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MODIFIED MAXWELL EQUATIONS IN QUANTUM ELECTRODYNAMICS

Henning F Harmuth
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$$-\operatorname{curl}\mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} + \mathbf{g}_m$$

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To the memory of Max Planck (1858 – 1947)

We owe respect to the living; to the dead we owe only truth.
(Françoise Marie Arouet de Voltaire)

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Preface

Electromagnetic theory has been based on Maxwell's equations for about a century. There is no need to elaborate the successes but from 1986 on we find publications claiming that Maxwell's equations generally do not have solutions that satisfy the causality law. Two scientists working independently and using different approaches arrived at the same result, which gives it great credibility. The mathematical investigations that uncovered the lack of causal solutions are necessarily complicated, otherwise it would not have taken a century to find this shortcoming of Maxwell's equations.

The problem could be corrected by the modification of Maxwell's equations with an added magnetic current density term. Initially this caused concern since magnetic charges or charge carriers have not been observed reliably even though there are good theoretical arguments for their existence, e.g., the quantization of the electric charge. However, it was soon realized that there was no need for *magnetic monopole currents* but that *magnetic dipole currents* were sufficient. The existence of magnetic dipoles is not disputed and their rotation can cause magnetic dipole currents just as the rotation of electric dipoles in a material like Barium-Titanate can cause *electric dipole currents*.

Electric dipole currents were always an important part of Maxwell's equations but they were called polarization currents and this choice of words obscured the unequal treatment of electric and magnetic dipoles. Electric dipole currents are needed to explain how an electric current can flow through the dielectric of a capacitor, which is an insulator for *electric monopole currents*.

The causality law is of no significance for the transmission of power and energy, or generally for steady state solutions of Maxwell's equations. But it is a must for the transmission of electromagnetic signals. Signal transmission without causality is a contradiction in terms.

We define a classical electromagnetic signal as a propagating wave that is zero before a certain time and has finite energy. All produced or observed propagating electromagnetic waves are of this type even though we often approximate them for mathematical convenience by infinitely extended sinusoidal waves. Signals are represented mathematically by functions or *signal solutions* that are zero before a certain time and are quadratically integrable. Such signal solutions satisfy the causality law and the conservation law of energy while infinitely extended periodic sinusoidal solutions satisfy neither.

The modified Maxwell equations have been applied in four books and numerous papers to problems ranging from the propagation of electromagnetic signals in seawater to their interstellar propagation over distances of billions of light years. The time has come to advance from classical physics to quantum physics.

It is one of the most basic principles of quantum mechanics that an observation interferes with what is being observed. In other words, a signal received during an observation changes what created the signal. Quantum electrodynamics should thus be a good field of application for an electromagnetic theory that permits signal solutions.

This expectation turned out to be fully justified. The first success of the use of the modified Maxwell equations was the elimination of the *infinite zero-point energy* that has been a problem for 70 years; the conventional theory can correct it only by renormalization, a process that is generally considered unsatisfactory. The infinite zero-point energy is shown to be reduced to a finite energy for both the pure radiation field and the Klein-Gordon equation.

The Hamilton function of a charged particle in an electromagnetic field derived with the modified Maxwell equations contains many more terms than the conventional Hamilton function. This provides the basis for new results, a fact of interest to those who look for topics for PhD theses.

The authors want to thank Humboldt University in Berlin for providing the computer power required for a number of complicated plots.

Contents

Equations are numbered consecutively within each of Sections 1.1 to 6.12. Reference to an equation in a different section is made by writing the number of the section in front of the number of the equation, e.g., Eq.(2.1-5) for Eq.(5) in Section 2.1.

Illustrations are numbered consecutively within each section, with the number of the section given first, e.g., Fig.1.1-1.

References are listed by the name of the author(s), the year of publication, and a lowercase Latin letter if more than one reference by the same author(s) is listed for that year.

5 · Klein-Gordon Equation and Vacuum Constants

6 Appendix

List of Frequently Used Symbols

$\mathbf{A}_\mathrm{e}$	As/m	electric vector potential
A_ec, A_es	-	Eqs.(6.12-44), (6.12-45), (6.12-126)
A_ev	As/m	Eq.(4.1-31)
$\mathbf{A}_\mathrm{m}$	Vs/m	magnetic vector potential
A_mv	Vs/m	Eq.(4.1-31)
a	-	$1+\omega^2$, Eq.(6.1-24)
$\mathbf{B}$	Vs/m^2	magnetic flux density
b^2	-	$(2\pi\kappa)^2+4\omega^2$, Eq.(6.1-24)
c	m/s	299 792 458; velocity of light
$\mathbf{D}$	As/m^2	electric flux density
d^2	-	$4[(2\pi\kappa)^2+\rho_2^2]$, Eq.(4.1-73)
$\mathbf{E}$, E	V/m	electric field strength
E_E	V/m	electric field strength due to electric excitation
E_H	V/m	electric field strength due to magnetic excitation
E	VAs	energy, Eq.(3.3-12)
e	As	electric charge
$\mathbf{g}_\mathrm{e}$	A/m^2	electric current density, Eq.(1.1-2)
$\mathbf{g}_\mathrm{m}$	V/m^2	magnetic current density, Eq.(1.1-9)
$\mathbf{H}$, H	A/m	magnetic field strength
H_E	A/m	magnetic field strength due to electric excitation
H_H	A/m	magnetic field strength due to magnetic excitation
$\mathcal{H}$	-	Hamilton function
h	Js	$6.626\,075\,5\times10^{-34}$, Planck's constant
$\hbar=h/2\pi$	Js	$1.054\,572\,7\times10^{-34}$
j_0	-	Eq.(6.11-42)
K	-	$c^2T\lvert(\sigma\mu-s\epsilon)\rvert/4\pi$, Eq.(4.1-87); or $\ll\lambda_1/2\pi$, Eq.(5.2-40)
$\mathcal{L}$	-	Lagrange function
m	kg	mass
m_0	kg	rest mass
p	-	τ_mp/τ
$\mathbf{p}$	kg m/s	momentum
q	-	τ_p/τ
q_1, q_2	-	Eq.(4.1-106)
q_3, q_4	-	Eq.(4.1-112)
q_m	Vs	hypothetical magnetic charge
s	V/Am	magnetic conductivity
t	s	time variable
T	s	time interval
v	m/s	velocity
V_e, V_e0	As/m^3	Eqs.(4.1-36), (4.1-103)
V_m	As/m^3	Eq.(4.1-38)
$Z=\mu c$	V/A	376.730 314; wave impedance of empty space
Zec	Vm	$1.809\,513\,6\times10^{-8}$

α	-	$Ze^2/2h \approxeq 7.297\,535 \times 10^{-3}$, Eq.(3.3-49)
α_e	-	$ZecA_e/m_0c^2$; Eq.(3.3-49)
γ_0	-	Eq.(5.2-27)
γ_1, γ_2	-	Eqs.(4.1-73), (6.1-24)
$\epsilon = 1/Zc$	As/Vm	$1/\mu c^2$; permittivity
ζ	-	normalized distance, Eq.(1.3-7)
$\eta = 2\pi\kappa$	-	Eq.(6.1-40)
θ	-	normalized time, Eqs.(1.3-7)–(1.3-10)
θ_{mp}, θ'_{mp}	-	Eq.(6.10-51)
θ_p, θ'_p	-	Eq.(6.10-51)
ι	-	Eq.(6.4-12)
ι_0	-	Eq.(5.2-26)
κ	-	normalized wave number; Eqs.(4.1-68),(6.1-18)
$\kappa_0 \ldots \kappa_4$	-	Eqs.(6.11-42), (6.11-47), (6.11-48), (6.11-52), (6.11-53)
λ_1, λ_2, λ_3	-	Eq.(5.2-8)
$\mu = Z/c$	Vs/Am	$4\pi \times 10^{-7}$; permeability
ρ^2	-	$\sigma s/c^2(\sigma\mu + s\epsilon)^2$, Eq.(1.3-12)
ρ_1	-	$c^2T(\sigma\mu + s\epsilon)$, Eq.(4.1-41)
ρ_2^2	-	$c^2T^2\sigma s$, Eq.(4.1-41)
ρ_e	As/m^3	electric charge density
ρ_m	Vs/m^3	hypothetical magnetic charge density
ρ_{em}, ρ_{ep}	-	Eq.(6.10-67)
ρ_{pm}, ρ_{mm}	-	Eq.(6.10-67)
ρ_{mp}, ρ_p	-	Eq.(6.10-67)
ρ_{pp}	-	Eq.(6.10-67)
ρ_s	-	Z/scT, Eq.(4.1-47)
ρ_σ	-	$ZTc\sigma$, Eq.(4.1-49)
σ	A/Vm	electric conductivity
ϕ_e	V	electric scalar potential
ϕ_m	A	magnetic scalar potential
ω^2	-	$s\epsilon/\sigma\mu$, Eq.(1.3-11)
ω_1	-	$[(1-\omega^2)^2 - \eta^2]^{1/2}$, Eq.(6.7-2)
ω_2	-	$[\eta^2 - (1-\omega^2)^2]^{1/2}$, Eq.(6.7-2)

1 Introduction

1.1 Maxwell's Equations

Maxwell's equations have been the basis of electromagnetic theory for more than a century. Their original formulation as a continuum theory with electric charges, electric current densities, field strengths, and flux densities was extended by Lorentz to include particles and the concept of mass. Quantum theory required a further extension to include quantization effects. The application of group theory to Maxwell's equations showed that the physically required conservation laws were satisfied as a result of certain symmetries but did not bring any further extension of Maxwell's equations (Fushich and Nikitin 1987). After a century of scrutiny and extensions, Maxwell's equations had become one of the more solid pillars of physics.

With this background it is not surprising that publications claiming that Maxwell's equations have generally no solutions that satisfy the causality law were not well received[1]. Indeed, it was next to impossible to publish this result[2]. The mathematical methods used were difficult to follow, otherwise it would not have taken a century to recognize the problem. But two scientists working independently, using different approaches, and arriving at the same result give that result great credibility. Furthermore, none of the many attempts by opponents since 1986 to obtain a correct solution of Maxwell's equations that satisfied the causality law was successful. The question of causality for Maxwell's equations soon became a more fundamental question: How does the causality law enter the mathematical formulation of a physical problem?

To answer this question we first spell out the causality law. It is rarely found or discussed in books on physics despite its undisputed importance to physics[3]. Let us state it in the following form:

Every effect requires a sufficient cause that occurred a finite time earlier.

It is sometimes believed that the causality law is a mathematical axiom that is automatically satisfied if one calculates correctly (Toll 1956). The word *time* in the causality law shows that this cannot be so. Pure mathematics

[1]Harmuth 1986a, b, c; Hillion 1990, 1991, 1992 a, b, 1993

[2]The credit for having the courage to publish goes to Peter W. Hawkes, editor of *Advances in Electronics and Electron Physics* (Academic Press) and the late Richard B. Schulz, editor of *IEEE Transactions on Electromagnetic Compatibility.*

[3]The causality law was recognized by the Greeks since it is impossible to think rationally without using it. The conservation laws of physics were accepted after 1800.

does not have the concept of time and there are neither a time variable nor spatial variables since these are concepts of physics. Instead we have complex variables, real variables, rational variables, integer variables, random variables, etc. No variable in pure mathematics has a physical dimension like meter or second. Only when mathematics is applied to physics can we have variables with physical dimension. But in this case both the mathematical axioms and the physical laws must be satisfied.

The word *earlier* in the causality law introduces the universally observed distinguished direction of time: the effect comes after the cause. Nothing equivalent exists for spatial variables since there is no general law that demands something must always happen in front, to the right, or above a certain spatial point[4].

To see how the causality law enters when a physical process is described by a partial differential equation in a coordinate system at rest we note that in this case one must find a function that satisfies three requirements:

1. The function satisfies the partial differential equation(s).
2. The function satisfies an initial condition that holds at a certain time t_0 for all values of the spatial variable(s).
3. The function satisfies a boundary condition that holds at all times t for certain values of the spatial variable(s).

A solution that satisfies the causality law requires that the initial condition at the time $t = t_0$ is independent of the boundary condition at the time $t > t_0$. Without this requirement a cause at the time $t > t_0$ could have an effect at the earlier time $t = t_0$.

Steady state equations and their solutions are always outside the causality law since the concept of cause and effect has no meaning in the steady state[5]. This explains why one can obtain useful results when ignoring the causality law. If one is interested only in power, energy, or their transmission one will usually be able to ignore the causality law. *The exact opposite is true if one is interested in the transmission of information or the detection of signals.* Information is transmitted by signals and detected by the reception of signals. The energy of a signal is of little interest as long as there is enough energy to make it detectable. But different signals may cause different effects and the propagation velocity of signals determines the time of an effect. Any serious study of information or signal transmission requires equations and solutions that satisfy the causality law.

[4]For a discussion of the concepts of space and time, particularly why we use one time variable but several space variables rather than one space variable and several time variables, see Harmuth (1989, 1992).

[5]Sometimes one reads that a process can go forward or backward in time. This is wrong, if the process is subject to the causality law, but it is only meaningless rather than wrong in the steady state ($t \to \infty$) since the causality law is meaningless in the steady state. The claim *irreversible processes have a distinguished direction of time but reversible processes do not* can be rephrased as follows: An increase of entropy is the cause for the effect of irreversibility. The distinguished direction of time is due to the causality law applied to the particular physical concept of entropy. *Reversible* is another word for the concept of *steady state.*

The special theory of relativity requires the concepts of signals and propagation velocity of signals. We learned that signals cannot propagate faster than with the velocity c of light, but there is no theory of signal propagation based on Maxwell's equations that tells us more. The inability of Maxwell's equations to satisfy the causality law made a theory of electromagnetic signal propagation impossible. This changed immediately when the problem of Maxwell's equations with causality was recognized and corrected (Harmuth 1986a; Harmuth, Boules, and Hussain 1999; Harmuth and Lukin 2000).

Let us turn to the concept of an electromagnetic signal. We define it as a propagating electromagnetic wave that is zero before a certain finite time[6] and has finite energy. The word *propagating* is important since there are standing waves and captive waves that do not propagate. *All observed or produced propagating waves are signals.* They satisfy both the causality law and the conservation law of energy. Only in theoretical work do we encounter waves that are not zero before a certain time, such as periodic sinusoidal waves, or do not have finite energy, again like periodic sinusoidal waves. Mathematically a signal is represented by a function that is zero before a certain time and quadratically integrable. There are also constraints on the peak amplitude of the function since field strengths, fluxes, voltages, currents, etc. are never infinite. It is usual to think of a signal as a field strength, a voltage, or a current at a certain location as function of time, but a signal could also be something observable at a certain time as function of one or more spatial variables.

Let us write Maxwell's equations in a coordinate system at rest using international units. We use the old-fashioned notation with curl and div since this notation was used when the lack of causal solutions was discovered. The symbols ∇ and $\Box$ will be used when mathematical compactness is moie important than physical lucidity:

$$\operatorname{curl} \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{g}_\mathrm{e} \tag{1}$$

$$-\operatorname{curl} \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \tag{2}$$

$$\operatorname{div} \mathbf{D} = \rho_\mathrm{e} \tag{3}$$

$$\operatorname{div} \mathbf{B} = 0 \tag{4}$$

Here $\mathbf{E}$ and $\mathbf{H}$ stand for the electric and magnetic field strength, $\mathbf{D}$ and $\mathbf{B}$ for the electric and magnetic flux density, $\mathbf{g}_\mathrm{e}$ and ρ_e for the electric current and charge density.

Maxwell's equations are augmented by *constitutive equations* that connect $\mathbf{D}$ with $\mathbf{E}$, $\mathbf{B}$ with $\mathbf{H}$, and $\mathbf{g}_\mathrm{e}$ with $\mathbf{E}$. In the simplest case this connection is

[6]When we produce a signal it will be zero before a finite time t_0 and again after a finite time $t_1 > t_0$. However, if the signal is received after passing through a lossy medium, it will generally only be zero before a finite time but may require an infinitely long time to drop to zero again. This means physically no more than the infinitely long time required to discharge a capacitor through a resistor. But it is the reason why we claim only finite energy and not finite duration for the wave

provided by scalar constants called permittivity ϵ, permeability μ, and conductivity σ:

$$\mathbf{D} = \epsilon \mathbf{E} \tag{5}$$

$$\mathbf{B} = \mu \mathbf{H} \tag{6}$$

$$\mathbf{g}_e = \sigma \mathbf{E} \tag{7}$$

The scalar constants ϵ, μ, σ may be functions of time t and location $\mathbf{r}$ in a time-variable, inhomogeneous medium, and they may become variable tensors for an anisotropic medium. In more general cases Eqs.(5)–(7) are replaced by partial differential equations.

When it was recognized that the set of Eqs.(1)–(7) generally had no solutions that could satisfy the causality law, a magnetic current density term $\mathbf{g}_m$ was added in Eq.(2) based strictly on mathematical considerations (Harmuth 1986a, b, c). This modification produces equations associated with the group symmetry $SU(2)$ rather than $U(1)$ as the original Maxwell equations (Barrett 1989c; 1990 a, b; 1993; 1995b). Solutions satisfying the causality law are obtained. One may make the transition $\mathbf{g}_m \to 0$ at the end of the calculation and retain the causality of the solution. This decisive difference between choosing $\mathbf{g}_m = 0$ at the beginning or the end of the calculation may be explained by the different symmetries $U(1)$ and $SU(2)$ or a singular behavior of the partial differential equations. It will become evident during the later calculations that one obtains different partial differential equations and that there is nothing surprising if one obtains different solutions.

From 1990 on it was understood that a magnetic current density term added to Eq.(2) did not imply the existence of magnetic charges or monopoles. Magnetic dipoles can cause magnetic dipole currents and a magnetic (dipole) current density term is required to represent such dipole currents. The electric current density term $\mathbf{g}_e$ in Eq.(1) always represented monopole currents carried by electric charges and dipole currents carried by dipoles. Maxwell called the dipole currents *polarization currents* since today's atomistic thinking did not exist in his time. Without a polarization or dipole current one cannot explain how an electric current could flow through a capacitor whose dielectric is an insulator—for monopole currents. If the term dipole current rather than polarization current had been used, the question would have been raised long ago why electric dipoles should cause electric dipole currents but magnetic dipoles should *not* cause magnetic dipole currents.

For a brief discussion of dipole currents consider Fig.1.1-1. On the left we see a negative and a positive charge carrier between two metal plates with positive and negative voltage. The charge carriers move toward the plate with opposite polarity. An electric monopole current is flowing as long as the charge carriers move.

In Fig.1.1-1b we see how an induced dipole can produce a dipole current. A neutral particle, such as a hydrogen atom, is not pulled in any direction by voltages at the two metal plates. However, the positive nucleus moves toward

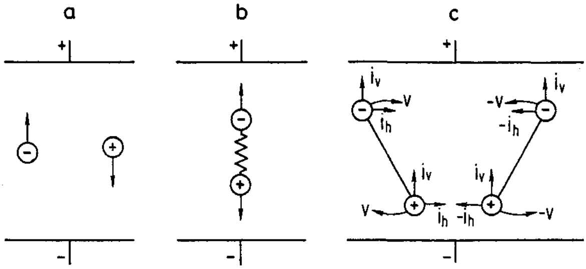

FIG.1.1-1. Current carried by independent positive and negative charges (a). Dipole current due to an induced dipole (b). Dipole current due to orientation polarization of inherent dipoles (c).

the plate with negative voltage and the negative electron toward the plate with positive voltage. A restoring force, symbolized by a coil spring, will pull nucleus and electron together once the voltage at the plates is switched off. A dipole current is flowing as long as the positive and the negative charge carriers are moving either apart or back together again. This simple model becomes more complicated if we say that the probability density function for the location of the electron looses its spherical symmetry and is deformed into the shape of an American football with the nucleus off-center in the elongated direction.

We note that a dipole current can become a monopole current if the field strength between the plates exceeds what is usually referred to as the ionization field strength. One cannot tell at the beginning whether a dipole current will become a monopole current or not, since this depends not only on the magnitude of the field strength but also on its duration. As a result a term in an equation representing a dipole current must be so that it can change to a monopole current. Vice versa, a term representing monopole currents must be so that it can change to a dipole current, since two particles having charges with opposite polarity may get close enough to become a neutral particle. The term $\mathbf{g}_e$ in Eq.(1) satisfies this requirement.

Most molecules, from H_2O to Barium-Titanate, are subject to electric *orientation polarization* in addition to the induced polarization of their atoms. Figure 1.1-1c shows charges with opposite polarity at the ends of rigid rods. A positive and a negative voltage applied to the metal plates will rotate these inherent dipoles to line up with the electric field strength. Dipole currents $2i_v$ are carried by each rotating dipole. There are also dipole currents $2i_h$ perpendicular to the field strength but they compensate if there are counter-rotating dipoles as shown. Only the currents in the direction of the field strength will remain observable macroscopically if there are many dipoles with random orientation. Dipole currents due to orientation polarization exist for magnetic dipoles too, which may range from the hydrogen atom to the magnetic compass needle.

In order to include magnetic dipole current densities one must add a term $\mathbf{g}_m$ in Eq.(2). Equation (4) may be left unchanged for dipole currents, but the zero must be replaced by ρ_m if there are magnetic charges or monopoles. This has been a disputed matter for more than 50 years. We do not have an acceptable explanation for the quantization of electric charges without admitting magnetic monopoles, but we do not have a direct experimental proof for the existence of magnetic monopoles either.

Equations (5)–(7) have to be augmented by a relation between $\mathbf{g}_m$ and $\mathbf{H}$ similar to the one for $\mathbf{g}_e$ and $\mathbf{E}$ in Eq.(7). Hence, we must use the following system of equations if we want solutions that satisfy the causality law:

$$\operatorname{curl}\mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{g}_e \tag{8}$$

$$-\operatorname{curl}\mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} + \mathbf{g}_m \tag{9}$$

$$\operatorname{div}\mathbf{D} = \rho_e \tag{10}$$

$$\operatorname{div}\mathbf{B} = 0 \quad \text{or} \quad \operatorname{div}\mathbf{B} = \rho_m \tag{11}$$

$$\mathbf{D} = \epsilon\mathbf{E} \tag{12}$$

$$\mathbf{B} = \mu\mathbf{H} \tag{13}$$

$$\mathbf{g}_e = \sigma\mathbf{E} \tag{14}$$

$$\mathbf{g}_m = s\mathbf{H} \tag{15}$$

For the magnetic conductivity s with dimension V/Am apply comments corresponding to those made for ϵ, μ, σ in connection with Eqs.(5)–(7). We use s here as a scalar constant but it may be a function of time t and location $\mathbf{r}$ in a time variable, inhomogeneous medium. For an anisotropic medium s becomes a tensor. In more general cases Eq.(15) is replaced by a partial differential equation.

1.2 Step Function Excitation of Planar TEM Wave

We want to obtain the field strengths $\mathbf{E}$ and $\mathbf{H}$ due to the excitation by a planar electric step function for both Eqs.(1.1-1)–(1.1-7) and (1.1-8)–(1.1-15). The coefficients ϵ, μ, σ, s shall all be scalar constants. It is sufficient to solve Eqs.(1.1-8)–(1.1-15) since the solution of Eqs.(1.1-1)–(1.1-7) can be obtained by the substitution $s = 0$. Consider a planar, transverse electromagnetic (TEM) wave propagating in the direction y. A TEM wave requires

$$E_y = 0, \quad H_y = 0 \tag{1}$$

while a planar wave calls for the following relations:

$$\partial E_x/\partial x = \partial E_x/\partial z = \partial E_z/\partial x = \partial E_z/\partial z = 0 \tag{2}$$

$$\partial H_x/\partial x = \partial H_x/\partial z = \partial H_z/\partial x = \partial H_z/\partial z = 0 \tag{3}$$

Writing the operator curl in Cartesian coordinates and introducing the conditions of Eqs.(1)–(3) brings Eqs.(1.1-8) and (1.1-9) into the following form:

$$-\partial H_x/\partial y = \epsilon\partial E_z/\partial t + \sigma E_z \quad (4)$$
$$\partial H_z/\partial y = \epsilon\partial E_x/\partial t + \sigma E_x \quad (5)$$
$$\partial E_x/\partial y = \mu\partial H_z/\partial t + sH_z \quad (6)$$
$$-\partial E_x/\partial y = \mu\partial H_x/\partial t + sH_x \quad (7)$$

With the substitutions

$$E = E_x = E_z, \quad H = H_x = -H_z \quad (8)$$

one may rewrite the two pairs of Eqs.(4) and (7) as well as (5) and (6) as one pair:

$$\partial E/\partial y + \mu\partial H/\partial t + sH = 0 \quad (9)$$
$$\partial H/\partial y + \epsilon\partial E/\partial t + \sigma E = 0 \quad (10)$$

Instead of using the substitutions of Eq.(8) we may make the more general substitutions

$$E_x = E\cos\chi, \quad E_z = E\sin\chi$$
$$H_x = H\sin\chi, \quad H_z = -H\cos\chi$$

where χ is the *polarization angle*[1] measured from the positive x-axis to the vector **E** of the electric field strength or from the negative z-axis to the vector **H** of the magnetic field strength, to obtain Eqs.(9) and (10). Hence, E and H in Eqs.(9) and (10) represent the magnitude of the field strengths **E** and **H**. Since the polarization angle χ is constant, the time variation of the field strengths **E** and **H** is the same as that of their magnitudes E and H. Hence, we can write our equations for E and H rather than for **E** and **H**.

Circularly polarized waves can be obtained by replacing the constant polarization angle χ by a time variable angle ωt:

$$E_x = E\cos\omega t, \quad E_z = E\sin\omega t$$
$$H_x = H\sin\omega t, \quad H_z = -H\cos\omega t$$

Substitution into Eqs.(4)–(7) yields again Eqs.(9) and (10). In this form one emphasizes functions with sinusoidal time variation. This distinction vanishes if one does not choose χ as a linear function of time, $\chi = \omega t$, but as a general function $\chi = f(t)$:

[1]Some authors distinguish between a polarization angle and a rotation angle of a wave. They would call χ a rotation angle.

$$E_x = E\cos[f(t)], \quad E_z = E\sin[f(t)]$$
$$H_x = H\sin[f(t)], \quad H_z = -H\cos[f(t)]$$

The substitution of E_x, E_z, H_x, and H_z into Eqs.(4)–(7) produces once more Eqs.(9) and (10).

Equations (9) and (10) were derived from Eqs.(1.1-8)–(1.1-15). The corresponding equations derived from Eqs.(1.1-1)–(1.1-7) are obtained by choosing $s = 0$ in Eq.(9):

$$\partial E/\partial y + \mu \partial H/\partial t = 0 \tag{11}$$
$$\partial H/\partial y + \epsilon \partial E/\partial t + \sigma E = 0 \tag{12}$$

Elimination of H from Eqs.(11) and (12) yields a second order equation for E:

$$\partial^2 E/\partial y^2 - \mu\epsilon\partial^2 E/\partial t^2 - \mu\sigma\partial E/\partial t = 0 \tag{13}$$

If E is found from this equation one may obtain H from either Eq.(11) or (12)

$$H(y,t) = -\frac{1}{\mu}\int \frac{\partial E}{\partial y} dt + H_t(y) \tag{14}$$
$$H(y,t) = -\int \left(\epsilon\frac{\partial E}{\partial t} + \sigma E\right) dy + H_y(t) \tag{15}$$

where $H_t(y)$ is an integration constant independent of t and $H_y(t)$ an integration constant independent of y.

Let us assume that a boundary condition $E(y,t) = E(0,t)$ and an initial condition $E(y,t) = E(y,0)$ are given for the electric field strength. One may then solve Eq.(13) for these electric boundary and initial conditions and obtain a function $E(y,t) = E_E(y,t)$. Substitution into Eqs.(14) and (15) yields the *associated magnetic field strength* $H(y,t)$ with undetermined functions $H_t(y)$ and $H_y(t)$. These two functions can be determined by the requirement that Eqs.(14) and (15) must yield the same magnetic field strength $H(y,t) = H_E(y,t)$. All this assumes, of course, that a solution $E_E(y,t)$ of Eq.(13) and an associated solution $H_E(y,t)$ of Eqs.(14) and (15) exist. No problem of existence seems to have been encountered if the time variation of $E_E(y,t)$ and $H_E(y,t)$ was that of a periodic or analytic function. However, a problem arose when $E_E(y,t)$ and $H_E(y,t)$ were required to represent signals. Observable or producible electromagnetic waves are always signals, periodic or everywhere analytic functions can only be used for their mathematical approximation if causality is not important.

Next let us assume that the boundary condition $H(y,t) = H(0,t)$ and the initial condition $H(y,t) = H(y,0)$ are given for the magnetic field strength.

Our method of solving first Eq.(13) and then Eqs.(14) and (15) fails. However, one can eliminate E from Eqs.(11) and (12) to obtain an equation for the magnetic field strength H:

$$\partial^2 H/\partial y^2 - \mu\epsilon\partial^2 H/\partial t^2 - \mu\sigma\partial H/\partial t = 0 \tag{16}$$

If the magnetic field strength H is found from this equation one may obtain the electric field strength E from either Eq.(11) or (12):

$$E(y,t) = -\mu \int \frac{\partial H}{\partial t} dy + E_{\mathrm{y}}(t) \tag{17}$$

$$E(y,t) = e^{-\sigma t/\epsilon}\left(-\frac{1}{\epsilon}\int \frac{\partial H}{\partial y} e^{\sigma t/\epsilon} dt + E_{\mathrm{t}}(y)\right) \tag{18}$$

With the boundary and initial conditions for the magnetic field strength one may solve Eq.(16) and obtain a function $H(y,t) = H_{\mathrm{H}}(y,t)$. Substitution into Eqs.(17) and (18) then yields the *associated electric field strength* $E(y,t)$ with undetermined functions $E_{\mathrm{t}}(y)$ and $E_{\mathrm{y}}(t)$. These two functions can be determined by the requirement that Eqs.(17) and (18) must yield the same electric field strength $E(y,t) = E_{\mathrm{H}}(y,t)$. Again it is assumed that a solution $H_{\mathrm{H}}(y,t)$ of Eq.(16) and an associated solution $E_{\mathrm{H}}(y,t)$ of Eqs.(17) and (18) exist.

In the general case, initial and boundary conditions will be given both for the electric and the magnetic field strength. In this case the magnitudes E_{G} and H_{G} of the combined field strengths are given by the sum of the electric and magnetic field strengths obtained from Eqs.(13) and (16), plus the associated electric and magnetic field strengths obtained from Eqs.(14), (15) and (17), (18):

$$E_{\mathrm{G}}(y,t) = E_{\mathrm{E}}(y,t) + E_{\mathrm{H}}(y,t) \tag{19}$$

$$H_{\mathrm{G}}(y,t) = H_{\mathrm{H}}(y,t) + H_{\mathrm{E}}(y,t) \tag{20}$$

We will show later on that the associated magnetic field strengths of Eqs.(14) and (15) remain undetermined for an electric excitation force $E(0,t)$ as boundary condition that has the time variation of a step function $E_0 S(t) = 0$ for $t < 0$ and $E_0 S(t) = E_0$ for $t \geq 0$. This result can be extended and holds generally for excitation forces with the time variation $(t/T)^n E_0 S(t)$ with $n = 0$, 1, 2, ... First we introduce Maxwell's equations modified by a magnetic current density that overcomes the problem of undetermined associated field strengths.

1.3 Solutions for the Electric Field Strength

The problem of undefined associated field strengths was first overcome by adding the term $\mathbf{g}_{\mathrm{m}}$ in Eq.(1.1-9). Later on it was realized that a magnetic dipole current caused by rotating magnetic dipoles demanded such a term on physical grounds even if magnetic charges and magnetic monopole currents should not exist. It is sometimes claimed that the term $\mathbf{g}_{\mathrm{m}}$ can be transformed to zero (Jackson 1975) but this is not so due to a singularity for $\mathbf{g}_{\mathrm{m}} = 0$ in Eq.(1.1-9) as will be seen later on. As a result of this singularity there is a difference whether one chooses $\mathbf{g}_{\mathrm{m}} = 0$ at the beginning of the calculation or makes the transition $\mathbf{g}_{\mathrm{m}} \rightarrow 0$ at the end.

We start with Eqs.(1.2-9) and (1.2-10). The elimination of H yields a second order differential equation for E alone:

$$\partial^2 E/\partial y^2 - \mu\epsilon\partial^2 E/\partial t^2 - (\mu\sigma + \epsilon s)\partial E/\partial t - s\sigma E = 0 \tag{1}$$

A comparison of this equation with Eq.(1.2-13) shows that the fourth term $s\sigma E$ is added and one obtains a significantly different equation for $s = 0$. The third term $(\mu\sigma + \epsilon s)\partial E/\partial t$ becomes only insignificantly different for $s = 0$.

The magnetic field strength $H = H_{\mathrm{E}}$ associated with the electric field strength $E = E_{\mathrm{E}}$ follows from either Eq.(1.2-9) or (1.2-10) by the method of variation of the constant:

$$H(y,t) = e^{-st/\mu}\left(-\frac{1}{\mu}\int \frac{\partial E}{\partial y} e^{st/\mu} dt + H_{\mathrm{t}}(y)\right) \tag{2}$$

$$H(y,t) = -\int \left(\epsilon\frac{\partial E}{\partial t} + \sigma E\right) dy + H_{\mathrm{y}}(t) \tag{3}$$

We may also eliminate E from Eqs.(1.2-9) and (1.2-10) and obtain a differential equation for $H = H_{\mathrm{H}}$ alone:

$$\partial^2 H/\partial y^2 - \mu\epsilon\partial^2 H/\partial t^2 - (\mu\sigma + \epsilon s)\partial H/\partial t - s\sigma H = 0 \tag{4}$$

The associated electric field strength $E = E_{\mathrm{H}}$ follows from either Eq.(1.2-9) or (1.2-10) by variation of the constant:

$$E(y,t) = e^{-\sigma t/\epsilon}\left(-\frac{1}{\epsilon}\int \frac{\partial H}{\partial y} e^{\sigma t/\epsilon} dt + E_{\mathrm{t}}(y)\right) \tag{5}$$

$$E(y,t) = -\int \left(\mu\frac{\partial H}{\partial t} + sH\right) dy + E_{\mathrm{y}}(t) \tag{6}$$

We replace the time variable t and the space variable y by the normalized variables θ and ζ:

$$\theta = \frac{\sigma t}{2\epsilon} = \frac{Zc\sigma}{2}t, \quad \zeta = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}\sigma y = \frac{Z\sigma}{2}y, \quad Z = \sqrt{\frac{\mu}{\epsilon}} \tag{7}$$

The electric conductivity σ is distinguished in Eq.(7) over the magnetic conductivity s. However, we could normalize the time variable t and the space variable y by

$$\theta = \frac{st}{2\mu} = \frac{cs}{2Z}t, \quad \zeta = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}sy = \frac{s}{2Z}y \tag{8}$$

to distinguish s over σ. Equation (7) is particularly useful if we want to investigate the limit $s \to 0$, $\sigma =$ constant, while Eq.(8) is useful for $\sigma \to 0$, $s =$ constant. A normalization that treats σ and s equally is

$$\theta = \left(\frac{\sigma}{\epsilon} + \frac{s}{\mu}\right)t, \quad \zeta = \frac{1}{c}\left(\frac{\sigma}{\epsilon} + \frac{s}{\mu}\right)y \tag{9}$$

These three normalizations permit one to study the limits $\sigma \to 0$ or $s \to 0$ but they are not readily usable if *both* σ and s approach zero. In this case one may use the normalization

$$\theta = t/T, \quad \zeta = y/cT \tag{10}$$

which gives no hint where the distinguished time T could come from. We shall see later on that the calculation provides an automatic answer.

The normalization of Eq.(7), which works well for the limit $s \to 0$, brings Eq.(1) into the form

$$\partial^2 E/\partial\zeta^2 - \partial^2 E/\partial\theta^2 - 2(1+\omega^2)\partial E/\partial\theta - 4\omega^2 E = 0, \quad \omega^2 = \frac{s\epsilon}{\sigma\mu} \tag{11}$$

while the normalization of Eq.(9) yields

$$\partial^2 E/\partial\zeta^2 - \partial^2 E/\partial\theta^2 - \partial E/\partial\theta - \rho^2 E = 0, \quad \rho^2 = \frac{\sigma s}{c^2(\sigma\mu + s\epsilon)^2} \tag{12}$$

and the normalization of Eq.(10) provides:

$$\begin{gathered}\partial^2 E/\partial\zeta^2 - \partial^2 E/\partial\theta^2 - \rho_1 \partial E/\partial\theta - \rho_2^2 E = 0 \\ \rho_1 = c^2 T(\mu\sigma + \epsilon s), \ \rho_2^2 = c^2 T^2 s\sigma\end{gathered} \tag{13}$$

An electric force function with the time variation of a step function is introduced as boundary condition:

$$\begin{aligned}E(0,\theta) = E_0 S(\theta) &= 0 \quad &&\text{for } \theta < 0 \\ &= E_0 &&\text{for } \theta \geq 0\end{aligned} \tag{14}$$

As initial condition we choose that the electric field strength should be zero for all values $\zeta > 0$ at $\theta = 0$:

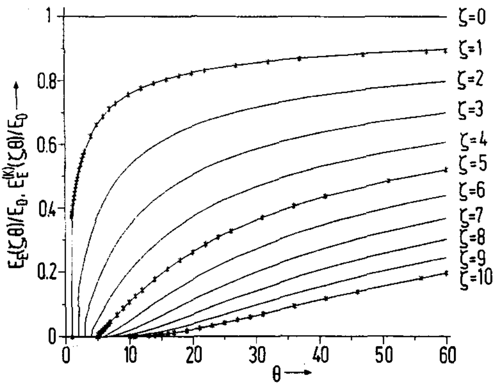

FIG.1.3-1. Electric field strengths of Eqs.(16) and (17) as functions of the normalized time θ with the normalized distance ζ as parameter. The solid lines represent Eq.(16), the stars Eq.(17). (Courtesy M.G.M.Hussain, University of Kuwait)

$$E(\zeta, 0) = 0 \qquad \text{for } \zeta > 0 \tag{15}$$

For $\zeta = 0$ the initial condition is already defined by the boundary condition of Eq.(14).

We solve Eq.(1) for these initial and boundary conditions by means of Fourier's method of standing waves and make the transition $s \to 0$ at the end of the calculation. The calculation is carried out in Appendix 6.1. In terms of the normalization of Eq.(7) the result is

$$\begin{aligned} E(\zeta,\theta) = E_{\mathrm{E}}(\zeta,\theta) &= 0 \qquad \text{for } \theta < \zeta \\ &= E_0\left[1 - e^{-\theta}\int_0^{\zeta}\left(\frac{\theta I_1\left(\sqrt{\theta^2-\eta^2}\right)}{(\theta^2-\eta^2)^{1/2}} + I_0\left(\sqrt{\theta^2-\eta^2}\right)\right)d\eta\right] \quad \text{for } \theta > \zeta \end{aligned} \tag{16}$$

where I_0 and I_1 are modified Bessel functions of the first kind. The field strength $E(\zeta,\theta)/E_0$ is plotted for the locations $\zeta = 0, 1, 2, \ldots, 10$ in the time interval $0 \le \theta \le 60$ by the solid lines in Fig.1.3-1.

If one starts from the original Maxwell equations and solves Eq.(1.2-13) rather than Eq.(1) by a Laplace transform one obtains (Kuester and Harmuth 1987):

$$\begin{aligned} E_{\mathrm{E}}^{(K)} &= 0 \qquad \text{for } \theta < \zeta \\ &= E_0\left(e^{-\zeta} + \zeta\int_{\zeta}^{\theta}\frac{I_1\left(\sqrt{\eta^2-\zeta^2}\right)e^{-\eta}}{(\eta^2-\zeta^2)^{1/2}}d\eta\right) \quad \text{for } \theta > \zeta \end{aligned} \tag{17}$$

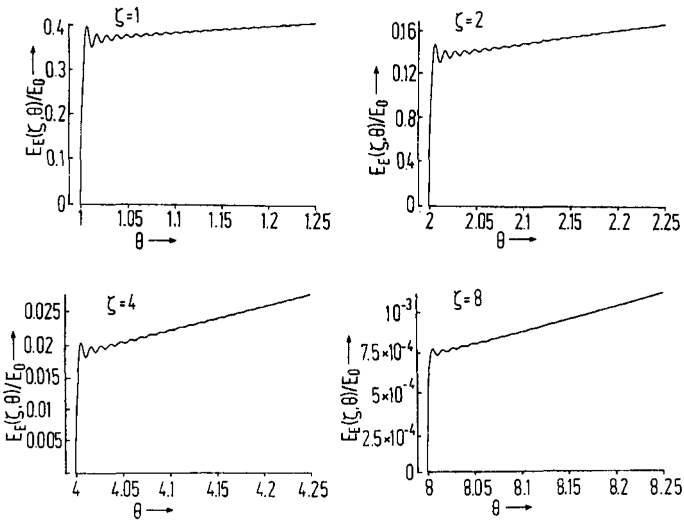

FIG.1.3-2. Plots according to Eq.(16) in the vicinity of $\theta = \zeta$ with a large scale for θ; $\zeta = 1, 2, 4, 8$. (Courtesy M.G.M.Hussain, University of Kuwait)

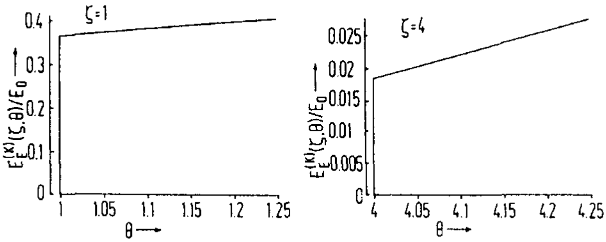

FIG.1.3-3. Plots according to Eq.(17) in the vicinity of $\theta = \zeta$ with a large time scale for θ; $\zeta = 1, 4$. (Courtesy M.G.M.Hussain, University of Kuwait)

where I_1 is again a modified Bessel function of first kind. At first glance Eqs.(16) and (17) look very different but the values $E_{\mathrm{E}}^{(K)}(\zeta, \theta)/E_0$ represented by the stars for $\zeta = 1, 5, 10$ in Fig.1.3-1 show that they are actually quite similar. The plots of Eq.(16) in Fig.1.3-2 and of Eq.(17) in Fig.1.3-3 in the vicinity of $\theta = \zeta$ with a much enlarged time scale for θ show that there is a difference (Hussain 1992).

1.4 Associated Magnetic Field Strength

Either Eq.(1.3-16) or (1.3-17) substituted into Eqs.(1.2-14) and (1.2-15) should give the associated magnetic field strength for Maxwell's original equations, but it turns out that the magnetic field strength remains undefined. This unexpected result was the origin for the claim that Maxwell's equations require a modification to permit signal solutions rather than the usual solutions that

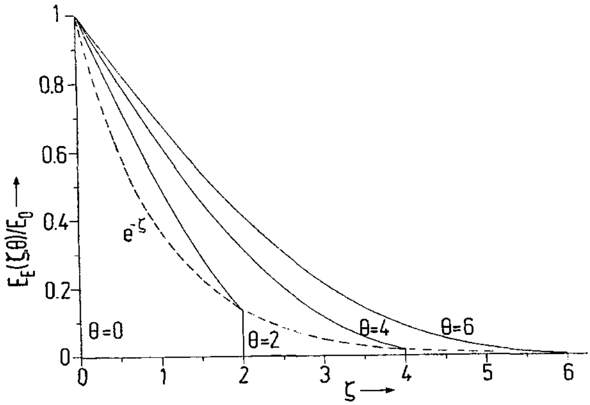

FIG.1.4-1. The electric field strength according to Eqs.(1.3-16) or (1.3-17) as function of ζ with parameter $\theta = 2, 4, 6$. Note that the function $e^{-\zeta}$ yields the amplitude of the jumps at $\theta = \zeta$. (Courtesy R.N.Boules, Towson State University, Maryland, USA)

cannot satisfy the causality law. For a closer analysis we rewrite Eqs.(1.2-14) and (1.2-15) with the normalized variables ζ and θ of Eq.(1.3-7):

$$H(\zeta,\theta) = -\frac{1}{Z}\int \frac{\partial E}{\partial \zeta} d\theta + H_\theta(\zeta) \tag{1}$$

$$H(\zeta,\theta) = -\frac{1}{Z}\int \left(\frac{\partial E}{\partial \theta} + 2E\right) d\zeta + H_\zeta(\theta) \tag{2}$$

For $\theta < 0$ we have $E = 0$ and $\partial E/\partial\theta = 0$ according to Fig.1.3-1. The derivative $\partial E/\partial\theta$ is not defined[1] for $\theta = \zeta$ but E can be defined as a finite right limit. Both E and $\partial E/\partial\theta$ are defined and finite for $\theta > \zeta$. One might claim that $\partial E/\partial\theta$ is infinite for $\theta = \zeta$, but the integration with respect to ζ over such an infinity would yield an undefined value. Hence, the same result is obtained for both points of view.

Let us see what happens if a sum contains an undefined, non-negligible term A. The sum $A + 1 + 1/2$ is not defined if A is non-negligible. The same holds true for the infinite sum $A+1/2+1/4+\cdots = A+2$ with denumerably many terms. If an integral—which is a sum with nondenumerable many terms—yields a defined value and we add a non-negligible, undefined value, we obtain an undefined value. Hence, if $\partial E/\partial\theta$ is not defined and not negligible for just one point $\theta = \zeta$, the integral of Eq.(2) is not defined for $\theta \geq \zeta$, but it is defined and zero for $\theta < \zeta$ if the electric excitation force is applied at the time $\theta = 0$ at the plane $\zeta = 0$.

A similar argument holds for the integral of Eq.(1). First we plot the electric field strength $E(\zeta,\theta)$ of Eqs.(1.3-16) or (1.3-17) as function of ζ with

[1]The discontinuities in Fig.1.3-1 for $\theta = \zeta$ can be shown analytically to exist for any finite distance (Boules 1989, p. 22).

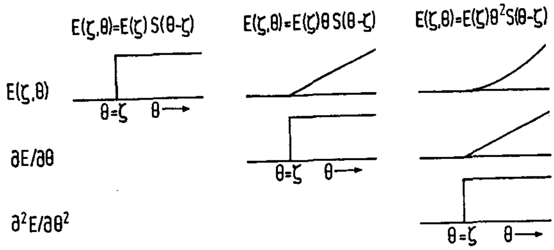

FIG.1.4-2. Time variation of three electric field strengths $E(\zeta, \theta)$ and their derivatives $\partial E/\partial\theta$, $\partial^2 E/\partial\theta^2$. The function $S(\theta - \zeta)$ is the unit step function which is zero for $\theta < \zeta$ and one for $\theta \geq \zeta$.

parameter θ as shown in Fig.1.4-1 rather than as function of θ with parameter ζ as in Fig.1.3-1. We note that the amplitude of the jumps varies like $e^{-\zeta}$ (Harmuth, Boules, and Hussain 1999, p. 160). According to this illustration the derivative $\partial E/\partial\zeta$ is defined and finite for $\zeta < \theta$, undefined for $\zeta = \theta$, and defined as well as zero for $\zeta > \theta$. The integral with respect to θ is not defined for $\zeta \geq \theta$, but it is defined for $\zeta < \theta$. There is no pair of values $\zeta \geq 0$, $\theta \geq 0$ for which the integrals in both Eq.(1) and (2) are defined. The functions $H_\zeta(\theta)$ or $H_y(t)$ and $H_\theta(\zeta)$ or $H_t(y)$ cannot be determined without additional information.

Our proof that the associated magnetic field strength $H_E(\zeta, \theta)$ cannot be obtained from the electric field strength $E_E(\zeta, \theta)$ does not depend in any way on how the electric field strength $E = E_E$ of Eq.(1.2-13) is obtained. The proof depends on only two propositions:

a) Equations (1) and (2) are correct and describe the associated magnetic field strength in a realistic medium for the propagation of electromagnetic waves.
b) Maxwell's equations yield electric field strengths that have undefined derivatives[2] $\partial E/\partial t$ and $\partial E/\partial y$ in at least one point such as $y = ct$ that make a non-negligible contribution to the integrals in Eqs.(1) and (2).

We note that our proof does not depend on the assumption of a wave excited at an infinite plane $\zeta = 0$. Equations (1.2-9)–(1.2-12) are obtained for spherical waves too (Harmuth 1986a, p. 231). Furthermore, we do not need to claim that Maxwell's equations will *never* yield a defined associated field strength for a signal solution. It is sufficient to show that Maxwell's equations fail in one example of physical interest. But it is prudent to extend the proof to excitation functions that do not have the time variation of the step function $S(\theta)$.

The column headed by $E(\zeta, \theta) = E(\zeta)S(\theta - \zeta)$ in Fig.1.4-2 shows an electric field strength with the time variation of a step function. We want to generalize to electric field strengths that vary linearly like $\theta S(\theta - \zeta)$, quadratically like $\theta^2 S(\theta - \zeta)$, or generally like $\theta^n S(\theta - \zeta)$ with $n = 0, 1, 2, \ldots$, for $\theta \geq \zeta$. The

[2]The magnetic field strength remains undefined in a loss-free medium too, since the condition $\sigma = 0$ in Eq.(1.2-15) does not change the argument.

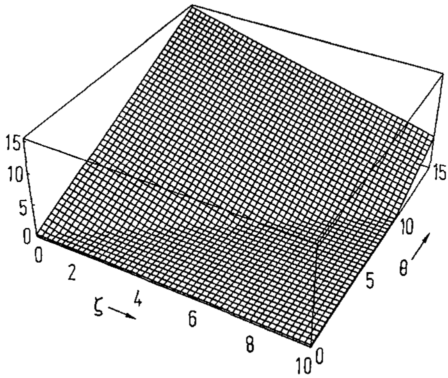

FIG.1.4-3. A continuous function $F(\zeta,\theta)$ of two variables having linear ramp functions $F(\zeta_0,\theta) = \theta S(\theta-\zeta)$ for any fixed value $\zeta = \zeta_0$ cannot have functions $F(\zeta,\theta_0)$ with a jump for a fixed value of $\theta = \theta_0$.

linearly and quadratically varying field strengths and their existing derivatives are shown in the second and third column of Fig.1.4-2. Consider the linearly varying field strength $E(\zeta,\theta) = E(\zeta)\theta S(\theta-\zeta)$. Its first derivative $\partial E/\partial\theta$ is now defined[3] and the integral in Eq.(2) becomes defined for $\theta \geq \zeta$. The linear variation of $E(\zeta)\theta S(\theta-\zeta)$ with θ for $\theta \geq \zeta$ implies that $E(\zeta)$ cannot have a jump at $\theta = \zeta$ as in Fig.1.4-1 but must have a bounded derivative $\partial E/\partial\zeta$ for any $\zeta \geq 0$ as is made evident by Fig.1.4-3. We may thus calculate $H(\zeta,\theta)$ from Eqs.(1) and (2). However, a problem is encountered if the derivative $\partial H/\partial\theta$ is then calculated from Eqs.(1) or (2):

$$\frac{\partial H}{\partial\theta} = -\frac{1}{Z}\frac{\partial E}{\partial\zeta} \tag{3}$$

$$\frac{\partial H}{\partial\theta} = -\frac{1}{Z}\int\left(\frac{\partial^2 E}{\partial\theta^2} + 2\frac{\partial E}{\partial\theta}\right)d\zeta + \frac{\partial H_\zeta(\theta)}{\partial\theta} \tag{4}$$

According to Fig.1.4-2 the second derivative $\partial^2 E/\partial\theta^2$ is not defined for $\theta = \zeta$, or it is infinite, and the integral of Eq.(4) remains undefined. Hence, Eq.(4) states that $\partial H/\partial\theta$ is not defined, but Eq.(3) states that $\partial H/\partial\theta$ has a defined value[4]. This is a contradiction and the assumptions leading to Eqs.(1)–(4) must be wrong.

Let us advance to the quadratic variation $E(\zeta)\theta^2 S(\theta-\zeta)$ shown in Fig.1.4-2. The first derivative $\partial H/\partial\theta$ can be obtained without difficulty since $\partial^2 E/\partial\theta^2$ is defined, but the second derivative $\partial^2 H/\partial\theta^2$ yields:

[3]One may question whether $\partial E/\partial\theta$ at $\theta = \zeta$ is zero or has a finite value $E(\zeta) > 0$. But a finite value in one point makes only an infinitesimal contribution to the integral of Eq.(2). This is the reason why the term *non-negligible* has been used repeatedly.

[4]One cannot assign a value to $\partial^2 E/\partial\theta^2$ to make Eq.(4) yield the same value for $\partial H/\partial\theta$ as Eq.(3) since this would also change $\partial E/\partial\theta$ and E.

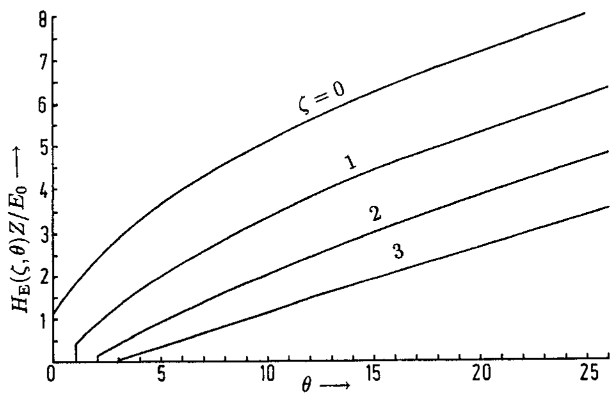

FIG.1.4-4. Normalized magnetic field strengths $H_E(\zeta,\theta)/E_0Z^{-1}$, $Z = (\mu/\epsilon)^{1/2}$, associated with the electric field strengths of Fig.1.3-1. The normalized time θ and the normalized distance ζ are defined in Eq.(1.3-7).

$$\frac{\partial^2 H}{\partial\theta^2} = -\frac{1}{Z}\frac{\partial^2 E}{\partial\zeta\partial\theta} \tag{5}$$

$$\frac{\partial^2 H}{\partial\theta^2} = -\frac{1}{Z}\int\left(\frac{\partial^3 E}{\partial\theta^3} + 2\frac{\partial^2 E}{\partial\theta^2}\right)d\zeta + \frac{\partial^2 H_\zeta(\theta)}{\partial\theta^2} \tag{6}$$

In Eq.(5) the derivative $\partial E/\partial\theta$ has the same time variation as the function E in the linear case $E(\zeta)\theta S(\theta-\zeta)$ according to Fig.1.4-2. Hence, the derivative $\partial^2 E/\partial\zeta\partial\theta$ exists. On the other hand, $\partial^3 E/\partial\theta^3$ is not defined and Eqs.(5) and (6) contradict each other.

The argument can be extended to any finite value of n in $\theta^n S(\theta-\zeta)$. The result is strictly due to the use of a signal as excitation force at the plane $\zeta = 0$ or $y = 0$, which is a time function that is zero before a certain time. Excitation forces with the time variation of periodic or analytic functions in the whole interval $-\infty < \theta < \infty$ would not yield such a result. *But we need solutions of Maxwell's equations or of some modification of these equations for signals, if we want to study signals and information transmission.* Putting it differently, we need signal solutions if we want to introduce the causality law into electrodynamics.

It is important to understand that Maxwell's equations do not yield a *wrong* result for the associated field strengths but an *undefined* one. As a result there are infinitely many solutions that will satisfy Maxwell's equations, and make one believe to have found *the* solution. But Maxwell's equations cannot tell us which of these many solutions is the one and only correct solution.

The failure to obtain the magnetic field strength from Eqs.(1) and (2) for a time variation $\theta^n S(\theta)$ with $n = 0$ for the electric force function at the boundary plane $\zeta = 0$ is instantly attention getting, while for larger values of n the problem shows up only for higher and higher derivatives $\partial^n H/\partial\theta^n$. We infer

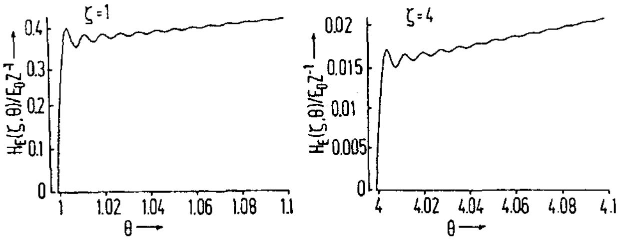

FIG.1.4-5. Magnetic field strengths as in Fig.1.4-4 for $\zeta = 1$, 4 but using a much larger scale for θ in the vicinity of $\theta = \zeta$.

from Fig.1.4-2 that the time $\theta = \zeta$ when the associated magnetic field strength begins at the distance ζ from the excitation plane $\zeta = 0$ is most conspicuous for $n = 0$, but become less and less conspicuous and thus harder to observe experimentally for increasing values of n. As n increases, $\theta^n S(\theta - \zeta)$ looks less like a signal and more like an analytic function. The failure of Maxwell's equations to yield a defined associated magnetic field strength is most evident for $n = 0$ and disappears for $n \to \infty$, while the beginning of a signal is most conspicuous for $n = 0$ and becomes unobservable for $n \to \infty$. If a signal connects a cause with its effect at another location, the time difference will be best defined for $n = 0$ and it becomes undefined for $n \to \infty$. The failure of Maxwell's equations should show up best where causality is most important, while there should be no problem when causality is unimportant.

We turn to the associated magnetic field strength of the modified Maxwell equations as written in Eqs.(1.3-2) and (1.3-3). With the help of Eq.(1.3-7) we write them in normalized form with ζ and θ:

$$H(\zeta,\theta) = e^{-2\omega^2\theta}\left[-\frac{1}{Z}\int \frac{\partial E}{\partial \zeta} e^{2\omega^2\theta} d\theta + H_\theta(\zeta)\right], \quad \omega^2 = \frac{\epsilon s}{\mu\sigma} \tag{7}$$

$$H(\zeta,\theta) = -\frac{1}{Z}\int \left(\frac{\partial E}{\partial \theta} + 2E\right) d\zeta + H_\zeta(\theta) \tag{8}$$

Equations (2) and (8) are identical, but there is enough difference between Eqs.(1) and (7) to solve the problem of undefined associated field strengths. Barrett (1989c; 1990a, b; 1993; 1995b) has explained this difference in terms of symmetries of group theory. The transition $s \to 0$, $\mathbf{g}_{\mathrm{m}} \to 0$ at the end of the calculation is equivalent to the concept of *symmetry breaking*.

Figure 1.4-4 shows plots of the associated magnetic field strength $H(\zeta,\theta) = H_{\mathrm{E}}(\zeta,\theta)$ obtained by substituting the electric field strength $E(\zeta,\theta) = E_{\mathrm{E}}(\zeta,\theta)$ of Eq.(1.3-16) into Eqs.(7) and (8). The calculation is presented in Appendix 6.2 and $H(\zeta,\theta)$ is defined by Eq.(6.2-41) for $H_\theta(\zeta) = 0$.

The plot of Fig.1.4-4 for $\zeta = 1$ and a new plot for $\zeta = 4$ are shown with a large scale of θ in Fig.1.4-5 in the vicinity of $\theta = \zeta$. There are damped oscillations of the magnetic field strength just like those of the electric field strength in Fig.1.3-2.

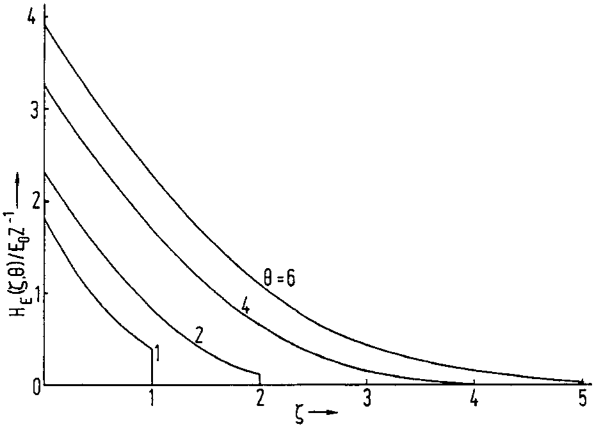

FIG.1.4-6. The magnetic field strength according to Eq.(6.2-41) as function of ζ with parameter $\theta = 2,\ 4,\ 6$.

The field strength $H(\zeta,\theta)$ as function of ζ with θ as parameter will be needed in Section 1.6. Figure 1.4-6 shows such a plot.

A completely different proof for the non-existence of causal solutions of Maxwell's equations is due to Hillion (1990, 1991, 1992a, b). Hillion knew that certain partial differential equations did not permit solutions with independent initial and boundary conditions in a coordinate system at rest. He recognized that Maxwell's equations were of this type. Hence, we have here one of those rare cases when two scientists working independently and using different approaches arrived at the same result.

Let us emphasize once more that our proof of the failure of Maxwell's equations to yield certain signal solutions of interest is based solely on the derivation of the associated field strength H from the electric field strength E defined by the differential equation (1.2-13) plus initial and boundary conditions. There never was a claim that one could not find solutions of Eq.(1.2-13). Such a claim would be extremely hard to prove unless it is restricted to certain methods of solution of partial differential equations, which would make it useless for our purpose. Proving a contradiction between Eqs.(1) and (2) avoids any discussion of how to solve partial differential equations. A number of authors derived solutions for Eq.(1.2-13), but by doing so they only proved that they had not read carefully what had been claimed. The method of deriving two contradicting equations will be used again in Section 3.1 to show that a generally accepted result cannot be correct.

The fact that the problem of signal solutions of Maxwell's equations shows up for the associated magnetic field strengths in Eqs.(1) and (2) but not for the electric field strength of Eq.(1.2-13) explains why it took a century to recognize the problem. Anyone satisfied with the electric field strength derived from Eq.(1.2-13) and ignoring the associated magnetic field strength never noticed that anything was amiss.

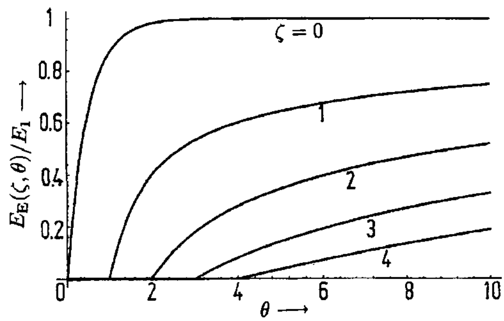

FIG.1.5-1. Electric field strengths $E_E(\zeta,\theta)/E_1$ according to Eq.(2) as function of the normalized time θ with the normalized distance ζ as parameter.

1.5 Field Strengths with Continuous Time Variation

The plots of the electric and magnetic field strength in Figs.1.3-1 to 1.3-3, 1.4-1, and 1.4-4 to 1.4-6 have jumps or steps at $\theta = \zeta$ where the derivatives with respect to θ or ζ are not defined. This creates problems in certain cases which are readily avoided by replacing the step function $E_0 S(\theta)$ of Eq.(1.3-14) with an exponential ramp function:

$$\begin{aligned} E(0,\theta) = E_1 S(\theta)(1-e^{-\iota\theta}) &= 0 && \text{for } \theta < 0 \\ &= E_1(1-e^{-\iota\theta}) && \text{for } \theta \geq 0 \end{aligned} \quad (1)$$

Instead of the field strength $E_E(\zeta,\theta)$ of Eq.(1.3-16) we derive in Section 6.4 the following field strength:

$$E(\zeta,\theta) = E_E(\zeta,\theta) = E_1[(1-e^{-2(1+\omega^2)\theta})e^{-2\omega\zeta} + u(\zeta,\theta)], \quad \omega^2 = \epsilon s/\mu\sigma \quad (2)$$

The special value $\omega = 0$ used in Sections 1.3 and 1.4 is replaced by the general value $\omega \geq 0$. Certain plots of E_E/E_1 for $\omega > 0$ are shown in Figs.6.4-1 and 6.4-2. Here we will concentrate on the special case $\omega = 0$ which reduces Eq.(6.4-28) to the following simpler form:

$$u(\zeta,\theta) = -\frac{4}{\pi}e^{-\theta}\left(\int_0^1 \frac{\operatorname{sh}(1-\eta^2)^{1/2}\theta}{(1-\eta^2)^{1/2}}\frac{\sin\zeta\eta}{\eta}d\eta + \int_1^\infty \frac{\sin(\eta^2-1)^{1/2}\theta}{(\eta^2-1)^{1/2}}\frac{\sin\zeta\eta}{\eta}d\eta\right) \quad (3)$$

Plots of $E_E(\zeta,\theta)/E_1$ as function of the time θ at various distances ζ are shown in Fig.1.5-1. The functions rise like $E(\zeta)\theta S(\theta-\zeta)$ in Fig.1.4-2. The derivative $\partial E_E/\partial\theta$ is defined everywhere in the interval $0 \leq \omega < \infty$.

Figure 1.5-2 shows the field strengths $E_E(\zeta,\theta)/E_1$ as function of the distance ζ for various times θ. There are no steps as in Fig.1.4-1 and the derivative

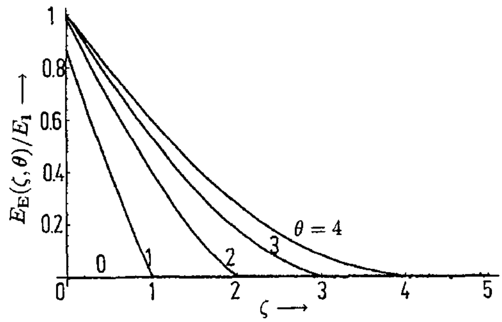

FIG.1.5-2. Electric field strengths $E_E(\zeta,\theta)/E_1$ according to Eq.(2) as function of the normalized distance ζ with the normalized time θ as parameter.

$\partial E_E/\partial\zeta$ is defined everywhere in the interval $0 \leq \zeta < \infty$. We note that the parameter $\zeta = 0$ in Fig.1.5-1 shows the boundary condition while the parameter $\theta = 0$ in Fig.1.5-2 shows the initial condition which is zero.

The magnetic field strength $H_E(\zeta,\theta)$ associated with the electric field strength $E_E(\zeta,\theta)$ of Eq.(2) is derived in Section 6.5 and represented by Eqs.(6.5-10), (6.5-15) for the general case $\omega \geq 0$. Certain plots of $H_E Z/E_1$ for $\omega > 0$ are shown in Figs.6.5-1 and 6.5-2. The simpler special case $\omega = 0$ suffices here:

$$H_E(\zeta,\theta) = Z^{-1}E_1[-2\zeta + I'_{E3}(\zeta,\theta) - I_{E4}(\zeta,\theta)] \tag{4}$$

$$\begin{aligned} I'_{E3}(\zeta,\theta) = \frac{2}{\pi}\Bigg\{ & 2\Bigg[\frac{1}{d} - \int_0^d [\exp(-\eta^2\theta/2) - 1]\,\frac{d\eta}{\eta^2} \\ & - e^{-\theta}\int_d^1 \left(\operatorname{ch}(1-\eta^2)^{1/2}\theta + \frac{\operatorname{sh}(1-\eta^2)^{1/2}\theta}{(1-\eta^2)^{1/2}}\right)\frac{d\eta}{\eta^2}\Bigg] \\ & + e^{-\theta}\int_0^1 \left(\operatorname{ch}(1-\eta^2)^{1/2}\theta + \frac{\operatorname{sh}(1-\eta^2)^{1/2}\theta}{(1-\eta^2)^{1/2}}\right)\left(\frac{\sin(\zeta\eta/2)}{\eta/2}\right)^2 d\eta\Bigg\} \end{aligned} \tag{5}$$

$$I_{E4}(\zeta,\theta) = \frac{4}{\pi}e^{-\theta}\int_1^\infty \left(\cos(\eta^2-1)^{1/2}\theta + \frac{\sin(\eta^2-1)^{1/2}\theta}{(\eta^2-1)^{1/2}}\right)\frac{\cos\zeta\eta}{\eta^2}\,d\eta \tag{6}$$

The function $I'_{E3}(\zeta,\theta)$ holds for the limit $d \to 0$. This means no more than to say that $(\sin x)/x$ must be evaluated numerically for the limit $x \to 0$ if one wants its value at $x = 0$. By trial and error one finds that a reduction of d below 10^{-5} yields changes of less than the line width of the plots.

Plots of $H_E Z/E_1$ as function of θ at various distances ζ are shown in Fig.1.5-3. Again there is no step where the functions begin to rise from zero and the derivative $\partial H_E/\partial\theta$ is defined everywhere in the interval $0 \leq \theta < \infty$.

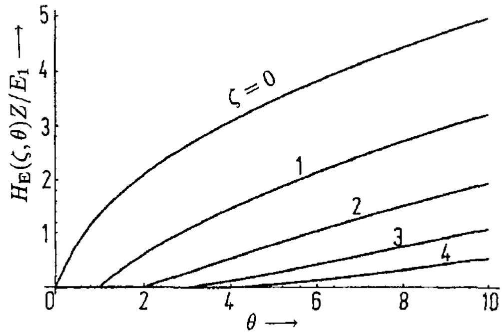

FIG.1.5-3. Magnetic field strengths $H_E(\zeta,\theta)Z/E_1$ according to Eq.(4) as function of the normalized time θ with the normalized distance ζ as parameter.

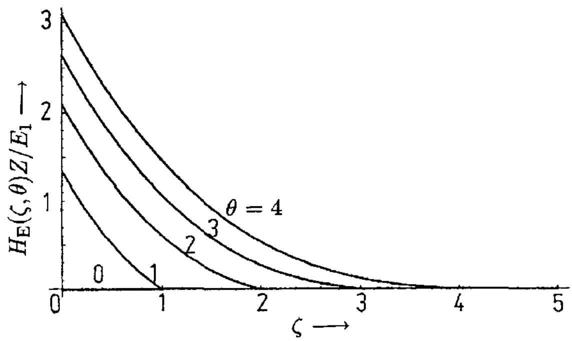

FIG.1.5-4. Magnetic field strengths $H_E(\zeta,\theta)Z/E_1$ according to Eq.(4) as function of the normalized distance ζ with the normalized time θ as parameter.

Figure 1.5-4 shows the associated magnetic field strength $H_E(\zeta,\theta)Z/E_1$ as function of the distance ζ for various times θ. Again there are no steps at $\zeta = 1$, 2, 3, 4 and the derivative $\partial H_E/\partial\zeta$ is defined everywhere in the interval $0 \leq \zeta < \infty$.

1.6 Modified Maxwell Equations in Potential Form

Since Maxwell's equations are often used in potential form we want to extend the results of Sections 1.1 to 1.5 about the existence of solutions satisfying the causality law to the potential form. We start from Eqs.(1.1-8) to (1.1-15) but substitute $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$. The velocity c of light and the wave impedance Z are used rather than μ and ϵ:

$$\operatorname{curl}\mathbf{H} = \frac{1}{Zc}\frac{\partial\mathbf{E}}{\partial t} + \mathbf{g}_{\mathrm{e}} \tag{1}$$

$$-\operatorname{curl}\mathbf{E} = \frac{Z}{c}\frac{\partial\mathbf{H}}{\partial t} + \mathbf{g}_{\mathrm{m}} \tag{2}$$

$$\operatorname{div}\mathbf{E} = Zc\rho_\mathrm{e} \tag{3}$$

$$\operatorname{div}\mathbf{H} = 0 \quad \text{or} \quad \operatorname{div}\mathbf{H} = \frac{c}{Z}\rho_\mathrm{m} \tag{4}$$

$$\mathbf{g}_\mathrm{e} = \sigma\mathbf{E} \tag{5}$$

$$\mathbf{g}_\mathrm{m} = s\mathbf{H} \tag{6}$$

$$Z = \sqrt{\mu/\epsilon} = \mu c,\ c = 1/\sqrt{\mu\epsilon},\ \epsilon = 1/\mu c^2 = 1/Zc,\ \mu = Z/c$$
$$\mu = 4\pi \times 10^{-7}\ [\mathrm{Vs/Am}], \quad c = 299\,792\,458\ [\mathrm{m/s}] \quad \text{(definitions)}$$
$$Z = \mu c \doteq 376.730\,314\ [\mathrm{V/A}] \tag{7}$$

The operator div applied to Eq.(1) and differentiation of Eq.(3) with respect to time yields the electric continuity equation:

$$\operatorname{div}\mathbf{g}_\mathrm{e} + \partial\rho_\mathrm{e}/\partial t = 0 \tag{8}$$

If there is a magnetic monopole current—which is a controversial matter—we may also obtain a corresponding magnetic continuity equation by application of the operator div to Eq.(2) and differentiation of Eq.(4) with respect to time:

$$\operatorname{div}\mathbf{g}_\mathrm{m} + \partial\rho_\mathrm{m}/\partial t = 0 \tag{9}$$

If there are no magnetic monopoles or charges ρ_m we obtain the reduced equation for dipole and multipole currents, which are not controversial:

$$\operatorname{div}\mathbf{g}_\mathrm{m} = 0 \tag{10}$$

Using both intuition and experience we assume that the magnetic field strength of a solution of Eqs.(1) and (2) can be written in the form

$$\mathbf{H} = \frac{c}{Z}\operatorname{curl}\mathbf{A}_\mathrm{m} - \frac{\partial\mathbf{A}_\mathrm{e}}{\partial t} - \operatorname{grad}\phi_\mathrm{m} \tag{11}$$

where $\mathbf{A}_\mathrm{m}$ and $\mathbf{A}_\mathrm{e}$ are the *magnetic* and the *electric vector potential*, while ϕ_m is the *magnetic scalar potential*. Substitution of Eq.(11) into Eq.(2) yields:

$$-\operatorname{curl}\mathbf{E} = \operatorname{curl}\frac{\partial\mathbf{A}_\mathrm{m}}{\partial t} - \frac{Z}{c}\left(\frac{\partial^2\mathbf{A}_\mathrm{e}}{\partial t^2} + \operatorname{grad}\frac{\partial\phi_\mathrm{m}}{\partial t}\right) + \mathbf{g}_\mathrm{m} \tag{12}$$

With some more intuition and experience we define a new vector $\mathbf{G}$ by the relation

$$\operatorname{curl}\operatorname{curl}\mathbf{G} = \frac{1}{c^2}\left(\frac{\partial^2\mathbf{A}_\mathrm{e}}{\partial t^2} + \operatorname{grad}\frac{\partial\phi_\mathrm{m}}{\partial t}\right) - \frac{1}{Zc}\mathbf{g}_\mathrm{m} \tag{13}$$

and substitute it into Eq.(12). We obtain $\operatorname{curl}\mathbf{E}$:

$$\operatorname{curl}\mathbf{E} = Zc\operatorname{curl}\operatorname{curl}\mathbf{G} - \operatorname{curl}\frac{\partial\mathbf{A}_\mathrm{m}}{\partial t} \tag{14}$$

The operator curl can be dropped. Since the curl of the gradient of a scalar is zero we must add such a gradient to maintain generality. Intuition and experience are needed once more to choose the scalar to be ϕ_e:

$$\mathbf{E} = Zc\,\mathrm{curl}\,\mathbf{G} - \frac{\partial \mathbf{A}_m}{\partial t} - \mathrm{grad}\,\phi_e \tag{15}$$

The comparison of Eqs.(11) and (15) suggests the choice

$$\mathrm{curl}\,\mathbf{G} = -\,\mathrm{curl}\,\mathbf{A}_e \tag{16}$$

and Eq.(15) is brought into the form of Eq.(11):

$$\mathbf{E} = -Zc\,\mathrm{curl}\,\mathbf{A}_e - \frac{\partial \mathbf{A}_m}{\partial t} - \mathrm{grad}\,\phi_e \tag{17}$$

Substitution of Eqs.(11) and (17) into Eq.(1) yields

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{A}_m = -\frac{1}{c^2}\left(\frac{\partial^2 \mathbf{A}_m}{\partial t^2} + \mathrm{grad}\,\frac{\partial \phi_e}{\partial t}\right) + \frac{Z}{c}\mathbf{g}_e \tag{18}$$

while the substitution of Eq.(16) into Eq.(13) brings:

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{A}_e = -\frac{1}{c^2}\left(\frac{\partial^2 \mathbf{A}_e}{\partial t^2} + \mathrm{grad}\,\frac{\partial \phi_m}{\partial t}\right) + \frac{1}{Zc}\mathbf{g}_m \tag{19}$$

We further substitute Eq.(17) into Eq.(3)

$$-\frac{1}{Zc}\left(\mathrm{div}\,\frac{\partial \mathbf{A}_m}{\partial t} + \nabla^2\phi_e\right) = \rho_e \tag{20}$$

and Eq.(11) into Eq.(4):

$$-\frac{Z}{c}\left(\mathrm{div}\,\frac{\partial \mathbf{A}_e}{\partial t} + \nabla^2\phi_m\right) = \rho_m \tag{21}$$

In the absence of magnetic charges we get the simpler equation

$$\mathrm{div}\,\frac{\partial \mathbf{A}_e}{\partial t} + \nabla^2\phi_m = 0 \tag{22}$$

The vector potentials are not completely specified since Eq.(11) only defines $\mathrm{curl}\,\mathbf{A}_m$ and Eq.(16) $\mathrm{curl}\,\mathbf{A}_e$. We may choose two additional conditions that we call the *extended Lorentz convention*:

$$\mathrm{div}\,\mathbf{A}_m + \frac{1}{c^2}\frac{\partial \phi_e}{\partial t} = 0 \tag{23}$$

$$\mathrm{div}\,\mathbf{A}_e + \frac{1}{c^2}\frac{\partial \phi_m}{\partial t} = 0 \tag{24}$$

Using the vector relation

$$\operatorname{curl}\operatorname{curl}\mathbf{f} = \operatorname{grad}\operatorname{div}\mathbf{f} - \nabla^2\mathbf{f} \tag{25}$$

we obtain from Eqs.(18)–(21), (23), and (24):

$$\nabla^2\mathbf{A}_\mathrm{e} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}_\mathrm{e}}{\partial t^2} \equiv \Box\mathbf{A}_\mathrm{e} = -\frac{1}{Zc}\mathbf{g}_\mathrm{m} \tag{26}$$

$$\nabla^2\mathbf{A}_\mathrm{m} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}_\mathrm{m}}{\partial t^2} \equiv \Box\mathbf{A}_\mathrm{m} = -\frac{Z}{c}\mathbf{g}_\mathrm{e} \tag{27}$$

$$\nabla^2\phi_\mathrm{e} - \frac{1}{c^2}\frac{\partial^2\phi_\mathrm{e}}{\partial t^2} \equiv \Box\phi_\mathrm{e} = -Zc\rho_\mathrm{e} \tag{28}$$

$$\nabla^2\phi_\mathrm{m} - \frac{1}{c^2}\frac{\partial^2\phi_\mathrm{m}}{\partial t^2} \equiv \Box\phi_\mathrm{m} = -\frac{c}{Z}\rho_\mathrm{m} \tag{29}$$

Particular solutions of these partial differential equations may be represented by integrals taken over the whole space[1]:

$$\mathbf{A}_\mathrm{e}(x,y,z,t) = \frac{1}{4\pi Zc}\iiint \frac{\mathbf{g}_\mathrm{m}(\xi,\eta,\zeta,t-r/c)}{r}\,d\xi\,d\eta\,d\zeta \tag{30}$$

$$\mathbf{A}_\mathrm{m}(x,y,z,t) = \frac{Z}{4\pi c}\iiint \frac{\mathbf{g}_\mathrm{e}(\xi,\eta,\zeta,t-r/c)}{r}\,d\xi\,d\eta\,d\zeta \tag{31}$$

$$\phi_\mathrm{e}(x,y,z,t) = \frac{Zc}{4\pi}\iiint \frac{\rho_\mathrm{e}(\xi,\eta,\zeta,t-r/c)}{r}\,d\xi\,d\eta\,d\zeta \tag{32}$$

$$\phi_\mathrm{m}(x,y,z,t) = \frac{c}{4\pi Z}\iiint \frac{\rho_\mathrm{m}(\xi,\eta,\zeta,t-r/c)}{r}\,d\xi\,d\eta\,d\zeta \tag{33}$$

Here r is the distance between the coordinates ξ, η, ζ of the current and charge densities and the coordinates x, y, z of the potentials:

$$r = \left[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\right]^{1/2} \tag{34}$$

The electric and magnetic field strengths are here determined by currents and charges that excite them. If there is no magnetic charge, the scalar potential ϕ_m drops out; if in addition there are no magnetic dipole currents exciting $\mathbf{E}$ and $\mathbf{H}$, the vector potential $\mathbf{A}_\mathrm{e}$ drops out too and only the conventional Eqs.(27) and (28) remain. Equations (11) and (17) are reduced to their usual form:

$$\mathbf{H} = \frac{c}{Z}\operatorname{curl}\mathbf{A}_\mathrm{m}, \quad \mathbf{E} = -\frac{\partial\mathbf{A}_\mathrm{m}}{\partial t} - \operatorname{grad}\phi_\mathrm{e} \tag{35}$$

[1]The proof may be found in Abraham and Becker (1932, Part III, Ch.X, Sec.10) or in Abraham and Becker (1950, Part III, Ch.X, Sec.9). Both books call these solutions for $\mathbf{A} = \mathbf{A}_\mathrm{m}$ and $\phi = \phi_\mathrm{e}$ *general solutions*, but later editions point out that they are only particular solutions to which one can add general solutions of the homogeneous versions of Eqs.(26)–(29) to obtain the general solutions (Becker 1957, 1964a, b).

We have not used Eqs.(5) and (6). These equations state the field strengths $\mathbf{E}$ and $\mathbf{H}$ required to drive the current densities $\mathbf{g}_e$ and $\mathbf{g}_m$. We usually associate the equation $\mathbf{g}_e = \sigma\mathbf{E}$ with ohmic losses but near zone radiation or interaction in vacuum would call for the same equation.

2 Monopole, Dipole, and Multipole Currents

2.1 Electric Monopoles and Dipoles With Constant Mass

We have seen in Sections 1.3 and 1.4 that the choice $\mathbf{g}_m = 0$ in Eq.(1.1-9) *at the beginning* of the calculation does not yield an associated magnetic field strength due to excitation by an electric field strength while the transition $\mathbf{g}_m \to 0$ *at the end* of the calculation does yield the associated magnetic field strength. An equivalent result is obtained for $\mathbf{g}_e$ in Eq.(1.1-8) if one uses excitation by a magnetic field strength and wants to calculate the associated electric field strength (Harmuth 1986a, Sections 2.6 and 2.7).

In quantum field theory it is usual to quantize Maxwell's equations in 'empty space' by choosing $\mathbf{g}_e = 0$ and $\mathbf{g}_m = 0$ at the beginning of the calculation. Since this cannot be done in the classical theory we will not expect it to work in quantum theory. This is a strictly mathematical objection to the choice $\mathbf{g}_e = 0$ and $\mathbf{g}_m = 0$.

There is also a physical objection. We can define 'empty space' by the absence of particles and charges. But we cannot exclude dipoles unless we are prepared to claim that a capacitor with vacuum or 'empty space' as dielectric cannot be charged and does not permit an electric current with sinusoidal time variation to pass through. Such a claim would contradict observation. Since $\mathbf{g}_e$ and $\mathbf{g}_m$ in Eqs.(1.1-8) and (1.1-9) stand for any kind of electric and magnetic current densities we must use them whenever dipole or higher order multipole currents are possible, regardless of whether these dipoles and multipoles are *real* or *virtual*. The only media in which real or virtual electric dipoles do not exist seem to be electric superconductors. Real or virtual magnetic dipoles appear to be even more ubiquitous. If there are no magnetic charges or monopoles there would be no magnetic superconductors that exclude magnetic dipoles.

We shall need the current densities produced by moving monopoles and dipoles. Furthermore, we shall need equations of motion. The dipoles can be either induced as in Fig.2.1-1b or inherent as in Fig.2.1-1c. We begin with monopoles.

An electric charge density ρ_e moving with the velocity $\mathbf{v}$ produces an electric current density $\mathbf{g}_e$ as shown in Fig.2.1-1a:

$$\mathbf{g}_e = \rho_e \mathbf{v} \tag{1}$$

The motion of a charge in an electromagnetic field is given by the Lorentz equation

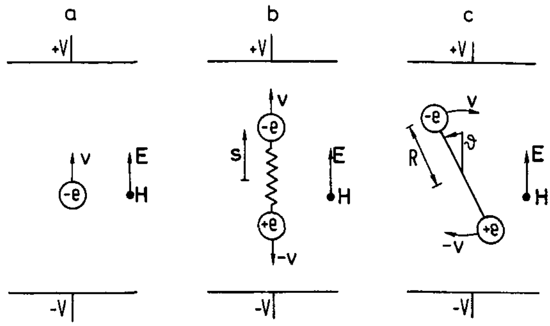

FIG.2.1-1. Particle with negative charge moving in an homogeneous electromagnetic field (a). Induced electric dipole in an homogeneous electromagnetic field (b). Inherent electric dipole rotating in an homogeneous electromagnetic field (c).

$$\mathbf{k}_\mathrm{m} = \rho_\mathrm{e}\left(\mathbf{E} + \frac{Z}{c}\mathbf{v} \times \mathbf{H}\right) \tag{2}$$

where $\mathbf{k}_\mathrm{m}$ is a *force density* that acts on the electric charge density ρ_e. For a particle with charge e rather than a charge density ρ_e we replace the force density $\mathbf{k}_\mathrm{m}$ in Eq.(2) by a force $\mathbf{K}_\mathrm{m}$:

$$\mathbf{K}_\mathrm{m} = e\left(\mathbf{E} + \frac{Z}{c}\mathbf{v} \times \mathbf{H}\right) \tag{3}$$

An equation of motion is obtained be equating $\mathbf{K}_\mathrm{m}$ with the force of inertia, which yields in Newton's mechanics for a particle with mass m:

$$\mathbf{K}_\mathrm{m} = m\frac{d\mathbf{v}}{dt} \tag{4}$$

If the current density $\mathbf{g}_\mathrm{e}$ of Eq.(1.1-1) refers to an electric dipole current density rather than a monopole current density we must replace Eqs.(1)–(4) by equations holding for dipoles. Figure 2.1-1b shows an electric dipole represented by two particles with mass m each and opposite charge. A spring between them represents a restorative force that is proportionate to the distance $\mathbf{s}$ of the particles from their common center of mass. The current density $\mathbf{g}_\mathrm{e} = \rho_\mathrm{e}\mathbf{v}$ of Eq.(1) for a monopole becomes

$$\mathbf{g}_\mathrm{e} = 2\rho_\mathrm{e}\mathbf{v} \tag{5}$$

for a dipole since both charged particles in Fig.2.1-1b have the same mass. For a dipole induced from atomic hydrogen we would get

$$\mathbf{g}_\mathrm{e} = (1 + 1/1836)\rho_\mathrm{e}\mathbf{v} \tag{6}$$

since the current is due mainly to the movement of the electron while the heavy proton barely moves. Our investigation is simplified if we use generally the relation

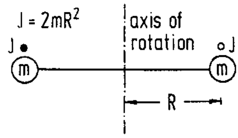

FIG.2.1-2. Dumb-bell model of a rotating dipole with two masses m at the ends of a thin rod of length $2R$.

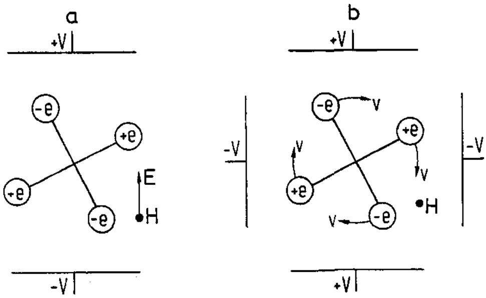

FIG.2.1-3. Electric quadrupole in an homogeneous (a) and an inhomogeneous (b) electric field.

$$\mathbf{g}_{\mathrm{e}} = \rho_{\mathrm{e}}\mathbf{v} \tag{7}$$

but make ρ_{e} either twice as large or $1 + 1/1836 \doteq 1.0005$ times as large as the charge of one part of the dipole. We get then the same relation $\mathbf{g}_{\mathrm{e}} = \rho_{\mathrm{e}}\mathbf{v}$ for dipoles as for monopoles in Eq.(1).

The equation of motion of Eq.(4) gets an additional term $m\mathbf{s}/\tau_{\mathrm{p}}^2$ that represents the restorative force proportionate to $\mathbf{s}$:

$$\mathbf{K}_{\mathrm{m}} = m\frac{d\mathbf{v}}{dt} + \frac{m\mathbf{s}}{\tau_{\mathrm{p}}^2} = m\left(\frac{d^2\mathbf{s}}{dt^2} + \frac{\mathbf{s}}{\tau_{\mathrm{p}}^2}\right) = m\left(\frac{d\mathbf{v}}{dt} + \frac{1}{\tau_{\mathrm{p}}^2}\int \mathbf{v}\,dt\right) \tag{8}$$

Besides the induced electric dipole of Fig.2.1-1b we have inherent[1] dipoles as represented by the two charges $+e$ and $-e$ at the fixed distance $2R$ in Fig.2.1-1c. Most molecules are inherent dipoles, e.g., H_2O, HCl, NH_3. Using the convention of Eq.(7) the current density of rotating inherent dipoles, which flows along a circle rather than a straight line as in Figs.2.1-1a and b, equals

$$\mathbf{g}_{\mathrm{e}} = \rho_{\mathrm{e}}\mathbf{v} = \rho_{\mathrm{e}}R\frac{d\vartheta}{dt} \tag{9}$$

If we denote with J the moment of inertia of the rotating dipole of Fig.2.1-1c and with $2R$ its length we obtain the equation of motion

[1]Reitz, Milford, and Christy (1979) use the term *polar* instead of *inherent*. This book discusses electric dipoles in great detail.

$$\mathbf{K}_{\mathrm{m}}R = \mathbf{J}\frac{d^2\vartheta}{dt^2} \tag{10}$$

For the dumb-bell model of a rotating dipole with two masses m at the end of a thin rod of length $2R$ as shown in Fig.2.1-2 we have the moment of inertia $\mathbf{J}$ with magnitude $J = 2mR^2$ and direction as shown in Fig.2.1-2.

We note that a dipole can be both inherent and induced. For instance a polarized molecule such as H_2O can be rotated by an electric field strength but in addition the electrons and the nuclei can be pulled apart.

Beyond induced and inherent dipoles we have quadrupoles and higher order multipoles. Figure 2.1-3a shows an electric quadrupole in an homogeneous electric field. The field strength $\mathbf{E}$ produces neither a shift nor a rotation of the quadrupole. A more complicated electric field as shown in Fig.2.1-3b produces a rotation. We do not derive any relations for quadrupoles since they will not be needed in this book. But it is important to understand that monopoles and dipoles are not the only possible carriers of an electric current.

From Fig.2.1-1b one may see that the distinction between electric dipoles and monopoles depends on the magnitude and the duration of the field strength $\mathbf{E}$. A neutral hydrogen atom will first become a dipole if an electric field strength $\mathbf{E}$ is applied. If this field strength is larger than the ionization field strength and it is applied long enough we get a negative and a positive monopole. A larger field strength can achieve the same effect in a shorter time. This possible transition from dipole to monopoles implies that any useful theory must treat monopoles and dipoles similarly since a dipole does not have *a priori* information whether an applied field strength will last long enough or will reach a sufficient magnitude to produce ionization.

We turn to the currents carried by electric monopoles and dipoles. This investigation is important since we shall need a simple representation of dipole currents in Section 4.1 to derive partial differential equations of fourth order that can be solved analytically without excessive mathematical effort. Without such a simplification one obtains partial differential equations of higher order that are hard to solve analytically or transcendental equations that can be solved numerically only. At this stage of the development of quantum electrodynamics based on the modified Maxwell equations it is more important to derive basic results analytically from simple equations rather than highly accurate results numerically from complicated equations. Complicated partial differential equations will be derived in Section 6.10 and they will demonstrate the virtue of simplicity very convincingly.

For the simplest case of a monopole current we have Ohm's law connecting an electric field strength and an electric current density:

$$\mathbf{g}_{\mathrm{e}} = \sigma\mathbf{E} \tag{11}$$

This law assumes that the current follows the field strength without delay, which implies current or charge carriers with negligible mass. Following Becker (1964a vol. I, b, § 58) we note that a current carrier with mass m_0, velocity $\mathbf{v}$,

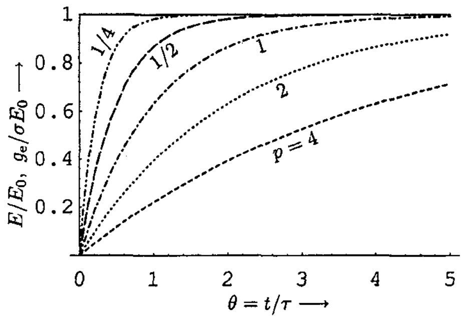

FIG.2.1-4. The step function excitation $\mathbf{E}/\mathbf{E}_0 = S(\theta)$ (solid line) and the lagging current densities $\mathbf{g}_e(t)/\sigma\mathbf{E}_0$ according to Eq.(14) due to a finite mass of the charge or current carriers for p=1/4, 1/2, 1, 2, 4.

and charge e is pulled by an electric field strength $\mathbf{E}$ with the force $e\mathbf{E}$. Using Newton's mechanic we obtain the following equation of motion:

$$m_0 \frac{d\mathbf{v}}{dt} = e\mathbf{E} - \xi_e \mathbf{v} \tag{12}$$

The term $\xi_e\mathbf{v}$ represents losses proportionate to the velocity. The term ξ_e is usually referred to as Stokes' friction constant due to its original use in fluid mechanics. In electrodynamics any losses proportionate to $\mathbf{v}$ are more likely to come from near zone radiation that is absorbed by surrounding matter. The term $\xi_e\mathbf{v}$ is clearly the simplest one that can account for losses. We do not have to decide what causes these losses unless we want to represent losses by more complicated terms than $\xi_e\mathbf{v}$.

If we obtain the derivative $d\mathbf{v}/dt$ from Eq.(1) and substitute it into Eq.(12) we obtain an extension of Ohm's law to electric monopole currents having current carriers with finite, constant mass:

$$\begin{gathered} \mathbf{g}_e + \tau_{mp}\frac{d\mathbf{g}_e}{dt} = \sigma\mathbf{E} \\ \tau_{mp} = \frac{m_0}{\xi_e}, \quad \sigma = \frac{\rho_e e}{\xi_e} = \frac{\rho_e e \tau_{mp}}{m_0} \end{gathered} \tag{13}$$

If the term $\tau_{mp} d\mathbf{g}_e/dt$ can be neglected we obtain the usual Ohm's law with conductivity σ. To recognize the effect of Eq.(13) consider the electric field strength $\mathbf{E}$ with the time variation of a step function $\mathbf{E}(t) = \mathbf{E}_0 S(t)$ as shown in Fig.2.1-4. In order to avoid the point $t = 0$ for which the step function $S(t)$ is not differentiable we consider the infinitesimally larger time $t = +0$. If we require a current density $\mathbf{g}_e(+0)$ to be zero we obtain from Eq.(13) the current density

$$\mathbf{g}_e(t) = \sigma\mathbf{E}_0(1 - e^{-t/\tau_{mp}})S(t) = \sigma\mathbf{E}_0(1 - e^{-\theta/p})S(\theta), \ \theta = t/\tau, \ p = \tau_{mp}/\tau \tag{14}$$

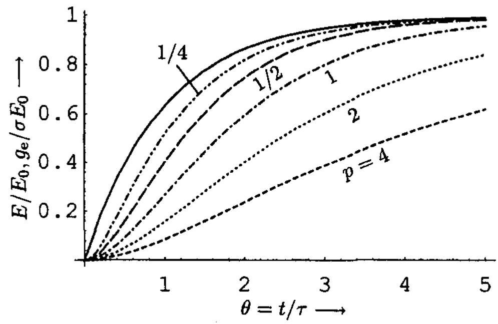

FIG.2.1-5. The normalized electric field strength E/E_0 represented by the exponential ramp function of Eq.(15) (solid line) and current densities $g_e/\sigma E_0$ according to Eq.(17) for $p = 1/4$, $1/2$, 1, 2, 4.

which shows the current density lagging behind the field strength $\mathbf{E}_0 S(t)$. The normalized current density $\mathbf{g}_e/\sigma\mathbf{E}_0$ of Eq.(14) is plotted in Fig.2.1-4. The lagging effect is seen to become negligible for $p \ll 1$.

In addition to the step function $S(t)$ we shall need the exponential ramp function

$$\mathbf{E}(t)/\mathbf{E}_0 = 1 - e^{-t/\tau} \tag{15}$$

shown in Fig.2.1-5 by the solid line. Equation (13) is replaced by

$$\mathbf{g}_e + \tau_{mp}\frac{d\mathbf{g}_e}{dt} = \sigma\mathbf{E}_0(1 - e^{-t/\tau}) \tag{16}$$

and the current density of Eq.(14) is replaced by

$$\begin{aligned}\mathbf{g}_e &= \sigma\mathbf{E}_0\left[1 - e^{-t/\tau_{mp}} + \frac{\tau}{\tau - \tau_{mp}}\left(e^{-t/\tau_{mp}} - e^{-t/\tau}\right)\right], && \tau_{mp} \neq \tau\\ &= \sigma\mathbf{E}_0\left[1 - \left(1 + \frac{t}{\tau_{mp}}\right)e^{-t/\tau_{mp}}\right], && \tau_{mp} = \tau \end{aligned} \tag{17}$$

Figure 2.1-5 shows plots of $\mathbf{g}_e$ for $\tau = 2\tau_{mp}$, τ_{mp}, and $\tau_{mp}/2$. The difference between the plot of the field strength $\mathbf{E}(t)/\mathbf{E}_0$ and the plots for the current densities is never as large as in Fig.2.1-4 for the step function.

For a dipole we obtain from Eqs.(8) and (12) the following equation of motion:

$$m_0\frac{d\mathbf{v}}{dt} + \frac{m_0}{\tau_p^2}\int \mathbf{v}\,dt = e\mathbf{E} - \xi_e\mathbf{v} \tag{18}$$

For the current density $\mathbf{g}_e$ we obtain with Eq.(7) in analogy to Eq.(13):

$$\mathbf{g}_{\mathrm{e}} + \tau_{\mathrm{mp}}\frac{d\mathbf{g}_{\mathrm{e}}}{dt} + \frac{\tau_{\mathrm{mp}}}{\tau_{\mathrm{p}}^2}\int \mathbf{g}_{\mathrm{e}}\,dt = \sigma_{\mathrm{p}}\mathbf{E}$$
$$\tau_{\mathrm{mp}} = \frac{m_0}{\xi_{\mathrm{e}}},\ \sigma_{\mathrm{p}} = \sigma = \frac{\rho_{\mathrm{e}}e}{\xi_{\mathrm{e}}} = \frac{\rho_{\mathrm{e}}e\tau_{\mathrm{mp}}}{m_0} \tag{19}$$

The field strength **E** is replaced in liquids or solids by a smaller *effective field strength* **F** but this is of no concern here[1]. The conductivity σ is often denoted by σ_{p} to indicate it is an *electric polarization current conductivity* or an *electric dipole current conductivity.* Equation (19) is Ohm's law extended to electric dipole currents with constant mass of the current carriers. A comparison with Eq.(13) shows that the integral is characteristic for dipole currents, while a term $d\mathbf{g}_{\mathrm{e}}/dt$ occurs in Ohm's law for monopole as well as for dipole currents if a delay is caused by the need to accelerate current carriers to give them a velocity.

Let the electric field strength **E** in Eq.(19) have the time variation of the step function in Fig.2.1-4

$$\mathbf{E} = \frac{2p}{q}\mathbf{E}_0 S(t) \tag{20}$$

where $2p/q$ is a factor that will be explained presently. The inhomogeneous term in Eq.(19) is removed by differentiation:

$$\mathbf{g}_{\mathrm{e}} + \frac{\tau_{\mathrm{p}}^2}{\tau_{\mathrm{mp}}}\frac{d\mathbf{g}_{\mathrm{e}}}{dt} + \tau_{\mathrm{p}}^2\frac{d^2\mathbf{g}_{\mathrm{e}}}{dt^2} = 0 \quad \text{for } t > 0 \tag{21}$$

The solutions are:

$$\begin{aligned}
\mathbf{g}_{\mathrm{e}} &= \mathbf{g}_{\mathrm{e1}}e^{-t/\tau_{\mathrm{e1}}} + \mathbf{g}_{\mathrm{e2}}e^{-t/\tau_{\mathrm{e2}}} && \text{for } \tau_{\mathrm{mp}}/\tau_{\mathrm{p}} \neq 1/2\\
\mathbf{g}_{\mathrm{e}} &= (\mathbf{g}_{\mathrm{e3}} + \mathbf{g}_{\mathrm{e4}}t)e^{-t/\tau_{\mathrm{p}}} && \text{for } \tau_{\mathrm{mp}}/\tau_{\mathrm{p}} = 1/2
\end{aligned} \tag{22}$$

$$\begin{aligned}
\tau_{\mathrm{e1,e2}} &= \frac{1}{2}\frac{\tau_{\mathrm{p}}^2}{\tau_{\mathrm{mp}}}\left[1 \pm \left(1 - \frac{4\tau_{\mathrm{mp}}^2}{\tau_{\mathrm{p}}^2}\right)^{1/2}\right] && \text{for } \frac{\tau_{\mathrm{mp}}}{\tau_{\mathrm{p}}} < \frac{1}{2}\\
&= \frac{1}{2}\frac{\tau_{\mathrm{p}}^2}{\tau_{\mathrm{mp}}}\left[1 \pm i\left(\frac{4\tau_{\mathrm{mp}}^2}{\tau_{\mathrm{p}}^2} - 1\right)^{1/2}\right] && \text{for } \frac{\tau_{\mathrm{mp}}}{\tau_{\mathrm{p}}} > \frac{1}{2}\\
\tau_{\mathrm{e}} &= \frac{1}{2}\frac{\tau_{\mathrm{p}}^2}{\tau_{\mathrm{mp}}} = \tau_{\mathrm{p}} && \text{for } \frac{\tau_{\mathrm{mp}}}{\tau_{\mathrm{p}}} = \frac{1}{2}
\end{aligned} \tag{23}$$

[1]See, e.g., Harmuth, Boules, and Hussain (1999), Section 1.3 or Harmuth and Lukin (2000), Section 1.2.

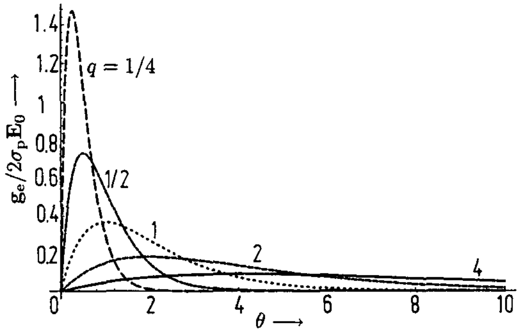

FIG.2.1-6. Time variation of dipole current densities according to Eq.(24) that transport equal charges through a certain cross section of the current path for $p = 1/2$, $q = 1/4, 1/2, 1, 2, 4$ in the interval $0 \le \theta \le 10$.

Substitution of the conditions for $\mathbf{g}_e(+0) = 0$ yields:

$$\mathbf{g}_e = 2\sigma_p \mathbf{E}_0 \frac{1}{q^2}\theta e^{-\theta/q} \quad \text{for } p = \frac{1}{2},\ \theta > 0$$
$$t/\tau = \theta,\ \tau_p/\tau = q,\ \tau_{mp}/\tau_p = p \tag{24}$$

The integral

$$\int_0^\infty \frac{1}{q^2}\theta e^{-\theta/q} d\theta = 1 \tag{25}$$

explains the use of the factor $1/q$ in Eq.(20) to obtain 1 or generally $1/2p$ in Eq.(25). The same charge will pass through a certain cross section of the path for the current density $\mathbf{g}_e$ during the time $0 < t < \infty$. The function $\mathbf{g}_e/2\sigma_p\mathbf{E}_0$ is plotted in Fig.2.1-6 for various values of q. The areas under the plots are all equal; they represent the constant charge passing through a certain cross section of the current path. As q decreases, the current density increases for an ever shorter time and approaches for $q \to 0$ an infinitely large and infinitely short needle pulse similar to Dirac's delta function. On the other hand, for large values of q we get a time variation similar to that of a monopole current according to Fig.2.1-4 up to fairly large values of θ.

For $p = \tau_{mp}/\tau_p \ne 1/2$ we obtain with $\mathbf{g}_e(+0) = 0$ the following relations for the current density[2]:

$$\mathbf{g}_e = 2\sigma_p \mathbf{E}_0 \frac{p\left(e^{-\theta/\theta_1} - e^{-\theta/\theta_2}\right)}{q\left(1 - 4p^2\right)^{1/2}} \qquad \text{for } p < \frac{1}{2} \tag{26}$$

$$= 2\sigma_p \mathbf{E}_0 \frac{2pe^{-\theta/2pq}}{q\left(4p^2 - 1\right)^{1/2}} \sin\frac{(4p^2-1)^{1/2}\theta}{2pq} \qquad \text{for } p > \frac{1}{2} \tag{27}$$

[2]Harmuth, Boules, and Hussain (1999), Section 1.3.

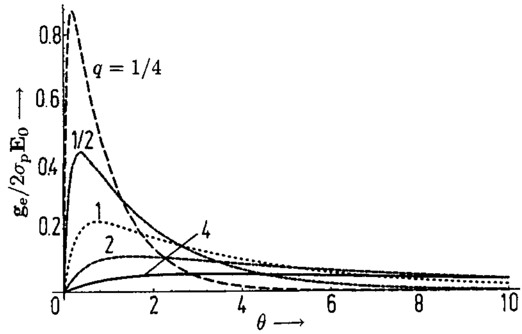

FIG.2.1-7. Time variation of dipole current densities according to Eq.(26) that transport equal charges through a certain cross section of the current path for $p = 1/4$ and $q = 1/4$, $1/2$, 1, 2, 4. The areas under the plots in the interval $0 \leq t < \infty$ are equal. They are also equal to the corresponding areas in Fig.2.1-6.

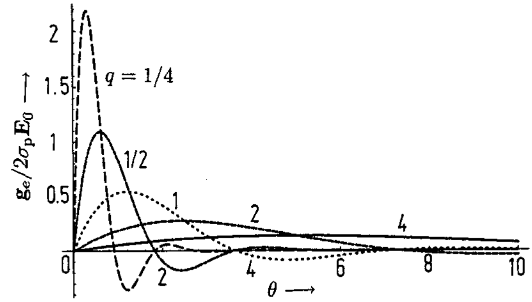

FIG.2.1-8. Time variation of dipole current densities according to Eq.(27) that transport equal charges through a certain cross section of the current path for $p = 1$ and $q = 1/4$, $1/2$, 1, 2, 4. The integrals over the plots in the interval $0 \leq \theta < \infty$ are equal. They equal $1/2p = 1/2$ the value of the corresponding integrals in Figs.2.1-6 and 2.1-7.

$$\theta_1 = q\left[1 + (1 - 4p^2)^{1/2}\right]/2p,\ \theta_2 = q\left[1 - (1 - 4p^2)^{1/2}\right]/2p$$

Plots of $\mathbf{g}_e/2\sigma_p\mathbf{E}_0$ are shown in Fig.2.1-7 for Eq.(26) and in Fig.2.1-8 for Eq.(27). The areas under the plots are all equal:

$$\frac{p}{q\,(1 - 4p^2)^{1/2}} \int_0^\infty \left(e^{-\theta/\theta_1} - e^{-\theta/\theta_2}\right) d\theta = 1 \tag{28}$$

$$\frac{2p}{q\,(4p^2 - 1)^{1/2}} \int_0^\infty e^{-\theta/2pq} \sin \frac{(4p^2 - 1)^{1/2}\theta}{2pq} d\theta = 1 \tag{29}$$

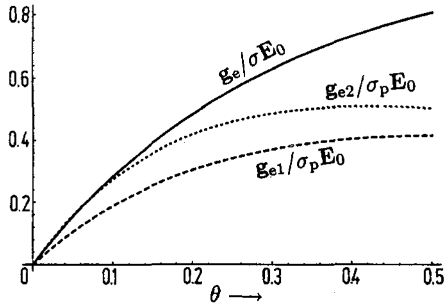

FIG.2.1-9. Normalized monopole current density $\mathbf{g}/\sigma\mathbf{E}_0$ according to Eq.(11) for an electric field strength with the time variation of the exponential ramp function of Eq.(15) (solid line) as well as normalized dipole current densities according to Eqs.(30) (dashed line) and (31) (dotted line).

Figures 2.1-7 and 2.1-8 show that for large values of $q = \tau_p/\tau$ we get a time variation of the current density similar to that of a monopole current in Fig.2.1-4 up to fairly large values of θ.

