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# Physical Space as a Quaternion Structure, I: Maxwell Equations. A Brief Note.

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This paper shows how to write Maxwell's Equations in Hamilton's Quaternions. The fact that the quaternion product is non-commuting leads to distinct left and right derivatives which must both be included in the theory. A new field component is then revealed, which reduces part of the degree of freedom found in the gauge, but which can then be used to explain thermoelectricity, suggesting that the theory of heat has just as fundamental a connection to electromagnetism as the magnetic field has to the electric field, for the new theory now links thermal, electric, and magnetic phenomena altogether in one set of elementary equations. This result is based on an initial hypothesis, named "The Quaternion Axiom," that postulates physical space is a quaternion structure.

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## I. THE QUATERNION AXIOM.

The quaternions

$$a = a0 + a1.i + a2.j + a3.k \tag{1}$$

where i, j, k are the anti-commuting hypercomplex roots of -1, and the a0, a1, a2, a3 are elements of the real number set, can be used to write down Maxwell's Equations. We postulate that physical space is a quaternion structure, so that the units { i, j, k} represent space dimensions, while the scalar { 1} represents time, and the space units obey the product rules given by W. R. Hamilton in 1843[1] ;

$$\begin{gathered} i^2 = j^2 = k^2 = -1 \\ i = jk = -kj, j = ki = -ik, k = ij = -ji \end{gathered} \tag{2}$$

We shall refer to this postulate as The Quaternion Axiom. A position vector in this quaternion space will take the form, $r = ct + ix + jy + kz$, with c the characteristic speed that links clock ticks to metre rules, equivalent here to the speed of light, so that all measures within this quaternion 4-vector are ultimately in the same units-of-length.

### A. Ambiguous Product.

Now, because quaternions don't commute, we need to recognise the distinction between left and right actions. Consider two quaternion variables a & b. Let us define the symbols '→' and '←' to mean 'operate to the right' and 'operate to the left', respectively. Then, for example, if the 'a' term is the operator, and the 'b' term is the variable acted upon by the operator, we have;

$$\begin{aligned} a \rightarrow b \;=\; & a0.b0 - a1.b1 - a2.b2 - a3.b3 \\ & + \; a0.(b1.i + b2.j + b3.k) \\ & + \; (a1.i + a2.j + a3.k).b0 \\ & + \; (a2.b3 - a3.b2).i \\ & + \; (a3.b1 - a1.b3).j \\ & + \; (a1.b2 - a2.b1).k \end{aligned} \tag{3}$$

$$\begin{aligned} b \leftarrow a \;=\; & a0.b0 - a1.b1 - a2.b2 - a3.b3 \\ & + \; a0.(b1.i + b2.j + b3.k) \\ & + \; (a1.i + a2.j + a3.k).b0 \\ & - \; (a2.b3 - a3.b2).i \\ & - \; (a3.b1 - a1.b3).j \\ & - \; (a1.b2 - a2.b1).k \end{aligned} \tag{4}$$

Note, we only need the symbols, → and ← , between the quaternions themselves. Once we have resolved the algebra to the level of the components, we can revert to the usual convention of putting the operator on the left and the variable acted upon on the right.

If there is no physical reason to select one product over the other, both products must be accounted for equally in the expressions used to model phenomena in any given theory, otherwise such expressions will have an inherent left-hand or right-hand bias.

When dealing with operator products therefore, we define the right product by, $a \rightarrow b$, and the left product by, $b \leftarrow a$. Then the two combinations of these, the symmetric product, {a,b}, and the antisymmetric product, [a,b], are defined correspondingly by;

$$\{a, b\} = (1/2)(a \rightarrow b + b \leftarrow a) \tag{5}$$

$$[a, b] = (1/2)(a \rightarrow b - b \leftarrow a) \tag{6}$$

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Which product we employ in our theory, and where, is dictated by the symmetries inherent in the physical problem.

