\rfloor u)v + \hat{u}(\mathbf{x}\rfloor v)$. Provided with the scalar multiplication $(u\wedge v)\rfloor w = u\rfloor(v\rfloor w)$, the exterior algebra $\bigwedge V$ and the Clifford algebra $C\ell(Q)$ are linearly isomorphic as left $\bigwedge V$-modules.

References

Boudet Roger: 1992, 'Les algèbres de Clifford et les transformations des multivecteurs', in A. Micali et al. (eds.): *Proceedings of the Second Workshop on "Clifford Algebras and their Applications in Mathematical Physics," Montpellier, France, 1989*, Kluwer, Dordrecht, pp. 343-352.

Chevalley Claude: 1954, 'The Algebraic Theory of Spinors', Columbia University Press, New York.

Crumeyrolle Albert: 1990, 'Orthogonal and Symplectic Clifford Algebras, Spinor Structures', Kluwer, Dordrecht.

Greub Werner: 1978, 'Multilinear Algebra' 2nd Ed., Springer, New York.

Helmstetter Jacques: 1982, 'Algèbres de Clifford et algèbres de Weyl', Cahiers Math. **25**, Montpellier.

Hestenes David, Sobczyk Garret: 1984, 1987, 'Clifford Algebra to Geometric Calculus', Reidel, Dordrecht.

Kähler Erich: 1962, 'Der innere Differentialkalkül', *Rendiconti di Matematica e delle sue Applicazioni (Roma)* **21**, 425-523.

Oziewicz Zbigniew: 1986, 'From Grassmann to Clifford', in J.S.R. Chisholm, A.K. Common (eds.): *Proceedings of the NATO and SERC Workshop on "Clifford Algebras and Their Applications in Mathematical Physics," Canterbury, England, 1985*, Reidel, Dordrecht, pp. 245-255.

Riesz Marcel: 1958, 'Clifford Numbers and Spinors', Univ. of Maryland.

MONOGENIC FORMS ON MANIFOLDS

VLADIMÍR SOUČEK
Mathematical Institute
Charles University
Sokolovská 83
18600 Praha
Czechoslovakia

Abstract. A generalization of holomorphic forms on Riemann surfaces to higher-dimensional manifolds with a spin-structure (so called monogenic forms) is described. Monogenic forms on R_m were defined by Delanghe, Sommen and Souček in recently published monograph on Clifford analysis. In the paper, the definition is generalized to the curved case (i.e. to manifolds with a spin-structure). Monogenic forms of any degree are defined in such a way that monogenic 0-forms are harmonic spinors, i.e. solutions of the Dirac equation. An analogue of the Cauchy theorem is proved for monogenic forms.

Key words: Clifford analysis – monogenic forms – Dirac operator – Cauchy theorem

1. Introduction

Starting from 30's the Dirac equation was established as the most appropriate generalization of Cauchy-Riemann equations to higher dimensions by effort of many people (see references in (Brackx, Delanghe, Sommen 1982), (Delanghe, Sommen, Souček 1992a)). Complex-valued functions were replaced in higher dimensions by spinor-valued maps (the spinor space in dimension 2 being the space of complex numbers).

We shall discuss here the next natural question, namely what is a natural generalization of holomorphic forms on Riemannian surfaces to higher dimensions. Note that this generalization is going to a quite different direction than holomorphic forms in several complex variables. The forms discussed here are defined on (real) manifolds with a spin structure, not on complex manifolds; they are not ordinary forms, but they have values in the corresponding spinor bundle and 0-forms are solutions of the Dirac equation instead of being holomorphic functions of several complex variables. Nevertheless, in the special case of the general situation – in the plane, i.e. for $m = 2$ – everything is reduced back to the standard case of holomorphic differential forms.

Such a generalization of holomorphic forms – so called monogenic forms – was described in flat case, i.e. on subset of R_m, in (Delanghe, Sommen, Souček 1992a), (Sommen 1984), (Sommen, Souček 1985), (Delanghe, Souček 1992), (Sommen 1992), (Sommen, Souček 1992). The purpose of the paper is to show that this definition can be extended from the flat case to any manifold endowed with a spin structure and to prove a basic property of monogenic forms – a generalization of the Cauchy theorem for monogenic functions.

2. A decomposition of spinor-valued forms

The object of our study is the space of all spinor-valued differential forms on a spin-manifold M. This is a space of sections of the bundle $\Lambda^*(T_c^*) \otimes S$, where T_c^* is the complexified cotangent bundle and S is the spinor bundle. It is a vector bundle associated to the representation $\Lambda^*(C_m^*) \otimes \mathbf{S}$, where $\mathbf{S}$ is a basic representation of the group Spin(m). We want to decompose the space of S-valued forms into smaller pieces while keeping the invariance properties with respect to the Spin group.

A systematic way how to describe all possible pieces in the decomposition is to consider the representation $\Lambda^i(C_m^*) \otimes \mathbf{S}$, to decompose it into irreducible pieces $E^{i,j}$ and then to define the subspaces of S-valued forms by considering the associated bundles to the representations $E^{i,j}$. The decomposition of the tensor product $\Lambda^i(C_m^*) \otimes \mathbf{S}$ was described in (Delanghe, Souček 1992) (and in the language of Clifford algebras in (Sommen, Souček 1992), (Delanghe, Sommen, Souček 1992a)). Using the standard characterization of Spin-modules by the highest weights (for more details see (Bröcker, tom Dieck 1985), (Delanghe, Souček 1992)), the result can be summarized as follows.

THEOREM 1. *Let* $\mathbf{S}$ *denote a basic spinor representation.*

The product $\Lambda^j(C_m^*) \otimes \mathbf{S} \cong \Lambda^{m-j}(C_m^*) \otimes \mathbf{S}, j = 1, \dots, [m/2]$ *decomposes into* $j+1$ *irreducible parts*

$$E_{\mu_0} \oplus \dots \oplus E_{\mu_j}.$$

The weights $\mu_l, l = 0, \dots, j$ *are given by:*

1. *If* m *is even and* $\mathbf{S} \cong (\frac{1}{2}, \dots, \frac{1}{2})$,
   then $\mu_l = \left(\frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2}, \dots \frac{1}{2}, (-1)^{j-l}\frac{1}{2}\right)$.
2. *If* m *is even and* $\mathbf{S} \cong (\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})$,
   then $\mu_l = \left(\frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2}, \dots \frac{1}{2}, (-1)^{j-l+1}\frac{1}{2}\right)$.
3. *If* m *is odd, then* $\mu_l = \left(\frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right)$.

The component $\frac{3}{2}$ *appears* l *times in all cases.*

Let us denote for simplicity the part E_{μ_l} in the decomposition of the product $\Lambda^j(C_m^*) \otimes \mathbf{S}$, resp. $\Lambda^{m-j}(C_m^*) \otimes \mathbf{S}$, $j = 1, \dots, [m/2]$ by $E^{j,l}$, resp. $E^{m-j,l}$.

3. Monogenic forms on a spin-manifold

Let (M, g) be an oriented Riemannian manifold of dimension m. Let us choose a spin-structure on M, i.e. let us suppose that we have chosen a principal fibre bundle $\tilde{P}$ over M with the group $G =$ Spin(m) together with the corresponding 2:1 covering map $\tilde{P} \to P$ onto the bundle P of oriented orthonormal frames. The Levi-Civita connection on P induces then a covariant derivative D on the spinor bundle $S = \tilde{P} \times_{Spin} \mathbf{S}$ associated to the basic spinor representation $\mathbf{S}$.

Let us denote the space of smooth S-valued differential forms of degree j on M by $\mathcal{E}^j(S)$. The covariant derivative D maps $\mathcal{E}^0(S)$ to $\mathcal{E}^1(S)$ and can be extended (see e.g. (Wells 1973)) to the maps $D : \mathcal{E}^k(S) \mapsto \mathcal{E}^{k+1}(S)$ for all $k = 1, \dots, m-1$.

To define monogenic differential forms, we are going to define a splitting of S-valued k-forms on M into two parts

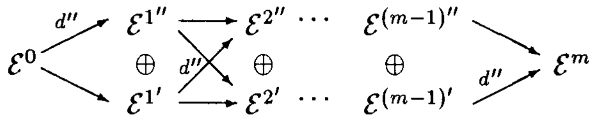
$$\mathcal{E}^k = \mathcal{E}^{k'} \oplus \mathcal{E}^{k''}.$$

It imitates the definition of holomorphic forms in dimension 2. Let us recall the definition of holomorphic forms in the plane. The space of 1-forms can be split into a direct sum of two pieces

$$\mathcal{E}^1 = \mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1}$$

and the value of the $\bar{\partial}$ -operator on a function f is defined to be the composition of the de Rham operator d and the projection onto the (0,1)-part of df. Holomorphic functions and holomorphic 1-forms are elements of the kernels of the maps $\bar{\partial}$.

Using the splitting $\mathcal{E}^k = \mathcal{E}^{k'} \oplus \mathcal{E}^{k''}$, we are going to consider the diagram



The operators d'' are defined as the composition of d with the projection onto $\mathcal{E}^{k''}$. Monogenic forms will be defined as elements of the kernels of the operators d''.

The requirement of invariance tells us that we have to choose for the primed and double primed parts sums of pieces in the decomposition described above. Let $\mathcal{E}^{j,k}$ denote the bundle on M associated to the representations $E^{j,k}$ described in the Theorem 1. Then we have the following definition.

DEFINITION 1. *(i) Let us define for $k \leq [m/2]$*

$$\mathcal{E}^{k'} = \bigoplus_{\substack{0\leq j\leq k\\ j \text{ even}}} \mathcal{E}^{k,k-j}; \quad \mathcal{E}^{k''} = \bigoplus_{\substack{0\leq j\leq k\\ j \text{ odd}}} \mathcal{E}^{k,k-j} \tag{1}$$

and for $k < [m/2]$

$$\mathcal{E}^{(m-k)'} = \bigoplus_{\substack{0\leq j\leq k\\ j \text{ odd}}} \mathcal{E}^{m-k,k-j}; \quad \mathcal{E}^{(m-k)''} = \bigoplus_{\substack{0\leq j\leq k\\ j \text{ even}}} \mathcal{E}^{m-k,k-j} \tag{2}$$

(ii) Let us define spaces $\widetilde{\mathcal{E}^{k'}}, \widetilde{\mathcal{E}^{k''}}, \widetilde{\mathcal{E}^{(m-k)'}}, \widetilde{\mathcal{E}^{(m-k)''}}$ by (1) for $k < [m/2]$ and by (2) for $k \leq [m/2]$.

Note that the both spaces (with or without tilde) coincide in any odd dimension and that they are different only in the middle dimension if the dimension is even.

As an illustration, let us consider the cases of dimension $m = 6$ and $m = 7$. In the odd-dimensional case, we have the following picture (the numbers indicate the dimension of $E^{k,j}$ counted in multiples of dim S). The pieces, belonging to $\mathcal{E}^{k'}$, are indicated by boxes. In the top row the dimension of the full spaces $\mathcal{E}^k$ is written. The spaces $\mathcal{E}^{k'}$ and $\tilde{\mathcal{E}}^{k'}$ coincide.

1	7	21	35	35	21	7	1
$\boxed{1}$	1	$\boxed{1}$	1	$\boxed{1}$	1	$\boxed{1}$	1
	$\boxed{6}$	6	$\boxed{6}$	6	$\boxed{6}$	6	
		$\boxed{14}$	14	$\boxed{14}$	14		
			$\boxed{14}$	14			

In the even-dimensional case ($m = 6$), we get two possibilities:

(i) The spaces $\mathcal{E}^{k'}$:

1	6	15	20	15	6	1
$\boxed{1}$	1	$\boxed{1}$	1	1	$\boxed{1}$	1
	$\boxed{5}$	5	$\boxed{5}$	$\boxed{5}$	5	
		$\boxed{9}$	9	9		
			$\boxed{5}$			

(ii) The spaces $\tilde{\mathcal{E}}^{k'}$:

1	6	15	20	15	6	1
$\boxed{1}$	1	$\boxed{1}$	$\boxed{1}$	1	$\boxed{1}$	1
	$\boxed{5}$	5	5	$\boxed{5}$	5	
		$\boxed{9}$	$\boxed{9}$	9		
			5			

The monogenic forms are now defined in the following way.

DEFINITION 2. *Let us define*

$$\mathcal{M}^k = \{\omega \in \mathcal{E}^{k'} \mid d''\omega = 0\}$$

and

$$\widetilde{\mathcal{M}^k} = \{\omega \in \tilde{\mathcal{E}}^{k'} \mid \tilde{d}''\omega = 0\}.$$

The forms $\omega \in \mathcal{M}^k$ (resp. $\omega \in \widetilde{\mathcal{M}^k}$) will be called monogenic k-forms.

As was shown in (Fegan 1976) and (Bureš, Souček 1986), the space $\mathcal{M}^0$ is just the usual space of solutions of the Dirac equation (i.e. the space of harmonic spinors).

Monogenic differential forms on domain in R_m are defined and studied in (Delanghe, Sommen, Souček 1992a), (Sommen, Souček 1992) and (Delanghe, Souček 1992). The main properties proved there are the description of homology of domains in R_m by homology of the sequence of monogenic forms and an analogue of the Cauchy theorem. In the next section, we are going to show how the Cauchy theorem can be proved for monogenic forms on spin-manifolds.

4. The Cauchy theorem

Let $e_1, \ldots, e_m$ be an orthonormal basis in R_m, the vectors e_j are considered as elements in the corresponding Cifford algebra $R_{0,m}$. Let us define an $(m-1)$-form $d\sigma = \sum (-1)^{j+1} e_j d\hat{x}_j$ with values in $R_{0,m}$.

The standard form of the Cauchy theorem for monogenic functions in Clifford analysis is the statement that the Clifford algebra valued differential form $\omega = f\, d\sigma\, g$ is closed whenever the functions g and f are left, resp. right, monogenic.

In the flat case, the space $\mathcal{M}^{m-1}$ of (left) monogenic $(m-1)$-forms consists of all forms $\omega = d\sigma f$, where f is a solution of the Dirac equation. Hence the form ω is a product of a monogenic 0-form and a monogenic $(m-1)$-form. More generally, it is proved in (Delanghe, Sommen, Souček 1992a) that a product of a (right) monogenic j-form and a (left) monogenic $(m-j-1)$-form is a closed form. We are going to show that this generalized Cauchy theorem holds for monogenic forms on manifolds as well.

In this more general situation, the notion of multiplication of Clifford-valued forms will be substituted by a duality given by an invariant Hermitean form on the spinor space S. It is well known that such an invariant Hermitean form on S exists, it induces then a Hermitean structure on the associated bundle S. To see the connection with left and right multiplication used in Clifford analysis on flat space, it is convenient to use a realization of the spinor space as a left ideal in the complexified Clifford algebra C_m described in (Delanghe, Sommen, Souček 1992a), Sect. I.4.7. It is showed there that a dual space to S can be realized conveniently as a right ideal $\bar{S}$ in C_m, where the bar map is the composition of a main antiinvolution in the Clifford algebra C_m and the complex conjugation. The product $\bar{S} \cdot S$ is one-dimensional complex space and when we identify it with C, an invariant Hermitean scalar product on S is given by the Clifford multiplication $< s, s' >= \bar{s}' s$. In our general situation, we shall use the scalar product instead of the multiplication. As a consequence, we shall not need to introduce left and right monogenic forms.

So having available the Hermitean scalar product on $< .,. >_x$ in each fiber $S_x, x \in M$, we can define a map $< .,. >$ from $\mathcal{E}^j(S) \times \mathcal{E}^k(S)$ into $\mathcal{E}^{j+k}$ by

$$< \omega \otimes s, \omega' \otimes s' >_x = \omega \wedge \omega' < s, s' >_x, \omega \in \mathcal{E}^j{}_x, \omega' \in \mathcal{E}^k{}_x; s, s' \in S_x.$$

The covariant derivative D induced by the Levi-Civita connection is compatible with the Hermitean structure, i.e. we have

$$d < \omega, \tau >=< D\omega, \tau > +(-1)^j < \omega, D\tau >, \omega \in \mathcal{E}^j(S), \tau \in \mathcal{E}^k(S)$$

(for 0-forms it is proved e.g. in (Lawson, Michelsohn 1989), it can be checked that it is true for general forms).

Now we can formulate the Cauchy theorem.

THEOREM 2. *For any $k = 1, \ldots, m-1$ and for all $\omega \in \mathcal{M}^k(S)$ and $\tau \in \tilde{\mathcal{M}}^{m-k-1}(S)$*

$$d < \omega, \tau >= 0.$$

The proof of the theorem is based on the multiplicative properties of elements in $\Lambda^*(R^*_m) \otimes$ S.

LEMMA 1. *Let $E^{j,k}$ be the decomposition of $\Lambda^j(C^*_m) \otimes$ S described in Theorem 1 and let $k = 1, \ldots, m-1$; $i, j < k$; $i \neq j$. Then for each $\omega \in E^{k,i}$and $\tau \in E^{m-k,j}$*

$$< \omega, \tau >= 0.$$

Proof

Using the realisation of the spinor space as a left ideal in the Clifford algebra and using the realization of the Hermitean scalar product described above, the lemma follows from the Lemma 5.2 in (Delanghe, Sommen, Souček 1992). ■

Proof of the Cauchy theorem

Using the defintion of $\mathcal{E}^{j'}$ and $\widetilde{\mathcal{E}^{m-j}}'$, it is easy to check that they never contain elements at the same row in the decomposition diagrams (see e.g. diagrams shown above).

Then it is sufficient to observe that

$$d < \omega, \tau >=< D\omega, \tau > +(-1)^j < \omega, D\tau >$$

and that the monogenicity conditions implies the condition $D\omega \in \mathcal{E}^{(j+1)'}$ and $D\tau \in \tilde{\mathcal{E}}^{(m-j)'}$. ■

The Cauchy theorem for monogenic forms is the key property which made possible to define a general notion of residue for monogenic forms for higher dimensional singularities introduced in (Delanghe, Sommen, Souček 1992a). So the generalization proved here will make possible to define the notion of a residue of a monogenic form with a higher dimensional singularity on spin-manifolds. This topic will be described, however, in another paper.

References

Brackx, F., Delanghe, R., Sommen, F.: 1982, *Clifford analysis*, Pitman: London

Bröcker, T., tom Dieck, T.: 1985, *Representations of Compact Lie Groups*, Springer-Verlag: Berlin

Delanghe, R., Souček, V.: 1992, 'On the structure of spinor-valued differential forms', *Complex Variables* **18**, 223-236

Delanghe, R., Sommen, F., Souček, V.: 1992a, *Clifford Algebra and Spinor-valued Functions*, Kluwer Academic Publishers, 485 pp.

Delanghe, R., Sommen, F., V.Souček, V.: 1992b, 'An explicit realization of spinor spaces and its application to Clifford analysis', , to appear
Fegan, H.D.: 1976, 'Conformally invariant first order differential operators', *Q.Jour.Math.* **27**, 371-378
Gentili, G., Mariconda, C., Tarallo, M.: 1990, 'Quaternionic regular maps and $\bar{\partial}$-type operators', , SISSA preprint
Lawson, H.B., Michelsohn, M.-L.: 1989, *Spin Geometry*, Princeton Univ. Press
Sommen, F.: 1984, 'Monogenic differential forms and homology theory', *Proc. Royal Irish Ac.* **84A**, 87-109
Sommen, F.: 1982, 'Monogenic differential calculus', , to appear
Sommen, F., Souček, V.: 1985, 'Hypercomplex differential forms applied to the de Rham and the Dolbeault complex', *Sem.Geom. 1984, Univ. Bologna* , 177-192
Sommen, F., Souček, V.: 1982, 'Monogenic differential forms', , to appear
Souček, V.: 1984, 'H-valued differential forms on H,', *Suppl.Rend.Circolo Mat.Palermo, ser.II* **3**, 293-299
Wells, R.O., jr.: 1973, *Differential analysis on complex manifolds*, Prentice.Hall, N.J.

ON INVERTIBILITY OF CLIFFORD ALGEBRAS ELEMENTS WITH DISJOINT SUPPORTS

PAVEL SEMENOV
Moscow State Pedagogical University
Kosmodemjanski 9A, kv. 27
125130 Moscow, Russia

Let $Cl(n)$ be the classical associative Clifford algebra over field $\mathbb{R}$ with generators $e_1, e_2, \ldots, e_n$ and relations

$$e_i e_j + e_j e_i = 0, i \neq j$$

$$e_i^2 = -1.$$

It's well known (see [1]) fact that algebras $Cl(n)$ for different n are isomorphic to some matrix $\mathbb{R}$-algebra or to direct sum of some matrix $\mathbb{R}$-algebras. Therefore, from the formal point of view, the question about invertibility in $Cl(n)$ is equivalent to the question about calculating of determinants of matrices. But,these matrices have sizes approximately equal to $2^{[n/2]} \times 2^{[n/2]}$ and really such calculatings are impossible. But for some classes of elements of algebra $Cl(n)$ a criteria of invertibility may be obtained without above mentioned matrix realizibility of Clifford algebra $Cl(n)$. The trivial example of such a class is the set of all vectors

$$x = \sum x_i e_i \in \mathbb{R}^n \subset Cl(n).$$

Indeed we have

$$x^2 = -\sum x_i^2$$

and hence vector x is invertible in $Cl(n)$ iff $x \neq 0$.

The main emphasis of the present lecture is given to questions about invertibility of bivectors $x = \sum x_{ij} e_{ij}$ and of elements with "disjoint supports", namely of elements $x = \sum x_i e_{\alpha_i}$ with nonempty, mutually disjoint sets α_i.

Case n=3. Let $x = \alpha e_{12} + \beta e_{23} + \gamma e_{13}$. It is easy to see that $x^2 = -(\alpha^2 + \beta^2 + \gamma^2)$ and hence x is noninvertible iff $x = 0$.

Case n=4.

THEOREM 1. **Let $x = \sum x_{ij} e_{ij}$ be a bivector in Cl(4). Then x is noninvertible iff**

$$[x_{12} = x_{34}, x_{13} = -x_{24}, x_{14} = x_{23}] \text{ or}$$

$$[x_{12} = -x_{34}, x_{13} = x_{24}, x_{14} = -x_{23}]$$

Case n=5. For any bivector $x = \sum x_{ij} e_{ij}$ in $Cl(n)$ we denote

$$\| x \|^2 = \sum x_{ij}^2$$

and for any $\alpha = \{i < j < k < n\} \subset \{1,2,3,4,5\}$

we denote $s_\alpha(x) = 2(x_{ij}x_{kn} - x_{ik}x_{jn} + x_{in}x_{jk})$

THEOREM 2. **Bivector x is non-invertible in Cl(5) iff**

$$\| x \|^4 = \sum s_\alpha^2(x)$$

where the sum is taken over all α consisting of four elements.

Case n=6. In $Cl(6) I$ dont know a criteria for invertibility of bivectors in general case. But for bivectors with disjoint sup ports the following theorem may be obtained.

THEOREM 3. **Let $x = \alpha e_{12} + \beta e_{34} + \gamma e_{56}$ and let**

$$y = \alpha(-\alpha^2 + \beta^2 + \gamma^2)e_{12} + \beta(\alpha^2 - \beta^2 + \gamma^2)e_{34}^{+}$$

$$+\gamma(\alpha^2 + \beta^2 - \gamma^2)e_{56} + 2\alpha\beta\gamma e_{123456}.$$

Then

$$xy = (\alpha + \beta + \gamma)(\alpha + \beta - \gamma)(\alpha - \beta + \gamma)(\alpha - \beta - \gamma)$$

Hence, the bivector x is non-invertible

$$\text{iff } (\alpha + \beta + \gamma)(\alpha + \beta - \gamma)(\alpha - \beta + \gamma)(\alpha - \beta - \gamma) = 0.$$

This theorem answers Prof. P. Lounesto's hypothesis (Montpellier, 1989, private communication). The proof of theorem 3 may be easily obtained if we remark that x^2 lies in subspace $V_4 = =$ span $\{1, e_{1234}, e_{1256}, e_{3456}\}$ and that subspace V_4 is in fact subalgebra of algebra $Cl(6)$.

Theorem 3 (but not its proof) has the following generalization for elements of Clifford algebra $Cl(n)$. For any set $\alpha \subset \{1,2,\dots,n\}$ we denote $|\alpha|$ the number of elements of this set.

THEOREM 4. **Let $x = x^1 + x^2 + x^3 + x^4$ where:**

$$x^1 = \sum_{i=1}^{k} a_i e_{\alpha_i}, x^2 = \sum_{j=1}^{l} b_j e_{\beta_j}, x^3 = \sum_{p=1}^{m} c_p e_{\gamma_p}, x^4 = \sum_{q=1}^{s} d_q e_{\tau_q}$$

sets $\{\alpha_1, ..., \alpha_k, \beta_1, ..., \tau_s\}$ are nonempty, mutually disjoint and $|\alpha_i| \equiv 1(\bmod 4), |\beta_j| \equiv 2) \bmod 4) \mid \gamma_p| \equiv 3(\bmod 4), |\tau_q| \equiv 0(\bmod 4)$. Then

a) in the case $M = \sum a_i^2 - \sum c_p^2 > 0$ the element x is noninvertible iff

$$\prod_{\epsilon_j, \delta_q = \pm 1} \{(b_1 - \epsilon_1\sqrt{M} - \epsilon_2 b_2 - \dots - \epsilon_l b_l)^2 + (d_1 - \delta_2 d_2 - \dots - \delta_s d_s)^2\} = 0$$

b) in the case $M = \sum a_i^2 - \sum c_p^2 < 0$ the element x is non–invertible iff

$$\prod_{\epsilon_j, \delta_q = \pm 1} \{(b_1 - \epsilon_2 b_2 - \dots - \epsilon_l b_l)^2 + (d_1 - \delta_1\sqrt{-M} - \delta_2 d_2 - \dots - \delta_s d_s)^2\} = 0$$

c) in the case M = 0 the element x is non-invertible iff

$$\prod_{\epsilon_j, \delta_q = \pm 1} \{(b_1 - \epsilon_2 b_2 - \dots - \epsilon_l b_l)^2 + (d_1 - \delta_2 d_2 - \dots - \delta_s d_s)^2\} = 0$$

Open problems

Problem 1. Find an analog of theorem 4 for elements with property of "small intersection" of supports.

Problem 2. What about an invertibility in algebras $Cl(p,q)$?

Problem 3. When may be an element of $Cl(n)$ written as a product of some elements, each of them is an element with disjoint supports ?

References

M. Karoubi **K**–Theory, Springer Verlag. 1978.

CLIFFORD ALGEBRAS AND ALGEBRAIC STRUCTURE OF FUNDAMENTAL FERMIONS

WOJCIECH KRÓLIKOWSKI
Institute of Theoretical Physics, Warsaw University
Hoża 69, PL - 00 - 681 Warszawa, Poland

Dedicated to Jan Rzewuski
on the happy occasion of
his 75^{th} *birthday*

Abstract. We show, how an idea of leptons and quarks composed of algebraic partons (defined by a sequence of Clifford algebras) can explain the existence of three and only three families of these fundamental fermions. In this argument, the theory of relativity, the probability interpretation of quantum mechanics and the Pauli exclusion principle, all extended to the algebraic partons, play a crucial role. As a consequence, a semiempirical mass spectral formula for charged leptons is discussed. In terms of experimental m_e and m_μ, it gives successfully $m_\tau = 1783.47$ MeV or 1776.80 MeV (two options, the second fitting excellently to new measurements of m_τ).

1. Introduction

The most puzzling feature of today's particle physics is perhaps the phenomenon of three families of leptons:

$$\begin{matrix} \nu_e & \nu_\mu & \nu_\tau\,(?) & (\text{charge} \quad 0) \\ e^- & \mu^- & \tau^- & (\text{charge} \, -1) \end{matrix} \tag{1}$$

and quarks:

$$\begin{matrix} u & c & t\,(?) & (\text{charge} \quad 2/3) \\ d & s & b & (\text{charge} \, -1/3) \end{matrix} \tag{2}$$

differing by nothing but their masses. Among them, the tauonic neutrino ν_τ and top quark t are not yet observed directly, though indirect evidence leaves practically no doubt as to their existence. In particular, the recent CERN measurements of total decay width for Z^0 gauge boson manifesting itself as a resonance at ca. 91 GeV of CM energy in the process

$$e^+ e^- \longrightarrow Z^0 \longrightarrow \text{anything}, \tag{3}$$

have shown that the number of different neutrino versions lighter than $\frac{1}{2} m_Z \simeq 46$ GeV is just three. Moreover, this result strongly suggests that the number of all lepton and quark families is equal to three if all neutrinos are light (or massless).

In this lecture, we are going to show that there are three different, physically distinguished versions of the Dirac equation

$$[\Gamma \cdot (p - gA) - M]\,\psi = 0, \tag{4}$$

where

$$\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu}. \tag{5}$$

Here, $g\Gamma \cdot A$ symbolizes the standard-model coupling, identical for all three versions, while the mass operator M may depend on the version. So, we shall be tempted to connect these versions with the three experimental families of leptons and quarks.

Our argument will express an idea of algebraic compositness of fundamental fermions that accepts an act of algebraic abstraction from the familiar notion of spatial compositness (so useful, for instance, in the case of pseudoscalar and vector mesons built up of quark-antiquark pairs moving in the physical space).

2. An example of algebraic compositness

Let us start from the familiar Duffin-Kemmer-Petiau equation describing a particle with spin $0 \oplus 1$ (for instance, a pseudoscalar or vector meson). In the free case, it can be written in the form

$$[\frac{1}{2}(\gamma_1 + \gamma_2) \cdot P - M]\psi(X) = 0, \tag{6}$$

where γ_1^μ and γ_2^μ are two sets of commuting Dirac matrices,

$$\{\gamma_i^\mu, \gamma_i^\nu\} = 2g^{\mu\nu} \;,\; [\gamma_1^\mu, \gamma_2^\nu] = 0. \tag{7}$$

Here, $\frac{1}{2}(\gamma_1^\mu + \gamma_2^\mu)$ are the 16×16 Duffin-Kemmer-Petiau matrices.

It can be readily seen that Eq. (6) may be considered as a point-like limiting form of the following two-body wave equation (Królikowski 1987, 1988):

$$\left[\gamma_1 \cdot (\frac{1}{2}P + p) + \gamma_2 \cdot (\frac{1}{2}P - p) - m_1 - m_2 - S(x)\right] \psi(X, x) = 0, \tag{8}$$

where (for simplicity) masses are assumed equal: $m_1 = m_2$ (what, for instance, is the case for a pair of a quark and an antiquark of the same sort). The internal interaction $S(x)$ in Eq. (8) can be related to the more familiar internal interaction $I(x)$ appearing in the Bethe-Salpeter equation (Bethe 1957) through the formula

$$S(x) = \left[\frac{1}{\gamma_1 \cdot \left(\frac{1}{2}P + p\right) - m_1 + i\varepsilon} + \frac{1}{\gamma_2 \cdot \left(\frac{1}{2}P - p\right) - m_2 + i\varepsilon}\right] I(x)\,. \tag{9}$$

Then, any of these two four-dimensional integral operators allows to calculate perturbatively, step by step, a three-dimensional integral operator playing the role of internal interaction $V(\mathbf{x})$ (internal interaction energy) in the one-time two-body wave equation having the conventional form of the state equation. This equation, derived many years ago by Jan Rzewuski and myself (Królikowski 1955, 1956), reduces to the familiar Salpeter equation (Salpeter 1952) in the case of an instantaneous internal interaction.

Now, an intriguing question might be asked, what would happen, if in Eq. (6) the commuting γ_1^μ and γ_2^μ were replaced by anticommuting γ_1^μ and γ_2^μ (Królikowski 1986). Then, there would be

$$\left\{\gamma_i^\mu, \gamma_j^\nu\right\} = 2\delta_{ij} g^{\mu\nu}, \tag{10}$$

instead of Eq. (7). Note that the Clifford algebra (10) could be represented by

$$\gamma_1^\mu = \gamma^\mu \otimes \mathbf{1} \;,\; \gamma_2^\mu = \gamma^5 \otimes i\gamma^5\gamma^\mu \tag{11}$$

with $\gamma^\mu, \mathbf{1}$ and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ being the usual Dirac 4×4 matrices.

In the case of Eq. (10), the counterpart of the Duffin-Kemmer-Petiau equation (6) (with the convenient coefficient $1/\sqrt{2}$ in place of $1/2$),

$$\left[\frac{1}{\sqrt{2}}(\gamma_1^\mu + \gamma_2^\mu) \cdot P - M\right] \psi(X) = 0, \tag{12}$$

might be considered as a point-like limiting form of the two-body wave equation

$$\begin{aligned}&\Big[\sqrt{2}\gamma_1 \cdot \left(\tfrac{1}{2}P + p\right) + \\ &+\sqrt{2}\gamma_2 \cdot \left(\tfrac{1}{2}P - p\right) - m_1 - m_2 - S(x)\Big]\, \psi(X, x) = 0,\end{aligned} \tag{13}$$

but the latter, in contrast to Eq. (8), could not be derived from the conventional quantum field theory. This is a consequence of the fact that the particle kinetic-energy operators in the Fock space $\gamma_i^0 (\boldsymbol{\gamma}_i \cdot \mathbf{p}_i + m)$ all commute, if they are derived from the field kinetic-energy operator $\int d^3\mathbf{x}\, \psi^+(x)\gamma^0 (\boldsymbol{\gamma} \cdot \mathbf{p} + m)\psi(x)$, so, in such a case, all γ_i^μ must commute for different i (at least, when massive particles are considered; if an interaction with an external scalar field is introduced, also massless particles cannot escape from this conclusion).

Thus, while Eq. (12) (with Eq. (10)) may be investigated for some hypothetical particles, it cannot be considered as a point-like limiting form of a two-body wave equation following from the conventional field theory. So, $\psi = (\psi_{\alpha_1\alpha_2})$ displays an algebraic structure that, now, does not coexist with any spatial internal structure, at any rate, in the framework of the conventional quantum field theory (Królikowski 1991). This illustrates, therefore, the notion of *algebraic compositness.* In Eq. (12) the Dirac bispinor indices α_1 and α_2 describe *"algebraic partons"*, agents of the idea of this compositness.

Let us emphasize that the logical relationship between the notions of spatial compositness and algebraic compositness reminds the logical relationship between the notions of orbital angular momentum and spin. In fact, in these cases we have to do with similar acts of algebraic abstraction from some notions of spatial character.

It is important to note that due to the Clifford algebra (10) the matrices

$$\Gamma^\mu = \frac{1}{\sqrt{2}}(\gamma_1^\mu + \gamma_2^\mu) \tag{14}$$

appearing in Eq. (12) satisfy the Dirac algebra (5). This implies that Eq. (12) has the form of the Dirac equation (4) (in the free case). Thus, the hypothetical particles described by Eq. (12), when coupled to the magnetic field, should display (magnetically "visible") spin $1/2$ though any of them is a composite of two algebraic partons of spin $1/2$. There exists, therefore, another (magnetically "hidden") spin $1/2$. It is related to the matrices $\left(1/\sqrt{2}\right)(\gamma_1^\mu - \gamma_2^\mu)$ also fulfilling the Dirac algebra (5) and anticommuting with the matrices Γ^μ.

Note further that the matrices (14) may be represented in the convenient form

$$\Gamma^\mu = \gamma^\mu \otimes \mathbf{1}, \tag{15}$$

if the representation (11) is changed into

$$\gamma_{1,2}^\mu = \frac{1}{\sqrt{2}}\left(\gamma^\mu \otimes \mathbf{1} \pm \gamma^5 \otimes i\gamma^5\gamma^\mu\right). \tag{16}$$

So, Eq. (12) can be rewritten as

$$(\gamma_{\alpha_1\beta_1} \cdot P - \delta_{\alpha_1\beta_1} M)\ \psi_{\beta_1\alpha_2}(X) = 0, \tag{17}$$

where the second Dirac bispinor index α_2 is free. Such an equation is known as the Dirac form (Banks 1982) of the Kähler equation (Kähler 1962).

3. A sequence of Dirac-type equations

As can be easily seen, the Dirac algebra (5) admits the remarkable sequence $N = 1, 2, 3, \ldots$ of representations

$$\Gamma^\mu = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\gamma_i^\mu, \tag{18}$$

where the matrices γ_i^μ, $i = 1, 2, 3, \ldots, N$, satisfy the sequence $N = 1, 2, 3, \ldots$ of Clifford algebras

$$\left\{\gamma_i^\mu, \gamma_j^\nu\right\} = 2\delta_{ij}g^{\mu\nu}. \tag{19}$$

With the matrices (18), Eq. (4) gives us a sequence $N = 1, 2, 3, \ldots$ of Dirac-type equations (Królikowski 1990, 1992). Of course, for $N = 1$ Eq. (4) (with the matrices (18) inserted) is the usual Dirac equation, while for $N = 2$ it is equivalent to the Dirac form of the Kähler equation already discussed in Section 2 (in the free case). For $N = 3, 4, 5, \ldots$ it provides us with new Dirac-type equations.

Except for $N = 1$, the representations (18) are reducible since they may be realized in the convenient form

$$\Gamma^\mu = \gamma^\mu \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{(N-1)times} \tag{20}$$

with γ^μ and $\mathbf{1}$ standing for the usual Dirac 4×4 matrices. In fact, for any $N > 1$ one can introduce, beside $\Gamma_1^\mu \equiv \Gamma^\mu$ given in Eq. (18) $N - 1$ other Jacobi-type independent combinations $\Gamma_2^\mu, \ldots, \Gamma_N^\mu$,

$$\Gamma_2^\mu = \frac{1}{\sqrt{2}}(\gamma_1^\mu - \gamma_2^\mu)\ ,\ \Gamma_3^\mu = \frac{1}{\sqrt{6}}(\gamma_1^\mu + \gamma_2^\mu - 2\gamma_3^\mu)\ , \ldots, \tag{21}$$

such that

$$\left\{\Gamma_i^\mu, \Gamma_j^\nu\right\} = 2\delta_{ij}g^{\mu\nu} \tag{22}$$

(in consequence of Eq. (19)). In particular, for $N = 3$ one may use the representation

$$\Gamma_1^\mu = \gamma^\mu \otimes \mathbf{1} \otimes \mathbf{1}\ ,\ \Gamma_2^\mu = \gamma^5 \otimes i\gamma^5\gamma^\mu \otimes \mathbf{1}\ ,\ \Gamma_3^\mu = \gamma^5 \otimes \gamma^5 \otimes \gamma^\mu. \tag{23}$$

In the representation (20), the Dirac-type equation (4) for any N can be rewritten as

$$[\gamma \cdot (p - g\,A) - M]_{\alpha_1\beta_1}\ \psi_{\beta_1\alpha_2\ldots\alpha_N} = 0, \tag{24}$$

where $M_{\alpha_1\beta_1} = \delta_{\alpha_1\beta_1} M$. Here, $\psi = (\psi_{\alpha_1\alpha_2\ldots\alpha_N})$ carries N Dirac bispinor indices α_i, $i = 1,2,\ldots,N$, of which only the first one is affected by the Dirac matrices γ^μ and so is coupled to the particle's momentum and to the standard-model gauge fields (among others, to the electromagnetic field). The rest of them are free. Thus, only α_1 is "visible", say, in the magnetic field, while $\alpha_2,\ldots,\alpha_N$ are "hidden". In consequence, a particle described by Eq. (4) or (24) can display , say, in the magnetic field only a "visible" spin 1/2, though it possesses also $N-1$ "hidden" spins 1/2.

Our *first crucial assumption* will be that the physical Lorentz group of the theory of relativity, if applied to the particle described by Eq. (4) or (24) for any N, is generated both by the particle's visible and hidden degrees of freedom. Then, the form $\psi^+\Gamma_1^0\Gamma_1^\mu\psi$ is no relativistic covariant for $N > 1$, though Eq. (4) with $\Gamma^\mu \equiv \Gamma_1^\mu$ implies that always

$$\partial_\mu \psi^+\Gamma_1^0\Gamma_1^\mu\psi = 0. \tag{25}$$

In contrast, the form $\psi^+\Gamma_1^0\Gamma_2^0\ldots\Gamma_N^0\Gamma_1^\mu\psi$ is a relativistic vector for any N, but Eq. (4) with $\Gamma^\mu \equiv \Gamma_1^\mu$ shows that

$$\partial_\mu \psi^+\Gamma_1^0\Gamma_2^0\ldots\Gamma_N^0\Gamma_1^\mu\psi = 0 \tag{26}$$

only for N odd.

Thus, the interplay of the theory of relativity and the probability interpretation of quantum mechanics requires that (i) only the odd terms

$$N = 1,3,5,\ldots \tag{27}$$

should be present in the sequence of the Dirac-type equation (4) (if these are considered as wave equations), and (ii) the probability current should have the form

$$j^\mu = \eta_N\, \psi^+\Gamma_1^0\Gamma_2^0\ldots\Gamma_N^0\Gamma_1^\mu\psi\,. \tag{28}$$

Here, η_N is a phase factor making the matrix of hidden internal parity

$$P_{\text{hidden}} = \eta_N\Gamma_2^0\ldots\Gamma_N^0 \tag{29}$$

Hermitian. Since due to Eq. (26) P_{hidden} is a constant of motion, one can consistently impose on the wave function ψ in the wave equation (4) the constraint

$$P_{\text{hidden}}\,\psi = \psi \tag{30}$$

in order to guarantee the probability density to be positive :

$$j^0 = \eta_N\psi^+\Gamma_2^0\ldots\Gamma_N^0\psi > 0. \tag{31}$$

4. Hidden exclusion principle

The Dirac-type equation (4) with $\Gamma^\mu \equiv \Gamma_1^\mu$ distinguishes the visible bispinor index α_1 from $N-1$ hidden bispinor indices $\alpha_2,\ldots,\alpha_N$. About the latter indices, appearing in this scheme on the equal footing, we will make our *second crucial assumption* that they represent physically nondistinguishable degrees of freedom obeying the Fermi statistics along with the Pauli exclusion principle. Then, the wave functions $\psi = (\psi_{\alpha_1\alpha_2\ldots\alpha_N})$ in the sequence (27) of the Dirac-type wave equations (4) or (24) should be completely antisymmetric with respect to the hidden indices $\alpha_2,\ldots,\alpha_N$. This implies that the sequence (27) must terminate at $N=5$,

$$N = 1,3,5, \tag{32}$$

leaving us with three and only three terms (32) in the sequence of the Dirac-type wave equations (4) or (24).

In the case of $N = 5$ our exclusion principle requires that

$$\psi_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} \equiv \varepsilon_{\alpha_2\alpha_3\alpha_4\alpha_5}\psi^{(5)}_{\alpha_1}. \tag{33}$$

Thus, in this case there are $4! = 24$ equivalent nonzero components (carrying the index α_1), all equal (up to the sign) to one Dirac function $\psi^{(5)}_{\alpha_1}$. This reduces the Dirac-type equation (4) or (24) to the usual Dirac equation. Here, of course, spin is 1/2 and it is provided by the visible spin, while four hidden spins sum up to zero.

The case of $N = 3$ is more complicated since then one should consider five candidates for relativistic covariants, viz.

$$p_{\alpha_1} = \left(C^{-1}\right)_{\alpha_2\alpha_3}\psi_{\alpha_1\alpha_2\alpha_3}\,,\; s_{\alpha_1} = \left(C^{-1}\gamma^5\right)_{\alpha_2\alpha_3}\psi_{\alpha_1\alpha_2\alpha_3}\,, \tag{34}$$

$$a^\mu_{\alpha_1} = \left(C^{-1}\gamma^\mu\right)_{\alpha_2\alpha_3}\psi_{\alpha_1\alpha_2\alpha_3}\,,\; v^\mu_{\alpha_1} = \left(C^{-1}\gamma^5\gamma^\mu\right)_{\alpha_2\alpha_3}\psi_{\alpha_1\alpha_2\alpha_3}\,, \tag{35}$$

$$t^{\mu\nu}_{\alpha_1} = \left(C^{-1}\gamma^5\frac{i}{2}[\gamma^\mu,\gamma^\nu]\right)_{\alpha_2\alpha_3}\psi_{\alpha_1\alpha_2\alpha_3}\,. \tag{36}$$

Here, C denotes the usual charge conjugation matrix that in the chiral representation, where $\gamma^5 = \mathrm{diag}(1,1,-1,-1)$, may be written as

$$C = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} = C^{-1}. \tag{37}$$

Making use of Eq. (23), one can write the hidden internal parity (29) in the form

$$P_{\text{hidden}} = i\Gamma_2^0\Gamma_3^0 = 1 \otimes \gamma^0 \otimes \gamma^0, \tag{38}$$

where in the chiral representation

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{39}$$

Then, the constraint (30) implies that

$$\begin{aligned} \psi_{\alpha_1 11} &= \psi_{\alpha_1 33}, \ \psi_{\alpha_1 22} = \psi_{\alpha_1 44}, \ \psi_{\alpha_1 12} = \psi_{\alpha_1 34}, \ \psi_{\alpha_1 21} = \psi_{\alpha_1 43}, \\ \psi_{\alpha_1 13} &= \psi_{\alpha_1 31}, \ \psi_{\alpha_1 24} = \psi_{\alpha_1 42}, \ \psi_{\alpha_1 14} = \psi_{\alpha_1 32}, \ \psi_{\alpha_1 41} = \psi_{\alpha_1 23}. \end{aligned} \tag{40}$$

Thus, the constraint (30) and our exclusion principle (requiring that $\psi_{\alpha_1\alpha_2\alpha_3} = -\psi_{\alpha_1\alpha_3\alpha_2}$) leads to the conclusion that from all components $\psi_{\alpha_1\alpha_2\alpha_3}$ only

$$\psi_{\alpha_1 12} = -\psi_{\alpha_1 21} = \psi_{\alpha_1 34} = -\psi_{\alpha_1 43} \equiv \psi_{\alpha_1}^{(3)} \tag{41}$$

and

$$\psi_{\alpha_1 14} = -\psi_{\alpha_1 41} = \psi_{\alpha_1 32} = -\psi_{\alpha_1 23} \tag{42}$$

may be nonzero. Then, after a simple calculation,

$$p_{\alpha_1} = 0 \quad , \quad s_{\alpha_1} = -4\,i\,\psi_{\alpha_1 12}, \tag{43}$$

$$a^\mu_{\alpha_1} = 0 \quad , \quad v^\mu_{\alpha_1} = \begin{cases} -4i\psi_{\alpha_1 14} & \text{for } \mu = 0 \\ 0 & \text{for } \mu = 1,2,3 \end{cases}, \tag{44}$$

$$t^{\mu\nu}_{\alpha_1} = 0. \tag{45}$$

But, the theory of relativity applied to the vector $v^\mu_{\alpha_1}$ given in Eq. (44) requires that $v^0_{\alpha_1} = 0$ since $v^\mu_{\alpha_1} = 0$ for $\mu = 1,2,3$. Hence, $\psi_{\alpha_1 14} = 0$. In this way, we can see that all components $\psi_{\alpha_1\alpha_2\alpha_3}$ must vanish except those in Eq. (41). So, in this case there are 4 equivalent nonzero components (carrying the index α_1), all equal (up to the sign) to the Dirac function $\psi^{(3)}_{\alpha_1}$. This reduces the Dirac-type equation (4) or (24) to the usual Dirac equation. Here, spin is evidently 1/2 and it is given by the visible spin, two hidden spins being summed up to zero.

Concluding, in each of the three allowed cases $N = 1,3,5$ there exists one and only one Dirac particle (for any given color and up/down weak flavor described by the standard model). So, it is natural to connect these three versions of the Dirac particle with the three experimental families of leptons and quarks. This happy existence of three and only three versions of the Dirac particle is a consequence of an interplay of the theory of relativity, the probability interpretation of quantum mechanics and the Pauli exclusion principle, all extended to the particle's hidden degrees of freedom.

As for the wave functions with $N = 1,3,5$ the number of equivalent nonzero components (carrying the visible bispinor index) is 1, 4, 24 , respectively, the following overall wave function comprising three sectors $N = 1,3,5$ (or three fundamental-fermion families) may be constructed:

$$\Psi = \frac{1}{\sqrt{29}} \begin{pmatrix} \psi^{(1)}_{\alpha_1} \\ \sqrt{4}\psi^{(3)}_{\alpha_1} \\ \sqrt{24}\psi^{(5)}_{\alpha_1} \end{pmatrix} = \hat{\rho} \begin{pmatrix} \psi^{(1)}_{\alpha_1} \\ \psi^{(3)}_{\alpha_1} \\ \psi^{(5)}_{\alpha_1} \end{pmatrix}. \tag{46}$$

Here, the sector-weighting (or family-weighting) matrix

$$\hat{\rho} = \frac{1}{\sqrt{29}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{4} & 0 \\ 0 & 0 & \sqrt{24} \end{pmatrix} \tag{47}$$

appears.

5. Mass spectral formula for charged leptons

The three-family wave function (46) implies the following form of the mass matrix for any triple of fundamental fermions ordered in one line in Eqs. (1) and (2):

$$\hat{M} = \hat{\rho}\,\hat{h}\,\hat{\rho}. \tag{48}$$

Here, $\hat{h}$ denotes a Higgs coupling strength matrix, while $\hat{\rho}$ is given as in Eq. (47). So, there are four different matrices (48) corresponding to triples of neutrinos, charged leptons, up quarks and down quarks, respectively.

Among all 12 fundamental-fermion masses, the masses m_e , m_μ , m_τ of charged leptons e^- , μ^- , τ^- are the best known. On the base of some numerical experience, we can propose the following phenomenological ansatz (in two options) for the matrix $\hat{h}$ in the case of charged leptons :

$$\hat{h} = \begin{pmatrix} h^{(1)} & 0 & 0 \\ 0 & h^{(3)} & 0 \\ 0 & 0 & h^{(5)} \end{pmatrix} \tag{49}$$

with

$$h^{(N)} = M_0 \left(N^2 - \frac{1 \pm \varepsilon^2}{N^2} \right) , \tag{50}$$

where $N = 1,3,5$. Here, $M_0 > 0$ and ε^2 denote two real constants independent of N. Then, the eigenvalues of the mass matrix (48) take the form

$$\begin{aligned} \mp m_e \equiv M^{(1)} &= \mp \frac{M_0}{29} \varepsilon^2 , \\ m_\mu \equiv M^{(3)} &= \frac{4}{9} \frac{M_0}{29} \left(80 \mp \varepsilon^2 \right) , \\ m_\tau \equiv M^{(5)} &= \frac{24}{25} \frac{M_0}{29} \left(624 \mp \varepsilon^2 \right) , \end{aligned} \tag{51}$$

since the Dirac masses are defined as nonnegative (*a priori*, the second option seems to be more attractive). From the system of three equations (51) we obtain in terms of experimental m_e and m_μ the predictions (in two options) for the mass m_τ,

$$m_\tau = \frac{6}{125} (351 m_\mu \pm 136\, m_e) = \begin{cases} 1783.47 \text{ MeV} \\ 1776.80 \text{ MeV} \end{cases} , \tag{52}$$

and for the parameters M_0 and ε^2,

$$M_0 = \frac{29}{320} (9 m_\mu \pm 4 m_e) = \begin{cases} 86.3629 \text{ MeV} \\ 85.9924 \text{ MeV} \end{cases} \tag{53}$$

and

$$\varepsilon^2 = \frac{320 m_e}{9 m_\mu \pm 4 m_e} = \begin{cases} 0.171590 \\ 0.172329 \end{cases} . \tag{54}$$

We can see an excellent agreement between the predictions (52) for m_τ and its experimental value

$$m_\tau = 1784.1^{+2.7}_{-3.6} \text{ MeV} \tag{55}$$

cited for several years by Particle Data Group (1992) or

$$m_\tau = (1776.9 \pm 0.4 \pm 0.3) \text{MeV} \ , \ m_\tau = (1776.3 \pm 2.4 \pm 1.4) \text{MeV} \tag{56}$$

reported recently by Beijing Electron-Positron Collider Group (Qi 1992) and ARGUS Collaboration (Albrecht 1992), respectively.

This strongly supports the phenomenological ansatz (50) operating with the number N of *"algebraic partons"* involved in the families $N = 1,3,5$ and described by the Dirac bispinor indices as these appear in the Clifford algebras (19) or, more conveniently, (22). The algebraic partons are agents of the idea of *algebraic compositness.* In the picture which emerges from our argument, any fundamental fermion with $N = 1,3,5$ is composed of one "visible" algebraic parton of spin 1/2 and $N-1 = 0,2,4$ "hidden" algebraic partons of spins 1/2, the latter forming relativistic scalars.

References

H. Albrecht *et al.* (ARGUS Collaboration), DESY report 92-086 (June 1992).

T. Banks, Y. Dothan and D. Horn, *Phys. Lett.* **B 117**, 413 (1982).

Cf. e.g. H.A. Bethe and E.E. Salpeter, in *Encyclopedia of Physics*, Vol. **35**, Springer, Berlin-Göttingen-Heidelberg, 1957.

E. Kähler, *Rendiconti di Matematica* **21**, 425 (1962); cf. also D. Ivanenko and L. Landau, *Z. Phys.* **48**, 341 (1928), published soon after the discovery of Dirac equation (P.A.M. Dirac, *Proc. Roy. Soc.* **117**, 610 (1928)). I am indebted to Andrzej Trautman for his calling my attention to this early paper.

W. Królikowski and J. Rzewuski, *Nuovo Cim.* **2**, 203 (1955); **4**, 974 (1956).

W. Królikowski, *Acta Phys. Pol.* **B 17**, 813 (1986).

W. Królikowski, *Acta Phys. Pol.* **B 18**, 111 (1987); *Nuovo Cim.* **100 A**, 701 (1988).

W. Królikowski, *Acta Phys. Pol.* **B 21**, 871 (1990); *Phys. Rev.* **D 45**, 3222 (1992).

For a formal quantization scheme beyond the conventional quantum field theory cf. W. Królikowski, *Acta Phys. Pol.* **B 22**, 613 (1991); [E., *Acta Phys. Pol.* **B 23**, 83 (1992)].

Particle Data Group, Review of Particle Properties, *Phys.Rev.* **D 45**, Part 2 (June 1992).

Nading Qi, *Bull. Am. Phys. Soc.* **37**, 957 (April 1992), caption only; information in: *CERN Courier* **32**, 13 (July 1992).

E.E. Salpeter, *Phys. Rev.* **87**, 328 (1952).

CLIFFORD ALGEBRA OF TWO-FORMS, CONFORMAL STRUCTURES, AND FIELD EQUATIONS

INGEMAR BENGTSSON
Institute of Theoretical Physics
S-41296 Göteborg,
Sweden

Abstract.
I review the equivalence between duality operators on two-forms and conformal structures in four dimensions, from a Clifford algebra point of view (due to Urban tke and Harnett). I also review an application, which leads to a set of "neighbo urs" of Einstein's equations. An attempt to formulate reality conditions for the "neighbours" is discussed.

There is a deep theory for how to solve the self-dual Yang-Mills equations

$$*F_{\alpha\beta i} = g^{-1/2} g_{\alpha\gamma} g_{\beta\delta} \tilde{F}_i^{\gamma\delta} = 1/2 g^{-1/2} g_{\alpha\gamma} g_{\beta\delta} \epsilon^{\gamma\delta\mu\nu} F_{\mu\nu i} = F_{\alpha\beta i} \quad (1)$$

where the duality operator is defined with respect to some fixed conformal structure, i.e. a metric up to a conformal factor (and some useful notation - the twiddle - has been introduced as well). Some time ago it occurred to Urbantke (1984) to pose this problem backwards: Given a field strength, with respect to which conformal structure is it self-dual? There is an elegant solution to this curious question, and an elegant proof - due to Urbantke and Harnett (1991) - based on the Clifford algebra of two-forms in four dimensional spaces. For the moment, let me state the result and then indicate how I want to use it. We need a triplet of two-forms, which is non-degenerate in the sense that it may serve as a basis in the three-dimensional space of self-dual two-forms. In particular, the index i ranges from one to three. Then

$$g_{\alpha\beta} = -2/3 \eta f_{ijk} F_{\alpha\gamma i} \tilde{F}^{\gamma\delta j} F_{\delta\beta k} \quad (2)$$

is Urbantke's formula. It gives the metric with respect to which $F_{\alpha\beta i}$ is automatically self-dual (the f_{ijk} are the structure constants of SO(3), and the conformal factor η is so far arbitrary).

Some work by Capovilla, Jacobson and Dell (1989) may be regarded as a more ambitious version of Urbantke's formula. (See also Plebanski 1977, Capovilla, Dell, Jacobson and Mason 1991.) We may regard $F_{\alpha\beta i}$ as the self-dual part of the Riemann tensor, considered - at the outset - as just an SO(3) field strength, with no connection to the metric. Then the question arises whether it is possible to formulate a set of differential equations, using the SO(3) connection (and the Levi-Civita tensor densities) alone, such that the above metric becomes Ricci flat. The answer turns out to be yes; more specifically, the answer is the field equations following from the action

$$S = 1/8 \int \eta (Tr\Omega^2 - 1/2(Tr\Omega)^2). \tag{3}$$

where η is a Lagrange multiplier and

$$\Omega_{ij} = \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta i} F_{\gamma\delta j} \tag{4}$$

The existence of this action is closely related to Ashtekar's (1987) formulation of the 3+1 version of Einstein's theory - in fact the CDJ action is a natural Lagrangian formulation of Ashtekar's variables. The action which leads to Einstein's equations including a cosmological constant is less elegant.

The next question is: What happens if we use the above building blocks to write an arbitrary action

$$S = \int \mathcal{L}(\eta; Tr\Omega, Tr\Omega^2, Tr\Omega^3), \tag{5}$$

where the only restriction on $\mathcal{L}$ is that it has density weight one? (Due to the characteristic equation for three-by-three matrices, there are only three independent traces.) The action is certainly generally covariant. Suppose that we solve the field equations and use Urbantke's formula to define a metric. Is that reasonable, and relevant for physics? What happens if we change the structure group from SO(3) to something else?

Now that you know where I am going, we return to prove Urbantke's formula. For any four-dimensional vector space **V**, the two-forms give a six-dimensional vector space **W**, with a natural metric

$$(\Sigma_1, \Sigma_2) = 1/2 \epsilon^{\alpha\beta\gamma\delta} \Sigma_{1\alpha\beta} \Sigma_{2\gamma\delta}. \tag{6}$$

There is a corresponding Clifford map to the space of endomorphisms on $\mathbf{V} \oplus \mathbf{V}^*$:

$$\gamma(\Sigma) = 2 \begin{pmatrix} 0 & \tilde{\Sigma}^{\alpha\beta} \\ \Sigma_{\alpha\beta} & 0 \end{pmatrix}; \quad \gamma(\Sigma)^2 = -(\Sigma, \Sigma)1. \tag{7}$$

We see that the original vector space **V** now becomes the space of Weyl spin ors for the Clifford algebra of two-forms.

Now we introduce a metric on **V**, so that we can define the duality operator *. **W** then splits into two orthogonal subspaces $\mathbf{W}^+$ (self-dual forms) and $\mathbf{W}^-$ (anti-self-dual forms). We choose Euclidean signature, so that ** = 1, and without loss of essential generality we choose the determinant of the metric to equal one. Then

$$\gamma(*\Sigma) = \gamma(Z)\gamma(\Sigma)\gamma(Z) \tag{8}$$

where

$$\gamma(Z) = \begin{pmatrix} 0 & g^{\alpha\beta} \\ g_{\alpha\beta} & 0 \end{pmatrix}. \tag{9}$$

Using a well-known property of six-dimensional γ-matrices, and Swedish indices in **W**, we can find a totally anti-symmetric tensor $Z^{\ddot{u}\ddot{a}\ddot{o}}$ such that

$$g_{\alpha\beta} = Z^{\ddot{u}\ddot{a}\ddot{o}} \Sigma_{\alpha\gamma\ddot{u}} \tilde{\Sigma}^{\gamma\delta}_{\ddot{a}} \Sigma_{\delta\beta\ddot{o}}. \tag{10}$$

This determines Z uniquely, and we observe that

$$*\Sigma = Z\Sigma Z \quad \Rightarrow \quad Z\Sigma = \Sigma Z \quad (\Sigma \in W^+)$$
$$Z\Sigma = -\Sigma Z \quad (\Sigma \in W^-). \tag{11}$$

We need a little bit more information about Z.

To prove the result we are after, we will commit the atrocity of choosing a basis in **W**. First we choose an ON-basis in **V**, and then we set

$$M_i = e_0 \wedge e_i \qquad N_i = 1/2 f_{ijk} e_j \wedge e_k;$$
$$X_i = M_i - N_i \qquad Y_i = M_i + N_i. \tag{12}$$

Clearly, the X's (Y's) form a basis for $\mathbf{W}^-$ ($\mathbf{W}^+$), and

$$(X_i, X_j) = -\delta_{ij} \quad (Y_i, Y_j) = \delta_{ij}. \tag{13}$$

Looking back on eq. (11), we see that we can set

$$Z = Y_1 Y_2 Y_3 \tag{14}$$

- that is to say that Z is the unit volume element of $\mathbf{W}^+$. But, since $\mathbf{W}^+$ is three-dimensional, this is all we need. In terms of an arbitrary basis $\Sigma_{\alpha\beta i}$ on $\mathbf{W}^+$, eq. (10) now becomes

$$g_{\alpha\beta} \propto \epsilon^{ijk} \Sigma_{\alpha\gamma i} \tilde{\Sigma}^{\gamma\delta}_j \Sigma_{\delta\beta k}. \tag{15}$$

This is Urbantke's formula.

When the metric on **V** has neutral signature, the metric on $\mathbf{W}^+$ bec omes indefinite, but the discussion is similar, while it becomes slightly more subtle if the metric on **V** is Lorentzian.

With this understanding of eq. (2), let us return to the action (5). Our main result so far (Capovilla 1992, Bengtsson and Peldán 1992, Bengtsson 1991, Peldán 1992) is that this action admits a 3+1 decomposition, and that the resulting formalism is a natural generalization of "Ashtekar's variables" for gravity. As is well known, the constraint algebra of general relativity actually singles out the space-time metric by means of its structure functions. For the SO(3) case, it turns out that - up to some ambiguity concerning the conformal factor - the "Hamiltonian" metric is precisely the same as Urbantke's. We refer to the models in this class as "neighbours of Einstein's equations", since they all have the same number of degrees of freedom. I will not discuss the case of arbitrary structure groups here.

There are several holes that have to be filled before we can claim that we have really been able to generalize Einstein's equations in an unsuspected way. For the case of Euclidean signatures, we have to show that the field equations derived from the action (5) ensure that the metric (2) is positive definite, rather than neutral. This can be done in specific cases. As an example, consider the action

$$S = 1/8 \int \eta(Tr\Omega^2 + \alpha(Tr\Omega)^2). \tag{16}$$

As is clear from the preceding discussion, there must be some property of the field equations that ensure that the matrix Ω_{ij} has definite signature. To see this, choose a gauge such that the matrix becomes diagonal. Then it is a straightforward exercise to show that the constraint that results when varying the action with respect to η implies that the matrix Ω_{ij} has definite signature if and only if

$$\alpha \geq -1/2. \tag{17}$$

In particular, $\alpha = -1/2$, which leads to Einstein's equations, is all r ight. (I owe this observation to Ted Jacobson.) Although it is not quite clear what a general statement is, it is clear that, in general, the requirement that the metric should have Euclidean signature will lead to some restrictions on the allowed actions.

A similar discussion can be given for neutral signature, provided that the de finition of the traces in the action is appropriately changed.

Our understanding of the Lorentzian case is in much worse shape. It is necess ary to show that propagation is causal with respect to the metric that we have defined. Moreover (since self-dual two-forms are necessarily

complex in this case) the variables in the action are complex valued, and one must show how to impose restrictions that imply that the metric is real in any solution. I believe that the latter problem is the crucial one, and that the former property somehow follows from the latter. It will not come as a surprise if I state that the conformal structure is real if and only if

$$(F_i, \bar{F}_j) = 0, \tag{18}$$

where the bar denotes complex conjugation. However, this condition is not very helpful in itself. It is not difficult to write down solutions with real Lorentzi an metrics - a small zoo of real solutions is already known, for various "neighb ours" (generalizations of Schwarzschild, de Sitter, Kasner, ...). On the other hand, there will always be some solutions for which the metric is not real - also in the Einstein case. The correct formulation of the problem is presumably to require that the space of real solutions should be "reasonably" big - of the same order as the space of solutions of Einstein's equations, say. It seems natural to switch to the Hamiltonian form of the equations, and to address the problem from an initial data point of view. Unfortunately, as soon as this is done, one discovers that the reality properties of the metric can be discussed easily (Ashtekar 1987) if and only if we deal with the Einstein case - for the more general models contained in the action (5), the calculations tend to b.

Which is where the matter stands at the moment. It is perhaps appropriate to add that we have investigated, in a preliminary way, whether the "neighbours" can be used to explain any property of the real world. The preliminary answer was not very encouraging, but perhaps the final verdict is not in yet. Certainly the more difficult case of arbitrary structure groups (Peldán 1992), which was not discussed here, should be carefully studied in this regard.

Acknowledgements: I thank Helmuth Urbantke and Ted Jacobson for explaining things, and the organizers for a nice stay in the castle.

References

A. Ashtekar (1987): New Hamiltonian Formulation of General Relativity, Phys. Rev. **D36** 1587.

I. Bengtsson (1991): Self-Duality and the Metric in a Family of Neighbours of Einstein's Equations, J. Math. Phys. **32** 3158.

I. Bengtsson and P. Peldán (1992): Another "Cosmological" Constant, Int. J.Mod. Phys. **A7** 1287.

R. Capovilla (1992): Generally Covariant Gauge Theories, Nucl. Phys. **B373** 233.

R. Capovilla, J. Dell and T. Jacobson (1989): General Relativity without the Metric, Phys. Rev. Lett. **63** 2325.

R. Capovilla, J. Dell, T. Jacobson and L. Mason (1991): Self-dual 2-forms and Gravity, Class. Quant. Grav. **8** 41.

G. Harnett (1991): The Bivector Clifford Algebra, unpublished manuscript.

P. Peldán (1992): Ashtekar's Variables for Arbitrary Gauge Group, Phys. Rev. **D**, to appear.
J. Plebanski (1977): On the Separation of Einsteinian Substructures, J. Math. Phys. **18** 2511.
H. Urbantke (1984): On Integrability Properties of SU(2) Yang-Mills Fields. I. Infinitesimal Part, J. Math. Phys. **25** 2321.

DIRAC FORM OF MAXWELL EQUATION $\mathbb{Z}_n$-GRADED ALGEBRAS

JAIME KELLER
División de Ciencias Básicas, F.Q., and Quantum Theory of Matter Project
Facultad de Estudios Superiores-Cuautitlán, Universidad Nacional Autónoma de México
Apartado Postal 70-528, 04510 México, D. F., Keller@UNAMVM1

(Received: March 31, 1993)

Abstract. A Dirac form of Maxwell's equation is derived. The interrelationship between tensors and spinors is discussed in terms of $\mathbb{Z}_2$-graded algebras. A generalization to $\mathbb{Z}_n$-graded algebras is given.

Key words: Clifford algebra – twistor – Maxwell equation – $\mathbb{Z}_n$-graded algebra

1. Introduction

This paper consists of two separate parts as reflected in the title.

Elié Cartan and then Marcel Riesz (in 1946) defined spinors as ideals in Clifford algebras[1]. Twistors as ideals in Clifford algebra have been considered by Chevalley in 1954, by Atiyah, Bott and Shapiro in 1964, by Crumeyrolle since 1974 and also by Abłamowicz, Oziewicz and Rzewuski (1982). We use their results in the derivation of a Dirac form of the Maxwell equation.

The Cartan and Riesz definition of spinors suggests that tensors are the more fundamental. The tensor product of spinors (or twistors) is mapped to tensors by the Cartan map. These simple facts are best expressed in terms of $\mathbb{Z}_2$-graded ($\equiv$super) algebras. The $\mathbb{Z}_2$-graded *Lie* algebras were introduced by Volkov and Akulov in 1973 and independently by Wess and Zumino in 1974. Here we generalize this construction to $\mathbb{Z}_n$-graded algebras. We are inspired by the lecture by Kerner on $\mathbb{Z}_3$-graded algebras in these proceedings.

One can consider $\mathbb{Z}_n$-graded algebras in which the subalgebra of zero grade consists of the direct sum of n Clifford algebras generated by the set of n different pseudo-riemannian spaces. In the case of $\mathbb{Z}_2$-grading, the first space can be chosen as Minkowski space-time and the second space as an internal space. A similar theory (for groups rather than algebras) was developed over the last ten years by Professor Jan Rzewuski (see his

[1] Spinors and twistors as ideals *must be* inhomogeneous combinations of skewsymmetric tensors of the even and odd degrees.

contribution in this volume).

2. Twistors

Let $\{M, g\}$ be the real pseudo-riemannian space-time with signature $(+---)$, and let Λ^c be a module of differential one-forms over an algebra of complex valued functions on M. The Witt decomposition (1937) of the differential one-forms is $\Lambda^c = F_1 \oplus F_2$ where the F_i are maximal (two-dimensional) isotropic $\mathbb{C}$-spaces, $g|F_i \equiv 0$. Denote by f_i the two-forms $f_i \in \wedge^2 F$ $(f_i^2 = 0)$. The Clifford $\mathbb{C}$-algebra of $\{M, g\}$ is denoted by $\mathcal{C}l^c_{1,3}$.

DEFINITION 1 (Twistor). *The left principal ideals of* $\mathcal{C}l^c_{1,3}$,

$$\mathcal{T}(f_i) \equiv \mathcal{C}l^c_{1,3} f_i \quad \triangleleft \quad \mathcal{C}l^c_{1,3},$$

generated by the two-forms $f_i \in \wedge^2 F_i \subset \mathcal{C}l^c_{1,3}$, *are called the* twistor *spaces.*

A twistor space $\mathcal{T}$ is a four dimensional linear $\mathbb{C}$-space *with* a hermitian correlation ($\equiv$ Dirac conjugation) of signature $(++--)$ (Crumeyrolle 1974 and 1990).

Let Lin $\mathcal{T}$ denote the algebra of linear endomorphisms of the twistor $\mathcal{C}l$-module and let γ be a Dirac representation ($\equiv$ algebra homomorphism) of the Clifford algebra $\mathcal{C}l_{1,3}$ in the twistor space,

$$\gamma \; : \; \mathcal{C}l^c_{1,3} \longrightarrow \text{Lin}\,\mathcal{T}. \tag{1}$$

This is preciesly the meaning of the Dirac γ-matrices (Oziewicz 1986). For the coordinate frames

$$\gamma_\mu \equiv \gamma(\partial_\mu), \;\; \gamma^\mu \equiv \gamma(dx^\mu) \quad \in \quad \text{Lin}\,\mathcal{T}.$$

Let $\{e^\mu\}$ be an g-orthonormal co-frame in Minkowski space-time.

The de Witt co-frame is the Sachs co-frame $e^{A\dot{B}} \equiv \sigma^{A\dot{B}}_\mu e^\mu$ such that $F_1 \equiv \text{span}\left\{e^{1\dot{1}}, e^{1\dot{2}}\right\}$, $F_2 \equiv \text{span}\left\{e^{2\dot{1}}, e^{2\dot{2}}\right\}$.

Let $\{\chi_a\}$ be a basis in a twistor left $\mathcal{C}l^c$-module $\mathcal{T} \equiv {}_{\mathcal{C}l^c}\mathcal{M}_{\mathbb{R}}$, and let $\{\chi^a\}$ be dual basis in the dual twistor right $\mathcal{C}l^c$-module $\mathcal{T}^* \equiv_{\mathbb{R}} \mathcal{M}_{\mathcal{C}l^c}$,

$$\chi^b \chi_a = \delta^b_a.$$

Following Abłamowicz, Oziewicz and Rzewuski (1982) we choose the bivector f_2 and a basis in the twistor space $\{\chi_a\} \in \mathcal{T}(f_2)$ in terms of the co-frame $\{e^\mu\}$,

$$\begin{aligned} \chi_1 &\equiv (e^0 - e^3)(ie^1 - e^2) \equiv f_2, \\ \chi_2 &\equiv (1 - e^0 e^3)(ie^1 e^2 - 1), \\ \chi_3 &\equiv (1 - e^0 e^3)(ie^1 - e^2), \\ \chi_4 &\equiv (e^0 - e^3)(ie^1 e^2 - 1). \end{aligned} \tag{2}$$

The first two and last two spinors *span* the two-dimensional $\mathbb{C}$-spaces of the Weyl spinors $\mathcal{W}$.

Let

$$\vartheta^b_a \equiv \chi_a \otimes \chi^b \;\in\; \text{Lin}\,\mathcal{T},$$

be the spinorial basis in the matrix algebra Lin $\mathcal{T}$.

A Dirac representation (1), $\gamma^\mu \equiv \gamma(e^\mu) \in \text{Lin}\,\mathcal{T}$, in terms of the spinor basis is (Keller and Rodríguez 1992)

$$\begin{aligned} \gamma(e^0) &= \vartheta^1_3 + \vartheta^2_4 + \vartheta^3_1 + \vartheta^4_2, & \gamma(e_{1\dot{1}}) &= \sqrt{2}\,(\vartheta^3_1 + \vartheta^2_4), \\ \gamma(e^1) &= i(-\vartheta^1_4 + \vartheta^2_3 + \vartheta^3_2 - \vartheta^4_1), & \gamma(e_{1\dot{2}}) &= \sqrt{2}\,(\vartheta^4_2 + \vartheta^1_3), \\ \gamma(e^2) &= \vartheta^1_4 + \vartheta^2_3 - \vartheta^3_2 - \vartheta^4_1, & \gamma(e_{2\dot{1}}) &= \sqrt{2}\,(-\vartheta^4_1 + \vartheta^2_3), \\ \gamma(e^3) &= \vartheta^1_3 - \vartheta^2_4 - \vartheta^3_1 + \vartheta^4_2, & \gamma(e_{2\dot{2}}) &= \sqrt{2}i(-\vartheta^3_2 + \vartheta^1_4). \end{aligned} \tag{3}$$

For every representation γ and one-forms α and $\beta \in \mathcal{C}l$ we have,

$$\gamma(\alpha \wedge \beta) = \frac{1}{2}[\gamma(\alpha), \gamma(\beta)]. \tag{4}$$

From this we calculate

$$\begin{aligned} \gamma(e^0 \wedge e^1) &= i(-\vartheta^1_2 + \vartheta^2_1 + \vartheta^3_4 - \vartheta^4_3), \\ \gamma(e^0 \wedge e^2) &= \vartheta^1_2 + \vartheta^2_1 - \vartheta^3_4 - \vartheta^4_3, \\ \gamma(e^0 \wedge e^3) &= \vartheta^1_1 - \vartheta^2_2 - \vartheta^3_3 + \vartheta^4_4, \\ \gamma(e^1 \wedge e^2) &= i(-\vartheta^1_1 + \vartheta^2_2 - \vartheta^3_3 + \vartheta^4_4), \\ \gamma(e^3 \wedge e^1) &= i(-\vartheta^1_2 - \vartheta^2_1 - \vartheta^3_4 - \vartheta^4_3), \\ \gamma(e^2 \wedge e^3) &= -\vartheta^1_2 + \vartheta^2_1 - \vartheta^3_4 + \vartheta^4_3. \end{aligned} \tag{5}$$

3. Dirac-Kähler operator

Let g denote a pseudo-riemannian structure. We have *two* mutually dual Clifford algebras: the Clifford algebra of the multivector fields, and the Clifford algebra of the differential forms $\mathcal{C}l \equiv \mathcal{C}l_g$. Let γ be the unique left adjoint representation[2] (Oziewicz 1986),

$$\gamma \; : \; \mathcal{C}l \longrightarrow \text{Lin}\,\mathcal{C}l. \tag{6}$$

This means that *the Clifford product* of two arbitrary differential forms α, $\beta \in \mathcal{C}l$ is not denoted by juxtaposition, but by

$$\gamma(\alpha, \beta) \equiv (\gamma\alpha)\beta \equiv \gamma_\alpha \beta \;\in\; \mathcal{C}l.$$

If α and β are differential 1-forms, then

$$\gamma_\alpha \beta \equiv \alpha \wedge \beta + g(\alpha, \beta).$$

[2] Note that (1) is a irreducible summand of (6): $\mathcal{C}l = \mathcal{T} \oplus \ldots$

For arbitrary differential forms α and β, and multivectors X and Y, we use "e" for *exterior* (Grassmann) product and "i" for *interior* product

$$e_\alpha\beta \equiv \alpha\wedge\beta, \qquad (i_X\alpha)Y \equiv \alpha(X\wedge Y).$$

It follows that if α is a differential 1-form (covector) then

$$\gamma_\alpha \equiv e_\alpha + i_{g\alpha} \;\in\; \mathrm{Lin}\,\mathcal{C}l \tag{7}$$

(cf. Chevalley 1954, pp. 38-42). Let $\{X_a\}$ be *a Cartan frame* of vector fields and $\{\omega^a\}$ a dual *co-frame* of the differential one-forms,

$$\omega^a X_b = \delta^a_b.$$

We use the notation

$$\gamma_a \equiv \gamma(X_a), \;\; \gamma^b \equiv \gamma(\omega^b).$$

When the representation γ (6) is restricted to one-vector fields we get, in these dual frames, the *operator valued differential form*

$$\gamma = \gamma(X_a)\otimes\omega^a = \gamma_\mu\otimes dx^\mu.$$

Similarly when we restrict γ to differential one-forms we get the operator valued vector field

$$\gamma = \gamma(\omega^a)\otimes X_a = \gamma^\mu\otimes\partial_\mu.$$

This Lin$\mathcal{C}l$-*valued* differential form was introduced by Fock and Ivanenko in 1929 as "a line element". For differential *one*-forms α and β we have

$$\gamma_\alpha\circ\gamma_\beta + \gamma_\beta\circ\gamma_\alpha = 2\,g(\alpha,\beta)\cdot \mathrm{id}_{\mathcal{C}l} \;\in\; \mathrm{Lin}\mathcal{C}l.$$

If we define $(\gamma\otimes\gamma)(\alpha\otimes\beta) \equiv \gamma_\alpha\circ\gamma_\beta$, then on symmetric second degree tensor fields, $\gamma\otimes\gamma = g\cdot\mathrm{id}_{\mathcal{C}l}$.

DEFINITION 2. The Dirac-Kähler operator *on a Clifford algebra of the differential forms is* defined *by*[3]

$$\mathcal{D} \equiv \gamma_{\omega^a}\circ\nabla_{X_a}: \;\; \mathcal{C}l_g \longrightarrow \mathcal{C}l_g.$$

This operator is both frame and coordinate *independent* because the definition uses dual frames. The Dirac-Kähler operator depends on the scalar

[3] The early versions of the Dirac-Kähler operator was considered by Darwin (1928), Landau and Ivanenko (1928) and by Marcel Riesz (1958). The most adequate references are (Kähler 1962 and Hestenes 1966). David Hestenes denotes the Dirac-Kähler operator by $\Box$ and calls it the *gradient*.

product g and on the connection ∇. From the definition of the Clifford product (7) it is clear that the Dirac-Kähler operator consists of the sum of two frame-independent parts

$$\mathcal{D} = d_\nabla + \delta_{(\nabla,g)},$$

where

$$\begin{aligned} d_\nabla &\equiv e_{\omega^a}\circ\nabla_{X_a}, \\ \delta_{(\nabla,g)} &\equiv i_{g\omega^a}\circ\nabla_{X_a}. \end{aligned} \tag{8}$$

One can show that

$$d_\nabla = d + T,$$

where d is the connection independent Cartan exterior differential, ($d^2\equiv 0$), and T is the *torsion* of the connection. For the co-differential ($\equiv$*divergence*) $\delta_{(\nabla,g)}$ in (8), we also separate the connection independent part $\delta \equiv \delta_g$, ($\delta^2\equiv 0$) (cf. Tucker 1986, p. 180),

$$\delta_{(\nabla,g)} = \delta + \text{co-Torsion} + \text{“}\nabla g - \text{dependent term”}.$$

DEFINITION 3. *The* Dirac operator *is the restriction of the Dirac-Kähler operator to the ideal (spinor or twistor fields) in the Clifford algebra of the differential forms* (Tucker 1986, Secs 6 and 8).

4. Dirac form of the Maxwell equation

The Maxwell equations $dF = 0$ and $\delta F = j$ were presented by Marcel Riesz (1958) in the Dirac-Kähler form

$$\mathcal{D}F = j \quad\in\quad \mathcal{C}l_g,$$

where $\mathcal{D}\equiv d+\delta$ is the Dirac-Kähler operator for the torsion-less riemannian connection, $\nabla g = 0$,

$$\mathcal{D}F \equiv \gamma_{\omega^a}(\nabla_{X_a}F).$$

Consider the Dirac representation (1) of this equation in the twistor $\mathcal{C}l$-module,

$$(\gamma_{\omega^a})\circ\gamma(\nabla_{X_a}F) = \gamma_j \quad\in\quad \mathrm{Lin}\mathcal{T}. \tag{9}$$

Here $\nabla_{X_a}F$ is a two-form:

$$\begin{aligned} \nabla_{X_a}F &\equiv \frac{1}{2}\{(\nabla_{X_a}F)(X_b\wedge X_c)\}\,\omega^b\wedge\omega^c \\ &\equiv \frac{1}{2}F_{bc,a}\,\omega^b\wedge\omega^c. \end{aligned}$$

Therefore equations (4) and (9) gives

$$\frac{1}{4}F_{bc,a}\gamma(\omega^a)\circ[\gamma(\omega^b),\gamma(\omega^c)] = \gamma(j) \quad \in \quad \mathrm{Lin}\,\mathcal{T}. \tag{10}$$

If $t \in \mathcal{T}$, then $\gamma_j t \in \mathcal{T}$. Equation (10) evaluated on twistor t is said to be (t-dependent) *Dirac form* of a Maxwell equation. In this way every twistor $t \in \mathcal{T}$ linearly maps differential forms into subspaces in twistor $\mathbb{C}$-space $\mathcal{T}$,

$$\gamma_\bullet t \;:\; \Lambda \ni \alpha \longmapsto \gamma_\alpha t \in \mathcal{T}.$$

Let $j \equiv j_\mu e^\mu \in \Lambda^1$. In basis (2) and using (3) we get

$$\begin{aligned} j &\longmapsto \gamma_j\chi_1 = (j_0+j_3)\chi_3 - (ij_1 - j_2)\chi_4, \\ j &\longmapsto \gamma_j\chi_2 = (ij_1+j_2)\chi_3 + (j_0 - j_3)\chi_4, \\ j &\longmapsto \gamma_j\chi_3 = (j_0-j_3)\chi_1 + (ij_1 - j_2)\chi_2, \\ j &\longmapsto \gamma_j\chi_4 = -(ij_1+j_2)\chi_1 + (j_0 + j_3)\chi_2. \end{aligned} \tag{11}$$

This means that every basis twistor $\chi_a \in \mathcal{T}$ maps *non* injectively co-vectors into a Weyl $\mathbb{C}$-spinors in $\mathcal{W}$. Moreover

$$\dim_{\mathbb{C}}\{\mathrm{image}(\gamma_\bullet t|\Lambda^1)\} \leq 3.$$

Let $F \equiv \frac{1}{2}F_{\mu\nu}e^\mu \wedge e^\nu \in Cl$ and $F_k \equiv F_{0k} + i\varepsilon_{ijk}F_{ij}$. We get

$$\begin{aligned} F &\longmapsto \gamma_F\chi_1 = \bar{F}_3\chi_1 - (\bar{F}_2 + i\bar{F}_1)\chi_2, \\ F &\longmapsto \gamma_F\chi_2 = (i\bar{F}_1 + \bar{F}_2)\chi_1 - \bar{F}_3\chi_2, \\ F &\longmapsto \gamma_F\chi_3 = -F_3\chi_3 + (iF_1 - F_2)\chi_4, \\ F &\longmapsto \gamma_F\chi_4 = -(iF_1+F_2)\chi_3 + F_3\chi_4. \end{aligned} \tag{12}$$

The Dirac representation of the Maxwell equation (10) was considered by Oppenheimer (1931), Ohmura (1956), Moses (1958) and by Keller and Rodríguez-Romo (1991).