Consider the exponential ramp function of Eq.(15) for $\mathbf{E}$ in Eq.(19). The monopole current density for current carriers with negligible mass has the same time variation except for a factor σ according to Eq.(11). This normalized current density $\mathbf{g}_e/\sigma\mathbf{E}_0$ is shown by the plot with solid line in Fig.2.1-9. If we substitute the exponential ramp function for $\mathbf{E}$ into Eq.(19) for the dipole current density we can obtain with $\theta = t/\tau$ functions

$$\mathbf{g}_{e1}/\sigma_p\mathbf{E}_0 = e^{-\theta} - e^{-3.333\theta} \qquad \text{for } p = 0.499 \qquad (30)$$

$$\mathbf{g}_{e2}/\sigma_p\mathbf{E}_0 = e^{-\theta} - e^{-4.605\theta} \qquad \text{for } p = 0.498 \qquad (31)$$

that are shown in Fig.2.1-9 by the plots with the dashed and the dotted lines. We see that the monopole current density $\mathbf{g}_e = \sigma\mathbf{E}$ can again be used as a crude but radically simpler representation of dipole currents.

A monopole current requires charge or current carriers such as electrons, ions, holes in semiconductors, alpha particles, etc. The two parameters τ_{mp} and σ in Eq.(13) are thus material constants. This is quite different for the constants τ_{mp}, τ_p, and σ_p of Eq.(19) since a dipole current can flow in vacuum, which implies that τ_{mp}, τ_p, and σ_p should have values for vacuum just like the permittivity ϵ. Originally, the permittivity ϵ of vacuum had to be obtained directly from observation. When Maxwell derived the relation $c^2 = 1/\mu\epsilon$ it became possible to calculate ϵ from the definition $\mu = 4\pi \times 10^{-7}$ [Vs/Am] of the permeability for vacuum and the observed[3] value $c = 299\,792\,458$ [m/s] of the velocity of light.

For a measurement of τ_{mp}, τ_p, and σ_p of Eq.(19) we need a circuit as shown in Fig.2.1-10. By closing the switch S one applies a step voltage $VS(t)$

[3]This observed value has become a definition.

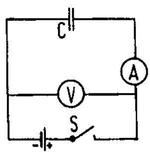

FIG.2.1-10. Principle of a circuit for the determination of the parameters τ_{mp}, τ_p, and σ_p of an electric dipole current in vacuum. The ampere meter A represents a single-shot sampling oscilloscope. The voltmeter V is only needed to observe initially the switching characteristics of the switch S. It must also be a single-shot sampling oscilloscope.

to the capacitor C with vacuum as dielectric. If the time variation of the current is observed—which would have to be done with a single-shot sampling oscilloscope rather than a conventional ampere meter—one can compare it with plots according to Figs.2.1-6 to 2.1-8 and determine which values of τ_{mp},τ_p, and σ_p give the best fit. The time resolution of the fastest sampling oscilloscopes is currently about 10 ps. This may not be short enough to make the circuit of Fig.2.1-10 practical.

2.2 Magnetic Monopoles and Dipoles With Constant Mass

We must develop relations for the magnetic current density $\mathbf{g}_m$ and the hypothetical charge density ρ_m in analogy to the relations for their electric equivalents $\mathbf{g}_e$ and ρ_e in Section 2.1. The equivalent of the electric current density $\mathbf{g}_e$ in Eq.(2.1-1) for a hypothetical magnetic current density follows readily from Eqs.(1.6-2) and (1.6-4):

$$\mathbf{g}_m = \rho_m \mathbf{v} \tag{1}$$

The force density $\mathbf{k}_e$ of the motion of a hypothetical magnetic charge in an electromagnetic field is given by the *magnetic Lorentz equation*

$$\mathbf{k}_e = \rho_m \left(\mathbf{H} - \frac{1}{Zc} \mathbf{v} \times \mathbf{E} \right) \tag{2}$$

where $\mathbf{k}_e$ is a force density that acts on the magnetic charge density ρ_m; the negative sign is due to the negative sign in Eq.(1.6-2). For a particle with a magnetic charge q rather than a charge density ρ_m we replace the force density $\mathbf{k}_e$ in Eq.(2) by a force $\mathbf{K}_e$:

$$\mathbf{K}_e = q \left(\mathbf{H} - \frac{1}{Zc} \mathbf{v} \times \mathbf{E} \right) \tag{3}$$

As before an equation of motion is obtained by demanding that $\mathbf{K}_e$ equals the force of inertia. Using again Newton's mechanic we get:

$$\mathbf{K}_e = m \frac{d\mathbf{v}}{dt} \tag{4}$$

Even though a magnetic charge density $\rho_{\rm m}$ and a magnetic monopole current density $\mathbf{g}_{\rm m}$ associated with it are strictly hypothetical they can serve a practical purpose. In Eq.(1.1-15) we connected the magnetic current density $\mathbf{g}_{\rm m}$ via a conductivity s to the magnetic field strength $\mathbf{H}$. The transition $s \rightarrow 0$ in Eq.(6.1-40) and $\omega = \sqrt{\epsilon s/\mu\sigma} \rightarrow 0$ in Eq.(6.2-41) means that the current density $\mathbf{g}_{\rm m}$ becomes zero, but one does not need to specify whether $\mathbf{g}_{\rm m}$ is a monopole, dipole, or higher order multipole current density and can think of it as a monopole current density.

An induced magnetic dipole according to Fig.2.1-1b but with the electric charges $\pm e$ replaced by magnetic charges $\pm q$ is just as hypothetical as a magnetic monopole but it can lead to equations that can be solved analytically while inherent, rotating dipoles according to Fig.2.1-1c have never yet lead to an analytically solvable equation. Furthermore, the induced dipole of Fig.2.1-1b has only one rest position defined by $\mathbf{s} = 0$ while the inherent dipole of Fig.2.1-1c has a rest position for any value of the angle ϑ. This requires averaging over many values of ϑ for numerical solutions.

The magnetic current density $\mathbf{g}_{\rm m}$ for an induced dipole becomes in analogy to Eq.(2.1-7)

$$\mathbf{g}_{\rm m} = \rho_{\rm m}\mathbf{v} \tag{5}$$

and the equation of motion has the form of Eq.(2.1-8) with $\mathbf{K}_{\rm m}$ replaced by $\mathbf{K}_{\rm e}$:

$$\mathbf{K}_{\rm e} = m\left(\frac{d^2\mathbf{s}}{dt^2} + \frac{\mathbf{s}}{\tau_{\rm p}^2}\right) = m\left(\frac{d\mathbf{v}}{dt} + \frac{1}{\tau_{\rm p}^2}\int \mathbf{v}\,dt\right) \tag{6}$$

The comparison of Eqs.(2.1-8) and (6) shows that the equation for the hypothetical induced magnetic dipole current follows from Eq.(2.1-19):

$$\mathbf{g}_{\rm m} + \tau_{\rm mp}\frac{d\mathbf{g}_{\rm m}}{dt} + \frac{\tau_{\rm mp}}{\tau_{\rm p}^2}\int \mathbf{g}_{\rm m}\,dt = 2s_{\rm p}\mathbf{H} \tag{7}$$

We have changed the notation $\sigma_{\rm p}$ to $2s_{\rm p}$, where $\sigma_{\rm p}$ has the physical dimension A/Vm while $s_{\rm p}$ has the dimension V/Am. The factor 2 takes into account that each dipole consists of two particles with equal mass. The results of Section 2.1 may readily be used with a changed notation.

Let us turn from the hypothetical induced magnetic dipole to the well-established inherent magnetic dipoles. Consider the ferromagnetic bar magnet of length $2R$ in an homogeneous magnetic field of strength $\mathbf{H}$ and flux density $\mathbf{B}$ shown in Fig.2.2-1. We introduce the magnetic dipole moment $m_{\rm mo}$ with dimension[1] Am^2 and the mechanical moment of inertia J with dimension Nms^2 = kg m^2 of the bar magnet. The equation of motion equals

[1]If we write $m_{\rm mo}\mathbf{B} = m_{\rm mo}\mu\mathbf{H}$, the term $m_{\rm mo}\mu$ has the dimension Vsm and the symmetry with the electric dipole moment eR [Asm] is maintained, if the electric charge $\pm e$ replaces $\pm q_{\rm m}$ in Fig.2.2-1. The product $(eR)\mathbf{E}$ [Asm× V/m] is then in complete analogy to the product $m_{\rm mo}\mu\mathbf{H}$ [Vsm×A/m].

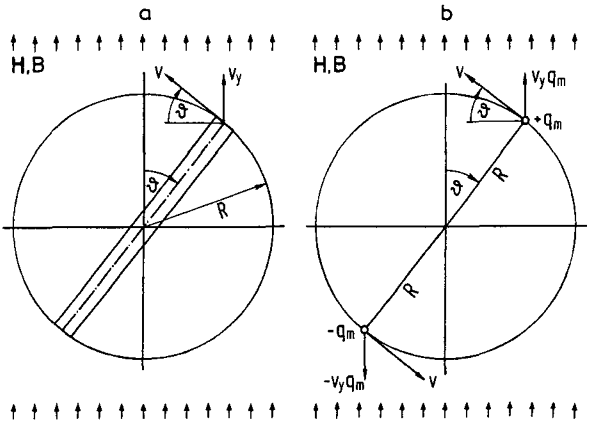

FIG.2.2-1. Ferromagnetic bar magnet in an homogeneous magnetic field (a) and its replacement by a thin rod with magnetic charges $\pm q_{\mathrm{m}}$ at its ends (b).

$$J\frac{d^2\vartheta}{dt^2} = -m_{\mathrm{mo}}B\sin\vartheta \tag{8}$$

where ϑ is the angle between the direction of the field strength and the bar magnet. The velocity of the end points of the bar has the value

$$v(t) = -R\frac{d\vartheta}{dt} \tag{9}$$

which suggests to introduce a friction or generally an attenuation term ξ_{m} into Eq.(8):

$$J\frac{d^2\vartheta}{dt^2} + \xi_{\mathrm{m}}R\frac{d\vartheta}{dt} + m_{\mathrm{mo}}B\sin\vartheta = 0 \tag{10}$$

An analytic solution of this differential equation is generally impossible due to the term $\sin\vartheta$. For small angles $\vartheta \approx \sin\vartheta$ it was already studied by Gauss[2]. The computer has freed us from the restrictions of the days of Gauss. We rewrite Eq.(10) in normalized form:

$$\frac{d^2\vartheta}{d\theta^2} + \frac{1}{pq}\frac{d\vartheta}{d\theta} + \frac{1}{q^2}\sin\vartheta = 0$$

$$q = \frac{1}{\tau}\sqrt{\frac{J}{m_{\mathrm{mo}}B}} = \frac{\tau_{\mathrm{p}}}{\tau}, \quad p = \frac{J}{\xi_{\mathrm{m}}R\tau_{\mathrm{p}}} = \frac{\sqrt{Jm_{\mathrm{mo}}B}}{\xi_{\mathrm{m}}R} = \frac{\tau_{\mathrm{mp}}}{\tau_{\mathrm{p}}}$$

$$\tau_{\mathrm{p}} = \sqrt{J/m_{\mathrm{mo}}B}, \quad \tau_{\mathrm{mp}} = J/\xi_{\mathrm{m}}R, \quad pq = \tau_{\mathrm{mp}}/\tau = J/\xi_{\mathrm{m}}R\tau \tag{11}$$

This differential equation can be solved numerically for the initial conditions $\vartheta(0) = n\vartheta_0 = \vartheta_n$ and $d\vartheta(0)/d\theta = 0$. The numerical values obtained for

[2] A recent analysis may be found in Harmuth, Boules, and Hussain (1999), Section 1.4.

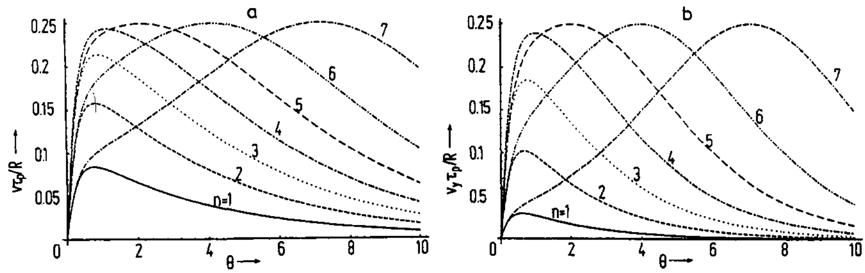

FIG.2.2-2. The functions $v(\theta)\tau/R$ (a) and $v_y(\theta)\tau/R$ (b) according to Eqs.(12) and (13) for $p = 1/4$, $\tau = \tau_p$, and $\vartheta_n = n\pi/8$ with $n = 1, 2, \dots, 7$ in the interval $0 \le \theta \le 10$.

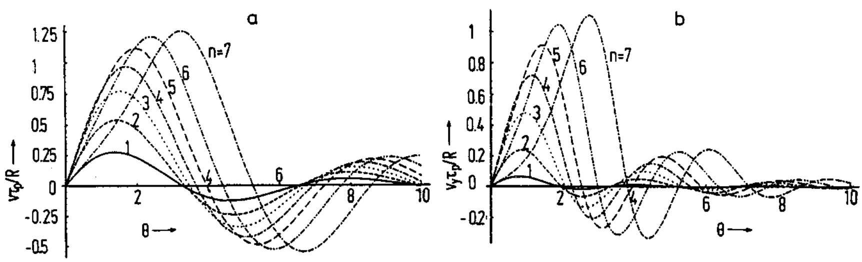

FIG.2.2-3. The functions $v(\theta)\tau/R$ (a) and $v_y(\theta)\tau/R$ (b) according to Eqs.(12) and (13) for $p = 2$, $\tau = \tau_p$, and $\vartheta_n = n\pi/8$ with $n = 1, 2, \dots, 7$ in the interval $0 \le \theta \le 10$.

$\vartheta(\theta)$ and $d\vartheta(\theta)/d\theta$ may then be used to calculate the velocities $v(\theta)$ and $v_y(\theta)$ of Fig.2.2-1:

$$v(\theta) = -R\frac{d\vartheta}{dt} = -\frac{R}{\tau}\frac{d\vartheta}{d\theta} \tag{12}$$

$$v_y(\theta) = v(\theta)\sin\vartheta = -\frac{R}{\tau}\frac{d\vartheta}{d\theta}\sin\vartheta \tag{13}$$

Plots of $v(\theta)$ and $v_y(\theta)$ are shown in Figs.2.2-2a and b for $p = 1/4$, $\tau_p = \tau$, and various values of ϑ_n. Figures 2.2-3a and b show the same plots for p=2.

We must connect the velocities $v_y(t)$ with the current density $\mathbf{g}_m(t)$ of a magnetic dipole current. First we replace the bar magnet in Fig.2.2-1a by a thin rod with fictitious magnetic charges $\pm q_m$ at its ends as shown in Fig.2.2-1b. The magnetic dipole moment m_{mo} equals $2q_mR$. The charge q_m must be connected with the magnetic dipole moment by the relation

$$q_m\,[\mathrm{Vs}] = \frac{\mu m_{mo}}{2R}\left[\frac{\mathrm{Vs}}{\mathrm{Am}}\frac{\mathrm{Am}^2}{\mathrm{m}}\right] \tag{14}$$

where μ is the magnetic permeability, to obtain the dimension Vs for q_m. The magnetic dipole current produced by such a bar magnet is $2q_mv_y(t)$. If there

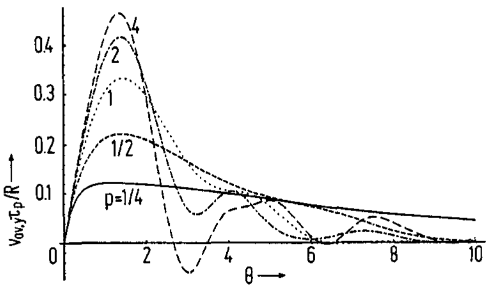

FIG.2.2-4. The average velocity $v_{\text{av},y}(\theta)\tau/R$ according to Eqs.(12)–(15) of the ends of randomly oriented bar magnets of length $2R$ for $q = 1$, $p = 1/4$, $1/2$, 1, 2, 4, and initial velocity $v(\theta) = 0$ for $\theta = 0$ in the interval $0 \le \theta \le 10$.

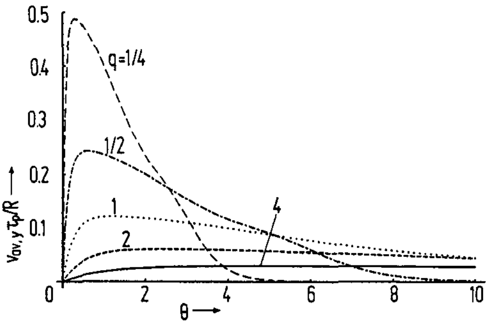

FIG.2.2-5. The average velocity $v_{\text{av},y}(\theta)\tau/R$ according to Eqs.(12)–(15) of the ends of randomly oriented bar magnets of length $2R$ for $p = 1/4$, $q = 1/4$, $1/2$, 1, 2, 4, and initial velocity $v(\theta) = 0$ for $\theta = 0$ in the interval $0 \le \theta \le 10$.

are N_0 bar magnets in a unit volume, all having the direction ϑ_0 and the velocity $v_y(t) = 0$ at $t = 0$, we would get the dipole current density $\mathbf{g}_{\text{m}}(t) = 2N_0 q_{\text{m}} v_y(t)\mathbf{y}/y$ for the current flowing in the direction of the y-axis. For many randomly oriented bar magnets we must average all velocities $v_y(t)$ in the sector $0 \le \vartheta \le \pi$ as well as in the sector $\pi \le \vartheta \le 2\pi$, which yields the same result:

$$v_{\text{av},y}(\theta) = \frac{1}{N}\sum_{n=1}^{N} v_y(\vartheta_n, \theta), \quad 0 \le \vartheta = \vartheta_n \le \pi, \quad \theta = t/\tau \tag{15}$$

Figure 2.2-4 shows the average velocity $v_{\text{av},y}(t)$ for $q = 1$ and various values of p in the interval $0 \le \theta \le 10$. Multiplication with $2N_0 q_{\text{m}}$ yields the magnitude of the current density $\mathbf{g}_{\text{m}}$ as function of time. For a vanishing moment of inertial J the plots in Fig.2.2-4 become similar to a Dirac delta function.

Figure 2.2-5 shows $v_{\text{av},y}(\theta)$ for $p = 1/4$ and various values of q, while Fig.2.2-6 shows the same but $p = 1/4$ is replaced by $p = 2$.

The magnetic dipole current density in the direction of the y-axis becomes

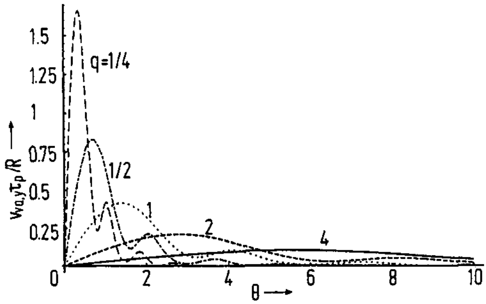

FIG.2.2-6. The average velocity $v_{\mathrm{av},y}(\theta)\tau/R$ according to Eqs.(12)–(15) of the ends of randomly oriented bar magnets of length $2R$ for $p=2$, $q=1/4$, $1/2$, 1, 2, 4, and initial velocity $v(\theta)=0$ for $\theta=0$ in the interval $0\le\theta\le 10$.

for pairs with magnetic charge $\pm q_{\mathrm{m}}$

$$\mathbf{g}_{\mathrm{m}} = 2N_0 q_{\mathrm{m}} v_{\mathrm{av},y}(t)\frac{\mathbf{y}}{y} \tag{16}$$

where N_0 is the number of magnetic dipoles.

Let us attempt to write the extension of Ohm's law for rotating magnetic dipoles in a form comparable to Eq.(7). We obtain from Eq.(9)

$$\frac{d\vartheta}{dt} = -\frac{v(t)}{R},\ \frac{d^2\vartheta}{dt^2} = -\frac{1}{R}\frac{dv}{dt},\ \vartheta = -\frac{1}{R}\int v(t)\,dt \tag{17}$$

and we rewrite Eq.(10):

$$\frac{J}{R}\frac{dv}{dt} + \xi_{\mathrm{m}} v + m_{\mathrm{mo}} B \sin\left(\frac{1}{R}\int v(t)\,dt\right) = 0 \tag{18}$$

Then we substitute Eq.(13) with $t=\theta\tau$:

$$v(t) = \frac{v_y(t)}{\sin\vartheta(t)} \tag{19}$$

to obtain:

$$\frac{J}{R}\frac{d}{dt}\left(\frac{v_y}{\sin\vartheta}\right) + \xi_{\mathrm{m}}\frac{v_y}{\sin\vartheta} + m_{\mathrm{mo}} B \sin\left(\frac{1}{R}\int\frac{v_y}{\sin\vartheta}dt\right) = 0 \tag{20}$$

Next we substitute

$$v_y = \frac{g_{\mathrm{m},\vartheta}}{2N_0 q_{\mathrm{m}}} \tag{21}$$

and multiply with $2N_0 q_{\mathrm{m}}$:

$$\frac{J}{R}\frac{d}{dt}\left(\frac{g_{\mathrm{m},\vartheta}}{\sin\vartheta}\right) + \xi_{\mathrm{m}}\frac{g_{\mathrm{m},\vartheta}}{\sin\vartheta} + 2N_0 q_{\mathrm{m}} m_{\mathrm{mo}} B \sin\left(\frac{1}{2N_0 q_{\mathrm{m}} R}\int\frac{g_{\mathrm{m},\vartheta}}{\sin\vartheta}dt\right) = 0 \tag{22}$$

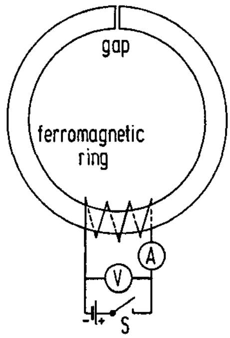

FIG.2.2-7. Principle of a circuit for the determination of the parameters τ_{mp}, τ_p, and s_p of a magnetic dipole current in vacuum. The ampere and volt meters represent single-shot sampling oscilloscopes.

This differential equation yields the current density $g_{m,\vartheta}$ for a certain initial angle $\vartheta = \vartheta_0$ of the dipoles as function of the magnetic flux density $\mathbf{B}$ or field strength $\mathbf{H}$. The direction of $g_{m,\vartheta}$ is that of the positive y-axis:

$$\mathbf{g}_{m,\vartheta} = g_{m,\vartheta} \frac{\mathbf{y}}{y} \tag{23}$$

Equations (21) and (22) are the extension of Ohm's law to magnetic dipole currents caused by inherent dipoles with certain initial orientation angle ϑ. We may substitute $\mathbf{g}_{m,\vartheta}$ for $\mathbf{g}_m$ in Eq.(1.1-15) and—at least in principle—calculate $\mathbf{E}$ and $\mathbf{H}$ for the initial angle ϑ_0. If we calculate $\mathbf{E}$ and $\mathbf{H}$ for many possible angles ϑ_0 and average in analogy to Eq.(15) we get the field strengths $\mathbf{E}$ and $\mathbf{H}$ for randomly oriented dipoles.

One will not likely proceed in this way since Eq.(22) does not offer much hope for an analytic solution of Eqs.(1.1-8)–(1.1-15). For a numerical solution one will generally prefer to use Eqs.(11), (13), and (21) rather than Eq.(22). For an analytic investigation one has little choice but to use Eq.(7) holding for a hypothetical induced magnetic dipole.

In Fig.2.1-10 we have shown a circuit for the measurement of the parameters τ_{mp}, τ_p, and σ_p of an electric dipole current in vacuum. A corresponding circuit for magnetic dipole currents is shown in Fig.2.2-7. The most important difference is the replacement of the wires for an electric monopole current leading to the capacitor by a ring of ferromagnetic material with large cross section. The reason for this is the lack of magnetic monopole currents that could be fed to the 'gap' in Fig.2.2-7 and produce a magnetic dipole current across the gap. Since permalloy has a relative permeability between 10 000 and 70 000 it is a good conductor for magnetic dipole currents compared with vacuum. But it would not work for switching times shorter than 1 μs. A ferrite material would work with switching times as low as 1 ns, but its relative permeability is in the range from 10 to 100. In the absence of electric monopole currents we could replace the circuit of Fig.2.1-10 by one according to Fig.2.2-7 with the ferromagnetic ring replaced by a barium-titanate ring with relative permittivity between 1000 and 9000.

2.3 Monopoles and Dipoles With Relativistic Variable Mass

For the transition from a fixed to a relativistic variable mass in Sections 2.1 and 2.2 we write for the mass

$$m = \frac{m_0}{(1 - v^2/c^2)^{1/2}} \tag{1}$$

and replace $m\,d\mathbf{v}/dt$ by $d(m\mathbf{v})/dt$:

$$\begin{aligned}
\frac{d(m\mathbf{v})}{dt} &= \frac{dm}{dt}\mathbf{v} + m\frac{d\mathbf{v}}{dt} \\
\frac{dm}{dt} &= \frac{dm}{dv}\frac{dv}{dt} = \frac{v}{c^2}\frac{m_0}{(1 - v^2/c^2)^{3/2}}\frac{dv}{dt} \\
\frac{d(m\mathbf{v})}{dt} &= \frac{m_0}{(1 - v^2/c^2)^{3/2}}\frac{d\mathbf{v}}{dt}
\end{aligned} \tag{2}$$

Equation (2.1-4) becomes

$$\mathbf{K}_{\mathrm{m}} = \frac{m_0}{(1 - v^2/c^2)^{3/2}}\frac{d\mathbf{v}}{dt} \tag{3}$$

while Eq.(2.2-4) requires the subscript e in place of m.

For the extension of the induced dipole equation to relativistic velocities we differentiate Eq.(2.1-8) with respect to t:

$$\frac{d\mathbf{K}_{\mathrm{m}}}{dt} = \frac{d^2(m\mathbf{v})}{dt^2} + \frac{m\mathbf{v}}{\tau_{\mathrm{p}}^2} \tag{4}$$

$$\frac{d^2(mv)}{dt^2} = \frac{d^2m}{dt^2}v + 2\frac{dm}{dt}\frac{dv}{dt} + m\frac{d^2v}{dt^2} \tag{5}$$

For d^2m/dt^2 we get

$$\begin{aligned}
\frac{d^2m}{dt^2} &= \frac{d}{dt}\left(\frac{dm}{dt}\right) = \frac{d}{dt}\left(\frac{dm}{dv}\right)\frac{dv}{dt} + \frac{dm}{dv}\frac{d^2v}{dt^2} = \frac{d^2m}{dv^2}\left(\frac{dv}{dt}\right)^2 + \frac{dm}{dv}\frac{d^2v}{dt^2} \\
&= \frac{m_0}{c^2(1 - v^2/c^2)^{3/2}}\left[\frac{1 + 2v^2/c^2}{1 - v^2/c^2}\left(\frac{dv}{dt}\right)^2 + v\frac{d^2v}{dt^2}\right]
\end{aligned} \tag{6}$$

and $d^2(mv)/dt^2$ becomes:

$$\frac{d^2(mv)}{dt^2} = \frac{3m_0}{c^2(1 - v^2/c^2)^{5/2}}v\left(\frac{dv}{dt}\right)^2 + \frac{m_0}{(1 - v^2/c^2)^{3/2}}\frac{d^2v}{dt^2} \tag{7}$$

Since $\mathbf{K}_\mathrm{m}$ and $\mathbf{v}$ in Eq.(3) have the same direction we may use K_m rather than $\mathbf{K}_\mathrm{m}$:

$$\frac{dK_\mathrm{m}}{dt} = \frac{m_0}{(1-v^2/c^2)^{1/2}}\left[\frac{3}{c^2(1-v^2/c^2)^2}v\left(\frac{dv}{dt}\right)^2 + \frac{1}{1-v^2/c^2}\frac{d^2v}{dt^2} + \frac{1}{\tau_\mathrm{p}^2}v\right] \tag{8}$$

This provides the extension of Eqs.(2.1-8) and (2.2-6) to relativistically variable masses.

For nonrelativistic velocities we had shown by Eq.(2.1-6) that the contribution of the proton to the induced current of a proton-electron dipole was very small. This changes when the masses increase relativistically. Conservation of momentum demands

$$m_\mathrm{p}\mathbf{v}_\mathrm{p} = m_\mathrm{e}\mathbf{v}_\mathrm{e} \tag{9}$$

if the subscripts p and e denote proton and electron. With the rest masses denoted $m_{0\mathrm{p}}$ and $m_{0\mathrm{e}}$ we get

$$\frac{m_{0\mathrm{p}}v_\mathrm{p}}{\left(1-v_\mathrm{p}^2/c^2\right)^{1/2}} = \frac{m_{0\mathrm{e}}v_\mathrm{e}}{(1-v_\mathrm{e}^2/c^2)^{1/2}} \tag{10}$$

since the direction of $\mathbf{v}_\mathrm{p}$ and $\mathbf{v}_\mathrm{e}$ is always opposite. The velocity v_p of the proton is derived as a function of the velocity v_e of the electron:

$$v_\mathrm{p} = \frac{m_{0\mathrm{e}}v_\mathrm{e}}{m_{0\mathrm{p}}\left[1-(1-m_{0\mathrm{e}}^2/m_{0\mathrm{p}}^2)v_\mathrm{e}^2/c^2\right]^{1/2}} \tag{11}$$

For $v_\mathrm{e} \to c$ we get $v_\mathrm{p} \to v_\mathrm{e}$. In this case the proton contributes as much as the electron to the dipole current.

The velocity v_e of the electron must be very close to the velocity of light before the proton contributes significantly to the current density. For instance, a proton velocity $v_\mathrm{p}/c = 0.01$ requires an electron velocity $v_\mathrm{e}/c = 0.99852$, which implies a ratio of the current densities $g_\mathrm{e}/g_\mathrm{p} = v_\mathrm{e}/v_\mathrm{p} = 0.99852/0.01 = 99.85$; hence, the proton contributes about 1% to the total current density $g_\mathrm{e} + g_\mathrm{p}$. For $v_\mathrm{p}/c = 0.1$ we get $v_\mathrm{e}/c = 0.9999853$ and $g_\mathrm{e}/g_\mathrm{p} = 0.9999853/0.1 = 9.999853$, which means the proton contributes about 9% to the total current density.

For the rotating dipole we start from Eq.(2.1-10) and substitute $J = 2mR^2$ for the magnitude of the moment of inertia of the dumb-bell model of Fig.2.1-2:

$$K_\mathrm{m} = 2Rm\frac{d^2\vartheta}{dt^2} \tag{12}$$

For the transition to the relativistic version of this equation we consider the velocity $Rd\vartheta/dt$

$$R\frac{d\vartheta}{dt} = -v, \quad R\frac{d^2\vartheta}{dt^2} = -\frac{dv}{dt} \tag{13}$$

and rewrite Eq.(12):

$$\mathbf{K}_{\mathrm{m}} = -2m\frac{d\mathbf{v}}{dt} \to -2\frac{d(m\mathbf{v})}{dt} = -\frac{2m_0}{(1-v^2/c^2)^{3/2}}\frac{d\mathbf{v}}{dt}$$
$$K_{\mathrm{m}} = \frac{2m_0 R}{[1-R^2(d\vartheta/dt)^2/c^2]^{3/2}}\frac{d^2\vartheta}{dt^2} \tag{14}$$

The extensions of Ohm's law derived in Sections 2.1 and 2.2 require a further extension if the relativistic variation of the mass of the current carriers is taken into account. We start with Eq.(2.1-12). The force $m\,d\mathbf{v}/dt$ in Newton's mechanic becomes $d(m\mathbf{v})/dt$ in relativistic mechanic. The term $\xi_{\mathrm{e}}\mathbf{v}$ in Eq.(2.1-12) represents somewhat of a problem. If we leave it as it is we get evidently wrong results. Acceptable results are obtained if $\mathbf{v}$ is replaced by $m\mathbf{v}/m_0$, where m/m_0 is defined by Eq.(1). The product $m\mathbf{v}/m_0$ is reduced to $\mathbf{v}$ for small velocities $v \ll c$. We replace Eq.(2.1-12) by the following relativistic equation:

$$\frac{d(m\mathbf{v})}{dt} = e\mathbf{E} - \frac{\xi_{\mathrm{e}} m\mathbf{v}}{m_0} \tag{15}$$

The term $d(m\mathbf{v})/dt$ is defined by Eq.(2). Substitution of Eq.(2) into Eq.(15) brings:

$$m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2}\left(\frac{1}{1-v^2/c^2}\frac{d\mathbf{v}}{dt} + \frac{\xi_{\mathrm{e}}\mathbf{v}}{m_0}\right) = e\mathbf{E} \tag{16}$$

Since $\mathbf{v}$ and $\mathbf{E}$ have the same direction we may use their magnitudes and rewrite Eq.(16) as follows:

$$(1-\beta^2)^{-1/2}\left(\frac{1}{1-\beta^2}\frac{d\beta}{d\theta} + \frac{1}{pq}\beta\right) = \frac{\gamma_{\mathrm{e}}E}{E_0}$$
$$\beta = v/c = N_0ev/N_0ec,\ \gamma_{\mathrm{e}} = \tau_{\mathrm{mp}}eE_0/m_0c,\ \tau_{\mathrm{mp}} = m_0/\xi_{\mathrm{e}}$$
$$q = \tau_{\mathrm{p}}/\tau,\ p = \tau_{\mathrm{mp}}/\tau_{\mathrm{p}},\ pq = \tau_{\mathrm{mp}}/\tau,\ \theta = t/\tau \tag{17}$$

We note that β represents either the normalized velocity v/c or the normalized current density N_0ev/N_0ec. For small values of β we obtain the normalized form of Eq.(2.1-13) with $m = m_0$:

$$\frac{d\beta}{d\theta} + \frac{1}{pq}\beta = \frac{\gamma_{\mathrm{e}}E}{E_0} \quad \text{or} \quad \tau_{\mathrm{mp}}\frac{dg_{\mathrm{e}}}{dt} + g_{\mathrm{e}} = \sigma E,\ \ \sigma = \frac{N_0e^2\tau_{\mathrm{mp}}}{m_0},\ g_{\mathrm{e}} = \beta N_0ec \tag{18}$$

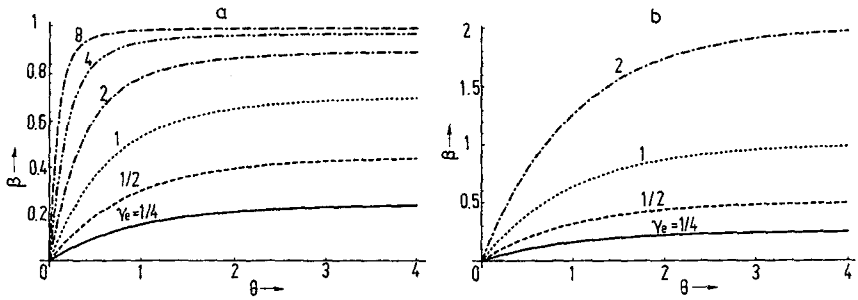

FIG.2.3-1. Plots of the normalized electric monopole current density $\beta(\theta)$ for $E/E_0 = S(\theta)$, $pq = 1$ and various values of γ_e according to Eqs.(17) and (18). (a) Relativistic plots using Eq.(17). (b) Nonrelativistic plots using Eq.(18).

Equation (18) yields for a step function $E = E_0 S(\theta)$ and the initial condition $\beta(0) = 0$ the solution

$$\beta(\theta) = pq\gamma_e \left(1 - e^{-\theta/pq}\right) \tag{19}$$

Plots of Eq.(19) are shown in Fig.2.3-1b for $pq = 1$ and various values of the parameter γ_e. For large times θ the plots approach γ_e. On the other hand, the amplitude of $\beta(\theta)$ according to the relativistic Eq.(17) shown in Fig.2.3-1a never exceeds 1.

Let us turn to the relativistic correction of the extension of Ohm's law to electric dipole currents. For a fixed charge carrier density N_0 in vacuum, a monopole current can be increased only by an increased velocity of the charge carriers. Dipole currents can be increased by either increasing the velocity of the charge carriers or the number of dipoles. There is no conservation law that prohibits an increase of dipoles. Only experimental work can decide whether there is a relativistic limitation for electric dipole current densities. We start from Eqs.(2.1-18) and (15):

$$\frac{d(m\mathbf{v})}{dt} + \frac{m\mathbf{v}}{\tau_{mp}} + \frac{1}{\tau_p^2}\int m\mathbf{v}\,dt = e\mathbf{E} \tag{20}$$

We differentiate with respect to t in order to eliminate the integral. Since $\mathbf{E}$ and $\mathbf{v}$ have the same direction we can simplify the calculation by using the magnitudes E and v:

$$\frac{d^2(mv)}{dt^2} + \frac{1}{\tau_{mp}}\frac{d(mv)}{dt} + \frac{1}{\tau_{p^2}}mv = e\frac{dE}{dt} \tag{21}$$

The terms $d(mv)/dt$ and $d^2(mv)/dt^2$ were evaluated in Eqs.(2) and (7). Equation (21) assumes the following form:

$$\frac{m_0}{(1-v^2/c^2)^{1/2}}\left[\frac{3}{c^2(1-v^2/c^2)^2}v\left(\frac{dv}{dt}\right)^2+\frac{1}{1-v^2/c^2}\frac{d^2v}{dt^2}\right.$$
$$\left.+\frac{1}{\tau_{\mathrm{mp}}}\frac{1}{1-v^2/c^2}\frac{dv}{dt}+\frac{1}{\tau_{\mathrm{p}}^2}v\right]=e\frac{dE}{dt} \quad (22)$$

Multiplication with τ^2/c brings Eq.(22) into the following normalized form

$$(1-\beta^2)^{-1/2}\left[\frac{1}{1-\beta^2}\left(\frac{d^2\beta}{d\theta^2}+\frac{1}{pq}\frac{d\beta}{d\theta}\right)+\frac{3}{(1-\beta^2)^2}\beta\left(\frac{d\beta}{d\theta}\right)^2+\frac{1}{q^2}\beta\right]$$
$$=\frac{\gamma_{\mathrm{e}}}{pq}\frac{1}{E_0}\frac{dE}{d\theta} \quad (23)$$

which uses the definitions listed in Eq.(17).

The nonrelativistic limit of this equation is obtained for $\beta^2 \to 0$ and $\beta(d\beta/d\theta)^2 \to 0$. We observe that $d\beta/d\theta$ represents the acceleration of a charge carrier:

$$\frac{d^2\beta}{d\theta^2}+\frac{1}{pq}\frac{d\beta}{d\theta}+\frac{1}{q^2}\beta=\frac{\gamma_{\mathrm{e}}}{pq}\frac{1}{E_0}\frac{dE}{d\theta}=\frac{\sigma_{\mathrm{p}}}{N_0ecpq}\frac{dE}{d\theta} \quad (24)$$

This equation should be compared with Eq.(2.1-21) for excitation by a step function $E = E_0S(\theta)$.

For the numerical evaluation of Eqs.(23) and (24) we have the initial conditions $\beta(\theta) = 0$ for $\theta = 0$ since the current or current density should be zero initially. For a second initial condition at $\theta = 0$ we rewrite Eq.(23) for $\beta = 0$

$$\frac{d^2\beta}{d\theta^2}+\frac{1}{pq}\frac{d\beta}{d\theta}=\frac{\gamma_{\mathrm{e}}}{pq}\frac{1}{E_0}\frac{dE}{d\theta} \quad (25)$$

and integrate:

$$\frac{d\beta}{d\theta}+\frac{1}{pq}\beta=\frac{\gamma_{\mathrm{e}}}{pq}\frac{1}{E_0}E(\theta) \quad (26)$$

For an electric step function excitation

$$E(\theta)=\frac{1}{q}E_0S(\theta) \quad (27)$$

we get

$$\frac{d\beta(0)}{d\theta}=\frac{\gamma_{\mathrm{e}}}{pq^2} \quad (28)$$

as the second initial condition since the term β/pq is zero due to the first initial condition $\beta(0) = 0$. Hence, for a step function excitation we obtain from Eqs.(23) and (24) the following equations and initial conditions:

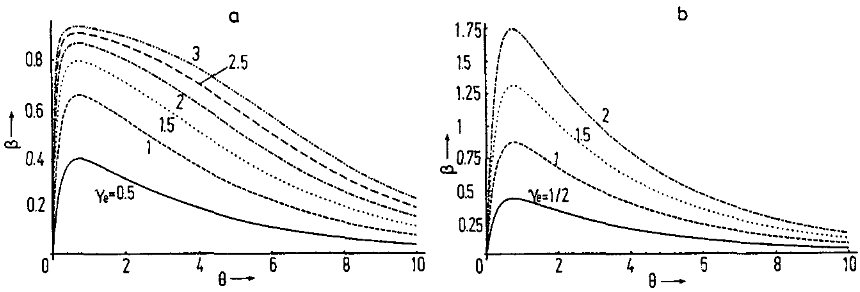

FIG.2.3-2. Plots of the normalized electric dipole current density $\beta(\theta)$ for $p = 1/4$, $q = 1$, and various values of γ_e according to Eqs.(29) to (31). (a) Relativistic plots using Eq.(29). (b) Nonrelativistic plots using Eq.(30).

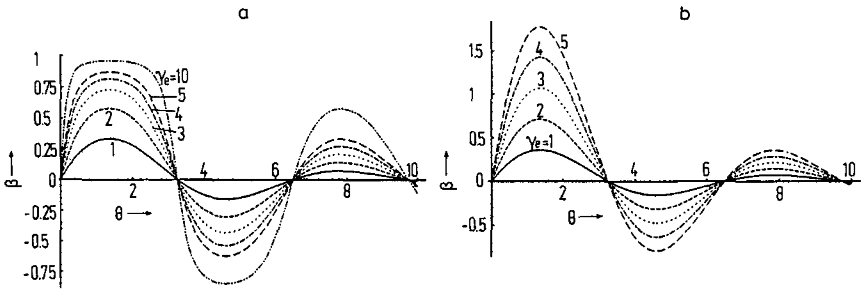

FIG.2.3-3. Plots of the normalized electric current density $\beta(\theta)$ for $p = 2$, $q = 1$, and various values of γ_e according to Eqs.(29) to (31). (a) Relativistic plots using Eq.(29). (b) Nonrelativistic plots using Eq.(30).

$$\frac{1}{1-\beta^2}\left(\frac{d^2\beta}{d\theta^2} + \frac{1}{pq}\frac{d\beta}{d\theta}\right) + \frac{3}{(1-\beta^2)^2}\beta\left(\frac{d\beta}{d\theta}\right)^2 + \frac{1}{q^2}\beta = 0 \qquad (29)$$

$$\frac{d^2\beta}{d\theta^2} + \frac{1}{pq}\frac{d\beta}{d\theta} + \frac{1}{q^2}\beta = 0 \qquad (30)$$

$$\beta(0) = 0, \; d\beta(0)/d\theta = \gamma_e/pq^2 \qquad (31)$$

Computer plots of $\beta(\theta)$ according to Eq.(30) are shown for $p = 1/4$, $q = 1$, and various values of γ_e in Fig.2.3-2b. Corresponding plots for $p = 2$ are shown in Fig.2.3-3b. As one would expect, the amplitude of the normalized current density $\beta(0)$ varies proportionate to γ_e.

Equation (29) does not lend itself to an analytic solution but plots of $\beta(\theta)$ can readily be obtained numerically. Figure 2.3-2a shows plots for $p = 1/2$, $q = 1$, and various values of γ_e, while Fig.2.3-3a shows corresponding plots for

$p = 2$. A comparison with the nonrelativistic plots shows first of all that the normalized current density $\beta(\theta)$ does not vary proportionate to γ_e and that it does not exceed 1. We further note that the values of θ for which $\beta(\theta)$ equals zero are the same in the relativistic and the nonrelativistic cases of Fig.2.3-3.

We turn to the rotating magnetic dipole described by Eq.(2.2-10). In order to obtain the relativistic form of the moment of inertia J we must specify in more detail where the mass of the bar magnet of Fig.2.2-1a is located. We assume the bar magnet can be represented by the dumb-bell shown in Fig.2.1-2 with the masses m at the ends of a thin rod of length $2R$, just as was assumed in Fig.2.2-1b for the magnetic charges $\pm q_m$. Introduction of $J = 2mR^2$ into Eq.(2.2-10) yields:

$$2mR^2\frac{d^2\vartheta}{dt^2} + \xi_m R\frac{d\vartheta}{dt} + m_{mo}B\sin\vartheta = 0 \tag{32}$$

For the transition to the relativistic version of this equation we consider the velocity $v(t)$ of Eq.(13) and rewrite Eq.(32) as follows:

$$2mR\frac{dv}{dt} + \xi_m v = m_{mo}B\sin\vartheta \tag{33}$$

The term on the right represents the force due to the flux density B which is imposed on the magnetic dipole. This force is independent of any relativistic change of the mass of the dipole. Terms $m\,dv/dt$ and $\xi_e v$ were rewritten into relativistic form in Eq.(15). In analogy, Eq.(33) is rewritten:

$$2R\frac{d(mv)}{dt} + \frac{\xi_m}{m_0}(mv) = m_{mo}B\sin\vartheta \tag{34}$$

With m and $d(mv)/dt$ from Eqs.(1) and (2) we obtain:

$$\frac{2Rm_0}{(1-v^2/c^2)^{3/2}}\frac{dv}{dt} + \frac{\xi_m}{(1-v^2/c^2)^{1/2}}v = m_{mo}B\sin\vartheta \tag{35}$$

We re-substitute for v and dv/dt from Eq.(13):

$$\frac{2R^2m_0}{[1-(R/c)^2(d\vartheta/dt)^2]^{3/2}}\frac{d^2\vartheta}{dt^2} + \frac{R\xi_m}{[1-(R/c)^2(d\vartheta/dt)^2]^{1/2}}\frac{d\vartheta}{dt} = -m_{mo}\sin\vartheta \tag{36}$$

Using the notation $\theta = t/\tau$ and $\rho = R/c\tau$ one can bring Eq.(36) into the following form:

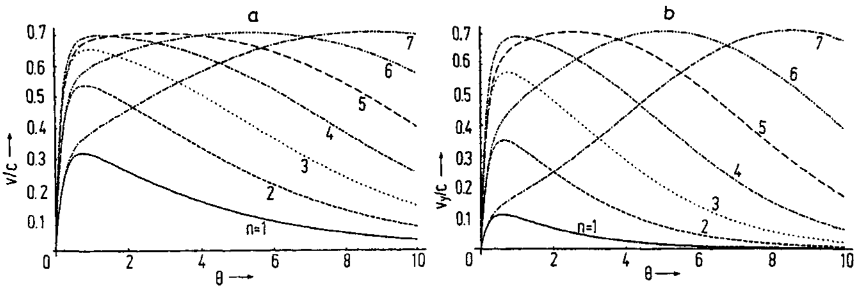

FIG.2.3-4. The normalized velocities $v(\theta)/c$ (a) and $v_y(\theta)/c$ (b) according to Eqs.(37), (40), and (41) for $p = 1/4$, $q = 1$, $\rho = 4$, and $\vartheta(0) = \vartheta_n = n\pi/8$ with $n = 1, 2, \ldots, 7$ in the interval $0 \leq \theta \leq 10$.

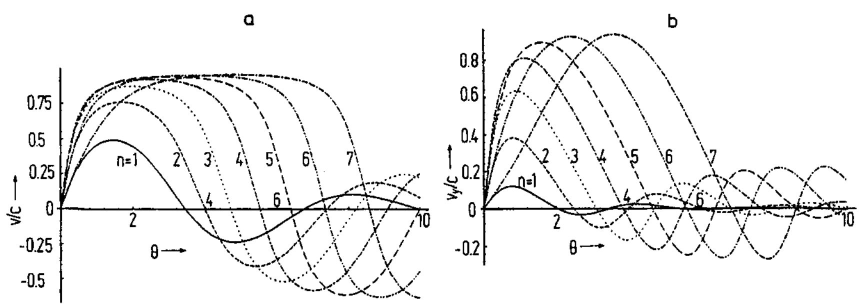

FIG.2.3-5. The normalized velocities $v(\theta)/c$ (a) and $v_y(\theta)/c$ (b) according to Eqs.(37), (40), and (41) for $p = 2$, $q = 1$, $\rho = 2$, and $\vartheta(0) = \vartheta_n = n\pi/8$ with $n = 1, 2, \ldots, 7$ in the interval $0 \leq \theta \leq 10$.

$$\left[1 - \rho^2 \left(\frac{d\vartheta}{d\theta}\right)^2\right]^{-1/2} \left(\frac{1}{1 - \rho^2 (d\vartheta/d\theta)^2} \frac{d^2\vartheta}{d\theta^2} + \frac{1}{pq}\frac{d\vartheta}{d\theta}\right) + \frac{1}{q^2}\sin\vartheta = 0$$

$$\theta = \frac{t}{\tau},\ \rho = \frac{R}{c\tau},\ q = \frac{R}{\tau}\sqrt{\frac{2m_0}{m_{\mathrm{mo}}B}} = \frac{\tau_{\mathrm{p}}}{\tau},\ p = \frac{\sqrt{2m_0 m_{\mathrm{mo}}B}}{\xi_{\mathrm{m}}} = \frac{\tau_{\mathrm{mp}}}{\tau_{\mathrm{p}}}$$

$$pq = \frac{2Rm_0}{\tau\xi_{\mathrm{m}}},\ \tau_{\mathrm{p}} = R\sqrt{\frac{2m_0}{m_{\mathrm{mo}}B}},\ \tau_{\mathrm{mp}} = \frac{2m_0 R}{\xi_{\mathrm{m}}} \tag{37}$$

For $\rho^2(d\vartheta/d\theta)^2 \to 0$ we obtain from Eq.(37) the nonrelativistic limit:

$$\frac{d^2\vartheta}{d\theta^2} + \frac{1}{pq}\frac{d\vartheta}{d\theta} + \frac{1}{q^2}\sin\vartheta = 0 \tag{38}$$

The initial conditions for both Eqs.(37) and (38) are

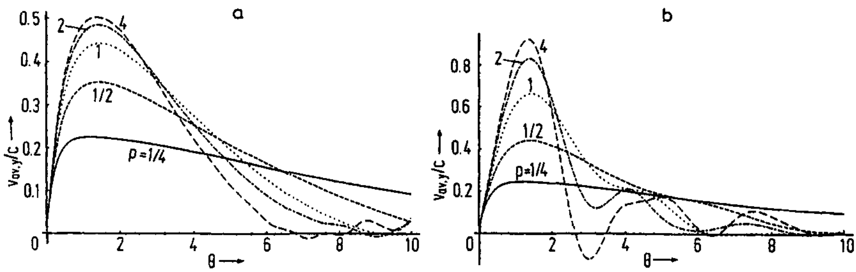

FIG.2.3-6. Plots of the normalized magnetic dipole current density $v_{\mathrm{av},y}(\theta)/c = \beta_{\mathrm{av},y}(\theta)$ for $q = 1$, $\rho = 2$, and $p = 1/4$, $1/2$, 1, 2, 4 according to Eqs.(41)–(43). (a) Relativistic plots using Eq.(37). (b) Nonrelativistic plots using Eq.(38).

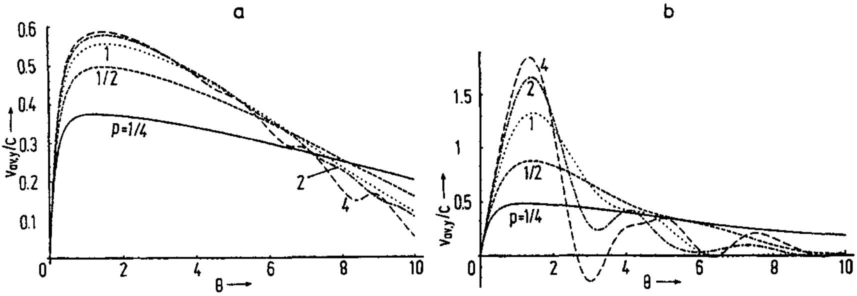

FIG.2.3-7. Plots of the normalized magnetic dipole current density $v_{\mathrm{av},y}(\theta)/c = \beta_{\mathrm{av},y}(\theta)$ for $q = 1$, $\rho = 4$, and $p = 1/4$, $1/2$, 1, 2, 4 according to Eqs.(41)–(43). (a) Relativistic plots using Eq.(37). (b) Nonrelativistic plots using Eq.(38).

$$\vartheta(0) = n\vartheta_0 = \vartheta_n, \quad d\vartheta(0)/d\theta = 0 \tag{39}$$

where ϑ_0 is the initial angle ϑ in Fig.2.2-1 before the flux density **B** is applied.

We still need the velocities $v(\theta)$ and $v_y(\theta)$ of Eqs.(2.2-12) and (2.2-13) in a rewritten form using the normalized distance $\rho = R/c\tau$:

$$\frac{v(\theta)}{c} = -\frac{R}{c\tau}\frac{d\vartheta}{d\theta} = -\rho\frac{d\vartheta}{d\theta} \tag{40}$$

$$\frac{v_y(\theta)}{c} = -\rho\frac{d\vartheta}{d\theta}\sin\vartheta \tag{41}$$

The average velocity $v_{\mathrm{av},y}$ of magnetic dipoles with random initial orientation is again obtained with Eq.(2.2-15):

$$\frac{v_{\mathrm{av},y}(\theta)}{c} = \beta_{\mathrm{av},y}(\theta) = \frac{1}{N}\sum_{n=1}^{N}\frac{v_y(\vartheta_n,\theta)}{c} = -\frac{\rho}{N}\sum_{n=1}^{N}\frac{d\vartheta(\vartheta_n,\theta)}{d\theta}\sin\vartheta(\vartheta_n,\theta) \tag{42}$$

The notation $v_y(\vartheta_n, \theta)$ and $\vartheta(\vartheta_n, \theta)$ indicates that the initial angle $\vartheta(0) = \vartheta_0$ in Eq.(39) has to be replaced by $\vartheta(0) = (n - 1/2)\pi/N$. The normalized magnetic current density follows from Eq.(2.2-16):

$$\frac{\mathbf{g}_{\rm m}}{2N_0 q_{\rm m} c} = \frac{2N_0 q_{\rm m} v_{{\rm av},y}(\theta)}{2N_0 q_{\rm m} c}\frac{\mathbf{y}}{y} = \frac{v_{{\rm av},y}(\theta)}{c}\frac{\mathbf{y}}{y} = \beta_{{\rm av},y}(\theta)\frac{\mathbf{y}}{y} \tag{43}$$

Equation (37) can readily be solved numerically and plots according to Eqs.(40)–(42) can be derived. Figure 2.3-4 shows $v(\theta)/c$ and $v_y(\theta)/c$ for $p = 1/4$, $q = 1$, $\rho = 4$ and various values of $\vartheta(0)$. The relativistic limitation $v/c < 1$ is readily recognizable in Fig.2.3-4a while the limitation $v_y/c < 1$ in Fig.2.3-4b is less conspicuous. Figure 2.3-5 shows the same plots but $p = 1/4$, $\rho = 4$ is replaced by $p = 2$, $\rho = 2$. Again, the relativistic limitation $v/c < 1$ is conspicuous in Fig.2.3-5a.

Figure 2.3-6 shows the normalized magnetic current density $\beta_{{\rm av},y}(\theta)$ according to Eq.(43) for $q = 1$, $\rho = 2$, and various values of p. On the left are relativistic plots derived from Eq.(37) while on the right are nonrelativistic plots derived from Eq.(38). Figure 2.3-7 shows the same plots but $\rho = 2$ is replaced by $\rho = 4$ to make the relativistic limitation $\beta_{{\rm av},y}(\theta) < 1$ more conspicuous.

2.4 Covariance of the Modified Maxwell Equations

We start with the continuity equation for an electric current density and charge density defined by Eq.(1.6-8):

$$\operatorname{div}\mathbf{g}_{\rm e} + \frac{\partial \rho_{\rm e}}{\partial t} \equiv \nabla\cdot\mathbf{g}_{\rm e} + \frac{\partial \rho_{\rm e}}{\partial t} = 0 \tag{1}$$

A four-vector current density $\mathfrak{g}_{\rm e}$ with the components $\mathfrak{g}_{\rm e1} = \mathfrak{g}_{{\rm e}x}$, $\mathfrak{g}_{\rm e2} = \mathfrak{g}_{{\rm e}y}$, $\mathfrak{g}_{\rm e3} = \mathfrak{g}_{{\rm e}z}$, $\mathfrak{g}_{\rm e4} = ic\rho_{\rm e}$ permits to write this equation in covariant form:

$$\sum_{\nu=1}^{4}\frac{\partial \mathfrak{g}_{{\rm e}\nu}}{\partial x_\nu} \equiv \frac{\partial \mathfrak{g}_{{\rm e}\nu}}{\partial x_\nu} \equiv \operatorname{Div}\mathfrak{g}_{\rm e} \equiv \Box\cdot\mathfrak{g}_{\rm e} = 0$$
$$x_1 = x,\ x_2 = y,\ x_3 = z,\ x_4 = ict \tag{2}$$

A corresponding four-vector current density $\mathfrak{g}_{\rm m}$ with the components $\mathfrak{g}_{\rm m1} = \mathfrak{g}_{{\rm m}x}$, $\mathfrak{g}_{\rm m2} = \mathfrak{g}_{{\rm m}y}$, $\mathfrak{g}_{\rm m3} = \mathfrak{g}_{{\rm m}z}$, $\mathfrak{g}_{\rm m4} = ic\rho_{\rm m}$ permits to write Eq.(1.6-9) in covariant form:

$$\frac{\partial \mathfrak{g}_{{\rm m}\nu}}{\partial x_\nu} \equiv \operatorname{Div}\mathfrak{g}_{\rm m} \equiv \Box\cdot\mathfrak{g}_{\rm m} = 0 \tag{3}$$

Equations (1.6-27) and (1.6-28) define a four-potential $\mathfrak{A}_{\rm m}$ with the components $\mathfrak{A}_{\rm m1} = A_{\rm m1}$, $\mathfrak{A}_{\rm m2} = A_{\rm m2}$, $\mathfrak{A}_{\rm m3} = A_{\rm m3}$, $\mathfrak{A}_{\rm m4} = i\phi_{\rm e}/c$. These two equations may be combined into one:

$$\frac{\partial^2 \mathfrak{A}_{\mathrm{m}\lambda}}{\partial x_\nu \partial x_\nu} = -\frac{Z}{c} \mathfrak{g}_{\mathrm{e}\lambda} \quad \text{or} \quad \Box \mathfrak{A}_{\mathrm{m}} = -\frac{Z}{c} \mathfrak{g}_{\mathrm{e}}$$

$$\Box \equiv \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \tag{4}$$

The extended Lorentz convention of Eq.(1.6-23) becomes:

$$\frac{\partial \mathfrak{A}_{\mathrm{m}\nu}}{\partial x_\nu} = 0 \quad \text{or} \quad \Box \cdot \mathfrak{A}_{\mathrm{m}} = 0 \tag{5}$$

From Eqs.(1.6-26) and (1.6-29) we obtain the equivalent relations for the four-potential $\mathfrak{A}_{\mathrm{e}}$ with the components $\mathfrak{A}_{\mathrm{e}1} = A_{\mathrm{e}1}$, $\mathfrak{A}_{\mathrm{e}2} = A_{\mathrm{e}2}$, $\mathfrak{A}_{\mathrm{e}3} = A_{\mathrm{e}3}$, $\mathfrak{A}_{\mathrm{e}4} = i\phi_{\mathrm{m}}/c$:

$$\frac{\partial^2 \mathfrak{A}_{\mathrm{e}\lambda}}{\partial x_\nu \partial x_\nu} = -\frac{1}{Zc} \mathfrak{g}_{\mathrm{m}\lambda} \quad \text{or} \quad \Box \mathfrak{A}_{\mathrm{e}} = -\frac{1}{Zc} \mathfrak{g}_{\mathrm{m}} \tag{6}$$

The extended Lorentz convention of Eq.(1.6-24) becomes:

$$\frac{\partial \mathfrak{A}_{\mathrm{e}\nu}}{\partial x_\nu} = 0 \quad \text{or} \quad \Box \cdot \mathfrak{A}_{\mathrm{e}} = 0 \tag{7}$$

We rewrite Maxwell's modified equations into four-vector form. This is more readily done from the basic Eqs.(1.6-1)–(1.6-4) than from the expressions for **E** and **H** by means of potentials in Eqs.(1.6-11) and (1.6-17). First we write Eqs.(1.6-1) and (1.6-3) in component form:

$$\operatorname{curl} \mathbf{H} - \frac{1}{Zc} \frac{\partial \mathbf{E}}{\partial t} = f_{\mathrm{m}1} \mathbf{e}_1 + f_{\mathrm{m}2} \mathbf{e}_2 + f_{\mathrm{m}3} \mathbf{e}_3 = g_{\mathrm{e}x} \mathbf{e}_1 + g_{\mathrm{e}y} \mathbf{e}_2 + g_{\mathrm{e}z} \mathbf{e}_3$$

$$f_{\mathrm{m}1} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - \frac{1}{Zc} \frac{\partial E_x}{\partial t} = g_{\mathrm{e}x} = g_{\mathrm{e}1}$$

$$f_{\mathrm{m}2} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - \frac{1}{Zc} \frac{\partial E_y}{\partial t} = g_{\mathrm{e}y} = g_{\mathrm{e}2}$$

$$f_{\mathrm{m}3} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \frac{1}{Zc} \frac{\partial E_z}{\partial t} = g_{\mathrm{e}z} = g_{\mathrm{e}3} \tag{8}$$

$$\operatorname{div} \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = Zc\rho_{\mathrm{e}} = -iZg_{\mathrm{e}4} \tag{9}$$

Next we introduce the electromagnetic field tensor $\mathbf{F}_{\mathrm{m}}$ with components $f_{\mathrm{m}\mu\nu}$:

$$f_{\mathrm{m}14} = -f_{\mathrm{m}41} = -\frac{i}{c} E_x, \; f_{\mathrm{m}24} = -f_{\mathrm{m}42} = -\frac{i}{c} E_y, \; f_{\mathrm{m}34} = -f_{\mathrm{m}43} = -\frac{i}{c} E_z$$

$$f_{\mathrm{m}12} = -f_{\mathrm{m}21} = B_z, \; f_{\mathrm{m}23} = -f_{\mathrm{m}32} = B_x, \; f_{\mathrm{m}31} = -f_{\mathrm{m}13} = B_y$$

$$f_{\mathrm{m}11} = f_{\mathrm{m}22} = f_{\mathrm{m}33} = f_{\mathrm{m}44} = 0 \tag{10}$$

Written as a matrix $\mathbf{F}_{\mathrm{m}}$ has the following form:

$$\mathbf{F}_{\mathrm{m}} = \begin{pmatrix} 0 & B_z & -B_y & -\frac{i}{c}E_x \\ -B_z & 0 & B_x & -\frac{i}{c}E_y \\ B_y & -B_x & 0 & -\frac{i}{c}E_z \\ \frac{i}{c}E_x & \frac{i}{c}E_y & \frac{i}{c}E_z & 0 \end{pmatrix} \tag{11}$$

Consider the derivative $\partial f_{\mathrm{m}\mu\nu}/\partial x_\nu$ of $\mathbf{F}_{\mathrm{m}}$. With $\mu = Z/c$ we get:

$$\begin{aligned}
\frac{\partial f_{\mathrm{m}11}}{\partial x_1}+\frac{\partial f_{\mathrm{m}12}}{\partial x_2}+\frac{\partial f_{\mathrm{m}13}}{\partial x_3}+\frac{\partial f_{\mathrm{m}14}}{\partial x_4} &= \frac{Z}{c}\left(0+\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}-\frac{1}{Zc}\frac{\partial E_x}{\partial t}\right)\\
\frac{\partial f_{\mathrm{m}21}}{\partial x_1}+\frac{\partial f_{\mathrm{m}22}}{\partial x_2}+\frac{\partial f_{\mathrm{m}23}}{\partial x_3}+\frac{\partial f_{\mathrm{m}24}}{\partial x_4} &= \frac{Z}{c}\left(-\frac{\partial H_z}{\partial x}+0+\frac{\partial H_x}{\partial z}-\frac{1}{Zc}\frac{\partial E_y}{\partial t}\right)\\
\frac{\partial f_{\mathrm{m}31}}{\partial x_1}+\frac{\partial f_{\mathrm{m}32}}{\partial x_2}+\frac{\partial f_{\mathrm{m}33}}{\partial x_3}+\frac{\partial f_{\mathrm{m}34}}{\partial x_4} &= \frac{Z}{c}\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}+0-\frac{1}{Zc}\frac{\partial E_z}{\partial t}\right)\\
\frac{\partial f_{\mathrm{m}41}}{\partial x_1}+\frac{\partial f_{\mathrm{m}42}}{\partial x_2}+\frac{\partial f_{\mathrm{m}43}}{\partial x_3}+\frac{\partial f_{\mathrm{m}44}}{\partial x_4} &= \frac{i}{c}\left(\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}+0\right)
\end{aligned} \tag{12}$$

A comparison of these equations with Eqs.(8) and (9) shows that we may write Eqs.(8) and (9) in covariant form:

$$\frac{\partial f_{\mathrm{m}\mu\nu}}{\partial x_\nu} = \frac{Z}{c} g_{\mathrm{e}\mu} \tag{13}$$

The remaining two modified Maxwell equations are also written in the component form of Eqs.(8) and (9):

$$\begin{aligned}
-\operatorname{curl}\mathbf{E} - \frac{Z}{c}\frac{\partial \mathbf{H}}{\partial t} &= f_{\mathrm{m}4}\mathbf{e}_1 + f_{\mathrm{m}5}\mathbf{e}_2 + f_{\mathrm{m}6}\mathbf{e}_3 = g_{\mathrm{m}x}\mathbf{e}_1 + g_{\mathrm{m}y}\mathbf{e}_2 + g_{\mathrm{m}z}\mathbf{e}_3\\
f_{\mathrm{m}4} &= -\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}+\frac{Z}{c}\frac{\partial H_x}{\partial t}\right) = g_{\mathrm{m}x} = g_{\mathrm{m}1}\\
f_{\mathrm{m}5} &= -\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}+\frac{Z}{c}\frac{\partial H_y}{\partial t}\right) = g_{\mathrm{m}y} = g_{\mathrm{m}2}\\
f_{\mathrm{m}6} &= -\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}+\frac{Z}{c}\frac{\partial H_z}{\partial t}\right) = g_{\mathrm{m}z} = g_{\mathrm{m}3}
\end{aligned} \tag{14}$$

$$\operatorname{div}\mathbf{H} = \frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z} = \frac{c}{Z}\rho_{\mathrm{m}} = -\frac{i}{Z}g_{\mathrm{m}4} \tag{15}$$

Consider now the following sum of three derivatives of the tensor $\mathbf{F}_{\mathrm{m}}$:

$$\frac{\partial f_{\mathrm{m}\mu\nu}}{\partial x_\lambda} + \frac{\partial f_{\mathrm{m}\nu\lambda}}{\partial x_\mu} + \frac{\partial f_{\mathrm{m}\lambda\mu}}{\partial x_\nu} = -(-1)^\kappa \frac{i}{c} g_{\mathrm{m}\kappa}$$
$$\kappa = 1,\ 2,\ 3,\ 4;\ \lambda = 2,\ 3,\ 4,\ 1;\ \mu = 3,\ 4,\ 1,\ 2;\ \nu = 4,\ 1,\ 2,\ 3; \qquad (16)$$

We obtain with the help of Eqs.(10) and (11) the following four equations:

$$\begin{aligned}
&\frac{\partial f_{\mathrm{m}34}}{\partial x_2}+\frac{\partial f_{\mathrm{m}42}}{\partial x_3}+\frac{\partial f_{\mathrm{m}23}}{\partial x_4}=-\frac{i}{c}\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}+\frac{Z}{c}\frac{\partial H_x}{\partial t}\right)=+\frac{i}{c}g_{\mathrm{m}x}\\
&\frac{\partial f_{\mathrm{m}41}}{\partial x_3}+\frac{\partial f_{\mathrm{m}13}}{\partial x_4}+\frac{\partial f_{\mathrm{m}34}}{\partial x_1}=\ \frac{i}{c}\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}+\frac{Z}{c}\frac{\partial H_y}{\partial t}\right)=-\frac{i}{c}g_{\mathrm{m}y}\\
&\frac{\partial f_{\mathrm{m}12}}{\partial x_4}+\frac{\partial f_{\mathrm{m}24}}{\partial x_1}+\frac{\partial f_{\mathrm{m}41}}{\partial x_2}=-\frac{i}{c}\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}+\frac{Z}{c}\frac{\partial H_z}{\partial t}\right)=+\frac{i}{c}g_{\mathrm{m}z}\\
&\frac{\partial f_{\mathrm{m}23}}{\partial x_1}+\frac{\partial f_{\mathrm{m}31}}{\partial x_2}+\frac{\partial f_{\mathrm{m}12}}{\partial x_3}=\ \frac{Z}{c}\left(\frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}\right)=-\frac{i}{c}g_{\mathrm{m}4}=\rho_{\mathrm{m}}
\end{aligned} \qquad (17)$$

A comparison with Eqs.(14) and (15) shows that Eq.(16) is the covariant four-vector equation of the modified Maxwell equations (14) and (15). The magnetic charge ρ_{m} may or may not be zero. Equations (13) and (16) are equal to the corresponding equations in the conventional theory of Maxwell's equations without magnetic current densities except for the term $-(-1)^\kappa(i/c)g_{\mathrm{m}\kappa}$ instead of 0 in Eq.(16).

Equations (10) and (11) favor the magnetic flux density **B** and the electric field strength **E**. The introduction of a new field tensor $\mathbf{F}_{\mathrm{e}}$ with the components $f_{\mathrm{e}\mu\nu}$ favors instead the electric flux density **D** and the magnetic field strength **H**:

$$\begin{aligned}
&f_{\mathrm{e}14} = -f_{\mathrm{e}41} = -\frac{i}{c}H_x,\ f_{\mathrm{e}24} = -f_{\mathrm{e}42} = -\frac{i}{c}H_y,\ f_{\mathrm{e}34} = -f_{\mathrm{e}43} = -\frac{i}{c}H_z\\
&f_{\mathrm{e}12} = -f_{\mathrm{e}21} = -D_z,\ f_{\mathrm{e}23} = -f_{\mathrm{e}32} = -D_x,\ f_{\mathrm{e}31} = -f_{\mathrm{e}13} = -D_y\\
&f_{\mathrm{e}11} = f_{\mathrm{e}22} = f_{\mathrm{e}33} = f_{\mathrm{e}44} = 0
\end{aligned} \qquad (18)$$

In matrix form we get for the tensor $\mathbf{F}_{\mathrm{e}}$:

$$\mathbf{F}_{\mathrm{e}} = \begin{pmatrix} 0 & -D_z & D_y & -\frac{i}{c}H_x \\ D_z & 0 & -D_x & -\frac{i}{c}H_y \\ -D_y & D_x & 0 & -\frac{i}{c}H_z \\ \frac{i}{c}H_x & \frac{i}{c}H_y & \frac{i}{c}H_z & 0 \end{pmatrix} \qquad (19)$$

Let us consider the following relation of the derivatives $\partial f_{e\mu\nu}$ of the electromagnetic field tensor $\mathbf{F}_e$ with $\epsilon = 1/Zc$:

$$\frac{\partial f_{e\mu\nu}}{\partial x_\nu} = \frac{1}{Zc} g_{m\mu}$$

$$\frac{\partial f_{e11}}{\partial x_1}+\frac{\partial f_{e12}}{\partial x_2}+\frac{\partial f_{e13}}{\partial x_3}+\frac{\partial f_{e14}}{\partial x_4}=-\frac{1}{Zc}\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}+\frac{Z}{c}\frac{\partial H_x}{\partial t}\right)=\frac{1}{Zc}g_{mx}$$

$$\frac{\partial f_{e21}}{\partial x_1}+\frac{\partial f_{e22}}{\partial x_2}+\frac{\partial f_{e23}}{\partial x_3}+\frac{\partial f_{e24}}{\partial x_4}=-\frac{1}{Zc}\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}+\frac{Z}{c}\frac{\partial H_y}{\partial t}\right)=\frac{1}{Zc}g_{my}$$

$$\frac{\partial f_{e31}}{\partial x_1}+\frac{\partial f_{e32}}{\partial x_2}+\frac{\partial f_{e33}}{\partial x_3}+\frac{\partial f_{e34}}{\partial x_4}=-\frac{1}{Zc}\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}+\frac{Z}{c}\frac{\partial H_z}{\partial t}\right)=\frac{1}{Zc}g_{mz}$$

$$\frac{\partial f_{e41}}{\partial x_1}+\frac{\partial f_{e42}}{\partial x_2}+\frac{\partial f_{e43}}{\partial x_3}+\frac{\partial f_{e44}}{\partial x_4}=\frac{i}{c}\left(\frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}\right)=\frac{i}{c}\frac{c}{Z}\rho_m \qquad (20)$$

We recognize Eq.(17) as well as Eqs.(14) and (15). In analogy to Eq.(16) we write the following additional four-vector equation:

$$\frac{\partial f_{e\mu\nu}}{\partial x_\lambda}+\frac{\partial f_{e\nu\lambda}}{\partial x_\mu}+\frac{\partial f_{e\lambda\mu}}{\partial x_\nu}=(-1)^\kappa\frac{i}{c}g_{e\kappa}$$

$$\kappa=1,\ 2,\ 3,\ 4;\ \lambda=2,\ 3,\ 4,\ 1;\ \mu=3,\ 4,\ 1,\ 2;\ \nu=4,\ 1,\ 2,\ 3;$$

$$\frac{\partial f_{e34}}{\partial x_2}+\frac{\partial f_{e42}}{\partial x_3}+\frac{\partial f_{e23}}{\partial x_4}=-\frac{i}{c}\left(\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}-\frac{1}{Zc}\frac{\partial E_x}{\partial t}\right)=-\frac{i}{c}g_{ex}$$

$$\frac{\partial f_{e41}}{\partial x_3}+\frac{\partial f_{e13}}{\partial x_4}+\frac{\partial f_{e34}}{\partial x_1}=\frac{i}{c}\left(\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}-\frac{1}{Zc}\frac{\partial E_2}{\partial t}\right)=\frac{i}{c}g_{ey}$$

$$\frac{\partial f_{e12}}{\partial x_4}+\frac{\partial f_{e24}}{\partial x_1}+\frac{\partial f_{e41}}{\partial x_2}=-\frac{i}{c}\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}-\frac{1}{Zc}\frac{\partial E_z}{\partial t}\right)=-\frac{i}{c}g_{ez}$$

$$\frac{\partial f_{e23}}{\partial x_1}+\frac{\partial f_{e31}}{\partial x_2}+\frac{\partial f_{e12}}{\partial x_3}=-\frac{1}{Zc}\left(\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}\right)=\frac{i}{c}g_{e4}=-\rho_e \qquad (21)$$

Here we recognize Eq.(13) as well as Eqs.(8) and (9). Hence, the tensor $\mathbf{F}_e$ is an alternate representation to the tensor $\mathbf{F}_m$ of Maxwell's modified equations.

The derivatives $\partial f_{m\lambda\mu}/\partial x_\nu = t_{m\lambda\mu\nu}$ of the components of the tensor $\mathbf{F}_m$ form a tensor of rank three. The same holds true for the derivatives $\partial f_{e\lambda\mu}/\partial x_\nu = t_{e\lambda\mu\nu}$. Equations (17) and (21) may be written in the form

$$t_{m\lambda\mu\nu}+t_{m\mu\nu\lambda}+t_{m\nu\lambda\mu}=-(-1)^\kappa\frac{i}{c}g_{m\kappa},\quad t_{m\lambda\mu\nu}=-t_{m\mu\lambda\nu}=\frac{\partial f_{m\lambda\mu}}{\partial x_\nu} \qquad (22)$$

$$t_{e\lambda\mu\nu}+t_{e\mu\nu\lambda}+t_{e\nu\lambda\mu}=(-1)^\kappa\frac{i}{c}g_{e\kappa},\qquad t_{e\lambda\mu\nu}=-t_{e\mu\lambda\nu}=\frac{\partial f_{e\lambda\mu}}{\partial x_\nu} \qquad (23)$$

Our next task is to write the Lorentz forces of Eqs.(2.1-2) and (2.2-2) in covariant notation. With the components $g_{e1} = g_{ex}$, $g_{e2} = g_{ey}$, $g_{e3} = g_{ez}$, $g_{e4} = ic\rho_e$ of $\mathbf{g}_e = \rho_e \mathbf{v}$ and $B = (Z/c)H$ we obtain from Eq.(2.1-2):

$$\begin{aligned}\mathbf{k}_m &= \frac{Z}{c}\mathbf{g}_e \times \mathbf{H} + \rho_e \mathbf{E} = k_{m1}\mathbf{e}_1 + k_{m2}\mathbf{e}_2 + k_{m3}\mathbf{e}_3 \\ k_{m1} &= g_{e2}B_z - g_{e3}B_y - \frac{i}{c}g_{e4}E_x \\ k_{m2} &= g_{e3}B_x - g_{e1}B_z - \frac{i}{c}g_{e4}E_y \\ k_{m3} &= g_{e1}B_y - g_{e2}B_x - \frac{i}{c}g_{e4}E_z \end{aligned} \tag{24}$$

According to Eq.(11) we may write the components k_{mi} with the help of the tensor $\mathbf{F}_m$:

$$\begin{aligned} k_{m1} &= 0 && + g_{e2}f_{m12} && + g_{e3}f_{m13} && + g_{e4}f_{m14} \\ k_{m2} &= g_{e1}f_{m21} && + 0 && + g_{e3}f_{m23} && + g_{e4}f_{m24} \\ k_{m3} &= g_{e1}f_{m31} && + g_{e2}f_{m32} && + 0 && + g_{e4}f_{m34} \end{aligned} \tag{25}$$

Equation (11) suggests to produce a four-vector by adding a component k_{m4}:

$$k_{m4} = \frac{i}{c}(g_{e1}f_{m41} + g_{e2}f_{m42} + g_{e3}f_{m43} + 0) \tag{26}$$

Using the relation $g_{e\mu} = (c/Z)\partial f_{m\mu\nu}/\partial x_\nu$ from Eq.(13) we may write Eqs.(25) and (26) in the following form (Note that a double sum over $\nu = 1, 2, 3, 4$ and $\lambda = 1, 2, 3, 4$ is required):

$$k_{m\mu} = \frac{c}{Z} f_{m\mu\nu} \frac{\partial f_{m\nu\lambda}}{\partial x_\lambda} \tag{27}$$

One may rewrite Eq.(27) some more by means of a new symmetric tensor $T_{\mu\nu}$:

$$\begin{aligned} T_{\mu\nu} &= \frac{c}{Z}\left(f_{m\mu\lambda}f_{m\lambda\nu} + \frac{1}{4}\delta_{\mu\nu}f_{m\lambda\iota}f_{m\lambda\iota} \right) \\ \delta_{\mu\nu} &= 1 \quad \text{for } \mu = \nu \\ &= 0 \quad \text{for } \mu \neq \nu \end{aligned} \tag{28}$$

This tensor is called *energy-momentum tensor*. Since it is the same tensor as used in the conventional theory we will not discuss it in any detail. Here it is sufficient to mention that it can be written in the form

$$T_{\mu\nu} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} & -\frac{i}{c}(\mathbf{E}\times\mathbf{H})_x \\ T_{yx} & T_{yy} & T_{yz} & -\frac{i}{c}(\mathbf{E}\times\mathbf{H})_y \\ T_{zx} & T_{zy} & T_{zz} & -\frac{i}{c}(\mathbf{E}\times\mathbf{H})_z \\ -\frac{i}{c}(\mathbf{E}\times\mathbf{H})_x & -\frac{i}{c}(\mathbf{E}\times\mathbf{H})_y & -\frac{i}{c}(\mathbf{E}\times\mathbf{H})_z & \frac{1}{2c}(\frac{E^2}{Z}+ZH^2) \end{pmatrix} \tag{29}$$

The space part T_{xx} to T_{zz} is equal to the Maxwell tensor that represents the flow of momentum. The time part $(E^2/Z + ZH^2)/2c$ represents the energy density and the space-time part the energy flow. Equation (27) may be written in terms of $T_{\mu\nu}$:

$$k_{\mathrm{m}\mu} = \frac{\partial T_{\mu\nu}}{\partial x_\nu} \tag{30}$$

Everything in Eqs.(21) to (30) is equal to the conventional theory of Maxwell's equations without magnetic (dipole) currents. We shall need the Maxwell tensor with its components T_{xx} to T_{zz}, written here with the help of $D = (1/Zc)E$ and $B = (Z/c)H$ to make it simple, but the replacement of D and B by E/Zc and ZH/c is often advantageous:

$$\begin{aligned}
T_{xx} &= \frac{1}{2}(+D_xE_x - D_yE_y - D_zE_z + B_xH_x - B_yH_y - B_zH_z) \\
T_{yy} &= \frac{1}{2}(-D_xE_x + D_yE_y - D_zE_z - B_xH_x + B_yH_y - B_zH_z) \\
T_{zz} &= \frac{1}{2}(-D_xE_x - D_yE_y + D_zE_z - B_xH_x - B_yH_y + B_zH_z) \\
T_{xy} &= T_{yx} = D_xE_y + B_xH_y \\
T_{xz} &= T_{zx} = D_xE_z + B_xH_z \\
T_{yz} &= T_{zy} = D_yE_z + B_yH_z
\end{aligned} \tag{31}$$

Let us turn to Eq.(2.2-2) and write it in the form of Eq.(24) with the components $g_{\mathrm{m}1}$, $g_{\mathrm{m}2}$, $g_{\mathrm{m}3}$, $g_{\mathrm{m}4} = ic\rho_{\mathrm{m}}$ of $\mathbf{g}_{\mathrm{m}} = \rho_{\mathrm{m}}\mathbf{v}$ and with $D = (1/Zc)E$:

$$\begin{aligned}
\mathbf{k}_{\mathrm{e}} &= -\frac{1}{Zc}\mathbf{g}_{\mathrm{m}} \times \mathbf{E} + \rho_{\mathrm{m}}\mathbf{H} = k_{\mathrm{e}1}\mathbf{e}_1 + k_{\mathrm{e}2}\mathbf{e}_2 + k_{\mathrm{e}3}\mathbf{e}_3 \\
k_{\mathrm{e}1} &= -g_{\mathrm{m}2}D_z + g_{\mathrm{m}3}D_y - \frac{i}{c}g_{\mathrm{m}4}H_x \\
k_{\mathrm{e}2} &= -g_{\mathrm{m}3}D_x + g_{\mathrm{m}1}D_z - \frac{i}{c}g_{\mathrm{m}4}H_y \\
k_{\mathrm{e}3} &= -g_{\mathrm{m}1}D_y + g_{\mathrm{m}2}D_x - \frac{i}{c}g_{\mathrm{m}4}H_z
\end{aligned} \tag{32}$$

We use Eq.(19) to write the components k_{ei} in terms of the tensor $\mathbf{F}_{\mathrm{e}}$:

$$\begin{aligned}
k_{\mathrm{e}1} &= 0 && + g_{\mathrm{m}2} f_{\mathrm{e}12} && + g_{\mathrm{m}3} f_{\mathrm{e}13} && + g_{\mathrm{m}4} f_{\mathrm{e}14} \\
k_{\mathrm{e}2} &= g_{\mathrm{m}1} f_{\mathrm{e}21} && + 0 && + g_{\mathrm{m}3} f_{\mathrm{e}23} && + g_{\mathrm{m}4} f_{\mathrm{e}24} \\
k_{\mathrm{e}3} &= g_{\mathrm{m}1} f_{\mathrm{e}31} && + g_{\mathrm{m}2} f_{\mathrm{e}32} && + 0 && + g_{\mathrm{m}4} f_{\mathrm{e}34}
\end{aligned} \tag{33}$$

A four-vector is produced by adding a component $k_{\mathrm{e}4}$ according to Eq.(26):

$$k_{\mathrm{e}4} = \frac{i}{c}(g_{\mathrm{m}1} f_{\mathrm{e}41} + g_{\mathrm{m}2} f_{\mathrm{e}42} + g_{\mathrm{m}3} f_{\mathrm{e}43} + 0) \tag{34}$$

Using the relation $g_{\mathrm{m}\mu} = Zc\partial f_{\mathrm{e}\mu\nu}/\partial x_\nu$ from Eq.(20) we may combine Eqs.(32) and (34) as follows, with a double sum over ν and λ:

$$k_{\mathrm{e}\mu} = Zc f_{\mathrm{e}\mu\nu} \frac{\partial f_{\mathrm{e}\nu\lambda}}{\partial x_\lambda} \tag{35}$$

In analogy to Eq.(28) we may rewrite $k_{\mathrm{e}\mu}$ by means of a symmetric tensor $\mathfrak{T}_{\mu\nu}$:

$$\mathfrak{T}_{\mu\nu} = Zc\left(f_{\mathrm{e}\mu\lambda} f_{\mathrm{e}\lambda\nu} + \frac{1}{4}\delta_{\mu\nu} f_{\mathrm{e}\lambda\iota} f_{\mathrm{e}\lambda\iota}\right) \tag{36}$$

This tensor is equal to the tensor $T_{\mu\nu}$ of Eq.(29) except that the terms T_{xx} to T_{zz} have to be replaced by $\mathfrak{T}_{xx}$ to $\mathfrak{T}_{zz}$:

$$\mathfrak{T}_{\mu\nu} = \begin{pmatrix}
\mathfrak{T}_{xx} & \mathfrak{T}_{xy} & \mathfrak{T}_{xz} & -\frac{i}{c}(\mathbf{E}\times\mathbf{H})_x \\
\mathfrak{T}_{yx} & \mathfrak{T}_{yy} & \mathfrak{T}_{yz} & -\frac{i}{c}(\mathbf{E}\times\mathbf{H})_y \\
\mathfrak{T}_{zx} & \mathfrak{T}_{zy} & \mathfrak{T}_{zz} & -\frac{i}{c}(\mathbf{E}\times\mathbf{H})_z \\
-\frac{i}{c}(\mathbf{E}\times\mathbf{H})_x & -\frac{i}{c}(\mathbf{E}\times\mathbf{H})_y & -\frac{i}{c}(\mathbf{E}\times\mathbf{H})_z & \frac{1}{2c}(\frac{E^2}{Z} + ZH^2)
\end{pmatrix} \tag{37}$$

Equation (35) may be rewritten with $\mathfrak{T}_{\mu\nu}$:

$$k_{\mathrm{e}\mu} = \frac{\partial \mathfrak{T}_{\mu\nu}}{\partial x_\nu} \tag{38}$$

The calculation of the tensor components $\mathfrak{T}_{xx}$ to $\mathfrak{T}_{zz}$ is straightforward but lengthy[1]. We outline the process by writing T_{xx} of Eq.(31) with D and H replaced by E/Zc and ZH/c:

[1]The derivation of the tensor of Eq.(29) is shown in Becker (1964a), § 83. The tensor of Eq.(36) is obtained with the substitutions $H \to -D$, $E \to H/c$, and a change from the Gaussian system to the International System.

$$T_{xx} = \frac{1}{2c}\left[\frac{1}{Z}\left(E_x^2 - E_y^2 - E_z^2\right) + Z\left(H_x^2 - H_y^2 - H_z^2\right)\right] \tag{39}$$

Equation (28) requires the multiplication of T_{xx} with Z/c. Then we compare Eqs.(11) and (19), and make the substitutions

$$B_. \to -D_. \quad \text{or} \quad H_. \to -E_./Z^2 \quad \text{and} \quad E_. \to H_. \tag{40}$$

to obtain $\mathfrak{T}_{xy}/Zc$ according to Eq.(36):

$$\frac{\mathfrak{T}_{xy}}{Zc} = \frac{Z}{2c^2}\left[\frac{1}{Z}\left(H_x^2 - H_y^2 - H_z^2\right) + \frac{1}{Z^3}\left(E_x^2 - E_y^2 - E_z^2\right)\right] \tag{41}$$

Multiplication with Zc finally yields $\mathfrak{T}_{xx}$ which is equal to T_{xx} of Eq.(39). Generally we get:

$$\mathfrak{T}_{\mu\nu} = T_{\mu\nu} \tag{42}$$

2.5 Energy and Momentum With Dipole Current Correction

We return to Section 1.6 which derived the potential form of the modified Maxwell equations. We had derived Eqs.(1.6-26)–(1.6-29) for the extended Lorentz convention of Eqs.(1.6-23) and (1.6-24) from Eqs.(1.6-18)–(1.6-21). The particular solutions of Eqs.(1.6-30)–(1.6-33) had been obtained for the potentials $\mathbf{A}_e$, $\mathbf{A}_m$, ϕ_e, and ϕ_m. We note once more that ϕ_m is zero if there are no magnetic monopoles but $\mathbf{A}_e$ requires only the existence of magnetic dipoles, which is not disputed. General solutions of the potentials are obtained if one adds general solutions of the homogeneous equations

$$\Box\mathbf{A}_e = 0, \quad \Box\mathbf{A}_m = 0, \quad \Box\phi_e = 0, \quad \Box\phi_m = 0 \tag{1}$$

to the particular solutions and observes the extended Lorentz convention of Eqs.(1.6-23) and (1.6-24). This yields the following generalizations of the potentials:

$$\mathbf{A}_e \to \mathbf{A}_e - \operatorname{grad}\chi_e \tag{2}$$
$$\mathbf{A}_m \to \mathbf{A}_m - \operatorname{grad}\chi_m \tag{3}$$
$$\phi_e \to \phi_e + d\chi_m/dt \tag{4}$$
$$\phi_m \to \phi_m + d\chi_e/dt \tag{5}$$

These generalizations of the potentials leave the field strengths $\mathbf{H}$ of Eq.(1.6-11) and $\mathbf{E}$ of Eq.(1.6-17) unchanged. The various choices of potentials for fixed field strengths are called *gauges*. The class of gauges satisfying the extended Lorentz convention of Eqs.(1.6-23) and (1.6-24) is called the *extended Lorentz gauge*. It has the two independent functions χ_e and χ_m, while in the conventional theory we have only the one function χ. There are two functions χ_e and χ_m whether or not magnetic monopoles exist.

Besides the extended Lorentz gauge we introduce the *extended Coulomb gauge* by the definitions

$$\operatorname{div} \mathbf{A}_m = 0 \tag{6}$$
$$\operatorname{div} \mathbf{A}_e = 0 \tag{7}$$

Substitution into Eqs.(1.6-18)–(1.6-21) yields the field equations

$$\Box \mathbf{A}_e - \frac{1}{c^2} \operatorname{grad} \frac{\partial \phi_m}{\partial t} = -\frac{1}{Zc} \mathbf{g}_m \tag{8}$$
$$\Box \mathbf{A}_m - \frac{1}{c^2} \operatorname{grad} \frac{\partial \phi_e}{\partial t} = -\frac{Z}{c} \mathbf{g}_e \tag{9}$$
$$\nabla^2 \phi_e = -Zc\rho_e \tag{10}$$
$$\nabla^2 \phi_m = -\frac{c}{Z} \rho_m \tag{11}$$

It is usual to state that the electromagnetic field in free space without current or charge densities is represented by Eqs.(1.6-26)–(1.6-29) or Eqs.(8)–(11) with $\mathbf{g}_m = \mathbf{g}_e = \rho_e = \rho_m = 0$. We must be cautious with such choices. One may state that in the absence of electric and magnetic charges at the beginning one may choose $\rho_e = \rho_m = 0$ since the conservation law of charges prohibits the generation of charges from nothing. This reasoning does not apply to the current densities $\mathbf{g}_e$ and $\mathbf{g}_m$ since they do not have to represent monopole current densities that require charges. Dipole and higher order multipole currents do not require net charges and they are thus not precluded by the conservation law of charge. If we accept the principle of vacuum polarization we must accept the creation of dipoles which in turn produces dipole current densities $\mathbf{g}_e$ and $\mathbf{g}_m$. The different rôle of charges and dipole currents is shown by Eqs.(1.2-9) and (1.2-10). If we choose there $s = 0$ or $\sigma = 0$ we eliminate the current densities $\mathbf{g}_m$ or $\mathbf{g}_e$ according to Eqs.(1.1-14) and (1.1-15). But the choice $\rho_e = \rho_m = 0$ in Eq.(1.1-10) and (1.1-11) has no effect on Eqs.(1.2-9) and (1.2-10). The choice of either $s = 0$ or $\sigma = 0$ in Eqs.(1.2-9) and (1.2-10) eliminates the term $s\sigma E$ in Eq.(1.3-1) and the resulting equation does not yield solutions that satisfy the causality law. Hence, the transition $\mathbf{g}_e \to 0$ and $\mathbf{g}_m \to 0$ in Eqs.(8) and (9) can be made only at the end of the calculation.

The energy U of the electric and the magnetic field strength in a certain volume is given by the integral over the energy density u in that volume; dV denotes the three-dimensional volume element[1]:

$$U = \frac{1}{2} \iiint u \, dV = \frac{1}{2} \iiint \left(\frac{1}{Zc} E^2 + \frac{Z}{c} H^2 \right) dV \tag{12}$$

[1]The analysis follows closely Chapter I, § 1 of the renowned book by Heitler (1954) in order to facilitate comparison.

The product *momentum* × *c* will be referred to as usual as momentum. The momentum of the field defined in this way in a certain volume equals

$$\mathbf{G} = \iiint \mathbf{g}\, dV = \frac{1}{c} \iiint \mathbf{E} \times \mathbf{H}\, dV \tag{13}$$

From Eqs.(2.1-2) and (2.1-3) as well as (2.2-2) and (2.2-3) we obtain the forces acting on electric and magnetic charges in a certain volume:

$$\mathbf{K}_\mathrm{m} = \iiint \mathbf{k}_\mathrm{m}\, dV = \iiint \rho_\mathrm{e} \left(\mathbf{E} + \frac{Z}{c} \mathbf{v} \times \mathbf{H} \right) dV \tag{14}$$

$$\mathbf{K}_\mathrm{e} = \iiint \mathbf{k}_\mathrm{e}\, dV = \iiint \rho_\mathrm{m} \left(\mathbf{H} - \frac{1}{Zc} \mathbf{v} \times \mathbf{E} \right) dV \tag{15}$$

Since $\mathbf{K}_\mathrm{m}$ and $\mathbf{K}_\mathrm{e}$ equal the force of inertia of electric and magnetic monopoles they must also equal the change of mechanical momentum of the electric and magnetic monopoles per unit time:

$$\mathbf{K}_\mathrm{m} + \mathbf{K}_\mathrm{e} = \frac{1}{c} \frac{d\mathbf{u}}{dt} \tag{16}$$

If there are no electric or magnetic charges but induced electric dipoles and inherent magnetic dipoles one must use Eqs.(2.1-8) and (2.2-6) for $\mathbf{K}_\mathrm{m}$ and $\mathbf{K}_\mathrm{e}$. For a mixture of electric monopoles and dipoles one must use the sum of Eqs.(2.1-3) and (2.1-8) using m_1, $\mathbf{v}_1$ or m_2, $\mathbf{v}_2$ for m and $\mathbf{v}$.

The change of the kinetic energy T_m and T_e of electric and magnetic monopoles is defined by

$$\frac{dT_\mathrm{m}}{dt} = \iiint \mathbf{k}_\mathrm{m} \cdot \mathbf{v}\, dV \tag{17}$$

$$\frac{dT_\mathrm{e}}{dt} = \iiint \mathbf{k}_\mathrm{e} \cdot \mathbf{v}\, dV \tag{18}$$

where $\mathbf{k}_\mathrm{m}$ represents the density of force defined by Eq.(2.1-2) and $\mathbf{k}_\mathrm{e}$ the one defined by Eq.(2.2-2).

Consider first Eq.(17). With the help of Eqs.(2.1-1) and (1.6-1) we may write

$$\frac{dT_\mathrm{m}}{dt} = \iiint \mathbf{k}_\mathrm{m} \cdot \mathbf{v}\, dV = \iiint \rho_\mathrm{e} \mathbf{v} \cdot \left(\mathbf{E} + \frac{Z}{c} \mathbf{v} \times \mathbf{H} \right) dV$$
$$= \iiint \mathbf{E} \cdot \left(\operatorname{curl} \mathbf{H} - \frac{1}{Zc} \frac{d\mathbf{E}}{dt} \right) dV \tag{19}$$

With the relation

$$\frac{1}{2}\frac{d}{dt}\left(\frac{E^2}{Z}+ZH^2\right)=\frac{1}{Z}\mathbf{E}\cdot\frac{d\mathbf{E}}{dt}+Z\mathbf{H}\cdot\frac{d\mathbf{H}}{dt} \tag{20}$$

and Eq.(1.6-2) we obtain

$$\frac{1}{Zc}\mathbf{E}\cdot\frac{d\mathbf{E}}{dt}=\frac{1}{2}\frac{d}{dt}\left(\frac{E^2}{Zc}+\frac{Z}{c}H^2\right)+\mathbf{H}\cdot\operatorname{curl}\mathbf{E}+\mathbf{H}\cdot\mathbf{g}_{\mathrm{m}} \tag{21}$$

Equation (19) can be rewritten with the help of Eq.(12):

$$\frac{dT_{\mathrm{m}}}{dt}=-\frac{dU}{dt}+\iiint(\mathbf{E}\cdot\operatorname{curl}\mathbf{H}-\mathbf{H}\cdot\operatorname{curl}\mathbf{E})dV-\iiint\mathbf{H}\cdot\mathbf{g}_{\mathrm{m}}\,dV \tag{22}$$

For dT_{e}/dt of Eq.(18) we obtain with the help of Eq.(2.2-2) the following relation in analogy to Eq.(19):

$$\begin{aligned}\frac{dT_{\mathrm{e}}}{dt}&=\iiint\mathbf{k}_{\mathrm{e}}\cdot\mathbf{v}\,dV=\iiint\rho_{\mathrm{m}}\mathbf{v}\cdot\left(\mathbf{H}-\frac{1}{Zc}\mathbf{v}\times\mathbf{E}\right)dV\\&=-\iiint\mathbf{H}\cdot\left(\operatorname{curl}\mathbf{E}+\frac{\mathrm{Z}}{\mathrm{c}}\frac{d\mathbf{H}}{dt}\right)dV\end{aligned} \tag{23}$$

We may rewrite $\mathbf{H}\cdot d\mathbf{H}/dt$ with the help of Eqs.(20) and (1.6-1):

$$\frac{Z}{c}\mathbf{H}\cdot\frac{d\mathbf{H}}{dt}=\frac{1}{2}\frac{d}{dt}\left(\frac{E^2}{Zc}+\frac{Z}{c}H^2\right)-\mathbf{E}\cdot\operatorname{curl}\mathbf{H}+\mathbf{E}\cdot\mathbf{g}_{\mathrm{e}} \tag{24}$$

Equation (23) can be rewritten with the help of Eq.(12):

$$\frac{dT_{\mathrm{e}}}{dt}=-\frac{dU}{dt}+\iiint(\mathbf{E}\cdot\operatorname{curl}\mathbf{H}-\mathbf{H}\cdot\operatorname{curl}\mathbf{E})dV-\iiint\mathbf{E}\cdot\mathbf{g}_{\mathrm{e}}\,dV \tag{25}$$

Using the relations

$$\operatorname{div}(\mathbf{E}\times\mathbf{H})=\mathbf{E}\cdot\operatorname{curl}\mathbf{H}-\mathbf{H}\cdot\operatorname{curl}\mathbf{E} \tag{26}$$

$$\int\operatorname{div}(\mathbf{E}\times\mathbf{H})dV=-\oint(\mathbf{E}\times\mathbf{H})_{\mathrm{n}}\cdot d\mathbf{F} \tag{27}$$

where n denotes the component of $\mathbf{E}\times\mathbf{H}$ normal to the surface of the volume, we obtain from Eqs.(22) and (25):

$$\frac{d(T_{\mathrm{m}}+U)}{dt}=-\oint(\mathbf{E}\times\mathbf{H})_{\mathrm{n}}\cdot d\mathbf{F}-\iiint\mathbf{H}\cdot\mathbf{g}_{\mathrm{m}}\,dV \tag{28}$$

$$\frac{d(T_{\mathrm{e}}+U)}{dt}=-\oint(\mathbf{E}\times\mathbf{H})_{\mathrm{n}}\cdot d\mathbf{F}-\iiint\mathbf{E}\cdot\mathbf{g}_{\mathrm{e}}\,dV \tag{29}$$

The sum of these two equations yields:

$$\frac{dU}{dt}+\frac{1}{2}\left(\frac{dT_{\mathrm{m}}}{dt}+\iiint \mathbf{E}\cdot\mathbf{g}_{\mathrm{e}}\,dV\right)+\frac{1}{2}\left(\frac{dT_{\mathrm{e}}}{dt}+\iiint \mathbf{H}\cdot\mathbf{g}_{\mathrm{m}}\,dV\right)$$
$$=-\oint(\mathbf{E}\times\mathbf{H})_{\mathrm{n}}\cdot d\mathbf{F} \quad (30)$$

With the help of Eqs.(14) and (17) we get

$$\iiint \mathbf{E}\cdot\mathbf{g}_{\mathrm{e}}\,dV=\iiint \mathbf{E}\rho_{\mathrm{e}}\cdot\mathbf{v}\,dV=\iiint \mathbf{k}_{\mathrm{m}}\cdot\mathbf{v}\,dV=\frac{dT_{\mathrm{m}}}{dt} \quad (31)$$

while Eqs.(15) and (18) yield

$$\iiint \mathbf{H}\cdot\mathbf{g}_{\mathrm{m}}\,dV=\iiint \mathbf{H}\rho_{\mathrm{m}}\cdot\mathbf{v}\,dV=\iiint \mathbf{k}_{\mathrm{e}}\cdot\mathbf{v}\,dV=\frac{dT_{\mathrm{e}}}{dt} \quad (32)$$

and Eq.(30) assumes the final form:

$$\frac{d(U+T_{\mathrm{m}}+T_{\mathrm{e}})}{dt}=-\oint(\mathbf{E}\times\mathbf{H})_{\mathrm{n}}\cdot d\mathbf{F} \quad (33)$$

The right side of this equation represents the electromagnetic power flowing through the surface of a certain volume. The left side represents the change with time of the electromagnetic energy U as well as the kinetic energy T_{m} and T_{e} of the electric and magnetic monopoles in the volume.

Let us turn to Eq.(16). If we have electric and magnetic monopoles we obtain with the help of Eqs.(2.1-1), (2.2-1), and (1.6-1)–(1.6–4):

$$\frac{1}{c}\frac{d\mathbf{u}}{dt}=\iiint\left(\frac{1}{Zc}\mathbf{E}\operatorname{div}\mathbf{E}-\frac{Z}{c}\mathbf{H}\times\operatorname{curl}\mathbf{H}-\frac{1}{c^2}\frac{d\mathbf{E}}{dt}\times\mathbf{H}\right)dV$$
$$+\iiint\left(\frac{Z}{c}\mathbf{H}\operatorname{div}\mathbf{H}-\frac{1}{Zc}\mathbf{E}\times\operatorname{curl}\mathbf{E}+\frac{1}{c^2}\frac{d\mathbf{H}}{dt}\times\mathbf{E}\right)dV \quad (34)$$

The first line in Eq.(34) is conventional, the second line shows the contribution of magnetic monopoles. With the help of the relation

$$\frac{d}{dt}(\mathbf{E}\times\mathbf{H})=\frac{d\mathbf{E}}{dt}\times\mathbf{H}+\mathbf{E}\times\frac{d\mathbf{H}}{dt} \quad (35)$$

as well as Eqs.(1.6-1) and (1.6-2) we may write

$$\frac{d\mathbf{E}}{dt}\times\mathbf{H}=\frac{d}{dt}(\mathbf{E}\times\mathbf{H})+\frac{1}{Zc}\mathbf{E}\times(\operatorname{curl}\mathbf{E}+\mathbf{g}_{\mathrm{m}}) \quad (36)$$

$$\frac{d\mathbf{H}}{dt}\times\mathbf{E}=-\frac{d}{dt}(\mathbf{E}\times\mathbf{H})-Zc\mathbf{H}\times(\operatorname{curl}\mathbf{H}-\mathbf{g}_{\mathrm{e}}) \quad (37)$$

Using Eq.(13) we obtain the following form of Eq.(34):

$$\begin{aligned}\frac{d\mathbf{u}}{dt} = -\frac{d\mathbf{G}}{dt} - \iiint \left(-\frac{1}{Z}\mathbf{E}\operatorname{div}\mathbf{E} + Z\mathbf{H}\times\operatorname{curl}\mathbf{H} + \frac{1}{Z}\mathbf{E}\times\operatorname{curl}\mathbf{E}\right)dV \\ -\frac{1}{Z}\iiint \mathbf{E}\times\mathbf{g}_{\mathrm{m}}\,dV \\ -\frac{d\mathbf{G}}{dt} - \iiint \left(-Z\mathbf{H}\operatorname{div}\mathbf{H} + Z\mathbf{H}\times\operatorname{curl}\mathbf{H} + \frac{1}{Z}\mathbf{E}\times\operatorname{curl}\mathbf{E}\right)dV \\ +Z\iiint \mathbf{H}\times\mathbf{g}_{\mathrm{e}}\,dV \qquad (38)\end{aligned}$$

The first line of this equation is equal to that of the conventional theory. The third line is due to the contribution of the magnetic monopoles. The second and the fourth line contain extra terms for lines one and two.

From Eqs.(14) and (15) we get with the substitutions $\mathbf{g}_{\mathrm{e}} = \rho_{\mathrm{e}}\mathbf{v}$ and $\mathbf{g}_{\mathrm{m}} = \rho_{\mathrm{m}}\mathbf{v}$:

$$\begin{aligned}\frac{1}{Z}\iiint \mathbf{E}\times\mathbf{g}_{\mathrm{m}}\,dV &= c\iiint \mathbf{k}_{\mathrm{e}}\,dV - c\iiint \rho_{\mathrm{m}}\mathbf{H}\,dV \\ &= c\mathbf{K}_{\mathrm{e}} - Z\iiint \mathbf{H}\operatorname{div}\mathbf{H}\,dV \qquad (39)\\ Z\iiint \mathbf{H}\times\mathbf{g}_{\mathrm{e}}\,dV &= -c\iiint \mathbf{k}_{\mathrm{m}}\,dV + c\iiint \rho_{\mathrm{e}}\mathbf{E}\,dV \\ &= -c\mathbf{K}_{\mathrm{m}} + \frac{1}{Z}\iiint \mathbf{E}\operatorname{div}\mathbf{E}\,dV \qquad (40)\end{aligned}$$

Using Eq.(16) we may now rewrite Eq.(38) in a shorter form:

$$\frac{d(\mathbf{u}+\mathbf{G})}{dt} = \iiint \left(\frac{1}{Z}\mathbf{E}\operatorname{div}\mathbf{E} + Z\mathbf{H}\operatorname{div}\mathbf{H} - Z\mathbf{H}\times\operatorname{curl}\mathbf{H} - \frac{1}{Z}\mathbf{E}\times\operatorname{curl}\mathbf{E}\right)dV \qquad (41)$$

In order to show the physical meaning of Eq.(41) we introduce the Maxwell tensor discussed in Section 2.4:

$$T = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

$$T_{xx} = \frac{1}{2c}\left[\frac{1}{Z}(+E_x^2 - E_y^2 - E_z^2) + Z(+H_x^2 - H_y^2 - H_z^2)\right]$$

$$T_{yy} = \frac{1}{2c}\left[\frac{1}{Z}(-E_x^2 + E_y^2 - E_z^2) + Z(-H_x^2 + H_y^2 - H_z^2)\right]$$

$$T_{zz} = \frac{1}{2c}\left[\frac{1}{Z}(-E_x^2 - E_y^2 + E_z^2) + Z(-H_x^2 - H_y^2 + H_z^2)\right]$$

$$
\begin{aligned}
T_{xy} &= T_{yx} = \frac{1}{c}\left(\frac{1}{Z}E_xE_y + ZH_xH_y\right)\\
T_{xz} &= T_{zx} = \frac{1}{c}\left(\frac{1}{Z}E_xE_z + ZH_xH_z\right)\\
T_{yz} &= T_{zy} = \frac{1}{c}\left(\frac{1}{Z}E_yE_z + ZH_yH_z\right)
\end{aligned} \tag{42}
$$

For the x-component of the tensor divergence $\operatorname{Div} T$ we get:

$$
\begin{aligned}
\mathrm{Div_x}\, T &\equiv \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z}\\
&= \frac{1}{c}\left[\frac{1}{Z}\left(E_x\frac{\partial E_x}{\partial x} - E_y\frac{\partial E_y}{\partial x} - E_z\frac{\partial E_z}{\partial x}\right)\right.\\
&\qquad + Z\left(H_x\frac{\partial H_x}{\partial x} - H_y\frac{\partial H_y}{\partial x} - H_z\frac{\partial H_z}{\partial x}\right)\\
&\quad + \frac{1}{Z}\left(E_x\frac{\partial E_y}{\partial y} + E_y\frac{\partial E_x}{\partial y}\right) + Z\left(H_x\frac{\partial H_y}{\partial y} + H_y\frac{\partial H_x}{\partial y}\right)\\
&\quad \left. + \frac{1}{Z}\left(E_x\frac{\partial E_z}{\partial z} + E_z\frac{\partial E_x}{\partial z}\right) + Z\left(H_x\frac{\partial H_z}{\partial z} + H_z\frac{\partial H_x}{\partial z}\right)\right]\\
&= \frac{1}{c}\left[\frac{1}{Z}E_x \operatorname{div}\mathbf{E} + ZH_x \operatorname{div}\mathbf{H}\right.\\
&\qquad \left. - Z(\mathbf{H}\times\operatorname{curl}\mathbf{H})_x - \frac{1}{Z}(\mathbf{E}\times\operatorname{curl}\mathbf{E})_x\right]
\end{aligned} \tag{43}
$$

This is the x-component of the kernel on the right side of Eq.(41) multiplied by $1/c$. Applying Gauss' formula to Eq.(43)

$$
\int \mathrm{Div_x}\, T\, dV = \oint T_{x\mathrm{n}}\, dF \tag{44}
$$

we may write the x-component of Eq.(41) in the following form:

$$
\frac{d(u_x + G_x)}{dt} = c\oint T_{x\mathrm{n}}\, dF \tag{45}
$$

The right side of this equation represents the x-component of the momentum flowing through the surface of a certain volume into that volume. The left side represents the change with time of the x-components of the mechanical momentum $\mathbf{u}$ and the momentum $\mathbf{G}$ of the electromagnetic field.