## II. MAXWELL EQUATIONS.

Now, let the Electromagnetic Potential be

$$A = U + A1.i + A2.j + A3.k \quad (7)$$

and the differential operator $(d/dr)$ be defined by,

$$\frac{d}{dr} = \frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\ i + \frac{\partial}{\partial y}\ j + \frac{\partial}{\partial z}\ k \quad (8)$$

then by inspection, the Electric and Magnetic fields as quaternions are,

$$E = -\{d/dr, A\} = -(1/2)(d/dr \rightarrow A + A \leftarrow d/dr) \quad (9)$$

$$B = +[d/dr, A] = +(1/2)(d/dr \rightarrow A - A \leftarrow d/dr) \quad (10)$$

That is, the electric field is the negative symmetric derivative of the potential, and the magnetic field is the positive antisymmetric derivative of the potential. The space components of these quaternion fields correspond exactly to the electric and magnetic fields in the usual 3-vector calculus. However, the electric quaternion field now has a time component, which we label, T, so that, $E = T + \mathbf{E}$, while the magnetic quaternion field has no time component, so that, $B = 0 + \mathbf{B}$. And if we allow our notation to alternate between Heaviside-Gibbs 3-vector and that of Hamilton's Quaternion 3-vector, taking care to match up only the components of the appropriate expressions, we can write the quaternion derivative in terms of the more familiar vector notation,

$$\frac{d}{dr} \rightarrow A = \frac{1}{c}\frac{\partial U}{\partial t} - div(\mathbf{A}) + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + grad(U) + curl(\mathbf{A}) \quad (11)$$

$$A \leftarrow \frac{d}{dr} = \frac{1}{c}\frac{\partial U}{\partial t} - div(\mathbf{A}) + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + grad(U) - curl(\mathbf{A}) \quad (12)$$

where we observe that the quaternion derivative has "five" distinct parts, much like the five fingers on the human hand. The electric field is then obtained from clapping the hands together, so that the distinction between the two vanish (and consequently giving rise, as we shall see, to "heat" as well as electricity). While, the magnetic field is obtained from the opposite action of pulling the hands apart, so that the distinction between the two become more pronounced.

Now, although we could invent constructions using the quaternion conjugate to eliminate the time component from all our calculations and so obtain the exact results of the 3-vector calculus, this would make the calculations more contrived and less natural than the simple structure presented here. We therefore let the natural structure of the algebra lead us instead, to the results that follow from accepting this new field component, rather than try to eliminate it from our framework. Then, by inspection, the reformulated **Maxwell Field Equations** are,

$$[d/dr, B] = +\{d/dr, E\} \quad (13)$$

$$[d/dr, E] = -\{d/dr, B\} \quad (14)$$

The antisymmetric derivative of the magnetic field is the positive of the symmetric derivative of the electric field. And, the antisymmetric derivative of the electric field is the negative of the symmetric derivative of the magnetic field. The first represents a real physical law, while the second is easily proven to be an algebraic identity when given the definitions of the electric and magnetic fields above.

When written in the vector notation of Heaviside-Gibbs, these two 4-vector (quaternion) equations become the usual four 3-vector equations,

$$\begin{aligned} curl(\mathbf{B}) &= +1/c.\partial\mathbf{E}/\partial t + grad(T) && (15)\\ curl(\mathbf{E}) &= -1/c.\partial\mathbf{B}/\partial t && (16)\\ div(\mathbf{E}) &= +1/c.\partial T/\partial t && (17)\\ div(\mathbf{B}) &= 0 && (18) \end{aligned}$$

$$\begin{aligned} T &= -1/c.\partial U/\partial t + div(\mathbf{A}) && (19)\\ \mathbf{E} &= -grad(U) - 1/c.\partial\mathbf{A}/\partial t && (20)\\ \mathbf{B} &= curl(\mathbf{A}) && (21) \end{aligned}$$

provided we now identify the electric charge density, $\rho$, and electric current density, $\mathbf{J}$, with the terms involving T. Thus, $4\pi\rho = +1/c.\partial T/\partial t$, and $4\pi\mathbf{J}/c = grad(T)$.

The scalar quantity, T, we shall call the "Temporal Field." This scalar field has the same units-of-dimension as the electric and magnetic vector fields in our Gaussian system of units. And thus for a given charge, q, the quantity, qT, has units of force, similar to the electric force, $q\mathbf{E}$, and the magnetic force, $q\mathbf{v}/c \times \mathbf{B}$. However, this scalar force has no space direction. Instead, it acts along the timeline, since that is the scalar axis under The Quaternion Axiom used as the basis of this derivation.