5. $\mathbb{Z}_n$-graded algebras

DEFINITION 4. *An $\mathbb{R}$-algebra $\mathcal{K}$ is said to be $\mathbb{Z}_n$-graded if*

$$\mathcal{K} = \oplus_{i\in\mathbb{Z}_n}\mathcal{K}_i,$$

and if the algebra multiplication is a zero grade map

$$\mathcal{K}_i \otimes \mathcal{K}_j \longrightarrow \mathcal{K}_{i+j}.$$

We are abreviating this last property by $\mathcal{K}_i\mathcal{K}_j \subset \mathcal{K}_{i+j}$.

Next we define the *matrix* $\mathbb{Z}_n$-graded algebra $\mathcal{K}$. Let $\{\mathcal{A}_k,\ k = 1,\ldots,n\}$ be a sequence of associative $\mathbb{R}$-algebras with units. Let $\mathcal{K}$ be a direct sum of the $(\mathcal{A}_k - \mathcal{A}_l)$-bimodules

$$\mathcal{K} \equiv \oplus\,({}_k\mathcal{M}_l) \equiv \oplus\ ({}_{\mathcal{A}_k}\mathcal{M}_{\mathcal{A}_l}).$$

The $\mathbb{Z}_n$-grading in $\mathcal{K}$ can be introduced in different ways. Every bimodule ${}_k\mathcal{M}_l$ is $\mathbb{Z}_n$-homogeneous. The simplest possibility is to put

$$\deg\,({}_k\mathcal{M}_l) = l - k \mod n.$$

The multiplication in $\mathcal{K}$ is *defined* by means of the Cartan map

$$\bigoplus\nolimits_k ({}_i\mathcal{M}_k \otimes {}_k\mathcal{M}_j) \longrightarrow {}_i\mathcal{M}_j.$$

The Cartan map is equivalent to matrix multiplication if we display $\mathcal{K}$ in the matrix form

$$\mathcal{K} \equiv \begin{pmatrix} {}_1\mathcal{M}_1 & {}_1\mathcal{M}_2 & \ldots & {}_1\mathcal{M}_n \\ {}_2\mathcal{M}_1 & {}_2\mathcal{M}_2 & \ldots & {}_2\mathcal{M}_n \\ \vdots & \vdots & \vdots & \vdots \\ {}_n\mathcal{M}_1 & {}_n\mathcal{M}_2 & \ldots & {}_n\mathcal{M}_n \end{pmatrix}.$$

We are assuming that all $(\mathcal{A}_k - \mathcal{A}_l)$-bimodules $\{{}_k\mathcal{M}_l\}$ are the tensor product of left and right modules,

$${}_k\mathcal{M}_l \equiv ({}_k\mathcal{M}) \otimes (\mathcal{M}_l),$$

and that the left and right $\mathcal{A}_i$-modules are mutually $\mathbb{R}$ or $\mathbb{C}$-dual,

$$\mathrm{ev}:\ \mathcal{M}_i \otimes {}_i\mathcal{M} \longrightarrow \mathbb{R} \quad \text{or} \quad \mathbb{C}.$$

If

$$\dim(\mathcal{A}_i) = (\dim\,{}_i\mathcal{M})^2,$$

then Cartan map is a linear (or an algebra) isomorphism and we can identify ${}_k\mathcal{M}_k \equiv \mathcal{A}_k$. In this case $(\mathcal{K}_k)^m = \mathcal{K}_{mk}$. It follows that if k does not divide n, the algebra $\mathcal{K}$ is generated by every subspace $\mathcal{K}_k$,

$$\mathcal{K} = \mathrm{gen}\,\{\mathcal{K}_k\}. \quad \text{In the simplest case} \quad \mathcal{K} = \mathrm{gen}\{\oplus_i({}_i\mathcal{M}_{i+1}) \oplus ({}_n\mathcal{M}_1)\}.$$

The general $\mathbb{Z}_2$-graded algebra is

$$\mathcal{K} \equiv \begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{pmatrix} \oplus \begin{pmatrix} 0 & {}_1\mathcal{M}_2 \\ {}_2\mathcal{M}_1 & 0 \end{pmatrix}.$$

We can now let: $\mathcal{A}_1$ be the *complexified* Dirac-Clifford algebra of Minkowski space-time (or real Clifford algebra of the de Sitter space) $Cl^c_{3,1} = Cl_5$ and $\mathcal{A}_2 \equiv \mathbb{R}$. This example of $\mathbb{Z}_2$-graded algebra is called *a geometric superalgebra* in (Keller and Rodríguez 1992).

Acknowledgements

The technical assistance of Mrs. Irma Aragón is greatly appreciated.

References

Abłamowicz Rafal, Zbigniew Oziewicz and Jan Rzewuski: 1982, 'Clifford algebra approach to twistors', *J. Math. Phys.* **23**, 231-242
Chevalley Claude: 1954, *The Algebraic Theory of Spinors*, Columbia University Press, New York, 38-42
Crumeyrolle, A.: 1990, *Orthogonal and Symplectic Clifford Algebras. Spinor Structures*, Kluwer A.P.: Dordrecht
Fock V. and D. Ivanenko: 1929, *C.R. Acad. Sci. (Paris)* **188**, 1470
Hestenes David: 1966, *Space-Time Algebra*, Gordon and Breach, 26 and 64
Keller Jaime: 1991, 'Spinors and mutivectors as a unified tool for spacetime geometry and for elementary particle physics', *International Journal Theoretical Physics* **30**, (2) 137-184
Keller Jaime and Rodríguez–Romo Suemi: 1991, *International Journal of Theoretical Physics* **30**, 185-196
Keller Jaime and Adán Rodríguez: 1992, 'Geometric superalgebra and the Dirac equation', *Journal of Math. Phys.* **33** **(1)**, 161
Kerner Richard: 1992, '$\mathbb{Z}_3$-graded algebras', *Journal of Math. Phys.* **33** (1), 403
Kerner Richard: 1993, '$\mathbb{Z}_3$-graded structures', *These Proceedings*, p.
Moses H. E.: 1958, *Nuovo Cimento* **8** **(Suppl. 1)**, 18
Ohmura T.: 1956, *Progress of Theoretical Physics* **16**, 604
Oppenheimer R.: 1931, *Physical Review* **38**, 725
Oziewicz Zbigniew: 1986, in Chisholm J.S.R. and A.K. Common, ed(s)., *Clifford Algebras and Their Applications in Mathematical Physics*, From Grassman to Clifford, Reidel: Dordrecht, NATO ASI C-183, pp. 245-255
Riesz Marcel: 1946, 'Sur certaines notions fondamentales en théorie quantique relativiste', *Dixième Congrès des Mathematiciens Scandinaves, Copenhaque* , 123-143
Riesz Marcel: 1958, *Clifford Numbers and Spinors*, University of Maryland, The Institute for Fluid Dynamics, Lecture Series #38
Rzewuski Jan: 1993, 'Structure of matrix manifolds and a particle model', *These Proceedings* ,
Tucker R.W.: 1986, in Chisholm J.S.R. and A.K. Common, ed(s)., *Clifford Algebras and Their Applications in Mathematical Physics*, A Clifford calculus for physical field theories, Reidel: Dordrecht, NATO ASI C-183, pp. 177-199
Volkov D.V. and V.P. Akulov: 1973, *Physics Letters* **46B**, 109
Wess Julius and Bruno Zumino: 1974, *Nuclear Physics* **B70**, 39

TRAVELLING WAVES WITHIN THE CLIFFORD ALGEBRA

BERNARD JANCEWICZ
Institute of Theoretical Physics, University of Wrocław,
pl. Maksa Borna 9, PL-50-204 Wrocław, Poland

Abstract. It is shown that travelling electromagnetic waves (radiation fields) cannot exist in a conducting medium. With the aid of algebra $C\ell_3$, however, an arbitrary plane wave can be decomposed into a sum of two electromagnetic fields travelling in opposite directions. These fields separately do not satisfy the Maxwell equations.

1. Introduction

David Hestenes in his works (1966, 1971, 1974a, 1974b, 1986) has demonstrated the importance of Clifford algebras in various branches of classical physics. Among others electrodynamics obtains a beneficial synthesis when expressed in terms of $C\ell_{1,3}$, the Clifford algebra of the Minkowski space (Jouvet and Schidlof 1932, Mercier 1935, Riesz 1958) or $C\ell_3$, the Clifford algebra of E^3 (Hestenes 1966, Jancewicz 1988).

As was shown in Hestenes 1966, p. 29, when discussing the Maxwell equations within $C\ell_3$, it is useful to form the Clifford number (we propose the term *cliffor*) $\mathbf{E}+\mathbf{B}e_{123}$, but this is practical only in empty space. In the presence of a material medium (and when a system of units is used in which $\mathbf{E}$ and $\mathbf{B}$ have different physical dimensions) one has to take into account the electric permittivity ε and the magnetic permeability μ. The best possibility is $f=\sqrt{\varepsilon}\mathbf{E}+\frac{1}{\sqrt{\mu}}\mathbf{B}e_{123}$ which has the dimension $\sqrt{J/m^3}$ in SI system of units, that is square root of the energy density. In this respect it resembles the wave function of quantum mechanics which has the dimension of square root of the probability density. We call this combination an *electromagnetic cliffor.* Having denoted $\mathbf{e}=\sqrt{\varepsilon}\mathbf{E}$, $\star\mathbf{b}=\frac{1}{\sqrt{\mu}}\mathbf{B}e_{123}$ we we write it explicitly as a sum of vector and bivector:

$$f=\mathbf{e}+(\star\mathbf{b}).$$

With the notation $f^\sim$ for the *reversion* of cliffors one obtains

$$\frac{1}{2}ff^\sim = w+\frac{1}{u}\mathbf{S}$$

where

$$w = \frac{1}{2}(\varepsilon \mathbf{E}^2 + \frac{1}{\mu}\mathbf{B}^2) = \frac{1}{2}(|\mathbf{e}|^2 + |\star\mathbf{b}|^2) = \frac{1}{2}|f|^2$$

is the energy density of the electromagnetic field, $\mathbf{S} = \mathbf{E}\times\mathbf{H} = u(\star\mathbf{b})\cdot\mathbf{e}$ is the Poynting vector and $u = \frac{1}{\sqrt{\varepsilon\mu}}$ is the phase velocity of light in the medium.

Having introduced $\mathcal{D} = \nabla + \frac{1}{u}\frac{\partial}{\partial t}$, the *Fueter operator* (Lounesto 1984) and $J = \frac{\rho}{\sqrt{\varepsilon}} + \sqrt{\mu}\mathbf{j}$ (here ρ is the charge density and $\mathbf{j}$ – the current density) one may write the Maxwell equations in the following single formula

$$\mathcal{D}f = J \tag{1}$$

if the medium is uniform in space and constant in time. We call Eq. (1) the *synthetic Maxwell equation.* When $J = 0$ we call f a *free electromagnetic field*; then

$$\mathcal{D}f = 0. \tag{2}$$

There exist solutions to Eq. (2) in form of the harmonic plane waves:

$$f(\mathbf{r},t) = e^{\pm I(\mathbf{k}\cdot\mathbf{r}-\mathbf{n}\omega t)}N \tag{3a}$$

or

$$f(\mathbf{r},t) = e^{\pm I[\omega t-\mathbf{n}(\mathbf{k}\cdot\mathbf{r})]}N \tag{3b}$$

where $I = e_{123}$, $\mathbf{k} = |\mathbf{k}|\mathbf{n}$ is a constant vector, $\omega = u|\mathbf{k}|$ and N is a vector-plus-bivector such that $N\mathbf{k} = -\mathbf{k}N$. The solutions (3) are *plane waves.* They were found in a quaternionic language by Imaeda (1983) and written in the above form in Jancewicz 1988. They were also considered independently by Baylis and Jones (1989) in the Pauli algebra, the matrix representation of $C\ell_3$. . The novelty of expressions (3) lies in the combination of scalar and vector present in the exponents of the exponential functions. This is the reason why the phase velocity can not be introduced for them. Solution (3a) can be called a plane wave with a *round polarization*, (3b) – a plane wave with a *spiral polarization* (Jancewicz 1988, Section 4.4).

2. Travelling plane waves

It has been proved (Jancewicz 1988, Section 2.5) that the inequality

$$|\mathbf{S}| \leq uw$$

holds for an arbitrary electromagnetic field f. If one introduces the velocity $\mathbf{v}$ of energy transport through the relation $\mathbf{S} = w\mathbf{v}$, the above inequality shows that $\mathbf{v}$ is bounded from above:

$$|\mathbf{v}| \leq u.$$

One can prove that equality occurs in the above relation iff a unit vector $\mathbf{n}$ exists such that $\mathbf{n}f = -f\mathbf{n}$ and the following equivalent conditions are satisfied:

(i) $\mathbf{e} = \mathbf{n}(\star\mathbf{b})$,
(ii) $\star\mathbf{b} = \mathbf{n}\mathbf{e}$,
(iii) $\mathbf{n}f = f$,
(iv) $f = (1+\mathbf{n})\mathbf{e} = (1+\mathbf{n})(\star\mathbf{b})$.

In this case we say that the electromagnetic field f is *travelling* in the direction $\mathbf{n}$. Conditions (i) - (iv) are also equivalent to

$$f^2 = 0. \tag{4}$$

Such a field is also known as a *radiation field.*

Waves (3) generally are not travelling fields in this sense. This fits the observation that the phase velocity can not be defined for them. However, with the aid of two idempotents $P_{\pm\mathbf{n}} = \frac{1}{2}(1 \pm \mathbf{n})$ satisfying $P_{\mathbf{n}} + P_{-\mathbf{n}} = 1$ and $P_{\mathbf{n}}P_{-\mathbf{n}} = 0$, one can decompose (3) into sums

$$f = P_{\mathbf{n}}f + P_{-\mathbf{n}}f = f_+ + f_-$$

of two fields travelling in opposite directions $\pm\mathbf{n}$ (Jancewicz 1988, Section 4.2). They separately satisfy the synthetic Maxwell equation Eq. (2).

Solutions (3) are decomposed into the form

$$f = e^{\pm I(\mathbf{k}\cdot\mathbf{r}-\omega t)}N_+ + e^{\pm I(\mathbf{k}\cdot\mathbf{r}+\omega t)}N_- \tag{5a}$$

$$f = e^{\pm I(\omega t-\mathbf{k}\cdot\mathbf{r})}N_+ + e^{\pm I(\omega t+\mathbf{k}\cdot\mathbf{r})}N_- \tag{5b}$$

where $N_\pm = P_{\pm\mathbf{n}}N$. The exponents show now that the phase velocities can be introduced and they are opposite: $\pm u\mathbf{n}$ in the two terms of both sums.

3. Plane waves in a conducting medium

We assume now $\rho = 0$ and $\mathbf{j} = \sigma\mathbf{E}$ where σ is the *conductivity* of a medium. Then the synthetic Maxwell equation assumes the form

$$\mathcal{D}f = -\sqrt{\frac{\mu}{\varepsilon}}\sigma\mathbf{e} \tag{6}$$

and has a particular solution in form of the harmonic plane wave:

$$f(\mathbf{r},t) = e^{-\gamma\mathbf{n}\cdot\mathbf{r}}e^{I\mathbf{n}(\mathbf{k}\cdot\mathbf{r}-\omega t)}(\frac{\omega}{u} + \mathbf{k} + \gamma I)\mathbf{C} \tag{7}$$

where $k = \frac{\omega}{\sqrt{2}u}\sqrt{\sqrt{1+\kappa^2}+1}$, $\mathbf{k} = k\mathbf{n}$, $\gamma = \frac{\omega}{\sqrt{2}u}\sqrt{\sqrt{1+\kappa^2}-1}$, $\mathbf{C}$ is a constant vector orthogonal to $\mathbf{n}$, $\kappa = \sigma/\varepsilon\omega$ (Jancewicz 1988, Section 5.1).

The exponent in the periodic exponential in (7) shows that a phase velocity can be introduced and is equal to $\frac{\omega}{k}\mathbf{n}$. The field (7), however, does not satisfy (4), so it is not the travelling field in our sense. The same is valid for any other plane wave solutions if $\sigma \neq 0$, therefore we claim that travelling waves can not exist in a conducting medium. Some authors (e.g. Jackson 1975, p. 270) consider that the possibility of introducing phase velocitites is sufficient to call a solution the travelling wave. In our opinion such solution should be called differently, let it be an *advancing-phase wave.*

With the aid of the same idempotents $P_{\pm\mathbf{n}}$, field (7) can be decomposed:

$$f = P_{\mathbf{n}} f + P_{-\mathbf{n}} f = f_+ + f_-$$

into two travelling fields:

$$f_\pm(\mathbf{r},t) = \frac{1}{2} e^{-\gamma\mathbf{n}\cdot\mathbf{r}} e^{I\mathbf{n}(\mathbf{k}\cdot\mathbf{r}-\omega t)}(\frac{\omega}{u} \pm k + \gamma I \pm \gamma I\mathbf{n} + \mathbf{k} \pm \frac{\omega}{u}\mathbf{n})\mathbf{C}$$

which, however, separately do not satisfy the Maxwell equation (6). What is striking, the both fields travelling in opposite directions $\pm\mathbf{n}$ have the same phase velocity in the direction $+\mathbf{n}$.

The ratio of energy fluxes of the two travelling fields is independent of time and position:

$$R = \frac{| S_-(\mathbf{r},t) |}{| S_+(\mathbf{r},t) |} = \frac{k - \frac{\omega}{u}}{k + \frac{\omega}{u}} \qquad (8).$$

This yields $R \cong \frac{\kappa^2}{16}$ for small conductivity σ and $R \cong 1$ for large σ.

If there is an interface perpendicular to $\mathbf{n}$ at $\mathbf{n}\cdot\mathbf{r} = x_0$ and a dielectric medium with the same ε and μ is present for $\mathbf{n}\cdot\mathbf{r} < x_0$, the continuity of the electromagnetic field implies that fields $f_\pm$ pass smoothly into free travelling electromagnetic waves:

$$f_\pm(\mathbf{r},t) = \frac{1}{2} e^{-\gamma x_0} e^{I\mathbf{n}(-\omega t \pm \mathbf{k}\cdot\mathbf{r})}(\frac{\omega}{u} \pm k + \gamma I \pm \gamma I\mathbf{n} + \mathbf{k} \pm \frac{\omega}{u}\mathbf{n})\mathbf{C}$$

with the same ratio (8) of the intensities of the two waves. Here f_+ can be interpreted as the incident wave and f_- as the reflected wave from the conductor. Thus (8) can be viewed as the *reflection coefficient* of the conducting medium.

It is, moreover, possible to show (Jancewicz 1991) that travelling electromagnetic waves also can not exist in a nonhomogeneous medium.

References

W. E. Baylis and G. Jones: "Relativistic dynamics of charges in external fields: the Pauli algebra approach", *J. Phys.* **A22**,17-29(1989).

D. Hestenes: *Space-Time Algebra*, Gordon and Breach, New York 1966, 1987.

D. Hestenes: "Vectors, spinors and complex numbers in classical and quantum physics", *Am. J. Phys.* **39**,1013-1027(1971).

D. Hestenes: "Proper particle mechanics" *J. Math. Phys.* **15**, 1768-1777(1974a).

D. Hestenes: "Proper dynamics of a rigid point particle" *J. Math. Phys.* **15**,1778-1786(1974b).

D. Hestenes: *New Foundations for Classical Mechanics*, D. Reidel, Dordrecht 1986.

J.D. Jackson: *Classical Electrodynamics*, Wiley, New York 1962, 1975.

B. Jancewicz: *Multivectors and Clifford Algebra in Electrodynamics*, World Scientific, Singapore 1988.

B. Jancewicz: "Plane electromagnetic wave propagating parallel to the gradient of the refractive index" *J. Opt. Soc. Am.* **A 8**, 1529-1535(1991).

G. Jouvet and A. Schidlof: "Sur les nombres hypercomplexes de Clifford et leurs applications à l'analyse vectorielle ordinaire, à l'électromagnetisme de Minkowski et à la theorie de Dirac", *Bull. Soc. Neuchatel Sci. Nat.* **57**,127-147(1932).

P. Lounesto: "Spinor function theory" *Simon Stevin Quart. J. Pure and Appl. Math.* **58**,193-218(1984).

A. Mercier: *Expression des équations de l'électromagnetisme au moyen des nombres de Clifford.* Thèse no 935, Université de Geneève 1935.

M. Riesz: *Clifford Numbers and Spinors*, Lecture Series No. 38, University of Maryland, College Park 1958.

HAMILTONIAN MECHANICS WITH GEOMETRIC CALCULUS

DAVID HESTENES
Department of Physics and Astronomy
Arizona State University
Tempe, Arizona 85287
USA

Abstract. Hamiltonian mechanics is given an invariant formulation in terms of Geometric Calculus, a general differential and integral calculus with the structure of Clifford algebra. Advantages over formulations in terms of differential forms are explained.

INTRODUCTION

In the recent renaissance of Analytic Mechanics, the calculus of differential forms has become the dominant mathematical language of practitioners. However, the physics community at large has been slow to adopt the language. This reluctance should not be attributed solely to the usual resistance of communities to innovation, for the calculus of forms has some serious deficiencies. For one thing, it does not articulate smoothly with vector calculus, and it is inferior to vector calculus for many applications to Newtonian mechanics. Another drawback is that the calculus of forms has accreted a veritable orgy of definitions and notations which make the preparation required to address even the simplest problems in mechanics inordinately excessive. This is evident, for example, in the pioneering textbook of Abraham and Marsden (1967), which provides nearly 200 pages of preparation before attacking any significant problem in mechanics. The same high ratio of formalism to results is characteristic of more recent books in the field, such as Libermann and Marle (1987). All this goes to show that the calculus of forms is not quite the right tool for mechanics.

Without denying that valuable insights have been gained with differential forms, the contention of this paper is that a better mathematical system is available for application to analytical mechanics; namely, the Geometric Calculus expounded by Hestenes and Sobczyk (1984, henceforth referred to as [GC]). In contrast to differential forms, this calculus includes and generalizes standard vector calculus with no need to change standard notation, and it has proven advantages in applications throughout Newtonian mechanics, most notably in rigid-body mechanics (Hestenes, 1985). Geometric Calculus also includes and generalizes the calculus of differential forms, as explained in [GC]. In particular, it embraces the quaternion theory of rotations and the entire theory of spinors, which are completely outside the purview of differential forms. This apparatus is crucial to the efficient development of rigid-body dynamics (Hestenes, 1985).

This paper shows how to employ Geometric Calculus in the formulation of Hamiltonian mechanics, though space limitations preclude the discussion of applications or advanced theory. However, the fundamentals are discussed in sufficient detail with supplementary references to make translation of standard results in symplectic geometry and Hamiltonian mechanics into the language of Geometric Calculus fairly straightforward.

1. VECTOR SPACE VERSION

The reader is presumed to be familiar with Clifford algebra and Hamiltonian mechanics, but familiarity with [GC] will be needed for full comprehension of the ideas, as well as for their applications. Definitions, notations, and results from [GC] will be employed without explanation. Though Geometric Calculus makes a completely coordinate-free approach possible, it also facilitates computations with coordinates. Coordinates are employed here primarily to establish a relation to conventional formulations.

For a mechanical system described by coordinates $\{q_1, \ldots, q_n\}$ and corresponding momenta $\{p_1, \ldots, p_n\}$, we first define *configuration space* as an n-dimensional real vector space $\mathcal{R}^n$ spanned by an orthonormal basis $\{e_k\}$ with

$$e_j \cdot e_k = \tfrac{1}{2}(e_j e_k + e_k e_j) = \delta_{jk} \tag{1.1}$$

for $j, k = 1, 2, \ldots, n$. The state of the system can then be represented by the pair of vectors

$$q = \sum_k q_k e_k, \quad p = \sum_k p_k e_k. \tag{1.2}$$

The vectors in configuration space generate a real Geometric Algebra, $\mathcal{R}_n = \mathcal{G}(\mathcal{R}^n)$, with geometric product

$$qp = q \cdot p + q \wedge p. \tag{1.3}$$

Differentiation with respect to vectors is defined in [GC, Chap.2] along with the necessary apparatus to perform computations without resorting to coordinates. However, it will suffice here to introduce the *vector derivative* ∂_q by specifying its relation to the coordinates:

$$\partial_q = \sum_k e_k \frac{\partial}{\partial q_k}. \tag{1.4}$$

Equation (1.2) can be solved to express the coordinates as *functions* of the vector q instead of as independent variables; thus

$$q_k = q_k(q) = q \cdot e_k. \tag{1.5}$$

Then the basis vectors e_k are given as gradients

$$e_k = \partial_q q_k. \tag{1.6}$$

The simple linear form (1.5) for the coordinate functions obtains only for orthogonal coordinates, but the general case is treated in [GC]. It should be noted, also, that the "inner product" in (1.1) and (1.5) has no physical significance as a "metric tensor." It is merely an algebraic mechanism for expressing functional relations. Among other things, it performs the role of *contraction* in the calculus of differential forms.

For a Hamiltonian, $H = H(q, p)$, Hamilton's equations of motion can be expressed in configuration space as the pair of equations

$$\dot{q} = \partial_p H, \tag{1.7}$$

$$\dot{p} = -\partial_q H. \tag{1.8}$$

Since p and q are independent variables, we can reduce this pair of coupled equations to a single equation in a space of higher dimension. However, to be useful, the extension to higher dimension must preserve the essential structure of Hamilton's equations in a way which facilitates computation. We now show how such a computationally efficient extension can be achieved with Geometric Calculus.

To that end, we define *momentum space* as an n-dimensional real vector space $\widetilde{\mathcal{R}}^n$ spanned by an orthonormal basis $\{\tilde{e}_k\}$ with

$$\tilde{e}_j \cdot \tilde{e}_k = \tfrac{1}{2}(\tilde{e}_j \tilde{e}_k + \tilde{e}_k \tilde{e}_j) = \delta_{jk}, \tag{1.9}$$

so the momentum of our mechanical system can be expressed as a vector

$$\tilde{p} = \sum_k p_k \tilde{e}_k. \tag{1.10}$$

Now we define *phase space* $\mathcal{R}^{2n}$ as the direct sum

$$\mathcal{R}^{2n} = \mathcal{R}^n \oplus \widetilde{\mathcal{R}}^n. \tag{1.11}$$

This generates the *phase space* (geometric) *algebra* $\mathcal{R}_{2n} = \mathcal{G}(\mathcal{R}^{2n})$, which is completely defined by supplementing (1.1) and (1.9) with the orthogonality relations

$$e_j \cdot \tilde{e}_k = \tfrac{1}{2}(e_j \tilde{e}_k + \tilde{e}_k e_j) = 0. \tag{1.12}$$

The *symplectic structure* of phase space is best described by introducing a *symplectic bivector*

$$J = \sum_k J_k \tag{1.13}$$

with component 2-blades

$$J_k = e_k \tilde{e}_k = e_k \wedge \tilde{e}_k. \tag{1.14}$$

The bivector J determines a unique pairing of directions in configuration space with directions in momentum space, as expressed by

$$\tilde{e}_k = e_k \cdot J = e_k \cdot J_k = e_k J_k = -J_k e_k, \tag{1.15}$$

$$e_k = J \cdot \tilde{e}_k = J_k \cdot \tilde{e}_k = J_k \tilde{e}_k = -\tilde{e}_k J_k. \tag{1.16}$$

Each blade J_k pairs a coordinate q_k with its corresponding momentum p_k. Moreover, since each J_k satisfies

$$J_k^2 = -1, \tag{1.17}$$

it functions as a "unit imaginary" relating q_k to p_k. Thus, the bivector J determines a unique *complex structure* for phase space. The symplectic structure on phase space can be described without the reference (1.14) to basis vectors by defining the symplectic bivector J through a specification of its general properties. The symplectic bivector determines a skew-symmetric linear transformation $\underline{J}$ which maps each phase space vector x into a vector

$$\tilde{x} = \underline{J}\, x = x \cdot J. \tag{1.18}$$

This, in turn, defines a skew-symmetric bilinear form

$$\tilde{x} \cdot y = y \cdot (\underline{J} x) = x \cdot J \cdot y = J \cdot (y \wedge x) = -\tilde{y} \cdot x. \tag{1.19}$$

This bilinear form is nondegenerate if and only if $\tilde{x}$ is nonzero whenever x is nonzero, or, equivalently, if and only if $< J^n >_{2n}$ is a nonvanishing pseudoscalar. With respect to the basis specified by (1.14),

$$< J^n >_{2n} = \underbrace{J \wedge \ldots \wedge J}_{n\ times} = n!(-1)^{[n/2]} E_n \tilde{E}_n, \tag{1.20}$$

where $E_n = e_1 e_2 \ldots e_n = e_1 \wedge e_2 \wedge \ldots e_n$, $\tilde{E}_n = \tilde{e}_1 \tilde{e}_2 \ldots \tilde{e}_n$, and $[n/2]$ is the greatest integer in $n/2$. The "complex structure" expressed by (1.17) can be characterized more generally by

$$(\tilde{x})^\sim = \underline{J}^2 x = -x. \tag{1.21}$$

It follows that

$$(\tilde{x})^2 = x^2, \tag{1.22}$$

which can be regarded as a "hermitian form" associated with the complex structure.

The group of linear transformations on phase space which preserves symplectic structure is called the *symplectic group.* It has recently been shown that the symplectic group has a natural representation as a "spin group" (Doran *et. al.,* 1992). This promises to be the ideal vehicle for characterizing symplectic transformations.

Now, to define Hamiltonian mechanics on phase space, from the "position and momentum vectors" (1.2) we can describe the state of our physical system by a single point x in phase space defined by

$$x = \tilde{q} + p = p + q \cdot J. \tag{1.23}$$

The derivative with respect to a phase space point is then given by

$$\partial \equiv \partial_x = \partial_{\tilde{q}} + \partial_p, \tag{1.24}$$

and we have

$$\tilde{\partial} = \tilde{\partial}_x = -J \cdot \partial_x = -\partial_q + \tilde{\partial}_p. \tag{1.25}$$

The Hamiltonian of the system is a scalar-valued function on phase space

$$H = H(x) = H(q, p). \tag{1.26}$$

Accordingly, Hamilton's equation for a phase space trajectory, $x = x(t)$, of the system assumes the simple form

$$\dot{x} = \tilde{\partial} H. \tag{1.27}$$

The transcription of the entire theory of Hamiltonian systems into this invariant formulation is now straightforward. For example, for any scalar-valued phase space function, $G = G(x)$, the *Poisson bracket* can be defined by

$$\{H, G\} = (\tilde{\partial} H) \cdot \partial G = -\{G, H\}. \tag{1.28}$$

Its equivalence to the conventional definition in terms of coordinates is provided by

$$\begin{aligned} \{H, G\} &= (\tilde{\partial}_p H - \partial_q H) \cdot (\partial_p + \tilde{\partial}_q) G \\ &= (\partial_p H) \cdot \partial_q G - (\partial_q H) \cdot (\partial_p G) \\ &= \textstyle\sum_k \left[\left(\frac{\partial H}{\partial p_k} \right) \left(\frac{\partial G}{\partial q_k} \right) - \left(\frac{\partial H}{\partial q_k} \right) \left(\frac{\partial G}{\partial p_k} \right) \right] \end{aligned} \tag{1.29}$$

The definition (1.28) does not actually require that G be scalar-valued, so it can be applied to any multivector-valued function, $M = M(x)$, describing some "observable" property of the system. It follows that the equation of motion for the observable is given by

$$\dot{M} = \dot{x}\cdot\partial M = (\tilde{\partial} H)\cdot\partial M = \{H, M\}. \tag{1.30}$$

For $M = x$ we have

$$\{H, x\} = (\tilde{\partial} H)\cdot\partial x = \tilde{\partial} H, \tag{1.31}$$

so Hamilton's equation (1.27) can be expressed in the form

$$\dot{x} = \{H, x\}. \tag{1.32}$$

According to (1.30), M is a *constant of the motion* if $\{H, M\} = 0$. It follows that H is a constant of the motion, since

$$\{H, H\} = (\tilde{\partial} H)\cdot(\partial H) = J\cdot(\partial H \wedge \partial H) = 0. \tag{1.33}$$

Our next task is to generalize this approach to Hamiltonian mechanics on manifolds.

2. VECTOR MANIFOLD VERSION

The initial characterization of configuration space in the preceding section depends on the choice of coordinates. There is a "canonical" choice, though. For a system of N particles a configuration space of dimension $n = 3N$ is naturally defined by

$$\mathcal{R}^{3N} = \underbrace{\mathcal{R}^3 \oplus \cdots \oplus \mathcal{R}^3}_{N\ times}, \tag{2.1}$$

where a separate copy of the 3-dimensional "physical space" is alloted to each particle. Whatever the choice of "generalized coordinates," its relation to physical space must be maintained, so a mapping to the choice (2.1) must be specified. For many purposes, however, this mapping is not of interest, so we desire a formulation of mechanics where it can be suppressed or resurrected as needed.

For a system of particles or rigid bodies with constraints, the space of allowable states is a manifold of dimension $2n$ equal to the number of independent degrees of freedom. Although this manifold can be mapped locally into the vector space representation of phase space in the preceding section, this is awkward if the system has cyclic coordinates. Alternatively, we can describe here the representation of *phase space* as a $2n$-dimensional *vector manifold* $\mathcal{M}^{2n}$. The mathematical apparatus needed for differential and integral calculus on vector manifolds has already been developed in [GC]. The phase space manifold $\mathcal{M}^{2n}$ can be regarded as embedded in a vector space of higher dimension (e.g., of dimension $6N$ for an N particle system), but this is not required except, perhaps, to describe the relation to physical space expressed by (2.1).

The mathematical apparatus in [GC] enables us to adapt our vector space version of Hamiltonian mechanics to a vector manifold version with comparatively minor alterations. The main difference is that the algebraic relations of interest will be defined on the tangent spaces of the manifold instead of on the manifold itself.

Each point x on the phase space manifold $\mathcal{M}^{2m}$ represents an allowable state of the system. The symplectic bivector J of the preceding section becomes a nondegenerate bivector field $J = J(x)$ on $\mathcal{M}^{2n}$ with values in the tangent algebra [GC, Chap.4]. For vector fields $v = v(x)$ and $u = u(x)$, in the tangent space at each point x, $J(x)$ determines a linear transformation

$$\tilde{v} = \underline{J}v = v\cdot J \tag{2.2}$$

and a corresponding nondegenerate bilinear form

$$u\cdot\tilde{v} = -v\cdot\tilde{u}, \tag{2.3}$$

just as in (1.18) and (1.19). However, a direct analogue of (1.21) is not feasible, because it may conflict with requirements on the derivatives of J. Instead, however, we can introduce another bivector field $K = K(x)$ with the property

$$\underline{K}\tilde{v} = \tilde{v}\cdot K = v. \tag{2.4}$$

Thus, $\underline{K} = \underline{J}^{-1}$ is the inverse of $\underline{J}$. Now the Jacobi identity [GC, p.14] implies that

$$\underline{K}\,\underline{J}v = K\cdot(v\cdot J) = (K\cdot v)\cdot J + v\cdot(K\times J),$$

where $K\times J = \frac{1}{2}(KJ - JK)$ is the *commutator product*. So if $\underline{J}$ is to be the inverse of $\underline{K}$, we must have

$$K\times J = 0, \tag{2.5}$$

or the equivalent operator equation

$$\underline{K}\,\underline{J} = \underline{J}\,\underline{K} = 1. \tag{2.6}$$

To specify the relation of K to J more precisely, we note that, as in (1.13), they can each be expressed as a sum of n commuting blades.

$$J = \sum_k J_k, \quad K = \sum_k K_k. \tag{2.7}$$

Moreover, we can select each K_k to be proportional to J_k. Then the condition $\underline{K} = \underline{J}^{-1}$ can be expressed by the more specific condition

$$K_k = J_k^{-1} \tag{2.8}$$

for each k. This generalizes the condition $J_k^{-1} = -J_k$ in (2.8). Incidentally, we note that

$$J \cdot K = \sum_k K_k \cdot J_k = n. \tag{2.9}$$

Modern approaches to Hamiltonian mechanics (Abraham and Marsden, 1967; Libermann and Marle, 1987) begin with symplectic manifolds. A manifold $\mathcal{M}^{2n}$ is said to be *symplectic* if it admits a *closed, nondegenerate* 2-form ω. As shown in [GC], this is equivalent to admitting a closed, nondegenerate bivector field K on the vector manifold. Indeed, the 2-form can be defined by

$$\omega = K \cdot (dx \wedge dy), \tag{2.10}$$

where dx and dy are tangent vectors. The 2-form is said to be closed if its "exterior differential" vanishes, that is, if

$$d\omega = (dx \wedge dy \wedge dz) \cdot (\partial \wedge K) = 0. \tag{2.11}$$

This condition is obviously satisfied if K has vanishing *curl:*

$$\partial \wedge K = 0. \tag{2.12}$$

Actually, though, (2.11) implies only the weaker condition of vanishing *cocurl:*

$$\nabla \wedge K = \underline{P}(\partial \wedge K) = 0, \tag{2.13}$$

where $\underline{P}$ is the projection into the *tangent algebra* of $\mathcal{M}^{2n}$ (see [GC, p.140]). The tangent algebra is essentially the same thing as the "Clifford bundle" which "pastes" Clifford algebras on manifolds, instead of generating them from a vector manifold as in [GC]. The *coderivative* ∇ as well as the *derivative* ∂ is an essential concept for calculus on vector manifolds, and its properties are thoroughly discussed in [GC, Chapt.4], so we can exploit some of its properties without establishing them here.

Instead of translating the "differential forms approach" into geometric algebra, it is more enlightening to ascertain directly what condition on the bivector field $J = J(x)$ are required to ensure the essential features of Hamiltonian mechanics on $\mathcal{M}^{2n}$. Hamilton's equation (1.27) can be adopted without change. The Hamiltonian $H(x)$ determines a vector field $\tilde{h} = \tilde{h}(x)$ on $\mathcal{M}^{2n}$ given by

$$\tilde{h} = \tilde{\partial} H = (\partial H) \cdot J. \tag{2.14}$$

Hamilton's equation

$$\dot{x}(t) = \tilde{h}(x(t)) \tag{2.15}$$

determines *integral curves* of this vector field. This condition that these curves describe an "incompressible flow" is given by *Liouvilles Theorem*

$$\nabla \cdot \tilde{h} = \partial \cdot \tilde{h} = 0. \tag{2.17}$$

Since

$$\partial \cdot \tilde{h} = \partial \cdot (-J \cdot h) = -(\partial \cdot J) \cdot h + J \cdot (\partial \wedge h).$$

and $\partial \wedge h = \partial \wedge \partial H = 0$, the condition

$$\nabla \cdot J = \underline{P}(\partial \cdot J) = 0 \tag{2.18}$$

suffices to imply Liouvilles Theorem. We adopt (2.18) instead of the weaker condition $h \cdot (\partial \cdot J) = (h \wedge \partial) \cdot J = 0$, because it appears to be essential for the theory of canonical transformations outlined below.

The definition (1.28) for the Poisson bracket can be taken over to $\mathcal{M}^{2n}$ without change. However, the role of J in determining its properties must be examined. Scalar-valued functions $F = F(x)$, $G = G(x)$, $H = H(x)$ determine vector fields

$$\tilde{f} = \tilde{\partial} F, \quad \tilde{g} = \tilde{\partial} G, \quad \tilde{h} = \tilde{\partial} H. \tag{2.19}$$

Let us refer to such fields as *symplectic vector fields.* It follows from (2.19) that

$$\tilde{\partial} \cdot f = -J \cdot (\partial \wedge f) = 0, \tag{2.20}$$

but (2.18) implies the stronger condition

$$\partial \cdot \tilde{f} = -\tilde{\partial} \cdot f = 0. \tag{2.21}$$

Therefore, all nonvanishing symplectic vector fields generate incompressible flows on (or automorphisms of) $\mathcal{M}^{2n}$.

The Poisson bracket can be written in a variety of forms, including

$$\begin{aligned} \{F, G\} &= -J \cdot (f \wedge g) = \tilde{f} \cdot g = -\tilde{g} \cdot f \\ &= \tilde{\partial} \cdot (Fg) = -\tilde{\partial} \cdot (Gf). \end{aligned} \tag{2.22}$$

Alternatively, using (2.4), one can write

$$\{F, G\} = K \cdot (\tilde{f} \wedge \tilde{g}), \tag{2.23}$$

which, according to (2.10), expresses the bracket as a 2-form evaluated on symplectic vector fields. This is closer to conventional formulations in terms of differential forms. However, (2.22) is simpler because K is not involved.

An essential property of the Poisson bracket is the *Jacobi identity*

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0. \tag{2.24}$$

Using (2.22) to express the left side of (2.24) in terms of vector fields, we obtain

$$-\tilde{\partial} \cdot [(\tilde{g} \cdot h)f + (\tilde{h} \cdot f)g + (\tilde{f} \cdot g)h]$$

$$= \tilde{\partial}\cdot[J\cdot(f\wedge g\wedge h)]$$
$$= (f\wedge g\wedge h)\cdot(\tilde{\partial}\wedge J) + (J\wedge\tilde{\partial})\cdot(f\wedge g\wedge h)$$
$$= (f\wedge g\wedge h)\cdot(\tilde{\partial}\wedge J) - \tfrac{1}{2}(J\wedge J)\cdot[\partial\wedge(f\wedge g\wedge h)]. \qquad (2.25)$$

This computation employed the algebraic identities

$$\begin{aligned} J\cdot(f\wedge g\wedge h) &= J\cdot(f\wedge g)h - J\cdot(f\wedge h)g + J\cdot(g\wedge h)f \\ &= (\tilde{g}\cdot f)h + (\tilde{f}\cdot h)g + (\tilde{h}\cdot g)f \end{aligned} \qquad (2.26)$$

[GC, eqn.(1-1.40)], and

$$(J\wedge J)\cdot\partial = 2J\wedge(J\cdot\partial) = -2J\wedge\tilde{\partial}, \qquad (2.27)$$

$$(J\wedge J)\cdot[\partial\wedge(f\wedge g\wedge h)] = [(J\wedge J)\cdot\partial]\cdot(f\wedge g\wedge h) \qquad (2.28)$$

[GC, eqn. (1-1.25b) or (1-4.6)].

The last term in (2.25) vanishes identically when f, g and h are gradients. Therefore, from (2.25) it follows that the Jacobi identity (2.24) obtains if and only if

$$\tilde{\nabla}\wedge J = \underline{P}(\tilde{\partial}\wedge J) = 0. \qquad (2.29)$$

This condition is not independent of the "incompressibility condition" (2.18), for from (2.27) we obtain the relation

$$\begin{aligned} \tfrac{1}{2}\nabla\cdot(J\wedge J) &= (\nabla\cdot J)\wedge J - (J\cdot\nabla)\wedge J \\ &= J\wedge(\nabla\cdot J) + \tilde{\nabla}\wedge J. \end{aligned} \qquad (2.30)$$

Thus, (2.24) and (2.29) together imply

$$\nabla\cdot(J\wedge J) = 0. \qquad (2.31)$$

In analogy with (1.30), a multivector field $M = M(x)$ which is invariant under the flow generated by a symplectic vector field $\tilde{f} = \tilde{\partial}F$ satisfies

$$\{F, M\} = \tilde{f}\cdot\nabla M = 0. \qquad (2.32)$$

Note the use of $\tilde{f}\cdot\nabla$ instead of $\tilde{f}\cdot\partial$ when M is not scalar-valued. A flow is said to be a *canonical transformation* when it leaves the symplectic bivector J invariant, that is, when

$$\{F, J\} = \tilde{f}\cdot\nabla J = 0. \qquad (2.33)$$

The differentiable vector fields on a manifold compose a Lie algebra under the *Lie bracket* defined for vector fields $u = u(x)$ and $v = v(x)$ by

$$[u, v] = u\cdot\partial v - v\cdot\partial u = \nabla\cdot(u\wedge v) + u\nabla\cdot v - v\nabla\cdot u. \qquad (2.34)$$

The properties of the Lie bracket are studied at length in [GC]. For symplectic fields we derive the identity

$$\begin{aligned} [\tilde{f},\tilde{g}] &= \tilde{f}\cdot\partial\tilde{g} - \tilde{g}\cdot\partial\tilde{f} = \{F,\tilde{\partial}G\} - \{G,\tilde{\partial}F\} \\ &= \tilde{\partial}\{F,G\} + f\cdot(\tilde{g}\cdot\partial J) - g\cdot(\tilde{f}\cdot\partial J). \end{aligned} \qquad (2.35)$$

According to (2.33), the last two terms in (2.35) vanish for canonical transformations. Therefore, the canonical transformations compose a closed Lie algebra on $\mathcal{M}^{2n}$, and the Poisson bracket of "canonical generators" F and G is also a canonical generator. This should suffice to show how the general theory of canonical transformations can be developed on vector manifolds.

As a final point, the crucial role of the symplectic bivector J in canonical transformations suggests that it should be more intimately linked with the Hamiltonian H in the theory. One attractive possiblility for linking them is to introduce a bivector field Ω given by

$$\Omega = HJ. \qquad (2.36)$$

Then (2.18) implies

$$\tilde{h} = (\partial H)\cdot J = \nabla\cdot(HJ) = \nabla\cdot\Omega, \qquad (2.37)$$

and Hamilton's equation (2.15) takes the form

$$\dot{x} = \nabla\cdot\Omega. \qquad (2.38)$$

Thus, Ω is a *bivector potential* for Hamiltonian flow, and H plays the role of an integrating factor for this bivector field. This is very suggestive!

3. CONCLUSIONS

Experts will have noted that phase space is identified with its own dual space in the preceding formulation of Hamiltonian mechanics. Some may claim that the conventional formulation in terms of differential forms is preferable because it does not make that identification. On the contrary, it can be argued that such generality is excessive, contributing little if anything to deepening analytical mechanics, while introducing unnecessary complications. Be that as it may, it should be recognized that the identification of phase space with its dual is a deliberate choice and not an intrinsic limitation of geometric algebra. Indeed, the geometric algebra apparatus needed to separate phase space from its dual is available in Doran *et. al.* (1992) and ready to be applied to mechanics. Ironically, that apparatus automatically produces a kind of quantization, something which can only be imposed artificially in conventional approaches. It remains to be seen if that fact has significant physical import.

The purpose of this short paper has been to lay the foundation for a reformulation of analytical mechanics in the language of geometric calculus. Translation of standard results into this language is not difficult, but it will not be without surprises and new insights as the treatment above already suffices to show. Though the emphasis here has been on an invariant methodology, a powerful apparatus for dealing with coordinates is available in [GC]. One especially promising possibility is an extention of the invariant formulation for rigid-body mechanics in Hestenes (1985) to a phase space formulation for systems of linked rigid bodies. That is likely to have important applications to robotics.

References

R. Abraham and J. Marsden (1967), *Foundations of Mechanics*, W.A.Benjamin, New York.

C. Doran, D. Hestenes, F. Sommen, and N. VanAcker (1992), Lie Groups as Spin Groups, *J.Math.Phys.* (submitted).

D. Hestenes (1985), *New Foundations for Classical Mechanics*, Kluwer, Dordrecht/Boston; paperback ed. (1987); fourth printing with corrections (1992).

D. Hestenes and G. Sobczyk (1984), *Clifford Algebra to Geometric Calculus*, Kluwer, Dordrecht/Boston; paperback ed. (1987); third printing with corrections (1992).

P. Libermann and C-M Marle (1987), *Symplectic Geometry and Analytical Mechanics*, Reidel, Dordrecht/Boston.

GRASSMANN MECHANICS, MULTIVECTOR DERIVATIVES AND GEOMETRIC ALGEBRA

CHRIS DORAN *
MRAO, Cavendish Laboratories, Madingley Road, Cambridge CB3 0HE, UK

ANTHONY LASENBY
DAMTP, Silver Street, Cambridge, CB3 9EW, UK

and

STEVE GULL
DAMTP, Silver Street, Cambridge, CB3 9EW, UK

Abstract. A method of incorporating the results of Grassmann calculus within the framework of geometric algebra is presented, and shown to lead to a new concept, the multivector Lagrangian. A general theory for multivector Lagrangians is outlined, and the crucial role of the multivector derivative is emphasised. A generalisation of Noether's theorem is derived, from which conserved quantities can be found conjugate to discrete symmetries.

1. Introduction

Grassmann variables enjoy a key role in many areas of theoretical physics, second quantization of spinor fields and supersymmetry being two of the most significant examples. However, not long after introducing his anticommuting algebra, Grassmann himself [Grassmann, 1877] introduced an inner product which he unified with his exterior product to give the familiar Clifford multiplication rule

$$ab = a{\cdot}b + a{\wedge}b. \tag{1}$$

What is surprising is that this idea has been lost to future generations of mathematical physicists, none of whom (to our knowledge) have investigated the possibility of recovering this unification, and thus viewing the results of Grassmann algebra as being special cases of the far wider mathematics that can be carried out with geometric (Clifford) algebra [Hestenes & Sobczyk, 1984].

There are a number of benefits to be had from this shift of view. For example it becomes possible to "geometrize" Grassmann algebra, that is, give the results a significance in real geometry, often in space or spacetime. Also by making available the associative Clifford product, the possibility of

* Supported by a SERC studentship.

generating new mathematics is opened up, by taking Grassmann systems further than previously possible. It is an example of this second possibility that we will illustrate in this paper.

A detailed introduction to these ideas is contained in [Lasenby *et al.*, 1992b], which is the first of a series of papers [Lasenby *et al.*, 1992a; Lasenby *et al.*, 1993; Lasenby *et al.*, 1992c; Doran *et al.*, 1993] in which we aim to show that many of concepts of modern physics, including 2-spinors, twistors, Grassmann dimensions, supersymmetry and internal symmetry groups, can be expressed purely in terms of the real geometric algebras of space and spacetime. This, coupled with David Hestenes' demonstration that the Dirac and Pauli equations can also be expressed in the same real algebras [Hestenes, 1975], has led us to believe that these algebras (with multiple copies for many particles) are all that are required for fundamental physics.

This paper starts with a brief survey of the translation between Grassmann and geometric algebra, which is used to motivate the concept of a multivector Lagrangian. The rest of the paper develops this concept, making full use of the multivector derivative [Hestenes & Sobczyk, 1984]. The point to stress is that as a result of the translation we have gained something new, which can then only be fully developed outside Grassmann algebra, within the framework of geometric algebra. This is possible because geometric algebra provides a richer algebraic structure than pure Grassmann algebra.

Throughout we have used most of the conventions of [Hestenes & Sobczyk, 1984], so that vectors are written in lower case, and multivectors in upper case. The Clifford product of the multivectors A and B is written as AB. The subject of Clifford algebra suffers from a nearly stifling plethora of conventions and notations, and we have settled on the one that, if it is not already the most popular, we believe should be. A full introduction to our conventions is provided in [Lasenby *et al.*, 1992b].

2. Translating Grassmann Algebra into Geometric Algebra

Given a set of n Grassmann generators $\{\zeta_i\}$, satisfying

$$\{\zeta_i, \zeta_j\} = 0, \tag{2}$$

we can map these into geometric algebra by introducing a set of n independent vectors $\{e_i\}$, and replacing the product of Grassmann variables by the exterior product,

$$\zeta_i\zeta_j \quad \leftrightarrow \quad e_i \wedge e_j. \tag{3}$$

In this way any combination of Grassmann variables can be replaced by a multivector. Note that nothing is said about the interior product of the e_i vectors, so the $\{e_i\}$ frame is completely arbitrary.

In order for the above scheme to have computational power, we need a translation for the second ingredient that is crucial to modern uses of Grassmann algebra, namely Berezin calculus [Berezin, 1966]. Looking at differentiation first, this is defined by the rules,

$$\frac{\partial \zeta_j}{\partial \zeta_i} = \delta_{ij} \tag{4}$$

$$\zeta_j \frac{\overleftarrow{\partial}}{\partial \zeta_i} = \delta_{ij}, \tag{5}$$

(together with the graded Leibnitz rule). This can be handled entirely within the algebra generated by the $\{e_i\}$ frame by introducing the reciprocal frame $\{e^i\}$, defined by

$$e_i \cdot e^j = \delta_i^j. \tag{6}$$

Berezin differentiation is then translated to

$$\frac{\partial}{\partial \zeta_i}(\quad \leftrightarrow \quad e^i \cdot (\tag{7}$$

so that

$$\frac{\partial \zeta_j}{\partial \zeta_i} \quad \leftrightarrow \quad e^i \cdot e_j = \delta_j^i. \tag{8}$$

Note that we are using lower and upper indices to distinguish a frame from its reciprocal, rather than to simply distinguish metric signature.

Integration is defined to be equivalent to right differentiation, i.e.

$$\int f(\zeta) d\zeta_n d\zeta_{n-1} \dots d\zeta_1 = f(\zeta) \frac{\overleftarrow{\partial}}{\partial \zeta_n} \frac{\overleftarrow{\partial}}{\partial \zeta_{n-1}} \cdots \frac{\overleftarrow{\partial}}{\partial \zeta_1}. \tag{9}$$

In this expression $f(\zeta)$ translates to a multivector F, so the whole expression becomes

$$(\dots((F \cdot e^n) \cdot e^{n-1}) \dots) \cdot e^1 = \langle F E^n \rangle, \tag{10}$$

where E^n is the pseudoscalar for the reciprocal frame,

$$E^n = e^n \wedge e^{n-1} \dots \wedge e^1, \tag{11}$$

and $\langle F E^n \rangle$ denotes the scalar part of the multivector FE^n.

Thus we see that Grassmann calculus amounts to no more than Clifford contraction, and the results of "Grassmann analysis" [de Witt, 1984; Berezin, 1966] can all be expressed as simple algebraic identities for multivectors. Furthermore these results are now given a firm geometric significance through the identification of Clifford elements with directed line, plane segments *etc.* Further details and examples of this are given in [Lasenby *et al.*, 1992b].

It is our opinion that this translation shows that the introduction of Grassmann variables to physics is completely unnecessary, and that instead genuine Clifford entities should be employed. This view results not from a mathematical prejudice that Clifford algebras are in some sense "more fundamental" than Grassmann algebras (such statements are meaningless), but is motivated by the fact that physics clearly does involve Clifford algebras at its most fundamental level (the electron). Furthermore, we believe that a systematic use of the above translation would be of great benefit to areas currently utilising Grassmann variables, both in geometrizing known results, and, more importantly, opening up possibilities for new mathematics. Indeed, if new results cannot be generated, the above exercise would be of very limited interest.

It is one of the possibilities for new mathematics that we wish to illustrate in the rest of this paper. The idea has its origin in pseudoclassical mechanics, and is illustrated with one of the simplest Grassmann Lagrangians,

$$L = \tfrac{1}{2}\zeta_i\dot{\zeta}_i - \tfrac{1}{2}\epsilon_{ijk}\omega_i\zeta_j\zeta_k, \tag{12}$$

where ω_i are a set of three scalar consants. This Lagrangian is supposed to represent the "pseudoclassical mechanics of spin" [Berezin & Marinov, 1977; Freund, 1986]. Following the above procedure we translate this to

$$L = \tfrac{1}{2}e_i\wedge\dot{e}_i - \omega, \tag{13}$$

where

$$\omega = \omega_1(e_2\wedge e_3) + \omega_2(e_3\wedge e_1) + \omega_3(e_1\wedge e_2), \tag{14}$$

which gives a *bivector* - valued Lagrangian. This is typical of Grassmann Lagrangians, and can be easily extended to supersymmetric Lagrangians, which become mixed grade multivectors. This raises a number of interesting questions; what does it mean when a Lagrangian is multivector-valued, and do all the usual results for scalar Lagrangians still apply? In the next section we will provide answers to some of these, illustrating the results with the Lagrangian of (13). In doing so we will have thrown away the origin of the Lagrangian in Grassmann algebra, and will work entirely within the framework of geometric algebra, where we hope it is evident that the possibilities are far greater.

3. The Variational Principle for Multivector Lagrangians

Before proceeding to derive the Euler-Lagrange equations for a multivector Lagrangian, it is necessary to first recall the definition of the multivector

derivative ∂_X, as introduced in [Hestenes, 1968; Hestenes & Sobczyk, 1984]. Let X be a mixed-grade multivector

$$X = \sum_r X_r, \tag{15}$$

and let $F(X)$ be a general multivector valued function of X. The A derivative of F is defined by

$$A*\partial_X F(X) = \frac{\partial}{\partial\tau}F(X+\tau A)\bigg|_{\tau=0}, \tag{16}$$

where $*$ denotes the scalar product

$$A*B = \langle AB\rangle. \tag{17}$$

We now introduce an arbitrary vector basis $\{e_j\}$, which is extended to a basis for the entire algebra $\{e_J\}$, where J is a general index. The multivector derivative is defined by

$$\partial_X = \sum_J e^J e_J*\partial_X. \tag{18}$$

∂_X thus inherits the multivector properties of its argument X, so that in particular it contains the same grades. A simple example of a multivector derivative is when X is just a position vector x, in which case ∂_x is the usual vector derivative (sometimes referred to as the Dirac operator). A special case is provided when the argument is a scalar, α, when we continue to write ∂_α.

A useful result of general applicability is

$$\partial_X\langle XA\rangle = P_X(A) \tag{19}$$

where $P_X(A)$ is the projection of A onto the terms containing the same grades as X. More complicated results can be derived by expanding in a basis, and repeatedly applying (19).

Now consider an initially scalar-valued function $L = L(X_i, \dot{X}_i)$ where X_i are general multivectors, and $\dot{X}_i$ denotes differentiation with respect to time. We wish to extremise the action

$$S = \int_{t_1}^{t_2} dt\, L(X_i, \dot{X}_i). \tag{20}$$

Following e.g. [Goldstein, 1950], we write,

$$X_i(t) = X_i^0(t) + \epsilon Y_i(t) \tag{21}$$

where Y_i is a multivector containing the same grades as X_i, ϵ is a scalar, and X_i^0 represents the extremal path. With this we find

$$\partial_\epsilon S = \int_{t_1}^{t_2} dt \left(Y_i * \partial_{X_i} L + \dot{Y}_i * \partial_{\dot{X}_i} L\right) \tag{22}$$

$$= \int_{t_1}^{t_2} dt Y_i * \left(\partial_{X_i} L - \partial_t(\partial_{\dot{X}_i} L)\right) \tag{23}$$

(summation convention implied), and from the usual argument about stationary paths, we can read off the Euler-Lagrange equations

$$\partial_{X_i} L - \partial_t(\partial_{\dot{X}_i} L) = 0. \tag{24}$$

We now wish to extend this argument to a multivector-valued L. In this case taking the scalar product of L with an arbitrary constant multivector A produces a scalar Lagrangian $\langle LA \rangle$, which generates its own Euler-Lagrange equations,

$$\partial_{X_i}\langle LA \rangle - \partial_t(\partial_{\dot{X}_i}\langle LA \rangle) = 0. \tag{25}$$

An 'allowed' multivector Lagrangian is one for which the equations from each A are mutually consistent. This has the consequence that if L is expanded in a basis, each component is capable of simultaneous extremisation.

From (25), a necessary condition on the dynamical variables is

$$\partial_{X_i} L - \partial_t(\partial_{\dot{X}_i} L) = 0. \tag{26}$$

For an allowed multivector Lagrangian this equation is also *sufficient* to ensure that (25) is satisfied for all A. We will take this as part of the definition of a multivector Lagrangian. To see how this can work, consider the bivector-valued Lagrangian of (13). From this we can construct the scalar Lagrangian $\langle LB \rangle$, where B is a bivector, and we can derive the equations of motion

$$\partial_{e_i}\langle LB \rangle - \partial_t(\partial_{\dot{e}_i}\langle LB \rangle) = 0 \tag{27}$$

$$\Rightarrow (\dot{e}_i + \epsilon_{ijk}\omega_j e_k)\cdot B = 0. \tag{28}$$

For this to be satisfied for all B, we simply require that the bracket vanishes. If instead we use (26), together with the 3-d result

$$\partial_a a \wedge b = 2a, \tag{29}$$

we find the equations of motion

$$\dot{e}_i + \epsilon_{ijk}\omega_j e_k = 0. \tag{30}$$

Thus, for the Lagrangian of (13), equation (26) is indeed sufficient to ensure that (27) is satisfied for all B.

Recalling (14), equations (30) can be written compactly as [Lasenby *et al.*, 1992b]

$$\dot{e}_i = e^i \cdot \omega, \tag{31}$$

which are a set of three coupled vector equations — nine scalar equations for nine unknowns. This illustrates how multivector Lagrangians have the potential to package up large numbers of equations into a single entity, in a highly compact manner. Equations (31) are studied and solved in [Lasenby *et al.*, 1992b].

This example also illustrates a second point, which is that, for a fixed A, (25) does not always lead to the full equations of motion. It is only by allowing A to vary that we arrive at (26). Thus it is crucial to the formalism that L is a multivector, and that (25) holds for all A, as we shall see in the following section, where we consider symmetries.

4. Noether's Theorem for Multivector Lagrangians

One of the most powerful ways of analysing the equations of motion resulting from a Lagrangian is via the symmetry properties of the Lagrangian itself. The general tool for doing this is Noether's theorem, and it is important that an analogue of this can be found for the case of multivector Lagrangians. There turn out to be two types of symmetry to be considered, depending on whether the transformation of variables is governed by a scalar or by a multivector parameter. We will look at these separately.

It should be noted that as all our results are expressed in the language of geometric algebra, we are explicitly working in a *coordinate-free* way, and thus all the symmetry transformations considered are *active.* Passive transformations have no place in this scheme, as the introduction of an arbitrary coordinate system is an unnecessary distraction.

4.1. Scalar Controlled Transformations

Given an allowed multivector Lagrangian of the type $L = L(X_i, \dot{X}_i)$, we wish to consider variations of the variables X_i controlled by a single scalar parameter, α. We thus write $X_i' = X_i'(X_i, \alpha)$, and define $L' = L(X_i', \dot{X}_i')$, so that L' has the same functional dependence as L. Making use of the identity $L' = \langle L'A \rangle \partial_A$, we proceed as follows:

$$\partial_\alpha L' = (\partial_\alpha X_i') * \partial_{X_i'}\langle L'A \rangle \partial_A + (\partial_\alpha \dot{X}_i') * \partial_{\dot{X}_i'}\langle L'A \rangle \partial_A \tag{32}$$

$$= (\partial_\alpha X_i') * \left(\partial_{X_i'}\langle L'A \rangle - \partial_t(\partial_{\dot{X}_i'}\langle L'A \rangle)\right) \partial_A + \partial_t \left((\partial_\alpha X_i') * \partial_{\dot{X}_i'} L'\right). \tag{33}$$

If we now assume that the equations of motion are satisfied for the X_i' (which must be checked for any given case), we have

$$\partial_\alpha L' = \partial_t \left((\partial_\alpha X_i') * \partial_{\dot{X}_i'} L'\right), \tag{34}$$

and if L' is independent of α, the corresponding conserved current is $(\partial_\alpha X_i')*\partial_{\dot{X}_i'}L'$. Note how important it was in deriving this that (25) be satisfied for all A. Equation (34) is valid whatever the grades of X_i and L, and in (34) there is no need for α to be infinitesimal. If L' is not independent of α, we can still derive useful consequences from,

$$\partial_\alpha L'\big|_{\alpha=0} = \partial_t \left((\partial_\alpha X_i')*\partial_{\dot{X}_i'}L'\right)\Big|_{\alpha=0}. \tag{35}$$

As a first application of (35), consider time translation,

$$X_i'(t,\alpha) = X_i(t+\alpha) \tag{36}$$
$$\Rightarrow \partial_\alpha X_i'\big|_{\alpha=0} = \dot{X}_i, \tag{37}$$

so (35) gives (assuming there is no explicit time-dependence in L)

$$\partial_t L = \partial_t(\dot{X}_i*\partial_{\dot{X}_i}L). \tag{38}$$

Hence we can define the conserved Hamiltonian by

$$H = \dot{X}_i*\partial_{\dot{X}_i}L - L. \tag{39}$$

Applying this to (13), we find

$$H = \dot{e}_i*\partial_{\dot{e}_i}L - L \tag{40}$$
$$= \tfrac{1}{2}e_i\wedge\dot{e}_i - L \tag{41}$$
$$= \omega, \tag{42}$$

so the Hamiltonian is, of course, a bivector, and conservation implies that $\dot{\omega} = 0$, which is easily checked from the equations of motion.

There are two further applications of (35) that are worth detailing here. First, consider dilations

$$X_i' = e^{\alpha}X_i, \tag{43}$$

so (35) gives

$$\partial_\alpha L'\big|_{\alpha=0} = \partial_t(X_i*\partial_{\dot{X}_i}L). \tag{44}$$

For the Lagrangian of (13), $L' = e^{2\alpha}L$, and we find that

$$2L = \partial_t(\tfrac{1}{2}e_i\wedge e_i) \tag{45}$$
$$= 0, \tag{46}$$

so when the equations of motion are satisfied, the Lagrangian vanishes. This is quite typical of first order Lagrangians.

Second, consider rotations

$$X_i' = e^{\alpha B/2}X_i e^{-\alpha B/2}, \tag{47}$$

where B is an arbitrary constant bivector specifying the plane(s) in which the rotation takes place. Equation (35) now gives

$$\partial_\alpha L'\big|_{\alpha=0} = \partial_t\left((B\times X_i)*\partial_{\dot{X}_i}L\right), \tag{48}$$

where $B\times X_i$ is one half the commutator $[B, X_i]$. Applying this to (13), we find

$$B\times L = \partial_t(\tfrac{1}{2}e_i\wedge(B\cdot e_i)). \tag{49}$$

However, since $L=0$ when the equations of motion are satisfied, we see that

$$e_i\wedge(B\cdot e_i) \tag{50}$$

must be constant for all B. In [Lasenby *et al.*, 1992b] it is shown that this is equivalent to conservation of the metric tensor g, defined by

$$g(e^i) = e_i. \tag{51}$$

4.2. Multivector Controlled Transformations

The most general transformation we can write down for the variables X_i governed by a single multivector M is

$$X_i' = f(X_i, M), \tag{52}$$

where f and M are time-independent functions and multivectors respectively. In general f need not be grade preserving, which opens up a route to considering analogues of supersymmetric transformations.

In order to write down the equivalent equation to (34), it is useful to introduce the differential notation of [Hestenes & Sobczyk, 1984],

$$A*\partial_M f(X_i, M) = \underline{f}_A(X_i, M). \tag{53}$$

We can now proceed in a similar manner to the preceding section, and derive,

$$A*\partial_M L' = \underline{f}_A(X_i,M)*\partial_{X_i'}L' + \underline{f}_A(\dot{X}_i,M)*\partial_{\dot{X}_i'}L' \tag{54}$$
$$= \underline{f}_A(X_i,M)*\left(\partial_{X_i'}\langle L'B\rangle - \partial_t(\partial_{\dot{X}_i'}\langle L'B\rangle)\right)\partial_B$$
$$+ \partial_t\left(\underline{f}_A(X_i,M)*\partial_{\dot{X}_i'}L'\right) \tag{55}$$
$$= \partial_t\left(\underline{f}_A(X_i,M)*\partial_{\dot{X}_i'}L'\right), \tag{56}$$

where again we have assumed that the equations of motion are satisfied for the transformed variables. We can remove the A dependence from this by differentiating, to yield

$$\partial_M L' = \partial_t\left(\partial_A\underline{f}_A(X_i,M)*\partial_{\dot{X}_i'}L'\right), \tag{57}$$

and if L' is independent of M, the corresponding conserved quantity is

$$\partial_A \underline{f}_A(X_i, M) * \partial_{\dot{X}_i'} L' = \hat{\partial}_M f(X_i, \hat{M}) * \partial_{\dot{X}_i'} L', \tag{58}$$

where the hat on $\hat{M}$ denotes that this is the M acted on by ∂_M. Which form of (58) is appropriate to any given problem will depend on the context. Nothing much is gained by setting $M = 0$ in (57), as usually multivector controlled transformations are not simply connected to the identity.

In order to illustrate (57), consider reflection symmetry applied to the Lagrangian of (13), that is

$$f(e_i, n) = -ne_in^{-1} \tag{59}$$

$$\Rightarrow L' = nLn^{-1}. \tag{60}$$

Since $L = 0$ when the equations of motion are satisfied, the left hand side of (57) vanishes, and we find that

$$\tfrac{1}{2}\partial_a \underline{f}_a(e_i, n) \wedge (ne_in^{-1}) \tag{61}$$

is conserved. Now

$$\underline{f}_a(e_i, n) = -ae_in^{-1} + ne_in^{-1}an^{-1}, \tag{62}$$

so (61) becomes

$$\tfrac{1}{2}\partial_a \langle -e_i^2 an^{-1} + ne_in^{-1}ae_in^{-1}\rangle_2 = -e_i^2 n^{-1} - e_i\cdot n^{-1} ne_in^{-1} \tag{63}$$

$$= -n(e_i^2 n^{-1} + e_i\cdot n^{-1} e_i)n^{-1}. \tag{64}$$

This is basically the same as was found for rotations, and again the conserved quantity is the metric tensor g. This is no surprise since rotations can be built out of reflections, so it is natural to expect the same conserved quantities for both.

Equation (57) is equally valid for scalar Lagrangians, and for the case of reflections will again lead to conserved quantities which are those that are usually associated with rotations. For example considering

$$L = \dot{x}^2 - \omega^2 x^2, \tag{65}$$

it is not hard to show from (57) that the angular momentum $x \wedge \dot{x}$ is conserved. This shows that many standard treatments of Lagrangian symmetries [Goldstein, 1950] are unnecessarily restrictive in only considering infinitesimal transformations. The subject is richer than this suggests, but without the powerful multivector calculus the necessary formulae are simply not available.

5. Conclusions

Grassmann calculus finds a natural setting within geometric algebra, where the additional mathematical structure allows for a number of generalisations. This is illustrated by Grassmann (pseudoclassical) mechanics, which opens up a new field — that of the multivector Lagrangian. In order to carry out such generalisations, it is necessary to have available the most powerful techniques of geometric algebra. For Lagrangian mechanics it turns out that the multivector derivative fulfills this role, allowing for tremendous compactness and clarity. Elsewhere [Lasenby *et al.*, 1993] the multivector derivative is developed and presented as the natural tool for the study of Lagrangian field theory.

It is our opinion that the translation of Berezin calculus into geometric algebra will be of great benefit in other fields where Grassmann variables are routinely employed. A start on this has been made in [Lasenby *et al.*, 1992b; Lasenby *et al.*, 1992a], but clearly the potential subject matter is vast, and much work remains.

Acknowledegment

Chris Doran would like to acknowledge his gratitude to Sidney Sussex College for financial support in attending this conference.

References

Barut, A.O., & Zanghi, N. 1984. Classical Models of the Dirac Electron. *Phys. Rev. Lett.*, **52**(23), 2009.
Berezin, F.A. 1966. *The Method of Second Quantisation.* Academic Press.
Berezin, F.A., & Marinov, M.S. 1977. Particle Spin Dynamics as the Grassmann Variant of Classical Mechanics. *Annals of Physics*, **104**, 336.
de Witt, B. 1984. *Supermanifolds.* Cambridge University Press.
Doran, C.J.L., Lasenby, A.N., & Gull, S.F. 1993. *States and Operators in the Spacetime Algebra.* To appear in: *Foundations of Physics.*
Freund, P.G.O. 1986. *Supersymmetry.* Cambridge University Press.
Goldstein, H. 1950. *Classical Mechanics.* Addison Wesley.
Grassmann, H. 1877. Der Ort der Hamilton'schen Quaternionen in der Ausdehnungslehre. *Math. Ann.*, **12**, 375.
Gull, S.F. 1990. Charged Particles at Potential Steps. *In:* Weingartshofer, A., & Hestenes, D. (eds), *The Electron 1990.* Kluwer.
Hestenes, D. 1968. Multivector Calculus. *J. Math. Anal. Appl.*, **24**, 313.
Hestenes, D. 1975. Observables, Operators, and Complex Numbers in the Dirac Theory. *J. Math. Phys.*, **16**(3), 556.
Hestenes, D., & Sobczyk, G. 1984. *Clifford Algebra to Geometric Calculus.* D. Reidel Publishing.
Lasenby, A.N., Doran, C.J.L., & Gull, S.F. 1992a. *2-Spinors, Twistors and Supersymmetry in the Spacetime Algebra.* These Proceedings.
Lasenby, A.N., Doran, C.J.L., & Gull, S.F. 1992b. *Grassmann Calculus, Pseudoclassical Mechanics and Geometric Algebra.* Submitted to: *J. Math. Phys.*
Lasenby, A.N., Doran, C.J.L., & Gull, S.F. 1992c. *Twistors and Supersymmetry in the Spacetime Algebra.* In Preparation.

Lasenby, A.N., Doran, C.J.L., & Gull, S.F. 1993. *A Multivector Derivative Approach to Lagrangian Field Theory*. To appear in: *Foundations of Physics*.

INTRINSIC NON-INVARIANT FORMS OF DIRAC EQUATION

JOSEP MANEL PARRA *
Laboratori de Física Matemàtica, I.E.C.
Departament de Física Fonamental
Universitat de Barcelona
Diagonal 647, E-08028 Barcelona SPAIN

Abstract. Several results that seem to arise quite naturally from Hestenes geometric formulation of Dirac's equation, and that conflict with the standard view on the relativistic invariance of it, are openly discussed. The result is a better understanding of all quantum theory. On one hand the mathematics of relativistic quantum mechanics is made fully compatible with classical physical theories. On the other hand, the geometrical content of these mathematical operations, involving in an intrinsic manner the observer's frame, elucidates some of the most fundamental problems and profound mathematical results of quantum mechanics.

Key words: Clifford - Dirac - Hestenes - equation - spinor - relativistic - invariance - observer - intrinsic

1. The frame-dependent intrinsic Dirac-Hestenes equation

The problem we want to address here is the study of the relativistic invariance of the Dirac-Hestenes equation: $\hbar DX\mathbf{e}_{12} + \frac{e}{c}AX + mcX\mathbf{e}_0 = 0$. This is a crucial point in Hestenes' theory in spite of being dismissed in Hestenes (1990a, p. 1221) saying that:

> "Equation (33) is Lorentz invariant, despite the explicit appearance of the constants γ_0 and $\mathbf{i} = \gamma_2\gamma_1$ in it. These constants are arbitrarily specified by writing (33). They need not be identified with the vectors of a particular coordinate system, though it is often convenient to do so."

We hold that, as seems almost obvious, the explicit appearance of the constant multivectors $\mathbf{e}_0 \equiv \gamma_0$ and $\mathbf{e}_{12} \equiv \gamma_{12}$ factors makes the equation **non-Lorentz invariant** and that these factors **must** be interpreted as belonging to the particular inertial reference frame in which the equation is written.

In spite of this disagreement, which motivates the full analysis that follows, we stress from the very beginning that the final results of our analysis

* This work has received financial support from the D.G.C y T. under contract No. PB90-0482-C02-01

of the problem give additional support to Hestenes' geometric interpretation of Dirac's theory.

We start with Dirac-Hestenes' equation written in the inertial frame Σ of a Minkowski's space of metric diag$(-g,g,g,g)$with$g = \pm 1$ (Parra, 1992a)

$$\hbar(-g\mathbf{e}_0\partial_{x^0} + g(\mathbf{e}_1\partial_{x^1} + \mathbf{e}_2\partial_{x^2} + \mathbf{e}_3\partial_{x^3})X\mathbf{e}_{12} + \frac{e}{c}(\phi\mathbf{e}_0 + A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3)X + mcX\mathbf{e}_0 = 0 \tag{1}$$

where $X = \alpha + g(E_1\mathbf{e}_{01} + E_2\mathbf{e}_{02} + E_3\mathbf{e}_{03} + B_1\mathbf{e}_{23} + B_2\mathbf{e}_{31} + B_3\mathbf{e}_{12}) + \lambda\mathbf{e}_{0123}$ is the geometric Dirac field in that frame.

Now we consider another inertial frame Σ' obtained from Σ by means of a constant Lorentz transformation L. The geometric algebra formulation of the (passive) transformation law of the components of any geometric object Z is

$$Z = (Z^a\mathbf{e}_a) = Z^a(L^{b'}\mathbf{e}'_b)\mathbf{e}'_a(L^{c'}\mathbf{e}'_c)^{-1} = (L^{b'}\mathbf{e}'_b)(Z^a\mathbf{e}'_a)(L^{c'}\mathbf{e}'_c)^{-1} = (Z^{b'}\mathbf{e}'_b) \tag{2}$$

where the subindexes a, b, c cover the sixteen linear dimensions of the exterior algebra and L is the Lorentz transformation operator given by

$$L = (L^a\mathbf{e}'_a) = \beta + K_1\mathbf{e}_{01} + K_2\mathbf{e}_{02} + K_3\mathbf{e}_{03} + R_1\mathbf{e}_{23} + R_2\mathbf{e}_{31} + R_3\mathbf{e}_{12} + \sigma\mathbf{e}_{0123} \tag{3}$$

with $\beta\sigma = \mathbf{K}\cdot\mathbf{R}$ and $\beta^2 - \mathbf{K}\cdot\mathbf{K} + \mathbf{R}\cdot\mathbf{R} - \sigma^2 = 1$, in such a way that only six constant parameters are needed to specify it. Scalar 1x1, vector 4x4, and bivector 6x6 matrix representations of the Lorentz group are replaced by the single operational law (2).

The transformation of equation (1) in the Σ' frame (in fact, a mere rewriting, because no geometric object is transformed) gives:

$$\hbar(-g\mathbf{e}'_0\partial_{x'^0} + g(\mathbf{e}'_1\partial_{x'^1} + \mathbf{e}'_2\partial_{x'^2} + \mathbf{e}'_3\partial_{x'^3})(X^{a'}\mathbf{e}'_a)(L^b\mathbf{e}'_b)\mathbf{e}'_{12}(L^c\mathbf{e}'_c)^{-1} + \frac{e}{c}(A^{\mu'}\mathbf{e}'_\mu)(X^{a'}\mathbf{e}'_a) + mc(X^{a'}\mathbf{e}'_a)(L^b\mathbf{e}'_b)\mathbf{e}'_0(L^c\mathbf{e}'_c)^{-1} = 0, \tag{4}$$

which retains, expressed in Σ' accordingly to eq. (2), the constant frame fields $\mathbf{e}_{12}$ and $\mathbf{e}_0$ of Σ. This means that it is not a form-invariant equation. This result is not a defect of Hestenes' theory, but a characteristic of Dirac's equation itself. This fact was indeed clearly appreciated by Darwin (1928, p. 657) " ... here we have a system invariant in fact but not in form". Hestenes' equation being a geometric intrinsic formulation of Dirac's equation, its lack of form-invariance implies its non-invariance in fact. All terms and operators in it possess well-defined transformation properties that follow from its geometric nature, and no miraculous remedy is possible.

What happens, then, with the well-established validity of the original Dirac equation in all inertial frames? Does Hestenes' formulation deny this fact? These apparently strong objections are easily resolved in the geometric formulation. We need only multiply eq. (4) at right by the constant Lorentz factor $(L^a\mathbf{e}'_a)$ to obtain the Dirac equation that in Σ' describes the same physics that is described by (1) in Σ. We obtain:

$$\hbar(-g\mathbf{e}'_0\partial_{x'^0} + g(\mathbf{e}'_1\partial_{x'^1} + \mathbf{e}'_2\partial_{x'^2} + \mathbf{e}'_3\partial_{x'^3})(L^a\mathbf{e}'_a)(X^b\mathbf{e}'_b)\mathbf{e}'_{12} + \frac{e}{c}(A^{\mu'}\mathbf{e}'_\mu)(L^a\mathbf{e}'_a)(X^b\mathbf{e}'_b) + mc(L^a\mathbf{e}'_a)(X^b\mathbf{e}'_b)\mathbf{e}'_0 = 0 \tag{5}$$

Equation (5) is effectively the Dirac equation in Σ' we were looking for. In this equation the geometric element that plays the role of the Dirac geometric field is not X

$$X = X^b\mathbf{e}_b = (L^a\mathbf{e}'_a)(X^b\mathbf{e}'_b)(L^c\mathbf{e}'_c)^{-1} = (X^{b'}\mathbf{e}'_b) \tag{6}$$

but a different intrinsic geometric object Y

$$Y = Y^b\mathbf{e}_b = Y^{b'}\mathbf{e}'_b = (L^a\mathbf{e}'_a)(X^b\mathbf{e}'_b) = (X^{b'}\mathbf{e}'_b)(L^c\mathbf{e}'_c) \tag{7}$$

This last equation (7) also explains the reasons for our apparently cumbersome notation. When intrinsic geometric language does not suffice to clarify the points at issue, as is the case here, then the safest way is to write the objects in their full operational form including the specific set of basis operators $\mathbf{e}_a$ used. Writing the essential formula (7) in the "compact" form $Y = Y' = LX = X'L$ will fit at the same time the schemes of both Bourbakists and physicists infected by the "coordinate virus MV/C", described in Hestenes, 1992. But it will be misunderstood by both.