We note that $\operatorname{div}\mathbf{H} = 0$ according to Eq.(1.6-4) will remove the term $Z\mathbf{H}\operatorname{div}\mathbf{H}$ from Eq.(41) and the term $ZH_x \operatorname{div}\mathbf{H}$ from Eq.(43). Hence, the conservation of momentum applies both for $\operatorname{div}\mathbf{H} = 0$ and $\operatorname{div}\mathbf{H} \neq 0$. The same independence from $\operatorname{div}\mathbf{H}$ holds true for Eq.(33); since $\operatorname{div}\mathbf{H}$ was not used for its derivation it makes no difference whether it is zero or not.

3 Hamiltonian Formalism

3.1 Undefined Potentials and Divergent Integrals

If we choose $\mathbf{A}_e = 0$ and $\phi_m = 0$ in Eqs.(1.6-17) and (1.6-11) we obtain the field strengths $\mathbf{E}$ and $\mathbf{H}$ from the potentials $\mathbf{A}_m$ and ϕ_e for the original Maxwell equations as previously shown by Eq.(1.6-35):

$$\mathbf{E} = -\frac{\partial \mathbf{A}_m}{\partial t} - \operatorname{grad} \phi_e \tag{1}$$

$$\mathbf{H} = \frac{c}{Z} \operatorname{curl} \mathbf{A}_m \tag{2}$$

We are going to show that $\mathbf{A}_m$ is generally not determined or is represented by a divergent integral. One will suspect that this is a sufficient cause for at least some of the divergencies that plague quantum field theory and that are sidestepped currently by means of renormalization[1].

Let the electric field strength $\mathbf{E}$ of Eq.(1) be linearly polarized and point in the direction z as shown in Fig.3.1-1. The magnetic field strength $\mathbf{H}$ of Eq.(2) shall point in the direction of x:

$$\mathbf{E} = E_E(\zeta, \theta)\mathbf{e}_z = E_E \mathbf{e}_z \tag{3}$$

$$\mathbf{H} = H_E(\zeta, \theta)\mathbf{e}_x = H_E \mathbf{e}_x \tag{4}$$

Equations (1) and (2) are rewritten in component form:

$$\frac{\partial A_{mx}}{\partial t} + \frac{\partial \phi_e}{\partial x} = 0 \tag{5}$$

$$\frac{\partial A_{my}}{\partial t} + \frac{\partial \phi_e}{\partial y} = 0 \tag{6}$$

$$\frac{\partial A_{mz}}{\partial t} + \frac{\partial \phi_e}{\partial z} = E_E \tag{7}$$

$$\frac{c}{Z}\left(\frac{\partial A_{mz}}{\partial y} - \frac{\partial A_{my}}{\partial z}\right) = H_E \tag{8}$$

[1]The results of this section were first mentioned on pages 2 and 6 of Harmuth and Lukin (2000).

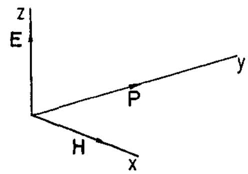

FIG.3.1-1. Orientation of the field strengths **E** and **H** of Eqs.(3) and (4)

$$\frac{c}{Z}\left(\frac{\partial A_{\mathrm{m}x}}{\partial z} - \frac{\partial A_{\mathrm{m}z}}{\partial x}\right) = 0 \tag{9}$$

$$\frac{c}{Z}\left(\frac{\partial A_{\mathrm{m}y}}{\partial x} - \frac{\partial A_{\mathrm{m}x}}{\partial y}\right) = 0 \tag{10}$$

Since **E** and **H** in Fig.3.1-1 represent a planar wave propagating in the direction y all derivatives with respect to x or z are zero:

$$\frac{\partial \phi_{\mathrm{e}}}{\partial x} = \frac{\partial \phi_{\mathrm{e}}}{\partial z} = \frac{\partial A_{\mathrm{m}x}}{\partial z} = \frac{\partial A_{\mathrm{m}y}}{\partial x} = \frac{\partial A_{\mathrm{m}y}}{\partial z} = \frac{\partial A_{\mathrm{m}z}}{\partial x} = 0 \tag{11}$$

Using the normalized notation of Eq.(1.3-7)

$$t = \frac{2\epsilon}{\sigma}\theta = \frac{2}{Zc\sigma}\theta, \qquad y = \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}\zeta = \frac{2}{Z\sigma}\zeta \tag{12}$$

we obtain:

$$\frac{\partial A_{\mathrm{m}x}}{\partial \theta} = 0 \tag{13}$$

$$\frac{Z\sigma}{2}\left(c\frac{\partial A_{\mathrm{m}y}}{\partial \theta} + \frac{\partial \phi_{\mathrm{e}}}{\partial \zeta}\right) = 0 \tag{14}$$

$$\frac{Zc\sigma}{2}\frac{\partial A_{\mathrm{m}z}}{\partial \theta} = E_{\mathrm{E}}(\zeta,\theta) \tag{15}$$

$$\frac{c\sigma}{2}\frac{\partial A_{\mathrm{m}z}}{\partial \zeta} = H_{\mathrm{E}}(\zeta,\theta) \tag{16}$$

$$\frac{c\sigma}{2}\frac{\partial A_{\mathrm{m}x}}{\partial \zeta} = 0 \tag{17}$$

We see from Eqs.(13) and (17) that $A_{\mathrm{m}x}$ is independent of ζ and θ, which means it is a constant. Furthermore, Eqs.(14)–(16) yield:

$$\phi_{\mathrm{e}} = -c\int \frac{\partial A_{\mathrm{m}y}}{\partial \theta} d\zeta \tag{18}$$

$$A_{\mathrm{m}z} = \frac{2}{Zc\sigma}\int E_{\mathrm{E}}(\zeta,\theta)d\theta \tag{19}$$

$$A_{\mathrm{m}z} = \frac{2}{c\sigma}\int H_{\mathrm{E}}(\zeta,\theta)d\zeta \tag{20}$$

In Eq.(18) we may choose any value we want for $A_{\mathrm{m}y}$ and get a defined value of ϕ_{e}. This freedom to choose reflects the fact that a variety of gauges for $\mathbf{A}_{\mathrm{m}}$ yield the same field strengths $\mathbf{E}$ and $\mathbf{H}$. In Eq.(19) we may substitute for $E_{\mathrm{E}}(\zeta,\theta)$ the function $E_{\mathrm{E}}^{(K)}$ of Eq.(1.3-17) and obtain a defined result for $A_{\mathrm{m}z}$ according to the plots of $E_{\mathrm{E}}^{(K)}$ in Figs.1.3-1 and 1.3-3. But for $H_{\mathrm{E}}(\zeta,\theta)$ in Eq.(20) we must substitute $H_{\mathrm{E}}(\zeta,\theta)$ of Eq.(6.2-51). This equation contains a divergent integral and leaves the magnetic field strength $H_{\mathrm{E}}(\zeta,\theta)$ undefined. Hence, Eqs.(19) and (20) contradict each other and Eqs.(1) and (2) must be wrong.

The divergence of Eq.(6.2-51) is discussed in detail in Section 6.2. It is due to the first integral of Eq.(6.2-51), which may be written for values of the variable η close to zero in the form

$$\int_0^\epsilon \frac{\cos\eta\zeta}{\eta^2}d\eta \approx \frac{1}{\eta}\bigg|_0^\epsilon, \quad \epsilon \ll 1$$

which shows the type of divergence.

This result holds so far only for excitation by an electric step function according to Eq.(1.3-14) or an electric current density with the same time variation according to Eq.(1.6-5). However, the result derived for a step function has been generalized in Section 1.4 in connection with Fig.1.4-2 and this generalization applies here too. Using series expansions in terms of time-shifted step functions we can obtain the electric and magnetic field strengths for electric excitation functions with any time variation that is of physical interest. It is possible that there are cases where Eqs.(19) and (20) do not contradict each other, but this must be shown rather than taken for granted. Of course, whenever the effect of magnetic dipole currents cannot be neglected one must use Eq.(1.1-9) with $\mathbf{g}_{\mathrm{m}} \neq 0$ and Eqs.(1), (2) are replaced by Eqs.(21), (22) below regardless of the relation between current densities $\mathbf{g}_{\mathrm{e}}$, $\mathbf{g}_{\mathrm{m}}$ and field strengths $\mathbf{E}$, $\mathbf{H}$.

It has been known at least since Weisskopf and Wigner (1930a, b) that the quantization of the electromagnetic field based on Eqs.(1) and (2) leads to divergent integrals but it was generally believed that the problem was caused by the quantization process rather than by Maxwell's equations[2]. We shall discuss this point in more detail from Section 3.4 on.

To see how the same calculation works with the potentials of the modified Maxwell equations we start with Eqs.(1.6-17) and (1.6-11):

$$\mathbf{E} = -Zc\,\mathrm{curl}\,\mathbf{A}_{\mathrm{e}} - \frac{\partial \mathbf{A}_{\mathrm{m}}}{\partial t} - \mathrm{grad}\,\phi_{\mathrm{e}} \tag{21}$$

$$\mathbf{H} = \frac{c}{Z}\,\mathrm{curl}\,\mathbf{A}_{\mathrm{m}} - \frac{\partial \mathbf{A}_{\mathrm{e}}}{\partial t} - \mathrm{grad}\,\phi_{\mathrm{m}} \tag{22}$$

[2]Weisskopf and Wigner (1930a) start on p. 59 with our Eqs.(1) and (2) for $\phi_{\mathrm{e}} = 0$ and arrive at an integral denoted (17) by them. In a footnote on p. 64 they point out that the integral diverges unless one limits the integration interval. But a skillful choice of the limits produces the desired result.

The equations are rewritten in component form using again the field strengths **E** and **H** of Eqs.(3) and (4):

$$-Zc\left(\frac{\partial A_{\mathrm{e}z}}{\partial y}-\frac{\partial A_{\mathrm{e}y}}{\partial z}\right)-\frac{\partial A_{\mathrm{m}x}}{\partial t}-\frac{\partial \phi_{\mathrm{e}}}{\partial x}=0 \tag{23}$$

$$-Zc\left(\frac{\partial A_{\mathrm{e}x}}{\partial z}-\frac{\partial A_{\mathrm{e}z}}{\partial x}\right)-\frac{\partial A_{\mathrm{m}y}}{\partial t}-\frac{\partial \phi_{\mathrm{e}}}{\partial y}=0 \tag{24}$$

$$-Zc\left(\frac{\partial A_{\mathrm{e}y}}{\partial x}-\frac{\partial A_{\mathrm{e}x}}{\partial y}\right)-\frac{\partial A_{\mathrm{m}z}}{\partial t}-\frac{\partial \phi_{\mathrm{e}}}{\partial z}=E_{\mathrm{E}} \tag{25}$$

$$\frac{c}{Z}\left(\frac{\partial A_{\mathrm{m}z}}{\partial y}-\frac{\partial A_{\mathrm{m}y}}{\partial z}\right)-\frac{\partial A_{\mathrm{e}x}}{\partial t}-\frac{\partial \phi_{\mathrm{m}}}{\partial x}=H_{\mathrm{E}} \tag{26}$$

$$\frac{c}{Z}\left(\frac{\partial A_{\mathrm{m}x}}{\partial z}-\frac{\partial A_{\mathrm{m}z}}{\partial x}\right)-\frac{\partial A_{\mathrm{e}y}}{\partial t}-\frac{\partial \phi_{\mathrm{m}}}{\partial y}=0 \tag{27}$$

$$\frac{c}{Z}\left(\frac{\partial A_{\mathrm{m}y}}{\partial x}-\frac{\partial A_{\mathrm{m}x}}{\partial y}\right)-\frac{\partial A_{\mathrm{e}z}}{\partial t}-\frac{\partial \phi_{\mathrm{m}}}{\partial z}=0 \tag{28}$$

The conditions of Eq.(11) for a planar wave propagating in the direction y are augmented for ϕ_{m} and $\mathbf{A}_{\mathrm{e}}$:

$$\frac{\partial \phi_{\mathrm{m}}}{\partial x}=\frac{\partial \phi_{\mathrm{m}}}{\partial z}=\frac{\partial A_{\mathrm{e}x}}{\partial z}=\frac{\partial A_{\mathrm{e}y}}{\partial x}=\frac{\partial A_{\mathrm{e}y}}{\partial z}=\frac{\partial A_{\mathrm{e}z}}{\partial x}=0 \tag{29}$$

Using the normalized variables of Eq.(12) we get:

$$Z\frac{\partial A_{\mathrm{e}z}}{\partial \zeta}+\frac{\partial A_{\mathrm{m}x}}{\partial \theta}=0 \tag{30}$$

$$c\frac{\partial A_{\mathrm{m}y}}{\partial \theta}+\frac{\partial \phi_{\mathrm{e}}}{\partial \zeta}=0 \tag{31}$$

$$\frac{Zc\sigma}{2}\left(Z\frac{\partial A_{\mathrm{e}x}}{\partial \zeta}-\frac{\partial A_{\mathrm{m}z}}{\partial \theta}\right)=E_{\mathrm{E}}(\zeta,\theta) \tag{32}$$

$$\frac{c\sigma}{2}\left(\frac{\partial A_{\mathrm{m}z}}{\partial \zeta}-Z\frac{\partial A_{\mathrm{e}x}}{\partial \theta}\right)=H_{\mathrm{E}}(\zeta,\theta) \tag{33}$$

$$c\frac{\partial A_{\mathrm{e}y}}{\partial \theta}+\frac{\partial \phi_{\mathrm{m}}}{\partial \zeta}=0 \tag{34}$$

$$\frac{\partial A_{\mathrm{m}x}}{\partial \zeta}+Z\frac{\partial A_{\mathrm{e}z}}{\partial \theta}=0 \tag{35}$$

Equations (31) and (34) yield ϕ_{e} and ϕ_{m} as functions of $A_{\mathrm{m}y}$ or $A_{\mathrm{e}y}$:

$$\phi_{\mathrm{e}}=-c\int\frac{\partial A_{\mathrm{m}y}}{\partial \theta}d\zeta \tag{36}$$

$$\phi_{\mathrm{m}}=-c\int\frac{\partial A_{\mathrm{e}y}}{\partial \theta}d\zeta \tag{37}$$

From Eqs.(30) and (35) one gets

$$A_{ez} = -\frac{1}{Z}\int \frac{\partial A_{mx}}{\partial\theta}d\zeta = -\frac{1}{Z}\int \frac{\partial A_{mx}}{\partial\zeta}d\theta \tag{38}$$

or

$$A_{mx} = -Z\int \frac{\partial A_{ez}}{\partial\zeta}d\theta = -Z\int \frac{\partial A_{ez}}{\partial\theta}d\zeta \tag{39}$$

Differentiation of Eq.(32) with respect to θ and of Eq.(33) with respect to ζ yields the one-dimensional, inhomogeneous wave equation for A_{mz}

$$\frac{\partial^2 A_{mz}}{\partial\zeta^2} - \frac{\partial^2 A_{mz}}{\partial\theta^2} = \frac{2}{Zc\sigma}\left(\frac{\partial E_E}{\partial\theta} + Z\frac{\partial H_E}{\partial\zeta}\right) \tag{40}$$

and the values of A_{ex} as function of A_{mz}:

$$A_{ex} = \frac{1}{Z}\int\left(\frac{\partial A_{mz}}{\partial\theta} + \frac{2}{Zc\sigma}E_E\right)d\zeta = \frac{1}{Z}\int\left(\frac{\partial A_{mz}}{\partial\zeta} - \frac{2}{c\sigma}H_E\right)d\theta \tag{41}$$

Instead of Eqs.(40) and (41) one may also derive the following relations from Eqs.(32) and (33):

$$\frac{\partial^2 A_{ex}}{\partial\zeta^2} - \frac{\partial^2 A_{ex}}{\partial\theta^2} = \frac{2}{Zc\sigma}\left(\frac{1}{Z}\frac{\partial E_E}{\partial\zeta} + \frac{\partial H_E}{\partial\theta}\right) \tag{42}$$

$$A_{mz} = Z\int\left(\frac{\partial A_{ex}}{\partial\zeta} + \frac{c\sigma}{2}E_E\right)d\theta = Z\int\left(\frac{\partial A_{ex}}{\partial\theta} + \frac{2}{Zc\sigma}H_E\right)d\zeta \tag{43}$$

Equations (40) and (42) are inhomogeneous wave equations with one spatial variable. Their solution is known[3]:

$$A_{mz}(\zeta,\theta) = -\frac{1}{Zc\sigma}\int_0^\theta\left[\int_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')}\left(\frac{\partial E_E(\zeta',\theta')}{\partial\theta'} + Z\frac{\partial H_E(\zeta',\theta')}{\partial\zeta'}\right)d\zeta'\right]d\theta' \tag{44}$$

$$A_{ex}(\zeta,\theta) = -\frac{1}{Zc\sigma}\int_0^\theta\left[\int_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')}\left(\frac{1}{Z}\frac{\partial E_E(\zeta',\theta')}{\partial\zeta'} + \frac{\partial H_E(\zeta',\theta')}{\partial\theta'}\right)d\zeta'\right]d\theta' \tag{45}$$

The notation $E_E(\zeta',\theta')$ and $H_E(\zeta',\theta')$ means that the variables ζ and θ of E_E in Eqs.(6.1-39), (6.4-29), (1.5-2) and for H_E in Eqs.(6.2-41), (6.5-15), (1.5-4) are replaced by ζ' and θ'.

[3] Smirnov 1961, vol. II, Cha. VII, § 1, Sec. 174, Eq. 95

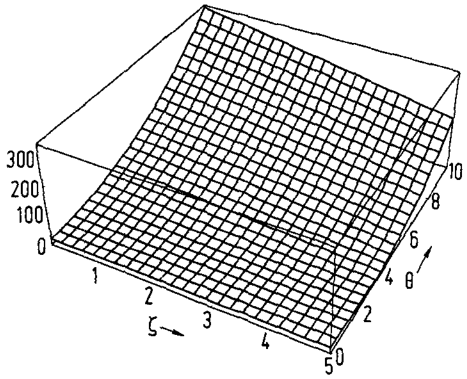

FIG.3.1-2. Plot of $Zc\sigma A_{mz}(\zeta,\theta)/E_1$ according to Eq.(46) for $\omega = 0.01$ in the interval $0 < \zeta < 5$, $0 < \theta < 10$.

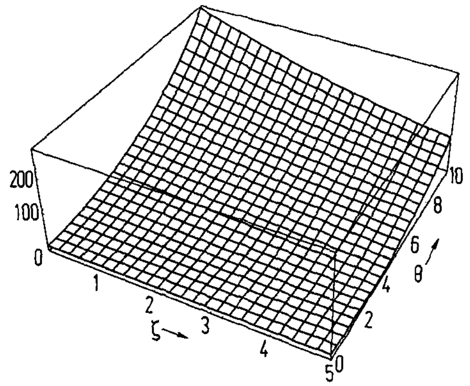

FIG.3.1-3. Plot of $Zc\sigma A_{mz}(\zeta,\theta)/E_1$ according to Eq.(46) for $\omega = 0.1$ in the interval $0 < \zeta < 5$, $0 < \theta < 10$.

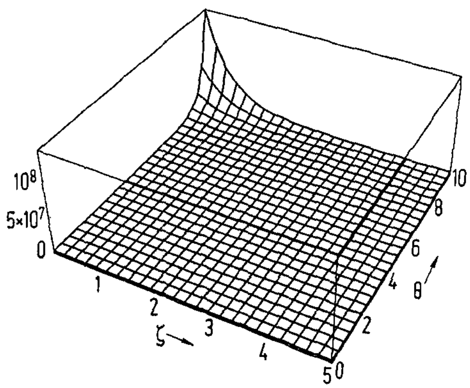

FIG.3.1-4. Plot of $Zc\sigma A_{mz}(\zeta,\theta)/E_1$ according to Eq.(46) for $\omega = 0.99$ in the interval $0 < \zeta < 5$, $0 < \theta < 10$.

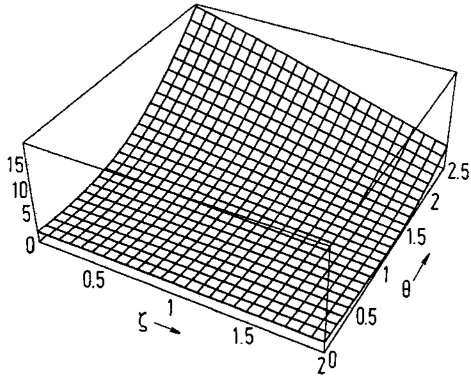

FIG.3.1-5. Plot of $Zc\sigma A_{mz}(\zeta,\theta)/E_1$ according to Eq.(46) for $\omega = 0.01$ in the interval $0 < \zeta < 2$, $0 < \theta < 2.5$.

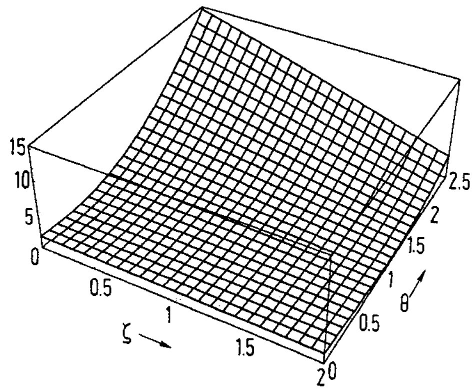

FIG.3.1-6. Plot of $Zc\sigma A_{mz}(\zeta,\theta)/E_1$ according to Eq.(46) for $\omega = 0.1$ in the interval $0 < \zeta < 2$, $0 < \theta < 2.5$.

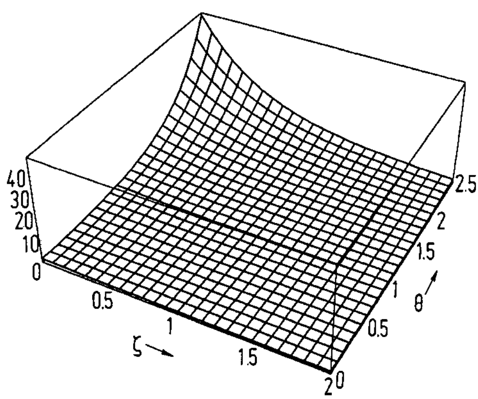

FIG.3.1-7. Plot of $Zc\sigma A_{mz}(\zeta,\theta)/E_1$ according to Eq.(46) for $\omega = 0.99$ in the interval $0 < \zeta < 2$, $0 < \theta < 2.5$.

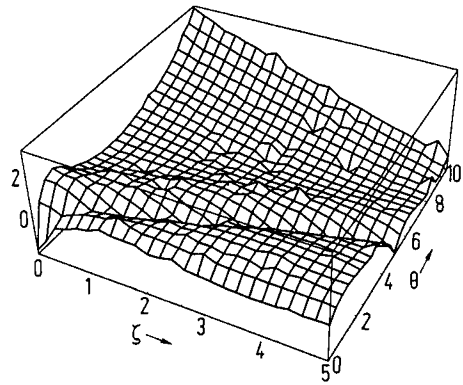

FIG.3.1-8. Plot of $Z^2 c\sigma A_{ez}(\zeta,\theta)/E_1$ according to Eq.(47) for $\omega = 0.05$ in the interval $0 < \zeta < 5$, $0 < \theta < 10$.

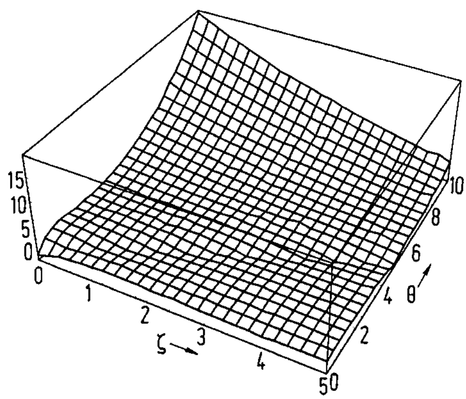

FIG.3.1-9. Plot of $Z^2 c\sigma A_{ez}(\zeta,\theta)/E_1$ according to Eq.(47) for $\omega = 0.1$ in the interval $0 < \zeta < 5$, $0 < \theta < 10$.

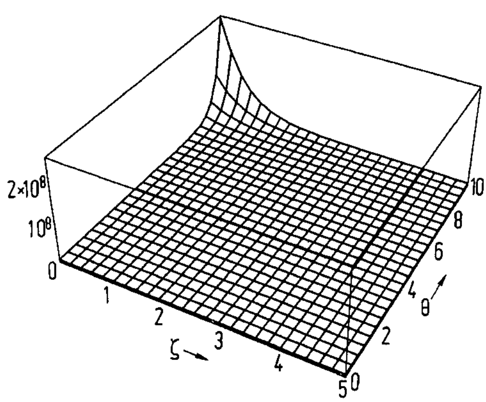

FIG.3.1-10. Plot of $Z^2 c\sigma A_{ez}(\zeta,\theta)/E_1$ according to Eq.(47) for $\omega = 0.99$ in the interval $0 < \zeta < 5$, $0 < \theta < 10$.

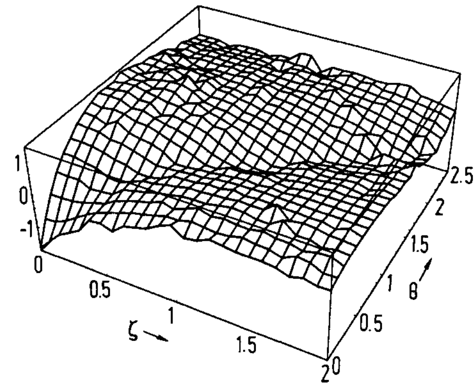

FIG.3.1-11. Plot of $Z^2 c\sigma A_{ez}(\zeta,\theta)/E_1$ according to Eq.(47) for $\omega = 0.05$ in the interval $0 < \zeta < 2$, $0 < \theta < 2.5$.

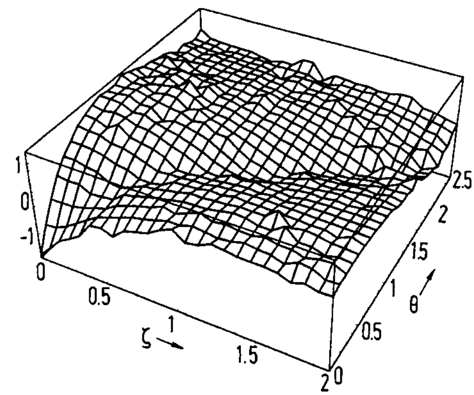

FIG.3.1-12. Plot of $Z^2 c\sigma A_{ez}(\zeta,\theta)/E_1$ according to Eq.(47) for $\omega = 0.1$ in the interval $0 < \zeta < 2$, $0 < \theta < 2.5$.

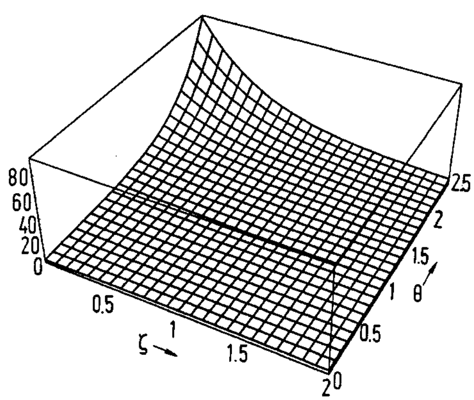

FIG.3.1-13. Plot of $Z^2 c\sigma A_{ez}(\zeta,\theta)/E_1$ according to Eq.(47) for $\omega = 0.99$ in the interval $0 < \zeta < 2$, $0 < \theta < 2.5$.

We may now recognize why the electric field strength of Figs.1.3-1 and 1.4-1 as well as the associated magnetic field strength of Figs.1.4-4 and 1.4-6 were not satisfactory. The steps or jumps at $\theta = \zeta$ yield undefined derivatives which create a problem in Eqs.(44) and (45). The field strength due to exponential ramp function excitation in Figs.1.5-1 to 1.5-4 have defined derivatives $\partial E_E/\partial\theta$, $\partial E_E/d\zeta$ and $\partial H_E/\partial\theta$, $\partial H_E/\partial\zeta$ for all values of ζ and θ in the interval $0 \leq \zeta < \infty$, $0 \leq \theta < \infty$. These field strengths can be substituted into Eqs.(44) and (45). The evaluation offers not particular mathematical difficulties but it is very laborious. For this reason it is carried out in Sections 6.6 and 6.7.

The two integrations of Eq.(44) can be done analytically and the normalized component $A_{mz}(\zeta,\theta)$ of the potential $\mathbf{A}_m$ can be brought into the form of Eq.(6.6-27):

$$\frac{Zc\sigma}{E_1}A_{mz}(\zeta,\theta) = -\frac{e^{-2\omega\zeta}}{\omega}\left(\frac{1-\operatorname{ch}2\omega\theta}{\omega} - \frac{\omega^3}{(1+\omega^2)^2-\omega^2}e^{-2(1+\omega^2)\theta} + \frac{(1+\omega^2)\operatorname{sh}2\omega\theta - \omega\operatorname{ch}2\omega\theta}{(1+\omega^2)^2-\omega^2}\right)$$
$$-\frac{16(1+\omega^2)}{\pi}\left[\int_0^{1-\omega^2}\frac{\sin\theta\eta}{\omega_1(\eta^2+4\omega^2)}(L_{21}\sin\zeta\eta - L_{22}\cos\zeta\eta)d\eta + \int_{1-\omega^2}^{\infty}\frac{\sin\theta\eta}{\omega_2(\eta^2+4\omega^2)}(L_{25}\sin\zeta\eta - L_{26}\cos\zeta\eta)d\eta\right]$$

$$\omega_1 = \left[(1-\omega^2)^2-\eta^2\right]^{1/2},\quad \omega_2 = \left[\eta^2-(1-\omega^2)^2\right]^{1/2},\quad \omega^2 = \epsilon s/\mu\sigma \qquad (46)$$

The functions $L_{21} = L_{21}(\zeta,\theta)$, L_{22} , L_{25}, and L_{26} are defined by Eqs.(6.6-15), (6.6-16), (6.6-19), and (6.6-20).

Equation (45) for the component $A_{ex}(\zeta,\theta)$ of the potential $\mathbf{A}_e$ can be brought into the form of Eq.(6.7-18):

$$\frac{Z^2c\sigma}{E_1}A_{ex}(\zeta,\theta) = -e^{-2\omega\zeta}\left\{\frac{1}{\omega}(1-\operatorname{ch}2\omega\theta) + \frac{1}{(1+\omega^2)^2-\omega^2}\times\left[[(1+\omega^2)^2+\omega](e^{-2(1+\omega^2)\theta}-\operatorname{ch}2\omega\theta) + (1+\omega^2)(1+\omega)\operatorname{sh}2\omega\theta\right]\right\}$$
$$+\frac{8(1+\omega^2)}{\pi}\left\{\int_0^{1-\omega^2}\frac{\cos\zeta\eta}{\eta^2+4\omega^2}\left[\left(\frac{\eta^2-1+\omega^4+\omega_1^2}{\eta\omega_1}L_{21} - \frac{2\omega^2}{\eta}L_{23}\right)\sin\theta\eta - \left(\frac{\eta^2-1+\omega^4+\omega_1^2}{\eta\omega_1}L_{22} - \frac{2\omega^2}{\eta}L_{24}\right)\cos\theta\eta\right]d\eta\right.$$
$$+\int_{1-\omega^2}^{\infty}\frac{\cos\zeta\eta}{\eta^2+4\omega^2}\left[\left(\frac{\eta^2-1+\omega^4-\omega_2^2}{\eta\omega_2}L_{25} - \frac{2\omega^2}{\eta}L_{27}\right)\sin\theta\eta\right.$$

$$- \left(\frac{\eta^2 - 1 + \omega^4 - \omega_2^2}{\eta\omega_2} L_{26} - \frac{2\omega^2}{\eta} L_{28} \right) \cos\theta\eta \Bigg] d\eta \Bigg\}$$

$$\omega_1 = \left[(1-\omega^2)^2 - \eta^2\right]^{1/2}, \quad \omega_2 = \left[\eta^2 - (1-\omega^2)^2\right]^{1/2}, \quad \omega^2 = \epsilon s/\mu\sigma \qquad (47)$$

The function L_{21} to L_{28} are defined by Eqs.(6.6-15)–(6.6-22).

We still have to show that Eqs.(46) and (47) are actual solutions rather than formal solutions. This can be done most convincingly by producing plots. Figures 3.1-2 to 3.1-4 show three-dimensional plots of $Zc\sigma A_{mz}(\zeta, \theta)/E_1$ according to Eq.(46) for $\omega = \sqrt{s\epsilon/\sigma\mu} = 0.01$, 0.1, and 0.99 in the interval $0 \leq \zeta \leq 5$, $0 \leq \theta \leq 10$. We note that the peak amplitude in Fig.3.1-3 is smaller than the peak amplitudes in either Fig.3.1-2 or 3.1-4. Figures 3.1-5 to 3.1-7 show the same function with the same values of ω but in the smaller interval $0 \leq \zeta \leq 2$, $0 \leq \theta \leq 2.5$. Again the peak amplitude in Fig.3.1-6 is smaller than in Figs.3.1-5 or 3.1-7.

Figures 3.1-8 to 3.10 show $Z^2 c\sigma A_{ex}(\zeta, \theta)/E_1$ according to Eq.(47) for $\omega =$ 0.05, 0.1, and 0.99. The value of ω in Fig.3.1-8 was increased from 0.01 to 0.05 since the function varies rapidly for small values of ω. Figures 3.1-11 to 3.1-13 show again $Z^2 c\sigma A_{ex}(\zeta, \theta)$ for $\omega = 0.05$, 0.1, and 0.99 but in the smaller interval $0 \leq \zeta \leq 2$, $0 \leq \theta \leq 2.5$.

The derived results apply to the field strengths of Eqs.(3) and (4) excited by the exponential ramp function of Eq.(1.5-1) only. However, the purpose of this section is strictly to give a convincing example that Eqs.(21) and (22) rather than Eqs.(1) and (2) should be used. There is no need to demonstrate the general proof of Section 1.4 with more examples, particularly since we have the independent and different proof of Hillion mentioned in the third paragraph from the end of Section 1.4.

The smoothness of the plots of Figs.3.1-2 to 3.1-7, 3.1-9, 3.1-10, and 3.1-13 implies that the derivatives of the potentials with respect to ζ and θ exist and that the field strengths according to Eqs.(32) and (33) can indeed be derived from the potentials. Figures 3.1-8, 3.1-11, and 3.1-12 are not smooth due to the rounding errors of the computation for small values of ω, ζ, and θ.

3.2 Charged Particle in an Electromagnetic Field

We start with the Lorentz equation (2.1-3) and use the force of inertia $d(m\mathbf{v})/dt$ rather than $md\mathbf{v}/dt$ as in Eq.(2.1-4):

$$\frac{d}{dt}(m\mathbf{v}) = e\mathbf{E} + \frac{Ze}{c}\mathbf{v} \times \mathbf{H} \qquad (1)$$

The field strengths $\mathbf{E}$ and $\mathbf{H}$ are written in the potential form of Eqs.(1.6-17) and (1.6-11):

$$\mathbf{E} = -Zc \operatorname{curl} \mathbf{A}_e - \frac{\partial \mathbf{A}_m}{\partial t} - \operatorname{grad} \phi_e \qquad (2)$$

$$\mathbf{H} = \frac{c}{Z} \operatorname{curl} \mathbf{A}_m - \frac{\partial \mathbf{A}_e}{\partial t} - \operatorname{grad} \phi_m \qquad (3)$$

Substitution of **E** and **H** into Eq.(1) yields:

$$\frac{d}{dt}(m\mathbf{v}) = e(-\operatorname{grad}\phi_{\mathrm{e}} - \frac{\partial \mathbf{A}_{\mathrm{m}}}{\partial t} + \mathbf{v}\times\operatorname{curl}\mathbf{A}_{\mathrm{m}}) \\ - \frac{Ze}{c}(\mathbf{v}\times\operatorname{grad}\phi_{\mathrm{m}} + \mathbf{v}\times\frac{\partial \mathbf{A}_{\mathrm{e}}}{\partial t} + c^2\operatorname{curl}\mathbf{A}_{\mathrm{e}}) \quad (4)$$

The first line is the conventional equation of motion of an electrically charged particle in an EM field. The second line contains additional terms due to magnetic currents ($\mathbf{A}_{\mathrm{e}}$) and hypothetical magnetic charges (ϕ_{m}).

We follow the usual course in rewriting Eq.(4) with the help of a Lagrange function L, spatial variables $x_j = x_1,\ x_2,\ x_3 = x,\ y,\ z$ and velocity variables $\partial x_j/\partial t = \dot{x}_j$. The Lagrange function must satisfy the Euler equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) = \frac{\partial L}{\partial x_j}, \quad j = 1,\ 2,\ 3; \quad x_1 = x,\ x_2 = y,\ x_3 = z \quad (5)$$

which derive the equation of motion from the principle of least action by Maupertuis and Hamilton.

In order to get the terms $\partial\mathbf{A}_{\mathrm{m}}/\partial t$ and $\mathbf{v}\times\partial\mathbf{A}_{\mathrm{e}}/\partial t$ in Eq.(4) to the left side we write:

$$\frac{d\mathbf{A}_{\mathrm{m}}}{dt} = \frac{\partial \mathbf{A}_{\mathrm{m}}}{\partial t} + \frac{\partial \mathbf{A}_{\mathrm{m}}}{\partial x}\dot{x} + \frac{\partial \mathbf{A}_{\mathrm{m}}}{\partial y}\dot{y} + \frac{\partial \mathbf{A}_{\mathrm{m}}}{\partial z}\dot{z} \quad (6)$$

$$\frac{d}{dt}(\mathbf{v}\times\mathbf{A}_{\mathrm{e}}) = \frac{d\mathbf{v}}{dt}\times\mathbf{A}_{\mathrm{e}} + \mathbf{v}\times\frac{d\mathbf{A}_{\mathrm{e}}}{dt} \\ = \frac{\partial\mathbf{v}}{\partial t}\times\mathbf{A}_{\mathrm{e}} + \mathbf{v}\times\frac{\partial \mathbf{A}_{\mathrm{e}}}{\partial t} + \left(\dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z}\right)(\mathbf{v}\times\mathbf{A}_{\mathrm{e}}) \quad (7)$$

Substitution of $\partial\mathbf{A}_{\mathrm{m}}/\partial t$ and $\mathbf{v}\times\partial\mathbf{A}_{\mathrm{e}}/\partial t$ into Eq.(4) brings:

$$\frac{d}{dt}(m\mathbf{v} + e\mathbf{A}_{\mathrm{m}}) + \frac{Ze}{c}\frac{d}{dt}(\mathbf{v}\times\mathbf{A}_{\mathrm{e}}) \\ = e\left(-\operatorname{grad}\phi_{\mathrm{e}} + \frac{\partial \mathbf{A}_{\mathrm{m}}}{\partial x}\dot{x} + \frac{\partial \mathbf{A}_{\mathrm{m}}}{\partial y}\dot{y} + \frac{\partial \mathbf{A}_{\mathrm{m}}}{\partial z}\dot{z} + \mathbf{v}\times\operatorname{curl}\mathbf{A}_{\mathrm{m}}\right) \\ + \frac{Ze}{c}\left[-\mathbf{v}\times\operatorname{grad}\phi_{\mathrm{m}} + \frac{\partial\mathbf{v}}{\partial t}\times\mathbf{A}_{\mathrm{e}} + \left(\dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z}\right)(\mathbf{v}\times\mathbf{A}_{\mathrm{e}}) \right. \\ \left. - c^2\operatorname{curl}\mathbf{A}_{\mathrm{e}}\right] \quad (8)$$

We break Eq.(8) into two equations. The first is the same as obtained from the classical Maxwell equations, the second contains the terms due to the modification of Maxwell's equations:

$$\frac{d}{dt}(m\mathbf{v}+e\mathbf{A}_\mathrm{m}) = e\bigg(-\operatorname{grad}\phi_\mathrm{e} + \frac{\partial \mathbf{A}_\mathrm{m}}{\partial x}\dot{x} + \frac{\partial \mathbf{A}_\mathrm{m}}{\partial y}\dot{y} + \frac{\partial \mathbf{A}_\mathrm{m}}{\partial z}\dot{z} + \mathbf{v}\times\operatorname{curl}\mathbf{A}_\mathrm{m}\bigg) \tag{9}$$

$$\frac{Ze}{c}\frac{d}{dt}(\mathbf{v}\times\mathbf{A}_\mathrm{e}) = \frac{Ze}{c}\bigg[-\mathbf{v}\times\operatorname{grad}\phi_\mathrm{m} + \frac{\partial\mathbf{v}}{\partial t}\times\mathbf{A}_\mathrm{e} + \left(\dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z}\right)(\mathbf{v}\times\mathbf{A}_\mathrm{e}) - c^2\operatorname{curl}\mathbf{A}_\mathrm{e}\bigg] \tag{10}$$

Writing $\operatorname{grad}\phi_\mathrm{e}$ and $\mathbf{v}\times\operatorname{curl}\mathbf{A}_\mathrm{m}$ in component form in a Cartesian coordinate system yields the following well-known x-component of Eq.(9):

$$\frac{d}{dt}(m\dot{x}+eA_{\mathrm{m}x}) = e\frac{\partial}{\partial x}(-\phi_\mathrm{e} + A_{\mathrm{m}x}\dot{x} + A_{\mathrm{m}y}\dot{y} + A_{\mathrm{m}z}\dot{z}) \tag{11}$$

The y- and z-components follow with the substitutions $x\to y\to z\to x$. These substitutions leave the term in parenthesis on the right side of Eq.(11) unchanged.

The Lagrange function $\mathfrak{L}_\mathrm{M}$ for Eq.(11) is found in many textbooks; the subscript M refers to 'Maxwell':

$$\mathfrak{L}_\mathrm{M} = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) + e(-\phi_\mathrm{e} + A_{\mathrm{m}x}\dot{x} + A_{\mathrm{m}y}\dot{y} + A_{\mathrm{m}z}\dot{z}) \tag{12}$$

Differentiation of $\mathfrak{L}_\mathrm{M}$ with respect to $\dot{x}$ yields

$$\frac{\partial\mathfrak{L}_\mathrm{M}}{\partial\dot{x}} = m\dot{x} + eA_{\mathrm{m}x} \tag{13}$$

and the substitution of $\mathfrak{L}_\mathrm{m}$ for L in Eq.(5) yields Eq.(11).

Next we write $\mathbf{v}\times\mathbf{A}_\mathrm{e}$, $\operatorname{grad}\phi_\mathrm{m}$, $(\partial\mathbf{v}/\partial t)\times\mathbf{A}_\mathrm{e}$, and $\operatorname{curl}\mathbf{A}$ of Eq.(10) in component form. The x-component of Eq.(10) becomes:

$$\frac{Ze}{c}\frac{d}{dt}(A_{\mathrm{e}z}\dot{y} - A_{\mathrm{e}y}\dot{z}) = \frac{Ze}{c}\frac{\partial}{\partial x}\bigg\{(A_{\mathrm{e}z}\dot{y} - A_{\mathrm{e}y}\dot{z})\dot{x} + \int\bigg[-\left(\frac{\partial\phi_\mathrm{m}}{\partial z}\dot{y} - \frac{\partial\phi_\mathrm{m}}{\partial y}\dot{z}\right) + A_{\mathrm{e}z}\ddot{y} - A_{\mathrm{e}y}\ddot{z} - c^2\left(\frac{\partial A_{\mathrm{e}z}}{\partial y} - \frac{\partial A_{\mathrm{e}y}}{\partial z}\right) + \left(\dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z}\right)(A_{\mathrm{e}z}\dot{y} - A_{\mathrm{e}y}\dot{z})\bigg]dx\bigg\} \tag{14}$$

As before, the substitutions $x\to y\to z\to x$ yield he y- and z-components.

We try the following Lagrange function $\mathfrak{L}_{\mathrm{c}x}$ for Eq.(14); the subscript c refers to 'correction':

$$\begin{aligned}
\mathfrak{L}_{cx} &= \frac{Ze}{c}\Bigg\{(A_{ez}\dot{y} - A_{ey}\dot{z})\dot{x} + \int\Bigg[-\left(\frac{\partial\phi_m}{\partial z}\dot{y} - \frac{\partial\phi_m}{\partial y}\dot{z}\right) + A_{ez}\ddot{y} - A_{ey}\ddot{z} \\
&\qquad - c^2\left(\frac{\partial A_{ez}}{\partial y} - \frac{\partial A_{ey}}{\partial z}\right) + \left(\dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z}\right)(A_{ez}\dot{y} - A_{ey}\dot{z})\Bigg]dx\Bigg\} \\
&= \frac{Ze}{c}\Bigg\{(\dot{\mathbf{r}} \times \mathbf{A}_e)_x\dot{x} + \int\Bigg[-\left(\dot{\mathbf{r}} \times \frac{\partial\phi_m}{\partial\mathbf{r}}\right)_x + (\ddot{\mathbf{r}} \times \mathbf{A}_e)_x - c^2(\operatorname{curl}\mathbf{A}_e)_x \\
&\qquad + \left(\dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z}\right)(\dot{\mathbf{r}} \times \mathbf{A}_e)_x\Bigg]dx\Bigg\} \\
&\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z
\end{aligned} \tag{15}$$

Differentiation of $\mathfrak{L}_{cx}$ with respect to $\dot{x}$ yields

$$\frac{\partial\mathfrak{L}_{cx}}{\partial\dot{x}} = \frac{Ze}{c}(A_{ez}\dot{y} - A_{ey}\dot{z}) \tag{16}$$

and the substitution of $\mathfrak{L}_{cx}$ for L in Eq.(5) yields Eq.(14).

In the case of the Lagrange function $\mathfrak{L}_M$ of Eq.(12) we could differentiate with respect to $\dot{x}$, $\dot{y}$, or $\dot{z}$ to satisfy Euler's equation (5). This is no longer so. If we want to satisfy Eq.(5) for $x_2 = y$ and $x_3 = z$ we must replace Eq.(15) by $\mathfrak{L}_{cy}$ and $\mathfrak{L}_{cz}$. These functions are obtained from $\mathfrak{L}_{cx}$ by the substitutions $x \to y \to z \to x$. We write them here explicitly due to their fundamental importance:

$$\begin{aligned}
\mathfrak{L}_{cy} &= \frac{Ze}{c}\Bigg\{(A_{ex}\dot{z} - A_{ez}\dot{x})\dot{y} + \int\Bigg[-\left(\frac{\partial\phi_m}{\partial x}\dot{z} - \frac{\partial\phi_m}{\partial z}\dot{x}\right) + A_{ex}\ddot{z} - A_{ez}\ddot{x} \\
&\qquad - c^2\left(\frac{\partial A_{ex}}{\partial z} - \frac{\partial A_{ez}}{\partial x}\right) + \left(\dot{z}\frac{\partial}{\partial z} + \dot{x}\frac{\partial}{\partial x}\right)(A_{ex}\dot{z} - A_{ez}\dot{x})\Bigg]dy\Bigg\} \\
&= \frac{Ze}{c}\Bigg\{(\dot{\mathbf{r}} \times \mathbf{A}_e)_y\dot{y} + \int\Bigg[-\left(\dot{\mathbf{r}} \times \frac{\partial\phi_m}{\partial\mathbf{r}}\right)_y + (\ddot{\mathbf{r}} \times \mathbf{A}_e)_y - c^2(\operatorname{curl}\mathbf{A}_e)_y \\
&\qquad + \left(\dot{z}\frac{\partial}{\partial z} + \dot{x}\frac{\partial}{\partial x}\right)(\dot{\mathbf{r}} \times \mathbf{A}_e)_y\Bigg]dy\Bigg\}
\end{aligned} \tag{17}$$

$$\begin{aligned}
\mathfrak{L}_{cz} &= \frac{Ze}{c}\Bigg\{(A_{ey}\dot{x} - A_{ex}\dot{y})\dot{z} + \int\Bigg[-\left(\frac{\partial\phi_m}{\partial y}\dot{x} - \frac{\partial\phi_m}{\partial x}\dot{y}\right) + A_{ey}\ddot{x} - A_{ex}\ddot{y} \\
&\qquad - c^2\left(\frac{\partial A_{ey}}{\partial x} - \frac{\partial A_{ex}}{\partial y}\right) + \left(\dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y}\right)(A_{ey}\dot{x} - A_{ex}\dot{y})\Bigg]dz\Bigg\} \\
&= \frac{Ze}{c}\Bigg\{(\dot{\mathbf{r}} \times \mathbf{A}_e)_z\dot{z} + \int\Bigg[-\left(\dot{\mathbf{r}} \times \frac{\partial\phi_m}{\partial\mathbf{r}}\right)_z + (\ddot{\mathbf{r}} \times \mathbf{A}_e)_z - c^2(\operatorname{curl}\mathbf{A}_e)_z \\
&\qquad \left(\dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y}\right)(\dot{\mathbf{r}} \times \mathbf{A}_e)_z\Bigg]dz\Bigg\}
\end{aligned} \tag{18}$$

Let us observe that we added one term $\mathbf{g}_{\mathrm{m}}$ in Eq.(1.1-9) to change the Maxwell equations to the modified Maxwell equations. But Eqs.(15), (17), and (18) written in vector form like Eqs.(1.1-8) to (1.1-11) add six terms to the conventional Lagrange function of Eq.(12). This increase of the number of terms is characteristic for the use of the modified Maxwell equations in quantum electrodynamics.

For an understanding of the Lagrange function consider a spinning bullet shot upwards. The kinetic energy will be transformed into potential energy on the way up and back to kinetic energy on the way down. Such transformable energies are represented by the Lagrange function $\mathcal{L}_{\mathrm{M}}$ of Eq.(12). The rotational energy of the spinning bullet will not be transformed into either kinetic or potential energy. Since the axis of the spin does not change, the energies of the x-, y-, and z-components of the rotational energy will not be transformed either. Such non-transformable energies are represented by the Lagrange function with the components $\mathcal{L}_{cx}$, $\mathcal{L}_{cy}$, and $\mathcal{L}_{cz}$ of Eqs.(15), (17), and (18). One must be careful not to read too much into this analogy of a spinning bullet. The term $\operatorname{curl}\mathbf{A}_{\mathrm{e}}$ in Eq.(10) represents a rotation but it is not evident why all the other terms should be interpreted as rotations. At this time it is best to think of transformable and non-transformable energy and avoid any more detailed interpretation.

We may write the Lagrange function for Eqs.(4) or (8) as a vector $\boldsymbol{\mathcal{L}}$ that is the sum of two vectors $\boldsymbol{\mathcal{L}}_{\mathrm{M}}$ and $\boldsymbol{\mathcal{L}}_{\mathrm{c}}$

$$\boldsymbol{\mathcal{L}} = \boldsymbol{\mathcal{L}}_{\mathrm{M}} + \boldsymbol{\mathcal{L}}_{\mathrm{c}}$$
$$\mathcal{L}_k = \mathcal{L}_{\mathrm{M}} + \mathcal{L}_{ck}, \quad k = x,\ y,\ z \tag{19}$$

with the components $\mathcal{L}_k$, $k = 1, 2, 3$ or $\mathcal{L}_x$, $\mathcal{L}_y$, $\mathcal{L}_z$. The three components $\mathcal{L}_{cx}$, $\mathcal{L}_{cy}$, $\mathcal{L}_{cz}$ of $\boldsymbol{\mathcal{L}}_{\mathrm{c}}$ are defined by Eqs.(15), (17), and (18), while all three components of $\boldsymbol{\mathcal{L}}_{\mathrm{M}}$ are defined by Eq.(12).

We shall need relations between the moments p_x, p_y, p_z and the variables $\dot{x}$, $\dot{y}$, $\dot{z}$. The moments are the derivatives of the components $\mathcal{L}_j = \mathcal{L}_x$, $\mathcal{L}_y$, $\mathcal{L}_z$ with respect to $\dot{x}_j = \dot{x}$, $\dot{y}$, $\dot{z}$. From Eqs.(12) and (15) we get:

$$p_x = \frac{\partial \mathcal{L}_x}{\partial \dot{x}} = \frac{\partial(\mathcal{L}_{\mathrm{M}} + \mathcal{L}_{cx})}{\partial \dot{x}} = m\dot{x} + eA_{\mathrm{m}x} + \frac{Ze}{c}(A_{\mathrm{e}z}\dot{y} - A_{\mathrm{e}y}\dot{z}) \tag{20}$$

$$p_y = \frac{\partial \mathcal{L}_y}{\partial \dot{y}} = m\dot{y} + eA_{\mathrm{m}y} + \frac{Ze}{c}(A_{\mathrm{e}x}\dot{z} - A_{\mathrm{e}z}\dot{x}) \tag{21}$$

$$p_z = \frac{\partial \mathcal{L}_z}{\partial \dot{z}} = m\dot{z} + eA_{\mathrm{m}z} + \frac{Ze}{c}(A_{\mathrm{e}y}\dot{x} - A_{\mathrm{e}x}\dot{y}) \tag{22}$$

In order to obtain $\dot{x}$, $\dot{y}$, $\dot{z}$ as functions of p_x, p_y, p_z we must solve these equations for $\dot{x}$, $\dot{y}$, $\dot{z}$. First we define a common denominator D:

$$D = m\left[\left(\frac{Ze}{c}\right)^2 (A_{\mathrm{e}x}^2 + A_{\mathrm{e}y}^2 + A_{\mathrm{e}z}^2) + m^2\right] = m\left[m^2 + \left(\frac{Ze}{c}\mathbf{A}_{\mathrm{e}}\right)^2\right] \tag{23}$$

Then we write the solutions for $D\dot{x}$, $D\dot{y}$, $D\dot{z}$:

$$\begin{aligned}
D\dot{x} = {} & \left(\frac{Ze}{c}\right)^2 A_{\mathrm{e}x}[A_{\mathrm{e}x}(p_x - eA_{\mathrm{m}x}) + A_{\mathrm{e}y}(p_y - eA_{\mathrm{m}y}) + A_{\mathrm{e}z}(p_z - eA_{\mathrm{m}z})] \\
& + \frac{Zem}{c}[A_{\mathrm{e}y}(p_z - eA_{\mathrm{m}z}) - A_{\mathrm{e}z}(p_y - eA_{\mathrm{m}y})] + m^2(p_x - eA_{\mathrm{m}x}) \\
= {} & m^2(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_x + \left(\frac{Ze}{c}\right)^2 A_{\mathrm{e}x}\mathbf{A}_{\mathrm{e}} \cdot (\mathbf{p} - e\mathbf{A}_{\mathrm{m}}) \\
& + \frac{Zem}{c}[\mathbf{A}_{\mathrm{e}} \times (\mathbf{p} - e\mathbf{A}_{\mathrm{m}})]_x \qquad (24)
\end{aligned}$$

$$\begin{aligned}
D\dot{y} = {} & \left(\frac{Ze}{c}\right)^2 A_{\mathrm{e}y}[A_{\mathrm{e}x}(p_x - eA_{\mathrm{m}x}) + A_{\mathrm{e}y}(p_y - eA_{\mathrm{m}y}) + A_{\mathrm{e}z}(p_z - eA_{\mathrm{m}z})] \\
& + \frac{Zem}{c}[A_{\mathrm{e}z}(p_x - eA_{\mathrm{m}x}) - A_{\mathrm{e}x}(p_z - eA_{\mathrm{m}z})] + m^2(p_y - eA_{\mathrm{m}y}) \\
= {} & m^2(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_y + \left(\frac{Ze}{c}\right)^2 A_{\mathrm{e}y}\mathbf{A}_{\mathrm{e}} \cdot (\mathbf{p} - e\mathbf{A}_{\mathrm{m}}) \\
& + \frac{Zem}{c}[\mathbf{A}_{\mathrm{e}} \times (\mathbf{p} - e\mathbf{A}_{\mathrm{m}})]_y \qquad (25)
\end{aligned}$$

$$\begin{aligned}
D\dot{z} = {} & \left(\frac{Ze}{c}\right)^2 A_{\mathrm{e}z}[A_{\mathrm{e}x}(p_x - eA_{\mathrm{m}x}) + A_{\mathrm{e}y}(p_y - eA_{\mathrm{m}y}) + A_{\mathrm{e}z}(p_z - eA_{\mathrm{m}z})] \\
& + \frac{Zem}{c}[A_{\mathrm{e}x}(p_y - eA_{\mathrm{m}y}) - A_{\mathrm{e}y}(p_x - eA_{\mathrm{m}x})] + m^2(p_z - eA_{\mathrm{m}z}) \\
= {} & m^2(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_z + \left(\frac{Ze}{c}\right)^2 A_{\mathrm{e}z}\mathbf{A}_{\mathrm{e}} \cdot (\mathbf{p} - e\mathbf{A}_{\mathrm{m}}) \\
& + \frac{Zem}{c}[\mathbf{A}_{\mathrm{e}} \times (\mathbf{p} - e\mathbf{A}_{\mathrm{m}}]_z \qquad (26)
\end{aligned}$$

For $\mathbf{A}_{\mathrm{e}} = 0$ we get the usual equations $m\dot{x} = p_x - eA_{\mathrm{m}x}$, etc. In addition to the first derivatives $\dot{x}$, $\dot{y}$, $\dot{z}$ we will also need the second derivatives $\ddot{x}$, $\ddot{y}$, $\ddot{z}$. We multiply the second derivatives with the common denominator D^2, where D is defined by Eq.(23). The vector notation rather than the component notation is used since it is shorter:

$$\begin{aligned}
D^2\ddot{x} = {} & D\left(\frac{\partial(D\dot{x})}{\partial t} - \dot{x}\frac{\partial D}{\partial t}\right) = m\left[m^2 + \left(\frac{Ze}{c}\right)^2 \mathbf{A}_{\mathrm{e}}^2\right]\left\{m^2(\dot{\mathbf{p}} - e\dot{\mathbf{A}}_{\mathrm{m}})_x\right. \\
& + \left(\frac{Ze}{c}\right)^2 [(\dot{A}_{\mathrm{e}x}\mathbf{A}_{\mathrm{e}} + A_{\mathrm{e}x}\dot{\mathbf{A}}_{\mathrm{e}}) \cdot (\mathbf{p} - e\mathbf{A}_{\mathrm{m}}) + A_{\mathrm{e}x}\mathbf{A}_{\mathrm{e}} \cdot (\dot{\mathbf{p}} - e\dot{\mathbf{A}}_{\mathrm{m}})] \\
& \left. + \frac{Zem}{c}[\dot{\mathbf{A}}_{\mathrm{e}} \times (\mathbf{p} - e\mathbf{A}_{\mathrm{m}}) + \mathbf{A}_{\mathrm{e}} \times (\dot{\mathbf{p}} - e\dot{\mathbf{A}}_{\mathrm{m}})]_x\right\} \\
& - 2m\left(\frac{Ze}{c}\right)^2 \mathbf{A}_{\mathrm{e}} \cdot \dot{\mathbf{A}}_{\mathrm{e}}\left\{m^2(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_x + \left(\frac{Ze}{c}\right)^2 A_{\mathrm{e}x}\mathbf{A}_{\mathrm{e}} \cdot (\mathbf{p} - e\mathbf{A}_{\mathrm{m}})\right. \\
& \left. + \frac{Zem}{c}[\mathbf{A}_{\mathrm{e}} \times (\mathbf{p} - e\mathbf{A}_{\mathrm{m}})]_x\right\} \qquad (27)
\end{aligned}$$

$$D^2\ddot{y} = D\left(\frac{\partial(D\dot{y})}{\partial t} - \dot{y}\frac{\partial D}{\partial t}\right) = m\left[m^2 + \left(\frac{Ze}{c}\right)^2 \mathbf{A}_\mathrm{e}^2\right]\left\{m^2(\dot{\mathbf{p}} - e\dot{\mathbf{A}}_\mathrm{m})_y\right.$$
$$+ \left(\frac{Ze}{c}\right)^2 [(\dot{A}_{\mathrm{e}y}\mathbf{A}_\mathrm{e} + A_{\mathrm{e}y}\dot{\mathbf{A}}_\mathrm{e})\cdot(\mathbf{p} - e\mathbf{A}_\mathrm{m}) + A_{\mathrm{e}y}\mathbf{A}_\mathrm{e}\cdot(\dot{\mathbf{p}} - e\dot{\mathbf{A}}_\mathrm{m})]$$
$$\left. + \frac{Zem}{c}[\dot{\mathbf{A}}_\mathrm{e}\times(\mathbf{p} - e\mathbf{A}_\mathrm{m}) + \mathbf{A}_\mathrm{e}\times(\dot{\mathbf{p}} - e\dot{\mathbf{A}}_\mathrm{m})]_y\right\}$$
$$- 2m\left(\frac{Ze}{c}\right)^2 \mathbf{A}_\mathrm{e}\cdot\dot{\mathbf{A}}_\mathrm{e}\left\{m^2(\mathbf{p} - e\mathbf{A}_\mathrm{m})_y + \left(\frac{Ze}{c}\right)^2 A_{\mathrm{e}y}\mathbf{A}_\mathrm{e}\cdot(\mathbf{p} - e\mathbf{A}_\mathrm{m})\right.$$
$$\left. + \frac{Zem}{c}[\mathbf{A}_\mathrm{e}\times(\mathbf{p} - e\mathbf{A}_\mathrm{m})]_y\right\} \tag{28}$$

$$D^2\ddot{z} = D\left(\frac{\partial(D\dot{z})}{\partial t} - \dot{z}\frac{\partial D}{\partial t}\right) = m\left[m^2 + \left(\frac{Ze}{c}\right)^2 \mathbf{A}_\mathrm{e}^2\right]\left\{m^2(\dot{\mathbf{p}} - e\dot{\mathbf{A}}_\mathrm{m})_z\right.$$
$$+ \left(\frac{Ze}{c}\right)^2 [(\dot{A}_{\mathrm{e}z}\mathbf{A}_\mathrm{e} + A_{\mathrm{e}z}\dot{\mathbf{A}}_\mathrm{e})\cdot(\mathbf{p} - e\mathbf{A}_\mathrm{m}) + A_{\mathrm{e}z}\mathbf{A}_\mathrm{e}\cdot(\dot{\mathbf{p}} - e\dot{\mathbf{A}}_\mathrm{m})]$$
$$\left. + \frac{Zem}{c}[\dot{\mathbf{A}}_\mathrm{e}\times(\mathbf{p} - e\mathbf{A}_\mathrm{m}) + \mathbf{A}_\mathrm{e}\times(\dot{\mathbf{p}} - e\dot{\mathbf{A}}_\mathrm{m})]_z\right\}$$
$$- 2m\left(\frac{Ze}{c}\right)^2 \mathbf{A}_\mathrm{e}\cdot\dot{\mathbf{A}}_\mathrm{e}\left\{m^2(\mathbf{p} - e\mathbf{A}_\mathrm{m})_z + \left(\frac{Ze}{c}\right)^2 A_{\mathrm{e}z}\mathbf{A}_\mathrm{e}\cdot(\mathbf{p} - e\mathbf{A}_\mathrm{m})\right.$$
$$\left. + \frac{Zem}{c}[\mathbf{A}_\mathrm{e}\times(\mathbf{p} - e\mathbf{A}_\mathrm{m})]_z\right\} \tag{29}$$

In order to obtain the Hamilton function $\mathcal{H}$ from the Lagrange function $\mathcal{L}$ we observe that $\mathcal{L}$ is a vector with three components. Hence, $\mathcal{H}$ must be a vector with three components $\mathcal{H}_k$:

$$\mathcal{H}_k(p_j, x_j, t) = \sum_{j=1}^{3} p_j\dot{x}_j - \mathcal{L}_k = \mathbf{p}\cdot\dot{\mathbf{r}} - \mathcal{L}_k, \quad k = x,\ y,\ z \tag{30}$$

From Eqs.(20)–(22) we get

$$\sum_{j=1}^{3} p_j\dot{x}_j = p_x\dot{x} + p_y\dot{y} + p_z\dot{z}$$
$$= m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + e(A_{\mathrm{m}x}\dot{x} + A_{\mathrm{m}y}\dot{y} + A_{\mathrm{m}z}\dot{z})$$
$$= m\dot{\mathbf{r}}^2 + e\mathbf{A}_\mathrm{m}\cdot\dot{\mathbf{r}} \tag{31}$$

since the terms multiplied by Ze/c cancel. The following three components $\mathcal{H}_k$ are obtained from Eqs.(19), (12), (15), (17), and (18) due to the cancellation of the terms $A_{\mathrm{m}x}\dot{x} + A_{\mathrm{m}y}\dot{y} + A_{\mathrm{m}z}\dot{z}$:

$$\mathcal{H}_x = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + e\phi_\mathrm{e} - \mathcal{L}_{cx} = \frac{1}{2}m\dot{\mathbf{r}}^2 + e\phi_\mathrm{e} - \mathcal{L}_{cx} \tag{32}$$

$$\mathcal{H}_y = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + e\phi_\mathrm{e} - \mathcal{L}_{cy} = \frac{1}{2}m\dot{\mathbf{r}}^2 + e\phi_\mathrm{e} - \mathcal{L}_{cy} \tag{33}$$

$$\mathcal{H}_z = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + e\phi_\mathrm{e} - \mathcal{L}_{cz} = \frac{1}{2}m\dot{\mathbf{r}}^2 + e\phi_\mathrm{e} - \mathcal{L}_{cz} \tag{34}$$

The Hamilton function should be written in terms of the momentum $\mathbf{p}$ and the potentials ϕ_e, ϕ_m, $\mathbf{A}_\mathrm{m}$, $\mathbf{A}_\mathrm{e}$. One can do so by substituting $\dot{x}$, $\dot{y}$, $\dot{z}$, $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ from Eqs.(24)–(29) into Eqs.(32)–(34) as well as Eqs.(15), (17), and (18). This is quite tedious. Let us first gain some understanding of the Hamilton function. To this end we return to Eqs.(20)–(22) and see under what conditions the terms multiplied by Ze/c will be small. From Eq.(20) we get a first condition

$$\frac{Ze}{c}A_{\mathrm{e}z}\dot{y} \ll m\dot{x} \quad \text{and} \quad \frac{Ze}{c}A_{\mathrm{e}y}\dot{z} \ll m\dot{x} \tag{35}$$

which may be rewritten as follows:

$$mc^2 \gg ZecA_{\mathrm{e}z}\dot{y}/\dot{x} \quad \text{and} \quad mc^2 \gg ZecA_{\mathrm{e}y}\dot{z}/\dot{x} \tag{36}$$

In case $\dot{x}$ is close to zero while $\dot{y}$ and $\dot{z}$ are not we require an alternate condition for Eq.(20):

$$A_{\mathrm{m}x} \gg ZA_{\mathrm{e}z}\dot{y}/c \quad \text{and} \quad A_{\mathrm{m}x} \gg ZA_{\mathrm{e}y}\dot{z}/c \tag{37}$$

Equations (36) and (37) state in essence that either the energy mc^2 should be large compared with the energy due to the potential $\mathbf{A}_\mathrm{e}$ or the magnitude of the potential $\mathbf{A}_\mathrm{m}$ should be large compared with the magnitude of $\mathbf{A}_\mathrm{e}$. More detailed statements referring to the terms $\dot{y}/\dot{x}$, $\dot{z}/\dot{x}$, $\dot{y}/c$, and $\dot{z}/c$ are not of interest here. Relations equivalent to Eqs.(35)–(37) may be derived from Eqs.(21) and (22) too. If these conditions are satisfied we may rewrite Eqs.(20)–(22) in a simplified form:

$$p_x = m\dot{x} + eA_{\mathrm{m}x}, \; p_y = m\dot{y} + eA_{\mathrm{m}y}, \; p_z = m\dot{z} + eA_{\mathrm{m}z} \tag{38}$$

$$D = m^3 \tag{39}$$

Equations (24)–(26) become:

$$\dot{x} = \frac{1}{m}(\mathbf{p} - e\mathbf{A}_\mathrm{m})_x, \; \dot{y} = \frac{1}{m}(\mathbf{p} - e\mathbf{A}_\mathrm{m})_y, \; \dot{z} = \frac{1}{m}(\mathbf{p} - e\mathbf{A}_\mathrm{m})_z \tag{40}$$

The three components of the Hamilton function become:

$$\mathcal{H}_x = \frac{1}{2m}(\mathbf{p} - e\mathbf{A}_{\rm m})^2 + e\phi_e - \mathcal{L}_{cx} \tag{41}$$
$$\mathcal{H}_y = \frac{1}{2m}(\mathbf{p} - e\mathbf{A}_{\rm m})^2 + e\phi_e - \mathcal{L}_{cy} \tag{42}$$
$$\mathcal{H}_z = \frac{1}{2m}(\mathbf{p} - e\mathbf{A}_{\rm m})^2 + e\phi_e - \mathcal{L}_{cz} \tag{43}$$

These are the terms of the conventional Hamilton function of a charged particle in an electromagnetic field plus correcting terms $\mathcal{L}_{cx}$, $\mathcal{L}_{cy}$, $\mathcal{L}_{cz}$. The correcting terms will be negligible according to Eqs.(15), (17), and (18) if $A_{\rm e}$, $\partial A_{\rm e}/\partial r$, $\partial\phi_{\rm m}/\partial r$, $\dot{r}$, and $\ddot{r}$ are sufficiently small.

Let us observe that the variability of the mass m with velocity is not taken into account in Eqs.(41) to (43). This simplification permits us to obtain the components $\mathcal{H}_x$, $\mathcal{H}_y$, $\mathcal{H}_z$ of the Hamilton function and the correcting terms $\mathcal{L}_{cx}$, $\mathcal{L}_{cy}$, $\mathcal{L}_{cz}$ explicitly. In the following Section 3.3 we will take into account that the mass m depends on the velocities $\dot{x}$, $\dot{y}$, $\dot{z}$. The Hamilton function can then be obtained by means of series expansions only.

We note that all deviations from the conventional values in Eqs.(20)–(22) are due to the potential $\mathbf{A}_{\rm e}$ that is produced by magnetic monopole, dipole, or multipole currents $\mathbf{g}_{\rm m}$ according to Eq.(1.6-26). The correcting terms $\boldsymbol{\mathcal{L}}_{\rm c}$ of Eqs.(15), (17), and (18) contain mainly terms $\mathbf{A}_{\rm e}$ but the hypothetical magnetic charge $\rho_{\rm m}$ enters through the terms $\partial\phi_{\rm m}/\partial r$ according to Eq.(1.6-29).

We turn to the evaluation of Eqs.(32)–(34) without any approximation. Using Eqs.(23)–(26) and observing the relation $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) \equiv 0$ we obtain the following form for Eqs.(32)–(34):

$$\mathcal{H} = \frac{1}{2m}\left[(\mathbf{p}-e\mathbf{A}_{\rm m})^2 + \left(\frac{Ze}{mc}\right)^2 \{2[\mathbf{A}_{\rm e}\cdot(\mathbf{p}-e\mathbf{A}_{\rm m})]^2 + [\mathbf{A}_{\rm e}\times(\mathbf{p}-e\mathbf{A}_{\rm m})]^2\} + \left(\frac{Ze}{mc}\right)^4 \mathbf{A}_{\rm e}^2[\mathbf{A}_{\rm e}\cdot(\mathbf{p}-e\mathbf{A}_m)]^2\right]\left[1+\left(\frac{Ze}{mc}\right)^2\mathbf{A}_{\rm e}^2\right]^{-2} + e\phi_{\rm e} - \boldsymbol{\mathcal{L}}_{\rm c} \tag{44}$$

The vector $\boldsymbol{\mathcal{L}}_{\rm c}$ has the three components $\mathcal{L}_{cx}$, $\mathcal{L}_{cy}$, and $\mathcal{L}_{cz}$ of Eqs.(32)–(34). Each of these three terms consists of five components. For the first component of $\mathcal{L}_{cx}$ we obtain according to Eq.(15):

$$\begin{aligned}\mathcal{L}_{cx1} &= \frac{Ze}{c}(A_{{\rm e}z}\dot{y} - A_{{\rm e}y}\dot{z})\dot{x} = \frac{Ze}{m^2c}\bigg(A_{{\rm e}z}(\mathbf{p}-e\mathbf{A}_{\rm m})_y - A_{{\rm e}y}(\mathbf{p}-e\mathbf{A}_{\rm m})_z \\ &\quad + \frac{Ze}{mc}\Big\{A_{{\rm e}z}[\mathbf{A}_{\rm e}\times(\mathbf{p}-e\mathbf{A}_{\rm m})]_y - A_{{\rm e}y}[\mathbf{A}_{\rm e}\times(\mathbf{p}-e\mathbf{A}_{\rm m})]_z\Big\}\bigg) \\ &\quad \times\left[(\mathbf{p}-e\mathbf{A}_{\rm m})_x + \left(\frac{Ze}{mc}\right)^2 A_{{\rm e}x}\mathbf{A}_{\rm e}\cdot(\mathbf{p}-e\mathbf{A}_{\rm m}) + \frac{Ze}{mc}[\mathbf{A}_{\rm e}\times(\mathbf{p}-e\mathbf{A}_{\rm m})]_x\right]\left[1+\left(\frac{Ze}{mc}\right)^2\mathbf{A}_{\rm e}^2\right]^{-2}\end{aligned} \tag{45}$$

A second component $\mathfrak{L}_{cx2}$ is defined according to Eq.(15) as follows:

$$\begin{aligned}\mathfrak{L}_{cx2} &= -\frac{Ze}{c}\int\left(\frac{\partial\phi_{\mathrm{m}}}{\partial z}\dot{y} - \frac{\partial\phi_{\mathrm{m}}}{\partial y}\dot{z}\right)dx \\ &= -\frac{Ze}{m^2c}\int\left[\frac{\partial\phi_{\mathrm{m}}}{\partial z}(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})_y - \frac{\partial\phi_{\mathrm{m}}}{\partial y}(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})_z\right. \\ &\quad + \frac{Ze}{mc}\left(\frac{\partial\phi_{\mathrm{m}}}{\partial z}[\mathbf{A}_{\mathrm{e}}\times(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})]_y - \frac{\partial\phi_{\mathrm{m}}}{\partial y}[\mathbf{A}_{\mathrm{e}}\times(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})]_z\right) \\ &\quad \left. + \left(\frac{Ze}{mc}\right)^2\left(\frac{\partial\phi_{\mathrm{m}}}{\partial z}A_{\mathrm{e}y} - \frac{\partial\phi_{\mathrm{m}}}{\partial y}A_{\mathrm{e}z}\right)\mathbf{A}_{\mathrm{e}}\cdot(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})\right] \\ &\qquad \times\left[1+\left(\frac{Ze}{mc}\right)^2\mathbf{A}_{\mathrm{e}}^2\right]^{-1}dx \qquad (46)\end{aligned}$$

For the third component $\mathfrak{L}_{cx3}$ we obtain from Eqs.(15), (23), (28), and (29):

$$\begin{aligned}\mathfrak{L}_{cx3} &= \frac{Ze}{c}\int(A_{\mathrm{e}z}\ddot{y} - A_{\mathrm{e}y}\ddot{z})dx \\ &= \int\left\{\left[1+\left(\frac{Ze}{mc}\right)^2\mathbf{A}_{\mathrm{e}}^2\right]\left[A_{\mathrm{e}z}(\dot{\mathbf{p}}-e\dot{\mathbf{A}}_{\mathrm{m}})_y - A_{\mathrm{e}y}(\dot{\mathbf{p}}-e\dot{\mathbf{A}}_{\mathrm{m}})_z + \frac{Ze}{mc}\times\right.\right. \\ &\{A_{\mathrm{e}z}[\dot{\mathbf{A}}_{\mathrm{e}}\times(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})+\mathbf{A}_{\mathrm{e}}\times(\dot{\mathbf{p}}-e\dot{\mathbf{A}}_{\mathrm{m}})]_y - A_{\mathrm{e}y}[\dot{\mathbf{A}}_{\mathrm{e}}\times(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})+\mathbf{A}_{\mathrm{e}}\times(\dot{\mathbf{p}}-e\dot{\mathbf{A}}_{\mathrm{m}})]_z\} \\ &\quad \left. + \left(\frac{Ze}{mc}\right)^2[(A_{\mathrm{e}z}\dot{A}_{\mathrm{e}y} - A_{\mathrm{e}y}\dot{A}_{\mathrm{e}z})\mathbf{A}_{\mathrm{e}}\cdot(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})]\right] \\ &-2\left(\frac{Ze}{mc}\right)^2\mathbf{A}_{\mathrm{e}}\cdot\dot{\mathbf{A}}_{\mathrm{e}}\left(A_{\mathrm{e}z}(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})_y - A_{\mathrm{e}y}(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})_z + \frac{Ze}{mc}\{A_{\mathrm{e}z}[\mathbf{A}_{\mathrm{e}}\times(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})]_y\right. \\ &\quad \left.\left. - A_{\mathrm{e}y}[\mathbf{A}_{\mathrm{e}}\times(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})]_z\}\right)\right\}\left[1+\left(\frac{Ze}{mc}\right)^2\mathbf{A}_{\mathrm{e}}^2\right]^{-2}dx \qquad (47)\end{aligned}$$

The fourth component $\mathfrak{L}_{cx4}$ in Eq.(15) remains unchanged:

$$\mathfrak{L}_{cx4} = Zec\int\left(\frac{\partial A_{\mathrm{e}y}}{\partial z} - \frac{\partial A_{\mathrm{e}z}}{\partial y}\right)dx \qquad (48)$$

The fifth and last component $\mathfrak{L}_{cx5}$ in Eq.(15) is particularly long. It follows with Eqs.(13), (23), (25), and (26):

$$\begin{aligned}\mathfrak{L}_{cx5} &= \frac{Ze}{c}\int\left(\dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z}\right)(A_{\mathrm{e}z}\dot{y} - A_{\mathrm{e}y}\dot{z})\,dx \\ &= \frac{Ze}{m^2c}\int\left\{\left[(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})_y^2 + 2\frac{Ze}{mc}(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})_y[\mathbf{A}_{\mathrm{e}}\times(\mathbf{p}-e\mathbf{A}_{\mathrm{m}})]_y\right.\right.\end{aligned}$$

$$+\left(\frac{Ze}{mc}\right)^2\{[\mathbf{A}_\mathrm{e}\times(\mathbf{p}-e\mathbf{A}_\mathrm{m})]_y^2+2A_{\mathrm{e}y}(\mathbf{p}-e\mathbf{A}_\mathrm{m})_y\mathbf{A}_\mathrm{e}\cdot(\mathbf{p}-e\mathbf{A}_\mathrm{m})\}$$
$$+2\left(\frac{Ze}{mc}\right)^3 A_{\mathrm{e}y}[\mathbf{A}_\mathrm{e}\times(\mathbf{p}-e\mathbf{A}_\mathrm{m})]_y\mathbf{A}_\mathrm{e}\cdot(\mathbf{p}-e\mathbf{A}_\mathrm{m})$$
$$+\left(\frac{Ze}{mc}\right)^4 A_{\mathrm{e}y}^2[\mathbf{A}_\mathrm{e}\cdot(\mathbf{p}-e\mathbf{A}_\mathrm{m})]^2\Bigg]\frac{\partial A_{\mathrm{e}z}}{\partial y}$$
$$+\Bigg[(\mathbf{p}-e\mathbf{A}_\mathrm{m})_y(\mathbf{p}-e\mathbf{A}_\mathrm{m})_z+\frac{Ze}{mc}\{(\mathbf{p}-e\mathbf{A}_\mathrm{m})_y[\mathbf{A}_\mathrm{e}\times(\mathbf{p}-e\mathbf{A}_\mathrm{m})]_z$$
$$+(\mathbf{p}-e\mathbf{A}_\mathrm{m})_z[\mathbf{A}_\mathrm{e}\times(\mathbf{p}-e\mathbf{A}_\mathrm{m})]_y\}$$
$$+\left(\frac{Ze}{mc}\right)^2\{\mathbf{A}_\mathrm{e}\times(\mathbf{p}-e\mathbf{A}_\mathrm{m})]_y[\mathbf{A}_\mathrm{e}\times(\mathbf{p}-e\mathbf{A}_\mathrm{m})]_z$$
$$+\mathbf{A}_{\mathrm{e}z}[(\mathbf{p}-e\mathbf{A}_\mathrm{m})_y+(\mathbf{p}-e\mathbf{A}_\mathrm{m})_z]\mathbf{A}_\mathrm{e}\cdot(\mathbf{p}-e\mathbf{A}_\mathrm{m})\}$$
$$+\left(\frac{Ze}{mc}\right)^3\{A_{\mathrm{e}y}[\mathbf{A}_\mathrm{e}\times(\mathbf{p}-e\mathbf{A}_\mathrm{m})]_z+A_{\mathrm{e}z}[\mathbf{A}_\mathrm{e}\times(\mathbf{p}-e\mathbf{A}_\mathrm{m})]_y\}\mathbf{A}_\mathrm{e}\cdot(\mathbf{p}-e\mathbf{A}_\mathrm{m})$$
$$+\left(\frac{Ze}{mc}\right)^4 A_{\mathrm{e}y}A_{\mathrm{e}z}[\mathbf{A}_\mathrm{e}\cdot(\mathbf{p}-e\mathbf{A}_\mathrm{m})]^2\Bigg]\left(\frac{\partial A_{\mathrm{e}z}}{\partial y}-\frac{\partial A_{\mathrm{e}y}}{\partial z}\right)$$
$$+\Bigg[(\mathbf{p}-e\mathbf{A}_\mathrm{m})_z^2+2\frac{Ze}{mc}(\mathbf{p}-e\mathbf{A}_\mathrm{m})_z[\mathbf{A}_\mathrm{e}\times(\mathbf{p}-e\mathbf{A}_\mathrm{m})]_z$$
$$+\left(\frac{Ze}{mc}\right)^2\{[\mathbf{A}_\mathrm{e}\times(\mathbf{p}-e\mathbf{A}_\mathrm{m})]_z^2+2A_{\mathrm{e}z}(\mathbf{p}-e\mathbf{A}_\mathrm{m})_z\mathbf{A}_\mathrm{e}\cdot(\mathbf{p}-e\mathbf{A}_\mathrm{m})\}$$
$$+2\left(\frac{Ze}{mc}\right)^3 A_{\mathrm{e}z}[\mathbf{A}_\mathrm{e}\times(\mathbf{p}-e\mathbf{A}_\mathrm{m})]_z\mathbf{A}_\mathrm{e}\cdot(\mathbf{p}-e\mathbf{A}_\mathrm{m})$$
$$+\left(\frac{Ze}{mc}\right)^4 A_{\mathrm{e}z}^2[\mathbf{A}_\mathrm{e}\cdot(\mathbf{p}-e\mathbf{A}_\mathrm{m})]^2\Bigg]\frac{\partial A_{\mathrm{e}y}}{\partial z}\Bigg\}\left[1+\left(\frac{Ze}{mc}\right)^2\mathbf{A}_\mathrm{e}^2\right]^{-2}dx \tag{49}$$

The correcting terms $\mathfrak{L}_{cy}$ and $\mathfrak{L}_{cz}$ for Eqs.(42) and (43) may be obtained from Eqs.(45)–(49) by the substitutions $x \to y \to z \to x$.

In the text following Eq.(18) we had pointed out that the one added term $\mathbf{g}_\mathrm{m}$ in Eq.(1.1-9) had grown to six added terms in Eqs.(15), (17), and (18). The replacement of $\dot{z}$, $\dot{y}$, $\dot{z}$, $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ according to Eqs.(24) to (29) produces an avalanche of additional terms.

3.3 Variability of the Mass of a Charged Particle

The variability of the mass m of a particle with the velocities $\dot{x}$, $\dot{y}$, $\dot{z}$ is not shown explicitly in Eqs.(3.2-32) to (3.2-34) or (3.2-41) to (3.2-47), and (3.2-49). These equations thus hold for a particle with essentially constant mass. The extension of the results of Section 3.2 to a variable mass is possible by means of series expansions only. We start with Eq.(3.2-1):

$$\frac{d}{dt}(m\mathbf{v}) = e\mathbf{E} + \frac{Ze}{c}\mathbf{v}\times\mathbf{H} \tag{1}$$

The velocity $v_j = dx_j/dt$ is replaced by the four-velocity $u_\mu = dx_\mu/d\tau$ with the proper time differential

$$d\tau = (1 - v^2/c^2)^{1/2} dt \tag{2}$$

and with $x_\mu = x,\ y,\ z,\ ict$. Since the variables x_μ form a four-vector and $d\tau$ is a scalar, the components u_μ form a four-vector too:

$$u_j = \frac{v_j}{(1 - v^2/c^2)^{1/2}}, \quad u_4 = \frac{ic}{(1 - v^2/c^2)^{1/2}}, \quad j = 1,\ 2,\ 3$$

$$u_\mu u_\mu = \sum_{\mu=1}^{4} u_\mu^2 = -c^2 \tag{3}$$

We define the four-vector of a force K_ν as the product of eu_μ with the tensor $\mathbf{F}_{\mathrm{m}}$ of Eq.(2.4-11) having the components $f_{\mathrm{m}\mu\nu}$. Using the relation $\mathbf{B} = Z\mathbf{H}/c$ we get:

$$K_\nu = e f_{\mathrm{m}\mu\nu} u_\mu \tag{4}$$

$$K_\nu = e(\mathbf{E} + \frac{Z}{c}\mathbf{v} \times \mathbf{H})_\nu \left(1 - \frac{v^2}{c^2}\right)^{-1/2}, \quad \nu = 1,\ 2,\ 3 \text{ or } x,\ y\ ,z \tag{5}$$

$$K_4 = i\frac{e}{c}\mathbf{E} \cdot \mathbf{v} \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \tag{6}$$

Equations (4)–(6) are the relativistic generalization of the right side of Eq.(1). On the left side of Eq.(1) one must replace the time differential dt by the proper time differential $d\tau$ of Eq.(2), the velocity $\mathbf{v}$ by the four-velocity $\mathbf{u}$ of Eq.(3), and the mass m by the rest mass m_0. The relativistic version of Eq.(1) becomes:

$$\frac{d}{d\tau} m_0 u_\nu = e f_{\mathrm{m}\mu\nu} u_\mu \tag{7}$$

$$\frac{d}{dt} \frac{m_0 \mathbf{v}}{(1 - v^2/c^2)^{1/2}} = e\left(\mathbf{E} + \frac{Z}{c}\mathbf{v} \times \mathbf{H}\right) \tag{8}$$

$$\frac{d}{dt} \frac{m_0 c^2}{(1 - v^2/c^2)^{1/2}} = e\mathbf{E} \cdot \mathbf{v} \tag{9}$$

Equation (8) represents the three spatial components of the equation of motion while Eq.(9) represents the law of conservation of energy.

The product of the rest mass m_0 with the four-vector u_μ produces a new four-vector

$$p_\mu = m_0 u_\mu \tag{10}$$

with the three spatial components

$$p_j = \frac{m_0 v_j}{(1 - v^2/c^2)^{1/2}} \quad \text{or} \quad \mathbf{p} = \frac{m_0 \mathbf{v}}{(1 - v^2/c^2)^{1/2}}, \quad j = 1,\ 2,\ 3 \tag{11}$$

and the component p_4

$$p_4 = \frac{icm_0}{(1 - v^2/c^2)^{1/2}} = \frac{i}{c}\frac{m_0 c^2}{(1 - v^2/c^2)^{1/2}} = \frac{i}{c}\mathsf{E} \tag{12}$$

where E denotes an energy rather than the magnitude E of an electric field strength. Using Eq.(3) one obtains

$$\mathbf{p} \cdot \mathbf{p} - \frac{\mathsf{E}^2}{c^2} = -m^2 c^2 \quad \text{or} \quad \mathsf{E} = (p^2 c^2 + m^2 c^4)^{1/2} \tag{13}$$

as connection between energy E and momentum $\mathbf{p}$.

For the relativistic generalization of the Lagrange function of Section 3.2 we start with the conventional part $\mathcal{L}_M$ as defined by Eq.(3.2-12) and rewritten as follows:

$$\mathcal{L}_M = \frac{1}{2} m \mathbf{v}^2 + e(-\phi_e + \mathbf{A}_m \cdot \mathbf{v}) \tag{14}$$

The relativistic generalization of $\mathcal{L}_M$ is found in many textbooks:

$$\begin{aligned} \mathcal{L}_M &= -m_0 c^2 \left(1 - v^2/c^2\right)^{1/2} + e(-\phi_e + \mathbf{A}_m \cdot \mathbf{v}) \\ &= -m_0 c^2 \left(1 - \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{c^2}\right)^{1/2} + e(-\phi_e + A_{mx}\dot{x} + A_{my}\dot{y} + A_{mz}\dot{z}) \\ &\doteq -m_0 c^2 + \frac{1}{2} m_0 v^2 + e(-\phi_e + \mathbf{A}_m \cdot \mathbf{v}) \quad \text{for } \mathbf{v}^2/c^2 \ll 1 \end{aligned} \tag{15}$$

We adopt this generalization of the part $\mathcal{L}_M$ of the Lagrange function $\mathcal{L}$ of Eq.(3.2-19). The components of the correcting term $\mathcal{L}_c$ are left unchanged from their definitions in Eqs.(3.2-15), (3.2-17), and (3.2-18) since the mass m does not occur there and the potentials ϕ_m, $\mathbf{A}_e$ come from a relativistic theory; we note that ϕ_e and $\mathbf{A}_m$ are the same in Eq.(15) as in Eq.(14) too. Hence, the relativistic generalization of the Lagrange function of Eq.(3.2-19) is

$$\mathcal{L} = -m_0 c^2 (1 - v^2/c^2)^{1/2} + e(-\phi_e + \mathbf{A}_m \cdot \mathbf{v}) + \mathcal{L}_c \tag{16}$$

where $\mathcal{L}_c$ has the three components $\mathcal{L}_{cx}$, $\mathcal{L}_{cy}$, $\mathcal{L}_{cz}$ and $\mathcal{L}$ has the three components $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ or $\mathcal{L}_x$, $\mathcal{L}_y$, $\mathcal{L}_z$.

The nonrelativistic momenta p_x, p_y, p_z of Eqs.(3.2-20)–(3.2-22) must be generalized to 'relativistic canonical momenta'. They assume the following form:

$$p_x = \frac{\partial \mathcal{L}_x}{\partial \dot{x}} = \frac{\partial(\mathcal{L}_{\mathrm{M}} + \mathcal{L}_{\mathrm{c}})}{\partial \dot{x}}$$

$$= \frac{m_0 \dot{x}}{(1 - v^2/c^2)^{1/2}} + eA_{\mathrm{m}x} + \frac{Ze}{c}(A_{\mathrm{e}z}\dot{y} - A_{\mathrm{e}y}\dot{z}) \tag{17}$$

$$p_y = \frac{\partial \mathcal{L}_y}{\partial \dot{y}} = \frac{m_0 \dot{y}}{(1 - v^2/c^2)^{1/2}} + eA_{\mathrm{m}y} + \frac{Ze}{c}(A_{\mathrm{e}x}\dot{z} - A_{\mathrm{e}z}\dot{x}) \tag{18}$$

$$p_z = \frac{\partial \mathcal{L}_{\mathrm{y}}}{\partial \dot{z}} = \frac{m_0 \dot{z}}{(1 - v^2/c^2)^{1/2}} + eA_{\mathrm{m}z} + \frac{Ze}{c}(A_{\mathrm{e}y}\dot{x} - A_{\mathrm{e}x}\dot{y}) \tag{19}$$

For the derivation of the Hamilton function from the Lagrange function we have Eqs.(3.2-30) and (3.2-31) but we must now use Eqs.(17)–(19) for p_x, p_y, and p_z

$$\sum_{j=1}^{3} p_j \dot{x}_j = \frac{m_0}{(1 - v^2/c^2)^{1/2}}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + e(A_{\mathrm{m}x}\dot{x} + A_{\mathrm{m}y}\dot{y} + A_{\mathrm{m}z}\dot{z})$$

$$= \frac{m_0}{(1 - v^2/c^2)^{1/2}}\dot{\mathbf{r}}^2 + e\mathbf{A}_{\mathrm{m}} \cdot \dot{\mathbf{r}} \tag{20}$$

and the three components $\mathcal{H}_k$ of the Hamilton function are obtained in analogy to Eqs.(3.2-32)–(3.2-34):

$$\mathcal{H}_k(p_j, x_j, t) = \sum_{j=1}^{3} p_j \dot{x}_j - \mathcal{L}_k = \frac{m_0 c^2}{(1 - v^2/c^2)^{1/2}} + e\phi_{\mathrm{e}} - \mathcal{L}_k, \quad k = x,\ y,\ z$$

$$\mathcal{H}_x = \frac{m_0 c^2}{[1 - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)/c^2]^{1/2}} + e\phi_{\mathrm{e}} - \mathcal{L}_{cx} \tag{21}$$

$$\mathcal{H}_y = \frac{m_0 c^2}{[1 - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)/c^2]^{1/2}} + e\phi_{\mathrm{e}} - \mathcal{L}_{cy} \tag{22}$$

$$\mathcal{H}_z = \frac{m_0 c^2}{[1 - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)/c^2]^{1/2}} + e\phi_{\mathrm{e}} - \mathcal{L}_{cz} \tag{23}$$

We must eliminate the variables $\dot{x}$, $\dot{y}$, $\dot{z}$ and their derivatives $\ddot{x}$, $\ddot{y}$, $\ddot{z}$. In analogy to Section 3.2 the variables $\dot{x}$, $\dot{y}$, $\dot{z}$ are determined by Eqs.(17)–(19), but these are no longer three linear equations due to the term

$$(1 - v^2/c^2)^{1/2} = \left[1 - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)/c^2\right]^{1/2}$$

and the replacement of $\dot{x}$, $\dot{y}$, $\dot{z}$ by $\mathbf{p}$, $\mathbf{A}_{\mathrm{m}}$, ϕ_{e}, $\mathbf{A}_{\mathrm{e}}$, and ϕ_{m} becomes even more tedious than in Section 3.2. As before we shall first gain an understanding by using simplifying assumptions for Eqs.(17)–(19). The terms multiplied by Ze/c will be small in Eq.(17) if a first condition

$$\frac{Ze}{c}A_{ez}\dot{y} \ll \frac{m_0\dot{x}}{(1-v^2/c^2)^{1/2}} \quad \text{and} \quad \frac{Ze}{c}A_{ey}\dot{z} \ll \frac{m_0\dot{x}}{(1-v^2/c^2)^{1/2}} \tag{24}$$

is satisfied that may be rewritten as follows:

$$\frac{m_0c^2}{(1-v^2/c^2)^{1/2}} \gg ZecA_{ez}\dot{y}/\dot{x} \quad \text{and} \quad \frac{m_0c^2}{(1-v^2/c^2)^{1/2}} \gg ZecA_{ey}\dot{z}/\dot{x} \tag{25}$$

For small values of $\dot{x}$ but not $\dot{y}$ and $\dot{z}$ we require an alternate condition for Eq.(17):

$$A_{mx} \gg ZA_{ez}\dot{y}/c \quad \text{and} \quad A_{mx} \gg ZA_{ey}\dot{z}/c \tag{26}$$

Equation (25) states in essence that the energy due to the potential $\mathbf{A}_e$ should be small compared with the energy $m_0c^2/(1-v^2/c^2)^{1/2}$ while Eq.(26) demands that the magnitude of $\mathbf{A}_e$ should be small compared with the magnitude of $\mathbf{A}_m$. With these simplifying assumptions we obtain from Eqs.(17)–(19):

$$\begin{aligned} p_x &= \frac{m_0\dot{x}}{(1-v^2/c^2)^{1/2}} + eA_{mx} \\ p_y &= \frac{m_0\dot{y}}{(1-v^2/c^2)^{1/2}} + eA_{my} \\ p_z &= \frac{m_0\dot{z}}{(1-v^2/c^2)^{1/2}} + eA_{mz} \end{aligned} \tag{27}$$

Solution of these equations for $\dot{x}$, $\dot{y}$, $\dot{z}$ yields:

$$\dot{x} = \frac{(1-v^2/c^2)^{1/2}}{m_0}(\mathbf{p} - e\mathbf{A}_m)_x \tag{28}$$

$$\dot{y} = \frac{(1-v^2/c^2)^{1/2}}{m_0}(\mathbf{p} - e\mathbf{A}_m)_y \tag{29}$$

$$\dot{z} = \frac{(1-v^2/c^2)^{1/2}}{m_0}(\mathbf{p} - e\mathbf{A}_m)_z \tag{30}$$

Squaring and summing $\dot{x}$, $\dot{y}$, and $\dot{z}$ yields:

$$\begin{aligned} \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = v^2 &= \frac{1-v^2/c^2}{m_0^2}(\mathbf{p} - e\mathbf{A}_m)^2 \\ (\mathbf{p} - e\mathbf{A}_m)^2 &= \frac{m_0^2v^2}{1-v^2/c^2} = m_0^2c^2\left(\frac{1}{1-v^2/c^2} - 1\right) \\ \frac{m_0c^2}{(1-v^2/c^2)^{1/2}} &= \frac{m_0c^2}{[1-(\dot{x}^2+\dot{y}^2+\dot{z}^2)/c^2]^{1/2}} \\ &= c\left[(\mathbf{p} - e\mathbf{A}_m)^2 + m_0^2c^2\right]^{1/2} \end{aligned} \tag{31}$$

Substitution of the last line of Eq.(31) into Eqs.(21)–(23) yields:

$$\mathcal{H}_x = c\left[(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 + m_0^2c^2\right]^{1/2} + e\phi_\mathrm{e} - \mathcal{L}_{cx} \tag{32}$$

$$\mathcal{H}_y = c\left[(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 + m_0^2c^2\right]^{1/2} + e\phi_\mathrm{e} - \mathcal{L}_{cy} \tag{33}$$

$$\mathcal{H}_z = c\left[(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 + m_0^2c^2\right]^{1/2} + e\phi_\mathrm{e} - \mathcal{L}_{cz} \tag{34}$$

If we leave out the correcting terms $\mathcal{L}_{cx}$, $\mathcal{L}_{cy}$, $\mathcal{L}_{cz}$ we have the conventional relativistic Hamilton function for a charged particle in an electromagnetic field. The assumption we had to make to obtain Eqs.(32)–(34) was that A_e must be sufficiently small. If we want to leave out the correcting terms $\mathcal{L}_\mathrm{c}$ we get more complicated conditions since Eqs.(3.2-15), (3.2-17), and (3.2-18) contain not only A_e but also its derivatives and derivatives of ϕ_m.

Let us turn to the solution of Eqs.(17)–(19) for $\dot{x}$, $\dot{y}$, $\dot{z}$ without approximations. A difficulty is created by the term

$$(1 - v^2/c^2)^{1/2} = \left[1 - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)/c^2\right]^{1/2}$$

which makes these equations nonlinear while the corresponding Eqs.(3.2-20)–(3.2-22) of the nonrelativistic theory were linear. There is no standard method of solution for a system of nonlinear equations and we must find a method suitable for this specific case. As a first step we ignore that v^2 is a function of $\dot{x}$, $\dot{y}$, $\dot{z}$ and treat Eqs.(17)–(19) as a system of linear equations. We note that Eqs.(3.2-20)–(3.2-22) are transformed into Eqs.(17)–(19) by the substitution

$$m \to m_0/(1 - v^2/c^2)^{1/2}$$

and we get our 'first step solution' of Eqs.(17)–(19) by making the same substitution in Eqs.(3.2-23)–(3.2-26). First we write the common denominator D according to Eq.(3.2-23):

$$D = \frac{m_0^3}{(1 - v^2/c^2)^{3/2}}\left[1 + \alpha_\mathrm{e}^2\left(1 - \frac{v^2}{c^2}\right)\right] \tag{35}$$

$$\alpha_\mathrm{e} = \frac{ZecA_\mathrm{e}}{m_0c^2}, \quad \alpha_\mathrm{e}\left(1 - \frac{v^2}{c^2}\right)^{1/2} = \frac{ZecA_\mathrm{e}}{m_0c^2/\left(1 - v^2/c^2\right)^{1/2}} \tag{36}$$

The constant α_e represents the ratio of the energy due to the electric vector potential $\mathbf{A}_\mathrm{e}$ and the rest energy of the particle. The reference energy is actually $m_0c^2/(1-v^2/c^2)^{1/2}$ rather than m_0c^2 but we need v^2 as explicit variable. Let us note that α_e has no physical dimension but it is variable due to its component A_e.

The solutions for $\dot{x}$, $\dot{y}$, and $\dot{z}$ as functions of p_x, p_y, and p_z have the following form if we use the factor α_e of Eq.(36):

$$\dot{x} = \frac{(1-v^2/c^2)^{1/2}}{m_0}(\mathbf{p}-e\mathbf{A}_\mathrm{m})_x\left[1+\alpha_\mathrm{e}\left(1-\frac{v^2}{c^2}\right)^{1/2}\frac{[\mathbf{A}_\mathrm{e}\times(\mathbf{p}-e\mathbf{A}_\mathrm{m}]_x}{A_\mathrm{e}(\mathbf{p}-e\mathbf{A}_\mathrm{m})_x}\right.$$
$$\left.+\alpha_\mathrm{e}^2\left(1-\frac{v^2}{c^2}\right)\frac{A_{\mathrm{e}x}\mathbf{A}_\mathrm{e}\cdot(\mathbf{p}-e\mathbf{A}_\mathrm{m})}{A_\mathrm{e}^2(\mathbf{p}-e\mathbf{A}_\mathrm{m})_x}\right]\left[1+\alpha_\mathrm{e}^2\left(1-\frac{v^2}{c^2}\right)\right]^{-1} \quad (37)$$

$$\dot{y} = \frac{(1-v^2/c^2)^{1/2}}{m_0}(\mathbf{p}-e\mathbf{A}_\mathrm{m})_y\left[1+\alpha_\mathrm{e}\left(1-\frac{v^2}{c^2}\right)^{1/2}\frac{[\mathbf{A}_\mathrm{e}\times(\mathbf{p}-e\mathbf{A}_\mathrm{m}]_y}{A_\mathrm{e}(\mathbf{p}-e\mathbf{A}_\mathrm{m})_y}\right.$$
$$\left.+\alpha_\mathrm{e}^2\left(1-\frac{v^2}{c^2}\right)\frac{A_{\mathrm{e}y}\mathbf{A}_\mathrm{e}\cdot(\mathbf{p}-e\mathbf{A}_\mathrm{m})}{A_\mathrm{e}^2(\mathbf{p}-e\mathbf{A}_\mathrm{m})_y}\right]\left[1+\alpha_\mathrm{e}^2\left(1-\frac{v^2}{c^2}\right)\right]^{-1} \quad (38)$$

$$\dot{z} = \frac{(1-v^2/c^2)^{1/2}}{m_0}(\mathbf{p}-e\mathbf{A}_\mathrm{m})_z\left[1+\alpha_\mathrm{e}\left(1-\frac{v^2}{c^2}\right)^{1/2}\frac{[\mathbf{A}_\mathrm{e}\times(\mathbf{p}-e\mathbf{A}_\mathrm{m}]_z}{A_\mathrm{e}(\mathbf{p}-e\mathbf{A}_\mathrm{m})_z}\right.$$
$$\left.+\alpha_\mathrm{e}^2\left(1-\frac{v^2}{c^2}\right)\frac{A_{\mathrm{e}z}\mathbf{A}_\mathrm{e}\cdot(\mathbf{p}-e\mathbf{A}_\mathrm{m})}{A_\mathrm{e}^2(\mathbf{p}-e\mathbf{A}_\mathrm{m})_z}\right]\left[1+\alpha_\mathrm{e}^2\left(1-\frac{v^2}{c^2}\right)\right]^{-1} \quad (39)$$

Equations (37)–(39) are squared and summed:

$$\dot{x}^2+\dot{y}^2+\dot{z}^2 = v^2 = \frac{1-v^2/c^2}{m_0^2}(\mathbf{p}-e\mathbf{A}_\mathrm{m})^2\left[1\right.$$
$$+2\alpha_\mathrm{e}\left(1-\frac{v^2}{c^2}\right)^{1/2}\frac{[\mathbf{A}_\mathrm{e}\cdot(\mathbf{p}-e\mathbf{A}_\mathrm{m})]^2}{\mathbf{A}_\mathrm{e}^2(\mathbf{p}-e\mathbf{A}_\mathrm{m})^2}$$
$$+\alpha_\mathrm{e}^2\left(1-\frac{v^2}{c^2}\right)\frac{[\mathbf{A}_\mathrm{e}\times(\mathbf{p}-e\mathbf{A}_\mathrm{m})]^2+2[\mathbf{A}_\mathrm{e}\cdot(\mathbf{p}-e\mathbf{A}_\mathrm{m})]^2}{\mathbf{A}_\mathrm{e}^2(\mathbf{p}-e\mathbf{A}_\mathrm{m})^2}$$
$$+2\alpha_\mathrm{e}^3\left(1-\frac{v^2}{c^2}\right)^{3/2}\frac{\mathbf{A}_\mathrm{e}\cdot[\mathbf{A}_\mathrm{e}\times(\mathbf{p}-e\mathbf{A}_\mathrm{m})]\mathbf{A}_\mathrm{e}\cdot(\mathbf{p}-e\mathbf{A}_\mathrm{m})}{A_\mathrm{e}^3\,(\mathbf{p}-e\mathbf{A}_\mathrm{m})^2}$$
$$\left.+\alpha_\mathrm{e}^4\left(1-\frac{v^2}{c^2}\right)^2\frac{[\mathbf{A}_\mathrm{e}\cdot(\mathbf{p}-e\mathbf{A}_\mathrm{m})]^2}{\mathbf{A}_\mathrm{e}^2\,(\mathbf{p}-e\mathbf{A}_\mathrm{m})^2}\right]\left[1+\alpha_\mathrm{e}^2\left(1-\frac{v^2}{c^2}\right)\right]^{-2} \quad (40)$$

To find an approximate solution of this equation for

$$\alpha_\mathrm{e}(1-v^2/c^2)^{1/2} \ll 1 \quad (41)$$

we use only the first term on the right side of Eq.(40):

$$v^2 = \frac{1-v^2/c^2}{m_0^2}(\mathbf{p}-e\mathbf{A}_\mathrm{m})^2$$
$$\left(1-\frac{v^2}{c^2}\right)^{1/2} = \frac{m_0c}{\left[m_0^2c^2+(\mathbf{p}-e\mathbf{A}_\mathrm{m})^2\right]^{1/2}} \quad (42)$$

This is the same relation as shown in the first line of Eq.(31) and we get again the common term of the three components $\mathcal{H}_x$, $\mathcal{H}_y$, $\mathcal{H}_z$ of Eqs.(32)–(34). We call this the zero order solution in α_e of Eq.(40).