What effect is produced by a scalar force acting along the timeline?

## III. THERMOELECTRICITY.

The electric force moving a charge, q, through a displacement, $d\mathbf{x}$, does work, $dW = q\mathbf{E}{\cdot}d\mathbf{x}$. And work is a form of energy. This energy is delivered to the external environment, and is available, for example, as mechanical work, capable of moving the parts of a mechanical system against a frictional resistance.

### A. Heat.

Similarly, the temporal force acting on a charge, q, for a time interval, cdt, produces an energy-like term, $dW = -qTcdt$ (the minus sign reflecting the opposite signs in the squares of the unit time, $1^2 = +1$, and unit space, $i^2 = j^2 = k^2 = -1$, measures, in the quaternion spacetime, needed here because $\mathbf{E}$ and $d\mathbf{x}$ are Heaviside-Gibbs vectors; thus a positive charge, $q > 0$, under the influence of a positive value temporal field, $T > 0$, produces the equivalent of negative work, i.e. the charge-field interacting system will absorb energy from its surroundings, positive charges thus effectively appearing "cold," while negative charges effectively appearing "hot").

In this case, over the given time interval, energy is absorbed or evolved from the charge-field interacting system accordingly as the signs of the charge and the temporal field are the same or opposite. Since this scalar energy does not require the charge to move in space, in order to materialize as some observable physical phenomena the energy that is absorbed and/or evolved must manifest as a form of heat. Moreover, this heat is proportional to the first power of the charge, and thus reverses sign with the change in sign of the charge, or correspondingly a change in sign of the electric current, making this a reversible heat, corresponding to the experimental observations already known as Peltier and Thomson Heats in thermoelectricity.

Thus, the temporal field, $T = -1/c.\partial U/\partial t + div(\mathbf{A})$, represents the total heat energy per unit charge evolving per unit time (i.e. time measured in units of length) from the charge-field interacting system due to both the loss in electrostatic potential energy at the location and the flow of electrodynamic momentum out of the same location, much like the diffusion heat flow being due to the sum of two different processes, conduction and convection, in nonequilibrium thermodynamics.

Now, instead of using the symmetric and antisymmetric derivatives, the two quaternion electromagnetic equations can also be written using the left and right derivatives of the potential.

$$d/dr \rightarrow (d/dr \rightarrow A) \; + (A \leftarrow d/dr) \leftarrow d/dr \; = 0 \quad (22)$$
$$d/dr \rightarrow (A \leftarrow d/dr) \; - (d/dr \rightarrow A) \leftarrow d/dr \; = 0 \quad (23)$$

When we introduce the electric charge density and electric current source terms, as additional inhomogeneous parameters to the temporal terms, instead of identifying the electric source directly with the temporal terms themselves, the second equation is unchanged, and the first equation says,

$$d/dr \rightarrow (d/dr \rightarrow A) + (A \leftarrow d/dr) \leftarrow d/dr = 8\pi J \quad (24)$$

where, $J = (\rho, \mathbf{J}/c)$. In the Heaviside-Gibbs 3-vector format, the new inhomogeneous electromagnetic equations therefore become,

$$curl(\mathbf{B}) = +1/c.\partial\mathbf{E}/\partial t + grad(T) + 4\pi\mathbf{J}/c \quad (25)$$
$$curl(\mathbf{E}) = -1/c.\partial\mathbf{B}/\partial t \quad (26)$$
$$div(\mathbf{E}) = +1/c.\partial T/\partial t + 4\pi\rho \quad (27)$$
$$div(\mathbf{B}) = 0 \quad (28)$$

$$T = -1/c.\partial U/\partial t + div(\mathbf{A}) \quad (29)$$
$$\mathbf{E} = -grad(U) - 1/c.\partial\mathbf{A}/\partial t \quad (30)$$
$$\mathbf{B} = curl(\mathbf{A}) \quad (31)$$

These equations then make a clearer distinction between the "thermal" and the "electric" source contributions to the electromagnetic fields.