In this process of obtaining Hestenes' equation in Σ', the Lorentz transformation that relates Σ and Σ' performs two different kinds of transformation. From (1) to (4) it does nothing to the geometrical entities but changes their components (a passive transformation). From (4) to (5) it seems to do hardly anything on the algebra of the equations, but in fact it "actively" transforms the differential form in which the Dirac field consists. Said otherwise, while (1) and (4) are different frame representations of the same intrinsic equation, (4) and (5) are different intrinsic equations expressed in the same frame: Equation (5) has an extra geometric factor L at right.

We have then a nice explanation of the apparent paradox that puzzled Darwin, and also those that approach Hestenes' theory coming from the field of differential geometry: Dirac's equation in Σ is not Dirac's equation in Σ'. They imply each other, so there is no problem about their experimental validity or the actual working of the original matrix form in any inertial frame. But the Dirac field is, as an intrinsic geometrical object, different in each inertial frame. It is X in Σ and Y in Σ'. There is an intrinsic Dirac equation attached to each inertial reference system. Thus, precisely because it is intrinsically different in each system, it is meaningless to speak about invariance. Imposing it may produce meaningless results.

We have seen in Dirac's equation the emergence of the inertial reference system as an intrinsic ingredient of the theory. Moreover, put into mathematically compelling terms, this is Bohr's answer to all interpretational problems of quantum mechanics: until the full experimental arrangement has been set out and properly taken into account, one cannot ask meaningful questions in quantum mechanics. Hestenes theory, a transcription of Dirac's theory in terms of sound, powerful, and well–understood mathematics, shows precisely of what this experimental arrangement consists. It consists of the orthogonal tetrad attached to the inertial observer who writes the equation and performs the experiment. This orthonormal tetrad enters into the quantum mechanical description as an essential part of the wave function.

This also answers the longstanding and misleading philosophical debate about the involvement of the observer and his/her consciousness in the quantum mechanical description of nature: all that matters and finds its place into the equations is the orthonormal tetrad, that is the observer's time and oriented length standards that will be used in the experiment, and from which the standards and space-time splitting of all other physical magnitudes are derived.

All the preceeding considerations, based on mathematical expressions that follow naturally from equation (1), using only the geometric algebra, are not only in full agreement with Hestenes geometric interpretation of the Dirac field. They provide its mathematical foundation. Full understanding of the relationship between the two Dirac equations, of eq. (1) for X in Σ and eq.(5) for Y in Σ', is achieved considering how the wave function generates field observables through the same kind of bilinear transformation formula (2) that led from (1) to (5):

$$\mathbf{u}_a \equiv (X^b \mathbf{e}_b)\mathbf{e}_a(X^c\mathbf{e}_c)^t = (Y^{b'}\mathbf{e}'_b)\mathbf{e}'_a(Y^{c'}\mathbf{e}'_c)^t \tag{8}$$

where X^t means the reversed or trasposed element of X, which differs from X only in the sign of its bivector component. For the Lorentz transformation (3) the reversion is equivalent to the inversion, so (2) and (8) are really the same. Variation of the index a in (8) produces essentially all the sixteen bilinear covariants of Dirac theory, from which all physical observables are constructed. The geometric field X, whose components are in one-to-one correspondence with the eight real components or degrees of freedom of the complex Dirac spinor, acts then as a field of Lorentz transformations. A space-time dependent field that connects a space-time dependent tetrad, attached to the electron in some way to be further elucidated (Hestenes, 1990a,1990b), and the constant tetrad attached to the inertial frame of observation.

Leaving aside the problems related to the two remaining degrees of freeedom of the Dirac's wave function (Hestenes, 1990a, p. 1220), and limiting our analysis to the well–established kinematical factor or Lorentz rotation, we can describe the physical content of Hestenes theory in a more formal geometrical way.

To each observer there corresponds, for the same electron, a different field of Lorentz transformations, a different section of the Lorentz bundle over the Minkowski space-time. This should not be confused with a section of the frame bundle, that would give the wave function a status independent of the observer, a fact incompatible with the analysis performed above. Dirac's wave function is the operator that globally relates the constant (flat) frame section of the observer to the space-time dependent (twisted) frame section of the electron. The (passive) change of inertial observers, implied in any consideration of relativistic invariance, is precisely a change of the flat section of the frame bundle in which the electron's proper frame is to be expressed according to (8). The left translation of the Dirac field by a space-time constant Lorentz transformation relating two "flat" inertial frames, makes it possible for the second observer to find his own relationship with the same electron's proper frame from the data used by the first.

The physical content of Wigner's representation theory is then fully understood, and can no longer stand as a paradigmatic example of an "unreasonable effectiveness of mathematics". The left action of the full Lorentz group upon the wave function of an electron described in one inertial frame gives, as its "orbit", the set of all inertially-equivalent descriptions of the electron. And this set, considered as a whole, obviously constitutes an observer-free description of an electron state.

2. Spinor theory versus geometric algebra

It is the (active) left-translation expressed in (7) that has been mistakenly conceived and "understood" as the spin–1/2 (passive) transformation formula. The origin of this confusion was the unjustified belief that Dirac's equation is an invariant equation. The analysis we have performed shows that it is "intrinsically" not invariant. Further, there is no need to postulate such invariance for obtaining full agreement not only with the experimental evidence of its validity in all inertial frames but also with some of the most fundamental points of Bohr's interpretation of quantum mechanics. Plain acceptance of this fact renders the introduction of spinor fields truly meaningless: they were defined in terms of a (passive) transformation law that assured the invariant character of Dirac's equation. In the Hestenes formulation of Dirac theory the spinor fields are simply absent.

It is now possible to express clearly the fact that the even multivectors called spinors by Hestenes, of which the X in the Dirac equation is an example, have nothing to do with the "mathematical" spinors. Their components behave, under any passive transformation, as those belonging to an aggregate of differential forms of different degree (Graf, 1978), not as any spin-1/2 representation. Hestenes' spinors are not a space of representation of constant (passive) Lorentz transformations expressed in the matrix form of their covering group. Hestenes' spinors are, essentially, space-time dependent Lorentz transformation operators expressed in a direct intrinsic (and tensorial) geometrical form, and it is obviously misleading to persist in calling them spinors. We end by an attempt to clarify some misunderstandings that this misuse of the word "spinor" may have produced (see also Parra, 1992b).

Any claimed **physical equivalence** between the passive transformation law (6) and the active transformation law (7) can no longer be supported. They are not **alternative views** that can be held in a coherent geometric interpretation of Dirac theory. Both, applied in succession, were required to establish the agreement of the theory with the special relativity requirements. Geometrical understanding of the theory is greatly improved when each is conceived as it is. That the physical system of observation is intrinsically present in the wave function is not a result that can be dismissed as a matter of convention. Nor is it a matter of opinion, requiring no closer examination, that there may be no "spinor matter" and that the physical application of the extensive mathematical research on spinors and twistors may be 'flawed' from the very beginning. A sufficient reason for these almost unbelievable developments may well be that in 1928, when physicists had to tackle with the problem of the relativistic invariance of Dirac's equation, geometric Clifford algebra was, still, unbelievably absent!

References

Darwin, C. G.: 1928, 'The wave equations of the electron' *Proc. Royal Soc. London* **118**, 654-680.

Graf, W.: 1978, 'Differential forms as spinors', *Ann. Inst. Henri Poincaré Sec. A* **29**, 85-109.

Hestenes, D.: 1990 a, 'The Zitterbewegung Interpretation of Quantum Mechanics' *Found. Phys.* **20**, 1213-1232.

Hestenes, D.: 1990 b, 'Real Dirac Theory', Tempe, Arizona.(Draft, 89 pp. private communication).

Hestenes, D.: 1992, in A. Micali, R. Boudet and J. Helmstetter, ed(s)., *Proceedings of the Second International Conference on Clifford Algebras and Their Applications to Physics, 3-16*, Mathematical viruses, Kluwer:Dordrecht

Parra, J. M.: 1992 a, in A. Micali, R. Boudet and J. Helmstetter, ed(s)., *Proceedings of the Second International Conference on Clifford Algebras and Their Applications to Physics, 463-467*, On Dirac and Dirac-Darwin-Hestenes equations, Kluwer:Dordrecht

Parra J. M.: 1992 b, 'The Dirac-Hestenes equation and the algebraic structure of the Minkowski space-time', XIX ICGTMP, Salamanca (Spain) July-1992, to appear.

2-SPINORS, TWISTORS AND SUPERSYMMETRY IN THE SPACETIME ALGEBRAS

ANTHONY LASENBY
MRAO, Cavendish Laboratories, Madingley Road, Cambridge CB3 0HE, UK
CHRIS DORAN*
DAMTP, Silver Street, Cambridge, CB3 9EW, UK
and
STEVE GULL
MRAO, Cavendish Laboratories, Madingley Road, Cambridge CB3 0HE, UK

Abstract. We present a new treament of 2-spinors and twistors, using the spacetime algebra. The key rôle of bilinear covariants is emphasized. As a by-product, an explicit representation is found, composed entirely of real spacetime vectors, for the Grassmann entities of supersymmetric field theory.

1. Introduction

The aim of this presentation is to give a new translation of 2-spinors and twistors into the language of Clifford algebra. This has certainly been considered before [Ablamowicz *et al.* , 1982; Ablamowicz & Salingaros , 1985], but we differ from previous approaches by using the language of a particular form of Clifford algebra, the spacetime algebra (henceforth STA), in which the stress is on working in real 4-dimensional spacetime, with no use of a commutative scalar imaginary i. Moreover, the quantities which are Clifford multiplied together are always taken to be real geometric entities (vectors, bivectors, *etc.*), living in spacetime, rather than complex entities living in an abstract or internal space. Thus the real space geometry involved in any equation is always directly evident.

That such a translation can be achieved may seem surprising. It is generally believed that complex space notions and a unit imaginary i are fundamental in areas such as quantum mechanics, complex spin space, and 2-spinor and twistor theory. However using the spacetime algebra, it has already shown [Hestenes , 1975] how the i appearing in the Dirac, Pauli and Schrödinger equations has a geometrical explanation in terms of rotations in

* Supported by a SERC studentship.

real spacetime. Here we extend this approach to 2-spinors and twistors, and thereby achieve a reworking that we believe is mathematically the simplest yet found, and which lays bare very clearly the real (rather than complex) geometry involved.

As another motivation for what follows, we should point out that the scheme we present has great computational power, both for hand working, and on computers. Every time two entities are written side by side algebraically a Clifford product is implied, thus all our expressions can be programmed into a computer in a completely definite and explicit fashion. There is no need either for an abstract spin space, containing objects which have to be operated on by operators, or for an abstract index convention. The requirement for an explicit matrix representation is also avoided, and all equations are automatically Lorentz invariant since they are written in terms of geometric objects.

Due to the restriction on space, we will only consider the most basic levels of 2-spinor and twistor theory. There are many more results in our translation programme for 2-spinors and twistors that have already been obtained, in particular for higher valence twistors, the conformal group on spacetime, twistor geometry and curved space differentiation, and these will be presented with proper technical details in a forthcoming paper [Lasenby *et al.* , 1992c]. However, by spending some time being precise about the nature of our translation, we hope that even the basic level results presented here will still be of use and interest. A short introduction is also given of the equivalent process for field supersymmetry, and we end by discussing some implications for the rôle of 2-spinors and twistors in physics.

2. The Spacetime Algebra

The spacetime algebra is the geometric (Clifford) algebra of real 4–dimensional spacetime. Geometric algebra and the geometric product are described in detail in [Hestenes & Sobczyk , 1984]. Our own conventions follow those of this reference, and are also described in [Lasenby *et al.* , 1992a]. Briefly we define a *multivector* as a sum of Clifford objects of arbitrary grade (grade 0 = scalar, grade 1 = vector, grade 2 = bivector, *etc.*). These are equipped with an associative (geometric) product. We will also need the operation of *reversion* which reverses the order of multivectors,

$$(AB)\tilde{} = \tilde{B}\tilde{A}, \tag{1}$$

but leaves vectors (and scalars) unchanged, so it simply reverses the order of the vectors in any product.

The Clifford algebra for 3-dimensional Euclidean space is generatated by three orthonormal vectors $\{\sigma_k\}$, and is spanned by

$$1, \qquad \{\sigma_k\}, \qquad \{i\sigma_k\}, \qquad i \tag{2}$$

where $i = \sigma_1\sigma_2\sigma_3$ is the *pseudoscalar* (highest grade multivector) for the space. The pseudoscalar i squares to -1, and commutes with all elements of the algebra in this 3-dimensional case, so is given the same symbol as the unit imaginary. Note, however, that it has a definite geometrical rôle as on oriented volume element, rather than just being an imaginary scalar. For future clarity, we will reserve the symbol j for the uninterpreted commutative imaginary i, as used for example in conventional quantum mechanics and electrical engineering. The algebra (2) is the Pauli algebra, but in geometric algebra the three Pauli σ_k are no longer viewed as three matrix-valued components of a single isospace vector, but as three independent basis vectors for real space.

A quantum spin state contains a pair of complex numbers, ψ_1 and ψ_2

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{3}$$

and has a one to one correspondence with an even multivector ψ. A general even element can be written as $\psi = a^0 + a^k i\sigma_k$, where a^0 and the a^k are scalars (summation convention assumed), and the correspondence works via the basic identification

$$|\psi\rangle = \begin{pmatrix} a^0 + ja^3 \\ -a^2 + ja^1 \end{pmatrix} \leftrightarrow \psi = a^0 + a^k i\sigma_k. \tag{4}$$

We will call ψ a *spinor*, as one of its key properties is that it has a single-sided transformation law under rotations (section 3).

To show that this identification works, we also need the translation of the angular momentum operators on spin space. We will denote these operators $\hat{\sigma}_k$, where as usual

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

The translation scheme is then

$$|\phi\rangle = \hat{\sigma}_k |\psi\rangle \leftrightarrow \phi = \sigma_k \psi \sigma_3 \qquad (k = 1, 2, 3). \tag{6}$$

Verifying that this works is a matter of computation, e.g.

$$\hat{\sigma}_x |\psi\rangle = \begin{pmatrix} -a^2 + ja^1 \\ a^0 + ja^3 \end{pmatrix} \leftrightarrow -a^2 + a^3 i\sigma_1 - a^0 i\sigma_2 + a^1 i\sigma_3 = \sigma_1 \left(a^0 + a^k i\sigma_k\right) \sigma_3, \tag{7}$$

demonstrates the correspondence for $\hat{\sigma}_x$. Finally we need the translation for the action of j upon a state $|\psi\rangle$. This can be seen to be

$$|\phi\rangle = j\,|\psi\rangle \leftrightarrow \phi = \psi i\sigma_3. \tag{8}$$

We note this operation acts solely to the *right* of ψ. The significance of this will be discussed later.

An implicit notational convention should be apparent above. Conventional quantum states will always appear as bras or kets, while their STA equivalents will be written using the same letter but without the brackets. Operators (e.g. upon spin space) will be denoted by carets. We do not at this stage need a special notation for operators in STA, because the rôle of operators is taken over by right or left multiplication by elements from the same Clifford algebra as the spinors themselves are taken from. This is the first example of a conceptual unification afforded by STA — 'spin space' and 'operators upon spin space' become united, with both being just multivectors in real space. Similarly the unit imaginary j is disposed of to become another element of the same kind, which in the next section we show has a clear geometrical meaning.

In order to extend these results to 4-dimensional spacetime, we need the full 16-component STA, which is generated by four vectors γ_μ. This has basis elements 1 (scalar), γ_μ (vectors), $i\sigma_k$ and σ_k (bivectors), $i\gamma_\mu$ (pseudovectors) and i (pseudoscalar) ($\mu = 0, \ldots, 3$; $k = 1, 2, 3$). The even elements of this space, 1, σ_k, $i\sigma_k$ and i, coincide with the full Pauli algebra. Thus vectors in the Pauli algebra become bivectors as viewed from the Dirac algebra. The precise definitions are

$$\sigma_k \equiv \gamma_k\gamma_0 \quad \text{and} \quad i \equiv \gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_1\sigma_2\sigma_3. \tag{9}$$

Note that though these algebras share the same pseudoscalar i, this *anti*-commutes with the spacetime vectors γ_μ. Note also that reversion in this algebra (also denoted by a tilde — $\tilde{R}$), reverses the sign of all bivectors, so does not coincide with Pauli reversion. In matrix terms this is the difference between the Hermitian and Dirac adjoints. It should be clear from the context which is implied.

A 4-component Dirac column spinor $|\psi\rangle$ is put into a one to one correspondence with an even element of the Dirac algebra ψ [Gull , 1990] via

$$|\psi\rangle = \begin{pmatrix} a^0 + ja^3 \\ -a^2 + ja^1 \\ -b^3 + jb^0 \\ -b^1 - jb^2 \end{pmatrix} \leftrightarrow \psi = a^0 + a^k i\sigma_k + i(b^0 + b^k i\sigma_k). \tag{10}$$

The resulting translation for the action of the operators $\hat{\gamma}_\mu$ is

$$\hat{\gamma}_\mu\,|\psi\rangle \leftrightarrow \gamma_\mu\psi\gamma_0 \quad (\mu = 0, \ldots, 3), \tag{11}$$

which follows if the $\hat{\gamma}$ matrices are defined in the standard Dirac-Pauli representation [Bjorken & Drell , 1964]. Verification is again a matter of computation, and further details will be given in [Doran *et al.* , 1993]. The action of j is the same as in the Pauli case,

$$j\,|\psi\rangle \leftrightarrow \psi\, i\sigma_3. \tag{12}$$

3. Rotations and Bilinear Covariants

In STA, the vectors σ_k are simply the basis vectors for 3-dimensional space, which means that the translation (6) for the action of the $\hat{\sigma}_k$ can be recast in a particularly suggestive form. Let n be a unit vector, then the eigenvalue equation for the measurement of spin in a direction n is conventionally

$$n\cdot\hat{S}\,|\psi\rangle = \pm\frac{\hbar}{2}\,|\psi\rangle, \tag{13}$$

where in this scheme $\hat{S}$ is a 'vector', with 'components' $\hat{S}_k = (\hbar/2)\hat{\sigma}_k$. Now $n\cdot\hat{S} = \frac{\hbar}{2}n^k\hat{\sigma}_k$, so the STA translation for this equation is just

$$n\psi\sigma_3 = \pm\psi, \tag{14}$$

where n is a (true) vector in ordinary 3-dimensional space. Multiplying on the right by $\sigma_3\tilde{\psi}$ ($\tilde{\psi} = a^0 - a^k i\sigma_k$), yields

$$n\psi\tilde{\psi} = \pm\psi\sigma_3\tilde{\psi}. \tag{15}$$

Now $\psi\tilde{\psi}$ is a scalar in the Pauli case

$$|\psi|^2 \equiv \psi\tilde{\psi} = \tilde{\psi}\psi \tag{16}$$

$$= (a^0)^2 + (a^1)^2 + (a^2)^2 + (a^3)^2, \tag{17}$$

so we can write

$$n = \pm\frac{\psi\sigma_3\tilde{\psi}}{|\psi|^2}. \tag{18}$$

This shows that the wavefunction ψ is in fact an instruction on how to rotate the fixed reference direction σ_3 and align it parallel or anti-parallel with the desired direction n. The amplitude just gives a change of scale. This idea, of taking a fixed or 'fiducial' direction, and transforming it to give the particle spin axis, is a central one for the development of our physical interpretation of quantum mechanics.

In the relativistic case, $\psi\tilde{\psi}$ is not necessarily a pure scalar, and we have $\psi\tilde{\psi} = \tilde{\psi}\psi = \rho e^{i\beta}$. The relativistic wavefunction ψ now specifies a spin axis s via $s = \rho^{-1}\psi\gamma_3\tilde{\psi}$, and a complete set of body axes e_μ via

$$e_\mu = \rho^{-1}\psi\gamma_\mu\tilde{\psi}. \tag{19}$$

$e_0 = v$ is interpreted as the particle 4-velocity, while ρv is the standard Dirac probabilty current — see [Doran *et al.* , 1993] for further details. The main change in viewpoint on going to the STA should now be apparent — instead of the discrete and discontinous language of operators, eigenstates and eigenvalues we now have the idea of continuous families of transformations. This enables us to give a realistic physical description of particle tracks and spin directions in interaction with external apparatus [Lasenby *et al.* , 1992b].

One of the great advantages of geometric algebra is the way that rotation of a general multivector is achieved in exactly the same fashion as for a single vector. Thus to discuss Lorentz rotations for example, let us write $\psi = \rho^{1/2}e^{i\beta/2}R$. Then R is an even multivector satisfying $R\tilde{R} = \tilde{R}R = 1$ and therefore corresponds to a Lorentz rotation (combination of pure boost and spatial rotation). To rotate an arbitrary multivector M we just form the analogue of (19) and write

$$M' = RM\tilde{R}. \tag{20}$$

This is a very quick way of obtaining the transformation formulae for electric and magnetic fields for example. If we use the whole wavefunction, which incorporates information about the particle density, ρ, and also the β factor, and use it to rotate a given fixed Clifford entity such as the γ_0 and γ_3 considered above, then we get a physical density for some quantity. For example, the spin angular momentum density for a Dirac particle is the bivector $\frac{1}{2}\hbar\psi i\sigma_3\tilde{\psi}$. (Note the combination $\psi\ldots\tilde{\psi}$ preserves grade for objects of grade 1, 2 and 3.) Such expressions can generally be written equivalently as bilinear covariants in conventional Dirac theory notation — for example, $\rho v = \psi\gamma_0\tilde{\psi}$, the Dirac current, would be written conventionally as $j^\mu = \langle\overline{\psi}|\hat{\gamma}^\mu|\psi\rangle$ — but in the STA version the meaning of the expression is usually much clearer. We mention this point, since it will transpire that many of the quantities of importance for 2-spinors and twistors turn out to be bilinear covariants of the above kind, which could therefore in principle also be translated into the Dirac notation, but again, look more straightforward in our version.

As a final comment, we should discuss the way in which specific Clifford elements such as γ_0 and $i\sigma_3$ enter expressions such as $\rho v = \psi\gamma_0\tilde{\psi}$, and why general Lorentz covariance is not compromised by this. What is happening [Lasenby *et al.* , 1992c] is that the wavefunction ψ is an instruction to rotate *from* some fixed set of multivectors *to* the configuration required (by the Dirac equation for example) at some given spacetime point. If we desire the final configurations (at all positions) to be rotated an extra amount R, then we must use a new wavefunction $\psi' = R\psi$. This of course explains the usual spinor transformation law under a global rotation of space, but also shows us why we do *not* want to rotate the elements we started from as well. Thus general covariance and invariance under global Lorentz rotations is assured if all quantities appearing to the *left* of the wavefunction make no mention of specific axes, directions *etc.*, while those to the *right* are allowed to do so, but must remain fixed under such a rotation.

As a complementary exercise, one might decide to rotate the elements (such as γ_0, $i\sigma_3$, *etc.*) we start from, by R say, leaving the *final* configuration fixed. In this case we have $\psi' = \psi\tilde{R}$. This is what happens under a change of

'phase' for example, where $|\psi\rangle \mapsto e^{j\theta}|\psi\rangle$. Here the STA equivalent undergoes $\psi \mapsto \psi e^{\theta i\sigma_3}$, which thus corresponds to a rotation of starting orientation through 2θ radians about the fiducial σ_3 direction. The action of j itself is thus a rotation through π about the σ_3 axis. Note particularly that only one copy of real spacetime is necessary to represent what is going on in this process.

4. 2-spinors

Having been explicit about our translation of quantum Dirac and Pauli spinors, we are now in a position to begin the translation of 2-spinor theory. For the latter we adopt the notation and conventions of the standard exposition, [Penrose & Rindler , 1984; Penrose & Rindler , 1986].

The basic translation is as follows. In 2-spinor theory, a spinor can be written either as an abstract index entity κ^A, or as a complex spin vector in spin-space (just like a quantum Pauli spinor) $\underline{\boldsymbol{\kappa}}$. We put a 2-spinor κ^A in 1-1 correspondence with a Clifford spinor κ via

$$\kappa^A \leftrightarrow \kappa(1+\sigma_3), \tag{21}$$

where κ is the Clifford Pauli spinor in one to one correspondence with the column spinor $\underline{\boldsymbol{\kappa}}$ (via 4). The function of the 'fiducial projector' $(1+\sigma_3)$ (actually half this must be taken to get a projection operator) relates to what happens under a 'spin transformation' represented by an arbitrary complex spin matrix $\underline{\boldsymbol{R}}$. The new spin vector is $\underline{\boldsymbol{R\kappa}}$ and has only 4 real degrees of freedom, whereas an arbitrary Lorentz rotation specified by a Clifford R applied to a Clifford κ gives the quantity $R\kappa$, which contains 8 degrees of freedom. However, applying R to $\kappa(1+\sigma_3)$ limits the degrees of freedom back to 4 again, in conformity with what happens in the 2-spinor formulation.

The complex conjugate spinor $\overline{\kappa}^{A'}$ belongs to the opposite ideal under the action of the projector $(1+\sigma_3)$,

$$\overline{\kappa}^{A'} \leftrightarrow -\kappa i\sigma_2(1-\sigma_3). \tag{22}$$

This explains why κ^A and its complex conjugate have to be treated as belonging to different 'modules' in the Penrose and Rindler theory. Note that in more conventional quantum notation our projectors $(1\pm\sigma_3)$ would correspond to the chirality operators $(1\pm j\hat{\gamma}_5)$, or in the notation of the appendix of [Penrose & Rindler , 1986], to (multiples of) $\underline{\boldsymbol{\Pi}}$ and $\underline{\tilde{\boldsymbol{\Pi}}}$. We do not use these alternative notations since it is a vital part of what we are doing that the projection operators should be constructed from ordinary spacetime entities.

The most important quantities associated with a single 2-spinor κ^A are its *flagpole* $K^a = \kappa^A\overline{\kappa}^{A'}$, and the *flagplane* determined by the bivector $P^{ab} =$

$\kappa^A\kappa^B\epsilon^{A'B'} + \epsilon^{AB}\bar{\kappa}^{A'}\bar{\kappa}^{B'}$. Here we use the Penrose notation in which a is a 'lumped index' representing the spinor indices AA' *etc.* Now in order to get a precise translation for quantities like $\kappa^A\bar{\kappa}^{A'}$, or $\kappa^A\kappa^B\epsilon^{A'B'}$, it is necessary to develop 'multiparticle STA' [Lasenby *et al.* , 1992c]. This still involves real spacetime, but with a separate copy for each particle. We have carried this out and thereby found the STA equivalents of 2-spinor outer product expressions. However, we have also discovered a mapping from the spin-$\frac{1}{2}$ space of a single spinor to the spin-1 space of general complex world vectors (as Penrose & Rindler call them), which applied in reverse enables us to find 'spin-$\frac{1}{2}$' (*i.e.* just one copy of spacetime) equivalents for the lumped index expressions. It is these equivalents we give now, and proper proofs are contained in [Lasenby *et al.* , 1992c].

Firstly, if we write $\psi = \kappa(1+\sigma_3)$, the flagpole of the 2-spinor κ^A is just (up to a factor 2) the Dirac current associated with the wavefunction ψ,

$$K = \tfrac{1}{2}\psi\gamma_0\tilde{\psi} = \kappa(\gamma_0+\gamma_3)\tilde{\kappa}. \tag{23}$$

We see that the projector $(1+\sigma_3)$ has produced a massless (null) current.

Secondly, the flagplane bivector is a rotated version of the fiducial bivector σ_1:

$$P = \tfrac{1}{2}\psi\sigma_1\tilde{\psi} = \kappa(\gamma_1\wedge(\gamma_0+\gamma_3))\tilde{\kappa}. \tag{24}$$

Since σ_1 anticommutes with $i\sigma_3$, while γ_0 commutes, P responds at double rate to phase rotations $\kappa \mapsto \kappa e^{i\sigma_3\theta}$, whilst the flagpole is unaffected. A convenient spacelike vector L, perpendicular to the flagpole and satisfying $P = L\wedge K$, is $L = (\kappa\tilde{\kappa})^{-1/2}\kappa\gamma_1\tilde{\kappa}$, that is, just the 'body' 1-direction.

In 2-spinor theory, a 'spin-frame' is usually written o^A, ι^A, but for notational reasons, and to draw out the parallel with twistors, we prefer to write these as ω^A, π^A. In our translation, a spin-frame ω^A, π^A is packaged together to form a Clifford Dirac spinor ϕ via

$$\phi = \omega\tfrac{1}{2}(1+\sigma_3) - \pi i\sigma_2\tfrac{1}{2}(1-\sigma_3). \tag{25}$$

Now

$$\phi\tilde{\phi} = \tfrac{1}{2}\kappa(1+\sigma_3)i\sigma_2\tilde{\omega} + \text{ reverse} = \lambda + i\mu \qquad \text{say.} \tag{26}$$

If one now calculates the 2-spinor inner product for the same spin-frame one finds

$$\{\underline{\omega}, \underline{\pi}\} = \omega_A\pi^A = -(\lambda + j\mu). \tag{27}$$

Thus the complex 2-spinor inner product is in fact a disguised version of the quantity $\phi\tilde{\phi}$. The 'disguise' consists of representing something that is in fact a pseudoscalar (the i in $\lambda+i\mu$) as an uninterpreted scalar j. The condition for

a spin frame to be normalized, $\omega_A\pi^A = 1$, is in our approach the condition for ϕ to be a Lorentz transformation, that is $\phi\tilde{\phi} = 1$ (except for a change of sign which in twistor terms corresponds to negative helicity). We can thus say "*a normalized spin frame is equivalent to a Lorentz transformation*".

The orthonormal real tetrad, t^a, x^a, y^a, z^a, determined by such a spin-frame [Penrose & Rindler , 1984, p120], is in fact the same (up to signs) as the frame of 'body axes' $e_\mu = \phi\gamma_\mu\tilde{\phi}$ which we drew attention to in standard Dirac theory, whilst the null tetrad is just a rotated version of a certain 'fiducial' null tetrad as follows:

$$l^a = \frac{1}{\sqrt{2}}(t^a + z^a) = \omega^A\bar{\omega}^{A'} \leftrightarrow \phi(\gamma_0+\gamma_3)\tilde{\phi}, \tag{28}$$

$$n^a = \frac{1}{\sqrt{2}}(t^a - z^a) = \pi^A\bar{\pi}^{A'} \leftrightarrow \phi(\gamma_0-\gamma_3)\tilde{\phi}, \tag{29}$$

$$m^a = \frac{1}{\sqrt{2}}(x^a - jy^a) = \omega^A\bar{\pi}^{A'} \leftrightarrow -\phi(\gamma_1+i\gamma_2)\tilde{\phi}, \tag{30}$$

$$\bar{m}^a = \frac{1}{\sqrt{2}}(x^a + jy^a) = \pi^A\bar{\omega}^{A'} \leftrightarrow -\phi(\gamma_1-i\gamma_2)\tilde{\phi}. \tag{31}$$

Note that the x or y axis is inverted with respect to the world vector equivalents, which is a feature that occurs throughout our translation of 2-spinor theory. Note also that $\gamma_1 - i\gamma_2$ and $\gamma_1 + i\gamma_2$ involve *trivector* components. This is how complex world vectors in the Penrose & Rindler formalism appear when translated down to equivalent objects in a single-particle STA space. We shall find a use for these shortly as supersymmetry generators.

5. Valence-1 Twistors

On page 47 of [Penrose & Rindler , 1986] the authors state '*Any temptation to identify a twistor with a Dirac spinor should be resisted. Though there is a certain formal resemblance* at one point, *the vital twistor dependence on position has no place in the Dirac formalism.*' We argue on the contrary that a twistor *is* a Dirac spinor, with a particular dependence on position imposed. Our fundamental translation is

$$Z = \phi - r\phi\gamma_0 i\sigma_3\tfrac{1}{2}(1+\sigma_3), \tag{32}$$

where ϕ is an arbitrary constant relativistic STA spinor, and $r = x^\mu\gamma_\mu$ is the position vector in 4-dimensions. To start making contact with the Penrose notation, we decompose the Dirac spinor Z, quite generally, as

$$Z = \omega\tfrac{1}{2}(1+\sigma_3) - \pi i\sigma_2\tfrac{1}{2}(1-\sigma_3). \tag{33}$$

Then the pair of Pauli spinors ω and π are the translations of the 2-spinors ω^A and $\pi_{A'}$ appearing in the usual Penrose representation

$$Z^\alpha = (\omega^A, \pi_{A'}). \tag{34}$$

In (34) $\pi_{A'}$ is constant and ω^A is meant to have the fundamental twistor dependence on position

$$\omega^A = \omega_0^A - jx^{AA'}\pi_{A'}, \tag{35}$$

where ω_0^A is constant. We thus see that the arbitary constant spinor ϕ in (32) is

$$\phi = \omega_0 \tfrac{1}{2}(1+\sigma_3) - \pi\, i\sigma_2 \tfrac{1}{2}(1-\sigma_3). \tag{36}$$

We note this is identical to the STA representation of a spin-frame.

This ability, in the STA, to package the two parts of a twistor together, and to represent the position dependence in a straightforward fashion, leads to some remarkable simplifications in twistor analysis. This applies both with regard to connecting the twistor formalism with physical properties of particles (spin, momentum, helicity, etc.), and to the sort of computations required for establishing the geometry associated with a given twistor.

For present purposes, we confine ourselves to establishing the link with massless particles, and define a set of quantities to represent various properties of such particles (most of which are useful in the formulation of twistor geometry as well). These are basically just the bilinear covariants of Dirac theory, adapted to the massless case. Firstly, the null momentum associated with the particle is

$$p = Z\,(\gamma_0 - \gamma_3)\,\tilde{Z}. \tag{37}$$

This is constant (independent of spacetime position), since

$$Z\,(\gamma_0 - \gamma_3)\,\tilde{Z} = \phi\,(\gamma_0 - \gamma_3)\,\tilde{\phi} = \pi\,(1+\sigma_3)\tilde{\pi}\,\gamma_0. \tag{38}$$

p thus points in the flagpole direction of π. Secondly, the flagpole of the twistor itself, defined as the flagpole of its principal part ω^A, is the null vector

$$w = Z\,(\gamma_0 + \gamma_3)\,\tilde{Z}. \tag{39}$$

Evaluated at the origin, this becomes

$$w_0 = \phi\,(\gamma_0 + \gamma_3)\,\tilde{\phi} = \omega_0\,(1+\sigma_3)\,\tilde{\omega}_0\,\gamma_0. \tag{40}$$

Thirdly, we define an angular momentum bivector in the usual way for Dirac theory (see above)

$$M = Z\, i\sigma_3\,\tilde{Z}. \tag{41}$$

Substituting from (32) for Z yields (in two lines)

$$M = M_0 + r\wedge p, \tag{42}$$

where the constant part M_0 is given by

$$M_0 = \phi\, i\sigma_3\,\tilde{\phi}. \tag{43}$$

This angular momentum coincides with the real skew tensor field

$$M^{ab} = i\omega^{(A}\overline{\pi}^{B)}\epsilon^{A'B'} - i\overline{\omega}^{(A'}\pi^{B')}\epsilon^{AB}, \tag{44}$$

on page 68 of [Penrose & Rindler , 1986], who have

$$M^{ab} = M_0^{ab} - x^a p^b + x^b p^a. \tag{45}$$

The key calculation showing that (41) is the correct angular momentum, is to demonstrate that the Pauli-Lubanski vector for this massless case is proportional to the momentum. In the STA, the Pauli-Lubanski vector (the non-orbital part of the angular momentum, expressed as a vector) is given generally by

$$S = p\cdot(iM). \tag{46}$$

Now $p\cdot(iM) = p\cdot(iM_0 + ir\wedge p)$ and $p\cdot(ir\wedge p) = -i(p\wedge r\wedge p) = 0$. Also

$$p\,iM_0 = \phi\,(\gamma_0 - \gamma_3)\,\tilde{\phi}\, i\phi\, i\sigma_3\,\tilde{\phi}, \tag{47}$$

so that writing $\phi\tilde{\phi} = \tilde{\phi}\phi = \rho e^{i\beta}$, we have

$$p\,iM_0 = -\rho e^{-i\beta}\,\phi(-\gamma_3 + \gamma_0)\,\tilde{\phi} \tag{48}$$

and therefore

$$S = -\rho\cos\beta\, p. \tag{49}$$

The helicity s is thus just minus the scalar part of the product $\phi\tilde{\phi}$.

6. Field Supersymmetry Generators

A common version of the field supersymmetry generators required for the Poincaré super-Lie algebra uses 2-spinors Q_α with Grassmann entries:

$$Q_\alpha = -i\left(\frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\alpha'}\overline{\theta}^{\alpha'}\partial_\mu\right), \tag{50}$$

where the θ^α and $\overline{\theta}^\alpha$ are Grassmann variables, and μ is a spatial index [Freund , 1986; Srivastava , 1986; Müller-Kirsten & Wiedemann , 1987]. A translation of Q_α into STA basically amounts to finding real spacetime representations for the θ^α variables. Using 2-particle STA we have found such representations, and they turn out to be two distinct copies of the complex null tetrad discussed above. The two copies arise in a natural fashion in our version of 2-spinor theory, but are harder to spot in a conventional approach.

This has an interesting 'single particle' equivalent, using the 4 quantities $\gamma_0 \pm \gamma_3$ and $\gamma_1 \pm i\gamma_2$ as effective Grassmann variables, with the anticommutator $\{A, B\}$ replaced by the symmetric product $\langle \tilde{A}B \rangle$. With

$$\theta_1 = \gamma_0 + \gamma_3 \quad \overline{\theta}_1 = \gamma_0 - \gamma_3$$
$$\theta_2 = \gamma_1 + i\gamma_2 \quad \overline{\theta}_2 = -\gamma_1 + i\gamma_2$$

it is a simple exercise to verify that the θ_α satisfy the required supersymmetry algebra (with $\{A, B\} \equiv \langle \tilde{A}B \rangle$)

$$\{\theta_\alpha, \theta_\beta\} = \{\overline{\theta}_\alpha, \overline{\theta}_\beta\} = 0, \qquad \{\theta_\alpha, \overline{\theta}_\beta\} = 2\delta_{\alpha\beta}. \tag{51}$$

This raises interesting new possibilites, similar to those outlined in [Doran *et al.* , 1992], of being able to reduce the arena of 'superspace' to ordinary spacetime, without in any way diminishing its richness or interest.

7. Conclusions

When 2-spinors and twistors are absorbed into the framework of spacetime algebra, they become both easier to manipulate and interpret, and many parallels are revealed with ordinary Dirac theory. In particular the bilinear covariants of Dirac theory (expressed in STA), turn out to be precisely those needed to understand the rôle of higher valence spinors and twistors. As a byproduct of the translation we have shown that a commutative scalar imaginary is unnecessary in the formulation of 2-spinor and twistor theory. Furthermore, had space permitted, we would have presented a discussion of the mapping we have constructed between lumped vector index expressions, and spin-$\frac{1}{2}$ equivalents. This would have made it evident that the notion that 2-spinor or twistor space is more fundamental than the space of ordinary vectors or tensors, is misplaced. In our version the spinor space itself is imbued with all the metrical properties of spacetime, and the construction of vectors and tensors using outer products of spinors (as given in Penrose & Rindler for example) can be shown via our translation to use precisely the metrical properties already present at the so-called spinor level (which is in fact just ordinary spacetime).

Normalized spin-frames have been shown to be identical to Lorentz transforms, with spin frames in general identical to constant Dirac spinors (even multivectors in the STA approach). Twistors themselves have been shown to be Dirac spinors, with a particular position dependence imposed, and the physical quantities constructed from them to be just the standard Dirac bilinear covariants. It is therefore clear that some of the claims of the 'strong twistor' programme, as described in e.g. [Penrose , 1975], must appear in a new light, though the full implications remain to be worked out.

References

Ablamowicz, R., & Salingaros, N. 1985. On the Relationship Between Twistors and Clifford Algebras. *Lett. in Math. Phys.*, **9**, 149.

Ablamowicz, R., Oziewicz, Z., & Rzewuski, J. 1982. Clifford Algebra Approach to Twistors. *J. Math. Phys.*, **23**(2), 231.

Bjorken, J.D., & Drell, S.D. 1964. *Relativistic Quantum Mechanics, vol 1.* McGraw-Hill, New York.

Doran, C.J.L., Lasenby, A.N., & Gull, S.F. 1992. *Grassmann Mechanics, Multivector Derivatives and Geometric Algebra.* These Proceedings.

Doran, C.J.L., Lasenby, A.N., & Gull, S.F. 1993. *States and Operators in the Spacetime Algebra.* To appear in: *Foundations of Physics.*

Freund, P.G.O. 1986. *Supersymmetry.* Cambridge University Press.

Gull, S.F. 1990. Charged Particles at Potential Steps. *In:* Weingartshofer, A., & Hestenes, D. (eds), *The Electron 1990.* Kluwer.

Hestenes, D. 1975. Observables, Operators, and Complex Numbers in the Dirac Theory. *J. Math. Phys.*, **16**(3), 556.

Hestenes, D., & Sobczyk, G. 1984. *Clifford Algebra to Geometric Calculus.* D. Reidel Publishing.

Lasenby, A.N., Doran, C.J.L., & Gull, S.F. 1992a. *Grassmann Calculus, Pseudoclassical Mechanics and Geometric Algebra.* Submitted to: *J. Math. Phys.*

Lasenby, A.N., Gull, S.F., & Doran, C.J.L. 1992b. *Particle and Multiparticle Trajectories, Tunnelling Times and Spacetime Algebra.* In Preparation.

Lasenby, A.N., Doran, C.J.L., & Gull, S.F. 1992c. *Twistors and Supersymmetry in the Spacetime Algebra.* In Preparation.

Müller-Kirsten, H.J.W., & Wiedemann, A. 1987. *Supersymmetry.* World Scientific.

Penrose, R. 1975. Twistor Theory, its Aims and Achievements. *In:* Isham, C.J., Penrose, R., & Sciama, D.W. (eds), *Quantum Gravity — an Oxford Symposium.* Oxford University Press.

Penrose, R., & Rindler, W. 1984. *Spinors and space-time, Volume I: two-spinor calculus and relativistic fields.* Cambridge University Press.

Penrose, R., & Rindler, W. 1986. *Spinors and space-time, Volume II: spinor and twistor methods in space-time geometry.* Cambridge University Press.

Srivastava, P. 1986. *Supersymmetry, Superfields and Supergravity.* Adam Hilger.

QUANTUM DEFORMATIONS

QUANTIZED MINKOWSKI SPACE

JULIUS WESS, BRUNO ZUMINO[1] *, OLEG OGIEVETSKY ** and W.B. SCHMIDKE

Max-Planck-Institut für Physik und Astrophysik
Werner-Heisenberg-Institut für Physik
P.O. Box 40 12 12 , D - 8000 Munich 40, Germany

[1] *Department of Physics, University of California, Berkeley*

and
Theoretical Physics Group, Physics Division
Lawrence Berkeley Laboratory
1 Cyclotron Road
Berkeley, California 94720

The concept of symmetry groups has a mathematically well defined generalization in the framework of Hopf algebras. Such generalizations have become known as quantum groups- these are Hopf algebras with an algebraic structure which depends on one or more parameters q ($q \in \mathbf{C}, q \neq 0$), such that for a particular value of these parameters, say $q = 1$, the quantum group coincides with the group. In this sense a quantum group is a deformation of a group, q being a deformation parameter. With the concept of a group goes the concept of representations and representation spaces. These representation spaces find a natural generalization as well, called quantum spaces. These are algebraic structures that depend on the deformation parameter q and for $q = 1$ coincide with the linear space in which the corresponding group is represented. For $q \neq 1$, the quantum group acts as a linear morphism of the algebraic structure of the quantum space. The algebraic structure of the group and the quantum space are closely related.

One of the best known examples of a quantum group is the group $SL_q(2, \mathbf{C})$, a deformation of the group $SL(2, \mathbf{C})$ which is the covering group of the Lorentz group. Its representations are known as spinors and tensors. The

* This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY-90-21139

** On leave of absence from P.N.Lebedev Physical Institute, Theoretical Department, 117924 Moscow, Leninsky prospect 53, USSR

corresponding quantum spaces are called q-spinors and q-tensors. In this lecture I am going to to discuss the algebraic structure of a q-deformed four-vector space.It serves as a good example of quantizing Minkowski space. To give a physical interpretation of such a quantized Minkowski space we construct the Hilbert space representation and find that the relevant time and space operators have a discrete spectrum. Thus the q-deformed Minkowski space has a lattice structure. Nevertheless this lattice structure is compatible with the operation of q-deformed Lorentz transformations.The generators of the q-deformed Lorentz group can be represented as linear operators in the same Hilbert space.To be more specific and to illustrate the general concepts just mentioned let us study the 2×2 matrix A:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{1}$$

If the entries a, b, c and d are real numbers, A is an element of $GL(2, \mathbf{R})$, and if they are complex numbers, A is an element of $GL(2, \mathbf{C})$. A is an element of $GL_q(2)$ if the entries a, b, c, and d satisfy the following algebraic relations:

$$\begin{array}{lll} ab = qba & bd = qdb & ad = da + \lambda cb \\ ac = qca & cd = qdc & bc = cb \end{array} \tag{2}$$

where $\lambda = q - q^{-1}$. These relations seem quite arbitrary, they are, however, determined by the following properties.

The most important property of (2) is that matrix multiplication preserves these relations. Take a second matrix A', with entries which commute with the entries of A, and by themselves they satisfy the relations (2) as well. Take the matrix product $AA' = A''$ and you will find that the entries of A'' satisfy the relations (2) again.

The relations (2) have some further properties:

1) They allow an ordering. The left hand side is alphabetically, the right hand side antialphabetically ordered.

2) The ordering is invertible. An antialphabetic order can be alphabetically rearranged. It is possible to order a polinomial in any desired order without changing its degree.

3) The relations are consistent. By this we mean that they do not create higher order relations. The following example illustrates that this is not trivial. Take the same relations as in (2) except for $bd = q'db$, $q' \neq q$. Try to put abd into the order dba by starting to exchange first ab or by first exchanging bd. Comparing these two calculations you will find the third order relation $b^2c = 0$. The relations (2) are chosen in such a way that no such new relations will arise.

4) The relations (2) depend on the deformation parameter q and for $q = 1$ a, b, c and d commute.

All algebraic structures that we need in the context of quantum groups have these four properties. It is property 3 that is most restrictive and leads to the quantum Yang-Baxter equation. The relations (2) can be written in the form

$$A^i{}_a A^j{}_b \hat{R}^{ab}{}_{lm} = \hat{R}^{ij}{}_{cd} A^c{}_l A^d{}_m \tag{3}$$

with the 4×4 matrix

$$\hat{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \tag{4}$$

Rows and columns are labelled by (11), (12), (21), and (22). If we define the 8×8 matrix

$$\hat{R}_{12} = \hat{R}^{i_1 i_2}{}_{j_1 j_2} \delta^{i_3}{}_{j_3} \tag{5}$$

and similarly $\hat{R}_{23}$ they will satisfy the Yang-Baxter equation

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}. \tag{6}$$

This relation arises from the requirement that the product of any three elements of the matrix A can be rearranged by either first changing the first two or the last two elements. By reduction it can then be shown that the consistency condition as formulated in 3 holds for all orders.

Having in mind the importance of the Yang-Baxter equation (6) it is natural to proceed as follows. First try to find a solution of (6). Note that in the general case when A is an $n \times n$ matrix these are n^6 equations for n^4 variables. Next impose the 'RTT' relations (in general the matrix A is called T) to find a deformed group. There will always be a solution to the RTT equation, a multiple of the unit matrix will always do. We of course are interested in non-trivial solutions.

The $\hat{R}$-matrix also tells us how to define quantum spaces. The matrix $\hat{R}$ has eigenvalues and therefore satisfies a characteristic equation. From this equation we can find the projectors on the eigenvalue. These projectors will be polynomials in $\hat{R}$. In our example, the matrix (4) has the eigenvalues q and $-q^{-1}$. The characteristic equation is

$$(\hat{R} - q)(\hat{R} + q^{-1}) = 0. \tag{7}$$

The projectors are:

$$\begin{aligned} P_A &= \frac{1}{q + q^{-1}}(q - \hat{R}) \\ P_S &= \frac{1}{q + q^{-1}}(q^{-1} + \hat{R}). \end{aligned} \tag{8}$$

They satisfy $P_A P_S = 0$, $P_A^2 = P_A$, $P_S^2 = P_S$, and $P_A + P_S = \mathbb{1}$. In the limit $q \to 1$ these projectors become:

$$P_A^{ij}{}_{lm} \to \frac{1}{2}(\delta_l^i \delta_m^j - \delta_m^i \delta_l^j)$$
$$P_S^{ij}{}_{lm} \to \frac{1}{2}(\delta_l^i \delta_m^j + \delta_m^i \delta_l^j). \tag{9}$$

As both projectors are polynomials in $\hat{R}$ it follows from (3) that the following equation holds for both projectors:

$$AAP = PAA. \tag{10}$$

The matrices are as in (3). If we now define the algebraic relations of a quantum space by the condition

$$P^{ij}{}_{km} x^k x^m = 0, \tag{11}$$

it follows from (10) that the linear map

$$x'^i = A^i{}_l x^l \tag{12}$$

(where x^l and the entries of A commute) preserves the algebraic structure:

$$P^{ij}{}_{km} x'^k x'^m = 0. \tag{13}$$

From (9) follows that for $q \to 1$ the space defined with P_A becomes commutative whereas defined with P_S the coordinates will anticommute. It is natural to identify the anticommuting space with the differentials.

The $\hat{R}$-matrix can be expressed through the projectors:

$$\hat{R} = qP_S - q^{-1}P_A. \tag{14}$$

It therefore follows that a matrix that satisfies (10) for both projectors will also satisfy the RTT relations (3). In other words, if we find a linear morphism for the quantum plane and the differentials then we know that this must be an element of the quantum group defined through (3).

We now aplly another linear morphism

$$x''^i = A'^i{}_l x'^l = A'^i{}_l A^l{}_k x^k. \tag{15}$$

(A'' commutes with x', the entries of A' therefore commute with the entries of A). It follows that x'' will satisfy the same relations as x (for the coordinates and the differentials). $A'' = A'A$ will be an element of the quantum group again.

This demonstrates the power of the $\hat{R}$-matrix approach to quantum groups and quantum planes. Our aim is to construct an $\hat{R}$-matrix that allows us to deform the concept of the Minkowski space. To this end we shall construct a four-vector as a bi-spinor and use the relations (11) for spinors to derive the commutation relations for the four-vector. To be able to define reality properties we first have to define a conjugation operation on spinors.

We define the conjugate spinor:

$$\overline{x^i} = \bar{x}_i \tag{16}$$

and demand that conjugation is an involution ($\overline{\bar{x}_i} = x^i$) with the following property:

$$\overline{x^i x^j} = \bar{x}_j \bar{x}_i \qquad \bar{q} = q^* \text{ (complex conjugation)}. \tag{17}$$

From (11) follows for the conjugate quantum plane for real q:

$$\bar{x}_j \bar{x}_i = q^{-1} \hat{R}^{kl}{}_{ij} \bar{x}_l \bar{x}_k \tag{18}$$

where we have used the property of the $\hat{R}$-matrix $\hat{R}^{ij}{}_{kl} = \hat{R}^{kl}{}_{ij}$, the matrix (4) is symmetric. The $x - \bar{x}$ commutation relations have to be invented. They have to be consistent in the sense of property 3. A possible solution is [Wess 1990]

$$x^i \bar{x}_j = q \hat{R}^{-1\,li}{}_{kj} \bar{x}_l x_k. \tag{19}$$

The relations (11), (18) and (19) define the complex q-spinors, and their linear morphisms are elements of $SL_q(2, \mathbf{C})$.

Finally, we need a second copy of spinors. By themselves, they are supposed to satisfy (11), (18) and (19). Their commutation relations with x,$\bar{x}$ have to be consistent and covariant under $SL_q(2, \mathbf{C})$. There are two obvious choices:

$$\begin{aligned} x^i y^j &= \hat{R}^{ij}{}_{kl} y^k x^l \\ x^i \bar{y}_j &= \hat{R}^{-1\,li}{}_{kj} \bar{y}_l x^k \end{aligned} \tag{20}$$

or

$$\begin{aligned} x^i v^j &= \hat{R}^{-1\,ij}{}_{kl} v^k x^l \\ x^i \bar{v}_j &= \hat{R}^{li}{}_{kj} \bar{v}_l x^k. \end{aligned} \tag{21}$$

The four-vectors of the Minkowski plane are represented as

$$X^i{}_k = \bar{x}_k x^i \tag{22}$$

and we compute the commutations relation of $X^i{}_k$ with the four-vectors $\bar{y}_j y^l$ and $\bar{y}_j v^l$. This yields two different $\hat{\mathcal{R}}$-matrices, both 16×16 matrices with the same eigenvalues q^2, q^{-2} and -1. Their characteristic equation is

$$(\hat{\mathcal{R}} + 1)(\hat{\mathcal{R}} - q^2)(\hat{\mathcal{R}} - q^{-2}) = 0. \tag{23}$$

Each of them gives rise to three projectors. In each case, one of the subspaces can be further decomposed and we obtain four independent projectors altogether. Both $\hat{\mathcal{R}}$-matrices can be expressed in terms of these projectors:

$$\begin{aligned} \hat{\mathcal{R}}_{\mathrm{I}} &= q^2 P_+ + q^{-2} P_- - P_S - P_T \\ \hat{\mathcal{R}}_{\mathrm{II}} &= q^2 P_T + q^{-2} P_S - P_+ - P_- . \end{aligned} \tag{24}$$

To identify these projectors we give their values for $q = 1$:

$$\begin{aligned} P_T{}^{ij}{}_{kl} &= \frac{1}{4} g^{ij} g_{kl} \\ P_S{}^{ij}{}_{kl} &= \frac{1}{2}(\delta^i_k \delta^j_l + \delta^i_l \delta^j_k) - \frac{1}{4} g^{ij} g_{kl} \\ P_+{}^{ij}{}_{kl} &= \frac{1}{4}(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k) - \frac{i}{4} \epsilon^{ij}{}_{kl} \\ P_-{}^{ij}{}_{kl} &= \frac{1}{4}(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k) + \frac{i}{4} \epsilon^{ij}{}_{kl} . \end{aligned} \tag{25}$$

They reflect the property that in four dimensions a tensor of second rank can be decomposed into four irreducable subspaces.

The q-deformed Minkowski space is now naturally defined by

$$P_+ XX = 0 \qquad P_- XX = 0. \tag{26}$$

If we combine this relation with (24) and the fact that $P_+ + P_- + P_T + P_S = \mathbb{1}$ we obtain

$$X^i X^j = -\hat{\mathcal{R}}_{\mathrm{I}}{}^{ij}{}_{kl} X^k X^l . \tag{27}$$

This is the basic relation for our study of the quantized Minkowski space.

For convenience, we introduce the notation:

$$\bar{x}_1 x^2 = A, \qquad \bar{x}_2 x^1 = B, \qquad \bar{x}_1 x^1 = C, \qquad \bar{x}_2 x^2 = D \tag{28}$$

and we find from (27)

$$\begin{aligned} AB &= BA - q^{-1}\lambda DC + q\lambda D^2 & BC &= CB - q\lambda DB \\ AC &= CA + q^{-1}\lambda DA & BD &= q^2 DB \\ AD &= q^{-2} DA & CD &= DC. \end{aligned} \tag{29}$$

These relations are in agreement with our properties 1 to 4. The projector P_T projects on a one dimensional subspace the invariant Minkowski length:

$$L^2 = AB - q^{-2} CD. \tag{30}$$

It turns out to be central, i.e. L^2 commutes with A, B, C and D.

A set of commuting variables is

$$\begin{aligned} T &= \frac{q}{q+q^{-1}}(C+D) \\ X^3 &= \frac{1}{q+q^{-1}}(qD - q^{-1}C) \\ R_\perp^2 &= BA. \end{aligned} \tag{31}$$

T is the 'time' operator, it is central. X^3 is the operator associated with the 3-coordinate and $R_\perp^2$ can be interpreted as the radius in the 1-2 plane. To specify a point in the Minkowski plane, we need one more operator. It turns out that the angular momentum operator in the 3-direction commutes with all the operators of (31), so it can be simultaneously diagonalized.

As a physicist we finally have to relate the derived relations to numbers. From quantum mechanics we have learned that Hilbert space representations are a proper tool to derive physical consequences. Thus we study the Hilbert space representations of (29). We choose C, D, $R_\perp^2$ and τ_3 to be be diagonal and label the states with the respective eigenvalues: $|c, d, \rho^2, j\rangle$. From the commutations relations it follows that A shifts the eigenvalues:

$$A\left|c,d,\rho^2,j\right\rangle = \alpha(c,d,\rho^2,j)\left|c - q\lambda d, q^2 d, \rho^2 + \lambda d(q^{-1}c - qd), q^{-4} j\right\rangle \tag{32}$$

where α is a normalization constant. Starting from a particular state with eigenvalues c_0, d_0, ρ_0^2 and j_0 we find that all the eigenvalues of the spectrum generated by A and $B = \overline{A}$ are:

$$\begin{aligned} c_n &= c_0 + d_0 - d_0 q^{2n} \\ d_n &= d_0 q^{2n} \\ \rho_n^2 &= \rho_0^2 + d_0 q^{-2}((c_0 + d_0)(q^{2n} - 1) + d_0(1 - q^{4n})) \\ j_n &= j_0 q^{-4n}. \end{aligned} \tag{33}$$

The nomalization can be related to ρ_n^2 because $\overline{A}A = BA = R_\perp^2$. We find

$$|\alpha_n|^2 = \rho_n^2 = \rho_0^2 + d_0 q^{-2}((c_0 + d_0)(q^{2n} - 1) + d_0(1 - q^{4n})). \tag{34}$$

This expression has to be greater than or equal to zero. For $q > 1$, $d_0 > 0$ and $c_0 > 0$, ρ_n^2 will tend to minus infinity for $n \to \infty$. There has to be a largest n such that for $n = N$ $\alpha_N = 0$. This leads to

$$\rho_n^2 = d_0 q^{-2}(q^{2N} - q^{2n})(d_0(q^{2N} + q^{2n}) - (c_0 + d_0)) \tag{35}$$

where $n \leq N$.

If we try to find a representation which allows a clock to be at the origin of the space coordinates we have to demand that ρ^2 and the eigenvalue of X^3 take the value zero for some n. This leads to the condition

$$c_0 + d_0 = d_0(q^{2K} + q^{2N}) \tag{36}$$

where K is an integer $K < N$ and $K \leq n \leq N$. The eigenvalues of X^3 will be

$$\frac{d_0}{q+q^{-1}}(q^{2n+1} + q^{2n-1} - q^{2K-1} - q^{2N-1}). \tag{37}$$

which is zero for $n = K = N - 1$. The eigenvalues of T for these states are:

$$\tau_N = d_0 q^{2N}. \tag{38}$$

This exhibits the discrete spectrum of the time coordinate. Up to now we have only used the q-Minkowski space relations (29). For a full discussion of the homogeneous Minkowski space we have to represent the quantum derivatives as well as the Lorentz transformations. This would glue together various representations of the q-deformed algebra (29). This has been done for the q-Lorentz algebra in [Pillin 1992].

References

J. Wess and B. Zumino: Covariant Differential Calculus on the Quantum Hyperplane. Nucl. Phys. B (Proc. Suppl.) **18 B**, 302 (1990)

J. Wess: Differential Calculus on Quantum Planes and Applications. Talk given at Third Centenary Celebrations of the Mathematische Gesellschaft, March 1990, based on work with B. Zumino, preprint KA-THEP-1990-22 (1990)

U. Carow-Watamura, M. Schlieker, M. Scholl and S. Watamura,: Tensor Representation of the Quantum Group $SL_q(2, \mathbf{C})$ and and Quantum Minkowski Space. Z. Phys. C-Particles and Fields **48**, 159 (1990); A Quantum Lorentz Group. Int. Jour. of Mod. Phys. A **6**, 3081 (1991)

W.B. Schmidke, J. Wess and B. Zumino: A q-deformed Lorentz Algebra. Z. Phys. C-Particles and Fields **52**, 471 (1991)

O. Ogievetsky, W.B. Schmidke, J. Wess and B. Zumino: Six Generator q-deformed Lorentz Algebra. Lett. Math. Phys. **23**, 233 (1991)

O. Ogievetsky, W.B. Schmidke, J. Wess and B. Zumino: q-Deformed Poincaré Algebra. preprint MPI-Ph/91-98 (1991), to be published in Commun. Math. Phys.

M. Pillin, W.B. Schmidke and J. Wess: q-Deformed Relativistic One-Particle States. preprint MPI-Ph/92-75 (1992), submitted to Nucl. Phys. B

D = 4 QUANTUM POINCARÉ ALGEBRAS AND FINITE DIFFERENCE TIME DERIVATIVES

JERZY LUKIERSKI* **

Institute for Theoretical Physics, University of Wrocław, pl. Maxa Borna 9, 50-204 Wrocław, Poland

ANATOL NOWICKI**

Institute of Physics, Pedagogical University, Plac Słowiański 6, 65029 Zielona Góra, Poland

and

HENRI RUEGG***

Département de Physique Théorique, Université de Genève 24, quai Ernest-Ansermet, CH-1211 Genève 4, Switzerland

Abstract. The contraction of quantum Lie algebras providing real $D = 4$ quantum Poincaré algebras are briefly reviewed. The case of κ–deformation of $D = 4$ Poincaré algebra with flat nonrelativistic sector is described in some detail. The κ–modification of relativistic dynamics consists in introducing in consistent way the finite difference time derivatives. The κ–Lorentz group has a quasigroup structure introduced by Batalin.

1. Introduction

Firstly we recall that recently several ways of obtaining the quantum deformation of $D = 4$ of Poincaré algebra were proposed:

a) By considering quantum anti–de–Sitter algebra $U_q(O(3,2))$ and performing the "quantum " de–Sitter contraction [1,2].[1]

$$\begin{Bmatrix} R \to \infty \\ q \to 1 \end{Bmatrix} : \qquad R \ln q \xrightarrow[R\to\infty]{} -i^{\epsilon}\kappa^{-1} \tag{1}$$

where $\epsilon = 1$ for $|q| = 1$ (see [1]) and $\epsilon = 0$ for q real (see [2]). It appears that one obtains in such a way the κ–deformation of Poincaré algebra which is

* Presented at II-nd Max Born Symposium by J. L.

** Partially supported by KBN grant Nr. 2/0124/91/01

*** Partially supported by Swiss National Science Foundation

[1] The contraction (1) for rank one quantum algebras $SU_q(2)$ and $SU_q(1.1)$ were firstly introduced by the Firenze group [3,4].

a Hopf algebra, with commuting fourmomenta and the Lorentz generators not forming a Hopf subalgebra.

b) The q-deformed $D = 4$ conformal algebra $U_q(O(4,2))$ has the following chain of Hopf algebras [5-9].[2]

$$U_q(O(4,2)) \supset U_q(\mathcal{P}_4 \oplus\!\!\!\!\supset D) \supset U_q(O(3,1)). \qquad (2)$$

The q-deformation of Poincaré algebra obtained in such a way is embedded in quantum Hopf algebra $U_q(\mathcal{P}_4 \oplus\!\!\!\!\supset D)$ where $\mathcal{P}_4$ denotes fourdimensional Poincaré algebra and D the eleventh dilatation generator, and it is characterized by noncommuting fourmomenta (forming quadratic algebra) and closed quantum Lorentz subalgebra.

c) Following the q-differential calculus on q-deformed Minkowski space [10,11] another q-deformation of Poincaré algebra has been recently obtained [12]. It appears that for this deformation again q-Lorentz algebra is a Hopf subalgebra, and the eleventh dilatation generator is needed for defining the coproducts for ten q-Poincaré generators.

d) In [6,7] the following contraction limit of $U_q(O(4,2))$ (q-conformal algebra) was proposed

$$\begin{bmatrix} R \to \infty \\ q \to 1 \end{bmatrix} : \qquad R^2 \ln q \xrightarrow[R\to\infty]{} -i\kappa^{-2} \qquad (|q| = 1) \qquad (3)$$

and supplemented by the rescaling of the fourmomentum and dilation generators. In such a way it was obtained a new κ-deformation of the Poincaré algebra, embedded in 11-dimensional Hopf algebra containing besides Poincaré generators an additional central generator.

In the following we shall present the κ-deformation of $D = 4$ Poincaré algebra with standard real structure and flat $O(3)$ sector [2]. It appears that in our quantum Poincaré algebra the fourmomenta are commuting, i.e. one can introduce four continuous space-time coordinates in conventional way, as canonically conjugated variables related by a Fourier transform

$$[X_\mu, P_\nu] = i\eta_{\mu\nu} \qquad (4)$$

The κ-deformation enters if we wish to define free fields by the differential operators invariant under κ-Poincaré transformations. It appears that in these operators the continuous time derivative is replaced by finite difference time derivatives.

It should be mentioned that our quantum deformation of relativistic physics is milder than the one proposed by Wess et all [10-13] where the q-Minkowski space is described by noncommutative geometry. In such an

[2] In [8] the real form of $U_q(Sl(4;C))$ providing $U_q(O(4,2))$ was not given explicitly. The discussion of all real forms of $U_q(Sl(4;C))$ is given in [9].

approach even classically the measurable values of four-positions and fourmomenta form a discrete lattice, given by the eigenvalue conditions of the operator-valued q-Minkowski coordinates and momenta.

2. The standard real quantum Poincaré algebra.

In order to obtain standard real κ-Poincaré algebra we proceed as follows:

i) Using the formulae for the commutators and coproducts of antipode-extended Cartan-Weyl basis of $U_q(O(3,2))$ (see [2]) we can write the q-deformation of the $O(3,2)$ Lie algebra as well as the coproduct relations for the q-deformed $O(3,2)$ generators.

ii) We perform further the quantum de-Sitter contraction, obtained by the conventional rescaling of the $O(3,2)$ rotation generators

$$\begin{aligned} &M_{\mu\nu} \quad \text{unchanged} && (M_{\mu\nu}^+ = M_{\mu\nu}) \\ &M_{4\mu} = RP_\mu && (P_\mu^+ = P_\mu) \end{aligned} \qquad (5)$$

and the $q \to 1$ limit described by (1).

As a result we obtain the following q-deformed Poincaré algebra:

a) **Three-dimensional O(3) rotations** ($M_\pm = M_1 + iM_2 \equiv M_{23} \pm iM_{31}$; $M_3 = M_{12}$)

i) commutation relations:

$$[M_+, M_-] = 2M_3 \qquad [M_3, M_\pm] = \pm M_\pm \qquad (6a)$$

ii) coproducts:

$$\Delta M_i = M_i \otimes I + I \otimes M_i \qquad (6b)$$

iii) antipode:

$$S(M_i) = -M_i \qquad (6c)$$

b) **Boosts sector O(3,1)** ($L_\pm = M_{14} \pm iM_{24}$, $L_3 = M_{34}$)

i) commutation relations:

$$[L_+, L_-] = -2M_3 \, cosh\frac{P_0}{\kappa} + \frac{1}{2\kappa^2} P_3^2 + \frac{1}{\kappa^2} M_3 P_3^2 - sinh\frac{P_0}{\kappa}$$

$$\begin{aligned} [L_+, L_3] = {} & e^{-\frac{P_0}{\kappa}} M_+ + \frac{1}{2\kappa}(iP_3L_+ + L_3P_-) + \\ & - \frac{i}{2\kappa^2} M_3P_3P_- + \frac{1}{4\kappa^2}(2-i)P_3P_- \end{aligned} \qquad (7a)$$

$$\begin{aligned} [L_-, L_3] = {} & -e^{-\frac{P_0}{\kappa}} M_- + \frac{1}{2\kappa}(iL_-P_3 - P_+L_3) - \frac{i}{2\kappa^2} P_3P_+M_3 + \\ & - \frac{1}{4\kappa^2}(2+i)P_3P_+ \end{aligned}$$

$$
\begin{aligned}
&[M_3, L_3] = 0 \qquad [M_3, L_\pm] = \pm L_\pm \\
&[M_\pm, L_\pm] = \mp \tfrac{1}{2\kappa} M_\pm P_\mp \\
&[M_+, L_-] = 2L_3 - \tfrac{1}{2\kappa} P_+ M_+ + \tfrac{i}{\kappa} M_3 P_3 + \tfrac{1}{\kappa} P_3 \\
&[M_-, L_+] = -2L_3 + \tfrac{1}{2\kappa} M_- P_- + \tfrac{i}{\kappa} P_3 M_3 - \tfrac{1}{\kappa} P_3 \\
&[M_+, L_3] = -L_+ + \tfrac{1}{2\kappa} M_3 P_- + \tfrac{i}{2\kappa} P_- \\
&[M_-, L_+] = L_- - \tfrac{1}{2\kappa} P_+ M_3 + \tfrac{i}{2\kappa} P_+
\end{aligned}
\tag{7b}
$$

ii) coproducts:

$$
\begin{aligned}
\Delta L_3 &= L_3 \otimes e^{\frac{P_0}{2\kappa}} + e^{-\frac{P_0}{2\kappa}} \otimes L_3 + \frac{1}{2\kappa} e^{-\frac{P_0}{2\kappa}} (M_+ \otimes P_+ + M_- \otimes P_-) \\
\Delta L_\pm &= L_\pm \otimes e^{\frac{P_0}{2\kappa}} + e^{-\frac{P_0}{2\kappa}} \otimes L_\pm + \frac{1}{2\kappa} \left(P_\mp \otimes M_3 e^{\frac{P_0}{2\kappa}} - e^{-\frac{P_0}{2\kappa}} M_3 \otimes P_\mp \right) \\
&\quad \mp \frac{i}{\kappa} e^{-\frac{P_0}{2\kappa}} M_\pm \otimes P_3
\end{aligned}
\tag{7c}
$$

iii) antipode:

$$
\begin{aligned}
S(L_3) &= -L_3 + \frac{i}{2\kappa} P_3 + \frac{1}{2\kappa} (M_+ P_+ + M_- P_-) \\
S(L_\pm) &= -L_\pm \mp \frac{1}{\kappa} P_\mp \mp \frac{i}{\kappa} M_\pm P_3
\end{aligned}
\tag{7d}
$$

c) **Translations sector** ($P_\pm = P_2 \pm i P_1, P_3, P_0$)
i) commutation relations ($\mu, \nu = 0, 1, 2, 3$)

$$
\begin{aligned}
&[P_\mu, P_\nu] = 0 \\
&[M_i, P_j] = i\epsilon_{ijk} P_k \qquad [M_i, P_0] = 0
\end{aligned}
\tag{8a}
$$

$$
\begin{aligned}
&[L_3, P_0] = iP_3 && [L_3, P_3] = i\kappa \, sinh \, \tfrac{P_0}{\kappa} - \tfrac{i}{2\kappa} P_+ P_- \\
&[L_3, P_2] = \tfrac{i}{2\kappa} P_3 P_2 && [L_3, P_1] = \tfrac{i}{2\kappa} P_3 P_1 \\
&[L_\pm, P_0] = iP_1 \mp P_2 && [L_\pm, P_2] = \mp \kappa \, sinh \, \tfrac{P_0}{\kappa} \pm \tfrac{1}{2\kappa} P_3^2 \\
&[L_\pm, P_3] = \mp \tfrac{1}{2\kappa} P_3 P_\mp && [L_\pm, P_1] = i\kappa \, sinh \, \tfrac{P_0}{\kappa} + \tfrac{1}{2i\kappa} P_3^2
\end{aligned}
\tag{8b}
$$

ii) coproducts

$$
\begin{aligned}
\Delta P_0 &= P_0 \otimes I + I \otimes P_0 \\
\Delta P_i &= P_i \otimes e^{\frac{P_0}{2\kappa}} + e^{-\frac{P_0}{2\kappa}} \otimes P_i \qquad (i = 1, 2, 3)
\end{aligned}
\tag{8c}
$$

iii) antipode

$$
S(P_\mu) = -P_\mu \tag{8d}
$$

Following [14] we have introduced in [2] a nonlinear transformation of the boost generators

$$
\begin{aligned}
\widetilde{L}_+ &= L_+ + \frac{i}{2\kappa} M_+ P_3 - \frac{1}{2\kappa} P_- \\
\widetilde{L}_- &= L_- - \frac{i}{2\kappa} P_3 M_- - \frac{1}{2\kappa} P_+ \\
\widetilde{L}_3 &= L_3 - \frac{i}{4\kappa} (M_+ P_+ + P_- M_-) + \frac{1}{2\kappa} P_3
\end{aligned}
\tag{9}
$$

simplifying the κ–Poincaré algebra substantially. The new boosts satisfy the following relations:

$$
\begin{aligned}
\left[M_l, \widetilde{L}_j \right] &= i\epsilon_{ijk} \widetilde{L}_k \\
\left[P_0, \widetilde{L}_k \right] &= -iP_k \\
\left[P_k, \widetilde{L}_j \right] &= -i\kappa \delta_{kj} sinh \tfrac{P_0}{\kappa} \\
\left[\widetilde{L}_l, \widetilde{L}_j \right] &= -i\epsilon_{ikj} \left(M_k cosh \frac{P_0}{\kappa} - \frac{1}{4\kappa^2} P_k (\mathbf{PM}) \right)
\end{aligned}
\tag{10}
$$

It is interesting to observe that the algebra (10) differs from the one obtained in [10] only by the replacement $\kappa \to i\kappa$. The same holds for the coproduct formulae.

$$
\Delta \left(\widetilde{L}_i \right) = \widetilde{L}_i \otimes e^{\frac{P_0}{2\kappa}} + e^{-\frac{P_0}{2\kappa}} \otimes \widetilde{L}_i + \tfrac{1}{2\kappa} \epsilon_{ijk} \left(P_j \otimes M_k e^{\frac{P_0}{2\kappa}} - e^{-\frac{P_0}{2\kappa}} M_j \otimes P_k \right) \tag{11}
$$

The coproducts (11) which satisfies the relation (1) permit to define the tensor product representations in Hilbert space. For completeness we give also the antipodes:

$$
S \left(\widetilde{L}_i \right) = -\widetilde{L}_i + \frac{3}{2} \frac{i}{\kappa} P_i \tag{12}
$$

One can construct the quantum deformation of quadratic Casimir, describing quantum relativistic mass square operator. One gets

$$C_1 = P_1^2 + P_2^2 + P_3^2 + 2\kappa^2\left(1 - \cosh\frac{P_0}{\kappa}\right) = \mathbf{P}^2 - \left(2\kappa\sinh\frac{P_0}{2\kappa}\right)^2 . \quad (13)$$

It should be mentioned that recently the $D = 4$ mass square Casimir was proposed in [4,15] as the extension of the results obtained for $D = 3$ Poincaré algebra.

The second Casimir can be obtained by introducing the κ–deformed Pauli–Lubanski fourvector

$$W_0 = \mathbf{PM}$$
$$W_k = \kappa M_k sinh\frac{P_0}{\kappa} + \epsilon_{kij}P_i\widetilde{L}_j \quad (14)$$

where $\widetilde{L}_i$ is defined by the formulae (9).

The formula for the second Casimir takes the form:

$$C_2 = \left(cosh\frac{P_0}{\kappa} - \frac{\mathbf{P}^2}{4\kappa^2}\right)W_0^2 - \mathbf{W}^2 \quad (15)$$

3. Finite difference time derivatives from κ–Poincaré .

Let us consider the simplest realization of the algebra (10) on the scalar functions $\phi(\mathbf{x},t)$

$$P_\mu = \frac{1}{i}\frac{\partial}{\partial x^\mu} \qquad M_i = \frac{1}{i}\epsilon_{ijk}x_jP_k$$
$$\tilde{L}_i = \frac{1}{i}\left(x_0P_i - \kappa x_i \sin h\frac{P_0}{\kappa}\right) \quad (16)$$

generalizing for $\kappa < \infty$ the spinless realization of the Poincaré algebra, for which $\mathbf{P}\cdot\mathbf{M} = 0$ and $W_\mu = 0$. The generators L_i act explicitly as follows:

$$\widetilde{L}_i\phi(\mathbf{x},t) = -x_0\frac{\partial}{\partial x_i}\phi(\mathbf{x},t) - ix_i\left(D^t_{\frac{\kappa}{2}}\phi(\mathbf{x},t)\right) \quad (17)$$

where [2]

$$\widetilde{D}^t_\kappa\phi(\mathbf{x},t) = \left.\frac{\phi(\mathbf{x},t+\Delta t) - \phi(\mathbf{x},t-\Delta t)}{2\Delta t}\right|_{\Delta t = \frac{i}{2\kappa}} = \left[2\kappa\sin\frac{\partial_t}{2\kappa}\right]\phi(\mathbf{x},t) \quad (18)$$

and of course

$$\lim_{\kappa\to\infty} D^t_\kappa\phi(\mathbf{x},t) = \partial_t\phi(\mathbf{x},t) \quad (19)$$

Using the realization (16) one can obtain also the κ–deformed Klein–Gordon equation (see [2])

$$\left[\Delta - \left(2\kappa\sin\frac{\partial_t}{2\kappa}\right)^2\right]\phi(\mathbf{x},t) = \left[\Delta - 2\kappa^2\left(1 - \cos\frac{\partial_t}{\kappa}\right)\right]\phi(\mathbf{x},t) = m^2\phi(\mathbf{x},t) \quad (20a)$$

which can be written as follows

$$\left(\Delta - \left(\widetilde{D}^t_\kappa\right)^2\right)\phi(\mathbf{x},t) = m^2\phi(\mathbf{x},t) \quad (20b)$$

One can introduce at least three forms of κ–deformed Dirac operators, defining three different κ–deformed Dirac equations:

i) The Dirac equation obtained by taking square root of the κ–deformed Klein–Gordon operator (see [2]). It is the simplest one, but its invariance properties under κ–Poincaré transformations are quite obscure.

ii) The one derived from the three–dimensional realization of κ–Poincaré algebra with spin $\frac{1}{2}$ (see [16]) acting on the functions only depending on the three–momenta coordinates [3] Such a Dirac operator by construction is on–shell κ–Poincaré – invariant.

iii) Recently there were found [18,19] the Dirac operators which commute off–shell with the four–dimensional realization of κ–Poincaré algebra [4].

The linearization of KG operator leads also to the free Hamiltonian H_0 describing scalar particles with relativistic kinematics:

$$H_0 = \sqrt{\mathbf{p}^2 + m^2} = \omega_m(\mathbf{p}) \quad (21)$$

where

$$m^2 + \mathbf{p}^2 - p_0^2 = (H_0 + p_0)(H_0 - p_0) \quad (22)$$

After κ–deformation one obtains ($C_1 = -M^2$)

$$M^2 + \mathbf{p}^2 - \left(2\kappa\sinh\frac{p_0}{2\kappa}\right)^2 = \left(\omega_M + 2\kappa\sinh\frac{p_0}{2\kappa}\right)\left(\omega_M - 2\kappa\sinh\frac{p_0}{2\kappa}\right) \quad (23)$$

The relativistic Schrödinger equation

$$i\partial_t\psi = \omega_m\left(\frac{1}{i}\nabla\right)\psi \quad (24)$$

is replaced by

$$iD^t_\kappa\psi_\kappa = \omega_M\left(\frac{1}{i}\nabla\right)\psi_\kappa \quad (25)$$

[3] These realizations in the classification of the forms of relativistic dynamics "given by Dirac [17] are called "instant forms" and describe Hamiltonian dynamics with relativistic kinematics.

[4] The derivation in [18] is using the κ–Poincaré algebra, and in [19] one employs the finite κ–Lorentz transformation.

The equation (25) can be described equivalently by the following modification of the free relativistic Hamiltonian:

$$i\partial_t \psi_\kappa = \omega_M^\kappa \left(\frac{1}{i}\nabla\right)\psi_\kappa \tag{26}$$

where

$$\omega_M^\kappa = 2\kappa\ arcsinh\frac{\omega_M}{2\kappa} = \omega_M + \frac{1}{6\kappa^2}\omega_M^3 + \mathcal{O}\left(\frac{1}{\kappa^4}\right) \tag{27}$$

In particular for the large values of energies one obtains

$$\omega_M^\kappa \xrightarrow[\omega_M \to \infty]{} 2\ ln\omega_M + \mathcal{O}\left(\frac{1}{\omega_M}\right) \tag{28}$$

i.e. the high energy behaviour is drastically modified. At present we are studying the consequences of such a modification e.g. for the description of confinement.

It should be mentioned here that the modification similar to the one given by eq. (25) was proposed by Caldirola [20] and studied by his followers. We would like to point out however two basic differences:

a) In our formalism the finite difference derivative (see (18)) contains elementary time shift *in purely imaginary direction*. This property has important consequences e.g. the κ–deformed kinematics respects the light velocity as the maximal one (see [21]). In Caldirola formalism there is a maximal energy at which the velocity achieves infinite value.

b) In the ref. [20] one proposes the replacement of the ordinary time derivative by finite difference time derivative in nonrelativistic Schrödinger equation. From our considerations it follows that the κ– deformation can be achieved only when the light velocity is finite - i.e. it seems that the κ–Galilei algebra, obtained from the κ–Poincaré algebra (c finite, κ finite) by the limit $c \to \infty$ does not exist.[5]

4. From κ–Poincaré algebra to κ–Poincaré group

Let us observe that the κ–Poincaré algebra is a special example of the following class of nonlinear algebras:

$$\begin{aligned} &[M_{\mu\nu}, M_{\rho\tau}] = f_{\mu\nu,\rho\tau}^{\ \ \sigma\rho}(P;\kappa)M_{\sigma\rho} \\ &[M_{\mu\nu}, M_\rho] = f_{\mu\nu,\rho}(P;\kappa) \\ &[P_\mu, P_\nu] = 0 \end{aligned} \tag{29}$$

[5] We would like to mention that in the deformation of Galilei algebra given in [14] one performs the limit $\kappa \to 0$, $c \to \infty$ ($\kappa \cdot c$ fixed) which is not a proper κ–Galilei limit.