To obtain the first order solution in α_e of Eq.(40) we use the first two terms on the right of Eq.(40). Since $\mathbf{p}$ will eventually be replaced by differential operators we are careful to preserve the sequence of factors:

$$v^2 = \frac{1 - v^2/c^2}{m_0^2}(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 \left[1 + 2\alpha_\mathrm{e}\left(1 - \frac{v^2}{c^2}\right)^{1/2} \frac{[\mathbf{A}_\mathrm{e}\cdot(\mathbf{p} - e\mathbf{A}_\mathrm{m})]^2}{\mathbf{A}_\mathrm{e}^2(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2}\right] \quad (43)$$

The term $(1-v^2/c^2)^{1/2}$ multiplied by $2\alpha_e$ is replaced by the zero order solution of Eq.(42). The resulting equation is solved for $1 - v^2/c^2$ and the following improvement of Eq.(42) in first order of α_e is obtained:

$$\begin{aligned} 1 - \frac{v^2}{c^2} = {} & \frac{m_0^2c^2}{m_0^2c^2 + (\mathbf{p} - e\mathbf{A}_\mathrm{m})^2} \\ & \times \left(1 - \frac{2\alpha_\mathrm{e} m_0 c(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2}{[m_0^2c^2 + (\mathbf{p} - e\mathbf{A}_\mathrm{m})^2]^{3/2}} \frac{[\mathbf{A}_\mathrm{e}\cdot(\mathbf{p} - e\mathbf{A}_\mathrm{m})]^2}{\mathbf{A}_\mathrm{e}^2(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2}\right) + O(\alpha_\mathrm{e}^2) \end{aligned} \quad (44)$$

The first order approximation in α_e of Eq.(31) follows from Eq.(44):

$$\begin{aligned} \frac{m_0c^2}{(1 - v^2/c^2)^{1/2}} = {} & c\left[(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 + m_0^2c^2\right]^{1/2} \\ & \times \left(1 + \frac{\alpha_\mathrm{e} m_0 c(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2}{[(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 + m_0^2c^2]^{3/2}} \frac{[\mathbf{A}_\mathrm{e}\cdot(\mathbf{p} - e\mathbf{A}_\mathrm{m})]^2}{\mathbf{A}_\mathrm{e}^2(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2}\right) + O(\alpha_\mathrm{e}^2) \end{aligned} \quad (45)$$

The three components of the Hamilton function of Eqs.(21)–(23) become in this approximation:

$$\mathcal{H}_x = c[(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 + m_0^2c^2]^{1/2}(1 + \alpha_\mathrm{e}Q) + e\phi_\mathrm{e} - \mathcal{L}_{cx} \quad (46)$$

$$\mathcal{H}_y = c[(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 + m_0^2c^2]^{1/2}(1 + \alpha_\mathrm{e}Q) + e\phi_\mathrm{e} - \mathcal{L}_{cy} \quad (47)$$

$$\mathcal{H}_z = c[(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 + m_0^2c^2]^{1/2}(1 + \alpha_\mathrm{e}Q) + e\phi_\mathrm{e} - \mathcal{L}_{cz} \quad (48)$$

$$Q = \frac{1}{m_0^2c^2}\frac{(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2[\mathbf{A}_\mathrm{e}\cdot(\mathbf{p} - e\mathbf{A}_\mathrm{m})]^2}{[1 + (\mathbf{p} - e\mathbf{A}_\mathrm{m})^2/m_0^2c^2]^{3/2}\,\mathbf{A}_\mathrm{e}^2(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2}$$

$$\alpha_\mathrm{e} = \frac{ZecA_\mathrm{e}}{m_0c^2} = 2\frac{Ze^2}{2h}\frac{h}{m_0c}\frac{A_\mathrm{e}}{e} = 2\alpha\frac{\lambda_\mathrm{C}A_\mathrm{e}}{e}$$

$$\alpha = \frac{Ze^2}{2h} \doteq 7.297\,535 \times 10^{-3} \text{ fine structure constant, } \lambda_\mathrm{C} = \frac{h}{m_0c}$$

$$\alpha_\mathrm{e} = 2.210 \times 10^5 A_\mathrm{e} \text{ for electron, } \alpha_\mathrm{e} = 1.204 \times 10^2 A_\mathrm{e} \text{ for proton} \quad (49)$$

For the evaluation of the terms $\mathcal{L}_{cx}$, $\mathcal{L}_{cy}$, $\mathcal{L}_{cz}$ we need $\dot{x}$, $\dot{y}$, $\dot{z}$ of Eqs.(37)–(39) with the terms $1 - v^2/c^2$ eliminated by means of Eq.(44). Only the zero order approximation in α_{e} will be required:

$$\dot{x} = \frac{c(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_x}{[m_0^2c^2 + (\mathbf{p} - e\mathbf{A}_{\mathrm{m}})^2]^{1/2}} + O(\alpha_{\mathrm{e}}) \tag{50}$$

$$\dot{y} = \frac{c(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_y}{[m_0^2c^2 + (\mathbf{p} - e\mathbf{A}_{\mathrm{m}})^2]^{1/2}} + O(\alpha_{\mathrm{e}}) \tag{51}$$

$$\dot{z} = \frac{c(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_z}{[m_0^2c^2 + (\mathbf{p} - e\mathbf{A}_{\mathrm{m}})^2]^{1/2}} + O(\alpha_{\mathrm{e}}) \tag{52}$$

For the first component $\mathcal{L}_{cx1}$ of $\mathcal{L}_{cx}$ we obtain from Eq.(3.2-15):

$$\mathcal{L}_{cx1} = \frac{Ze}{c}(A_{\mathrm{e}z}\dot{y} - A_{\mathrm{e}y}\dot{z})\dot{x} = \frac{\alpha_{\mathrm{e}}}{A_{\mathrm{e}}m_0}\frac{A_{\mathrm{e}z}(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_y - A_{\mathrm{e}y}(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_z}{[1 + (\mathbf{p} - e\mathbf{A}_{\mathrm{m}})^2/m_0^2c^2]^{1/2}}$$
$$\times \frac{(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_x}{[1 + (\mathbf{p} - e\mathbf{A}_{\mathrm{m}})^2/m_0^2c^2]^{1/2}} + O(\alpha_{\mathrm{e}}^2) \tag{53}$$

The unusual way of writing the denominators is due to the replacement of **p** by operators at a later time and the resulting non-commutability of the factors of a product. This will become important in Section 5.2. The second component $\mathcal{L}_{cx2}$ is defined according to Eq.(3.2-15) as follows:

$$\mathcal{L}_{cx2} = \frac{Ze}{c}\int\left(\frac{\partial\phi_{\mathrm{m}}}{\partial y}\dot{z} - \frac{\partial\phi_{\mathrm{m}}}{\partial z}\dot{y}\right)dx$$
$$= \frac{\alpha_{\mathrm{e}}}{A_{\mathrm{e}}}\int\left(\frac{\partial\phi_{\mathrm{m}}}{\partial y}(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_z - \frac{\partial\phi_{\mathrm{m}}}{\partial z}(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_y\right)$$
$$\times\left(1 + \frac{(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})^2}{m_0^2c^2}\right)^{-1/2} dx \tag{54}$$

For the third component $\mathcal{L}_{cx3}$ we require the time derivative of $\dot{y}$ and $\dot{z}$ of Eqs.(50)–(52):

$$\mathcal{L}_{cx3} = \frac{Ze}{c}\int(A_{\mathrm{e}z}\ddot{y} - A_{\mathrm{e}y}\ddot{z})dx$$
$$= \frac{\alpha_{\mathrm{e}}}{A_{\mathrm{e}}}\int\left[A_{\mathrm{e}z}\frac{\partial}{\partial t}\left(\frac{(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_y}{[1 + (\mathbf{p} - e\mathbf{A}_{\mathrm{m}})^2/m_0^2c^2]^{1/2}}\right)\right.$$
$$\left. - A_{\mathrm{e}y}\frac{\partial}{\partial t}\left(\frac{(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_z}{[1 + (\mathbf{p} - e\mathbf{A}_{\mathrm{m}})^2/m_0^2c^2]^{1/2}}\right)\right]dx \tag{55}$$

The fourth component $\mathcal{L}_{cx4}$ of Eq.(3.2-15) remains unchanged since there are no time derivatives of x, y, or z:

$$\mathcal{L}_{cx4} = Zec \int \left(\frac{\partial A_{ey}}{\partial z} - \frac{\partial A_{ez}}{\partial y} \right) dx = \frac{\alpha_e m_0 c^2}{A_e} \int \left(\frac{\partial A_{ey}}{\partial z} - \frac{\partial A_{ez}}{\partial y} \right) dx \qquad (56)$$

The fifth and last component $\mathcal{L}_{cx5}$ of Eq.(3.2-15) equals:

$$\begin{aligned}\mathcal{L}_{cx5} &= \frac{Ze}{c} \int \left(\dot{y} \frac{\partial}{\partial y} + \dot{z} \frac{\partial}{\partial z} \right) (A_{ex}\dot{y} - A_{ey}\dot{z}) dx \\ &= \frac{\alpha_e}{A_e m_0} \int \left(\frac{(\mathbf{p} - e\mathbf{A}_m)_y \partial/\partial y + (\mathbf{p} - e\mathbf{A}_m)_z \partial/\partial z}{[1 + (\mathbf{p} - e\mathbf{A}_m)^2/m_0^2c^2]^{1/2}} \right) \\ &\qquad \times \frac{A_{ez}(\mathbf{p} - e\mathbf{A}_m)_y - A_{ey}(\mathbf{p} - e\mathbf{A}_m)_z}{[1 + (\mathbf{p} - e\mathbf{A}_m)^2/m_0^2c^2]^{1/2}} dx \qquad (57)\end{aligned}$$

The correcting terms $\mathcal{L}_{cy}$ and $\mathcal{L}_{cz}$ for Eqs.(47) and (48) are obtained from Eqs.(53)–(57) by the substitutions $x \to y \to z \to x$.

The Klein-Gordon and the Dirac equations are derived from the Hamilton functions in Eqs.(32)–(34) without the correcting terms $\mathcal{L}_{cx}$, $\mathcal{L}_{cy}$, and $\mathcal{L}_{cz}$. The Hamilton functions of Eqs.(46)–(48) with first order correction in α_e will yield first order corrections to the Klein-Gordon and the Dirac equations, while higher order solutions in α_e of Eq.(40) will yield higher order corrections.

It is important to keep in mind that the term $\mathbf{g}_m$ in the Maxwell equation (1.1-9) had an effect on the solution even if the transition $\mathbf{g}_m \to 0$ was made at the end of the calculation since the term $\mathbf{g}_m$ produced the term $s\sigma E$ in Eq.(1.3-1). Without $\mathbf{g}_m$ one obtains a different differential equation that leads to divergent integrals as discussed in Section 1.4 and in Section 6.2 from Eq.(6.2-46) on. According to Eq.(1.6-26) the potential $\mathbf{A}_e$ represents the magnetic current density $\mathbf{g}_m$ here. It is prudent to expect that the transition $\mathbf{A}_e \to 0$ at the end of the calculation may have a similar effect as the transition $\mathbf{g}_m \to 0$. Hence, the terms with a factor α_e in Eq.(46)–(48) cannot be ignored even if one takes the limit $\mathbf{A}_e \to 0$ and $\mathbf{g}_m \to 0$ at the end of the calculation.

If we want to carry the theory to second order in α_e we must use the first three terms on the right side of Eq.(40). For the term $\alpha_e^2(1 - v^2/c^2)$ we must use $1 - v^2/c^2$ of Eq.(42) while for $2\alpha_e(1 - v^2/c^2)^{1/2}$ we must use the better approximation $1 - v^2/c^2$ of Eq.(44). We may proceed in this way to the third and fourth order approximation in α_e. But the process does not stop there. For the fourth order approximation we use for the term $\alpha_e^4(1 - v^2/c^2)^2$ in Eq.(40) the value of $1 - v^2/c^2$ of Eq.(42), but for the fifth order approximation we use $1 - v^2/c^2$ of Eq.(44). There is no end. Every improved approximation yields new terms and presumably new effects. The one term $\mathbf{g}_m$ in Eq.(1.1-9) of Maxwell's modified equations produces denumerably many terms in the Hamilton function of a charged, relativistic particle.

3.4 Steady State Solutions of the Modified Maxwell Equations

It is usual to derive the Hamiltonian form of Maxwell's equations for steady state solutions only. This is acceptable since we have seen that Maxwell's equations generally do not have solutions that satisfy the causality law and thus they can be valid for the steady state only. If infinitely extended periodic sinusoidal waves are used to represent a steady state solution one must observe that such solutions usually represent waves with infinite energy. Hence, they are outside both the causality law and the conservation law of energy; one should not be surprised if this leads to problems with infinite energy. It is possible to derive solutions of Maxwell's equations that satisfy the conservation law of energy but not the causality law. Such solutions based on the Gaussian bell function $e^{-\theta^2}$ have been developed (King and Harrison 1968; King 1993). They could be extended to the complete orthogonal system of parabolic cylinder functions, but few people seem to be aware of these non-sinusoidal solutions of Maxwell's equations.

Since we do not want to loose the causality of the modified Maxwell equations we will have to find a way to derive the Hamiltonian form in a more general way. In order to do so we derive first the Hamiltonian form in the conventional way developed for the original Maxwell equations and obtain a steady state theory. But this steady state theory will show what changes have to be made to derive the Hamiltonian form for a theory that satisfies the causality law. The corrected quantization will be carried out in Chapter 4.

In order to rewrite Maxwell's equations (1.1-1)–(1.1-7) in Hamiltonian form it is usual to choose $\rho_e = 0$ and $\mathbf{g}_e = 0$. According to the discussion following Eq.(2.5-11) it is permissible to choose $\rho_e = 0$ as well as $\rho_m = 0$ due to the law of conservation of charge, which we extend here to apply to electric and (hypothetical) magnetic charges. The choice $\mathbf{g}_e = 0$ and $\mathbf{g}_m = 0$ cannot be justified by any widely accepted physical law since current densities may be due to dipole and higher order multipole currents in the absence of monopole currents carried by charges. Indeed, the choice $\mathbf{g}_e = 0$ implies that a capacitor with vacuum as dielectric cannot be charged, that it cannot pass an alternating current, that the permittivity of vacuum is zero, and the velocity of light infinite. If we choose either $\mathbf{g}_e = 0$ or $\mathbf{g}_m = 0$ in the modified Maxwell equations (1.1-8) and (1.1-9) we get $s\sigma E = 0$ in Eq.(1.3-1) and obtain a differential equation that does not lead to solutions that satisfy the causality law. Hence, the choice $\mathbf{g}_e = 0$ and $\mathbf{g}_m = 0$ at the beginning of a calculation yields unacceptable equations, but one may try the transition $\mathbf{g}_e \rightarrow 0$ and $\mathbf{g}_m \rightarrow 0$ at the end of the calculation as shown by Eq.(6.2-41) for $\omega = \sqrt{\epsilon s/\mu\sigma} \rightarrow 0$. The closest we can come to the conventional derivation without giving up the causality law is to use Maxwell's modified equations with Coulomb gauge as shown by Eqs.(2.5-6)–(2.5-11) and choose $\rho_e = \rho_m = 0$. The electric and magnetic field strengths are then defined by Eqs.(1.6-17) and (1.6-11).

First we write the modified Maxwell equations for vacuum in accordance with Eqs.(1.6-1)–(1.6-4):

$$\operatorname{curl}\mathbf{H} = \frac{1}{Zc}\frac{\partial\mathbf{E}}{\partial t} + \mathbf{g}_{\mathrm{e}}, \qquad \operatorname{div}\mathbf{H} = 0 \tag{1}$$

$$-\operatorname{curl}\mathbf{E} = \frac{Z}{c}\frac{\partial\mathbf{H}}{\partial t} + \mathbf{g}_{\mathrm{m}}, \qquad \operatorname{div}\mathbf{E} = 0 \tag{2}$$

Then we write **H** and **E** according to Eqs.(1.6-11) and (1.6-17) for $\phi_{\mathrm{e}} = \phi_{\mathrm{m}} = 0$:

$$\mathbf{H} = \frac{c}{Z}\operatorname{curl}\mathbf{A}_{\mathrm{m}} - \frac{\partial\mathbf{A}_{\mathrm{e}}}{\partial t} \tag{3}$$

$$\mathbf{E} = -Zc\operatorname{curl}\mathbf{A}_{\mathrm{e}} - \frac{\partial\mathbf{A}_{\mathrm{m}}}{\partial t} \tag{4}$$

The potentials $\mathbf{A}_{\mathrm{m}}$ and $\mathbf{A}_{\mathrm{e}}$ in Coulomb gauge for $\rho_{\mathrm{e}} = \rho_{\mathrm{m}} = 0$ and thus $\phi_{\mathrm{e}} = \phi_{\mathrm{m}} = 0$ follow from Eqs.(2.5-6)–(2.5-9):

$$\nabla^2\mathbf{A}_{\mathrm{e}} - \frac{1}{c^2}\ddot{\mathbf{A}}_{\mathrm{e}} = -\frac{1}{Zc}\mathbf{g}_{\mathrm{m}}, \qquad \operatorname{div}\mathbf{A}_{\mathrm{e}} = 0 \tag{5}$$

$$\nabla^2\mathbf{A}_{\mathrm{m}} - \frac{1}{c^2}\ddot{\mathbf{A}}_{\mathrm{m}} = -\frac{Z}{c}\mathbf{g}_{\mathrm{e}}, \qquad \operatorname{div}\mathbf{A}_{\mathrm{m}} = 0 \tag{6}$$

We want to solve the system of partial differential equations (5) and (6), writing the result in Hamiltonian form. For general solutions we may follow the calculations of Sections 1.2 to 1.5 as well as Sections 6.1 and 6.ں. If one is satisfied with steady state solutions one may follow the conventional calculation and make a Fourier representation of $\mathbf{A}_{\mathrm{m}}(\mathbf{r},t)$. By implication we have to represent $\mathbf{A}_{\mathrm{e}}(\mathbf{r},t)$ in the same way. This is done by introducing a finite interval or a box with finite dimensions in order to be able to use a Fourier series with denumerable terms rather than a Fourier integral with non-denumerable terms. The introduction of boundary values on the surface of a box has the character of a cooking recipe: it works and people accept it. A more physical approach to obtain denumerable terms is provided by noting that infinitesimal distances dx or dt cannot be observed and thus can be only mathematical approximations for observable finite distances Δx and Δt. There is no need to require some minimum value for Δx and Δt, one only must demand that they are finite. A distance of 10^{-100} m in space or 10^{-100} s in time is still finite but completely different from dx and dt. It has been shown that the distinction between infinitesimal differences dx, dt and arbitrarily small but finite differences Δx, Δt yields no significant effect in non-relativistic quantum mechanics but this is not so in relativistic quantum mechanics (Harmuth 1992, pp. 228, 244). Hence, we must try to introduce finite intervals Δx, Δt even though we use infinitesimal intervals dx, dt for computational convenience.

For the introduction of finite intervals Δx consider the function $A(x) = A(m)$ defined at the points $m\Delta x$ with $m = -n, -n+1, \ldots, n$ as shown

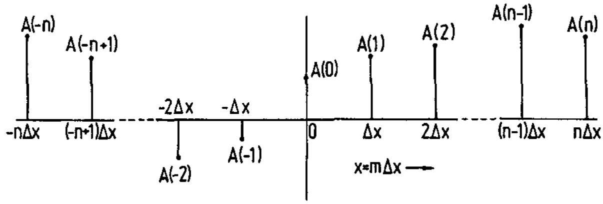

FIG.3.4-1. A function $A(x) = A(m)$ defined at the $2n+1$ points $x = m\Delta x$ with $m = -n, -n+1, \dots, n$; $L = 2n\Delta x$.

in Fig.3.4-1. We may represent the $2n+1$ samples or discrete values of this function by a series with $2n+1$ 'Fourier functions'

$$\begin{gathered} 1, \quad \sqrt{2}\cos(2\pi mx/L), \quad \sqrt{2}\sin(2\pi mx/L) \\ m = 0,\ 1, \dots,\ n; \quad -n\Delta x \le x \le n\Delta x; \quad L = 2n\Delta x \end{gathered} \tag{7}$$

where $m = 0$ denotes the constant function with amplitude 1 in the interval $-L/2 \le x \le L/2$. For $n \to \infty$ we obtain a series with denumerable terms. Let us observe that the representation of a discrete function $A(m)$ with $2n+1$ values of m by a Fourier series or other series of orthogonal functions with $2n+1$ functions is exact. The concept of mean-square-approximation enters only if one approximates a function $A(x)$ with non-denumerable values of x by a system of denumerable, orthogonal or linearly independent functions.

Let[1] the vector potential $\mathbf{A}_{\mathrm{m}}(\mathbf{r},t)$ and the (dipole) current density $\mathbf{g}_{\mathrm{e}}(\mathbf{r},t)$ of Eq.(6) be represented by a Fourier series in complex notation in the three-dimensional interval $-L/2 \le x \le L/2$, $-L/2 \le y \le L/2$, and $-L/2 \le z \le L/2$:

$$\mathbf{A}_{\mathrm{m}}(\mathbf{r},t) = \frac{1}{L^{3/2}} \sum_{\lambda=2}^{3} \sum_{\mathbf{k}} A_{\mathrm{m},\mathbf{k}\lambda} \mathbf{s}_{\mathbf{k}\lambda} \exp(i2\pi\mathbf{k}\cdot\mathbf{r}/L) \tag{8}$$

$$\mathbf{g}_{\mathrm{e}}(\mathbf{r},t) = \frac{1}{L^{3/2}} \sum_{\lambda=2}^{3} \sum_{\mathbf{k}} g_{\mathrm{e},\mathbf{k}\lambda} \mathbf{s}_{\mathbf{k}\lambda} \exp(i2\pi\mathbf{k}\cdot\mathbf{r}/L) \tag{9}$$

$$\mathbf{s}_{\mathbf{k}\lambda} = \mathbf{s}_{-\mathbf{k}\lambda} \tag{10}$$

The values $\lambda = 2,\ 3$ are used to denote the two orthogonal components of transversal waves, while $\lambda = 1$ is reserved for longitudinal waves that do not occur in Coulomb gauge. The factors $\mathbf{s}_{\mathbf{k}\lambda}$ are linearly independent unit vectors that define one of the possible two orthogonal polarizations of the wave component $\mathbf{k}$:

$$\mathbf{s}_{\mathbf{k}\lambda} = s_{\mathbf{k}\lambda x}\mathbf{e}_x + s_{\mathbf{k}\lambda y}\mathbf{e}_y + s_{\mathbf{k}\lambda z}\mathbf{e}_z, \quad s_{\mathbf{k}\lambda x}^2 + s_{\mathbf{k}\lambda y}^2 + s_{\mathbf{k}\lambda z}^2 = 1 \tag{11}$$

[1]Section 3.4 follows closely § 51 of vol. 2 of the book by Becker (1963, 1964 a) to facilitate comparison with the conventional theory.

The use of $\mathbf{s}_{\mathbf{k}\lambda}$ in both Eq.(8) and (9) does not imply that $\mathbf{A}_{\mathrm{m}}$ and $\mathbf{g}_{\mathrm{e}}$ have the same direction, only that they are composed with the same vectors $\mathbf{s}_{\mathbf{k}\lambda}$. The terms $\mathbf{k}$ and $\mathbf{r}$ stand for the vectors

$$
\begin{aligned}
&\mathbf{k} = k_x\mathbf{e}_x + k_y\mathbf{e}_y + k_z\mathbf{e}_z, \quad k_x,\ k_y,\ k_z = 0,\ \pm1,\ \pm2,\ \dots \\
&\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z
\end{aligned} \tag{12}
$$

and the sum over $\mathbf{k}$ in Eqs.(8) and (9) represents the three sums over k_x, k_y, and k_z. We use the factor $L^{-3/2}$ rather than L^{-3} for normalization since the square of this factor will be needed. Equation (10) states that the polarization remains unchanged if we replace $\mathbf{k}$ by $-\mathbf{k}$ or k_x, k_y, k_z by $-k_x$, $-k_y$, $-k_z$. With this choice for the polarization we can make the right sides of Eqs.(8) and (9) real like the left sides by demanding that a change of the sign of $\mathbf{k}$ produces the complex conjugate:

$$
\begin{aligned}
A_{\mathrm{m},-\mathbf{k}\lambda} &= A_{\mathrm{m},-\mathbf{k},\lambda} = A^*_{\mathrm{m},\mathbf{k},\lambda} = A^*_{\mathrm{m},\mathbf{k}\lambda} \\
g_{\mathrm{e},-\mathbf{k}\lambda} &= g_{\mathrm{e},-\mathbf{k},\lambda} = g^*_{\mathrm{e},\mathbf{k},\lambda} = g^*_{\mathrm{e},\mathbf{k}\lambda}
\end{aligned} \tag{13}
$$

The divergence of $\mathbf{A}_{\mathrm{m}}(\mathbf{r},t)$ is zero according to Eq.(6)

$$
\begin{aligned}
\operatorname{div}\mathbf{A}_{\mathrm{m}}(\mathbf{r},t) &= \frac{\partial A_{\mathrm{m}x}}{\partial x} + \frac{\partial A_{\mathrm{m}y}}{\partial y} + \frac{\partial A_{\mathrm{m}z}}{\partial z} = 0 \\
&= \frac{2\pi i}{L^{5/2}}\sum_{\lambda=2}^{3}\sum_{\mathbf{k}} A_{\mathrm{m},\mathbf{k}\lambda}(s_{\mathbf{k}\lambda x}k_x + s_{\mathbf{k}\lambda y}k_y + s_{\mathbf{k}\lambda z}k_z)\exp(i2\pi\mathbf{k}\cdot\mathbf{r}/L)
\end{aligned} \tag{14}
$$

which implies

$$
\mathbf{s}_{\mathbf{k}\lambda}\cdot\mathbf{k} = 0 \tag{15}
$$

The waves are thus transverse and λ can have only two values as assumed in Eqs.(8) and (9). Substitution of Eqs.(8) and (9) into Eq.(6) yields:

$$
\begin{aligned}
\ddot{A}_{\mathrm{m},\mathbf{k}\lambda} + \omega_{\mathbf{k}}^2 A_{\mathrm{m},\mathbf{k}\lambda} &= -\frac{Z}{c}g_{\mathrm{e},\mathbf{k}\lambda} \\
\omega_{\mathbf{k}}^2 = (2\pi c\mathbf{k}/L)^2 &= (2\pi c/L)^2(k_x^2 + k_y^2 + k_z^2)
\end{aligned} \tag{16}
$$

The general solution of the homogeneous equation is given by:

$$
A_{\mathrm{m},\mathbf{k}\lambda}(t) = B_{\mathrm{m},\mathbf{k}\lambda}e^{i\omega_{\mathbf{k}}t} + C_{\mathrm{m},\mathbf{k}\lambda}e^{-i\omega_{\mathbf{k}}t} \tag{17}
$$

A particular solution of the inhomogeneous equation is obtained by the method of variation of the constant:

$$A_{\mathrm{m},\mathbf{k}\lambda}(t) = B_{\mathrm{m},\mathbf{k}\lambda}(t)e^{i\omega_{\mathbf{k}}t} + C_{\mathrm{m},\mathbf{k}\lambda}(t)e^{-i\omega_{\mathbf{k}}t} \tag{18}$$

Differentiation $\partial^2 A_{\mathrm{m},\mathbf{k}\lambda}(t)/\partial t^2$ and substitution into Eq.(16) yields:

$$\frac{\partial}{\partial t}\left(\frac{\partial B_{\mathrm{m},\mathbf{k}\lambda}(t)}{\partial t}e^{i\omega_{\mathbf{k}}t} + \frac{\partial C_{\mathrm{m},\mathbf{k}\lambda}(t)}{\partial t}e^{-i\omega_{\mathbf{k}}t}\right) + i\omega_{\mathbf{k}}\left(\frac{\partial B_{\mathrm{m},\mathbf{k}\lambda}(t)}{\partial t}e^{i\omega_{\mathbf{k}}t} - \frac{\partial C_{\mathrm{m},\mathbf{k}\lambda}(t)}{\partial t}e^{-i\omega_{\mathbf{k}}t}\right) = -\frac{Z}{c}g_{\mathrm{e},\mathbf{k}\lambda} \tag{19}$$

This equation is satisfied if the following two equations hold:

$$\frac{\partial B_{\mathrm{m},\mathbf{k}\lambda}(t)}{\partial t}e^{i\omega_{\mathbf{k}}t} + \frac{\partial C_{\mathrm{m},\mathbf{k}\lambda}(t)}{\partial t}e^{-i\omega_{\mathbf{k}}t} = 0 \tag{20}$$

$$\frac{\partial B_{\mathrm{m},\mathbf{k}\lambda}(t)}{\partial t}e^{i\omega_{\mathbf{k}}t} - \frac{\partial C_{\mathrm{m},\mathbf{k}\lambda}(t)}{\partial t}e^{-i\omega_{\mathbf{k}}t} = -\frac{Z}{c}g_{\mathrm{e},\mathbf{k}\lambda} \tag{21}$$

Addition and subtraction of Eqs.(20) and (21) yield $B_{\mathrm{m},\mathbf{k}\lambda}(t)$ and $C_{\mathrm{m},\mathbf{k}\lambda}(t)$:

$$B_{\mathrm{m},\mathbf{k}\lambda}(t) = +i\frac{Z}{2c\omega_{\mathbf{k}}}\int g_{\mathrm{e},\mathbf{k}\lambda}e^{-i\omega_{\mathbf{k}}t}dt \tag{22}$$

$$C_{\mathrm{m},\mathbf{k}\lambda}(t) = -i\frac{Z}{2c\omega_{\mathbf{k}}}\int g_{\mathrm{e},\mathbf{k}\lambda}e^{i\omega_{\mathbf{k}}t}dt \tag{23}$$

The particular solution of Eq.(17) becomes

$$A_{\mathrm{m},\mathbf{k}\lambda}(t) = i\frac{Z}{2c\omega_{\mathbf{k}}}\left(e^{i\omega_{\mathbf{k}}t}\int g_{\mathrm{e},\mathbf{k}\lambda}e^{-i\omega_{\mathbf{k}}t}dt - e^{-i\omega_{\mathbf{k}}t}\int g_{\mathrm{e},\mathbf{k}\lambda}e^{i\omega_{\mathbf{k}}t}dt\right) \tag{24}$$

and the general solution of Eq.(16) equals

$$A_{\mathrm{m},\mathbf{k}\lambda}(t) = B_{\mathrm{m},\mathbf{k}\lambda}e^{i\omega_{\mathbf{k}}t} + C_{\mathrm{m},\mathbf{k}\lambda}e^{-i\omega_{\mathbf{k}}t} + i\frac{Z}{2c\omega_{\mathbf{k}}}\left(e^{i\omega_{\mathbf{k}}t}\int g_{\mathrm{e},\mathbf{k}\lambda}e^{-i\omega_{\mathbf{k}}t}dt - e^{-i\omega_{\mathbf{k}}t}\int g_{\mathrm{e},\mathbf{k}\lambda}e^{i\omega_{\mathbf{k}}t}dt\right) \tag{25}$$

where $B_{\mathrm{m},\mathbf{k}\lambda}$ and $C_{\mathrm{m},\mathbf{k}\lambda}$ are two arbitrary constants.

For Eq.(5) we obtain equivalent equations by exchanging the subscripts e and m as well as the factors Z/c and $1/Zc$:

$$\mathbf{A}_{\mathrm{e}}(\mathbf{r},t) = \frac{1}{L^{3/2}}\sum_{\lambda=2}^{3}\sum_{\mathbf{k}} A_{\mathrm{e},\mathbf{k}\lambda}\mathbf{s}_{\mathbf{k}\lambda}\exp(i2\pi\mathbf{k}\cdot\mathbf{r}/L) \tag{26}$$

$$\mathbf{g}_{\mathrm{m}}(\mathbf{r},t) = \frac{1}{L^{3/2}}\sum_{\lambda=2}^{3}\sum_{\mathbf{k}} g_{\mathrm{m},\mathbf{k}\lambda}\mathbf{s}_{\mathbf{k}\lambda}\exp(i2\pi\mathbf{k}\cdot\mathbf{r}/L) \tag{27}$$

$$\mathbf{s}_{\mathbf{k}\lambda} = \mathbf{s}_{-\mathbf{k}\lambda} \tag{28}$$

$$A_{\mathrm{e},\mathbf{k}\lambda} = A_{\mathrm{e},\mathbf{k},\lambda} = A^*_{\mathrm{e},-\mathbf{k},\lambda}, \quad g_{\mathrm{m},\mathbf{k}\lambda} = g_{\mathrm{m},\mathbf{k},\lambda} = g^*_{\mathrm{m},-\mathbf{k},\lambda} \tag{29}$$

$$\begin{aligned} \ddot{A}_{\mathrm{e},\mathbf{k}\lambda} + \omega_{\mathbf{k}}^2 A_{\mathrm{e},\mathbf{k}\lambda} &= -\frac{1}{Zc} g_{\mathrm{m},\mathbf{k}\lambda} \\ \omega_{\mathbf{k}}^2 = (2\pi c\mathbf{k}/L)^2 &= (2\pi c/L)^2(k_x^2 + k_y^2 + k_z^2) \end{aligned} \tag{30}$$

$$\begin{aligned} A_{\mathrm{e},\mathbf{k}\lambda}(t) &= B_{\mathrm{e},\mathbf{k}\lambda} e^{i\omega_{\mathbf{k}}t} + C_{\mathrm{e},\mathbf{k}\lambda} e^{-i\omega_{\mathbf{k}}t} \\ &+ i\frac{1}{2Zc\omega_{\mathbf{k}}}\left(e^{i\omega_{\mathbf{k}}t}\int g_{\mathrm{m},\mathbf{k}\lambda} e^{-i\omega_{\mathbf{k}}t}dt - e^{-i\omega_{\mathbf{k}}t}\int g_{\mathrm{m},\mathbf{k}\lambda} e^{i\omega_{\mathbf{k}}t}dt\right) \end{aligned} \tag{31}$$

We note that the system of functions $\{L^{-3/2}\exp(i2\pi\mathbf{kr}/L)\}$ is normalized as well as orthogonal:

$$\begin{aligned} L^{-3}\int_{-L/2}^{L/2}\int_{-L/2}^{L/2}\int_{-L/2}^{L/2} \exp(i2\pi\mathbf{k}\cdot\mathbf{r}/L)\exp(-i2\pi\mathbf{j}\cdot\mathbf{r}/L)dx\,dy\,dz &= \delta_{\mathbf{kj}} \\ \delta_{\mathbf{kj}} &= 1 \ \text{ for } \mathbf{k} = \mathbf{j} \\ &= 0 \ \text{ for } \mathbf{k} \neq \mathbf{j} \end{aligned} \tag{32}$$

Let us calculate the energy U of the electric and magnetic field strength in the volume defined by the limits of the integral in Eq.(32). From Eq.(2.5-12) we get:

$$U = \frac{1}{2}\iiint\left(\frac{1}{Zc}E^2 + \frac{Z}{c}H^2\right)dV \tag{33}$$

Equations (3) and (4) yield with the help of Eqs.(13) and (29):

$$\begin{aligned} E^2 &= \left(Zc\,\mathrm{curl}\,\mathbf{A}_{\mathrm{e}} + \frac{\partial\mathbf{A}_{\mathrm{m}}}{\partial t}\right)\cdot\left(Zc\,\mathrm{curl}\,\mathbf{A}_{\mathrm{e}} + \frac{\partial\mathbf{A}_{\mathrm{m}}}{\partial t}\right)^* \\ &= \frac{\partial\mathbf{A}_{\mathrm{m}}}{\partial t}\cdot\frac{\partial\mathbf{A}^*_{\mathrm{m}}}{\partial t} + Z^2c^2\,\mathrm{curl}\,\mathbf{A}_{\mathrm{e}}\cdot\mathrm{curl}\,\mathbf{A}^*_{\mathrm{e}} + 2Zc\frac{\partial\mathbf{A}_{\mathrm{m}}}{\partial t}\cdot\mathrm{curl}\,\mathbf{A}^*_{\mathrm{e}} \end{aligned} \tag{34}$$

$$H^2 = \frac{\partial\mathbf{A}_{\mathrm{e}}}{\partial t}\cdot\frac{\partial\mathbf{A}^*_{\mathrm{e}}}{\partial t} + \frac{c^2}{Z^2}\,\mathrm{curl}\,\mathbf{A}_{\mathrm{m}}\cdot\mathrm{curl}\,\mathbf{A}^*_{\mathrm{m}} - \frac{2c}{Z}\frac{\partial\mathbf{A}_{\mathrm{e}}}{\partial t}\cdot\mathrm{curl}\,\mathbf{A}^*_{\mathrm{m}} \tag{35}$$

Since only $A_{\mathrm{m},\mathbf{k}\lambda}$ in Eq.(8) is a function of time according to Eq.(25) we obtain:

$$\iiint\frac{\partial\mathbf{A}_{\mathrm{m}}}{\partial t}\cdot\frac{\partial\mathbf{A}^*_{\mathrm{m}}}{\partial t}dV = \sum_{\lambda=2}^{3}\sum_{\mathbf{k}}\dot{A}_{\mathrm{m},\mathbf{k}\lambda}\dot{A}^*_{\mathrm{m},\mathbf{k}\lambda} \tag{36}$$

The second term in Eq.(34) can be simplified by partial integration (Heitler 1954, p. 40):

$$\iiint \operatorname{curl} \mathbf{A}_{\mathrm{e}} \cdot \operatorname{curl} \mathbf{A}_{\mathrm{e}}^{*} \, dV = \oint (\mathbf{A}_{\mathrm{e}} \times \operatorname{curl} \mathbf{A}_{\mathrm{e}}^{*})_{\mathrm{n}} dO + \iiint \mathbf{A}_{\mathrm{e}} \cdot \operatorname{curl} \operatorname{curl} \mathbf{A}_{\mathrm{e}}^{*} \, dV \quad (37)$$

The surface integral vanishes either because our functions are zero outside the box with dimension L^3 or because they repeat periodically[2]. The last term of Eq.(37) becomes with the help of Eq.(6):

$$\iiint \mathbf{A}_{\mathrm{e}} \cdot \operatorname{curl} \operatorname{curl} \mathbf{A}_{\mathrm{e}}^{*} \, dV = \iiint \mathbf{A}_{\mathrm{e}} \cdot (\operatorname{grad} \operatorname{div} \mathbf{A}_{\mathrm{e}}^{*} - \nabla^2 \mathbf{A}_{\mathrm{e}}^{*}) dV = -\iiint \mathbf{A}_{\mathrm{e}} \cdot \left(\frac{1}{c^2} \mathbf{A}_{\mathrm{e}}^{*} - \frac{1}{Zc} \mathbf{g}_{\mathrm{m}}^{*} \right) dV \quad (38)$$

If we substitute Eqs.(26) and (29) we get a cancellation of the term $\mathbf{g}_{\mathrm{m}}^{*}$ and obtain the final result:

$$\iiint \operatorname{curl} \mathbf{A}_{\mathrm{e}} \cdot \operatorname{curl} \mathbf{A}_{\mathrm{e}}^{*} \, dV = \sum_{\lambda=2}^{3} \sum_{\mathbf{k}} \frac{\omega_{\mathbf{k}}^2}{c^2} A_{\mathrm{e},\mathbf{k}\lambda} A_{\mathrm{e},\mathbf{k}\lambda}^{*} \quad (39)$$

The third term of Eq.(34) requires $\operatorname{curl} \mathbf{A}_{\mathrm{e}}^{*}$. Using Eqs.(26), (11), and (12) one obtains:

$$\operatorname{curl} \mathbf{A}_{\mathrm{e}}^{*} = \frac{2\pi i}{L^{5/2}} \sum_{\lambda=2}^{3} \sum_{\mathbf{k}} A_{\mathrm{e},\mathbf{k}\lambda} [(s_{\mathbf{k}\lambda y} k_z - s_{\mathbf{k}\lambda z} k_y)\mathbf{e}_x + (s_{\mathbf{k}\lambda z} k_x - s_{\mathbf{k}\lambda x} k_z)\mathbf{e}_y + (s_{\mathbf{k}\lambda x} k_y - s_{\mathbf{k}\lambda y} k_x)\mathbf{e}_z] \exp(-i2\pi \mathbf{k} \cdot \mathbf{r}/L) \quad (40)$$

For $\partial \mathbf{A}_{\mathrm{m}}/\partial t$ we get from Eqs.(8) and (11):

$$\frac{\partial \mathbf{A}_{\mathrm{m}}}{\partial t} = \frac{1}{L^{3/2}} \sum_{\lambda=2}^{3} \sum_{\mathbf{k}} \dot{A}_{\mathrm{m},\mathbf{k}\lambda} (s_{\mathbf{k}\lambda x} \mathbf{e}_x + s_{\mathbf{k}\lambda y} \mathbf{e}_y + s_{\mathbf{k}\lambda z} \mathbf{e}_z) \exp(i2\pi \mathbf{k} \cdot \mathbf{r}/L) \quad (41)$$

The inner product of Eqs.(40) and (41) yields zero:

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[2]The Fourier functions of Eq.(7) may be assumed to be zero outside the interval $-L/2 \leq x \leq L/2$ or periodically continued. The assumption of periodic continuation is more usual but in a science based on observation it seems to be preferable not to claim that every assumption and every result is continued periodically to infinity since infinite distances in time or space are beyond observation.

$$\frac{\partial \mathbf{A}_{\mathrm{m}}}{\partial t} \cdot \operatorname{curl} \mathbf{A}_{\mathrm{e}}^{*} = 0 \tag{42}$$

The three terms of H^2 in Eq.(35) yield the following result in analogy to Eqs.(36), (39), and (42):

$$\iiint \frac{\partial \mathbf{A}_{\mathrm{e}}}{\partial t} \cdot \frac{\partial \mathbf{A}_{\mathrm{e}}^{*}}{\partial t}\, dV = \sum_{\lambda=2}^{3} \sum_{\mathbf{k}} \dot{A}_{\mathrm{e},\mathbf{k}\lambda} \dot{A}_{\mathrm{e},\mathbf{k}\lambda}^{*} \tag{43}$$

$$\iiint \operatorname{curl} \mathbf{A}_{\mathrm{m}} \cdot \operatorname{curl} \mathbf{A}_{\mathrm{m}}^{*}\, dV = \sum_{\lambda=2}^{3} \sum_{\mathbf{k}} \frac{\omega_{\mathbf{k}}^{2}}{c^{2}} A_{\mathrm{m},\mathbf{k}\lambda} A_{\mathrm{m},\mathbf{k}\lambda}^{*} \tag{44}$$

$$\frac{\partial \mathbf{A}_{\mathrm{e}}}{\partial t} \cdot \operatorname{curl} \mathbf{A}_{\mathrm{m}}^{*} = 0 \tag{45}$$

Substitution of Eqs.(36), (39), and (42)–(45) into Eqs.(34), (35) and then into Eq.(33) yields the energy U:

$$U = \frac{1}{2} \sum_{\lambda=2}^{3} \sum_{\mathbf{k}} \left[\frac{1}{Zc} \left(\dot{A}_{\mathrm{m},\mathbf{k}\lambda} \dot{A}_{\mathrm{m},\mathbf{k}\lambda}^{*} + \omega_{\mathbf{k}}^{2} A_{\mathrm{m},\mathbf{k}\lambda} A_{\mathrm{m},\mathbf{k}\lambda}^{*} \right) \right.$$
$$\left. + \frac{Z}{c} \left(\dot{A}_{\mathrm{e},\mathbf{k}\lambda} \dot{A}_{\mathrm{e},\mathbf{k}\lambda}^{*} + \omega_{\mathbf{k}}^{2} A_{\mathrm{e},\mathbf{k}\lambda} A_{\mathrm{e},\mathbf{k}\lambda}^{*} \right) \right] \tag{46}$$

The two terms in the first parentheses have the same form as in the conventional theory, but one must observe that $A_{\mathrm{m},\mathbf{k}\lambda}$ contains the electric dipole current density terms $g_{\mathrm{e},\mathbf{k}\lambda}$ according to Eq.(25). The two terms in the second parentheses are caused by the potential $\mathbf{A}_{\mathrm{e}}$ that is due to the magnetic dipole current density $\mathbf{g}_{\mathrm{m}}$ according to Eqs.(1.6-26) and (31).

We must rewrite the field energy U in the form of a Hamilton function $\mathcal{H}(p_j, q_j)$ so that the equations of motion (16) and (30) follow from the equations

$$\dot{p}_j = -\frac{\partial \mathcal{H}}{\partial q_j}, \quad \dot{q}_j = \frac{\partial \mathcal{H}}{\partial p_j} \tag{47}$$

The variables $A_{\mathrm{m},\mathbf{k}\lambda}$ and $A_{\mathrm{e},\mathbf{k}\lambda}$ cannot be used as coordinates in the Hamiltonian representation due to the restrictions imposed on them by Eqs.(13) and (29). But one may use the ansatz

$$a_{\mathrm{m},\mathbf{k}\lambda} = \frac{1}{2} \left(A_{\mathrm{m},\mathbf{k}\lambda} + \frac{i}{\omega_{\mathbf{k}}} \dot{A}_{\mathrm{m},\mathbf{k}\lambda} \right), \quad a_{\mathrm{m},-\mathbf{k}\lambda}^{*} = \frac{1}{2} \left(A_{\mathrm{m},\mathbf{k}\lambda} - \frac{i}{\omega_{\mathbf{k}}} \dot{A}_{\mathrm{m},\mathbf{k}\lambda} \right) \tag{48}$$

to introduce the new amplitudes $a_{\mathrm{m},\mathbf{k}\lambda}$ as well as $a_{\mathrm{m},-\mathbf{k}\lambda}^{*}$ and to obtain

$$A_{\mathrm{m},\mathbf{k}\lambda} = a_{\mathrm{m},\mathbf{k}\lambda} + a_{\mathrm{m},-\mathbf{k}\lambda}^{*}, \quad \dot{A}_{\mathrm{m},\mathbf{k}\lambda} = -i\omega_{\mathbf{k}} (a_{\mathrm{m},\mathbf{k}\lambda} - a_{\mathrm{m},-\mathbf{k}\lambda}^{*}) \tag{49}$$

Furthermore, we write for $g_{e,\mathbf{k}\lambda}$:

$$h_{e,\mathbf{k}\lambda} = \frac{1}{2} g_{e,\mathbf{k}\lambda}, \quad h^*_{e,-\mathbf{k}\lambda} = \frac{1}{2} g_{e,\mathbf{k}\lambda}$$
$$g_{e,\mathbf{k}\lambda} = h_{e,\mathbf{k}\lambda} + h^*_{e,-\mathbf{k}\lambda} \tag{50}$$

The conditions of Eq.(13) are now automatically satisfied while $a_{m,\mathbf{k}\lambda}$, $a^*_{m,\mathbf{k}\lambda}$, $h_{e,\mathbf{k}\lambda}$, and $h^*_{e,\mathbf{k}\lambda}$ are no longer subject to any restriction. This suggests the possibility of using $a_{m,\mathbf{k}\lambda}$ for the Hamiltonian representation of Eq.(46). Substitution of Eqs.(48) and (50) into Eq.(16) yields

$$-i\omega_{\mathbf{k}}(\dot{a}_{m,\mathbf{k}\lambda} - \dot{a}^*_{m,-\mathbf{k}\lambda}) + \omega_{\mathbf{k}}^2(a_{m,\mathbf{k}\lambda} + a^*_{m,-\mathbf{k}\lambda}) = Zc(h_{e,\mathbf{k}\lambda} + h^*_{e,-\mathbf{k}\lambda}) \tag{51}$$

and we get two differential equations of first order instead of one of second order:

$$\dot{a}_{m,\mathbf{k}\lambda} + i\omega_{\mathbf{k}} a_{m,\mathbf{k}\lambda} = i\frac{Zc}{\omega_{\mathbf{k}}} h_{e,\mathbf{k}\lambda}$$
$$\dot{a}^*_{m,-\mathbf{k}\lambda} - i\omega_{\mathbf{k}} a^*_{m,-\mathbf{k}\lambda} = -i\frac{Zc}{\omega_{\mathbf{k}}} h^*_{e,-\mathbf{k}\lambda}$$
$$\text{or} \qquad \dot{a}^*_{m,\mathbf{k}\lambda} - i\omega_{\mathbf{k}} a^*_{m,\mathbf{k}\lambda} = -i\frac{Zc}{\omega_{\mathbf{k}}} h^*_{e,\mathbf{k}\lambda} \tag{52}$$

In analogy to Eqs.(48)–(52) we may write for $\mathbf{A}_e$ and $\mathbf{g}_m$:

$$a_{e,\mathbf{k}\lambda} = \frac{1}{2}\left(A_{e,\mathbf{k}\lambda} + \frac{i}{\omega_{\mathbf{k}}}\dot{A}_{e,\mathbf{k}\lambda}\right), \quad a^*_{e,-\mathbf{k}\lambda} = \frac{1}{2}\left(A_{e,\mathbf{k}\lambda} - \frac{i}{\omega_{\mathbf{k}}}\dot{A}_{e,\mathbf{k}\lambda}\right) \tag{53}$$
$$A_{e,\mathbf{k}\lambda} = a_{e,\mathbf{k}\lambda} + a^*_{e,-\mathbf{k}\lambda}, \qquad \dot{A}_{e,\mathbf{k}\lambda} = -i\omega_{\mathbf{k}}(a_{e,\mathbf{k}\lambda} - a^*_{e,-\mathbf{k}\lambda}) \tag{54}$$
$$h_{m,\mathbf{k}\lambda} = \frac{1}{2} g_{m,\mathbf{k}\lambda}, \qquad h^*_{m,-\mathbf{k}\lambda} = \frac{1}{2} g_{m,\mathbf{k}\lambda} \tag{55}$$
$$\dot{a}_{e,\mathbf{k}\lambda} + i\omega_{\mathbf{k}} a_{e,\mathbf{k}\lambda} = \frac{ic}{Z\omega_{\mathbf{k}}} h_{m,\mathbf{k}\lambda}$$
$$\dot{a}^*_{e,\mathbf{k}\lambda} - i\omega_{\mathbf{k}} a^*_{e,\mathbf{k}\lambda} = -\frac{ic}{Z\omega_{\mathbf{k}}} h^*_{m,\mathbf{k}\lambda} \tag{56}$$

The field energy U of Eq.(46) may now be expressed in terms of $a_{m,\mathbf{k}\lambda}$, $a^*_{m,\mathbf{k}\lambda}$, $a_{e,\mathbf{k}\lambda}$, and $a^*_{e,\mathbf{k}\lambda}$:

$$U = \frac{1}{2}\sum_{\lambda=2}^{3}\sum_{\mathbf{k}}\left[\frac{\omega_{\mathbf{k}}^2}{Zc}\left(a^*_{m,\mathbf{k}\lambda}a_{m,\mathbf{k}\lambda} + a^*_{m,-\mathbf{k}\lambda}a_{m,-\mathbf{k}\lambda}\right) + \frac{Z\omega_{\mathbf{k}}^2}{c}\left(a^*_{e,\mathbf{k}\lambda}a_{e,\mathbf{k}\lambda} + a^*_{e,-\mathbf{k}\lambda}a_{e,-\mathbf{k}\lambda}\right)\right]$$
$$= \sum_{\lambda=2}^{3}\sum_{\mathbf{k}}\frac{\omega_{\mathbf{k}}^2}{c}\left(\frac{1}{Z}a^*_{m,\mathbf{k}\lambda}a_{m,\mathbf{k}\lambda} + Za^*_{e,\mathbf{k}\lambda}a_{e,\mathbf{k}\lambda}\right) \tag{57}$$

The terms with subscript m are separated from the terms with subscript e. This is a result of Eqs.(42) and (45), but more fundamentally it is due to Eqs.(5) and (6). The potentials $\mathbf{A}_\mathrm{m}$ are due to the electric current density $\mathbf{g}_\mathrm{e}$ and the potentials $\mathbf{A}_\mathrm{e}$ are due to the magnetic current density $\mathbf{g}_\mathrm{m}$. There is no interaction and the Hamiltonian formalism cannot change that. We may thus represent Eq.(57) by the sum

$$\mathcal{H} = \mathcal{H}_\mathrm{m} + \mathcal{H}_\mathrm{e} \tag{58}$$

and the canonical coordinates according to Eq.(47):

$$q_{\mathrm{m}j} = a_{\mathrm{m},\mathbf{k}\lambda}, \qquad p_{\mathrm{m}j} = \frac{i\omega_\mathbf{k}}{Zc} a^*_{\mathrm{m},\mathbf{k}\lambda} \tag{59}$$

$$q_{\mathrm{e}j} = a_{\mathrm{e},\mathbf{k}\lambda}, \qquad p_{\mathrm{e}j} = \frac{iZ\omega_\mathbf{k}}{c} a^*_{\mathrm{e},\mathbf{k}\lambda} \tag{60}$$

The functions $\mathcal{H}_\mathrm{m}$ and $\mathcal{H}_\mathrm{e}$ become

$$\mathcal{H}_\mathrm{m} = -i\sum_{\lambda=2}^{3}\sum_{\mathbf{k}} \omega_\mathbf{k} q_{\mathrm{m}j} p_{\mathrm{m}j} = \sum_{\lambda=2}^{3}\sum_{\mathbf{k}} \frac{\omega_\mathbf{k}^2}{Zc} a^*_{\mathrm{m},\mathbf{k}\lambda} a_{\mathrm{m},\mathbf{k}\lambda} \tag{61}$$

$$\mathcal{H}_\mathrm{e} = -i\sum_{\lambda=2}^{3}\sum_{\mathbf{k}} \omega_\mathbf{k} q_{\mathrm{e}j} p_{\mathrm{e}j} = \sum_{\lambda=2}^{3}\sum_{\mathbf{k}} \frac{Z\omega_\mathbf{k}^2}{c} a^*_{\mathrm{e},\mathbf{k}\lambda} a_{\mathrm{e},\mathbf{k}\lambda} \tag{62}$$

and their sum equals U of Eq.(57).

A simplification will be achieved later if we replace the variables $a_{\mathrm{m},\mathbf{k}\lambda}$ and $a_{\mathrm{e},\mathbf{k}\lambda}$ by new variables $b_{\mathrm{m},\mathbf{k}\lambda}$ and $b_{\mathrm{e},\mathbf{k}\lambda}$:

$$a_{\mathrm{m},\mathbf{k}\lambda} = \left(\frac{Zc\hbar}{\omega_\mathbf{k}}\right)^{1/2} b_{\mathrm{m},\mathbf{k}\lambda}, \qquad a_{\mathrm{e},\mathbf{k}\lambda} = \left(\frac{c\hbar}{Z\omega_\mathbf{k}}\right)^{1/2} b_{\mathrm{e},\mathbf{k}\lambda} \tag{63}$$

Equation (57) is rewritten:

$$U = \sum_{\lambda=2}^{3}\sum_{\mathbf{k}} \hbar\omega_\mathbf{k}(b^*_{\mathrm{m},\mathbf{k}\lambda} b_{\mathrm{m},\mathbf{k}\lambda} + b^*_{\mathrm{e},\mathbf{k}\lambda} b_{\mathrm{e},\mathbf{k}\lambda}) \tag{64}$$

The canonically conjugate coordinates and momenta are now:

$$q_{\mathrm{m}j} = b_{\mathrm{m},\mathbf{k}\lambda}, \qquad p_{\mathrm{m}j} = i\hbar b^*_{\mathrm{m},\mathbf{k}\lambda} \tag{65}$$

$$q_{\mathrm{e}j} = b_{\mathrm{e},\mathbf{k}\lambda}, \qquad p_{\mathrm{e}j} = i\hbar b^*_{\mathrm{e},\mathbf{k}\lambda} \tag{66}$$

The transition from the original Maxwell equations to the modified ones that yield solutions satisfying the causality law requires that one (a) writes the conventional equations with subscripts m or e added and (b) uses the inhomogeneous equations (16) and (30); the transition to the limits $g_{\mathrm{e},\mathbf{k}\lambda} \to 0$ and $g_{\mathrm{m},\mathbf{k}\lambda} \to 0$ may be made at the end of the calculation.

3.5 Steady State Quantization of the Modified Radiation Field

We rewrite Eqs.(3.4-61) and (3.4-62) with $a_{\mathrm{m},\mathbf{k}\lambda}$ and $a_{\mathrm{e},\mathbf{k}\lambda}$ replaced by $b_{\mathrm{m},\mathbf{k}\lambda}$ and $b_{\mathrm{e},\mathbf{k}\lambda}$ according to Eq.(3.4-63) and obtain the two Hamilton functions in the following form:

$$\mathcal{H}_{\mathrm{m}} = \sum_{\lambda=2}^{3} \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} b^{*}_{\mathrm{m},\mathbf{k}\lambda} b_{\mathrm{m},\mathbf{k}\lambda} \tag{1}$$

$$\mathcal{H}_{\mathrm{e}} = \sum_{\lambda=2}^{3} \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} b^{*}_{\mathrm{e},\mathbf{k}\lambda} b_{\mathrm{e},\mathbf{k}\lambda} \tag{2}$$

For quantization we replace the complex amplitudes $b^{*}_{\mathrm{m},\mathbf{k}\lambda}$, $b_{\mathrm{m},\mathbf{k}\lambda}$ and $b^{*}_{\mathrm{e},\mathbf{k}\lambda}$, $b_{\mathrm{e},\mathbf{k}\lambda}$ by operators $b^{+}_{\mathrm{m},\mathbf{k}\lambda}$, $b^{-}_{\mathrm{m},\mathbf{k}\lambda}$ and $b^{+}_{\mathrm{e},\mathbf{k}\lambda}$, $b^{-}_{\mathrm{e},\mathbf{k}\lambda}$:

$$b^{*}_{\mathrm{m},\mathbf{k}\lambda} \rightarrow b^{+}_{\mathrm{m},\mathbf{k}\lambda} = \frac{1}{\sqrt{2}}\left(\alpha\zeta - \frac{1}{\alpha}\frac{d}{d\zeta}\right), \quad b_{\mathrm{m},\mathbf{k}\lambda} \rightarrow b^{-}_{\mathrm{m},\mathbf{k}\lambda} = \frac{1}{\sqrt{2}}\left(\alpha\zeta + \frac{1}{\alpha}\frac{d}{d\zeta}\right) \tag{3}$$

$$b^{*}_{\mathrm{e},\mathbf{k}\lambda} \rightarrow b^{+}_{\mathrm{e},\mathbf{k}\lambda} = \frac{1}{\sqrt{2}}\left(\alpha\zeta - \frac{1}{\alpha}\frac{d}{d\zeta}\right), \quad b_{\mathrm{e},\mathbf{k}\lambda} \rightarrow b^{-}_{\mathrm{e},\mathbf{k}\lambda} = \frac{1}{\sqrt{2}}\left(\alpha\zeta + \frac{1}{\alpha}\frac{d}{d\zeta}\right) \tag{4}$$

These equations may be written with b^{*} and b interchanged:

$$b_{\mathrm{m},\mathbf{k}\lambda} \rightarrow b^{+}_{\mathrm{m},\mathbf{k}\lambda} = \frac{1}{\sqrt{2}}\left(\alpha\zeta - \frac{1}{\alpha}\frac{d}{d\zeta}\right), \quad b^{*}_{\mathrm{m},\mathbf{k}\lambda} \rightarrow b^{-}_{\mathrm{m},\mathbf{k}\lambda} = \frac{1}{\sqrt{2}}\left(\alpha\zeta + \frac{1}{\alpha}\frac{d}{d\zeta}\right) \tag{5}$$

$$b_{\mathrm{e},\mathbf{k}\lambda} \rightarrow b^{+}_{\mathrm{e},\mathbf{k}\lambda} = \frac{1}{\sqrt{2}}\left(\alpha\zeta - \frac{1}{\alpha}\frac{d}{d\zeta}\right), \quad b^{*}_{\mathrm{e},\mathbf{k}\lambda} \rightarrow b^{-}_{\mathrm{e},\mathbf{k}\lambda} = \frac{1}{\sqrt{2}}\left(\alpha\zeta + \frac{1}{\alpha}\frac{d}{d\zeta}\right) \tag{6}$$

We have left out the subscripts $m, \mathbf{k}\lambda$ and $e, \mathbf{k}\lambda$ from the differential operators in order to simplify them. The choice of one of the two possible replacements is a well known ambiguity that will be discussed briefly in Section 4.4 (Becker 1963, 1964a, vol. 2, § 52; Heitler 1954, p. 57). Here we choose Eqs.(5) and (6). Their substitution into Eqs.(1) and (2) yields:

$$\mathcal{H}_{\mathrm{m}} = \sum_{\lambda=2}^{3} \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} b^{-}_{\mathrm{m},\mathbf{k}\lambda} b^{+}_{\mathrm{m},\mathbf{k}\lambda} \tag{7}$$

$$\mathcal{H}_{\mathrm{e}} = \sum_{\lambda=2}^{3} \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} b^{-}_{\mathrm{e},\mathbf{k}\lambda} b^{+}_{\mathrm{e},\mathbf{k}\lambda} \tag{8}$$

The vector potential $\mathbf{A}_{\mathrm{m}}(\mathbf{r}, t)$ of Eq.(3.4-8) becomes an operator if we replace the coefficients $A_{\mathrm{m},\mathbf{k}\lambda}$ by $a_{\mathrm{m},\mathbf{k}\lambda} + a^{*}_{\mathrm{m},-\mathbf{k}\lambda}$ according to Eq.(3.4-49), then make the substitution of $b_{\mathrm{m},\mathbf{k}\lambda}$ for $a_{\mathrm{m},\mathbf{k}\lambda}$ according to Eq.(3.4-63), and finally replace the coefficients $b^*_{\mathrm{m},\mathbf{k}\lambda}$ and $b_{\mathrm{m},\mathbf{k}\lambda}$ by operators $b^+_{\mathrm{m},\mathbf{k}\lambda}$ and $b^-_{\mathrm{m},\mathbf{k}\lambda}$ according to Eq.(5):

$$\mathbf{A}_{\mathrm{m}}(\mathbf{r},t) = \left(\frac{Zc\hbar}{L^3}\right)^{1/2} \sum_{\lambda=2}^{3} \sum_{\mathbf{k}} s_{\mathbf{k}\lambda} \omega_{\mathbf{k}}^{-1/2} \Big(b^+_{\mathrm{m},\mathbf{k}\lambda} e^{i2\pi\mathbf{k}\cdot\mathbf{r}/L} + b^-_{\mathrm{m},\mathbf{k}\lambda} e^{-i2\pi\mathbf{k}\cdot\mathbf{r}/L} \Big) \quad (9)$$

The vector potential $\mathbf{A}_{\mathrm{e}}(\mathbf{r},t)$ of Eq.(3.4-26) is also rewritten with the help of Eqs.(3.4-54), (3.4-63), and (6). The main difference is that Z is shifted from the numerator to the denominator in Eq.(3.4-63):

$$\mathbf{A}_{\mathrm{e}}(\mathbf{r},t) = \left(\frac{c\hbar}{ZL^3}\right)^{1/2} \sum_{\lambda=2}^{3} \sum_{\mathbf{k}} s_{\mathbf{k}\lambda} \omega_{\mathbf{k}}^{-1/2} \Big(b^+_{\mathrm{e},\mathbf{k}\lambda} e^{i2\pi\mathbf{k}\cdot\mathbf{r}/L} + b^-_{\mathrm{e},\mathbf{k}\lambda} e^{-i2\pi\mathbf{k}\cdot\mathbf{r}/L} \Big) \quad (10)$$

If Eqs.(7) and (8) held generally and not for the steady state only, their quantization would require that we add a function $\Phi(\mathbf{r},t)$ on the right side and replace $\mathcal{H}$ by $-(\hbar/i)\partial\Phi(\mathbf{r},t)/\partial t$:

$$\sum_{\lambda=2}^{3} \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} b^-_{\mathrm{m},\mathbf{k}\lambda} b^+_{\mathrm{m},\mathbf{k}\lambda} \Phi_{\mathrm{m}} = \mathcal{H}_{\mathrm{m}} \Phi_{\mathrm{m}} = -\frac{\hbar}{i} \frac{\partial \Phi_{\mathrm{m}}}{\partial t} \quad (11)$$

$$\sum_{\lambda=2}^{3} \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} b^-_{\mathrm{e},\mathbf{k}\lambda} b^+_{\mathrm{e},\mathbf{k}\lambda} \Phi_{\mathrm{e}} = \mathcal{H}_{\mathrm{e}} \Phi_{\mathrm{e}} = -\frac{\hbar}{i} \frac{\partial \Phi_{\mathrm{e}}}{\partial t} \quad (12)$$

For the steady state case we must rewrite the right sides of Eqs.(11) and (12) for a sinusoidal time variation of Φ_{m} and Φ_{e}:

$$\Phi_{\mathrm{m}}(\mathbf{r},t) = \Phi_{\mathrm{m}}(\mathbf{r}) \exp(-i\mathsf{E}_{\mathrm{m}} t/\hbar) \quad (13)$$

$$\Phi_{\mathrm{e}}(\mathbf{r},t) = \Phi_{\mathrm{e}}(\mathbf{r}) \exp(-i\mathsf{E}_{\mathrm{e}} t/\hbar) \quad (14)$$

The general Eqs.(11) and (12) are then replaced by equations with eigenfunctions or eigenvectors Φ_{m} or Φ_{e} as well as eigenvalues E_{m} or E_{e}. We emphasize that the causality law has no meaning in the steady state and that the transition from Eqs.(11) and (12) to (15) and (16) implies a major reduction of the physical content:

$$\sum_{\lambda=2}^{3} \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} b^-_{\mathrm{m},\mathbf{k}\lambda} b^+_{\mathrm{m},\mathbf{k}\lambda} \Phi_{\mathrm{m}} = \mathcal{H}_{\mathrm{m}} \Phi_{\mathrm{m}} = \mathsf{E}_{\mathrm{m}} \Phi_{\mathrm{m}} \quad (15)$$

$$\sum_{\lambda=2}^{3} \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} b^-_{\mathrm{e},\mathbf{k}\lambda} b^+_{\mathrm{e},\mathbf{k}\lambda} \Phi_{\mathrm{e}} = \mathcal{H}_{\mathrm{e}} \Phi_{\mathrm{e}} = \mathsf{E}_{\mathrm{e}} \Phi_{\mathrm{e}} \quad (16)$$

Consider a certain term of the sum of Eq.(15) for specific values of the subscripts $\mathbf{k}$ and λ:

$$b^-_{\mathrm{m},\mathbf{k}\lambda} b^+_{\mathrm{m},\mathbf{k}\lambda} \Phi_\mathrm{m} = \frac{\mathsf{E}_{\mathrm{m},\mathbf{k}\lambda}}{\hbar\omega_\mathbf{k}} \Phi_\mathrm{m} \tag{17}$$

Using the explicit differential operators of Eq.(5) we obtain

$$\frac{1}{\sqrt{2}}\left(\alpha\zeta + \frac{1}{\alpha}\frac{d}{d\zeta}\right)\left[\frac{1}{\sqrt{2}}\left(\alpha\zeta - \frac{1}{\alpha}\frac{d}{d\zeta}\right)\Phi_\mathrm{m}\right] = \frac{\mathsf{E}_{\mathrm{m},\mathbf{k}\lambda}}{\hbar\omega_\mathbf{k}}\Phi_\mathrm{m}$$

$$\left(\alpha^2\zeta^2 - \frac{1}{\alpha^2}\frac{d^2}{d\zeta^2}\right)\Phi_\mathrm{m} = 2\left(\frac{\mathsf{E}_{\mathrm{m},\mathbf{k}\lambda}}{\hbar\omega_\mathbf{k}} - \frac{1}{2}\right)\Phi_\mathrm{m} = 2\lambda_{\mathrm{m},\mathbf{k}\lambda}\Phi_\mathrm{m} \tag{18}$$

$$\lambda_{\mathrm{m},\mathbf{k}\lambda} = \frac{\mathsf{E}_{\mathrm{m},\mathbf{k}\lambda}}{\hbar\omega_{\mathbf{k}\lambda}} - \frac{1}{2} \tag{19}$$

With the substitution

$$\xi = \alpha\zeta \tag{20}$$

we obtain a standard form of the differential equation of the parabolic cylinder functions

$$\frac{d^2\Phi_\mathrm{m}}{d\xi^2} + (2\lambda_{\mathrm{m},\mathbf{k}\lambda} - \xi^2)\Phi_\mathrm{m} = 0 \tag{21}$$

with the solution

$$\Phi_\mathrm{m} = e^{-\xi^2/2}\chi(\xi) \tag{22}$$

where $\chi(\xi)$ satisfies the differential equation of the Hermite polynomials[1]

$$\frac{d^2\chi}{d\xi^2} - 2\xi\frac{d\chi}{d\xi} + 2\lambda_{\mathrm{m},\mathbf{k}\lambda}\chi = 0 \tag{23}$$

Solutions for Φ_m that vanish for $\xi \to \pm\infty$ exist for $\lambda_{\mathrm{m},\mathbf{k}\lambda} = 0,\ 1,\ 2,\ \ldots$ only. Using Eq.(19) we obtain:

$$2\lambda_{\mathrm{m},\mathbf{k}\lambda} = 2\left(\frac{\mathsf{E}_{\mathrm{m},\mathbf{k}\lambda}}{\hbar\omega_{\mathbf{k}\lambda}} - \frac{1}{2}\right) = 2n$$

$$\mathsf{E}_{\mathrm{m},\mathbf{k}\lambda} = \omega_{\mathbf{k}\lambda}\hbar\left(n + \frac{1}{2}\right), \quad n = 0,\ 1,\ 2,\ \ldots \tag{24}$$

For $n = 0$ we obtain the so-called zero-point energy of the electromagnetic field in vacuum. The parameter λ has the two values 2 and 3 but $\mathbf{k}$ has

[1]See, e.g.: Becker 1964a, b, vol. II, § 15; Landau and Lifschitz 1966, vol. III, § 23; Abramovitz and Stegun 1964

denumerably infinite values according to Eq.(3.4-12) and the total zero-point energy becomes infinite.

We quote from the renowned book by Berestetzki, Lifschitz, and Pitajewski (1970, 1982; § 3. Photons, second paragraph): *But already in this state each oscillator has the 'zero-point energy' $2\pi f\hbar/2$, which differs from zero. The summation over the infinitely many oscillators yields an infinite result. We meet here one of the 'divergencies' that the existing theory contains because it is not complete and not logically consistent.*

We read further (§ 1. Uncertainty Relations in the Relativistic Theory, second paragraph from the end): *The lack of complete logical consistency shows in this theory by the existence of divergent expressions when the mathematical methods are directly applied; however, there are unambiguous methods for the removal of these divergencies. Nevertheless, these methods have largely the character of semi-empirical recipes and our belief in the correctness of the results obtained in this way is based in the end on their excellent agreement with experiment, but not on the inner consistency and the logic lucidity of the basic principles of the theory.*

Becker has the following to say: *The ground state represented by $[n = 0]$ corresponds to vacuum; it still contains zero-point vibrations, however, as in the case of the linear oscillator. Since we are dealing with an infinite number of oscillators the mean-square values of the field strengths, $\overline{\mathbf{E}^2}$, $\overline{\mathbf{H}^2}$, must also be infinitely great. A completely satisfactory treatment of this anomaly does not yet exist*[2]. The anomaly is directly associated with divergent integrals: *This divergence has for long been an insuperable difficulty of quantum theory; it has not yet been completely overcome, but has been ingeniously circumvented through the concept of the mass renormalization of the electron (Kramer, 1945)*[3].

A detailed historical review of the problem of infinities in quantum field theory, listing many references, may be found in a book by Weinberg (1995, pp. 31–48).

If we had used Eq.(16) instead of Eq.(15) we would have obtained again Eqs.(17) to (24) but the subscript m would have been replaced by the subscript e. The zero-point energy would again be infinite.

Let us see how we ended up with a theory yielding infinite energy. The transition from Eqs.(11) and (12) to (15) and (16) reduced the general theory to a steady state theory. But something more happened. In Section 3.4 we had the current densities $\mathbf{g}_\mathrm{m}$ and $\mathbf{g}_\mathrm{e}$ in Eqs.(3.4-5) and (3.4-6). These current densities lead to the terms $g_{\mathrm{e},\mathbf{k}\lambda}$ and $g_{\mathrm{m},\mathbf{k}\lambda}$ in Eqs.(3.4-25) and (3.4-31). Implicitly they are contained in the terms $b^*_{\mathrm{m},\mathbf{k}\lambda}$ to $b_{\mathrm{e},\mathbf{k}\lambda}$ of Eq.(3.4-64). But they are not contained in the operators $b^+_{\mathrm{m},\mathbf{k}\lambda}$ to $b^-_{\mathrm{e},\mathbf{k}\lambda}$ of Eqs.(15) and (16). If one chooses $\mathbf{g}_\mathrm{m} = 0$ in Eqs.(1.1-9) and (3.4-2) one obtains equations that yield divergent integrals for the associated field strength in transient solutions; this means the magnetic field strength for electric excitation and the electric field strength

[2]Becker 1964a, vol. II, § 52, footnote p. 311
[3]Becker 1964a, vol. II, § 53, p. 319, small print following Eq.(53.9)

for magnetic excitation. We have pointed out at the beginning of Section 3.4 that a possible transition $\mathbf{g}_m \rightarrow 0$ or $\mathbf{g}_e \rightarrow 0$ must be made at the end of the calculation to avoid divergent integrals. If the classical theory leads to divergencies one cannot expect that the quantized theory will do any better.

There is a second analogy with the classical Maxwell theory without modification. In Eqs.(1) and (15) we have functions, operators, or constants with subscript m while in Eqs.(2) and (16) we have the same terms but with subscript e. One might say there are electric photons and magnetic photons without any connection with each other. A similar situation existed in the classical theory when transient solutions for electric and magnetic field strengths were derived. Stratton (1941), who seems to be the only author of a text book that attempted to obtain such transient solutions, arrived at electric field strengths due to electric excitation and magnetic field strengths due to magnetic excitation without any connection between the two. It took more than 40 years before the modification of Maxwell's equations permitted the derivation of an electric field strength plus an associated magnetic field strength due to electric excitation or a magnetic field strength plus an associated electric field strength due to magnetic excitation.

4 Quantization of the Pure Radiation Field

4.1 Radiation Field in Extended Lorentz Gauge

We start from Eqs.(1.6-26)–(1.6-29) and assume there are neither electric nor magnetic charge densities ρ_e and ρ_m. From Eqs.(1.6-32) and (1.6-33) we get in this case

$$\phi_e(x,y,z,t) \equiv \phi_m(x,y,z,t) \equiv 0 \tag{1}$$

and only Eqs.(1.6-26), (1.6-27) remain:

$$\nabla^2 \mathbf{A}_e - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_e}{\partial t^2} = -\frac{1}{Zc}\mathbf{g}_m \tag{2}$$

$$\nabla^2 \mathbf{A}_m - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_m}{\partial t^2} = -\frac{Z}{c}\mathbf{g}_e \tag{3}$$

The current densities $\mathbf{g}_m$ and $\mathbf{g}_e$ can refer in a vacuum—which is assumed by the concept of a 'pure radiation field'—to dipole and higher order multipole currents only. We have derived in Sections 2.1 to 2.3 a variety of equations linking $\mathbf{g}_e$ and $\mathbf{g}_m$ to the field strengths $\mathbf{E}$ and $\mathbf{H}$. Here we are going to use the very simplest approximations $\mathbf{g}_e = \sigma_p \mathbf{E}$ and $\mathbf{g}_m = 2s_p\mathbf{H}$ that one can derive from Eqs.(2.1-19) and (2.2-7) for small values of τ_{mp} in order to obtain equations that can be solved analytically. The text following Eqs.(2.1-29) and (2.1-31) explains this some more. Better approximations will be developed in Section 6.10. To reduce subscripts, we write $\sigma = \sigma_p$ and $s = 2s_p$. Using Eqs.(1.6-11) and (1.6-17) we may then connect the current densities $\mathbf{g}_m$ and $\mathbf{g}_e$ with the vector potentials $\mathbf{A}_m$ and $\mathbf{A}_e$:

$$\mathbf{g}_m = s\mathbf{H} = s\left(\frac{c}{Z}\operatorname{curl}\mathbf{A}_m - \frac{\partial \mathbf{A}_e}{\partial t}\right) \tag{4}$$

$$\mathbf{g}_e = \sigma\mathbf{E} = -\sigma\left(Zc\operatorname{curl}\mathbf{A}_e + \frac{\partial \mathbf{A}_m}{\partial t}\right) \tag{5}$$

The current densities in Eqs.(2), (3) may be eliminated and two equations containing the vector potentials $\mathbf{A}_m$ and $\mathbf{A}_e$ only are obtained:

$$\nabla^2\mathbf{A}_\mathrm{e} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}_\mathrm{e}}{\partial t^2} + \frac{s}{Zc}\left(\frac{c}{Z}\operatorname{curl}\mathbf{A}_\mathrm{m} - \frac{\partial\mathbf{A}_\mathrm{e}}{\partial t}\right) = 0 \tag{6}$$

$$\nabla^2\mathbf{A}_\mathrm{m} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}_\mathrm{m}}{\partial t^2} - \frac{\sigma Z}{c}\left(Zc\operatorname{curl}\mathbf{A}_\mathrm{e} + \frac{\partial\mathbf{A}_\mathrm{m}}{\partial t}\right) = 0 \tag{7}$$

The potentials $\mathbf{A}_\mathrm{e}$ and $\mathbf{A}_\mathrm{m}$ are connected in these two equations while they are independent for $\mathbf{g}_\mathrm{m} = \mathbf{g}_\mathrm{e} = 0$ in Eqs.(1.6-26), (1.6-27). Hence, we can expect to obtain electromagnetic photons instead of independent electric and magnetic photons as in Section 3.5.

In Cartesian coordinates we write the vectors $\mathbf{A}$, $\operatorname{curl}\mathbf{A}$, and $\nabla^2\mathbf{A}$ in component form with the unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$:

$$\mathbf{A} = A_x\mathbf{e}_x + A_y\mathbf{e}_y + A_z\mathbf{e}_z \tag{8}$$

$$\operatorname{curl}\mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{e}_z \tag{9}$$

$$\nabla^2\mathbf{A} = \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right)\mathbf{e}_x + \left(\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}\right)\mathbf{e}_y + \left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right)\mathbf{e}_z \tag{10}$$

With the help of these relations we may rewrite Eqs.(6) and (7) in Cartesian coordinates:

$$\frac{\partial^2 A_{\mathrm{e}x}}{\partial x^2} + \frac{\partial^2 A_{\mathrm{e}x}}{\partial y^2} + \frac{\partial^2 A_{\mathrm{e}x}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 A_{\mathrm{e}x}}{\partial t^2} + \frac{s}{Zc}\left[\frac{c}{Z}\left(\frac{\partial A_{\mathrm{m}z}}{\partial y} - \frac{\partial A_{\mathrm{m}y}}{\partial z}\right) - \frac{\partial A_{\mathrm{e}x}}{\partial t}\right] = 0 \tag{11}$$

$$\frac{\partial^2 A_{\mathrm{e}y}}{\partial x^2} + \frac{\partial^2 A_{\mathrm{e}y}}{\partial y^2} + \frac{\partial^2 A_{\mathrm{e}y}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 A_{\mathrm{e}y}}{\partial t^2} + \frac{s}{Zc}\left[\frac{c}{Z}\left(\frac{\partial A_{\mathrm{m}x}}{\partial z} - \frac{\partial A_{\mathrm{m}z}}{\partial x}\right) - \frac{\partial A_{\mathrm{e}y}}{\partial t}\right] = 0 \tag{12}$$

$$\frac{\partial^2 A_{\mathrm{e}z}}{\partial x^2} + \frac{\partial^2 A_{\mathrm{e}z}}{\partial y^2} + \frac{\partial^2 A_{\mathrm{e}z}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 A_{\mathrm{e}z}}{\partial t^2} + \frac{s}{Zc}\left[\frac{c}{Z}\left(\frac{\partial A_{\mathrm{m}y}}{\partial x} - \frac{\partial A_{\mathrm{m}x}}{\partial y}\right) - \frac{\partial A_{\mathrm{e}z}}{\partial t}\right] = 0 \tag{13}$$

$$\frac{\partial^2 A_{\mathrm{m}x}}{\partial x^2} + \frac{\partial^2 A_{\mathrm{m}x}}{\partial y^2} + \frac{\partial^2 A_{\mathrm{m}x}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 A_{\mathrm{m}x}}{\partial t^2} - \frac{\sigma Z}{c}\left[Zc\left(\frac{\partial A_{\mathrm{e}z}}{\partial y} - \frac{\partial A_{\mathrm{e}y}}{\partial z}\right) + \frac{\partial A_{\mathrm{m}x}}{\partial t}\right] = 0 \tag{14}$$

$$\frac{\partial^2 A_{my}}{\partial x^2} + \frac{\partial^2 A_{my}}{\partial y^2} + \frac{\partial^2 A_{my}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 A_{my}}{\partial t^2} - \frac{\sigma Z}{c}\left[Zc\left(\frac{\partial A_{ex}}{\partial z} - \frac{\partial A_{ez}}{\partial x}\right) + \frac{\partial A_{my}}{\partial t}\right] = 0 \tag{15}$$

$$\frac{\partial^2 A_{mz}}{\partial x^2} + \frac{\partial^2 A_{mz}}{\partial y^2} + \frac{\partial^2 A_{mz}}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 A_{mz}}{\partial t^2} - \frac{\sigma Z}{c}\left[Zc\left(\frac{\partial A_{ey}}{\partial x} - \frac{\partial A_{ex}}{\partial y}\right) + \frac{\partial A_{mz}}{\partial t}\right] = 0 \tag{16}$$

Let us simplify these equations for a planar wave propagating in the direction of y. All derivatives with respect to x and z must be zero:

$$\frac{\partial A_{ex}}{\partial x} = \frac{\partial A_{ey}}{\partial x} = \frac{\partial A_{ez}}{\partial x} = \frac{\partial A_{ex}}{\partial z} = \frac{\partial A_{ey}}{\partial z} = \frac{\partial A_{ez}}{\partial z} = 0 \tag{17}$$

$$\frac{\partial A_{mx}}{\partial x} = \frac{\partial A_{my}}{\partial x} = \frac{\partial A_{mz}}{\partial x} = \frac{\partial A_{mx}}{\partial z} = \frac{\partial A_{my}}{\partial z} = \frac{\partial A_{mz}}{\partial z} = 0 \tag{18}$$

Equations (11)–(16) are reduced to the following form:

$$\frac{\partial^2 A_{ex}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{ex}}{\partial t^2} + \frac{s}{Zc}\left(\frac{c}{Z}\frac{\partial A_{mz}}{\partial y} - \frac{\partial A_{ex}}{\partial t}\right) = 0 \tag{19}$$

$$\frac{\partial^2 A_{ey}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{ey}}{\partial t^2} - \frac{s}{Zc}\frac{\partial A_{ey}}{\partial t} = 0 \tag{20}$$

$$\frac{\partial^2 A_{ez}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{ez}}{\partial t^2} - \frac{s}{Zc}\left(\frac{c}{Z}\frac{\partial A_{mx}}{\partial y} + \frac{\partial A_{ez}}{\partial t}\right) = 0 \tag{21}$$

$$\frac{\partial^2 A_{mx}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{mx}}{\partial t^2} - \frac{\sigma Z}{c}\left(Zc\frac{\partial A_{ez}}{\partial y} + \frac{\partial A_{mx}}{\partial t}\right) = 0 \tag{22}$$

$$\frac{\partial^2 A_{my}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{my}}{\partial t^2} - \frac{\sigma Z}{c}\frac{\partial A_{my}}{\partial t} = 0 \tag{23}$$

$$\frac{\partial^2 A_{mz}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{mz}}{\partial t^2} + \frac{\sigma Z}{c}\left(Zc\frac{\partial A_{ex}}{\partial y} - \frac{\partial A_{mz}}{\partial t}\right) = 0 \tag{24}$$

If we further specialize to a transverse electromagnetic (TEM) wave we may demand that E_y and H_y are zero. This implies the following conditions for the vector potentials $\mathbf{A}_e$ and $\mathbf{A}_m$ according to Eqs.(1.6-17) and (1.6-11) for $\phi_e = \phi_m = 0$:

$$E_y = -Zc\left(\frac{\partial A_{ex}}{\partial z} - \frac{\partial A_{ez}}{\partial x}\right) - \frac{\partial A_{my}}{\partial t} = 0 \tag{25}$$

$$H_y = \frac{c}{Z}\left(\frac{\partial A_{mx}}{\partial z} - \frac{\partial A_{mz}}{\partial x}\right) - \frac{\partial A_{ey}}{\partial t} = 0 \tag{26}$$

Due to Eqs.(17) and (18) these two equations are reduced to

$$\frac{\partial A_{\mathrm{m}y}}{\partial t} = 0 \tag{27}$$

$$\frac{\partial A_{\mathrm{e}y}}{\partial t} = 0 \tag{28}$$

Equations(20) and (23) assume the form

$$\frac{\partial^2 A_{\mathrm{e}y}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{\mathrm{e}y}}{\partial t^2} = 0 \tag{29}$$

$$\frac{\partial^2 A_{\mathrm{m}y}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{\mathrm{m}y}}{\partial t^2} = 0 \tag{30}$$

since Eqs.(27) and (28) do not imply that the second partial derivatives with respect to t are zero. We make the following substitutions where the subscript v alludes to 'variable':

$$A_{ex} = A_{ez} = A_{\mathrm{ev}}, \qquad A_{mx} = -A_{mz} = A_{m\mathrm{v}} \tag{31}$$

Equations (19) and (21) as well as Eqs.(22) and (24) are reduced to one equation each with the variables A_{ev} and A_{mv}:

$$\frac{\partial^2 A_{\mathrm{ev}}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{\mathrm{ev}}}{\partial t^2} - \frac{s}{Zc}\left(\frac{c}{Z}\frac{\partial A_{\mathrm{mv}}}{\partial y} + \frac{\partial A_{\mathrm{ev}}}{\partial t}\right) = 0 \tag{32}$$

$$\frac{\partial^2 A_{\mathrm{mv}}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{\mathrm{mv}}}{\partial t^2} - \frac{\sigma Z}{c}\left(Zc\frac{\partial A_{\mathrm{ev}}}{\partial y} + \frac{\partial A_{\mathrm{mv}}}{\partial t}\right) = 0 \tag{33}$$

The comments about polarization made in Section 1.2 following Eq.(1.2-10) apply again if E_x, H_x, E_z, H_z, E, and H are replaced by A_{ex}, A_{mx}, A_{ez}, A_{mz}, A_{ev}, and A_{mv}.

For Eqs.(29) and (30) we have the following general solution for $y \geq 0$ and $t \geq 0$, where $f_{\mathrm{e}0}$, $f_{\mathrm{e}1}$, $f_{\mathrm{m}0}$, and $f_{\mathrm{m}1}$ denote arbitrary functions:

$$A_{\mathrm{e}y}(y,t) = A_{\mathrm{e}0} f_{\mathrm{e}}(y - ct), \qquad y \geq 0,\ t \geq 0 \tag{34}$$

$$A_{\mathrm{m}y}(y,t) = A_{\mathrm{m}0} f_{\mathrm{m}}(y - ct) \tag{35}$$

These solutions hold for excitation functions or boundary conditions $f_{\mathrm{e}}(0,t)$ and $f_{\mathrm{m}}(0,t)$ at the plane $y = 0$ for all times $t \geq 0$ as well as initial conditions $f_{\mathrm{e}}(y,0)$ and $f_{\mathrm{m}}(y,0)$ for $t = 0$ at all locations $y \geq 0$.

In order to separate the variables A_{ev} and A_{mv} in Eqs.(32) and (33) we differentiate Eq.(33) with respect to y, express $\partial A_{\mathrm{mv}}/\partial y$ by Eq.(32), differentiate it as often as needed with respect to t and y, and substitute into the differentiated Eq.(33). We obtain:

$$\frac{\partial^2 V_e}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 V_e}{\partial t^2} - \frac{1}{c}\left(\sigma Z + \frac{s}{Z}\right)\frac{\partial V_e}{\partial t} - \sigma s V_e = 0 \tag{36}$$

$$\frac{\partial^2 A_{ev}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{ev}}{\partial t^2} = V_e(y,t) \tag{37}$$

The dimension of A_{ev} is As/m and the dimension of V_e is As/m^3, which is an electric charge density.

With the substitutions $A_{ev} \leftrightarrow A_{mv}$, $c/Z \leftrightarrow Zc$, $s \leftrightarrow \sigma$ one may transform Eq.(32) into Eq.(33) and vice versa. Equations (36) and (37) are then replaced by:

$$\frac{\partial^2 V_m}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 V_m}{\partial t^2} - \frac{1}{c}\left(\sigma Z + \frac{s}{Z}\right)\frac{\partial V_m}{\partial t} - \sigma s V_m = 0 \tag{38}$$

$$\frac{\partial^2 A_{mv}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{mv}}{\partial t^2} = V_m(y,t) \tag{39}$$

The dimension of A_{mv} is Vs/m and the dimension of V_m is Vs/m^3, which is an hypothetical magnetic charge density.

From the various normalizations for y and t discussed in Section 1.3 we choose the one of Eq.(1.3-10)

$$\theta = t/T, \quad \zeta = y/cT \tag{40}$$

and rewrite Eqs.(36) and (37):

$$\frac{\partial^2 V_e}{\partial \zeta^2} - \frac{\partial^2 V_e}{\partial \theta^2} - \rho_1 \frac{\partial V_e}{\partial \theta} - \rho_2^2 V_e = 0$$

$$\rho_1 = cT(\sigma Z + s/Z) = c^2T(\sigma\mu + s\epsilon),\ \rho_2^2 = c^2T^2\sigma s \tag{41}$$

$$\frac{\partial^2 A_{ev}}{\partial \zeta^2} - \frac{\partial^2 A_{ev}}{\partial \theta^2} = c^2T^2 V_e(\zeta,\theta) \tag{42}$$

Equation (41) is equal to Eqs.(6.1-1) or (6.4-6) if V_e is written for E and the normalization of Eq.(1.3-7) is used as one may infer from Eqs.(1.3-11) and (1.3-13). Hence, everything said there about the electric field strength E applies to the auxiliary function V_e. Equation (42) is the inhomogeneous wave equation shown by Eq.(3.1-40) and solved by Eq.(3.1-44). We obtain as solution of Eq.(42):

$$A_{ev}(\zeta,\theta) = -\frac{c^2T^2}{2}\int_0^\theta \left(\int_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} V_e(\zeta',\theta')d\zeta'\right)d\theta' \tag{43}$$

The variables ζ and θ of $V_e(\zeta,\theta)$ in Eq.(42) have to be replaced by ζ' and θ' when Eq.(43) is used.

For the variable $V_m(y,t)$ of Eqs.(38) and (39) we obtain the following three equations in analogy to Eqs.(41)-(43):

$$\frac{\partial^2 V_m}{\partial\zeta^2} - \frac{\partial^2 V_m}{\partial\theta^2} - \rho_1\frac{\partial V_m}{\partial\theta} - \rho_2^2 V_m = 0 \tag{44}$$

$$\frac{\partial^2 A_{mv}}{\partial\zeta^2} - \frac{\partial^2 A_{mv}}{\partial\theta^2} = c^2T^2 V_m(\zeta,\theta) \tag{45}$$

$$A_{mv}(\zeta,\theta) = -\frac{c^2T^2}{2}\int_0^\theta\left(\int_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} V_m(\zeta',\theta')d\zeta'\right)d\theta' \tag{46}$$

Again, ζ and θ in Eqs.(44) and (45) must be replaced by ζ' and θ' when Eq.(46) is used.

If $A_{ev}(\zeta,\theta)$ is found from Eq.(43) for certain boundary and initial conditions, one may obtain the component $A_{mv}(\zeta,\theta)$ of an *associated* potential from either Eq.(32) or (33). Consider Eq.(32) first:

$$A_{mv}(\zeta,\theta) = Z\rho_s\int\left(\frac{\partial^2 A_{ev}}{\partial\zeta^2} - \frac{\partial^2 A_{ev}}{\partial\theta^2} - \frac{1}{\rho_s}\frac{\partial A_{ev}}{\partial\theta}\right)d\zeta,\quad \rho_s = \frac{Z}{scT} = \frac{\mu}{sT} \tag{47}$$

A second expression for $A_{mv}(\zeta,\theta)$ is obtained from Eq.(33) by treating this equation as an inhomogeneous equation for A_{mv} with a known term $\partial A_{ev}/\partial y$ or $\partial A_{ev}/\partial\zeta$:

$$\frac{\partial^2 A_{mv}}{\partial\zeta^2} - \frac{\partial^2 A_{mv}}{\partial\theta^2} - \rho_\sigma\frac{\partial A_{mv}}{\partial\theta} = Z\rho_\sigma\frac{\partial A_{ev}}{\partial\zeta} \tag{48}$$

$$\rho_\sigma = ZTc\sigma = \frac{\sigma T}{\epsilon},\quad \rho_s = \frac{Z}{scT} = \frac{\mu}{sT},\quad \rho_s\rho_\sigma = \frac{\sigma\mu}{s\epsilon} = \frac{1}{\omega^2} \tag{49}$$

The integration of Eq.(47) is much simpler than the solution of Eq.(48), but we cannot ignore Eq.(48). Since Eqs.(47) and (48) must yield the same result for $A_{mv}(\zeta,\theta)$ we generally need the solution of Eq.(48) to determine integration constants.

Alternately, if $A_{mv}(\zeta,\theta)$ is found from Eq.(46) for certain boundary and initial conditions one may obtain an *associated* potential $A_{ev}(\zeta,\theta)$ from either Eq.(32) or (33). First we get from Eq.(33):

$$A_{ev}(\zeta,\theta) = \frac{1}{Z\rho_\sigma}\int\left(\frac{\partial^2 A_{mv}}{\partial\zeta^2} - \frac{\partial^2 A_{mv}}{\partial\theta^2} - \rho_\sigma\frac{\partial A_{mv}}{\partial\theta}\right)d\zeta \tag{50}$$

The second expression for $A_{ev}(\zeta,\theta)$ is obtained from Eq.(32) by treating it as an inhomogeneous equation for A_{ev} with a known term $\partial A_{mv}/\partial y$ or $\partial A_{mv}/\partial\zeta$:

$$\frac{\partial^2 A_{\text{ev}}}{\partial \zeta^2} - \frac{\partial^2 A_{\text{ev}}}{\partial \theta^2} - \frac{1}{\rho_{\text{s}}} \frac{\partial A_{\text{ev}}}{\partial \theta} = \frac{1}{Z\rho_{\text{s}}} \frac{\partial A_{\text{mv}}}{\partial \zeta} \tag{51}$$

Again, one must generally obtain $A_{\text{ev}}(\zeta, \theta)$ from both Eq.(50) and (51) in order to determine the integration constants.

If we denote the solution of A_{ev} derived from Eq.(43) by A_{eve} and the associated solution obtained from A_{mv} via Eqs.(50) and (51) by A_{evm} we obtain the general solution of A_{ev} as the sum

$$A_{\text{ev}}(\zeta, \theta) = A_{\text{eve}}(\zeta, \theta) + A_{\text{evm}}(\zeta, \theta) \tag{52}$$

Similarly, if we denote the solution derived for A_{mv} from Eq.(46) by A_{mvm} and the associated solution obtained from A_{ev} via Eqs.(47) and (48) by A_{mve} we obtain the general solution of A_{mv} as the sum

$$A_{\text{mv}}(\zeta, \theta) = A_{\text{mvm}}(\zeta, \theta) + A_{\text{mve}}(\zeta, \theta) \tag{53}$$

This means we can choose initial and boundary conditions independently for A_{eve} and A_{mvm}, but the associated potentials A_{mve} and A_{evm} are always automatically excited with A_{eve} or A_{mvm}. We can never excite A_{ev} without exciting A_{mv} and vice versa.

For a first solution of Eq.(41) we assume as boundary condition at $\zeta = 0$ a step function

$$\begin{aligned} V_{\text{e}}(0, \theta) = V_{\text{e0}} S(\theta) &= 0 \quad &&\text{for } \theta < 0 \\ &= V_{\text{e0}} \quad &&\text{for } \theta \geq 0 \end{aligned} \tag{54}$$

It is usual to assume a further boundary condition for $\zeta \to \infty$

$$V_{\text{e}}(\infty, \theta) = \text{finite} \tag{55}$$

We cannot use it since θ and ζ will be restricted to the finite intervals $0 \leq \theta \leq 1$ and $0 \leq \zeta \leq 1$. An alternative condition will be introduced presently.

The boundary condition of Eq.(54) uses a step function that is not quadratically integrable. This may cause concern that an infinite energy is introduced, but there is no such problem. The boundary condition of Eq.(54) excites an electromagnetic wave with finite energy. Of course, one can easily eliminate the concern about quadratical integrability by subtracting from $V_{\text{e0}}S(\theta)$ a delayed step function $V_{\text{e0}}S(\theta - \theta_1)$ and thus replacing the step function of Eq.(54) by a rectangular pulse. We shall discuss that in Section 4.6.

We turn to the initial condition(s). As initial condition at $\theta = 0$ we assume the relation

$$V_{\text{e}}(\zeta, 0) = 0 \tag{56}$$

but observe that this condition implies $V_{\text{e}}(\zeta, \theta) = 0$ for $\theta < 0$ due to Eq.(54). Hence, the potential $\mathbf{A}_{\text{e}}$ derived from A_{ev} will be zero for $\theta < 0$. We note that a function of time that describes a physical process subject to the causality law must be zero before a finite time[1].

If $V_e(\zeta, 0)$ is zero for all values $\zeta > 0$, its derivatives with respect to ζ must be zero too

$$\partial^n V_e(\zeta, 0)/\partial\zeta^n = 0$$

which produces with the help of Eqs.(56) and (41) the equation

$$\frac{\partial}{\partial\theta}\left(\frac{\partial V_e(\zeta,0)}{\partial\theta} + \rho_1 V_e(\zeta,0)\right) = 0$$

that is satisfied by $V_e(\zeta, 0) = 0$ of Eq.(56) and the additional condition

$$\partial V_e(\zeta,\theta)/\partial\theta = 0 \quad \text{for } \theta = 0 \tag{57}$$

We assume that the general solution of Eq.(41) can be written as the sum of a steady state solution $F(\zeta)$ plus a deviation $w(\zeta, \theta)$ from it:

$$V_e(\zeta,\theta) = V_{e0}[F(\zeta) + w(\zeta,\theta)] \tag{58}$$

Substitution of $F(\zeta)$ into Eq.(41) yields:

$$\begin{aligned} d^2F/d\zeta^2 - \rho_2^2 F &= 0 \\ F(\zeta) &= A_{10}e^{-\rho_2\zeta} + A_{11}e^{\rho_2\zeta} \end{aligned} \tag{59}$$

The boundary condition of Eq.(55) would demand $A_{11} = 0$; the value $A_{10} = 1$ would then follow from Eq.(54). Since we cannot use Eq.(55) we substitute the following reasoning: The function $F(\zeta)$ represents an amplitude and $F^2(\zeta)$ an energy density. Since the energy comes strictly from the boundary $\zeta = 0$ the energy in a certain interval $\Delta\zeta$ cannot increase with increasing values of ζ; hence, A_{11} must be zero. Alternately, if the energy is fed in at the boundary $\zeta = 1$ one must choose A_{10} equal to zero. This is an issue introduced by the replacement of infinite intervals for time and space by arbitrarily large but finite intervals. A more general discussion will be found in the small printed text following Eq.(75) below. One may also claim that we do not need the most general solution but only a sufficiently general solution, which is provided by the choice $A_{11} = 0$. Since $A_{11} = 0$ implies $A_{10} = 1$ we write:

$$F(\zeta) = e^{-\rho_2\zeta} \tag{60}$$

The introduction of $F(\zeta)$ transforms the boundary condition of Eq.(54) for V_e into an homogeneous boundary condition for w, which is the reason for using Eq.(58):

[1]Mathematicians use sometimes the expression *causal function* for such functions.

$$V_e(0,\theta) = V_{e0}[F(0) + w(0,\theta)] = V_{e0} \quad \text{for } \theta \geq 0 \tag{61}$$
$$w(0,\theta) = 0 \quad \text{for } \theta \geq 0 \tag{62}$$

The boundary condition of Eq.(55) becomes

$$w(\infty,\theta) = \text{finite} \tag{63}$$

Again, the use of a finite interval $0 \leq \zeta \leq 1$ prevents us from using this boundary condition. It turns out not to be required anywhere. The initial conditions of Eqs.(56) and (57) yield:

$$F(\zeta) + w(\zeta,0) = 0, \quad w(\zeta,0) = -e^{-\rho_2\zeta} \tag{64}$$
$$\partial w(\zeta,\theta)/\partial\theta = 0 \quad \text{for } \theta = 0,\ \zeta > 0 \tag{65}$$

Substitution of Eq.(58) into Eq.(41) yields for $w(\zeta,\theta)$ the same equation as for $V_e(\zeta,\theta)$:

$$\partial^2 w/\partial\zeta^2 - \partial^2 w/\partial\theta^2 - \rho_1 \partial w/\partial\theta - \rho_2^2 w = 0 \tag{66}$$

Particular solutions of this equation denoted $w_\kappa(\zeta,\theta)$ are obtained by means of Bernoulli's product method for the separation of variables

$$w_\kappa(\zeta,\theta) = \phi(\zeta)\psi(\theta) \tag{67}$$
$$\frac{1}{\phi}\frac{\partial^2\phi}{\partial\zeta^2} = \frac{1}{\psi}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\rho_1}{\psi}\frac{\partial\psi}{\partial\theta} + \rho_2^2 = -(2\pi\kappa)^2 \tag{68}$$

which yields two ordinary differential equations

$$d^2\phi/d\zeta^2 + (2\pi\kappa)^2\phi = 0 \tag{69}$$
$$d^2\psi/d\theta^2 + \rho_1 d\psi/d\theta + [(2\pi\kappa)^2 + \rho_2^2]\psi = 0 \tag{70}$$

with the solutions

$$\phi(\zeta) = A_{20}\sin 2\pi\kappa\zeta + A_{21}\cos 2\pi\kappa\zeta \tag{71}$$
$$\psi(\theta) = A_{30}\exp(\gamma_1\theta) + A_{31}\exp(\gamma_2\theta) \tag{72}$$

The coefficients γ_1 and γ_2 are the roots of the equation

$$\gamma^2 + \rho_1\gamma + [(2\pi\kappa)^2 + \rho_2^2] = 0$$
$$\gamma_1 = \frac{1}{2}[-\rho_1 + (\rho_1^2 - d^2)^{1/2}] \quad \text{for } d^2 < \rho_1^2$$
$$\gamma_2 = \frac{1}{2}[-\rho_1 - (\rho_1^2 - d^2)^{1/2}]$$
$$\gamma_1 = \frac{1}{2}[-\rho_1 + i(d^2 - \rho_1^2)^{1/2}] \quad \text{for } d^2 > \rho_1^2$$
$$\gamma_2 = \frac{1}{2}[-\rho_1 - i(d^2 - \rho_1^2)^{1/2}]$$
$$\rho_1 = c^2T(\sigma\mu + s\epsilon),\ d^2 = 4[(2\pi\kappa)^2 + \rho_2^2],\ \rho_2^2 = c^2T^2\sigma s \tag{73}$$

The boundary condition of Eq.(62) requires $A_{21} = 0$ in Eq.(71) and the particular solution $w_\kappa(\zeta,\theta)$ becomes:

$$w_\kappa(\zeta,\theta) = [A_1 \exp(\gamma_1\theta) + A_2 \exp(\gamma_2\theta)] \sin 2\pi\kappa\zeta \tag{74}$$

The solution $w_\kappa(\zeta,\theta)$ is usually generalized by making A_1 and A_2 functions of κ, and integrating over all values of κ as shown by Eq.(6.1-26). From this point we follow a path that is different from the usual one.

We may generalize w_κ of Eq.(74) by making A_1 and A_2 functions of κ and taking the sum of denumerably many values of κ. In essence the Fourier sine integral of Eq.(6.1-26) is replaced by a Fourier sine series. The constant term and the terms multiplied with $\cos 2\pi\kappa\zeta$ of the usual Fourier series have been eliminated by the boundary condition of Eq.(62). A Fourier series requires a finite interval for ζ in Eq.(74) which we must define. This problem does not occur for the integral of Eq.(6.1-26) since the interval of the sine transform always runs from zero to infinity for both ζ and κ as shown by Eq.(6.1-30). We choose the finite interval for ζ to be

$$0 \le \zeta = y/cT \le 1 \tag{75}$$

where the time interval T is arbitrarily large but finite.

Equation (75) makes it impossible to use the boundary conditions of Eqs.(55) and (63). Instead we had to use the argument in the text following Eq.(59) to replace Eq.(55); no replacement has been needed for Eq.(63). A completely different possibility is to replace the normalization $\theta = t/T$ and $\zeta = y/cT$ of Eq.(40) by the normalizations $\theta = t/\Delta t$ and $\zeta = y/c\Delta t$, where Δt is an arbitrarily small but finite time interval. We then get the finite intervals

$$0 \le t \le T,\ 0 \le y \le cT \text{ or } 0 \le \theta \le T/\Delta t,\ 0 \le \zeta \le T/\Delta t, \quad T/\Delta t = N \gg 1$$

and we can justify $A_{11} = 0$ in Eqs.(59), (60) with Eq.(55) rewritten $V_e(N,\theta) =$ finite and $\zeta \to N \gg 1$. We consider this an important improvement of the theory developed here since it eliminates both infinitely large time and space intervals as well as infinitely small or infinitesimal intervals dt, dr. Both "infinitely large" and "infinitesimal" are beyond observation and can exist in a science based on observation for mathematical convenience only. We note that the upper limit of κ in Eq.(76) below becomes very large but finite rather than denumerably infinite for a finite value of Δt. Since the largest value of κ will represent the number of photons, an arbitrarily large but finite value for it is clearly preferable to either a nondenumerable or a denumerable infinite value.

It is known that the use of arbitrarily small but finite intervals Δt, $\Delta \mathbf{r}$ produces changes in relativistic quantum mechanics but not in the nonrelativistic theory (Harmuth 1989, pp. 236–303; 1992, pp. 197–269) and it is very satisfying that our theory suggests to use Δt, $\Delta \mathbf{r}$. But we must limit the scope of this book and we sidestep the use of Δt. Hence, we use the normalization $\theta = t/T$, $\zeta = y/cT$ and the intervals $0 \leq \theta \leq 1$, $0 \leq \zeta \leq 1$. We hope to elaborate the use of $\theta = t/\Delta t$, $\zeta = y/c\Delta t$ in a future book.

It is usual to continue a Fourier series outside its finite interval of definition periodically to $\pm\infty$, but there is no need to do so. Since we defined the boundary condition in Eq.(54) for $\zeta = 0$ we are not interested in the interval $\zeta < 0$ and can ignore it[2]. Similarly, we can ignore the interval $\zeta > 1$ since we can choose T as large as we want. A problem would only occur if we let T go to infinity, but this would take us beyond the realm of physics since we cannot make observations at an infinite distance in space or time.

We generalize Eq.(74) by a sum with denumerably many terms of the variable κ and a finite interval for the variable ζ:

$$w(\zeta,\theta) = \sum_{\kappa=1}^{\infty}[A_1(\kappa)\exp(\gamma_1\theta) + A_2(\kappa)\exp(\gamma_2\theta)]\sin 2\pi\kappa\zeta \tag{76}$$

Mathematicians call the sum over non-denumerably many values of κ represented by the integral of Eq.(6.1-26) a generalization of Eq.(76), but in terms of physics it is an abstraction that may or may not be usable to simplify a calculation. Simplification of calculations is the only justification for the use of differentials $d\kappa$, dx, dt, ... and non-denumerably many values for a physical observable. No observation can distinguish a wave with wave number κ from one with wave number $\kappa + d\kappa$. A finite difference $\Delta\kappa$ is needed for an observation. We had discussed in the text following Eq.(3.4-6) that finite space and time differences Δx, Δt can be used instead of a finite interval or a box to obtain a Fourier series instead of a Fourier integral. Hence, we will use finite values $\Delta\kappa = 1$ to obtain the Fourier series of Eq.(76) instead of the Fourier integral of Eq.(6.1-26).