P. W. Bridgman[2] observed, in 1961, that thermoelectric phenomena require the phenomenological description of e.m.f to allow for two different kinds of electromotive force, one that provides what he calls the "working" e.m.f, and the other that provides the "driving" e.m.f, for the thermoelectric system. The "working" e.m.f is responsible for the production of the total energy that emerges from the system, while the "driving" e.m.f is responsible for moving the charges in the system, giving rise to the electric current.

These two e.m.fs, traditionally considered the same normally in electricity, are not the same when including thermoelectric effects.

Bridgman invents a thermodynamic construction to define these two phenomenologically required e.m.fs, but he emphasises that since these are constructions they are not directly observable. Here we find an alternative explanation of Bridgman's idea of the two e.m.fs, on grounds much more fundamentally linked to the electromagnetic equations and not requiring his ad hoc thermodynamic arguments.

If we take our new Coulomb-Gauss law, eqn (27), and separate the sources so that we isolate each effect in order to consider the impact of one independently of the other, we find the corresponding result to Bridgman's two e.m.fs.

We set the free electric charge density to zero and label the electric field, whose source is given by the temporal term, $+1/c.\partial T/\partial t$, alone, with a subscript to indicate this part arises from the temporal source; $\mathbf{E}_T$.

$$div(\mathbf{E}_T) = +1/c.\partial T/\partial t \tag{32}$$

Then, by subtracting this electric field from the total field we obtain the Coulomb-Gauss law we are more familiar with, consisting of just the free electric charge density as source.

$$div(\mathbf{E} - \mathbf{E}_T) = 4\pi\rho \tag{33}$$

Here, we can define $\mathbf{E}$, as the "driving" field, since this is clearly responsible for moving the charges through the region of space. Then, $\mathbf{E}-\mathbf{E}_T$, is the "working" field, that determines the total energy (i.e. net energy) delivered by the field in the act of moving those charges—part of the energy that would normally be observed due to a charge moving under the influence of that total electric field being now reabsorbed instead by the mechanics of the reversible thermoelectric heat in the internal energy conversion process.

It is not clear to me, however, whether these two e.m.fs should simply replace Bridgman's constructions, or whether Bridgman's e.m.fs should be considered to exist independently, thus masking the presence of the quaternion results.

Bridgman also notes that symmetric thermoelectric arguments require that a time varying temperature will induce a related thermoelectric electromotive force in the material undergoing the variation of temperature, an effect that has not yet been experimentally observed.

Now, we would expect that the temporal field, T, in an homogeneous material, would itself be uniform throughout the material medium, and show no direct independent variation in space or time, except that, given the close relation to heat energy, the temporal field would vary directly with temperature, K, and would consequently show an indirect space variation and time variation to the extent that the temperature itself varies with space and time.[3]

### B. dT/dK

This relationship between the temporal field and temperature can be more clearly expressed functionally, $T = T(K)$. And with this in mind, we conclude that the time rate of change of the temporal field is proportional to the time rate of change of the temperature,

$$\frac{1}{c}\frac{\partial T}{\partial t} = \frac{1}{c}\frac{dT}{dK}\frac{\partial K}{\partial t} \tag{34}$$

In fact, for any given homogeneous material medium, the measure of the temporal field is simply a measure of the temperature, because the parameter dT/dK is a characteristic of the medium. Given that we can write, $dT/dK = -1/(qc).d(-qTc)/dK$, and -qTc is the heat energy absorbed per unit time by the charge, q, we see that dT/dK is effectively the measure of a type of "heat capacity" of the unit charge within the particular material.

Consequently, from the new Coulomb-Gauss law, we infer that an electric field, $E_T$, is induced by a time rate of change in temperature, the magnitude of the field being determined by this special "heat capacity" of a unit charge within the material and the rate of change of temperature with time. The direction of the field is radially away from the point at which the temperature change occurs.

In an homogeneous isotropic medium, the neighbouring points also produce similar fields, but being oppositely directed, they cancel each other, leaving no net field within the medium. However, where there is an anisotropy introduced into the medium, such as a temperature gradient within the material, a net field will emerge along the direction of that anisotropy. This is consistent with Bridgman's thermoelectric e.m.f induced by changing temperature with time, which requires the presence of an isothermal electric current, thus providing the requisite anisotropy, and also consistent with the pyroelectric effect where the mechanical stress and strain provide the anisotropic conditions. Thus, here again, we find the effects of the temporal field being masked by other theoretically anticipated and experimentally known phenomena.