If the deformation parameter $\kappa \to \infty$ one recovers the standard Poincaré algebra.

In [19] we show that our choice given by the eq. (10) is a unique possible solution for large class of the fourmomentum–dependent structure constants in (29).

In order to discuss the finite κ–Poincaré transformations one should introduce the fourmomentum realization dual (in the sense of the Fourier transform) to the one given by formula (16). Introducing

$$P_\mu = p_\mu \qquad\qquad x_\mu = \frac{1}{i}\frac{\partial}{\partial p^\mu} \tag{30}$$

we see from (16) that the realization of M_i and L_i is described by a linear differential operators in fourmomentum space. The infinitesimal κ–Lorentz transformations of the fourmomenta are given by the formulae (α_i–space rotations, β_i - boosts):

$$\begin{aligned} \delta p_i &= \epsilon_{ij\kappa}\delta\alpha_j p_\kappa + \kappa \sinh\frac{p_0}{\kappa}\delta\beta_i \\ \delta p_0 &= p_i \delta\beta_i \end{aligned} \tag{31}$$

The infinitesimal transformations (31) can be integrated and one obtains the nonlinear formulae, preserving the κ–deformed length of the fourvector, given by (13).[6] Our main observation here is that the general formalism of finite transformations with the generators corresponding to the infinitesimal transformations (31) was elaborated by Batalin [24]. It appears that

i) The nonlinear functions on rhs of (31) imply the generalization of the composition law of two κ–Lorentz transformations. If the integrated form of (31) looks as follows

$$p'_\mu = \phi_\mu(p_\mu, \alpha_A) \tag{32}$$

where $\alpha_A \equiv (\alpha_i, \beta_i)$ describe the κ–Lorentz parameters, one obtains that

$$\phi_\mu\left(\phi(p,\alpha),\alpha'\right) = \phi_\mu\left(p, \varphi(\alpha,\alpha';p)\right)$$

We would like to stress that for the usual Lie groups the function φ describing the composition law does not depend on the group element (p in our case).

ii) One can relate the nonlinearities of (32) and the momentum–dependent structure constants.

iii) There is a natural generalization of the Maurer–Cartan equations for κ–Poincaré algebra.

For more details see [19] and further publications.

[6] In particular one can obtain the κ–generalization of one–parameter Lorentz transformations, which are described by elliptic functions ([19]).

References

[1] J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy, Phys. Lett **B264**, 331 (1991)
[2] J. Lukierski, A. Nowicki and H. Ruegg, Phys. Lett. **B 293**, 344 (1992)
[3] E. Celeghini, R. Giacchetti, E. Sorace and M. Tarlini, J. Math. Phys. **32** 1155 (1991); ibid. 1159
[4] E. Celeghini, R. Giacchetti, E. Sorace and M. Tarlini, "Contraction of quantum groups", Proceeding of First EIMI Workshop on Quantum Groups", Leningrad, October - December 1990, ed. P. Kulish, Springer Verlag 1991
[5] V. Dobrev, "Canonical q-Deformations of Noncompact Lie (Super)-Algebras", Göttingen Univ. preprint, July 1991
[6] J. Lukierski and A. Nowicki, Phys. Lett. **B 279**, 299 (1992)
[7] J. Lukierski, A. Nowicki and H. Ruegg, Boston Univ. preprint BUHEP-91-21 November 1991; to be published in the Proceedings of II Wigner Symposium (Goslar 11.07–16.07.1991), vol. "Quantum Groups", ed. H.D. Doebner and V. Dobrev, Springer Verlag (1992)
[8] V. Dobrev, Trieste preprint IC/92/13, January 1992; to be published in the Proceedings of II Wigner Symposium (Goslar 11.07–16.07.1991), vol. "Quantum Groups", ed. H.D. Doebner and V. Dobrev, Springer Verlag (1992)
[9] J. Lukierski, A. Nowicki and J. Sobczyk, Wrocław Univ. preprint ITP UWr 1992 807/92/Bochum Univ. preprint Math. 156/92 (June 1992)
[10] W.B. Schmidke, J. Wess and B. Zumino, Z. Phys. **C52**, 471 (1991)
[11] O. Ogievetsky, W.B. Schmidke, J. Wess and B. Zumino, Lett. Math. Phys. **23**, 233 (1991)
[12] O. Ogievetsky, W.B. Schmidke, J. Wess and B. Zumino, "q- Deformed Poincaré algebra", Max Planck and Berkeley preprint MPI-Ph/91-98 LBL - 31 703 UCB 92/04, November 1991
[13] J. Wess, B. Zumino, D. Ogievetsky and W.B. Schmidke, to be published in these Proceedings
[14] S. Giller, J. Kunz, P. Kosinski, M. Majewski and P. Maslanka, Phys. Lett. **B286**, 57 (1992)
[15] E. Celeghini, R. Giacchetti, E. Sorace and M. Tarlini, Univ. Firenze preprint DFF 151/11/91
[16] S. Giller, J. Kunz, P. Kosinski, M. Majewski and P. Maslanka, "On q-covariant wave functions", Lodz, Univ. preprint, August 1992
[17] P.A. Dirac, Rev. Mod. Phys. **21**, 392 (1949)
[18] A. Nowicki, E. Sorace and M. Tarlini, Firenze Univ. preprint DFF 117 / 12 / 92 , December 1992
[19] J. Lukierski, H. Ruegg and W. Rühl, Kaiserslautern Univ. preprint KL - TH - 92 / 22, December 1992.
[20] F. Caldirola, Lett. Nuovo Cim. **16**, 151 (1976); **17**, 461 (1976)
[21] J. Lukierski, H. Ruegg and A. Nowicki, talk at Int. Symposium Symmetry VI, Bregenz, 2–7. 08. 92, to be published by Plenum Press

QUANTUM LORENTZ GROUP AND q-DEFORMED CLIFFORD ALGEBRA

URSULA CAROW-WATAMURA
Department of Physics, Faculty of Science
Tohoku University, Sendai 980, Japan

Abstract. We explain the construction of the quantum Lorentz Group $Fun_q(SO(3,1))$, the quantum Minkowski space and the q-deformed Dirac γ matrices.

1. Introduction

When investigating the quantum gravity we have serious problems of conceptual nature, since on one hand the laws of nature have to be covariant with respect to the group of diffeomorphisms of the spacetime manifold M, $Diff(M)$. On the other hand from the point of view of particle physics the metric itself becomes a dynamical variable and we have to define the dynamical variables before the spacetime points have a physical identity. The algebraic approach to the quantum gravity suggests that the distinction between spacelike and timelike directions becomes established at scales large compared to the Planck scale [Fredenhagen 1987]. The standard canonical quantization approach concludes that the concept of time emerges only at a classical level [Halliwell 1992], and the idea that the concept of space and time has to be modified drastically when going beyond the Planck scale, which is considered as the quantum regime of gravity, is not unfamiliar to physicists. However we do not have a description of how the classical concepts of space and time may be modified in the transition to the quantum theory.

The motivation to study the quantum group structure is that it opens a way to investigate a theory based on a geometry with non-commutative coordinate function algebra. Therefore, this new class of non-commutative, non-cocommutative Hopf algebras being available it is interesting to study the q-deformation of the spacetime symmetry, and as a first step to investigate the quantum Lorentz group.

When talking about quantum groups we always think of the q-deformed algebra of functions on a certain group G, $Fun_q(G) : G \to C$. So we are using the approach dual to the one given by [Jimbo 1986].

2. The q-Plane Approach to the Quantum Lorentz Group

To construct the quantum Lorentz group, instead of starting from the definition of the $Fun_q(SL(2,C))$ [see Carow-Watamura 1991a] we report here about the construction [Carow-Watamura 1990], using the approach of the q-deformation of the quantum space algebra [Manin 1988]. We introduce the q-spinor $z^\rho = \binom{x}{y}$, with the q-deformed commutation relation

$$xy = qyx, \tag{1}$$

and the anticommuting q-plane $\xi^\rho = \binom{\xi^1}{\xi^2}$, with $\xi^1\xi^2 + q^{-1}\xi^2\xi^1 = 0$, and $(\xi^1)^2 = (\xi^2)^2 = 0$.

In the q-plane approach the quantum group commutation relation

$$\hat{R}_{12}M_1M_2 = M_2M_1\hat{R}_{12} \tag{2}$$

is obtained by the requirement that the coaction $\Delta_L(z^\rho) = M^\rho_\sigma \otimes z^\sigma$, is an algebra homomorphism of the comodule algebra generated by z^ρ and ξ^ρ, where $M^\rho_\sigma \in Fun_q(SL(2,C))$. Δ_L provides z^ρ with a left $Fun_q(SL(2,C))$ comodule structure.

The definition of $Fun_q(SL(2,C))$ given in [Reshetikhin 1989] was not complete as a complex quantum group, see also [Carow-Watamura 1991d], and thus the $Fun_q(SO(3,1))$ was not constructed. * The construction of the $Fun_q(SO(3,1))$**, the construction of a possible candidate of the q-deformed Minkowski space and the q-deformed analog of the Clifford algebra satisfied by the corresponding 'Dirac' matrices have been given in [Carow-Watamura 1990,1991a] and will be shown here briefly.

To construct the $Fun_q(SO(3,1))$ our strategy is to build a real vector representation out of the tensor product of two spinor representations. For this purpose we take a second 'copy' of the algebra of $Fun_q(SL(2,C))$ denoted by $\widetilde{Fun_q}(SL(2,C))$, and consider the bigger algebra $Fun_q(SL(2,C))\otimes \widetilde{Fun_q}(SL(2,C))$. Then the reality condition is defined by an inner involution of this bigger algebra, i.e. we construct a real form of the quantum double [Faddeev, pivate communication]. One may associate with this the method to contruct a scalar quantity from two spinors of opposite chirality in ordinary field theory.

Another property in our construction is that in order to obtain four dimensional non-null planes, we need two pairs of q-spinors: one pair transforming under the $Fun_q(SL(2,C))$ and a second pair transforming under the $\widetilde{Fun_q}(SL(2,C))$. They are denoted by z^ρ_i and $\tilde{z}^\rho_j$, respectively, $i,j = 1,2$.

* The authors in [Reshetikhin 1989] themselves have pointed out that the quantum group $Fun_q(SO(n,m))$ for $|n-m| \geq 2$ is not obtained from their construction.

** The quantum Lorentz group has also been constructed independently by [Podlés 1990]. The algebra of the q-Minkowski space is first constructed in [Carow-Watamura 1990].

Similarly we denote the q-matrix of $\widetilde{Fun_q}(SL(2,C))$ by $\tilde{M}$ where the $\tilde{M}$ satisfy the same commutation relation as eq.(2), and $\Delta_L(\tilde{z}^\rho_i) = \tilde{M}^\rho_\sigma \otimes \tilde{z}^\sigma_i$. The q-analog of the inner product between two q-spinors can be used to define the q-analog of a "length" which has to be a central element in our algebra. Therefore we require the invariant products

$$q^{1/2}\epsilon_{\rho\sigma}z^\rho_1 z^\sigma_2 = x_1y_2 - qy_1x_2 = \lambda, \quad q^{1/2}\epsilon_{\rho\sigma}\tilde{z}^\rho_1\tilde{z}^\sigma_2 = \tilde{x}_1\tilde{y}_2 - q\tilde{y}_1\tilde{x}_2 = \lambda', \tag{3}$$

to be central where λ and λ' are complex numbers. This definition has a correspondence in twistor theory where it is equivalent to the existence of a non-zero inner product between two spinors [Wald 1984].

From the requirement of covariance and the consistency with their q-spinor relations eq.(1) the commutation relation among the z_i (and the $\tilde{z}_i$) must be nontrivial. Taking into consideration that the products eq.(3) have to be central these commutation relations are obtained as

$$z_1z_2 = \hat{R}z_2z_1, \quad \text{and} \quad \tilde{z}_1\tilde{z}_2 = \hat{R}\tilde{z}_2\tilde{z}_1. \tag{4}$$

The $\hat{R}$-matrix is the one of the $Fun_q(SU(2))$ [Jimbo 1986].

To construct the tensor representation we have to fix the commutation relations of the q-spinors z^ρ with the $\tilde{z}^\rho$. It is clear that there are two possibilities: z^ρ and $\tilde{z}^\rho$ are either commuting or non-commuting.

3. Non-Commuting Case

The quantum group $Fun_q(SO(3,1))$ is obtained by non-trivial commutation relations between z^ρ_i and $\tilde{z}^\rho_i$, i.e. we take the twisted tensor product of the two algebras. They are given as

$$z_1\tilde{z}_2 = k'\hat{R}\tilde{z}_2z_1\ , \quad z_2\tilde{z}_1 = k'\hat{R}\tilde{z}_1z_2\ ,$$
$$z_1\tilde{z}_1 = \frac{1}{k'q}\hat{R}\tilde{z}_1z_1\ , \quad z_2\tilde{z}_2 = \frac{1}{k'q}\hat{R}\tilde{z}_2z_2\ . \tag{5}$$

The parameters k' and q are introduced such that the requirement of centrality of the products eq.(3) holds. The choice of the $\hat{R}$-matrix in the commutation relations eq.(5) as well as in eq.(4) is a matter of convention. We could as well have taken $\hat{R}^{-1}$. However we will see below that the $\hat{\mathbf{R}}$ of the tensor representation is now determined.

The eq.(5) also fixes the commutation relation between M and $\tilde{M}$ as

$$M^\rho{}_{\rho'}\tilde{M}^\sigma{}_{\sigma'}\hat{R}^{\rho'\sigma'}{}_{\sigma''\rho''} = \hat{R}^{\rho\sigma}{}_{\sigma'\rho'}\tilde{M}^{\sigma'}{}_{\sigma''}M^{\rho'}{}_{\rho''} \tag{6}$$

The left coaction on the tensor representation is given by $\Delta_L(\tilde{z}^\rho z^\sigma) = \mathbf{T}^{(\rho\sigma)}{}_{(\nu\mu)} \otimes \tilde{z}^\nu z^\mu = \tilde{M}^\rho{}_\nu M^\sigma{}_\mu \otimes \tilde{z}^\nu z^\mu$.

Our matrix $\mathbf{T}^{(\rho\sigma)}{}_{(\nu\mu)}$ satisfies a relation $\mathbf{R}_{12}\mathbf{T}_1\mathbf{T}_2 = \mathbf{T}_2\mathbf{T}_1\mathbf{R}_{12}$. However, due to the two choices of the R-matrix in eq.(2) and in the corresponding equation for $\tilde{M}$, i.e. either R or R^{-1}, we have in principle four possibilities to compose this $\mathbf{R}_{12}$. These four choices reduce to two inequivalent choices since pairwise two combimations are the inverse of each other. So we are left with two candidates for the $Fun_q(SO(3,1))$. It turns out that the R-matrix corresponding to the quantum Lorentz group can be specified by investigating the projector expansion [Carow-Watamura 1991a]. The solution is

$$\hat{\mathbf{R}}^{(\sigma\rho)(\omega\mu)}{}_{(\omega''\mu'')(\sigma''\rho'')} = \hat{R}^{\rho\omega}{}_{\omega'\rho'}\hat{R}^{\sigma\omega'}{}_{\omega''\sigma'}\hat{R}^{\rho'\mu}{}_{\mu'\rho''}\hat{R}^{-1\sigma'\mu'}{}_{\mu''\sigma''} \tag{7}$$

the projector expansion of which coincides with the classical result in the limit $q \to 1$.

It can also be verified easily that this $\widehat{\mathbf{R}}$-matrix satisfies the Yang-Baxter equation. The second possible candidate for the R-matrix was excluded from our present considerations since its projector composition has no classical analog in the limit $q \to 1$. ***

4. Real Representation

In order to construct a real representation one way which is often used is the operation of the hermitian conjugation and in fact it will also do the job here. Mathematically we look for a $*$-conjugation which is an involution of the algebra and consistent with the Hopf algebra structure of the quantum group. What we will get is a $*$-Hopf algebra and our reality condition is obtained with this definition of the $*$-conjugation: The hermitian conjugate of a q-spinor is $\bar{z}_\rho = (\bar{x}, \bar{y})$. The left comodule structure of the q-spinors z_ρ induces a corresponding comodule structure on its hermitian conjugate as $\Delta_L(\bar{z}_\rho) = M^{\dagger\sigma}{}_\rho \otimes \bar{z}_\sigma$. The symbol $*$ acts as the complex conjugation for a usual complex number. Note that $S(M^\rho_\nu) = \epsilon^{\rho\sigma} M^\mu{}_\sigma \epsilon_{\mu\nu}$. Thus including the operation of hermitian conjugation, the quantum space algebra of z^ρ and of $\bar{z}_\rho$ are generated by two elements, namely x and y. Including the spinor metric we find another two: $\Delta_L(z^\rho\epsilon_{\rho\sigma}) = S(M^\rho{}_\sigma) \otimes (z^\nu\epsilon_{\nu\rho})$ and $\Delta_L(\epsilon^{*\rho\sigma}\bar{z}_\sigma) = S(M^{\dagger\rho}{}_\sigma) \otimes (\epsilon^{*\sigma\nu}\bar{z}_\nu)$.

In terms of spinors the reality condition needed to reduce to the real representations is to identify these algebras. It turns out that it is sufficient to consider q real. ‡ Then we have the following identification

$$\tilde{z}^\rho_i = \epsilon^{\rho\sigma}\bar{z}_{i,\sigma} \tag{8}$$

This implies that $S(\tilde{M}) = M^\dagger$. It is straightforward to derive the commutation relation for the q-spinor and its adjoint as

$$z(\epsilon\bar{z}) = k'\hat{R}(\epsilon\bar{z})z \tag{9}$$

*** For the quantum Lorentz group case see below. For the other case excluded here the projector expansion has been given in [Carow-Watamura 1991c].

‡ For q being a pure phase we cannot get non-null space relations.

In order to define the commutation relation between M and $M^\dagger$ a heuristic argument to find this relation has been given in [Carow-Watamura 1991d]. The result is

$$M^{\dagger i}_{i'} M^{j'}_k \hat{R}^{-1}{}_{j'l}{}^{i'j} = \hat{R}^{-1}{}_{ki'}{}^{ij'} M^j_{j'} M^{\dagger i'}_l \tag{10}$$

Note that with the unitarity condition we can consistently restrict our system to $Fun_q(SU(2))$ since it preserves the substructure $Fun_q(SL(2,C)) \supset Fun_q(SU(2))$.

The central term in our algebra which can be identified with the length is obtained by substituting eq.(8) into eq.(3) and taking λ' as the complex conjugate of λ:

$$x_1y_2 - qy_1x_2 = \lambda \quad \text{and} \quad \bar{y}_2\bar{x}_1 - q\bar{x}_2\bar{y}_1 = \lambda^*. \tag{11}$$

The commutation relations of all q-spinors are given by eq.(4) and

$$z_1(\epsilon\bar{z}_2) = k'\hat{R}(\epsilon\bar{z}_2)z_1\ , \quad z_1(\epsilon\bar{z}_1) = \frac{1}{k'q}\hat{R}(\epsilon\bar{z}_1)z_1\ ,$$
$$z_2(\epsilon\bar{z}_2) = \frac{1}{k'q}\hat{R}(\epsilon\bar{z}_2)z_2\ . \tag{12}$$

and their hermitian conjugates.

Now all relations among the q-spinors are defined. With a tedious but straightforward calculation one can show that with the following definition of the 4-dimensional q-space coordinate functions $A = \bar{x}_1y_2 + \bar{x}_2y_1$, $B = \bar{y}_1x_2 + \bar{y}_2x_1$, $C = \bar{x}_1x_2 + \bar{x}_2x_1$ and $D = \bar{y}_1y_2 + \bar{y}_2y_1$ we obtain a closed algebra:

$$DC = CD, \quad CA - AC = (1-q^2)AD,$$
$$BC - CB = (1-q^2)DB, \quad DA = q^2AD,\ DB = q^{-2}BD,$$
$$AB - BA = (q^{-2}-1)CD + (q^2-1)D^2. \tag{13}$$

The central terms yield:$BA - CD - (1-q^2)D^2 = \frac{1}{k'q}\lambda\lambda^*$. Note that $A = B^*$, C and D are hermitian and thus eq.(13) as well as the central term are invariant under the $*$-operation.

The structure of the above algebra is preserved under the coaction of the $Fun_q(SL(2,C))$ since $\Delta_L(\bar{z}_{\rho,i}z^\sigma{}_j + \bar{z}_{\rho,j}z^\sigma{}_i) = \bar{M}^{\rho'}{}_\rho M^\sigma{}_{\sigma'} \otimes (\bar{z}_{\rho',i}z^{\sigma'}{}_j + \bar{z}_{\rho',j}z^{\sigma'}{}_i)$.

We complete our considerations with showing that the 'rotation group' $Fun_q(SO(3))$ appears as a subalgebra of the $Fun_q(SO(3,1))$. For this end we redefine $T = \frac{C+D}{q\sqrt{q+q^{-1}}}$, $X_+ = q^{-1/2}A$, $Z = \frac{q^{-1}C-qD}{\sqrt{q+q^{-1}}}$, $X_- = q^{1/2}B$. These coordinate functions also generate a closed algebra with the commutation relations given by

$$\begin{aligned} &[T,X_+] = [T,X_-] = [T,Z] = 0 \\ &ZX_+ - q^2X_+Z = (1-q^2)X_+T \\ &X_-Z - q^2ZX_- = (1-q^2)TX_- \\ &[X_+,X_-] = (q-q^{-1})(Z^2-ZT) \end{aligned} \tag{14}$$

The central product is given by: $qX_+X_- + Z^2 + q^{-1}X_-X_+ - T^2 = \frac{q+q^{-1}}{k'q}\lambda\lambda^*$. In this algebra T is a central element and may be put to zero. In this case the other quantities form the 3-dimensional comodule of the quantum group $Fun_{q^2}(SO(3))$.

Let us define the 'four-vector' $\mathbf{U} = (X_-, Z, X_+, T)$. We can prove by direct computation that the matrix $\mathbf{\Lambda}$ of the coaction of $Fun_q(SO(3,1))$ on $\mathbf{U}$, $\Delta_L(\mathbf{U}) = \mathbf{\Lambda} \otimes \mathbf{U}$ satisfies the orthogonality relation ${}^t\mathbf{\Lambda C \Lambda} = \mathbf{C}$, where $\mathbf{C}$ is

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & q^{-1} & 0 \\ 0 & 1 & 0 & 0 \\ q & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{15}$$

and $\mathbf{\Lambda}$ is

$$\mathbf{\Lambda} = \begin{pmatrix} \bar{d}a & \frac{q(\bar{c}a-\bar{d}b)}{Q} & q\bar{c}b & \frac{q(q^2\bar{c}a+\bar{d}b)}{Q} \\ \frac{\bar{b}a-\bar{d}c}{qQ} & \frac{\bar{a}a-q^2\bar{c}c-\bar{b}b+q^2\bar{d}d}{Q^2} & \frac{\bar{a}b-q^2\bar{c}d}{Q} & \frac{q^2\bar{a}a-q^4\bar{c}c+\bar{b}b-q^2\bar{d}d}{Q^2} \\ \frac{\bar{b}c}{q} & \frac{\bar{a}c-\bar{b}d}{Q} & \bar{a}d & \frac{q^2\bar{a}c+\bar{b}d}{Q} \\ \frac{\bar{b}a+\bar{d}c}{qQ} & \frac{\bar{a}a+\bar{c}c-\bar{b}b-\bar{d}d}{Q^2} & \frac{\bar{a}b+\bar{c}d}{Q} & \frac{q^2\bar{a}a+q^2\bar{c}c+\bar{b}b+\bar{d}d}{Q^2} \end{pmatrix} \tag{16}$$

with $Q = \sqrt{1+q^2}$.

In this basis the reality condition for the four-vector is $\mathbf{U}^\dagger\eta = {}^t\mathbf{UC}$ where $\eta = diag(1,1,1,-1)$ and the reality condition for the q-matrix $\mathbf{\Lambda}$ is $\mathbf{\Lambda}^\dagger = \eta\mathbf{\Lambda}^{-1}\eta$. Restricting to the substructure $Fun_q(SU(2))$, i.e. substituting the condition $M^\dagger = S(M)$ into eq.(16) the matrix $\mathbf{\Lambda}$ splits into a 3×3 and a 1×1 part. The 3×3 matrix M_3 gives the matrix corepresentation of the $Fun_q(SO(3))$ in terms of the matrix elements of the corepresentation of the $Fun_q(SU(2))$.

$$M_3 = \begin{pmatrix} aa & -Qba & -bb \\ -Qca & (da+qbc) & Qdb \\ -cc & Qdc & dd \end{pmatrix} \tag{17}$$

satisfying the orthogonality equation ${}^tM_3C_3M_3 = C_3$.

We conclude that our above defined 4-dimensional q-space eq.(13) is a real left comodule of $Fun_q(SL(2,C))$ and can be considered as the quantum Minkowski space. The corresponding q-matrices of the left coaction on this comodule satisfy the properties of the quantum group $Fun_q(SO(3,1))$ and are referred to as quantum Lorentz group in the following.

Since the $\hat{R}$-matrices of the $Fun_q(SL(2,C))$ are the building blocks of the quantum Lorentz group $\hat{\mathbf{R}}$-matrix we use this fact to derive the projector decomposition of the $Fun_q(SO(3,1))$-$\hat{\mathbf{R}}$ matrix. With the graphical technique developed in [Carow-Watamura 1991a] we can derive the characteristic equation of the $\hat{\mathbf{R}}$ matrix as

$$(\hat{\mathbf{R}} - q)(\hat{\mathbf{R}} + q^{-1})(\hat{\mathbf{R}} - q^{-3}) = 0 \tag{18}$$

From eq.(18) the projectors are found as $P_S = N_S(\hat{\mathbf{R}} - q^{-3})(\hat{\mathbf{R}} + q^{-1})$, $P_A = N_A(\hat{\mathbf{R}} - q^{-3})(\hat{\mathbf{R}} - q)$, $P_1 = N_1(\hat{\mathbf{R}} + q^{-1})(\hat{\mathbf{R}} - q)$, where the N_S, N_A, N_1 are normalization constants. In terms of these projectors the $\hat{\mathbf{R}}$-matrix is given as $\hat{\mathbf{R}} = qP_S - q^{-1}P_A + q^{-3}P_1$. Using the projector expansion we can write our equations in a very compact form. Identifying $(A,B,C,D) = (U_1^2, U_2^1, U_1^1, U_2^2)$ where $\mathbf{U}_i{}^j = \epsilon_{ki}\mathbf{U}^{kj}$, we can write the algebra eq.(13) as $P_A(\mathbf{U}\otimes\mathbf{U}) = 0$.

From the explicit form of the P_1 we know that the length L is proportional to $P_1(\mathbf{U}\otimes\mathbf{U})$, $L = \mathbf{C}_{(ij)(kl)}\mathbf{U}^{ij}\mathbf{U}^{kl} = -(q+q^{-1})(BA - CD - (1-q^2)D^2)$.

The projector to the symmetic part gives us the commutation relations of the q-deformed Clifford algebra.

5. q-Deformed Clifford Algebra

In the non-deformed case the connection between spinor and vector representation is given by defining the γ matrices. Since one of our requirements is that in the limit $q \to 1$ the ordinary Lorentz group has to be recovered we expect that for the quantum Lorentz group a corresponding Clifford algebra exists. However due to the non-commutativity properties it is not clear whether indices can be raised and lowered in the ordinary fashion. Thus we introduce four sets of Pauli matrices $\sigma^\mu_{\tilde{k}l}$, $\bar{\sigma}^\mu_{k\tilde{l}}$, $\sigma^{\tilde{k}l}_\mu$, and $\bar{\sigma}^{k\tilde{l}}_\mu$, requiring orthogonality $\sigma^\mu_{\tilde{k}l}\sigma^{\tilde{k}l}_\nu = \delta^\mu_\nu$, $\bar{\sigma}^\mu_{k\tilde{l}}\bar{\sigma}^{k\tilde{l}}_\nu = \delta^\mu_\nu$ and completeness $\sigma^\mu_{\tilde{k}l}\sigma^{\tilde{k}'l'}_\mu = \delta^{\tilde{k}'}_{\tilde{k}}\delta^{l'}_l$, $\bar{\sigma}^\mu_{k\tilde{l}}\bar{\sigma}^{k'\tilde{l}'}_\mu = \delta^{l'}_l\delta^{\tilde{k}'}_{\tilde{k}}$. Using these Pauli matrices the four dimensional generators of the quantum Lorentz group can be represented as

$$\mathbf{U}^\mu = \sigma^\mu_{\tilde{k}l}(\bar{z}\epsilon)^{\tilde{k}}z^l \tag{19}$$

The left coaction on $\mathbf{U}^\mu$ is $\Delta_L(\sigma^\mu_{\tilde{k}l}(\bar{z}\epsilon)^{\tilde{k}}z^l) = \sigma^\mu_{\tilde{k}l}\tilde{M}^{\tilde{k}}{}_{\tilde{k}'}M^l{}_{l'}\sigma^{\tilde{k}'l'}_\nu \otimes \sigma^\nu_{\tilde{s}t}(\bar{z}\epsilon)^{\tilde{s}}z^t$.

The the relation between the Pauli matrices σ and $\bar{\sigma}$ can be found as follows. First we observe that we can build another four vector by using the $\bar{\sigma}$ as

$$\mathbf{W}^{\mu} = \bar{\sigma}^{\mu}_{i\bar{j}} z^{i} (\bar{z}\epsilon)^{\bar{j}}. \tag{20}$$

Since this four vector $\mathbf{W}^{\mu}$ belongs to the same representation space as $\mathbf{U}^{\mu}$, the coaction on them must be the same. This leads us to the equations

$$\bar{\sigma}^{\mu}_{l\bar{k}} = \sigma^{\mu}_{\bar{k}'l'} \hat{R}^{-1\bar{k}'l'}{}_{l\bar{k}} \quad \text{and} \quad \bar{\sigma}^{l\bar{k}}_{\mu} = \hat{R}^{l\bar{k}}{}_{\bar{k}'l'} \sigma^{\bar{k}'l'}_{\mu} \tag{21}$$

There is a choice of an overall factor in eq.(21) which we have set to 1. Using the orthogonality we can rewrite this as $\sigma^{\bar{k}l}_{\mu} \bar{\sigma}^{\mu}_{l'\bar{k}'} = \hat{R}^{-1\bar{k}l}{}_{l'\bar{k}'}$ and $\bar{\sigma}^{l\bar{k}}_{\mu} \sigma^{\mu}_{\bar{k}'l'} = \hat{R}^{l\bar{k}}{}_{\bar{k}'l'}$.

These q-deformed Pauli matrices can be used as a basis for representing tensors.

With the properties of the q-Pauli matrices we can write down the analogous equations of the ordinary $SL(2,C)$ spinor calculus for the case of the quantum Lorentz group, for examle $Tr\{\sigma^{\mu}\bar{\sigma}^{\nu}\} \equiv \epsilon^{\bar{k}\bar{k}'} \sigma^{\mu}_{\bar{k}l} \epsilon^{ll'} \sigma^{\nu}_{l'\bar{k}'} = q^{-1}\mathbf{C}^{\mu\nu}$. Furthermore one can prove that the condition for the anticommutation relation is

$$P_S{}^{\mu\nu}{}_{\nu'\mu'} \sigma^{\nu'}_{\bar{k}l} \epsilon^{ll'} \bar{\sigma}^{\mu'}_{l'\bar{k}'} = 0 \, . \tag{22}$$

The validity of eq.(22) is most easily understood by using the diagrammatics [Carow-Watamura 1991a].

With the projector decomposition of the $\hat{\mathbf{R}}$-matrix as $q\hat{\mathbf{R}} + \mathbf{1} = qQP_S + q^{-1}QP_1$ and using eq.(22) we derive

$$\sigma^{\mu}_{\bar{k}l} \epsilon^{ll'} \bar{\sigma}^{\nu}_{l'\bar{m}} + q\hat{\mathbf{R}}^{\mu\nu}{}_{\nu'\mu'} \sigma^{\nu'}_{\bar{k}l} \epsilon^{ll'} \bar{\sigma}^{\mu'}_{l'\bar{m}} = -q^{-2}\mathbf{C}^{\mu\nu}\epsilon_{\bar{k}\bar{m}}. \tag{23}$$

$$\bar{\sigma}^{\mu}_{l\bar{k}} \epsilon^{\bar{k}\bar{k}'} \sigma^{\nu}_{\bar{k}'m} + q\hat{\mathbf{R}}^{\mu\nu}{}_{\nu'\mu'} \bar{\sigma}^{\nu'}_{l\bar{k}} \epsilon^{\bar{k}\bar{k}'} \sigma^{\mu'}_{\bar{k}'m} = -q^{-2}\mathbf{C}^{\mu\nu}\epsilon_{lm}. \tag{24}$$

The eq.(23)) and eq.(24) suggest to define the Dirac matrix γ as follows

$$\gamma^{\mu} \equiv \sqrt{Q} \begin{pmatrix} 0 & \epsilon\bar{\sigma}^{\mu} \\ -q\epsilon\sigma^{\mu} & 0 \end{pmatrix} \tag{25}$$

Then eqs.(23) and (24) can be combined as

$$\gamma^{\mu}\gamma^{\nu} + q\hat{\mathbf{R}}^{\mu\nu}{}_{\nu'\mu'} \gamma^{\nu'}\gamma^{\mu'} = q^{-1}Q\mathbf{C}^{\mu\nu} \tag{26}$$

The eq.(26) is equivalent to the condition $P_S(\Gamma \otimes \Gamma) = 0$, where Γ symbolizes the basis of the Clifford algebra. For the proof see [Carow-Watamura 1991a].

Finally we present an explicit form of the q-deformed Pauli matrices. With $\sigma^{\bar{k}}_{l} \equiv \epsilon^{\bar{k}\bar{k}'} \sigma_{\bar{k}'l}$ we obtain

$$\sigma^{0\bar{k}}{}_{l} = \frac{1}{q\sqrt{Q}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad \sigma^{+\bar{k}}{}_{l} = \frac{1}{\sqrt{q}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ,$$
$$\sigma^{-\bar{k}}{}_{l} = \sqrt{q} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \quad \sigma^{3\bar{k}}{}_{l} = \frac{1}{\sqrt{Q}} \begin{pmatrix} q^{-1} & 0 \\ 0 & -q \end{pmatrix} , \tag{27}$$

which in the limit $q \to 1$ coincide with the conventional Pauli matrices.

With this basis we can represent our 'Minkowski four vector' as

$$\mathbf{U}^{\mu} = \bar{z}_1 \sigma^{\mu} z_2 + \bar{z}_2 \sigma^{\mu} z_1 = \bar{z}_{1,\bar{k}} \sigma^{\mu\bar{k}}{}_{l} z_2^{l} + \bar{z}_{2,\bar{k}} \sigma^{\mu\bar{k}}{}_{l} z_1^{l} \tag{28}$$

where the index $\mu = 0, +, 3, -$ and the components of $\mathbf{U}^{\mu}$ are related to the quantum four plane by the identification $(\mathbf{U}^0, \mathbf{U}^+, \mathbf{U}^3, \mathbf{U}^-) = (T, X, Z, Y)$. Using the spinors of the $Fun_q(SL(2,C))$ we can represent a Dirac spinor Ψ_{Dirac} as

$$\Psi_{Dirac} = \begin{pmatrix} z_1^k \\ (\bar{z}_2\epsilon)^{\bar{k}} \end{pmatrix} , \tag{29}$$

and the conjugate spinor is $\bar{\Psi} = \Psi^{\dagger}\gamma^0$. The element $\bar{\Psi}_1\Psi_2 = \Psi_1^{\dagger}\gamma^0\Psi_2$ is central with respect to the algebra of coordinate functions.

The relations among the q-Dirac matrices also fixes the reality condition of the Lorentz vectors. From $\gamma^{\mu\dagger}\gamma^{0\dagger} = \gamma^0\gamma^{\rho}\mathbf{C}_{\rho\nu}\eta^{\nu\mu}$ with the metric $\mathbf{C}_{\mu\nu}$ in this basis given as

$$\mathbf{C}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q \\ 0 & 0 & -1 & 0 \\ 0 & -q^{-1} & 0 & 0 \end{pmatrix} \tag{30}$$

and $\eta^{\mu\nu} = diag(1,-1,-1,-1)$ we obtain the following condition for $\mathbf{U}$: $\mathbf{U}^{\rho\dagger} = \mathbf{U}^{\mu}\mathbf{C}_{\mu\nu}\eta^{\nu\rho}$. For the transformation matrix we get $\Lambda^{\dagger} = \eta\Lambda^{-1}\eta$.

In order to compare eq.(30) with the metric of the quantum Minkowski space we simply have to change the overall sign in the definition of the metric of the q-Minkowski space.

6. Conclusion

As already pointed out in the introduction the quantum group gives us a possibility to investigate a theory, the algebra of coordinate functions of which is non-commuting. In order to reach such a stage one way is to study the q-deformed generalizations of the known theory. With this aim we have also investigated the differential calculus on the q-Euclidian space

[Carow-Watamura 1991b] which gave some encouraging results involving the q-deformed polynomials. It is our hope that this new approach will give us a better insight into the problems of formulating a quantum theory of gravity based on the non-commutative geometry.

References

Carow-Watamura, U., Schlieker, M., Scholl, M. and Watamura, S.: 1990, *Z. Phys. C - Particles and Fields* **48**, 159

Carow-Watamura, U., Schlieker, M., Scholl, M. and Watamura, S.: 1991a, *Int. Jour. of Mod. Phys.* **A6**, 3081

Carow-Watamura, U., Schlieker M. and Watamura, S.: 1991b, *Z. Phys. C - Particles and Fields* **49**, 439

Carow-Watamura, U., Schlieker, M., Watamura, S. and Weich,W.,: 1991c, *Commun. Math. Phys.* **142**, 605

Carow-Watamura, U. and Watamura, S.: 1991d 'Complex quantum group, dual algebra and bicovariant differential calculus', preprint-TU-382(1991), to be published in Commun. Math. Phys.

Fredenhagen, K. and Haag, R.,: 1987, *Commun. Math. Physics* **108**, 91

Halliwell, J.: 1992, 'Quantum cosmology and time asymmetry', Proceedings of the NATO Workshop, 'Physical Origins of Time Asymmetry', eds. J. Halliwell, J. Perez-Mercader and W. Zurek (Cambridge University Press, Cambridge, 1992).

Jimbo, M.: 1986, *Lett. Math. Phys.* **10**, 63

Manin, Yu. I.:1988, 'Quantum groups and non-commutative geometry', Centre de Recherches Mathematiques, Université de Montreal.

Podlés, P. and Woronowicz, S.L.: 1990, *Commun. Math. Phys.* **130**, 381

Reshetikhin, N.Yu., Takhtadzhyan L.A. and Faddeev, L.D.: 1989 Algebra and Analysis **1** 178; translation Leningrad Math. J. **1**, 193-225 (1990).

ISOTROPIC q-LORENTZ GROUP

JAKUB REMBIELIŃSKI

Department of Theoretical Physics
University of Łódź
ul. Pomorska 149/153, 90-236 Łódź, Poland

Abstract. A new q-deformation of the Lorentz group is proposed and investigated. In this Hopf algebra the rotation group $SO(3)$ is an authomorphism group.

Quantum deformations of the Lorentz group were considered by number of authors [?, ?, ?]. In particular the q-Lorentz group considered by Wess et al. [?, ?] corresponds to a q-Minkowski space-time with a non-commutative (non-isotropic) space sector. On the other hand the deformation proposed by Lukierski et al. [?], although isotropic, acts in a fully commutative Minkowski space-time.

In this paper we propose another deformation of the Lorentz symmetry acting in an isotropic but non-commutative space-time. Isotropy means that the standard rotation group $SO(3)$ is an authomorphism group both the q-Minkowski space and time and the q-Lorentz group. Now, under the isotropy condition, the hermitean coordinate generators x^μ satisfy

$$x^i x^j = x^j x^i \tag{1}$$

for $i, j = 1, 2, 3$ and

$$x^0 \mathbf{x} = q \mathbf{x} x^0 \tag{2}$$

with $|q| = 1$. The co-module action of q-Lorentz group reads

$$\delta(x) = \Lambda \otimes x \tag{3}$$

where the isotropy condition implies the following form for Λ

$$\Lambda = L_w R \tag{4}$$

wher R is a rotation

$$R = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \boldsymbol{R} \end{array}\right) \tag{5}$$

i.e. $\boldsymbol{R}^{\mathrm{T}}\boldsymbol{R} = I$ and $\boldsymbol{R}^i{}_j$ belong to the center of algebra generated by $\Lambda^\mu{}_\nu$. The q-matrix L_w corresponds to a quantum boosts. It can be parametrized as follows

$$L_w = \left(\begin{array}{c|c} w^0 & \mathbf{w}_+^{\mathrm{T}} \\ \hline \mathbf{w}_- & \alpha + \frac{q}{w^0+\alpha}\mathbf{w}_- \times \mathbf{w}_+^{\mathrm{T}} \end{array}\right) \tag{6}$$

with

$$\begin{cases} w^{0\,2} - q\mathbf{w}_-\mathbf{w}_+ = \alpha^2 & \text{(metric condition)} \\ w_+^i w_-^j - w_+^j w_-^i = 0 & \text{(co-linearity condition)} \end{cases} \tag{7}$$

Here $\mathbf{w}_\pm^i = w_\pm^i$, w^0 and α are hermitean and satisfy t he following rules

$$\left.\begin{array}{l} w_\pm^i w_\pm^j = w_\pm^j w_\pm^i \\ w_+^i w_-^j = q^2 w_-^j w_+^i \\ \mathbf{w}_\pm \alpha = q^{\pm 1}\alpha \mathbf{w}_\pm \\ \mathbf{w}_\pm w^0 = q^{\pm 1} w^0 \mathbf{w}_\pm \\ w^0\alpha = \alpha w^0 \end{array}\right\} \tag{8}$$

The co-unity is defined by

$$\epsilon(\boldsymbol{R}) = I, \quad \epsilon(\mathbf{w}_\pm) = 0, \quad \epsilon(w^0) = \epsilon(\alpha) = 1 \tag{9}$$

The co-product reads

$$\left.\begin{array}{l} \Delta(w^\circ) = w^\circ \otimes w^\circ + \mathbf{w}_+ \otimes \mathbf{w}_- \\ \Delta(\mathbf{w}_-) = \mathbf{w}_- \otimes w^\circ + \alpha \otimes \mathbf{w}_- + \frac{q}{w^\circ+\alpha}\mathbf{w}_-(\mathbf{w}_+ \otimes \mathbf{w}_-) \\ \Delta(\alpha) = \alpha \otimes \alpha \end{array}\right\} \tag{10}$$

The $\Delta(\mathbf{w}_+)$ can be calculated by means of the form of product

$$L_w \otimes L_w \equiv L_{\Delta(w)} R_w \tag{11}$$

where the Thomas precession R_w is given explicitly by

$$\begin{aligned} R_w \;=\; & (\alpha^{-1} \otimes \alpha^{-1})\Big\{\mathbf{w}_- \otimes \mathbf{w}_+^{\mathrm{T}} + \\ & + \left(\alpha + \frac{q}{w^0+\alpha}\mathbf{w}_- \times \mathbf{w}_+^{\mathrm{T}}\right) \otimes \left(\alpha + \frac{q}{w^0+\alpha}\mathbf{w}_- \times \mathbf{w}_+^{\mathrm{T}}\right) + \\ & - \frac{q}{\Delta(w^\circ) + \Delta(\alpha)}\Delta(\mathbf{w}_-) \times \\ & \times \left[w^0 \otimes \mathbf{w}_+^{\mathrm{T}} + \mathbf{w}_+^{\mathrm{T}} \otimes \left(\alpha + \frac{q}{w^0+\alpha}\mathbf{w}_- \times \mathbf{w}_+^{\mathrm{T}}\right)\right]\Big\} \end{aligned} \tag{12}$$

From the Eq. (11)

$$\Delta(\mathbf{w}_+) = R_w\left[w^\circ \otimes \mathbf{w}_+ + \mathbf{w}_+ \otimes \left(\alpha + \frac{q}{w^\circ+\alpha}\mathbf{w}_- \times \mathbf{w}_+\right)\right] \tag{13}$$

Finally, the antipode has the form

$$L_w^{-1} = \alpha^{-2}\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & q^2 I \end{array}\right)\left(\begin{array}{c|c} w^0 & \mathbf{w}_+^{\mathrm{T}} \\ \hline \mathbf{w}_- & \alpha + \frac{q}{w^0+\alpha}\mathbf{w}_- \times \mathbf{w}_+^{\mathrm{T}} \end{array}\right) \tag{14}$$

The above introduced q-Lorentz group can be extended to q-Poincaré group. Its form, properties and representations will be given in the forthcoming papers.

Acknowledgements

I am grateful to Prof. W. Tybor, Dr. P. Kosiński and Mr. K.A. Smoliński for interesting discussions.

This work is supported by University of Łódź grant No. 505/771.

References

[1] Schmidke, W.B., Wess, J., Zumino, B.: 1991, preprint MPI–Ph/91–15; Ogievetsky, O., Schmidke, W.B., Wess, J., Zumino, B.: 1 991, preprint MPI–Ph/91–51.

[2] Carow-Watamura, U., Schlieker, M., Scholl, M., Watamura, S.: 1990, *Z. Phys.* **C48**, 150; Carow-Watamura, U., Schlieker, M., Scholl, M., Watamura, S.: 1991, *Int. J. Mod. Phys.* **A6**, 3081.

[3] Lukierski, J., Nowicki, A., Ruegg, H., Tolstoy, V.N.: 1991, *Phys. Lett.* **B264**, 331; Lukierski, J., Nowicki , A., Ruegg, H.: 1992, preprint UGVA–DTP 1992/07–776.

LORENTZ ALGEBRA AND TWISTS

RALF ENGELDINGER, MICHAEL SCHLIEKER* and WOLFGANG WEICH
Sektion Physik der Universität München
Lehrstuhl Professor Wess, Theresienstr. 37, 8000 München 2
Federal Republic of Germany

The aim of the following paper is to classify the relation between the deformation of SO (4) and the q-deformation of the Lorentzgroup.

The starting point of our investigation is the universal enveloping algebra U_q equiped with comultiplication $\Delta(a) = a_1 \otimes a_2$ $(a \in U_q)$, antipode S, and counit ε. This Hopfalgebra should be of standard-type described in ref. [Drinfel'd 1986, Faddeev et al 1987]. It is coassociative but not cocommutative.

It holds

$$\begin{aligned} \sigma \circ \Delta &= R \, \Delta \, R^{-1} \\ &= R_{21}^{-1} \, \Delta \, R_{21} \end{aligned} \tag{1}$$

On the tensorproduct over $\mathbb{C}$ $U_q \otimes U_q$ it is possible to define a natural Hopfalgebrastructure by the following definitions:

$$\begin{aligned} \Box &:= (id \otimes \sigma \otimes id) \, \Delta \otimes \Delta \\ \boldsymbol{S} &:= S \otimes S \\ \boldsymbol{\varepsilon} &:= \varepsilon \otimes \varepsilon \end{aligned}$$

This Hopfalgebra is called $\mathcal{A}$ in the following.

The Hopfalgebra $\mathcal{A}$ has the property that the diagonal embedding

$$\Delta : \; U_q \to U_q \otimes U_q, \; \Delta(a) = a_1 \otimes a_2 \tag{2}$$

is not a subbialgebra in $\mathcal{A}$, because

$$\Box(\Delta(a)) \in \Delta(U_q) \otimes \Delta(U_q) \tag{3}$$

The mapping $\Delta \otimes \Delta$ would trivially make this diagonal embedding a subbialgebra but $\Delta \otimes \Delta$ is coassociative outside $\Delta(U_q)$. Starting from that observation it is possible to introduce a new comultiplication which respects the diagonal embedding by coassociative continuation of $\Delta \otimes \Delta$ in the following way: on $\Delta(U_q)$ one has the following relation between $\Box$ and $\Delta \otimes \Delta$:

$$\Box(a) = R_{23} \Delta \otimes \Delta R_{23}^{-1}, \quad a \in \Delta(U_q) \tag{4}$$

* Talk presented by M. Schlieker

The proof of (4) is based on the coassociativity of Δ. Therefore one defines the following new Hopfalgebra $\tilde{\mathcal{A}}$ by

$$\tilde{\mathcal{A}} = (U_q \otimes U_q,\ \tilde{\Box} = R_{23}^{-1}\Box R_{23},\ \tilde{S} = R_{21}(S\otimes S)R_{21}^{-1},\ \tilde{\varepsilon} = \varepsilon\otimes\varepsilon) \quad (5)$$

It is easy to proove that $\tilde{\mathcal{A}}$ is indeed a Hopfalgebra.
This new Hopfalgebra $\tilde{\mathcal{A}}$ now contains $\Delta(U_q)$ as subhopfalgebra because:

$$\begin{array}{llcl} 1., & \tilde{\Box}(a) & = & \Delta\otimes\Delta(a) \\ & & if & a\in\Delta(U_q) \\ 2., & \text{using } \sigma\circ\Delta & = & R\Delta R^{-1} \\ & \text{and } \sigma\circ\Delta & = & R_{21}^{-1}\ \Delta\ R_{21} \\ & \text{it holds} & & \\ & \tilde{S}(\Delta(a)) & = & \Delta(s(a)) \\ & & \text{if} & a\in\Delta(U_q) \end{array}$$

Because of the definition of $\sigma\circ\Delta$ there exists a second twist by replacing R by R_{21}. These two twisted Hopfalgebras only coincide on $\Delta(U_q)$.

Using the definition of $\tilde{\mathcal{A}}$ it is now easy to calculate the **universal** R-matrix for $\tilde{\mathcal{A}}$. in order to do that we define the following exchange operation

$$\tilde{\sigma}\circ(a\otimes b\otimes c\otimes d) = (c\otimes d\otimes a\otimes b) \quad (6)$$

Then it holds:

$$\begin{aligned} \tilde{\sigma}\circ\tilde{\Box} &= \tilde{\sigma}(R_{23}^{-1}\Box R_{23} \\ &= R_{41}^{-1}R_{13}R_{24}R_{23}\tilde{\Box}R_{23}^{-1}R_{24}^{-1}R_{13}^{-1}R_{41} \end{aligned}$$

Therefore the universal R-matrix of $\tilde{\mathcal{A}}$ is given by

$$\boldsymbol{R} = R_{41}^{-1}R_{13}R_{24}R_{23}$$

If R is the R-matrix of $SL_q\,(2)$, then $\boldsymbol{R}$ is a reparametrization of the Lorentz-group R-matrix given in ref. [Carow-Watamura et al 1990 and 1991].

If one introduces a pairing one obtains a modification of the quantum group Hopfalgebra $\tilde{\mathcal{A}}'$. The reason for that is the twist of the comultiplication:

i., the comultiplication on $\tilde{\mathcal{A}}'$ is unchanged:

$$\Box(s\otimes t) = s_1\otimes t_1\otimes s_2\otimes t_2$$

ii., the tensor multiplication is modified in the following form:

$$f\in\tilde{\mathcal{A}};\ s\otimes t, g\otimes k\in\tilde{\mathcal{A}}$$

$$\begin{aligned} &< f,(s\otimes t)(g\otimes k) > \\ &=< \tilde{\Box}(f), s\otimes t\otimes g\otimes k > \\ &=< f, s\cdot\sigma(R*(t\otimes g)*R^{-1})\cdot k > \end{aligned}$$

$$\Rightarrow\quad (s\otimes t)\cdot(g\otimes k) = s\cdot\sigma(R*(t\otimes g)*R^{-1})\cdot k \quad (7)$$

with the '$*$'-action of the universal enveloping algebra on the dual defined by: $f\in U_q$

$$f*s = s_1 < f, s_2 > \quad (8)$$
$$s*f = < f, s_1 > s_2 \quad (9)$$

Modified tensor multiplication laws have been considered in great detail by Majid in refs.: [Majid 1990, Majid 1992].

Now it possible to introduce a more canonical system of generators in $\tilde{\mathcal{A}}$ by the following definition:

$$L_+{}^{ab}{}_{cd} := (id\otimes id\otimes T^a{}_c\otimes T^b{}_d)\boldsymbol{R} \quad (10)$$
$$L_-{}^{ab}{}_{cd} := (T^a{}_c\otimes T^b{}_d\otimes id\otimes id)\boldsymbol{R}^{-1} \quad (11)$$

Using the definition of the generators of $U_q : l_\pm$ in terms of the R-matrix of U_q one obtains the following expression:

$$L_+{}^{ab}{}_{cd} := l_-{}^b{}_s\, l_+{}^a{}_t\otimes l_+{}^s{}_d\, l_+{}^t{}_c \quad (12)$$
$$L_-{}^{ab}{}_{cd} := l_-{}^b{}_s\, l_-{}^a{}_t\otimes l_-{}^s{}_d\, l_+{}^t{}_c \quad (13)$$

In the case $U_q = U_q(su_q(2))$ one can explicitely show, that the above formulas for L_+, L_- are invertible.

The comultiplication of the $L_\pm$ is given by:

$$\tilde{\Box}(L_\pm{}^{ab}{}_{cd}) = L_\pm{}^{ab}{}_{st}\otimes L_\pm{}^{st}{}_{cd} \quad (14)$$

This has to be expected from their definition, but can also be calculated by using identities of the form:

$$R^{-1}(l_+{}^s{}_a\otimes l_-{}^b{}_s)R = l_+{}^b{}_s\otimes l_-{}^s{}_a \quad (15)$$

The complex conjugation on these twisted algebras has not yet been investigated.

The easiest example of the above developed procedure is given by the relation between the universal enveloping algebra of the SO(4) and the Lorentz-group. To see that we start from

$$U_q = U_q\ (su_q(2))$$

the untwisted Hopfalgebra $\mathcal{A}$ equals $U_q(so_q(4))$.

The twisted Hopfalgebra $\tilde{A}$ corresponds to the deformation of the enveloping algebra of the Lorentzgroup which can be seen from example by defining a pairing through he generators $L_{\pm}$ and comparing this pairing with the standard FRT-pairing using the R-matrix of [Carow-Watamura 1990 and 1991] (and a reparametrization).

The relation between this algebra and the one considered in [Drabant et al 1992] is now under investigation.

References

Carow-Watamura, U., Schlieker, M., Scholl, M., Watamura, S.: 1990, 'Tensor representation of the quantum group $SL_q(2,\mathbb{C})$ and quantum Minkowski space', *Z.Phys.C.* **48**, 159–165

Carow-Watamura, U., Schlieker, M., Scholl, M., Watamura, S.: 1991, ''A quantum Lorentz group', *Int.J.Mod.Phys.A.* **6**, **17**, 3081–3108

Drabant, B., Schlieker, M., Weich, W., Zumino, B.: 1992, *Commun. Math.Phys.* **174**, 625–633

Drinfel'd, V.G.: 1986, 'Quantum Groups' In: *Proceedings of the International Congress of Mathematicians*, Berkeley 1986, 798–820

Faddeev, L.D., Reshetikhin, N.Yu., Takhtajan, L.A.: 1987, 'Quantization of Lie groups and Lie algebras' *Algebra Analysis* 1, 178

Majid, S.: 1990, *J.Algebra* **130**, 17–64

Majid, S.: 1992, *DAMTPI* **92/12**

ON A NONCOMMUTATIVE EXTENSION OF ELECTRODYNAMICS

JOHN MADORE
*Laboratoire de Physique Théorique et Hautes Energies**
Université de Paris-Sud, Bât. 211, F-91405 ORSAY
France

Abstract. The Maxwell vector potential and the Dirac spinor used to describe the classical theory of electrodynamics both have components which are considered to be ordinary smooth functions on space-time. We reformulate electrodynamics by adding an additional structure to the algebra of these functions in the form of the algebra M_n of $n \times n$ complex matrices. This involves a generalization of the notions of geometry to include the geometry of matrices. Some rather general constraints on the reformulation are imposed which can be motivated by considering matrix geometry in the limit of very large n. A few of the properties of the resulting models are given for the values $n = 2, 3$. One of the more interesting is the existence of several distinct stable phases or vacua.

Key words: noncommutative geometry

1. Introduction

In the usual formulation of electrodynamics the Maxwell potential and the Dirac spinor are constructed with components which lie in the algebra $\mathcal{C}$ of smooth functions on space-time. We wish to extend the construction to the algebra $\mathcal{A} = \mathcal{C} \otimes M_n$, where M_n is the algebra of $n \times n$ complex matrices. The Maxwell potential is a 1-form on space-time. We must therefore be able to define differential forms on the geometric structure defined by $\mathcal{A}$. This involves generalizing the notions of geometry to include the geometry of matrices. We give a brief review of matrix geometry in Section 2. In Section 3 a noncommutative generalization of the Maxwell-Dirac action is given. There are several possible generalizations, depending principally on the structure of the spinors. At the end of Section 3 we shall make some assumptions which reduce the possibilities to a set of models parametrized uniquely by the integer n, a mass scale m and the analog g of the electric charge. These can be partially motivated by considering matrix geometry in the limit of very large n, which in a sense which can be made explicit tends to the geometry of the ordinary 2-sphere. In Section 4 the properties of the models are sketched for $n = 2, 3$.

* *Laboratoire associé au CNRS.*

2. Matrix Geometry

We recall briefly here some of the details of matrix geometry (Dubois-Violette *et al.* 1989b, 1990a). For an introduction to noncommutative geometry in general we refer to the work of Connes (1986, 1990). An essential element in differential geometry is the notion of a vector field or derivation. It is an elementary fact of algebra that all derivations of M_n are interior. A derivation X is therefore necessarily of the form $X = \mathrm{ad}\, f$ for some f in M_n. The vector space D_n of all derivations of M_n is of dimension $n^2 - 1$.

Let λ_a, for $1 \leq a \leq n^2 - 1$, be an antihermitian basis of the Lie algebra of the special unitary group in n dimensions chosen with units of a mass scale m. The product $\lambda_a \lambda_b$ can be written in the form

$$\lambda_a \lambda_b = \frac{1}{2} C^c{}_{ab} \lambda_c + \frac{1}{2} D^c{}_{ab} \lambda_c - \frac{1}{n} m^2 g_{ab}. \tag{1}$$

The structure constants $C^c{}_{ab}$ are real and have also units of mass. The Killing metric is given by $k_{ab} = -C^c{}_{ad} C^d{}_{bc}$. It is related to g_{ab} by

$$k_{ab} = 2nm^2 g_{ab}.$$

The tensor $k_{ad} C^d{}_{bc}$ is completely antisymmetric. We shall raise and lower indices with g_{ab}. Then C_{abc} is also completely antisymmetric. We shall normalize the λ_a such that g_{ab} is the ordinary euclidean metric in $n^2 - 1$ dimensions.

The set λ_a is a set of generators of M_n. It is not a minimal set but it is convenient because of the fact that the derivations

$$e_a = \mathrm{ad}\, \lambda_a$$

form a basis over the complex numbers for D_n. Any element X of D_n can be written as a linear combination of the e_a: $X = X^a e_a$, where the X^a are complex numbers. The vector space D_n has a Lie-algebra structure. In particular the derivations e_a satisfy the commutation relations

$$[e_a, e_b] = C^c{}_{ab}\, e_c.$$

We define differential forms on M_n just as one does in the commutative case (Dubois-Violette 1988). For each matrix f we define the differential of f by the formula

$$df(e_a) = e_a(f). \tag{2}$$

This means in particular that

$$d\lambda^a(e_b) = [\lambda_b, \lambda^a] = C_{cb}{}^a \lambda^c. \tag{3}$$

We define the set of 1-forms $\Omega^1(M_n)$ to be the set of all elements of the form fdg or the set of all elements of the form $(dg)f$, with f and g in M_n. The

two definitions coincide because of the relation $d(fg) = f(dg) + (df)g$. The p-forms are defined exactly as in commutative case (Dubois-Violette *et al.* 1990a) with the product given as usual. The set of all differential forms is a differential algebra.

There is a basis θ^a of the 1-forms dual to the derivations e_a:

$$\theta^a(e_b) = \delta^a_b. \tag{4}$$

We have here suppressed the unit matrix which should appear as a factor of δ^a_b on the right-hand side. The θ^a are related to the $d\lambda^a$ by the equations

$$d\lambda^a = C^a{}_{bc} \lambda^b \theta^c, \tag{5}$$

and their inverse

$$\theta^a = m^{-4} \lambda_b \lambda^a d\lambda^b. \tag{6}$$

They satisfy the same structure equations as the components of the Maurer-Cartan form on the special unitary group SU_n:

$$d\theta^a = -\frac{1}{2} C^a{}_{bc}\, \theta^b \theta^c. \tag{7}$$

The product on the right-hand side of this formula is the product in the algebra of forms. Using the θ^a the exterior derivative can be written as $df = e_a \theta^a$. We shall consider the θ^a as the analog of a moving frame. They constitute a set of $n^2 - 1$ elements each of which is an $(n^2 - 1) \times n^2$ matrix. Each θ^a takes in fact D_n, of dimension $n^2 - 1$, into M_n, of dimension n^2. The interior product and the Lie derivative are defined as usual.

From the generators θ^a we can construct a 1-form θ in $\Omega^1(M)$ which will play an important role in the study of gauge fields. We set

$$\theta = -\lambda_a \theta^a.$$

From equation (5) we see that it can be written in the forms

$$\theta = -\frac{1}{nm^2} \lambda_a d\lambda^a = \frac{1}{nm^2} d\lambda_a \lambda^a.$$

Using θ we can rewrite equation (6) as

$$\theta^a = m^{-4} C^a{}_{bc} \lambda^b d\lambda^c - nm^{-2} \lambda^a \theta. \tag{8}$$

From equations (5) and (7) one sees that θ satisfies the zero-curvature condition:

$$d\theta + \theta^2 = 0. \tag{9}$$

It satisfies with respect to the algebraic exterior derivative the same condition which the Maurer-Cartan form satisfies with respect to ordinary exterior derivation on the group SU_n.

We shall introduce a metric on D_n by the requirement that the frame e_a be orthonormal. For $X = X^a e_a$ and $Y = Y^a e_a$ we define

$$g(X,Y) = g_{ab} X^a Y^b.$$

To within a rescaling $g(X,Y)$ is the unique metric on D_n with respect to which all the derivations e_a are Killing derivations. The first structure equations for the frame e_a and a linear connection $\omega^a{}_b$ can be now written down:

$$d\theta^a + \omega^a{}_b \theta^b = \Theta^a.$$

We see then that if we require the torsion form Θ^a to vanish then the internal structure is like a curved space with a linear connection given by

$$\omega^a{}_b = -\frac{1}{2} C^a{}_{bc}\, \theta^c. \tag{10}$$

The second structure equation defines the curvature form $\Omega^a{}_b$, which satisfies the Bianchi identities as before.

The complete set of all derivations of M_n is the natural analog of the space of all smooth vector fields $D(V)$ on a manifold V. If V is parallizable then $D(V)$ is a free module over the algebra of smooth functions with a set of generators e_α which is closed under the Lie bracket and which has the property that if $e_\alpha f = 0$ for all e_α then f is a constant function. The matrix algebra M_n has in general several Lie algebras of derivations D with this property. The smallest such one, D_2, is obtained by considering three matrices λ_a which form the irreducible n-dimensional representation of SU_2. These matrices generate the algebra M_n. The most general element of M_n is a polynomial in the λ_a. The equations

$$e_a f = 0, \qquad e_a = \mathrm{ad}\,\lambda_a, \quad 1 \le a \le 3,$$

imply that f is proportional to the unit element. The set D_2 could also be considered as the natural counterpart of a moving frame on a manifold (Madore 1991).

With a restricted set of derivations, one can define the exterior differential exactly as before using equation (2). However now the set of e_a is a basis of $D \subseteq D_n$. The derivations are taken, so to speak, only along the preferred directions. Equation (3) remains valid, the only change being that the structure constants are those of the algebra of derivations. A difference lies in the fact that the forms are of course multilinear maps on the preferred derivations and are not defined on all elements of D_n. The formula (4) which defines the dual forms is as before but the meaning of the expression θ^a changes. If we choose for example D_2 as the derivations then θ^a is a $3 \times n^2$ matrix. It takes the vector space D_2 into M_n and it is not defined on the $n^2 - 4$ remaining generators of D_n. Equation (5) remains unchanged but equations (6) and (8) will have to be modified.

Consider the case $D = D_2 \subseteq D_n$. Let e_r for $1 \le r \le n^2 - 1$ be a basis of D_n and let e_a for $1 \le a \le 3$ be a basis of D_2. We can choose the e_a to be the first 3 elements of the e_r. Then the SU_2 structure constants $C^a{}_{bc}$ are the restriction of the SU_n structure constants $C^r{}_{st}$. Equation (1) will be therefore written

$$\lambda_a \lambda_b = \frac{1}{2} C^c{}_{ab} \lambda_c + \frac{1}{2} D^r{}_{ab} \lambda_r - \frac{1}{n} m^2 g_{ab}. \tag{11}$$

If a basis J_a of D_2 satisfies the commutation relations $[J_a, J_b] = 2i\epsilon_{abc} J^c$ then $J_a J^a = n^2 - 1$. On the other hand, from equation (11) we see that $\lambda_a \lambda^a = -3m^2/n$. If we write then

$$\lambda_a = -\frac{i}{2r} J_a,$$

we find that r is related to m by the equation $12 m^2 r^2 = n(n^2 - 1)$ and that the SU_2 structure constants are given by

$$C_{abc} = r^{-1} \epsilon_{abc}.$$

Let θ^r be the dual basis of e_r and let θ^a be the dual basis of e_a. We have then 2 possible expressions for θ^a. We have

$$\theta'^a = m^{-4} C^a{}_{rs} \lambda^r d\lambda^s - n m^{-2} \lambda^a \theta',$$

with θ' constructed using λ^r and we have

$$\theta^a = \frac{n}{3m^2} (r^2 C^a{}_{bc} \lambda^b d\lambda^c + \theta \lambda^a), \tag{12}$$

with θ constructed using λ^a. Both definitions satisfy the equation (4). That is, they coincide as $3 \times n^2$ matrices. In equation (10) each of the λ_r can be expanded as a polynomialin terms of the 3 elements λ_a and using the Leibnitz rule this yields a long complicated expression for θ'^a in terms of the $d\lambda^a$. In equation (12) the expression $d\lambda^a$ is a $3 \times n^2$ matrix but it has a natural extension to an $(n^2 - 1) \times n^2$ matrix in which case it coincides with the definition of $d\lambda^r$ for the first three values of the index r. The two expressions for θ^a can be compared therefore as forms on the complete set of derivations. Whereas by construction $\theta'^a(e_r) = 0$ for $r \ge 4$, in general the corresponding equation for θ^a would not be satisfied.

Using the basis of D_2 and its dual we can write the differential of a matrix f as

$$df = e_a f \theta^a. \tag{13}$$

The complete differential is given by $df = e_r f \theta^r$. If $df(e_a) = 0$ then $e_a f = 0$. This means that f is proportional to the unit element and therefore that $df = 0$. However if α is a general 1-form then the condition $\alpha(e_a) = 0$ does not imply that $\alpha = 0$. For example any basis element θ^r for $r \geq 4$ satisfies the equation $\theta^r(e_a) = 0$. When we consider the restricted set D_2 of derivations we shall choose the algebra of forms to be the differential algebra generated by the forms (12). In this case if α is a 1-form which satisfies the condition $\alpha(e_a) = 0$ then $\alpha = 0$.

Using the 1-form θ we can write the differential of a matrix f as

$$df = -[\theta, f].$$

If we consider the algebra of all forms as a $\mathbf{Z}_2$-graded algebra then we can define another d acting on any form α by the formula (Connes 1986, 1990)

$$d\alpha = -[\eta, \alpha], \tag{14}$$

where η is some 1-form and the bracket is $\mathbf{Z}_2$-graded. See also Quillen (1985) and Dubois-Violette *et al.* (1991). If $\eta^2 = -1$ we have $d^2 = 0$. Equation (9) becomes

$$d\eta + \eta^2 = 1.$$

The definition (14) is interesting in that it does not use derivations and thus can be used when considering the case of more abstract algebras which have none.

We shall now consider an extension of matrix geometry by considering the algebra of matrix-valued functions on space-time (Dubois-Violette *et al.* 1989a, 1989b, 1990b). Let x^μ be coordinates of space-time. Then the set (x^μ, λ^a) is a set of generators of the algebra $\mathcal{A}$ which is the tensor product

$$\mathcal{A} = \mathcal{C} \otimes M_n, \tag{15}$$

of $\mathcal{C}$ the algebra of smooth real-valued functions on space-time and M_n. The tensor product is over the complex numbers. Let $e_\alpha = e^\mu_\alpha \partial_\mu$ be a moving frame on space-time and e_a with $1 \leq a \leq 3$ a basis of D_2. Let $i = (\alpha, a)$. Then $1 \leq i \leq 7$. We shall refer to the set $e_i = (e_\alpha, e_a)$ as a moving frame on the algebra $\mathcal{A}$.

For $f \in \mathcal{A}$ we define df by equation (2) but with the index a replaced by i. Choose a basis $\theta^\alpha = \theta^\alpha_\lambda dx^\lambda$ of the 1-forms on space-time dual to the e_α and introduce $\theta^i = (\theta^\alpha, \theta^a)$ as generators of the 1-forms $\Omega^1(\mathcal{A})$ as a left or right $\mathcal{A}$-module. Then if we define

$$\Omega^1_H = \Omega^1(\mathcal{C}) \otimes M_n, \qquad \Omega^1_V = \mathcal{C} \otimes \Omega^1(M_n),$$

we can write $\Omega^1(\mathcal{A})$ as a direct sum:

$$\Omega^1(\mathcal{A}) = \Omega^1_H \oplus \Omega^1_V.$$

The differential df of a matrix function is given by

$$df = d_H f + d_V f.$$

We have written it as the sum of two terms, the horizontal and vertical parts, using notation from Kaluza-Klein theory. The horizontal component is the usual exterior derivative $d_H f = e_\alpha f \theta^\alpha$. The vertical component d_V, given by equation (13), is purely algebraic and it is what replaces the derivative in the hidden compactified dimensions. The algebra $\Omega^*(\mathcal{A})$ of all differential forms is defined as usual. It is again a differential algebra.

3. The Maxwell-Dirac action

We shall now write down the analog of the Maxwell-Dirac action in the geometry defined by the algebra (15). We shall identify a connection with an anti-hermitian element ω of $\Omega^1(\mathcal{A})$. We saw above that it can be split as the sum of two parts which we called horizontal and vertical. We write then

$$\omega = A + \omega_V, \tag{16}$$

where A is an element of Ω^1_H and ω_V is an element of Ω^1_V.

In Section 2 we introduced a 1-form θ in $\Omega^1(M) \subseteq \Omega^1_V$. We shall use this 1-form as a preferred origin for the elements of Ω^1_V. We write accordingly

$$\omega_V = \theta + \phi. \tag{17}$$

The field ϕ is the Higgs field.

We have noted previously that θ resembles a Maurer-Cartan form. Formula (3.1) with $\phi = 0$ is therefore formally similar to the connection form on a trivial principal U_1-bundle. We have in fact a bundle over a space which itself resembles a bundle. This double-bundle structure, which is what gives rise to a quartic Higgs potential as we shall see below, has been investigated previously, by Manton (1979), Harnad *et al.* (1980), Chapline and Manton (1980), and, more recently, by Kerner *et al.* (1987) and by Coquereaux and Jadczyk (1988). The $\mathcal{A}$-modules which we shall consider are the natural generalization of the space of sections of a trivial U_1-bundle since M_n has replaced $\mathbf{C}$ in our models. So the U_n gauge symmetry comes not from the number of generators of the module, which we shall always choose to be equal to 1, but rather from the factor M_n in our algebra $\mathcal{A}$.

Let U_n be the unitary elements of the matrix algebra M_n and let $\mathcal{U}_n$ be the group of unitary elements of $\mathcal{A}$, considered as the algebra of functions on space-time with values in M_n. We shall choose $\mathcal{U}_n$ to be the group of local gauge transformations. A gauge transformation defines a mapping of $\Omega^1(\mathcal{A})$ into itself of the form

$$\omega' = g^{-1}\omega g + g^{-1}dg.$$

We define

$$\theta' = g^{-1}\theta g + g^{-1}d_V g,$$
$$A' = g^{-1}Ag + g^{-1}d_H g,$$

and so ϕ transforms under the adjoint action of $\mathcal{U}_n$:

$$\phi' = g^{-1}\phi g.$$

It can be readily seen that in fact θ is invariant under the action of $\mathcal{U}_n$:

$$\theta' = \theta.$$

Therefore the transformed potential ω' is again of the form (17).

The fact that θ is invariant under a gauge transformation means in particular that it cannot be made to vanish by a choice of gauge. We have then a connection with vanishing curvature but which is not gauge-equivalent to zero. If M_n were an algebra of functions over a compact manifold, the existence of such a 1-form would be due to the non-trivial topology of the manifold.

We define the curvature 2-form Ω and the field strength F as usual:

$$\Omega = d\omega + \omega^2, \quad F = d_H A + A^2.$$

In terms of components, with $\phi = \phi_a\theta^a$ and $A = A_\alpha\theta^\alpha$ and with

$$\Omega = \frac{1}{2}\Omega_{ij}\theta^i \wedge \theta^j, \quad F = \frac{1}{2}F_{\alpha\beta}\theta^\alpha \wedge \theta^\beta,$$

we find

$$\Omega_{\alpha\beta} = F_{\alpha\beta}, \quad \Omega_{\alpha a} = D_\alpha\phi_a, \quad \Omega_{ab} = [\phi_a, \phi_b] - C^c{}_{ab}\,\phi_c. \tag{18}$$

The analog of the Maxwell action is given by

$$S_B = \int \mathcal{L}_B, \tag{19}$$

where

$$\mathcal{L}_B = \frac{1}{4g^2}Tr(F_{\alpha\beta}F^{\alpha\beta}) + \frac{1}{2g^2}Tr(D_\alpha\phi_a D^\alpha\phi^a) - V(\phi). \tag{20}$$

The Higgs potential $V(\phi)$ is given by

$$V(\phi) = -\frac{1}{4g^2}Tr(\Omega_{ab}\Omega^{ab}). \tag{21}$$

It is a quartic polynomial in ϕ which is fixed and has no free parameters apart from the mass scale m. The trace is the equivalent of integration on the

matrix factor in the algebra. The constant g is the gauge coupling constant. We see then that the analog of the Maxwell action describes the dynamics of a U_n gauge fields unified with a set of Higgs fields which take their values in the adjoint representation of the gauge group.

The lagrangian (20) is the standard lagrangian chosen for all gauge theories which use the Higgs mechanism. Given a gauge group the theories differ according to the representation in which the Higgs particles lie and the form of the Higgs potential. The particular expression to which we have been lead has been also found by slightly different, group theoretical, considerations in the context of dimensional reduction by Harnad *et al.* (1980) and by Chapline and Manton (1980). What our formalism shows is that the Higgs potential is itself the action of a gauge potential on a purely algebraic structure. The Ω_{ab} are in fact the components of the curvature Ω_V of the connection (17):

$$\Omega_V = d\omega_V + \omega_V^2 = \frac{1}{2}\Omega_{ab}\theta^a \wedge \theta^b.$$

The connection determines a covariant derivative on an associated $\mathcal{A}$-module (Connes 1986, 1990). See also Dubois-Violette *et al.* (1991). Let H be a M_n-module. It inherits therefore a U_n-module structure. Define $\mathcal{H} = \mathcal{C} \otimes H$. Then $\mathcal{H}$ is an $\mathcal{A}$-module as well as a $\mathcal{U}_n$-module. The form of the covariant derivative depends on the module structure of H. The covariant derivative of $\psi \in \mathcal{H}$ is of the form

$$D\psi = d\psi + \omega\psi.$$

The action of ω on ψ is determined by the action of U_n on H. We have only then to define the vertical derivatives $e_a\psi$ of ψ. Since $\mathcal{H}$ is a $\mathcal{A}$-module, for any f in $\mathcal{A}$ we must have the relation

$$e_i(f\psi) = (e_i f)\psi + f e_i\psi. \tag{22}$$

Suppose that H is a left module. We shall consider only the case $H = \mathbf{C}^n$. From equation (22) we see that we must set

$$e_a\psi = \lambda_a\psi.$$

The action of U_n can only be left multiplication. We find then that

$$D_a\psi = \phi_a\psi.$$

Suppose that H is a bimodule. We shall consider only the case $H = M_n$. From equation (22) we see that we must set

$$e_a\psi = [\lambda_a, \psi].$$

There are now two possibilities for the action of U_n. We can choose H to be a bimodule with the adjoint action or a left module with left multiplication. We find then in the first case

$$D_a\psi = [\phi_a, \psi].$$

This is invariant under the adjoint action of U_n. In the second case we find

$$D_a\psi = \phi_a\psi - \psi\lambda_a. \tag{23}$$

This is invariant only under the left action of U_n.

With the frame θ^i which was introduced above the geometry of the algebra $\mathcal{A}$ resembles in some aspects ordinary commutative geometry in dimension 7. As $n \to \infty$ it resembles more and more ordinary commutative geometry in dimension 6 and the frame θ^i becomes a redundant one in the limit. Let g_{kl} be the Minkowski metric in dimension 7 and γ^k the associated Dirac matrices which we shall take to be given by

$$\gamma^k = (1 \otimes \gamma^\alpha, \sigma^a \otimes \gamma^5).$$

The space of spinors must be a left module with respect to the Clifford algebra. It is therefore a space of functions with values in a vector space P of the form

$$P = H \otimes \mathbf{C}^2 \otimes \mathbf{C}^4.$$

The Dirac operator is a linear first-order operator of the form

$$\not{D} = \gamma^k D_k,$$

where D_k is the appropriate covariant derivative, which we must now define. The space-time components are the usual ones:

$$D_\alpha\psi = e_\alpha\psi + A_\alpha\psi + \frac{1}{4}\omega_\alpha{}^\beta{}_\gamma\gamma_\beta\gamma^\gamma\psi.$$

The $\omega_\alpha{}^\beta{}_\gamma$ are the coefficients of a linear connection defined over space-time:

$$\omega_\alpha{}^\beta = \omega_\alpha{}^\beta{}_\gamma\theta^\gamma.$$

By analogy we have to add to the covariant derivative given above a term which reflects the fact that the algebraic structure resembles a curved space with a linear connection given by equation (10). We make then the replacement (Madore 1989)

$$D_a\psi \to D_a\psi - \frac{1}{8}C^b{}_{ca}\gamma_b\gamma^c\psi. \tag{24}$$

The analog of the Dirac action is given by

$$S_F = \int \mathcal{L}_F, \tag{25}$$

where

$$\mathcal{L}_F = Tr(\bar\psi\not{D}\psi).$$

We have therefore defined a set of theories which are generalizations of electrodynamics to the algebra $\mathcal{A}$. In order to restrict the generality we shall make three assumptions. First, we shall suppose there is no explicit mass term in the classical Dirac action. We have already supposed that the derivations to be used are the algebra D_2. Last, we shall suppose also that H has the module structure which leads to the covariant derivative defined by equation (23) to which we add the curvature term as in (24). The last two assumptions can be motivated by showing that in the limit for large n the covariant derivative tends in a sense which can be made explicit to that used in the Schwinger model (Grosse & Madore 1992). With the restrictions we have a set of classical models which for each integer n depend only on the coupling constant g and the mass scale m and given by the classical action

$$S = S_B + S_F, \tag{26}$$

where S_B is defined by equation (19) and S_F is defined by equation (25).

Different restrictions result in different models (Dubois-Violette *et al.* 1989a, 1989b, 1990, 1991, Madore 1989, 1991, Balakrishna *et al.* 1991a, 1991b). If one uses the exterior derivative (14) one obtains yet different models (Connes & Lott 1989, Coquereaux *et al.* 1991) but which are similar at least in the bosonic sector. The main difference lies in the form of the Higgs potential which is in fact closer in form to that used in the standard electroweak model.

4. Models

We shall now consider the action (26) in the case $n = 2$ (Dubois-Violette *et al.* 1989b, 1990b) and examine the resulting physical spectrum. The lagrangian (20) is a generalization of the Yang-Mills-Higgs-Kibble lagrangian, with a more elaborate Higgs sector. The most original part is the potential term $V(\phi)$ which comes from the curvature of the vertical part of the connection. It is not the most general gauge-invariant polynomial in the Higgs field which would be allowed and there is no reason to suppose that its form remains invariant under renormalization effects. The fermions are Dirac fermions which take their values in the space $M_2 \otimes \mathbf{C}^2$ and the gauge group is U_2. There are therefore four U_2 doublets.

From equation (21) and the definition (18) of Ω_{ab} we see that the vacuum configurations are given by the values μ_a of ϕ_a which satisfy the equation

$$[\mu_a, \mu_b] - C^c{}_{ab}\mu_c = 0. \tag{27}$$

The number of solutions to this equation is given by the partition function, the number of ways one can partition the integer n into a set of decreasing positive integers. Two obvious solutions are $\mu_a = 0$ which corresponds to the partition $(1, \dots, 1)$ and $\mu_a = \lambda_a$ which corresponds to the partition (n). If $n = 2$ there are no others. Matter can exist then in two phases. In the symmetric phase all the gauge bosons are massless and three of them are gluon-like. The fermions are quark-like. In units of $(1/2\sqrt{2})m$ there are two doublets of mass 3 and two of mass 5. We call this phase the hadronic phase. We shall suppress in an *ad hoc* way the U_1 component of the gauge group and reduce it to SU_2. There is no photon then and the fermions are all neutral. In the broken phase, the gauge bosons are all massive if we suppress the U_1 component. The fermions are again neutral but of different masses. There are now two doublets of mass 5, a doublet of mass 7 and a doublet of split mass 5 and 7 units. We call this phase the third phase, for reasons to be made clear below.

In the case $n = 3$ the fermions are Dirac fermions which take their values in the space $M_3 \otimes \mathbf{C}^2$ and the gauge group is U_3. There are therefore six U_3 triplets. Matter can exist now in 3 phases corresponding to the 3 partitions of 3. In the symmetric phase all the gauge bosons are massless and eight of them are gluon-like. The fermions are quark-like. In the units given above they have all masses of the order of one. This is the hadronic phase. Since we do not wish to interpret the U_1 component of the gauge group as the photon, the fermions are neutral. In the broken phase which corresponds to the $n = 2$ case the gauge bosons are all massive if we suppress the U_1 component. The fermions are then again neutral and again of different masses. But in the units given above they still have masses of the order of one. This is the third phase.