We need the derivative $\partial w(\zeta,\theta)/\partial\theta$. Instead of the integral of Eq.(6.1-27) we get now a sum with denumerably many terms:

$$\frac{\partial w}{\partial \theta} = \sum_{\kappa=1}^{\infty}[A_1(\kappa)\gamma_1\exp(\gamma_1\theta) + A_2(\kappa)\gamma_2\exp(\gamma_2\theta)]\sin 2\pi\kappa\zeta \tag{77}$$

[2]Instead of finding a solution in the interval $\zeta \geq 0$ and ignoring the interval $\zeta < 0$ we could develop a solution for $\zeta \leq 0$ and ignore the interval $\zeta > 0$. Equation (75) would then have to be replaced by $-1 \leq \zeta \leq 0$.

The functions $A_1(\kappa)$ and $A_2(\kappa)$ can be obtained from Eqs.(76) and (77) with the help of Eqs.(64) and (65):

$$w(\zeta,0) = \sum_{\kappa=1}^{\infty}[A_1(\kappa)+A_2(\kappa)]\sin 2\pi\kappa\zeta = -e^{-\rho_2\zeta} \tag{78}$$

$$\frac{\partial w(\zeta,0)}{\partial\theta} = \sum_{\kappa=1}^{\infty}[A_1(\kappa)\gamma_1+A_2(\kappa)\gamma_2]\sin 2\pi\kappa\zeta = 0 \tag{79}$$

These two equations must be solved for the functions $A_1(\kappa)$ and $A_2(\kappa)$. To this end consider the Fourier sine series in the following form:

$$g_s(\kappa) = 2\int_0^1 f_s(\zeta)\sin 2\pi\kappa\zeta\, d\zeta \tag{80}$$

$$f_s(\zeta) = \sum_{\kappa=1}^{\infty} g_s(\kappa)\sin 2\pi\kappa\zeta \tag{81}$$

The boundary condition of Eq.(65) avoids a term $\cos 2\pi\kappa\zeta$ in Eq.(78) and leads to the Fourier sine series of Eq.(81) rather than the general Fourier series with sine and cosine functions. The absence of a term $\cos 2\pi\kappa\zeta$ in Eq.(81) suggests to use the real rather than the complex form of the Fourier series.

If we identify the function $g_s(\kappa)$ first with $A_1(\kappa)+A_2(\kappa)$ we obtain from Eqs.(78) and (80)

$$A_1(\kappa)+A_2(\kappa) = -2\int_0^1 e^{-\rho_2\zeta}\sin 2\pi\kappa\zeta\, d\zeta \tag{82}$$

while identification of $g_s(\kappa)$ with $A_1(\kappa)\gamma_1+A_2(\kappa)\gamma_2$ of Eq.(79) yields

$$A_1(\kappa)\gamma_1+A_2(\kappa)\gamma_2 = 0 \tag{83}$$

Using the tabulated integral (Gradshtein and Ryzhik 1980, p. 196, 2663/1)

$$\int e^{px}\sin qx\, dx = \frac{e^{px}(p\sin qx - q\cos qx)}{p^2+q^2} \tag{84}$$

one obtains from Eq.(82):

$$\begin{aligned} A_1(\kappa)+A_2(\kappa) &= -\frac{4\pi\kappa(1-e^{-\rho_2})}{(2\pi\kappa)^2+\rho_2^2} \doteq -\frac{4\pi\kappa\rho_2}{(2\pi\kappa)^2+\rho_2^2} \doteq -\frac{2\rho_2}{2\pi\kappa} \quad \text{for } \rho_2 \ll 1 \\ &\doteq -\frac{4\pi\kappa}{(2\pi\kappa)^2+\rho_2^2} \quad \text{for } \rho_2 \gg 1 \\ \rho_2^2 &= c^2T^2\sigma s \end{aligned} \tag{85}$$

If we had used the Fourier transform of Eq.(6.1-30) rather than the Fourier series of Eqs.(80), (81) we would have obtained an upper limit of infinity in Eq.(82) in analogy to Eq.(6.1-31)[3]; the solution represented by Eq.(6.1-42) is readily extended to $\sigma = 0$ (Harmuth 1986, pp. 54, 55). It is the use of denumerably rather than non-denumerably many values of κ that introduced the arbitrarily large but finite time interval T and the very different results for $\rho_2 \ll 1$ and $\rho_2 \gg 1$ in Eq.(85). We shall show in Section 6.8 that the choice $\rho_2 \ll 1$, which implies $\sigma s \to 0$, leads to a strange result. Here we will persue the result for $\rho_2 = cT\sqrt{\sigma s} \gg 1$, which implies that the conductivities σ and s cannot be zero. The choice $\rho_2 = 7$ yields already $e^{-\rho_2} < 0.001$ and is thus sufficiently large compared with 1 for results represented by plots.

We had discussed in Section 3.5 in the second paragraph from the end that the current densities $\mathbf{g}_{\mathrm{m}}$ and $\mathbf{g}_{\mathrm{e}}$ cannot be chosen to be zero which is the same as claiming σ and s cannot be zero. We see here that this result follows directly from the choice of denumerably many values of the wave number κ. If there were only monopole currents one could choose $\mathbf{g}_{\mathrm{m}}$ and $\mathbf{g}_{\mathrm{e}}$ zero in the absence of charge carriers, since the conservation law of charge would preclude their generation. But $\mathbf{g}_{\mathrm{m}}$ and $\mathbf{g}_{\mathrm{e}}$ can also stand for dipole current densities and there is no conservation law that prohibits the generation of dipoles in vacuum. Indeed we could not explain how a capacitor with vacuum as dielectric could be charged without the help of an electric dipole current.

We return to the determination of the function $w(\zeta, \theta)$. Equations (83) and (85) are solved for $A_1(\kappa)$ and $A_2(\kappa)$:

$$
\begin{aligned}
A_1(\kappa) &= -\frac{4\pi\kappa(1-e^{-\rho_2})}{(2\pi\kappa)^2+\rho_2^2}\frac{\gamma_2}{\gamma_2-\gamma_1} \\
&= -\frac{2\pi\kappa(1-e^{-\rho_2})}{(2\pi\kappa)^2+\rho_2^2}\left(1+\frac{\rho_1}{(\rho_1^2-d^2)^{1/2}}\right) \quad \text{for } \rho_1^2 > d^2 \\
&= -\frac{2\pi\kappa(1-e^{-\rho_2})}{(2\pi\kappa)^2+\rho_2^2}\left(1-\frac{i\rho_1}{(d^2-\rho_1^2)^{1/2}}\right) \quad \text{for } d^2 > \rho_1^2 \\
A_2(\kappa) &= -\frac{4\pi\kappa(1-e^{-\rho_2})}{(2\pi\kappa)^2+\rho_2^2}\frac{\gamma_1}{\gamma_1-\gamma_2} \\
&= -\frac{2\pi\kappa(1-e^{-\rho_2})}{(2\pi\kappa)^2+\rho_2^2}\left(1-\frac{\rho_1}{(\rho_1^2-d^2)^{1/2}}\right) \quad \text{for } \rho_1^2 > d^2 \\
&= -\frac{2\pi\kappa(1-e^{-\rho_2})}{(2\pi\kappa)^2+\rho_2^2}\left(1+\frac{i\rho_1}{(d^2-\rho_1^2)^{1/2}}\right) \quad \text{for } d^2 > \rho_1^2
\end{aligned}
$$

$$
\rho_2^2 = c^2T^2\sigma s, \ \rho_1 = c^2T(\sigma\mu + s\epsilon), \ d^2 = 4[(2\pi\kappa)^2+\rho_2^2] \tag{86}
$$

[3]We note that $A_1(\kappa)$ and $A_2(\kappa)$ refer here to the solution of Eq.(41) while in Section 6.1 they refer to Eq.(6.1-1).

Substitution of Eq.(86) for $A_1(\kappa)$, $A_2(\kappa)$ and of Eq.(73) for γ_1, γ_2 brings Eq.(76) into the following form:

$$
\begin{aligned}
w(\zeta,\theta) = -e^{-\rho_1\theta/2}(1-e^{-\rho_2})\Bigg\{ \sum_{\kappa=1}^{<K} \Bigg[\left(1+\frac{\rho_1}{(\rho_1^2-d^2)^{1/2}}\right) \exp\frac{(\rho_1^2-d^2)^{1/2}\theta}{2} \\
+ \left(1-\frac{\rho_1}{(\rho_1^2-d^2)^{1/2}}\right)\exp\frac{-(\rho_1^2-d^2)^{1/2}\theta}{2}\Bigg] \frac{2\pi\kappa}{(2\pi\kappa)^2+\rho_2^2}\sin 2\pi\kappa\zeta \\
+ \sum_{\kappa>K}^{\infty}\Bigg[\left(1-\frac{i\rho_1}{(d^2-\rho_1^2)^{1/2}}\right)\exp\frac{i(d^2-\rho_1^2)^{1/2}\theta}{2} \\
+ \left(1+\frac{i\rho_1}{(d^2-\rho_1^2)^{1/2}}\right)\exp\frac{-i(d^2-\rho_1^2)^{1/2}\theta}{2}\Bigg]\frac{2\pi\kappa}{(2\pi\kappa)^2+\rho_2^2}\sin 2\pi\kappa\zeta\Bigg\} \\
K = c^2T|(\sigma\mu - s\epsilon)|/4\pi,\ d^2 = 4[(2\pi\kappa)^2+\rho_2^2] \qquad (87)
\end{aligned}
$$

The notation $< K$ and $> K$ in the limits of the sums means that the largest integer smaller than K or the smallest integer larger than K should be used.

The exponential terms in Eq.(87) may be eliminated with the help of hyperbolic and trigonometric functions:

$$
\begin{aligned}
w(\zeta,\theta) = -2e^{-\rho_1\theta/2}(1-e^{-\rho_2})\Bigg[\sum_{\kappa=1}^{<K}\Bigg(\operatorname{ch}[(\rho_1^2-d^2)^{1/2}\theta/2] \\
+ \frac{\rho_1\operatorname{sh}[(\rho_1^2-d^2)^{1/2}\theta/2]}{(\rho_1^2-d^2)^{1/2}}\Bigg)\frac{2\pi\kappa}{(2\pi\kappa)^2+\rho_2^2}\sin 2\pi\kappa\zeta \\
+ \sum_{\kappa>K}^{\infty}\Bigg(\cos[(d^2-\rho_1^2)^{1/2}\theta/2] \\
+ \frac{\rho_1\sin[(d^2-\rho_1^2)^{1/2}\theta/2]}{(d^2-\rho_1^2)^{1/2}}\Bigg)\frac{2\pi\kappa}{(2\pi\kappa)^2+\rho_2^2}\sin 2\pi\kappa\zeta\Bigg] \\
\rho_1 = c^2T(\sigma\mu+s\epsilon),\ \rho_2^2 = c^2T^2\sigma s \\
d^2 = 4[(2\pi\kappa)^2+\rho_2^2],\ K = c^2T|(\sigma\mu-s\epsilon)|/4\pi \qquad (88)
\end{aligned}
$$

Equations (60) for $F(\zeta)$ and (88) for $w(\zeta,\theta)$ define $V_{\mathrm{e}}(\zeta,\theta)$ according to Eq.(58). In order to obtain $A_{\mathrm{ev}}(\zeta,\theta)$ according to Eq.(43) we must replace the variables ζ, θ by ζ', θ' and integrate over ζ', θ'. First we integrate over ζ' and denote the result by $\partial A_{\mathrm{ev}}(\zeta,\theta')/\partial\theta'$:

$$
\frac{\partial A_{\mathrm{ev}}(\zeta,\theta')}{\partial\theta'} = -\frac{c^2T^2V_{\mathrm{e}0}}{2}\int_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')}[F(\zeta')+w(\zeta',\theta')]d\zeta' \qquad (89)
$$

Only two simple integrals need to be evaluated

$$\int_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} e^{-\rho_2\zeta'}d\zeta' = -\frac{1}{\rho_2}(e^{-\rho_2(\zeta+\theta)}e^{\rho_2\theta'} - e^{-\rho_2(\zeta-\theta)}e^{-\rho_2\theta'}) \tag{90}$$

$$\int_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} \sin 2\pi\kappa\zeta'\, d\zeta' = \frac{\sin 2\pi\kappa\zeta}{\pi\kappa}(\sin 2\pi\kappa\theta \cos 2\pi\kappa\theta' - \cos 2\pi\kappa\theta \sin 2\pi\kappa\theta') \tag{91}$$

and we obtain:

$$\begin{aligned}
\frac{\partial A_{\mathrm{ev}}(\zeta,\theta')}{\partial\theta'} &= \frac{c^2T^2V_{\mathrm{e}0}}{2}\bigg\{\frac{1}{\rho_2}(e^{-\rho_2(\zeta+\theta)}e^{\rho_2\theta'} - e^{-\rho_2(\zeta-\theta)}e^{-\rho_2\theta'}) \\
&+ 4e^{-\rho_1\theta'/2}(1-e^{-\rho_2})\bigg[\sum_{\kappa=1}^{<K}\bigg(\mathrm{ch}[(\rho_1^2-d^2)^{1/2}\theta'/2]+\frac{\rho_1\,\mathrm{sh}[(\rho_1^2-d^2)^{1/2}\theta'/2]}{(\rho_1^2-d^2)^{1/2}}\bigg) \\
&\times \frac{\sin 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2}(\sin 2\pi\kappa\theta\cos 2\pi\kappa\theta' - \cos 2\pi\kappa\theta\sin 2\pi\kappa\theta') \\
&+ \sum_{\kappa>K}^{\infty}\bigg(\cos[(d^2-\rho_1^2)^{1/2}\theta'/2] + \frac{\rho_1\sin[(d^2-\rho_1^2)^{1/2}\theta'/2]}{(d^2-\rho_1^2)^{1/2}}\bigg) \\
&\times \frac{\sin 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2}(\sin 2\pi\kappa\theta\cos 2\pi\kappa\theta' - \cos 2\pi\kappa\theta\sin 2\pi\kappa\theta')\bigg]\bigg\}
\end{aligned} \tag{92}$$

The terms in Eq.(92) have been written in a form that will facilitate the following integration over θ':

$$A_{\mathrm{ev}}(\zeta,\theta) = \int_0^\theta \frac{\partial A_{\mathrm{ev}}(\zeta,\theta')}{\partial\theta'}d\theta' \tag{93}$$

We recognize the following integrals over θ' in Eq.(92):

$$\frac{1}{\rho_2}\int_0^\theta (e^{-\rho_2(\zeta+\theta)}e^{\rho_2\theta'} - e^{-\rho_2(\zeta-\theta)}e^{-\rho_2\theta'})d\theta' = \frac{2}{\rho_2^2}e^{-\rho_2\zeta}(1-\mathrm{ch}\,\rho_2\theta) \tag{94}$$

$$L_{11}(\theta,\kappa) = \int_0^\theta e^{-\rho_1\theta'/2}\,\mathrm{sh}[(\rho_1^2-d^2)^{1/2}\theta'/2]\cos 2\pi\kappa\theta'\,d\theta' \tag{95}$$

$$L_{12}(\theta,\kappa) = \int_0^\theta e^{-\rho_1\theta'/2}\,\mathrm{sh}[(\rho_1^2 - d^2)^{1/2}\theta'/2]\sin 2\pi\kappa\theta'\,d\theta' \tag{96}$$

$$L_{13}(\theta,\kappa) = \int_0^\theta e^{-\rho_1\theta'/2}\,\mathrm{ch}[(\rho_1^2 - d^2)^{1/2}\theta'/2]\cos 2\pi\kappa\theta'\,d\theta' \tag{97}$$

$$L_{14}(\theta,\kappa) = \int_0^\theta e^{-\rho_1\theta'/2}\,\mathrm{ch}[(\rho_1^2 - d^2)^{1/2}\theta'/2]\sin 2\pi\kappa\theta'\,d\theta' \tag{98}$$

$$L_{15}(\theta,\kappa) = \int_0^\theta e^{-\rho_1\theta'/2}\sin[(d^2 - \rho_1^2)^{1/2}\theta'/2]\cos 2\pi\kappa\theta'\,d\theta' \tag{99}$$

$$L_{16}(\theta,\kappa) = \int_0^\theta e^{-\rho_1\theta'/2}\sin[(d^2 - \rho_1^2)^{1/2}\theta'/2]\sin 2\pi\kappa\theta'\,d\theta' \tag{100}$$

$$L_{17}(\theta,\kappa) = \int_0^\theta e^{-\rho_1\theta'/2}\cos[(d^2 - \rho_1^2)^{1/2}\theta'/2]\cos 2\pi\kappa\theta'\,d\theta' \tag{101}$$

$$L_{18}(\theta,\kappa) = \int_0^\theta e^{-\rho_1\theta'/2}\cos[(d^2 - \rho_1^2)^{1/2}\theta'/2]\sin 2\pi\kappa\theta'\,d\theta' \tag{102}$$

The integrals of Eqs.(95)–(102) are either tabulated or can readily be rewritten into a tabulated form. Equation (93) assumes the following form:

$$\begin{aligned}
A_{\mathrm{ev}}(\zeta,\theta) = c^2T^2V_{\mathrm{e0}}\Bigg(&\frac{1}{\rho_2^2}e^{-\rho_2\zeta}(1-\mathrm{ch}\,\rho_2\theta)\\
&+2(1-e^{-\rho_2})\Bigg\{\sum_{\kappa=1}^{<K}\Bigg[\Bigg(L_{13}(\theta,\kappa)+\frac{\rho_1L_{11}(\theta,\kappa)}{(\rho_1^2-d^2)^{1/2}}\Bigg)\sin 2\pi\kappa\theta\\
&-\Bigg(L_{14}(\theta,\kappa)+\frac{\rho_1L_{12}(\theta,\kappa)}{(\rho_1^2-d^2)^{1/2}}\Bigg)\cos 2\pi\kappa\theta\Bigg]\frac{\sin 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2}\\
&+\sum_{\kappa>K}^{\infty}\Bigg[\Bigg(L_{17}(\theta,\kappa)+\frac{\rho_1L_{15}(\theta,\kappa)}{(d^2-\rho_1^2)^{1/2}}\Bigg)\sin 2\pi\kappa\theta\\
&-\Bigg(L_{18}(\theta,\kappa)+\frac{\rho_1L_{16}(\theta,\kappa)}{(d^2-\rho_1^2)^{1/2}}\Bigg)\cos 2\pi\kappa\theta\Bigg]\frac{\sin 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2}\Bigg\}\Bigg)
\end{aligned} \tag{103}$$

The constant V_{e0} has the dimension As/m^3 like V_{e} in Eq.(37). We note that all terms of the two sums contain products that represent propagating sinusoidal waves:

$$\sin 2\pi\kappa\theta \sin 2\pi\kappa\zeta = \frac{1}{2}[\cos 2\pi\kappa(\zeta - \theta) - \cos 2\pi\kappa(\zeta + \theta)]$$
$$\cos 2\pi\kappa\theta \sin 2\pi\kappa\zeta = \frac{1}{2}[\sin 2\pi\kappa(\zeta - \theta) + \sin 2\pi\kappa(\zeta + \theta)]$$

In order to simplify Eq.(103) we choose K to be smaller than 1:

$$K = c^2T|(\sigma\mu - s\epsilon)|/4\pi < 1, \quad T < 2\pi/c^2|(\sigma\mu - s\epsilon)| \tag{104}$$

The first sum in Eq.(103) is eliminated while the second sum runs from $\kappa = 1$ to infinity. All denominators $d^2 - \rho_1^2$ are positive:

$$A_{\text{ev}}(\zeta, \theta) = c^2T^2V_{\text{e0}}\Bigg\{\frac{1}{\rho_2^2}e^{-\rho_2\zeta}(1 - \operatorname{ch}\rho_2\theta)$$
$$+ 2(1 - e^{-\rho_2})\sum_{\kappa=1}^{\infty}\Bigg[\left(L_{17}(\theta, \kappa) + \frac{\rho_1 L_{15}(\theta, \kappa)}{(d^2 - \rho_1^2)^{1/2}}\right)\sin 2\pi\kappa\theta$$
$$- \left(L_{18}(\theta, \kappa) + \frac{\rho_1 L_{16}(\theta, \kappa)}{(d^2 - \rho_1^2)^{1/2}}\right)\cos 2\pi\kappa\theta\Bigg]\frac{\sin 2\pi\kappa\zeta}{(2\pi\kappa)^2 + \rho_2^2}\Bigg\}$$

$$d^2 = 4[(2\pi\kappa)^2 + \rho_2^2], \ \rho_1 = c^2T(\sigma\mu + s\epsilon), \ \rho_2^2 = c^2T^2\sigma s \tag{105}$$

The integrals $L_{15}(\theta, \kappa)$ to $L_{18}(\theta, \kappa)$ of Eqs.(99)–(102) can readily be rewritten with the help of two new variables $q_1 = q_1(\kappa)$ and $q_2 = q_2(\kappa)$:

$$q_1 = \frac{1}{2}(d^2 - \rho_1^2)^{1/2} + 2\pi\kappa, \ q_2 = \frac{1}{2}(d^2 - \rho_1^2)^{1/2} - 2\pi\kappa, \quad d^2 - \rho_1^2 > 0 \tag{106}$$

We obtain:

$$L_{15}(\theta, \kappa) = \frac{1}{2}\Bigg[\frac{q_1}{(\rho_1/2)^2 + q_1^2} + \frac{q_2}{(\rho_1/2)^2 + q_2^2} - e^{-\rho_1\theta/2}$$
$$\times \left(\frac{(\rho_1/2)\sin q_1\theta + q_1\cos q_1\theta}{(\rho_1/2)^2 + q_1^2} + \frac{(\rho_1/2)\sin q_2\theta + q_2\cos q_2\theta}{(\rho_1/2)^2 + q_2^2}\right)\Bigg] \tag{107}$$

$$L_{16}(\theta, \kappa) = \frac{1}{2}\Bigg[-\frac{\rho_1}{2}\left(\frac{1}{(\rho_1/2)^2 + q_1^2} - \frac{1}{(\rho_1/2)^2 + q_2^2}\right) + e^{-\rho_1\theta/2}$$
$$\times \left(\frac{(\rho_1/2)\cos q_1\theta - q_1\sin q_1\theta}{(\rho_1/2)^2 + q_1^2} - \frac{(\rho_1/2)\cos q_2\theta - q_2\sin q_2\theta}{(\rho_1/2)^2 + q_2^2}\right)\Bigg] \tag{108}$$

$$L_{17}(\theta, \kappa) = \frac{1}{2}\Bigg[\frac{\rho_1}{2}\left(\frac{1}{(\rho_1/2)^2 + q_1^2} + \frac{1}{(\rho_1/2)^2 + q_2^2}\right) - e^{-\rho_1\theta/2}$$
$$\times \left(\frac{(\rho_1/2)\cos q_1\theta - q_1\sin q_1\theta}{(\rho_1/2)^2 + q_1^2} + \frac{(\rho_1/2)\cos q_2\theta - q_2\sin q_2\theta}{(\rho_1/2)^2 + q_2^2}\right)\Bigg] \tag{109}$$

$$L_{18}(\theta,\kappa) = \frac{1}{2}\left[\frac{q_1}{(\rho_1/2)^2+q_1^2} - \frac{q_2}{(\rho_1/2)^2+q_2^2} - e^{-\rho_1\theta/2} \right.$$
$$\left. \times \left(\frac{(\rho_1/2)\sin q_1\theta + q_1\cos q_1\theta}{(\rho_1/2)^2+q_1^2} - \frac{(\rho_1/2)\sin q_2\theta + q_2\cos q_2\theta}{(\rho_1/2)^2+q_2^2}\right)\right] \quad (110)$$

If the relation

$$\sigma\mu = s\epsilon \quad \text{or} \quad \omega^2 = \frac{s\epsilon}{\sigma\mu} = 1 \quad (111)$$

is satisfied the simplified Eq.(105) will hold for any value of T according to Eq.(104). In all other cases the first sum in Eq.(103) will apply to the smallest values of the wave number κ for a sufficiently large value of T. The integrals $L_{11}(\theta,\kappa)$ to $L_{14}(\theta,\kappa)$ of Eqs.(95)–(98) will then be needed. With the two new variables $q_3 = q_3(\kappa)$ and $q_4 = q_4(\kappa)$

$$q_3 = \frac{1}{2}[(\rho_1^2 - d^2)^{1/2} - \rho_1], \; q_4 = \frac{1}{2}[(\rho_1^2 - d^2)^{1/2} + \rho_1], \quad \rho_1^2 - d^2 > 0 \quad (112)$$

we obtain the following four integrals:

$$L_{11}(\theta,\kappa) = \frac{1}{2}\left(\frac{e^{q_3\theta}(q_3\cos 2\pi\kappa\theta + 2\pi\kappa\sin 2\pi\kappa\theta) - q_3}{q_3^2+(2\pi\kappa)^2} + \frac{e^{-q_4\theta}(q_4\cos 2\pi\kappa\theta - 2\pi\kappa\sin 2\pi\kappa\theta) - q_4}{q_4^2+(2\pi\kappa)^2}\right) \quad (113)$$

$$L_{12}(\theta,\kappa) = \frac{1}{2}\left(\frac{e^{q_3\theta}(q_3\sin 2\pi\kappa\theta - 2\pi\kappa\cos 2\pi\kappa\theta) + 2\pi\kappa}{q_3^2+(2\pi\kappa)^2} + \frac{e^{-q_4\theta}(q_4\sin 2\pi\kappa\theta + 2\pi\kappa\cos 2\pi\kappa\theta) - 2\pi\kappa}{q_4^2+(2\pi\kappa)^2}\right) \quad (114)$$

$$L_{13}(\theta,\kappa) = \frac{1}{2}\left(\frac{e^{q_3\theta}(q_3\cos 2\pi\kappa\theta + 2\pi\kappa\sin 2\pi\kappa\theta) - q_3}{q_3^2+(2\pi\kappa)^2} - \frac{e^{-q_4\theta}(q_4\cos 2\pi\kappa\theta - 2\pi\kappa\sin 2\pi\kappa\theta) - q_4}{q_4^2+(2\pi\kappa)^2}\right) \quad (115)$$

$$L_{14}(\theta,\kappa) = \frac{1}{2}\left(\frac{e^{q_3\theta}(q_3\sin 2\pi\kappa\theta - 2\pi\kappa\cos 2\pi\kappa\theta) + 2\pi\kappa}{q_3^2+(2\pi\kappa)^2} - \frac{e^{-q_4\theta}(q_4\sin 2\pi\kappa\theta + 2\pi\kappa\cos 2\pi\kappa\theta) - 2\pi\kappa}{q_4^2+(2\pi\kappa)^2}\right) \quad (116)$$

We turn to the potential $A_{\rm mv}(\zeta,\theta)$ that is associated with $A_{\rm ev}(\zeta,\theta)$ according to Eq.(47). There are three integrals that have to be evaluated. Consider the first one:

$$\begin{aligned}
A_{\rm mv1}(\zeta,\theta) &= Z\rho_{\rm s}\int\frac{\partial^2 A_{\rm ev}(\zeta,\theta)}{\partial\zeta^2}d\zeta = Z\rho_{\rm s}\frac{\partial A_{\rm ev}(\zeta,\theta)}{\partial\zeta}\\
&= c^2T^2Z\rho_{\rm s}V_{\rm e0}\Bigg(-\frac{1}{\rho_2}e^{-\rho_2\zeta}(1-\operatorname{ch}\rho_2\theta)\\
&+2(1-e^{-\rho_2})\Bigg\{\sum_{\kappa=1}^{<K}\Bigg[\bigg(L_{13}(\theta,\kappa)+\frac{\rho_1L_{11}(\theta,\kappa)}{(\rho_1^2-d^2)^{1/2}}\bigg)\sin 2\pi\kappa\theta\\
&-\bigg(L_{14}(\theta,\kappa)+\frac{\rho_1L_{12}(\theta,\kappa)}{(\rho_1^2-d^2)^{1/2}}\bigg)\cos 2\pi\kappa\theta\Bigg]\frac{2\pi\kappa\cos 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2}\\
&+\sum_{\kappa>K}^{\infty}\Bigg[\bigg(L_{17}(\theta,\kappa)+\frac{\rho_1L_{15}(\theta,\kappa)}{(d^2-\rho_1^2)^{1/2}}\bigg)\sin 2\pi\kappa\theta\\
&-\bigg(L_{18}(\theta,\kappa)+\frac{\rho_1L_{16}(\theta,\kappa)}{(d^2-\rho_1^2)^{1/2}}\bigg)\cos 2\pi\kappa\theta\Bigg]\frac{2\pi\kappa\cos 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2}\Bigg\}\Bigg)
\end{aligned}\tag{117}$$

The second and third integral in Eq.(47) are more difficult to evaluate due to the differentiations with respect to θ:

$$A_{\rm mv2}(\zeta,\theta) = -Z\rho_{\rm s}\int\frac{\partial^2 A_{\rm ev}(\zeta,\theta)}{\partial\theta^2}d\zeta = -Z\rho_{\rm s}\frac{\partial^2}{\partial\theta^2}\int A_{\rm ev}(\zeta,\theta)d\zeta \tag{118}$$

$$A_{\rm mv3}(\zeta,\theta) = -Z\int\frac{\partial A_{\rm ev}(\zeta,\theta)}{\partial\theta}d\zeta = -Z\frac{\partial}{\partial\theta}\int A_{\rm ev}(\theta,\zeta)d\zeta \tag{119}$$

The integral $\int A_{\rm ev}d\zeta$ follows readily from Eq.(103):

$$\begin{aligned}
\int A_{\rm ev}(\zeta,\theta)d\zeta &= c^2T^2V_{\rm e0}\Bigg(-\frac{1}{\rho_2^3}e^{-\rho_2\zeta}(1-\operatorname{ch}\rho_2\theta)\\
&-2(1-e^{-\rho_2})\Bigg\{\sum_{\kappa=1}^{<K}\Bigg[\bigg(L_{13}(\theta,\kappa)+\frac{\rho_1L_{11}(\theta,\kappa)}{(\rho_1^2-d^2)^{1/2}}\bigg)\sin 2\pi\kappa\theta\\
&-\bigg(L_{14}(\theta,\kappa)+\frac{\rho_1L_{12}(\theta,\kappa)}{(\rho_1^2-d^2)^{1/2}}\bigg)\cos 2\pi\kappa\theta\Bigg]\frac{\cos 2\pi\kappa\zeta}{2\pi\kappa[(2\pi\kappa)^2+\rho_2^2]}\\
&+\sum_{\kappa>K}^{\infty}\Bigg[\bigg(L_{17}(\theta,\kappa)+\frac{\rho_1L_{15}(\theta,\kappa)}{(d^2-\rho_1^2)^{1/2}}\bigg)\sin 2\pi\kappa\theta\\
&-\bigg(L_{18}(\theta,\kappa)+\frac{\rho_1L_{16}(\theta,\kappa)}{(d^2-\rho_1^2)^{1/2}}\bigg)\cos 2\pi\kappa\theta\Bigg]\frac{\cos 2\pi\kappa\zeta}{2\pi\kappa[(2\pi\kappa)^2+\rho_2^2]}\Bigg\}\Bigg)
\end{aligned}\tag{120}$$

The component A_{mv3} of the potential defined by Eq.(119) can still be written without undue effort:

$$
\begin{aligned}
A_{\mathrm{mv3}}(\zeta,\theta) = -c^2T^2ZV_{\mathrm{e0}}\Bigg(&\frac{1}{\rho_2^2}e^{-\rho_2\zeta}\,\mathrm{sh}\,\rho_2\theta \\
-2(1-e^{-\rho_2})\Bigg\{&\sum_{\kappa=1}^{<K}\Bigg[\left(\frac{\partial L_{13}}{\partial\theta}+\frac{\rho_1}{(\rho_1^2-d^2)^{1/2}}\frac{\partial L_{11}}{\partial\theta}\right)\frac{\sin 2\pi\kappa\theta}{2\pi\kappa} \\
&+\left(L_{13}+\frac{\rho_1 L_{11}}{(\rho_1^2-d^2)^{1/2}}\right)\cos 2\pi\kappa\theta \\
&-\left(\frac{\partial L_{14}}{\partial\theta}+\frac{\rho_1}{(\rho_1^2-d^2)^{1/2}}\frac{\partial L_{12}}{\partial\theta}\right)\frac{\cos 2\pi\kappa\theta}{2\pi\kappa} \\
&+\left(L_{14}+\frac{\rho_1 L_{12}}{(\rho_1^2-d^2)^{1/2}}\right)\sin 2\pi\kappa\theta\Bigg]\frac{\cos 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2} \\
&+\sum_{\kappa>K}^{\infty}\Bigg[\left(\frac{\partial L_{17}}{\partial\theta}+\frac{\rho_1}{(d^2-\rho_1^2)^{1/2})}\frac{\partial L_{15}}{\partial\theta}\right)\frac{\sin 2\pi\kappa\theta}{2\pi\kappa} \\
&+\left(L_{17}+\frac{\rho_1 L_{15}}{(d^2-\rho_1^2)^{1/2}}\right)\cos 2\pi\kappa\theta \\
&-\left(\frac{\partial L_{18}}{\partial\theta}+\frac{\rho_1}{(d^2-\rho_1^2)^{1/2}}\frac{\partial L_{16}}{\partial\theta}\right)\frac{\cos 2\pi\kappa\theta}{2\pi\kappa} \\
&+\left(L_{18}+\frac{\rho_1 L_{16}}{(d^2-\rho_1^2)^{1/2}}\right)\sin 2\pi\kappa\theta\Bigg]\frac{\cos 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2}\Bigg\}\Bigg)
\end{aligned}
\tag{121}
$$

The derivatives $\partial L_{11}/\partial\theta$ to $\partial L_{18}/\partial\theta$ are the kernels of the integrals of Eqs.(95)–(102) with θ' replaced by θ:

$$\frac{\partial L_{11}}{\partial\theta} = e^{-\rho_1\theta/2}\,\mathrm{sh}\,\frac{(\rho_1^2-d^2)^{1/2}\theta}{2}\cos 2\pi\kappa\theta \tag{122}$$

$$\vdots$$

$$\frac{\partial L_{18}}{\partial\theta} = e^{-\rho_1\theta/2}\cos\frac{(d^2-\rho_1^2)^{1/2}\theta}{2}\sin 2\pi\kappa\theta \tag{123}$$

Writing the last component A_{mv2} of the associated potential defined by Eq.(118) is a challenge. We multiply Eq.(121) by ρ_{s} and differentiate with respect to θ:

$$A_{\mathrm{mv2}}(\zeta,\theta) = -c^2T^2Z\rho_{\mathrm{s}}V_{\mathrm{e0}}\Bigg(\frac{1}{\rho_2}e^{-\rho_2\zeta}\,\mathrm{ch}\,\rho_2\theta$$

$$
\begin{aligned}
&-2(1-e^{\rho_2})\Bigg\{\sum_{\kappa=1}^{<K}\Bigg[\left(\frac{\partial^2 L_{13}}{\partial\theta^2}+\frac{\rho_1}{(\rho_1^2-d^2)^{1/2}}\frac{\partial^2 L_{11}}{\partial\theta^2}\right)\frac{\sin 2\pi\kappa\theta}{2\pi\kappa}\\
&\qquad+2\left(\frac{\partial L_{13}}{\partial\theta}+\frac{\rho_1}{(\rho_1^2-d^2)^{1/2}}\frac{\partial L_{11}}{\partial\theta}\right)\cos 2\pi\kappa\theta\\
&\qquad-2\pi\kappa\left(L_{13}+\frac{\rho_1 L_{11}}{(\rho_1^2-d^2)^{1/2}}\right)\sin 2\pi\kappa\theta\\
&\qquad-\left(\frac{\partial^2 L_{14}}{\partial\theta^2}+\frac{\rho_1}{(\rho_1^2-d^2)^{1/2}}\frac{\partial^2 L_{12}}{\partial\theta^2}\right)\frac{\cos 2\pi\kappa\theta}{2\pi\kappa}\\
&\qquad+2\left(\frac{\partial L_{14}}{\partial\theta}+\frac{\rho_1}{(\rho_1^2-d^2)^{1/2}}\frac{\partial L_{12}}{\partial\theta}\right)\sin 2\pi\kappa\theta\\
&\qquad+2\pi\kappa\left(L_{14}+\frac{\rho_1 L_{12}}{(\rho_1^2-d^2)^{1/2}}\right)\cos 2\pi\kappa\theta\Bigg]\frac{\cos 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2}\\
&\quad+\sum_{\kappa>K}^{\infty}\Bigg[\left(\frac{\partial^2 L_{17}}{\partial\theta^2}+\frac{\rho_1}{(d^2-\rho_1^2)^{1/2}}\frac{\partial^2 L_{15}}{\partial\theta^2}\right)\frac{\sin 2\pi\kappa\theta}{2\pi\kappa}\\
&\qquad+2\left(\frac{\partial L_{17}}{\partial\theta}+\frac{\rho_1}{(d^2-\rho_1^2)^{1/2}}\frac{\partial L_{15}}{\partial\theta}\right)\cos 2\pi\kappa\theta\\
&\qquad-2\pi\kappa\left(L_{17}+\frac{\rho_1 L_{15}}{(d^2-\rho_1^2)^{1/2}}\right)\sin 2\pi\kappa\theta\\
&\qquad-\left(\frac{\partial^2 L_{18}}{\partial\theta^2}+\frac{\rho_1}{(d^2-\rho_1^2)^{1/2}}\frac{\partial^2 L_{16}}{\partial\theta^2}\right)\frac{\cos 2\pi\kappa\theta}{2\pi\kappa}\\
&\qquad+2\left(\frac{\partial L_{18}}{\partial\theta}+\frac{\rho_1}{(d^2-\rho_1^2)^{1/2}}\frac{\partial L_{16}}{\partial\theta}\right)\sin 2\pi\kappa\theta\\
&\qquad+2\pi\kappa\left(L_{18}+\frac{\rho_1 L_{16}}{(d^2-\rho_1^2)^{1/2}}\right)\cos 2\pi\kappa\theta\Bigg]\frac{\cos 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2}\Bigg\}\Bigg)
\end{aligned}
\tag{124}
$$

The second derivatives $\partial^2 L_{11}/\partial\theta^2$ to $\partial^2 L_{18}/\partial\theta^2$ follow from Eqs.(122) and (123). We write only $\partial^2 L_{11}/\partial\theta^2$:

$$
\begin{aligned}
\frac{\partial^2 L_{11}}{\partial\theta^2} &= -\frac{\rho_1}{2}e^{-\rho_1\theta/2}\,\mathrm{sh}\,\frac{(\rho_1^2-d^2)^{1/2}\theta}{2}\cos 2\pi\kappa\theta\\
&\quad+\frac{1}{2}(\rho_1^2-d^2)^{1/2}e^{-\rho_1\theta/2}\,\mathrm{ch}\,\frac{(\rho_1^2-d^2)^{1/2}\theta}{2}\cos 2\pi\kappa\theta\\
&\quad-2\pi\kappa e^{-\rho_1\theta/2}\,\mathrm{sh}\,\frac{(\rho_1^2-d^2)^{1/2}\theta}{2}\sin 2\pi\kappa\theta
\end{aligned}
\tag{125}
$$

If we substitute Eqs.(117), (121), and (124) into Eq.(47) we obtain the component $A_{\mathrm{mv}} = A_{\mathrm{mve}}$ of the potential associated with $\mathbf{A}_{\mathrm{e}}$:

$$
\begin{aligned}
A_{\mathrm{mv}}(\zeta,\theta) &= A_{\mathrm{mv1}}(\zeta,\theta) + A_{\mathrm{mv2}}(\zeta,\theta) + A_{\mathrm{mv3}}(\zeta,\theta) \\
&= c^2T^2 Z\rho_{\mathrm{s}} V_{\mathrm{e0}}\Bigg(-\frac{1}{\rho_2}\left(1+\frac{\operatorname{sh}\rho_2\theta}{\rho_s\rho_2}\right)e^{-\rho_2\zeta} \\
&+ 2(1-e^{-\rho_2})\Bigg\{ \sum_{\kappa=1}^{<K} \Bigg[\left(\frac{\partial^2 L_{13}}{\partial\theta^2} + \frac{\rho_1}{(\rho_1^2-d^2)^{1/2}}\frac{\partial^2 L_{11}}{\partial\theta^2}\right)\frac{\sin 2\pi\kappa\theta}{2\pi\kappa} \\
&+ \left(\frac{\partial L_{13}}{\partial\theta} + \frac{\rho_1}{(\rho_1^2-d^2)^{1/2}}\frac{\partial L_{11}}{\partial\theta}\right)\left(2\cos 2\pi\kappa\theta + \frac{\sin 2\pi\kappa}{2\pi\kappa\rho_2}\right) \\
&+ \left(L_{13} + \frac{\rho_1 L_{11}}{(\rho_1^2-d^2)^{1/2}}\right)\frac{\cos 2\pi\kappa\theta}{\rho_2} \\
&- \left(\frac{\partial^2 L_{14}}{\partial\theta^2} + \frac{\rho_1}{(\rho_1^2-d^2)^{1/2}}\frac{\partial^2 L_{12}}{\partial\theta^2}\right)\frac{\cos 2\pi\kappa\theta}{2\pi\kappa} \\
&+ \left(\frac{\partial L_{14}}{\partial\theta} + \frac{\rho_1}{(\rho_1^2-d^2)^{1/2}}\frac{\partial L_{12}}{\partial\theta}\right)\left(2\sin 2\pi\kappa\theta - \frac{\cos 2\pi\kappa\theta}{2\pi\kappa\rho_s}\right) \\
&+ \left(L_{14} + \frac{\rho_1 L_{12}}{(\rho_1^2-d^2)^{1/2}}\right)\frac{\sin 2\pi\kappa\theta}{\rho_s}\Bigg]\frac{\cos 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2} \\
&+ \sum_{\kappa>K}^{\infty} \Bigg[\left(\frac{\partial^2 L_{17}}{\partial\theta^2} + \frac{\rho_1}{(d^2-\rho_1^2)^{1/2}}\frac{\partial^2 L_{15}}{\partial\theta^2}\right)\frac{\sin 2\pi\kappa\theta}{2\pi\kappa} \\
&+ \left(\frac{\partial L_{17}}{\partial\theta} + \frac{\rho_1}{(d^2-\rho_1^2)^{1/2}}\frac{\partial L_{15}}{\partial\theta}\right)\left(2\cos 2\pi\kappa\theta + \frac{\sin 2\pi\kappa\theta}{2\pi\kappa\rho_s}\right) \\
&+ \left(L_{17} + \frac{\rho_1 L_{15}}{(d^2-\rho_1^2)^{1/2}}\right)\frac{\cos 2\pi\kappa\theta}{\rho_s} \\
&- \left(\frac{\partial^2 L_{18}}{\partial\theta^2} + \frac{\rho_1}{(d^2-\rho_1^2)^{1/2}}\frac{\partial^2 L_{16}}{\partial\theta^2}\right)\frac{\cos 2\pi\kappa\theta}{2\pi\kappa} \\
&+ \left(\frac{\partial L_{18}}{\partial\theta} + \frac{\rho_1}{(d^2-\rho_1^2)^{1/2}}\frac{\partial L_{16}}{\partial\theta}\right)\left(2\sin 2\pi\kappa\theta - \frac{\cos 2\pi\kappa\theta}{2\pi\kappa\rho_s}\right) \\
&+ \left(L_{18} + \frac{\rho_1 L_{16}}{(d^2-\rho_1^2)^{1/2}}\right)\frac{\sin 2\pi\kappa\theta}{\rho_s}\Bigg]\frac{\cos 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_s^2}\Bigg\}\Bigg) \qquad (126)
\end{aligned}
$$

A significant simplification of this equation can be achieved by choosing K according to Eq.(104). We may then leave out the first sum and let the second sum run from $\kappa = 1$ to infinity just as in Eq.(105).

One still must solve Eq.(48) if one wants to determine integration constants. We shall forgo this task at this time in order to stick to the most important goals and avoid getting overwhelmed by details.

For the general solution of Eqs.(52) and (53) one must still solve Eqs.(38), (39), (46), and (50), which implies a repetition of our calculations with the changed notation $V_{\mathrm{e}} \rightarrow V_{\mathrm{m}}$, $V_{\mathrm{e0}} \rightarrow V_{\mathrm{m0}}$, $A_{\mathrm{ev}} \rightarrow A_{\mathrm{mv}}$, and $A_{\mathrm{mv}} \rightarrow A_{\mathrm{ev}}$.

4.2 Simplification of $A_{\mathrm{ev}}(\zeta,\theta)$ and $A_{\mathrm{mv}}(\zeta,\theta)$

The functions $A_{\mathrm{ev}}(\zeta,\theta)$ and $A_{\mathrm{mv}}(\zeta,\theta)$ of Eqs.(4.1-103) and (4.1-126) define the components $A_{\mathrm{e}x}$, $A_{\mathrm{e}z}$, $A_{\mathrm{m}x}$, and $A_{\mathrm{m}z}$ of the potentials $\mathbf{A}_{\mathrm{e}}$ and $\mathbf{A}_{\mathrm{m}}$ according to Eqs.(4.1-2), (4.1-3), and (4.1-31) for a step function excitation $V_{\mathrm{e}}(0,\theta)$ according to Eq.(4.1-54) at the boundary $\zeta = 0$. The functions $A_{\mathrm{ev}}(\zeta,\theta)$ and $A_{\mathrm{mv}}(\zeta,\theta)$ are rather complicated and must be simplified for $\theta > 0$, $\zeta > 0$.

One might want to ignore the terms multiplied by $e^{-\rho_1\theta/2}$ in Eqs.(107) to (110) as well as the terms multiplied by $e^{q_3\theta}$ and $e^{-q_4\theta}$ in Eqs.(113) to (116), which can be rewritten to show a factor $e^{-\rho_1\theta/2}$ explicitly. But this does not work since the energy of the wave is reduced to zero. In order to retain these terms we start out by rewriting Eqs.(4.1-106) to (4.1-110) and (4.1-112) to (4.1-116) in a slightly changed notation:

$$q_1 = \frac{1}{2}(d^2-\rho_1^2)^{1/2} + 2\pi\kappa,\ q_2 = \frac{1}{2}(d^2-\rho_1^2)^{1/2} - 2\pi\kappa, \quad d^2-\rho_1^2 > 0 \tag{1}$$

$$\begin{aligned} L_{15}(\theta,\kappa) &= L_{15\mathrm{A}}(\kappa) + e^{-\rho_1\theta/2}L_{15\mathrm{B}}(\theta,\kappa) \\ &= \frac{1}{2}\bigg[\frac{q_1}{(\rho_1/2)^2+q_1^2} + \frac{q_2}{(\rho_1/2)^2+q_2^2} - e^{-\rho_1\theta/2} \\ &\quad\times\bigg(\frac{(\rho_1/2)\sin q_1\theta + q_1\cos q_1\theta}{(\rho_1/2)^2+q_1^2} + \frac{(\rho_1/2)\sin q_2\theta + q_2\cos q_2\theta}{(\rho_1/2)^2+q_2^2}\bigg)\bigg] \end{aligned} \tag{2}$$

$$\begin{aligned} L_{16}(\theta,\kappa) &= L_{16\mathrm{A}}(\kappa) + e^{-\rho_1\theta/2}L_{16\mathrm{B}}(\theta,\kappa) \\ &= \frac{1}{2}\bigg[-\frac{\rho_1}{2}\bigg(\frac{1}{(\rho_1/2)^2+q_1^2} - \frac{1}{(\rho_1/2)^2+q_2^2}\bigg) + e^{-\rho_1\theta/2} \\ &\quad\times\bigg(\frac{(\rho_1/2)\cos q_1\theta - q_1\sin q_1\theta}{(\rho_1/2)^2+q_1^2} - \frac{(\rho_1/2)\cos q_2\theta - q_2\sin q_2\theta}{(\rho_1/2)^2+q_2^2}\bigg)\bigg] \end{aligned} \tag{3}$$

$$\begin{aligned} L_{17}(\theta,\kappa) &= L_{17\mathrm{A}}(\kappa) + e^{-\rho_1\theta/2}L_{17\mathrm{B}}(\theta,\kappa) \\ &= \frac{1}{2}\bigg[\frac{\rho_1}{2}\bigg(\frac{1}{(\rho_1/2)^2+q_1^2} + \frac{1}{(\rho_1/2)^2+q_2^2}\bigg) - e^{-\rho_1\theta/2} \\ &\quad\times\bigg(\frac{(\rho_1/2)\cos q_1\theta - q_1\sin q_1\theta}{(\rho_1/2)^2+q_1^2} + \frac{(\rho_1/2)\cos q_2\theta - q_2\sin q_2\theta}{(\rho_1/2)^2+q_2^2}\bigg)\bigg] \end{aligned} \tag{4}$$

$$
\begin{aligned}
L_{18}(\theta,\kappa) &= L_{18\mathrm{A}}(\kappa) + e^{-\rho_1\theta/2}L_{18\mathrm{B}}(\theta,\kappa)\\
&= \frac{1}{2}\bigg[\frac{q_1}{(\rho_1/2)^2+q_1^2} - \frac{q_2}{(\rho_1/2)^2+q_2^2} - e^{-\rho_1\theta/2}\\
&\quad\times\left(\frac{(\rho_1/2)\sin q_1\theta + q_1\cos q_1\theta}{(\rho_1/2)^2+q_1^2} - \frac{(\rho_1/2)\sin q_2\theta + q_2\cos q_2\theta}{(\rho_1/2)^2+q_2^2}\right)\bigg] \qquad (5)
\end{aligned}
$$

Equations (4.1-112) to (4.1-116) assume the following form

$$
q_3 = \frac{1}{2}[(\rho_1^2-d^2)^{1/2}-\rho_1],\ q_4 = \frac{1}{2}[(\rho_1^2-d^2)^{1/2}+\rho_1],\quad \rho_1^2-d^2>0 \qquad (6)
$$

$$
\begin{aligned}
L_{11}(\theta,\kappa) &= L_{11\mathrm{A}}(\kappa) + e^{-\rho_1\theta/2}L_{11\mathrm{B}}(\theta,\kappa)\\
&= \frac{1}{2}\bigg[-\frac{q_3}{q_3^2+(2\pi\kappa)^2} - \frac{q_4}{q_4^2+(2\pi\kappa)^2} + e^{-\rho_1\theta/2}\\
&\quad\times\bigg(\frac{\exp[(\rho_1^2-d^2)^{1/2}\theta/2](q_3\cos 2\pi\kappa\theta + 2\pi\kappa\sin 2\pi\kappa\theta)}{q_3^2+(2\pi\kappa)^2}\\
&\quad+\frac{\exp[-(\rho_1^2-d^2)^{1/2}\theta/2](q_4\cos 2\pi\kappa\theta - 2\pi\kappa\sin 2\pi\kappa\theta)}{q_4^2+(2\pi\kappa)^2}\bigg)\bigg] \qquad (7)
\end{aligned}
$$

$$
\begin{aligned}
L_{12}(\theta,\kappa) &= L_{12\mathrm{A}}(\kappa) + e^{-\rho_1\theta/2}L_{12\mathrm{B}}(\theta,\kappa)\\
&= \frac{1}{2}\bigg[\frac{2\pi\kappa}{q_3^2+(2\pi\kappa)^2} - \frac{2\pi\kappa}{q_4^2+(2\pi\kappa)^2} + e^{-\rho_1\theta/2}\\
&\quad\times\bigg(\frac{\exp[(\rho_1^2-d^2)^{1/2}\theta/2](q_3\sin 2\pi\kappa\theta - 2\pi\kappa\cos 2\pi\kappa\theta)}{q_3^2+(2\pi\kappa)^2}\\
&\quad+\frac{\exp[-(\rho_1^2-d^2)^{1/2}\theta/2](q_4\sin 2\pi\kappa\theta + 2\pi\kappa\cos 2\pi\kappa\theta)}{q_4^2+(2\pi\kappa)^2}\bigg)\bigg] \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
L_{13}(\theta,\kappa) &= L_{13\mathrm{A}}(\kappa) + e^{-\rho_1\theta/2}L_{13\mathrm{B}}(\theta,\kappa)\\
&= \frac{1}{2}\bigg[-\frac{q_3}{q_3^2+(2\pi\kappa)^2} + \frac{q_4}{q_4^2+(2\pi\kappa)^2} + e^{-\rho_1\theta/2}\\
&\quad\times\bigg(\frac{\exp[(\rho_1^2-d^2)^{1/2}\theta/2](q_3\cos 2\pi\kappa\theta + 2\pi\kappa\sin 2\pi\kappa\theta)}{q_3^2+(2\pi\kappa)^2}\\
&\quad-\frac{\exp[-(\rho_1^2-d^2)^{1/2}\theta/2](q_4\cos 2\pi\kappa\theta - 2\pi\kappa\sin 2\pi\kappa\theta)}{q_4^2+(2\pi\kappa)^2}\bigg)\bigg] \qquad (9)
\end{aligned}
$$

$$\begin{aligned}
L_{14}(\theta,\kappa) &= L_{14\mathrm{A}}(\kappa) + e^{-\rho_1\theta/2} L_{14\mathrm{B}}(\theta,\kappa) \\
&= \frac{1}{2}\Bigg[\frac{2\pi\kappa}{q_3^2+(2\pi\kappa)^2} + \frac{2\pi\kappa}{q_4^2+(2\pi\kappa)^2} + e^{-\rho_1\theta/2} \\
&\quad \times \bigg(\frac{\exp[(\rho_1^2-d^2)^{1/2}\theta/2](q_3\sin 2\pi\kappa\theta - 2\pi\kappa\cos 2\pi\kappa\theta)}{q_3^2+(2\pi\kappa)^2} \\
&\quad - \frac{\exp[-(\rho_1^2-d^2)^{1/2}\theta/2](q_4\sin 2\pi\kappa\theta + 2\pi\kappa\cos 2\pi\kappa\theta)}{q_4^2+(2\pi\kappa)^2}\bigg)\Bigg] \qquad (10)
\end{aligned}$$

Equation (4.1-103) may be written in the following form:

$$\begin{aligned}
A_{ev}(\zeta,\theta) &= c^2T^2V_{e0}\Bigg(\frac{1}{\rho_2^2}e^{-\rho_2\zeta}(1-\operatorname{ch}\rho_2\theta) \\
&+ 2(1-e^{-\rho_2})\sum_{\kappa=0}^{<K}\Bigg\{\bigg(L_{13\mathrm{A}}(\kappa)+\frac{\rho_1 L_{11\mathrm{A}}(\kappa)}{(\rho_1^2-d^2)^{1/2}}\bigg)\sin 2\pi\kappa\theta \\
&- \bigg(L_{14\mathrm{A}}(\kappa)+\frac{\rho_1 L_{12\mathrm{A}}(\kappa)}{(\rho_1^2-d^2)^{1/2}}\bigg)\cos 2\pi\kappa\theta \\
&+ e^{-\rho_1\theta/2}\bigg[\bigg(L_{13\mathrm{B}}(\theta,\kappa)+\frac{\rho_1 L_{11\mathrm{B}}(\theta,\kappa)}{(\rho_1^2-d^2)^{1/2}}\bigg)\sin 2\pi\kappa\theta \\
&- \bigg(L_{14\mathrm{B}}(\theta,\kappa)+\frac{\rho_1 L_{12\mathrm{B}}(\theta,\kappa)}{(\rho_1^2-d^2)^{1/2}}\bigg)\cos 2\pi\kappa\theta\bigg]\Bigg\}\frac{\sin 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2} \\
&+ 2(1-e^{-\rho_2})\sum_{\kappa>K}^{\infty}\Bigg\{\bigg(L_{17\mathrm{A}}(\kappa)+\frac{\rho_1 L_{15\mathrm{A}}(\kappa)}{(d^2-\rho_1^2)^{1/2}}\bigg)\sin 2\pi\kappa\theta \\
&- \bigg(L_{18\mathrm{A}}(\kappa)+\frac{\rho_1 L_{16\mathrm{A}}(\kappa)}{(d^2-\rho_1^2)^{1/2}}\bigg)\cos 2\pi\kappa\theta \\
&+ e^{-\rho_1\theta/2}\bigg[\bigg(L_{17\mathrm{B}}(\theta,\kappa)+\frac{\rho_1 L_{15\mathrm{B}}(\theta,\kappa)}{(d^2-\rho_1^2)^{1/2}}\bigg)\sin 2\pi\kappa\theta \\
&- \bigg(L_{18\mathrm{B}}(\theta,\kappa)+\frac{\rho_1 L_{16\mathrm{B}}(\theta,\kappa)}{(d^2-\rho_1^2)^{1/2}}\bigg)\cos 2\pi\kappa\theta\bigg]\Bigg\}\frac{\sin 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2}\Bigg) \qquad (11)
\end{aligned}$$

We recognize that the terms $L_{..\mathrm{A}}(\kappa)$ do not contain the variable θ and are connected with ζ and θ only via the products with $\sin 2\pi\kappa\theta$, $\cos 2\pi\kappa\theta$, and $\sin 2\pi\kappa\zeta$. The terms $L_{..\mathrm{B}}(\theta,\kappa)$ contain θ and they are multiplied in addition by $e^{-\rho_1\theta/2}$. If we succeed in eliminating θ from $L_{..\mathrm{B}}(\theta,\kappa)$ and further eliminate $e^{-\rho_1\theta/2}$ we can use the usual quantization process. This elimination is possible by means of Fourier series expansions. The calculations are straight forward but lengthy. They will be found in Section 6.12. We recognize in Eq.(6.12-43) the last two lines of Eq.(11), and in Eq.(6.12-60) lines 4 and 5 of Eq.(11).

The whole Eq.(11) is shown in Eq.(6.12-61), still with the factors $e^{-\rho_1\theta/2}$ and other features that need reworking. The required changes are carried out in the calculations that lead from Eq.(6.12-61) to (6.12-108) and its radically simplified form shown by Eq.(6.12-110). We copy this equation:

$$A_{\mathrm{ev}}(\zeta,\theta) = c^2T^2V_{\mathrm{e0}}\sum_{\kappa=1}^{\infty} C_{\mathrm{e}\kappa}(\theta)\sin 2\pi\kappa\zeta$$

$$C_{\mathrm{e}\kappa}(\theta) = 2\Big(A_{\mathrm{es}}(\kappa)\sin 2\pi\kappa\theta + A_{\mathrm{ec}}(\kappa)\cos 2\pi\kappa\theta + \sum_{\nu=1}^{\infty}[B_{\mathrm{es}}(\kappa,\nu)\sin 2\pi\nu\theta + B_{\mathrm{ec}}(\kappa,\nu)\cos 2\pi\nu\theta]\Big) \tag{12}$$

Let us turn to the potential $A_{\mathrm{mv}}(\zeta,\theta)$ that is associated with $A_{\mathrm{ev}}(\zeta,\theta)$ according to Eq.(4.1-47). Three integrals have to be evaluated. Here is the first one:

$$\begin{aligned}A_{\mathrm{mv1}}(\zeta,\theta) &= Z\rho_{\mathrm{s}}\int\frac{\partial^2 A_{\mathrm{ev}}(\zeta,\theta)}{\partial\zeta^2}d\zeta = Z\rho_{\mathrm{s}}\frac{\partial A_{\mathrm{ev}}(\zeta,\theta)}{\partial\zeta}\\ &= c^2T^2V_{\mathrm{e0}}Z\rho_{\mathrm{s}}\frac{\partial}{\partial\zeta}\sum_{\kappa=1}^{\infty}C_{\mathrm{e}\kappa}(\theta)\sin 2\pi\kappa\zeta\\ &= c^2T^2V_{\mathrm{e0}}Z\rho_{\mathrm{s}}\sum_{\kappa=1}^{\infty}2\pi\kappa C_{\mathrm{e}\kappa}(\theta)\cos 2\pi\kappa\zeta\end{aligned} \tag{13}$$

The third integral in Eq.(4.1-47) yields:

$$\begin{aligned}A_{\mathrm{mv3}}(\zeta,\theta) &= -Z\int\frac{\partial A_{\mathrm{ev}}}{\partial\theta}d\zeta\\ &= c^2T^2V_{\mathrm{e0}}Z\frac{\partial}{\partial\theta}\sum_{\kappa=1}^{\infty}C_{\mathrm{e}\kappa}(\theta)\frac{\cos 2\pi\kappa\zeta}{2\pi\kappa}\\ &= c^2T^2V_{\mathrm{e0}}Z\sum_{\kappa=1}^{\infty}\frac{\partial C_{\mathrm{e}\kappa}(\theta)}{\partial\theta}\frac{\cos 2\pi\kappa\zeta}{2\pi\kappa}\end{aligned}$$

$$\frac{\partial C_{\mathrm{e}\kappa}(\theta)}{\partial\theta} = 2\Big(2\pi\kappa[A_{\mathrm{es}}(\kappa)\cos 2\pi\kappa\theta - A_{\mathrm{ec}}(\kappa)\sin 2\pi\kappa\theta] + \sum_{\nu=1}^{\infty}2\pi\nu[B_{\mathrm{es}}(\kappa,\nu)\cos 2\pi\nu\theta - B_{\mathrm{ec}}(\kappa,\nu)\sin 2\pi\nu\theta]\Big) \tag{14}$$

The second integral in Eq.(4.1-47) calls for a second differentiation of $A_{\mathrm{mv3}}(\zeta,\theta)$ with respect to θ:

$$
\begin{aligned}
A_{\text{mv2}}(\zeta,\theta) &= -Z\rho_{\text{s}}\int \frac{\partial^2 A_{\text{ev}}(\zeta,\theta)}{\partial\theta^2}d\zeta \\
&= c^2T^2V_{\text{e0}}Z\rho_{\text{s}}\frac{\partial^2}{\partial\theta^2}\sum_{\kappa=1}^{\infty} C_{\text{e}\kappa}(\theta)\frac{\cos 2\pi\kappa\zeta}{2\pi\kappa} \\
&= c^2T^2V_{\text{e0}}Z\rho_{\text{s}}\sum_{\kappa=1}^{\infty}\frac{\partial^2 C_{\text{e}\kappa}(\theta)}{\partial\theta^2}\frac{\cos 2\pi\kappa\zeta}{2\pi\kappa}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 C_{\text{e}\kappa}(\theta)}{\partial\theta^2} = -2\bigg((2\pi\kappa)^2[A_{\text{es}}(\kappa)\sin 2\pi\kappa\theta + A_{\text{ec}}(\kappa)\cos 2\pi\kappa\theta] \\
+\sum_{\kappa=1}^{\infty}(2\pi\nu)^2[B_{\text{es}}(\kappa,\nu)\sin 2\pi\nu\theta + B_{\text{ec}}(\kappa,\nu)\cos 2\pi\nu\theta]\bigg) \qquad (15)
\end{aligned}
$$

The potential $A_{\text{mv}}(\zeta,\theta) = A_{\text{mve}}(\zeta,\theta)$ associated with $A_{\text{ev}}(\zeta,\theta)$ of Eq.(12) is the sum of A_{mv1}, A_{mv2}, and A_{mv3}:

$$
\begin{aligned}
A_{\text{mv}}(\zeta,\theta) &= A_{\text{mv1}}(\zeta,\theta) + A_{\text{mv2}}(\zeta,\theta) + A_{\text{mv3}}(\zeta,\theta) \\
&= c^2T^2V_{\text{e0}}Z\sum_{\kappa=1}^{\infty} C_{\text{m}\kappa}(\theta)\cos 2\pi\kappa\zeta
\end{aligned}
$$

$$
\begin{aligned}
C_{\text{m}\kappa}(\theta) &= 2\pi\kappa\rho_{\text{s}}C_{\text{e}\kappa}(\theta) + \frac{1}{2\pi\kappa}\frac{\partial C_{\text{e}\kappa}(\theta)}{\partial\theta} + \frac{\rho_{\text{s}}}{2\pi\kappa}\frac{\partial^2 C_{\text{e}\kappa}(\theta)}{\partial\theta^2} \\
&= 2\Big\{-A_{ec}(\kappa)\sin 2\pi\kappa\theta + A_{es}(\kappa)\cos 2\pi\kappa\theta \\
&\qquad + \sum_{\nu=1}^{\infty}[C_{\text{es}}(\kappa,\nu)\sin 2\pi\nu\theta + C_{\text{ec}}(\kappa,\nu)\cos 2\pi\nu\theta]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
C_{\text{es}}(\kappa,\nu) &= 2\pi\kappa\rho_{\text{s}}\left(1-\frac{(2\pi\nu)^2}{(2\pi\kappa)^2}\right)B_{\text{es}}(\kappa,\nu) - \frac{2\pi\nu}{2\pi\kappa}B_{\text{ec}}(\kappa,\nu) \\
C_{\text{ec}}(\kappa,\nu) &= 2\pi\kappa\rho_{\text{s}}\left(1-\frac{(2\pi\nu)^2}{(2\pi\kappa)^2}\right)B_{\text{ec}}(\kappa,\nu) + \frac{2\pi\nu}{2\pi\kappa}B_{\text{es}}(\kappa,\nu) \qquad (16)
\end{aligned}
$$

With have now simplified the expressions for $A_{\text{ev}}(\zeta,\theta)$ and $A_{\text{mv}}(\zeta,\theta)$ sufficiently to be able to derive some useable results for the Hamilton function of the pure radiation field.

4.3 Hamilton Function for Planar Wave

Our Fourier series expansion permits a largest time T and a largest distance cT in the direction y with the intervals $0 \le t \le T$, $0 \le y \le cT$. In the directions x and z we have not specified any intervals and we shall follow Eq.(3.4-32) to make them $-L/2 \le x \le L/2$, $-L/2 \le z \le L/2$. The energy U of the electric and magnetic field strength in this volume follows from Eq.(3.4-33):

$$U = \frac{1}{2}\int_{-L/2}^{L/2}\int_{-L/2}^{L/2}\left[\int_0^{cT}\left(\frac{1}{Zc}E^2 + \frac{Z}{c}H^2\right)dy\right]dx\,dz \tag{1}$$

$$E^2 = \left(-Zc\operatorname{curl}\mathbf{A}_\mathrm{e} - \frac{\partial \mathbf{A}_\mathrm{m}}{\partial t}\right)^2 \tag{2}$$

$$H^2 = \left(\frac{c}{Z}\operatorname{curl}\mathbf{A}_\mathrm{m} - \frac{\partial \mathbf{A}_\mathrm{e}}{\partial t}\right)^2 \tag{3}$$

We do not need the complex notation of Eqs.(3.4-34) and (3.4-35) since the potentials or their components were derived in real form in Section 4.1.

From Eqs.(4.1-31), (4.1-34), and (4.1-35) we obtain the following values for the components of $\mathbf{A}_\mathrm{e}$ and $\mathbf{A}_\mathrm{m}$ in Eqs.(2) and (3):

$$\begin{aligned}
A_{\mathrm{e}x}(\zeta,\theta) &= A_{\mathrm{ev}}(\zeta,\theta) & A_{\mathrm{m}x}(\zeta,\theta) &= A_{\mathrm{mv}}(\zeta,\theta)\\
A_{\mathrm{e}y}(\zeta,\theta) &= A_{\mathrm{e}0}f_\mathrm{e}(\zeta-\theta) & A_{\mathrm{m}y}(\zeta,\theta) &= A_{\mathrm{m}0}f_\mathrm{m}(\zeta-\theta)\\
A_{\mathrm{e}z}(\zeta,\theta) &= A_{\mathrm{ev}}(\zeta,\theta) & A_{\mathrm{m}z}(\zeta,\theta) &= -A_{\mathrm{mv}}(\zeta,\theta)
\end{aligned} \tag{4}$$

The functions $A_{\mathrm{ev}}(\zeta,\theta)$ and $A_{\mathrm{mv}}(\zeta,\theta)$ are defined by Eqs.(4.1-103) and (4.1-126), while $f_\mathrm{e}(\zeta-\theta)$ and $f_\mathrm{m}(\zeta-\theta)$ are arbitrary functions.

For the vector components of the expansion of the right sides of Eqs.(2) and (3)

$$\left(-Zc\operatorname{curl}\mathbf{A}_\mathrm{e} - \frac{\partial \mathbf{A}_\mathrm{m}}{\partial t}\right)^2 = Z^2c^2\operatorname{curl}^2\mathbf{A}_\mathrm{e} + 2Zc\operatorname{curl}\mathbf{A}_\mathrm{e}\cdot\frac{\partial \mathbf{A}_\mathrm{m}}{\partial t} + \left(\frac{\partial \mathbf{A}_\mathrm{m}}{\partial t}\right)^2 \tag{5}$$

$$\left(\frac{c}{Z}\operatorname{curl}\mathbf{A}_\mathrm{m} - \frac{\partial \mathbf{A}_\mathrm{e}}{\partial t}\right)^2 = \frac{c^2}{Z^2}\operatorname{curl}^2\mathbf{A}_\mathrm{m} - \frac{2c}{Z}\operatorname{curl}\mathbf{A}_\mathrm{m}\cdot\frac{\partial \mathbf{A}_\mathrm{e}}{\partial t} + \left(\frac{\partial \mathbf{A}_\mathrm{e}}{\partial t}\right)^2 \tag{6}$$

we obtain with the help of Eqs.(4.1-17), (4.1-18), (4.1-27), (4.1-28), and (4.1-31) the following relations :

$$\operatorname{curl}^2\mathbf{A}_\mathrm{e} = 2\left(\frac{\partial A_{\mathrm{ev}}}{\partial y}\right)^2 = \frac{2}{c^2T^2}\left(\frac{\partial A_{\mathrm{ev}}}{\partial \zeta}\right)^2 \tag{7}$$

$$\operatorname{curl} \mathbf{A}_{\mathrm{e}} \cdot \frac{\partial \mathbf{A}_{\mathrm{m}}}{\partial t} = 2\frac{\partial A_{\mathrm{ev}}}{\partial y}\frac{\partial A_{\mathrm{mv}}}{\partial t} = \frac{2}{cT^2}\frac{\partial A_{\mathrm{ev}}}{\partial \zeta}\frac{\partial A_{\mathrm{mv}}}{\partial \theta} \tag{8}$$

$$\left(\frac{\partial \mathbf{A}_{\mathrm{m}}}{\partial t}\right)^2 = 2\left(\frac{\partial A_{\mathrm{mv}}}{\partial t}\right)^2 = \frac{2}{T^2}\left(\frac{\partial A_{\mathrm{mv}}}{\partial \theta}\right)^2 \tag{9}$$

$$\operatorname{curl}^2 \mathbf{A}_{\mathrm{m}} = 2\left(\frac{\partial A_{\mathrm{mv}}}{\partial y}\right)^2 = \frac{2}{c^2T^2}\left(\frac{\partial A_{\mathrm{mv}}}{\partial \zeta}\right)^2 \tag{10}$$

$$\operatorname{curl} \mathbf{A}_{\mathrm{m}} \cdot \frac{\partial \mathbf{A}_{\mathrm{e}}}{\partial t} = -2\frac{\partial A_{\mathrm{mv}}}{\partial y}\frac{\partial A_{\mathrm{ev}}}{\partial t} = -\frac{2}{cT^2}\frac{\partial A_{\mathrm{mv}}}{\partial \zeta}\frac{\partial A_{\mathrm{ev}}}{\partial \theta} \tag{11}$$

$$\left(\frac{\partial \mathbf{A}_{\mathrm{e}}}{\partial t}\right)^2 = 2\left(\frac{\partial A_{\mathrm{ev}}}{\partial t}\right)^2 = \frac{2}{T^2}\left(\frac{\partial A_{\mathrm{ev}}}{\partial \theta}\right)^2 \tag{12}$$

We get for E^2 and H^2 in Eqs.(2) and (3)

$$\begin{aligned} E^2 &= \frac{2}{T^2}\left[Z^2\left(\frac{\partial A_{\mathrm{ev}}}{\partial \zeta}\right)^2 + 2Z\frac{\partial A_{\mathrm{ev}}}{\partial \zeta}\frac{\partial A_{\mathrm{mv}}}{\partial \theta} + \left(\frac{\partial A_{\mathrm{mv}}}{\partial \theta}\right)^2\right] \\ &= \frac{2}{T^2}\left(Z\frac{\partial A_{\mathrm{ev}}}{\partial \zeta} + \frac{\partial A_{\mathrm{mv}}}{\partial \theta}\right)^2 \end{aligned} \tag{13}$$

$$\begin{aligned} H^2 &= \frac{2}{Z^2T^2}\left[\left(\frac{\partial A_{\mathrm{mv}}}{\partial \zeta}\right)^2 + 2Z\frac{\partial A_{\mathrm{mv}}}{\partial \zeta}\frac{\partial A_{\mathrm{ev}}}{\partial \theta} + Z^2\left(\frac{\partial A_{\mathrm{ev}}}{\partial \theta}\right)^2\right] \\ &= \frac{2}{Z^2T^2}\left(\frac{\partial A_{\mathrm{mv}}}{\partial \zeta} + Z\frac{\partial A_{\mathrm{ev}}}{\partial \theta}\right)^2 \end{aligned} \tag{14}$$

while the energy U becomes:

$$\begin{aligned} U = \frac{c^2T}{Z}\int\limits_{-L/2cT}^{L/2cT}\int\limits_{-L/2cT}^{L/2cT}\Bigg\{\int\limits_0^1\Bigg[\left(Z\frac{\partial A_{\mathrm{ev}}}{\partial \zeta} + \frac{\partial A_{\mathrm{mv}}}{\partial \theta}\right)^2 & \\ + \left(\frac{\partial A_{\mathrm{mv}}}{\partial \zeta} + Z\frac{\partial A_{\mathrm{ev}}}{\partial \theta}\right)^2\Bigg]d\zeta\Bigg\}d\left(\frac{x}{cT}\right)d\left(\frac{z}{cT}\right) & \\ = \frac{c^2T}{Z}\left(\frac{L}{cT}\right)^2\int\limits_0^1\Bigg[Z^2\left(\frac{\partial A_{\mathrm{ev}}}{\partial \zeta}\right)^2 + Z^2\left(\frac{\partial A_{\mathrm{ev}}}{\partial \theta}\right)^2 & \\ + 2Z\left(\frac{\partial A_{\mathrm{ev}}}{\partial \zeta}\frac{\partial A_{\mathrm{mv}}}{\partial \theta} + \frac{\partial A_{\mathrm{mv}}}{\partial \zeta}\frac{\partial A_{\mathrm{ev}}}{\partial \theta}\right) & \\ + \left(\frac{\partial A_{\mathrm{mv}}}{\partial \zeta}\right)^2 + \left(\frac{\partial A_{\mathrm{mv}}}{\partial \theta}\right)^2\Bigg]d\zeta & \end{aligned} \tag{15}$$

We write $A_{\mathrm{ev}}(\zeta,\theta)$ and $A_{\mathrm{em}}(\zeta,\theta)$ in the short form presented in Eqs.(4.2-12) and (4.2-16). The functions $C_{\mathrm{e}\kappa}(\theta)$ and $C_{\mathrm{m}\kappa}(\theta)$ are defined there:

$$A_{\mathrm{ev}}(\zeta,\theta) = c^2T^2V_{\mathrm{e0}} \sum_{\kappa=1}^{\infty} C_{\mathrm{e}\kappa}(\theta) \sin 2\pi\kappa\zeta \tag{16}$$

$$A_{\mathrm{mv}}(\zeta,\theta) = c^2T^2V_{\mathrm{e0}}Z \sum_{\kappa=1}^{\infty} C_{\mathrm{m}\kappa}(\theta) \cos 2\pi\kappa\zeta \tag{17}$$

Equation (16) is a Fourier sine series as in Eq.(4.1-81) while Eq.(17) is a Fourier cosine series with the constant term for $\kappa = 0$ equal to zero.

Since the sine and cosine functions with different values of κ in Eqs.(16) and (17) are orthogonal we get the following results for the terms of Eq.(15):

$$\begin{aligned}\int_0^1 \left(\frac{\partial A_{\mathrm{ev}}}{\partial\zeta}\right)^2 d\zeta &= (c^2T^2V_{\mathrm{e0}})^2 \int_0^1 \left(\sum_{\kappa=1}^{\infty} 2\pi\kappa C_{\mathrm{e}\kappa}(\theta)\cos 2\pi\kappa\zeta\right)^2 d\zeta \\ &= \frac{1}{2}(c^2T^2V_{\mathrm{e0}})^2 \sum_{\kappa=1}^{\infty} (2\pi\kappa)^2 C_{\mathrm{e}\kappa}^2(\theta)\end{aligned} \tag{18}$$

$$\begin{aligned}\int_0^1 \left(\frac{\partial A_{\mathrm{ev}}}{\partial\theta}\right)^2 d\zeta &= (c^2T^2V_{\mathrm{e0}})^2 \int_0^1 \left(\sum_{\kappa=1}^{\infty} \frac{\partial C_{\mathrm{e}\kappa}(\theta)}{\partial\theta}\sin 2\pi\kappa\zeta\right)^2 d\zeta \\ &= \frac{1}{2}(c^2T^2V_{\mathrm{e0}})^2 \sum_{\kappa=1}^{\infty} \left(\frac{\partial C_{\mathrm{e}\kappa}(\theta)}{\partial\theta}\right)^2\end{aligned} \tag{19}$$

$$\begin{aligned}\int_0^1 \frac{\partial A_{\mathrm{ev}}}{\partial\zeta}\frac{\partial A_{\mathrm{mv}}}{\partial\theta} d\zeta &= (c^2T^2V_{\mathrm{e0}})^2 Z \int_0^1 \sum_{\kappa=1}^{\infty} 2\pi\kappa C_{\mathrm{e}\kappa}(\theta)\frac{\partial C_{\mathrm{m}\kappa}(\theta)}{\partial\theta}\cos^2 2\pi\kappa\zeta\, d\zeta \\ &= \frac{1}{2}(c^2T^2V_{\mathrm{e0}})^2 Z \sum_{\kappa=1}^{\infty} 2\pi\kappa C_{\mathrm{e}\kappa}(\theta)\frac{\partial C_{\mathrm{m}\kappa}(\theta)}{\partial\theta}\end{aligned} \tag{20}$$

$$\begin{aligned}\int_0^1 \frac{\partial A_{\mathrm{mv}}}{\partial\zeta}\frac{\partial A_{\mathrm{ev}}}{\partial\theta} d\zeta &= (c^2T^2V_{\mathrm{e0}})^2 Z \int_0^1 \sum_{\kappa=1}^{\infty} -2\pi\kappa C_{\mathrm{m}\kappa}(\theta)\frac{\partial C_{\mathrm{e}\kappa}(\theta)}{\partial\theta}\sin^2 2\pi\kappa\zeta\, d\zeta \\ &= -\frac{1}{2}(c^2T^2V_{\mathrm{e0}})^2 Z \sum_{\kappa=1}^{\infty} 2\pi\kappa C_{\mathrm{m}\kappa}(\theta)\frac{\partial C_{\mathrm{e}\kappa}(\theta)}{\partial\theta}\end{aligned} \tag{21}$$

$$\begin{aligned}\int_0^1 \left(\frac{\partial A_{\mathrm{mv}}}{\partial\zeta}\right)^2 d\zeta &= (c^2T^2V_{\mathrm{e0}}Z)^2 \int_0^1 \left(\sum_{\kappa=1}^{\infty} 2\pi\kappa C_{\mathrm{m}\kappa}(\theta)\sin 2\pi\kappa\zeta\right)^2 d\zeta \\ &= \frac{1}{2}(c^2T^2V_{\mathrm{e0}}Z)^2 \sum_{\kappa=1}^{\infty} (2\pi\kappa)^2 C_{\mathrm{m}\kappa}^2(\theta)\end{aligned} \tag{22}$$

$$\int_0^1 \left(\frac{\partial A_{\rm mv}}{\partial\theta}\right)^2 d\zeta = (c^2T^2V_{\rm e0}Z)^2 \int_0^1 \left(\sum_{\kappa=1}^{\infty} \frac{\partial C_{\rm m\kappa}(\theta)}{\partial\theta}\cos 2\pi\kappa\zeta\right)^2 d\zeta$$
$$= \frac{1}{2}(c^2T^2V_{\rm e0}Z)^2 \sum_{\kappa=1}^{\infty}\left(\frac{\partial C_{\rm m\kappa}(\theta)}{\partial\theta}\right)^2 \qquad (23)$$

Substitution of Eqs.(18)–(23) into Eq.(15) yields the energy U as the sum of the energy of the components $C_{\rm e\kappa}(\theta)$ and $C_{\rm m\kappa}(\theta)$:

$$U = \frac{ZV_{\rm e0}^2L^2T^3c^4}{2}\sum_{\kappa=1}^{\infty}\left\{\left[\left(\frac{\partial C_{\rm m\kappa}(\theta)}{\partial\theta}\right)^2 + (2\pi\kappa)^2C_{\rm m\kappa}^2(\theta)\right]\right.$$
$$- 4\pi\kappa\left(C_{\rm m\kappa}(\theta)\frac{\partial C_{\rm e\kappa}(\theta)}{\partial\theta} - C_{\rm e\kappa}(\theta)\frac{\partial C_{\rm m\kappa}(\theta)}{\partial\theta}\right)$$
$$\left. + \left(\frac{\partial C_{\rm e\kappa}(\theta)}{\partial\theta}\right)^2 + (2\pi\kappa)^2C_{\rm e\kappa}^2(\theta)\right\} \qquad (24)$$

Note that neither $C_{\rm e\kappa}(\theta)$ nor $C_{\rm m\kappa}(\theta)$ have a dimension. All physical dimensions are contained in the factor $ZV_{\rm e0}^2L^2T^3c^4$ with the dimension VAs as required for an electromagnetic energy; the text following Eq.(4.1-37) gives the dimension of $V_{\rm e}$ as As/m^3 and $V_{\rm e0}$ has the same dimension according to Eq.(4.1-58).

When we compare Eq.(24) with Eq.(3.4-46) we recognize the first and the last two terms, except for the change from real to complex notation and the difference between a solution satisfying the causality law versus a steady state solution. But the center terms containing products of $C_{\rm m\kappa}(\theta)$ and $C_{\rm e\kappa}(\theta)$ in Eq.(24) with the other's derivative represent now an interaction between electric and magnetic terms that is missing in Eq.(3.4-46).

Equation (24) can be further simplified. However, the easy comparison with Eq.(3.4-46) is lost in the process:

$$U = \frac{ZV_{\rm e0}^2L^2T^3c^4}{2}\sum_{\kappa=1}^{\infty}\left[\left(2\pi\kappa C_{\rm e\kappa}(\theta) + \frac{\partial C_{\rm m\kappa}(\theta)}{\partial\theta}\right)^2\right.$$
$$\left. + \left(2\pi\kappa C_{\rm m\kappa}(\theta) - \frac{\partial C_{\rm e\kappa}(\theta)}{\partial\theta}\right)^2\right] \qquad (25)$$

Substitution of $C_{\rm e\kappa}$ and $C_{\rm m\kappa}$ from Eqs.(4.2-12) and (4.2-16) produces the following result for the two terms in large parentheses in Eq.(25):

$$2\pi\kappa C_{\rm e\kappa}(\theta) + \frac{\partial C_{\rm m\kappa}(\theta)}{\partial\theta} = 4\pi\sum_{\nu=1}^{\infty}\Big\{[\kappa B_{\rm es}(\kappa,\nu) - \nu C_{\rm ec}(\kappa,\nu)]\sin 2\pi\nu\theta$$
$$+ [\kappa B_{\rm ec}(\kappa,\nu) + \nu C_{\rm es}(\kappa,\nu)]\cos 2\pi\nu\theta\Big\} \qquad (26)$$

$$2\pi\kappa C_{\mathrm{m}\kappa}(\theta) - \frac{\partial C_{\mathrm{e}\kappa}(\theta)}{\partial\theta} = 4\pi\sum_{\nu=1}^{\infty}\Big\{[\nu B_{\mathrm{ec}}(\kappa,\nu) + \kappa C_{\mathrm{es}}(\kappa,\nu)]\sin 2\pi\nu\theta$$
$$- [\nu B_{\mathrm{es}}(\kappa,\nu) - \kappa C_{\mathrm{ec}}(\kappa,\nu)]\cos 2\pi\nu\theta\Big\} \quad (27)$$

Squaring and summing these two expressions yields an impressive formula that we can show only in part:

$$\left(2\pi\kappa C_{\mathrm{e}\kappa}(\theta) + \frac{\partial C_{\mathrm{m}\kappa}(\theta)}{\partial\theta}\right)^2 + \left(2\pi\kappa C_{\mathrm{m}\kappa}(\theta) - \frac{\partial C_{\mathrm{e}\kappa}(\theta)}{\partial\theta}\right)^2$$
$$= U_{\mathrm{c}\kappa} + U_{\mathrm{v}\kappa}(\theta) \quad (28)$$

$$U_{\mathrm{c}\kappa} = (4\pi)^2\sum_{\nu=1}^{\infty}\Big\{2\kappa\nu[B_{\mathrm{ec}}(\kappa,\nu)C_{\mathrm{es}}(\kappa,\nu) - B_{\mathrm{es}}(\kappa,\nu)C_{\mathrm{ec}}(\kappa,\nu)]$$
$$+ \frac{1}{2}(\kappa^2+\nu^2)[[B_{\mathrm{es}}^2(\kappa,\nu) + B_{\mathrm{ec}}^2(\kappa,\nu) + C_{\mathrm{es}}^2(\kappa,\nu) + C_{\mathrm{ec}}^2(\kappa,\nu)]\Big\}$$
$$= 2(2\pi\kappa)^2[U_{\mathrm{cs}}^2(\kappa) + U_{\mathrm{cc}}^2(\kappa)] \quad (29)$$

$$U_{\mathrm{cs}}^2(\kappa) = \sum_{\nu=1}^{\infty}\left[\left(B_{\mathrm{ec}}(\kappa,\nu) + \frac{\nu}{\kappa}C_{\mathrm{es}}(\kappa,\nu)\right)^2\right.$$
$$\left. + \left(B_{\mathrm{es}}(\kappa,\nu) - \frac{\nu}{\kappa}C_{\mathrm{ec}}(\kappa,\nu)\right)^2\right] \quad (30)$$
$$U_{cc}^2(\kappa) = \sum_{\nu=1}^{\infty}\left[\left(\frac{\nu}{\kappa}B_{\mathrm{ec}}(\kappa,\nu) + C_{\mathrm{es}}(\kappa,\nu)\right)^2\right.$$
$$\left. + \left(\frac{\nu}{\kappa}B_{\mathrm{es}}(\kappa,\nu) - C_{\mathrm{ec}}(\kappa,\nu)\right)^2\right] \quad (31)$$

The time variable term $U_{\mathrm{v}\kappa}(\theta)$ is much more complicated. We introduce four new functions $U_{\mathrm{v}1}(\theta)$ to $U_{\mathrm{v}4}(\theta)$ but do not write them explicitly:

$$U_{\mathrm{v}\kappa}(\theta) = \sum_{\nu=1}^{\infty}U_{\mathrm{v}1}(\kappa,\nu)\cos 4\pi\nu\theta$$
$$+ \sum_{\substack{\nu=1\\ \nu\neq\lambda}}^{\infty}\sum_{\lambda=1}^{\infty}[U_{\mathrm{v}2}(\kappa,\nu,\lambda)\sin 2\pi\nu\theta\sin 2\pi\lambda\theta + U_{\mathrm{v}3}(\kappa,\nu,\lambda)\cos 2\pi\nu\theta\cos 2\pi\lambda\theta]$$
$$+ \sum_{\nu=1}^{\infty}\sum_{\lambda=1}^{\infty}U_{\mathrm{v}4}(\kappa,\nu,\lambda)\sin 2\pi\nu\theta\cos 2\pi\lambda\theta \quad (32)$$

The term $U_{c\kappa}$ represents an energy that does not depend on the time θ. The term $U_{v\kappa}$ depends on θ but its time average is zero. If we write $U_{v\kappa}(\theta) = U_{v\kappa}(t)/T$ we have a time variable power with average power or energy equal to zero. This is a generally encountered result of the theory. For its physical explanation we may think of two plates of a capacitor with vacuum between them. An electric field strength drives an electric dipole current through this vacuum. We may assume that the dipoles are created by the field strength. But it is also possible that they are created and annihilated all the time in the absence of an electric field strength. The random orientation of the dipoles prevents an observable macroscopic effect. An applied field strength orients the dipoles and a macroscopic effect is produced in the form of a dipole current.

The non-fluctuating part U_c of the energy U in Eq.(25) may be written in the following form:

$$U_c = ZV_{e0}^2 L^2 T^3 c^4 \sum_{\kappa=1}^{\infty} (2\pi\kappa)^2 [U_{cs}^2(\kappa) + U_{cc}^2(\kappa)] \tag{33}$$

For the derivation of the Hamilton function $\mathcal{H}$ we first normalize the energy U_c in Eq.(33)

$$U_c / ZV_{e0}^2 L^2 T^3 c^4 = \mathcal{H} \tag{34}$$

$$\mathcal{H} = \sum_{\kappa=1}^{\infty} \mathcal{H}_\kappa = \sum_{\kappa=1}^{\infty} (2\pi\kappa)^2 [U_{cs}^2(\kappa) + U_{cc}^2(\kappa)] \tag{35}$$

and then rewrite $\mathcal{H}_\kappa$ as follows:

$$\begin{aligned}
\mathcal{H}_\kappa &= (2\pi\kappa)^2 \Big\{ [U_{cs}^2(\kappa) + U_{cc}^2(\kappa)] \sin^2 2\pi\kappa\theta + [U_{cs}^2(\kappa) + U_{cc}^2(\kappa)] \cos^2 2\pi\kappa\theta \Big\} \\
&= (2\pi\kappa)^2 [U_{cs}(\kappa) + iU_{cc}(\kappa)](\sin 2\pi\kappa\theta - i\cos 2\pi\kappa\theta) \\
&\qquad \times [U_{cs}(\kappa) - iU_{cc}(\kappa)](\sin 2\pi\kappa\theta + i\cos 2\pi\kappa\theta) \\
&= -2\pi i\kappa p_\kappa(\theta) q_\kappa(\theta)
\end{aligned} \tag{36}$$

For $p_\kappa(\theta)$ and $q_\kappa(\theta)$ we get:

$$\begin{aligned}
p_\kappa(\theta) &= \sqrt{2\pi i\kappa}[U_{cs}(\kappa) + iU_{cc}(\kappa)](\sin 2\pi\kappa\theta - i\cos 2\pi\kappa\theta) \\
&= \sqrt{2\pi i\kappa}[U_{cc}(\kappa) - iU_{cs}(\kappa)]e^{2\pi i\kappa\theta}
\end{aligned} \tag{37}$$

$$\dot{p}_\kappa = \frac{\partial p_\kappa(\theta)}{\partial\theta} = (2\pi i\kappa)^{3/2}[U_{cc}(\kappa) - iU_{cs}(\kappa)]e^{2\pi i\kappa\theta} \tag{38}$$

$$\begin{aligned}
q_\kappa(\theta) &= \sqrt{2\pi i\kappa}[U_{cs}(\kappa) - iU_{cc}(\kappa)](\sin 2\pi\kappa\theta + i\cos 2\pi\kappa\theta) \\
&= \sqrt{2\pi i\kappa}[U_{cc}(\kappa) + iU_{cs}(\kappa)]e^{-2\pi i\kappa\theta}
\end{aligned} \tag{39}$$

$$\dot{q}_\kappa = \frac{\partial q_\kappa(\theta)}{\partial \theta} = -(2\pi i\kappa)^{3/2}[U_{cc}(\kappa) + iU_{cs}(\kappa)]e^{-2\pi i\kappa\theta} \tag{40}$$

The derivatives $\partial\mathcal{H}_\kappa/\partial q_\kappa$ and $\partial\mathcal{H}_\kappa/\partial p_\kappa$ equal:

$$\frac{\partial\mathcal{H}_\kappa}{\partial q_\kappa} = -2\pi i\kappa p_\kappa(\theta) = -(2\pi i\kappa)^{3/2}[U_{cc}(\kappa) - iU_{cs}(\kappa)]e^{2\pi i\kappa\theta} \tag{41}$$

$$\frac{\partial\mathcal{H}_\kappa}{\partial p_\kappa} = -2\pi i\kappa q_\kappa(\theta) = -(2\pi i\kappa)^{3/2}[U_{cc}(\kappa) + iU_{cs}(\kappa)]e^{-2\pi i\kappa\theta} \tag{42}$$

The comparison of Eqs.(41) and (42) with Eqs.(38) and (40) yields the proper relations for the components $\mathcal{H}_\kappa$ of the Hamilton function:

$$\frac{\partial\mathcal{H}_\kappa}{\partial q_\kappa} = -\dot{p}_\kappa, \quad \frac{\partial\mathcal{H}_\kappa}{\partial p_\kappa} = \dot{q}_\kappa \tag{43}$$

Equation (35) may be rewritten in analogy to Eq.(3.4-61) by means of the definitions

$$\begin{aligned} a_\kappa &= [U_{cc}(\kappa) - iU_{cs}(\kappa)]e^{2\pi i\kappa\theta} \\ a_\kappa^* &= [U_{cc}(\kappa) + iU_{cs}(\kappa)]e^{-2\pi i\kappa\theta} \end{aligned} \tag{44}$$

to yield:

$$\begin{gathered} \mathcal{H} = -i\sum_{\kappa=1}^{\infty} 2\pi\kappa\, p_\kappa q_\kappa = \sum_{\kappa=1}^{\infty}(2\pi\kappa)^2 a_\kappa a_\kappa^* = \sum_{\kappa=1}^{\infty}\frac{2\pi\kappa}{T}\hbar\, b_\kappa b_\kappa^* \\ b_\kappa = \left(\frac{2\pi\kappa T}{\hbar}\right)^{1/2} a_\kappa, \quad b_\kappa^* = \left(\frac{2\pi\kappa T}{\hbar}\right)^{1/2} a_\kappa^* \end{gathered} \tag{45}$$

Let us check whether the energy U_c of Eq.(33) is finite since an infinite energy would violate the conservation law of energy. The proof of finite energy is also required to permit in the following section the claim that mass renormalization is not needed. An infinitely large amplitude V_{e0} of the excitation force $V_e(0,\theta)$ in Eq.(4.1-54) or an infinitely large excitation area L^2 according to Eq.(1) would make U_c infinite, but this is of no interest. The factors T and c in Eq.(33) are finite. A finite value of T is assumed by the definition of Eq.(4.1-40) as well as by Eq.(4.1-75) and the text discussing it. The sum U_c in Eq.(33) and $\mathcal{H}$ in Eq.(35) must be finite too, but this is something one should show explicitly. To do so we must show that U_{cs}^2 and U_{cc}^2 of Eqs.(30) and (31) approach zero sufficiently fast for $\kappa \to \infty$ to yield a convergent series for $\mathcal{H}$.

It is shown in Section 6.12 that U_{es}^2 decreases like $1/(2\pi\kappa)^6$ for $\kappa \to \infty$ while U_{cc}^2 decreases like $1/(2\pi\kappa)^4$. For the terms in the sum of Eq.(35) we obtain the decrease

$$\mathcal{H}_\kappa = (2\pi\kappa)^2[U_{cs}^2(\kappa) + U_{cc}^2(\kappa)] \propto 1/(2\pi\kappa)^2 \quad \text{for } \kappa \to \infty \tag{46}$$

from Eq.(6.12-136). Hence, the terms of $\mathcal{H}$ in Eq.(35) decrease fast enough to make the sum convergent.

The results derived here for a planar wave may readily be applied to a spherical wave. This is shown in Section 6.9.

4.4 Quantization of a Planar Wave

We start with the Hamilton function $\mathcal{H}$ of Eq.(4.3-45) using the functions b_κ and b_κ^*

$$\mathcal{H} = \sum_{\kappa=1}^{\infty} \mathcal{H}\kappa = \frac{\hbar}{T}\sum_{\kappa=1}^{\infty} 2\pi\kappa\, b_\kappa b_\kappa^* \tag{1}$$

and follow the conventional procedure for quantization. The conjugate complex functions b_κ and b_κ^* are replaced by operators b_κ^+ and b_κ^-. There are two ways of making this replacement

$$b_\kappa^* \to b_\kappa^+, \quad b_\kappa \to b_\kappa^- \tag{2}$$

or

$$b_\kappa^* \to b_\kappa^-, \; b_\kappa \to b_\kappa^+ \tag{3}$$

The choice of one of these replacements is a known arbitrariness or ambiguity of the conventional theory as pointed out in Section 3.5 following Eq.(3.5-6). We will first use Eq.(2) and later on Eq.(3). For the component κ in Eq.(1) we obtain:

$$b_\kappa^- b_\kappa^+ = \frac{\mathcal{H}_\kappa T}{2\pi\kappa\hbar} \equiv \frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar} \tag{4}$$

$$b_\kappa^- = \frac{1}{\sqrt{2}}\left(\alpha\zeta + \frac{1}{\alpha}\frac{d}{d\zeta}\right), \quad b_\kappa^+ = \frac{1}{\sqrt{2}}\left(\alpha\zeta - \frac{1}{\alpha}\frac{d}{d\zeta}\right) \tag{5}$$

If we apply the operators b_κ^- and b_κ^+ to a function Φ we get:

$$\frac{1}{\sqrt{2}}\left(\alpha\zeta + \frac{1}{\alpha}\frac{d}{d\zeta}\right)\left[\frac{1}{\sqrt{2}}\left(\alpha\zeta - \frac{1}{\alpha}\frac{d}{d\zeta}\right)\Phi\right] = \frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar}\Phi$$

$$\left(\alpha^2\zeta^2 - \frac{1}{\alpha^2}\frac{d^2}{d\zeta^2}\right)\Phi = 2\left(\frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar} - \frac{1}{2}\right)\Phi = 2\lambda_\kappa\Phi \tag{6}$$

$$\lambda_\kappa = \frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar} - \frac{1}{2} \tag{7}$$

These equations are essentially the same as Eqs.(3.5-18) and (3.5-19). We may thus use Eq.(3.5-24) to write E_κ of Eq.(7) in the form

$$\mathsf{E}_\kappa = \mathsf{E}_{\kappa n} = \frac{2\pi\kappa\hbar}{T}\left(n+\frac{1}{2}\right), \quad n = 0,\ 1,\ 2,\ \ldots \tag{8}$$

This result is very similar to the conventional one shown by Eq.(3.5-24)

$$\mathsf{E}_{\mathrm{m,k}\lambda} = \omega_{\mathbf{k}\lambda}\hbar\left(n+\frac{1}{2}\right) = 2\pi f\hbar\left(n+\frac{1}{2}\right), \ n = 0,\ 1,\ 2,\ \ldots;\ \omega_{\mathbf{k}\lambda} = 2\pi f \tag{9}$$

but the time interval T in Eq.(8) is finite while Eq.(9) assumes an infinite time interval since a frequency f is defined only for periodic sinusoidal functions in the whole interval $-\infty < t < +\infty$. The *period number* κ in Eq.(8) gives the number of periods in the time interval of duration T or a spatial interval of length cT.

Let us turn to Eq.(8). If κ runs from 1 to infinity we obtain according to Eqs.(4.3-33), (4.3-35), and (4.3-46) a finite energy. The energy of a photon increases proportionately to κ according to Eq.(8) but their number must decrease like $1/\kappa^3$ to make $\mathcal{H}_\kappa$ of Eq.(4.3-46) decrease like $1/\kappa^2$. Hence, this particular and historically first divergency is eliminated.

We want to discuss this important result in a non-mathematical way. In Section 4.1 we started with vector potentials $\mathbf{A}_\mathrm{e}$ and $\mathbf{A}_\mathrm{m}$, while the scalar potentials ϕ_e and ϕ_m were assumed to be zero. These potentials define electric and magnetic field strengths $\mathbf{E}$ and $\mathbf{H}$ according to Eqs.(1.6-17) and (1.6-11). The substitutions $A_{\mathrm{e}x} = A_{\mathrm{e}z} = A_{\mathrm{ev}}$ and $A_{\mathrm{m}x} = -A_{\mathrm{m}z} = A_{\mathrm{mv}}$ of Eqs.(4.1-31) as well as $V_\mathrm{e}(y,t)$ and $V_\mathrm{m}(y,t)$ in Eqs.(4.1-37) and (4.1-39) tend to obscure that $w(\zeta,\theta)$ in Eqs.(4.1-76) and (4.1-88) represents an electromagnetic wave by a superposition of standing sinusoidal pulses. But the two components $A_{\mathrm{e}y}$, $A_{\mathrm{m}y}$ of the vector potentials $\mathbf{A}_\mathrm{e}$ and $\mathbf{A}_\mathrm{m}$ are shown in Eqs.(4.1-34), (4.1-35) while the remaining four components $A_{\mathrm{e}x} = A_{\mathrm{e}z} = A_{\mathrm{ev}}$ and $A_{\mathrm{m}x} = -A_{\mathrm{m}z} = A_{\mathrm{mv}}$ are defined by Eqs.(4.1-103) and (4.1-126). These components of the vector potentials define via Eqs.(1.6-17) and (1.6-11) the electric and magnetic field strength of a wave. This wave is zero for $\theta = t/T < 0$ according to Eq.(4.1-54). Hence, it will satisfy both the causality law and the conservation law of energy if its energy is finite. The step function excitation of Eq.(4.1-54) does not tell us whether the energy of the excited wave is finite. The finite energy follows from Eqs.(4.2-12) and (4.2-16) that define A_{ev} and A_{mv} in terms of $A_{\mathrm{es}}(\kappa)$, $A_{\mathrm{ec}}(\kappa)$, $B_{\mathrm{es}}(\kappa,\nu)$, $B_{\mathrm{ec}}(\kappa,\nu)$, $C_{\mathrm{es}}(\kappa,\nu)$, and $C_{\mathrm{ec}}(\kappa,\nu)$, which are shown in Eqs.(6.12-126) and (6.12-145)–(6.12-147) to decrease with increasing values of κ sufficiently fast to make the energy of the wave excited by the step function of Eq.(4.1-54) finite. The quantization by the substitutions of Eq.(5) replaces the superposition of standing sinusoidal pulses representing a classical wave by a superposition of operators representing photons. The finite energy of the wave remains unchanged by this change of representation. The 'zero-point energy' for $n = 0$ remains finite because the finite energy of the wave makes the energy of any component of the wave finite.

We pointed out in the first paragraph of Section 3.4 that solutions with finite energy of Maxwell's original equations were developed from 1968 on, but these solutions do not seem to have ever reached quantum electrodynamis.

We turn to the substitution of Eq.(3). It is usually ignored in the conventional theory since an infinite energy is bad but an infinite negative energy is even worse. We obtain in analogy to Eqs.(4), (6), and (7):

$$b_\kappa^+ b_\kappa^- = \frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar} \tag{10}$$

$$\frac{1}{\sqrt{2}}\left(\alpha\zeta - \frac{1}{\alpha}\frac{d}{d\zeta}\right)\left[\frac{1}{\sqrt{2}}\left(\alpha\zeta + \frac{1}{\alpha}\frac{d}{d\zeta}\right)\Phi\right] = \frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar}\Phi$$

$$\left(\alpha^2\zeta^2 - \frac{1}{\alpha^2}\frac{d^2}{d\zeta^2}\right)\Phi = 2\left(\frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar} + \frac{1}{2}\right)\Phi = 2\lambda_\kappa\Phi \tag{11}$$

$$\lambda_\kappa = \frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar} + \frac{1}{2} \tag{12}$$

For $n = 0$ we obtain now a finite negative energy. It is not clear whether this result has any significance but we cannot simply ignore it:

$$\mathsf{E}_\kappa = \mathsf{E}_{\kappa n} = \frac{2\pi\kappa\hbar}{T}\left(n - \frac{1}{2}\right), \quad n = 0,\ 1,\ 2,\ldots \tag{13}$$

Equations (8) and (13) yield for $n = 0$ energies with the same magnitude but opposite sign. We have encountered in Eq.(4.3-32) time variable terms with positive and negative energy having the time average zero. They may be related to $n = 0$ in Eqs.(8) and (13).

It is reasonably evident how one has to proceed if one wants to modify conventional quantum electrodynamics to make it correspond to the modified Maxwell equations. A lot of work is required but there are many scientists and PhD students qualified to do it. We want to concentrate here on tasks for which a detailed knowledge of the conventional theory is not of much help. The first such task is to investigate the replacement of the step function excitation of Eq.(4.1-54) by an *exponential ramp function* excitation.

The excitation by an exponential ramp function is important because the convergence of the solution due to the step function excitation of Eq.(4.1-54) is not always fast enough. The step function must then be replaced by an exponential ramp function. Such a case will be encountered in Section 5.3. We shall elaborate the exponential ramp function solution in Section 4.5 in order to have it ready when needed.

In Section 4.6 we study the replacement of the step function for excitation by a rectangular pulse with finite duration. The finite duration is an important improvement over the infinitely extended step function. However, it inherently produces equations twice as long as the ones due to step function excitation.

4.5 Exponential Ramp Function Excitation

We have solved Eqs.(4.1-41) and (4.1-42) for a step function $V_e(0,\theta)$ as boundary condition or excitation function defined by Eq.(4.1-54). Equation (4.1-41) has the same form as Eq.(6.1-1), which is solved in Section 6.1 for a step function $E(0,\theta)$ and in Section 6.4 for an exponential ramp function $E(0,\theta)$ as boundary condition. The solution of Eq.(4.1-41) for an exponential ramp function differs only in the notation from the calculations in Section 6.4 until Eq.(6.4-19) is reached and the Fourier sine integral is replaced by the Fourier sine series in analogy to Eq.(4.1-76). Hence, we may rewrite the calculations of Section 6.4 with the new notation in short form.