### C. Seebeck Effect

Now, consider an electric charge moving from one material medium into another, say two different metals. From the rest frame of the moving charge, we find that there is a jump in the temporal field, T, at the time the charge is seen to cross the boundary from one metal and enter the other metal in the lab frame. This time rate of change of temporal field is seen as an electric field, $E_T$, by the charge, according to our new Coulomb-Gauss law, and this produces an electric force on the charge that accelerates it from rest in its instantaneous rest frame, or just accelerates the charge in the lab frame, and this is the source of the Seebeck thermoelectric e.m.f.

Note, no static surface charges are required at the boundary between the two metals to establish an electrostatic field that will then accelerate the charges to sustain the current. The jump in the temporal field between the two metals, which is essentially related to the jump in the specific heat capacities per unit charge and the jump in electrical conductivity, is the sole source of the manifest e.m.f. Randomly moving charges that cross the boundary will be accelerated by this e.m.f, and a closed circuit will thus sustain a current.

### D. Thomson Effect

Consider the new form of Ampere's law. If we multiply both sides of the equation (25) by the ratio of current to electrical conductivity, $\mathbf{J}/\sigma$, using the vector "dot product," then write the gradient of the temporal field in terms of the temperature gradient, and re-arrange the terms, we obtain four quantities that sum to zero,

$$\begin{aligned} \frac{\mathbf{J}^2}{\sigma} &+ \frac{c}{4\pi\sigma}\frac{dT}{dK}\mathbf{J}\cdot grad(K) \\ &- \frac{c}{4\pi\sigma}\mathbf{J}\cdot curl(\mathbf{B}) + \frac{1}{4\pi\sigma}\mathbf{J}\cdot\partial\mathbf{E}/\partial t = 0. \end{aligned} \tag{35}$$

The first term we recognise as the Joule Heat produced by an electrical current. It is proportional to the square of the electrical current density, $\mathbf{J}^2$, and is thus independent of the direction of the current or the sign of the charge carriers.

The second term we recognise has the form of the Thomson Heat produced by an electrical current flowing up a temperature gradient. It is linear in the electrical current density, $\mathbf{J}$, and thus reverses sign when the current reverses direction or the signs of the charge carriers are reversed.

These two terms must just balance the last two terms in this equation, for a thermally isolated electromagnetic system. But if we design an experiment to reduce the latter two terms to zero, say, for example, we arrange things so that the electric field is constant in time, $\partial\mathbf{E}/\partial t = 0$, and the circulation of the magnetic field is perpendicular to the current flow, $\mathbf{J}\cdot curl(\mathbf{B}) = 0$, then the system can no longer be thermally isolated, in general, and we must either pump heat energy into or extract heat out to maintain our particular requirements.

When placing this electromagnetic system in contact with heat reservoirs, therefore, and allowing heat to be exchanged between the system and the reservoirs, in such a way that the last two terms vanish, we obtain the conditions that characterize the Thomson Heat experiment. The rate of net heat energy exchanged with the reservoirs, $dQ/dt$, is now equal to the sum of the first two heats, and the equation becomes the usual Thomson Heat equation of thermoelectricity, which says the rate of production of heat by the system is the sum of the irreversible Joule Heat and the reversible Thomson Heat,

$$\frac{dQ}{dt} = \mathbf{J}^2/\sigma - h_T\mathbf{J}\cdot grad(K) \tag{36}$$

$$h_T = -\frac{c}{4\pi\sigma}\frac{dT}{dK} \tag{37}$$

Where now, $h_T$, is the Thomson Specific Heat of the material, which is defined to be the quantity of "reversible heat" absorbed per unit time by an electrical current of unit strength flowing up a temperature gradient of one degree per unit length in a wire of unit area cross section. The "reversible heat" is separated from the "irreversible heat" due to Joule heating by measuring the total heat produced by the current flowing in one direction, then reversing the direction of the current and measuring the total heat again. The difference between the two total heats is then twice the Thomson Heat.