The extra phase for $n = 3$ we call the leptonic phase. It is given by the solution to the equation (27) of the form

$$\mu_a = -\frac{i}{\sqrt{2}} m \begin{pmatrix} 0 & 0 \\ 0 & \sigma_a \end{pmatrix}.$$

As we shall see below, in this phase there are two massless gauge modes. We must identify one of the corresponding fields with the photon and again in an *ad hoc* way suppress the other mode. Define the matrices κ_4 and κ_5 by

$$\kappa_4 = i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \kappa_5 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We write the gauge potential in the form

$$A = A^4\kappa_4 + A^5\kappa_5 + \begin{pmatrix} 0 & W^- \\ W^+ & Z \end{pmatrix}.$$

Here A^4 and A^5 are ordinary 1-forms, W^+ is a 1-form with values in $\mathbf{C}^2$, $W^- = -(W^+)^*$ and Z is a 1-form with values in the Lie algebra of SU_2. In this phase there are therefore 2 charged gauge bosons and 3 neutral ones. Their masses are given by

$$m_W^2 = \frac{3}{2}m^2, \qquad m_Z^2 = 4m^2.$$

There are two massless bosons. We shall set

$$A^5 = 0$$

and choose A^4 to represent the photon then the unit of charge is given by $e = g$. All of the 6 triplets of fermions have again masses of the order of m and these masses are again different from the corresponding masses in the hadronic and the third phases. Two triplets have charge 1 and the other 4 are neutral. We write then the spinor field in the form

$$\psi = \begin{pmatrix} e & \mu & \tau \\ \nu_e & \nu_\mu & \nu_\tau \\ l_1 & l_2 & l_3 \end{pmatrix}.$$

Here, e, μ and τ are charged doublets; each ν and l is a neutral doublet.

The coupling of the Higgs field to the fermions is not constrained by gauge invariance and so there is no reason why the corresponding coefficient in equation (23) for example should be equal to 1. We could have for any real number x:

$$D_a\psi = x\phi_a\psi - \psi\lambda_a - \frac{1}{8}C^b{}_{ca}\gamma_b\gamma^c\psi.$$

The same argument applies to the curvature terms. Under an arbitrary change of frame,

$$\theta^\alpha \mapsto \theta'^\alpha = \Lambda^\alpha_\beta\theta^\beta,$$

the spinor ψ transforms to $\psi' = S^{-1}(\Lambda)\psi$ and the Dirac matrices transform as

$$\gamma^\alpha \mapsto \gamma'^\alpha = S^{-1}(\Lambda)\gamma^\alpha S(\Lambda).$$

The space-time components $D_\alpha\psi$ of the covariant derivative of ψ have been constructed so that they transform correctly. The same behaviour must be required of the algebraic components $D_a\psi$. The covariant derivative we have used transforms as it should. But in fact each term transforms correctly and we could have more generally for any real numbers x, y

$$D_a\psi = x\phi_a\psi - \psi\lambda_a - \frac{1}{8}yC^b{}_{ca}\gamma_b\gamma^c\psi. \tag{28}$$

There is no way then to fix the renormalized values of the masses of the fermions. They will depend on the mass scale m and the two parameters x and y.

5. Conclusions

We have presented some of the details of the simplest noncommutative extension of electrodynamics, which we have completed in an *ad hoc* way by suppressing the abelian component of U_n gauge potentials. Even with this modification none of the models we have presented has the correct phenomenology. There are in general unobserved particles and the mass spectrum is too rigid. There is only one mass parameter m and all of the particles either have mass zero or a mass or order m. Renormalization effects could however within the context of the model introduce a modification of the mass spectrum according to equation (28).

Acknowledgements

Much of the content of this article was obtained in collaboration with Michel Dubois-Violette and Richard Kerner.

References

Balakrishna B. S., Gürsey F., Wali K. C.: 1991a, Phys. Lett. **B254** 430.
Balakrishna B. S., Gürsey F., Wali K. C.: 1991b, Phys. Rev. **D44** 3313.
Chapline G., Manton N. S.: 1980, Nucl. Phys. **B184** 391.
Connes A.: 1986, Publications of the I.H.E.S. **62** 257.
Connes A.: 1990, *Géométrie noncommutative'*, InterEditions: Paris
Connes A., Lott J.: 1989, Nucl. Phys Proc. Suppl. **B18** 29.
Coquereaux R., Esposito-Farèse G., Vaillant G.: 1991, Nucl. Phys. **B353** 689.
Coquereaux R., Jadczyk A.: 1988, World Scientific Lecture Notes in Physics **16**.
Dubois-Violette M.: 1988, C. R. Acad. Sci. Paris **307** Série I 403.
Dubois-Violette M., Kerner R., Madore J.: 1989a, Phys. Lett. **B217** 485.
Dubois-Violette M., Kerner R., Madore J.: 1989b, Class. Quant. Grav. **6** 1709.
Dubois-Violette M., Kerner R., Madore J.: 1990a, J. Math. Phys. **31** 316.
Dubois-Violette M., Kerner R., Madore J.: 1990b, J. Math. Phys. **31** 323.
Dubois-Violette M., Kerner R., Madore J.: 1991, Class. Quant. Grav. **8** 1077.
Grosse H., Madore J.: 1992, Phys. Lett. **B283** 218.
Harnad J., Shnider S., Tafel J.: 1980, Lett. in Math. Phys. **4** 107.
Kerner R., Nikolova L., Rizov V.: 1987, Lett. in Math. Phys. **14** 333.
Madore J.: 1989, Mod. Phys Lett. A **4** 2617.
Madore J.: 1991, Int. Jour. of Mod. Phys. A **6** 1287.
Manton N.S.: 1979, Nucl. Phys. **B158** 141.
Quillen D.: 1985, Topology **24** 89.

BICOVARIANT DIFFERENTIAL CALCULUS AND q-DEFORMATION OF GAUGE THEORY

SATOSHI WATAMURA
Department of Physics
College of General Education
Tohoku University
Kawauchi, Aobaku, Sendai 980, JAPAN

Abstract. The q-deformation of the BRST algebra, the algebra of the ghost, matter and gauge field on one spacetime point is constructed using the result of the bicovariant differential calculus. We define the covariant commutation relation among the fields and their derivatives consistently with the two nilpotent operation the spacetime derivative and the BRST operation.

1. Introduction

It is an interesting question whether one can construct a q-analogue of the gauge theory by taking the quantum group [Drinfeld 1986, Reshetikhin 1986, Jimbo 1986, Woronowicz 1987] as a symmetry. One of the interesting possibilities of such a q-deformed theory is that the deformation parameter q may play the role of a regularization parameter. Furthermore, since the quantum group is provided by a noncommutative algebra, in such a theory the noncommutative geometry plays a basic role like the differential geometry in the usual gauge theory.

There are some proposals to this problem [Aref'eva 1991, Bernard 1990, Hirayama 1992, Isaev 1992, Wu 1992]. However, it seems that there are still conceptual problems concerning the definition of the gauge transformation when we take the quantum group as an algebraic object of the gauge symmetry. Since the quantum group is formulated in the language of the Hopf algebra, it forces us to formulate the whole theory in an appropriate algebraic language [Brzeziński 1992]. Therefore, the gauge transformation has to be represented in this abstract language and the notion of the transformation parameter becomes obscure. Even when we consider only the infinitesimal transformation, we still have to clarify the definition of the infinitesimal parameters.

One of the alternative formulations of the gauge theory is given by the BRST formalism [Becchi 1976, Tyuitin 1975]. There, the gauge transformation parameter is replaced by the ghost field and becomes an object of equal level with the matter and the gauge fields. Therefore, when we consider the q-deformation of the gauge theory, it is very natural to consider the q-deformed field algebra starting with the BRST formalism.

Here we report recent results about q-deformation of the BRST algebra which is the algebra of the gauge fields, the ghost fields and appropriate matter fields on one spacetime point [Watamura 1992]. The gauge transformation of the theory is replaced by the BRST transformation which is represented by a nilpotent "differential operator" δ_B.

For the notation concerning the Hopf algebra [Abe 1980], we take: the coproduct Δ, the antipode κ and the counit ε. Through this talk the upper case roman index I, J, K, L runs $0, -, 3, +$ and the lower case index like a, b, c, d runs over the label of the adjoint representation, $-, 3, +$, otherwise we specify explicitly.

2. Bicovariant Differential Calculus

Before we start to construct the BRST algebra, let us briefly recall some results of the bicovariant differential calculus [Woronowicz 1989, Jurco 1991, Carow-Watamura 1991a]. The one forms are defined by the right invariant bases θ^i_j $(i,j = 1,2)$ where $\theta^{i*}_j = \theta^j_i$. Using the spinor metric $\epsilon^{kl} = \begin{pmatrix} 0 & -q^{-\frac{1}{2}} \\ q^{\frac{1}{2}} & 0 \end{pmatrix}$, we define $\theta^{ij} = \theta^i_k \epsilon^{kj}$ then they have the commutation relation

$$a\theta^{ij} = \theta^{kl}(a * \mathbf{L}^{ij}_{kl}) \tag{1}$$

for $\forall a \in Fun_q(SU(2))$. $\mathbf{L}$ is the functional $Fun_q(SU(2)) \to C$ defined by

$$\mathbf{L}^{ij}_{kl} = (L_{-}{}^{j}_{l} * L_{+}{}^{i}_{k}) \circ \kappa \quad , \tag{2}$$

where the functionals [1] $L_\pm$ are defined using $\widehat{\mathbf{R}}$-matrix as $L^i_{\pm j}(M^k_l) = \widehat{\mathbf{R}}^{\pm ik}_{\ \ lj}$ for generators M^i_j of $Fun_q(SU(2))$, and the convolution product is $(f * g) = (f \otimes g)\Delta$.

The right invariant basis θ^{ij} can be split into two parts according to its transformation property, the adjoint representation θ^a $(a = -, 3, +)$ and the singlet θ^0, by using the q-Pauli matrices σ^I_{ij} and σ^{ij}_I where $\sigma^I_{ij}\sigma^{ij}_J = \delta^I_J$ $(I = 0, -, 3, +)$ and $\sigma^0_{kl} = \frac{q}{\sqrt{Q}}\epsilon_{kl}$, $\sigma^+_{kl} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma^-_{kl} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\sigma^3_{kl} = \frac{-1}{\sqrt{Q}}\begin{pmatrix} 0 & q^{\frac{1}{2}} \\ q^{-\frac{1}{2}} & 0 \end{pmatrix}$ with $Q = q + q^{-1}$ and $\epsilon_{kl} = -\epsilon^{kl}$. The projectors can be

[1] The functionals $f_\pm$ appearing in ref.[Carow-Watamura 1991a] are equivalent to the $L_\pm$ in ref.[Reshetikhin 1989] which we use here. Thus, $\mathbf{L}^{ij}_{kl}$ is equivalent to the functional $f^{ij}_{\mathrm{Ad}\,kl} \circ \kappa$ in ref.[Carow-Watamura 1991a].

written as $\mathcal{P}^{ij}_{A\,kl} = \sigma^{ij}_0\sigma^0_{kl}$, $\mathcal{P}^{ij}_{S\,kl} = \sigma^{ij}_a\sigma^a_{kl}$ where $\mathcal{P}_S$ ($\mathcal{P}_A$) is the projector for the q-(anti)symmetric product and $\widehat{\mathbf{R}} = q^{\frac{1}{2}}\mathcal{P}_S - q^{\frac{-3}{2}}\mathcal{P}_A$.

The q-deformed exterior derivative $\mathbf{d}$ is defined as a map from $Fun_q(G)$ to the bimodule defined with the basis θ requiring that the nilpotency and the Leibniz rule hold in the standard way [Woronowicz 1989]. Such an operation can be defined simply as a commutator with the singlet component θ^0 as [Carow-Watamura 1991a]

$$\mathbf{d}a = \frac{ig}{\omega}[\theta^0, a]_- \tag{3}$$

for any element $a \in Fun_q(SU(2))$, where $\omega = q - q^{-1}$, $i = \sqrt{-1}$ and g is a non-zero real constant. [2] Since $\mathbf{d}a$ is an element of the bicovariant bimodule, we can expand it with the basis as

$$\mathbf{d}a = \theta^I(a * \chi_I) \quad , \tag{4}$$

where the right invariant vector field χ_{ij} is given by

$$\chi_I \equiv \sigma^{ij}_I\chi_{ij} = \sigma^{ij}_I\frac{ig}{\omega}(\sigma^0_{ij}\varepsilon - \sigma^0_{kl}\mathbf{L}^{kl}_{ij}) \quad . \tag{5}$$

One of the suggesting relation given by the bicovariant differential calculus is the q-analogue of the Maurer-Cartan equation. We gave the expression in a more familiar form in ref.[Carow-Watamura 1991a]:

$$\mathbf{d}\theta^0 = 0 \quad , \quad \mathbf{d}\theta^a = \frac{-ig}{q^2 + q^{-2}}f^a_{bc}\theta^b \wedge \theta^c \quad , \tag{6}$$

where $\wedge$ is the q-deformed exterior product. The f^a_{bc} is the q-analogue of the structure constants. Using the general formula for the structure constants in ref.[Carow-Watamura 1991a] (See also ref.[Carow-Watamura 1992]), we obtain them for the $Fun_q(SU(2))$ as

$$\begin{aligned} f^+_{+3} = f^-_{3-} = q \ , \quad f^+_{3+} = f^-_{-3} = -q^{-1} \quad , \\ f^3_{+-} = -f^3_{-+} = 1 \ , \quad f^3_{33} = q - q^{-1} \quad . \end{aligned} \tag{7}$$

The commutation relation of the right invariant basis are $[\chi_0, \chi_a] = 0$ and

$$\mathbf{P}^{b'c'}_{Ad\,bc}(\chi_{b'} * \chi_{c'}) = \frac{-1}{q^2 + q^{-2}}f^a_{bc}\mathbf{p} * \chi_a \quad , \tag{8}$$

where the functional $\mathbf{p}$ is $\mathbf{p} = ig\varepsilon - \omega\chi_0$ and $\mathbf{P}^{ab}_{Ad\ cd} = \frac{-1}{q^2+q^{-2}}f^{a'}_{cd}f^{ab}_{a'}$, where $f^{bc}_a \equiv -f^a_{bc}$ is the projector onto the antisymmetric product of two adjoint representations. (see also section 5 of ref.[Carow-Watamura 1991b].)

[2] The relation of the constant g with the constant N_0 in ref.[Carow-Watamura 1991a] is $g = \frac{-i\sqrt{Q}}{qN_0}$.

3. Gauge Transformation and BRST Formalism

As we explained in the introduction, we need to represent the gauge theory using an appropriate algebra language which fits to the Hopf algebra structure. Thus, let us first reconsider the gauge and the BRST transformation in the non-deformed theory.

When we consider the usual non-deformed gauge theory with a symmetry group $SU(2)$, the matter like a lepton is represented by the field which is the section of the associated fiber bundle of the structure group $SU(2)$ with the spacetime as a base manifold B. Thus the algebra of the matter fields is the algebra of all possible sections.

Giving the $SU(2)$ valued function $g(x) \in SU(2)$ on the base manifold $B \ni x$, when the matter is of the fundamental representation the gauge transformation of the matter $\Psi^i(x)$ can be written as

$$[\Psi^i(x)]^g = M^i_j(g(x))\Psi^j(x) \quad , \tag{9}$$

where $(i, j = 1, 2)$. We wrote the gauge transformation matrix as $M^i_j(g(x))$ to clarify the algebraic structure. The matrix element M^i_j maps the $g(x)$ to the complex valued function on the base manifold and thus pointwise M^i_j is an element of the $Fun(SU(2))$. Therefore, the gauge transformation property of the matter field can be translated into the algebraic language such that the algebra of matter fields is the (left)comodule algebra, and there is a pointwise (left)coaction Δ_L of $Fun(SU(2))$ on the field Ψ:

$$\Delta_L(\Psi) = \sum_s T_s \otimes \Psi^s \quad , \tag{10}$$

where $T_s \in Fun(SU(2))$ are matrix elements of the representation corresponding to the matter Ψ. For the fundamental representation eq.(10) is $\Delta_L(\Psi^i) = M^i_j \otimes \Psi^j$ and with the corresponding argument we get eq.(9).

The infinitesimal transformation corresponding to the transformation (9) can be written as

$$\delta_\xi(\Psi^i(x)) = \xi^a(x)\chi_a(M^i_j)\Psi^j(x) \quad , \tag{11}$$

where $a = -, 3, +$ is the label of the adjoint representation of $SU(2)$ and ξ^a is the gauge parameter which is the real function of the spacetime and $\chi_a(M^i_j)$ is a 2×2 matrix. In the non-deformed case we can identify χ_a with the right invariant vector fields and the evaluation $\chi_a(M^i_j)$ gives the Pauli matrix for the $SU(2)$ case; thus eq.(11) is the familiar infinitesimal transformation. In general, the infinitesimal transformation δ_ξ of the matter field Ψ can be represented by the vector fields χ_a and the infinitesimal parameter ξ^a as

$$\delta_\xi\Psi = \xi^a(\Psi * \chi_a) \equiv \xi^a(\chi_a \otimes id)\Delta_L(\Psi) \quad , \tag{12}$$

where $(\cdot * \cdot)$ denotes the convolution product of a comodule element with a functional. The infinitesimal transformation of the quantum group coaction is also investigated in section 5 of ref.[Carow-Watamura 1991b].

Using the above algebraic formulation, we may consider the q-analogue of the finite and the infinitesimal gauge transformation which we will discuss elsewhere.[3]

Here we want to concentrate on the q-deformation of the BRST algebra which seems the most appropriate algebra to consider the q-deformation of the gauge theory.

The BRST transformation of the matter field is defined by replacing the gauge parameter ξ^a by the ghost field C^a [Faddeev 1967]. Thus the BRST transformation can be written as

$$\delta_B\Psi = C^a(\Psi * \chi_a) \quad . \tag{13}$$

To define the q-deformed BRST algebra we extract appropriate properties from the non-deformed BRST formalism and impose them as the condition. We also require that in general under the limit $q \to 1$ the algebra always reduces to the non-deformed one.

The BRST algebra is the algebra which contains the matter fields Ψ and the gauge fields A^I and the ghosts C^I which are the standard field contents of the BRST formalism. The suffix I corresponds to the adjoint representation in non-deformed case. However in the q-deformed case we only require that it contains the adjoint representation and allow to add a singlet component like the right invariant basis θ^i_j in the bicovariant differential calculus.

We also have to consider the spacetime derivative. In the BRST algebra we introduce the spacetime derivative d as the formal mapping:

$$(\Psi,\ A^I,\ C^I) \ \xrightarrow{d}\ (d\Psi,\ dA^I,\ dC^I) \ \xrightarrow{d}\ 0. \tag{14}$$

The fields $d\Psi$, dA^I and dC^I must be treated as independent generators from the original fields.

Definition 1 : *The BRST algebra $\mathcal{A}_B$ is a comodule algebra over $Fun_q(G)$ which is generated by the following set of fields:*

$$\mathcal{A}_B = \mathrm{C} < C^I, \Psi, A^I, dC^I, d\Psi, dA^I > /\mathcal{I} \quad , \tag{15}$$

where C^I represents the ghost, Ψ the matter and A^I the gauge fields. $\mathcal{I}$ is a set of the covariant commutation relations among these comodules, which we shall determine in the next section.

In the non-deformed BRST formalism of the gauge theory, the exterior derivative d and the BRST transformation δ_B are nilpotent operators. Thus

[3] The global transformations are considered in ref.[Brzeziński 1992].

we require the nilpotency and also the Leibniz rule for each operator. Furthermore we require the following properties: $d\delta_B + \delta_B d = 0$ and under the $*$ conjugation $\delta_B \circ * = * \circ \delta_B$, and $d \circ * = * \circ d$.

In order to define the properties of the ghost in the BRST algebra we identify them with the right invariant one-form θ as a comodule. In the q-deformed case, the result of the bicovariant bimodule calculus says that the number of independent bases of the invariant one-forms is 4 for the calculus on $Fun_q(SU(2))$. They include both the adjoint and singlet representation. Therefore, in the q-deformed BRST algebra, we introduce the four ghosts C^I where the suffix I runs $0, -, 3, +$.

Definition 2 : *In the q-deformed BRST algebra based on the bicovariant differential calculus on $Fun_q(SU(2))$, we define the ghost field as a comodule represented by a 2×2 matrix $C^i{}_j$. The left-coaction on it is*

$$\Delta_L(C^i{}_j) = M^i_{i'}\kappa(M^{j'}_j) \otimes C^{i'}{}_{j'} \quad , \tag{16}$$

and under the $*$-conjugation it transforms as a hermitian field: $(C^i{}_j)^* = C^j{}_i$.

From the properties of the ghosts in the non-deformed case we also require that they are q-anticommuting and that the BRST transformation δ_B of the ghosts has the same form as the Maurer-Cartan equation obtained by the bicovariant differential calculus.

$$\delta_B C^0 = 0 \quad , \delta_B C^a = \frac{-ig}{q^2 + q^{-2}} f^a_{bc} C^b C^c \quad , \tag{17}$$

where g is an arbitrary non-zero real constant. Note that we also decompose the ghost fields into singlet and adjoint representation as $C^I = \sigma^I_{i_1 i_2} C^{i_1 i_2}$ where $I = 0, -, 3, +$.

Then we define the q-deformed BRST transformation of the matter analogously to eq.(13) as

$$\delta_B \Psi = C^I(\Psi * \chi_I) = C^a(\Psi * \chi_a) + C^0(\Psi * \chi_0) \quad , \tag{18}$$

where $\chi_I \in \mathcal{U}_q(SU(2))$ is the one given in eq.(5). Note that the last term does not have a corresponding term in the non-deformed case, and it goes to zero in the limit $q \to 1$. The singlet component of the ghost is not desirable from the physical point of view. On the other hand, as we shall discuss it seems it is necessary to include it in order to put the algebra in a simple form.

Finally we require the existence of the covariant derivative which is represented by the derivative d and the gauge field A^I. The coupling of the gauge field to the matter field is determined naturally by the structure of the BRST transformation of the matter field given in eq.(18). Therefore, our requirement concerning the covariant derivative is:

There exists a covariant derivative ∇ which acts on the matter as

$$\nabla\Psi = d\Psi + A^I(\Psi * \chi_I) \quad , \tag{19}$$

where $(A^i_j)^* = A^j_i$. The covariant derivative transforms with the same rule as the corresponding matter

$$\delta_B \nabla\Psi = C^I(\nabla\Psi * \chi_I) \quad . \tag{20}$$

and $\nabla \circ * = * \circ \nabla$

Requiring the above conditions and the covariance, we define the comodule algebra $\mathcal{A}_B$. The main part of the construction is to define the commutation relations $\mathcal{I}$.

4. The BRST Algebra

Here we give the commutation among the elements and the BRST transformation of the gauge fields and other relations without proof. The commutation relation of each field among itself can be defined by taking the q-antisymmetric (q-symmetric) product to vanish if it is a bosonic(fermionic) field in the limit $q \to 1$.

The ghosts are q-anticommuting fields by definition. The gauge fields are also q-anticommuting since they are 1-forms in the limit of $q \to 1$. We define the q-anticommutation relation of these field using the same formula used to define the $\wedge$ product in ref.[Carow-Watamura 1991a]:

$$(\mathcal{P}_S, \mathcal{P}_S)^{IJ}_{KL} C^K C^L = 0, \text{ and } (\mathcal{P}_A, \mathcal{P}_A)^{IJ}_{KL} C^K C^L = 0 \quad , \tag{21}$$

$$(\mathcal{P}_S, \mathcal{P}_S)^{IJ}_{KL} A^K A^L = 0, \text{ and } (\mathcal{P}_A, \mathcal{P}_A)^{IJ}_{KL} A^K A^L = 0 \quad . \tag{22}$$

The pair of projectors $(\mathcal{P}, \mathcal{Q})$ with $\mathcal{P}^{ij}_{kl}$ and $\mathcal{Q}^{ij}_{kl}$ is given by $(\mathcal{P}, \mathcal{Q})^{i_1 i_2 j_1 j_2}_{k_1 k_2 l_1 l_2} = \hat{\mathbf{R}}^{-}{}^{i_2 j_1}_{j'_1 i'_2} \mathcal{P}^{i_1 j'_1}_{k_1 l'_1} \mathcal{Q}^{i'_2 j_2}_{k'_2 l_2} \hat{\mathbf{R}}^{l'_1 k'_2}_{k_2 l_1}$.

The other relations including the derivative of the fields have to be also defined. Since the operation d relates some of the relations, they are not all independent, i.e. some of them can be obtained from others by the operation d. The independent commutation relations are the ones between the following pairs: $(\{C^I\}, \{dC^I\})$, $(\{A^I\}, \{dA^I\})$, $(\{C^I\}, \{A^I\})$, $(\{C^I\}, \{\Psi\})$, $(\{A^I\}, \{\Psi\})$, $(\{\Psi\}, \{\Psi\})$, and $(\{\Psi\}, \{d\Psi\})$. When we require the consistency with other structures, we can fix all these relations. For the derivation, we refer to the paper [Watamura 1992]. The resulting relations except the $(\{\Psi\}, \{\Psi\})$ and the $(\{\Psi\}, \{d\Psi\})$ relations are given by

Proposition 1 : *Define the ordering of the fields as*

$$\{\Psi, d\Psi\} > \{dA^I\} > \{A^I\} > \{dC^I\} > \{C^I\} \quad , \tag{23}$$

then if $X > Y^I$, the commutation relation is given by

$$XY^I = \pm Y^J(X * \mathbf{L}^I_J) \quad , \tag{24}$$

where the sign is taken as $+(-)$ if they are commuting(anticommuting) in the limit $q \to 1$ and $\mathbf{L}^I_J$ is the functional defined in eq.(2). Note that we take the 1-forms and the ghosts anticommuting with each other.

The relations of dA^I and dC^I are

$$(\mathcal{P}_S, \mathcal{P}_A)^{IJ}_{KL} dA^K dA^L = 0, \text{ and } (\mathcal{P}_A, \mathcal{P}_S)^{IJ}_{KL} dA^K dA^L = 0 \quad , \tag{25}$$
$$(\mathcal{P}_S, \mathcal{P}_A)^{IJ}_{KL} dC^K dC^L = 0, \text{ and } (\mathcal{P}_A, \mathcal{P}_S)^{IJ}_{KL} dC^K dC^L = 0 \quad , \tag{26}$$

which simply mean that dA^I and dC^I are q-commuting fields.

The algebra of the matter fields can be defined like a quantum plane, since the quantum plane algebra is the algebra generated by the comodule imposing an appropriate commutation relation [Manin 1988]. The algebra depends on the representation of the matter fields in the model. In our construction, we do not need to specify the representations of the matter. The algebra of the ghost and gauge fields which is defined here is applicable for any representation of the matter. This property provides the flexibility to consider the model with various matter fields. One can find an example of the commutation relation of the matter fields in ref.[Watamura 1992].

With the above relations we can find the BRST transformation of the gauge fields by using the standard logic to define it in the field theory: The derivative of the field is not covariant under the BRST transformation. Its transformation is

$$\delta_B d\Psi = -d\delta_B\Psi = C^I(d\Psi * \chi_I) - (dC^I)(\Psi * \chi_I) \quad . \tag{27}$$

On the one hand the covariance under the BRST transformation (20) can be rewritten as

$$\delta_B \nabla\Psi = C^I(\nabla\Psi * \chi_I) = C^I(d\Psi * \chi_I) + C^I A^J(\Psi * \chi_I * \chi_J) \quad . \tag{28}$$

On the other hand taking the BRST transformation of the *r.h.s.* of eq.(19) we get

$$\begin{aligned}\delta_B \nabla\Psi &= -(dC^I)\Psi * \chi_I + C^I(d\Psi * \chi_I) + (\delta_B A^I)(\Psi * \chi_I) \\ &\qquad - A^I C^J(\Psi * \chi_I * \chi_J) \quad .\end{aligned} \tag{29}$$

The BRST transformation of the gauge field can be defined by requiring the equivalence of the eqs.(28) and (29). Thus we get

$$(\delta_B A^I)(\Psi * \chi_I) = (dC^I)(\Psi * \chi_I) + (A^I C^J + C^I A^J)(\Psi * \chi_I * \chi_J) \quad . \tag{30}$$

Using the commutation relation of A^I and C^I we see that the $(A^I C^J + C^I A^J)$ term is proportional to the projector $\mathbf{P}_{Ad}$. Then, applying eq.(8) and comparing the coefficient of the functionals χ_I we get

Proposition 2 : *The BRST transformation of the gauge field is given by*

$$\delta_B A^0 = dC^0, \quad \text{and} \quad \delta_B A^a = dC^a - ig(\omega C^0 A^a + f^a_{bc} C^b A^c) \quad , \tag{31}$$

and it is nilpotent.

We can also define the field strength using the above algebra as:

Proposition 3 : *The field strength is given by*

$$F^0 = dA^0, \quad \text{and} \quad F^a = dA^a - \frac{ig}{q^2 + q^{-2}} f^a_{bc} A^b A^c \quad . \tag{32}$$

The field strength is covariant under the BRST transformation: $\delta_B F^I = C^J(F^I * \chi_J)$, and satisfies the Bianchi identity:

$$dF^0 = 0, \quad \text{and} \quad dF^a = \frac{ig}{q^2 + q^{-2}} f^a_{bc}[A^b F^c - F^b A^c] \quad . \tag{33}$$

In order to obtain the q-deformation of the BRST formulation of the gauge field theory, we have to take the structure of the base manifold into consideration. Using the result here, one may take the base manifold as a usual spacetime but a more interesting possibility is the one when the base manifold is also described by the non-commutative function algebra. In both cases, we have to reconsider the meaning of the usual quantization so that it fit to the pure algebraic formulation.

References

Abe, E.: 1980, 'Hopf Algebras', Cambridge Tracts in Math., vol. 74, (Cambridge Univ. Press, 1980).
Aref'eva, I.Ya. and Volovich, I.V.: 1991, *Mod. Phys. Lett.* **A6**, 893
Bernard, D.: 1990, *Prog. Theor. Phys. Suppl.* **102**, 49
Brzeziński, T. and Majid, S.: 1992 'Quantum group gauge theory on quantum spaces.' Preprint DAMTP/92-27, ; 'Quantum Group Gauge Theory on Classical Spaces' Preprint DAMTP/92-51.
C. Becchi, A. Rouet and R. Stora,: 1976, *Ann. Phys.* , 287
Tyuitin, I.V.: 1975 Lebedev preprint FIAN 39 (1975), unpublished.
Carow-Watamura, U., Schlieker, M., Watamura, S. and Weich, W.: 1991a, *Commun. Math. Physics* **142** , 605
Carow-Watamura, U. and Watamura, S.: 1991b "Complex Quantum Group, Dual Algebra and Bicovariant Differential Calculus," preprint TU-382(1991) to be published in Commun. Math. Physics.
Carow-Watamura, U.: 1992, Ph.D. Thesis, 'The quantum group symmetry in the models of elementary particle physics.'
Drinfeld, V.G.: 1986, 'Quantum Groups', Proceedings of the International Congress Mathematicians, Vol.1, 798.
Faddeev, L.D. and Popov, V.: 1967, *Phys. Lett.* **25B**, 29
Hirayama, M.: 1992, 'Gauge Field Theory of the Quantum Group $SU_q(2)$', preprint TOYAMA-74(1992).
Isaev, A.P. and Popowicz, Z.: 1992, *Phys. Lett.* **281B**, 271
Jimbo, M.: 1986, *Lett. Math. Phys.* **10**, 63

Jurco, B.: 1991, *Lett. Math. Phys.* **22**, 177
Manin, Yu.I.: 1988, 'Quantum groups and non-commutative geometry', Centre de Recherches Mathematiques, Universite de Montreal.
Reshetikhin, N.Yu., Takhtajan, L.A. and Faddeev, L.D.:1989, 'Quantization of Lie groups and Lie algebras', Algebra i Analiz, 1 178.
Watamura, S.: 1992, 'Quantum Deformation of BRST Algebra', preprint HD-THEP-92-39, TU-411(1992).
Woronowicz, S.L.: 1987, *Commun. Math. Phys.* **111**, 613
Woronowicz, S.L.: 1989, *Commun. Math. Phys.* **122**, 122
Wu, K. and Zhang, R.: 1992, *Commun. Theor. Phys.* **17**, 175

CYCLIC PARAGRASSMANN REPRESENTATIONS FOR COVARIANT QUANTUM ALGEBRAS

ALEXEY ISAEV
Laboratory of Theoretical Physics, JINR, Dubna
SU-101 000 Moscow, Russia

Abstract. This report is devoted to the consideration from the algebraic point of view the paragrassmann algebras with one and many paragrassmann generators θ_i, $\theta_i^{p+1} = 0$. We construct the paragrassmann versions of the Heisenberg algebra. For the special case, this algebra is nothing but the algebra for coordinates and derivatives considered in the context of covariant differential calculus on quantum hyperplane. The parameter of deformation q in our case is (p+1)-root of unity. Our construction is nondegenerate only for even p. Taking bilinear combinations of paragrassmann derivatives and coordinates we realize generators for the covariant quantum algebras as tensor products of $(p+1) \times (p+1)$ matrices. There is now the extensive literature about finite dimensional cyclic representations for quantum algebras with q being a root of unity (see e.g. [2],[24]). It is rather interesting to relate our paragrassmann representations with representations explored in [2],[24]. At the end of our talk we discuss the paragrassmann extensions of the Virasoro algebra. This report is largely based on the papers [25-27].

Paragrassmann algebras (PGA) are interesting for several reasons. They are relevant to conformal field theories [1,2] and to unusual statistics [3], in particular, to the Green-Volkov parastatistics which was earlier discussed mainly in the context of the standard field theory [4]. There are also some hints (e.g., Ref.[5]) that PGA have a connection to quantum groups. Finally, it looks aesthetically appealing to find a generalization of the Grassmann analysis [6] that proved to be so successful in describing supersymmetry.

Recently, some applications of PGA have been discussed in literature. In Ref.[7], a parasupersymmetric generalization of quantum mechanics has been proposed. Refs.[8],[21],[22] have attempted at a more systematic consideration of the algebraic aspects of PGA. Using the Green ansatz [4] and fractional supersymmetry approach a sort of paragrassmann generalizations of the conformal algebra have been introduced in [8],[28]. Applications to the relativistic theory of the first-quantized spinning particles have been discussed in Ref.[9]. Further references can be found in [2],[5],[7],[8],[28].

We start by defining the PGA $\Gamma_p(1)$ (or simply Γ), generated by one nilpotent variable θ ($\theta^{p+1} = 0$, p is some positive integer). Any element of the algebra, $a \in \Gamma$, is a polynomial in θ of the degree p,

$$a = a_0 + a_1\theta + \ldots + a_p\theta^p , \quad (1)$$

where a_i are real or complex numbers or, more generally, elements of some commutative ring (say, a ring of complex functions) [10]. It is useful to have a matrix realization of this algebra. One may regard a_i as coordinates of the vector a in the basis $(1, \theta, \ldots, \theta^p)$. Defining the operator of multiplication by θ, $\theta(a) = a_0\theta + \ldots + a_{p-1}\theta^p$, we see that it can be represented by the triangular $(p+1)\times(p+1)$-matrix acting on the coordinates of the vector a:

$$(\theta)_{mn} = \delta_{m,n+1} , \ (\theta^k)_{mn} = \delta_{m,n+k} , \quad (2)$$

$m, n = 0, 1, \ldots, p$. We may now treat elements of the algebra as matrices.

The next step is the definition of the derivative with respect to θ. We expect a differentiation $\partial \equiv \partial/\partial\theta$ to act as

$$\partial(1) = 0, \ \partial(\theta) = 1, \ \partial(\theta^n) \propto \theta^{n-1} , \ n > 1, \quad (3)$$

It is easy to see that the condition $\partial(\theta) = 1$ together with the standard Leibniz rule, $\partial(ab) = \partial(a)\cdot b + a\cdot\partial(b)$, completely define the action of ∂ on any $a \in \Gamma$, but this immediately leads to a contradiction $0 \equiv \partial(\theta^{p+1}) =$ (via Leibniz rule) $= (p+1)\theta^p$. This is a manifestation of the general fact about nilpotent algebras known even for the Grassmann case: once the normalization conditions of the type (3) are established, the Leibniz rule is to be deformed.

To introduce a useful definition of ∂ we suggest a generalized Leibniz rule (g-Leibniz rule)

$$\partial(ab) = \partial(a)\cdot b + g(a)\cdot\partial(b) , g(\theta) = \sum_{m=0}^{p-1} \gamma_m \theta^{m+1} , \quad (4)$$

where γ_m are some numbers and g is an automorphism of the algebra Γ_p , $g(ab) = g(a)g(b)$. For the Grassmann case ($p = 1$) we have $g(a) = (-1)^{(a)}a$ where (a) is the Grassmann parity of the element a . The automorphism g and, hence, the derivative ∂ are completely fixed by the normalization conditions $\partial(\theta) = 1$ and $\partial(\theta^2) \propto \theta$. These, by (4), give $\gamma_m = 0$ for $m > 0$, $\partial(1) = 0$, $\partial(\theta^n) = (1 + \gamma_0 + \ldots + \gamma_0^{n-1})\theta^{n-1}$ and from $\partial(\theta^{p+1}) \equiv 0$ we get $1+\gamma_0+\ldots+\gamma_0^p = 0$, so that γ_0 is fixed to be a root of unity. For the moment, we choose γ_0 to be the *prime* root i.e.: $\gamma_0 = q \equiv e^{2\pi i/(p+1)} = (-1)^{2/(p+1)}$. By introducing the notation $(n)_q \equiv 1 + q + \ldots + q^{n-1} = \frac{1-q^n}{1-q}$, the action of ∂ can be performed as $\partial(\theta^n) = (n)_q\theta^{n-1}$, and so the matrix elements of ∂ in the basis $\{\theta^m\}, m = 0, \ldots, p$ are

$$(\partial)_{mn} = (m+1)_q\delta_{m+1,n} . \quad (5)$$

Since $(p+1)_q = 0$, the operator ∂ is nilpotent, $\partial^{p+1} = 0$. It is not difficult to see that ∂ and θ satisfy the q-deformed commutation relation

$$[\partial, \theta]_q \equiv \partial\theta - q\theta\partial = 1 . \quad (6)$$

The Grassmann case for $p = 1$ and the classical one in the limit $p \to \infty$ are evidently reproduced. The last equation is suggestive of a relation between PGA and much discussed q-deformed oscillators and quantum groups (see, e.g. Refs.[12] — [14],[18],[23],[29],[30]) with the deformation parameter q being a root of unity.

Consider now the algebra $\Pi_p(1)$ (or, simply Π) generated by both θ and ∂. One can show (see [25]) that the algebra Π has the basis $\{\theta^m\partial^n\}$, $m, n = 0, \ldots, p$ and is isomorphic to the algebra $Mat(p+1)$ with natural "along-diagonal" grading $deg(\theta^m\partial^n) = m - n$.

The automorphism g from Eq.(4) is expressed in the operator form as

$$g = \partial\theta - \theta\partial = 1 + (q-1)\theta\partial , \ g^{p+1} = 1 . \quad (7)$$

Its matrix elements are $(g)_{mn} = q^m\delta_{mn}$. In the mathematical literature (see, e.g. Ref.[11]), our generalized differentiation (4) is called g-differentiation. Mathematicians also consider a further generalization, called $(g, \bar{g})$-differentiation that satisfies the rule

$$\partial(ab) = \partial(a)\cdot\bar{g}(b) + g(a)\cdot\partial(b) . \quad (8)$$

Although we think that Eq.(4) looks more natural than Eq.(8), the latter can be used to define "real" differentiation, i.e., the one with real matrix elements. In fact, choosing for g and $\bar{g}$ the automorphisms defined by $g(\theta) = q^{1/2}\theta$, $\bar{g}(\theta) = q^{-1/2}\theta$, we find that

$$\partial(\theta^n) = [n]_{\sqrt{q}}\theta^{n-1} , \ [n]_{\sqrt{q}} \equiv \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} . \quad (9)$$

This is obviously a real representation of ∂. The operators g and $\bar{g}$ have the matrix elements $(g)_{mn} = q^{m/2}\delta_{mn}$, $(\bar{g})_{mn} = q^{-m/2}\delta_{mn}$, and the following operator expressions in terms of θ and ∂ $g = \partial\theta - q^{-1/2}\theta\partial$, $\bar{g} = \partial\theta - q^{1/2}\theta\partial$. One can easily recognize in these formulas the definition of the quantum oscillator in the MacFarlane-Biedenharn form (see [12],[18]).

In addition to the g-differentiation, one can also define the integration over θ $\int d\theta a(\theta) = a_p$ that generalizes the Grassmann integration to the paragrassmann one (see e.g. [15])

In some applications (e.g., in constructing parasupersymmetries) one has to deal with a_n (Eq.(1)) taken from the ring of the differentiable functions of a real or complex variable z i.e., $a_n = a_n(z)$. For such an algebra, it is possible to define a sort of "covariant derivative"

$$D = \partial_\theta + \frac{1}{(p)_q!}\theta^p\partial_z , \quad (10)$$

where $\partial_\theta \equiv \partial$ and the standard notation is used $(p)_q! = (p)_q(p-1)_q \dots (1)_q$. This derivative obviously satisfies the g-Leibniz rule (4) and may be considered as a root of ∂_z since $D^{p+1}a(z;\theta) = \partial_z a(z;\theta)$.

Our discussion of the PGA $\Gamma_p(1)$ and $\Pi_p(1)$ was completely general and did not rely on special matrix representations for θ and ∂. In fact, different representations could be classified if we relaxed our assumption for q to be the prime root of unity, $q_p = \exp(2\pi i/(p+1))$. Then, one would find that the structure of the extended algebras $\Gamma_p(N)$ and $\Pi_p(N)$ depend on the arithmetic properties of $(p+1)$ (see [25]).

We present here just the explicit inductive construction of $\Gamma_p(N)$. Starting with $N = 2$, define

$$\theta_1 = g \otimes \theta \ , \ \theta_2 = \theta \otimes 1 \ , \tag{11}$$

where θ and g have been defined in (2), (5) and (7). It is easy to see that $\theta_1\theta_2 = q\theta_2\theta_1$, $\theta_i^{p+1} = 0$. The crucial fact is that the definition (11) allows for nilpotency of any linear combination of θ_1 and θ_2

$$(a_1\theta_1 + a_2\theta_2)^{p+1} = 0 \ , \tag{12}$$

as long as q is a primitive root of unity (see for details [25]).

Suppose now that we have constructed the algebra $\Gamma_p(N)$ satisfying the relations

$$\theta_i\theta_j = q\theta_j\theta_i \ , \ i < j \ , \ i,j = 1\dots N \ , \tag{13}$$

$$(\sum_{i=1}^{N} a_i\theta_i)^{p+1} = 0 \ . \tag{14}$$

Then, $N+1$ matrices ϑ_i satisfying (13) and (14) can be constructed by analogy with (11)

$$\vartheta_i = g \otimes \theta_i \ , \ i = 1\dots N \ , \vartheta_{N+1} = \theta \otimes 1. \tag{15}$$

The proof of the identity (14) is performed in full analogy with the $N = 2$ case.

It is rather amusing that the consideration of PGA naturally leads to the objects introduced in the context of quantum groups. In fact, the generators of the algebra $\Gamma_p(N)$, determined by the relations of type (13) and (14), might be considered as coordinates of a certain nilpotent Manin's quantum hyperplane similar to those of Refs.[13], [14]. Such an object and, especially, its ∂-extensions (defined by its automorphisms) look rather interesting both from algebraic [17] and quantum-geometric [16] points of view.

Let us consider an algebra $\Gamma_p(N)$ with the commutation relations $\theta_i\theta_j = q^{\rho_{ij}}\theta_j\theta_i$, $i,j = 1\dots N$, where q denotes the prime root of unity. The requirement for $q^{\rho_{ij}}$ to be a primitive root is equivalent to the requirement for ρ_{ij} to

be invertible elements of the ring $\mathbf{Z}_{p+1}$. Then, let us define differentiations ∂_i satisfying the normalization conditions $\partial_i(\theta_k) = \delta_{ik}$, and the g-Leibniz rule

$$\partial_i(ab) = \partial_i(a)\cdot b + g_i(a)\cdot\partial_i(b) \tag{16}$$

where the action of the automorphisms g_i on θ_k is $g_i(\theta_k) = q^{\nu_{ik}}\theta_k$. These conditions determine the commutation relations in the operator form $\partial_i\theta_k = \delta_{ik} + q^{\nu_{ik}}\theta_k\partial_i$, $\partial_i\partial_j = q^{\rho_{ij}}\partial_j\partial_i$, and, for $i \neq k$, we have $\nu_{ik} = \rho_{ki} = -\rho_{ik}$, while the diagonal ν_{ii} remains nonspecified.

It is possible to construct another interesting extension of $\Gamma_p(N)$ (where p is an even number) with the generators θ_i and ∂_i if we even further relax the g-Leibniz rule (16) to the form familiar from the theory of quantum groups [16] $\partial_i(ab) = \partial_i(a)\cdot b + g_i^j(a)\cdot\partial_j(b)$. This makes it possible to construct the operators ∂_i by the inductive procedure similar to (15)

$$\tilde{\partial}_i = g \otimes \partial_i, \ \ i = 1\dots N, \ \ \tilde{\partial}_{N+1} = \partial \otimes 1 \ , \tag{17}$$

where we have also slightly modified the definition of ∂ and g

$$\partial\theta - q^2\theta\partial = 1 \ , \ \ \partial\theta - \theta\partial = g^2 \ . \tag{18}$$

¿From these equations and from definitions of θ_i and ∂_i ($i = 1,\dots,N$) we obtain the following algebra

$$\begin{aligned} &\theta_i\theta_j = q\theta_j\theta_i \quad i<j \ , \ \partial_i\partial_j = q^{-1}\partial_j\partial_i \quad i<j \ , \\ &\partial_i\theta_j = q\theta_j\partial_i \quad i \neq j \ , \ \partial_i\theta_i - q^2\theta_i\partial_i = 1 + (q^2-1)\sum_{k>i}\theta_k\partial_k \ . \end{aligned} \tag{19}$$

These are the well known formulas for $GL_q(N)$-covariant differential calculus on the quantum hyperplane [16]. These formulas may also be interpreted as the definition of the covariant q-oscillators [18],[23] or, else, as the central extension of the quantum symplectic space relations for the quantum group $Sp_q(2N)$ (see [13]). Note that nilpotency of the linear combinations $a_i\theta_i$ and $b_i\partial_i$ as well as nondegeneracy of ∂ (18) are guaranteed since both q and q^2 are primitive roots of unity (for p even integer only). Here we would like to stress that the representations (15), (17) can be extracted from the first paper of [18], where the analogous representations have been considered in the context of q-oscillators. Using the matrix representations (2), (5) for θ and ∂ one can realize $2N$ variables θ_i and ∂_i as $(p+1)^N \times (p+1)^N$ matrices.

This example demonstrates a deep relation between PGA and quantum groups.

Indeed, relations (19) are invariant under the coaction of the $GL_q(N)$ group

$$\theta_i \mapsto \theta'_i = T_{ij}\theta_j \equiv \theta_j \otimes T_{ij} \ , \qquad \partial_i \mapsto \partial'_i = \partial_j T^{-1}_{ji} \equiv \partial_j \otimes S(T_{ji}) \ , \tag{20}$$

where $\{T_{ij}\}$ are generators of $GL_q(N)$ and $S(.)$ is an antipode. Now, we would like to consider the composite operators

$$E_{ij} = \theta_i \partial_j \tag{21}$$

with the coadjoint transformation rule (see (20))

$$E_{ij} \mapsto E'_{ij} = T_{ik} E_{kl} T^{-1}_{lj} \equiv E_{kl} \otimes T_{ik} T^{-1}_{lj} \ . \tag{22}$$

For $q = 1$ these operators realize the Jordan-Schwinger construction for the generators of the usual $gl(N)$ algebra. It is natural to expect that the operators E_{ij} (for $q \neq 1$) generate the deformed algebra $U_q(gl(N))$. However, it is known [29], that q-deformed commutation relations for E_{ij} defined in (21) are not represented in a unique form. In this situation, there are two ways to write down the commutation relations for E_{ij} uniquely. First, the covariant algebra $\{\theta_i,\ \partial_j\}$ can be realized in terms of the MacFarlein-Biedenharn oscillators (see [18]). Then, these oscillators can be used (via Jordan-Schwinger construction) to construct the quantum algebra in the Drinfeld-Jimbo form. But it is a rather long way. Second, one can use the covariancy of the algebra under the coadjoint transformations (22). It is possible to prove [27] that q-deformed $GL_q(N)$-covariant commutation relations for E_{ij} are unique (up to some inessential rescaling factors).

To present these relations, let us rewrite the algebra (19) in the R-matrix form [16],[30]

$$R_{12}\theta_1\theta_2 = q\theta_2\theta_1 \ , \quad \partial_1\partial_2 R_{12} = q\partial_2\partial_1 \ , \quad \partial_1\theta_2 = \kappa\delta_{12} + q\theta_2 R_{12}\partial_1 \ , \tag{23}$$

where κ is an arbitrary parameter (in Eqs.(19) we have put $\kappa = 1$), R_{12} is the R-matrix satisfying the Hecke relation $(P_{12}R_{12})^2 - (q - q^{-1})(P_{12}R_{12}) - 1 = 0$ and P_{12} is a permutation matrix (for the notations see [13]). Using the algebra (23) one can obtain the complete set of covariant relations for E_{ij} (21) (see [27])

$$R_{12}E_1R_{21}E_2 - E_2R_{12}E_1R_{21} = \frac{\kappa}{q}(P_{12}E_1R_{21} - R_{12}E_1P_{12}) \ , \tag{24}$$

$$(R_{12} - qP_{12})(qE_1R_{21}E_2 + \kappa E_1P_{12})(R_{12} - qP_{12}) = 0 \tag{25}$$

The equations (24) give us for $q = 1$ usual $gl(N)$-commutators and it is natural to consider Eqs.(24) as structure relations for the covariant quantum deformation of the $gl(N)$-algebra. The Casimir operators c_k for this algebra are expressed via q-deformed trace: $c_k = Tr_q(E^k) = Tr(DE^k)$, $D = diag(1, q^2, q^4, \ldots)$. After shifting $Y = E - \frac{\kappa}{q^2-1}$, one can rewrite eqs.(24) in the concise form

$$R_{12}Y_1R_{21}Y_2 = Y_2R_{12}Y_1R_{21} \ . \tag{26}$$

This is nothing but the well known commutation relations for the operator $Y = (L^-)^{-1}L^+$ (here $L^\pm$ are the Borel subalgebras of $U_q(gl(N))$ [13]) interpreted also as a differential operator in the bicovariant differential calculus on the quantum group $GL_q(N)$ [31]. These also are the structure relations for the braided algebras [30],[20]. As it was shown in [27], the second set of the covariant relations (25) in the limit $\kappa = 0$ gives us the part of q-deformed anticommutation relations for 1-forms $d(T)T^{-1}$ defined for the $GL_q(N)$-group. The other part of such relations can be obtained by considering the covariant differential calculus on the fermionic quantum hyperplane [27]. The realization of $E_{ij} = \theta_i\partial_j$ in terms of the $(p+1)^N \times (p+1)^N$ matrices θ_i, ∂_i (15), (17) leads us to the matrix paragrassmann representations for the covariant quantum algebras with commutation relations (24), (26) ($q^{p+1} = 1$).

At the end of this report we would like to present a paragrassmann extension Vir_p of the Virasoro algebra. This extension has been discussed in [26]. We define this algebra denoted by Vir_p as the algebra of generators for the parasuperconformal transformations $z \mapsto \tilde{z}(z,\ \theta)$,$\theta \mapsto \tilde{\theta}(z,\ \theta)$, conserving the form of the covariant derivative (10). It means that we have $Da(\tilde{z},\ \tilde{\theta}) = D(\tilde{\theta})\tilde{D}a(\tilde{z},\ \tilde{\theta})$ for an arbitrary parasuperfunction $a(z,\ \theta)$. As it was shown in [26], the algebra Vir_p has generators T_n and G_r with the following commutation relations (we also present here the possible central extensions)

$$[T_n,\ T_m] = (n-m)T_{n+m} + \tfrac{2}{p+1}\textstyle\sum_j c_j(n^3 - n)\delta_{n+m,0},$$
$$[T_n,\ G_r] = (\tfrac{n}{p+1} - r)G_{n+r} \ ,$$
$$\{G_{r_0}, \ldots,\ G_{r_p}\}_c = (p+1)T_{\sum r} - \textstyle\sum c_j[\sum_i r_i r_{i+j} + \frac{1}{p+1}]\delta_{\sum r,0} \ ,$$

where $\{\ldots\}_c$ is the cyclic sum of the $p+1$ linear monomials $\{G_0, \ldots, G_p\}_c = G_0 \cdots G_p + G_pG_0 \cdots G_{p-1} + \ldots G_1 \cdots G_pG_0$, and the number of the central charges c_j is equal to $[\frac{p+1}{2}]$. The algebra $Vir_p(p > 1)$ has the multilinear commutation relations and, in fact, is not a Lie algebra. Note, that the special case of this algebra has been considered in [28].

As a final remark, we would like to mention a possible relation of PGA to the finite-dimensional quantum models introduced by H.Weyl in his famous book and further studied by J.Schwinger (Refs.[19]). They considered quantum variables described by unitary finite matrices U_i satisfying the relations: $U_iU_j = qU_jU_i$ and $(U_i)^{p+1} = 1$. (Obviously, q must be a root of unity). They realized that the $p = 1$ case is relevant for describing the spin variables and treated the infinite-dimensional limit $p \to \infty$ as a limit in which usual commutative geometry is restored.

References

[1] A.B. Zamolodchikov and V.A. Fateev, *Sov.Phys. JETP* **62**(1985)215.

[2] V. Pasquer and H. Saleur, *Nucl.Phys.* **B330**(1990)523.
[3] C. Aneziris, A.P. Balachandran and D. Sen, *Statistics in one dimension,* Prep. SU-4228-453, Syracuse, 1990.
[4] A.B. Govorkov, *Sov. J. Part. Nucl.* **14**(1983)520; Y. Ohnuki and S. Kamefuchi, *Quantum Field Theory and Parastatistics,* Univ. of Tokyo Press, 1982.
[5] V.P. Spiridonov, *in: Proc. of the Intern. Seminar Quarks-90,* Telavi, USSR, May 1990; eds. V.A. Matveev et al., World Sci., Singapore, 1990.
[6] F. Berezin, *Intr. into Algebra and Analysis with Anticommuting Variables,* Moscow State Univ. Press, 1983; R. Casalbuoni, *Nuovo Cim.,* **A33**(1976)389.
[7] V.A. Rubakov and V.P. Spiridonov, *Mod. Phys. Lett.* **A3**(1988)1337.
[8] S. Durand, R. Floreanini, M. Mayrand, L. Vinet, *Phys. Lett.* **233B** (1989)158.
[9] V. Gershun and V. Tkach, *Problems of Nucl. Physics and Cosmic Rays (Kharkov Univ. Press),* **23**(1985)42; G. Korchemsky, *Phys.Lett.* **267B** (1991)497.
[10] N. Jacobson, *Structure of Rings,* Am. Math. Soc. Publ., Providence, 1956.
[11] N. Jacobson, *Lie Algebras,* Interscience Publish, N.-Y.-London, 1962.
[12] A. MacFarlane, *J.Phys.* **A22**(1989)4581; L. Biedenharn, *ibid.* L873.
[13] L. Faddeev, N. Reshetikhin and M. Takhtajan, *Alg. i Anal.* **1**(1989)178.
[14] Yu. Manin, *Quantum Groups and Noncommutative Geometry,* Prep. Montreal Univ., CRM-1561 (1988); E. Corrigan, D. Fairlie, P. Fletcher, *J.Math.Phys.* **31**(1990)776.
[15] M. Rausch de Traubenberg and N. Fleury, *in: Leite Lopes Festschrift,* World Sci., Singapore, 1988.
[16] J. Wess and B. Zumino, *Nucl. Phys. (Proc. Suppl.)* **18B** (1990)302.
[17] A. Borowiec, W. Marcinek and Z. Oziewicz, *On multigraded differential calculus,* in: Proc. of 1st Max Born Sympos., Wroclaw 1991, (World Scient., 1992).
[18] M. Chaichian, P. Kulish and J. Lukierski, *Phys.Lett.* **262B**(1991)43; P. Kulish and E. Damaskinsky, *J.Phys.* **A23**(1990)L415; D. Fairlie and C. Zachos, *Phys.Lett.* **256B**(1991)43.
[19] H. Weyl, *Theory of Groups and Quantum Mechanics,* Dover. Publ., N.Y., 1931; J. Schwinger, *Proc. Natl. Acad. of Sci.,* **46**(1960)570.
[20] Sh. Madjid, *J.Math.Phys.* **32**(1991)3246.
[21] C. Ahn, D. Bernard and A. LeClair, *Nucl. Phys.* **B346** (1990)409.
[22] R. Kerner, *J. Math. Phys.,* **33** (1992)403.
[23] W. Pusz and S.L. Woronowicz, *Reports Math. Phys.* **27** (1989)231.
[24] C. deConcun and V. Kac, *Progr. in Math.* **92**(1990)471; D. Arnaudon and A. Chakrabarti, *Comm.Math.Phys.***139**(1991)461,605; E. Date, M. Jimbo, K. Miki and T. Miwa, *ibid.* **137**(1991)133; V. Bazhanov, R. Kashaev, V. Mangazeev and Yu. Stroganov *ibid.* **138**(1991)393.
[25] A. Filippov, A. Isaev and A. Kurdikov, *Mod.Phys.Lett.* **A7**(1992)2129.
[26] A. Filippov, A. Isaev and A. Kurdikov, *Paragrassmann extensions of Virasoro algebra,* Prep. JINR E2-92-392,393 (1992).
[27] A. Isaev and P. Pyatov, *$GL_q(N)$-covariant algebras and covariant differential calculus,* Prep. JINR E-2-92 (1992).
[28] S. Durand, R. Floreanini, M. Mayrand, V. Spiridonov and L. Vinet, *Parasupersymmetries and fractional supersymetries,* in: Proc. of 18th Int. Colloquium on Group Theor. Meth. in Physics, Moscow (1991); S. Durand, McGill Univ. Prep. 92-30(1992).
[29] S.P. Vokos, *J.Math.Phys.* **32**(1991)2979.
[30] P. Kulish, *Quantum groups and quantum algebras as symmetries of dynamical systems,* Prep. YITP/K-959(1991).
[31] B. Zumino, *Introduction to the differential geometry of quantum groups,* Prep. LBL-31432(1991).

HECKE SYMMETRIES AND BRAIDED LIE ALGEBRAS

DMITRI GUREVICH
Max-Planck-Institute für Mathematik, 26 Gottfried-Claren Strasse, 5300 Bonn 3

Abstract. We consider Hecke symmetries of minimal type, i.e., solutions of the QYBE with two eigenvalues and such that the Poicaré series of the corresponding exterior algebras are polynomials of degree 2. We construct the corresponding quantum cogroups and introduce notion of braided Lie algebra. The examples of Hecke symmetries of minimal type and of braided Lie algebras are given.

Key words: Quantum Yang-Baxter Equation, Hecke symmetry, bi-rank, quantum cogroup, braided Lie algebra

Generalized Lie algebras connected with involutive ($S^2 = 1$) solution of the quantum Yang-Baxter equation (QYBE) have been introduced in our paper [3]. In [5] (see also references therein for our previous papers) we have constructed some explicit examples of generalized Lie algebras (or in other words S-Lie algebras) of *gl* and *sl* types, connected with involutive non-quasiclassical (or non-deformation) solutions of the QYBE. The problem of a proper generalization of this notion to the non-involutive case was open though a lot of papers were devoted to the problem.

This paper is devoted to two questions. On the one hande we continue to study some non-quasiclassical non-involutive solutions S of the QYBE (so called Hecke symmetries). On the other hand we propose the definition of S-Lie algebras (called here braided Lie algebras to stress non-involutivity of the operator S) connected with Hecke symmetries.

The paper consists of three Sections. In Section 1 we recall some usuful facts about Hecke symmetries. We put emphasis on Hecke symmetries of minimal type, i.e. such that the Poincaré series of corresponding exterior algebras are polynomials of degree 2 with leading coefficient 1. Some of such type solutions of the QYBE have been independently constructed in [1].

In Section 2 we introduce quantum cogroups connected with Hecke symmetries of minimal type and compare these objects with Hopf algebras arising from non-degenerated bilinear forms defined in [1]. In Section 3 we introduce a notion of braided groups and give their examples connected with Hecke symmetries of minimal type.

1. HECKE SYMMETRIES: STRUCTURE, EXAMPLES

Let V be a finite-dimensional vector space over a field $\mathbf{k}$ of characteristic 0 and $S: V^{\otimes 2} \to V^{\otimes 2}$ a solution of the QYBE

$$(S \otimes \mathrm{id})(\mathrm{id} \otimes S)(S \otimes \mathrm{id}) = (\mathrm{id} \otimes S)(S \otimes \mathrm{id})(\mathrm{id} \otimes S).$$

Among all the solutions of the QYBE, the most interesting are the so called *closed* ones. Fix a base $\{e_i, 1 \le i \le n = \mathrm{dimV}\}$ in the space V and put $S(e_i \otimes e_j) = S_{ij}^{kl}\, e_k \otimes e_l$. Consider the operator T which in the base $\{e_i\}$ is defined by $S_{ij}^{kl}\, T_{kn}^{im} = \delta_j^m \delta_n^l$. We call the solution S of the QYBE *closed* if T exists.

It is not difficult to show that a closed solution of the QYBE can be extended up to a braiding operator in a *rigid* quasitensor category $\mathfrak{A}$ containing the space V. According to generally accepted terminology, a quasitensor category is called *rigid* if it satisfies the condition $U \in Ob\,\mathfrak{A} \to U^* \in Ob\,\mathfrak{A}$. The braiding operator S (or in other words, "commutativity morphism") is a mophism in the category $\mathfrak{A}$ but it is not involutive in general.

In this paper, we deal only with solutions of the QYBE wich have two eigenvalues. We call them Hecke symmeties. More precisely we call a solution S of the QYBE a *Hecke symmetry* if S satisfies the equation

$$(q\,\mathrm{id} - S)(\mathrm{id} + S) = 0.$$

We assume that $q \neq 0$ and $q^n \neq 1, n = 2; 3; \ldots$

The Hecke symmetries have a great advantage: it is possible to define for them an analogue of the symmetric and exterior algebras. Namely we put

$$\wedge_\pm(V) = T(V)/\{I_\mp\}$$

where $T(V) = \oplus V^{\otimes k}$ is the tensor algebra of V and $\{I_+\}$ (resp., $\{I_-\}$) is the ideal in $T(V)$ generated by the image I_+ (resp.,I_-) of $S + \mathrm{id}$ (resp., $q\,\mathrm{id} - S$). Denote $\wedge_\pm^k(V)$ the homogeneous component of degree k of these algebras and consider the Poincaré series $\mathcal{P}_\pm(t)$ of the algebras $\wedge_\pm(V)$:

$$\mathcal{P}_\pm(t) = \sum \mathrm{dim}\wedge_\pm^k(V) t^k.$$

We call a Hecke symmetry S (and the corresponding space V) *even* if it is closed and the Poincaré serie $\mathcal{P}_-(t)$ is a polynomial (as it was shown in [5] this condition is equivalent to following one: $\mathcal{P}_-(t)$ is a polynomial with leading coefficient 1). If this polynomial is of degree k we say that V (or S) has bi-rank $k|0$ and denote it bi-rk V. [1]

[1] Note that bi-rank is well-defined for odd objects of Hecke type (it is left to the reader to give a definition of odd spaces). For them we say that bi-rank is equal to $0|l$ and for some objects V composed in some sens from even and odd spaces it is natural to put bi-rank $V = k|l$. We dont want to examine this problem in more detail but stress only that it is not clear yet, whether all *involutive* closed solutions of the QYBE have a bi-rank.

Now we introduce two important operators $B = B(S): V \to V$ and $C = C(S): V \to V$ as follows

$$B(e_i) = B_i^j e_j = T_{ik}^{jk} e_j, \;\; C(e_i) = C_i^j e_j = T_{ki}^{kj} e_j,$$

where $\{e_i\}$ is the fixed base in V.

It is easy to see that this definition does not depend on the choice of the base. These operators can be defined for any object in any rigid quasitensor category but we need them only for an inicial space V equipped with a Hecke symmetry S.

The following statements are proved in, or can be easily deduced from, [5].

PROPOSITION 1. *1. For any Hecke symmetry S the relation*

$$\mathcal{P}_+(t)\,\mathcal{P}_-(-t) = 1$$

holds.

2. If S is even then the polynomial $\mathcal{P}_-(t)$ is reciprocal.

3. Moreover if bi-rk $V = k|0$ then the operators B and C satisfy the relation

$$\mathrm{tr}\, B = \mathrm{tr}\, C = q^{-k} k_q$$

(we denote here and below $k_q = 1 + q + \ldots + q^{k-1}$).

4. If bi-rk $V = 2|0$ then $BC = CB = q^{-3}\mathrm{id}$ and the operators $b = Bq^2$ and $c = Cq^2$ satisfy the following condition

J) if Jordan form of b or c contains a cell with eigenvalue x it contains another cell with eigenvalue qx^{-1} (with the same multiplicity).

5. If an operator $c: V \to V$ satisfies the conditions J) and $\mathrm{tr}\, c = 1 + q$ then there exists an even closed Hecke symmetry $S: V^{\otimes 2} \to V^{\otimes 2}$ of bi-rank $2|0$ such that $C = C(S) = q^{-2}c$. There exists the one-to-one correspondance between the family of all such Hecke symmetries and matrices v satisfying the condition

$$(c^*)^{-1} q = v^{-1} c v, \;\; v = (v^{ij})$$

(c^* denotes the matrix conjugated to c). If such v is fixed then the corresponding Hecke symmetry is of the form

$$S_{ij}^{kl} = q\delta_i^k \delta_j^l - (1+q) u_{ij} v^{kl},$$

where $u = (u_{ij})$ can be deduced from the equality

$$c = (1+q) v u^* \;\; \text{i.e.} \;\; c_i^j = (1+q)\, v^{jk} u_{ik}.$$

Remark that the quantity $\operatorname{tr} B = \operatorname{tr} C$, which can be defined for any element of a rigid category, is usually called its rank (see for example [6]). So the statement 3 of Proposition 1 establishes the relation for even Hecke symmetries between rank in this sens and bi-rank in our sense. Here and further on, we say that a Hecke symmetry is of *minimal type* if it is even and has bi-rank $2|0$.

Stress also that bi-rank does not change under deformation and therefore, a quasiclassical Hecke symmetry (i.e.,a deformation of the usual tranposition) must have bi-rank $n|0, n = \dim V$.

Let us give two examples of minimal Hecke symmetries.

EXAMPLE 1. *Let* $\dim V = 2$ *and* $q \neq 1$. *Then any pair* (c, v) *satysfying the conditions above has in some base form*

$$c = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \quad v = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}.$$

Then

$$S = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & qm^{-1} & 0 \\ 0 & m & q-1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \quad \text{where } m = -a/b.$$

Stress that the operator $N = uv$ *is scalar iff* $m^2 = q$ *(the role of this operator will be explained in Proposition 2).*

EXAMPLE 2. *Let* $\dim V = 3$. *We put* $c = \operatorname{diag}(x, t, q/x)$ *where* t *is one of roots* $\pm\sqrt{q}$ *and* x *satisfies the equation* $x + t + q/x = 2$. *Then assuming* v *to be as follows we obtain* S

$$v = \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q-x & 0 & -bx/a & 0 & -cx/a & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -at/b & 0 & q-t & 0 & -tc/b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q & 0 & 0 & 0 \\ 0 & 0 & -qa/cx & 0 & -qb/cx & 0 & q-q/x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \end{pmatrix}.$$

For this example the operator $N = uv$ *is scalar if* $a/c = x/t$.

Stress that the last example can be easily generalized to arbitrary dimension $n = \dim V$.

2. HECKE SYMMETRIES ARISING FROM BILINEAR FORM AND QUANTUM COGROUPS

In [1] a method have been introduced to construct a solution of the QYBE by means of a non-degenerated bilinear form. In this Section we show that the family of such solutions coincides with subset of Hecke symmetries of minimal type. We introduce also the quantum groups connected with Hecke symmetries of minimal type and compare them with Hopf algebras defined in [1].

Consider a linear space $L = V \otimes V^*$ with base $\{e_i^j = e_i \otimes e^j\}$ equipped with the operator

$$S_Q : L^{\otimes 2} \to L^{\otimes 2}, \; S_Q(e_i^j \otimes e_k^l) = S_{ik}^{ab}\, (S^{-1})_{mn}^{jl}\, e_a^m \otimes e_b^n.$$

Stress here that V^* differs from the dual space (right or left one) in the rigid category mentioned above and moreover the space L does not belong to this category (wich will appear below as the category of comoduls of a quantum cogroup).

It is obvious that this operator S_Q satisfies the QYBE and has eigenvalue 1.

Consider the algebra $A(S) = T(L)/\{I\}$ where $\{I\}$ is the ideal generated by the image I of the operator $\mathrm{id} - S_Q$. Suppose now that the initial operator S is Hecke symmetry of minimal type and introduce the so called *quantum determinant* $det = u_{jl} v^{ik} e_i^j \otimes e_k^l$ (in [5] it was defined for any even Hecke symmetry).

One can see that

$$S_Q(det \otimes e_i^j) = M_{ik}^{jl} e_l^k \otimes det$$

for some operator $M : L \to L$. Introduce the formal inverse element det^{-1} and put

$$S_Q(det^{-1} \otimes e_i^j) = (M^{-1})_{ik}^{jl} e_l^k \otimes det^{-1}$$

(so the element $det\, det^{-1}$ is central) and define the algebra $\mathbf{k}[GL(S)]$ as the quotient of $A(S)$ with the additional generator det^{-1} by the ideal generated by elements

$$det^{-1} \otimes e_i^j - S_Q(det^{-1} \otimes e_i^j).$$

It is natural to do this because

$$S_Q^2(det \otimes e_i^j) = det \otimes e_i^j$$

(see [5]).

If det is a central element of $A(S)$, we introduce also the following algebra $\mathbf{k}[SL(S)] = A(S)/\{I_{det}\}$ where $\{I_{det}\}$ is the ideal in $A(S)$ generated by *det* − 1. The algebras $\mathbf{k}[GL(S)]$ and $\mathbf{k}[SL(S)]$, being equipped with the usual comultiplication ($\Delta e_i^j = e_i^k \otimes e_k^j$) the usual counit ($\gamma e_i^j = \delta_i^j$) and some antipod, are Hopf algebras. We call them *quantum cogroups* because, like in deformation case, it is more natural to use the terme *quantum groups* for dual objects (although we do not have their description similar quasiclassical quatum groups $U_q(\mathfrak{g})$).

These quantum cogroups have been introduced in [4] and [5].

PROPOSITION 2. *(see [4],[5]) If S is Hecke symmetries of minimal type then the element det $\in A(S)$ is central iff the operator $N = uv(N_i^j = u_{ik}v^{kj})$ is scalar.*

Represent now the construction of [1] in a form convenient for our aims.

PROPOSITION 3. *Let $B = (B_{ij})$ be a non-degenerated bilinear form. Then the operator S_{DL}*

$$(S_{DL})_{ij}^{kl} = \delta_i^k \delta_j^l + aB_{ij}(B^{-1})^{kl}$$

where $B_{ik}(B^{-1})^{kl} = \delta_i^l$ is a solution of the QYBE iff $a + a^{-1} + B_{ij}(B^{-1})^{ij} = 0$.

To establish the relation between the construction from [1] and ours, consider the operator

$$S = qS_{DL} = q\,\mathrm{id} + qaB \otimes B^{-1} \qquad (S_{ij}^{kl} = q\delta_i^k\delta_j^l + qaB_{ij}(B^{-1})^{kl})$$

and put $u_{ij} = B_{ij}, v^{kl} = -qa(1+q)^{-1}(B^{-1})^{kl}$. It is easy to see that the operator S satisfies the conditions of Proposition 1 iff for B, a and q the relation above and relation $qa^2 = 1$ hold.

Hence S is a Hecke symmetry of minimal type with eigenvalues -1 and a^{-2} and the operator S_{DL} has eigenvalues $-a^2$ and 1. The operator $N = uv$ is scalar in case under consideration. Therefore the map

$$\{S_{DL}\} \to \{\text{Hecke symmetries of minimal type with central } det\}$$

is constructed. It is invertible because assuming *det* to be central we have $v = bu^{-1}$ with some $b \in \mathbf{k}$.

PROPOSITION 4. *In the algebra $\mathbf{k}[SL(S)]$ the relations*

$$u_{kl}e_i^k \otimes e_j^l = u_{ij}, \quad v^{ij}e_i^k \otimes e_j^l = v^{kl}$$

hold.

The first relation arrises from the follow chain of equalities

$$u_{kl}e_i^k \otimes e_j^l = (1+q)^{-1}(q\mathrm{id} + S_Q)u_{kl}e_i^k \otimes e_j^l =$$

$$(1+q)^{-1}(qu_{kl}e_i^k \otimes e_j^l + u_{kl}S_{ij}^{ab}e_a^m \otimes e_b^n(S^{-1})_{mn}^{kl}) =$$

$$(1+q)^{-1}(qu_{kl}e_i^k \otimes e_j^l - u_{mn}S_{ij}^{ab}e_a^m \otimes e_b^n) = u_{ij}det = u_{ij}.$$

The second relation can be proved in the similar way. Here we use the following lemma.

LEMMA 1. *The relations*

$$S(u_{kl}e^k \otimes e^l) = u_{kl}S_{ab}^{kl}e^a \otimes e^b = -u_{kl}e^k \otimes e^l,$$

$$S(v^{ij}e_i \otimes e_j) = v^{ij}S_{ij}^{ab}e_a \otimes e_b = -v^{ij}e_i \otimes e_j$$

hold

Vice versa any of the relations from Proposition 4 yields the equality $det = 1$.

In [1] some Hopf algebras have been introduced as quotients of $T(L)$ by the relations from Proposition 4. Due to this Proposition we can conclude that these algebras coincid with quantum cogroups $\mathbf{k}[SL(S)]$ defined above.

3. BRAIDED LIE ALGEBRAS

Let us recall first the definition of S-Lie algebras in the case when the operator S is an involutive solution of the QYBE. We say that the space V is equipped with a structure of S-Lie algebra if there exists an operator (S-Lie bracket) $[,]: V^{\otimes 2} \to V$ satisfying the axioms

1. $[,]S = -[,]$;
2. $[,][,]^{12}(\mathrm{id} + S^{12}S^{23} + S^{23}S^{12}) = 0$;
3. $S[,]^{12} = [,]^{23}S^{12}S^{23}$ with usual notation $S^{12} = S \otimes \mathrm{id}$ and so on.

To introduce a braded counterpart of this notion we consider first a notion of quadratic algebras. Let the space V be fixed. Consider a subspace $I \subset V^{\otimes 2}$ and so called *quadratic algebra* corresponding to $I : \wedge_+(V) = T(V)/\{I\}$ where $\{I\}$ is the ideal in $T(V)$ generated by I.

Recall now that a quadratic algebra $\wedge_+(V)$ is called Koszul algebra if the complex

$$\ldots \xrightarrow{d} \wedge_+^k(V) \otimes \wedge_-^l(V) \xrightarrow{d} \wedge_+^{k+1}(V) \otimes \wedge_-^{l-1}(V) \xrightarrow{d} \ldots$$

is exact. [2] Here $\wedge_+^k(V)$ is as usually the k−homogenous component of $\wedge_+(V)$; $\wedge_-^l(V)$ are defined as follows $\wedge_-^1(V) = V, \wedge_-^2(V) = I, \wedge_-^3(V) = I \otimes V \cap V \otimes I$ and so on, and d is a natural differential (see [7] for details).

Let a map $[,] : I \to V$ be given. Define *a quadratic-linear* algebra (an analogue of envelopping algebra) in the natural way $U(\mathfrak{g}) = T(V)/\{J\}$ where

[2] In some papers another complex connected with quadratic algebra is considered and the algebra is called Koszul algebra if the last complex is exact (see [5] where the both complexes are considered).

$\{J\} \subset T(V)$ is ideal generated by elements $I - [,]I$. Since in this algebra there exists a natural filtration, it is possible to consider the graded algebra $\mathrm{Gr}U(\mathfrak{g})$.

PROPOSITION 5. *Let us assume that the algebra $\wedge_+(V)$ is Koszul algebra and that the following conditions*

$$([,]^{12} - [,]^{23})(I \otimes V \cap V \otimes I) \subset I$$

and

$$[,]([,]^{12} - [,]^{23})(I \otimes V \cap V \otimes I) = 0$$

hold. Then $GrU(\mathfrak{g})$ is isomorphic to $\wedge_+(V)$.

This Proposition is proved in [7] where the first condition is called *correctness* and the second one is called *Jacoby identity.*

Suppose now that we have an algebra $A = \mathbf{k}[GL(S)]$ or $A = \mathbf{k}[SL(S)]$ as above. Consider the category $\mathfrak{A}$ of left comodules of A, i.e., for any $V \in \mathfrak{A}$ there exists a coaction $\Delta : V \to A \otimes V$ with usual properties.

Let $V \in \mathfrak{A}$. Suppose that there exists a map $[,] : V^{\otimes 2} \to V$.

DEFINITION 1. *The agregate $(V, I \oplus I^* = V^{\otimes 2}, [,])$ will be called a braided Lie algebra if the following axioms hold*

0. the algebra $\wedge_+(V) = T(V)/\{I\}$ is Koszul algebra;
1. $[,]I^* = 0$;
2. the relations from Proposition 5 are satisfied;
3. I, $I^* \in Ob\,\mathfrak{A}$ and the map $[,]$ is a morphism in $\mathfrak{A}$.

Let us explain that the last condition means that

$$\Delta[,] = (\mu \otimes [,])(\Delta \otimes \Delta)$$

where $\Delta \otimes \Delta : V^{\otimes 2} \to A^{\otimes 2} \otimes V^{\otimes 2}$ and $\mu : A^{\otimes 2} \to A$ is the multiplication in the algebra A.

Stress that a S-Lie algebra for involutive S is a particular case of a braided Lie algebra. If we put $I = I_-$ and $I^* = I_+$ where $I_\pm \in V^{\otimes 2}$ is as in Section 1 (assuming $q = 1$), all axioms of braided Lie algebras are satisfied for any S-Lie algebra. The verification of this fact is left to the reader. We note only that "koszulity" of the algebras $\wedge_+(V)$ have been proved (in more general context) in [5].

Note also that it is natural to introduce the axiom 0 if we want to obtain a "good" envelopping algebra (see Proposition 5). In the forthcoming publications we hope to elucidate the important role of this axiom in the quantization procedure.

Consider now an example of a braided Lie algebra constructed by means of a Hecke symmetry of minimal type.

Let $S : V^{\otimes 2} \to V^{\otimes 2}$ be a Hecke symmetry of minimal type such that *det* is central and put $A = \mathbf{k}[SL(S)]$. Fix the base $\{e_i,\ 1 \le i \le n = \dim V\}$. Consider one-dimensional A-comodule $V_0 = \mathbf{k}\, e_0$ $(\Delta e_0 = 1 \otimes e_0)$ and denote $V' = V \oplus V_0$. We put $I = I_- \oplus I_0$ (resp., $I^* = I_+ \oplus I_0^*$) where $I_\pm \subset V^{\otimes 2}$ are the same spaces as in Section 1 and $I_0, I_0^* \subset V_0^{\otimes 2} \oplus V_0 \otimes V \oplus V \otimes V_0$ are generated by elements $\{e_0 \otimes e_i - e_i \otimes e_0\}$ (resp., $\{e_0 \otimes e_0,\ e_0 \otimes e_i + e_i \otimes e_0\}$).

In [5] we have proved that $\wedge_+(V)$ is Koszul algebra. Using this result it is not difficult to show that the algebra $\wedge_+(V') = T(V')/\{I\}$ is Koszul algebra as well. We introduce in V' an A-modul structure putting $\Delta e_i = e_i^p \otimes e_p$, $\Delta e_0 = 1 \otimes e_0$ and extend this structure on $T(V')$ in a natural way. It is obvious that I, I^* are A-comodules and $I \oplus I^* = V'^{\otimes 2}$. Introduce a bracket:

$$[e_i, e_j] = g u_{ij} e_0,\ 1 \le i, j \le n, [e_i, e_0] = -[e_0, e_i] = c_i e_i$$

where $u = (u_{ij})$ is as in Proposition 1 and $g, c_i \in \mathbf{k}$.

Verify now that this bracket is a morphism in the category $\mathfrak{A}$ of A-comodules. First we will check compatibility of the bracket $[e_i, e_j]$ with coaction of A. Indeed by virtue of Proposition 4

$$(\mu \otimes [,])(\Delta e_i, \Delta e_j) = (\mu \otimes [,])(e_i^p \otimes e_p, e_j^q \otimes e_q) = \mu(e_i^p \otimes e_j^q) \otimes [e_p, e_q] =$$

$$g\mu(e_i^p \otimes e_j^q) u_{pq} \otimes e_0 = g u_{ij} v^{mn} e_m^p \otimes e_n^q u_{pq} \otimes e_0 = 1 \otimes g u_{ij} e_0 = \Delta[e_i, e_j].$$

It is obvious that the bracket $[e_i, e_0]$ is compatible with coaction of A iff $c_i = c$ for any i. The verification of the axiom 1 is left to the reader. Verify now the axiom 2. Since S is a Hecke symmetry of minimal type one has $I_- \otimes V \cap V \otimes I_- = \{0\}$. Hence the space $I \otimes V' \cap V' \otimes I$ is generated be the elements

$$\{v^{ij}(e_i \otimes e_j \otimes e_0 - e_i \otimes e_0 \otimes e_j + e_0 \otimes e_i \otimes e_j)\}.$$

Applying the operator $[,]^{12} - [,]^{23}$ to an element from this family we have

$$([,]^{12} - [,]^{23}) v^{ij}(e_i \otimes e_j \otimes e_0 - e_i \otimes e_0 \otimes e_j + e_0 \otimes e_i \otimes e_j) =$$

$$v^{ij}(g u_{ij} e_0 \otimes e_0 - c e_i \otimes e_j - c e_i \otimes e_j - c e_i \otimes e_j - c e_i \otimes e_j - g u_{ij} e_0 \otimes e_0) =$$

$$-4 v^{ij} c e_i \otimes e_j \in I.$$

Axiom 2 is satisfied if $cg = 0$. Therefore under this condition all axioms of braided Lie algebra are satisfied.