The partial differential equation for V_e as function of ζ and θ defined by Eq.(4.1-41)

$$\frac{\partial^2 V_e}{\partial \zeta^2} - \frac{\partial^2 V_e}{\partial \theta^2} - \rho_1 \frac{\partial V_e}{\partial \theta} - \rho_2^2 V_e = 0 \tag{1}$$

is to be solved for the boundary conditions

$$\begin{aligned} V_e(0,\theta) = V_{e1} S(\theta)(1-e^{-\iota\theta}) &= 0 && \text{for } \theta < 0 \\ &= V_{e1}(1-e^{-\iota\theta}) && \text{for } \theta \geq 0 \end{aligned} \tag{2}$$

for $\zeta = 0$. The usual further boundary condition for $\zeta \to \infty$

$$V_e(\infty,\theta) = \text{finite} \tag{3}$$

is avoided just like in the case of Eq.(4.1-55). There are further the following initial conditions according to Eqs.(6.1-4)–(6.1-6):

$$V_e(\zeta,0) = 0 \tag{4}$$

$$\partial V_e(\zeta,0)/\partial\zeta = 0 \tag{5}$$

$$\partial V_e(\zeta,\theta)/\partial\theta = 0 \quad \text{for } \theta = 0,\ \zeta \geq 0 \tag{6}$$

Equation (1) is satisfied by the function

$$V_e(\zeta,\theta) = V_{e1}\left[u(\zeta,\theta) + (1-e^{-\iota\theta})F(\zeta)\right] \tag{7}$$

Substitution of $V_{e1}(1-e^{-\iota\theta})F(\zeta)$ into Eq.(1) yields:

$$F(\zeta) = A_{10}e^{-\rho_2\zeta} + A_{11}e^{\rho_2\zeta} \tag{8}$$

$$\iota = \rho_1 \quad \text{or } \iota = 0 \tag{9}$$

The choice $\iota = 0$ yields a trivial solution. Using the argument that led from Eq.(4.1-59) to (4.1-60) we obtain:

$$V_e(\zeta,\theta) = V_{e1}\left[u(\zeta,\theta) + (1 - e^{-\rho_1\theta})e^{-\rho_2\zeta}\right] \tag{10}$$

Since the function $V_e(0,\theta)$ must satisfy the boundary condition according to Eq.(2)

$$u(0,\theta) + 1 - e^{-\rho_1\theta} = 1 - e^{-\rho_1\theta} \tag{11}$$

we obtain for $\zeta = 0$ the homogeneous boundary condition

$$u(0,\theta) = 0 \tag{12}$$

For $\zeta \to \infty$ we obtain with $F(\zeta \to \infty) = 0$ from Eq.(3) a second boundary condition

$$u(\infty,\theta) = \text{finite} \tag{13}$$

which we do not use due to the restricted interval $0 \leq \zeta \leq 1$. It is not required anywhere. The initial conditions of Eqs.(4) and (6) yield:

$$V_e(\zeta,0) = V_{e1}u(\zeta,0) = 0 \tag{14}$$

$$\partial u/\partial\theta + \rho_1 e^{-\rho_2\zeta} = 0, \quad \partial u/\partial\theta = -\rho_1 e^{-\rho_2\zeta} \quad \text{for } \theta = 0,\ \zeta \geq 0 \tag{15}$$

The calculation of $u(\zeta,\theta)$ proceeds as in Section 4.1 from Eq.(4.1-66) on with $w(\zeta,\theta)$ replaced by $u(\zeta,\theta)$ until Eqs.(4.1-76) and (4.1-77) are reached:

$$u(\zeta,\theta) = \sum_{\kappa=1}^{\infty}[A_1(\kappa)\exp(\gamma_1\theta) + A_2(\kappa)\exp(\gamma_2\theta)]\sin 2\pi\kappa\zeta \tag{16}$$

$$\frac{\partial u}{\partial\theta} = \sum_{\kappa=1}^{\infty}[A_1(\kappa)\gamma_1\exp(\gamma_1\theta) + A_2(\kappa)\gamma_2\exp(\gamma_2\theta)]\sin 2\pi\kappa\zeta \tag{17}$$

Substitution of $u(\zeta,0)$ and $\partial u/\partial\theta$ from Eqs.(14) and (15) into Eqs.(16) and (17) for $\theta = 0$ yields equations for the determination of $A_1(\kappa)$ and $A_2(\kappa)$:

$$u(\zeta,0) = \sum_{\kappa=1}^{\infty}[A_1(\kappa) + A_2(\kappa)] = 0 \tag{18}$$

$$\frac{\partial u}{\partial\theta} = \sum_{\kappa=1}^{\infty}[A_1(\kappa)\gamma_1 + A_2(\kappa)\gamma_2]\sin 2\pi\kappa\zeta = -\rho_1 e^{-\rho_2\zeta} \tag{19}$$

Using the Fourier sine series of Eqs.(4.1-80) and (4.1-81) we obtain the following two equations for the determination of $A_1(\kappa)$ and $A_2(\kappa)$:

$$A_1(\kappa) + A_2(\kappa) = 0 \tag{20}$$

$$A_1(\kappa)\gamma_1 + A_2(\kappa)\gamma_2 = -2\rho_1 \int_0^1 e^{-\rho_2\zeta} \sin 2\pi\kappa\zeta \, d\zeta \tag{21}$$

We evaluate the integral with the help of Eq.(4.1-84)

$$\int_0^1 e^{-\rho_2\zeta} \sin 2\pi\kappa\zeta \, d\zeta = \frac{2\pi\kappa(1 - e^{-\rho_2})}{(2\pi\kappa)^2 + \rho_2^2} \tag{22}$$

and obtain:

$$\begin{aligned} A_1(\kappa)\gamma_1 + A_2(\kappa)\gamma_2 &= -\frac{4\pi\kappa\rho_1(1 - e^{-\rho_2})}{(2\pi\kappa)^2 + \rho_2^2} \\ &\approx -\frac{4\pi\kappa\rho_1\rho_2}{(2\pi\kappa)^2 + \rho_2^2} \quad \text{for } \rho_2 \ll 1 \\ &\approx -\frac{4\pi\kappa\rho_1}{(2\pi\kappa)^2 + \rho_2^2} \quad \text{for } \rho_2 \gg 1 \end{aligned} \tag{23}$$

Following Eq.(4.1-85) we choose $\rho_2 \gg 1$. Equation (4.1-73) is then used to obtain $A_1(\kappa)$ and $A_2(\kappa)$:

$$\begin{aligned} A_1(\kappa) = -A_2(\kappa) &= -\frac{4\pi\kappa\rho_1}{(2\pi\kappa)^2 + \rho_2^2} \frac{1}{\gamma_1 - \gamma_2} \\ &= -\frac{4\pi\kappa\rho_1}{(4\pi\kappa)^2 + \rho_2^2} \frac{1}{(\rho_1^2 - d^2)^{1/2}} \quad \text{for } d^2 < \rho_1^2 \\ &= +\frac{4\pi\kappa\rho_1}{(4\pi\kappa)^2 + \rho_2^2} \frac{i}{(d^2 - \rho_1^2)^{1/2}} \quad \text{for } d^2 > \rho_1^2 \end{aligned}$$

$$d^2 = 4[(2\pi\kappa)^2 + \rho_2^2], \; \rho_2^2 = c^2T^2\sigma s, \; \rho_1 = c^2T(\sigma\mu + s\epsilon) \tag{24}$$

Substitution of Eq.(24) for $A_1(\kappa)$, $A_2(\kappa)$ and of Eq.(4.1-73) for γ_1, γ_2 brings Eq.(16) into the following form:

$$\begin{aligned} u(\zeta,\theta) = -2\rho_1 e^{-\rho_1\theta/2} \Bigg(\sum_{\kappa=1}^{<K} & \frac{\exp[(\rho_1^2 - d^2)^{1/2}\theta/2] - \exp[-(\rho_1^2 - d^2)^{1/2}\theta/2]}{(\rho_1^2 - d^2)^{1/2}} \\ & \times \frac{2\pi\kappa \sin 2\pi\kappa\zeta}{(2\pi\kappa)^2 + \rho_2^2} \\ + \sum_{\kappa>K}^{\infty} & \frac{\exp[i(d^2 - \rho_1^2)^{1/2}\theta/2] - \exp[-i(d^2 - \rho_1^2)^{1/2}\theta/2]}{i\,(d^2 - \rho_1^2)^{1/2}} \frac{2\pi\kappa \sin 2\pi\kappa\zeta}{(2\pi\kappa)^2 + \rho_2^2} \Bigg) \end{aligned}$$

$$K = c^2T|(\sigma\mu - s\epsilon)|/4\pi \tag{25}$$

Using hyperbolic and trigonometric functions we may rewrite Eq.(25) in real form:

$$u(\zeta,\theta) = -4\rho_1 e^{-\rho_1\theta/2}\Bigg(\sum_{\kappa=1}^{<K} \frac{\operatorname{sh}[(\rho_1^2-d^2)^{1/2}\theta/2]}{(\rho_1^2-d^2)^{1/2}} \frac{2\pi\kappa\sin 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2} + \sum_{\kappa>K}^{\infty} \frac{\sin[(d^2-\rho_1^2)^{1/2}\theta/2]}{(d^2-\rho_1^2)^{1/2}} \frac{2\pi\kappa\sin 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2}\Bigg) \tag{26}$$

Equation (8) for $F(\zeta)$ with $A_{10} = 1$, $A_{11} = 0$, and Eq.(26) for $u(\zeta,\theta)$ define $V_e(\zeta,\theta)$ according to Eq.(7). We need $A_{ev}(\zeta,\theta)$ of Eq.(4.1-43). In order to obtain it we must replace the variables ζ, θ by ζ', θ' and integrate over ζ', θ'. We integrate first over ζ' and denote the result by $\partial A_{ev}(\zeta,\theta')/\partial\theta'$:

$$\frac{\partial A_{ev}(\zeta,\theta')}{\partial\theta'} = -\frac{1}{2}c^2T^2V_{e1}\int\limits_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} [(1-e^{-\rho_1\theta'})F(\zeta') + u(\zeta',\theta')]d\zeta' \tag{27}$$

As in the case of Eq.(4.1-89) we need to evaluate only two simple integrals:

$$\int\limits_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} e^{-\rho_2\zeta'}d\zeta' = -\frac{1}{\rho_2}e^{-\rho_2\zeta}\left(e^{-\rho_2\theta}e^{\rho_2\theta'} - e^{\rho_2\theta}e^{-\rho_2\theta'}\right) \tag{28}$$

$$\int\limits_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} \sin 2\pi\kappa\zeta'\,d\zeta' = \frac{\sin 2\pi\kappa\zeta}{\pi\kappa}\big(\sin 2\pi\kappa\theta\cos 2\pi\kappa\theta' - \cos 2\pi\kappa\theta\sin 2\pi\kappa\theta'\big) \tag{29}$$

Substitution of Eqs.(26),(28), and (29) into Eqs.(27) yields:

$$\begin{aligned}\frac{\partial A_{ev}(\zeta,\theta')}{\partial\theta'} = \frac{c^2T^2V_{e1}}{2}\Bigg\{&\frac{1-e^{-\rho_1\theta'}}{\rho_2}\left(e^{-\rho_2(\zeta+\theta)}e^{\rho_2\theta'} - e^{-\rho_2(\zeta-\theta)}e^{-\rho_2\theta'}\right)\\ &+ 8\rho_1 e^{-\rho_1\theta'/2}\Bigg[\sum_{\kappa=1}^{<K}\frac{\operatorname{sh}[(\rho_1^2-d^2)^{1/2}\theta'/2]}{(\rho_1^2-d^2)^{1/2}}\\ &\times\big(\sin 2\pi\kappa\theta\cos 2\pi\kappa\theta' - \cos 2\pi\kappa\theta\sin 2\pi\kappa\theta'\big)\frac{\sin 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2}\\ &+ \sum_{\kappa>K}^{\infty}\frac{\sin[(d^2-\rho_1^2)^{1/2}\theta'/2]}{(d^2-\rho_1^2)^{1/2}}\\ &\times\big(\sin 2\pi\kappa\theta\cos 2\pi\kappa\theta' - \cos 2\pi\kappa\theta\sin 2\pi\kappa\theta'\big)\frac{\sin 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2}\Bigg]\Bigg\}\end{aligned} \tag{30}$$

The terms in Eq.(30) have been written in an expanded form that will facilitate the following integrations over θ':

$$A_{\mathrm{ev}}(\zeta,\theta)=\int_0^\theta \frac{\partial A_{\mathrm{ev}}(\zeta,\theta')}{\partial\theta'}d\theta' \tag{31}$$

One may readily recognize the following five integrals over θ' in Eq.(30):

$$\begin{aligned}\frac{e^{-\rho_2\zeta}}{\rho_2}\int_0^\theta (1-e^{\rho_1\theta'})\big(e^{-\rho_2\theta}e^{\rho_2\theta'}-e^{\rho_2\theta}e^{-\rho_2\theta'}\big)d\theta' &= \frac{2}{\rho_2}e^{-\rho_2\zeta}\\ \times\left[\frac{1}{\rho_2}(1-\operatorname{ch}\rho_2\theta)+\frac{1}{\rho_1^2-\rho_2^2}(\rho_2 e^{-\rho_1\theta}-\rho_2\operatorname{ch}\rho_2\theta+\rho_1\operatorname{sh}\rho_2\theta)\right]\end{aligned} \tag{32}$$

$$L_{11}(\theta,\kappa)=\int_0^\theta e^{-\rho_1\theta'/2}\operatorname{sh}[(\rho_1^2-d^2)^{1/2}\theta'/2]\cos 2\pi\kappa\theta'\,d\theta' \tag{33}$$

$$L_{12}(\theta,\kappa)=\int_0^\theta e^{-\rho_1\theta'/2}\operatorname{sh}[(\rho_1^2-d^2)^{1/2}\theta'/2]\sin 2\pi\kappa\theta'\,d\theta' \tag{34}$$

$$L_{15}(\theta,\kappa)=\int_0^\theta e^{-\rho_1\theta'/2}\sin[(d^2-\rho_1^2)^{1/2}\theta'/2]\cos 2\pi\kappa\theta'\,d\theta' \tag{35}$$

$$L_{16}(\theta,\kappa)=\int_0^\theta e^{-\rho_1\theta'/2}\sin[(d^2-\rho_1^2)^{1/2}\theta'/2]\sin 2\pi\kappa\theta'\,d\theta' \tag{36}$$

The integrals of Eqs.(33)–(36) have been evaluated in Eqs.(4.1-113), (4.1-114), (4.1-107), and (4.1-108). Equation (31) becomes:

$$\begin{aligned}A_{\mathrm{ev}}(\zeta,\theta)=c^2T^2V_{\mathrm{e}1}\bigg\{&\frac{1}{\rho_2}e^{-\rho_2\zeta}\bigg[\frac{1}{\rho_2}(1-\operatorname{ch}\rho_2\theta)\\ &+\frac{1}{\rho_1^2-\rho_2^2}(\rho_2 e^{-\rho_1\theta}-\rho_2\operatorname{ch}\rho_2\theta+\rho_1\operatorname{sh}\rho_2\theta)\bigg]\\ &+4\rho_1\bigg(\sum_{\kappa=1}^{<K}\frac{L_{11}(\theta,\kappa)\sin 2\pi\kappa\theta-L_{12}(\theta,\kappa)\cos 2\pi\kappa\theta}{(\rho_1^2-d^2)^{1/2}}\frac{\sin 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2}\\ &+\sum_{\kappa>K}^{\infty}\frac{L_{15}(\theta,\kappa)\sin 2\pi\kappa\theta-L_{16}(\theta,\kappa)\cos 2\pi\kappa\theta}{(d^2-\rho_1^2)^{1/2}}\frac{\sin 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2}\bigg)\bigg\}\end{aligned} \tag{37}$$

We turn to the potential $A_{\rm mv}(\zeta,\theta)$ that is associated with $A_{\rm ev}(\zeta,\theta)$ according to Eq.(4.1-47). Three integrals must be evaluated. Consider the first one:

$$
\begin{aligned}
A_{\rm mv1}(\zeta,\theta) &= Z\rho_{\rm s}\int\frac{\partial^2 A_{\rm ev}(\zeta,\theta)}{\partial\zeta^2}d\zeta = Z\rho_s\frac{\partial A_{\rm ev}(\zeta,\theta)}{\partial\zeta}\\
&= c^2T^2Z\rho_{\rm s}V_{\rm e1}\bigg\{-e^{-\rho_2\zeta}\bigg[\frac{1}{\rho_2}(1-\operatorname{ch}\rho_2\theta)\\
&\qquad+\frac{1}{\rho_1^2-\rho_2^2}(\rho_2e^{-\rho_1\theta}-\rho_2\operatorname{ch}\rho_2\theta+\rho_1\operatorname{sh}\rho_2\theta)\bigg]\\
&\quad+4\rho_1\bigg(\sum_{\kappa=1}^{<K}\frac{L_{11}(\theta,\kappa)\sin 2\pi\kappa\theta-L_{12}(\theta,\kappa)\cos 2\pi\kappa\theta}{(\rho_1^2-d^2)^{1/2}}\frac{2\pi\kappa\cos 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2}\\
&\quad+\sum_{\kappa>K}^{\infty}\frac{L_{15}(\theta,\kappa)\sin 2\pi\kappa\theta-L_{16}(\theta,\kappa)\cos 2\pi\kappa\theta}{(d^2-\rho_1^2)^{1/2}}\frac{2\pi\kappa\cos 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2}\bigg)\bigg\}
\end{aligned}
\tag{38}
$$

The differentiation with respect to θ of the second and third integral in Eq.(4.1-47) makes the evaluation of these integrals more difficult:

$$A_{\rm mv2}(\zeta,\theta) = -Z\rho_{\rm s}\int\frac{\partial^2A_{\rm ev}(\zeta,\theta)}{\partial\theta^2}d\zeta = -Z\rho_{\rm s}\frac{\partial^2}{\partial\theta^2}\int A_{\rm ev}(\zeta,\theta)d\zeta \tag{39}$$

$$A_{\rm mv3}(\zeta,\theta) = -Z\int\frac{\partial A_{\rm ev}(\zeta,\theta)}{\partial\theta}d\zeta = -Z\frac{\partial}{\partial\theta}\int A_{\rm ev}(\theta,\zeta)d\zeta \tag{40}$$

The integral $\int A_{\rm ev}d\zeta$ follows readily from Eq.(37):

$$
\begin{aligned}
\int A_{\rm ev}(\zeta,\theta)d\zeta &= c^2T^2V_{\rm e1}\bigg\{-\frac{1}{\rho_2^2}e^{-\rho_2\zeta}\bigg[\frac{1}{\rho_2}(1-\operatorname{ch}\rho_2\theta)\\
&\qquad+\frac{1}{\rho_1^2-\rho_2^2}\bigg[\rho_2e^{-\rho_1\theta}-\rho_2\operatorname{ch}\rho_2\theta+\rho_1\operatorname{sh}\rho_2\theta)\bigg]\\
&\quad-4\rho_1\bigg(\sum_{\kappa=1}^{<K}\frac{L_{11}(\theta,\kappa)\sin 2\pi\kappa\theta-L_{12}(\theta,\kappa)\cos 2\pi\kappa\theta}{(\rho_1^2-d^2)^{1/2}}\frac{\cos 2\pi\kappa\zeta}{2\pi\kappa[(2\pi\kappa)^2+\rho_2^2]}\\
&\quad+\sum_{\kappa>K}^{\infty}\frac{L_{15}(\theta,\kappa)\sin 2\pi\kappa\theta-L_{16}(\theta,\kappa)\cos 2\pi\kappa\theta}{(d^2-\rho_1^2)^{1/2}}\frac{\cos 2\pi\kappa\zeta}{2\pi\kappa[(2\pi\kappa)^2+\rho_2^2]}\bigg)\bigg\}
\end{aligned}
\tag{41}
$$

The component $A_{\rm mv3}(\zeta,\theta)$ of the potential defined by Eq.(40) is readily derived from Eq.(41):

$$\begin{aligned}
A_{\mathrm{mv3}}(\zeta,\theta) = c^2T^2ZV_{\mathrm{el}}\Bigg\{ & -\frac{1}{\rho_2^2}e^{-\rho_2\zeta}\Bigg[\operatorname{sh}\rho_2\theta \\
& + \frac{\rho_2}{\rho_1^2-\rho_2^2}(\rho_1 e^{-\rho_1\theta} + \rho_2\operatorname{sh}\rho_2\theta - \rho_1\operatorname{ch}\rho_2\theta)\Bigg] \\
& + 4\rho_1\Bigg[\sum_{\kappa=1}^{<K}\frac{1}{(\rho_1^2-d^2)^{1/2}}\bigg(\frac{\partial L_{11}}{\partial\theta}\frac{\sin 2\pi\kappa\theta}{2\pi\kappa} + L_{11}\cos 2\pi\kappa\theta \\
& - \frac{\partial L_{12}}{\partial\theta}\frac{\cos 2\pi\kappa\theta}{2\pi\kappa} + L_{12}\sin 2\pi\kappa\theta\bigg)\frac{\cos 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2} \\
& + \sum_{\kappa>K}^{\infty}\frac{1}{(d^2-\rho_1^2)^{1/2}}\bigg(\frac{\partial L_{15}}{\partial\theta}\frac{\sin 2\pi\kappa\theta}{2\pi\kappa} + L_{15}\cos 2\pi\kappa\theta \\
& - \frac{\partial L_{16}}{\partial\theta}\frac{\cos 2\pi\kappa\theta}{2\pi\kappa} + L_{16}\sin 2\pi\kappa\theta\bigg)\frac{\cos 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2}\Bigg]\Bigg\} \qquad (42)
\end{aligned}$$

The derivatives $\partial L_{11}/\partial\theta$ to $\partial L_{16}/\partial\theta$ are the kernels of the integrals of Eqs.(33)–(36) with θ' replaced by θ:

$$\frac{\partial L_{11}}{\partial\theta} = e^{-\rho_1\theta/2}\operatorname{sh}[(\rho_1^2-d^2)^{1/2}\theta/2]\cos 2\pi\kappa\theta \qquad (43)$$

$$\frac{\partial L_{12}}{\partial\theta} = e^{-\rho_1\theta/2}\operatorname{sh}[(\rho_1^2-d^2)^{1/2}\theta/2]\sin 2\pi\kappa\theta \qquad (44)$$

$$\frac{\partial L_{15}}{\partial\theta} = e^{-\rho_1\theta/2}\sin[(d^2-\rho_1^2)^{1/2}\theta/2]\cos 2\pi\kappa\theta \qquad (45)$$

$$\frac{\partial L_{16}}{\partial\theta} = e^{-\rho_1\theta/2}\sin[(d^2-\rho_1^2)^{1/2}\theta/2]\sin 2\pi\kappa\theta \qquad (46)$$

To obtain the last component $A_{\mathrm{mv2}}(\zeta,\theta)$ of the associated potential defined by Eq.(39) requires multiplication of Eq.(42) by ρ_s and differentiation with respect to θ:

$$\begin{aligned}
A_{\mathrm{mv2}}(\zeta,\theta) = c^2T^2Z\rho_s V_{\mathrm{el}}\Bigg\{ & -\frac{1}{\rho_2}e^{-\rho_2\zeta}\Bigg[\operatorname{ch}\rho_2\theta \\
& + \frac{1}{\rho_1^2-\rho_2^2}(-\rho_1^2 e^{\rho_1\theta} + \rho_2^2\operatorname{ch}\rho_2\theta - \rho_1\rho_2\operatorname{sh}\rho_2\theta)\Bigg] \\
+4\Bigg[\sum_{\kappa=1}^{<K}\frac{\rho_1}{(\rho_1^2-d^2)^{1/2}}\bigg(& \frac{\partial^2 L_{11}}{\partial\theta^2}\frac{\sin 2\pi\kappa\theta}{2\pi\kappa} + 2\frac{\partial L_{11}}{\partial\theta}\cos 2\pi\kappa\theta - 2\pi\kappa L_{11}\sin 2\pi\kappa\theta \\
- \frac{\partial^2 L_{12}}{\partial\theta^2}\frac{\cos 2\pi\kappa\theta}{2\pi\kappa} & + 2\frac{\partial L_{12}}{\partial\theta}\sin 2\pi\kappa\theta + 2\pi\kappa L_{12}\cos 2\pi\kappa\theta\bigg)\frac{\cos 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2} \\
+\sum_{\kappa>K}^{\infty}\frac{\rho_1}{(d^2-\rho_1^2)^{1/2}}\bigg(& \frac{\partial^2 L_{15}}{\partial\theta^2}\frac{\sin 2\pi\kappa\theta}{2\pi\kappa} + 2\frac{\partial L_{15}}{\partial\theta}\cos 2\pi\kappa\theta - 2\pi\kappa L_{15}\sin 2\pi\kappa\theta
\end{aligned}$$

$$-\frac{\partial^2 L_{16}}{\partial \theta^2}\frac{\cos 2\pi\kappa\theta}{2\pi\kappa}+2\frac{\partial L_{16}}{\partial \theta}\sin 2\pi\kappa\theta+2\pi\kappa L_{16}\cos 2\pi\kappa\theta\Big)\frac{\cos 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2}\Big]\Big\} \quad (47)$$

The second derivatives $\partial^2 L_{11}/\partial\theta^2$ to $\partial^2 L_{16}/\partial\theta^2$ follow from Eqs.(43)–(46). The term $\partial^2 L_{11}/\partial\theta^2$ is written in detail in Eq.(4.1-125).

If we substitute Eqs.(38), (42), and (47) into Eq.(4.1-47) we obtain the component $A_{\mathrm{mv}} = A_{\mathrm{mve}}$ of the potential associated with $\mathbf{A}_{\mathrm{e}}$:

$$\begin{aligned}
A_{\mathrm{mv}}(\zeta,\theta) &= A_{\mathrm{mv1}}(\zeta,\theta)+A_{\mathrm{mv2}}(\zeta,\theta)+A_{\mathrm{mv3}}(\zeta,\theta)\\
&= -c^2T^2 Z\rho_{\mathrm{s}}V_{\mathrm{e1}}\Bigg(e^{-\rho_2\zeta}\Bigg\{\frac{1}{\rho_2}\left(1+\frac{1}{\rho_2\rho_{\mathrm{s}}}\,\mathrm{sh}\,\rho_2\theta\right)\\
&\quad+\frac{1}{\rho_1^2-\rho_2^2}\Bigg[\left(\rho_2-\rho_1^2+\frac{\rho_1\rho_2}{\rho_{\mathrm{s}}}\right)e^{-\rho_1\theta}+\rho_2\left(1+\rho_1-\frac{\rho_1}{\rho_{\mathrm{s}}}\right)\mathrm{ch}\,\rho_2\theta\\
&\qquad+\left(\rho_1+\rho_2^2+\frac{\rho_2^2}{\rho_{\mathrm{s}}}\right)\mathrm{sh}\,\rho_2\theta\Bigg]\Bigg\}\\
&\quad+4\Bigg\{\sum_{\kappa=1}^{<K}\frac{\rho_1}{(\rho_1^2-d^2)^{1/2}}\Bigg[\frac{1}{2\pi\kappa}\left(\frac{\partial^2 L_{11}}{\partial\theta^2}\sin 2\pi\kappa\theta-\frac{\partial^2 L_{12}}{\partial\theta^2}\cos 2\pi\kappa\theta\right)\\
&\quad+\left(2\cos 2\pi\kappa\theta+\frac{\sin 2\pi\kappa\theta}{2\pi\kappa\rho_{\mathrm{s}}}\right)\frac{\partial L_{11}}{\partial\theta}+\left(2\sin 2\pi\kappa\theta-\frac{\cos 2\pi\kappa\theta}{2\pi\kappa\rho_{\mathrm{s}}}\right)\frac{\partial L_{12}}{\partial\theta}\\
&\qquad+\frac{1}{\rho_{\mathrm{s}}}(L_{11}\cos 2\pi\kappa\theta+L_{12}\sin 2\pi\kappa\theta)\Bigg]\frac{\cos 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2}\\
&\quad+\sum_{\kappa>K}^{\infty}\frac{\rho_1}{(d^2-\rho_1^2)^{1/2}}\Bigg[\frac{1}{2\pi\kappa}\left(\frac{\partial^2 L_{15}}{\partial\theta^2}\sin 2\pi\kappa\theta-\frac{\partial^2 L_{16}}{\partial\theta^2}\cos 2\pi\kappa\theta\right)\\
&\quad+\left(2\cos 2\pi\kappa\theta+\frac{\sin 2\pi\kappa\theta}{2\pi\kappa\rho_{\mathrm{s}}}\right)\frac{\partial L_{15}}{\partial\theta}+\left(2\sin 2\pi\kappa\theta-\frac{\cos 2\pi\kappa\theta}{2\pi\kappa\rho_{\mathrm{s}}}\right)\frac{\partial L_{16}}{\partial\theta}\\
&\qquad+\frac{1}{\rho_{\mathrm{s}}}(L_{15}\cos 2\pi\kappa\theta+L_{16}\sin 2\pi\kappa\theta)\Bigg]\frac{\cos 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2}\Bigg\}\Bigg)
\end{aligned} \quad (48)$$

In order to simplify $A_{\mathrm{ev}}(\zeta,\theta)$ and $A_{\mathrm{mv}}(\zeta,\theta)$ of Eqs.(37) and (48) for $\theta > 0$, $\zeta > 0$ one must go through the calculations of Sections 4.2, 4.3, and 6.12. These calculations are too long and their reliability is currently too low to justify printing them. We only state that these calculations yielded the result

$$\mathcal{H} \propto 1/(2\pi\kappa)^4 \quad \text{for } \kappa \to \infty \quad (49)$$

instead of Eq.(4.3-46). Assuming that the calculations leading to Eq.(49) will stand up to further scrutiny we may conclude that the exponential ramp function of Eq.(2) yields a much faster decrease of $\mathcal{H}_\kappa$ with κ than the step function of Eq.(4.1-54). One would expect that a sudden step function excitation would produce more "high frequency" terms—or more photons with large period number κ—than the more slowly rising exponential ramp function excitation.

The calculations of Section 4.4 apply to exponential ramp function excitation as well as to step function excitation. Hence, we get again the eigenvalues E_κ of Eqs.(4.4-8) and (4.4-13).

Let us reflect how the infinite zero-point energy of the quantized pure wave field was replaced by a finite energy. The excitation of a transient either by a step function or an exponential ramp function with finite amplitude V_{e0} or V_{e1} in an area of finite size L^2 produces an EM wave with finite energy. This energy will remain finite if we represent the EM wave by a sum of photons rather than a sum of sinusoidal waves with finite extension. Furthermore, the energy will remain finite if the simple approximations of dipole currents in Eqs.(4.1-4) and (4.1-5) are replaced by the more sophisticated approximations or representations of Sections 2.1 and 2.2. The modification of Maxwell's equations not only permits to satisfy the causality law but also the conservation law of energy.

The use of excitation functions with other time variation does not change anything. Such excitation functions have to be represented by sums of time shifted step functions or exponential ramp functions[2] to produce solutions for the excited waves and any excited wave with finite energy will be represented by photons with finite energy.

The excitation of an EM wave at a certain time $t = 0$ at a plane $y = 0$ implies that there are no photons for $t < 0$. A wave excited at a plane $y = 0$ travels both in the direction $y > 0$ and $y < 0$ even though we ignored the wave for $y < 0$ since it produces the same results except for a change of sign. However, a reflector or absorber can eliminate the wave for $y < 0$. An arbitrarily large but finite time T had to be introduced to permit only denumerably many values of κ. The finite time T implies a finite spatial interval cT or $2cT$. There is no question that infinite times and distances should not exist in a science based on observation except for mathematical convenience. But it seems to be the first time that a mathematical theory rather than philosophical considerations demanded a finite time and spatial interval.

4.6 Excitation With Rectangular Pulse

In Section 4.1 we used the step function excitation $V_e(0,\theta)$ of Eq.(4.1-54) and in Section 4.5 we used the exponential ramp function excitation of Eq.(4.5-2). In both cases the excitation is applied in the whole time interval $0 \le t \le T$ or $0 \le \theta \le 1$. According to Fig.4.6-1 we may produce an excitation function of shorter duration $\Delta\theta$ by using the difference of two step functions with delay $\Delta\theta$:

$$R_e(0,\theta) = V_e(0,\theta) - V_e(0,\theta - \Delta\theta) = V_{e0}[S(\theta) - S(\theta - \Delta\theta)] \qquad (1)$$

The concept applies to the ramp function excitation of Eq.(4.5-2) too.

The solution $V_e(\zeta,\theta)$ defined by Eq.(4.1-58) for the step function excitation of Eq.(4.1-54) assumes the following form for the rectangular excitation function $R_e(0,\theta)$:

[2]See, e.g., Harmuth and Lukin (2000), Section 1.6.

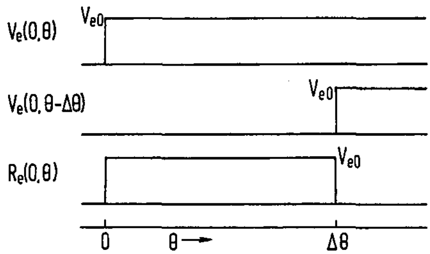

FIG.4.6-1. A step function $V_e(0,\theta)$ minus a delayed step function $V_e(0,\theta-\Delta\theta)$ yields a rectangular function $R_e(0,\theta)$.

$$\begin{aligned} R_e(\zeta,\theta) &= V_e(\zeta,\theta) = V_{e0}[F(\zeta)+w(\zeta,\theta)] && \text{for} \quad 0 \le \theta \le \Delta\theta \\ &= V_e(\zeta,\theta) - V_e(\zeta,\theta-\Delta\theta) && \text{for} \quad \Delta\theta < \theta \le 1 \end{aligned} \tag{2}$$

The function $F(\zeta)$ is defined by Eq.(4.1-60) while the function $w(\zeta,\theta)$ is defined by Eq.(4.1-88) and $V_e(\zeta,\theta-\Delta\theta)$ follows readily from Eq.(4.1-58) by the substitution of $\theta-\Delta\theta$ for θ:

$$V_e(\zeta,\theta-\Delta\theta) = V_{e0}[F(\zeta) - w(\zeta,\theta-\Delta\theta)] \tag{3}$$

We obtain $w(\zeta,\theta-\Delta\theta)$ by replacing θ with $\theta-\Delta\theta$ in Eq.(4.1-88). Equations (4.1-43) and (4.1-47) yield then $A_{ev}(\zeta,\theta-\Delta\theta)$ and $A_{mv}(\zeta,\theta-\Delta\theta)$. The simplifications of $A_{ev}(\zeta,\theta-\Delta\theta)$ and $A_{mv}(\zeta,\theta-\Delta\theta)$ according to Eqs.(4.2-12) and (4.2-16) requires the replacement of $\sin 2\pi\kappa\theta$, $\cos 2\pi\kappa\theta$, $\sin 2\pi\nu\theta$, and $\cos 2\pi\nu\theta$ by the time-shifted functions $\sin 2\pi\kappa(\theta-\Delta\theta)$, $\cos 2\pi\kappa(\theta-\Delta\theta)$, $\sin 2\pi\mu(\theta-\Delta\theta)$, and $\cos 2\pi\nu(\theta-\Delta\theta)$. The coefficients $A_{es}(\kappa)$, $A_{ec}(\kappa)$, $B_{es}(\kappa,\nu)$, $B_{ec}(\kappa,\nu)$, $C_{es}(\kappa,\nu)$, and $C_{ec}(\kappa,\nu)$ are not affected.

In Section 4.3 the functions $\sin 2\pi\kappa\theta$ and $\cos 2\pi\kappa\theta$ vanish after Eq.(4.3-17) while $\sin 2\pi\nu\theta$ and $\cos 2\pi\nu\theta$ vanish after Eq.(27), except for their use in Eq.(4.3-32). This implies that Eq.(4.3-46) also holds for the rectangular pulse of Fig.4.6-1:

$$\mathcal{H}_\kappa \propto 1/(2\pi\kappa)^2 \quad \text{for } \kappa \to \infty \tag{3}$$

The relative energy represented by $\mathcal{H}$ and its components $\mathcal{H}_\kappa$ will be less for the rectangular pulse than for the step function excitation, but Eqs.(4.3-46) and (3) refer only to the decrease of the components $\mathcal{H}_\kappa$ proportionate to $1/(2\pi\kappa)^2$, nothing is claimed about the actual value of $\mathcal{H}_\kappa$.

Let us point out that terms $U_{v\kappa}(\theta)$ according to Eq.(4.3-32) occur for the ramp function excitation of Section 4.5 and the rectangular pulse excitation of the current section too. There is always a fluctuating power with time average equal to zero.

5 Klein-Gordon Equation and Vacuum Constants

5.1 Modified Klein-Gordon Equation

In Section 3.3 we derived two approximations of the Hamilton function for a charged particle in an electromagnetic field. The first approximation is provided by Eqs.(3.3-32)–(3.3-34), the second by Eqs.(3.3-46)–(3.3-49). The terms $\mathcal{L}_{cx}$, $\mathcal{L}_{cy}$, and $\mathcal{L}_{cz}$ in Eqs.(3.2-32)–(3.2-34) are defined by the non-relativistic Eqs.(3.2-45)–(3.2-49), while the equally denoted terms in Eqs.(3.3-46)–(3.3-48) stand for the relativistic Eqs.(3.3-53)–(3.3-57).

We try to use the approximation of Eqs.(3.3-46)–(3.3-48). If we leave out the correcting terms $\alpha_e Q$, $\mathcal{L}_{cx}$, $\mathcal{L}_{cy}$, and $\mathcal{L}_{cz}$ we reduce these three equations to one:

$$\mathcal{H} = c[(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 + m_0^2c^2]^{1/2} + e\phi_\mathrm{e} \tag{1}$$

We rewrite this equation:

$$(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 - \frac{1}{c^2}(\mathcal{H} - e\phi_\mathrm{e})^2 = -m_0^2c^2$$

$$(p_x - eA_{\mathrm{m}x})^2 + (p_y - eA_{\mathrm{m}y})^2 + (p_z - eA_{\mathrm{m}z})^2 - \frac{1}{c^2}(\mathcal{H} - e\phi_\mathrm{e})^2 = -m_0^2c^2 \tag{2}$$

Using the substitutions

$$p_x \to \frac{\hbar}{i}\frac{\partial}{\partial x}, \quad p_y \to \frac{\hbar}{i}\frac{\partial}{\partial y}, \quad p_z \to \frac{\hbar}{i}\frac{\partial}{\partial z}, \quad \mathcal{H} \to -\frac{\hbar}{i}\frac{\partial}{\partial t} \tag{3}$$

and applying the resulting operators to a function Ψ transforms Eq.(2) into the Klein-Gordon equation:

$$\left[\left(\frac{\hbar}{i}\frac{\partial}{\partial x} - eA_{\mathrm{m}x}\right)^2 + \left(\frac{\hbar}{i}\frac{\partial}{\partial y} - eA_{\mathrm{m}y}\right)^2 + \left(\frac{\hbar}{i}\frac{\partial}{\partial z} - eA_{\mathrm{m}z}\right)^2 - \frac{1}{c^2}\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - e\phi_\mathrm{e}\right)^2\right]\Psi = -m_0^2c^2\Psi$$

$$\left[\sum_{j=1}^{3}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j}\right)^2 - \frac{1}{c^2}\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - e\phi_\mathrm{e}\right)^2\right]\Psi = -m_0^2c^2\Psi \tag{4}$$

With the relation

$$\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j}\right)^2 = -\hbar^2\frac{\partial^2}{\partial x_j^2} + 2ie\hbar A_{\mathrm{m}x_j}\frac{\partial}{\partial x_j} + ie\hbar\frac{\partial A_{\mathrm{m}x_j}}{\partial x_j} + e^2A_{\mathrm{m}x_j}^2 \quad (5)$$

we may rewrite Eq.(4) into the following alternate forms:

$$\left[\sum_{j=1}^{3}\left(-\hbar^2\frac{\partial^2}{\partial x_j^2} + 2ie\hbar A_{\mathrm{m}x_j}\frac{\partial}{\partial x_j} + ie\hbar\frac{\partial A_{\mathrm{m}x_j}}{\partial x_j} + e^2A_{\mathrm{m}x_j}^2\right) - \frac{1}{c^2}\left(-\hbar^2\frac{\partial^2}{\partial t^2} - 2ie\hbar\phi_{\mathrm{e}}\frac{\partial}{\partial t} - ie\hbar\frac{\partial\phi_{\mathrm{e}}}{\partial t} + e^2\phi_{\mathrm{e}}^2\right)\right]\Psi = -m_0^2c^2\Psi \quad (6)$$

$$\left(\hbar^2\frac{\partial^2}{\partial t^2} + 2ie\hbar\phi_{\mathrm{e}}\frac{\partial}{\partial t} + ie\hbar\frac{\partial\phi_{\mathrm{e}}}{\partial t} - e^2\phi_{\mathrm{e}}^2\right)\Psi = c^2(\hbar^2\nabla^2 - 2ie\hbar\mathbf{A}_{\mathrm{m}}\,\mathrm{grad} - ie\hbar\,\mathrm{div}\,\mathbf{A}_{\mathrm{m}} - e^2\mathbf{A}_{\mathrm{m}}^2 - m_0^2c^2)\Psi \quad (7)$$

We replace Eq.(1) by the complete Eq.(3.3-46) and proceed in analogy to Eq.(2):

$$\mathcal{H}_x = c\left[(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})^2 + m_0^2c^2\right]^{1/2}(1 + \alpha_{\mathrm{e}}Q) + e\phi_{\mathrm{e}} - \mathcal{L}_{\mathrm{c}x} \quad (8)$$

$$(\mathcal{H}_x - e\phi_{\mathrm{e}} + \mathcal{L}_{\mathrm{c}x})^2 = c^2\left[(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})^2 + m_0^2c^2\right](1 + \alpha_{\mathrm{e}}Q)^2 \quad (9)$$

Since our calculation holds only in first order of α_{e} we write $(1 + \alpha_{\mathrm{e}}Q)^2 \approx 1 + 2\alpha_{\mathrm{e}}Q$. Furthermore, we leave out the term $\mathcal{L}_{\mathrm{c}x}^2$ since all its components are multiplied by α_{e}^2 according to Eqs.(3.3-53)–(3.3-57):

$$(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})^2 - \frac{1}{c^2}(\mathcal{H}_x - e\phi_{\mathrm{e}})^2 + \alpha_{\mathrm{e}}\left\{2\left[(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})^2 + m_0^2c^2\right]Q - \frac{1}{c^2}\left[(\mathcal{H}_x - e\phi_{\mathrm{e}})\frac{\mathcal{L}_{\mathrm{c}x}}{\alpha_{\mathrm{e}}} + \frac{\mathcal{L}_{\mathrm{c}x}}{\alpha_{\mathrm{e}}}(\mathcal{H}_x - e\phi_{\mathrm{e}})\right]\right\} = -m_0^2c^2$$

$$\alpha_{\mathrm{e}} = \frac{ZecA_{\mathrm{e}}}{m_0c^2} \ll 1 \quad (10)$$

Instead of the function Ψ in Eq.(4) we use the function

$$\Psi_x = \Psi_{x0} + \alpha_{\mathrm{e}}\Psi_{x1} \quad (11)$$

and obtain

$$\Big\{(\mathbf{p}-e\mathbf{A}_{\rm m})^2 - \frac{1}{c^2}(\mathcal{H}_x - e\phi_{\rm e})^2 + \alpha_{\rm e}\Big[2[(\mathbf{p}-e\mathbf{A}_{\rm m})^2 + m_0^2c^2]Q - \frac{1}{c^2}\Big((\mathcal{H}_x - e\phi_{\rm e})\frac{\mathfrak{L}_{cx}}{\alpha_{\rm e}} + \frac{\mathfrak{L}_{cx}}{\alpha_{\rm e}}(\mathcal{H}_x - e\phi_{\rm e})\Big)\Big]\Big\}(\Psi_{x0} + \alpha_{\rm e}\Psi_{x1}) = -m_0^2c^2(\Psi_{x0} + \alpha_{\rm e}\Psi_{x1}) \quad (12)$$

Since the five components of the correcting term $\mathfrak{L}_{cx}$

$$\mathfrak{L}_{cx} = \mathfrak{L}_{cx1} + \mathfrak{L}_{cx2} + \mathfrak{L}_{cx3} + \mathfrak{L}_{cx4} + \mathfrak{L}_{cx5} \quad (13)$$

are all multiplied by $\alpha_{\rm e}$ in Eqs.(3.3-53)–(3.3-57) we may break Eq.(12) into one part of order $O(1)$ and a second one of order $O(\alpha_{\rm e})$:

$$\Big((\mathbf{p}-e\mathbf{A}_{\rm m})^2 - \frac{1}{c^2}(\mathcal{H}_x - e\phi_{\rm e})^2 + m_0^2c^2\Big)\Psi_{x0} = 0 \quad (14)$$

$$\Big((\mathbf{p}-e\mathbf{A}_{\rm m})^2 - \frac{1}{c^2}(\mathcal{H}_x - e\phi_{\rm e})^2 + m_0^2c^2\Big)\Psi_{x1} = -\Big[2[(\mathbf{p}-e\mathbf{A}_{\rm m})^2 + m_0^2c^2]Q - \frac{1}{c^2}\Big((\mathcal{H}_x - e\phi_{\rm e})\frac{\mathfrak{L}_{cx}}{\alpha_{\rm e}} + \frac{\mathfrak{L}_{cx}}{\alpha_{\rm e}}(\mathcal{H}_x - e\phi_{\rm e})\Big)\Big]\Psi_{x0} \quad (15)$$

Equation (14) is essentially the old Eq.(4) while Eq.(15) is an inhomogeneous variant of that equation.

The factor Q in Eq.(15) contains a term $[1+(\mathbf{p}-e\mathbf{A}_{\rm m})^2/m_0^2c^2]^{-3/2}$ according to Eq.(3.3-49) while $\mathfrak{L}_{cx1}$ to $\mathfrak{L}_{cx3}$ and $\mathfrak{L}_{cx5}$ according to Eqs.(3.3-53)–(3.3-55) and (3.3-57) have the same factor with the exponent $\kappa = -3/2$ replaced by $\kappa = -1/2$. If we want to replace the momentum $\mathbf{p}$ by the differential operators of Eq.(3) we must explain what the resulting operators mean. This may be done by series expansion in the case of a denominator $[1+(\mathbf{p}-e\mathbf{A}_{\rm m})^2/m_0^2c^2]^{\kappa}$:

$$[1+(\mathbf{p}-e\mathbf{A}_{\rm m})^2/m_0^2c^2]^{\kappa} \approx 1 + \kappa(\mathbf{p}-e\mathbf{A}_{\rm m})^2/m_0^2c^2 \quad (16)$$

The term $\mathbf{A}_{\rm e}^2(\mathbf{p}-e\mathbf{A}_{\rm m})^2$ in the denominator of Q in Eq.(3.3-49) is more difficult to deal with. First, we see how it can be explained as operator if there is only one spatial variable x. We obtain:

$$\begin{aligned}\mathbf{A}_{\rm e}^2 &= \mathbf{A}_{\rm e}\cdot\mathbf{A}_{\rm e} = A_{{\rm e}x}\mathbf{e}_x \cdot A_{{\rm e}x}\mathbf{e}_x = A_{{\rm e}x}^2\\ (\mathbf{p}-e\mathbf{A}_{\rm m})^2 &= (p_x - eA_{{\rm m}x})^2\\ \mathbf{A}_{\rm e}^2(\mathbf{p}-e\mathbf{A}_{\rm m})^2 &= A_{{\rm e}x}^2(p_x - eA_{{\rm m}x})^2\end{aligned} \quad (17)$$

$$\begin{aligned}\mathbf{A}_{\rm e}\cdot(\mathbf{p}-e\mathbf{A}_{\rm m}) &= A_{{\rm e}x}(p_x - eA_{{\rm m}x})\\ [\mathbf{A}_{\rm e}\cdot(\mathbf{p}-e\mathbf{A}_{\rm m})]^2 &= A_{{\rm e}x}^2(p_x - eA_{{\rm m}x})^2\end{aligned} \quad (18)$$

Since Eqs.(17) and (18) are equal we get a cancellation of the terms

$$Q_{02} = \frac{[\mathbf{A}_\mathrm{e} \cdot (\mathbf{p} - e\mathbf{A}_\mathrm{m})]^2}{\mathbf{A}_\mathrm{e}^2(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2} = 1 \tag{19}$$

in Eq.(3.3-49) and we avoid the problem of having to explain the meaning of $1/\mathbf{A}_\mathrm{e}^2(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2$.

Consider now that the vectors in Eq.(19) are represented by matrices of rank 3 whose components are vectors:

$$\mathbf{A}_\mathrm{e} = \begin{pmatrix} A_{\mathrm{e}x}\mathbf{e}_x & 0 & 0 \\ 0 & A_{\mathrm{e}y}\mathbf{e}_y & 0 \\ 0 & 0 & A_{\mathrm{e}z}\mathbf{e}_z \end{pmatrix} \tag{20}$$

$$\mathbf{p} - e\mathbf{A}_\mathrm{m} = \begin{pmatrix} (p_x - eA_{\mathrm{m}x})\mathbf{e}_x & 0 & 0 \\ 0 & (p_y - eA_{\mathrm{m}y})\mathbf{e}_y & 0 \\ 0 & 0 & (p_z - eA_{\mathrm{m}z})\mathbf{e}_z \end{pmatrix} \tag{21}$$

We obtain

$$\begin{aligned}\mathbf{A}_\mathrm{e}^2 &= \begin{pmatrix} A_{\mathrm{e}x}\mathbf{e}_x & 0 & 0 \\ 0 & A_{\mathrm{e}y}\mathbf{e}_y & 0 \\ 0 & 0 & A_{\mathrm{e}z}\mathbf{e}_z \end{pmatrix}\begin{pmatrix} A_{\mathrm{e}x}\mathbf{e}_x & 0 & 0 \\ 0 & A_{\mathrm{e}y}\mathbf{e}_y & 0 \\ 0 & 0 & A_{\mathrm{e}z}\mathbf{e}_z \end{pmatrix} \\ &= \begin{pmatrix} A_{\mathrm{e}x}^2 & 0 & 0 \\ 0 & A_{\mathrm{e}y}^2 & 0 \\ 0 & 0 & A_{\mathrm{e}z}^2 \end{pmatrix}\end{aligned} \tag{22}$$

and

$$\begin{aligned}&\mathbf{A}_\mathrm{e}^2(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 \\ &\quad= \begin{pmatrix} A_{\mathrm{e}x}^2(p_x - eA_{\mathrm{m}x})^2 & 0 & 0 \\ 0 & A_{\mathrm{e}y}^2(p_y - eA_{\mathrm{m}y})^2 & 0 \\ 0 & 0 & A_{\mathrm{e}z}^2(p_z - eA_{\mathrm{m}z})^2 \end{pmatrix}\end{aligned} \tag{23}$$

$$\begin{aligned}&\mathbf{A}_\mathrm{e} \cdot (\mathbf{p} - e\mathbf{A}_\mathrm{m}) \\ &\quad= \begin{pmatrix} A_{\mathrm{e}x}(p_x - eA_{\mathrm{m}x}) & 0 & 0 \\ 0 & A_{\mathrm{e}y}(p_y - eA_{\mathrm{m}y}) & 0 \\ 0 & 0 & A_{\mathrm{e}z}(p_z - eA_{\mathrm{m}z}) \end{pmatrix}\end{aligned} \tag{24}$$

$$\begin{aligned}&[\mathbf{A}_\mathrm{e} \cdot (\mathbf{p} - e\mathbf{A}_\mathrm{m})]^2 \\ &\quad= \begin{pmatrix} A_{\mathrm{e}x}^2(p_x - eA_{\mathrm{m}x})^2 & 0 & 0 \\ 0 & A_{\mathrm{e}y}^2(p_y - eA_{\mathrm{m}y})^2 & 0 \\ 0 & 0 & A_{\mathrm{e}z}^2(p_z - eA_{\mathrm{m}z})^2 \end{pmatrix}\end{aligned} \tag{25}$$

Since Eqs.(23) and (25) are equal we get again a cancellation of the terms of Eq.(19) and $Q_{02} = 1$. Equation (3.3-49) may be simplified:

$$Q = \frac{1}{m_0^2c^2} \frac{(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2}{[1 + (\mathbf{p} - e\mathbf{A}_\mathrm{m})^2/m_0^2c^2]^{3/2}} \tag{26}$$

With the help of Eq.(16) we can fully explain the meaning of $Q = Q_\mathrm{r}$:

$$Q_\mathrm{r} = (\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 \left[m_0^2c^2 - \frac{3}{2}(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2\right] \tag{27}$$

However, the subscript r suggests that one can readily write two more variants of Q:

$$Q_\mathrm{l} = \left[m_0^2c^2 - \frac{3}{2}(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2\right](\mathbf{p} - e\mathbf{A}_\mathrm{m})^2 \tag{28}$$

$$Q_\mathrm{s} = \frac{1}{m_0^2c^2}\left[m_0^2c^2 - \frac{3}{4}(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2\right](\mathbf{p} - e\mathbf{A}_\mathrm{m})^2\left[m_0^2c^2 - \frac{3}{4}(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2\right] \tag{29}$$

These obvious expressions are not the only possible ones. A short reflection shows that one can produce arbitrarily many variations of Eqs.(27)–(29). We note that the potential $\mathbf{A}_\mathrm{e}$ that comes from the magnetic dipole current density term $\mathbf{g}_\mathrm{m}$ in the modified Maxwell equations has disappeared from Eqs.(27)–(29). But its influence has not disappeared because $\mathbf{p} - e\mathbf{A}_\mathrm{m}$ is now the matrix of Eq.(21) that was forced on us by $\mathbf{A}_\mathrm{e}$ in order to obtain a usable form of Q.

Having explained the term Q in Eq.(15) we turn to $\mathfrak{L}_{cx}$ and its components defined by Eq.(13). We shall only write the 'right' variant corresponding to Eq.(27). Equations (3.3-53)–(3.3-57) yield:

$$\begin{aligned}\frac{1}{\alpha_\mathrm{e}}\mathfrak{L}_{cx1} \approx \frac{1}{A_\mathrm{e}m_0}&[A_{\mathrm{e}z}(\mathbf{p} - e\mathbf{A}_\mathrm{m})_y - A_{\mathrm{e}y}(\mathbf{p} - e\mathbf{A}_\mathrm{m})_z] \\ &\times\left(1 - \frac{(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2}{2m_0^2c^2}\right)(\mathbf{p} - e\mathbf{A}_\mathrm{m})_x\left(1 - \frac{(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2}{2m_0^2c^2}\right)\end{aligned} \tag{30}$$

$$\begin{aligned}\frac{1}{\alpha_\mathrm{e}}\mathfrak{L}_{cx2} \approx \frac{1}{A_\mathrm{e}}\int&\left(\frac{\partial\phi_\mathrm{m}}{\partial y}(\mathbf{p} - e\mathbf{A}_\mathrm{m})_z - \frac{\partial\phi_\mathrm{m}}{\partial z}(\mathbf{p} - e\mathbf{A}_\mathrm{m})_y\right) \\ &\times\left(1 - \frac{(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2}{2m_0^2c^2}\right)dx\end{aligned} \tag{31}$$

$$\begin{aligned}\frac{1}{\alpha_\mathrm{e}}\mathfrak{L}_{cx3} \approx \frac{1}{A_\mathrm{e}}\int&\left\{A_{\mathrm{e}z}\frac{\partial}{\partial t}\left[(\mathbf{p} - e\mathbf{A}_\mathrm{m})_y\left(1 - \frac{(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2}{2m_0^2c^2}\right)\right]\right. \\ &\left.- A_{\mathrm{e}y}\frac{\partial}{\partial t}\left[(\mathbf{p} - e\mathbf{A}_\mathrm{m})_z\left(1 - \frac{(\mathbf{p} - e\mathbf{A}_\mathrm{m})^2}{2m_0^2c^2}\right)\right]\right\}dx\end{aligned} \tag{32}$$

$$\frac{1}{\alpha_{\mathrm{e}}}\mathcal{L}_{cx4} = \frac{m_0 c^2}{A_{\mathrm{e}}} \int \left(\frac{\partial A_{\mathrm{e}y}}{\partial z} - \frac{\partial A_{\mathrm{e}z}}{\partial y} \right) dx \tag{33}$$

$$\begin{aligned}\frac{1}{\alpha_{\mathrm{e}}}\mathcal{L}_{cx5} \approx \frac{1}{A_{\mathrm{e}} m_0} \int \Bigg[&(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_y \left(1 - \frac{(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})^2}{2m_0^2 c^2} \right) \frac{\partial}{\partial y} \\ &+ (\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_z \left(1 - \frac{(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})^2}{2m_0^2 c^2} \right) \frac{\partial}{\partial z} \Bigg] \\ &\times [A_{\mathrm{e}z}(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_y - A_{\mathrm{e}y}(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})_z] \left(1 - \frac{(\mathbf{p} - e\mathbf{A}_{\mathrm{m}})^2}{2m_0^2 c^2} \right) dx \end{aligned} \tag{34}$$

We substitute the operators of Eq.(3) into Eqs.(14) and (15). Equation (14) equals formally Eq.(4) but we must write matrices according to Eq.(21) as well as replace Ψ by Ψ_{x0}:

$$\begin{aligned}\Bigg[&\begin{pmatrix} \left(\frac{\hbar}{i}\frac{\partial}{\partial x} - eA_{\mathrm{m}x} \right)^2 & 0 & 0 \\ 0 & \left(\frac{\hbar}{i}\frac{\partial}{\partial y} - eA_{\mathrm{m}y} \right)^2 & 0 \\ 0 & 0 & \left(\frac{\hbar}{i}\frac{\partial}{\partial z} - eA_{\mathrm{m}z} \right)^2 \end{pmatrix} \\ &- \frac{1}{c^2} \begin{pmatrix} \left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - e\phi_{\mathrm{e}} \right)^2 & 0 & 0 \\ 0 & \left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - e\phi_{\mathrm{e}} \right)^2 & 0 \\ 0 & 0 & \left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - e\phi_{\mathrm{e}} \right)^2 \end{pmatrix} \\ &+ \begin{pmatrix} m_0^2 c^2 & 0 & 0 \\ 0 & m_0^2 c^2 & 0 \\ 0 & 0 & m_0^2 c^2 \end{pmatrix} \Bigg] \begin{pmatrix} \Psi_{x0x} & 0 & 0 \\ 0 & \Psi_{x0y} & 0 \\ 0 & 0 & \Psi_{x0z} \end{pmatrix} = 0 \end{aligned} \tag{35}$$

This is essentially three times Eq.(4) without the summation sign but with the index j retaining the values $j = 1, 2, 3$:

$$\left[\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j} \right)^2 - \frac{1}{c^2}\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - e\phi_{\mathrm{e}} \right)^2 + m_0^2 c^2 \right] \Psi_{x0x_j} = 0 \tag{36}$$

Using the notation of Eq.(36) we may write Eq.(15) with the help of Eq.(27) in the following form:

$$\begin{aligned}
&\left[\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j}\right)^2 - \frac{1}{c^2}\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - e\phi_{\mathrm{e}}\right)^2 + m_0^2c^2\right]\Psi_{x1x_j} \\
&\qquad = -\left\{2\left[\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j}\right)^2 + m_0^2c^2\right]\right. \\
&\qquad\times\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j}\right)^2\left[m_0^2c^2 - \frac{3}{2}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j}\right)^2\right] \\
&\qquad - \frac{1}{c^2}\left[\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - e\phi_{\mathrm{e}}\right)(\mathfrak{L}_{\mathrm{c}x1j} + \mathfrak{L}_{\mathrm{c}x2j} + \mathfrak{L}_{\mathrm{c}x3j} + \mathfrak{L}_{\mathrm{c}x4j} + \mathfrak{L}_{\mathrm{c}x5j})\right. \\
&\left.\left. + (\mathfrak{L}_{\mathrm{c}x1j} + \mathfrak{L}_{\mathrm{c}x2j} + \mathfrak{L}_{\mathrm{c}x3j} + \mathfrak{L}_{\mathrm{c}x4j} + \mathfrak{L}_{\mathrm{c}x5j})\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - e\phi_{\mathrm{e}}\right)\right]\right\}\Psi_{x0x_j} \qquad (37)
\end{aligned}$$

The operators $\mathfrak{L}_{\mathrm{c}x1j}$ to $\mathfrak{L}_{\mathrm{c}x5j}$ follow from Eqs.(30)–(34) with the help of Eq.(3) and the substitution

$$\frac{1}{\alpha_{\mathrm{e}}}\mathcal{L}_{\mathrm{c}xkj} \to \mathfrak{L}_{\mathrm{c}xkj}, \quad k = 1,\ 2,\ 3,\ 4,\ 5 \qquad (38)$$

The matrix $\mathfrak{L}_{\mathrm{c}x1}$ has the terms $\mathfrak{L}_{\mathrm{c}x1j}$ along its main diagonal and zeroes everywhere else:

$$\begin{aligned}
\mathfrak{L}_{\mathrm{c}x1j} = \frac{1}{A_{\mathrm{e}}m_0}&\left[A_{\mathrm{e}z}\left(\frac{\hbar}{i}\frac{\partial}{\partial y} - eA_{\mathrm{m}y}\right) - A_{\mathrm{e}y}\left(\frac{\hbar}{i}\frac{\partial}{\partial z} - eA_{\mathrm{m}z}\right)\right] \\
&\times\left[1 - \frac{1}{2m_0^2c^2}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j}\right)^2\right]\left(\frac{\hbar}{i}\frac{\partial}{\partial x} - eA_{\mathrm{m}x}\right) \\
&\qquad\qquad\times\left[1 - \frac{1}{2m_0^2c^2}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j}\right)^2\right] \\
&j = 1,\ 2,\ 3; \quad x_1 = x,\ x_2 = y,\ x_3 = z \qquad (39)
\end{aligned}$$

For clarification we observe that the terms

$$\frac{1}{2m_0^2c^2}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j}\right)^2$$

are the terms of a matrix with rank 3 like the first matrix in Eq.(35) with the terms along the main diagonal varying according to $j = 1,\ 2,\ 3$. On the other hand, the terms

$$A_{\mathrm{e}z}\left(\frac{\hbar}{i}\frac{\partial}{\partial y} - eA_{\mathrm{m}y}\right),\ A_{\mathrm{e}y}\left(\frac{\hbar}{i}\frac{\partial}{\partial z} - eA_{\mathrm{m}z}\right),\ \left(\frac{\hbar}{i}\frac{\partial}{\partial x} - eA_{\mathrm{m}x}\right)$$

form matrices of rank 3 with equal values for all elements in the main diagonal, like the second and third matrix in Eq.(35).

For the remaining terms $\mathfrak{L}_{cx2j}$ to $\mathfrak{L}_{cx5j}$ we obtain from Eqs.(31)–(34) in analogy to Eq.(39) the following relations:

$$\mathfrak{L}_{cx2j} = \frac{1}{A_\mathrm{e}} \int \left[\frac{\partial \phi_\mathrm{m}}{\partial y} \left(\frac{\hbar}{i} \frac{\partial}{\partial z} - eA_{\mathrm{m}z} \right) - \frac{\partial \phi_\mathrm{m}}{\partial z} \left(\frac{\hbar}{i} \frac{\partial}{\partial y} - eA_{\mathrm{m}y} \right) \right. \\ \left. \times \left[1 - \frac{1}{2m_0^2c^2} \left(\frac{\hbar}{i} \frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j} \right)^2 \right] dx \tag{40}$$

$$\mathfrak{L}_{cx3j} = \frac{1}{A_\mathrm{e}} \int \left(A_{\mathrm{e}z} \frac{\partial}{\partial t} \left\{ \left(\frac{\hbar}{i} \frac{\partial}{\partial y} - eA_{\mathrm{m}y} \right) \left[1 - \frac{1}{2m_0^2c^2} \left(\frac{\hbar}{i} \frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j} \right)^2 \right] \right\} \right. \\ \left. - A_{\mathrm{e}y} \frac{\partial}{\partial t} \left\{ \left(\frac{\hbar}{i} \frac{\partial}{\partial z} - eA_{\mathrm{m}z} \right) \left[1 - \frac{1}{2m_0^2c^2} \left(\frac{\hbar}{i} \frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j} \right)^2 \right] \right\} \right) dx \tag{41}$$

$$\mathfrak{L}_{cx4j} = \frac{m_0c^2}{A_\mathrm{e}} \int \left(\frac{\partial A_{\mathrm{e}y}}{\partial z} - \frac{\partial A_{\mathrm{e}z}}{\partial y} \right) dy \tag{42}$$

$$\mathfrak{L}_{cx5j} = \frac{1}{A_\mathrm{e} m_0} \int \left\{ \left(\frac{\hbar}{i} \frac{\partial}{\partial y} - eA_{\mathrm{m}y} \right) \left[1 - \frac{1}{2m_0^2c^2} \left(\frac{\hbar}{i} \frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j} \right)^2 \right] \frac{\partial}{\partial y} \right. \\ \left. + \left(\frac{\hbar}{i} \frac{\partial}{\partial z} - eA_{\mathrm{m}z} \right) \left[1 - \frac{1}{2m_0^2c^2} \left(\frac{\hbar}{i} \frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j} \right)^2 \right] \frac{\partial}{\partial z} \right\} \\ \times \left[A_{\mathrm{e}z} \left(\frac{\hbar}{i} \frac{\partial}{\partial y} - eA_{\mathrm{m}y} \right) - A_{\mathrm{e}y} \left(\frac{\hbar}{i} \frac{\partial}{\partial z} - eA_{\mathrm{m}z} \right) \right] \\ \times \left[1 - \frac{1}{2m_0^2c^2} \left(\frac{\hbar}{i} \frac{\partial}{\partial x_j} - eA_{\mathrm{m}x_j} \right)^2 \right] dx \tag{43}$$

Equation (37) is now completely defined in terms of time and spatial derivatives, the potentials $\mathbf{A}_\mathrm{m}$, $\mathbf{A}_\mathrm{e}$, ϕ_m, ϕ_e and the rest mass m_0 of a charged particle. Instead of the 'right' variant used from Eq.(30) to Eq.(43) we may write a 'left' and a 'symmetric' variant corresponding to Eqs.(28) and (29). Obvious questions are which variant should be used, are all three meaningful, and what shall we make of the infinitely many possible variations mentioned in the text following Eq.(29)? One definite result obtained is that Ψ_{x0x_j} in Eq.(36) is one of the three components of the last matrix in Eq.(35). This is different from the conventional theory which does not have that matrix.

Equation (36) can be solved for certain initial and boundary conditions just like partial differential equations for the field strengths $\mathbf{E}$ and $\mathbf{H}$ or the potentials $\mathbf{A}_\mathrm{e}$ and $\mathbf{A}_\mathrm{m}$ derived from the modified Maxwell equations can be solved. We note that a vector with three components that are scalars is formally similar to a matrix of rank 3 with three components in the main diagonal that are vectors. If the solution is done by Fourier's method of standing waves we are led to a quantization procedure as in Sections 4.3 and 4.4 for the pure radiation field without charged particles.

The solution of Eq.(37) requires a particular solution of the inhomogeneous equation since the homogeneous solution is the same as for Eq.(36). If $\mathbf{A}_{\rm m}$, $\phi_{\rm e}$, and m_0 are zero, Eq.(37) is reduced to the inhomogeneous wave equation with the very satisfactory solution used in Eqs.(3.1-44) and (3.1-45) or Eqs.(4.1-43) and (4.1-46). At this time we must hope that mathematicians will generalize that solution to $m_0 \neq 0$, $\phi_{\rm e} \neq 0$, and $\mathbf{A}_{\rm m} \neq 0$.

5.2 Planar Wave Solution

We have seen in Section 4.4 that the infinite zero-point energy for $n = 0$ as well as any other infinite energies for $n > 0$ are eliminated for the pure radiation field—or the pure electromagnetic wave—if the modified Maxwell equations are used as the basis of quantum electrodynamics. The need for renormalization was thus eliminated for this particular case. Here we will attempt to extend this result to the interaction of photons with bosons. To do so we must in essence replace Eqs.(4.1-38) and (4.1-39) by Eqs.(5.1-36) and (5.1-37). Hence, we turn to the solution of Eq.(5.1-36) by Fourier's method of standing waves. We write $x_j = y$ in order to connect with previously derived results:

$$\left[\left(\frac{\hbar}{i}\frac{\partial}{\partial y} - eA_{{\rm m}y}\right)^2 - \frac{1}{c^2}\left(-\frac{\hbar}{i}\frac{\partial}{\partial t} - e\phi_{\rm e}\right)^2 + m_0^2c^2\right]\Psi_{x0y} = 0 \tag{1}$$

To reduce the number of subscripts we write

$$\Psi = \Psi_{x0y} \tag{2}$$

and obtain with the help of

$$\left(\frac{\hbar}{i}\frac{\partial}{\partial y} - eA_{{\rm m}y}\right)^2\Psi = -\hbar^2\frac{\partial^2\Psi}{\partial y^2} + 2i\hbar eA_{{\rm m}y}\frac{\partial\Psi}{\partial y} + \left(e^2A_{{\rm m}y}^2 + i\hbar e\frac{\partial A_{{\rm m}y}}{\partial y}\right)\Psi \tag{3}$$

$$\left(\frac{\hbar}{i}\frac{\partial}{\partial t} + e\phi_{\rm e}\right)^2\Psi = -\hbar^2\frac{\partial^2\Psi}{\partial t^2} - 2i\hbar e\phi_{\rm e}\frac{\partial\Psi}{\partial t} + \left(e^2\phi_{\rm e}^2 - i\hbar e\frac{\partial\phi_{\rm e}}{\partial t}\right)\Psi \tag{4}$$

from Eq.(1):

$$\begin{aligned}\frac{\partial^2\Psi}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} - 2i\frac{e}{\hbar}\left(A_{{\rm m}y}\frac{\partial\Psi}{\partial y} - \frac{1}{c^2}\phi_{\rm e}\frac{\partial\Psi}{\partial t}\right)\\ - \frac{e^2}{\hbar^2}\left[A_{{\rm m}y}^2 - \frac{1}{c^2}\phi_{\rm e}^2 - i\frac{\hbar}{e}\left(\frac{\partial A_{{\rm m}y}}{\partial y} + \frac{1}{c^2}\frac{\partial\phi_{\rm e}}{\partial t}\right) - \frac{m_0^2c^2}{e^2}\right]\Psi = 0\end{aligned} \tag{5}$$

We choose A_{my} and ϕ_e to be independent of y and t. A partial differential equation with constant coefficients is obtained:

$$\frac{\partial^2\Psi}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} - 2i\frac{e}{\hbar}\left(A_{my}\frac{\partial\Psi}{\partial y} - \frac{1}{c^2}\phi_e\frac{\partial\Psi}{\partial t}\right) - \frac{e^2}{\hbar^2}\left(A_{my}^2 - \frac{1}{c^2}\phi_e^2 - \frac{m_0^2c^2}{e^2}\right)\Psi = 0 \quad (6)$$

Except for the term $\partial\Psi/\partial y$ and the imaginary constant i we have again Eq.(4.1-38). Using the normalization

$$\theta = t/T,\ \zeta = y/cT \quad (7)$$

we get

$$\frac{\partial^2\Psi}{\partial\zeta^2} - \frac{\partial^2\Psi}{\partial\theta^2} - 2i\lambda_1\left(\frac{\partial\Psi}{\partial\zeta} - \lambda_3\frac{\partial\Psi}{\partial\theta}\right) - \lambda_2^2\Psi = 0$$
$$\lambda_1 = \frac{ecTA_{my}}{\hbar},\ \lambda_2^2 = \lambda_1^2\left(1 - \frac{\phi_e^2}{c^2A_{my}^2} - \frac{m_0^2c^2}{e^2A_{my}}\right) < \lambda_1^2,\ \lambda_3 = \frac{\phi_e}{cA_{my}} \quad (8)$$

A comparison of Eq.(8) with Eq.(4.1-41) shows that they are very similar. This suggests finding a solution of Eq.(8) in terms of Fourier's method of standing waves that satisfied the causality law and the conservation law of energy.

For a first solution of Eq.(8) we follow Eq.(4.1-54) and assume as boundary condition at $\zeta = 0$ a step function

$$\Psi(0,\theta) = \Psi_0 S(\theta) = 0 \quad \text{for } \theta < 0$$
$$= \Psi_0 \quad \text{for } \theta \geq 0 \quad (9)$$

The usual boundary condition for $\zeta \to \infty$

$$\Psi(\infty,\theta) = \text{finite} \quad (10)$$

cannot be used just as in the case of Eq.(4.1-55).

For the initial condition(s) we follow Eq.(4.1-56) and assume at $\theta = 0$ the relation

$$\Psi(\zeta,0) = 0 \quad (11)$$

This initial condition implies $\Psi(\zeta,\theta) = 0$ for $\theta < 0$ due to Eq.(9). As in Section 4.1 we emphasize that a function of time that describes a physical process subject to the causality law must be zero before a finite time. The choice $\theta = 0$ for this time does not imply any loss of generality.

If $\Psi(\zeta, 0)$ is zero for all values $\zeta > 0$, all its derivatives with respect to ζ must be zero too:

$$\partial^n \Psi(\zeta, 0)/\partial \zeta^n = 0 \tag{12}$$

We assume that the general solution of Eq.(8) can be written as the sum of a steady state solution $F(\zeta)$ plus a deviation $w(\zeta, \theta)$ from it:

$$\Psi(\zeta, \theta) = \Psi_0[F(\zeta) + w(\zeta, \theta)] \tag{13}$$

Substitution of $F(\zeta)$ into Eq.(8) yields:

$$\partial^2 F/\partial \zeta^2 - 2i\lambda_1 \partial F/\partial \zeta - \lambda_2^2 F = 0$$

$$F(\zeta) = A_{10} \exp\left\{i\left[\lambda_1 + (\lambda_1^2 - \lambda_2^2)^{1/2}\right]\zeta\right\} + A_{11} \exp\left\{i\left[\lambda_1 - (\lambda_1^2 - \lambda_2^2)^{1/2}\right]\zeta\right\} \tag{14}$$

Due to the relation $\lambda_1^2 > \lambda_2^2$ in Eq.(8) the root $(\lambda_1^2 - \lambda_2^2)^{1/2}$ is always real. The boundary condition of Eq.(9) demands that $F(0)$ equals 1, which implies

$$F(0) = 1, \quad A_{10} + A_{11} = 1 \tag{15}$$

$$F(\zeta) = \exp\left\{i\left[\lambda_1 + (\lambda_1^2 - \lambda_2^2)^{1/2}\right]\zeta\right\} + A_{13} e^{i\lambda_1 \zeta} \sin\left[(\lambda_1^2 - \lambda_2^2)^{1/2}\zeta\right] \tag{16}$$

The constant A_{13} remains undetermined at this time.

Substitution of Eq.(13) into the boundary condition of Eq.(9) yields the boundary condition for $w(0, \theta)$:

$$\begin{aligned} \Psi(0, \theta) = \Psi_0[F(0) + w(0, \theta)] = \Psi_0 \quad &\text{for } \theta \geq 0 \\ w(0, \theta) = 0 \quad &\text{for } \theta \geq 0 \end{aligned} \tag{17}$$

For $\zeta \to \infty$ we obtain from Eq.(10) the boundary condition

$$w(\infty, \theta) = \text{finite} \tag{18}$$

which cannot be used but is not needed either, while the initial condition of Eq.(11) yields

$$F(\zeta) + w(\zeta, 0) = 0, \quad w(\zeta, 0) = -F(\zeta) \tag{19}$$

$$\partial w(\zeta, \theta)/\partial \theta = 0 \qquad \text{for } \theta = 0,\ \zeta > 0 \tag{20}$$

Substitution of Eq.(13) into Eq.(8) yields for $w(\zeta,\theta)$ the same equation as for Ψ since the purpose of the definition of Eq.(13) is strictly to obtain the homogeneous boundary condition of Eq.(17) for $w(0,\theta)$:

$$\frac{\partial^2 w}{\partial \zeta^2} - \frac{\partial^2 w}{\partial \theta^2} - 2i\lambda_1\left(\frac{\partial w}{\partial \zeta} - \lambda_3\frac{\partial w}{\partial \theta}\right) - \lambda_2^2 w = 0 \tag{21}$$

Particular solutions of this equation denoted $w_\kappa(\zeta,\theta)$ are obtained by separating the variables:

$$w_\kappa(\zeta,\theta) = \phi(\zeta)\psi(\theta) \tag{22}$$

$$\frac{1}{\phi}\left(\frac{\partial^2\phi}{\partial\zeta^2} - 2i\lambda_1\frac{\partial\phi}{\partial\zeta}\right) = \frac{1}{\psi}\left(\frac{\partial^2\psi}{\partial\theta^2} - 2i\lambda_1\lambda_3\frac{\partial\psi}{\partial\theta}\right) + \lambda_2^2 = -(2\pi\kappa)^2 \tag{23}$$

We get two ordinary differential equations

$$\frac{\partial^2\phi}{\partial\zeta^2} - 2i\lambda_1\frac{\partial\phi}{\partial\zeta} + (2\pi\kappa)^2\phi = 0 \tag{24}$$

$$\frac{\partial^2\psi}{\partial\theta^2} - 2i\lambda_1\lambda_3\frac{\partial\psi}{\partial\theta} + [(2\pi\kappa)^2 + \lambda_2^2]\psi = 0 \tag{25}$$

with the solutions:

$$\begin{aligned}\phi(\zeta) &= A_{20}e^{i(\lambda_1+\iota_0)\zeta} + A_{21}e^{i(\lambda_1-\iota_0)\zeta}\\ \iota_0 &= \left[(2\pi\kappa)^2 + \lambda_1^2\right]^{1/2}\end{aligned} \tag{26}$$

$$\begin{aligned}\psi(\theta) &= A_{30}e^{i(\lambda_1\lambda_3+\gamma_0)\theta} + A_{31}e^{i(\lambda_1\lambda_3-\gamma_0)\theta}\\ \gamma_0 &= \left[(2\pi\kappa)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\right]^{1/2}\end{aligned} \tag{27}$$

$$\lambda_1 = \frac{ecTA_{my}}{\hbar},\ \lambda_2^2 = \lambda_1^2\left(1 - \frac{\phi_e^2}{c^2A_{my}} - \frac{m_0^2c^2}{e^2A_{my}}\right),\ \lambda_3 = \frac{\phi_e}{cA_{my}}$$

The boundary condition $w(0,\theta) = 0$ in Eq.(17) requires in Eq.(26) the relation

$$A_{21} = -A_{20} \tag{28}$$

and the particular solution $w_\kappa(\zeta,\theta)$ becomes:

$$w_\kappa(\zeta,\theta) = \{A_1\exp[i(\lambda_1\lambda_3+\gamma_0)\theta] + A_2\exp[i(\lambda_1\lambda_3-\gamma_0]\theta\}e^{i\lambda_1\zeta}\sin\iota_0\zeta \tag{29}$$

The solution $w_\kappa(\zeta,\theta)$ is usually generalized by making A_1 and A_2 functions of κ and integrating over all values of κ. We follow the alternate route of Section 4.1 from Eq.(74) on in order to replace the nondenumerable terms of the Fourier integral by the denumerable terms of the Fourier series. We need again an arbitrarily large but finite spatial interval $0 \le y \le cT$ and time interval $0 \le t \le T$, where T is finite but otherwise as large as we need it to be. The normalized variables ζ and θ will be used:

$$0 \le \zeta = y/cT \le 1, \quad 0 \le \theta = t/T \le 1 \tag{30}$$

Instead of the Fourier sum of Eq.(4.1-76) we get:

$$w(\zeta,\theta) = \sum_{\kappa=1}^{\infty} \Big\{ A_1(\kappa)\exp[i(\lambda_1\lambda_3 + \gamma_0)\theta] + A_2(\kappa)\exp[i(\lambda_1\lambda_3 - \gamma_0)\theta] \Big\} \times e^{i\lambda_1\zeta} \sin \iota_0\zeta \tag{31}$$

The derivative of $w(\zeta,\theta)$ with respect to θ becomes:

$$\frac{\partial w}{\partial\theta} = \sum_{\kappa=1}^{\infty} i\Big\{ A_1(\kappa)(\lambda_1\lambda_3 + \gamma_0)\exp[i(\lambda_1\lambda_3 + \gamma_0)\theta] + A_2(\kappa)(\lambda_1\lambda_3 - \gamma_0)\exp[i(\lambda_1\lambda_3 - \gamma_0)\theta] \Big\} e^{i\lambda_1\zeta} \sin \iota_0\zeta \tag{32}$$

The functions $A_1(\kappa)$ and $A_2(\kappa)$ can be obtained from Eqs.(31) and (32) with the help of Eqs.(19) and (20):

$$w(\zeta,0) = \sum_{\kappa=1}^{\infty} [A_1(\kappa) + A_2(\kappa)] e^{i\lambda_1\zeta} \sin \iota_0\zeta = -F(\zeta) \tag{33}$$

$$\frac{\partial w(\zeta,0)}{\partial\theta} = \sum_{\kappa=1}^{\infty} i[A_1(\kappa)(\lambda_1\lambda_2 + \gamma_0) + A_2(\kappa)(\lambda_1\lambda_3 - \gamma_0)] \times e^{i\lambda_1\zeta} \sin \iota_0\zeta = 0 \tag{34}$$

In order to obtain the functions $A_1(\kappa)$ and $A_2(\kappa)$ we use the Fourier series according to Eqs.(4.1-80) and (4.1-81). The factor $e^{i\lambda_1\zeta}$ can be written in front of the summation sign since it does not contain κ according to Eq.(27):

$$\sum_{\kappa=1}^{\infty} [A_1(\kappa) + A_2(\kappa)] \sin\Big\{ \big[(2\pi\kappa)^2 + \lambda_1^2\big]^{1/2} \zeta \Big\} = -F(\zeta) e^{-i\lambda_1\zeta} \tag{35}$$

$$\sum_{\kappa=1}^{\infty} [A_1(\kappa)(\lambda_1\lambda_3 + \gamma_0) + A_2(\kappa)(\lambda_1\lambda_3 - \gamma_0)] \times \sin\Big\{ [(2\pi\kappa)^2 + \lambda_1^2]^{1/2}\zeta \Big\} = 0 \tag{36}$$

Equations (35) and (36) do not have quite the form of Eq.(4.1-81). To bring them into that form we represent $\sin\{[(2\pi\kappa)^2 + \lambda_1^2]^{1/2}\zeta\}$ by a Fourier series:

$$\sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\} = \frac{a(\kappa,0)}{2}+\sum_{n=1}^{\infty}[a(\kappa,n)\cos 2\pi n\zeta + b(\kappa,n)\sin 2\pi n\zeta] \tag{37}$$

$$a(\kappa,n) = 2\int_0^1 \sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}\cos 2\pi n\zeta\, d\zeta = \frac{2[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\left\{1-\cos[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\right\}}{(2\pi\kappa)^2+\lambda_1^2-(2\pi n)^2} \tag{38}$$

$$b(\kappa,n) = 2\int_0^1 \sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}\sin 2\pi n\zeta\, d\zeta = -\frac{2[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\sin[(2\pi\kappa)^2+\lambda_1^2]^{1/2}}{(2\pi\kappa)^2+\lambda_1^2-(2\pi n)^2} \tag{39}$$

This expansion is not needed for sufficiently large values of κ:

$$\sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\} \approx \sin 2\pi\kappa\zeta \quad \text{for } \kappa \geq K \gg \lambda_1/2\pi \tag{40}$$

We note that the constant K is here not as precisely defined as in Eq.(4.1-81) for the pure electromagnetic wave. The coefficients $a(\kappa,n)$ and $b(\kappa,n)$ can be computed for any κ and n. We may thus produce the following sums :

$$c(n)=\sum_{\kappa=1}^{<K} a(\kappa,n), \quad d(n)=\sum_{\kappa=1}^{<K} b(\kappa,n) \tag{41}$$

The summation limits $< K$ mean the largest integer smaller than K.

After the summation we may replace the variable $n = 1, 2, \ldots$ by the variable $\kappa = 1, 2, \ldots$

$$c(\kappa)=c(n), \quad d(\kappa)=d(n) \tag{42}$$

Equations (35) and (36) become with the help of Eq.(40):

$$\sum_{\kappa=1}^{\infty}[A_1(\kappa)+A_2(\kappa)][c_0(\kappa)\cos 2\pi\kappa\zeta + d_0(\kappa)\sin 2\pi\kappa\zeta] = -F(\zeta)e^{-2\lambda_1\zeta} \tag{43}$$

$$\sum_{\kappa=1}^{\infty}[A_1(\kappa)(\lambda_1\lambda_2+\gamma_0)+A_2(\kappa)(\lambda_1\lambda_2-\gamma_0)] \times [c_0(\kappa)\cos 2\pi\kappa\zeta + d_0(\kappa)\sin 2\pi\kappa\zeta] = 0 \tag{44}$$

$$c_0(\kappa)=c(\kappa),\ d_0(\kappa)=d(\kappa) \quad \text{for } \kappa < K \gg \lambda_1/2\pi = ecTA_{my}/2\pi\hbar \tag{45}$$

$$=0, \qquad =1 \qquad \text{for } \kappa \geq K \tag{46}$$

Using Eqs.(43) and (44) we obtain from the terms with $c_0(\kappa) = c(\kappa)$ and $d_0(\kappa) = d(\kappa)$ or $c_0(\kappa) = 0$ and $d_0(\kappa) = 1$ in analogy to Eqs.(4.1-80) and (4.1-81) the following relations:

$$\text{for } \kappa < K \gg \lambda_1/2\pi$$

$$[A_1(\kappa) + A_2(\kappa)]d(\kappa) = -2\int_0^1 F(\zeta)e^{-i\lambda_1\zeta}\sin 2\pi\kappa\zeta\, d\zeta = -2I_{\mathrm{F}} \tag{47}$$

$$[A_1(\kappa)(\lambda_1\lambda_3 + \gamma_0) + A_2(\kappa)(\lambda_1\lambda_2 - \gamma_0)]d(\kappa) = 0 \tag{48}$$

$$\text{for } \kappa \geq K$$

$$A_1(\kappa) + A_2(\kappa) = -2\int_0^1 F(\zeta)e^{-i\lambda_1\zeta}\sin 2\pi\kappa\zeta\, d\zeta = -2I_{\mathrm{F}} \tag{49}$$

$$A_1(\kappa)(\lambda_1\lambda_3 + \gamma_0) + A_2(\kappa)(\lambda_1\lambda_3 - \gamma_0) = 0 \tag{50}$$

If $\sin 2\pi\kappa\zeta$ in Eqs.(4.1-80) and (4.1-81) is replaced by $\cos 2\pi\kappa\zeta$ one obtains the essentially same result for $\kappa < K$ by means of the Fourier cosine series:

$$\text{for } \kappa < K$$

$$[A_1(\kappa) + A_2(\kappa)]d(\kappa) = -2\int_0^1 F(\zeta)e^{-i\lambda_1\zeta}\cos 2\pi\kappa\zeta\, d\zeta \tag{51}$$

$$[A_1(\kappa)(\lambda_1\lambda_3 + \gamma_0) + A_2(\kappa)(\lambda_1\lambda_2 - \gamma_0)]d(\kappa) = 0 \tag{52}$$

Equations (47) and (48) as well as Eqs.(49) and (50) are readily solved for $A_1(\kappa)$ and $A_2(\kappa)$:

$$\begin{aligned} A_1(\kappa) &= -\frac{I_{\mathrm{F}}}{d(\kappa)}\left(1 - \frac{\lambda_1\lambda_3}{\gamma_0}\right) \qquad \text{for } \kappa < K \gg \lambda_1/2\pi \\ A_2(\kappa) &= -\frac{I_{\mathrm{F}}}{d(\kappa)}\left(1 + \frac{\lambda_1\lambda_3}{\gamma_0}\right) \end{aligned} \tag{53}$$

$$\begin{aligned} A_1(\kappa) &= -I_{\mathrm{F}}\left(1 - \frac{\lambda_1\lambda_3}{\gamma_0}\right) \qquad \text{for } \kappa \geq K \\ A_2(\kappa) &= -I_{\mathrm{F}}\left(1 + \frac{\lambda_1\lambda_3}{\gamma_0}\right) \end{aligned} \tag{54}$$

For the evaluation of the integral I_{F} we start from Eq.(16):

$$F(\zeta)e^{-i\lambda_1\zeta} = \cos\left[(\lambda_1^2 - \lambda_2^2)^{1/2}\zeta\right] + (i + A_{13})\sin\left[(\lambda_1^2 - \lambda_2^2)^{1/2}\zeta\right] \quad (55)$$

If we choose $A_{13} = -i$ we obtain:

$$\begin{aligned} I_{\mathrm{F}} &= \int_0^1 F(\zeta)e^{-i\lambda_1\zeta}\sin 2\pi\kappa\zeta\, d\zeta = \int_0^1 \cos\left[(\lambda_1^2 - \lambda_2^2)^{1/2}\zeta\right]\sin 2\pi\kappa\zeta\, d\zeta \\ &= \frac{2\pi\kappa\left[1 - \cos(\lambda_1^2 - \lambda_2^2)^{1/2}\right]}{(2\pi\kappa)^2 - (\lambda_1^2 - \lambda_2^2)} \end{aligned} \quad (56)$$

The functions $A_1(\kappa)$ and $A_2(\kappa)$ become:

$$\begin{aligned} A_1(\kappa) &= -\frac{2\pi\kappa\left[1 - \cos(\lambda_1^2 - \lambda_2^2)^{1/2}\right](\gamma_0 - \lambda_1\lambda_3)}{\left[(2\pi\kappa)^2 - \lambda_1^2 + \lambda_2^2\right]\gamma_0 d(\kappa)} \quad \text{for } \kappa < K \\ A_2(\kappa) &= -\frac{2\pi\kappa\left[1 - \cos(\lambda_1^2 - \lambda_2^2)^{1/2}\right](\gamma_0 + \lambda_1\lambda_3)}{\left[(2\pi\kappa)^2 - \lambda_1^2 + \lambda_2^2\right]\gamma_0 d(\kappa)} \end{aligned} \quad (57)$$

$$\begin{aligned} A_1(\kappa) &= -\frac{2\pi\kappa\left[1 - \cos(\lambda_1^2 - \lambda_2^2)^{1/2}\right](\gamma_0 - \lambda_1\lambda_3)}{\left[(2\pi\kappa)^2 - \lambda_1^2 + \lambda_2^2\right]\gamma_0} \quad \text{for } \kappa \geq K \\ A_2(\kappa) &= -\frac{2\pi\kappa\left[1 - \cos(\lambda_1^2 - \lambda_2^2)^{1/2}\right](\gamma_0 + \lambda_1\lambda_3)}{\left[(2\pi\kappa)^2 - \lambda_1^2 + \lambda_2^2\right]\gamma_0} \end{aligned} \quad (58)$$

Substitution of $A_1(\kappa)$ and $A_2(\kappa)$ brings Eq.(31) into the following form:

$$\begin{aligned} w(\zeta,\theta) = \sum_{\kappa=1}^{<K} &\frac{4\pi\kappa\left[\cos(\lambda_1^2 - \lambda_2^2)^{1/2} - 1\right]}{\left[(2\pi\kappa)^2 - \lambda_1^2 + \lambda_2^2\right]d(\kappa)} \\ &\times \left(e^{i\lambda_1\lambda_3\theta}\cos\left\{\left[(2\pi\kappa)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\right]^{1/2}\theta\right\}\right. \\ &+ \left.\frac{i\lambda_1\lambda_3}{\gamma_0}e^{-i\lambda_1\lambda_3\theta}\sin\left\{\left[(2\pi\kappa)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\right]^{1/2}\theta\right\}\right) \\ &\times e^{i\lambda_1\zeta}\sin\left\{\left[(2\pi\kappa)^2 + \lambda_1^2\right]^{1/2}\zeta\right\} \\ + \sum_{\kappa>K}^{\infty} &\frac{4\pi\kappa\left[\cos(\lambda_1^2 - \lambda_2^2)^{1/2} - 1\right]}{(2\pi\kappa)^2 - \lambda_1^2 + \lambda_2^2} \\ &\times \left(e^{i\lambda_1\lambda_3\theta}\cos\left\{\left[(2\pi\kappa)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\right]^{1/2}\theta\right\}\right. \\ &+ \left.\frac{i\lambda_1\lambda_3}{\gamma_0}e^{-i\lambda_1\lambda_3\theta}\sin\left\{\left[(2\pi\kappa)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\right]^{1/2}\theta\right\}\right) \\ &\times e^{i\lambda_1\zeta}\sin\left\{\left[(2\pi\kappa)^2 + \lambda_1^2\right]^{1/2}\zeta\right\} \end{aligned} \quad (59)$$

For $\kappa \geq K$ the first term of the second sum varies like $\kappa^{-1}\cos 2\pi\kappa\theta \sin 2\pi\kappa\zeta$ if κ grows beyond all bounds, which means this is a conditionally convergent sum. The second term varies like $\kappa^{-1}\sin 2\pi\kappa\theta \sin 2\pi\kappa\theta$ for $\kappa \to \infty$ and is also conditionally convergent.

In order to improve the convergence we replace the step function of Eq.(9) for the boundary condition by an exponential ramp function according to Eq.(4.5-2):

$$\begin{aligned}\Psi(0,\theta) = \Psi_1 S(\theta)(1-e^{-\iota\theta}) &= 0 && \text{for } \theta < 0,\ \zeta = 0 \\ &= \Psi_1(1-e^{-\iota\theta}) && \text{for } \theta \geq 0\end{aligned} \tag{60}$$

Again we avoid using the boundary condition

$$\Psi(\infty,\theta) = \text{finite} \tag{61}$$

As initial condition we have in analogy to Eqs.(4.5-4)–(4.5-6):

$$\Psi(\zeta,0) = 0 \tag{62}$$

$$\partial\Psi(\zeta,0)/\partial\zeta = 0 \tag{63}$$

$$\partial\Psi(\zeta,\theta)/\partial\theta = 0 \quad \text{for } \theta = 0,\ \zeta \geq 0 \tag{64}$$

For the solution of Eq.(8) we try the function

$$\Psi(\zeta,\theta) = \Psi_1[u(\zeta,\theta) + (1-e^{-\iota\theta})F(\zeta)] \tag{65}$$

Substitution of $\Psi_1(1-e^{-\iota\theta})F(\zeta)$ into Eq.(8) yields:

$$(1-e^{-\iota\theta})\frac{\partial^2 F}{\partial\zeta^2} + \iota^2 e^{-\iota\theta}F - 2i\lambda_1\left[(1-e^{-\iota\theta})\frac{\partial F}{\partial\zeta} - \lambda_3\iota e^{-\iota\theta}F\right] - \lambda^2(1-e^{-\iota\theta})F = 0 \tag{66}$$

The terms with different functions of θ must vanish separately. Hence, we get two equations:

$$\frac{\partial^2 F}{\partial\zeta^2} - 2i\lambda_1\frac{\partial F}{\partial\theta} - \lambda_2^2 F = 0 \tag{67}$$

$$\iota^2 + 2i\lambda_1\lambda_3\iota = 0 \tag{68}$$

Equation (68) has the trivial solution $\iota = 0$ and a useful solution

$$\iota = -2i\lambda_1\lambda_3 \tag{69}$$

Equation (67) is the same as Eq.(14) and the solution for $F(\zeta)$ shown by Eq.(16) applies again.

Since Eq.(65) must satisfy the boundary condition of Eq.(60) we get with $F(0) = 1$ from Eq.(15):

$$u(0,\theta) + 1 - e^{-\iota\theta} = 1 - e^{-\iota\theta}$$
$$u(0,\theta) = 0 \tag{70}$$

As always we ignore the boundary condition

$$u(\infty,\theta) = \text{finite} \tag{71}$$

derivable from Eq.(61) for $\zeta \to \infty$. The initial conditions of Eqs.(62) and (63) yield with $F(0) = 1$:

$$\Psi(\zeta,0) = \Psi_1 u(\zeta,0) = 0 \tag{72}$$
$$\partial u/\partial\theta + \iota e^{-\iota\theta}F(\zeta) = 0, \quad \partial u/\partial\theta = -\iota F(\zeta) \quad \text{for } \theta = 0,\ \zeta \geq 0 \tag{73}$$

The calculation of $u(\zeta,\theta)$ proceeds as from Eq.(21) on with $w(\zeta,\theta)$ replaced by $u(\zeta,\theta)$ until Eqs.(31) and (32) are reached:

$$u(\zeta,\theta) = \sum_{\kappa=1}^{\infty}\{A_1(\kappa)\exp[i(\lambda_1\lambda_3 + \gamma_0)\theta] + A_2(\kappa)\exp[i(\lambda_1\lambda_3 - \gamma_0)\theta]\} \times e^{i\lambda_1\zeta}\sin\iota_0\zeta \tag{74}$$

$$\frac{\partial u(\zeta,\theta)}{\partial\theta} = \sum_{\kappa=1}^{\infty} i\{A_1(\kappa)(\lambda_1\lambda_3 + \gamma_0)\exp[i(\lambda_1\lambda_3 + \gamma_0)\theta] + A_2(\kappa)(\lambda_1\lambda_3 - \gamma_0)\exp[i(\lambda_1\lambda_3 - \gamma_0)\theta]\}e^{i\lambda_1\zeta}\sin\iota_0\zeta \tag{75}$$

Substitution of $u(\zeta,0)$ and $\partial u/\partial\theta$ from Eqs.(72) and (73) into Eqs.(74) and (75) yields equations for the determination of $A_1(\kappa)$ and $A_2(\kappa)$. The factor $e^{i\lambda_1\zeta}$ is moved to the right side since it does not depend on κ:

$$u(\zeta,0) = \sum_{\kappa=1}^{\infty}[A_1(\kappa) + A_2(\kappa)]\sin\left\{[(2\pi\kappa)^2 + \lambda_1^2]^{1/2}\zeta\right\} = 0 \tag{76}$$

$$\frac{\partial u}{\partial\theta} = \sum_{\kappa=1}^{\infty} i[A_1(\kappa)(\lambda_1\lambda_3 + \gamma_0) + A_2(\kappa)(\lambda_1\lambda_3 - \gamma_0)]\sin\left\{[(2\pi\kappa)^2 + \lambda_1^2]^{1/2}\zeta\right\} = -\iota F(\zeta)e^{-i\lambda_1\zeta} = 2i\lambda_1\lambda_3F(\zeta)e^{-i\lambda_1\zeta} \tag{77}$$

As in the case of Eqs.(35) and (36) our equations do not have quite the form of Eqs.(4.5-18) and (4.5-19). One must use the Fourier series of Eqs.(37)–(39) and the approximation of Eq.(40) to achieve the correct form. In analogy to Eqs.(47)–(50) we obtain:

$$\text{for } \kappa < K \gg \lambda_1/2\pi$$

$$[A_1(\kappa) + A_2(\kappa)]d(\kappa) = 0 \tag{78}$$

$$[A_1(\kappa)(\lambda_1\lambda_3 + \gamma_0) + A_2(\kappa)(\lambda_1\lambda_3 - \gamma_0)]d(\kappa)$$
$$= 2\lambda_1\lambda_3 \int_0^1 F(\zeta)e^{-i\lambda_1\zeta} \sin 2\pi\kappa\zeta \, d\zeta = 2\lambda_1\lambda_3 I_{\mathrm{F}} \tag{79}$$

$$\text{for } \kappa \geq K$$

$$A_1(\kappa) + A_2(\kappa) = 0 \tag{80}$$

$$A_1(\kappa)(\lambda_1\lambda_3 + \gamma_0) + A_2(\kappa(\lambda_1\lambda_3 - \gamma_0) = 2\lambda_1\lambda_3 I_{\mathrm{F}} \tag{81}$$

Equations (78) and (79) as well as Eqs.(80) and (81) are solved for $A_1(\kappa)$ and $A_2(\kappa)$:

$$A_1(\kappa) = -A_2(\kappa) = \frac{\lambda_1\lambda_3 I_{\mathrm{F}}}{\gamma_0 d(\kappa)} = \frac{2\pi\kappa\lambda_1\lambda_3\left[1 - \cos(\lambda_1^2 - \lambda_2^2)^{1/2}\right]}{[(2\pi\kappa)^2 - \lambda_1^2 + \lambda_2^2]\,\gamma_0 d(\kappa)} \quad \text{for } \kappa < K \tag{82}$$

$$A_1(\kappa) = -A_2(\kappa) = \frac{\lambda_1\lambda_3 I_{\mathrm{F}}}{\gamma_0} = \frac{2\pi\kappa\lambda_1\lambda_3\left[1 - \cos(\lambda_1^2 - \lambda_2^2)^{1/2}\right]}{[(2\pi\kappa)^2 - \lambda_1^2 + \lambda_2^2]\,\gamma_0} \quad \text{for } \kappa \geq K \tag{83}$$

Substitution of $A_1(\kappa)$ and $A_2(\kappa)$ into Eq.(74) produces the following relation for $u(\zeta, \theta)$:

$$u(\zeta,\theta) = \sum_{\kappa=1}^{K-1} \frac{4\pi\kappa\lambda_1\lambda_3[1 - \cos(\lambda_1^2 - \lambda_2^2)^{1/2}]}{[(2\pi\kappa)^2 - \lambda_1^2 + \lambda_2^2]\,[(2\pi\kappa)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2]^{1/2}\, d(\kappa)}$$
$$\times e^{i\lambda_1\lambda_3\theta} \sin\left\{[(2\pi\kappa)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2]^{1/2}\theta\right\}$$
$$\times e^{i\lambda_1\zeta} \sin\left\{[(2\pi\kappa)^2 + \lambda_1^2]^{1/2}\zeta\right\}$$
$$+ \sum_{\kappa=K}^{\infty} \frac{4\pi\kappa\lambda_1\lambda_3[1 - \cos(\lambda_1^2 - \lambda_2^2)^{1/2}]}{[(2\pi\kappa)^2 - \lambda_1^2 + \lambda_2^2]\,[(2\pi\kappa)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2]^{1/2}}$$
$$\times e^{i\lambda_1\lambda_3\theta} \sin\left\{[(2\pi\kappa)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2]^{1/2}\theta\right\}$$
$$\times e^{i\lambda_1\zeta} \sin\left\{[(2\pi\kappa)^2 + \lambda_1^2]^{1/2}\zeta\right\} \tag{84}$$

For $\kappa \geq K$ the terms of the second sum vary like $\kappa^{-2}\sin 2\pi\kappa\theta \sin 2\pi\kappa\zeta$ if κ grows beyond all bounds and the sum has absolute convergence.

Equations (65) and (55) with the substitution $A_{13} = -i$ yield the following solution for $\Psi(\zeta,\theta)$:

$$\Psi(\zeta,\theta) = \Psi_1\left\{u(\zeta,\theta) + \left(1 - e^{2i\lambda_1\lambda_3\theta}\right)e^{i\lambda_1\zeta}\cos\left[(\lambda_1^2 - \lambda_2^2)^{1/2}\zeta\right]\right\} \tag{85}$$

Let us observe that the term $m_0^2c^2/e^2$, which shows the presence of a particle with rest mass m_0 and charge e, occurs in λ_2^2 of Eq.(8) and in γ_0 of Eq.(27) via the term λ_2^2. It is through λ_2 and γ_0 that m_0 and e enter Eqs.(82) and (83) for $A_1(\kappa)$ and $A_2(\kappa)$ as well as Eqs.(84) and (85) for $u(\zeta,\theta)$ and $\Psi(\zeta,\theta)$. The term $d(\kappa)$ in Eqs.(82) and (84) does not contain m_0 or e according to Eqs.(42), (41), and (39).

5.3 Hamilton Function for the Planar Klein-Gordon Wave

The energy density of a wave according to the Klein-Gordon equation is defined by the term T_{00} of the energy-impulse tensor[1]:

$$T_{00} = \frac{1}{c^2}\frac{\partial\Psi^*}{\partial t}\frac{\partial\Psi}{\partial t} + \nabla\Psi^*\cdot\nabla\Psi + \frac{m_0^2c^2}{\hbar^2}\Psi^*\Psi \tag{1}$$

Since the dimension of T_{00} is J/m^3 the dimension of $\Psi^*\Psi$ must be J/m or VAs/m if one wants electromagnetic units for the energy. In the case of Eq.(5.2-1) for a planar wave propagating in the direction of y we have $\nabla = \partial/\partial y$.

The Fourier series expansion of Eq.(5.2-84) permits a largest time T and a largest distance cT in the direction of y with the intervals $0 \leq t \leq T$ and $0 \leq y \leq cT$. In the direction of x and y we have not specified any intervals and we shall follow Eq.(4.3-1) to make them $-L/2 \leq x \leq L/2$, $-L/2 \leq z \leq L/2$. The energy U of a Klein-Gordon wave in this interval becomes:

$$U = \int_{-L/2}^{L/2}\int_{-L/2}^{L/2}\left[\int_0^{cT}\left(\frac{1}{c^2}\frac{\partial\Psi^*}{\partial t}\frac{\partial\Psi}{\partial t} + \frac{\partial\Psi^*}{\partial y}\frac{\partial\Psi}{\partial y} + \frac{m_0^2c^2}{\hbar^2}\Psi^*\Psi\right)dy\right]dx\,dz \tag{2}$$

Using the normalized variables

$$t/T = \theta,\ y/cT = \zeta,\ x/cT,\ z/cT \tag{3}$$

we obtain U in the following form:

$$U = cT\int_{-L/2cT}^{L/2cT}\int_{-L/2cT}^{L/2cT}\left[\int_0^1\left(\frac{\partial\Psi^*}{\partial\theta}\frac{\partial\Psi}{\partial\theta} + \frac{\partial\Psi^*}{\partial\zeta}\frac{\partial\Psi}{\partial\zeta} + \frac{m_0^2c^4T^2}{\hbar^2}\Psi^*\Psi\right)d\zeta\right] \times d\left(\frac{x}{cT}\right)d\left(\frac{z}{cT}\right) \tag{4}$$

[1]Berestezki, Lifschitz, Pitajewski 1970, 1982; § 10, Eq.10.13.

We substitute Ψ from Eq.(5.2-85) rather than from Eq.(5.2-13) since $u(\zeta,\theta)$ of Eq.(5.2-84) is simpler and converges faster than $w(\zeta,\theta)$ of Eq.(5.2-59). The function $u(\zeta,\theta)$ of Eq.(5.2-84) is rewritten in a shorter form:

$$u(\zeta,\theta) = e^{i\lambda_1\lambda_3\theta}e^{i\lambda_1\zeta}\sum_{\kappa=1}^{\infty} B(\kappa)\sin\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\} \times \sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}$$

$$\begin{aligned} B(\kappa) &= \frac{4\pi\kappa\lambda_1\lambda_3[\cos(\lambda_1^2-\lambda_2^2)^{1/2}-1]}{[(2\pi\kappa)^2-\lambda_1^2+\lambda_2^2]\,[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\,d(\kappa)} \quad \text{for } 1\le\kappa<K \\ &= \frac{4\pi\kappa\lambda_1\lambda_3[\cos(\lambda_1^2-\lambda_2^2)^{1/2}-1]}{[(2\pi\kappa)^2-\lambda_1^2+\lambda_2^2]\,[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}} \quad \text{for } \kappa\ge K \end{aligned} \tag{5}$$

From Eqs.(5.2-85) and (5) we get:

$$\begin{aligned} \Psi^*\Psi &= \Psi_1^2\{u^*(\zeta,\theta)u(\zeta,\theta)+u^*(\zeta,\theta)(1-e^{2i\lambda_1\lambda_3\theta})e^{i\lambda_1\zeta}\cos[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta] \\ &\quad + u(\zeta,\theta)(1-e^{-2i\lambda_1\lambda_3\theta})e^{-i\lambda_1\zeta}\cos[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta] \\ &\quad + (1-e^{-2i\lambda_1\lambda_3\theta})(1-e^{2i\lambda_1\lambda_3\theta})\cos^2[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]\} \\ &= \Psi_1^2\Bigg[\Bigg(\sum_{\kappa=1}^{\infty} B(\kappa)\sin\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\} \\ &\qquad \times \sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}\Bigg)^2 \\ &\qquad + 2(1-\cos 2\lambda_1\lambda_3\theta)\cos^2[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]\Bigg] \end{aligned} \tag{6}$$

Differentiation of $\Psi(\zeta,\theta)$ of Eq.(5.2-85) with respect to θ or ζ yields:

$$\frac{\partial\Psi}{\partial\theta} = \Psi_1\left(\frac{\partial u}{\partial\theta} - 2i\lambda_1\lambda_3 e^{2i\lambda_1\lambda_3\theta}e^{i\lambda_1\zeta}\cos[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]\right) \tag{7}$$

$$\begin{aligned} \frac{\partial u}{\partial\theta} &= e^{i\lambda_1\lambda_3\theta}e^{i\lambda_1\zeta}\sum_{\kappa=1}^{\infty} B(\kappa)\sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\} \\ &\quad \times \Big(i\lambda_1\lambda_3\sin\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\} \\ &\quad + [(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\cos\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}\Big) \end{aligned} \tag{8}$$

$$\begin{aligned} \frac{\partial\Psi}{\partial\zeta} &= \Psi_1\Big(\frac{\partial u}{\partial\zeta} + (1-e^{2i\lambda_1\lambda_3\theta})e^{i\lambda_1\zeta}\{i\lambda_1\cos[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta] \\ &\quad - (\lambda_1^2-\lambda_2^2)^{1/2}\sin[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]\}\Big) \end{aligned} \tag{9}$$

$$\frac{\partial u}{\partial \zeta} = e^{i\lambda_1\lambda_3\theta}e^{i\lambda_1\zeta}\sum_{\kappa=1}^{\infty} B(\kappa)\sin\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}$$
$$\times\left(i\lambda_1\sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}\right.$$
$$\left.+[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\cos\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}\right) \quad (10)$$

We may now write the first and second term in Eq.(4):

$$\frac{\partial\Psi^*}{\partial\theta}\frac{\partial\Psi}{\partial\theta} = \Psi_1^2\left(\frac{\partial u^*}{\partial\theta}\frac{\partial u}{\partial\theta} - 2i\lambda_1\lambda_3\frac{\partial u^*}{\partial\theta}e^{2i\lambda_1\lambda_3\theta}e^{i\lambda_1\zeta}\cos[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]\right.$$
$$+2i\lambda_1\lambda_3\frac{\partial u}{\partial\theta}e^{-2i\lambda_1\lambda_3\theta}e^{-i\lambda_1\zeta}\cos[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]$$
$$\left.+4\lambda_1^2\lambda_3^2\cos^2[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]\right)$$
$$=\Psi_1^2\left[\left(\sum_{\kappa=1}^{\infty}B(\kappa)\lambda_1\lambda_3\sin\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}\right.\right.$$
$$\left.\times\sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}\right)^2$$
$$+\left(\sum_{\kappa=1}^{\infty}B(\kappa)[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\cos\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}\right.$$
$$\left.\times\sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}\right)^2$$
$$-4\lambda_1\lambda_3\cos[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]\sum_{\kappa=1}^{\infty}B(\kappa)\sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}$$
$$\times\left(\lambda_1\lambda_3\cos\lambda_1\lambda_3\theta\sin\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}\right.$$
$$\left.-[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\sin\lambda_1\lambda_3\theta\cos\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}\right)$$
$$\left.+4\lambda_1^2\lambda_3^2\cos^2[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]\right] \quad (11)$$

The second term is even longer:

$$
\begin{aligned}
\frac{\partial\Psi^*}{\partial\zeta}\frac{\partial\Psi}{\partial\zeta} = \Psi_1^2\bigg(&\frac{\partial u^*}{\partial\zeta}\frac{\partial u}{\partial\zeta} + \frac{\partial u^*}{\partial\zeta}(1-e^{2i\lambda_1\lambda_3\theta})e^{i\lambda_1\zeta} \\
&\times\left\{i\lambda_1\cos[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta] - (\lambda_1^2-\lambda_2^2)^{1/2}\sin[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]\right\} \\
&+\frac{\partial u}{\partial\zeta}(1-e^{-2i\lambda_1\lambda_3\theta})e^{-i\lambda_1\zeta} \\
&\times\left\{-i\lambda_1\cos[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta] - (\lambda_1^2-\lambda_2^2)^{1/2}\sin[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]\right\} \\
&+(1-e^{-2i\lambda_1\lambda_3\theta})e^{-i\lambda_1\zeta} \\
&\times\left\{-i\lambda_1\cos[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta] - (\lambda_1^2-\lambda_2^2)^{1/2}\sin[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]\right\} \\
&+(1-e^{-2i\lambda_1\lambda_3\theta})(1-e^{2i\lambda_1\lambda_3\theta}\{\lambda_1^2\cos^2[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta] \\
&+(\lambda_1^2-\lambda_2^2)\sin^2[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]\}\bigg) \\
= \Psi_1^2\Bigg[&\bigg(\sum_{\kappa=1}^{\infty}B(\kappa)\lambda_1\sin\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\} \\
&\times\sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}\bigg)^2 \\
&+\bigg(\sum_{\kappa=1}^{\infty}B(\kappa)[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\sin\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\} \\
&\times\cos\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}\bigg)^2 \\
&+4\lambda_1\sin\lambda_1\lambda_3\theta\sum_{\kappa=1}^{\infty}B(\kappa)\sin\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\} \\
&\times\Big((\lambda_1^2-\lambda_2^2)^{1/2}\sin[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]\sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\} \\
&+[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\cos[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]\cos\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}\Big) \\
&+2(1-\cos 2\lambda_1\lambda_3\theta)\left\{\lambda_1^2-\lambda_2^2\sin^2[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]\right\}
\end{aligned}
\tag{12}
$$

The next step is the integration of $(\partial\Psi^*/\partial\theta)(\partial\Psi/\partial\theta)$, $(\partial\Psi^*/\partial\zeta)(\partial\Psi/\partial\zeta)$, and $\Psi^*\Psi$ of Eqs.(11), (12), and (6) with respect to ζ according to Eq.(4). This requires straight forward but lengthy calculations that will be found in Section 6.11. Equations (6.11-23), (6.11-33), and (6.11-17) are obtained. Substitution of these equations into Eq.(4) yields the energy U. It is shown in Section 6.11 that the energy U consists of a constant term U_c, defined by Eq.(6.11-44), and a time-variable term U_v with a time average equal to zero, defined by Eq.(6.11-54). We use Eq.(6.11-44)

$$U_c = \frac{L^2\Psi_1^2}{2cT}\sum_{\kappa=0}^{\infty}D(\kappa) \tag{13}$$

where $D(\kappa)$ is defined for $\kappa = 0, 1, 2, \ldots$ by Eqs.(6.11-55) and (6.11-56). The factor Ψ_1^2 is the same as in Eq.(6), having the dimension J/m=VAs/m.