We are now in a position to interpret the special "heat capacity" parameter, dT/dK, introduced above. It is essentially equivalent to the product of the Thomson Specific Heat and the Electrical Conductivity of the medium, $h_T \cdot \sigma$, and thus it is indeed a characteristic of the material.

## IV. POLARIZATION AND MAGNETIZATION.

We can identify the net electric field with the electric displacement, $\mathbf{D} = \mathbf{E} - \mathbf{E}_T$. Then, if we define the magnetic field, $\mathbf{B}_T$, to be that which solves the equation, $curl(\mathbf{B}_T) = +1/c.\partial\mathbf{E}_T/\partial t + grad(T)$, we can write, $\mathbf{H} = \mathbf{B} - \mathbf{B}_T$, and obtain the more familiar macroscopic equations.

We can then argue that in a macroscopic medium, the effects due to the presence of these new terms, $grad(T)$ and $+1/c.\partial T/\partial t$, are normally hidden in the complexities of the Polarization and Magnetization parameters that describe the material, and thus, in many instances, the effects of the temporal field are not easily separated to be identified and measured as independent phenomena.

## V. CONCLUSION.

When James Clerk Maxwell[4] wrote the second edition of his *Treatise on Electricity and Magnetism* he included a quaternion representation of his electromagnetic equations, but he did not include both left-hand and right-hand derivatives, and the differential operator nabla was restricted to the 3-dimensional space form lacking a time component, and so his work is fundamentally different from that presented here.

Indeed, in the calculus of quaternions the differential operator almost always appears on the left acting towards the variable on the right, ignoring the other alternative. And even though, Charles Jasper Joly[5] notes the distinction in his book *A Manual of Quaternions*, the importance of the idea goes unnoticed, unexplored, and unused. As a consequence of this, an important field component went missing in Maxwell's Equations, and all of modern physics has developed from there perpetuating one of the consequences of this oversight, namely, that the electromagnetic field possesses six components, whereas, as we have shown, there should be seven.

Our macroscopic experience tells us that heat is produced by two opposing agents acting, the one against the other, rubbing as it were, as in the familiar case of mechanical friction, to produce the gross fire that manifests as heat when there is contact with matter. So, when we find the electric field is also the sum of two opposing principles, the left-handed tension acting against the right-handed tension, we should not be surprised to find a heat component, a more subtle fire, hiding within the field, and emerging as heat when the field comes into contact with charged matter.

Indeed, the action of a material body can be generally described in terms of three transformations: translations, rotations, and pulsations. The electric field, **E**, tends to induce translations in a test charge, while the magnetic field, **B**, tends to induce rotations when the charge is moving, we are then left to conclude that the temporal field, T, tends to induce the pulsations.

Thus, with the inclusion of the missing temporal field the description of the action of a charged material particle is complete—we infer such a particle must have an extended structure with a variable intrinsic pulse in addition to its quantum mechanically determined fixed intrinsic spin.

This brief note introduces the essential ideas that will be explored more fully in a future paper.

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[1] W. R. Hamilton, 1844, *On a new species of Imaginary Quantities connected with the Theory of Quaternions* [communicated November 13, 1843], Ir. Acad. Proc., II, 424-434.

[2] P. W. Bridgman, 1961, *The Thermodynamics of Electrical Phenomena in Metals and a Condensed Collection of Thermodynamic Formulas*, Dover Publications. – for definitions of "working" and "driving" electromotive forces, see comments in preface page v, and text pages 19,61,63,69,129ff. And for comments relating to the expected, but as yet unseen, "E.M.F. produced by temperature varying with time," see preface page vi, and text pages 144-145.

[3] We use the letter, K, for temperature, instead of the more usual letter, T, since the latter is being used here for the temporal field, and we can remember this as K for Kelvin instead of T for Thomson, given that William Thomson became Lord Kelvin, hence the promotion of the letter.

[4] J. C. Maxwell, 1954, *Treatise on Electricity and Magnetism*, 3rd ed., 2 vols, Dover, New York.

[5] C. J. Joly, 1905, *A Manual of Quaternions*, London Macmillan. In Art 57, Joly recognises the two different left and right differentiations; pp.74-77, and Exercises Ex.5, Ex.11, on pg.76.