Consider the particular case $n = 2$. In terms of the "envelopping algebra" the relations between the generators e_0, e_1, e_2 are of the form

$$a e_1 \otimes e_2 + b e_2 \otimes e_1 = g(1 + q) e_0,\ e_1 \otimes e_0 - e_0 \otimes e_1 = 2 c e_1,$$

$$e_2 \otimes e_0 - e_0 \otimes e_2 = 2 c e_2$$

where we assume that a, b from Example 1 satisfy the condition $m^2 = (a/b)^2 = q$ and eigher $c = 0$ or $g = 0$. As result we obtaine a braided deformation of usual Lie algebras, namely of Heisenberg algebra when $c = 0$ and of the algebra $[e_1, e_2] = 0$, $[e_1, e_0] = 2ce_1$, $[e_2, e_0] = 2ce_2$ when $g = 0$ (in fact only the first relation is deformed).

Stress that these relations differ from ones arising from representation of quantum group $U_q(sl_2)$ of spine 1 (see [2]). The last example will be considered elsewhere from the point of view of our definition of braided Lie algebras.

Acknowledgements. The athor wishes to thank Max-Planck-Institut für Mathematik for hospitality during the preparation of the paper. I am grateful to Organizing Committee of Second Max Born Symposium for invitation to give a talk. I am also very grateful to Professor J.Donin for stimulating discussions.

References

[1] M.Dubois-Violette and G.Launer, *The quantum group of a non-degenerated bilinear form*, Phisics Letters B, **245** (1990), no.2, pp.175-177

[2] I.Egusquiza, *Quantum mechanics on the quantum sphere*, Preprint, Cambridge, 1991

[3] D.Gurevich, *Generalized translation operators on Lie groups*, Soviet J. Contempory Math. Anal.,**18** (1983)

[4] D.Gurevich, *Hecke symmetries and quantum determinants*, Soviet Math. Dokl., **38** (1989), no.3, pp.555-559

[5] D.Gurevich, *Algebraic aspects of the quantum Yang-Baxter equation* , Leningrad Math.J., **2** (1991), no.4, pp.801-828

[6] S.Majid and J.Sobeilman, *Rank of quantized universal algebras and modular functions*, Comm. Math. Phys. **137** (1991), no.2, pp.249-262

[7] A.Polistchuk and L.Posicelsky *On quadratic algebras*, Preprint, Moscow, 1991

ANYONIC QUANTUM GROUPS

SHAHN MAJID*

Department of Applied Mathematics & Theoretical Physics
University of Cambridge, Cambridge CB3 9EW, U.K.

Abstract. We introduce non-standard quantum group structures on the finite groups $\mathbb{Z}_n$. These determine non-trivial braidings Ψ in the category of $\mathbb{Z}_n$-graded vector spaces. The braiding is an anyonic one, $\Psi(v \otimes w) = e^{\frac{2\pi i |v||w|}{n}} w \otimes v$ for homogeneous elements of degree $|v|$, $|w|$. This category of anyonic vector spaces generalizes that of super vector spaces, which are recovered as $n = 2$. We give examples of anyonic quantum groups. These are like super quantum groups with ± 1 statistics generalised to anyonic ones. They include examples obtained by transmutation of $u_q(sl_2)$ at a root of unity.

Key words: supersymmetry – anyonic symmetry – quantum groups – braided category

1. Introduction

One of the fascinating aspects of Clifford algebras is their close connection with fermionic statistics and supersymmetry. One might ask for corresponding algebraic structures that play a similar role when super statistics are replaced by more general anyonic statistics. In order to do this, one has to first understand the algebraic structure underlying anyonic symmetry itself, which is what we do here using the theory of quantum groups and braided categories. We then give several examples of 'anyonic' algebraic structures such as anyonic groups and anyonic quantum groups. The construction of anyonic matrices, anyonic harmonic oscillators etc. remain for further work.

Our study of anyonic symmetry is an application of the general theory of algebraic structures living in braided categories introduced in [6][7][8], recalled briefly in the Preliminaries. The role of ± 1 statistics in the super case is now played by the braiding or quasisymmetry Ψ. For example, our notion of *braided groups* and *quantum braided groups* are modelled on super groups and super quantum groups respectively. An interesting feature now, however, is that since the role of transposition is played by a braiding, many algebraic computations inevitably reduce to braid and tangle diagrams [7][8].

We begin with our formalization of *anyonic vector spaces*, which means for us nothing other than a vector space in which the finite group of order n, $\mathbb{Z}_n$, acts. The braiding Ψ in the category of anyonic vector spaces is the one familiar to physicists in the context of anyonic statistics, such as [17]. Hence

* SERC Fellow and Drapers Fellow of Pembroke College, Cambridge. This paper is in final form and no version of it will be submitted for publication elsewhere.

the name. The main result of Section 2 is to identify this as the category of representations of a certain quantum group $\mathbb{C}\mathbb{Z}'_n$, which we introduce. We also give formulae for the anyonic dimension and the anyonic trace.

We then proceed in Section 3 to construct our examples of groups and quantum groups living in such anyonic categories, i.e. anyonic groups and anyonic quantum groups. Our first example is an anyonic version of the symmetry group of an equilateral triangle. Our second is an anyonic version of the quantum group $u_q(sl_2)$ at a root of unity and leads to a simple formula for its universal R-matrix.

In Section 4 we give the general construction that was used to obtain the anyonic ones. We note that other generalizations of $\mathbb{Z}_n$-graded spaces have been considered in the literature, for example to G-graded spaces (where G may be non-Abelian), such as [1]. By contrast, our generalization by means of self-dual Hopf algebras appears to be in a new direction.

I thank A.J. Macfarlane and S. Shnider for comments. This is the final version of a May 1991 preprint of the same title and much the same content.

Preliminaries

A general introduction to quasitensor or braided categories[4] in the context of the representations of quantum groups is in [5, Sec. 7]. Briefly, a quasitensor category is $(\mathcal{C},\otimes,\underline{1},\Phi,\Psi)$ where $\mathcal{C}$ is a category (a collection of objects $X,Y,\cdots$, and morphisms or 'maps' between them) and $\otimes$ is a tensor product with unit object $\underline{1}$. $\Phi_{X,Y,Z}: X\otimes(Y\otimes Z)\to(X\otimes Y)\otimes Z$ are associativity isomorphisms for any three objects and $\Psi_{X,Y}: X\otimes Y\to Y\otimes X$, the *braiding* or 'quasisymmetry' between any two. Their appearance in physics in the statistics of quantum fields in low dimensions was recognized in [3]. The connection with quantum groups leads to link and 3-manifold invariants[14]. We suppress Φ, as well as isomorphisms associated with the unit. Then Ψ obeys

$$\Psi_{X,Y\otimes Z}=\Psi_{X,Z}\circ\Psi_{X,Y},\ \Psi_{X\otimes Y,Z}=\Psi_{X,Z}\circ\Psi_{Y,Z},\ \Psi_{X,\underline{1}}=\Psi_{\underline{1},X}=\mathrm{id}. \quad (1)$$

We work over a commutative field k. Our examples are over $\mathbb{C}$. A quantum group over k in the usual sense means for us a quasitriangular Hopf algebra $(H,\Delta,\epsilon,S,\mathcal{R})$ where H is an algebra over k, $\Delta: H\to H\otimes H$ the coproduct homomorphism, $\epsilon: H\to k$ the counit, $S: H\to H$ the antipode and $\mathcal{R}$ the quasitriangular structure or 'universal R-matrix' obeying [2]

$$(\mathrm{id}\otimes\Delta)(\mathcal{R})=\mathcal{R}_{13}\mathcal{R}_{12},\ (\Delta^{\mathrm{op}}\otimes\mathrm{id})(\mathcal{R})=\mathcal{R}_{23}\mathcal{R}_{13},\ \Delta^{\mathrm{op}}=\mathcal{R}(\Delta(\ ))\mathcal{R}^{-1} \quad (2)$$

where $\mathcal{R}_{12}=\mathcal{R}\otimes 1$ etc, and Δ^{op} is the opposite coproduct. We have written the middle axiom in a slightly unconventional form but one that generalizes immediately to quantum groups in quasitensor categories. For an introduction to quantum groups see [5].

The axioms of a quasitriangular Hopf algebra $(\underline{H},\underline{\Delta},\underline{\Delta}^{\mathrm{op}},\underline{\epsilon},\underline{S},\underline{\mathcal{R}})$ in a quasitensor category $\mathcal{C}$ (a quantum braided group) are just the same except that $\underline{\Delta}$ and $\underline{\mathcal{R}}$ are defined with respect to the braided tensor product algebra structure[6]. In the concrete cases below, this is

$$(a\otimes b)(c\otimes d)=a\Psi(b\otimes c)d,\qquad a,b,c,d\in\underline{H}. \quad (3)$$

By this we mean to first apply $\Psi_{\underline{H},\underline{H}}$ to $b\otimes c$ and then multiply the result on the left by a and on the right by d. The definition of $\underline{\Delta}^{\mathrm{op}}$ in the braided case when $\Psi^2\neq\mathrm{id}$ is rather more subtle[7] and not given simply by $\Psi\circ\underline{\Delta}$ or $\Psi^{-1}\circ\underline{\Delta}$. If $\underline{\Delta}^{\mathrm{op}}=\underline{\Delta}$ then we say that we have a braided group. Algebraic structures in symmetric (not braided) monoidal categories have been studied by many authors, such as [15]. The novel aspect of our work is to go further to the truly braided case.

2. Anyonic Vector Spaces

In this section we study quasitensor or 'braided' categories associated to $\mathbb{Z}_n$, the finite group of order n. Let g be the generator of $\mathbb{Z}_n$ with $g^n=1$. As a category of objects and morphisms we take the category $\mathrm{Rep}(\mathbb{Z}_n)$ of finite-dimensional representations of $\mathbb{Z}_n$. Given an object V of $\mathrm{Rep}(\mathbb{Z}_n)$ we can decompose it under the action of $\mathbb{Z}_n$ as

$$V=\oplus_{a=0}^{n-1}V_a,\qquad a=0,1,\cdots,n-1;\quad g\triangleright v=e^{\frac{2\pi\imath a}{n}}v,\qquad \forall v\in V_a.$$

Here a runs over the set of irreducible representations ρ_a of $\mathbb{Z}_n$ and V_a is the subspace of V where g acts as copies of ρ_a. The action is simply denoted $\triangleright$. If $v\in V_a$, we say that v is homogeneous of degree $|v|=a$.

On this category $\mathrm{Rep}(\mathbb{Z}_n)$ we can now define the non-standard braiding

$$\Psi_{V,W}(v\otimes w)=e^{\frac{2\pi\imath|v||w|}{n}}w\otimes v \quad (4)$$

on homogeneous elements of degree $|v|,|w|$. This is well known to physicists in the context of anyons[17]. The phase factors in (4) can be called fractional or anyonic statistics. We denote by $\mathcal{C}_n$ the category $\mathrm{Rep}(\mathbb{Z}_n)$ equipped with this anyonic braiding. The associativity Φ is the usual vector space one.

To my knowledge the structure of this quasitensor category $\mathcal{C}_n$ has not been systematically studied before. Our main result of this section is to identify it as the category of representations of a quantum group. Although it is well known that quantum groups (in the strict sense, with quasitriangular structures) have quasitensor or braided categories of representations, given such a category it may not come from a quantum group. Our result is,

PROPOSITION 2.1. *Let $\mathbb{C}\mathbb{Z}_n$ denote the group algebra of $\mathbb{Z}_n$. This is just the algebra over $\mathbb{C}$ generated by $1,g$ and the relation $g^n=1$. It is a Hopf* *algebra with* $\Delta g = g\otimes g$, $\epsilon g = 1$, $Sg = g^{-1}$. *Then* $\mathbb{C}\mathbb{Z}_n$ *has a non-trivial quasitriangular structure*

$$\mathcal{R} = \frac{1}{n}\sum_{a,b=0}^{n-1} e^{-\frac{2\pi\imath ab}{n}} g^a\otimes g^b. \tag{5}$$

We denote the Hopf algebra $\mathbb{C}\mathbb{Z}_n$ *equipped with this non-standard quasitriangular structure by* $\mathbb{C}\mathbb{Z}'_n$. *Moreover,* $\mathcal{C}_n = \mathrm{Rep}\,(\mathbb{C}\mathbb{Z}'_n)$. *If* $n\ge 3$ *then* $\mathbb{C}\mathbb{Z}'_n$ *is strictly quasitriangular and* $\mathcal{C}_n$ *is strictly braided.*

Proof It is easy to verify that the $\mathcal{R}$ shown obeys all the axioms (2) for a quantum group. The prime in $\mathbb{C}\mathbb{Z}'_n$ is to distinguish it from the usual group algebra $\mathbb{C}\mathbb{Z}_n$ with $\mathcal{R} = 1\otimes 1$. We compute the braiding in the category of representations of $\mathbb{C}\mathbb{Z}'_n$. Recall, e.g.[5, Sec. 7], that for any quantum group H, the category $\mathrm{Rep}\,(H)$ of representations becomes a quasitensor category as follows: The tensor product of representations V, W is $h\triangleright(v\otimes w) = \sum h_{(1)}\triangleright v\otimes h_{(2)}\triangleright w$ for $v\otimes w\in V\otimes W$ and the braiding is

$$\Psi_{V,W}(v\otimes w) = \sum \mathcal{R}^{(2)}\triangleright w\otimes\mathcal{R}^{(1)}\triangleright v, \qquad \mathcal{R}\equiv\sum\mathcal{R}^{(1)}\otimes\mathcal{R}^{(2)}$$

where $\triangleright$ is the relevant action. For us, the $\otimes$ is as for $\mathbb{Z}_n$-representations, and $\Psi_{V,W}(v\otimes w) = \frac{1}{n}\sum_{a,b} e^{\frac{2\pi\imath b|w|}{n}} w\otimes e^{\frac{2\pi\imath a|v|}{n}} v e^{-\frac{2\pi\imath ab}{n}} = \sum_b e^{\frac{2\pi\imath b|w|}{n}} w\otimes v\delta_{|v|,b}$ $= e^{\frac{2\pi\imath |v||w|}{n}} w\otimes v$ on homogeneous elements. We use here and below the orthogonality of $\mathbb{Z}_n$ representations in the form $\frac{1}{n}\sum_{a=0}^{n-1} e^{\frac{2\pi\imath a(b-c)}{n}} = \delta_{b,c}$ □

Now in any quasitensor category with duals (as there are here) there is an intrinsic notion of $\mathrm{rank}\,(V)$ for any object V, and of $\underline{\mathrm{Tr}}\, f$ for any endomorphism f. The $\underline{\mathrm{Tr}}\, f$ is defined as the morphism

$$\underline{1}\to V\otimes V^*\xrightarrow{f\otimes\mathrm{id}} V\otimes V^*\xrightarrow{\Psi_{V,V^*}} V^*\otimes V\to\underline{1}$$

and $\mathrm{rank}\,(V) = \underline{\mathrm{Tr}}\,\mathrm{id}_V$. Here $\underline{1}$ is the identity object in the category (in our case, the trivial representation $\mathbb{C}$). The definition of trace $\underline{\mathrm{Tr}}$ extends further as a morphism $\underline{\mathrm{Hom}}(V,V) = V\otimes V^*\xrightarrow{\Psi_{V,V^*}} V^*\otimes V\to\underline{1}$, where $\underline{\mathrm{Hom}}$ is the internal hom in the category. For the quasitensor categories $\mathrm{Rep}\,(H)$ where H is a quantum group, the rank was studied in [5][9]. For $H = U_q(sl_2)$ it comes out as a variant of the familiar q-dimension. In general it comes out as[5], $\mathrm{rank}\,(V) = \mathrm{Tr}\,\rho_V(\underline{u})$ where $\underline{u}\in H$ is $\underline{u} = \sum(S\mathcal{R}^{(2)})\mathcal{R}^{(1)}$ and $\rho_V(\underline{u})$ is the matrix of $\underline{u}$ acting on V. Likewise if $f: V\to V$ is an endomorphism or indeed any linear map (viewed as for vector spaces in $\underline{\mathrm{Hom}}(V,V) = V\otimes V^*$),

$$\underline{\mathrm{Tr}}\, f = \mathrm{Tr}\,\rho_V(\underline{u})f. \tag{6}$$

Because of Proposition 2.1 we can apply this general theory to the quasitensor categories $\mathcal{C}_n$. It is also evident from this formula that $\underline{\mathrm{Tr}}\, f\circ g = \underline{\mathrm{Tr}}\, g\circ f$

for endomorphisms f, g. Note that this is not necessarily true if f, g are not intertwiners for H but merely linear maps. For example, one can show that $\underline{\mathrm{Tr}}\,\rho_V(h)\circ\rho_V(g) = \underline{\mathrm{Tr}}\,\rho_V(S^{-2}g)\circ\rho_V(h)$ for $h, g\in H$. Because of Proposition 2.1 we can apply this general theory to $\mathcal{C}_n$. We have,

PROPOSITION 2.2. *The intrinsic category-theoretic rank or 'anyonic' dimension* $\underline{\dim}$ *of an anyonic vector space* V *in* $\mathcal{C}_n$, *and the 'anyonic' trace of a map* $f: V\to V$ *are*

$$\underline{\dim}(V)\equiv\mathrm{rank}\,(V) = \sum_{a=0}^{n-1} e^{-\frac{2\pi\imath a^2}{n}}\dim V_a, \quad \underline{\mathrm{Tr}}\, f = \sum_{a=0}^{n-1} e^{-\frac{2\pi\imath a^2}{n}}\mathrm{Tr}\, f|_{V_a} \tag{7}$$

where V_a *is the subspace of homogeneous degree* a *and* $f|_{V_a}: V_a\to V_a$ *is the restriction of* f *to degree* a. *If* f *is not degree-preserving we project it to each* V_a. *If* $n = 2$ *we recover the usual super-dimension and super-trace.*

Proof We compute

$$\underline{u} = \frac{1}{n}\sum_{a,b} g^{-b}g^a e^{-\frac{2\pi\imath ab}{n}} = \frac{1}{n}\sum_a g^a\theta_n(a); \qquad \theta_n(a) = \sum_b e^{-\frac{2\pi\imath(a+b)b}{n}}. \tag{8}$$

To compute $\underline{\mathrm{Tr}}\, f$, let $\{e_{a,\gamma_a}\}$ be a basis of V and $\{f^{a,\gamma_a}\}$ a dual basis, where the e_{a,γ_a} are homogeneous of degree a and $\gamma_a = 1,\cdots\dim V_a$. By cyclicity of the ordinary trace, we can apply $\underline{u}$ first. So $\underline{\mathrm{Tr}}(f) = \frac{1}{n}\sum_a\theta_n(a)\sum_b\sum_{\gamma_b} f^{b,\gamma_b}($ $f(g^a\triangleright e_{b,\gamma_b})) = \frac{1}{n}\sum_{a,b}\theta_n(a)e^{\frac{2\pi\imath ab}{n}}(\sum_{\gamma_b} f^{b,\gamma_b}(f(e_{b,\gamma_b})))$ giving the result □

3. Anyonic Quantum Groups

An elementary example of a quantum group living in the category of anyonic vectors spaces is what we call the *anyonic enveloping algebra of the line*, denoted $U_n(k)$. This by definition has one generator ξ with

$$|\xi| = 1, \quad \xi^n = 0, \quad \underline{\Delta}\xi = \xi\otimes 1 + 1\otimes\xi \tag{9}$$

extended to products of the generators as a homomorphism working in $\mathcal{C}_n$, i.e., remembering the anyonic statistics of ξ. For example, using Ψ to take one ξ past another ξ, we have $\underline{\Delta}\xi^2 = (\xi\otimes 1 + 1\otimes\xi)^2 = \xi^2\otimes 1 + 1\otimes\xi^2 + \xi\otimes\xi + \Psi(\xi\otimes\xi) = \xi^2\otimes 1 + 1\otimes\xi^2 + (1 + e^{\frac{2\pi\imath}{n}})\xi\otimes\xi$.

In the remainder of this section we show how to obtain further quantum groups in the category of anyonic vector spaces by means of the general transmutation theorem in [7]. This theory applies to quasitensor categories which are generated as the representations of some quantum group H_1. Proposition 2.1 says that $\mathcal{C}_n$ are of this type with generating quantum group $H_1 = \mathbb{C}\mathbb{Z}'_n$. The general transmutation theory says that if H is any ordinary

Hopf algebra into which H_1 maps by a Hopf algebra map $f: H_1 \to H$, then H acquires the additional structure of a Hopf algebra $\underline{H}$ in the quasitensor category $\mathrm{Rep}(H)$. It consists of the same vector space and algebra as H, but with a modified coproduct. The vector space of H becomes an object $\underline{H}$ in $\mathrm{Rep}(H)$ by means of the adjoint action via f. $\underline{H}$ also has a certain opposite coproduct and if H is a quantum group (with $\mathcal{R}$) then $\underline{H}$ has a quasitriangular structure $\underline{\mathcal{R}}$ in $\mathrm{Rep}(H)$. In our case we obtain,

PROPOSITION 3.1. *If $(H, \mathcal{R})$ is an ordinary quantum group containing a group-like element g of order n, then it has the additional structure of an anyonic quantum group $\underline{H}$ in the category $\mathcal{C}_n$. The product coincides with that of H. The anyonic quantum group structure of $\underline{H}$ is*

$$\underline{\Delta} b = \sum b_{(1)} g^{-|b_{(2)}|} \otimes b_{(2)}, \quad \underline{\epsilon} b = \epsilon b, \qquad \underline{S} b = g^{|b|} S b$$

$$\underline{\Delta}^{\mathrm{op}} b = \sum b_{(2)} g^{-2|b_{(1)}|} \otimes g^{-|b_{(2)}|} b_{(1)}, \quad \underline{\mathcal{R}} = \mathcal{R}_g^{-1} \sum \mathcal{R}^{(1)} g^{-|\mathcal{R}^{(2)}|} \otimes \mathcal{R}^{(2)}$$

where $\Delta b = \sum b_{(1)} \otimes b_{(2)}$ is the usual coproduct. The action of g on H is in the adjoint representation $g \triangleright b = g b g^{-1}$ for $b \in \underline{H}$ and defines the degree of homogeneous elements by $g \triangleright b = e^{\frac{2\pi\imath |b|}{n}} b$. The quantity $\mathcal{R}_g$ is the quasitriangular structure on $\mathbb{C}\mathbb{Z}'_n$ as given in Proposition 2.1.

Proof These formulae follow directly from the general formulae in [7]. In the notation there we are computing $\underline{H} = B(\mathbb{C}\mathbb{Z}'_n, H)$ where $\mathbb{C}\mathbb{Z}'_n$ is the Hopf subalgebra generated by g, equipped with the non-standard quasitriangular structure given in Section 2. In the result shown it is assumed that all tensor product decompositions are into homogeneous elements. The second coproduct $\underline{\Delta}^{\mathrm{op}}$ specified in [7] is not simply $\Psi^{-1} \circ \underline{\Delta}$ but has something of the character of this. It comes out as

$$\underline{\Delta}^{\mathrm{op}} b = \sum e^{-\frac{2\pi\imath |b_{\underline{(1)}}||b_{\underline{(2)}}|}{n}} b_{\underline{(2)}} g^{-2|b_{\underline{(1)}}|} \otimes b_{\underline{(1)}} \tag{10}$$

where $\underline{\Delta} b = \sum b_{\underline{(1)}} \otimes b_{\underline{(2)}}$. This then computes to the form stated. Note also that g itself appears in $\underline{H}$ with degree 0 □

The transmutation formulae in [7] hold slightly more generally in the situation where there is a Hopf algebra map $\mathbb{C}\mathbb{Z}'_n \to H$ that need not be an inclusion. We limit ourselves here to giving two examples of the transmutation procedure. In both of these the map is an inclusion.

Our first example is with H the group algebra of a finite non-Abelian group containing an element g of order n. To be concrete we take for our example the group S_3, the permutation group on three elements, regarded as the symmetries of an equilateral triangle with fixed vertices 0,1,2, numbered clockwise. Let g denote a clockwise rotation of the triangle by $\frac{2\pi}{3}$ and let

R_a denote reflections about the bisector through the fixed vertex a. Let $\mathbb{C}S_3$ denote the group Hopf algebra of S_3. It has basis $\{1, g, g^2, R_0, R_1, R_2\}$. Of course, there are many ways to work with S_3: we present it in a way that makes the generalization to higher n quite straightforward.

EXAMPLE 3.2. *Let $H = \mathbb{C}S_3$ where S_3 is the symmetry group of an equilateral triangle as described. Its transmutation $\underline{S_3}$ by Proposition 3.1 is the following anyonic group in $\mathcal{C}_3$. Firstly, some homogeneous elements are*

$$r_a = \frac{1}{3} \sum_{b=0}^{b=2} e^{-\frac{2\pi\imath ab}{3}} R_b, \qquad |r_a| = a.$$

Together with $1, g, g^2$ of degree zero they form a basis of $\underline{S_3}$ as an anyonic vector space. Its anyonic dimension is $\underline{\dim}\underline{S_3} = 2e^{-\frac{\pi\imath}{3}}$. Its algebra and counit are those of $\mathbb{C}S_3$ but now

$$\underline{\Delta} r_a = \underline{\Delta}^{\mathrm{op}} r_a = \sum_{c=0}^{c=2} e^{-\frac{2\pi\imath c(a-c)}{3}} r_c \otimes r_{a-c}, \quad \underline{S} r_a = e^{-\frac{2\pi\imath a^2}{3}} r_a, \quad \underline{\mathcal{R}} = \mathcal{R}_g^{-1}$$

Proof The reflections have the property that $g R_a g^{-1} = R_{a+1}$ (mod 3). Hence their inverse Fourier transforms r_a as shown are homogeneous of degree as stated. The Hopf algebra structure on g (of degree zero) is unmodified. The usual coproduct in the remainder of $\mathbb{C}S_3$ is $\Delta R_a = R_a \otimes R_a$, hence $\Delta r_a = \sum_c r_c \otimes r_{a-c}$. This then becomes modified as $\underline{\Delta} r_a = \sum_c r_c g^{c-a} \otimes r_{a-c}$. Now note that in S_3, $R_a g = g R_a g^{-1} = g \triangleright R_a$ for all a. Hence $r_a g = g \triangleright r_a = e^{\frac{2\pi\imath a}{3}} r_a$ giving the result shown. Likewise, the original antipode on the R_a is $S R_a = R_a^{-1} = R_a$. Hence $S r_a = r_a$ also. From this and $g^{-1} R_a = g \triangleright R_a$ for all a (so that $g^{-1} r_a = e^{\frac{2\pi\imath a}{3}} r_a$) we obtain $\underline{S}$ as shown. The computation for $\underline{\Delta}^{\mathrm{op}}$ is similar to that for $\underline{\Delta}$ and comes out the same. The unmodified $\mathcal{R}$ of $\mathbb{C}S_3$ is $\mathcal{R} = 1 \otimes 1$, so that $\underline{\mathcal{R}} = \mathcal{R}_g^{-1}$ □

For the second example we consider the quantum groups $H = u_q(sl_2)$ defined at q a root of unity as in [14]. Here we refer to the finite-dimensional versions. They are generated by K, X, Y with relations

$$K X K^{-1} = q^{\frac{1}{2}} X, \quad K Y K^{-1} = q^{-\frac{1}{2}} Y, \quad [X, Y] = \frac{K^2 - K^{-2}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$$

and $X^r = Y^r = 0$, $K^{4r} = 1$ with $q = e^{\frac{2\pi\imath}{r}}$. There is a coproduct $\Delta X = X \otimes K + K^{-1} \otimes X$ etc, and a quasitriangular structure[14]. We work with this quantum group in an equivalent form with new generators and the $[X, Y]$ relation taking the form

$$g = K, \qquad E = X K^3, \qquad F = Y K^{-1}, \qquad q E F - F E = \frac{g^4 - 1}{q - 1}.$$

EXAMPLE 3.3. *Let $n = 4r$, so $q = e^{\frac{8\pi\imath}{n}}$ and $H = u_q(sl_2)$ as described. Its transmutation $\underline{u_q(sl_2)}$ by Proposition 3.1 is the following anyonic quantum group in $\mathcal{C}_n$. As an anyonic algebra it has generators g, E, F with $|g| = 0$, $|E| = 2$, $|F|$=-2. The algebra and counit are those of $u_q(sl_2)$, but now*

$$\underline{\Delta}E = E\otimes g^4 + 1\otimes E,\ \underline{\Delta}F = F\otimes 1 + 1\otimes F,\ \underline{\mathcal{R}} = \sum_{m=0}^{r-1}\frac{E^m\otimes F^m(q-1)^{2m}}{(q^m-1)\cdots(q-1)}$$

$$\underline{\Delta}^{\rm op}E = E\otimes 1 + 1\otimes E,\ \underline{\Delta}^{\rm op}F = F\otimes 1 + g^4\otimes F,\ \underline{S}E = -Eg^{-4},\ \underline{S}F = -F.$$

Proof This follows from direct computation using the form of the generators shown. The degree of E, F is from $gEg^{-1} = e^{\frac{2\pi\imath|E|}{n}}E$ and similarly for $|F|$. Since g has degree 0 its structure is of course unchanged. The formula for $\underline{\mathcal{R}}$ was in fact obtained by direct computation from the axioms for an anyonic quasitriangular structure in $\underline{u_q(sl_2)}$. Proposition 3.1 can then be pushed backwards to obtain a new expression for $\mathcal{R}$ in $u_q(sl_2)$, namely

$$\mathcal{R}_g\sum_{m=0}^{r-1}\frac{E^m g^{-2m}\otimes F^m(q-1)^{2m}}{(q^m-1)\cdots(q-1)} = \mathcal{R}_K\sum_{m=0}^{r-1}\frac{(KX)^m\otimes(K^{-1}Y)^m(1-q^{-1})^{2m}}{(1-q^{-m})\cdots(1-q^{-1})}.$$

Its matrices in the standard representations coincide with those in [14]. $\mathcal{R}_g = \mathcal{R}_K$ comes from Proposition 2.1 □

Note in this example the general phenomenon of transmutation: it can trade a non-cocommutative object $u_q(sl_2)$ in an ordinary bosonic category into a more cocommutative one (see $\underline{\Delta}F, \underline{\Delta}^{\rm op}E$) in a more non-commutative (in this case anyonic) category. There are plenty of other examples along the lines of the two examples above.

4. General Construction Based on Self-Dual Hopf Algebras

In this section we briefly describe a general construction from which the results of Section 2 were obtained. For these purposes we work over an arbitrary field k of characteristic not 2. Let H be an arbitrary self-dual Hopf algebra. This means a Hopf algebra equipped with a bilinear form $<\ ,\ >$: $H\otimes H \to k$ such that $<\Delta g, a\otimes b> = <g, ab>$, $<h\otimes g, \Delta a> = <hg, a>$ etc hold, see [5, Sec. 1]. Now for any finite-dimensional Hopf algebra H there is a quasitriangular one, $D(H)$ (the quantum double of H) introduced in [2] and built on $H\otimes H^*$ with certain relations. In [10] we showed how to generalize this construction to the situation of dually paired Hopf algebras, and use this now in the self-dual case. Then $D(H)$ is built now on $H\otimes H$ with the product[10]

$$(h\otimes a)(g\otimes b) = \sum <Sh_{(1)}, b_{(1)}> (h_{(2)}g\otimes b_{(2)}a) <h_{(3)}, b_{(3)}> \qquad (11)$$

and the tensor product coalgebra structure. The antipode is $S(h\otimes a) = (Sh\otimes 1)(1\otimes S^{-1}a)$.

PROPOSITION 4.1. *Let H be an involutory self-dual Hopf algebra and $D(H)$ its quantum double as described on $H\otimes H$. Then*

$$H' = \frac{D(H)}{(h\otimes 1 - 1\otimes h\ :\ h\in H)},\qquad \mathcal{R} = \sum_a f^a\otimes e_a \in H'\otimes H'$$

is a commutative Hopf algebra, which is quasitriangular at least in the finite-dimensional case. Here $\{e_a\}$ is a basis of H and $\{f^a\}$ another, dual, basis of H and we use for $\mathcal{R}$ their projections to H'. If S does not act as the identity in H' then $\mathcal{R}$ is strictly quasitriangular (so that Rep(H) *is strictly braided).*

Proof On $D(H)$ we have $\Delta(h\otimes a) = \sum h_{(1)}\otimes a_{(1)}\otimes h_{(2)}\otimes a_{(2)}$, hence

$$\begin{aligned}\Delta(h\otimes 1 - 1\otimes h) \ =\ & \frac{1}{2}\sum(h_{(1)}\otimes 1 - 1\otimes h_{(1)})\otimes(1\otimes h_{(2)} + h_{(2)}\otimes 1)\\ & +(h_{(1)}\otimes 1 + 1\otimes h_{(1)})\otimes(h_{(2)}\otimes 1 - 1\otimes h_{(2)})\end{aligned}$$

where we use the same decomposition $\Delta h = \sum h_{(1)}\otimes h_{(2)}$ for the two h's. Hence the ideal generated by the relations $(h\otimes 1) = (1\otimes h)$ for all $h\in H$ is a biideal in the sense of [16, p. 87]. It is also respected by S if $S^2 = {\rm id}$ (the condition that H is involutory). Hence the quotient is a Hopf algebra. The $\mathcal{R}$ shown is just the projection of the one on $D(H)$ found by [2]. We used the conventions of [11]. That H' is commutative follows from the formula for the product (in our conventions, $D(H)$ includes H on the right with the opposite product). Note that in any quasitriangular Hopf algebra one has $(S\otimes{\rm id})(\mathcal{R}) = \mathcal{R}^{-1}$ which leads to $\mathcal{R}^{-1} = \sum Sf^a\otimes e_a$. Note that because H' is commutative, its antipode has square 1 □

We used this construction applied to $H = \mathbb{C}\mathbb{Z}_n$ to obtain the structure of $\mathbb{C}\mathbb{Z}'_n$ described above. To do this we take for H a basis $e_a = g^a$ for $a = 0, 1, \cdots, n-1$. The dual basis can be written in terms of $\hat{\mathbb{Z}}_n$ (the character group of $\mathbb{Z}_n$), which we identify with $\mathbb{Z}_n$ itself to obtain the self-pairing.

Of course, the input Hopf algebra H need not itself be commutative or cocommutative. For example, let T_n be the group of upper triangular matrices in $M_n(k)$ with 1 on the diagonal. Then in [12] we constructed a self-dual Hopf algebra $kT_n{}^{\beta}{\bowtie}_{\alpha}(kT_n)^*$ by means a certain action α and coaction β. The left T_n factor here plays the role of momentum group, the other of position space. The Hopf algebra itself is then the quantum algebra of observables in an algebraic approach to quantum gravity[12]. Physically, in this setting Hopf algebra duality corresponds to a reversal of the roles of observables and states in the quantum system, and in this class of models the dual Hopf algebra is of the same type with the roles of position and momentum interchanged.

Finally we mention a variant of Proposition 4.1 which avoids some of the restrictions there. It applies also to H infinite-dimensional provided the antipode is invertible and that $\mathcal{R}$ makes sense.

PROPOSITION 4.2. *Let H be a finite-dimensional antiself dual Hopf algebra. Then $H' = D(H)/(h\otimes 1 - 1\otimes h\ :\ h\in H)$ is a quasitriangular Hopf algebra (not necessarily commutative) with $\mathcal{R}$ as in Proposition 4.1.*

Proof This variant differs in that we now suppose that there is a pairing $<\ ,\ >: H\otimes H\to k$ that obeys $<\Delta h, a\otimes b> = <h, ba>$ and $<Sh, a> = <h, S^{-1}a>$ for all $h, a, b\in H$ (the rest as before). In the finite-dimensional case this says $H\cong H^{*\mathrm{op}}$ where the latter is H^* with the opposite product. The formulae for $D(H)$ are now similar but with $ab_{(2)}$ rather than $b_{(2)}a$ in (11) and $S(h\otimes a) = (Sh\otimes 1)(1\otimes Sa)$. This means that both H factors in $H\otimes H$ are sub-Hopf algebras. In this case H' is always a Hopf algebra and need not be commutative □

Few antiself-dual Hopf algebras are known so far. One example of H that is found to be antiself-dual (as well as self-dual) is U_+ in [7, Prop. 2.9]. In this example, H' coincides with H and the last proposition recovers its known quasitriangular structure as in [13].

References

[1] M. Cohen. *Contemp. Math.* **43** (1985) 49–61.
[2] V.G. Drinfeld. Quantum groups. In *Proc. ICM*, pages 798–820, AMS (1987).
[3] K. Fredenhagen, K.H. Rehren, and B. Schroer, *Comm. Math. Phys.* **125** (1989) 201–226; R. Longo, *Comm. Math. Phys.* **126** (1989) 217.
[4] A. Joyal and R. Street. Mathematics Reports 86008, Macquarie University (1986).
[5] S. Majid. *Int. J. Modern Physics A* **5** (1990) 1–91.
[6] S. Majid. In *Proc. XIXDGM, Rapallo (1990), Spring. Lect. Notes. Phys* **375**, pages 131–142; *Int. J. Mod. Phys.* **6** (1991) 4359–4374; *Lett. Math. Phys.* **22** (1991) 167–176; *J. Math. Phys.* **32** (1991) 3246–3253.
[7] S. Majid. Transmutation theory and rank for quantum braided groups. *Math. Proc. Camb. Phil. Soc*, in press.
[8] S. Majid. Cross products by braided groups and bosonization. *J. Algebra*, in press.
[9] S. Majid. *Comm. Algebra* **18** (1990) 3705–3712.
[10] S. Majid. *Isr. J. Math* **72** (1990) 133–148.
[11] S. Majid. *Comm. Algebra* **19** (1991) 3061-3073.
[12] S. Majid. Phd Thesis, Harvard, (1988); *J. Algebra* **130** (1990) 17–64; *J. Classical and Quantum Gravity* **5** (1988) 1587–1606; *J. Funct. Analysis* **95** (1991) 291–319.
[13] D. Radford. *J. Algebra* **141** (1991) 354–358.
[14] N.Yu Reshetikhin and V.G. Turaev. *Invent. Math.* **103** (1991) 547–597.
[15] M. Scheunert, *J. Math. Phys.* **20** (1979) 712-720; D.I. Gurevich, *Leningrad Math. J.* **2** (1991) 801–828.
[16] M.E. Sweedler. *Hopf Algebras.* Benjamin (1969).
[17] F. Wilczek, ed. *Fractional Statistics and Anyon Superconductivity,* World. Sci. (1990); J. Fröhlich and P.-A. Marchetti, *Comm. Math. Phys.* **121** (1989) 177.

ON S–LIE–CARTAN PAIRS

WLADYSLAW MARCINEK
Institute of Theoretical Physics,
University of Wroclaw
Pl. Maxa Borna 9, 50–204 Wroclaw, Poland

Abstract. The noncommutative differential geometry corresponding to an arbitrary triangular Yang–Baxter operator S is described in a purely algebraical way. The concept of Lie–Cartan pairs of Kastler and Stora is generalized to the noncommutative case. All vector spaces, algebras and modules considered in the paper are in th e symmetric monoidal category $\mathcal{C}(S)$ corresponding to S.

1. Introduction

It is known that in the noncommutative geometry the role of the algebra of smooth functions $C^\infty(M)$ on a smooth manifold M is played by a noncommutative abstract associative algebras F [1,2,3]. The noncommutative generalizations of operators d, i_X, L_X are defined exactly like in the commutative case [2,3]. The essential difference with commutative case is that the algebra *der* F of derivations of F is not an F–module in general. It is very interesting that for some special noncommutative algebras, namely for S–symmetric algebras F_S, where S is a Yang–Baxter operator, the algebra of generalized derivations (S–derivations) *der* F_S is an F_S–module. Hence for such algebras we can construct the noncommutative geometry in the complete analogical way like in the commutative case. In order to do it we use the concept of Lie–Cartan pairs. The concept of Lie–Cartan pairs has been considered by Kastler and Stora [4] as a purely algebraical frame for describing operators of classical differential geometry. The generalizations to the supersymmetric case has been presented by Jadczyk and Kastler [5], Coquereaux and Jadczyk [6] and generalized further to the colour symmetry by the authors [7,8,9]. Similar approach has been considered by Matthews [10]. In this paper we generalize the concept of Lie–Cartan pairs to the case of arbitrary triangular Yang–Baxter operator S. We describe the noncommutative geometry correspond to S in the purely algebraical way like in the commutative case. All vector spaces, algebras and modules considered here are in the symmetric monoidal category determined by S [11,12].

2. Symmetric monoidal categories

Let E be a vector space over the field $\mathbb{C}$ of complex numbers. A linear operator $S : E \otimes E \longrightarrow E \otimes E$ such that

$$S^{(1)}S^{(2)}S^{(1)} = S^{(2)}S^{(1)}S^{(2)} \tag{1}$$

and

$$S^2 = id \tag{2}$$

is said to be a triangular Yang–Baxter or symmetry on E, here $S^{(1)} = S \otimes id$, $S^{(2)} = id \otimes S$ [12,13]. An S–vector space or vector space with symmetry is a given vector space equipped with symmetry defined above. Let E be an S–vector space. Then there exist a series of representations

$$\rho : \pi \in S_n \longrightarrow \rho(\pi) \in end\ (E^{\otimes n}) \tag{3}$$

of the symmetric group S_n defined by

$$\rho(\pi) = \prod_{i=1}^{k} S^{(i)} , \tag{4}$$

where $S^{(i)} = id \otimes \ldots \otimes S \otimes \ldots \otimes id$, ($S$ on the i–th place), $\pi = \tau_1, \ldots, \tau_k$, τ_i are transpositions and $\rho(\tau_i) = S^{(i)}$. Moreover, there exist the rigid symmetric monoidal category $\mathcal{C}(S)$ determined by S. The construction of $\mathcal{C}(S)$ has been described by Lyubashenko in Ref. [12], see also References [14,15]. The fact that $\mathcal{C}(S)$ is a rigid symmetric monoidal category means that for every pair U, W of objects of $\mathcal{C}(S)$ we have a family of natural isomorphisms $S = S_{U,W}$. $U \otimes W \longrightarrow W \otimes U$ such that

$$S_{U\otimes V,W} = (S_{U,W} \otimes id_V) \circ (id_V \otimes S_{V,W}) \tag{5}$$

$$(S_{V,W} \otimes id_U) \circ S_{U,V\otimes W} = (id_W \otimes S_{U,V}) \circ S_{U\otimes V,W} \tag{6}$$

and

$$S_{U,W} \circ S_{W,U} = id_{W\otimes U} . \tag{7}$$

The category $\mathcal{C}(S)$ contains: the underground field $\mathbb{C}$, the given vector space E, the left and right duals of E, tensor products of such spaces, some algebras such as the S–symmetric algebra F, the algebra of S–derivations, S–Lie algebras, ... , and some F–modules.

3. S–Lie algebra

An algebra L in $\mathcal{C}(S)$ equipped with a bracket $[\ ,\ ]_S : \mathcal{L} \otimes \mathcal{L} \longrightarrow \mathcal{L}$ such that

$$[\ ,\ ]_S = -[\ ,\ ]_S \circ S \tag{8}$$

$$[\ ,\ ]_S \circ [\ ,\ ]_S^{(2)} = [\ ,\ ]_S \circ [\ ,\ ]_S^{(1)} + [\ ,\ ]_S \circ [\ ,\ ]_S^{(2)} \circ S^{(1)} , \tag{9}$$

$$S \circ [\ ,\ ]_S^{(1)} = [\ ,\ ]_S^{(2)} \circ S^{(1)} \circ S^{(2)} , \tag{10}$$

is said to be an S–Lie algebra in $\mathcal{C}(S)$ $[\ ,\ ]$, here we have the following notation $[\ ,\ ]_S^{(1)} = [\ ,\ ]_S \otimes id$, $[\ ,\ ]_S^{(2)} = id \otimes [\ ,\ ]_S$, see [13,15]. An $\mathcal{L}$–module is an S–vector space $E \in \mathcal{C}(S)$ equipped with an action $(X \otimes f) \in \mathcal{L} \otimes E \longrightarrow Xf \in E$ such that

$$[X \otimes Y]_S f = (\circ - \circ S)(X \otimes Y)(id \otimes f) \tag{11}$$

for $X, Y \in \mathcal{L}$, $f \in E$, $\circ$ is a composition map. Let F be an S–symmetric algebra in $\mathcal{C}(S)$ equipped with multiplication $m : F \otimes F \to F$. The S–symmetry of F means that $m = m \circ S$. An S–derivation of F is a linear mapping $X : F \longrightarrow F$ such that

$$X \circ m = m \circ (X \otimes id) \circ (id + S) . \tag{12}$$

see References [12,16]. Obviously the space of all S–derivations $der\ F$ of F is an S–Lie algebra

$$[X \otimes Y]_S = (\circ - \circ S)(X \otimes Y) . \tag{13}$$

We also have

$$f(gX) = (fg)X, \qquad f(Xg) = (fX)g, \tag{14}$$

for $g, f \in F$, $X \in der\ F$. This means that $der\ F$ is an F–module. Moreover we have

$$[X \otimes fY]_S = [\ ,\ ]_S^{(2)} \circ ev \circ S^{(1)}(X \otimes f \otimes Y) + (Xf)Y$$

for $X, Y \in derF$, $f \in F$ and $ev(f \otimes X) = fX$. Let $der\ F$ be an S–Lie algebra of S–derivations of an S–symmetric algebra F. A F–linear mapping $\omega : (der\ F)^{\otimes p} \longrightarrow F$ such that

$$\omega = (sgn\ \pi) f \circ \rho(\pi) \tag{15}$$

for every $\pi \in S_p$ is said to be skew–S–symmetric of degree p, here $\rho : S_p \longrightarrow end(der\ F)^{\otimes p}$ is the representation of the symmetric group S_p in the space

$(der F)^{\otimes p}$. The space of all F–linear and skew–S–symmetric mappings of degree p is denoted by $\Lambda^p(der F, F)$. If $\omega \in \Lambda^p(der F, F)$ and $\eta \in \Lambda^q(der\ F, F)$, then the S–exterior product $\omega \wedge \eta \in \Lambda^{p+q}(der\ F, F)$ is defined by

$$\begin{aligned}(\omega \wedge \eta)(X_1 \otimes, \ldots, \otimes X_{p+q}) \\ = \sum_{\pi \in S_{p,q}} sgn\pi\, (\omega \otimes \eta) \circ \rho(\pi)(X_1 \otimes, \ldots, \otimes X_{p+q})\end{aligned} \tag{16}$$

for $X_1, \ldots, X_{p+q} \in der\ F$, where $S_{p,q} = \{\pi \in S_{p+q} : \pi(1) < \ldots < \pi(p)$ and $\pi(p+1) < \ldots < \pi(p+q)\}$. The exterior derivative $d : \Lambda^p(der\ F, F) \longrightarrow \Lambda^{p+1}(der\ F, F)$ is defined by

$$\begin{aligned} d\omega(X_1 \otimes, \ldots, \otimes X_{p+1}) \\ = \sum_k (-1)^{k+1}(X_k\omega) \circ \rho(\pi_k)(X_1 \otimes, \ldots, X_{p+1}) \\ + \sum_{k<l} (-1)^{k+l}\omega \circ ([\ ,\ ]_S \otimes id_{p-1}) \\ \circ \rho(\pi_{kl})(X_1 \otimes, \ldots, \otimes X_{p+1}),\end{aligned} \tag{17}$$

where $\pi_k(1, \ldots, p+1) = (k, 1, \ldots, k-1, k+1, \ldots, p+1)$ and $\pi_{kl}(1, \ldots, k, \ldots, l, \ldots, p+1) = (k, l, 1, \ldots, k-1, k+1, \ldots, l-1, l+1, \ldots, p+1)$.

4. S–Lie–Cartan pairs

Let B be an S–Lie algebra and let F be an S–symmetric algebra, both algebras B and F are in the category $\mathcal{C}(S)$. The pair (B, F) is said to be an S–Lie–Cartan pair if the following linear mappings are given

$$ev : (X \otimes f) \in B \otimes F \longrightarrow Xf \in F, \tag{18}$$

$$\cdot : (f \otimes X) \in F \otimes B \longrightarrow f \cdot X \in B, \tag{19}$$

and (i) the mapping (18) defines an S–morphism ∂ of B into $der\ F$, $\partial : X \in B \longrightarrow \partial_X \in der\ F$ such that

$$\partial_X \circ m = m \circ (\partial_X \otimes id) \circ (id + S), \tag{20}$$

where $m : F \otimes F \longrightarrow F$ is the S–symmetric multiplication in F, and

$$\partial_{[X \otimes Y]_S} = \circ(\partial_X \otimes \partial_Y) - \circ S(\partial_X \otimes \partial_Y) S^{-1}, \tag{21}$$

where $\circ(\partial_X \otimes \partial_Y) = \partial_X \circ \partial_Y$, (ii) the mapping (19) makes B an unital F–module

$$f \circ (g \cdot X) = m(f \otimes g) \cdot X, \qquad 1_F \cdot X = X \tag{22}$$

for $f, g \in F$, $X \in B$, 1_F is the unit in F, (iii) in addition we have

$$(f \cdot X)g = f \cdot (Xg) \qquad \text{for} \qquad f, g \in F, \quad X \in B, \tag{23}$$

and

$$[X \otimes gY]_S = ev \circ [\ ,]_S^{(2)} \circ S^{(1)}(X \otimes g \otimes Y) + (Xg) \cdot Y \tag{24}$$

for $g \in F; X, Y \in B$. Let us take some examples.

Example 1. The pair $(der\ F, F)$, where F is an arbitrary S–symmetric algebra, is an S–Lie–Cartan pair.

Example 2. If S is colour symmetry [13], then the S–Lie–Cartan pair becomes graded Lie–Cartan pair of Refers [7,8,9].

Example 3. If $S \equiv T$ (the transposition), then we obtain the ordinary Lie–Cartan pair of Kastler and Stora [4].

Let (B, F) be an S–Lie–Cartan pair and let V be an F–module. An F–linear mapping $\nabla_X : V \longrightarrow V$ such that

$$\nabla_X fx = ev \circ (id \otimes \nabla \circ id) \circ S^{(1)}(X \otimes f \otimes x) + (Xf)x \tag{25}$$

for $X \in B$, $f \in F$, $x \in V$; $ev(f \otimes \nabla_X \otimes x) = f\nabla_X x$, and

$$\nabla_{f \cdot X} x = f \cdot \nabla_X x \tag{26}$$

for $f \in F, X \in B, x \in V$, is said to be a covariant S–derivative. The mapping $\nabla : X \in B \longrightarrow \nabla_X \in end(V)$ such that ∇_X is a covariant S–derivative for every $X \in B$, is said to be an S–connection on V. An F–linear mapping $R : (X \otimes Y) \in B \otimes B \longrightarrow R_{X \otimes Y} \in end(V)$ defined by

$$R_{X \otimes Y} = \circ(\nabla_X \otimes \nabla_Y) - \circ S(\nabla_X \otimes \nabla_Y) S^{-1} - \nabla_{[X \otimes Y]_S} \tag{27}$$

is said to be an S–curvature of ∇. Next one can define the covariant exterior S–derivation, the generalized inner derivation or covariant S–Lie derivative in a similar way.

References

[1] A. Connes, Non–commutative differential geometry, Publ. I.H.E.S. 62, 257 (1985).
[2] M. Dubois–Violette, Derivations et calculi differential noncommutative, C.R. Acad. Sci. Paris Ser I, 307, 403 (1988)
[3] M. Dubois–Violette, R. Kerner and J. Madore, Non–commutative differential geometry of matrix algebras, J. Math. Phys. **31**, 316 (1990).
[4] D. Kastler and R. Stora, Lie–Cartan pairs, J. Geom. and Phys. **2**, 1 (1985).

[5] A. Jadczyk and D. Kastler, Graded Lie–Cartan pairs II, Annals of Phys. **179**, 169 (1987).
[6] R. Coquereaux and A. Jadczyk, Differential and integral geometry of Grassmann algebras, Rev. Math. Phys. **3**, 63 (1991).
[7] W. Marcinek, Generalized Lie algebras and related topics, Preprint ITP UWr No 691 and 692 (1987).
[8] W. Marcinek, Generalized Lie-Cartan pairs, Rep. Math. Phys. **27**, 385 (1989).
[9] W. Marcinek, Graded algebras and geometry based on Yang–Baxter operators, J. Math. Phys. **33**, 1631 (1992).
[10] R. Matthes, A covariant differential calculus on the "quantum group" $\mathbb{C}_q^n$, Proceedings of the Wigner Symposium, Goslar.
[11] S. Mac Lane, Categories for Working Mathematician, Graduate Text in Mathematics 5, Springer–Verlag 1971.
[12] V.V. Lyubashenko, Vectorsymmetries, in Seminar on Supermanifolds, No 19, 1 (1987).
[13] D. Gurevich, A. Radul and V. Rubstov, Noncommutative differential geometry and Yang–Baxter equation, I.H.E.S./M/91/88.
[14] S. Majid, Braided groups, DAMTP/90–42.
[15] D. Gurevich, Quantum Yang–Baxter equation and a generalization of the formal Lie theory in Seminar on Supermanifolds, Stokholm University, Report No 19, 33 (1987).
[16] A. Borowiec, W. Marcinek and Z. Oziewicz, On multigraded differential calculus, Proceedings of the First Max Born Symposium, ed. by R. Gielerak et all. Kluwer Acad. Pub. 1992.

NEW REAL FORMS OF $U_q(\mathcal{G})$

JAN SOBCZYK
Institute of Theoretical Physics
Wrocław University
Pl. Maxa Borna 9, 50-205 Wrocław, Poland

Abstract. We consider different coalgebra structures in $U_q(\mathcal{G})$ induced by its (as algebra) automorphisms. We prove that for each obtained Hopf algebra a complete classification of real forms can be done.

1. Introduction

Recently many authors [1,4,5,6,7,11] addressed a question of defining real forms of complex Hopf algebras. If the Hopf algebra in consideration is a deformation of the universal enveloping algebra of a simple complex Lie algebra its real forms can be viewed as deformations of real Lie algebras. Taking into account a fundamental role played in physics by e.g. Poincare algebra or $su(n)$ such study may be relevant in future applications of quantum groups.

Most often (for other approach see [4]) by a real form one understoods a morphism Φ with the following properties:

$$\Phi^2 = 1 \tag{1}$$

$$\Phi(\alpha X + \beta Y) = \alpha^*\Phi(X) + \beta^*\Phi(Y) \tag{2}$$

$$\Phi(XY) = \Phi(Y)\,\Phi(X) \tag{3}$$

$$(\Phi \otimes \Phi)\Delta(X) = \Delta(\Phi(X)) \tag{4}$$

Then it can be shown [9] that Φ has to satisfy also

$$\Phi \circ S \circ \Phi \circ S = 1 \tag{5}$$

$$\varepsilon\,(\Phi(X)) = (\varepsilon(X))^* \tag{6}$$

In the case of quantum deformations of enveloping algebra of a complex simple Lie algebra $\mathcal{G}$ a complete classification of such transformations has been done by Twietmeyer [11] The classification has been done for a definite coproduct in $U_q(\mathcal{G})$ namely ($e_{\pm a}$ and h_a are elements of Cartan-Chevalley basis)

$$\Delta(e_a) = e_a \otimes 1 + q^{h_a} \otimes e_a \tag{7}$$

$$\Delta(e_{-a}) = e_{-a} \otimes q^{-h_a} + 1 \otimes e_{-a} \tag{8}$$

$$\Delta(h_a) = h_a \otimes 1 + 1 \otimes h_a \tag{9}$$

It is however well known that there are many coproducts in $U_q(\mathcal{G})$ which with the same algebra structure make it a Hopf algebra [2,3,8]). Thus a natural question arises if Twietmeyer classification can also be done for coproducts different from that given in eqs.(7-9).

2. Main theorem

Suppose we have algebra authomorphisms Ω and Ω^{-1} of $U_q(\mathcal{G})$ (here viewed only as an algebra). It can be easily seen that any such Ω defines a new coproduct in $U_q(\mathcal{G})$ from an "initial" one Δ. We put namely

$$\Delta^\Omega \equiv (\Omega \otimes \Omega) \circ \Delta \circ \Omega^{-1} \tag{10}$$

Since

$$(1 \otimes \Delta^\Omega) \circ \Delta^\Omega = (\Omega \otimes \Omega \otimes \Omega) \circ (1 \otimes \Delta) \circ \Delta \circ \Omega^{-1} \tag{11a}$$

$$(\Delta^\Omega \otimes 1) \circ \Delta^\Omega = (\Omega \otimes \Omega \otimes \Omega) \circ (\Delta \otimes 1) \circ \Delta \circ \Omega^{-1} \tag{11b}$$

Δ^Ω is coassociative whenever Δ is so.
We can now state a following

THEOREM

$\Phi^\Omega \equiv \Omega \circ \Phi \circ \Omega^{-1}$ defines a real form for $U_q(\mathcal{G})$ with the coproduct Δ^Ω.

Proof:
We calculate

$$\begin{aligned}(\Phi^\Omega \otimes \Phi^\Omega) \circ \Delta^\Omega &= (\Omega \otimes \Omega) \circ (\Phi \otimes \Phi) \circ (\Omega^{-1} \otimes \Omega^{-1}) \circ \Omega^\Omega = \\ &= (\Omega \otimes \Omega) \circ (\Phi \otimes \Phi) \circ \Delta\, \Omega^{-1} = \\ &= (\Omega \otimes \Omega) \circ \Delta \circ \Phi \circ \Omega^{-1} = \\ &= \Delta^\Omega \circ \Omega \circ \Phi \circ \Omega^{-1} = \Delta^\Omega \circ \Phi^\Omega\end{aligned} \tag{12}$$

Remaining requirements are trivially satisfied.

Let us ask now what is a relation between real forms defined by Φ and Φ^Ω. In the undeformed case they give rise to two real Lie algebras with generators

$$\mathcal{G}^\Phi = \{A : \Phi(A) = -A\} \tag{13a}$$

$$\mathcal{G}^{\Phi^\Omega} = \{B : \Phi^\Omega(B) = -B\} \tag{13b}$$

Observe now that $A \in \mathcal{G}^\Phi$ if and only if $\Omega(A) \in \mathcal{G}^{\Phi^\Omega}$:

$$\Phi^\Omega(\Omega(A)) = \Omega \circ \Phi \circ \Omega^{-1}(\Omega(A)) = \Omega(\Phi(A)) \tag{14}$$

We conclude that in the limit $q \to 1$ Ω becomes an isomorphism of two real Lie algebras. Therefore Φ and Φ^Ω correspond to deformations of the same real form of $\mathcal{G}$.

3. Example

As an illustration of the above scheme let us present an example of $U_q(sl(3))$. We start with ($e_{\pm 1}$ and $e_{\pm 2}$ are simple roots)

$$A_{ab} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$[h_a, e_{\pm b}] = \pm A_{ab} e_{\pm b} \tag{15a}$$

$$[e_a, e_b] = \delta_{ab} \frac{q^{h_a} - q^{-h_a}}{q - q^{-1}} \tag{15b}$$

$$e_{1+2} = [e_1, e_2]_q \equiv e_1 e_2 - q e_2 e_1 \tag{15c}$$

$$[e_1, e_{1+2}]_{q^{-1}} = [e_{1+2}, e_2]_{q^{-1}} = 0 \tag{15d}$$

The coproduct is given in eqs. (7-9). We can take as Ω

$$\Omega(e_{\pm 1}) = e_{\pm(1+2)} \tag{16a}$$

$$\Omega(e_2) = e_{-1} q^{-h_1}, \quad \Omega(e_{-2}) = q^{h_1} e_1 \tag{16b}$$

$$\Omega(h_1) = h_1 + h_2, \quad \Omega(h_2) = -h_1 \tag{16c}$$

It then follows that

$$\Delta^\Omega(e_{1+2}) = e_{1+2} \otimes 1 + q^{h_1+h_2} \otimes e_{1+2} \tag{17a}$$

$$\Delta^\Omega(e_{-(1+2)}) = e_{-(1+2)} \otimes q^{-(h_1+h_2)} + 1 \otimes e_{-(1+2)} \tag{17b}$$

$$\Delta^\Omega(e_{-1}) = e_{-1} \otimes q^{h_1} + 1 \otimes e_{-1} \tag{17c}$$

$$\Delta^\Omega(e_1) = e_1 \otimes 1 + q^{-h_1} \otimes e_1 \tag{17d}$$

If we look at Ω it should be clear that the new coproduct acts on e_{1+2} and $e_{-1}q^{-h_1}$ as if they were simple roots. In fact Ω as $q \to 1$ becomes an element of the Weyl group of $sl(3)$. It should also be added that Δ^Ω can be obtained [10] from Δ by a nontrivial twisting [2], [8]: $\Delta^\Omega(X) = F\,\Delta(X)\,F^{-1}$ (nontrivial in a sense that F is constructed not only from the elements of the Cartan subalgebra).

Acknowledgments

I am grateful to V.Tolstoy and J.Lukierski for many useful conversations.

References

[1] V. Dobrev, to appear in "*Quantum Groups*", eds. H.D. Doebner and V. Dobrev, Springer Verlag 1992 (series *Lecture Notes in Physics*).
[2] V.G. Drinfeld, *Quantum Groups*, Proceedings of the International Congress of Mathematicians, Berkeley 1986.
[3] S.M. Khoroshkin, V.N. Tolstoy, *Cartan-Weyl basis for quantized Kac-Moody (super) algebras and universal R-matrix*, to appear in "*Quantum Groups*", eds. H.D. Doebner, V. Dobrev, Springer Verlag 1992 (series *Lecture Notes in Physics*).
[4] V. Lyubashenko, *Real and imaginary forms of quantum groups*, Kiev preprint KPI-2606.
[5] J. Lukierski, A. Nowicki, H. Ruegg, Phys. Lett. **B 271**, 321 (1991).
[6] M. Mozrzymas, *Reality conditions for quantum algebras*, University Bordeaux 1 preprint, LPTB 92-2.
[7] T. Masuda, K. Mimachi, Y. Nakagami, M. Noumi, Y. Saburi, K.Ueno, Lett. Math. Phys. **19**, 187 (1990).
[8] N. Reshetikhin, Lett. Math. Phys. **20** 331 (1990).
[9] M. Scheunert, *The antipode of and star operations in Hopf algebra*, Bonn University preprint HE-92-13.
[10] V.N. Tolstoy, talk given at Max Born Symposium, Wroclaw - Sobotka, September 1992.
[11] E. Twietmeyer, Lett. Math. Phys. **24**, 49 (1992).

RELATED TOPICS

Z_3–GRADED STRUCTURES*

RICHARD KERNER
Laboratoire de Physique Théorique
GCR - Université Pierre et Marie Curie,
CNRS - URA 769
Tour 22, Boite 142
4, Place Jussieu
75005 Paris
France

Abstract. We investigate some consequences of imposing the Z_3–grading on algebraic structures. It leads to abandoning of algebras defined by binary relations, introducing the ternary relations instead. Possible analogs of Lie algebras and Grassmann algebras are discussed, as well as an example of a simple gauge theory on 3×3 matrices. We show also how a cubic root of super–translations can be defined.

1. Why is Z_3–grading beautiful?

Among the swarming multitude of new structures which are under investigation since a few years, such as non–commutative geometries, quantum groups, braid groups, and the like, all of which generalize the well known classical algebraic structures such as Lie algebras and Lie groups by transgressing one of the axioms, we would like to point out one of the possibilities which consists in replacing the Z_2–grading by Z_3–grading.

By Z_3 we mean the cyclic group of three elements, which can be represented on the complex plane $\mathbb{C}$ as multiplication by $j = e^{2\pi i/3}$, j^2 and $j^3 = 1$. This simple group is a subgroup of the group of permutations of three elements, S_3, which contains six elements. It can be also represented faithfully on complex plane $\mathbb{C}$ if we add the involution, which is complex conjugation. Then the two other involutions are generated by composition with cyclic elements. Here is the full representation of S_3 in the complex plane:

---

* Dedicated to Prof. Jan Rzewuski on his 75-th birthday

Permutation	$\binom{ABC}{ABC}$	$\binom{ABC}{BCA}$	$\binom{ABC}{CAB}$	$\binom{ABC}{CBA}$	$\binom{ABC}{BAC}$	$\binom{ABC}{ACB}$
Complex representation	1	j	j^2	$-$ complex conjugation	$\wedge$ reflexion in j^2	$*$ reflexion in j

There is another representation obtained by replacing j by j^2 and $\wedge$ by $*$.

S_3 is the last group of permutations that possesses faithful representations on the complex plane. S_4 has an unfaithful representation, whereas S_5 and higher do not have representations in $\mathbb{C}$ (besides the trivial one, reducing everything to S_2).

In the case of Z_2 we could not distinguish between Z_2 and S_2; here Z_3 is abelian, whereas S_3 is not. Nevertheless, the possibility of representing it faithfully in the complex numbers suggests that some important ternary symmetries of wave functions depending on three variables should be investigated. Perhaps these symmetries will be adequate to describe quarks?

2. Z_3–graded derivations, Z_3–commutators

Consider a free associative algebra with unit element, over complex numbers, generated by a finite number of elements $a, b, c, \ldots$. It can be naturally Z_3–graded if we define $\mathbb{C}$ and the unit element as grade 0, the generators as grade 1 elements, their binary products as grade 2, and afterwards, any product of p elements as being of grade $[p \bmod 3]$.

A Z_3–graded derivation of grade k of such an algebra is a linear mapping from this algebra into itself satisfying the generalized Leibniz rule:

$$\overset{(k)}{D}(ab) = \left(\overset{(k)}{D}\, a\right) b + j^{k\ grade(a)}\, a \left(\overset{(k)}{D}\, b\right). \tag{2.1}$$

with

$$\text{grade}\left(\overset{(k)}{D}\, a\right) = [\text{grade}(a) - k] \bmod 3 \tag{2.2}$$

$k = 0, 1, 2$.

It is easy to prove that for $k = 1$ or 2 the third power of any such derivation vanishes.

Now, contrary to the Z_2–case, no binary relations can be imposed on the products in our algebra. Indeed, suppose that among the generators a_α some binary relation exists, given by

$$\sum_{\alpha,\beta} U^{\alpha\beta}\, a_\alpha\, a_\beta = 0\,. \tag{2.3}$$

Then, deriving it with respect to a_α, one gets

$$U^{\alpha\beta} + jU^{\beta\alpha} = 0 \tag{2.4}$$

which implies

$$U^{\alpha\beta} = j^2 U^{\alpha\beta} \tag{2.5}$$

possible only if $U^{\alpha\beta} \equiv 0$.

On the contrary, ternary relations may be imposed, compatible with our derivation.

They should satisfy

$$\sum_{\alpha,\beta,\gamma} U^{\alpha\beta\gamma}\, a_\alpha\, a_\beta\, a_\gamma = 0 \tag{2.6}$$

$$U^{\alpha\beta\gamma} + jU^{\beta\alpha\gamma} + j^2 U^{\beta\gamma\alpha} = 0 \tag{2.7}$$

because the binary products $a_\alpha a_\beta$ are linearly independent by virtue of (2.5). One of the simplest solutions to (2.7) is of course $U^{\alpha\beta\gamma} \equiv 1$, because $1 + j + j^2 = 0$.

The binary relations that define the Grassmann algebra in the Z_2–graded case, i.e. $a_\alpha a_\beta + a_\beta a_\alpha = 0$ should be replaced by a ternary relation. We have the choice, depending on the interpretation of the antisymmetry as a Z_2 or an S_2 group average, between the following two possibilities of generalization:

$$a_\alpha a_\beta a_\gamma = j\, a_\beta a_\gamma a_\alpha = j^2 a_\gamma a_\alpha a_\beta = 0 \tag{2.8}$$

i.e. a Z_3 invariance property, or

$$a_\alpha a_\beta a_\gamma + a_\beta a_\gamma a_\alpha + a_\gamma a_\alpha a_\beta + a_\gamma a_\beta a_\alpha + a_\beta a_\alpha a_\gamma + a_\alpha a_\gamma a_\beta = 0 \tag{2.9}$$

which is an S_3–average without any weight.

The formula (2.8) gives rise to a finite algebra whose dimension depends on the number of generators N as $\frac{(N+1)^3-(N+1)}{3} + 1$; e.g.

$$\text{one generator} \;:\; \mathbb{1}, a, a^2; \quad a^3 = 0 \tag{2.10}$$

$$\text{two generators} : \mathbb{1}, a, b, a^2, ab, ba, b^2, ab^2, a^2 b \tag{2.11}$$

etc.

The formula (2.9) defines an infinite algebra. The finite algebras defined by (2.8) are incompatible with the graded derivations except for the simplest case (2.10) given by one generator; the structures defined by (2.9) are compatible with the graded Leibniz derivation.

The simplest algebra consisting of $\mathbb{1}, a, a^2$ admits three Z_3–graded derivations defined as follows:

$$\begin{aligned} \partial_1 \mathbb{1} &= 0 & \partial_1 a &= \mathbb{1} & \partial_1 a^2 &= -j^2 a, \\ \partial_2 \mathbb{1} &= 0 & \partial_2 a &= a^2 & \partial_2 a^2 &= 0, \\ \partial_3 \mathbb{1} &= 0 & \partial_3 a &= a & \partial_3 a^2 &= -j^2 a^2. \end{aligned} \tag{2.12}$$

The Z_3–grades of these derivations are 1,2 and 0 respectively. They do not form an algebra, i.e. no binary combination yields another Z_3–derivation; instead, they close under the following ternary rule:

$$\begin{aligned} \partial_1\partial_2\partial_1 + \partial_2\partial_1\partial_1 + \partial_1\partial_1\partial_2 &= -j^2\partial_1, \\ \partial_2\partial_1\partial_2 + \partial_1\partial_2\partial_2 + \partial_2\partial_2\partial_1 &= -j^2\partial_2. \end{aligned} \tag{2.13}$$

This is an example of a ternary algebra, that we shall encounter in other realizations, too.

Consider now a Z_3–graded commutator defined as follows:

$$[A, B]_{Z_3} = AB = j^{ab} BA \tag{2.14}$$

where A, B are elements of a Z_3–graded associative algebra, a, b their respective grades.

Let η be of grade 1. Then we can define a Z_3–grade 1 differential as

$$dA = [\eta, A]_{Z_3} = \eta A - j^a A \eta. \tag{2.15}$$

Of course, $d^2 \neq 0$, but $d^3 = 0$ if η^3 commutes with all element of our algebra.

There is no analog of Jacobi identity for the Z_3-graded commutator; nor the derivation d is a derivation of the commutator algebra. Instead, one has the following identity:

$$\begin{aligned} &[\{A, B, C\}\, D]_{Z_3} + [\{B, C, D\}, A]_{Z_3} + \\ &+ [\{C, D, A\}, B]_{Z_3} + [\{D, A, B\}, C]_{Z_3} = 0 \end{aligned} \tag{2.16}$$

where $\{A, B, C\} = \Big[[A, B]_{Z_3}, C\Big]_{Z_3} + \Big[[B, C]_{Z_3}, A\Big]_{Z_3} + \Big[[C, A]_{Z_3}, B\Big]_{Z_3}$, and only if all the items have the same grade.

The simplest representation of such an algebra is given by 3×3 complex matrices separated into three linear subspaces with grades 0, 1 and 2:

$$\mathcal{A}_0 : \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} ;\ \mathcal{A}_1 : \begin{pmatrix} 0 & 0 & \alpha \\ \beta & 0 & 0 \\ 0 & \gamma & 0 \end{pmatrix} ;\ \mathcal{A}_2 : \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \\ \gamma & 0 & 0 \end{pmatrix}. \tag{2.17}$$

There are three linearly independent grade 1 derivations,generated by a Z_3–commutator with the matrices

$$\eta_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ \eta_2 = \begin{pmatrix} 0 & 0 & 1 \\ j & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix}, \ \eta_3 = \begin{pmatrix} 0 & 0 & 1 \\ j^2 & 0 & 0 \\ 0 & j & 0 \end{pmatrix}. \tag{2.18}$$

If we put $\partial_1 A = \eta_1 A - j^a A \eta_1$ etc., we can easily prove that

$$\partial_1\partial_2\partial_3 + \partial_2\partial_3\partial_1 + \partial_3\partial_1\partial_2 + \partial_2\partial_1\partial_3 + \partial_1\partial_3\partial_2 + \partial_3\partial_2\partial_1 \equiv 0 \tag{2.19}$$

and, of course, $\partial_1^3 \equiv 0$, $\partial_2^3 \equiv 0$, $\partial_3^3 \equiv 0$.

Ternary composition rules displaying nice representation properties with respect to Z_3 or S_3 permutations can be easily defined on free associative algebras over $\mathbb{C}$. For example, one can define

$$\{a, b, c\} \stackrel{df}{=} abc + j\, bca + j^2 cab. \tag{2.20}$$

Obviously, $\{a, a, a\} \equiv 0$, and

$$\{a, b, c\} = j^2 \{b, c, a\} = j \{c, a, b\}. \tag{2.21}$$

We don't know if an analog of Ado's theorem for the Lie groups can be proved for the Z_3–case, namely, whether any ternary rule satisfying (2.21) and perhaps some 4–linear analog of Jacobi identity may be realized by embedding in some associative algebra like in the formula (2.20).

3. Z_3–graded gauge theory

Consider the simple model of a Z_3–graded algebra provided by the 3×3 complex matrices (2.17). Let us choose first one of our grade 1 differentials, e.g. the Z_3–graded commutator with η_1. One can easily prove that in our algebra

$$Im(d) = Ker(d^2), \qquad Im(d^2) = Ker(d). \tag{3.1}$$

A covariant differential can be introduced on a free left module over our algebra; in such a case, as any element of the module can be represented by an action of some algebra element on a fixed element of the module, it is enough to define our covariant differential on the algebra itself.

Let

$$DB = dB + AB \tag{3.2}$$

where $A \in \mathcal{A}_1$.

We have

$$\begin{aligned} D^2 B &= (d + A)(dB + AB) = \\ &= d^2 B + AdB + (dA)B + jAdB + A^2 B \end{aligned} \tag{3.3}$$

which can not be reduced to the left action of some element on B. However,

$$D^3B = \left[d^2A + d\left(A^2\right) + AdA + A^3\right]B \tag{3.4}$$

because of $d^3B \equiv 0$, and $1+j+j^2 = 0$. (This is to be composed with $dA+A^2$ in Z_3–graded case).

We can define the Z_3–curvature of the connection A as

$$\Omega = d^2A + d\left(A^2\right) + AdA + A^3; \tag{3.5}$$

Ω is an element of $\mathcal{A}_0$.

The natural question to ask is what are the connections that have no curvature, i.e. the flat ones? The answer is easy to compute: if

$$A = \begin{pmatrix} 0 & 0 & \alpha \\ \beta & 0 & 0 \\ 0 & \gamma & 0 \end{pmatrix},$$

then

$$\Omega = [(\alpha + 1)(\beta + 1)(\gamma + 1) - 1]\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}. \tag{3.6}$$

Consider now a gauge transformation defined by

$$B \longrightarrow U^{-1}B \tag{3.7}$$

and

$$A \longrightarrow U^{-1}AU + U^{-1}dU = A' \tag{3.8}$$

with U any non–singular 3×3 matrix of definite Z_3–grade. The covariant differential undergoes a usual transformation only if $U \in \mathcal{A}_0$; if not, it transforms as

$$U^{-1}(d + A)U = j^u d + A' \tag{3.9}$$

where $u =$ grade of U.

Nevertheless the curvature Ω transforms covariantly whatever the grade of U:

$$\Omega' = U^{-1}\Omega U. \tag{3.10}$$

The action is very poor: if $U \in \mathcal{A}_0$, $\Omega' = \Omega$; if not, the diagonal matrix Ω will undergo a cyclic permutation of its three entries.

One could imagine the generalization of action if a hermitian product could be introduced:

$$\langle\Omega|\Omega\rangle > 0 \quad \text{if} \quad \Omega \neq 0.$$

This can be done if we introduce the notion of hermiticity for our matrices.

This new 3–linear curvature may serve for defining a cubic root of a Z_2–graded derivation: suppose that the entries α, β, γ in the connection matrix A are replaced by some differential operators, and suppose that we want to keep only the linear part of (3.5). Now, as α, β, γ do not commute, our formula for Ω becomes

$$\Omega = \begin{pmatrix} (\alpha+1)(\beta+1)(\gamma+1)-1 & 0 & 0 \\ 0 & (\beta+1)(\gamma+1)(\alpha+1)-1 & 0 \\ 0 & 0 & (\gamma+1)(\alpha+1)(\beta+1)-1 \end{pmatrix}. \tag{3.6}$$

Keeping the linear part $\alpha + \beta + \gamma$ on the main diagonal means that the following identities must hold:

$$\begin{aligned} \alpha\beta + \beta\gamma + \alpha\gamma &= 0, & \alpha\beta\gamma &= 0, \\ \beta\gamma + \gamma\alpha + \beta\alpha &= 0, & \beta\gamma\alpha &= 0, \\ \gamma\alpha + \alpha\beta + \gamma\beta &= 0, & \gamma\alpha\beta &= 0. \end{aligned} \tag{3.7}$$

It is not difficult to realize these relations if we put

$$\alpha = \beta = \gamma = (\lambda\mathcal{D}_1 + \mu\mathcal{D}_2), \tag{3.8}$$

$\mathcal{D}_1$, $\mathcal{D}_2$ being the two Z_2–graded nilpotent supersymmetric derivations, λ and μ arbitrary numbers. Because of

$$\mathcal{D}_1^2 = 0, \quad \mathcal{D}_2^2 = 0, \quad \mathcal{D}_1\mathcal{D}_2 + \mathcal{D}_2\mathcal{D}_1 = 0 \tag{3.9}$$

we shall have

$$\Omega = (\lambda\mathcal{D}_1 + \mu\mathcal{D}_2)\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}. \tag{3.10}$$

This means that we have found the cubic root of the supersymmetry translations.

Note that we can generalize our scheme by considering three independent exterior derivations induced by η_1, η_2, and η_3, and the corresponding covariant derivations D_1, D_2, D_3.

There is enough space then to accommodate other Z_2–graded derivations $\mathcal{D}_{\dot{1}}$, $\mathcal{D}_{\dot{2}}$, and to find their cubic roots, too.

Our scheme can be now resumed as follows: D_1, D_2, D_3: Z_3–graded "cubic roots" of the Z_2–graded supersymmetric translations $\mathcal{D}_\alpha$, $\mathcal{D}_{\dot{\beta}}$ $(\alpha, \dot{\beta} = 1, 2)$. $\mathcal{D}_\alpha$, $\mathcal{D}_{\dot{\beta}}$: "square roots" of ordinary translations contained in the Dirac operator:

$$\mathcal{D}_\alpha\mathcal{D}_{\dot{\beta}} + \mathcal{D}_{\dot{\beta}}\mathcal{D}_\alpha = 2\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu \tag{3.11}$$

and finally, the Dirac operator being the "square root" of the Klein–Gordon operator.

It is tempting to think that the equations

$$D_1\chi = m\chi, \qquad D_2\chi = m\chi, \qquad \text{etc.} \tag{3.12}$$

are the analogs of Dirac equation for the entities that could be identified as quarks.

JORDAN FORM IN ASSOCIATIVE ALGEBRAS

GARRET SOBCZYK
Universidad Nacional Autónoma de México, FES-C
Apartado Postal #25, Cuautitlan Izcalli 54700, Estado de México
garret@unamvm1.bitnet

April 30, 1993

Abstract. The Jordan form of an element in an associative algebra over the real number field is uniquely determined by special generators of the factor algebra of its minimal polynomial.

Key words: Jordan form - Minimal polynomial - Factor Algebra

1. Introduction

Augustin Cauchy observed in 1847 that the real factor ring $\mathbb{R}[\lambda]/<\lambda^2+1>$ of the principal ideal $<\lambda^2+1>$ is isomorphic to the algebra of complex numbers. In (Sobczyk 1993), we defined a $\mathbb{C}$-algebra $\mathbb{C}\{m_1,\ldots,m_r\}$ which is isomorphic to the complex factor ring $\mathbb{C}[\lambda]/<\psi>$ for a given polynomial ψ, and used this result to find the Jordan form of an element in a $\mathbb{C}$-algebra $\mathcal{A}_{\mathbb{C}}$. In this paper, we define an $\mathbb{R}$-algebra which is isomorphic to the factor ring $\mathbb{R}[\lambda]/<\psi>$, and find the related Jordan forms.

As examples, we find relevant Jordan forms for elements having minimal polynomials of degree four or less. These canonical forms make it possible to extend the domain of any function f to a domain $D_{\mathcal{A}} \subset \mathcal{A}$, where $f: D_{\mathcal{A}} \to \mathcal{A}$, if the roots of the minimal polynomials of each $x \in D_{\mathcal{A}}$ are in D, (Sobczyk 1993).

2. The Algebra $\mathbb{R}[\lambda]/<\psi>$

Let $\{m_1,\ldots,m_r\}$ and $\{q_1,\ldots,q_s\}$ be two sets of non-decreasing positive integers with only the first h and l of them $\equiv 1$. We allow the possibility that one of these sets may be empty. For the set of r distinct real numbers $\{\lambda_i \in \mathbb{R};\ i=1,\ldots,r\}$ and the set of s distinct pairs of real numbers $\{(\alpha_j,\beta_j) \in \mathbb{R}^2;\ j=r+1,\ldots,r+s,\text{ and } \beta_j>0\}$, the most general polynomial $\psi \in \mathbb{R}[\lambda]$ can be written in the irreducible form

$$\psi \equiv \prod_{i=1}^{r}(\lambda-\lambda_i)^{m_i} \prod_{j=r+1}^{r+s} [(\lambda-\alpha_j)^2+\beta_j^2]^{q_j}. \tag{1}$$

We now extend $\mathbb{R}$ to an associative and commutative $\mathbb{R}$-algebra $\mathbb{R}\{m_1, \ldots, m_r; q_{r+1}, \ldots, q_{r+s}\}$ which we will show is isomorphic to the factor algebra $\mathbb{R}[\lambda]/<\psi>$.