Let us consider the constant energy U_c. Equation (13) corresponds to Eq.(4.3-33) for the pure radiation field. We follow the procedure developed from Eq.(4.3-33) on.

For the derivation of the Hamilton function $\mathcal{H}$ we first normalize the energy U_c in Eq.(13):

$$2cTU_c/L^2\Psi_1^2 = \mathcal{H} \tag{14}$$

$$\mathcal{H} = \sum_{\kappa=0}^{\infty} \mathcal{H}_\kappa = \sum_{\kappa=0}^{\infty} D(\kappa) \tag{15}$$

The component $\mathcal{H}_\kappa$ is rewritten in analogy to Eq.(4.3-36):

$$\begin{aligned}\mathcal{H}_\kappa &= (2\pi\kappa)^2 \frac{\sqrt{D(\kappa)}}{2\pi\kappa}(\sin 2\pi\kappa\theta - i\cos 2\pi\kappa\theta)\frac{\sqrt{D(\kappa)}}{2\pi\kappa}(\sin 2\pi\kappa\theta + i\cos 2\pi\kappa\theta)\\ &= -2\pi i\kappa p_\kappa(\theta)q_\kappa(\theta)\end{aligned} \tag{16}$$

$$p_\kappa(\theta) = \sqrt{2\pi i\kappa}\frac{\sqrt{D(\kappa)}}{2\pi\kappa}e^{2\pi i\kappa\theta} \tag{17}$$

$$\dot{p}_\kappa = \frac{\partial p_\kappa}{\partial\theta} = (2\pi i\kappa)^{3/2}\frac{\sqrt{D(\kappa)}}{2\pi\kappa}e^{2\pi i\kappa\theta} = 2\pi i\kappa p_\kappa(\theta) \tag{18}$$

$$q_\kappa(\theta) = \sqrt{2\pi i\kappa}\frac{\sqrt{D(\kappa)}}{2\pi\kappa}e^{-2\pi i\kappa\theta} \tag{19}$$

$$\dot{q}_\kappa = \frac{\partial q_\kappa}{\partial\theta} = -(2\pi i\kappa)^{3/2}\frac{\sqrt{D(\kappa)}}{2\pi\kappa}e^{-2\pi i\kappa\theta} = -2\pi i\kappa q_\kappa(\theta) \tag{20}$$

The derivatives $\partial\mathcal{H}_\kappa/\partial q_\kappa$ and $\partial\mathcal{H}_\kappa/\partial p_\kappa$ equal:

$$\frac{\partial\mathcal{H}_\kappa}{\partial q_\kappa} = -2\pi i\kappa p_\kappa = -\dot{p}_\kappa \tag{21}$$

$$\frac{\partial\mathcal{H}_\kappa}{\partial p_\kappa} = -2\pi i\kappa q_\kappa = +\dot{q}_\kappa \tag{22}$$

These are the proper relations for the components $\mathcal{H}_\kappa$ of the Hamilton function of Eq.(16).

Equation (15) may be rewritten in analogy to Eq.(3.4-61) by means of the definitions

$$a_\kappa = \frac{\sqrt{D(\kappa)}}{2\pi\kappa}e^{2\pi i\kappa\theta}, \quad a_\kappa^* = \frac{\sqrt{D(\kappa)}}{2\pi\kappa}e^{-2\pi i\kappa\theta} \tag{23}$$

to yield:

$$\mathcal{H} = -i\sum_{\kappa=0}^{\infty} 2\pi\kappa p_\kappa(\theta) q_\kappa(\theta) = \sum_{\kappa=0}^{\infty} (2\pi\kappa)^2 a_\kappa a_\kappa^* = \frac{\hbar}{T}\sum_{\kappa=0}^{\infty} 2\pi\kappa b_\kappa b_\kappa^*$$

$$b_\kappa = \left(\frac{2\pi\kappa T}{\hbar}\right)^{1/2} a_\kappa, \quad b_\kappa^* = \left(\frac{2\pi\kappa T}{\hbar}\right)^{1/2} a_\kappa^* \tag{24}$$

We check whether the energy U_c of Eq.(13) is finite to avoid violating the conservation law of energy and introducing divergencies. An infinite amplitude of Ψ_1 in Eq.(5.2-60) or an infinite excitation area L^2 according to Eq.(2) would make U_c infinite, but this is of no interest. The time T in Eq.(13) is finite. We have to show that the sum of $D(\kappa)$ is finite, which means that $D(\kappa)$ must decrease sufficiently fast for $\kappa \to \infty$ to yield a convergent series for U_c and $\mathcal{H}$.

Equation (6.11-55) yields for large values of κ the following relation for $D(\kappa)$:

$$\mathcal{H}_\kappa = D(\kappa) \approx (2\pi\kappa)^2 B^2(\kappa), \quad \kappa \gg 1 \tag{25}$$

From Eq.(5) we get for $\kappa \geq K$

$$B(\kappa) \approx \frac{2\lambda_1\lambda_3[\cos(\lambda_1^2 - \lambda_2^2)^{1/2} - 1]}{(2\pi\kappa)^2}, \quad \kappa \gg 1 \tag{26}$$

and $D(\kappa)$ becomes:

$$\mathcal{H}_\kappa = D(\kappa) \approx \frac{4\lambda_1^2\lambda_3^2[\cos(\lambda_1^2 - \lambda_2^2)^{1/2} - 1]^2}{(2\pi\kappa)^2}, \quad \kappa \gg 1 \tag{27}$$

Hence, the terms of $\mathcal{H}$ in Eq.(15) decrease as fast as in Eq.(4.3-46) and the sum is convergent.

5.4 Quantization of the Planar Klein-Gordon Wave

Following Section 4.4 we start with the Hamilton function $\mathcal{H}$ of Eq.(5.3-24) using the functions b_κ and b_κ^*:

$$\mathcal{H} = \sum_{\kappa=0}^{\infty} \mathcal{H}_\kappa = \frac{\hbar}{T}\sum_{\kappa=0}^{\infty} 2\pi\kappa b_\kappa b_\kappa^* \tag{1}$$

We follow the conventional procedure for quantization and replace the conjugate complex functions b_κ and b_κ^* by the operators b_κ^+ and b_κ^-:

$$b_\kappa^* \to b_\kappa^+, \quad b_\kappa \to b_\kappa^- \tag{2}$$

$$b_\kappa^* \to b_\kappa^-, \quad b_\kappa \to b_\kappa^+ \tag{3}$$

We have discussed the known arbitrariness of these two choices in Section 3.5 following Eq.(3.5-6). As before we use Eq.(2) first. For a particular value of κ in Eq.(1) we obtain:

$$b_\kappa^+ b_\kappa^- = \frac{\mathcal{H}_\kappa T}{2\pi\kappa\hbar} \equiv \frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar} \tag{4}$$

$$b_\kappa^- = \frac{1}{\sqrt{2}}\left(\alpha\zeta + \frac{1}{\alpha}\frac{d}{d\zeta}\right), \quad b_\kappa^+ = \frac{1}{\sqrt{2}}\left(\alpha\zeta - \frac{1}{\alpha}\frac{d}{d\zeta}\right) \tag{5}$$

Application of the operators b_κ^- and b_κ^+ to a function Φ yields:

$$\frac{1}{\sqrt{2}}\left(\alpha\zeta + \frac{1}{\alpha}\frac{d}{d\zeta}\right)\left[\frac{1}{\sqrt{2}}\left(\alpha\zeta - \frac{1}{\alpha}\frac{d}{d\zeta}\right)\Phi\right] = \frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar}\Phi$$

$$\left(\alpha^2\zeta^2 - \frac{1}{\alpha^2}\frac{d^2}{d\zeta^2}\right)\Phi = 2\left(\frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar} - \frac{1}{2}\right)\Phi = 2\lambda_\kappa\Phi \tag{6}$$

$$\lambda_\kappa = \frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar} - \frac{1}{2} \tag{7}$$

Using the solution of Eq.(3.5-18) we may write E_κ in the following form:

$$\mathsf{E}_\kappa = \mathsf{E}_{\kappa n} = \frac{2\pi\kappa\hbar}{T}\left(n + \frac{1}{2}\right), \quad n = 0,\ 1,\ 2,\ \ldots \tag{8}$$

The energy $\mathsf{E}_{\kappa n}$ increases with κ beyond all bounds for any value of n. But the total energy $\mathcal{H}_\kappa$ must decrease according to Eq.(5.3-27) like $1/\kappa^2$ for large values of κ. Hence, the number of particles with energy $\mathsf{E}_{\kappa n}$ according to Eq.(8) must decrease like $1/\kappa^3$ to make the total energy $\mathcal{H}_\kappa$ decrease like $1/\kappa^2$. As in the case of the pure radiation field of Section 4.4 there is no divergency and thus no need for renormalization.

We note that our result is very general since we only needed the functions b_κ and b_κ^* to derive it. If we had to use the functions a_κ and a_κ^* of Eqs.(5.3-24) and (4.3-45) we would have to use the different definitions of Eqs.(5.3-23) and (4.3-44) for a_κ and a_κ^* which would have produced different results. This equality does not carry over to the total energy $\mathcal{H}_\kappa$ since Eqs.(5.3-27) and (4.3-46) are quite different. Indeed, Eq.(4.3-46) holds for a step function excitation while Eq.(5.3-27) holds for excitation by an exponential ramp function. If we had used the solution $w(\zeta, \theta)$ of Eq.(5.2-59) holding for a step function excitation rather than $u(\zeta, \theta)$ of Eq.(5.2-84) for exponential ramp function excitation we would not have obtained a decrease like $1/\kappa^2$ for $\mathcal{H}_\kappa$ and we would have ended up with a divergency. It is not surprising that the presence of a mass m_0 excludes the sudden excitation of a step function but permits the continuously increasing excitation of the exponential ramp function.

The energies $\mathsf{E}_{\kappa n}$ in Eqs.(4.4-8) and (8) are the same since the photons of a pure EM wave are the same as that of an EM wave interacting with bosons.

What differs is the fraction of photons with one of the energies $\mathsf{E}_{\kappa n}$. These fractions depend, in a way that is too complicated to evaluate here, on the constants $A_1(\kappa)$ and $A_2(\kappa)$ in Eq.(4.1-86) or (4.5-24) for the pure EM wave while they depend on $A_1(\kappa)$ and $A_2(\kappa)$ in Eqs.(5.2-82), (5.2-83) for the EM wave interacting with bosons. The parameters λ_1 and γ_0 in Eqs.(5.2-82), (5.2-83) contain the term $m_0^2c^2/e^2$ according to Eqs.(5.2-8) and (5.2-27) with the rest mass m_0 as well as the electric charge e of the bosons. On the other hand, Eqs.(4.1-86) or (4.5-24) do not contain any rest mass or electric charge. This shows how the analysis of the pure EM wave and the EM wave interacting with bosons leads to different results even though the possible energies of the photons are equal in both cases.

A comparison of Eq.(8) with Eq.(4.4-9), $\mathsf{E}_{\mathrm{m,k}\lambda} = 2\pi f\hbar(n+1/2)$, shows that the frequency f of the conventional theory is replaced by κ/T. The use of the concept of frequency of a sinusoidal wave usually creates problems since it is defined for an indefinitely extended sinusoidal wave with infinitely many periods, which is outside the framework of an experimental science. If one truncates such a wave, its Fourier transform contains frequencies of the whole band $0 \leq f < \infty$ and one must explain which of the frequencies is to be used in the product $2\pi f\hbar$. No such explanation is needed in Eq.(8) since $\kappa = 1$, 2, ... is an integer number and T can be chosen, provided it is large enough and finite. But another fine detail occurs. If we have a certain value κ/T and increase T by an integer factor $m = 1, 2, \ldots$ we must increase κ by the same factor to get the old result $\kappa/T = m\kappa/mT$. However, this is not so if m is not an integer since $m\kappa$ may not be an integer in this case. Hence, the choice of T can have an effect on κ/T.

We have here considered the constant energy U_c of Eq.(5.3-13) only but mentioned in the text before Eq.(5.3-13) that there is also a variable energy U_v with time average equal to zero. Such a variable energy with time average zero also occurred for the pure EM wave excited by a rectangular pulse of finite duration in Section 4.3. This suggests that such variable energies are a significant part of the theory that is not likely to be eliminated by improved mathematical methods. We do not want to offer a physical explanation at this time going beyond the text following Eq.(4.3-32) since such explanations have a tendency to cause more harm than good if advocated too early.

We turn to the substitution of Eq.(3) which is usually ignored in order to avoid an infinite negative energy. In analogy to Eqs.(4), (6), and (7) we get:

$$b_\kappa^+ b_\kappa^- = \frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar} \tag{9}$$

$$\frac{1}{\sqrt{2}}\left(\alpha\zeta - \frac{1}{\alpha}\frac{d}{d\zeta}\right)\left[\frac{1}{\sqrt{2}}\left(\alpha\zeta + \frac{1}{\alpha}\frac{d}{d\zeta}\right)\Phi\right] = \frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar}\Phi$$

$$\left(\alpha^2\zeta^2 - \frac{1}{\alpha^2}\frac{d^2}{d\zeta^2}\right)\Phi = 2\left(\frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar} + \frac{1}{2}\right)\Phi = 2\lambda_\kappa\Phi \tag{10}$$

$$\lambda_\kappa = \frac{\mathsf{E}_\kappa T}{2\pi\kappa\hbar} + \frac{1}{2} \tag{11}$$

$$\mathsf{E}_\kappa = \mathsf{E}_{\kappa n} = \frac{2\pi\kappa\hbar}{T}\left(n - \frac{1}{2}\right), \quad n = 0,\ 1,\ 2,\ \ldots \tag{12}$$

For $n = 0$ we obtain a negative but finite energy. We avoid any explanation of this result as premature at this stage of the theory. But there is no obvious reason why Eq.(12) should be ignored.

5.5 Dipole Current Conductivities in Vacuum

Observable constants that characterize the vacuum are important to improve our understanding of the concept of *physical vacuum*. Initially the concept of vacuum seems to have come from Greek philosophers who associated *matter* with 'something' and *vacuum* with 'nothing'. This was strictly abstract thinking without any connection to observation. In physics the vacuum represents the absence of matter but this is quite different from 'nothing'. Electromagnetic waves are known to propagate through vacuum, so there is something at least some of the time. Mathematicians have freely assigned features to the physical vacuum: It is a continuum, homogeneous, isotropic, limitless, has n dimensions, etc. The concept of the continuum predates Aristotle (1930, Apostle 1969), the concept of limitless but finite is due to Riemann (1854), and the concept of n dimensions with flexible n grew out of the work of Bolyai (1832) and Lobachevskii (1840, 1856) on Euclid's parallel axiom.

These mathematical features do not mean more for the physical vacuum than the abstract claims of Greek philosophers. To prove the existence of a continuum one has to make an observation at the locations x and $x + dx$, and in addition at the times t and $t + dt$ if a space-time continuum is postulated. Nobody expects that such observations can actually be made. Apart from the infinitesimal distance dx and time interval dt one would need to assume that infinite information is represented by a finite distance ΔX or time interval ΔT. Infinite information is no more acceptable in a science based on observation than infinite energy. Neither can be produced or observed. If we can only observe at x and $x + \Delta x$ or t and $t + \Delta t$ the continuum disappears and is replaced by coordinate systems (Harmuth 1989, 1992). Limitless coordinate systems with n dimensions can readily be constructed by us not only in our mind or on paper but in reality. But these coordinate systems are evidently constructed by us, which makes them different from the continuum that mathematicians postulate to be provided by nature.

To distinguish between inventions provided by humans and the vacuum provided by nature one needs observable natural constants that characterize the vacuum. The velocity c of light is such a constant. But it is the only one within electromagnetic theory, since the permeability $\mu = 4\pi \times 10^{-7}$ [Vs/Am] is defined and the permittivity $\epsilon = 1/c^2\mu$ as well as the wave impedance $Z = \mu c$ are derived from μ and c. The parameters τ_{mp}, τ_{p}, and σ_{p} used in Eq.(2.1-19) should be fundamental physical constants of the electromagnetic field in the absence of matter like the velocity c of light. Let us see how one could obtain values for the constants of electric dipole currents in vacuum.

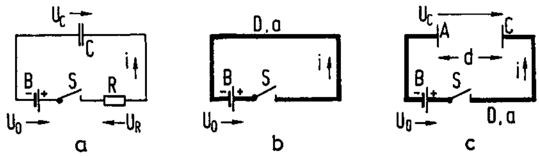

FIG.5.5-1. Circuit with resistor R, capacitor C, switch S, and battery B (a). The same circuit without C and R but with a conductor of length D and cross section a (b). A plate capacitor C with area A and distance d between the plates is added (c).

We have already shown the principle of circuits for the measurement of τ_{mp}, τ_{p}, σ_{p}, and s_{p} in Figs.2.1-10 and 2.2-7, but we want to add some details. Figure 5.5-1 shows a battery B with constant voltage U_0, a switch S, a resistor R, and a capacitor C. If the switch S is closed at the time $t = 0$ we obtain the following relation between U_0 and the current $i(t)$:

$$U_0 = iR + \frac{1}{C}\int_0^t i(t')\,dt' \tag{1}$$

The current and voltages according to Fig.5.5-1a become:

$$\begin{aligned} i &= \frac{U_0}{R}e^{-t/\tau}, \quad \tau = RC \\ U_{\mathrm{C}} &= U_0(1 - e^{-t/\tau}) \\ U_{\mathrm{R}} &= U_0 e^{-t/\tau} \end{aligned} \tag{2}$$

We ignore the term $\tau_{\mathrm{mp}} d\mathbf{g}_{\mathrm{e}}/dt$ of the monopole current in Eq.(2.1-13) and consider a conductor with length D as well as a cross section a. Since $\mathbf{g}_{\mathrm{e}}$ and $\mathbf{E}$ have the same direction we may use their magnitudes and we obtain:

$$\begin{aligned} ag_{\mathrm{e}} = i = \sigma a E = \sigma\frac{a}{D}U = \frac{U}{R} \\ R = \frac{D}{\sigma a} \end{aligned} \tag{3}$$

where U is the voltage drop over the length L of the conductor. The term iR in Eq.(1) is an approximation of Eq.(2.1-13).

We ignore now the terms $\mathbf{g}_{\mathrm{e}}$ and $\tau_{\mathrm{mp}} d\mathbf{g}_{\mathrm{e}}/dt$ in Eq.(2.1-19). Consider a plate capacitor with area A and the distance d between the plates. One may again write g_{e} and E for $\mathbf{g}_{\mathrm{e}}$ and $\mathbf{E}$:

$$A\frac{\tau_{\mathrm{mp}}}{\tau_{\mathrm{p}}^2}\int g_{\mathrm{e}}\,dt = \frac{\tau_{\mathrm{mp}}}{\tau_{\mathrm{p}}^2}\int i\,dt = \sigma_{\mathrm{p}}AE = \frac{\sigma_{\mathrm{p}}A}{d}U_{\mathrm{C}} \tag{4}$$

$$U_{\mathrm{C}} = \frac{d\tau_{\mathrm{mp}}}{\sigma_{\mathrm{p}}A\tau_{\mathrm{p}}^2}\int i\,dt = \frac{1}{C}\int i\,dt$$

$$C = \epsilon\frac{A}{d} = \frac{A\sigma_{\mathrm{p}}\tau_{\mathrm{p}}^2}{d\tau_{\mathrm{mp}}}$$

$$\frac{\sigma_{\mathrm{p}}\tau_{\mathrm{p}}^2}{\tau_{\mathrm{mp}}} = \epsilon \tag{5}$$

The term $C^{-1}\int i\,dt$ in Eq.(1) is an approximation of Eq.(2.1-19).

A measurement of U and i yields the conductivity σ in Eq.(3) for known values of the length L and the cross section a of the conductor. Equation (5) yields a value for $\sigma_{\mathrm{p}}\tau_{\mathrm{p}}^2/\tau_{\mathrm{mp}}$ that is independent of A or d, which is a necessity for a quantity that characterizes vacuum rather than a capacitor.

We turn to Fig.5.5-1b that shows a voltage U_0 drive a current i through a conductor of length D and cross section a. With $E_0 = U_0/D$ we get from Eq.(2.1-14):

$$ag_{\mathrm{e}}(t) = i(t) = \frac{a\sigma}{D}U_0\left(1 - e^{-t/\tau_{\mathrm{mp}}}\right) \tag{6}$$

Becker (1964a, 1964b, vol. 1, § 58) gives the value $\tau_{\mathrm{mp}} = 2.4 \times 10^{-14}$ s for copper, but no value for τ_{mp} was found for vacuum. The time resolution of the fastest sampling oscilloscopes for either periodic or single events is about 10 ps=10^{-11} s. If τ_{mp} for vacuum is anywhere close to the value for copper, the direct observation of plots like those of Fig.2.1-5 is beyond our current means but without a study of dipole currents we would not know why an integral is written in Eq.(1). Light would propagate in 2.4×10^{-14} s the distance $c\tau_{\mathrm{mp}} = 7.2 \times 10^{-6}$ m or 7.2 μm. The circuits of Fig.5.5-1 would thus have to have dimensions of the order of micrometer to make delay effects negligible. The technology of integrated circuits and chemical machining make this possible, but it is at the frontier of the current technology.

Consider the circuit of Fig.5.5-1c. The current $i = ag_{\mathrm{e}}$ in the conductor of length D and cross section a follows from Eq.(2.1-13):

$$i + \tau_{\mathrm{mp}}\frac{di}{dt} = \sigma a\frac{U_0 - U_{\mathrm{C}}}{D} \tag{7}$$

The current $i = Ag_{\mathrm{e}}$ in the plate capacitor with area A and distance d between the plates follows from Eq.(2.1-19):

$$i + \tau_{\mathrm{mp}}\frac{di}{dt} + \frac{\tau_{\mathrm{mp}}}{\tau_{\mathrm{p}}^2}\int i\,dt = \sigma_{\mathrm{p}}A\frac{U_{\mathrm{C}}}{d} \tag{8}$$

Elimination of U_{C} from Eqs.(7) and (8) yields:

$$\left(1+\frac{\sigma ad}{\sigma_{\mathrm{p}}AD}\right)i+\tau_{\mathrm{mp}}\left(1+\frac{\sigma ad}{\sigma_{\mathrm{p}}AD}\right)\frac{di}{dt}+\frac{\tau_{\mathrm{mp}}}{\tau_{\mathrm{p}}^{2}}\frac{\sigma ad}{\sigma_{\mathrm{p}}AD}\int i\,dt=\frac{\sigma a}{D}U_{0} \tag{9}$$

The solution of the homogeneous equation equals

$$i=I_{0}e^{-t/\tau}$$

$$\tau_{1,2}=\tau_{\mathrm{p}}\left\{\frac{1}{2}\frac{\tau_{\mathrm{p}}}{\tau_{\mathrm{mp}}}\left(\frac{\sigma_{\mathrm{p}}AD}{\sigma ad}+1\right)\pm\left[\frac{1}{4}\frac{\tau_{\mathrm{p}}^{2}}{\tau_{\mathrm{mp}}^{2}}\left(\frac{\sigma_{\mathrm{p}}AD}{\sigma ad}+1\right)^{2}-\frac{\sigma_{\mathrm{p}}AD}{\sigma ad}\right]^{1/2}\right\} \tag{10}$$

where τ_1 and τ_2 may be real or conjugated complex. A particular solution of the inhomogeneous Eq.(9)

$$\int i\,dt=\frac{\tau_{\mathrm{p}}^{2}}{\tau_{\mathrm{mp}}}\frac{\sigma_{\mathrm{p}}A}{d}U_{0} \tag{11}$$

yields the general solution

$$\int i\,dt=-\tau_{1}I_{1}e^{-t/\tau_{1}}-\tau_{2}I_{2}e^{-t/\tau_{2}}+\frac{\tau_{\mathrm{p}}^{2}}{\tau_{\mathrm{mp}}}\frac{\sigma_{\mathrm{p}}A}{d}U_{0} \tag{12}$$

With the initial conditions $i=0$ and $\int i\,dt=0$ for $t=0$ one obtains:

$$\int i\,dt=\frac{\tau_{\mathrm{p}}^{2}\sigma_{\mathrm{p}}A}{\tau_{\mathrm{mp}}d}U_{0}\left(1-\frac{\tau_{1}}{\tau_{1}-\tau_{2}}e^{-t/\tau_{1}}+\frac{\tau_{2}}{\tau_{1}-\tau_{2}}e^{-t/\tau_{2}}\right) \tag{13}$$

$$i=\frac{\sigma_{\mathrm{p}}A}{d}\frac{\tau_{\mathrm{p}}^{2}}{\tau_{\mathrm{mp}}(\tau_{1}-\tau_{2})}U_{0}\left(e^{-t/\tau_{1}}-e^{-t/\tau_{2}}\right) \tag{14}$$

For $t\gg\tau_1$ and $t\gg\tau_2$ we get from Eq.(13):

$$U_{0}=\frac{d\tau_{\mathrm{mp}}}{\sigma_{\mathrm{p}}A\tau_{\mathrm{p}}^{2}}\int i\,dt=\frac{1}{C}\int i\,dt \tag{15}$$

which leads back to Eq.(5). If we rewrite Eq.(5)

$$\sigma_{\mathrm{p}}=\epsilon\tau_{\mathrm{mp}}/\tau_{\mathrm{p}}^{2} \tag{16}$$

and substitute σ_{p} into Eqs.(10) as well as (14) we know the values of all the parameters except τ_{p}. In principle one could make many plots of Eq.(14) for various values of τ_{p} and observe which plot corresponds best to an experimentally obtained plot of $i(t)$ to determine τ_{p}. This will not be possible in the foreseeable future and a more practical method to obtain either σ_{p} or τ_{p} must be found.

Consider the replacement of the constant voltage U_0 in Fig.5.5-1 and Eq.(9) by a sinusoidal voltage:

$$v(t) = U_0 \sin \omega t \tag{17}$$

Equation (9) is rewritten:

$$(1+\delta)i + (1+\delta)\tau_{mp}\frac{di}{dt} + \frac{\tau_{mp}\delta}{\tau_p^2}\int i\,dt = \frac{\sigma a}{D}U_0 \sin \omega t$$

$$\delta = \frac{\sigma a d}{\sigma_p A D} \tag{18}$$

A particular solution of Eq.(18) may be written in the form

$$i(t) = I_1 \sin \omega t + I_2 \cos \omega t = I_0 \sin(\omega t + \varphi) \tag{19}$$

and the following two equations are obtained:

$$\left[(1+\delta)I_2 + (1+\delta)\tau_{mp}\omega I_1 - \frac{\tau_{mp}\delta}{\tau_p^2\omega}I_1\right]\cos \omega t = 0 \tag{20}$$

$$\left[(1+\delta)I_1 - (1+\delta)\tau_{mp}\omega I_2 + \frac{\tau_{mp}\delta}{\tau_p^2\omega}I_2\right]\sin \omega t = \frac{\sigma a}{D}U_0 \sin \omega t \tag{21}$$

From Eq.(20) we obtain the ratio

$$\frac{I_2}{I_1} = \frac{\tau_{mp}}{\tau_p^2\omega}\frac{\delta}{1+\delta} - \tau_{mp}\omega \tag{22}$$

Equation (19) yields a relation for φ:

$$\varphi = \operatorname{arctg}\frac{I_2}{I_1} = \operatorname{arctg}\left(\frac{\tau_{mp}}{\tau_p^2\omega}\frac{\delta}{1+\delta} - \tau_{mp}\omega\right) \tag{23}$$

The observation of the phase shift φ between the voltage $v(t)$ and the current $i(t)$ at various circular frequencies ω permits one to obtain τ_{mp}, τ_p, and σ_p. Since the technology for sinusoidal waves with very high frequencies is better developed than the technology of step or rectangular waves with short switching times, we see here a more promising approach for the measurement of the constants τ_{mp}, τ_p, and σ_p for vacuum.

The extension of these results from electric to magnetic dipole currents has not yet yielded any satisfactory results. One reason is the occurrence of the sinusoidal function $\sin\vartheta$ in Eq.(2.2-10) that shows up in variations all the way to Eq.(2.2-22). A second reason is that we can measure electric monopole currents directly but need a transducer to measure magnetic dipole currents. For electric dipole currents the problem of a transducer is solved by the dipole currents between the plates of a capacitor becoming monopole currents in the plates and their connecting leads.

6 Appendix

6.1 Electric Field Strength Due to Electric Step Function

We start with Eq.(1.3-1) and rewrite it in normalized form with the definitions of θ and ζ shown in Eq.(1.3-7):

$$\partial^2 E/\partial\zeta^2 - \partial^2 E/\partial\theta^2 - 2(1+\omega^2)\partial E/\partial\theta - 4\omega^2 E = 0$$
$$\omega^2 = \epsilon s/\mu\sigma,\ \theta = \sigma t/2\epsilon,\ \zeta = Z\sigma y/2,\ Z = \sqrt{\mu/\epsilon} \tag{1}$$

An electric force function with the time variation of a step function is introduced as boundary condition at the plane $\zeta = 0$:

$$\begin{aligned} E(0,\theta) = E_0 S(\theta) &= 0 \quad \text{for } \theta < 0 \\ &= E_0 \quad \text{for } \theta \geq 0 \end{aligned} \tag{2}$$

At a great distance $\zeta \gg 1$ or $\zeta \to \infty$ we have a further boundary condition:

$$E(\infty,\theta) = \text{finite} \tag{3}$$

Let E and H be zero for $\zeta > 0$ at the time $\theta = 0$. This yields the following initial conditions[1]:

$$E(\zeta,0) = 0,\ H(\zeta,0) = 0 \tag{4}$$

If $E(\zeta,0)$ and $H(\zeta,0)$ are zero for all values $\zeta > 0$, their derivatives with respect to ζ must be zero too:

$$\partial E(\zeta,0)/\partial\zeta = 0,\ \partial H(\zeta,0)/\partial\zeta = 0 \tag{5}$$

Equations (4) and (5) also imply the initial conditions

$$\partial E(\zeta,\theta)/\partial\theta = 0,\ \partial H(\zeta,\theta)/\partial\theta = 0 \tag{6}$$

for $\zeta > 0$ and $\theta = 0$ according to Eqs.(1.2-9) and (1.2-10).

[1]Even though we require $E(\zeta,\theta) = H(\zeta,\theta) = 0$ for $\theta = 0$ only, the implication is that $E(\zeta,\theta)$ and $H(\zeta,\theta)$ are zero for $\theta < 0$, since $E(0,\theta)$ equals zero for $\theta < 0$ according to Eq.(2).

We assume that the solution of Eq.(1) can be written as a sum of a steady state solution $F(\zeta)$ plus the deviation $w(\zeta, \theta)$ from it[2]

$$E(\zeta,\theta) = E_{\mathrm{E}}(\zeta,\theta) = E_0[F(\zeta) + w(\zeta,\theta)] \tag{7}$$

Substitution of $F(\zeta)$ into Eq.(1) yields the equation[3]

$$\partial^2 F/\partial\zeta^2 - 4\omega^2 F = 0 \tag{8}$$

with the general solution:

$$F(\zeta) = A_{00}e^{-2\omega\zeta} + A_{01}e^{2\omega\zeta} \tag{9}$$

The boundary conditions of Eqs.(2) and (3) require $A_{01} = 0$ and $A_{00} = 1$:

$$F(\zeta) = e^{-2\omega\zeta} \tag{10}$$

For the calculation of $w(\zeta, \theta)$ of Eq.(7) we observe that the introduction of the function $F(\zeta)$ transforms the boundary condition of Eq.(2) for $E = E_{\mathrm{E}}$ into an homogeneous boundary condition for w,

$$E_{\mathrm{E}}(0,\theta) = E_0 + E_0 w(0,\theta) = E_0 \quad \text{for } \theta \geq 0 \tag{11}$$

$$w(0,\theta) = 0 \quad \text{for } \theta \geq 0 \tag{12}$$

while Eq.(3) yields

$$w(\infty,\theta) = \text{finite} \tag{13}$$

The initial conditions of Eqs.(4) and (5) yield:

$$F(\zeta) + w(\zeta,0) = 0, \quad w(\zeta,0) = -e^{-2\omega\zeta} \tag{14}$$

$$\partial w(\zeta,\theta)/\partial\theta = 0 \quad \text{for } \theta = 0,\ \zeta > 0 \tag{15}$$

Substitution of Eq.(7) into Eq.(1) yields for $w(\zeta, \theta)$ the same equation as for $E(\zeta, \theta)$:

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[2]The assumption of a solution of the telegrapher's equation in the form $F(\zeta) + w(\zeta, \theta)$ for the voltages and currents along a transmission line of finite length is discussed by Smirnov (1964, vol. 2, ch. VII) who credits Krylov (1929) as the initiator of the method. The finite length of the transmission line leads to a Fourier series for $w(\zeta, \theta)$ rather than a Fourier transform, which we will use. The telegrapher's equation uses the parameters inductance L, capacitance C, conductance G, and resistance R instead of μ, ϵ, σ, and s in Eq.(1). The need to introduce a parameter R equivalent to the magnetic conductance s never arose, since the resistance R was always part of the telegrapher's equation, which predates Maxwell's equations.

[3]Equation (8) as written follows from Eq.(1), but it is an ordinary differential equation and should be rewritten with $d^2F/d\zeta^2$ replacing $\partial^2 F/\partial\zeta^2$. We shall forgo such strictly cosmetic steps.

$$\partial^2 w/\partial\zeta^2 - \partial^2 w/\partial\theta^2 - 2(1+\omega^2)\partial w/\partial\theta - 4\omega^2 w = 0 \tag{16}$$

Particular solutions $w_\kappa(\zeta,\theta)$ are obtained by the separation of variables using Bernoulli's product method

$$w_\kappa(\zeta,\theta) = \phi(\zeta)\psi(\theta) \tag{17}$$

$$\phi^{-1}\partial^2\phi/\partial\zeta^2 = \psi^{-1}\partial^2\psi/\partial\theta^2 + 2(1+\omega^2)\psi^{-1}\partial\psi/\partial\theta + 4\omega^2 = -(2\pi\kappa)^2 \tag{18}$$

which yields two ordinary differential equations

$$d^2\phi/d\zeta^2 + (2\pi\kappa)^2\phi = 0 \tag{19}$$

and

$$d^2\psi/d\theta^2 + 2(1+\omega^2)d\psi/d\theta + \left[(2\pi\kappa)^2 + 4\omega^2\right]\psi = 0 \tag{20}$$

with the solutions:

$$\phi(\zeta) = A_{10}\sin 2\pi\kappa\zeta + A_{11}\cos 2\pi\kappa\zeta \tag{21}$$

$$\psi(\theta) = A_{20}\exp(\gamma_1\theta) + A_{21}\exp(\gamma_2\theta) \tag{22}$$

The coefficients γ_1 and γ_2 are the roots of the equation

$$\gamma^2 + 2(1+\omega^2)\gamma + \left[(2\pi\kappa)^2 + 4\omega^2\right] = 0 \tag{23}$$

which we write in the following form:

$$\begin{aligned}
\gamma_1 &= -a + (a^2 - b^2)^{1/2} \quad \text{for } a^2 > b^2 \\
\gamma_2 &= -a - (a^2 - b^2)^{1/2} \\
\gamma_1 &= -a + i(b^2 - a^2)^{1/2} \quad \text{for } b^2 > a^2 \\
\gamma_2 &= -a - i(b^2 - a^2)^{1/2} \\
a &= 1 + \omega^2,\ b^2 = (2\pi\kappa)^2 + 4\omega^2
\end{aligned} \tag{24}$$

The boundary condition of Eq.(12) requires $A_{11} = 0$ in Eq.(21). The particular solution $w_\kappa(\zeta,\theta)$ becomes:

$$w_\kappa(\zeta,\theta) = [A_1\exp(\gamma_1\theta) + A_2\exp(\gamma_2\theta)]\sin 2\pi\kappa\zeta \tag{25}$$

A general solution $w(\zeta,\theta)$ is found by making A_1 and A_2 functions of the normalized wavenumber κ, and then integrating over all possible values of κ:

$$w(\zeta,\theta) = \int_0^\infty [A_1(\kappa)\exp(\gamma_1\theta) + A_2(\kappa)\exp(\gamma_2\theta)]\sin 2\pi\kappa\zeta\, d\kappa \tag{26}$$

The derivative $\partial w/\partial\theta$ equals:

$$\frac{\partial w}{\partial\theta} = \int_0^\infty [A_1(\kappa)\gamma_1\exp(\gamma_1\theta) + A_2(\kappa)\gamma_2\exp(\gamma_2\theta)]\sin 2\pi\kappa\zeta\, d\kappa \tag{27}$$

The initial conditions of Eqs.(14) and (15) demand:

$$\int_0^\infty [A_1(\kappa) + A_2(\kappa)]\sin 2\pi\kappa\zeta\, d\kappa = -e^{-2\omega\zeta} \tag{28}$$

$$\int_0^\infty [A_1(\kappa)\gamma_1 + A_2(\kappa)\gamma_2]\sin 2\pi\kappa\zeta\, d\kappa = 0 \tag{29}$$

These two equations must be solved for the functions $A_1(\kappa)$ and $A_2(\kappa)$. To this end consider the Fourier sine transform in the following form:

$$g_s(\kappa) = 2\int_0^\infty f_s(\zeta)\sin 2\pi\kappa\zeta\, d\zeta, \quad f_s(\zeta) = 2\int_0^\infty g_s(\kappa)\sin 2\pi\kappa\zeta\, d\kappa \tag{30}$$

If we identify $2g_s(\kappa)$ first with $A_1(\kappa)+A_2(\kappa)$ and then with $A_1(\kappa)\gamma_1+A_2(\kappa)\gamma_2$ we obtain from Eqs.(28) and (29):

$$A_1(\kappa) + A_2(\kappa) = 2g_s(\kappa) = -4\int_0^\infty e^{-2\omega\zeta}\sin 2\pi\kappa\zeta\, d\zeta \tag{31}$$

$$A_1(\kappa)\gamma_1 + A_2(\kappa)\gamma_2 = 2g_s(\kappa) = 0 \tag{32}$$

Using the tabulated integral (Gradshtein and Ryzhik 1980; p. 477, 3.893/1)

$$\int_0^\infty e^{-u\zeta}\sin 2\pi\kappa\zeta\, d\zeta = \frac{2\pi\kappa}{(2\pi\kappa)^2 + u^2} \tag{33}$$

one obtains from Eq.(31):

$$A_1(\kappa) + A_2(\kappa) = -\frac{8\pi\kappa}{(2\pi\kappa)^2 + 4\omega^2} = -\frac{8\pi\kappa}{b^2} \tag{34}$$

For the limit $s \to 0$ we get:

$$\omega^2 \to 0,\ b^2 \to (2\pi\kappa)^2,\ A_1(\kappa) + A_2(\kappa) = -2/\pi\kappa \tag{35}$$

Equations (32) and (34) are solved for $A_1(\kappa)$ and $A_2(\kappa)$:

$$\begin{aligned}
A_1(\kappa) &= -\frac{8\pi\kappa}{b^2}\frac{\gamma_2}{\gamma_2-\gamma_1}\\
&= -\frac{4\pi\kappa}{b^2}\left(1+\frac{a}{(a^2-b^2)^{1/2}}\right) \quad \text{for } a^2 > b^2\\
&= -\frac{4\pi\kappa}{b^2}\left(1-\frac{ia}{(b^2-a^2)^{1/2}}\right) \quad \text{for } b^2 > a^2\\
A_2(\kappa) &= -\frac{8\pi\kappa}{b^2}\frac{\gamma_1}{\gamma_1-\gamma_2}\\
&= -\frac{4\pi\kappa}{b^2}\left(1-\frac{a}{(a^2-b^2)^{1/2}}\right) \quad \text{for } a^2 > b^2\\
&= -\frac{4\pi\kappa}{b^2}\left(1+\frac{ia}{(b^2-a^2)^{1/2}}\right) \quad \text{for } b^2 > a^2
\end{aligned} \tag{36}$$

Substitution of Eqs.(24) and (36) into Eq.(26) yields:

$$\begin{aligned}
w(\zeta,\theta) = -e^{-a\theta}\Bigg\{ &\int_0^K \Bigg[\left(1+\frac{a}{(a^2-b^2)^{1/2}}\right)\exp\left[(a^2-b^2)^{1/2}\theta\right]\\
&+\left(1-\frac{a}{(a^2-b^2)^{1/2}}\right)\exp\left[-(a^2-b^2)^{1/2}\theta\right]\Bigg]\frac{\sin 2\pi\kappa\zeta}{b^2/4\pi\kappa}\,d\kappa\\
&+\int_K^\infty \Bigg[\left(1-\frac{ia}{(b^2-a^2)^{1/2}}\right)\exp\left[i(b^2-a^2)^{1/2}\theta\right]\\
&+\left(1+\frac{ia}{(b^2-a^2)^{1/2}}\right)\exp\left[-i(b^2-a^2)^{1/2}\theta\right]\Bigg]\frac{\sin 2\pi\kappa\zeta}{b^2/4\pi\kappa}\,d\kappa\Bigg\}
\end{aligned}$$

$$K = (1-\omega^2)/2\pi,\ a = 1+\omega^2,\ b^2 = (2\pi\kappa)^2 + 4\omega^2,\ \omega^2 = \epsilon s/\mu\sigma \tag{37}$$

The imaginary terms in the second integral may be rewritten in real form by means of the formulas

$$e^{iq} + e^{-iq} = 2\cos q, \quad -i\left(e^{iq} - e^{-iq}\right) = 2\sin q$$

while the first integral can be simplified with the help of hyperbolic functions:

$$e^q + e^{-q} = 2\,\mathrm{ch}\, q, \quad e^q - e^{-q} = 2\,\mathrm{sh}\, q$$

One obtains:

$$w(\zeta,\theta) = -\frac{2}{\pi}e^{-a\theta}\Bigg[\int_0^{1-\omega^2}\left(\mathrm{ch}\left[(a^2-b^2)^{1/2}\theta\right] + \frac{a\,\mathrm{sh}\left[(a^2-b^2)^{1/2}\theta\right]}{(a^2-b^2)^{1/2}}\right) \times \frac{2\pi\kappa\sin 2\pi\kappa\zeta}{b^2}d(2\pi\kappa)$$
$$+\int_{1-\omega^2}^{\infty}\left(\cos\left[(b^2-a^2)^{1/2}\theta\right] + \frac{a\sin\left[(b^2-a^2)^{1/2}\theta\right]}{(b^2-a^2)^{1/2}}\right) \times \frac{2\pi\kappa\sin 2\pi\kappa\zeta}{b^2}d(2\pi\kappa)\Bigg] \quad (38)$$

To obtain $E_E(\zeta,\theta)$ we still have to add $F(\zeta)$ to $w(\zeta,\theta)$ according to Eq.(7). With the help of Eq.(10) we get:

$$E_E(\zeta,\theta) = E_0\left[e^{-2\omega\zeta} + w(\zeta,\theta)\right] \quad (39)$$

We now make the transition to $s = 0$. From Eqs.(24) and (37) we get in this limit:

$$b = 2\pi\kappa = \eta,\ a = 1,\ 2\pi K = 1-\omega^2 = 1 \quad (40)$$

Equations (38) and (39) become:

$$w(\zeta,\theta) = -\frac{2}{\pi}e^{-\theta}\Bigg[\int_0^1\left(\mathrm{ch}\left[(1-\eta^2)^{1/2}\theta\right] + \frac{\mathrm{sh}\left[(1-\eta^2)^{1/2}\theta\right]}{(1-\eta^2)^{1/2}}\right)\frac{\sin\eta\zeta}{\eta}\,d\eta$$
$$+\int_1^{\infty}\left(\cos\left[(\eta^2-1)^{1/2}\theta\right] + \frac{\sin\left[(\eta^2-1)^{1/2}\theta\right]}{(\eta^2-1)^{1/2}}\right)\frac{\sin\eta\zeta}{\eta}\,d\eta\Bigg] \quad (41)$$

$$E_E(\zeta,\theta) = E_0[1 + w(\zeta,\theta)] \quad (42)$$

Equation (42) may be rewritten into a form that shows analytically that the field strength $E_E(\zeta,\theta)$ is zero for $\theta < \zeta$ (Boules 1989). Using the relations

$$\cos ix = \mathrm{ch}\,x, \quad \sin ix = \mathrm{sh}\,x$$

we rewrite $w(\zeta,\theta)$ as follows:

$$w(\zeta,\theta) = -\frac{2}{\pi}e^{-\theta}\int_0^{\infty}\left(\cos\left[(\eta^2-1)^{1/2}\theta\right] + \frac{\sin\left[(\eta^2-1)^{1/2}\theta\right]}{(\eta^2-1)^{1/2}}\right)\frac{\sin\zeta\eta}{\eta}\,d\eta \quad (43)$$

We further use the relation

$$\cos ax = \frac{d}{dx}\frac{\sin ax}{a} \tag{44}$$

to obtain

$$w(\zeta,\theta) = -\frac{2}{\pi}e^{-\theta}\left(\frac{\partial}{\partial\theta}+1\right)\int_0^\infty \frac{\sin\left[(\eta^2-1)^{1/2}\theta\right]}{(\eta^2-1)^{1/2}}\frac{\sin\zeta\eta}{\eta}\,d\eta \tag{45}$$

Using once more Eq.(44) we get:

$$\frac{\partial w}{\partial\zeta} = -\frac{2}{\pi}e^{-\theta}\left(\frac{\partial}{\partial\theta}+1\right)\int_0^\infty \frac{\sin\left[(\eta^2-1)^{1/2}\theta\right]}{(\eta^2-1)^{1/2}}\cos\zeta\eta\,d\eta \tag{46}$$

This integral is tabulated (Gradshteyn and Ryzhik 1980, p. 472, 3.876/1):

$$\begin{aligned}\int_0^\infty \frac{\sin\left[(\eta^2-1)^{1/2}\theta\right]}{(\eta^2-1)^{1/2}}\cos\zeta\eta\,d\eta \\ &= \frac{\pi}{2}J_0\big(i\sqrt{\theta^2-\zeta^2}\big) = \frac{\pi}{2}I_0\big(\sqrt{\theta^2-\zeta^2}\big) \quad \text{for } 0<\zeta<\theta \\ &= 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{for } 0<\theta<\zeta \end{aligned} \tag{47}$$

Here $J_0\big(i\sqrt{\theta^2-\zeta^2}\big)$ is the Bessel function of the first kind of order zero. Furthermore, $I_0\big(\sqrt{\theta^2-\zeta^2}\big)$ is the modified Bessel function (of first kind) of order zero. Using Eq.(44) we may rewrite Eq.(47):

$$\begin{aligned}\frac{\partial}{\partial\zeta}\int_0^\infty \frac{\sin\left[(\eta^2-1)^{1/2}\theta\right]}{(\eta^2-1)^{1/2}}\frac{\sin\zeta\eta}{\eta}\,d\eta &= \frac{\pi}{2}I_0\big(\sqrt{\theta^2-\zeta^2}\big) \quad \text{for } 0<\zeta<\theta \\ &= 0 \qquad\qquad\qquad\qquad \text{for } 0<\theta<\zeta\end{aligned} \tag{48}$$

Integration of Eq.(48) introduces integration constants that are arbitrary and may be chosen so that a desired result is obtained. The ability to choose these constants reflects the fact that there are infinitely many ways to write Eq.(43). Our goal is to obtain 0 for the interval $0 < \theta < \zeta$ in Eq.(51) below. By trial and error one finds that this calls for the integration constant $(\pi/2)\,\text{sh}\,\theta$ for $0 < \theta < \zeta$ in the following Eq.(49), which is the integral of Eq.(48) with respect to ζ. The integration constant for $0 < \zeta < \theta$ is of lesser importance:

$$\begin{aligned}\int_0^\infty \frac{\sin\left[(\eta^2-1)^{1/2}\theta\right]}{(\eta^2-1)^{1/2}}\frac{\sin\zeta\eta}{\eta}\,d\eta &= \frac{\pi}{2}\int_0^\zeta I_0\big(\sqrt{\theta^2-\zeta'^2}\big)d\zeta' \quad \text{for } 0<\zeta<\theta \\ &= \frac{\pi}{2}\,\text{sh}\,\theta \qquad\qquad\qquad\qquad \text{for } 0<\theta<\zeta\end{aligned} \tag{49}$$

Equation (45) may now be written as follows:

$$w(\zeta,\theta) = -e^{-\theta}\left(\frac{\partial}{\partial\theta}+1\right)\int_0^\zeta I_0\left(\sqrt{\theta^2-\zeta'^2}\right)d\zeta' \qquad \text{for } 0<\zeta<\theta$$

$$= -e^{-\theta}\left(\frac{\partial}{\partial\theta}+1\right)\operatorname{sh}\theta = -e^{-\theta}(\operatorname{ch}\theta+\operatorname{sh}\theta) = -1 \quad \text{for } 0<\theta<\zeta \qquad (50)$$

Substitution into Eq.(42) brings the desired result

$$E_{\mathrm{E}}(\zeta,\theta) = E_0\left[1 - e^{-\theta}\int_0^\zeta \left(\frac{\theta I_1\left(\sqrt{\theta^2-\zeta'^2}\right)}{(\theta^2-\zeta'^2)^{1/2}} + I_0\left(\sqrt{\theta^2-\zeta'^2}\right)\right)d\zeta'\right]$$

$$\text{for } 0<\zeta<\theta$$

$$= 0 \quad \text{for } 0<\theta<\zeta \qquad (51)$$

where $I_1\left(\sqrt{\theta^2-\zeta'^2}\right)$ denotes the modified Bessel function of first kind of order one.

6.2 Magnetic Field Strength Due to Electric Step Function

In Section 6.1 we derived the electric field strength caused by excitation with an electric excitation force having the time variation of a step function $S(\theta)$ at the plane $\zeta = 0$. For $\omega \neq 0$ or $s \neq 0$ the electric field strength is defined by Eq.(6.1-7):

$$E_{\mathrm{E}}(\zeta,\theta) = E_0[F(\zeta) + w(\zeta,\theta)] \qquad (1)$$

The associated magnetic field strength is defined by Eqs.(1.4-7) and (1.4-8):

$$H_{\mathrm{E}}(\zeta,\theta) = e^{-2\omega^2\theta}\left[-\frac{1}{Z}\int \frac{\partial E_{\mathrm{E}}}{\partial\zeta} e^{2\omega^2\theta}d\theta + H_\theta(\zeta)\right] \qquad (2)$$

$$H_{\mathrm{E}}(\zeta,\theta) = -\frac{1}{Z}\int\left(\frac{\partial E_{\mathrm{E}}}{\partial\theta} + 2E_{\mathrm{E}}\right)d\zeta + H_\zeta(\theta) \qquad (3)$$

$$\omega^2 = \epsilon s/\mu\sigma, \ Z = \sqrt{\mu/\epsilon}$$

Substitution of Eq.(1) into Eq.(2) yields with the help of Eqs.(6.1-26) and (6.1-10):

$$H_{\rm E}(\zeta,\theta) = -\frac{E_0}{Z}e^{-2\omega^2\theta}\int\left(-2\omega e^{-2\omega\zeta}+\frac{\partial w}{\partial\zeta}\right)e^{2\omega^2\theta}d\theta + H_\theta(\zeta)e^{-2\omega^2\theta}$$

$$= -\frac{E_0}{Z}e^{-2\omega^2\theta}\Big[-\frac{1}{\omega}e^{2\omega(\omega\theta-\zeta)} + 2\pi\int_0^\infty\left(\frac{A_1(\kappa)\kappa}{\gamma_1+2\omega^2}e^{(\gamma_1+2\omega^2)\theta} + \frac{A_2(\kappa)\kappa}{\gamma_2+2\omega^2}e^{(\gamma_2+2\omega^2)\theta}\right)\cos 2\pi\kappa\zeta\, d\kappa\Big] + H_\theta(\zeta)e^{-2\omega^2\theta}$$

$$= \frac{E_0}{Z}\Big[\frac{1}{\omega}e^{-2\omega\zeta} - \frac{1}{2\pi}\int_0^\infty\left(\frac{A_1(\eta)\eta}{\gamma_1+2\omega^2}e^{\gamma_1\theta} + \frac{A_2(\eta)\eta}{\gamma_2+2\omega^2}e^{\gamma_2\theta}\right)\cos\zeta\eta\, d\eta\Big] + H_\theta(\zeta)e^{-2\omega^2\theta}$$

$$\eta = 2\pi\kappa,\ \theta=\sigma t/2\epsilon,\ \zeta = Z\sigma y/2,\ \omega^2=\epsilon s/\mu\sigma \qquad (4)$$

With the help of Eqs.(6.1-24) and (6.1-36) we obtain:

$$\frac{A_1(\eta)\eta}{\gamma_1+2\omega^2} = -\frac{4\eta^2}{b^2}\frac{\gamma_2}{\gamma_2-\gamma_1}\frac{1}{\gamma_1+2\omega^2} = \frac{4}{b^2}(1+q_{\rm s}),\quad a^2>b^2$$

$$= \frac{4}{b^2}(1-iq_{\rm s}'),\quad b^2>a^2$$

$$\frac{A_2(\eta)\eta}{\gamma_2+2\omega^2} = -\frac{4\eta^2}{b^2}\frac{\gamma_1}{\gamma_1-\gamma_2}\frac{1}{\gamma_2+2\omega^2} = \frac{4}{b^2}(1-q_{\rm s}),\quad a^2>b^2$$

$$= \frac{4}{b^2}(1+iq_{\rm s}'),\quad b^2>a^2$$

$$q_{\rm s} = \frac{2(1-\omega^2)-\eta^2}{2\left[(1-\omega^2)^2-\eta^2\right]^{1/2}},\quad q_{\rm s}' = \frac{2(1-\omega)^2-\eta^2}{2\left[\eta^2-(1-\omega^2)^2\right]^{1/2}}$$

$$a = 1+\omega^2,\quad b^2=\eta^2+4\omega^2 \qquad (5)$$

Substitution into Eq.(4) yields:

$$H_{\rm E}(\zeta,\theta) = \frac{E_0}{Z}\Big[\frac{1}{\omega}e^{-2\omega\zeta} - \frac{2}{\pi}e^{-(1+\omega^2)\theta}\Big(\int_0^{1-\omega^2}\Big\{(1+q_{\rm s})\exp\left[(a^2-b^2)^{1/2}\theta\right] + (1-q_{\rm s})\exp\left[-(a^2-b^2)^{1/2}\theta\right]\Big\}\frac{\cos\eta\zeta}{b^2}d\eta$$

$$+\int_{1-\omega^2}^\infty\Big\{(1-iq_{\rm s}')\exp\left[i(b^2-a^2)^{1/2}\theta\right] + (1+iq_{\rm s}')\exp\left[-i(b^2-a^2)^{1/2}\theta\right]\Big\}\frac{\cos\eta\zeta}{b^2}d\eta\Big)\Big] + H_\theta(\zeta)e^{-2\omega^2\theta} \qquad (6)$$

Using hyperbolic and trigonometric functions we obtain:

$$H_{\mathrm{E}}(\zeta,\theta) = \frac{E_0}{Z}\Bigg\{\frac{1}{\omega}e^{-2\omega\zeta} - \frac{4}{\pi}e^{-(1+\omega^2)\theta}\Bigg[\int\limits_0^{1-\omega^2}\Bigg(\operatorname{ch}\{[(1-\omega^2)^2-\eta^2]^{1/2}\theta\}$$
$$+\frac{[2(1-\omega^2)-\eta^2]\operatorname{sh}\{[(1-\omega^2)^2-\eta^2]^{1/2}\theta\}}{2\,[(1-\omega^2)^2-\eta^2]^{1/2}}\Bigg)\frac{\cos\eta\zeta}{\eta^2+4\omega^2}d\eta$$
$$+\int\limits_{1-\omega^2}^{\infty}\Bigg(\cos\{[\eta^2-(1-\omega^2)^2]^{1/2}\theta\}$$
$$+\frac{[2(1-\omega^2)-\eta^2]\sin\{[\eta^2-(1-\omega^2)^2]^{1/2}\theta\}}{2\,[\eta^2-(1-\omega^2)^2]^{1/2}}\Bigg)\frac{\cos\eta\zeta}{\eta^2+4\omega^2}d\eta\Bigg]\Bigg\}$$
$$+H_\theta(\zeta)e^{-2\omega^2\theta} \tag{7}$$

We make the transition $\omega \to 0$ or $s \to 0$ for $\sigma \neq 0$. The first term of Eq.(7) yields:

$$\lim_{\omega\to 0}\frac{1}{\omega}e^{-2\omega\zeta} = \frac{1}{\omega} - 2\zeta \tag{8}$$

In order to calculate the first integral in Eq.(7) we observe that the denominator $b^2 = \eta^2 + 4\omega^2$ causes a problem for $\omega = 0$, but the denominator $2[(1-\omega^2)^2 - \eta^2]^{1/2}$ does not since we get

$$\lim_{\omega\to 0}\frac{2(1-\omega^2)-\eta^2}{2\,[(1-\omega^2)^2-\eta^2]^{1/2}}\operatorname{sh}\left\{\left[(1-\omega^2)^2-\eta^2\right]^{1/2}\theta\right\}$$
$$= \frac{2-\eta^2}{2\,(1-\eta^2)^{1/2}}\operatorname{sh}\left[(1-\eta^2)^{1/2}\theta\right] \tag{9}$$

which yields $\theta/2$ for $\eta^2 = 1$. The first integral of Eq.(7) becomes for small values of ω:

$$I_{\mathrm{E1}}(\zeta,\theta) = \frac{4}{\pi}e^{-\theta}\int\limits_0^1\Bigg(\operatorname{ch}\left[(1-\eta^2)^{1/2}\theta\right]$$
$$+\frac{(2-\eta^2)\operatorname{sh}\left[(1-\eta^2)^{1/2}\theta\right]}{2\,(1-\eta^2)^{1/2}}\Bigg)\frac{\cos\eta\zeta}{\eta^2-4\omega^2}d\eta \tag{10}$$

This integral may be split into two parts by means of the identity

$$\cos\eta\zeta \equiv 1 - 2\sin^2(\eta\zeta/2) \tag{11}$$

to yield:

$$I_{\mathrm{E1}}(\zeta,\theta) = \frac{4}{\pi}e^{-\theta}\Bigg\{ \int\limits_0^1 \Bigg(\mathrm{ch}\left[(1-\eta^2)^{1/2}\theta\right] + \frac{(2-\eta^2)\,\mathrm{sh}\left[(1-\eta^2)^{1/2}\theta\right]}{2\,(1-\eta^2)^{1/2}} \Bigg) \frac{d\eta}{\eta^2+4\omega^2}$$
$$- 2\int\limits_0^1 \Bigg(\mathrm{ch}\left[(1-\eta^2)^{1/2}\theta\right] + \frac{(2-\eta^2)\,\mathrm{sh}\left[(1-\eta^2)^{1/2}\theta\right]}{2\,(1-\eta^2)^{1/2}} \Bigg) \frac{\sin^2(\eta\zeta/2)}{\eta^2+4\omega^2} d\eta \Bigg\} \quad (12)$$

We split this integral into four components of which only one depends on ζ and θ, the other three are functions of θ only:

$$I_{\mathrm{E1}}(\zeta,\theta) = -I_{11}(\zeta,\theta) + I_{12}(\theta) + I_{13}(\theta) - I_{14}(\theta) \quad (13)$$

Here $I_{11}(\zeta,\theta)$ stands for the second integral in Eq.(12), which remains finite for $\omega = 0$,

$$I_{11}(\zeta,\theta) = \frac{2}{\pi}e^{-\theta} \int\limits_0^1 \Bigg(\mathrm{ch}\left[(1-\eta^2)^{1/2}\theta\right] + \frac{(2-\eta^2)\,\mathrm{sh}\left[(1-\eta^2)^{1/2}\theta\right]}{2\,(1-\eta^2)^{1/2}} \Bigg) \times \left(\frac{\sin(\eta\zeta/2)}{\eta/2}\right)^2 d\eta \quad (14)$$

and can be evaluated by computer. The other three components of Eq.(13) are:

$$I_{12}(\theta) = \frac{4}{\pi}e^{-\theta} \int\limits_0^1 \frac{\mathrm{ch}\left[(1-\eta^2)^{1/2}\theta\right]}{\eta^2+4\omega^2} d\eta \quad (15)$$

$$I_{13}(\theta) = \frac{4}{\pi}e^{-\theta} \int\limits_0^1 \frac{\mathrm{sh}\left[(1-\eta^2)^{1/2}\theta\right]}{(1-\eta^2)^{1/2}\,(\eta^2+4\omega^2)} d\eta \quad (16)$$

$$I_{14}(\theta) = \frac{2}{\pi}e^{-\theta} \int\limits_0^1 \frac{\eta^2\,\mathrm{sh}\left[(1-\eta^2)^{1/2}\theta\right]}{(1-\eta^2)^{1/2}\,(\eta^2+4\omega^2)} d\eta \quad (17)$$

Integral $I_{14}(\theta)$ remains finite for $\omega \to 0$

$$\lim_{\omega\to 0} I_{14}(\theta) = \frac{2}{\pi} e^{-\theta} \int_0^1 \frac{\operatorname{sh}\left[(1-\eta^2)^{1/2}\theta\right]}{(1-\eta^2)^{1/2}} d\eta \tag{18}$$

but $I_{12}(\theta)$ and $I_{13}(\theta)$ pose problems when η approaches zero. Hence, we divide the integration interval $0 < \eta < 1$ into two subintervals $0 < \eta < d$ and $d < \eta < 1$, where $d \ll 1$ is a small constant:

$$I_{12}(\theta) = \frac{4}{\pi} e^{-\theta} \left(\int_0^d \frac{\operatorname{ch}\left[(1-\eta^2)^{1/2}\theta\right]}{\eta^2 + 4\omega^2} d\eta + \int_d^1 \frac{\operatorname{ch}\left[(1-\eta^2)^{1/2}\theta\right]}{\eta^2} d\eta \right) \tag{19}$$

$$I_{13}(\theta) = \frac{4}{\pi} e^{-\theta} \left(\int_0^d \frac{\operatorname{sh}\left[(1-\eta^2)^{1/2}\theta\right]}{(1-\eta^2)^{1/2}(\eta^2+4\omega^2)} d\eta + \int_d^1 \frac{\operatorname{sh}\left[(1-\eta^2)^{1/2}\theta\right]}{(1-\eta^2)^{1/2}\eta^2} d\eta \right) \tag{20}$$

In the interval $d < \eta < 1$ the integrals are finite and they can be evaluated numerically by computer. In the interval $0 < \eta < d$ the variable η is small compared with 1 and we can resort to series expansions:

$$\begin{aligned}
(1-\eta^2)^{1/2} &\approx 1 - \eta^2/2 \\
(1-\eta^2)^{-1/2} &\approx 1 + \eta^2/2 \\
\operatorname{ch}\left[(1-\eta^2)^{1/2}\theta\right] &\approx \operatorname{ch}[(1-\eta^2/2)\theta] \\
&\approx \operatorname{ch}\theta \operatorname{ch}(\eta^2\theta/2) - \operatorname{sh}\theta \operatorname{sh}(\eta^2\theta/2) \\
\operatorname{sh}\left[(1-\eta^2)^{1/2}\theta\right] &\approx \operatorname{sh}[(1-\eta^2/2)\theta] \\
&\approx \operatorname{sh}\theta \operatorname{ch}(\eta^2\theta/2) - \operatorname{ch}\theta \operatorname{sh}(\eta^2\theta/2)
\end{aligned} \tag{21}$$

The two problem integrals in Eqs.(19) and (20) become:

$$\begin{aligned}
I_{12\mathrm{p}}(\theta) &= \frac{4}{\pi} e^{-\theta} \int_0^d \frac{\operatorname{ch}\left[(1-\eta^2)^{1/2}\theta\right]}{\eta^2 + 4\omega^2} d\eta \\
&= \frac{4}{\pi} e^{-\theta} \left(\operatorname{ch}\theta \int_0^d \frac{\operatorname{ch}(\eta^2\theta/2)}{\eta^2 + 4\omega^2} d\eta - \operatorname{sh}\theta \int_0^d \frac{\operatorname{sh}(\eta^2\theta/2)}{\eta^2 + 4\omega^2} d\eta \right)
\end{aligned} \tag{22}$$

$$\begin{aligned}
I_{13\mathrm{p}}(\theta) &= \frac{4}{\pi} e^{-\theta} \int_0^d \frac{\operatorname{sh}\left[(1-\eta^2)^{1/2}\theta\right]}{(1-\eta^2)^{1/2}(\eta^2 + 4\omega^2)} d\eta \\
&\approx \frac{4}{\pi} e^{-\theta} \int_0^d \left(1 + \frac{\eta^2}{2}\right) \frac{\operatorname{sh}\left[(1+\eta^2)^{1/2}\theta\right]}{\eta^2 + 4\omega^2} d\eta \quad \text{continued}
\end{aligned}$$

$$\approx \frac{4}{\pi} e^{-\theta} \Bigg(\operatorname{sh}\theta \int_0^d \frac{\operatorname{ch}(\eta^2\theta/2)}{\eta^2 + 4\omega^2} d\eta - \operatorname{ch}\theta \int_0^d \frac{\operatorname{sh}(\eta^2\theta/2)}{\eta^2 + 4\omega^2} d\eta + \frac{1}{2} \int_0^d \frac{\eta^2 \operatorname{sh}\left[(1-\eta^2)^{1/2}\theta\right]}{\eta^2 + 4\omega^2} d\eta \Bigg) \tag{23}$$

For two of these integrals we can make the transition $\omega \to 0$ without creating a pole:

$$\lim_{\omega \to 0} \int_0^d \frac{\eta^2 \operatorname{sh}\left[(1-\eta^2)^{1/2}\theta\right]}{\eta^2 + 4\omega^2} d\eta = \int_0^d \operatorname{sh}\left[(1-\eta^2)^{1/2}\theta\right] d\eta \tag{24}$$

$$\lim_{\omega \to 0} \int_0^d \frac{\operatorname{sh}(\eta^2\theta/2)}{\eta^2 + 4\omega^2} d\eta = \int_0^d \frac{\operatorname{sh}(\eta^2\theta/2)}{\eta^2} d\eta \tag{25}$$

The problem has been reduced to just one integral. Using the relation

$$\operatorname{ch} x \equiv 1 + 2\operatorname{sh}^2(x/2) \tag{26}$$

we obtain:

$$\int_0^d \frac{\operatorname{ch}(\eta^2\theta/2)}{\eta^2 + 4\omega^2} d\eta = \int_0^d \frac{d\eta}{\eta^2 + 4\omega^2} + 2\int_0^d \frac{\operatorname{sh}^2(\eta^2\theta/4)}{\eta^2 + 4\omega^2} d\eta \tag{27}$$

The second integral has again no pole for $\omega = 0$,

$$\lim_{\omega \to 0} \int_0^d \frac{\operatorname{sh}^2(\eta^2\theta/4)}{\eta^2 + 4\omega^2} d\eta = \int_0^d \frac{\operatorname{sh}^2(\eta^2\theta/4)}{\eta^2} d\eta \tag{28}$$

while the first is tabulated (Gradshtein and Ryzhik 1980, p. 60, 2.124/1):

$$\int_0^d \frac{d\eta}{\eta^2 + 4\omega^2} = \frac{1}{2\omega} \operatorname{arctg} \frac{\eta}{2\omega} \bigg|_0^d, \quad \omega^2 > 0 \tag{29}$$

With the series expansion

$$\operatorname{arctg} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^2} - \ldots, \quad x^2 \geq 1 \tag{30}$$

we get:

$$\lim_{\omega \to 0} \int_0^d \frac{d\eta}{\eta^2 + 4\omega^2} = \lim_{\omega \to 0} \frac{1}{2\omega} \left(\frac{\pi}{2} - \frac{2\omega}{d} \right) = \lim_{\omega \to 0} \frac{\pi}{4\omega} - \frac{1}{d} \tag{31}$$

Converting $\operatorname{ch}\theta$ and $\operatorname{sh}\theta$ to exponential functions one obtains for $I_{12\mathrm{p}}(\theta)$ and $I_{13\mathrm{p}}(\theta)$ the following expressions for $\omega \to 0$:

$$I_{12\mathrm{p}} = \frac{2}{\pi}\left[(1+e^{-2\theta})\left(\frac{\pi}{4\omega} - \frac{1}{d} + 2\int_0^d \frac{\operatorname{sh}^2(\eta^2\theta/4)}{\eta^2}d\eta\right)\right.$$
$$\left. - (1-e^{-2\theta})\int_0^d \frac{\operatorname{sh}(\eta^2\theta/2)}{\eta^2}d\eta\right] \quad (32)$$

$$I_{13\mathrm{p}}(\theta) = \frac{2}{\pi}\left\{(1-e^{-2\theta})\left(\frac{\pi}{4\omega} - \frac{1}{d} + 2\int_0^d \frac{\operatorname{sh}^2(\eta^2\theta/4)}{\eta^2}d\eta\right)\right.$$
$$\left. - (1+e^{-2\theta})\int_0^d \frac{\operatorname{sh}(\eta^2\theta/2)}{\eta^2}d\eta + \frac{1}{2}e^{-\theta}\int_0^d \operatorname{sh}\left[(1-\eta^2)^{1/2}\theta\right]d\eta\right\} \quad (33)$$

The last integral of $I_{13\mathrm{p}}(\theta)$ becomes insignificant for small values of d and we may simplify $I_{13\mathrm{p}}(\theta)$:

$$I_{13\mathrm{p}}(\theta) = \frac{2}{\pi}\left[(1-e^{-2\theta})\left(\frac{\pi}{4\omega} - \frac{1}{d} + 2\int_0^d \frac{\operatorname{sh}^2(\eta^2\theta/4)}{\eta^2}d\eta\right)\right.$$
$$\left. - (1+e^{-2\theta})\int_0^d \frac{\operatorname{sh}(\eta^2\theta/2)}{\eta^2}d\eta\right] \quad (34)$$

The integrals $I_{12\mathrm{p}}(\theta)$ and $I_{13\mathrm{p}}(\theta)$ become equal when θ approaches infinity.

We return to the integrals $I_{12}(\theta)$ and $I_{13}(\theta)$ of Eqs.(19) and (20). They may be rewritten as follows:

$$I_{12}(\theta) = \frac{2}{\pi}\left[(1+e^{-2\theta})\left(\frac{\pi}{4\omega} - \frac{1}{d} + 2\int_0^d \frac{\operatorname{sh}^2(\eta^2\theta/4)}{\eta^2}d\eta\right)\right.$$
$$\left. - (1-e^{-2\theta})\int_0^d \frac{\operatorname{sh}(\eta^2\theta/2)}{\eta^2}d\eta + 2e^{-\theta}\int_d^1 \frac{\operatorname{ch}\left[(1-\eta^2)^{1/2}\theta\right]}{\eta^2}d\eta\right] \quad (35)$$

$$I_{13}(\theta) = \frac{2}{\pi}\left[(1-e^{-2\theta})\left(\frac{\pi}{4\omega} - \frac{1}{d} + \int_0^d \frac{\operatorname{sh}^2(\eta^2\theta/4)}{\eta^2}d\eta\right)\right.$$
$$\left. - (1+e^{-2\theta})\int_0^d \frac{\operatorname{sh}(\eta^2\theta/2)}{\eta^2}d\eta + 2e^{-\theta}\int_d^1 \frac{\operatorname{sh}\left[(1-\eta^2)^{1/2}\theta\right]}{(1-\eta^2)^{1/2}\eta^2}d\eta\right] \quad (36)$$

The sum of $I_{12}(\theta)$ and $I_{13}(\theta)$ minus the integrals $I_{14}(\theta)$ of Eq.(18) and $I_{11}(\zeta,\theta)$ of Eq.(14) yields integral $I_{\mathrm{E1}}(\zeta,\theta)$ of Eq.(13). We sum first $I_{12}(\theta)$ and $I_{13}(\theta)$:

$$I_{12}(\theta)+I_{13}(\theta)=\frac{1}{2\omega}-\frac{2}{\pi d}+\frac{4}{\pi}e^{-\theta}\Bigg\{\int_0^d\left(2\,\mathrm{sh}^2\frac{\eta^2\theta}{4}-\mathrm{sh}\frac{\eta^2\theta}{2}\right)\frac{d\eta}{\eta^2}$$
$$+\int_d^1\left(\mathrm{ch}\left[(1-\eta^2)^{1/2}\theta\right]+\frac{\mathrm{sh}\left[(1-\eta^2)^{1/2}\theta\right]}{(1-\eta^2)^{1/2}}\right)\frac{d\eta}{\eta^2}\Bigg\} \quad (37)$$

With the relation $2\,\mathrm{sh}^2(x/2)\equiv\mathrm{ch}\,x-1$ we may rewrite the kernel of the first integral

$$2\,\mathrm{sh}^2(\eta^2\theta/4)-\mathrm{sh}(\eta^2\theta/2)=\mathrm{ch}(\eta^2\theta/2)-\mathrm{sh}(\eta^2\theta/2)-1=\exp(-\eta^2\theta/2)-1$$

and obtain for $I_{\mathrm{E1}}(\zeta,\theta)$:

$$I_{\mathrm{E1}}(\zeta,\theta)=-\frac{1}{\omega}-\frac{2}{\pi d}+\frac{2}{\pi}e^{-\theta}\Bigg\{2\int_0^d\frac{\exp(-\eta^2\theta/2)-1}{\eta^2}d\eta$$
$$+2\int_d^1\left(\mathrm{ch}\left[(1-\eta^2)^{1/2}\theta\right]+\frac{\mathrm{sh}\left[(1-\eta^2)^{1/2}\theta\right]}{(1-\eta^2)^{1/2}}\right)\frac{d\eta}{\eta^2}$$
$$-\int_0^1\left(\mathrm{ch}\left[(1-\eta^2)^{1/2}\theta\right]+\frac{(2-\eta^2)\,\mathrm{sh}\left[(1-\eta^2)^{1/2}\theta\right]}{2\,(1-\eta^2)^{1/2}}\right)\left(\frac{\sin(\eta\zeta/2)}{\eta/2}\right)^2d\eta$$
$$+\int_0^1\frac{\mathrm{sh}\left[(1-\eta^2)^{1/2}\theta\right]}{(1-\eta^2)^{1/2}}d\eta\Bigg\}$$
$$d\ll 1,\quad \omega\ll 1 \quad (38)$$

For finite values of θ and sufficiently small values of d one may rewrite the first integral in Eq.(38):

$$\int_0^d\frac{\exp(-\eta^2\theta/2)-1}{\eta^2}d\eta=-\frac{\theta}{2}\int_0^d d\eta=-\frac{\theta d}{2}$$

This integral can be neglected for finite values of θ and sufficiently small values of d.

We introduce the notation $-I'_{\mathrm{E1}}(\zeta,\theta)$ for the function $I_{\mathrm{E1}}(\zeta,\theta)$ in Eq.(38) without the term $1/\omega$:

$$I'_{\mathrm{E1}}(\zeta,\theta) = -I_{\mathrm{E1}}(\zeta,\theta) + 1/\omega$$
$$= \frac{2}{\pi d} - \frac{2}{\pi}e^{-\theta}\Bigg\{2\int\limits_d^1 \left(\mathrm{ch}\left[(1-\eta^2)^{1/2}\theta\right] + \frac{\mathrm{sh}\left[(1-\eta^2)^{1/2}\theta\right]}{(1-\eta^2)^{1/2}}\right)\frac{d\eta}{\eta^2}$$
$$-\int\limits_0^1 \left(\mathrm{ch}\left[(1-\eta^2)^{1/2}\theta\right] + \frac{(2-\eta^2)\,\mathrm{sh}\left[(1-\eta^2)^{1/2}\theta\right]}{2\,(1-\eta^2)^{1/2}}\right)\left(\frac{\sin(\eta\zeta/2)}{\eta/2}\right)^2 d\eta$$
$$+\int\limits_0^1 \frac{\mathrm{sh}\left[(1-\eta^2)^{1/2}\theta\right]}{(1-\eta^2)^{1/2}}d\eta\Bigg\} \quad (39)$$

Let us turn to the second integral in Eq.(7). It remains finite for $\omega \to 0$ and we get for this limit the function $I_{\mathrm{E2}}(\zeta,\theta)$:

$$I_{\mathrm{E2}}(\zeta,\theta) = \lim_{\omega\to 0}\frac{4}{\pi}e^{-(1+\omega^2)\theta}\int\limits_{1-\omega^2}^{\infty}\Bigg(\cos\left\{[\eta^2-(1-\omega^2)^2]^{1/2}\theta\right\}$$
$$+\frac{[2(1-\omega^2)-\eta^2]\sin\left\{[\eta^2-(1-\omega^2)^2]^{1/2}\theta\right\}}{2\,[\eta^2-(1-\omega^2)^2]^{1/2}}\Bigg)\frac{\cos\eta\zeta}{\eta^2+4\omega^2}d\eta$$
$$= \frac{4}{\pi}e^{-\theta}\int\limits_1^{\infty}\Bigg(\cos\left[(\eta^2-1)^{1/2}\theta\right]$$
$$+\frac{(2-\eta^2)\sin\left[(\eta^2-1)^{1/2}\theta\right]}{2\,(\eta^2-1)^{1/2}}\Bigg)\frac{\cos\eta\zeta}{\eta^2}d\eta \quad (40)$$

We may now rewrite $H_{\mathrm{E}}(\zeta,\theta)$ of Eq.(7) for $\omega \ll 1$:

$$H_{\mathrm{E}}(\zeta,\theta) = \frac{E_0}{Z}\left[\frac{1}{\omega} - 2\zeta + I'_{\mathrm{E1}}(\zeta,\theta) - \frac{1}{\omega} - I_{\mathrm{E2}}(\zeta,\theta)\right] + H_\theta(\zeta)$$
$$= \frac{E_0}{Z}\left[-2\zeta + I'_{\mathrm{E1}}(\zeta,\theta) - I_{\mathrm{E2}}(\zeta,\theta)\right] + H_\theta(\zeta) \quad (41)$$

We see that the two terms $1/\omega$ and $-1/\omega$ cancel and that $H_{\mathrm{E}}(\zeta,\theta)$ thus remains finite for $\omega \to 0$, or $s \to 0$ and $\mathbf{g}_{\mathrm{m}} \to 0$.

We turn to the integral of Eq.(3). Substitution of the electric field strength $E_{\mathrm{E}}(\zeta,\theta)$ of Eq.(1) yields:

$$H_{\mathrm{E}}(\zeta,\theta) = -\frac{E_0}{Z}\int\left(\frac{\partial w(\zeta,\theta)}{\partial\theta} + 2F(\zeta) + 2w(\zeta,\theta)\right)d\zeta + H_\zeta(\theta)$$
$$= -\frac{E_0}{Z}\left[-\frac{1}{\omega}e^{-2\omega\zeta} + \int\left(\frac{\partial w}{\partial\theta} + 2w\right)d\zeta\right] + H_\zeta(\theta) \quad (42)$$

Using Eqs.(6.1-26) and (6.1-27) we obtain:

$$H_{\mathrm{E}}(\zeta,\theta) = -\frac{E_0}{Z}\Big[-\frac{1}{\omega}e^{-2\omega\zeta} - \frac{1}{2\pi}\int\limits_0^\infty \left[A_1(\eta)\gamma_1 e^{\gamma_1\theta} + A_2(\eta)\gamma_2 e^{\gamma_2\theta}\right]\frac{\cos\eta\zeta}{\eta}d\eta$$

$$-\frac{1}{\pi}\int\limits_0^\infty \left[A_1(\eta)e^{\gamma_1\theta} + A_2(\eta)e^{\gamma_2\theta}\right]\frac{\cos\eta\zeta}{\eta}d\eta\Big] + H_\zeta(\theta)$$

$$= \frac{E_0}{Z}\Big[\frac{1}{\omega}e^{-2\omega\zeta} - \frac{1}{2\pi}\int\limits_0^\infty\Big(-\frac{A_1(\eta)(\gamma_1+2)}{\eta}e^{\gamma_1\theta}$$

$$-\frac{A_2(\eta)(\gamma_2+2)}{\eta}e^{\gamma_2\theta}\Big)\Big]\cos\eta\zeta\, d\eta + H_\zeta(\theta)$$

$$\eta = 2\pi\kappa \tag{43}$$

From Eqs.(6.1-36) and (6.1-24) as well as from the definition of q_{s} and q'_{s} in Eq.(5) follows:

$$-\frac{A_1(\eta)(\gamma_1+2)}{\eta} = -\frac{4\eta}{b^2}\frac{\gamma_2}{\gamma_2-\gamma_1}\frac{\gamma_1+2}{\eta} = -\frac{4}{b^2}(1+q_{\mathrm{s}}),\quad a^2 > b^2$$

$$= -\frac{4}{b^2}(1-iq'_{\mathrm{s}}),\quad b^2 > a^2$$

$$-\frac{A_2(\eta)(\gamma_2+2)}{\eta} = -\frac{4\eta}{b^2}\frac{\gamma_1}{\gamma_1-\gamma_2}\frac{\gamma_2+2}{\eta} = -\frac{4}{b^2}(1-q_{\mathrm{s}}),\quad a^2 > b^2$$

$$= -\frac{4}{b^2}(1+iq'_{\mathrm{s}}),\quad b^2 > a^2 \tag{44}$$

The comparison of Eqs.(4) and (5) with Eqs.(43) and (44) shows that they are equal for

$$H_\theta(\zeta)e^{-2\omega^2\theta} = H_\zeta(\theta) = H_{\mathrm{E}0}e^{-2\omega^2\theta} \tag{45}$$

The initial condition of Eq.(6.1-4) demands $H_{\mathrm{E}0} = 0$.

Plots of $H_{\mathrm{E}}(\zeta,\theta)$ according to Eq.(41) as function of θ for $H_\theta(\zeta) = 0$ are shown in Figs.1.4-4 and 1.4-5 for various values of ζ. Since $I'_{\mathrm{E}1}(\zeta,\theta)$ of Eq.(39) contains the small but otherwise undefined constant d one must choose successively smaller and smaller values of d until the change of the plots with d becomes less than the line width. The plots of Figs.1.4-4 and 1.4-5 were done with $d = 0.001$. The process is the same as used for the determination of the value of $(\sin x)/x$ for $x = 0$ by computer. For a discussion of the plots of Fig.1.4-4 see Harmuth (1986a). This reference also derived the first plots for magnetic rather than electric step function excitation and for electric as well as magnetic excitation with force functions having the time variation of an exponential ramp function $[1-\exp(-\theta)]S(\theta)$.

We have shown in Section 1.4 that no associated magnetic field strength can be obtained from Eqs.(1.4-1) and (1.4-2) that were derived from the original Maxwell equations. It is of interest to see how this general proof works out in an example. Using the electric field strength of Eq.(6.1-39) and (6.1-38)—without questioning how it was derived—we want to calculate the associated magnetic field strength from Eqs.(1.2-14) and (1.2-15) rather than from Eqs.(1.3-2) and (1.3-3). We see that Eqs.(1.2-15) and (1.3-3) are identical. Equation (1.3-2) becomes Eq.(1.2-14) if we choose $s = 0$. For $s = 0$ or $\omega = 0$ we obtain from Eqs.(1) and (2) with the help of Eq.(6.1-10):

$$E_{\mathrm{E}}(\zeta, \theta) = E_0[1 + w(\zeta, \theta)] \tag{46}$$

$$H_{\mathrm{E}}(\zeta, \theta) = -\frac{1}{Z} \int \frac{\partial E_{\mathrm{E}}(\zeta, \theta)}{\partial \zeta} d\theta + H_\theta(\zeta) \tag{47}$$

Substitution of Eq.(46) into Eq.(47) yields with the help of Eq.(6.1-26):

$$\begin{aligned} H_{\mathrm{E}}(\zeta, \theta) &= -\frac{E_0}{Z} \int \frac{\partial w(\zeta, \theta)}{\partial \zeta} d\theta + H_\theta(\zeta) \\ &= -\frac{E_0}{2\pi Z} \int_0^\infty \left(\frac{A_1(\eta)\eta}{\gamma_1} e^{\gamma_1 \theta} + \frac{A_2(\eta)\eta}{\gamma_2} e^{\gamma_2 \theta} \right) \cos \eta\zeta \, d\eta + H_\theta(\zeta) \\ &\qquad \eta = 2\pi\kappa \end{aligned} \tag{48}$$

From Eq.(5) we obtain for $\omega = 0$

$$\begin{aligned} \frac{A_1(\eta)\eta}{\gamma_1} &= \frac{4}{\eta^2} \left(1 + \frac{2 - \eta^2}{2(1 - \eta^2)^{1/2}} \right), \quad 1 > \eta \\ &= \frac{4}{\eta^2} \left(1 - i \frac{2 - \eta^2}{2(\eta^2 - 1)^{1/2}} \right), \quad \eta > 1 \\ \frac{A_2(\eta)\eta}{\gamma_2} &= \frac{4}{\eta^2} \left(1 - \frac{2 - \eta^2}{2(1 - \eta^2)^{1/2}} \right), \quad 1 > \eta \\ &= \frac{4}{\eta^2} \left(1 + i \frac{2 - \eta^2}{2(\eta^2 - 1)^{1/2}} \right), \quad \eta > 1 \end{aligned} \tag{49}$$

while Eq.(6.1-24) yields:

$$\begin{aligned} \gamma_1 &= -1 + (1 - \eta^2)^{1/2}, \quad 1 > \eta \\ &= -1 + i(\eta^2 - 1)^{1/2}, \quad \eta > 1 \\ \gamma_2 &= -1 - (1 - \eta^2)^{1/2}, \quad 1 > \eta \\ &= -1 - i(\eta^2 - 1)^{1/2}, \quad \eta > 1 \end{aligned} \tag{50}$$

Substitution into Eq.(48) yields:

$$H_{\mathrm{E}}(\zeta,\theta) = -\frac{E_0}{Z}\frac{4}{\pi}e^{-\theta}\Bigg\{\int_0^1\Bigg(\operatorname{ch}\left[(1-\eta^2)^{1/2}\theta\right] + \frac{(2-\eta^2)\operatorname{sh}\left[(1-\eta^2)^{1/2}\theta\right]}{2\,(1-\eta^2)^{1/2}}\Bigg)\frac{\cos\eta\zeta}{\eta^2}d\eta$$
$$+\int_1^\infty\Bigg(\cos\left[(\eta^2-1)^{1/2}\theta\right] + \frac{(2-\eta^2)\sin\left[(\eta^2-1)^{1/2}\theta\right]}{2\,(\eta^2-1)^{1/2}}\Bigg)\frac{\cos\eta\zeta}{\eta^2}d\eta\Bigg\} + H_\theta(\zeta) \quad (51)$$

The second integral equals $-I_{\mathrm{E}2}(\zeta,\theta)$ in Eq.(40) and (41) but the first integral is not defined. It equals $I_{\mathrm{E}1}(\zeta,\theta)$ of Eq.(10) for $\omega = 0$ and $I_{\mathrm{E}1}(\zeta,\theta)$ is shown in Eq.(38) to have a term $-1/\omega$ that is infinite for $\omega = 0$.

6.3 Excitation by a Magnetic Step Function

In Sections 1.3, 1.4, 6.1, and 6.2 we investigated the excitation of a TEM wave by an electric excitation force with the time variation of a step function in the plane $y = 0$ at the time $t = 0$. We turn to the case of a magnetic excitation force.

Using the normalized variables θ and ζ defined in Eq.(1.3-7) we consider the following step function for the magnetic field strength as a boundary condition:

$$\begin{aligned} H(0,\theta) = H_0S(\theta) &= 0 \quad \text{for } \theta < 0 \\ &= H_0 \quad \text{for } \theta \geq 0 \end{aligned} \quad (1)$$

At the plane $\zeta \to \infty$ we have again the boundary condition corresponding to Eq.(6.1-3):

$$H(\infty,\theta) = \text{finite} \quad (2)$$

Let $H(\zeta,\theta)$ be zero for $\zeta > 0$ at the time $\theta = 0$. We have then exactly the same problem as in Section 6.1, except that H, H_0, and H_{H} must be written for E, E_0, and E_{E}. Equation (6.1-39) assumes the form

$$H_{\mathrm{H}}(\zeta,\theta) = H_0\left[e^{-2\omega\zeta} + w(\zeta,\theta)\right], \quad \omega = \sqrt{\epsilon s/\mu\sigma} \quad (3)$$

where $w(\zeta,\theta)$ is defined by Eq.(6.1-38).

The associated electric field strength $E_{\mathrm{H}}(\zeta,\theta)$ due to the magnetic field strength $H(0,\theta)$ at the boundary plane $\zeta = 0$ follows from Eqs.(1.2-17) and (1.2-18):

$$E_{\mathrm{H}}(\zeta,\theta) = -Z\int\left(\frac{\partial H_{\mathrm{H}}}{\partial\theta} + 2\omega^2 H_{\mathrm{H}}\right)d\zeta + E_\zeta(\theta) \tag{4}$$

$$E_{\mathrm{H}}(\zeta,\theta) = -Ze^{-2\theta}\int\frac{\partial H_{\mathrm{H}}}{\partial\zeta}e^{2\theta}d\theta + E_\theta(\zeta)e^{-2\theta} \tag{5}$$

We do not get a repetition of the calculations of Section 6.2 with $E_{\mathrm{E}}(\zeta,\theta)$ replaced by $H_{\mathrm{H}}(\zeta,\theta)$ since we want a solution for $\sigma > 0$, $s = 0$ rather than $s > 0$, $\sigma = 0$.

Equations (3) and (5) yield:

$$E_{\mathrm{H}}(\zeta,\theta) = H_0 Z\left(\omega e^{-2\omega\zeta} - e^{-2\theta}\int\frac{\partial w}{\partial\zeta}e^{2\theta}d\theta\right) + E_\theta(\zeta)e^{-2\theta} \tag{6}$$

From Eq.(6.1-26) we get:

$$\begin{aligned}
&H_0 Ze^{-2\theta}\int\frac{\partial w}{\partial\zeta}e^{2\theta}d\theta\\
&\quad = 2\pi H_0 Ze^{-2\theta}\int_0^\infty\left[\int\Big(A_1(\kappa)\exp[(2+\gamma_1)\theta]\right.\\
&\qquad\qquad\left. + A_2(\kappa)\exp[(2+\gamma_2)\theta]\Big)d\theta\right]\kappa\cos 2\pi\kappa\zeta\, d\kappa\\
&\quad = \frac{H_0 Z}{2\pi}\int_0^\infty\left(\frac{A_1(\eta)}{2+\gamma_1}\exp(\gamma_1\theta) + \frac{A_2(\eta)}{2+\gamma_2}\exp(\gamma_2\theta)\right)\eta\cos\eta\zeta\, d\eta\\
&\qquad\qquad \eta = 2\pi\kappa
\end{aligned} \tag{7}$$

Using Eqs.(6.1-36) and (6.1-24) we obtain:

$$\begin{aligned}
\frac{A_1(\eta)\eta}{2+\gamma_1} &= \frac{4\omega^2}{\eta^2+4\omega^2}(1-p_1) \quad \text{for } (1-\omega^2)^2 > \eta^2\\
&= \frac{4\omega^2}{\eta^2+4\omega^2}(1+ip_2) \quad \text{for } (1-\omega^2)^2 < \eta^2\\
\frac{A_2(\eta)\eta}{2+\gamma_2} &= \frac{4\omega^2}{\eta^2+4\omega^2}(1+p_1) \quad \text{for } (1-\omega^2)^2 > \eta^2\\
&= \frac{4\omega^2}{\eta^2+4\omega^2}(1-ip_2) \quad \text{for } (1-\omega^2)^2 < \eta^2
\end{aligned}$$

$$p_1 = \frac{\eta^2 + 2\omega^2(1-\omega^2)}{2\omega^2\left[(1-\omega^2)^2-\eta^2\right]^{1/2}}, \quad p_2 = \frac{\eta^2 + 2\omega^2(1-\omega^2)}{2\omega^2\left[\eta^2-(1-\omega^2)^2\right]^{1/2}} \tag{8}$$

Equation (6) becomes:

$$E_{\mathrm{H}}(\zeta,\theta) = ZH_0\bigg\{\omega e^{-2\omega\zeta}$$
$$-\frac{2\omega^2}{\pi}e^{-(1+\omega^2)\theta}\bigg[\int_0^{1-\omega^2}\bigg((1-p_1)\exp\Big\{[(1-\omega^2)^2-\eta^2]^{1/2}\theta\Big\}$$
$$+(1+p_1)\exp\Big\{-[(1-\omega^2)^2-\eta^2]^{1/2}\theta\Big\}\bigg)\frac{\cos\eta\zeta}{\eta^2+4\omega^2}\,d\eta$$
$$+\int_{1-\omega^2}^{\infty}\bigg((1+ip_2)\exp\Big\{i[\eta^2-(1-\omega^2)^2]^{1/2}\theta\Big\}$$
$$+(1-ip_2)\exp\Big\{-i[\eta^2-(1-\omega^2)^2]^{1/2}\theta\Big\}\bigg)\frac{\cos\eta\zeta}{\eta^2+4\omega^2}d\eta\bigg]\bigg\}+E_\theta(\zeta)e^{-2\theta} \quad (9)$$

Equation (9) is rewritten with the help of trigonometric and hyperbolic functions:

$$E_{\mathrm{H}}(\zeta,\theta) = ZH_0\bigg\{\omega e^{-2\omega\zeta}-\frac{4\omega^2}{\pi}e^{-(1-\omega^2)\theta}\bigg[\int_0^{1-\omega^2}\bigg(\mathrm{ch}\Big\{[(1-\omega^2)^2-\eta^2]^{1/2}\theta\Big\}$$
$$-\frac{\eta^2+2\omega^2(1-\omega^2)}{2\omega^2\,[(1-\omega^2)^2-\eta^2]^{1/2}}\,\mathrm{sh}\Big\{[(1-\omega)^2-\eta^2]^{1/2}\theta\Big\}\bigg)\frac{\cos\eta\zeta}{\eta^2+4\omega^2}d\eta$$
$$+\int_{1-\omega^2}^{\infty}\bigg(\cos\Big\{[\eta^2-(1-\omega^2)^2]^{1/2}\theta\Big\}-\frac{\eta^2+2\omega^2(1-\omega^2)}{2\omega^2\,[\eta^2-(1-\omega^2)^2]^{1/2}}$$
$$\times\sin\Big\{[\eta^2-(1-\omega)^2]^{1/2}\theta\Big\}\bigg)\frac{\cos\eta\zeta}{\eta^2+4\omega^2}d\eta\bigg]\bigg\}+E_\theta(\zeta)e^{-2\theta} \quad (10)$$

We make the transition $s \to 0$ for $\sigma > 0$, or $\omega = \sqrt{\epsilon s/\mu\sigma} \to 0$. This result should be compared with Eq.(6.2-8). The first term in Eq.(10) vanishes. The remainder of Eq.(10) becomes:

$$E_{\mathrm{H}}(\zeta,\theta) = \frac{2}{\pi}ZH_0e^{-\theta}\bigg(\int_0^1\frac{\mathrm{sh}[(1-\eta^2)^{1/2}\theta]}{(1-\eta^2)^{1/2}}\cos\eta\zeta\,d\eta$$
$$+\int_1^{\infty}\frac{\sin[(\eta^2-1)^{1/2}\theta]}{(\eta^2-1)^{1/2}}\cos\eta\zeta\,d\eta\bigg)+E_\theta(\zeta)e^{-2\theta} \quad (11)$$

This is the electric field strength derived from Eq.(5). We turn now to the electric field strength defined by Eq.(4). Substitution of Eq.(3) yields:

$$E_{\mathrm{H}}(\zeta,\theta) = ZH_0\left(-e^{-2\omega\zeta} - \int \frac{\partial w}{\partial\theta} d\zeta - 2\omega^2 \int w\, d\zeta\right) + E_\zeta(\theta) \tag{12}$$

With the definitions of Eqs.(6.1-26) and (6.1-27) for $w(\zeta,\theta)$ and $\partial w/\partial\theta$ we obtain for $\eta = 2\pi\kappa$:

$$E_{\mathrm{H}}(\zeta,\theta) = ZH_0\Bigg\{ - e^{-2\omega\zeta} + \frac{1}{2\pi}\int_0^\infty \Big[(2\omega^2+\gamma_1)A_1(\eta)e^{\gamma_1\theta} + (2\omega^2+\gamma_2)A_2(\eta)e^{\gamma_2\theta}\Big]\frac{\cos\eta\zeta}{\eta}d\eta\Bigg\} + E_\zeta(\theta) \tag{13}$$

The following relations are obtained from Eqs.(6.1-36) and (6.1-24) for $\eta = 2\pi\kappa$ with p_1 and p_2 defined in Eq.(8):

$$\begin{aligned}
\frac{A_1(\eta)(2\omega^2+\gamma_1)}{\eta} &= -\frac{4\omega^2}{\eta^2+4\omega^2}(1-p_1) \quad \text{for } (1-\omega^2)^2 > \eta^2\\
&= -\frac{4\omega^2}{\eta^2+4\omega^2}(1+ip_2) \quad \text{for } (1-\omega^2)^2 < \eta^2\\
\frac{A_2(\eta)(2\omega^2+\gamma_2)}{\eta} &= -\frac{4\omega^2}{\eta^2+4\omega^2}(1+p_1) \quad \text{for } (1-\omega^2)^2 > \eta^2\\
&= -\frac{4\omega^2}{\eta^2+4\omega^2}(1-ip_2) \quad \text{for } (1-\omega^2)^2 < \eta^2
\end{aligned} \tag{14}$$

Substitution of Eq.(14) into Eq.(13) yields:

$$\begin{aligned}
E_{\mathrm{H}}(\zeta,\theta) = ZH_0\Bigg\{&\omega e^{-2\omega\zeta}\\
&- \frac{2\omega^2}{\pi}e^{-(1+\omega^2)\theta}\Bigg[\int_0^{1-\omega^2}\bigg((1-p_1)\exp\Big\{[(1-\omega^2)^2-\eta^2]^{1/2}\theta\Big\}\\
&+ (1+p_1)\exp\Big\{-[(1-\omega^2)^2-\eta^2]^{1/2}\theta\Big\}\bigg)\frac{\cos\eta\zeta}{\eta^2+4\omega^2}\,d\eta\\
&+ \int_{1-\omega^2}^{\infty}\bigg((1+ip_2)\exp\Big\{i[\eta^2-(1-\omega^2)^2]^{1/2}\theta\Big\}\\
&+ (1-ip_2)\exp\Big\{-i[\eta^2-(1-\omega^2)^2]^{1/2}\theta\Big\}\bigg)\frac{\cos\eta\zeta}{\eta^2+4\omega^2}d\eta\Bigg]\Bigg\} + E_\zeta(\theta)
\end{aligned} \tag{15}$$

Equations (9) and (15) are equal for any value of ω if we choose

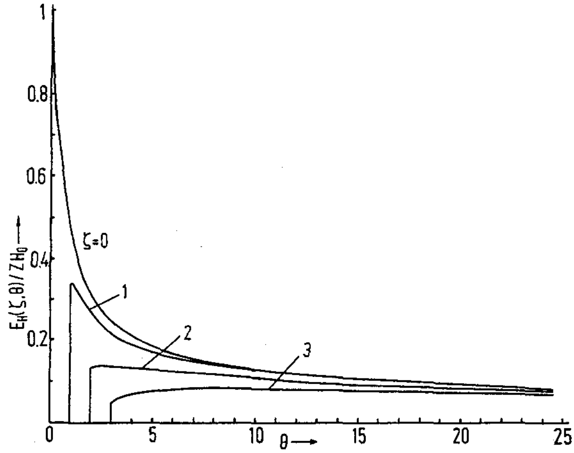

FIG.6.3-1. Magnitude E_H of the associated electric field strength according to Eq.(11) caused by a magnetic step function excitation at the plane $\zeta = 0$ of a lossy medium for $\zeta = 0$, 1, 2, 3. Note that the maximum of $E_H(\zeta, \theta)$ is very close to the jump for $\zeta = 0$—too close to show in our plot—but shifts to the right for $\zeta > 0$, as clearly shown by $E_H(3, \theta)$.

$$E_\theta(\zeta)e^{-2\theta} = E_\zeta(\theta) = E_{H0}e^{-2\theta} \tag{16}$$

The initial condition of Eq.(6.1-4) requires $E_{H0} = 0$.

The magnetic field strength of Eq.(3) assumes for $\omega = 0$ the form

$$H_H(\zeta, \theta) = H_0[1 + w(\zeta, \theta)] \tag{17}$$

where $w(\zeta, \theta)$ is defined by Eq.(6.1-41). It may be rewritten in the form of Eqs.(6.1-51) or (1.3-16):

$$H_H = H_0 \left[1 - e^{-\theta} \int_0^\zeta \left(\frac{\theta I_1(\sqrt{\theta^2 - \zeta'^2})}{(\theta^2 - \zeta'^2)^{1/2}} + I_0\left(\sqrt{\theta^2 - \zeta'^2}\right) \right) d\zeta' \right] \quad \text{for } \zeta < \theta$$
$$= 0 \quad \text{for } \zeta > \theta \tag{18}$$

Plots of $H_H(\zeta, \theta)$ are shown in Figs.1.3-1 and 1.3-2 if E_E and E_0 are replaced there by H_H and H_0. Plots of the associated field strength $E_H(\zeta, \theta)$ according to Eq.(11) are shown in Fig.6.3-1 for $\zeta = 0$, 1, 2, 3. The functions are zero for $\theta < 0$. There is a jump at $\theta = \zeta$. All plots approach zero for $\theta \to \infty$.

The delays and the jumps in Fig.6.3-1 correspond to the results for the associated magnetic field strength due to electric excitation in Fig.1.4-4. But the curves now drop to zero for $\theta \to \infty$ rather than increase linearly with θ. The reason is that the magnetic field strength is not attenuated directly by losses. It must first be transformed into an electric field strength, which is then attenuated by ohmic losses. As the electric field strength approaches zero, the ohmic losses approach zero too, and a time invariant magnetic field strength without electric field strength remains. This is, of course, what we expect to observe when an electromagnet is switched on.

It turns out that Eqs.(9) and (15) for $\omega = 0$ or $s = 0$ can be obtained by choosing $\omega = 0$ at the beginning of the calculation. This was not possible in Sections 6.1 and 6.2. We show here the simplified solution. Instead of Eq.(3) we use

$$H_{\mathrm{H}}(\zeta,\theta) = H_0[1 + w(\zeta,\theta)] \tag{19}$$

Substitution into Eq.(5) yields:

$$\begin{aligned} E_{\mathrm{H}}(\zeta,\theta) &= -ZH_0 e^{-2\theta}\int \frac{\partial w}{\partial \zeta} e^{2\theta} d\theta + E_\theta(\zeta)e^{-2\theta} \\ &= -\frac{ZH_0}{2\pi}\int_0^\infty \left(\frac{A_1(\eta)}{2+\gamma_1}e^{\gamma_1\theta} + \frac{A_2(\eta)}{2+\gamma_2}e^{\gamma_2\theta}\right)\eta\cos\eta\zeta\, d\eta + E_\theta(\zeta)e^{-2\theta} \end{aligned} \tag{20}$$

From Eqs.(8) and (6.1-24) we obtain for $\omega = 0$:

$$\begin{aligned} \frac{A_1(\eta)\eta}{2+\gamma_1} &= -\frac{2}{(1-\eta^2)^{1/2}} && \text{for } 1 > \eta^2 \\ &= +i\frac{2}{(\eta^2-1)^{1/2}} && \text{for } 1 < \eta^2 \\ \frac{A_2(\eta)\eta}{2+\gamma_2} &= +\frac{2}{(1-\eta^2)^{1/2}} && \text{for } 1 > \eta^2 \\ &= -i\frac{2}{(\eta^2-1)^{1/2}} && \text{for } 1 < \eta^2 \end{aligned} \tag{21}$$

$$\begin{aligned} &\gamma_1 = -1 + (1-\eta^2)^{1/2}, \quad \gamma_2 = -1 - (1-\eta^2)^{1/2} \quad \text{for } 1 > \eta^2 \\ &\gamma_1 = -1 + i(\eta^2-1)^{1/2}, \quad \gamma_2 = -1 - i(\eta^2-1)^{1/2} \quad \text{for } \eta^2 > 1 \end{aligned} \tag{22}$$

Substitution into Eq.(20) yields Eq.(11). This was much simpler than the calculation from Eq.(6) to Eq.(11). Let us see whether the transition $\omega \to 0$ at the beginning rather than at the end of the calculation can be made for Eq.(4) too. Substitution of Eq.(19) into Eq.(4) yields for $\omega = 0$:

$$\begin{aligned} E_{\mathrm{H}}(\zeta,\theta) &= -Z\int \frac{\partial w}{\partial \theta} d\zeta + E_\zeta(\theta) \\ &= \frac{ZH_0}{2\pi}\int_0^\infty \left[A_1(\eta)\gamma_1 e^{\gamma_1\theta} + A_2(\eta)\gamma_2 e^{\gamma_2\theta}\right]\frac{\cos\eta\zeta}{\eta} d\eta + E_\zeta(\theta) \end{aligned} \tag{23}$$

From Eq.(14) we obtain for $\omega = 0$:

$$\begin{aligned}
\frac{A_1(\eta)\gamma_1}{\eta} &= +\frac{2}{(1-\eta^2)^{1/2}} \quad \text{for } 1 > \eta^2 \\
&= -i\frac{2}{(\eta^2-1)^{1/2}} \quad \text{for } 1 < \eta^2 \\
\frac{A_2(\eta)\gamma_2}{\eta} &= -\frac{2}{(1-\eta^2)^{1/2}} \quad \text{for } 1 > \eta^2 \\
&= +i\frac{2}{(\eta^2-1)^{1/2}} \quad \text{for } 1 < \eta^2
\end{aligned} \tag{24}$$

Substitution into Eq.(23) yields again Eq.(11) with the term $E_\theta(\zeta)e^{-2\theta}$ replaced by $E_\zeta(\eta)$. Hence, the substitution $\omega = 0$ at the beginning of the calculation worked again.