DEFINITION 1. *The set of elements* $\{u_i, v_j, n_k^t, v_k n_k^t\}$, *for the range of indicies*

$$\{1 \le i \le r+s,\ r+1 \le j \le r+s,\ h+1 \le k \le r \text{ and } 1 \le t < m_k,$$

$$\text{or } r+l+1 \le k \le r+s \text{ and } 1 \le t < q_k\},$$

make up a basis of an associative and commutative $deg(\psi)$*-dimensional* $\mathbb{R}$*-algebra*

$$\mathbb{R}\{m_1, \ldots, m_r; q_{r+1}, \ldots, q_{r+s}\} \equiv span\{1, u_i, v_j, n_k^t, v_k n_k^t\}$$

where the operations of addition and multiplication of the basis elements are specified by

$$\{u_1 + \ldots + u_{r+s} = 1, \quad u_i u_j = \delta_{ij} u_i, \quad v_k^2 = -u_k, \quad v_j u_j = v_j,$$
$$n_k^{m_k - 1} \ne 0 \text{ but } n_k^{m_k} = 0, \text{ or } n_k^{q_k - 1} \ne 0 \text{ but } n_k^{q_k} = 0,\ n_k u_k = n_k\}.$$

The elements u_i make up a *partition of unity* and are *mutually annihilating idempotents.* The elements n_k are *nilpotents* with the respective *indexes* m_k or q_k. The nilpotents n_k are *projectively* related to the corresponding idempotents u_k. The *pseudo imaginaries* v_j are projectively related to the corresponding idempotents u_j. We adopt the convention that $n_i \equiv 0$, for $i = 1, \ldots, h$, and for $i = r+1, \ldots, r+l$.

We have $dim_{\mathbf{R}}\ \mathbb{R}\{m_1, \ldots, m_r; q_{r+1}, \ldots, q_{r+s}\} = \sum m_i + 2\sum q_j = deg(\psi)$. We have $\mathbb{R}\{1;\} = \mathbb{R}$, and $\mathbb{R}\{;1\} = \mathbb{C}$. Other examples are $\mathbb{R}\{1,1\}$, which are called the *real unipodal numbers* in (DGM-Sobczyk 1992, p.397), and $\mathbb{R}\{1, \ldots, 1; 1, \ldots, 1\} = \mathbb{R}^r \times \mathbb{C}^s$. It is interesting to note that $\mathbb{R}\{m_1, \ldots, m_r; q_{r+1}, \ldots, q_{r+s}\}$ can be algebraically extended to a Clifford algebra, (Sobczyk 1992, p.57).

3. The Isomorphism Theorem

The investigation of the structure of a linear operator has been considered by many authors, *e.g.*, (Gantmacher 1960, p.175-214), and (Greub 1967, p.368-427). The following Theorem and its Corollary provide new tools in this investigation.

THEOREM 1. *The algebra* $\mathbb{R}\{m_1, \ldots, m_r; q_{r+1}, \ldots, q_{r+s}\}$ *is isomorphic to the factor ring* $\mathbb{R}[\lambda]/<\psi>$ *for the polynomial* ψ *given in (1).*

Proof: Define $w \in \mathbb{R}\{m_1, \ldots, m_r; q_{r+1}, \ldots, q_{r+s}\}$ by

$$w \equiv \sum_{i=1}^{r} (\lambda_i u_i + n_i) + \sum_{j=r+1}^{r+s} (\alpha_j u_j + \beta_j v_j + n_j) \tag{2}$$

Let $\{\check{1}, \check{\lambda}, \ldots, \check{\lambda}^{-1+deg\psi}\}$ denote the standard basis of $\mathbb{R}[\lambda]/<\psi>$. The algebra isomorphism

$$\phi:\ \mathbb{R}[\lambda]/<\psi> \to \mathbb{R}\{m_1, \ldots, m_r; q_{r+1}, \ldots, q_{r+s}\}$$

is defined by

$$\check{\lambda}^k \overset{\phi}{\leftrightarrow} w^k$$

for $k = 0, 1, \ldots, -1 + deg\psi$.

The proof is completed by observing that the determinant of the mapping ϕ between the basis elements of $\mathbb{R}[\lambda]/<\psi>$ and $\mathbb{R}\{m_1, \ldots, m_r; q_{r+1}, \ldots, q_{r+s}\}$ is non-zero, which is a consequence of the fact that this determinant involves the first $m_k - 1$ linearly independent derivatives of the functions $\lambda^{m_k - 1}$, together with the first $q_k - 1$ linearly independent derivatives of the functions $\lambda^{q_k - 1}$, evaluated at the distinct roots of the polynomial ψ.

Q.E.D.

The importance of the isomorphism between these two algebras is that we can find elements in $\mathbb{R}[\lambda]/<\psi>$ which have the same multiplication rules as the basis elements $u_i, v_j, n_k \in \mathbb{R}\{m_1, \ldots, m_r; q_{r+1}, \ldots, q_{r+s}\}$. We state this in the following

COROLLARY 1. *Their exist polynomials*

$$u_i(\check{\lambda}),\ v_j(\check{\lambda}),\ n_k(\check{\lambda}) \in \mathbb{R}[\lambda]/<\psi>$$

which have the same multiplication rules as do the elements u_i, v_j, $n_k \in \mathbb{R}\{m_1, \ldots, m_r; q_{r+1}, \ldots, q_{r+s}\}$.

Proof: Define

$$u_i(\check{\lambda}) = \phi^{-1} u_i,\ v_j(\check{\lambda}) = \phi^{-1} v_j,\ n_k(\check{\lambda}) = \phi^{-1} n_k.$$

Q.E.D.

We classify the algebras $\mathbb{R}\{m_1, \ldots, m_r; q_{r+1}, \ldots, q_{r+s}\}$ into *Jordan types* according to the various possibilities for the sets $\{m_1, \ldots, m_r\}$ and $\{q_{r+1}, \ldots, q_{r+s}\}$. We say that $\mathbb{R}\{m_1, \ldots, m_r; q_{r+1}, \ldots, q_{r+s}\}$ is

TYPE a) if $\{q_{r+1},\ldots,q_{r+s}\} = \{\}$,

TYPE b) if $\{m_1,\ldots,m_r\} = \{\}$,

and of mixed

TYPE c) if $\{q_{r+1},\ldots,q_{r+s}\} \neq \{\}$, and $\{m_1,\ldots,m_r\} \neq \{\}$.

Each of these types is further broken into *Jordan subtypes* $I, II, III, \ldots$ according to

$$Subtype \;\equiv\; max\,\{m_1,\ldots,m_r;q_{r+1},\ldots,q_{r+s}\}.$$

Algebras of Jordan Type a) can be considered as a special case of the algebras $\mathbb{C}\{m_1,\ldots,m_r\}$ studied in (Sobczyk 1993), and the formulas derived there by a different method apply without modification. We shall use Theorem 1 and its Corollary to study representatives of the 17 algebras $\mathbb{R}\{m_1,\ldots,m_r;q_{r+1},\ldots,q_{r+s}\}$ for which $deg(\psi) \equiv \sum m_i + 2\sum q_j \leq 4$, and their relationship to the corresponding factor rings $\mathbb{R}[\lambda]/<\psi>$.

4. The algebras $\mathbb{R}\{m_1,\ldots,m_r;q_{r+1},\ldots,q_{r+s}\}$ for which $deg\ \psi \leq 4$.

The following is a complete list of the various possible 17 Jordan types:

$I\ a):\quad \{m_1,\ldots,m_r\} = \{1\},\ \{1,1\},\ \{1,1,1\},\ \{1,1,1,1\}$,

$II\ a):\quad \{m_1,\ldots,m_r\} = \{1,1,2\},\ \{1,2\},\ \{2,2\},\ \{2\}$,

$III\ a):\quad \{m_1,\ldots,m_r\} = \{1,3\},\ \{3\}$,

$IV\ a):\quad \{m_1,\ldots,m_r\} = \{4\}$.

$I\ b):\quad \{q_{r+1},\ldots,q_{r+s}\} = \{1,1\},\quad \{1\}$,

$II\ b):\quad \{q_{r+1},\ldots,q_{r+s}\} = \{2\}$.

$I\ c):\quad \{m_1,\ldots,m_r;q_{r+1},\ldots,q_{r+s}\} = \{1,1;1\},\quad \{1;1\}$

$II\ c):\quad \{2;1\}$.

TYPE I a): As representative of this class we choose $\{m_1,\ldots,m_r\} = \{1,1,1\}$. By theorem 1,

$$\mathbb{R}\{1,1,1\} \simeq \mathbb{R}[\lambda]/<(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)>,$$

for the algebra isomorphism

$$\check{\lambda} \overset{\phi}{\leftrightarrow} w \equiv \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 \in \mathbb{R}\{1,1,1\}.$$

Taking powers of $w \in \mathbb{R}\{1,1,1\}$ gives

$$\{u_1+u_2+u_3 = 1,\ \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = w,\ \lambda_1^2 u_1 + \lambda_2^2 u_2 + \lambda_3^2 u_3 = w^2,\}$$

which can be considered as a system of linear equations in u_i. Solving for $u_i \equiv u_i(w)$ gives

$$u_1(w) = \frac{(w-\lambda_2)(w-\lambda_3)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)},\quad u_2(w) = \frac{(w-\lambda_1)(w-\lambda_3)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)},$$

$$u_3(w) = \frac{(w-\lambda_1)(w-\lambda_2)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}. \tag{3}$$

By corollary 1, the elements

$$u_1(\check{\lambda}), u_2(\check{\lambda}), u_3(\check{\lambda}) \in \mathbb{R}[\lambda]/<(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)>$$

have the same multiplication rules as the corresponding mutually anihiliating idempotents $u_1, u_2, u_3 \in \mathbb{R}\{1,1,1\}$. A similar construction can be found in (Turnbull and Aitken 1955, p.163), and in (Herstein 1969, p.48).

TYPE III a): As representative of this class we choose $\{m_1,\ldots,m_r\} = \{1,3\}$. By theorem 1,

$$\mathbb{R}\{1,3\} \simeq \mathbb{R}[\lambda]/<(\lambda-\lambda_1)(\lambda-\lambda_2)^3>,$$

for the algebra isomorphism

$$\check{\lambda} \overset{\phi}{\leftrightarrow} \lambda_1 u_1 + \lambda_2 u_2 + n_2 \equiv w \in \mathbb{R}\{1,3\}.$$

Taking powers $w \in \mathbb{R}\{1,3\}$ gives

$$\{u_1+u_2 = 1,\ \lambda_1 u_1 + \lambda_2 u_2 + n_2 = w,$$

$$\lambda_1^2 u_1 + \lambda_2^2 u_2 + 2\lambda_2 n_2 + n_2^2 = w^2,\ \lambda_1^3 u_1 + \lambda_2^3 u_2 + 3\lambda_2^2 n_2 + 3\lambda_2 n_2^2 = w^3\},$$

which can be considered as a system of linear equations in u_1, u_2, n_2, n_2^2. Solving this system gives

$$u_1(w) = \frac{(w-\lambda_2)^3}{(\lambda_1-\lambda_2)^3},\quad u_2(w) = 1 - u_1(w),$$

$$n_2(w) = w - \lambda_1 u_1(w) - \lambda_2 u_2(w). \tag{4}$$

By corollary 1, the polynomials

$$u_1(\check{\lambda}), u_2(\check{\lambda}), n_2(\check{\lambda}) \in \mathbb{R}[\lambda]/<(\lambda - \lambda_1)(\lambda - \lambda_2)^3>$$

have the same multiplication rules as the corresponding elements $u_1, u_2, n_2 \in \mathbb{R}\{1,3\}$.

Alternative formulas for Jordan Type a) have been given in (Sobczyk 1993). We apply the more general techniques of this paper to an example of Jordan Types II c).

TYPE II c): The single member of this class is $\{q_{r+1}, \ldots, q_{r+s}\} = \{2;1\}$. By theorem 1,

$$\mathbb{R}\{2;1\} \simeq \mathbb{R}[\lambda]/<(\lambda - \lambda_1)^2[(\lambda - \alpha_2)^2 + \beta_2^2]>,$$

for the algebra isomorphism

$$\check{\lambda} \overset{\phi}{\leftrightarrow} w = \lambda_1 u_1 + n_1 + \alpha_2 u_2 + \beta_2 v_2 \equiv w \in \mathbb{R}\{2;1\}.$$

Taking powers of w leads to the system of linear equations

$$\{u_1 + u_2 = 1,\ \lambda_1 u_1 + n_1 + \alpha_2 u_2 + \beta_2 v_2 = w,$$

$$\lambda_1^2 u_1 + 2\lambda_1 n_1 + (\alpha_2^2 - \beta_2^2)u_2 + 2\alpha_2\beta_2 v_2 = w^2, \text{ and}$$

$$\lambda_1^3 u_1 + 3\lambda_1^2 n_1 + (\alpha_2^3 - 3\alpha_2\beta_2^2)u_2 + (3\alpha_2^2\beta_2 - \beta_2^3)v_2 = w^3\}$$

in u_1, v_1, u_2, v_2. Solving this system for $u_1 \equiv u_1(w)$, $n_1 \equiv n_1(w), u_2 = u_2(w)$, and $v_2 \equiv v_2(w)$ gives

$$u_1 = \frac{[(w-\alpha_2)^2 + \beta_2^2][(2w + \alpha_2 - 3\lambda_1)(\alpha_2 - \lambda_1) + \beta_2^2]}{[(\lambda_1 - \alpha_2)^2 + \beta_2^2]^2}$$

$$n_1 = \frac{(w-\lambda_1)[(w-\alpha_2)^2 + \beta_2^2][(\lambda_1 - \alpha_2)^2 + \beta_2^2]}{[(\lambda_1 - \alpha_2)^2 + \beta_2^2]^2}$$

$$u_2 = \frac{(w-\lambda_1)^2[(\lambda_1 - \alpha_2)(2w + \lambda_1 - 3\alpha_2) - \beta_2^2]}{[(\lambda_1 - \alpha_2)^2 + \beta_2^2]^2}$$

$$v_2 = \frac{(w-\lambda_1)^2\{(w-\alpha_2)[(\lambda_1 - \alpha_2)^2 - \beta_2^2] - 2(\lambda_1 - \alpha_2)\beta_2^2\}}{\beta_2[(\lambda_1 - \alpha_2)^2 + \beta_2^2]^2}$$

By corollary 1, the polynomials

$$u_1(\check{\lambda}), v_1(\check{\lambda}), u_2(\check{\lambda}), v_2(\check{\lambda}) \in \mathbb{R}[\lambda]/<(\lambda - \lambda_1)^2[(\lambda - \alpha_2)^2 + \beta_2^2]>,$$

have the same multiplication rules as the elements $u_1, u_2, n_2 \in \mathbb{R}\{1,3\}$.

5. A Matrix Example

We shall find the transition matrix and Jordan normal form of the matrix

$$a = \begin{pmatrix} 969 & -148 & -752 & 1150 \\ 40 & -5 & -32 & 46 \\ 937 & -143 & -729 & 1116 \\ -195 & 30 & 150 & -229 \end{pmatrix}$$

which has the minimal polynomial $\psi = (\lambda - 1)^2[(\lambda - 2)^2 + 3^2]$ of Jordan Type II c). Letting $w = a$ in the formulas given for this type in the previous section, we find the Jordan form

$$a = u_1 + n_1 + 2u_2 + 3v_2$$

for the matrices

$$u_1 = \begin{pmatrix} 54 & -8 & -42 & 64 \\ 39 & -5 & -30 & 46 \\ -209 & 32 & 163 & -248 \\ -177 & 27 & 138 & -210 \end{pmatrix}, n_1 = \begin{pmatrix} 118 & -18 & -92 & 140 \\ 472 & -72 & -368 & 560 \\ 59 & -9 & -46 & 70 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} -53 & 8 & 42 & -64 \\ -39 & 6 & 30 & -46 \\ 209 & -32 & -162 & 248 \\ 177 & -27 & -138 & 211 \end{pmatrix}, v_2 = \begin{pmatrix} 301 & -46 & -234 & 358 \\ -131 & 20 & 102 & -156 \\ 223 & -34 & -174 & 266 \\ -124 & 19 & 96 & -147 \end{pmatrix}$$

Denote the column vector $c_1 = [1,0,0,0]$. From this column vector we construct the generalized eigenvectors of the matrix a,

$$a_1 \equiv n_1 c_1,\ a_2 = u_1 c_1,\ a_3 = v_2 c_1,\ a_4 = u_2 c_1.$$

The transition matrix c is constructed from these eigenvectors;

$$c \equiv \{a_1, a_2, a_3, a_4\}.$$

Carrying out these calculations, we get

$$c = \begin{pmatrix} 118 & 54 & 301 & -53 \\ 472 & 39 & -131 & -39 \\ 59 & -209 & 223 & 209 \\ 0 & -177 & -124 & 177 \end{pmatrix}.$$

Using the transition matrix c we calculate the Jordan normal form

$$c^{-1}ac = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & -3 & 2 \end{pmatrix}$$

of the matrix a.

Acknowledgements

I greatly benefited from discussions with Professor Jaime Keller and want to thank him for inviting me to Universidad Nacional Autónoma de México where this work was written. Calculations were done with the help of Stephen Wolfram's MATHEMATICA. The author wishes to thank the Organizers for an excellent Conference.

References

Gantmacher, F.R.: 1960, *Matrix Theory*, Vol. 1, Chelsea Publishing Company, New York.
Greub, W.H.: 1967, *Linear Algebra*, Third Edition, Springer-Verlag, New York.
Herstein, I.N.: 1976 *Rings with Involution*, Chicago Lecture Series in Mathematics, University of Chicago Press, Chicago.
Sobczyk Garret: 1993, 'Jordan Form in Clifford Algebras', in *Clifford Algebras and their Applications in Mathematical Physics*, Proceedings of the Third International Clifford Algebras Workshop, Edited by Fred Brackx, Kluwer, Dordrect.
Sobczyk Garret: 1992, in Catto, S. and A. Rocha, ed(s)., *Differential Geometry Methods in Theoretical Physics*, , World Scientific Publ. Co., Singapore, pp.397-407.
Sobczyk Garret: 1992, 'Unipotents, Idempotents, and a Spinor Basis for Matrices', *Advances in Applied Clifford Algebras* [2], No. 1.
Turnbull H.W. and A.C. Aitken: 1969, *An Introduction to the Theory of Canonical Matrices*, Dover Publications, Inc..

UNIFIED THEORY OF SPIN AND ANGULAR MOMENTUM

LEOPOLD HALPERN
Department of Physics
Florida State University
Tallahassee, Florida 32306-3016
USA

The requirement that a physical theory be expressible in a mathematical form that is not contradictory within its system of axioms and the fact that we have no theory of truly everything, together, lend support to the assumption that we are bound to discover in any good (and therefore clear) theory "so great an absurdity that no reasonable person can believe in it." This should result from the limitations on mathematical axioms and physical assumptions we have to impose.

We consider Einstein's general theory of relativity as probably the best macroscopic physical theory and we assume here **tentatively** that an absurdity of the prescribed kind is found in its prediction, that a spherical cloud of dust of uniform density and sufficient extension will have to contract ultimately to a point, irrespective of what the physical short range interaction between the dust particles may be. The adopted point of view that situations should occur where either the physical assumptions or the mathematical axioms or both no longer apply, is in general not shared by most physicists – notably in the case of general relativity, where the Einstein-Hilbert field equations successfully predicted a large domain of physical reality.

Schrödinger's discovery of the "alarming phenomenon" of pair creation in the non-static external gravitational field of the expanding universe [E. Schrödinger, 1939] and the analogue of the effects of virtual elementary particle pairs which give rise to the Lamb shift [R. Utiyama, 1962],[R. Utiyama and B. De Witt, 1962] indicated a possible way to avoid the absurdity by supplementing the Lagrangian of general relativity: $\sqrt{g}\, R$ by a linear superposition with terms nonlinear in the curvature [L. Halpern, 1967]:

$$\sqrt{g}\,(aR + bR^2 + cR_{iklm}\,R^{iklm}) \tag{1}$$

The coefficients of this expression remain unknown – even if the removal of divergent terms had been solved unambiguously, one is not able to estimate the effects of all the elementary particles involved.

The field equations of the above modified Lagrangian admit all the vacuum solutions of general relativity. No other solutions are known that could remedy the discussed problem.

C.N. Yang suggested a gauge theory of gravitation with $GL(4,\mathbb{R})$ as the gauge group [C.N. Yang, 1974]. He considered field equations for the vacuum of the form:

$$R_{ij;k} - R_{ik;j} = 0 \qquad (i,j,k \,:\, 1,2\ldots r) \tag{2}$$

These equations admit obviously all the vacuum solutions of the Einstein-Hilbert equations; they were however shown to admit other, unphysical, solutions. Yang's equations, due to Bianchi identities, are also expressible in the form:

$$R^{hijk}{}_{;k} = 0 \tag{3}$$

Expressed in the curvature two-form of Riemannian geometry with the Christoffel connection, equation (3) may be called the Riemannian analogue of Maxwell's electromagnetic vacuum equations. Yang has somehow excluded torsion from his theory, which should however appear naturally in a gauge theory of $GL(4,\mathbb{R})$.

The present author suggested a gauge theory of gravitation (or more generally of space-time geometry) for which in its simplest form the gauge group is the subgroup $SO(3,1)$ of $GL(4,\mathbb{R})$, which results in the same curvature [L. Halpern, 1980],[L. Halpern, 1984].

The principal fibre bundle of the manifold of the anti-De Sitter group $G = SO(3,2)$ and its subgroup $H = SO(3,1)$ (proper Lorentz group) is:

$$P(G,H,G/H,\pi) \tag{4}$$

with G/H the space of left cosets and the natural projection $\pi : G \to G/H$.

With the Cartan-Killing metric γ on G the metric on the base manifold $B = G/H$ is obtained by the projection $g = \pi'\gamma$; with this metric B becomes the Anti-De Sitter universe.

The metric γ for any n-dimensional semi-simple group fulfills Einstein's equations with a cosmological member:

$$R_{uv} - \frac{1}{2}\gamma_{uv}R + \gamma_{uv}\frac{(n-2)}{8} = 0 \tag{5}$$

and in our case so does g (with a different value of the cosmological member). This suggests considering the group manifold with γ as the vacuum solution of a special kind of Kaluza-Klein theory which has the Anti-De Sitter universe as the base manifold with the metric g of space-time. There are

other vacuum solutions with non-vanishing torsion. Cartan has made the interesting suggestion to relate torsion to spin; the present formalism indicates a bold generalization of this idea, associating all properties of matter to torsion. The restrictions of this presentation do not, however, allow discussion of the lengthy general features of equation (5) in our theory with $n = 10$. We discuss here only the theory with vanishing torsion but with a right hand member of equation (5) for the matter source.

The Riemannian curvature is then the gauge field, yet expressible in terms of the metric g of the base manifold. The analogue of the charge in this theory is related to elementary particle spin which is convertible into orbital angular momentum. The connection as well as g is determined by the metric γ. Horizontal vectors are perpendicular to vertical vectors.

We express now the left hand side of equation (5) in terms of the curvature tensor on the base which we denote also by B. We work in an orthonormal frame in which horizontal vectors are labeled by capital indices $A\ldots K$ and vertical vectors by $L\ldots Q$. The Einstein summation convention is applied to each of these separately and also to indices $R\ldots Z$ which extend over all the ten components. This convention will henceforth be used without further warning.

We obtain for the vertical $M-N$ component of the expression (5) the term:

$$\frac{-9}{4}\, c^M_{AB}\, c^N_{DE}\, B^{AB}{}_{IJ}\, B^{DEIJ} - \frac{1}{2}\gamma^{MN}\left(\frac{-3}{4}B_{ABDE}\, B^{ABDE} + B - \frac{4}{3}\right) \tag{6}$$

c^M_{AB}, c^N_{DE} are structure constants of the group $G = SO(3,2)$.

The vertical component of the metric in the original version of the Kaluza-Klein theory was kept constant and thus the analogue of the above expression would not result from a variational principle and would not be considered for the field equations. Without sources it would also yield unphysical results. Requiring the present term (6) to vanish without source would simplify the remaining equations by elimination of many terms quadratic in the curvature. The gravitational collapse of a cloud of dust seems however not to be avoidable in this case.

The mixed vertical-horizontal components of equation (5) are expressible as

$$\frac{1}{2}R_{EM}\, c^M_{AB} \equiv \frac{1}{2}B_{ABE}{}^{I}{}_{;I} \tag{7}$$

this is nothing other than Yang's term (3) which thus forms part of the present equation (5). The equations (7) have in our case a source in the presence of elementary particle spin. This source is the analogue of the charge and (7) is the analogue of Maxwell's equations. A particle with spin has in the test particle approximation a vertical component of the world line's velocity vector.

The horizontal component of equation (5) is

$$B_{AE} - \frac{3}{2} B_{ADIJ}\, B_E{}^{DIJ} + \frac{1}{2}\gamma_{AE}(1 - B + \frac{3}{8} B_{DIJK}\, B^{DIJK}) \tag{8}$$

This expression contains besides the Einstein tensor of space-time with cosmological member, also nonlinear terms which are the analogue of the energy-momentum tensor of the electromagnetic gauge field. Likewise, this term is covariantly conserved only together with a term containing the interaction of the spin current with the curvature.

The appearance of our nonlinear term in (8) is nevertheless surprising because the Einstein term itself has been shown to decompose into the linear Fierz-Pauli spin two-wave operator with the energy-momentum complex of the self-interacting spin two-field as source [A. Papapetrou, 1954]. The present additional term bilinear in the curvature should be related to the interaction with the curvature of the gravitational field's own spin current.

The spherical symmetric vacuum solution of general relativity is only a solution of the horizontal and the mixed terms (7,8) but not of the vertical term (6). A similar feature occurs, as mentioned, in the five-dimensional theory. The mixed equations (7) and the nonlinear term in the horizontal equations (8) modify the solutions in the presence of matter, even if matter is spinless. Modified solutions of this kind have not yet been obtained.

Any representation of a group G is equivalent to a functional realization in terms of the parameters of the group manifold [F. Bopp and R. Haag, 1959]. Representations of integer as well as half integer spin can thus be expressed this way on the manifold of our pseudo orthogonal groups G and H.

Dirac has constructed De Sitter covariant wave equations on the manifold of the De Sitter universe, using only the generators of the group as differential operators and for half-integer spin, besides this matrix representations of the Clifford algebra in four-dimension [P.A.M. Dirac, 1935].

The scalar wave equation can in our formulation be expressed with the generators on the group manifold as:

$$\gamma^{RS}\, \bar{A}_R\, \bar{A}_S\, \psi = 0 \tag{9}$$

ψ is a trivial realization of H. Dirac showed that in the limit of increasing radius of the universe (decreasing magnitude of the cosmological member), only the terms with the generators of the De Sitter translations (in our case the $\bar{A}_E$) contribute significantly.

Our equation (9) is formulated on G, not on G/H; it is tempting to replace in this case also the generators of G for which Dirac used elements of the Clifford algebra, by generators on the group manifold. The left invariant generators A_M appear indeed suited for this purpose, as they act only on the fibres and not on the base and commute with all $\bar{A}_R$. The horizontal A_E have not the former property – but they are the only, remaining of importance after the limit. In order to have 10 generators of the required kind one would have to consider a higher dimensional group; $G = SO(4,2)$ and $H = (SO(3,2)$ with a five-dimensional base G/H on which electromagnetic theory can be formulated as a sub Kaluza-Klein theory as indicated in [L. Halpern, 1992]. In the present case of $G = SO(3,2)$ one can however use the six A_M to form the analogue of the Weyl spinor equation. The present complete formalism makes this equation appear in its covariant form as the [4,5] component of a set of equations:

$$\bar{A}_{[4,5]} = c_\epsilon{}^{bde}\, A_{[b,d]}\, \bar{A}_{[e,5]} \qquad (b,d,e\ :\ 1,2,3) \tag{10}$$

The field equations of the general theory of relativity determine even the geodesic motion of test particles [A. Papaetrou, 1951]. The same is still true for the original Kaluza-Klein theory. The conclusion for a higher dimensional generalization of such a theory is not unambiguous [L. Halpern, 1992]. In the present model we can identify P with the bundle of orthonormal frames of $SO(3,1)$ over the Anti De Sitter manifold. This allows even the following of the angular position of a spinning body in its rest frame along the world line; the vertical components of the tangent vector indicate angular velocity and linear acceleration of the rest frame. Elementary particle spin can in fact not be ascribed to such a simple model of angular momentum of a rotating rigid body [E. Schrödinger, 1930]. This justifies our assumption about the interaction of the spin charge with the curvature. Spin is however convertible into angular momentum, as demonstrated by the Einstein-DeHaas effect. We may expect it thus to have the same kind of interaction mechanism and strength with the curvature as a macroscopic spinning test body in general relativity [A. Papaetrou, 1951] and to assume here that it moves along a geodesic in ten-dimensional space. This assumption seemed to be justified by the projection on space-time of the orbits of geodesics with vertical components; it is however not correct for the spin precession. The correct non-geodesic orbit is given in [L. Halpern, 1992]. This apparent contradiction is one of the most interesting features of the model. The description seems to require modifications of the mathematical structure to be assumed.

The theory has been extended to the principal fibre bundle of the universal covering group of $SO(3,2)$ and its corresponding 6-parameter subgroup. The topology of the structure admits then a remarkable possibility of describing multiparticle systems which imply even the spin-statistics relations [L. Halpern, 1992]. This structure is somewhat of a generalization of features occurring already in the five-dimensional theory.

Acknowledgments

The author thanks the president of Florida State University and the University Foundation for their continuing support.

References

F. Bopp and R. Haag, *Z. Naturf.*, **5a**, p. 644-635, 1959.
P.A.M. Dirac, *Annals of Mathematics*, **36**, p. 657, 1935.
L. Halpern, *Arkiv F. Fysik*, **34**, p. 539, 1967.
L. Halpern, *Physics and Contemporary Needs.* **5**, Riazudding and Asghar Qadir editors, p.556, Phenum Press, 1980.
L. Halpern, *Internat. J. Theor. Phys.*, **23**, p. 848, 1984.
L. Halpern, *J. Korean Phys. Soc.*, **25**, p.S224-229, 1992.
A. Papaetrou, *Proc. R. Soc.*, **A209**, p. 248, 1951 and M. Mathisson, *Acta Physica Poland*, **6**, p. 167, 1937 and J. Lubański, ibid, p. 356.
A. Papapetrou, *Proc. R. Irish Acad.*, **52**, p. 1683, 1954.
R. Pavell, *Phys. Rev. Lett.*, **17**, p. 1114, 1975.
E. Schrödinger, *Sitzber. Pruss. Acad. Science*, **24**, 1930.
E. Schrödinger, *Physica*, **6**, p. 899-912, 1939 and *Proc. R. Irish Acad.*, **A46**, p. 25-47, 1940.
R. Utiyama, *Phys. Rev.*, **125**, p.1741, 1962.
R. Utiyama and B. De Witt, *J. Math. Phys.* **3**, p. 608, 1962.
C.N. Yang, *Phys. Rev. Lett.*, **33**, p. 445, 1974.

NOETHERIAN SYMMETRIES IN PARTICLE MECHANICS AND CLASSICAL FIELD THEORY

DAN RADU GRIGORE
Department of Theoretical Physics,
Institute of Atomic Physics
Bucharest-Magurele, P.O. Box MG6,
Romania

Abstract. A geometric generalization of the first-order Lagrangian formalism is proposed, following the original ideas of Poincaré and Cartan and the extension to field theory due to Krupka, Betounes and Rund.

The method is particulary suited for the study of Noetherian symmetries. This point is proved by explicit study of systems with gauge groups of symmetries.

1. Introduction

There are many advantages in using a geometric framework for the Lagrangian formalism. Most of the papers are based on the Poincaré-Cartan 1-form, but it was also realized that the formalism became more natural working with a 2-form having as associated system exactly the Euler-Lagrange equations (see e. g. [1]). This 2-form is defined on the projective tangent bundle over the space-time manifold of the system, called by Souriau [2] the evolution space of the system and can be used for an alternative definition of the phase space. This formulation also allows a very elegant treatment of the Noetherian symmetries and of the connection with the symplectic action of groups appearing in the Hamiltonian formalism .

In this paper we will present a generalization of these ideas to classical field theory closely related to the point of view of Krupka, Betounes and Rund [3-6]. The most important property of this generalization is the possibility of expressing in a geometric way the usual notion of Noetherian symmetry. This definition is very suitable for practical computations. Namely, we can solve, in principle, the classification problem of Lagrangian systems with Noetherian groups of symmetry for many important groups appearing in theoretical physics.

The general theory will be presented in Section 2 and in Section 3 we will illustrate the method on the case of Abelian gauge theories. The details of computation will appear elsewere [8].

2. A Geometric Formulation of the Lagrangian Formalism

2.1 Let S be a differentiable manifold of dimension $n+N$. The first order Lagrangian formalism is based on an auxiliary object, namely the bundle of 1-jets of n-dimensional submanifolds of S, denoted by $J^1_n(S)$. This differentiable manifold is, by definition:

$$J^1_n(S) \equiv \cup_{p\in S} J^1_n(S)_p$$

where $J^1_n(S)_p$ is the manifold of n-dimensional linear subspaces of the tangent space $T_p(S)$ at S in the point $p \in S$. This manifold is naturally fibered over S and we denote by π the canonical projection. Let us construct charts on $J^1_n(S)$ adapted to this fibered structure. We first choose a local coordinate system (x^μ, ψ^A) on the open set $U \subseteq S$; here $\mu = 1, ..., n$ and $A = 1, ..., N$. Then on the open set $V \subseteq \pi^{-1}(U)$, we shall choose the local coordinate system $(x^\mu, \psi^A, \chi^A{}_\mu)$, defined as follows: if (x^μ, ψ^A) are the coordinates of $p \in U$, then the n-dimensional plane in $T_p(S)$ corresponding to $(x^\mu, \psi^A, \chi^A{}_\mu)$ is spanned by the tangent vectors:

$$\frac{\delta}{\delta x^\mu} \equiv \frac{\partial}{\partial x^\mu} + \chi^A{}_\mu \frac{\partial}{\partial \psi^A}. \tag{2.1}$$

We will systematically use the summation convention over the dummy indices.

By an *evolution space* we mean any (open) subbundle E of $J^1_n(S)$.

2.2 Let us define for a given evolution space E:

$$\Lambda_{LS} \equiv \{\sigma \in \wedge^{n+1}(J^1_n(S)) | i_{Z_1} i_{Z_2}\sigma = 0, \forall Z_i,\ s.\ t.\ \pi_* Z_i = 0, i = 1,2\}. \tag{2.2}$$

Next, one defines the local operator K on Λ_{LS} by:

$$K\sigma \equiv i_{\frac{\delta}{\delta x^\mu}} i_{\frac{\partial}{\partial \chi^A{}_\mu}} (\delta\psi^A \wedge \sigma), \tag{2.3}$$

where

$$\delta\psi^A \equiv d\psi^A - \chi^A{}_\mu dx^\mu \ ,$$

and proves that K is in fact globally defined [7].

We say that $\sigma \in \Lambda_{LS}$ is a *Lagrange-Souriau form on* E if it verifies

$$K\sigma = 0. \tag{2.4}$$

and is also closed:

$$d\sigma = 0. \tag{2.5}$$

A *Lagrangian system over* S is a couple (E, σ) where $E \subseteq J^1_n(S)$ is some evolution space over S and σ is a Lagrange-Souriau form on E.

It is natural to call the Lagrangian systems (E_1, σ_1) and (E_2, σ_2) over the same manifold S *equivalent* if there exists $\alpha \in Diff(S)$ such that $\dot\alpha(E_1) = E_2$ and:

$$\dot\alpha^* \sigma_2 = \sigma_1. \tag{2.6}$$

Here $\dot\alpha \in Diff(J^1_n(S))$ is the natural lift of α.

2.3 The purpose of the Lagrangian formalism is to describe *evolutions* i.e. immersions $\Psi : M \to S$, where M is some n-dimensional manifold, usually interpreted as the space-time manifold of the system.

Let us note that frequently, one supposes that S is fibered over M, but we do not need this additional restriction in developing the general formalism. Let us denote by $\dot\Psi : M \to J^1_n(S)$ the natural lift of Ψ. If (E, σ) is a Lagrangian system over S, we say that $\Psi : M \to S$ verifies the *Euler-Lagrange equations* if:

$$\dot\Psi^* i_Z \sigma = 0. \tag{2.7}$$

for any vector field Z on E.

2.4 By a *symmetry* of the Euler-Lagrange equations we understand a map $\phi \in Diff(S)$ such that if $\Psi : M \to S$ is a solution of these equations, then $\phi \circ \Psi$ is a solution of these equations also.

It is easy to see that if $\phi \in Diff(S)$ is such that $\dot\phi$ leaves E invariant and:

$$\dot\phi^* \sigma = \sigma. \tag{2.8}$$

then it is a symmetry of the Euler-Lagrange equations (2.7). We call the symmetries of this type *Noetherian symmetries* for (E, σ).

If a group G act on S: $G \ni g \mapsto \phi_g \in Diff(S)$ then we say that G is a *group of Noetherian symmetries* for (E, σ) if for any $g \in G$, ϕ_g is a Noetherian symmetry. In particular we have:

$$(\dot\phi_g)^* \sigma = \sigma. \tag{2.9}$$

It is considered of physical interest to solve the following classification problem: given the manifold S with an action of some group G on S, find all Lagrangian systems (E, σ) where $E \subseteq J^1_n(S)$ is on open subset and G is a group of Noetherian symmetries for (E, σ). This goal will be achieved by solving simultaneously (2.4), (2.5) and (2.9) in local coordinates and then investigating the possibility of globalizing the result.

2.5 Now we make the connection with the usual Lagrangian formalism. We can consider that the open set $V \subseteq \pi^{-1}(U)$ is simply connected by choosing it small enough.

The first task is to exhibit somehow a Lagrangian. This can be done as follows [7]. Form (2.5) one has that

$$\sigma = d\theta. \tag{2.10}$$

Then one can show that by eventually redefining θ, one can exhibit it in the form:

$$\theta \equiv \varepsilon_{\mu_1,\dots,\mu_n} \sum_{k=0}^{n} \frac{1}{k!} C_n^k L^{\mu_1,\dots,\mu_k}_{A_1,\dots,A_k} \delta\psi^{A_1} \wedge \dots \wedge \delta\psi^{A_k} \wedge dx^{\mu_{k+1}} \wedge \dots \wedge dx^{\mu_n}. \quad (2.11)$$

where the smoth functions $L^{\mu_1,\dots,\mu_k}_{A_1,\dots,A_k}$ are completely antisymmetric in the upper indices and also in the lower indices.

Finally, using the structure equation (2.4) one shows that there exists a smooth function $L : V \to R$ such that:

$$L^{\mu_1,\dots,\mu_k}_{A_1,\dots,A_k} \equiv \frac{1}{k!} \sum_{\sigma \in P_k} (-1)^{|\sigma|} \frac{\partial^k L}{\partial \chi^{A_1}{}_{\mu_{\sigma(1)}} \dots \partial \chi^{A_k}{}_{\mu_{\sigma(k)}}}. \quad (2.12)$$

(P_k is the permutation group of the numbers $1,\dots,k$) and $|\sigma|$ is the signature of σ). L is called a *local Lagrangian.* The formulae (2.11)-(2.12) are exactly those of [3]-[6]. If σ is of the form (2.10)-(2.12) then we denote it by σ_L.

Now one can easily show the following facts. If $\sigma = \sigma_L$, then

1) the local form of the Euler-Lagrange equations (2.7) coincides with the usual one.

2) the Euler-Lagrange equations (2.7) are trivial *iff* $\sigma_L = 0$.

3) let us suppose now for the moment that σ is exact i.e. verifies (2.10) on the whole E. Then one can define the action functional (see e.g. [8]) and establish that the definition (2.8) is equivalent to the usual definition for the Noetherian symmetries.

3. Abelian Gauge Theories

3.1 We consider only the case without matter fields. If M the n-dimensional Minkowski space, then in the general framework of Section 2, we take $S = M \times M$ with coordinates (x^μ, A^ν); $\mu, \nu = 1, \dots, n$. A^ν are the components of the electromagnetic potential.

In the global coordinates $(x^\mu, A^\nu, \chi^\nu{}_\mu)$ on $E \equiv J^1_n(S)$, the expression of any $\sigma \in \Lambda_{LS}$ is:

$$\sigma = \varepsilon_{\mu_1,\dots,\mu_n} \sum_{k=0}^{n} \frac{1}{k!} C_n^k \sigma^{\mu_0,\dots,\mu_k}_{\nu_0,\dots,\nu_k} d\chi^{\nu_0}{}_{\mu_0} \wedge \delta A^{\nu_1} \wedge \dots \wedge \delta A^{\nu_k} \wedge dx^{\mu_{k+1}} \wedge \dots \wedge dx^{\mu_n} +$$

$$\varepsilon_{\mu_1,\dots,\mu_k} \sum_{k=0}^{n} \frac{1}{(k+1)!} C_n^k \tau^{\mu_1,\dots,\mu_k}_{\nu_0,\dots,\nu_k} \delta A^{\nu_0} \wedge \dots \wedge \delta A^{\nu_k} \wedge dx^{\mu_k} \wedge \dots \wedge dx^{\mu_n}. \quad (3.1)$$

with $\sigma^{\dots}_{\dots}$ and $\tau^{\dots}_{\dots}$ having apropriate antisymmetry properties and:

$$\delta A^\nu \equiv dA^\nu - \chi^\nu{}_\mu dx^\mu. \quad (3.2)$$

The structure relations (2.4) is in this case:

$$\sum_{i,j=1}^{k} (-1)^{i+j} \sigma^{\mu_j,\mu_1,\dots,\hat{\mu_j},\dots,\mu_k}_{\nu_i,\nu_1,\dots,\hat{\nu_i},\dots,\nu_k} = 0. \quad (3.3)$$

for $k = 1, \dots, n$ and the closedness condition (2.5) gives:

$$\frac{\partial \sigma^{\mu_0,\dots,\mu_k}_{\nu_0,\dots,\nu_k}}{\partial \chi^{\nu_{k+1}}{}_{\mu_{k+1}}} - \sigma^{\mu_0,\dots,\mu_{k+1}}_{\nu_0,\dots,\nu_{k+1}} - (\mu_0\nu_0 \leftrightarrow \mu_{k+1}\nu_{k+1}) = 0. \quad (3.4)$$

$$\frac{\delta \sigma^{\zeta,\rho,\mu_1,\dots,\mu_k}_{\omega,\nu_0,\dots,\nu_k}}{\delta x^\rho} - \sum_{i=0}^{k} (-1)^i \frac{\partial \sigma^{\zeta,\mu_1,\dots,\mu_k}_{\omega,\nu_0,\dots,\hat{\nu_i},\dots,\nu_k}}{\partial A^{\nu_i}} + \frac{\partial \tau^{\mu_1,\dots,\mu_k}_{\nu_0,\dots,\nu_k}}{\partial \chi^\omega{}_\zeta} - \tau^{\mu_1,\dots,\mu_k,\zeta}_{\nu_0,\dots,\nu_k,\omega} = 0. \quad (3.5)$$

$$\frac{\delta \tau^{\mu_0,\dots,\mu_k}_{\nu_0,\dots,\nu_{k+1}}}{\delta x^{\mu_0}} + \sum_{i=0}^{k+1} (-1)^i \frac{\partial \tau^{\mu_1,\dots,\mu_k}_{\nu_0,\dots,\hat{\nu_i},\dots,\nu_{k+1}}}{\partial A^{\nu_i}} = 0. \quad (3.6)$$

for $k = 0, \dots, n$.

Here:

$$\frac{\delta}{\delta x^\mu} \equiv \frac{\partial}{\partial x^\mu} + \chi^\nu{}_\mu \frac{\partial}{\partial A^\nu}. \quad (3.7)$$

3.2 We now impose the gauge invariance of the theory. If $\xi : M \to R$ is an infinitesimal gauge transformation let us define the following transformation on S:

$$\phi_\xi(x^\mu, A^\nu) = (x^\mu, A^\nu + (\partial^\nu \xi)(x)). \quad (3.8)$$

We say that the system is *gauge invariant iff*:

$$(\dot{\phi}_\xi)^* \sigma = \sigma. \quad (3.9)$$

One can prove that in the particular case when σ is exact and we have an action functional, this definition coincides with the usual definition of gauge invariance.

We also impose Poincaré invariance; the action of $P^\uparrow_+$ on S is:

$$\phi_{\Lambda,a}(x, A) = (\Lambda x + a, \Lambda A). \quad (3.10)$$

and we require that $\phi_{\Lambda,a}$ are Noetherian symmetries for any $(\Lambda, a) \in P^\uparrow_+$:

$$\dot{\phi}_{\Lambda,a} \sigma = \sigma. \quad (3.11)$$

3.3 One can easily translate (3.9) and (3.11) into conditions on the coefficients $\sigma^{\dots}_{\dots}$ and $\tau^{\dots}_{\dots}$ appearing in (3.1); namely these functions are dependent only of the field strength variable:

$$F_{\mu\nu} \equiv \chi_{\mu\nu} - \chi_{\nu\mu}, \quad (3.12)$$

and they are Lorentz invariant tensor functions.

From (3.3)-(3.6) it is clear that the functions $\sigma^{\cdots}_{\cdots}$ are constrained only by (3.3) and (3.4). Using induction we can prove that there exist a F-dependent function L_{YM} such that:

$$\sigma^{\mu_0,\ldots,\mu_k}_{\nu_0,\ldots,\nu_k} = \frac{\partial(L_{YM})^{\mu_1,\ldots,\mu_k}_{\nu_1,\ldots,\nu_k}}{\partial F^{\nu_0}{}_{\mu_0}} - (L_{YM})^{\mu_0,\ldots,\mu_k}_{\nu_0,\ldots,\nu_k}. \quad (3.13)$$

where:

$$(L_{YM})^{\mu_1,\ldots,\mu_k}_{\nu_1,\ldots,\nu_k} = \frac{1}{k!}\sum_{\sigma\in P_k}\frac{\partial^k L_{YM}}{\partial F^{\nu_1}{}_{\mu_{\sigma(1)}}\ldots\partial F^{\nu_k}{}_{\mu_{\sigma(k)}}}. \quad (3.14)$$

The Lorentz invariance of the tensor function $\sigma^{\cdots}_{\cdots}$ can be used to show that, without modifying σ, one can redefine L_{YM} such that it is a Lorentz invariant function:

$$L_{YM}(\Lambda\cdot F) = L_{YM}(F). \quad (3.15)$$

This fact is of cohomological nature.

3.4 Let us now turn to the functions $\tau^{\cdots}_{\cdots}$. We have from the structure equations (3.3)-(3.6) only:

$$\frac{\partial\tau^{\mu_1,\ldots,\mu_k,\nu_0,\ldots,\nu_k}}{\partial F_{\nu_{k+1}\mu_{k+1}}} = \tau^{\mu_1,\ldots,\mu_{k+1},\nu_0,\ldots,\nu_{k+1}}. \quad (3.16)$$

(for $k=0,\ldots,n$). The tensor $\tau^{\mu_1,\ldots,\mu_{k+1},\nu_0,\ldots,\nu_{k+1}}$ can be shown to be completely antisymmetric in all indices. Let us use the notation $m\equiv[\frac{n}{2}]$. Now one easily integrates (3.16) and gets that:

$$\tau^{\mu_1,\ldots,\mu_k,\nu_0,\ldots,\nu_k} = \sum_{p=k}^{m}\frac{1}{(p-k)!2^{p-k}}C^{\mu_1,\ldots,\mu_p,\nu_0,\ldots,\nu_p}\prod_{i=k+1}^{p}F_{\nu_i\mu_i}. \quad (3.17)$$

where $C^{\cdots}$ are some constants which are completely antisymmetric in all indices.

The Lorentz invariance of the tensor function $\tau^{\cdots}$ is equivalent to the Lorentz invariance of the tensors $C^{\cdots}$, so we get two distinct cases: (a) if $n=2m$ the tensors $\tau^{\cdots}$ are zero for any k, so we have $\sigma=\sigma_{L_{YM}}$; (b) if $n=2m+1$ then:

$$C^{\mu_1,\ldots,\mu_m,\nu_0,\ldots,\nu_m} = \kappa\varepsilon^{\mu_1,\ldots,\mu_m,\nu_0,\ldots,\nu_m}. \quad (3.18)$$

for some $\kappa\in R$ and all the others tensors $C^{\cdots}$ are zero.

If we define:

$$L_{CS}(x,A,\chi) = \frac{\kappa}{(m+1)!2^m}A^{\nu_0}\varepsilon_{\mu_1,\ldots,\mu_k,\nu_0,\ldots,\nu_k}\prod_{i=1}^{m}F^{\nu_i\mu_i}. \quad (3.19)$$

then in this case we have: $\sigma=\sigma_{L_{YM}+L_{CS}}$.

Here L_{CS} is the usual expression of the Chern-Simons Lagrangian.

4. Conclusions

The method of analysing Lagrangian systems with group of Noetherian symmetries illustrated above can be succesfully used for other interesting physical situations: non-Abelian gauge theories [8], Galilean invariant many-particles systems [9], string theory [10], gravitation theory [11], etc.

References

[1] J. Klein, Ann. Inst. Fourier (Grenoble) **12** (1962) 1-124
[2] J. M. Souriau, "Structure des Systémes Dynamiques", Dunod, Paris, 1970
[3] D. Krupka, Czech. Math. Journ. **27** (1977) 114-118
[4] D. Betounes, Phys. Rev. **D 29** (1984) 599-606
[5] D. Betounes, J. Math. Phys. **28** (1987) 2347-2353
[6] H. Rund, Lect. Notes in Pure and Appl. Math. **100** (1985) 455-469
[7] D. R. Grigore and O. T. Popp, " On the Souriau-Lagrange Form in Classical Field Theory", submitted for publication
[8] D. R. Grigore, "A Generalized Lagrangian Formalism in Particle Mechanics and Classical Field Theory", to appear in Fortschr. der Physik
[9] D.R.Grigore, "Generalized Lagrangian Dynamics and Noetherian Symmetries", to appear in International J. Mod. Phys. **A**
[10] D. R. Grigore, J. Phys. **A 25** (1992) 3797-3811
[11] D. R. Grigore, Class. Quant. Gravity **9** (1992) 1555-1571

LIÉNARD-WIECHERT YANG-MILLS FIELDS

K. PAUL TOD
Mathematical Institute and St John's College
Oxford

Abstract. We consider the problem of defining Liénard-Wiechert fields in Yang-Mills theory. Trautman (1981a,b) following Arodź (1978) defined them by the form of the potential, but we choose to define them in terms of the principal spinors of the curvature. This leads to essentially the same results as those of Trautman for gauge group $SU(n)$, but some different solutions arise for gauge group $SL(n,\mathbf{R})$.

1. Introduction

The Liénard-Wiechert solution of Maxwell's equations is the retarded solution corresponding to an electric monopole moving on an arbitrary world-line in Minkowski space. The charge is conserved, but the solution radiates energy at a rate proportional to the acceleration of the world-line. It is natural to ask if there is a solution of the (non-linear) Yang-Mills equations which corresponds to the Liénard-Wiechert solution in a suitable sense. One might then seek to see if it is possible to define conserved charges or whether charges can be radiated, and also whether the non-linearity of the Yang-Mills equations causes the world-line to be restricted in any way.

In such an investigation, one needs to decide which characterization of the Liénard-Wiechert solution to choose for generalization to the Yang-Mills theory. In his study of this problem, Trautman (1981a,b; see also Tafel and Trautman 1983) chose to characterize the field by its potential, and, following Arodź (1978), to take a form of Yang-Mills potential which generalized that. He concluded that, if the gauge group is any compact, semi-simple group (and so in particular if it is $SU(n)$), then there are conserved "colour" charges and the Yang-Mills Liénard-Wiechert field is a product of Maxwellian Liénard-Wiechert fields. However, for other gauge groups, he found solutions which do radiate colour charge.

It is possible to characterize the Maxwellian Liénard-Wiechert fields in spinorial terms by their principal null directions (Lind and Newman 1974): briefly, one principal spinor of the Maxwell spinor must be tangent to a twist-free, shear-free congruence of null geodesics (see e.g. Penrose and Rindler 1984 for the definitions of these terms). Such a congruence is necessarily generated by the future (or past) null cones springing from an arbitrary world-line in Minkowski space, and the Liénard-Wiechert field is based on this world-line. In this paper, my aim is to explore the consequences of using this other characterization of the Maxwellian Liénard-Wiechert fields as the characterization of Liénard-Wiechert solutions in Yang-Mills theory also. This study leads to a more general class of solutions, depending on the gauge group chosen, but the conclusions for $SU(n)$ are the same as Trautman's.

In the Section 2, I review the coordinate and tetrad system of Held et al (1970) and Lind and Newman (1974) which is adapted to a twist-free, shear-free congruence in Minkowski space. I solve the radial parts of the Yang-Mills equations in the Newman-Penrose formalism and obtain a reduced system of equations. In Section 3, I solve the reduced system. The spin-coefficients from Section 2 and the Yang-Mills equations in the NP formalism are given in an Appendix.

2. The Coordinate System and Tetrad

We begin by describing the coordinate and tetrad system of Held et al (1970) which is adapted to an arbitrary time-like world-line Γ in Minkowski space, M. Suppose Γ is given parametrically by

$$x^a = z^a(\tau) \tag{2.1}$$

where τ is the proper-time along Γ, so that

$$\eta_{ab}\dot{z}^a\dot{z}^b = 1 \tag{2.2}$$

where dot denotes differentiation with respect to τ. Next we coordinatize the null cone by θ, ϕ according to

$$L^a = (1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \tag{2.3}$$

Define

$$V = \eta_{ab}\dot{z}^a L^b \tag{2.4}$$

then the coordinate system (τ, r, θ, ϕ) adapted to Γ is defined implicitly by

$$x^a = z^a(\tau) + \frac{r}{V}L^a(\theta,\phi). \tag{2.5}$$

Here τ labels the future null-cones springing from Γ, θ and ϕ label the null geodesic generators of these null-cones, and r is an affine paramter along each generator. The generators taken together constitute the null geodesic congruence defined by Γ.

In these coordinates, the metric is

$$ds^2 = (1 - 2r\frac{\dot{V}}{V})d\tau^2 + 2d\tau dr - \frac{r^2}{V^2}(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.6}$$

A suitable null tetrad is defined by

$$\ell = d\tau;\ n = (1 - 2r\frac{\dot{V}}{V})d\tau + dr;\ m = \frac{-r}{V\sqrt{2}}(d\theta + i\sin\theta d\phi) \tag{2.7}$$

or in covariant form:

$$\ell = \frac{\partial}{\partial r};\ n = \frac{\partial}{\partial \tau} - \frac{1}{2}\left(1 - 2r\frac{\dot{V}}{V}\right)\frac{\partial}{\partial r};\ m = \frac{V}{r\sqrt{2}}\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right). \tag{2.8}$$

As usual, we write (o^A, ι^A) for the spinor dyad corresponding to this null tetrad. It is a straightforward matter to calculate the NP spin-coefficients for this dyad, and they are given in the Appendix.

Given a choice of gauge group G, a Yang-Mills field is a connection on a principal G-bundle, B, over M. For simplicity, I will assume that B is trivial, which corresponds to the assumption that there is no magnetic charge. Then the connection may be represented by a globally-defined, Lie-algebra-valued 1-form A_a. The gauge freedom in the potential is given by

$$A_a \to \hat{A}_a = (gA_a + \nabla_a g)g^{-1} \text{ for } g: M \to G. \tag{2.9}$$

We exploit this freedom to set $\ell^a A_a$ equal to zero. This requires the solution of the equation

$$\frac{\partial g}{\partial r} = -g(\ell^a A_a) \tag{2.10}$$

and then we may suppose that

$$\ell^a A_a \equiv \gamma_{oo'} = 0. \tag{2.11}$$

The curvature of the Yang-Mills connection is represented by a Lie-algebra-valued 2-form F_{ab}, which can be decomposed into spinors as

$$F_{ab} = \chi_{AB}\epsilon_{A'B'} + \tilde{\chi}_{A'B'}\epsilon_{AB} \tag{2.12}$$

The relation between the spinor fields χ_{AB} and $\tilde{\chi}_{A'B'}$ depends on the choice of gauge group G. Given the gauge condition (2.11), the potential can be expanded in the tetrad (2.7,8) as

$$A_a = \gamma_{11'}\ell_a - \gamma_{10'}m_a - \gamma_{01'}\bar{m}_a. \tag{2.13}$$

Again, there will be relations among the components of A_a depending on the choice of G.

The property of Liénard-Wiechert fields which we are regarding as characteristic is that the spinor o^A is a principal spinor of χ_{AB} (and similarly for $\bar{o}^{A'}$ and $\tilde{\chi}_{A'B'}$). In components:

$$\chi_{AB}o^A o^B \equiv \chi_0 = 0;\; \tilde{\chi}_{A'B'}\bar{o}^{A'}\bar{o}^{B'} \equiv \tilde{\chi}_0 = 0 \tag{2.14}$$

while

$$\chi_{AB}o^A \iota^B \equiv \chi_1;\; \tilde{\chi}_{A'B'}\bar{o}^{A'}\bar{\iota}^{B'} \equiv \tilde{\chi}_1$$

$$\chi_{AB}\iota^A \iota^B \equiv \chi_2;\; \tilde{\chi}_{A'B'}\bar{\iota}^{A'}\bar{\iota}^{B'} \equiv \tilde{\chi}_2. \tag{2.15}$$

The Yang-Mills equations in the NP formalism are given in the Appendix with the specializations (2.11) and (2.14). We proceed to solve these.

Equation (A.3a) implies

$$\chi_1 = \frac{1}{r^2}\chi(\tau,\theta,\phi);\; \tilde{\chi}_1 = \frac{1}{r2}\tilde{\chi}(\tau,\theta,\phi) \tag{2.16}$$

for some χ, $\tilde{\chi}$. Next (A.2a) implies

$$\gamma_{01'} = \frac{V}{r}\gamma(\tau,\theta,\phi);\; \gamma_{10'} = \frac{V}{r}\tilde{\gamma}(\tau,\theta,\phi) \tag{2.17}$$

for some γ, $\tilde{\gamma}$. From (A.3c) we find

$$\delta_0\chi + [\chi,\gamma] = 0;\; \bar{\delta}_0\tilde{\chi} + [\tilde{\chi},\tilde{\gamma}] = 0 \tag{2.18}$$

where

$$\delta_0 = \frac{r}{V}\delta = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right) \tag{2.19}$$

From (A.2b) we find

$$\gamma_{11'} = -\frac{1}{r}(\chi + \tilde{\chi}) \tag{2.20}$$

and

$$\bar{\eth}_0\gamma - \eth_0\tilde{\gamma} + [\gamma,\tilde{\gamma}] - \frac{\chi}{V^2} + \frac{\tilde{\chi}}{V^2} = 0 \tag{2.21}$$

where $\eth_0$ is the "eth" of Newman and Penrose (1966; see also Penrose and Rindler 1984), here defined on a spin-weight s quantity by

$$\eth_0 = \delta_0 - \frac{s}{\sqrt{2}}\cot\theta \tag{2.22}$$

From (A.2c) we find

$$\chi_2 = -\frac{V}{r}\dot{\tilde{\gamma}} - \frac{V}{r^2}(\bar{\delta}_0\chi + [\chi,\tilde{\gamma}])$$

$$\tilde{\chi}_2 = -\frac{V}{r}\dot{\gamma} - \frac{V}{r^2}(\delta_0\tilde{\chi} + [\tilde{\chi},\gamma]) \tag{2.23}$$

so that (2.16,17,20,23) give the radial dependence of all quantities in terms of $\gamma, \tilde{\gamma}, \chi$ and $\tilde{\chi}$, which in turn are functions on $S^2 \times \mathbf{R}$ coordinatized by (θ,ϕ) and τ, respectively. Equation (A.3b) is now identically satisfied, while (A.3d) reduces to

$$\eth_0\dot{\tilde{\gamma}} + [\dot{\tilde{\gamma}},\gamma] + \left(\frac{\chi}{V^2}\right)^{\cdot} = 0$$

$$\bar{\eth}_0\dot{\gamma} + [\dot{\gamma},\tilde{\gamma}] + \left(\frac{\tilde{\chi}}{V^2}\right)^{\cdot} = 0. \tag{2.24}$$

The reduced Yang-Mills equations are therefore (2.18) and (2.21), which may be viewed as constraints, and (2.24) which is the evolution (and which preserves (2.21)). There is residual gauge freedom, namely:

$$\chi \to \hat{\chi} = g\chi g^{-1};\; \tilde{\chi} \to \hat{\tilde{\chi}} = g\tilde{\chi}g^{-1}$$

$$\gamma \to \hat{\gamma} = (g\gamma + \eth_0 g)g^{-1};\; \tilde{\gamma} \to \hat{\tilde{\gamma}} = (g\tilde{\gamma} + \bar{\eth}_0 g)g^{-1} \tag{2.25}$$

where $g = g(\theta,\phi)$ is a function from $S^x 2$ to the gauge-group G.

For the gauge group $SU(n)$, γ, $\tilde{\gamma}$, χ and $\tilde{\chi}$ are $n \times n$ complex matrices with

$$\tilde{\chi} = -\chi^\dagger;\; \tilde{\gamma} = -\gamma^\dagger;\; tr\chi = 0;\; tr\gamma = 0 \tag{2.26}$$

while for gauge group $SL(n,\mathbf{R})$ γ, $\tilde{\gamma}$, χ and $\tilde{\chi}$ are $n \times n$ complex matrices with

$$\tilde{\chi} = \bar{\chi};\; \tilde{\gamma} = \bar{\gamma};\; tr\chi = 0;\; tr\gamma = 0. \tag{2.27}$$

In the next Section, we set about solving the remaining equations.

3. Solving the Reduced System

We begin by considering the equations

$$\eth_0 h + h\gamma = 0;\; \bar{\eth}_0\tilde{h} + \tilde{h}\tilde{\gamma} = 0 \tag{3.1}$$

where h, $\tilde{h}$, γ and $\tilde{\gamma}$ are $n \times n$ complex matrices. We may interpret each of the operators $\bar{\eth}_0 + \gamma$ and $\bar{\eth}_0 + \tilde{\gamma}$ as defining a $\bar{\partial}$-operator on the Yang-Mills bundle B, and then (3.1) is the condition for h and $\tilde{h}$ respectively to be n holomorphic sections. We can therefore solve these equations at each fixed τ for non-singular matrices h and $\tilde{h}$, provided the Yang-Mills bundle B is trivial as a holomorphic bundle at each τ. We will make this assumption, thereby extending the assumption that B is trivial as a smooth bundle, which was made in Section 2 to eliminate magnetic charges.

Having solved (3.1), we find that (2.18) is solved by

$$\chi = h^{-1}Ah; \ \tilde{\chi} = \tilde{h}^{-1}\tilde{A}\tilde{h} \tag{3.2}$$

where

$$\eth_0 A = 0; \ \eth_0 \tilde{A} = 0 \tag{3.3}$$

If χ and $\tilde{\chi}$ are to be globally regular on S^2 then (3.3) implies that A and $\tilde{A}$ are functions only of τ. They have the character of matrices of charges, which, at this stage, can apparently change with time. Note that the invariants of χ and $\tilde{\chi}$ are the same as those of A and $\tilde{A}$ respectively, so that these are also functions only of τ.

Now we introduce $\omega = h\tilde{h}^{-1}$, to find that (2.21) becomes

$$\bar{\eth}_0\, \eth_0\, \omega - \bar{\eth}_0 \omega \omega^{-1}\, \eth_0\, \omega + \frac{1}{V^2}(A\omega - \omega\tilde{A}) = 0. \tag{3.4}$$

As we shall see below, this equation has something of the character of an eigenvalue equation on S^2, with the matrices $A, \tilde{A}$ of charges as the eigenvalues.

The evolution equations (2.24) are solved with the aid of

$$p = -\dot{h}h^{-1}; \ \tilde{p} = -\dot{\tilde{h}}\tilde{h}^{-1} \tag{3.5}$$

as

$$\bar{\eth}_0(\omega^{-1}\, \eth_0\, p\, \omega) + \left(\frac{\tilde{A}}{V^2}\right)^{\cdot} + \left[\tilde{p}, \frac{\tilde{A}}{V^2}\right] = 0$$

$$\eth_0(\omega \bar{\eth}_0 \tilde{p}\, \omega{-}1) + \left(\frac{A}{V^2}\right)^{\cdot} + \left[p, \frac{A}{V^2}\right] = 0. \tag{3.6}$$

At this point we may summarize the freedom available. Gauge transformations (2.25) have the effect

$$h \to hg^{-1}; \ \tilde{h} \to \tilde{h}g^{-1} \tag{3.7}$$

so that $A, \tilde{A}, \omega, p$ and $\tilde{p}$ are all gauge-invariant. Thus (3.4,6) is a gauge-invariant formulation of the problem. However, there is also freedom in the choice of h and $\tilde{h}$ satisfying (3.1), namely

$$h \to bh; \ \tilde{h} \to \tilde{b}\tilde{h} \tag{3.8}$$

where b and $\tilde{b}$ are functions only of τ. Call this a b-transformation, then under a b-transformation we find

$$A \to bAb{-}1; \ \tilde{A} \to \tilde{b}\tilde{A}\tilde{b}^{-1}$$

$$\omega \to b\omega\tilde{b}^{-1}; \ p \to (bp - \dot{b})b^{-1}; \ \tilde{p} \to (\tilde{b}\tilde{p} - \dot{\tilde{b}})\tilde{b}^{-1} \tag{3.9}$$

We can exploit this freedom to make A and $\tilde{A}$ independent of τ. To do this, choose b and $\tilde{b}$ so that

$$\int_{S2} \frac{1}{V^2}p = 0; \ \int_{S^2} \frac{1}{V^2}\tilde{p} = 0. \tag{3.10}$$

then integrating (3.6) over the sphere and using

$$\int_{S^2} \frac{1}{V^2} = 4\pi \tag{3.11}$$

we find at once that $\dot{A}$ and $\dot{\tilde{A}}$ vanish, so that the matrices A and $\tilde{A}$, which we have identified intuitively as matrices of charges, are constant in time.

To make this intuitive identification tighter, we may recall that a definition of quasi-local charges for Yang-Mills fields was proposed in (Tod 1983). Briefly, given a $GL(n, \mathbf{C})$-Yang-Mills field, the definition associates a pair of $n \times n$ complex matrices up to similarity transformations with any topologically spherical, space-like 2-surface in Minkowski space. The construction mirrors Penrose's quasi-local mass construction (Penrose and Rindler 1984), and the eigenvalues of the given matrices can be regarded as quasi-local charges for the Yang-Mills field. In the present case, the matrices obtained at any 2-surface of constant τ and r are actually the matrices A and $\tilde{A}$. This strengthens the identification of the eigenvalues of these matrices with charges. Further, by the result above, in this case these quasi-local charges are constant. We may use a b-transformation with constant b and $\tilde{b}$ to put A and $\tilde{A}$ into canonical form.

How we proceed now depends on the choice of gauge group. For $SU(n)$, following (2.26), we find

$$\tilde{h}^{-1} = h^{\dagger}; \ \tilde{A} = -A^{\dagger}; \ \tilde{p} = -p^{\dagger}; \ \omega = \omega\dagger; \ trA = 0 \tag{3.12}$$

while for $SL(n, \mathbf{R})$

$$\tilde{h} = \bar{h};\ \tilde{A} = \bar{A};\ \tilde{p} = \bar{p};\ \tilde{\omega} = \omega^{-1};\ trA = 0 \tag{3.13}$$

Consider first the case of $SU(n)$. Then ω is positive-definite and Hermitian, and so it has a positive-definite square-root. Call this Ω, and integrate (3.4) over S^2 to find

$$\int_{S^2} E^+ E = \int_{S^2} \frac{1}{V^2}(A\omega + \omega A^+) \tag{3.14}$$

where

$$E = \Omega^{-1}\, \eth_0\, \omega.$$

Suppose A has been diagonalized, say $A = \text{diag}(\lambda_1, ..., \lambda_n)$, then a diagonal entry on the right-hand-side of (3.14) takes the form

$$(\lambda_i + \bar{\lambda}_i)\omega_{ii}$$

while the corresponding entry on the left-hand-side is non-negative. Since ω is positive-definite, ω_{ii} is positive and therefore $\lambda_i + \bar{\lambda}_i$ is non-negative for each i. Since trA is zero, this forces $\lambda_i + \bar{\lambda}_i$ to be zero for each i which in turn forces the left-hand-side in (3.14) to have zeroes on the diagonal, from which it follows that ω is constant on the sphere.

If A cannot be diagonalized, a similar argument applied to the Jordan canonical form leads to the same conclusion. With ω constant on the sphere, we may use a b-transformation to set it equal to the identity matrix. Many things now simplify; by (3.4) and (3.12), A is skew-Hermitian, so we may assume that it is diagonal; by (3.12) h is unitary; by (3.5) and (3.12) p is skew-Hermitian; finally, the evolution equation (3.6) reduces to

$$\bar{\eth}_0\, \eth_0\, p - \frac{2A\dot{V}}{V^3} + \frac{1}{V^2}[p, A] = 0. \tag{3.15}$$

Since A is diagonal, it follows rapidly from this that p is of the form

$$p = \dot{f}A + q \tag{3.16}$$

where

$$\eth_0 \bar{\eth}_0 \dot{f} = \frac{2\dot{V}}{V^3}$$

and q is a constant diagonal matrix with imaginary entries. From (3.9) we see that q can be eliminated by a suitable b-transformation, and then from (3.5) and (3.16) h is diagonal. From (3.1) this makes γ diagonal and the field is reduced to a product of Maxwellian Liénard-Wiechert fields.

This conclusion was also reached by Trautman (1981) but we have arrived at it, admittedly after more labour, by beginning with a more general notion of Liénard-Wiechert field. To find something new, we consider the case of gauge group $SL(2, \mathbf{R})$ and, for simplicity, we assume that the world-line Γ is straight, so that V is 1. Define

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix},\ \text{real constant}\ \lambda;\ h = \begin{pmatrix} 1 & i\eta \\ 0 & 1 \end{pmatrix},\ \text{real}\ \eta \tag{3.17}$$

then

$$\omega = \begin{pmatrix} 1 & 2i\eta \\ 0 & 1 \end{pmatrix}.$$

The constraint equation (3.4) reduces to

$$\bar{\eth}_0\, \eth_0\, \eta + 2\lambda\eta = 0 \tag{3.18}$$

while the evolution equation (3.6) becomes

$$\bar{\eth}_0\, \eth_0\, \dot{\eta} + 2\lambda\dot{\eta} = 0 \tag{3.19}$$

Thus η is an eigenfunction of the Laplacian on the sphere, with arbitrary time-dependence, and 4λ is the corresponding eigenvalue; for these solutions the "charges", that is the eigenvalues of the charge-matrix A, are discrete. The charges are conserved, but there is arbitrary time-dependence in the radiation field.

These solutions may not be the general solution for $SL(2, \mathbf{R})$, but they are the general solution with upper-triangular ω and, together with the corresponding lower-triangular solutions, they are the only solutions with a linearisation at fixed A.

These solutions have angular dependence and so it could be argued that they are not Liénard-Wiechert solutions since they are not monopoles but are higher multipoles. However, in Maxwell theory, the Maxwell spinor corresponding to a radiating dipole or higher multipole stationary at the origin will not have the radially out-going spinor as a principal spinor, that is to say it will not satisfy (2.14) (since this property precisely characterizes the Liénard-Wiechert field, which in this case is just the Coulomb field). The Yang-Mills fields which we have found here are purely outgoing, but have no counterpart in Maxwell theory.

Appendix

In this appendix we give the spin-coefficients for the tetrad of Section 2 and the Yang-Mills equations written out in the NP formalism with the simplifications (2.11) and (2.14). The Yang-Mills equations were given in (Newman and Tod 1980) but unfortunately with some sign errors.

The spin coefficients for the tetrad defined by (2.7) are

$$\kappa = \sigma = \epsilon = \tau = \pi = \lambda = 0$$

$$\rho = -\frac{1}{r}; \; \mu = -\frac{1}{2r}; \; \gamma = -\frac{\dot{V}}{2V} \tag{A.1}$$

$$\alpha = -\bar{\beta} = \frac{\bar{\delta} V}{2V} - \frac{1}{2\sqrt{2}}\frac{V}{r}\cot\theta.$$

With the definitions (2.13) and (2.15) and the conditions (2.11) and (2.14) the Yang-Mills equations reduce to two sets. The first set define the field from the potential;

$$0 = D\gamma_{01'} - \rho\gamma_{01'} \qquad \text{(A.2a)}$$
$$0 = D\gamma_{10'} - \rho\gamma_{10'}$$

$$2\chi_1 = D\gamma_{11'} + \bar{\delta}\gamma_{01'} - \delta\gamma_{10'} - [\gamma_{10'}, \gamma_{01'}] \qquad \text{(A.2b)}$$
$$2\tilde{\chi}_1 = D\gamma_{11'} + \delta\gamma_{10'} - \bar{\delta}\gamma_{01'} - [\gamma_{01'}, \gamma_{10'}]$$

$$\chi_2 = \bar{\delta}\gamma_{11'} + 2\alpha\gamma_{11'} - \Delta\gamma_{10'} - \mu\gamma_{10'} + [\gamma_{11'}, \gamma_{10'}] \qquad \text{(A.2c)}$$
$$\tilde{\chi}_2 = \delta\gamma_{11'} + 2\bar{\alpha}\gamma_{11'} - \Delta\gamma_{01'} - \mu\gamma_{01'} + [\gamma_{11'}, \gamma_{01'}].$$

and the second set are the field equations;

$$D\chi_1 - 2\rho\chi_1 = 0 \qquad \text{(A.3a)}$$
$$D\tilde{\chi}_1 - 2\rho\tilde{\chi}_1 = 0$$

$$D\chi_2 - \bar{\delta}\chi_1 - [\chi_1, \gamma_{10'}] - \rho\chi_2 = 0 \qquad \text{(A.3b)}$$
$$D\tilde{\chi}_2 - \delta\tilde{\chi}_1 - [\tilde{\chi}_1, \gamma_{01'}] - \rho\tilde{\chi}_2 = 0$$

$$\delta\chi_1 + [\chi_1, \gamma_{01'}] = 0 \qquad \text{(A.3c)}$$
$$\bar{\delta}\tilde{\chi}_1 + [\tilde{\chi}_1, \gamma_{10'}] = 0$$

$$\delta\chi_2 - \Delta\chi_1 + [\chi_2, \gamma_{01'}] - [\chi_1, \gamma_{11'}] + 2\beta\chi_2 - 2\mu\chi_1 = 0 \qquad \text{(A.3d)}$$
$$\bar{\delta}\tilde{\chi}_2 - \Delta\tilde{\chi}_1 + [\tilde{\chi}_2, \gamma_{10'}] - [\tilde{\chi}_1, \gamma_{11'}] + 2\bar{\beta}\tilde{\chi}_2 - 2\mu\tilde{\chi}_1 = 0$$

References

H. Arodź 1978 Phys. Lett. **78B** 129
A. Held, E.T. Newman and R. Posadas 1970 J. Math. Phys. **11** 3145
R.W. Lind and E.T. Newman 1974 J. Math. Phys. **15** 1103
E.T. Newman 1974 J. Math. Phys. **15** 44
E.T. Newman and R. Penrose 1966 J. Math. Phys. **7** 863
E.T. Newman and K.P. Tod 1980 in *General relativity and gravitation: one hundred years after the birth of Albert Einstein* vol.2, ed. A. Held, Plenum Press: New York
R. Penrose and W. Rindler 1984 *Spinors and space-time vol. 1* Cambridge: CUP
J. Tafel and A. Trautman 1983 J. Math. Phys. **24** 1087
K.P. Tod 1983 Proc. Roy. Soc. (Lond.) **A389** 369
A. Trautman 1981a Phys. Rev. Lett. **46** 875
1981b Acta Physica Austriaca Suppl. XXIII 401

THE TWIST PRESCRIPTION IN THE TOPOLOGICAL YANG-MILLS THEORY

SORIN MARCULESCU
Fachbereich Physik, Universität-GH-Siegen, Postfach 10 12 40, W-5900 Siegen, FRG

Abstract. The quantum properties of topological Yang-Mills theory are derived from the $N = 2$ supersymmetric Yang-Mills theory in flat space via the twist prescription.

Key words: Topological Field Theory

1. Introduction

The topological Yang-Mills theory (TYM) (Witten 1988) is described by fields of integer spin of both statistics. It includes the gauge field a_μ and is defined on a Riemannian manifold $\mathcal{M}$. TYM has a nilpotent fermionic symmetry, called BRS, such that physical states form BRS-cohomology classes. Moreover, the energy-momentum tensor is a BRS variation. The last condition is satisfied because the full quantum action of TYM can be written as a BRS variation. As a consequence, all local observables which represent the continuous symmetries of the action are trivial, i.e., their matrix elements between physical states vanish. However, this quantum field theory possesses local observables, the Donaldson polynomials, which have non-vanishing physical correlation functions.

In this contribution we show that, at least in perturbation theory, all the properties of TYM are preserved by the fully quantized theory. We accomplish this by fully exploiting the $N = 2$ supersymmetry present in flat TYM. At the quantum level the superconformal invariance of the flat theory is broken resulting in an $N = 2$ anomaly multiplet. We consider the components of the superconformal anomaly after coupling the theory minimally to euclidean gravity.

We change now the renormalization prescription inherited from $N = 2$ supersymmetry to one compatible with BRS symmetry in curved space. It turns out that the energy momentum tensor is not altered, while the BRS current attains an additional contribution keeping it conserved.

Furthermore we study the effects of the above procedure upon the superconformal anomaly. Some components, like the trace of the energy- momentum tensor, while receiving radiative corrections in each order of perturbation theory, are still BRS variations and represent therefore trivial local observables. The other components yield just the Donaldson polynomials (Donaldson 1990). They are observables but obey a non-renormalization theorem (Dahmen 1991).

2. $N = 2$ versus singlet supersymmetry

The prescription (twist) allowing for construction of TYM consists of identifying the isospinor index of $N = 2$ supersymmetric Yang-Mills theory with a dotted spinor index.

In the Wess-Zumino gauge the $N = 2$ Yang-Mills multiplet contains a gauge field $a_{\alpha\dot{\beta}}$, a pair of complex conjugate spinor-isospinor fields $\lambda_{\alpha B}$, a pair of complex conjugate scalars C and $\bar{C}$ and an auxiliary real isotriplet field H_{AB}. Spinor indices are denoted by dotted or undotted Greek letters from the beginning of the alphabet, isospinor indices are denoted by capital Latin letters. Whenever more indices of one sort are present full symmetry is assumed with the exception of the antisymmetric invariant tensors $\epsilon_{\alpha\beta}$, $\epsilon_{\dot{\alpha}\dot{\beta}}$ and g_{AB}. All the fields belong to a representation of the (compact) gauge group generated by the antihermitean matrices t_i are normalized appropriately. For TYM the twist procedure yields the following field content a_μ, ψ_μ, $\chi_{\mu\nu}$, η, ϕ, λ and $b_{\mu\nu}$, as follows:

$$\begin{array}{lll} a_{\alpha\dot{\beta}} \to a_{\alpha\dot{\beta}}\,; & \lambda_{\alpha B} \to -i\psi_{\alpha\dot{\beta}}\,; & \bar{\lambda}_{\dot{\alpha}B} \to i(\chi_{\dot{\alpha}\dot{\beta}} + \frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\eta)\,; \\ C \to \sqrt{2}\phi\,; & \bar{C} \to -\frac{1}{\sqrt{2}}\lambda\,; & H_{AB} \to -2ib_{\dot{\alpha}\dot{\beta}}\,. \end{array} \qquad (1)$$

The twist turns the half integer spins into integer ones without changing the original statistics, i.e. $\psi_\mu, \chi_{\mu\nu}$ and η remain anticommuting. Similarly, the $N = 2$ supersymmetry parameters $\zeta_{\alpha B}, \bar{\zeta}_{\dot{\alpha}B}$ are converted to anticommuting parameters with three different SO(4) structures ζ_μ, $\zeta_{\mu\nu}$ and ζ presented in spinorial notation

$$\zeta_{\alpha B} \to i\zeta_{\alpha\dot{\beta}}\,; \qquad \bar{\zeta}_{\dot{\alpha}B} \to i(\zeta_{\dot{\alpha}\dot{\beta}} - \frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\,\zeta)\,. \qquad (2)$$

If TYM is minimally coupled to a curved background, generally these anticommuting tensors become local, but constrained by the Killing conditions. Obviously, only the Killing condition for the scalar ζ does not restrict the background informing us in fact, that ζ remains constant.

Hence $N = 2$ supersymmetry is broken, leaving a symmetry parametrized by an anticommuting scalar ζ. This singlet supersymmetry takes the form (Galperin 1991)

$$\begin{array}{ll} \delta a_\mu = i\zeta\psi_\mu\,; & \delta\psi_\mu = \zeta D_\mu\phi\,; \\ \delta\phi = 0\,; & \delta\lambda = i\zeta\eta\,; \\ \delta\eta = -\zeta[\phi,\lambda]\,; & \delta\chi_{\mu\nu} = \zeta(b_{\mu\nu} + \frac{1}{2}f^-_{\mu\nu})\,; \end{array}$$
$$\delta b_{\mu\nu} = -i\zeta\left(\frac{1}{2}D_{[\mu}\psi_{\nu]}^- + [\phi,\chi_{\mu\nu}]\right)\,. \qquad (3)$$

One verifies that two transformations (3) commute up to gauge transformation of parameter ϕ.

The above construction is very similar to supersymmetry transformations in Wess-Zumino gauge and suggests a superspace description for TYM (Horne 1988). The superfields are A_μ, A_θ, $X_{\mu\nu}$ and Λ and depend on the coordinates (x^μ, θ), θ being a Grassmann variable. From the superconnections A_μ and A_θ one can easily construct the covariant derivatives D_μ, D_θ and the superfield strength $F_{\mu\nu}$, F_μ, F. The field components are defined by

$$\begin{array}{lll} A_\mu| = a_\mu\,; & F_\mu| = -\psi_\mu\,; & \\ X_{\mu\nu}| = \chi_{\mu\nu}\,; & D_\theta X_{\mu\nu}| = b_{\mu\nu} + \frac{1}{2}f^-_{\mu\nu}\,; & \\ \Lambda| = \lambda\,; & D_\theta\Lambda| = \eta\,; & F| = 2i\phi\,. \end{array} \qquad (4)$$

The action of $N = 2$ supersymmetric Yang-Mills theory is given, up to a total divergence, by (Grimm 1978)

$$\int d^4x Tr(-\bar{C}D^m D_m C - i\bar{\lambda}_{\dot{\alpha}C}D^{\dot{\alpha}\beta}\lambda^C_\beta - f^{\dot{\alpha}}{}_{\dot{\beta}}f^{\dot{\beta}}{}_{\dot{\alpha}} - \frac{1}{4}H^A{}_B H^B{}_A$$
$$-i\sqrt{2}\bar{C}\lambda^\alpha_B\lambda^B_\alpha + i\sqrt{2}C\bar{\lambda}^{\dot{\alpha}B}\bar{\lambda}_{\dot{\alpha}B} - \frac{1}{2}[C,\bar{C}]^2)\,. \qquad (5)$$

The twist prescription leads to the Witten action. We give the result in superspace

$$\mathcal{S} = \int_{\mathcal{M}} d_4x\sqrt{g}\partial_\theta Tr(iF_{\mu\nu}X^{\mu\nu} + \frac{1}{2}X_{\mu\nu}D_\theta X^{\mu\nu} + iF_\mu D^\mu\Lambda + \frac{1}{4}F[\Lambda, D_\theta\Lambda])\,. \qquad (6)$$

The superspace approach is suited very well also for discussing the properties of TYM as listed e.g. in (Birmingham 1991). For instance, the coupling constant and the metric independence of the partition function follows by integrating by parts with respect to θ and by assuming the BRS invariance of the superspace path integral measure.

Moreover, all correlation functions of TYM currents vanish. We call a TYM current any gauge invariant object which can be obtained by the Noether procedure modulo improvements from the Lagrangian. Obviously, it is a θ-component as the Lagrangian itself.

3. The superconformal current multiplet

An important class of TYM currents can be obtained by twist procedure from the $N = 2$ superconformal current multiplet.

It is known (Sohnius 1979) that the R-current $\jmath_{\alpha\dot\beta}$, the isospin current $\jmath_{\alpha\dot\beta CD}$, the supersymmetry current $\delta_{\alpha\beta\dot\gamma D}$ and its complex conjugate $\bar\delta_{\gamma\dot\alpha\dot\beta D}$ as well as the energy-momentum tensor $\jmath_{\alpha\beta\dot\gamma\dot\delta}$ are conserved and belong to an $N = 2$ supermultiplet. The corresponding objects $b_\mu, b_{\mu\nu\rho}$ (antiselfdual in μ,ν), $\Lambda_{\mu\nu}, \Lambda_{\mu\nu\rho}$ (antiselfdual in ν,ρ) and $\theta_{\mu\nu}$ (symmetric in μ,ν) of flat TYM are obtained using the twist prescription

$$\begin{aligned}
\jmath_{\alpha\dot\beta} &\to -ib_{\alpha\dot\beta}\,; & \jmath_{\alpha\dot\beta CD} &\to ib_{\dot\gamma\dot\delta\,\alpha\dot\beta}\,;\\
\delta_{\gamma\delta\dot\alpha B} &\to \Lambda_{\gamma\dot\beta\,\delta\dot\alpha}\,; & \bar\delta_{\alpha\dot\gamma\dot\delta B} &\to i\Lambda_{\alpha\dot\beta\,\dot\gamma\dot\delta}\,;\\
\jmath_{\alpha\beta\,\dot\gamma\dot\delta} &\to \theta_{\alpha\dot\gamma\,\beta\dot\delta}\,. & &
\end{aligned} \tag{7}$$

The above $N = 2$ currents have the following explicit form in terms of field components

$$\begin{aligned}
\jmath_{\beta\dot\alpha} &= -2Tr(\bar\lambda_{\dot\alpha C}\lambda^C_\beta + i\bar C \overleftrightarrow{D}_{\beta\dot\alpha}\, C)\,;\\
\jmath_{\beta\dot\alpha CD} &= 2Tr\bar\lambda_{\dot\alpha(C}\lambda_{\beta D)}\,;\\
\delta_{\gamma\delta\dot\alpha B} &= Tr(-4f_{\gamma\delta}\bar\lambda_{\dot\alpha B} - \frac{2\sqrt2}{3}\lambda_{(\gamma B}D_{\delta)\dot\alpha}\bar C + \frac{\sqrt2}{3}\bar C D_{(\gamma\dot\alpha}\lambda_{\delta)B})\,;\\
\bar\delta_{\dot\gamma\dot\delta\alpha B} &= Tr(4f_{\dot\gamma\dot\delta}\lambda_{\alpha B} + \frac{2\sqrt2}{3}\bar\lambda_{(\dot\gamma B}D_{\alpha\dot\delta)}C - \frac{\sqrt2}{3}CD_{\alpha(\dot\gamma}\bar\lambda_{\dot\delta)B})\,;\\
\jmath_{\alpha\beta\dot\gamma\dot\delta} &= Tr(-4f_{\alpha\beta}f_{\dot\gamma\dot\delta} + \frac{i}{4}\bar\lambda_{(\dot\gamma E}D_{(\alpha\dot\delta)}\lambda^E_{\beta)} + \frac{i}{4}\lambda^E_{(\gamma}D_{\delta)(\dot\alpha}\bar\lambda_{\dot\beta)E}\\
&\quad -\frac{1}{12}\bar C D_{(\alpha(\dot\gamma}D_{\beta)\dot\delta)}C - \frac{1}{12}CD_{(\alpha(\dot\gamma}D_{\beta)\dot\delta)}\bar C\\
&\quad +\frac{1}{3}D_{(\alpha(\dot\gamma}\bar C D_{\beta)\dot\delta)}C)\,.
\end{aligned} \tag{8}$$

By minimally coupling flat TYM to euclidean gravity, both b_μ and $b_{\mu\nu\rho}$ become generally covariant and, as a consequence of the equations of motion, covariantly conserved.

The corresponding procedure is less straightforward for $\Lambda_{\mu\nu}$ because $N = 2$ transformation properties lead to a traceless energy-momentum tensor incompatible with general covariance. Hence $\Lambda_{\mu\nu}$ must be redefined such as to allow for a nonvanishing trace. There is only one quantity

$$\Delta_\mu = 2i\,Tr\lambda\psi_\mu\,, \tag{9}$$

appearing in the $N = 2$ transformation laws, which can be used for this purpose. The object Δ_μ arises via the twisting procedure from the spinor-isospinor component of the supercurrent $\delta_{\alpha B}$. By imposing conservation and

selfduality of the antisymmetric part one can determine $s_{\mu\nu}$ as the improved $\Lambda_{\mu\nu}$. Up to an overall numerical factor we choose

$$s_{\mu\nu} = -\Lambda_{\mu\nu} - \frac{1}{3}(\partial_\mu\Delta_\nu - \delta_{\mu\nu}\partial_\rho\Delta_\rho + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial_\rho\Delta_\sigma) \tag{10}$$

and express it in terms of topological fields. After coupling to euclidean gravity and using equations of motion it becomes covariantly conserved.

We still have to determine the singlet supersymmetry partners of b_μ, $b_{\mu\nu\rho}$ and $s_{\mu\nu}$. As in the case of $s_{\mu\nu}$ we need the remaining components of the supercurrent multiplet. After twisting we call them S, $\Delta_{\mu\nu}$, Δ, $S^+_{\mu\nu}$ and $S^-_{\mu\nu}$ and present their topological field dependence in superfield form

$$S + \frac{1}{2}\theta\Delta = -iTr\Lambda F\,; \tag{11}$$

$$\Delta_{\mu\nu} - \frac{1}{4}\theta S^-_{\mu\nu} = -2iTrFX_{\mu\nu}\,; \tag{12}$$

$$S^+_{\mu\nu} - 4\theta(s_{\mu\nu} - s_{\nu\mu}) = 2iTr\Lambda F^+_{\mu\nu}\,. \tag{13}$$

Now, the supersymmetry transformation of b_μ, $s_{\mu\nu}$ and $b_{\mu\nu\rho}$ can be written as

$$\delta b_\mu = i\zeta\left(\frac{1}{2}\Lambda_{\nu\mu\nu} - \frac{2}{3}\partial_\nu\Delta_{\mu\nu}\right)\,; \tag{14}$$

$$\delta s_{\mu\nu} = i\zeta\left[\theta_{\mu\nu} - \frac{1}{4}\partial_\rho(b_{\mu\rho\nu} + b_{\nu\rho\mu}) + \frac{1}{6}(\delta_{\mu\nu}\Box S - \partial_\mu\partial_\nu S)\right]\,; \tag{15}$$

$$\begin{aligned}
\delta b_{\mu\nu\rho} &= i\zeta\left[\frac{1}{3}\partial_\mu\Delta_{\nu\rho} + \delta_{\mu\rho}(-s_\nu - \frac{1}{3}\partial_\sigma\Delta_{\nu\sigma} + \frac{1}{6}\partial_\nu\Delta) - (\mu\leftrightarrow\nu)\right.\\
&\quad \left. -\Lambda_{\rho\mu\nu} + \epsilon_{\mu\nu\rho\sigma}\left(s_\sigma - \frac{1}{6}\partial_\sigma\Delta\right) + \frac{2}{3}\partial_\rho\Delta_{\mu\nu}\right]\,.
\end{aligned} \tag{16}$$

We evaluate the RHS of eqs. (14) – (16) in terms of fields and couple then minimally to curved background. In this way we get three covariantly conserved superfield currents

$$H_\mu = iTr(2F^\nu X_{\mu\nu} - F_\mu D_\theta\Lambda + \Lambda\overleftrightarrow{D}_\mu F)\,; \tag{17}$$

$$\begin{aligned}
H_{\mu\nu} &= iTr\left\{\frac{1}{2}F^+_{\mu\rho}X_\nu{}^\rho + F_\mu D_\nu\Lambda + (\mu\leftrightarrow\nu)\right.\\
&\quad \left. -g_{\mu\nu}(F^\rho D_\rho\Lambda + \frac{1}{2}\Lambda D_\rho F^\rho)\right\} - \frac{1}{2}\partial_\theta TrF^+_{\mu\nu}\Lambda\,;
\end{aligned} \tag{18}$$

$$\begin{aligned}
H_{\mu\nu\rho} &= iTr\,\{2X_{\mu\rho}F_\nu + g_{\mu\nu}(2X_{\nu\sigma}F^\sigma - F_\nu D_\theta\Lambda) - (\mu\leftrightarrow\nu)\\
&\quad +2X_{\mu\nu}F_\rho + \eta_{\mu\nu\rho\sigma}F^\sigma D_\theta\Lambda\}\,.
\end{aligned} \tag{19}$$

The θ-components represent the BRS current s_μ, the energy-momentum tensor $t_{\mu\nu}$ and the antiselfdual supersymmetry current $s_{\mu\nu\rho}$, respectively.

At this point our procedure of constructing the superfield currents starting from the $N = 2$ superconformal current is completed.

Hitherto, we encountered singlet supersymmetry only. When the flat components of the superfield current (17) – (19) are transformed under the full $N = 2$ supersymmetry group second order derivatives occur. Without affecting the conservation rules such derivatives can be eliminated by further redefinitions

$$v_\mu \equiv s_\mu + \nabla^\nu \Delta_{\mu\nu} \; ; \tag{20}$$

$$v_{\mu\nu} \equiv s_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu\rho\sigma}\nabla^\rho \Delta^\sigma \; . \tag{21}$$

The resulting improved supercurrent in flat space will transform according to an irreducible representation of $N = 2$ supersymmetry, i.e. without derivatives of higher order than the first.

4. Superconformal anomaly multiplet

It is commonly assumed that flat TYM is a renormalizable theory. However, since the supersymmetry gauge theory in Minkowski space has been formulated in Wess-Zumino gauge there is no guarantee that both gauge invariance and $N = 2$ supersymmetry are preserved by quantum corrections. A detailed analysis of this question is presented in (Breitenlohner 1988).

Here we shall assume that this remains true even after the twist procedure, which means that TYM in curved space will be free of BRS anomalies.

Quantum corrections break superconformal invariance giving rise to an $N = 2$ anomaly supermultiplet. Of course, $N = 2$ supersymmetry is broken in curved space and some rearrangement of the anomaly component might become necessary. Unlike the case of the supercurrent multiplet additional modifications have to be made since quantizing the theory requires a certain renormalization prescription.

The superconformal anomaly in Minkowski space leads to the following $N = 2$ multiplet of TYM anomalies

$$\begin{array}{rclrcl} y &\to& L\,; & \bar{y} &\to& \frac{1}{2}M\,; \\ \eta_{\alpha B} &\to& iP_{\alpha\dot\beta}\,; & \bar\eta_{\dot\alpha B} &\to& \bar\eta_{\dot\alpha,\dot\beta} = P_{\dot\alpha\dot\beta} + \frac{1}{2}\epsilon_{\dot\alpha\dot\beta}P\,; \\ y_{AB} &\to& L_{\dot\alpha\dot\beta}\,; & \bar y_{AB} &\to& M_{\dot\alpha\dot\beta}\,; \\ y_{\alpha\beta} &\to& N_{\alpha\beta}\,; & y_{\dot\alpha\dot\beta} &\to& N_{\dot\alpha\dot\beta}\,; \\ \omega_{\alpha B} &\to& -i\Gamma_{\alpha\dot\beta}\,; & \bar\omega_{\dot\alpha B} &\to& \bar\omega_{\dot\alpha,\dot\beta} = \Gamma_{\dot\alpha\dot\beta} + \epsilon_{\dot\alpha\dot\beta}\Gamma\,; \\ a &\to& -iA\,; & t &\to& T\,. \end{array} \tag{22}$$

Depending on the form of singlet supersymmetry transformations one can group the components (22) into two classes, one which transforms without derivatives, the other one involving only the space derivative in the transformation laws.

By forming the linear combinations

$$M_{\pm\mu\nu} = -\frac{1}{2}\left(\frac{1}{2}N^-_{\mu\nu} \pm M_{\mu\nu}\right) \; ; \qquad A_+ = T + \frac{A}{2} \tag{23}$$

one can organize the components of the first class into singlet superfields. The field dependence of the relevant components can be obtained by twist from

$$\begin{aligned} \bar y_{AB} &= -\frac{c}{8\pi^2}\epsilon^{\dot\gamma\dot\delta}Tr\bar\lambda_{\dot\delta(A}\bar\lambda_{\dot\gamma B)} \; ; \\ y_{\dot\alpha\dot\beta} &= -\frac{c}{8\pi^2}Tr(\bar\lambda_{(\dot\alpha C}\bar\lambda_{\dot\beta)}{}^C - 4\sqrt{2}i\bar C f_{\dot\alpha\dot\beta}) \; ; \\ \omega_{\alpha B} &= -\frac{c}{2\pi^2}Tr(f_\alpha{}^\gamma\lambda_{\gamma B} - \frac{1}{2\sqrt2}CD_\alpha{}^{\dot\gamma}\bar\lambda_{\dot\gamma B}) \; ; \\ \bar\omega_{\dot\alpha B} &= -\frac{c}{2\pi^2}Tr(f^{\dot\gamma}{}_{\dot\alpha}\bar\lambda_{\dot\gamma B} - \frac{1}{2\sqrt2}\bar C D^\gamma{}_{\dot\alpha}\lambda_{\gamma B}) \; ; \\ a &= -\frac{ic}{4\pi^2}\left[Tr(f_\alpha{}^\beta f_\beta{}^\alpha - f^{\dot\alpha}{}_{\dot\beta}f^{\dot\beta}{}_{\dot\alpha}) + \partial^m Tr\bar C \overleftrightarrow{D}_m C\right] \; ; \\ t &= -\frac{c}{8\pi^2}Tr\{f_\alpha{}^\beta f_\beta{}^\alpha + f^{\dot\alpha}{}_{\dot\beta}f^{\dot\beta}{}_{\dot\alpha} + \frac{1}{2}(CD^mD_m\bar C + \bar C D_m D^m C) \\ &\quad + \frac{i}{4}(\lambda^C_\alpha D^{\dot\beta\alpha}\bar\lambda_{\dot\beta C} + \bar\lambda_{\dot\alpha C}D^{\dot\alpha\beta}\lambda^C_\beta)\} \; . \end{aligned} \tag{24}$$

The coefficient in front of the trace is taken from a one-loop $N = 2$ superspace calculation of the superconformal anomaly in Minkowski space (Marculescu 1987). We give here only the first class superfields we will be concerned with

$$\Omega_{\mu\nu} \equiv M_{+\mu\nu} + \theta\Gamma_{\mu\nu} = \frac{c}{4\pi^2}Tr(X_\mu{}^\rho X_{\nu\rho} - \frac{i}{2}\Lambda F^-_{\mu\nu}) \; ; \tag{25}$$

$$\Omega \equiv \Gamma + \theta A_+ = \frac{ic}{8\pi^2}Tr(X_{\mu\nu}F^{\mu\nu} - \Lambda D_\mu F^\mu) \; . \tag{26}$$

As before, the generally covariant form of RHS of (25), (26) is obtained after minimally coupling TYM to euclidean gravity.

The singlet supersymmetry transformations of the anomaly components of the second class can be written as

$$\begin{array}{lll} \delta L = 0\,; & \delta P_\mu = -i\zeta\nabla_\mu L\,; & \delta L_{-\mu\nu} = i\zeta\nabla_{[\mu}P_{\nu]}\,; \\ \delta\Gamma_\mu = -i\zeta\nabla^\nu L_{+\mu\nu}\,; & \delta A = i\zeta\nabla_\mu\Gamma^\mu & \end{array} \tag{27}$$

where

$$L_{\pm\mu\nu} \equiv -\frac{1}{2}\left(\frac{1}{2}N^+_{\mu\nu} \pm L_{\mu\nu}\right) \; . \tag{28}$$

The tensors $L_{\pm\mu\nu}$ are dual to each other. The twist prescription leads to the following solution of (27)

$$\begin{aligned} w_0^4 &= \frac{c}{4\pi^2}Tr\phi^2\,; \qquad w_1^3 = -\frac{ic}{2\pi^2}Tr\phi\psi\,;\\ w_2^2 &= -\frac{c}{4\pi^2}Tr(\psi^2 - i\phi f^+)\,;\\ w_3^1 &= -\frac{c}{4\pi^2}Tr(f^+\psi + \phi D\chi - \frac{1}{2}\phi * D\eta)\,;\\ w_4^0 &= -\frac{c}{4\pi^2}(Trf^2 + dTr\lambda * \overleftrightarrow{D}\,\eta)\,. \end{aligned} \tag{29}$$

where we introduced the space forms of various ghost numbers (upper index) for L, P_μ, $L_{-\mu\nu}$, Γ_μ and A. As for the currents discussed in the previous section we have to use the field equations of motion of TYM in order to verify that eqs. (29) satisfy the conditions (27). Except for w_0^4 and w_1^3 the forms differ from Donaldson polynomials despite the fact that they satisfy the same descent equations. The reason for this discrepancy can be traced back to a different renormalization procedure used in TYM.

The results obtained by the twisting method presupposed an $N = 2$ renormalization prescription in flat space. If we require now that quantized TYM in curved space is renormalized in agreement with BRS, the presence of various terms entering anomaly components has to be reconsidered. For instance, we modify the ghost number anomaly to read

$$\nabla_\mu b^{\mu\text{ren}} = -\frac{c}{8\pi^2}Tr f_{\mu\nu}\tilde{f}^{\mu\nu} + \text{grav. contr.}\,. \tag{30}$$

On the RHS of eq. (30) we included the appropriated gravitational contribution as calculated by (Dahmen 1991). The quantum ghost number current $b^{\mu\text{ren}}$ is expressed in terms of fields renormalized by a BRS prescription.

Let us define a renormalized BRS current s_μ^{ren} by

$$i\zeta s_\mu^{\text{ren}} = \delta b_\mu^{\text{ren}}\,. \tag{31}$$

On the basis of eqs. (30), (31) we introduce a modified BRS current

$$s_\mu^{\text{mod}} \equiv s_\mu^{\text{ren}} - w_\mu\,, \quad w_\mu \equiv -\frac{c}{2\pi^2}Tr\tilde{f}_{\mu\nu}\psi^\nu \tag{32}$$

which is conserved. As a consequence of this, the gauge invariant polynomials Γ_μ and A will be modified into w_μ and $-\frac{c}{8\pi^2}Trf_{\mu\nu}\tilde{f}^{\mu\nu}$, respectively. By subjecting the whole solution (29) to the change of renormalization prescription one arrives at precisely the Donaldson polynomials

$$\begin{aligned} W_0^4 &= w_0^4\,; \qquad W_1^3 = w_1^3\,;\\ W_2^2 &= -\frac{c}{2\pi^2}Tr\left(\frac{1}{2}\psi^2 - i\phi f\right)\,;\\ W_3^1 &= -\frac{c}{2\pi^2}Trf\psi\,; \qquad W_4^0 = -\frac{c}{4\pi^2}Trf^2\,. \end{aligned} \tag{33}$$

One can show (Dahmen 1991) that the whole set of Donaldson polynomials as well as the dimension of the instanton moduli space (the integral version of eq. (30)) remains non-renormalized beyond one-loop.

We turn now again to the first class superfields. To start with we assume that one can construct a renormalized energy-momentum tensor by means of the transformation rule

$$\delta(v_{\mu\nu}^{\text{ren}} - \frac{1}{2}\eta_{\mu\nu\rho\sigma}\nabla^\rho\Delta^{\sigma\text{ren}}) = i\zeta t_{\mu\nu}^{\text{ren}} \tag{34}$$

where Δ_μ^{ren} and $v_{\mu\nu}^{\text{ren}}$ are the quantum version of the quantities defined in eqs. (9) and (21), respectively. Note that from the conservation of $t_{\mu\nu}^{\text{ren}}$ it follows that $\nabla^\nu v_{\mu\nu}^{\text{ren}}$ is a BRS invariant.

Since $t_{\mu\nu}^{\text{ren}}$ is symmetric, eq. (34) implies that $v_{\mu\nu}^{\text{ren}} - v_{\nu\mu}^{\text{ren}} - \eta_{\mu\nu\rho\sigma}\nabla^\rho\Delta^{\sigma\text{ren}}$ is the θ component of some antisymmetric tensor. In the 'classical' TYM this antisymmetric tensor is selfdual. The quantum theory produces a certain antiselfdual contribution. Hence the transformation law following from (13) is changed to

$$i\zeta(v_{\mu\nu}^{\text{ren}} - v_{\nu\mu}^{\text{ren}} - \eta_{\mu\nu\rho\sigma}\nabla^\rho\Delta^{\sigma\text{ren}}) = -\frac{1}{4}\delta S_{\mu\nu}^+ + \frac{1}{2}\delta\Omega_{\mu\nu}|\,. \tag{35}$$

In passing to the BRS prescription we have to allow for an arbitrary relative factor between the two superfields on the RHS of eq. (25). We may, however, modify $v_{\mu\nu}^{\text{ren}} - v_{\nu\mu}^{\text{ren}}$ such that the antiselfdual anomaly takes the form

$$\Omega'_{\mu\nu} = \frac{c}{4\pi^2}TrX_\mu{}^\rho X_{\nu\rho}\,. \tag{36}$$

Let us now discuss the scale anomaly Ω. From eq. (34) we get

$$\delta v_\mu{}^{\mu\text{ren}} = i\zeta t_\mu{}^{\mu\text{ren}}\,. \tag{37}$$

Hence stipulating (37), $t_{\mu\nu}^{\text{ren}}$ cannot receive gravitational contributions while remaining conserved. An explicit one-loop computation (Dahmen 1991) confirms this assumption. The scale Ward identity can be written as

$$\delta\Omega| + \nabla_\mu\delta\Delta^{\mu\text{ren}} = i\zeta t_\mu{}^{\mu\text{ren}}\,. \tag{38}$$

The BRS prescription means that we allow for Ω the form

$$\frac{ic}{8\pi^2}Tr(\alpha F_{\mu\nu}X^{\mu\nu} - \gamma\Lambda D_\mu F^\mu)\,. \tag{39}$$

However, the scale anomaly is prescription independent to one-loop (Gross 1975). By using (26) one finds $\alpha^{(1)} = \gamma^{(1)} = 1$, where the superscript refers to the one-loop approximation.

At least in the background field quantization scheme, it follows that all one-loop wave-function renormalization constants are equal as for the $N = 2$

supersymmetric Yang-Mills theory in Minkowski space. This however, means that the normalization factors of the antiselfdual anomaly $\Omega_{\mu\nu}$, the trace anomaly Ω and the Donaldson polynomials are related by $N = 2$ supersymmetry even after changing to BRS renormalizing prescriptions. This fact has been already taken into account in formulae (30), (32), and (36).

Also the one-loop β-function of TYM coincides with that of $N = 2$ Yang-Mills in Minkowski space (Dahmen 1990). This is not in conflict with the properties of TYM as discussed above. Indeed, we have shown that both the radiatively corrected currents s_{μ}^{mod} and $t_{\mu\nu}^{\text{ren}}$ are BRS variations. Thus, their correlation functions vanish and the renormalized partition function is metric independent. Finally, its gauge coupling constant independence follows trivially from the absence of a genuine BRS anomaly.

5. Conclusions

In this work we attempted to explain all the properties of TYM in the light of the $N = 2$ supersymmetry observed for a flat metric. To this end we constructed a system of currents conserved in curved space which forms an $N = 2$ supermultiplet in the limit of flat space.

In passing to the quantum theory, superconformal invariance is broken and the system of currents develops anomalous Ward identities. As a consequence the BRS current and the energy-momentum tensor receive quantum corrections which can be represented as BRS variations. On this basis one can understand the metric independence of the partition function and the vanishing of correlation functions of the BRS current and the energy-momentum tensor.

By imposing BRS invariant renormalization prescriptions we were able to derive the Donaldson polynomials from the one-loop $N = 2$ superconformal anomaly.

The $N = 2$ supersymmetry of flat TYM is still present at the one-loop level showing in a common normalization factor of various anomalies. This explains why the β-function of TYM coincides in this approximation with that of $N = 2$ supersymmetric Yang-Mills in Minkowski space.

Acknowledgements

This paper is a somewhat simplified version of work done in collaboration with Hans Dahmen and Lech Szymanowski. I would like to thank them for their valuable contribution to all the results presented above. I am also particularly grateful to John Klein presently at the Mathematical Department of the University of Bielefeld for teaching me modern differential geometry.

References

Birmingham, D., Blau, M., Rakowski, M. and Thompson, G.: 1991 *Phys. Rep.* **209**, 129.
Breitenlohner, P.: 1988 in *Renormalization of quantum field theories with non-linear field transformations, Lecture Notes in Physics 303* P. Breitenlohner, D. Maison and K. Sibold eds., Springer-Verlag:Heidelberg, p. 64.
Dahmen, H.D., Marculescu, S. and Szymanowski, L.: 1990 *Phys. Lett. B* **252**, 591.
Dahmen, H.D., Marculescu, S. and Szymanowski, L.: 1991, 'Currents and anomalies in topological Yang-Mills theory', *Preprint SI-09*.
Donaldson, S.: 1990 *Topology* **29**, 257.
Galperin, A. and Ogievetsky, O.: 1991 *Commun. Math. Phys.* **139**, 377.
Grimm, R., Sohnius, M. and Wess, J.: 1978 *Nucl. Phys. B* **133**, 275.
Gross, D.J.: 1976, in *Methods in field theory* R. Balian and J. Zinn-Justin eds., North Holland:Amsterdam, p. 141.
Horne, J.H.: 1988 *Nucl. Phys. B* **318**, 22.
Marculescu, S.: 1987 *Phys. Lett. B* **188**, 203.
Sohnius, M.: 1979 *Phys. Lett. B* **81**, 8.
Witten, E.: 1988 *Commun. Math. Phys.* **117**, 353.

ON SYMMETRY PROPERTIES OF CERTAIN CLASSICAL LAGRANGE FUNCTIONS UNDER ROTATIONS

PETER STICHEL
University of Bielefeld,
Germany

and

JAN T. ŁOPUSZAŃSKI
Institute of Theoretical Physics, University of Wrocław,
pl. Maxa Borna 9, 50-204 Wrocław,
Poland

Preliminary version, presented by J.T. Łopuszański

Abstract. It is shown that for a one–particle Lagrange function in a 3–dimensional Euclidean space of the standard type, giving rise to Euler variation which is supposed to be a vector with respect to rotations, the difference between this Lagrange function and rotated one is equal to a time derivative. We discuss also the problem how to recover a rotationally invariant Lagrange function from a non–symmetric one, which however, gives rise to rotationally covariant Euler variations.

The aim of this talk is to convey to you two remarks related to the work done currently by Peter Stichel and myself. This work is still in progress.

The first remark refers to the following statement. We are going to show that for a one–particle Lagrange function in a 3–dimensional Euclidean space of the type

$$L(\underline{x},\dot{\underline{x}}) = \frac{1}{2}\,\dot{\underline{x}}^2 - V(\underline{x},\dot{\underline{x}}) \qquad \underline{x} = (x_1,x_2,x_3) \quad \text{etc.}$$

giving rise to Euler variation

$$f_j(\underline{x},\dot{\underline{x}}) \equiv \frac{d}{dt}\frac{\partial V}{\partial \dot{x}_j} - \frac{\partial V}{\partial x_j}, \qquad j = 1,2,3, \tag{1}$$

which is supposed to be a vector with respect to the rotations, we have

$$L(R\underline{x},R\dot{\underline{x}}) - L(\underline{x},\dot{\underline{x}}) = \frac{d\Phi}{dt}\,.$$

Here R stands for a 3–dimensional matrix representing an element of the rotation group, viz.

$$R = \bar{R}, \quad R^{-1} = R^T \quad \text{and} \quad det\ R = 1,$$

Φ is a function of R and $\underline{x}$ (it does not depend on $\dot{\underline{x}}$), viz.

$$\Phi = \Phi(R\underline{x}, R) ;$$

it may also depend on other parameters built in into the model. Notice that we do not require that V should be rotationally invariant, i.e. that V is a scalar. Since the Euler variation is linear with respect to L we may restrict ourselves to V only, as $\frac{1}{2}\dot{\underline{x}}^2$ is for sure rotationally invariant.

This assertion can be immediately extended to rotations in an n–dimensional Euclidean space $n = 2, 4, \ldots$.

The proof we are going to present here has the merit to be so elementary that it can be used in regular classes on classical mechanics for beginners at the University.

Before we enter, however, into the proof let me say few words about the setting of the story.

It is well known [1] that the necessary and sufficient condition for any trajectory to be an optimal one is that the Lagrange function is just a time derivative. But having two Lagrange functions which differ from each other and yield the same set of solutions of the Euler–Lagrange Equations (so called s–equivalence) or even yield the same Euler–Lagrange Equations, this does not yet imply that these two Lagrange functions, say,

$$L(\underline{x}, \dot{\underline{x}}, t) \qquad \text{and} \qquad L'(\underline{x}, \dot{\underline{x}}, t)$$

differ by a time derivative of a certain function, viz.

$$L - L' \not\equiv \frac{d\Phi}{dt}(\underline{x}, t) .$$

To see that take the example of the 1–dimensional harmonic oscillator [2]

$$L = \frac{1}{2}\left(\dot{x}^2 - x^2\right) \qquad \text{and} \qquad L' = \frac{1}{2}A^2\left(\dot{x}^2 - x^2\right) ;$$

A, a real constant, can e.g. be viewed as a scaling transformation $x \to x\,e^{-\alpha}$, $-\alpha$ - a real parameter ($e^{-\alpha} = A$). The Euler variations

$$\ddot{x} + x \qquad \text{and} \qquad A^2(\ddot{x} + x)$$

lead to the same Euler–Lagrange Equations. Nevertheless,

$$\left(A^2 - 1\right)\left(\dot{x}^2 - x^2\right) \neq \frac{d\Phi}{dt}(x) .$$

What, of course, is true is that on the so called "mass shell" (i.e. where the equation of motion is satisfied) we have

$$x = -\ddot{x}$$

and consequently

$$\dot{x}^2 - x^2 = \dot{x}^2 + x\ddot{x} = \frac{d}{dt}(x\dot{x}) .$$

There is a lore that for L and L' yielding the same Euler–Lagrange Equations or the same set of solutions [3] we have

$$L - L' = aL + \frac{d\Phi}{dt} ,$$

α being a constant.
But this conjecture is also not true as shown by the following example [4]. Let us take

$$L = \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) \qquad \text{and} \qquad L' = \dot{x}_1\dot{x}_2$$

in a 2–dimensional Euclidean space. Both Lagrange functions yield identical Euler–Lagrange Equations, viz.

$$\ddot{x}_1 = \ddot{x}_2 = 0$$

but

$$L - L' = \frac{1}{2}(\dot{x}_1 - \dot{x}_2)^2 \neq aL + \frac{d\Phi}{dt} .$$

What is essential in this example is that L and L' give rise to different set of Poisson' Brackets, as we have

$$p_1 \equiv \frac{\partial L}{\partial \dot{x}_1} = \dot{x}_1 \qquad p_2 = \dot{x}_2 \qquad \text{but} \qquad p_1' = \dot{x}_2, \quad p_2' = \dot{x}_1 .$$

It was shown by Henneaux [4] that if L and L' yield the same Euler–Lagrange Equations, as well as the same Poisson Brackets up to a multiplicative constant, viz. $[x_i, \dot{x}_j]_L = \alpha[x_i, \dot{x}_j]_{L'}$, $[\dot{x}_i, \dot{x}_j]_L = \alpha[\dot{x}_i, \dot{x}_j]_{L'}$ α a real constant, $\neq 0$ (assuming that canonical Poisson brackets are satisfied) then really,

$$L' = aL + \frac{d\Phi}{dt} \tag{2}$$

and vice versa, if (2) holds then L and L' are equivalent in the above sense.

Going back to the proof of our assertion, announced before, let us first present the following Lemma:

Given are three functions

$$f_j(\underline{x}, \dot{\underline{x}}), \qquad j = 1, 2, 3.$$

If there exists a Lagrange function $V^{(0)}(\underline{x},\dot{\underline{x}})$ such that f_j coincide with the Euler variations of $V^{(0)}$, then the most general expression for the Lagrange function $V(\underline{x},\dot{\underline{x}})$, which yields the same Euler variations, is

$$V = V^{(0)} + \frac{d\Phi}{dt} . \tag{3}$$

Proof: The function f_j do not, by assumption, depend on $\ddot{\underline{x}}$ and also, by assumption, are supposed to be Euler variations relatively $V^{(0)}$. If there are other Lagrange functions V, leading to the same Euler variations, then these $V's$ have to satisfy also the relations

$$f_j(\underline{x},\dot{\underline{x}}) = \sum_{k=1}^{3}\left(\frac{\partial^2 V}{\partial\dot{x}_j\partial\dot{x}_k}\ddot{x}_k + \frac{\partial^2 V}{\partial\dot{x}_j\partial x_k}\dot{x}_k\right) - \frac{\partial V}{\partial x_j} . \tag{4}$$

Hence

$$\frac{\partial^2 V}{\partial\dot{x}_j\partial\dot{x}_k} = 0$$

or

$$V(\underline{x}) = \sum_{j=1}^{3} A_j(\underline{x})\,\dot{x}_j + B(\underline{x}) . \tag{5}$$

Relation (5) concerns, of course, also the Lagrange function

$$V^{(0)} = \sum_{j=1}^{3} A_j^{(0)}\,\dot{x}_j + B^{(0)} .$$

If we insert (5) into (4) we get

$$f_j(\underline{x},\dot{\underline{x}}) = \sum_{k=1}^{3} a_{jk}(\underline{x})\dot{x}_k + b_j(\underline{x}) \tag{6}$$

where

$$\frac{\partial A_j}{\partial x_k} - \frac{\partial A_k}{\partial x_j} = a_{jk} = -a_{kj} \tag{7}$$

and

$$b_j = -\frac{\partial B}{\partial x_j} , \tag{8}$$

By assumption a_{jk} as well as b_j $(j,k = 1,2,3)$ are given. The most general solution of (7) and (8) reads

$$A_j = A_j^{(0)} + \frac{\partial\Phi'}{\partial x_j} \tag{9}$$

and

$$B = B^{(0)} + B' , \qquad B' \text{ being a constant.}$$

Then (3) follows immediately from (5) where

$$\Phi = \Phi' + B't .$$

This proves the Lemma.

This result can be immediately generalized for the case of an n–dimensional Euclidean space $n = 2,4,\ldots,$.

Let us now make a digression taking into account the assumptions of our main assertion as well as the just proven Lemma. According to our hypothesis as well as (1) and (6) a_{jk} behaves like a tensor and b_j – like a vector under rotations. Relation (7) is then similar to that obtained in electrodynamics. There too the so called vector potential $A(x)$ does not need to form a vector, in contradiction to the field strength, which forms a genuine, skew–symmetric tensor. The relation (9) resembles the gauge transformation. We have also the Hamilton function

$$H \equiv \sum_{k=1}^{3} p_k\dot{x}_k - L = \frac{1}{2}\left(\underline{p} + \underline{A}\right)^2 + B$$

where

$$p_k \equiv \frac{\partial L}{\partial\dot{x}_k} = \dot{x}_k - A_k$$

analogous to the Hamilton function of a charged particle in an external electromagnetic field. B plays here the role of the so called scalar potential.

After this digression let us tackle the problem of rotations.

Let us denote

$$x'_j = \sum_{k=1}^{3} R_{jk}x_k .$$

We have also, according to (5), for a Lagrange function $V(\underline{x},\dot{\underline{x}})$ arbitrarily chosen among the set (3)

$$V(\underline{x}',\dot{\underline{x}}') = \sum_{j=1}^{3}\sum_{k=1}^{3} A_j(\underline{x}')R_{jk}\dot{x}_k + B(x') .$$

Notice that the Euler–Lagrange Equations

$$f_j(\underline{x}',\dot{\underline{x}}') = \sum_{k=1}^{3} R_{jk}f_k(\underline{x},\dot{x}) = 0$$

remain unchanged, as f_j is a vector with respect to the rotations. Further we have

$$\frac{\partial A_j(\underline{x}')}{\partial x'_k} - \frac{\partial A_k(\underline{x}')}{\partial x'_j} = a_{jk}(\underline{x}') =$$

$$= \sum_{l=1}^{3}\sum_{m=1}^{3} R_{jl}R_{km}a_{lm}(\underline{x})$$

$$\frac{\partial B(\underline{x}')}{\partial x'_j} = -b_j(\underline{x}') = -\sum_{k=1}^{3} R_{jk}b_k(\underline{x})$$

or

$$\sum_{j=1}^{3}\left(\frac{\partial}{\partial x_s}A_j(\underline{x}')R_{jr} - \frac{\partial}{\partial x_r}A_j(\underline{x}')R_{js}\right) = a_{rs}(\underline{x}) \tag{10}$$

$$\frac{\partial B(\underline{x}')}{\partial x_t} = -b_t(\underline{x})\ . \tag{11}$$

Then by virtue of our Lemma

$$V(\underline{x}') = \sum_{j=1}^{3}\sum_{k=1}^{3} A_j(\underline{x}')R_{jr}\dot{x}_r + B(\underline{x}') =$$

$$= V(\underline{x}) + \tfrac{d\Phi}{dt}(\underline{x}', R)$$

This proves the assertion.

Turning to our second remark, it is well known that the symmetry of the Euler–Lagrange Equations can exceed the symmetry content of the Lagrange function from which they originate. Our second remark is related to this problem, namely how to recover a rotationally invariant Lagrange function from a non–symmetric Lagrange function, which, however, gives rise to rotationally covariant Euler variations.

As the rotational group is compact and consequently has a finite volume, the simplest way to solve this problem seems to be to integrate the non–symmetric Lagrange function over the group [2]. This procedure should not apparently affect the Euler variations, which were supposed to be covariant under rotations.

This procedure, although seemingly logically well founded, is, however, not as straightforward as can be seen on the following example in the 3–dimensional Euclidean space [5].

Let us inspect the Lagrange function

$$V(\underline{x}, \dot{\underline{x}}, \underline{b}) = \frac{(\underline{b}\underline{x})\,(\underline{b}\,(\underline{x}\wedge\dot{\underline{x}}))}{|\underline{x}|\ \ (\underline{b}\wedge\underline{x})^2} \tag{12}$$

where $\underline{b}$ is an arbitrary constant vector; we may, of course, restrict ourselves to

$$|\underline{b}| \equiv \left(b_1^2 + b_2^2 + b_2^2\right)^{1/2} = 1.$$

Here $(\underline{a}\underline{b})$ stands for $\sum_{i=1}^{3} a_i b_i$ and

$$(\underline{a}\wedge\underline{b})_i = \sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}a_j b_k\ .$$

Strange enough the Euler variations

$$\frac{d}{dr}\frac{\partial V}{\partial \dot{x}_i} - \frac{\partial V}{\partial x_i} = \sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}\,\frac{x_j\dot{x}_k}{|x|^3} \tag{13}$$

form a genuine vector with respect to rotations and consequently do not depend on the direction of $\underline{b}$.

Let us try to integrate V over the group of rotations, to obtain the rotationally invariant part of it, or, what turns out to lead to the same goal, average over all directions of $\underline{b}$.

To this aim we introduce the following notation

$$\underline{b}\,\underline{x} = r\cos\theta$$

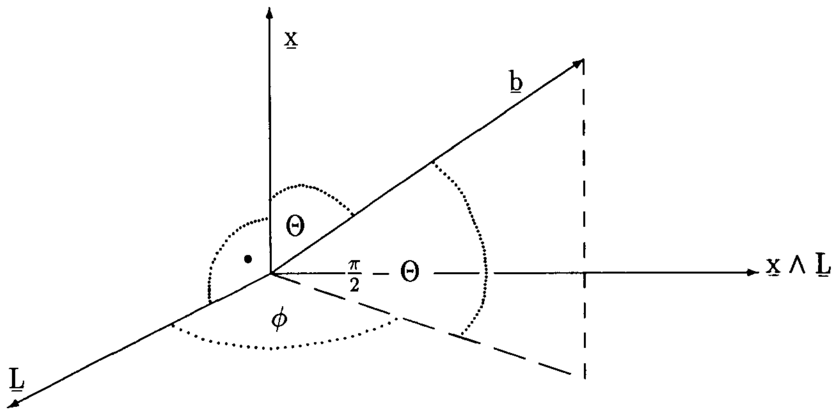
$$\underline{b}\,\underline{L} = L\sin\theta\cos\phi$$

$$|\underline{b}\wedge\underline{x}| = r\sin\theta$$

$$r \equiv |\underline{x}|, \qquad L \equiv |\underline{L}|$$

$$\underline{L} \equiv \underline{x}\wedge\dot{\underline{x}}$$

$$|\underline{b}| = 1$$



Then

$$V_{inv} \equiv \frac{1}{\Omega} \int V \, d\Omega = \frac{1}{\Omega} \int_0^{\pi} \int_0^{2\pi} \frac{L \cos\theta \sin\theta \cos\phi}{r \sin^2\theta} \sin\theta d\theta d\phi = 0 \, .$$

This Lagrange function, for sure, does not give rise to the Euler variation (13). The reason for the failure of the method of averaging, presented above, is the singular behavior of V for $r = 0$ and/or for $\underset{\sim}{b}$ being parallel to $\underset{\sim}{x}$.

This singularity prevents the interchanging of integration over the group with differentiation with respect to $\underset{\sim}{x}$ and $\dot{\underset{\sim}{x}}$.

We close our considerations with exemplifying our assertion on the model (13). For

$$\underset{\sim}{b} = (0,0,1)$$

we have

$$V = \frac{x_3(x_1\dot{x}_2 - \dot{x}_1 x_2)}{r\,(x_1^2 + x_2^2)} \, . \tag{14}$$

As the Euler variation is a genuine vector and so transforms covariantly under the rotations, we should have, taking into account our assertion

$$V(R\underset{\sim}{x}) - V(\underset{\sim}{x}) = \frac{d\Phi}{dt} \, . \tag{15}$$

Let us check (15) by inspection for a special case of transformation, namely the rotation

$$x_1' = x_2, \qquad x_2' = x_3, \qquad x_3' = x_1 .$$

Then we have

$$V(\underset{\sim}{x}') - V(\underset{\sim}{x}) = \frac{x_1(x_2\dot{x}_3 - \dot{x}_2 x_3)}{r\,(x_2^2 + x_3^2)} - \frac{x_3(x_1\dot{x}_2 - \dot{x}_1 x_2)}{r\,(x_1^2 + x_2^2)} =$$

$$\frac{d}{dt}\left(arc\ tan \frac{x_1 x_3}{x_2 r} \right) .$$

Notice that for this particular model we have

$$\frac{d\Phi(R\underset{\sim}{x}, \underset{\sim}{b}, \phi, \underset{\sim}{n})}{dt} = \frac{d\Phi(R'\underset{\sim}{x}, \underset{\sim}{b}, \phi', \underset{\sim}{n}')}{dt} - \tag{16}$$

$$- \frac{d\Phi(R'R\underset{\sim}{x}, \underset{\sim}{b}, \phi', \underset{\sim}{n}')}{dt} = \frac{d\Gamma(R\underset{\sim}{x})}{dt} - \frac{d\Gamma(\underset{\sim}{x})}{dt}$$

where

$$R = R(\phi, \underset{\sim}{n}) \quad |\underset{\sim}{n}| = 1$$

and $R' = R(\phi', \underset{\sim}{n}')$, $|\underset{\sim}{n}'| = 1$ is any rotation which connects $\underset{\sim}{b}$ and $\underset{\sim}{n}$, viz.

$$\underset{\sim}{b} = R(\phi', \underset{\sim}{n}')\underset{\sim}{n}. \tag{17}$$

Such a rotation is a product of an arbitrary fixed rotation transforming $\underset{\sim}{n}$ into $\underset{\sim}{b}$ and any element of the little group of $\underset{\sim}{b}$ or/and $\underset{\sim}{n}$. There are good reasons to expect that the relation of the type (16) does not only conforms to the considered model but is a general property of Lagrange functions subjected to rotations. To justify (16) let us start from the formula established before, namely (15), viz.

$$L(R\underset{\sim}{x}, R\dot{\underset{\sim}{x}}; \underset{\sim}{b}) - L(\underset{\sim}{x}, \dot{\underset{\sim}{x}}; \underset{\sim}{b}) = \tag{18}$$

$$\frac{d\Phi}{dt}(R\underset{\sim}{x}, \underset{\sim}{b}, \phi, \underset{\sim}{n}) \, .$$

Since

$$L(R\underset{\sim}{x}, R\dot{\underset{\sim}{x}}; \underset{\sim}{n}) = L\left(\underset{\sim}{x}, \dot{\underset{\sim}{x}}, R^{-1}\underset{\sim}{n}\right) = L(\underset{\sim}{x}, \dot{\underset{\sim}{x}}\underset{\sim}{n}) \tag{19}$$

as

$$R(\phi, \underset{\sim}{n})\underset{\sim}{n} = \underset{\sim}{n},$$

we may write (18), by virtue of (19) and (17), as

$$L(R\underline{x}, R\dot{\underline{x}}; \underline{b}) - L\left(R\underline{x}, R\dot{\underline{x}}; R'^{-1}\underline{b}\right) + L\left(\underline{x}, \dot{\underline{x}}; R'^{-1}\underline{b}\right)$$

$$- L(\underline{x}, \dot{\underline{x}}; \underline{b}) =$$

$$= -[L(R'R\underline{x}, R'R\dot{\underline{x}}; b) - L(R\underline{x}, R\dot{\underline{x}}; b)] +$$

$$+ [L(R'\underline{x}, R'\dot{\underline{x}}; \underline{b}) - L(\underline{x}, \dot{\underline{x}}; \underline{b})]$$

which proves the assertion (16).

The results presented here can be generalized to the case of systems of several particles as well as to the Galilei group symmetry.

The problems considered here are not new and numerous papers were devoted to them [6]. It is not excluded that the results reported here were obtained earlier by other authors by using a more sophisticated mathematical apparatus. As mentioned before, the main reason to present our results here was to emphasize the pedestrian way these results were derived by us.

We are grateful to Professor Jan Rzewuski and Dr. Marek Mozrzymas for comments made after my talk.

References

[1] see e.g. R Courant and D. Hilbert "Methoden der Mathematischen Physik" 2-nd edition, Berlin, 1931
[2] We are grateful to Mr. Jerzy Cisło for making this remark
[3] E.L. Hill, Revs. Mod. Phys. **23** (1951) 253
[4] M. Henneaux, Am. Phys. N.Y., **140** (1982) 45
[5] J.C. Houard, J. Math. Phys. **18** (1977) 502
[6] J.-M. Levy-Leblond, Commun. Math. Phys. **12** (1969) 64,
J.-M. Levy-Leblond, "Galilei-Group and Galilean Invariance" in "Group Theory and Its Applications", vol. II (edited E.M. Loebl), Academic Press, 1971
J.-M. Levy-Leblond, Am. J. Phys. **39/5** (1971) 502
M. Henneaux and J.C. Shepley, J. Math. Phys. **23** (1982) 2101

TUNNELLING OF NEUTRAL PARTICLE WITH SPIN 1/2 THROUGH MAGNETIC FIELD

M. MIJATOVIĆ, G. IVANOVSKI, B. VELJANOSKI AND G. APOSTOLOVSKA
Institute of Physics,
Faculty of Natural Sciences and Mathematics,
Skopje, Macedonia

Abstract. The main part of this lecture concerns work announced in Ref. [1]. The last part contains some new results, not previously reported. We have investigated the scattering and bound states of a nonrelativistic and relativistic spin-1/2 particle in the system of N magnetic barriers (or magnetic wells). We have studied two types of problems: tunnelling with spin and band structure.

1. Introduction

The motion of a neutral spin-1/2 particle through a magnetic field has been extensively studied in recent years, first motivated by the measurement of the final state polarization in the neutron-spin echo experiments, and also by the measurement of the final state of the neutron wave function in neutron interferometry.

The second approach to this consideration is a study of the anomalous magnetic moment of the neutrino in connection with the solar neutrino problem. Namely, it was argued that a neutrino magnetic moment of the order of 10^{-10} μ_B would be sufficient to flip a large number of the left-handed neutrinos into right-handed ones over length L in a magnetic field B such that BL $\sim 10^7$Tm, e.q. L $\sim 10^8$m, B $\sim 10^{-1}$T.

The one dimensional treatment of the motion of a quantum particle through the field of potential barriers is the simplest approximation. Since the historical paper of Kronig and Penney on electron motion in an infinite periodic chain, this model has served as a valuable tool in explaining several interesting properties of real materials as forbidden energy gaps.

Our paper has several aims. In Section 2 we will make a generalization of the paper [2] i.e. we will study the tunneling of a neutral spin-1/2 particle through a finite number of magnetic square wells (or barriers, depending on spin polarization).

In Section 3 we consider the bound states of an infinite chain of identical magnetic square wells.

The Section 4 contains the relativistic generalization of previous results.

2. Tunnelling with Spin

Let us study the scattering states of a spinless (for simplicity) particle which incidents from the left and transmits through N rectangular potential barriers of width a, separated by the gap b (the period has length $l = a + b$) (Fig. 1).

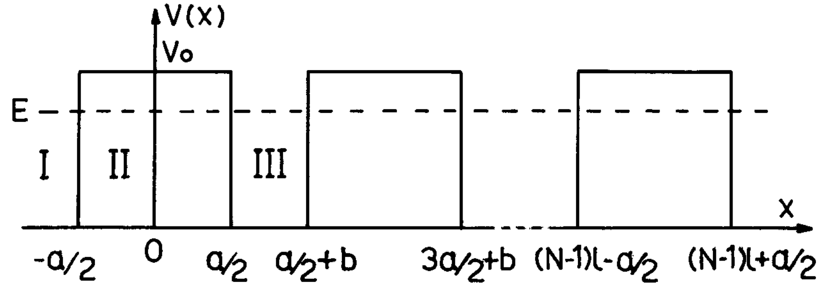


Fig. 1. Multiple magnetic potential barrier

In the beginning we restricted our consideration to two barriers. Then the solution of the Schrödinger equation can be written in the form (for $E < V_0 = \mu_B B$)

$$\Psi(x) = \begin{cases} Ae^{ik_0x} + Be^{-ik_0x}, & x < -a/2 \\ Ce^{-\nu x} + De^{\nu x}, & -a/2 < x < a/2 \\ A_1e^{ik_0x} + B_1e^{-ik_0x}, & a/2 < x < a/2 + b \\ C_1e^{-\nu x} + D_1e^{\nu x}, & a/2 + b < x < 3a/2 + b \\ Fe^{ik_0x} + Ge^{-ik_0x}, & x > 3a/2 + b. \end{cases} \qquad (1)$$

The connection between incoming and outcoming data for the scattering on two barriers follows from the relations

$$\begin{pmatrix} A \\ B \end{pmatrix} = \mathcal{M}^2 \begin{pmatrix} F \\ G \end{pmatrix}, \qquad (2)$$

where

$$\mathcal{M}^2 = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} M_{11} & M_{12}C^{-1} \\ M_{21}C & M_{22} \end{pmatrix}, \qquad (3)$$

and the matrix elements of so-called M-matrix are given by

$$M_{11} = e^{i\mu} \cosh\lambda, \qquad M_{12} = i \sinh\lambda,$$

$$M_{21} = -i \sinh\lambda, \qquad M_{22} = e^{-i\mu} \cosh\lambda,$$

$$\mu = k_0a - \arctan(\frac{\varepsilon}{2} \tan\nu a), \qquad \lambda = \sinh^{-1}(\frac{\eta}{2} \sin\nu a), \qquad (4)$$

$$\varepsilon = \frac{\nu}{k_0} - \frac{k_0}{\nu}, \qquad \eta = \frac{\nu}{k_0} + \frac{k_0}{\nu},$$

$$k_0 = \frac{\sqrt{2mE}}{\hbar}, \qquad v_0 = \frac{\sqrt{2m\mu_B B}}{\hbar},$$

$$\nu = \sqrt{{v_0}^2 - {k_0}^2}.$$

It is not difficult to show that for N barriers the expression for the M-matrix becomes

$$\mathcal{M}^N = \prod_{j=0}^{N-1} \begin{pmatrix} M_{11} & M_{12}C^{-j} \\ M_{21}C^j & M_{22} \end{pmatrix}. \qquad (5)$$

The matrix product (5) is calculated in the Appendix of Ref. [1] with the result

$$\mathcal{M}^N = (x_1)^{N-1} \begin{pmatrix} \mathcal{A} & \mathcal{D}C^{-(N-1)} \\ \mathcal{F} & \mathcal{H}C^{-(N-1)} \end{pmatrix} + (x_2)^{N-1} \begin{pmatrix} \mathcal{B} & \mathcal{E}C^{-(N-1)} \\ \mathcal{G} & \mathcal{J}C^{-(N-1)} \end{pmatrix}, \qquad (6)$$

where

$$\mathcal{A} = \frac{C + x_1M_{11}}{x_2 - x_1}, \qquad \mathcal{B} = \frac{-C + x_2M_{11}}{x_2 - x_1},$$

$$\mathcal{D} = -\frac{x_1M_{12}}{x_2 - x_1}, \qquad \mathcal{E} = \frac{x_2M_{12}}{x_2 - x_1},$$

$$\mathcal{F} = -\frac{x_1M_{21}}{x_2 - x_1}, \qquad \mathcal{G} = \frac{x_2M_{21}}{x_2 - x_1}, \qquad (7)$$

$$\mathcal{H} = \frac{1 - x_1M_{22}}{x_2 - x_1}, \qquad \mathcal{J} = \frac{1 + x_2M_{22}}{x_2 - x_1},$$

and x_1 and x_2 are the roots of the equation

$$x^2 = x(M_{11} + CM_{22}) - C. \qquad (8)$$

For the calculation of transmission coefficient T we need the matrix element $\mathcal{M}_{11}^N$ only

$$\mathcal{M}_{11}^N = (x_1)^{N-1}\mathcal{A} + (x_2)^{N-1}\mathcal{B}. \tag{9}$$

Using the notation (4) the equation (8) becomes

$$x^2 - 2x \cosh\lambda \cos(\mu - k_0 l) e^{ik_0 l} + e^{2ik_0 l} = 0. \tag{10}$$

Introducing the new variables γ and γ^* by

$$L = \cosh\lambda cos(\mu - k_0 l) = \begin{cases} \cos\gamma & \text{if} \quad L \le 1 \\ \cosh\gamma^* & \text{if} \quad L \ge 1 \end{cases}, \tag{11}$$

the roots of the equation (10) can be written in the form

$$x_{1/2} = e^{ik_0 l}\left(\begin{Bmatrix} \cos\gamma \\ \cosh\gamma^* \end{Bmatrix} \mp i \begin{Bmatrix} \sin\gamma \\ \sinh\gamma^* \end{Bmatrix}\right), \tag{12}$$

where the variables γ and γ^* are used according the condition (11). Then, transmission coefficient can be found in the form

$$T = \frac{1}{|\mathcal{M}_{11}^N|^2} = \frac{1}{1 + \sinh^2\lambda \begin{Bmatrix} \frac{\sin^2 N\gamma}{\sin^2\gamma} \\ \frac{\sinh^2 N\gamma^*}{\sinh^2\gamma^*} \end{Bmatrix}}. \tag{13}$$

The above method can be simply generalized to the cases $E > V_0$ and also to scattering of a spin-1/2 particle by a rectangular magnetic barriers. Here we present only the final formula for the transmission coefficient if $\bar{k} \ge 1$, ($\bar{k} = k_0/\alpha$, $\alpha = av_0$).

$$T = \frac{|\alpha_0|^2}{1 + \frac{1}{4\bar{k}^2(\bar{k}^2+1)} \sin^2 \alpha\sqrt{\bar{k}^2+1} \begin{Bmatrix} \frac{\sin^2 N\gamma_1}{\sin^2\gamma_1} \\ \frac{\sinh^2 N\gamma_1^*}{\sinh^2\gamma_1^*} \end{Bmatrix}} + \frac{|\beta_0|^2}{1 + \frac{1}{4\bar{k}^2(\bar{k}^2-1)} \sin^2 \alpha\sqrt{\bar{k}^2-1} \begin{Bmatrix} \frac{\sin^2 N\gamma_2}{\sin^2\gamma_2} \\ \frac{\sinh^2 N\gamma_2^*}{\sinh^2\gamma_2^*} \end{Bmatrix}} \tag{14}$$

where we have introduced the variables γ_j, γ_j^* $(j = 1, 2)$ in the following manner

$$L_1 = \cos(\alpha\sqrt{\bar{k}^2+1}) \cos\bar{k}\alpha\xi - \frac{2\bar{k}^2+1}{2\bar{k}\sqrt{\bar{k}^2+1}} \sin(\alpha\sqrt{\bar{k}^2+1}) \sin\bar{k}\alpha\xi = \begin{cases} \cos\gamma_1, & L_1 \le 1 \\ \cosh\gamma_1^*, & L_1 \ge 1 \end{cases},$$

$$L_2 = \cos(\alpha\sqrt{\bar{k}^2-1}) \cos\bar{k}\alpha\xi - \frac{2\bar{k}^2-1}{2\bar{k}\sqrt{\bar{k}^2-1}} \sin(\alpha\sqrt{\bar{k}^2-1}) \sin\bar{k}\alpha\xi = \begin{cases} \cos\gamma_2, & L_2 \le 1 \\ \cosh\gamma_2^*, & L_2 \ge 1 \end{cases}, \tag{15}$$

where $\xi = b/a$,

On the other hand if $0 < \bar{k} \le 1$, the following formula is satisfied

$$T = \frac{|\alpha_0|^2}{1 + \frac{1}{4\bar{k}^2(\bar{k}^2+1)} \sin^2 \alpha\sqrt{\bar{k}^2+1} \begin{Bmatrix} \frac{\sin^2 N\gamma_1}{\sin^2\gamma_1} \\ \frac{\sinh^2 N\gamma_1^*}{\sinh^2\gamma_1^*} \end{Bmatrix}} + \frac{|\beta_0|^2}{1 + \frac{1}{4\bar{k}^2(1-\bar{k}^2)} \sin^2 \alpha\sqrt{1-\bar{k}^2} \begin{Bmatrix} \frac{\sin^2 N\gamma}{\sin^2\gamma} \\ \frac{\sinh^2 N\gamma^*}{\sinh^2\gamma^*} \end{Bmatrix}} \tag{16}$$

where

$$L = \cosh(\alpha\sqrt{1-\bar{k}^2}) \cos\bar{k}\alpha\xi + \frac{1-2\bar{k}^2}{2\bar{k}\sqrt{1-\bar{k}^2}} \sinh(\alpha\sqrt{1-\bar{k}^2}) \sin\bar{k}\alpha\xi = \begin{cases} \cos\gamma, & L \le 1 \\ \cosh\gamma^*, & L \ge 1 \end{cases}. \tag{17}$$

The plot of the transmission coefficient obtained from the equations (14) and (16) is shown in Fig. 2. On the x-axis is the variable $\bar{k}$ (impact momentum k_0 divided by $v_0 = \sqrt{2m\mu_B B}/\hbar$). The white and black bands in the bottom of the graph describes the bound states of a infinite chain, which will be considered in Section 3 and 4.

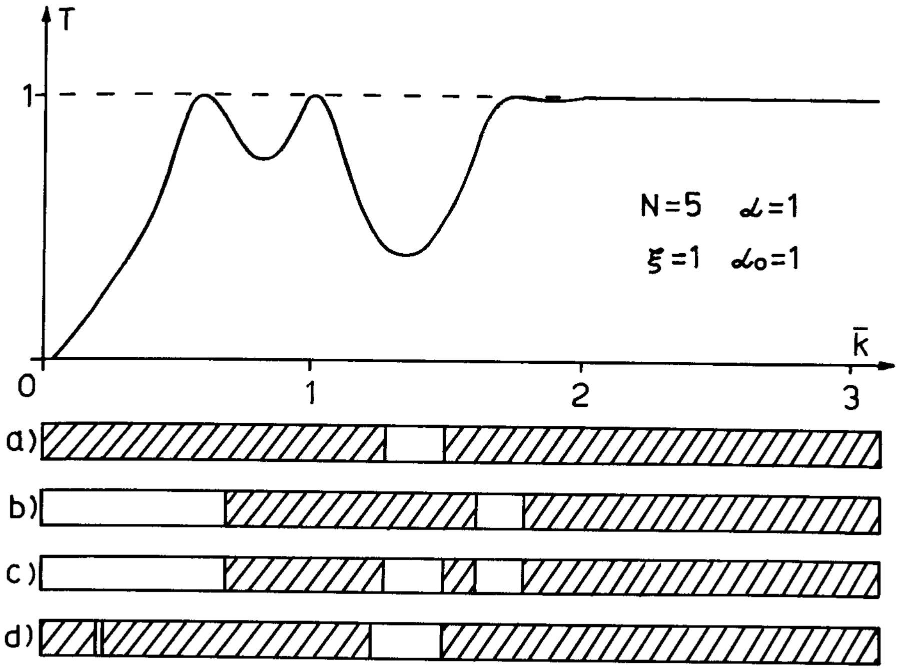


Fig. 2. *Above.* The transmission coefficient of the system of rectangular magnetic potential barriers as a function of impact momentum for fixed values: $N = 5$ (number of barriers), $\xi = 1$ (ratio between b and a), $\alpha = 1$ ($\alpha = av_0$) and $\alpha_0 = 1$ (spin polarization).
Below. The band structure of an infinite chain of rectangular magnetic potential barriers for fixed values: $\xi = 1$ and $\alpha = 1$ which correspond to (a) up-projection of spin; (b) down–projection of spin; (c) solution of the equations (18), (19) and (d) relativistical up-projection $\Pi = 1$.

3. Band Structure

The band structure of the infinite one-dimensional magnetic chain can be expressed quite simply in terms of the properties of a magnetic moment in the presence of a single magnetic barrier [1]. Here we present the final result for band energy equations only:

$$\cos\gamma = \cos(\alpha\sqrt{\overline{k}^2+1})\cos\overline{k}\alpha\xi - \frac{2\overline{k}^2+1}{2\overline{k}\sqrt{\overline{k}^2+1}}\sin(\alpha\sqrt{\overline{k}^2+1})\sin\overline{k}\alpha\xi, \quad (18a)$$

$$\cos\gamma = \cosh(\alpha\sqrt{1-\overline{k}^2})\cos\overline{k}\alpha\xi + \frac{1-2\overline{k}^2}{2\overline{k}\sqrt{1-\overline{k}^2}}\sinh(\alpha\sqrt{1-\overline{k}^2})\sin\overline{k}\alpha\xi, \quad (18b)$$

where $0 \le \overline{k} \le 1$. The first equation corresponds to the up-projection of spin (first band in Fig. 2.). The second equation corresponds to the down-projection of spin (second band in Fig. 2.). The third band in Fig. 2. in fact is folded over the second band and it corresponds to the common solutions of the equations (18a) and (18b).

In the case $\overline{k} \ge 1$ the equation (18b) transforms into

$$\cos\gamma = \cos(\alpha\sqrt{\overline{k}^2-1})\cos\overline{k}\alpha\xi - \frac{2\overline{k}^2-1}{2\overline{k}\sqrt{\overline{k}^2-1}}\sin(\alpha\sqrt{\overline{k}^2-1})\sin\overline{k}\alpha\xi, \quad (19)$$

while the equation (18a) remins the same. The folded over permitted bands for a different spin projection mean that a flip of the spin is possible. However, this process is forbidden according to the law of conservation of angular momentum.

4. Dirac Particle in Magnetic Field

Let us study the Dirac particle of mass m which incidents from the left and transmits through N magnetic barriers (Fig. 1.). In the case N=1 the eigenvectors are

$$\Psi_I = A\begin{pmatrix}\alpha_0\\ \beta_0\\ \frac{c\hbar k_0}{E+mc^2}\alpha_0\\ \frac{c\hbar k_0}{E+mc^2}\beta_0\end{pmatrix}e^{ik_0x} + B\begin{pmatrix}\alpha_1\\ \beta_1\\ -\frac{c\hbar k_0}{E+mc^2}\alpha_1\\ -\frac{c\hbar k_0}{E+mc^2}\beta_1\end{pmatrix}e^{-ik_0x}, \quad (20a)$$

$$\Psi_{II} = \begin{pmatrix}C'e^{-ik_1x}\\ C''e^{-ik_2x}\\ -\frac{c\hbar k_1}{E+mc^2+\mu_BB}C'e^{-ik_1x}\\ -\frac{c\hbar k_2}{E+mc^2-\mu_BB}C''e^{-ik_2x}\end{pmatrix} + \begin{pmatrix}D'e^{ik_1x}\\ D''e^{ik_2x}\\ \frac{c\hbar k_1}{E+mc^2+\mu_BB}D'e^{ik_1x}\\ \frac{c\hbar k_2}{E+mc^2-\mu_BB}D''e^{ik_2x}\end{pmatrix}, \quad (20b)$$

$$\Psi_{III} = F\begin{pmatrix}\alpha_0'\\ \beta_0'\\ \frac{c\hbar k_0}{E+mc^2}\alpha_0'\\ \frac{c\hbar k_0}{E+mc^2}\beta_0'\end{pmatrix}e^{ik_0x} + G\begin{pmatrix}\alpha_1'\\ \beta_1'\\ -\frac{c\hbar k_0}{E+mc^2}\alpha_1'\\ -\frac{c\hbar k_0}{E+mc^2}\beta_1'\end{pmatrix}e^{-ik_0x}. \quad (20c)$$

The joining conditions at $x = -a/2$ and $x = a/2$ determine the M-matrix which elements are same as (4), with a new variables

$$\frac{\nu}{k_0} = \sqrt{\left(\sqrt{1+\frac{1}{\Pi^2}} - \frac{\Pi}{2\bar{k}^2}\right)^2 - \frac{1}{\Pi^2}}, \tag{21}$$

$$\varepsilon = \frac{\nu}{k_0}\frac{e}{e_0} - \frac{k_0}{\nu}\frac{e_0}{e}, \qquad \eta = \frac{\nu}{k_0}\frac{e}{e_0} + \frac{k_0}{\nu}\frac{e_0}{e}, \tag{22}$$

where

$$\frac{e}{e_0} = \frac{\Pi+1}{\Pi+1-\frac{\Pi^2}{2\bar{k}^2}}, \tag{23}$$

and $\Pi = k_0\hbar/mc$.

References

M. Mijatović, G. Ivanovski, B. Veljanoski and K. Trenčevski, to be published
Barut, A. O., Božić, M., Marić, Z. and Rauch, H., Z. Phys. **A328**, 1 (1987)

LIST of PARTICIPANTS

1. Viktor ABRAMOV
Department of Applied Mathematics
Tartu University
Liivi 2-417
EE-2400 Tartu, ESTONIA
e-mail: ems@math.tartu.ew.su

2. William E. BAYLIS
Physics Department
University of Windsor
Windsor, Ontario N9B 3P4, CANADA
e-mail baylis@ucc.uwindsor.ca

3. Ingemar BENGTSSON
Institute of Theoretical Physics
Chalmers University of Technology
S-412 96 Göteborg, SWEDEN
e-mail: tfeib@fy.chalmers.se

4. Andrzej BOROWIEC
Instytut Fizyki Teoretycznej
Uniwersytet Wrocławski
pl. Maksa Borna 9
PL-50-204 Wrocław, POLAND
e-mail: borowiec@plwruw11

5. Krystyna BUGAJSKA
Department of Mathematics
York University
4700 Keele Street
North York, Ontario, CANADA M3J 1P3
e-mail: mathstat@yorkvm1.bitnet, for Dr. Bugajska

6. Ursula CAROW-WATAMURA
Department of Physics
Tohoku University
Aramaki, Aobaku
Sendai 980, JAPAN
e-mail watamura@jpntuvm0

7. Jerzy CISŁO
Instytut Fizyki Teoretycznej
Uniwersytet Wrocławski

pl. Maksa Borna 9
PL-50-204 Wrocław, POLAND

8. Ludwik DĄBROWSKI
Interdisciplinary Laboratory, SISSA
Strada Costiera 11
34014 Trieste, ITALY
e-mail: dabrowski@itssissa.bitnet; fax (+39)40-3787528

9. Geoffrey DIXON
Mathematics and Physics Department
Brandeis University
Waltham, Massachusetts 02254, USA
e-mail: dixon@binah.cc.brandeis.edu

10. Chris DORAN
Department of Applied Mathematics and Theoretical Physics
University of Cambridge
10 Silver Street
Cambridge CB3 9EW, GREAT BRITAIN
e-mail: cjld1@amtp.cam.ac.uk

11. Michel DUBOIS-VIOLETTE
Laboratoire de Physique Thèorique et Hautes Énergies
Université de Paris-Sud
Batiment 211
91400 Orsay Cedex, FRANCE
e-mail: flad@qcd.circe.fr

12. Andrzej FRYDRYSZAK
Instytut Fizyki Teoretycznej
Uniwersytet Wrocławski
pl. Maksa Borna 9
PL-50-204 Wrocław, POLAND

13. Gary W. GIBBONS
Department of Applied Mathematics and Theoretical Physics
University of Cambridge
10 Silver Street
Cambridge CB3 9EW, GREAT BRITAIN
e-mail: gwg1@hx.cam.ac.uk

14. Dan Radu GRIGORE
Department of Theoretical Physics
Institute of Atomic Physics

P.O.Box MG 06 Bucharest-Magurele, ROMANIA
e-mail:ifa@pi-bucuresti.th-darmstadt.de; telex: 11350 ifa r

15. Ralf GRUNEWALD
Humboldt Universität zu Berlin,
Sektion Mathematik
Unter den Linden 6, PSF - 1297,
O–1086 Berlin, GERMANY

16. Steve GULL
Department of Physics, Cavendish Laboratory
University of Cambridge
Madingsley Road
Cambridge CB3 0HE, GREAT BRITAIN
e-mail: steve@vax.ra.physics.cambridge.ac.uk

17. Dmitri GUREVICH
Max-Planck-Institut für Mathematik
Gottfried-Claren-Strasse 26
D-5300 Bonn 3, GERMANY
e-mail: Gurevich@mpim-bonn.mpg.de

18. Leopold HALPERN
Department of Physics
Florida State University
Tallahassee, Florida 32306-3016, USA

19. Jerzy HAŃĆKOWIAK
Instytut Fizyki
Wyższa Szkoła Pedagogiczna
pl. Słowiański 6
PL-65-069 Zielona Góra, POLAND

20. David HESTENES
Physics Department
Arizona State University
Tempe AZ 85287, USA
e-mail: brydon@phyast.la.asu.edu

21. Alexey P. ISAEV
Laboratory of Theoretical Physics
Joint Institute of Nuclear Research
Dubna
Moscow, SU 101000, RUSSIA
e-mail: isaevap@theor.jinr.dubna.su

22. Arkadiusz JADCZYK
Instytut Fizyki Teoretycznej
Uniwersytet Wrocławski
pl. Maksa Borna 9
PL-50-204 Wrocław, POLAND
e-mail: ajad@plwruw11

23. Bernard JANCEWICZ
Instytut Fizyki Teoretycznej
Uniwersytet Wrocławski
pl. Maksa Borna 9
PL-50-204 Wrocław, POLAND

24. Cezary JUSZCZAK
Instytut Fizyki Teoretycznej
Uniwersytet Wrocławski
pl. Maksa Borna 9
PL-50-204 Wrocław, POLAND

25. Andrzej KARPIO
Katedra Fizyki
Filia Uniwersytetu Warszawskiego
ul. Lipowa 4
15-441 Białystok, POLAND

26. Jaime KELLER
Div. de Ciencias Básicas, F.Q.,
Universidad Nacional Autonoma de Mexico (U.N.A.M.)
Apartado 70-528
04510 México, D.F., MEXICO
e-mail: keller@unamvm1.bitnet

27. Richard KERNER
Laboratoire de Physique Théorique
Institut Henri Poincaré
11, Rue Pierre et Marie Curie
F-75231 Paris, FRANCE
e-mail: rk@frunip62

28. Małgorzata KLIMEK
Instytut Matematyki
Politechnika Częstochowska
pl. Dąbrowskiego 73
PL-42-200 Częstochowa, POLAND

29. Wojciech KOPCZYŃSKI
Instytut Fizyki Teoretycznej
Uniwersytet Warszawski
ulica Hoża 69
00-681 Warszawa, POLAND
e-mail: kopcz@fuw.edu.pl

30. Anthony LASENBY
Department of Physics, Cavendish Laboratory
University of Cambridge
Madingsley Road
Cambridge CB3 0HE, GREAT BRITAIN
e-mail: anthony@vax.ra.physics.cambridge.ac.uk;
fax (+44)0223-354599

31. Jan ŁOPUSZAŃSKI
Instytut Fizyki Teoretycznej
Uniwersytet Wrocławski
pl. Maksa Borna 9
PL-50-204 Wrocław, POLAND

32. Pertti LOUNESTO
Department of Mathematics
Helsinki University of Technology
SF-02150 Espoo, FINLAND
e-mail: hhakula@dopey.hut.fi; fax 4871-36848

33. Jerzy LUKIERSKI
Instytut Fizyki Teoretycznej
Uniwersytet Wrocławski
pl. Maksa Borna 9
PL-50-204 Wrocław, POLAND
lukier@plwrtu11

34. Vladimir Dmitrievich LYAKHOVSKY
Department of Theoretical Physics
Sankt Petersburg State University
198904 Sankt Petersburg, RUSSIA
e-mail: LAM@onti.phys.lgu.spb.su
or: dvassil@sph.spb.su

35. John MADORE
Laboratoire de Physique Théorique et Hautes Énergies
Université de Paris-Sud
Batiment 211
91400 Orsay Cedex, FRANCE

36. Shahn MAJID
Dept. of Applied Mathematics and Theoretical Physics
University of Cambridge
Silver Street
Cambridge, CB3 9EW, GREAT BRITAIN
e-mail: majid@ampt.cam.ac.uk

37. Władysław MARCINEK
Instytut Fizyki Teoretycznej
Uniwersytet Wrocławski
pl. Maksa Borna 9
PL-50-204 Wrocław, POLAND

38. Sorin MARCULESCU
Fachbereich Physik
Universität - Gesamt Hohschule Siegen
Postfach 10-12-40
D-5900 Siegen, GERMANY
e-mail: marculescu@hrz.uni-siegen.dbp.de

39. Sergey A. MERKULOV
Department of Mathematics and Computer Sciences
Odense University
Campuswej 55
DK-5230 Odense, DENMARK
e-mail: merkulov@imada.ou.dk

40. Mijat MIJATOVIĆ
Institute of Physics
University "Cyril and Methodius"
P.O.Box 162
Skopje, MACEDONIA
fax: 38-91-228-141

41. Jan MILEWSKI
Instytut Fizyki
Uniwersytet Adama Mickiewicza
ul. Matejki 48/49
PL-60-769 Poznań, POLAND

42. Jan MOZRZYMAS
Instytut Fizyki Teoretycznej
Uniwersytet Wrocławski
pl. Maksa Borna 9
PL-50-204 Wrocław, POLAND

43. Marek MOZRZYMAS
Instytut Fizyki Teoretycznej
Uniwersytet Wrocławski
pl. Maksa Borna 9
PL-50-204 Wrocław, POLAND

44. Zbigniew OZIEWICZ
Instytut Fizyki Teoretycznej
Uniwersytet Wrocławski
pl. Maksa Borna 9
PL-50-204 Wrocław, POLAND
e-mail: oziewicz@plwruw11

45. Josep Manel PARRA
Departament de Fisica Fonamental
Universitat de Barcelona
Diagonal, 647
E-08028 Barcelona, SPAIN
e-mail: ubaffp01@ebcescal

46. William M. PEZZAGLIA
Department of Physics and Astronomy
San Francisco State University
1600 Holloway Avenue
San Francisco, California 94132, USA
e-mail: bpezzag@stars.sfsu.edu

47. Ziemowit POPOWICZ
Instytut Fizyki Teoretycznej
Uniwersytet Wrocławski
pl. Maksa Borna 9
PL-50-204 Wrocław, POLAND
e-mail: ziemek@plwruw11

48. Krzysztof RAPCEWICZ
Instytut Fizyki Teoretycznej
Uniwersytet Wrocławski
pl. Maksa Borna 9
PL-50-204 Wrocław, POLAND

49. Jerzy RAYSKI
Instytut Fizyki
Uniwersytet Jagielloński
ul. Reymonta 4
PL-30-059 Kraków, POLAND

50. Jakub REMBIELIŃSKI
Instytut Fizyki
Uniwersytet Łódzki
ul. Nowotki 149/153
PL-90-236 Łódź, POLAND

51. Jerzy RÓŻAŃSKI
Instytut Fizyki Teoretycznej
Uniwersytet Wrocławski
pl. Maksa Borna 9
PL-50-204 Wrocław, POLAND

52. Jan RZEWUSKI
Instytut Fizyki Teoretycznej
Uniwersytet Wrocławski
pl. Maksa Borna 9
PL-50-204 Wrocław, POLAND

53. Michael SCHLIEKER
Institute for Theoretical Physics
Karlsruhe University
Kaiserstrasse 12
D-7500 Karlsruhe, GERMANY

54. Pavel V. SEMENOV
Moscow State Pedagogical University
Kozmodemjanski 9A kv. 27
125130 Moscow, RUSSIA
e-mail: lysov@imb.msk.su (with subject to Semenov P.)

55. Uwe SEMMELMANN
Humboldt Universität zu Berlin,
Sektion Mathematik
Unter den Linden 6, PSF - 1297,
O–1086 Berlin, GERMANY

56. Przemysław SIEMION
Instytut Fizyki Teoretycznej
Uniwersytet Wrocławski
pl. Maksa Borna 9
PL-50-204 Wrocław, POLAND

57. Garret SOBCZYK
Facultad de Estudios Superiores Cuautitlan
Universidad Nacional Autonoma de Mexico (U.N.A.M.)

Apartado Postal # 25
Cuautitlan Izcalli 54700
Edo. de MEXICO
e-mail: garret@unamvm1.bitnet

58. Jan SOBCZYK
Instytut Fizyki Teoretycznej
Uniwersytet Wrocławski
pl. Maksa Borna 9
PL-50-204 Wrocław, POLAND

59. Vladimir SOUČEK
Mathematical Institute
Charles University
Sokolovská 83
18600 Praha, CZECHO-SLOVAKIA
e-mail: vsoucek@cspguk11

60. Wolfgang SPRÖSSIG
Fachbereich Mathematik
Bergakademie Freiberg
Bernhard-von-Cotta Str. 2
O-9200 Freiberg, GERMANY

61. Jan TARSKI
492 Michigan Avenue
Berkeley, CA 94707, USA

62. Ireneusz TOBIJASZEWSKI
Instytut Fizyki
Uniwersytet Adama Mickiewicza
ul. Matejki 48/49
PL-60-769 Poznań, POLAND

63. K. Paul TOD
Department of Mathematics
University of Oxford
St. John's College
Oxford OX1 3JP, GREAT BRITAIN
e-mail: tod@uk.ac.ox.vax

64. Valeriy TOLSTOY
Institute of Nuclear Physics
Moscow State Unversity
119899 Moscow, RUSSIA
e-mail: tolstoy@compnet.npimsu.msk.su

65. Helmut URBANTKE
Institute of Theoretical Physics
Universität Wien
Boltzmanngasse 5
A-1090 Wien, AUSTRIA

66. Dmitry V. VOLKOV
Kharkov Institute of Physics and Technology
ul. Akamicheskaya 1
310108 Kharkov, UKRAINE
telex: 311052 dekan su

67. Satoshi WATAMURA
Department of Physics
College of General Education
Tohoku University
Kawauchi, Sendai 980, JAPAN
e-mail: watamura@jpntuvm0; fax (+81)22-262-2429

68. Julius WESS
Max-Planck-Institut für Physik und Astrophysik
Föhringer Ring 6, Postfach 40-12-12
D-8000 München 40, GERMANY
e-mail: jup@dm0mpi11

69. Aleksandr A. ZHELTUKHIN
Kharkov Institute of Physics and Technology
ul. Akademicheskaya 1
Kharkov 310108, UKRAINE
telex: 311052 dekan su