We have assumed $\sigma > 0$, $s \to 0$ in Sections 6.1 to 6.3. The opposite case $s > 0$, $\sigma \to 0$ follows readily from the symmetry of Eqs.(1.2-9) and (1.2-10).

6.4 Electric Field Strength Due to Electric Ramp Function

We replace the electric step function of Eq.(6.1-2) by an electric exponential ramp function

$$\begin{aligned}
E(0,\theta) = E_1 S(\theta)(1-e^{-\iota\theta}) &= 0 \quad &\text{for } \theta < 0 \\
&= E_1(1-e^{-\iota\theta}) \quad &\text{for } \theta \geq 0
\end{aligned} \tag{1}$$

and repeat the calculation. At a great distance ζ we have the further boundary condition

$$E(\infty,\theta) = \text{finite} \tag{2}$$

If $E(\zeta,\theta)$ and $H(\zeta,\theta)$ are zero for $\zeta \geq 0$ at the time $\theta = 0$ we have the initial conditions

$$E(\zeta,0) = H(\zeta,0) = 0, \ \zeta \geq 0 \tag{3}$$

This equation implies that the derivatives with respect to ζ must be zero too:

$$\partial E(\zeta,0)/\partial\zeta = \partial H(\zeta,0)/\partial\zeta = 0 \tag{4}$$

According to Eqs.(1.2-9) and (1.2-10) the derivatives of the field strengths with respect to θ must vanish, if the field strengths and their derivatives with respect to ζ are zero:

$$\partial E(\zeta,\theta)/\partial\theta = \partial H(\zeta,\theta)/\partial\theta = 0 \quad \text{for } \theta = 0,\ \zeta \geq 0 \tag{5}$$

We assume that the solution of the previously derived Eq.(6.1-1)

$$\begin{gathered}\partial^2 E/\partial\zeta^2 - \partial^2 E/\partial\theta^2 - 2(1+\omega^2)\partial E/\partial\theta - 4\omega^2 E = 0 \\ \omega^2 = \epsilon s/\mu\sigma,\ \theta = \sigma t/2\epsilon.\ \zeta = Z\sigma y/2,\ Z = \sqrt{\mu/\epsilon}\end{gathered} \tag{6}$$

can be written in the form:

$$E(\zeta,\theta) = E_{\mathrm{E}}(\zeta,\theta) = E_1[u(\zeta,\theta) + (1 - e^{-\iota\theta})F(\zeta)] \tag{7}$$

Substitution of $E_1(1 - e^{-\iota\theta})F(\zeta)$ into Eq.(6) yields:

$$(1-e^{-\iota\theta})\frac{\partial^2 F}{\partial\zeta^2} + \iota^2 e^{-\iota\theta}F(\zeta) - 2\iota(1+\omega^2)e^{-\iota\theta}F(\zeta) - 4\omega^2(1-e^{-\iota\theta})F(\zeta) = 0 \tag{8}$$

Since $F(\zeta)$ is assumed to be a function of ζ but not of θ, the terms with different functions of θ must vanish separately. We get thus an equation for the first and the last term of Eq.(8)

$$\partial^2 F/\partial\zeta^2 - 4\omega^2 F = 0 \tag{9}$$

and a second equation for the two remaining terms

$$\iota - 2(1+\omega^2) = 0 \tag{10}$$

Equation (9) yields again the solution of Eq.(6.1-10)

$$F(\zeta) = e^{-2\omega\zeta} \tag{11}$$

while Eq.(10) yields ι as function of ϵ, s, μ, and σ:

$$\iota = 2(1+\omega^2) = 2a = 2(1 + \epsilon s/\mu\sigma) \tag{12}$$

Substitution of Eq.(11) into Eq.(7) yields:

$$E(\zeta,\theta) = E_{\mathrm{E}}(\zeta,\theta) = E_1[u(\zeta,\theta) + (1 - e^{-2a\theta})e^{-2\omega\zeta}] \tag{13}$$

We equate $E_{\mathrm{E}}(0,\theta)$ with the boundary condition of Eq.(1)

$$u(0,\theta) + 1 - e^{-2a\theta} = 1 - e^{-2a\theta} \tag{14}$$

and obtain the homogeneous boundary condition

$$u(0,\theta) = 0 \tag{15}$$

which is the goal of this mathematical method. At the plane $\zeta \to \infty$ we obtain from Eq.(2) with $F(\infty) = 0$ the boundary condition

$$u(\infty, \theta) = \text{finite} \tag{16}$$

The initial condition of Eq.(3) yields

$$E_{\mathrm{E}}(\zeta, 0) = E_1 u(\zeta, 0) = 0 \tag{17}$$

while the second initial condition of Eq.(5) requires according to Eq.(13):

$$\partial u/\partial\theta + 2ae^{-2a\theta}e^{-2\omega\zeta} = 0 \quad \text{for } \theta = 0,\ \zeta \geq 0 \tag{18}$$

The calculation of $u(\zeta, \theta)$ proceeds as in Section 6.1 for $w(\zeta, \theta)$ until Eqs.(6.1-26) and (6.1-27) are reached:

$$u(\zeta, \theta) = \int_0^\infty [A_1(\kappa)\exp(\gamma_1\theta) + A_2(\kappa)\exp(\gamma_2\theta)] \sin 2\pi\kappa\zeta \, d\kappa \tag{19}$$

$$\frac{\partial u}{\partial\theta} = \int_0^\infty [A_1(\kappa)\gamma_1\exp(\gamma_1\theta) + A_2(\kappa)\gamma_2\exp(\gamma_2\theta)] \sin 2\pi\kappa\zeta \, d\kappa \tag{20}$$

The coefficients γ_1 and γ_2 are defined by Eq.(6.1-24). Substitution of Eqs.(19) and (20) into Eqs. (17) and (18) yields:

$$\int_0^\infty [A_1(\kappa) + A_2(\kappa)] \sin 2\pi\kappa\zeta \, d\kappa = 0 \tag{21}$$

$$\int_0^\infty [A_1(\kappa)\gamma_1 + A_2(\kappa)\gamma_2] \sin 2\pi\kappa\zeta \, d\kappa = -2ae^{-2\omega\zeta} \tag{22}$$

These two equations should be compared to Eqs.(6.1-28) and (6.1-29). Using the Fourier sine transform pair of Eq.(6.1-30) we obtain from Eqs.(21) and (22):

$$A_1(\kappa) + A_2(\kappa) = 0 \tag{23}$$

$$A_1(\kappa)\gamma_1 + A_2(\kappa)\gamma_2 = -8a\int_0^\infty e^{-2\omega\zeta} \sin 2\pi\kappa\zeta \, d\zeta \tag{24}$$

Using Eq.(6.1-33) we obtain:

$$A_1(\kappa)\gamma_1 + A_2(\kappa)\gamma_2 = -\frac{16a\pi\kappa}{(2\pi\kappa)^2 + 4\omega^2} = -\frac{16a\pi\kappa}{b^2} \tag{25}$$

With the help of Eq.(6.1-24) we may solve Eqs.(23) and (25):

$$A_1(\kappa) = -A_2(\kappa) = -\frac{16a\pi\kappa}{b^2(\gamma_1 - \gamma_2)} = -\frac{8a\pi\kappa}{b^2\,(a^2-b^2)^{1/2}} \quad \text{for } a^2 > b^2$$
$$= +i\frac{8a\pi\kappa}{b^2\,(b^2-a^2)^{1/2}} \quad \text{for } a^2 < b^2 \quad (26)$$

We substitute Eqs.(26) and (6.1-24) into Eq.(19):

$$u(\zeta,\theta) = -4ae^{-a\theta}\left(\int_0^K \frac{\exp[(a^2-b^2)^{1/2}\theta] - \exp[-(a^2-b^2)^{1/2}\theta]}{(a^2-b^2)^{1/2}}\frac{\sin 2\pi\kappa\zeta}{b^2/2\pi\kappa}d\kappa\right.$$
$$\left. - i\int_K^\infty \frac{\exp[i(b^2-a^2)^{1/2}\theta] - \exp[-i(b^2-a^2)^{1/2}\theta]}{(b^2-a^2)^{1/2}}\frac{\sin 2\pi\kappa\zeta}{b^2/2\pi\kappa}d\kappa\right)$$
$$K = (1-\omega^2)/2\pi,\ a = 1+\omega^2,\ b^2 = (2\pi\kappa)^2 + 4\omega^2,\ \omega^2 = \epsilon s/\mu\sigma \quad (27)$$

This equation may be rewritten into a more compact form with the help of hyperbolic and trigonometric functions as well as the substitution $2\pi\kappa = \eta$:

$$u(\zeta,\theta) = -\frac{4}{\pi}(1+\omega^2)\exp[-(1+\omega^2)\theta]\left(\int_0^{1-\omega^2} \frac{\operatorname{sh}\{[(1-\omega^2)^2-\eta^2]^{1/2}\theta\}}{[(1-\omega^2)^2-\eta^2]^{1/2}}\frac{\eta\sin\zeta\eta}{\eta^2+4\omega^2}d\eta\right.$$
$$\left. + \int_{1-\omega^2}^\infty \frac{\sin\{[\eta^2-(1-\omega^2)^2]^{1/2}\theta\}}{[\eta^2-(1-\omega^2)^2]^{1/2}}\frac{\eta\sin\zeta\eta}{\eta^2+4\omega^2}d\eta\right)$$
$$\omega^2 = \epsilon s/\mu\sigma,\ 2\pi\kappa = \eta \quad (28)$$

Equation (7) may be rewritten as follows:

$$E(\zeta,\theta) = E_{\mathrm{E}}(\zeta,\theta) = E_1\left[\left(1 - e^{-2(1+\omega^2)\theta}\right)e^{-2\omega\zeta} + u(\zeta,\theta)\right] \quad (29)$$

Figure 6.4-1 shows plots of $E_{\mathrm{E}}(\zeta,\theta)/E_1$ for $\zeta = 1$ and $\zeta = 3$ as function of θ for various values of ω. The values of ω are chosen by observing that the relation

$$\omega^2 = \frac{\epsilon s}{\mu\sigma} = \frac{1}{Z^2}\frac{s}{\sigma} \quad (30)$$

implies $s/\sigma = Z^2\omega^2$.

Figure 6.4-2 shows plots of $E_{\mathrm{E}}(\zeta,\theta)/E_1$ for $\theta = 1$ and $\theta = 3$ as function of ζ. The number of values of the parameter ω has been reduced compared with

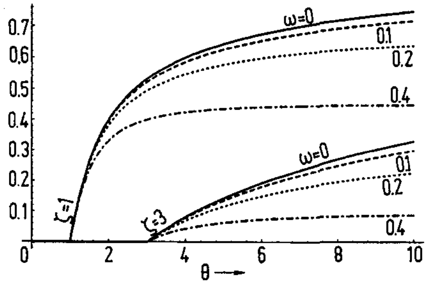

FIG.6.4-1. Electric field strengths $E_E(\zeta,\theta)/E_1$ according to Eq.(29) as function of the normalized time θ with the normalized distance $\zeta = 1$, 3 and $\omega = 0$, 0.1, 0.2, 0.4 as parameters.

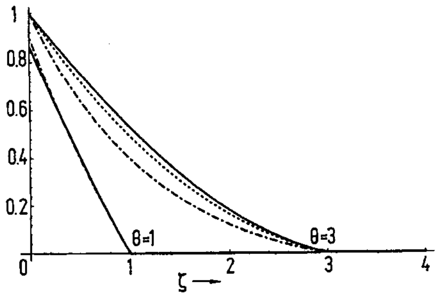

FIG.6.4-2. Electric field strengths $E_E(\zeta,\theta)/E_1$ according to Eq.(29) as function of the normalized distance ζ and the parameters $\theta = 1$, $\omega = 0$, 0.4 and $\theta = 3$, $\omega = 0$, 0.2, 0.4. The solid lines hold for $\omega = 0$, the dotted line for $\omega = 0.2$, and the dashed-dotted lines for $\omega = 0.4$.

Fig.6.4-1 since the plots are very close together. We note that $E_E(\zeta,\theta)$ has defined derivatives $\partial E_E/\partial\theta$ or $\partial E_E/\partial\zeta$ in the interval $0 \le \theta < \infty$, $0 \le \zeta < \infty$.

6.5 Magnetic Field Strength Due to Electric Ramp Function

From the electric field strength $E_E(\zeta,\theta)$ of Eq.(6.4-29) excited by an electric exponential ramp function follows the associated magnetic field strength $H_E(\zeta,\theta)$ by means of Eqs.(1.3-2) and (1.3-3) or Eqs.(1.4-7) and (1.4-8) for normalized notation:

$$H_E(\zeta,\theta) = e^{-2\omega^2\theta}\left(-\frac{1}{Z}\int\frac{\partial E_E}{\partial\zeta}e^{2\omega^2\theta}d\theta + H_\theta(\zeta)\right), \quad \omega^2 = \frac{\epsilon s}{\mu\sigma} \tag{1}$$

$$H_E(\zeta,\theta) = -\frac{1}{Z}\int\left(\frac{\partial E_E}{\partial\theta} + 2E_E\right)d\zeta + H_\zeta(\theta) \tag{2}$$

The various derivatives and integrals are calculated with the help of Eq.(6.4-29). First we determine $H_E(\zeta,\theta)$ of Eq.(1):

$$-\frac{E_1}{Z}e^{-2\omega^2\theta}\int\left(1-e^{-2(1+\omega^2)\theta}\right)\frac{\partial}{\partial\zeta}e^{-2\omega\zeta}e^{2\omega^2\theta}d\theta$$
$$=\frac{E_1}{Z}\frac{1}{\omega}\left(1+\omega^2e^{-2(1+\omega^2)\theta}\right)e^{-2\omega\zeta} \quad (3)$$

Using Eq.(6.4-19) we get another term of Eq.(1):

$$-\frac{E_1}{Z}e^{-2\omega^2\theta}\int\frac{\partial u(\zeta,\theta)}{\partial\zeta}e^{2\omega^2\theta}d\theta$$
$$=-\frac{2\pi E_1}{Z}\int_0^\infty\left(\frac{A_1(\kappa)\kappa}{\gamma_1+2\omega^2}e^{\gamma_1\theta}+\frac{A_2(\kappa)\kappa}{\gamma_2+2\omega^2}e^{\gamma_2\theta}\right)\cos 2\pi\zeta\kappa\,d\kappa \quad (4)$$

Substitution of Eqs.(3) and (4) into Eq.(1) brings:

$$H_{\mathrm{E}}(\zeta,\theta)=\frac{E_1}{Z}\left[\frac{1}{\omega}\left(1+\omega^2e^{-2(1+\omega^2)\theta}\right)e^{-2\omega\zeta}\right.$$
$$\left.-\int_0^\infty\left(\frac{2\pi\kappa A_1(\kappa)}{\gamma_1+2\omega^2}e^{\gamma_1\theta}+\frac{2\pi\kappa A_2(\kappa)}{\gamma_2+2\omega^2}e^{\gamma_2\theta}\right)\cos 2\pi\zeta\kappa\right]d\kappa$$
$$+H_\theta(\zeta)e^{-2\omega^2\theta} \quad (5)$$

We obtain with the help of Eqs.(6.4-26) and (6.1-24):

$$\frac{2\pi\kappa A_1(\kappa)}{\gamma_1+2\omega^2}=\frac{4a}{b^2}(1+q_{\mathrm{e}}),\quad \text{for } a^2>b^2 \text{ or } (1-\omega^2)^2>(2\pi\kappa)^2$$
$$=\frac{4a}{b^2}(1-iq'_{\mathrm{e}}),\quad \text{for } a^2<b^2 \text{ or } (1-\omega^2)^2<(2\pi\kappa)^2 \quad (6)$$
$$\frac{2\pi\kappa A_2(\kappa)}{\gamma_2+2\omega^2}=\frac{4a}{b^2}(1-q_{\mathrm{e}}),\quad \text{for } a^2>b^2$$
$$=\frac{4a}{b^2}(1+iq'_{\mathrm{e}}),\quad \text{for } a^2<b^2 \quad (7)$$

$$q_{\mathrm{e}}=\frac{1-\omega^2}{[(1-\omega^2)^2-(2\pi\kappa)^2]^{1/2}},\quad q'_{\mathrm{e}}=\frac{1-\omega^2}{[(2\pi\kappa)^2-(1-\omega^2)^2]^{1/2}}$$
$$a=1+\omega^2,\quad b^2=(2\pi\kappa)^2+4\omega^2,\quad \omega^2=\epsilon s/\mu\sigma \quad (8)$$

Substitution into Eq.(5) yields:

$$H_{\mathrm{E}}(\zeta,\theta) = \frac{E_1}{Z}\left\{\frac{1}{\omega}\left(1+\omega^2 e^{-2(1+\omega^2)\theta}\right)e^{-2\omega\zeta} - \frac{2a}{\pi}e^{-a\theta}\right.$$

$$\times\left[\int\limits_0^{1-\omega^2}\left((1+q_{\mathrm{e}})\exp\left[(a^2-b^2)^{1/2}\theta\right] + (1-q_{\mathrm{e}})\exp\left[-(a^2-b^2)^{1/2}\theta\right]\right)\frac{\cos\zeta\eta}{b^2}d\eta\right.$$

$$\left.\left.+\int\limits_{1-\omega^2}^{\infty}\left((1-iq_{\mathrm{e}}')\exp\left[i(b^2-a^2)^{1/2}\theta\right]+(1+iq_{\mathrm{e}}')\exp\left[-i(b^2-a^2)^{1/2}\theta\right]\right)\frac{\cos\zeta\eta}{b^2}d\eta\right]\right\}$$

$$+ H_\theta(\zeta)e^{-2\omega^2\theta} \quad (9)$$

Using hyperbolic and trigonometric functions we obtain:

$$H_{\mathrm{E}}(\zeta,\theta) = \frac{E_1}{Z}\left\{\frac{1}{\omega}\left(1+\omega^2 e^{-2(1+\omega^2)\theta}\right)e^{-2\omega\zeta}\right.$$

$$-\frac{4(1+\omega^2)}{\pi}e^{-(1+\omega^2)\theta}\left[\int\limits_0^{1-\omega^2}\left(\mathrm{ch}\left\{[(1-\omega^2)^2-\eta^2]^{1/2}\theta\right\}\right.\right.$$

$$\left.+\frac{1-\omega^2}{[(1-\omega^2)^2-\eta^2]^{1/2}}\,\mathrm{sh}\left\{[(1-\omega^2)^2-\eta^2]^{1/2}\theta\right\}\right)\frac{\cos\zeta\eta}{\eta^2+4\omega^2}d\eta$$

$$+\int\limits_{1-\omega^2}^{\infty}\left(\cos\left\{[\eta^2-(1-\omega^2)^2]^{1/2}\theta\right\}\right.$$

$$\left.\left.\left.+\frac{1-\omega^2}{[\eta^2-(1-\omega^2)^2]^{1/2}}\sin\left\{[\eta^2-(1-\omega^2)^2]^{1/2}\theta\right\}\right)\frac{\cos\zeta\eta}{\eta^2+4\omega^2}d\eta\right]\right\}$$

$$+ H_\theta(\zeta)e^{-2\omega^2\theta} \quad (10)$$

We turn to Eq.(2). From Eq.(6.4-29) we get:

$$H_{\mathrm{E}}(\zeta,\theta) = \frac{E_1}{Z}\left[\frac{1}{\omega}\left(1+\omega^2 e^{-2a\theta}\right)e^{-2\omega\zeta} + \int\left(\frac{\partial u}{\partial\theta}+2u\right)d\zeta\right] + H_\zeta(\theta) \quad (11)$$

Equations (6.4-19) and (6.4-20) yield:

$$\int\left(\frac{\partial u}{\partial\theta}+2u\right)d\zeta = -\int\limits_0^{\infty}\left(\frac{2+\gamma_1}{2\pi\kappa}A_1(\kappa)e^{\gamma_1\theta}+\frac{2+\gamma_2}{2\pi\kappa}A_2(\kappa)e^{\gamma_2\theta}\right)\cos 2\pi\zeta\kappa\,d\kappa \quad (12)$$

With γ_1, γ_2 from Eq.(6.1-24) and $A_1(\kappa)$, $A_2(\kappa)$ from Eq.(6.4-26) we obtain:

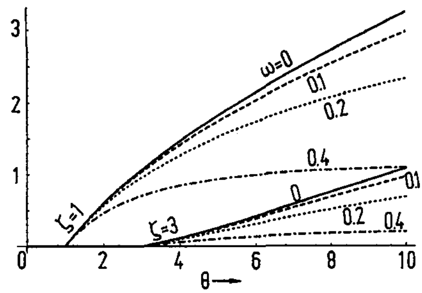

FIG.6.5-1. Associated magnetic field strength $H_{\mathrm{E}}(\zeta,\theta)Z/E_1$ according to Eq.(10) as function of the normalized time θ with the normalized distance $\zeta = 1$, 3 and $\omega = 0$, 0.1, 0.2, 0.4 as parameters.

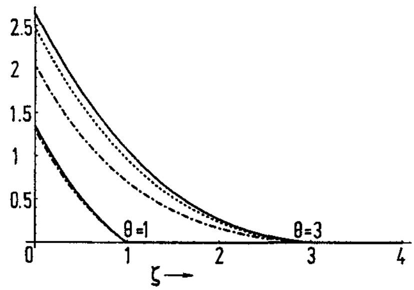

FIG.6.5-2. Associated magnetic field strength $H_{\mathrm{E}}(\zeta,\theta)Z/E_1$ according to Eq.(10) as function of the normalized distance ζ and the parameters $\theta = 1$, $\omega = 0$, 0.4 and $\theta = 3$, $\omega = 0$, 0.2, 0.4. The solid lines hold for $\omega = 0$, the dotted line for $\omega = 0.2$, and the dashed-dotted lines for $\omega = 0.4$.

$$
\begin{aligned}
-\frac{2+\gamma_1}{2\pi\kappa}A_1(\kappa) = \frac{2+\gamma_1}{2\pi\kappa}\frac{16a\pi\kappa}{b^2(\gamma_1-\gamma_2)} &= \frac{4a}{b^2}(1+q_{\mathrm{e}}) \quad \text{for } a^2 > b^2 \\
&= \frac{4a}{b^2}(1-iq'_{\mathrm{e}}) \quad \text{for } a^2 < b^2 \\
-\frac{2+\gamma_2}{2\pi\kappa}A_2(\kappa) = \frac{2+\gamma_2}{2\pi\kappa}\frac{16a\pi\kappa}{b^2(\gamma_2-\gamma_1)} &= \frac{4a}{b^2}(1-q_{\mathrm{e}}) \quad \text{for } a^2 > b^2 \\
&= \frac{4a}{b^2}(1+iq'_{\mathrm{e}}) \quad \text{for } a^2 < b^2 \qquad (13)
\end{aligned}
$$

The coefficients q_{e}, q'_{e} are defined by Eq.(8). Substitution of Eqs.(12) and (13) into Eq.(11) yields again Eq.(9) but the integration constant $H_\theta(\zeta)e^{-2\omega\theta}$ is replaced by $H_\zeta(\theta)$. Hence, the condition

$$
H_\theta(\zeta)e^{-2\omega\theta)} = H_\zeta(\theta) = H_{\mathrm{E}0}e^{-2\omega\theta} \qquad (14)
$$

must be satisfied. The initial condition of Eq.(6.4-3) requires $H_{\mathrm{E}0} = 0$.

For $\omega = 0$ one may rewrite Eq.(10) into the following form (Harmuth 1986, Eqs.2.9-28 to 2.9-30):

$$H_{\mathrm{E}}(\zeta,\theta) = \frac{E_1}{Z}[-2\zeta + I'_{\mathrm{E3}}(\zeta,\theta) - I_{\mathrm{E4}}(\zeta,\theta)] \tag{15}$$

$$\begin{aligned} I'_{\mathrm{E3}}(\zeta,\theta) = \frac{2}{\pi}\Bigg\{ 2\Bigg[\frac{1}{d} - \int_0^d [\exp(-\eta^2\theta/2) - 1]\frac{d\eta}{\eta^2} \\ - e^{-\theta}\int_d^1 \left(\mathrm{ch}(1-\eta^2)^{1/2}\theta + \frac{\mathrm{sh}(1-\eta^2)^{1/2}\theta}{(1-\eta^2)^{1/2}}\right)\frac{d\eta}{\eta^2}\Bigg] \\ + e^{-\theta}\int_0^1 \left(\mathrm{ch}(1-\eta^2)^{1/2}\theta + \frac{\mathrm{sh}(1-\eta^2)^{1/2}\theta}{(1-\eta^2)^{1/2}}\right)\left(\frac{\sin(\zeta\eta/2)}{\eta/2}\right)^2 d\eta\Bigg\} \\ d \ll 1 \end{aligned} \tag{16}$$

$$I_{\mathrm{E4}}(\zeta,\theta) = \frac{4}{\pi}e^{-\theta}\int_1^\infty \left(\cos(\eta^2-1)^{1/2}\theta + \frac{\sin(\eta^2-1)^{1/2}\theta}{(\eta^2-1)^{1/2}}\right)\frac{\cos\zeta\eta}{\eta^2}d\eta \tag{17}$$

The function $I'_{\mathrm{E3}}(\zeta,\theta)$ holds for the limit $d \to 0$. Plots made with decreasing values of d show that for $d < 10^{-5}$ one obtains changes that are of the order of the line width of the plot.

Figure 6.5-1 shows plots of the associated magnetic field strength $H_{\mathrm{E}}Z/E_1$ for $\zeta = 1$ and $\zeta = 3$ as function of θ for various values of ω. Figure 6.5-2 shows $H_{\mathrm{E}}(\zeta,\theta)Z/E_1$ for $\theta = 1$ and $\theta = 3$ as function of ζ. The number of values of the parameter ω has been reduced compared with Fig.6.5-1 since the plots are very close together. We note that $H_{\mathrm{E}}(\zeta,\theta)$ has defined derivatives $\partial H_{\mathrm{E}}/\partial\theta$ and $\partial H_{\mathrm{E}}/\partial\zeta$ in the interval $0 \leq \theta < \infty$, $0 \leq \zeta < \infty$.

6.6 Component A_{mz} of the Vector Potential

For the evaluation of the integral of Eq.(3.1-44) we obtain first the derivative $\partial E_{\mathrm{E}}(\zeta',\theta')/\partial\theta'$ from Eq.(1.5-2) with the substitutions $\zeta \to \zeta'$, $\theta \to \theta'$:

$$\frac{1}{E_1}E_{\mathrm{E}}(\zeta',\theta') = (1 - e^{-2(1+\omega^2)\theta'})e^{-2\omega\zeta'} + u(\zeta',\theta') \tag{1}$$

$$\frac{1}{E_1}\frac{\partial E_{\mathrm{E}}}{\partial\theta'} = 2(1+\omega^2)e^{-2(1+\omega^2)\theta'}e^{-2\omega\zeta'} + \frac{\partial u}{\partial\theta'} \tag{2}$$

Next we derive $\partial u(\zeta',\theta')/\partial\theta'$ from Eq.(6.4-28):

$$u(\zeta',\theta') = -\frac{4(1+\omega^2)}{\pi}e^{-(1+\omega^2)\theta'}\left(\int\limits_0^{1-\omega^2}\frac{\operatorname{sh}\omega_1\theta'}{\omega_1}\frac{\eta\sin\zeta'\eta}{\eta^2+4\omega^2}d\eta + \int\limits_{1-\omega^2}^{\infty}\frac{\sin\omega_2\theta'}{\omega_2}\frac{\eta\sin\zeta'\eta}{\eta^2+4\omega^2}d\eta\right)$$

$$\omega_1 = [(1-\omega^2)^2-\eta^2]^{1/2}, \quad \omega_2 = [\eta^2-(1-\omega^2)^2]^{1/2}, \quad \omega^2 = \epsilon s/\mu\sigma$$

$$\frac{\partial u}{\partial\theta'} = +\frac{4(1+\omega^2)}{\pi}e^{-(1+\omega^2)\theta'}\left[\int\limits_0^{1-\omega^2}\left(\frac{1+\omega^2}{\omega_1}\operatorname{sh}\omega_1\theta' - \operatorname{ch}\omega_1\theta'\right)\frac{\eta\sin\zeta'\eta}{\eta^2+4\omega^2}d\eta + \int\limits_{1-\omega^2}^{\infty}\left(\frac{1+\omega^2}{\omega^2}\sin\omega_2\theta' - \cos\omega_2\theta'\right)\frac{\eta\sin\zeta'\eta}{\eta^2+4\omega^2}d\eta\right] \quad (3)$$

The derivative $\partial H_{\mathrm{E}}(\zeta',\theta')/\partial\zeta'$ of Eq.(3.1-44) follows from Eq.(6.5-10) for $H_\theta(\zeta)e^{-2\omega^2\theta} = 0$:

$$\frac{Z}{E_1}H_{\mathrm{E}}(\zeta',\theta') = \frac{1}{\omega}\left(1+\omega^2e^{-2(1+\omega^2)\theta'}\right)e^{-2\omega\zeta'} + v(\zeta',\theta') \quad (4)$$

$$v(\zeta',\theta') = -\frac{4(1+\omega^2)}{\pi}e^{-(1+\omega^2)\theta'} \times\left[\int\limits_0^{1-\omega^2}\left(\operatorname{ch}\omega_1\theta' + \frac{1-\omega^2}{\omega_1}\operatorname{sh}\omega_1\theta'\right)\frac{\cos\zeta'\eta}{\eta^2+4\omega^2}d\eta + \int\limits_{1-\omega^2}^{\infty}\left(\cos\omega_2\theta' + \frac{1-\omega^2}{\omega_2}\sin\omega_2\theta'\right)\frac{\cos\zeta'\eta}{\eta^2+4\omega^2}d\eta\right] \quad (5)$$

The differentiation with respect to ζ' does not have to be carried out since it is followed immediately by an integration over ζ' in Eq.(3.1-44).

We turn to the integration over ζ' in Eq.(3.1-44). The first term on the right in Eq.(2) yields:

$$I_{\mathrm{e}1}(\zeta,\theta') = \int\limits_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} 2(1+\omega^2)e^{-2(1+\omega^2)\theta'}e^{-2\omega\zeta'}d\zeta' = \frac{1+\omega^2}{\omega}e^{-2\omega\zeta}\left(e^{2\omega\theta}e^{-2(1+\omega+\omega^2)\theta'} - e^{-2\omega\theta}e^{-2(1-\omega+\omega^2)\theta'}\right) \quad (6)$$

The final form of $I_{e1}(\zeta,\theta')$ was chosen to facilitate integration over θ' later on. Integral $I_{h1}(\zeta,\theta')$ is obtained from the first term on the right of Eq.(4):

$$I_{h1}(\zeta,\theta') = \int\limits_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} \frac{\partial}{\partial\zeta'}\left[\frac{1}{\omega}\left(1+\omega^2 e^{-2(1+\omega^2)\theta'}\right)e^{-2\omega\zeta'}\right]d\zeta'$$
$$= \frac{1}{\omega}\left(1+\omega^2 e^{-2(1+\omega^2)\theta'}\right)e^{-2\omega\zeta'}\Bigg|_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')}$$
$$= -\frac{1}{\omega}e^{-2\omega\zeta}\Big[e^{2\omega\theta}\left(e^{-2\omega\theta'}+\omega^2 e^{-2(1+\omega+\omega^2)\theta'}\right)$$
$$- e^{-2\omega\theta}\left(e^{2\omega\theta'}+\omega^2 e^{-2(1-\omega+\omega^2)\theta'}\right)\Big] \quad (7)$$

The integral of $\partial u(\zeta',\theta')/\partial\theta'$ over ζ' is derived from Eq.(3) and denoted $I_{e2}(\zeta,\theta')$:

$$I_{e2}(\zeta,\theta') = \int\limits_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} \frac{\partial u}{\partial\theta'}d\zeta' = \frac{4(1+\omega^2)}{\pi}e^{-(1+\omega^2)\theta'}$$
$$\times\Bigg\{\int\limits_0^{1-\omega^2}\Bigg[\int\limits_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')}\left(\frac{1+\omega^2}{\omega_1}\operatorname{sh}\omega_1\theta'-\operatorname{ch}\omega_1\theta'\right)\frac{\eta\sin\zeta'\eta}{\eta^2+4\omega^2}d\zeta'\Bigg]d\eta$$
$$+\int\limits_{1-\omega^2}^{\infty}\Bigg[\int\limits_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')}\left(\frac{1+\omega^2}{\omega_2}\sin\omega_2\theta'-\cos\omega_2\theta'\right)\frac{\eta\sin\zeta'\eta}{\eta^2+4\omega^2}d\zeta'\Bigg]d\eta\Bigg\}$$
$$= \frac{8(1+\omega^2)}{\pi}e^{-(1+\omega^2)\theta'}$$
$$\times\Bigg[\int\limits_0^{1-\omega^2}\sin\theta\eta\left(\frac{1+\omega^2}{\omega_1}\operatorname{sh}\omega_1\theta'-\operatorname{ch}\omega_1\theta'\right)\frac{\sin\zeta\eta\cos\theta'\eta-\cos\zeta\eta\sin\theta'\eta}{\eta^2+4\omega^2}d\eta$$
$$+\int\limits_{1-\omega^2}^{\infty}\sin\theta\eta\left(\frac{1+\omega^2}{\omega_2}\sin\omega_2\theta'-\cos\omega_2\theta'\right)\frac{\sin\zeta\eta\cos\theta'\eta-\cos\zeta\eta\sin\theta'\eta}{\eta^2+4\omega^2}d\eta\Bigg] \quad (8)$$

The terms $\sin\zeta\eta\cos\theta'\eta - \cos\zeta\eta\sin\theta'\eta$ are written in this expanded form to facilitate integration of $I_{e2}(\zeta,\theta')$ over θ'. Our last required integral over ζ' is that of $\partial v/\partial\zeta'$:

$$I_{h2}(\zeta,\theta') = \int\limits_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} \frac{\partial v}{\partial \zeta'} d\zeta' = v(\zeta',\theta')\Big|_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')}$$

$$= \frac{8(1+\omega)^2}{\pi} e^{-(1+\omega^2)\theta'}$$

$$\times\left[\int\limits_0^{1-\omega^2} \sin\theta\eta \left(\operatorname{ch}\omega_1\theta' + \frac{1-\omega^2}{\omega_1}\operatorname{sh}\omega_1\theta'\right) \frac{\sin\zeta\eta\cos\theta'\eta - \cos\zeta\eta\sin\theta'\eta}{\eta^2+4\omega^2} d\eta\right.$$

$$\left. + \int\limits_{1-\omega^2}^{\infty} \sin\theta\eta \left(\cos\omega_2\theta' + \frac{1-\omega^2}{\omega_2}\sin\omega_2\theta'\right) \frac{\sin\zeta\eta\cos\theta'\eta - \cos\zeta\eta\sin\theta'\eta}{\eta^2+4\omega^2} d\eta\right] \quad (9)$$

The next step is to integrate I_{e1}, I_{h1}, I_{e2}, and I_{h2} over θ' according to Eq.(3.1-44):

$$K_{e1}(\zeta,\theta) = \int\limits_0^{\theta} I_{e1}(\zeta,\theta')d\theta'$$

$$= \frac{1+\omega^2}{\omega\left[(1+\omega^2)^2-\omega^2\right]} e^{-2\omega\zeta}\left[(1+\omega^2)\operatorname{sh}2\omega\theta - \omega\operatorname{ch}2\omega\theta\right] \quad (10)$$

$$K_{h1}(\zeta,\theta) = \int\limits_0^{\theta} I_{h1}(\zeta,\theta')d\theta' = \frac{1}{\omega}e^{-2\omega\zeta}\left(\frac{1-\operatorname{ch}2\omega\theta}{\omega}\right.$$

$$\left. - \frac{\omega^3}{(1+\omega^2)^2-\omega^2}e^{-2(1+\omega^2)\theta} - \frac{\omega^2[(1+\omega^2)\operatorname{sh}2\omega\theta - \omega\operatorname{ch}2\omega\theta]}{(1+\omega^2)^2-\omega^2}\right) \quad (11)$$

$$K_{e1}(\zeta,\theta) + K_{h1}(\zeta,\theta)] = \frac{e^{-2\omega\zeta}}{\omega}\left(\frac{1-\operatorname{ch}2\omega\theta}{\omega}\right.$$

$$\left. - \frac{\omega^3}{(1+\omega^2)^2-\omega^2}e^{-2(1+\omega^2)\theta} + \frac{(1+\omega^2)\operatorname{sh}2\omega\theta - \omega\operatorname{ch}2\omega\theta}{(1+\omega^2)^2-\omega^2}\right) \quad (12)$$

$$\lim_{\omega\to 0}[K_{e1}(\zeta,\theta) + K_{h1}(\zeta,\theta)] = -1 + 2\theta - 2\theta^2 \quad (13)$$

The integral over $I_{e2}(\zeta,\theta')$ with respect to θ' follows from Eq.(8) and is denoted K_{e2}:

$$K_{e2}(\zeta,\theta) = \int_0^\theta I_{e2}(\zeta,\theta')d\theta' = \frac{8(1+\omega^2)}{\pi}\int_0^\theta e^{-(1+\omega^2)\theta'}$$

$$\times\left[\int_0^{1-\omega^2} \sin\theta\eta\left(\frac{1+\omega^2}{\omega_1}\operatorname{sh}\omega_1\theta' - \operatorname{ch}\omega_1\theta'\right)\frac{\sin\zeta\eta\cos\theta'\eta - \cos\zeta\eta\sin\theta'\eta}{\eta^2+4\omega^2}d\eta\right.$$

$$\left.+\int_{1-\omega^2}^\infty \sin\theta\eta\left(\frac{1+\omega^2}{\omega_2}\sin\omega_2\theta' - \cos\omega_2\theta'\right)\frac{\sin\zeta\eta\cos\theta'\eta - \cos\zeta\eta\sin\theta'\eta}{\eta^2+4\omega^2}d\eta\right]d\theta'$$

$$= \frac{8(1+\omega^2)}{\pi}\left\{\int_0^{1-\omega^2}\frac{\sin\theta\eta}{\eta^2+4\omega^2}\left[\sin\zeta\eta\left(\frac{1+\omega^2}{\omega_1}L_{21} - L_{23}\right)\right.\right.$$

$$\left.-\cos\zeta\eta\left(\frac{1+\omega^2}{\omega_1}L_{22} - L_{24}\right)\right]d\eta$$

$$+\int_{1-\omega^2}^\infty \frac{\sin\theta\eta}{\eta^2+4\omega^2}\left[\sin\zeta\eta\left(\frac{1+\omega^2}{\omega_2}L_{25} - L_{27}\right)\right.$$

$$\left.\left.-\cos\zeta\eta\left(\frac{1+\omega^2}{\omega_2}L_{26} - L_{28}\right)\right]d\eta\right\} \quad (14)$$

The integrals $L_{21} = L_{21}(\theta,\eta)$ to $L_{24} = L_{24}(\theta,\eta)$ are listed in Gradshteyn and Ryzhik (1980) as 2.674, 1-4 and L_{25}, L_{26} as 2.664, 1. Integrals L_{26} and L_{27} are readily transformed into tabulated integrals:

$$L_{21}(\theta,\eta) = \int_0^\theta e^{-(1+\omega^2)\theta'}\operatorname{sh}\omega_1\theta'\cos\eta\theta'\,d\theta'$$

$$= \frac{e^{(\omega_1-1-\omega^2)\theta}[(\omega_1-1-\omega^2)\cos\eta\theta + \eta\sin\eta\theta] - (\omega_1-1-\omega^2)}{2\left[(\omega_1-1-\omega^2)^2+\eta^2\right]}$$

$$+\frac{e^{-(\omega_1+1+\omega^2)\theta}[(\omega_1+1+\omega^2)\cos\eta\theta - \eta\sin\eta\theta] - (\omega_1+1+\omega^2)}{2\left[(\omega_1+1+\omega^2)^2+\eta^2\right]} \quad (15)$$

$$L_{22}(\theta,\eta) = \int_0^\theta e^{-(1+\omega^2)\theta'}\operatorname{sh}\omega_1\theta'\sin\eta\theta'\,d\theta'$$

$$= \frac{e^{(\omega_1-1-\omega^2)\theta}[(\omega_1-1-\omega^2)\sin\eta\theta - \eta\cos\eta\theta] + \eta}{2[(\omega_1-1-\omega^2)^2+\eta^2]}$$

$$+\frac{e^{-(\omega_1+1+\omega^2)\theta}[(\omega_1+1+\omega^2)\sin\eta\theta + \eta\cos\eta\theta] - \eta}{2[(\omega_1+1+\omega^2)^2+\eta^2]} \quad (16)$$

$$L_{23}(\theta,\eta)=\int_0^\theta e^{-(1+\omega^2)\theta}\,\mathrm{ch}\,\omega_1\theta'\cos\eta\theta'\,d\theta'$$
$$=\frac{e^{(\omega_1-1-\omega^2)\theta}[(\omega_1-1-\omega^2)\cos\eta\theta+\eta\sin\eta\theta]-(\omega_1-1-\omega^2)}{2[(\omega_1-1-\omega^2)^2+\eta^2]}$$
$$-\frac{e^{-(\omega_1+1+\omega^2)\theta}[(\omega_1+1+\omega^2)\cos\eta\theta-\eta\sin\eta\theta]-(\omega_1+1+\omega^2)}{2[(\omega_1+1+\omega^2)^2+\eta^2]} \tag{17}$$

$$L_{24}(\theta,\eta)=\int_0^\theta e^{-(1+\omega^2)\theta'}\,\mathrm{ch}\,\omega_1\theta'\sin\eta\theta'\,d\theta'$$
$$=\frac{e^{(\omega_1-1-\omega^2)\theta}[(\omega_1-1-\omega^2)\sin\eta\theta-\eta\cos\eta\theta]+\eta}{2[(\omega_1-1-\omega^2)^2+\eta^2]}$$
$$-\frac{e^{-(\omega_1+1+\omega^2)\theta}[(\omega_1+1+\omega^2)\sin\eta\theta+\eta\cos\eta\theta]-\eta}{2[(\omega_1+1+\omega^2)^2+\eta^2]} \tag{18}$$

$$L_{25}(\theta,\eta)=\int_0^\theta e^{-(1+\omega^2)\theta'}\sin\omega_2\theta'\cos\eta\theta'\,d\theta'$$
$$=-\frac{e^{-(1+\omega^2)\theta}[(1+\omega^2)\sin(\omega_2+\eta)\theta+(\omega_2+\eta)\cos(\omega_2+\eta)\theta]-(\omega_2+\eta)}{2[(1+\omega^2)^2+(\omega_2+\eta)^2]}$$
$$-\frac{e^{-(1+\omega^2)\theta}[(1+\omega^2)\sin(\omega_2-\eta)\theta+(\omega_2-\eta)\cos(\omega_2-\eta)\theta]-(\omega_2-\eta)}{2[(1+\omega^2)^2+(\omega_2-\eta)^2]} \tag{19}$$

$$L_{26}(\theta,\eta)=\int_0^\theta e^{-(1+\omega^2)\theta'}\sin\omega_2\theta'\sin\eta\theta'\,d\theta'$$
$$=\frac{e^{-(1+\omega^2)\theta}[(1+\omega^2)\cos(\omega_2+\eta)\theta-(\omega_2+\eta)\sin(\omega_2+\eta)\theta]-(1+\omega^2)}{2[(1+\omega^2)^2+(\omega_2+\eta)^2]}$$
$$-\frac{e^{-(1+\omega^2)\theta}[(1+\omega^2)\cos(\omega_2-\eta)\theta-(\omega_2-\eta)\sin(\omega_2-\eta)\theta]-(1+\omega^2)}{2[(1+\omega^2)^2+(\omega_2-\eta)^2]} \tag{20}$$

$$L_{27}(\theta,\eta)=\int_0^\theta e^{-(1+\omega^2)\theta'}\cos\omega_2\theta'\cos\eta\theta'\,d\theta'$$
$$=-\frac{e^{-(1+\omega^2)\theta}[(1+\omega^2)\cos(\omega_2+\eta)\theta-(\omega_2+\eta)\sin(\omega_2+\eta)\theta]-(1+\omega^2)}{2[(1+\omega^2)^2+(\omega_2+\eta)^2]}$$
$$-\frac{e^{-(1+\omega^2)\theta}[(1+\omega^2)\cos(\omega_2-\eta)\theta-(\omega_2-\eta)\sin(\omega_2-\eta)\theta]-(1+\omega^2)}{2[(1+\omega^2)^2+(\omega_2-\eta)^2]} \tag{21}$$

$$L_{28}(\theta,\eta)=\int_0^\theta e^{-(1+\omega^2)\theta'}\cos\omega_2\theta'\sin\eta\theta'\,d\theta'$$
$$=-\frac{e^{-(1+\omega^2)\theta}[(1+\omega^2)\sin(\omega_2+\eta)\theta+(\omega_2+\eta)\cos(\omega_2+\eta)\theta]-(\omega_2+\eta)}{2[(1+\omega^2)^2+(\omega_2+\eta)^2]}$$
$$+\frac{e^{-(1+\omega^2)\theta}[(1+\omega^2)\sin(\omega_2-\eta)\theta+(\omega_2-\eta)\cos(\omega_2-\eta)\theta]-(\omega_2-\eta)}{2[(1+\omega^2)^2+(\omega_2-\eta)^2]} \tag{22}$$

We turn to the integral over $I_{h2}(\zeta,\theta')$ with respect to θ' as defined by Eq.(9). It is denoted $K_{h2}(\zeta,\theta)$:

$$K_{h2}(\zeta,\theta)=\int_0^\theta I_{h2}(\zeta,\theta')\,d\theta'=\frac{8(1+\omega^2)}{\pi}\int_0^\theta e^{-(1+\omega^2)\theta'}$$
$$\times\left[\int_0^{1-\omega^2}\sin\theta\eta\left(\operatorname{ch}\omega_1\theta'+\frac{1-\omega^2}{\omega_1}\operatorname{sh}\omega_1\theta'\right)\frac{\sin\zeta\eta\cos\theta'\eta-\cos\zeta\eta\sin\theta'\eta}{\eta^2+4\omega^2}d\eta\right.$$
$$\left.+\int_{1-\omega^2}^{\infty}\sin\theta\eta\left(\cos\omega_2\theta'+\frac{1-\omega^2}{\omega_2}\sin\omega_2\theta'\right)\frac{\sin\zeta\eta\cos\theta'\eta-\cos\zeta\eta\sin\theta'\eta}{\eta^2+4\omega^2}d\eta\right]d\theta'$$
$$=\frac{8(1+\omega^2)}{\pi}\left\{\int_0^{1-\omega^2}\frac{\sin\theta\eta}{\eta^2+4\omega^2}\left[\sin\zeta\eta\left(L_{23}+\frac{1-\omega^2}{\omega_1}L_{21}\right)\right.\right.$$
$$\left.-\cos\zeta\eta\left(L_{24}+\frac{1-\omega^2}{\omega_1}L_{22}\right)\right]d\eta$$
$$+\int_{1-\omega^2}^{\infty}\frac{\sin\theta\eta}{\eta^2+4\omega^2}\left[\sin\zeta\eta\left(L_{27}+\frac{1-\omega^2}{\omega_2}L_{25}\right)\right.$$
$$\left.\left.-\cos\zeta\eta\left(L_{28}+\frac{1-\omega^2}{\omega_2}L_{26}\right)\right]d\eta\right\} \tag{23}$$

We may now rewrite Eq.(3.1-44) as a function of ζ and θ without the integrations over ζ' and θ'. First, Eq.(3.1-44) is normalized to bring it into conformity with our current notation:

$$\frac{Zc\sigma}{E_1}A_{mz}(\zeta,\theta)=-\int_0^\theta\left[\int_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')}\left(\frac{1}{E_1}\frac{\partial E_E}{\partial\theta'}+\frac{Z}{E_1}\frac{\partial H_E}{\partial\zeta'}\right)d\zeta'\right]d\theta' \tag{24}$$

We obtain with the help of Eqs.(10), (14) and (11), (23)

$$\int_0^\theta \left(\int_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} \frac{1}{E_\mathrm{E}} \frac{\partial E_\mathrm{E}}{\partial \theta'} d\zeta' \right) d\theta' = K_{\mathrm{e}1}(\zeta,\theta) + K_{\mathrm{e}2}(\zeta,\theta) \tag{25}$$

$$\int_0^\theta \left(\int_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} \frac{Z}{E_1} \frac{\partial H_\mathrm{E}}{\partial \zeta'} d\zeta' \right) d\theta' = K_{\mathrm{h}1}(\zeta,\theta) + K_{\mathrm{h}2}(\zeta,\theta) \tag{26}$$

and $A_{\mathrm{m}z}(\zeta,\theta)$ can be written in a form that requires only one numerical integration over the variable η:

$$\begin{aligned}
\frac{Zc\sigma}{E_1} A_{\mathrm{m}z}(\zeta,\theta) = &-\frac{e^{-2\omega\zeta}}{\omega}\left(\frac{1-\operatorname{ch} 2\omega\theta}{\omega} - \frac{\omega^3}{(1+\omega^2)^2-\omega^2} e^{-2(1+\omega^2)\theta} \right. \\
&\left. + \frac{(1+\omega^2)\operatorname{sh} 2\omega\theta - \omega \operatorname{ch} 2\omega\theta}{(1+\omega^2)^2-\omega^2} \right) \\
&- \frac{16(1+\omega^2)}{\pi} \left[\int_0^{1-\omega^2} \frac{\sin\theta\eta}{\omega_1(\eta^2+4\omega^2)} (L_{21}\sin\zeta\eta - L_{22}\cos\zeta\eta) d\eta \right. \\
&\left. + \int_{1-\omega^2}^{\infty} \frac{\sin\theta\eta}{\omega_2(\eta^2+4\omega^2)} (L_{25}\sin\zeta\eta - L_{26}\cos\zeta\eta) d\eta \right]
\end{aligned} \tag{27}$$

The terms L_{21}, L_{22}, L_{25}, and L_{26} are listed in Eqs.(15), (16), (19), and (20). There are no poles or singularities. Hence, we need to check only the limit $\eta \to \infty$ to confirm that Eq.(27) is an actual rather than a formal representation of $A_{\mathrm{m}z}(\zeta,\theta)$. We obtain the following variations of L_{21} to L_{28} for large values of η:

$$\begin{aligned}
&|L_{21}| \approx \frac{1}{\eta^2} \quad &&|L_{22}| \approx \frac{1}{\eta} \quad &&|L_{23}| \approx \frac{1}{\eta^2} \quad &&|L_{24}| \approx \frac{1}{\eta} \\
&|L_{25}| \approx \frac{1}{\eta} \quad &&|L_{26}| \approx \frac{1}{\eta^2} \quad &&|L_{27}| \approx \frac{1}{\eta^2} \quad &&|L_{28}| \approx \frac{1}{\eta}
\end{aligned} \tag{28}$$

Due to the factor $1/(\eta^2+4\omega^2)$ in the kernels of the integrals of Eq.(27) all terms of the integrals decrease like $1/\eta^3$ or $1/\eta^4$ for large values of η. Given the complexity of Eq.(27) such a fast convergence is very fortunate.

6.7 Component A_{ex} of the Vector Potential

We need to evaluate the integral of Eq.(3.1-45). The calculation is quite similar to that in Section 6.6 but we have now the derivatives $\partial E_\mathrm{E}(\zeta',\theta')/\partial\zeta'$ and $\partial H_\mathrm{E}(\zeta',\theta')/\partial\theta'$ instead of $\partial E_\mathrm{E}(\zeta',\theta')/\partial\theta'$ and $\partial H_\mathrm{E}(\zeta',\theta')/\partial\zeta'$. One does not have to carry out the differentiation of E_{E} with respect to ζ' since it is followed immediately by an integration with respect to ζ'. Hence, we get for the first term in Eq.(3.1-45) from Eqs.(6.4-28) or (6.6-1) and (6.6-3) for $\zeta \to \zeta'$, $\theta \to \theta'$:

$$\frac{1}{E_1}E_{\mathrm{E}}(\zeta',\theta') = \left(1 - e^{-2(1+\omega^2)\theta'}\right)e^{-2\omega\zeta'} + u(\zeta',\theta') \tag{1}$$

$$u(\zeta',\theta') = -\frac{4(1+\omega^2)}{\pi}e^{-(1+\omega^2)\theta'}\left(\int_0^{1-\omega^2}\frac{\operatorname{sh}\omega_1\theta'}{\omega_1}\frac{\eta\sin\zeta'\eta}{\eta^2+4\omega^2}d\eta + \int_{1-\omega^2}^{\infty}\frac{\sin\omega_2\theta'}{\omega_2}\frac{\eta\sin\zeta'\eta}{\eta^2+4\omega^2}d\eta\right)$$

$$\omega_1 = [(1-\omega^2)^2 - \eta^2]^{1/2}, \quad \omega_2 = [\eta^2 - (1-\omega^2)^2]^{1/2}, \quad \omega^2 = \epsilon s/\mu\sigma \tag{2}$$

The derivative $\partial H_{\mathrm{E}}(\zeta',\theta')/\partial\theta'$ of Eq.(3.1-45) follows from Eqs.(6.5-10) and (6.6-4) for $H_\theta(\zeta)e^{-2\omega^2\theta} = 0$:

$$\frac{Z}{E_1}H_{\mathrm{E}}(\zeta',\theta') = \frac{1}{\omega}\left(1 + \omega^2 e^{-2(1+\omega^2)\theta'}\right)e^{-2\omega\zeta'} + v(\zeta',\theta') \tag{3}$$

$$\frac{Z}{E_1}\frac{\partial H_{\mathrm{E}}}{\partial\theta'} = -2(1+\omega^2)\omega e^{-2(1+\omega^2)\theta'}e^{-2\omega\zeta'} + \frac{\partial v}{\partial\theta'} \tag{4}$$

The term $\partial v(\zeta',\theta')/\partial\theta'$ follows from Eq.(6.6-5):

$$\frac{\partial v}{\partial\theta'} = \frac{4(1+\omega^2)}{\pi}e^{-(1+\omega^2)\theta'} \times\left\{\int_0^{1-\omega^2}\left[2\omega^2\operatorname{ch}\omega_1\theta' + \left(\frac{1-\omega^4}{\omega_1} - \omega_1\right)\operatorname{sh}\omega_1\theta'\right]\frac{\cos\zeta'\eta}{\eta^2+4\omega^2}d\eta + \int_{1-\omega^2}^{\infty}\left[2\omega^2\cos\omega_2\theta' + \left(\frac{1-\omega^4}{\omega_2} + \omega_2\right)\sin\omega_2\theta'\right]\frac{\cos\zeta'\eta}{\eta^2+4\omega^2}d\eta\right\} \tag{5}$$

Let us turn to the integration over ζ' in Eq.(3.1-45). The first term on the right of Eq.(4) yields:

$$\begin{aligned}J_{\mathrm{h1}}(\zeta,\theta') &= -2(1+\omega^2)\omega e^{-2(1+\omega^2)\theta'}\int_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} e^{-2\omega\zeta'}d\zeta' \\ &= -(1+\omega^2)e^{-2\omega\zeta}\left(e^{2\omega\theta}e^{-2(1+\omega+\omega^2)\theta'} - e^{-2\omega\theta}e^{-2(1-\omega+\omega^2)\theta'}\right)\end{aligned} \tag{6}$$

As in Section 6.6 the specific form of $J_{h1}(\zeta,\theta')$ was chosen to facilitate integration over θ' later on. The integral $J_{e1}(\zeta,\theta')$ is obtained from the first term on the right of Eq.(1):

$$J_{e1}(\zeta,\theta') = \int\limits_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} \frac{\partial}{\partial\zeta'}\left[\left(1-e^{-2(1+\omega^2)\theta'}\right)e^{-2\omega\zeta'}\right]d\zeta'$$

$$= -e^{-2\omega\zeta}\Big[e^{2\omega\theta}\left(e^{-2\omega\theta'}-e^{-2(1+\omega+\omega^2)\theta'}\right)$$

$$-e^{-2\omega\theta}\left(e^{2\omega\theta'}-e^{-2(1-\omega+\omega^2)\theta'}\right)\Big] \qquad (7)$$

The integral of $\partial v/\partial\theta'$ over ζ' is derived from Eq.(5). It is denoted $J_{h2}(\zeta,\theta')$:

$$J_{h2}(\zeta,\theta') = \int\limits_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} \frac{\partial v}{\partial\theta'}d\zeta' = \frac{4(1+\omega^2)}{\pi}e^{-(1+\omega^2)\theta'}$$

$$\times\left\{\int\limits_0^{1-\omega^2}\left[\int\limits_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')}\left(2\omega^2\operatorname{ch}\omega_1\theta'+\frac{1-\omega^4-\omega_1^2}{\omega_1}\operatorname{sh}\omega_1\theta'\right)\frac{\cos\zeta'\eta}{\eta^2+4\omega^2}d\zeta'\right]d\eta\right.$$

$$\left.+\int\limits_{1-\omega^2}^{\infty}\left[\int\limits_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')}\left(2\omega^2\cos\omega_2\theta'+\frac{1-\omega^4+\omega_2^2}{\omega_2}\sin\omega_2\theta'\right)\frac{\cos\zeta'\eta}{\eta^2+4\omega^2}d\zeta'\right]d\eta\right\}$$

$$= \frac{8(1+\omega^2)}{\pi}e^{-(1+\omega^2)\theta'}$$

$$\times\left[\int\limits_0^{1-\omega^2}\cos\zeta\eta\left(\frac{1-\omega^4-\omega_1^2}{\omega_1}\operatorname{sh}\omega_1\theta'+2\omega^2\operatorname{ch}\omega_1\theta'\right)\frac{\sin\theta\eta\cos\theta'\eta-\cos\theta\eta\sin\theta'\eta}{\eta(\eta^2+4\omega^2)}d\eta\right.$$

$$\left.+\int\limits_{1-\omega^2}^{\infty}\cos\zeta\eta\left(\frac{1-\omega^4+\omega_2^2}{\omega_2}\sin\omega_2\theta'+2\omega^2\cos\omega_2\theta'\right)\frac{\sin\theta\eta\cos\theta'\eta-\cos\theta\eta\sin\theta'\eta}{\eta(\eta^2+4\omega^2)}d\eta\right] \qquad (8)$$

As in Section 6.6 the terms $\sin\theta\eta\cos\theta'\eta - \cos\theta\eta\sin\theta'\eta$ are written in this expanded form to facilitate integration of $J_{h2}(\zeta,\theta')$ over θ'. We still require the integral of $\partial u/\partial\zeta'$ over ζ':

$$J_{e2}(\zeta,\theta') = \int\limits_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} \frac{\partial u}{\partial \zeta'} d\zeta' = u(\zeta',\theta')\Big|_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')}$$
$$= -\frac{8(1+\omega^2)}{\pi} e^{-(1+\omega^2)\theta'}$$
$$\times \left(\int\limits_0^{1-\omega^2} \frac{\eta\cos\zeta\eta\,\mathrm{sh}\,\omega_1\theta'}{\omega_1} \frac{\sin\theta\eta\cos\theta'\eta - \cos\theta\eta\sin\theta'\eta}{\eta^2+4\omega^2} d\eta \right.$$
$$\left. + \int\limits_{1-\omega^2}^{\infty} \frac{\eta\cos\zeta\eta\sin\omega_2\theta'}{\omega_2} \frac{\sin\theta\eta\cos\theta'\eta - \cos\theta\eta\sin\theta'\eta}{\eta^2+4\omega^2} d\eta \right) \quad (9)$$

We must integrate J_{h1}, J_{e1}, J_{h2}, and J_{e2} with respect to θ' according to Eq.(3.1-45). Let us denote the resulting integrals by M_{h1} and M_{e2}:

$$M_{h1}(\zeta,\theta) = \int\limits_0^{\theta} J_{h1}(\zeta,\theta')d\theta'$$
$$= \frac{1+\omega^2}{(1+\omega^2)^2-\omega^2} e^{-2\omega\zeta}\left[(1+\omega^2)\left(e^{-2(1+\omega^2)\theta} - \mathrm{ch}\,2\omega\theta\right) + \omega\,\mathrm{sh}\,2\omega\theta\right] \quad (10)$$

$$M_{e1}(\zeta,\theta) = \int\limits_0^{\theta} J_{e1}(\zeta,\theta')d\theta' = e^{-2\omega\zeta}\left[\frac{1}{\omega}(1-\mathrm{ch}\,2\omega\theta)\right.$$
$$\left. + \frac{1+\omega^2}{(1+\omega^2)^2-\omega^2}\mathrm{sh}\,2\omega\theta + \frac{\omega}{(1+\omega^2)^2-\omega^2}\left(e^{-2(1+\omega^2)\theta} - \mathrm{ch}\,2\omega\theta\right)\right] \quad (11)$$

The sum of $M_{h1}(\zeta,\theta)$ and $M_{e1}(\zeta,\theta)$ becomes:

$$M_{h1}(\zeta,\theta) + M_{e1}(\zeta,\theta) = e^{-2\omega\zeta}\left\{\frac{1}{\omega}(1-\mathrm{ch}\,2\omega\theta) + \frac{1}{(1+\omega^2)^2-\omega^2}\right.$$
$$\left. \times \left[[(1+\omega^2)^2+\omega](e^{-2(1+\omega^2)\theta} - \mathrm{ch}\,2\omega\theta) + (1+\omega^2)(1+\omega)\,\mathrm{sh}\,2\omega\theta\right]\right\} \quad (12)$$

The integral over $J_{h2}(\zeta,\theta')$ with respect to θ' follows from Eq.(8). It is denoted $M_{h2}(\zeta,\theta)$:

$$M_{h2}(\zeta,\theta) = \int_0^\theta J_{h2}(\zeta,\theta')d\theta' = \frac{8(1+\omega^2)}{\pi}\int_0^\theta e^{-(1+\omega^2)\theta'}$$
$$\times\left\{\int_0^{1-\omega^2}\cos\zeta\eta\left(\frac{1-\omega^4-\omega_1^2}{\omega_1}\operatorname{sh}\omega_1\theta'+2\omega^2\operatorname{ch}\omega_1\theta'\right)\frac{\sin\theta\eta\cos\theta'\eta-\cos\theta\eta\sin\theta'\eta}{\eta(\eta^2+4\omega^2)}d\eta\right.$$
$$\left.+\int_{1-\omega^2}^\infty\cos\zeta\eta\left(\frac{1-\omega^2+\omega_2^2}{\omega_2}\sin\omega_2\theta'+2\omega^2\cos\omega_2\theta'\right)\frac{\sin\theta\eta\cos\theta'\eta-\cos\theta\eta\sin\theta'\eta}{\eta(\eta^2+4\omega^2)}d\eta\right\}d\theta'$$
$$=\frac{8(1+\omega^2)}{\pi}\left\{\int_0^{1-\omega^2}\frac{\cos\zeta\eta}{\eta(\eta^2+4\omega^2)}\left[\sin\theta\eta\left(\frac{1-\omega^4-\omega_1^2}{\omega_1}L_{21}+2\omega^2L_{23}\right)\right.\right.$$
$$\left.-\cos\theta\eta\left(\frac{1-\omega^4-\omega_1^2}{\omega_1}L_{22}+2\omega^2L_{24}\right)\right]d\eta$$
$$+\int_{1-\omega^2}^\infty\frac{\cos\zeta\eta}{\eta(\eta^2+4\omega^2)}\left[\sin\theta\eta\left(\frac{1-\omega^4+\omega_2^2}{\omega_2}L_{25}+2\omega^2L_{27}\right)\right.$$
$$\left.\left.-\cos\theta\eta\left(\frac{1-\omega^4+\omega_2^2}{\omega_2}L_{26}+2\omega^2L_{28}\right)\right]d\eta\right\} \quad (13)$$

The terms L_{21} to L_{28} are defined by Eqs.(6.6-15)–(6.6-22).

Let us turn to the integral over $J_{e2}(\zeta,\theta')$ with respect to θ' as defined in Eq.(9). We denote it $M_{e2}(\zeta,\theta)$:

$$M_{e2}(\zeta,\theta) = \int_0^\theta J_{e2}(\zeta,\theta')d\theta' = -\frac{8(1+\omega^2)}{\pi}\int_0^\theta e^{-(1+\omega^2)\theta'}$$
$$\times\left(\int_0^{1-\omega^2}\frac{\eta\cos\zeta\eta\operatorname{sh}\omega_1\theta'}{\omega_1}\frac{\sin\theta\eta\cos\theta'\eta-\cos\theta\eta\sin\theta'\eta}{\eta^2+4\omega^2}d\eta\right.$$
$$\left.+\int_{1-\omega^2}^\infty\frac{\eta\cos\zeta\eta\sin\omega_2\theta'}{\omega_2}\frac{\sin\theta\eta\cos\theta'\eta-\cos\theta\eta\sin\theta'\eta}{\eta^2+4\omega^2}d\eta\right)d\theta'$$
$$=-\frac{8(1+\omega^2)}{\pi}\left[\int_0^{1-\omega^2}\frac{\eta\cos\zeta\eta}{\omega_1(\eta^2+4\omega^2)}(L_{21}\sin\theta\eta-L_{22}\cos\theta\eta)d\eta\right.$$
$$\left.+\int_{1-\omega^2}^\infty\frac{\eta\cos\zeta\eta}{\omega_2(\eta^2+4\omega^2)}(L_{25}\sin\theta\eta-L_{26}\cos\theta\eta)d\eta\right] \quad (14)$$

We have accumulated enough integrals to rewrite Eq.(3.1-45) as a function of ζ and θ without the integrations over ζ' and θ'. First, we rewrite Eq.(3.1-45) for our current normalization:

$$\frac{Z^2c\sigma}{E_1}A_{ex}(\zeta,\theta)=-\int_0^\theta\left[\int_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')}\left(\frac{1}{E_1}\frac{\partial E_E(\zeta',\theta')}{\partial\zeta'}+\frac{Z}{E_1}\frac{\partial H_E(\zeta',\theta')}{\partial\theta'}\right)d\zeta'\right]d\theta' \quad (15)$$

Using Eqs.(1), (7), (11), (9), and (14) we obtain:

$$\int_0^\theta\left(\int_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')}\frac{1}{E_1}\frac{\partial E_E(\zeta',\theta')}{\partial\zeta'}d\zeta'\right)d\theta'=M_{e1}(\zeta,\theta)+M_{e2}(\zeta,\theta) \quad (16)$$

while Eqs.(3), (6), (10), (8), and (13) yield:

$$\int_0^\theta\left(\int_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')}\frac{Z}{E_1}\frac{\partial H_E(\zeta',\theta')}{\partial\theta'}d\zeta'\right)d\theta'=M_{h1}(\zeta,\theta)+M_{h2}(\zeta,\theta) \quad (17)$$

The component $A_{ex}(\zeta,\theta)$ of the potential $\mathbf{A}_e$ can be written in a form that requires only one numerical integration over the variable η:

$$\begin{aligned}\frac{Z^2c\sigma}{E_1}A_{ex}(\zeta,\theta) &= -[M_{h1}(\zeta,\theta)+M_{e1}(\zeta,\theta)+M_{h2}(\zeta,\theta)+M_{e2}(\zeta,\theta)]\\
&= -e^{-2\omega\zeta}\bigg\{\frac{1}{\omega}(1-\operatorname{ch}2\omega\theta)+\frac{1}{(1+\omega^2)^2-\omega^2}\\
&\quad\times\Big[[(1+\omega^2)^2+\omega](e^{-2(1+\omega^2)\theta}-\operatorname{ch}2\omega\theta)+(1+\omega^2)(1+\omega)\operatorname{sh}2\omega\theta\Big]\bigg\}\\
&+\frac{8(1+\omega^2)}{\pi}\bigg\{\int_0^{1-\omega^2}\frac{\cos\zeta\eta}{\eta^2+4\omega^2}\bigg[\left(\frac{\eta^2-1+\omega^4+\omega_1^2}{\eta\omega_1}L_{21}-\frac{2\omega^2}{\eta}L_{23}\right)\sin\theta\eta\\
&\quad-\left(\frac{\eta^2-1+\omega^4+\omega_1^2}{\eta\omega_1}L_{22}-\frac{2\omega^2}{\eta}L_{24}\right)\cos\theta\eta\bigg]d\eta\\
&\quad+\int_{1-\omega^2}^{\infty}\frac{\cos\zeta\eta}{\eta^2+4\omega^2}\bigg[\left(\frac{\eta^2-1+\omega^4-\omega_2^2}{\eta\omega_2}L_{25}-\frac{2\omega^2}{\eta}L_{27}\right)\sin\theta\eta\\
&\quad-\left(\frac{\eta^2-1+\omega^4-\omega_2^2}{\eta\omega_2}L_{26}-\frac{2\omega^2}{\eta}L_{28}\right)\cos\theta\eta\bigg]d\eta\bigg\} \quad (18)\end{aligned}$$

Equations (6.6-15)–(6.6-22) define the terms L_{21} to L_{28}.

The first integral of Eq.(18) requires some analysis of the limit $\eta \to 0$ before one can feed it into a computer. We separate the first integral into two terms, the first term for the interval $0 < \eta < \varepsilon \ll 1$ and the second term for the remaining interval $\varepsilon < \eta < 1 - \omega^2$. We obtain for $\eta \ll 1$:

$$\frac{\cos\zeta\eta}{\eta^2+4\omega^2}\Bigg[\left(\frac{\eta^2-1+\omega^4+\omega_1^2}{\eta\omega_1}L_{21}-\frac{2\omega^2}{\eta}L_{23}\right)\sin\theta\eta - \left(\frac{\eta^2-1+\omega^4+\omega_1^2}{\eta\omega_1}L_{22}-\frac{2\omega^2}{\eta}L_{24}\right)\cos\theta\eta\Bigg]$$
$$\approx \frac{1}{2}\left[-(L_{21}+L_{23})\theta+\frac{1}{\eta}(L_{22}+L_{24})\right],\quad \eta \ll 1 \quad (19)$$

From Eqs.(6.6-15)–(6.6-18) we obtain for $\eta \ll 1$:

$$L_{21}(\theta,\eta) \approx \frac{1-e^{-2\omega^2\theta}}{4\omega^2}-\frac{1-e^{-2\theta}}{4} \quad (20)$$

$$L_{23}(\theta,\eta) \approx \frac{1-e^{-2\omega^2\theta}}{4\omega^2}+\frac{1-e^{-2\theta}}{4} \quad (21)$$

$$L_{22}(\theta,\eta) \approx \eta\left(\frac{1-(1+2\omega^2\theta)e^{-2\omega^2\theta}}{8\omega^4}-\frac{1-(1+2\theta)e^{-2\theta}}{8}\right) \quad (22)$$

$$L_{24}(\theta,\eta) \approx \eta\left(\frac{1-(1+2\omega^2\theta)e^{-2\omega^2\theta}}{8\omega^4}+\frac{1-(1+2\theta)e^{-2\theta}}{8}\right) \quad (23)$$

Substitution into Eq.(19) brings:

$$\frac{1}{2}\left[-(L_{21}+L_{23})\theta+\frac{1}{\eta}(L_{22}+L_{24})\right]$$
$$= -\frac{\theta}{4\omega^2}\left(1-e^{-2\omega^2\theta}\right)+\frac{1}{8\omega^4}\left[1+(1+2\omega^2\theta)e^{-2\omega^2\theta}\right] \quad (24)$$

With

$$\int_0^{\varepsilon}\frac{1}{8}[1-(3-2\theta)e^{-2\theta}]\,d\eta = \frac{\varepsilon}{8}[1-(3-2\theta)e^{-2\theta}] \quad (25)$$

we may rewrite the first integral in Eq.(18) in the following computer friendly form:

$$\int_0^{1-\omega^2} \frac{\cos\zeta\eta}{\eta^2+4\omega^2}\Big[\dots\Big]d\eta \approx \frac{\varepsilon}{8}\big[1-(3-2\theta)e^{-2\theta}\big]$$

$$+\int_\varepsilon^{1-\omega^2} \frac{\cos\zeta\eta}{\eta^2+4\omega^2}\bigg[\bigg(\frac{\eta^2-1+\omega^4+\omega_1^2}{\eta\omega_1}L_{21}-\frac{2\omega^2}{\eta}L_{23}\bigg)\sin\theta\eta$$

$$-\bigg(\frac{\eta^2-1+\omega^4+\omega_1^2}{\eta\omega_1}L_{22}-\frac{2\omega^2}{\eta}L_{24}\bigg)\cos\theta\eta\bigg]d\eta, \qquad \varepsilon \ll 1 \quad (26)$$

Certain advanced computer programs do not require Eqs.(19)–(26).

6.8 Choice of $\rho_2 \ll 1$ in Eq.(4.1-85)

In Eq.(4.1-85) we pointed out that one has to choose $\rho_2 \gg 1$ rather than $\rho_2 \ll 1$ to obtain results in line with observation. The calculations of Section 4.1 hold generally for any value of ρ_2 but in the following Section 4.2 we used the simplification $\rho_2 \gg 1$ from Eq.(4.2-6) on. We return here to Eq.(4.1-103) and see what becomes of $A_{\text{ev}}(\zeta,\theta)$ for $\rho_2 \ll 1$:

$$A_{\text{ev}}(\zeta,\theta) = c^2T^2V_{\text{e0}}\Bigg(\frac{1}{\rho_2^2}e^{-\rho_2\zeta}(1-\operatorname{ch}\rho_2\theta)$$

$$+2(1-e^{-\rho_2})\Bigg\{\sum_{\kappa=1}^{<K}\Bigg[\bigg(L_{13}(\theta,\kappa)+\frac{\rho_1L_{11}(\theta,\kappa)}{(\rho_1^2-d^2)^{1/2}}\bigg)\sin 2\pi\kappa\theta$$

$$-\bigg(L_{14}(\theta,\kappa)+\frac{\rho_1L_{12}(\theta,\kappa)}{(\rho_1^2-d^2)^{1/2}}\bigg)\cos 2\pi\kappa\theta\Bigg]\frac{\sin 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2}$$

$$+\sum_{\kappa>K}^{\infty}\Bigg[\bigg(L_{17}(\theta,\kappa)+\frac{\rho_1L_{15}(\theta,\kappa)}{(d^2-\rho_1^2)^{1/2}}\bigg)\sin 2\pi\kappa\theta$$

$$-\bigg(L_{18}(\theta,\kappa)+\frac{\rho_1L_{16}(\theta,\kappa)}{(d^2-\rho_1^2)^{1/2}}\bigg)\cos 2\pi\kappa\theta\Bigg]\frac{\sin 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2}\Bigg\}\Bigg) \quad (1)$$

The condition $\rho_2 = cT\sqrt{\sigma s} \ll 1$ implies that at least one conductivity σ or s approaches zero. The first line of Eq.(1) becomes:

$$\frac{1}{\rho^2}e^{-\rho_2\zeta}(1-\operatorname{ch}\rho_2\theta) \approx -\frac{1}{2}\theta^2 \quad \text{for } \sigma\to 0 \text{ and/or } s\to 0 \qquad (2)$$

The value of K in Eq.(1) equals

$$K = c^2T|(\sigma\mu - s\epsilon)|/4\pi < 1 \quad \text{for } \sigma\to 0 \text{ and } s\to 0 \qquad (3)$$

the factor $1-e^{-\rho_2}$ is reduced to

$$1 - e^{-\rho_2} \approx \rho_2 = cT\sqrt{\sigma s} \quad \text{for } \sigma s \to 0 \tag{4}$$

and Eq.(1) assumes the following form:

$$A_{\text{ev}}(\zeta,\theta) = c^2T^2V_{\text{e0}}\bigg\{ -\frac{1}{2}\theta^2 + 2\rho_2 \sum_{\kappa=1}^{\infty} \bigg[\bigg(L_{17}(\theta,\kappa) + \frac{\rho_1 L_{15}(\theta,\kappa)}{(d^2-\rho_1^2)^{1/2}}\bigg)\sin 2\pi\kappa\theta$$
$$- \bigg(L_{18}(\theta,\kappa) + \frac{\rho_1 L_{16}(\theta,\kappa)}{(d^2-\rho_1^2)^{1/2}}\bigg)\cos 2\pi\kappa\theta\bigg] \frac{\sin 2\pi\kappa\zeta}{(2\pi\kappa)^2+\rho_2^2}\bigg\}, \quad d^2 > \rho_1^2 \tag{5}$$

The first sum in Eq.(1) is eliminated by the value $K < 1$ for the upper limit and the second sum runs from $\kappa = 1$ to infinity. We first consider the case where either σ or s approach zero, but not both. This implies $\rho_2 \to 0$ but $\rho_1 \neq 0$. A check of $L_{15}(\theta,\kappa)$ to $L_{18}(\theta,\kappa)$ in Eqs.(4.1-107)–(4.1-110) shows that the terms of the sum of Eq.(5) are finite and the sum vanishes for $\rho_2 \to 0$.

If both σ and s approach zero we have $\rho_2 \to 0$ and $\rho_1 \to 0$. In this case one must check the terms

$$\frac{q_2}{(\rho_1/2)^2 + q_2^2} \quad \text{and} \quad \frac{\rho_1/2}{(\rho_1/2)^2 + q_2^2}$$

in Eqs.(4.1-107)–(4.1-110). We get the following results for $\rho_1 \to 0$:

$$q_2 = 2\pi\kappa\bigg[1 - \frac{\rho_1^2}{4(2\pi\kappa)^2}\bigg] - 2\pi\kappa \to -\bigg(\frac{\rho_1}{4\pi\kappa}\bigg)^2$$
$$q_2^2 \to \bigg(\frac{\rho_1}{4\pi\kappa}\bigg)^4, \quad \bigg(\frac{\rho_1}{2}\bigg)^2 + q_2^2 \to \bigg(\frac{\rho_1}{2}\bigg)^2, \quad \rho_1 q_2 \to -\frac{\rho_1^3}{(4\pi\kappa)^2} \tag{6}$$

We obtain for the largest terms for $\rho_1 \to 0$ and $\rho_2 \to 0$ in Eq.(5) with the help of Eqs.(4.1-107)–(4.1-110):

$$L_{17}(\theta,\kappa): \quad \frac{\rho_1^2\theta/4}{(\rho_1/2)^2 + q_2^2} \approx \theta \tag{7}$$

$$\rho_1 L_{15}(\theta,\kappa): \quad \frac{\rho_1^2\theta q_2/2}{(\rho_1/2)^2 + q_2^2} \approx -\frac{\rho_1}{2\pi\kappa}\theta \tag{8}$$

$$L_{18}(\theta,\kappa): \quad \frac{\rho_1\theta q_2/2}{(\rho_1/2)^2 + q_2^2} \approx -2\bigg(\frac{\rho_1}{4\pi\kappa}\bigg)^2\theta \tag{9}$$

$$\rho_1 L_{11}(\theta,\kappa): \quad \frac{\rho_1^3\theta/4}{(\rho_1/2)^2 + q_2^2} \approx \rho_1\theta \tag{10}$$

All terms have either a value $0 \leq \theta \leq 1$ or approach zero for $\rho_1 \to 0$. Hence, Eq.(5) approaches

$$A_{\mathrm{ev}}(\zeta,\theta) = -\frac{1}{2}c^2T^2V_{\mathrm{e}0}\theta^2, \quad \text{for } s=0 \text{ and/or } \sigma = 0 \tag{11}$$

The spatial variable ζ has disappeared. We draw the conclusion that the conductivities σ and s cannot be assumed to be zero since we cannot exclude displacement or dipole currents in vacuum within either the original Maxwell theory or its modification by magnetic dipole currents.

6.9 Excitation of a Spherical Wave

The last two sentences in Section 4.3 state that the results obtained in Sections 4.1 to 4.3 for planar waves could readily be applied to spherical waves. To show this we write Eqs.(4.1-6) and (4.1-7) in spherical coordinates. First we rewrite **A** and certain functions derived from it in analogy to Eqs.(4.1-8)–(4.1-10):

$$\mathbf{A} = A_{\mathrm{r}}\mathbf{e}_{\mathrm{r}} + A_{\vartheta}\mathbf{e}_{\vartheta} + A_{\varphi}\mathbf{e}_{\varphi} \tag{1}$$

$$\begin{aligned}\operatorname{curl}\mathbf{A} = \frac{1}{r\sin\vartheta}\left(\frac{\partial(A_{\varphi}\sin\vartheta)}{\partial\vartheta} - \frac{\partial A_{\vartheta}}{\partial\varphi}\right)\mathbf{e}_{\mathrm{r}} + \frac{1}{r}\left(\frac{1}{\sin\vartheta}\frac{\partial A_{\mathrm{r}}}{\partial\varphi} - \frac{\partial(rA_{\varphi})}{\partial r}\right)\mathbf{e}_{\vartheta} \\ + \frac{1}{r}\left(\frac{\partial(rA_{\vartheta})}{\partial r} - \frac{\partial A_{\mathrm{r}}}{\partial\vartheta}\right)\mathbf{e}_{\varphi}\end{aligned} \tag{2}$$

$$\operatorname{div}\mathbf{A} = \frac{1}{r^2}\frac{\partial(r^2A_{\mathrm{r}})}{\partial r} + \frac{1}{r\sin\vartheta}\frac{\partial(A_{\vartheta}\sin\vartheta)}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial A_{\varphi}}{\partial\varphi} \tag{3}$$

The vector $\nabla^2\mathbf{A}$ of Eq.(4.1-10) is much more complicated in spherical coordinates than in Cartesian ones. We shall need only terms that are circularly symmetric in φ. This makes it possible to leave out terms $\partial A_{.}/\partial\varphi$ from the beginning and use the shortened version of $\nabla^2\mathbf{A}$:

$$\begin{aligned}\nabla^2\mathbf{A} &= \operatorname{grad}\operatorname{div}\mathbf{A} - \operatorname{curl}\operatorname{curl}\mathbf{A} \qquad \text{for } \partial A_{.}/\partial\varphi = 0 \\ &= \left\{\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial(r^2A_{\mathrm{r}})}{\partial r} + \frac{1}{r\sin\vartheta}\frac{\partial(A_{\vartheta}\sin\vartheta)}{\partial\vartheta}\right)\right. \\ &\qquad \left. - \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left[\left(\frac{\partial(rA_{\vartheta})}{\partial r} - \frac{\partial A_{\mathrm{r}}}{\partial\vartheta}\right)\sin\vartheta\right]\right\}\mathbf{e}_{\mathrm{r}} \\ &+ \left[\frac{1}{r}\frac{\partial}{\partial\vartheta}\left(\frac{1}{r^2}\frac{\partial(r^2A_{\mathrm{r}})}{\partial r} + \frac{1}{r\sin\vartheta}\frac{\partial(A_{\vartheta}\sin\vartheta)}{\partial\vartheta}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial(rA_{\vartheta})}{\partial r} - \frac{\partial A_{\mathrm{r}}}{\partial\vartheta}\right)\right]\mathbf{e}_{\vartheta} \\ &\qquad + \left(\frac{1}{r}\frac{\partial^2(rA_{\varphi})}{\partial r^2} + \frac{1}{r^2\sin\vartheta}\frac{\partial^2(A_{\varphi}\sin\vartheta)}{\partial\vartheta^2}\right)\mathbf{e}_{\varphi}\end{aligned} \tag{4}$$

Equations (4.1-6) and (4.1-7) assume the following form in analogy to Eqs.(4.1-11)–(4.1-16) for $\partial A_{.}/\partial\varphi = 0$:

$$\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial(r^2 A_{\mathrm{er}})}{\partial r}+\frac{1}{r\sin\vartheta}\frac{\partial(A_{\mathrm{e}\vartheta}\sin\vartheta)}{\partial\vartheta}\right)-\frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left[\left(\frac{\partial(rA_{\mathrm{e}\vartheta})}{\partial r}-\frac{\partial A_{\mathrm{er}}}{\partial\vartheta}\right)\sin\vartheta\right]-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{er}}}{\partial t^2}+\frac{s}{Zc}\left(\frac{c}{Zr\sin\vartheta}\frac{\partial(A_{\mathrm{m}\varphi}\sin\vartheta)}{\partial\vartheta}-\frac{\partial A_{\mathrm{er}}}{\partial t}\right)=0 \quad (5)$$

$$\frac{1}{r}\frac{\partial}{\partial\vartheta}\left(\frac{1}{r^2}\frac{\partial(r^2 A_{\mathrm{er}})}{\partial r}+\frac{1}{r\sin\vartheta}\frac{\partial(A_{\mathrm{e}\vartheta}\sin\vartheta)}{\partial\vartheta}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial(rA_{\mathrm{e}\vartheta})}{\partial r}-\frac{\partial A_{\mathrm{er}}}{\partial\vartheta}\right)-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{e}\vartheta}}{\partial t^2}+\frac{s}{Zc}\left(-\frac{c}{Zr}\frac{\partial(rA_{\mathrm{m}\varphi})}{\partial r}-\frac{\partial A_{\mathrm{e}\vartheta}}{\partial t}\right)=0 \quad (6)$$

$$\frac{1}{r}\frac{\partial^2(rA_{\mathrm{e}\varphi})}{\partial r^2}+\frac{1}{r^2\sin\vartheta}\frac{\partial^2(A_{\mathrm{e}\varphi}\sin\vartheta)}{\partial\vartheta^2}-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{e}\varphi}}{\partial t^2}+\frac{s}{Zc}\left[\frac{c}{Zr}\left(\frac{\partial(rA_{\mathrm{m}\vartheta})}{\partial r}-\frac{\partial A_{\mathrm{mr}}}{\partial\vartheta}\right)-\frac{\partial A_{\mathrm{e}\varphi}}{\partial t}\right]=0 \quad (7)$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial(r^2 A_{\mathrm{mr}})}{\partial r}+\frac{1}{r\sin\vartheta}\frac{\partial(A_{\mathrm{m}\vartheta}\sin\vartheta)}{\partial\vartheta}\right)-\frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left[\left(\frac{\partial(rA_{\mathrm{m}\vartheta})}{\partial r}-\frac{\partial A_{\mathrm{mr}}}{\partial\vartheta}\right)\sin\vartheta\right]-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{mr}}}{\partial t^2}-\frac{Z\sigma}{c}\left(\frac{Zc}{r\sin\vartheta}\frac{\partial(A_{\mathrm{e}\varphi}\sin\vartheta)}{\partial\vartheta}+\frac{\partial A_{\mathrm{mr}}}{\partial t}\right)=0 \quad (8)$$

$$\frac{1}{r}\frac{\partial}{\partial\vartheta}\left(\frac{1}{r^2}\frac{\partial(r^2 A_{\mathrm{mr}})}{\partial r}+\frac{1}{r\sin\vartheta}\frac{\partial(A_{\mathrm{m}\vartheta}\sin\vartheta)}{\partial\vartheta}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial(rA_{\mathrm{m}\vartheta})}{\partial r}-\frac{\partial A_{\mathrm{mr}}}{\partial\vartheta}\right)-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{m}\vartheta}}{\partial t^2}-\frac{Z\sigma}{c}\left(-\frac{Zc}{r}\frac{\partial(rA_{\mathrm{e}\varphi})}{\partial r}+\frac{\partial A_{\mathrm{m}\vartheta}}{\partial t}\right)=0 \quad (9)$$

$$\frac{1}{r}\frac{\partial^2(rA_{\mathrm{m}\varphi})}{\partial r^2}+\frac{1}{r^2\sin\vartheta}\frac{\partial^2(A_{\mathrm{m}\varphi}\sin\vartheta)}{\partial\vartheta^2}-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{m}\varphi}}{\partial t^2}-\frac{Z\sigma}{c}\left[\frac{Zc}{r}\left(\frac{\partial(rA_{\mathrm{e}\vartheta})}{\partial r}-\frac{\partial A_{\mathrm{er}}}{\partial\vartheta}\right)+\frac{\partial A_{\mathrm{m}\varphi}}{\partial t}\right]=0 \quad (10)$$

Two more equations are obtained from the extended Lorentz convention of Eqs.(1.6-23) and (1.6-24) for $\phi_{\mathrm{e}}=\phi_{\mathrm{m}}=0$ using Eq.(3) and $\partial A./\partial\varphi=0$:

$$\operatorname{div}\mathbf{A}_{\mathrm{m}}=\frac{1}{r^2}\frac{\partial(r^2 A_{\mathrm{mr}})}{\partial r}+\frac{1}{r\sin\vartheta}\frac{\partial(A_{\mathrm{m}\vartheta}\sin\vartheta)}{\partial\vartheta}=0 \quad (11)$$

$$\operatorname{div}\mathbf{A}_{\mathrm{e}}=\frac{1}{r^2}\frac{\partial(r^2 A_{\mathrm{er}})}{\partial r}+\frac{1}{r\sin\vartheta}\frac{\partial(A_{\mathrm{e}\vartheta}\sin\vartheta)}{\partial\vartheta}=0 \quad (12)$$

We want to derive a TEM wave, which means the components E_{r} and H_{r} of the electric and magnetic field strength should be zero. Equations (4.1-25) and (4.1-26) for Cartesian coordinates are replaced by equations for spherical coordinates with the help of Eqs.(1.6-11), (1.6-17), and (2):

$$E_{\mathrm{r}} = -\frac{Zc}{r\sin\vartheta}\frac{\partial(A_{\mathrm{e}\varphi}\sin\vartheta)}{\partial\vartheta} - \frac{\partial A_{\mathrm{mr}}}{\partial t} = 0, \qquad \frac{\partial A_{\mathrm{e.}}}{\partial\varphi} = 0 \tag{13}$$

$$H_{\mathrm{r}} = \frac{c}{Zr\sin\vartheta}\frac{\partial(A_{\mathrm{m}\varphi}\sin\vartheta)}{\partial\vartheta} - \frac{\partial A_{\mathrm{er}}}{\partial t} = 0, \qquad \frac{\partial A_{\mathrm{m.}}}{\partial\varphi} = 0 \tag{14}$$

Substitution of Eq.(13) into Eq.(8) and of Eq.(14) into Eq.(5) yields the following two equations:

$$\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial(r^2A_{\mathrm{mr}})}{\partial r} + \frac{1}{r\sin\vartheta}\frac{\partial(A_{\mathrm{m}\vartheta}\sin\vartheta)}{\partial\vartheta}\right) - \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left[\left(\frac{\partial(rA_{\mathrm{m}\vartheta})}{\partial r} - \frac{\partial A_{\mathrm{mr}}}{\partial\vartheta}\right)\sin\vartheta\right] - \frac{1}{c^2}\frac{\partial^2 A_{\mathrm{mr}}}{\partial t^2} = 0 \tag{15}$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial(r^2A_{\mathrm{er}})}{\partial r} + \frac{1}{r\sin\vartheta}\frac{\partial(A_{\mathrm{e}\vartheta}\sin\vartheta)}{\partial\vartheta}\right) - \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left[\left(\frac{\partial(rA_{\mathrm{e}\vartheta})}{\partial r} - \frac{\partial A_{\mathrm{er}}}{\partial\vartheta}\right)\sin\vartheta\right] - \frac{1}{c^2}\frac{\partial^2 A_{\mathrm{er}}}{\partial t^2} = 0 \tag{16}$$

A comparison of these two equations with $\nabla^2\mathbf{A}$ of Eq.(4) yields the following equations for the amplitudes of the vectors $\mathbf{e}_{\mathrm{r}}$ if $\mathbf{A}$ is replaced by $\mathbf{A}_{\mathrm{m}}$ or $\mathbf{A}_{\mathrm{e}}$:

$$\left(\nabla^2\mathbf{A}_{\mathrm{m}} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}_{\mathrm{m}}}{\partial t^2}\right)_{\mathrm{r}} = 0, \qquad \left(\nabla^2\mathbf{A}_{\mathrm{e}} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}_{\mathrm{e}}}{\partial t^2}\right)_{\mathrm{r}} = 0 \tag{17}$$

Hence, Eqs.(5) and (8) will be satisfied if the radial components of $\mathbf{A}_{\mathrm{m}}$ and $\mathbf{A}_{\mathrm{e}}$ satisfy the wave equation.

We have Eqs.(6), (7), (9), (10), (11), and (12) left to determine the six variables A_{mr}, $A_{\mathrm{m}\vartheta}$, $A_{\mathrm{m}\varphi}$, A_{er}, $A_{\mathrm{e}\vartheta}$, and $A_{\mathrm{e}\varphi}$. Let us substitute Eq.(12) into Eq.(6)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial(rA_{\mathrm{e}\vartheta})}{\partial r} - \frac{\partial A_{\mathrm{er}}}{\partial\vartheta}\right) - \frac{1}{c^2}\frac{\partial^2 A_{\mathrm{e}\vartheta}}{\partial t^2} - \frac{s}{Zc}\left(\frac{c}{Zr}\frac{\partial(rA_{\mathrm{m}\varphi})}{\partial r} + \frac{\partial A_{\mathrm{e}\vartheta}}{\partial t}\right) = 0 \tag{18}$$

and Eq.(11) into Eq.(9):

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial(rA_{\mathrm{m}\vartheta})}{\partial r} - \frac{\partial A_{\mathrm{mr}}}{\partial\vartheta}\right) - \frac{1}{c^2}\frac{\partial^2 A_{\mathrm{m}\vartheta}}{\partial t^2} + \frac{Z\sigma}{c}\left(\frac{Zc}{r}\frac{\partial(rA_{\mathrm{e}\varphi})}{\partial r} - \frac{\partial A_{\mathrm{m}\vartheta}}{\partial t}\right) = 0 \tag{19}$$

Equation (7) is rewritten

$$\frac{\partial(rA_{\mathrm{m}\vartheta})}{\partial r} - \frac{\partial A_{\mathrm{mr}}}{\partial\vartheta} = \frac{Z}{c}\left\{\frac{Zc}{s}\left[-\left(\frac{\partial^2(rA_{\mathrm{e}\varphi})}{\partial r^2} + \frac{1}{r\sin\vartheta}\frac{\partial^2(A_{\mathrm{e}\varphi}\sin\vartheta)}{\partial\vartheta^2}\right) + \frac{r}{c^2}\frac{\partial^2 A_{\mathrm{e}\varphi}}{\partial t^2}\right] + r\frac{\partial A_{\mathrm{e}\varphi}}{\partial t}\right\} \tag{20}$$

and substituted into Eq.(19):

$$\frac{1}{r}\frac{\partial}{\partial r}\frac{Z}{c}\left\{\frac{Zc}{s}\left[-\left(\frac{\partial^2(rA_{\mathrm{e}\varphi})}{\partial r^2} + \frac{1}{r\sin\vartheta}\frac{\partial^2(A_{\mathrm{e}\varphi}\sin\vartheta)}{\partial\vartheta^2}\right) + \frac{r}{c^2}\frac{\partial^2 A_{\mathrm{e}\varphi}}{\partial t^2}\right] + r\frac{\partial A_{\mathrm{e}\varphi}}{\partial t}\right\} - \frac{1}{c^2}\frac{\partial^2 A_{\mathrm{m}\vartheta}}{\partial t^2} + \frac{Z\sigma}{c}\left(\frac{Zc}{r}\frac{\partial(rA_{\mathrm{e}\varphi})}{\partial r} - \frac{\partial A_{\mathrm{m}\vartheta}}{\partial t}\right) = 0 \tag{21}$$

Equation (21) suggests the substitutions:

$$A_{\mathrm{e}\varphi}(r,t,\vartheta) = \frac{A'_{\mathrm{e}\varphi}(r,t)}{\sin\vartheta}, \quad A_{\mathrm{m}\vartheta}(r,t,\vartheta) = \frac{A'_{\mathrm{m}\vartheta}(r,t)}{\sin\vartheta} \tag{22}$$

We observe that $A'_{\mathrm{e}\varphi}(r,t)$ is not a function of ϑ and obtain from Eq.(21):

$$\frac{Z}{c}\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{Zc}{s}\left(-\frac{\partial^2(rA'_{\mathrm{e}\varphi})}{\partial r^2} + \frac{r}{c^2}\frac{\partial^2 A'_{\mathrm{e}\varphi}}{\partial t^2}\right) + r\frac{\partial A'_{\mathrm{e}\varphi}}{\partial t}\right] + \frac{Z^2\sigma}{r}\frac{\partial(rA'_{\mathrm{e}\varphi})}{\partial r} - \frac{1}{c^2}\frac{\partial^2 A'_{\mathrm{m}\vartheta}}{\partial t^2} - \frac{Z\sigma}{c}\frac{\partial A'_{\mathrm{m}\vartheta}}{\partial t} = 0 \tag{23}$$

In analogy to Eq.(20) we rewrite Eq.(10)

$$\frac{\partial(rA_{\mathrm{e}\vartheta})}{\partial r} - \frac{\partial A_{\mathrm{er}}}{\partial\vartheta} = \frac{1}{Zc}\left[\frac{c}{Z\sigma}\left(\frac{\partial^2(rA_{\mathrm{m}\varphi})}{\partial r^2} + \frac{1}{r\sin\vartheta}\frac{\partial^2(A_{\mathrm{m}\varphi}\sin\vartheta)}{\partial\vartheta} - \frac{r}{c^2}\frac{\partial^2 A_{\mathrm{m}\varphi}}{\partial t^2}\right) - r\frac{\partial A_{\mathrm{m}\varphi}}{\partial t}\right] \tag{24}$$

and substitute into Eq.(18):

$$\frac{1}{Zc}\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{c}{Z\sigma}\left(\frac{\partial^2(rA_{\mathrm{m}\varphi})}{\partial r^2} + \frac{1}{r\sin\vartheta}\frac{\partial^2(A_{\mathrm{m}\varphi}\sin\vartheta)}{\partial\vartheta} - \frac{r}{c^2}\frac{\partial^2 A_{\mathrm{m}\varphi}}{\partial t^2}\right) - r\frac{\partial A_{\mathrm{m}\varphi}}{\partial t}\right] - \frac{1}{c^2}\frac{\partial^2 A_{\mathrm{e}\vartheta}}{\partial t^2} - \frac{s}{Zc}\left(\frac{c}{Zr}\frac{\partial(rA_{\mathrm{m}\varphi})}{\partial r} + \frac{\partial A_{\mathrm{e}\vartheta}}{\partial t}\right) = 0 \tag{25}$$

The following substitutions for $A_{\mathrm{m}\varphi}(r,t,\vartheta)$ and $A_{\mathrm{e}\vartheta}(r,t,\vartheta)$ are suggested by Eq.(25):

$$A_{\mathrm{m}\varphi}(r,t,\vartheta) = \frac{A'_{\mathrm{m}\varphi}(r,t)}{\sin\vartheta}, \quad A_{\mathrm{e}\vartheta}(r,t,\vartheta) = \frac{A'_{\mathrm{e}\vartheta}(r,t)}{\sin\vartheta} \tag{26}$$

Since $A'_{\mathrm{m}\varphi}(r,t)$ is not a function of ϑ we obtain from Eq.(25):

$$\frac{1}{Zc}\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{c}{Z\sigma}\left(\frac{\partial^2(rA'_{\mathrm{m}\varphi})}{\partial r^2} - \frac{r}{c^2}\frac{\partial^2 A'_{\mathrm{m}\varphi}}{\partial t^2}\right) - r\frac{\partial A'_{\mathrm{m}\varphi}}{\partial t}\right] - \frac{s}{Z^2 r}\frac{\partial(rA'_{\mathrm{m}\varphi})}{\partial r}$$
$$- \frac{1}{c^2}\frac{\partial^2 A'_{\mathrm{e}\vartheta}}{\partial t^2} - \frac{s}{Zc}\frac{\partial A'_{\mathrm{e}\vartheta}}{\partial t} = 0 \tag{27}$$

Substitution of Eq.(22) into Eq.(11) and of Eq.(26) into Eq.(12) yields equations for the components A_{mr} and A_{er}:

$$\frac{\partial(r^2 A_{\mathrm{mr}})}{\partial r} = 0, \qquad \frac{\partial(r^2 A_{\mathrm{er}})}{\partial r} = 0, \tag{28}$$

These equations suggest to choose the following solutions for A_{mr} and A_{er} that are independent of both φ and ϑ and thus spherically symmetric:

$$A_{\mathrm{mr}} = A_{\mathrm{mr}}(r,t) = \frac{A''_{\mathrm{mr}}(t)}{r^2}, \qquad A_{\mathrm{er}} = A_{\mathrm{er}}(r,t) = \frac{A''_{\mathrm{er}}(t)}{r^2} \tag{29}$$

Since A_{mr} and A_{er} do not depend on ϑ we obtain the following two relations:

$$\frac{\partial A_{\mathrm{er}}}{\partial\vartheta} = 0, \qquad \frac{\partial A_{\mathrm{mr}}}{\partial\vartheta} = 0 \tag{30}$$

Equations (30) permit the simplification of Eqs.(18) and (19):

$$\frac{1}{r}\frac{\partial^2(rA'_{\mathrm{e}\vartheta})}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2 A'_{\mathrm{e}\vartheta}}{\partial t^2} - \frac{s}{Zc}\frac{\partial A'_{\mathrm{e}\vartheta}}{\partial t} - \frac{s}{Z^2 r}\frac{\partial(rA'_{\mathrm{m}\varphi})}{\partial r} = 0 \tag{31}$$

$$\frac{1}{r}\frac{\partial^2(rA'_{\mathrm{m}\vartheta})}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2 A'_{\mathrm{m}\vartheta}}{\partial t^2} - \frac{Z\sigma}{c}\frac{\partial A_{\mathrm{m}\vartheta}}{\partial t} + \frac{Z^2\sigma}{r}\frac{\partial(rA'_{\mathrm{e}\varphi})}{\partial r} = 0 \tag{32}$$

Equations (31) and (32) suggest to multiply by r and to replace the variables $A'_{\mathrm{e}\vartheta}$ and $A'_{\mathrm{m}\vartheta}$ by the new variables $rA'_{\mathrm{e}\vartheta}$ and $rA'_{\mathrm{m}\vartheta}$:

$$\frac{\partial^2(rA'_{\mathrm{e}\vartheta})}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2(rA'_{\mathrm{e}\vartheta})}{\partial t^2} - \frac{s}{Zc}\frac{\partial(rA'_{\mathrm{e}\vartheta})}{\partial t} - \frac{s}{Z^2}\frac{\partial(rA'_{\mathrm{m}\varphi})}{\partial r} = 0 \tag{33}$$

$$\frac{\partial^2(rA'_{\mathrm{m}\vartheta})}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2(rA'_{\mathrm{m}\theta})}{\partial t^2} - \frac{Z\sigma}{c}\frac{\partial(rA'_{\mathrm{m}\vartheta})}{\partial t} + Z^2\sigma\frac{\partial(rA'_{\mathrm{e}\varphi})}{\partial r} = 0 \tag{34}$$

Next we substitute Eq.(30) into Eqs.(23) and (27), and we replace the variables $A'_{\mathrm{e}\varphi}$ and $A'_{\mathrm{m}\varphi}$ by the new variables $rA'_{\mathrm{e}\varphi}$ and $rA'_{\mathrm{m}\varphi}$:

$$\frac{\partial^2}{\partial r^2}\left(\frac{\partial(rA'_{e\varphi})}{\partial r}\right)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{\partial(rA'_{e\varphi})}{\partial r}\right)-\frac{s}{Zc}\frac{\partial}{\partial t}\left(\frac{\partial(rA'_{e\varphi})}{\partial r}\right)-\sigma s\frac{\partial(rA'_{e\varphi})}{\partial r}$$
$$+\frac{s}{Z^2c^2}\frac{\partial^2(rA'_{m\vartheta})}{\partial t^2}+\frac{\sigma s}{Zc}\frac{\partial(rA'_{m\vartheta})}{\partial t}=0 \quad (35)$$

$$\frac{\partial^2}{\partial r^2}\left(\frac{\partial(rA'_{m\varphi})}{\partial r}\right)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{\partial(rA'_{m\varphi})}{\partial r}\right)-\frac{Z\sigma}{c}\frac{\partial}{\partial t}\left(\frac{\partial(rA'_{m\varphi})}{\partial r}\right)-\sigma s\frac{\partial(rA'_{m\varphi})}{\partial r}$$
$$-\frac{Z^2\sigma}{c^2}\frac{\partial^2(rA'_{e\vartheta})}{\partial t^2}-\frac{Z\sigma s}{c}\frac{\partial(rA'_{e\vartheta})}{\partial t}=0 \quad (36)$$

Substitution of $\partial(rA'_{m\varphi})/\partial r$ in Eq.(33) into Eq.(36) produces an equation for the variable $rA'_{e\vartheta}$ alone:

$$\frac{\partial^2 V_e}{\partial r^2}-\frac{1}{c^2}\frac{\partial^2 V_e}{\partial t^2}-\frac{1}{c}\left(Z\sigma+\frac{s}{Z}\right)\frac{\partial V_e}{\partial t}-\sigma s V_e=0 \quad (37)$$

$$\frac{\partial^2(rA'_{e\vartheta})}{\partial r^2}-\frac{1}{c^2}\frac{\partial(rA'_{e\vartheta})}{\partial t^2}=V_e(r,t) \quad (38)$$

These are essentially Eqs.(4.1-36) and (4.1-37). If we further substitute the derivative $\partial(rA'_{e\varphi})/\partial r$ of Eq.(34) into Eq.(35) we obtain an equation for the variable $rA'_{m\vartheta}$ alone:

$$\frac{\partial^2 V_m}{\partial r^2}-\frac{1}{c^2}\frac{\partial V_m}{\partial t^2}-\frac{1}{c}\left(Z\sigma+\frac{s}{Z}\right)\frac{\partial V_m}{\partial t}-\sigma s V_m=0 \quad (39)$$

$$\frac{\partial^2(rA'_{m\vartheta})}{\partial r^2}-\frac{1}{c^2}\frac{\partial(rA'_{m\vartheta})}{\partial t^2}=V_m(r,t) \quad (40)$$

These are essentially Eqs.(4.1-38) and (4.1-39).

The functions $rA'_{e\vartheta}$ and $rA'_{m\vartheta}$ can be obtained from Eqs.(38) and (40) in analogy to Eqs.(4.1-43) and (4.1-46). The final two functions $rA'_{m\varphi}$ and $rA'_{e\varphi}$ may then be obtained from Eqs.(33) and (34).

6.10 Better Approximations of Dipole Currents

We use again the first three equations of Section 4.1 and thus stay with the extended Lorentz gauge of Section 4.1:

$$\phi_e(x,y,z,t)\equiv\phi_m(x,y,z,t)\equiv 0 \quad (1)$$

$$\nabla^2\mathbf{A}_e-\frac{1}{c^2}\frac{\partial^2\mathbf{A}_e}{\partial t^2}=-\frac{1}{Zc}\mathbf{g}_m \quad (2)$$

$$\nabla^2\mathbf{A}_m-\frac{1}{c^2}\frac{\partial^2\mathbf{A}_m}{\partial t^2}=-\frac{Z}{c}\mathbf{g}_e \quad (3)$$

Equation (4.1-5) is replaced by Eq.(2.1-19) for electric dipole currents. The conductivity σ is written instead of σ_{p} to reduce the number of subscripts:

$$\mathbf{g}_{\text{e}} + \tau_{\text{mp}} \frac{\partial \mathbf{g}_{\text{e}}}{\partial t} + \frac{\tau_{\text{mp}}}{\tau_{\text{p}}^2} \int \mathbf{g}_{\text{e}} dt = \sigma \mathbf{E} = -\sigma \Big(Zc \operatorname{curl} \mathbf{A}_{\text{e}} + \frac{\partial \mathbf{A}_{\text{m}}}{\partial t} \Big) \tag{4}$$

In order to represent magnetic dipole currents we could rewrite Eq.(2.2-22):

$$\left[\frac{J}{R} \frac{d}{dt} \left(\frac{\mathbf{g}_{\text{m},\vartheta}}{\sin\vartheta} \right) + \xi_{\text{m}} \frac{\mathbf{g}_{\text{m},\vartheta}}{\sin\vartheta} \right] \left[2N_0 q_{\text{m}} \mu m_{\text{mo}} \sin \left(\frac{1}{2N_0 q_{\text{m}} R} \int \frac{\mathbf{g}_{\text{m},\vartheta}}{\sin\vartheta} dt \right) \right]^{-1}$$
$$= -\mathbf{H} = -\frac{c}{Z} \operatorname{curl} \mathbf{A}_{\text{m}} + \frac{\partial \mathbf{A}_{\text{e}}}{\partial t} \tag{5}$$

The result of the calculation would hold for a particular initial angle ϑ and one would have to average over all angles. A strictly numerical solution may one day proceed in this way. Here, we shall use the hypothetical induced magnetic current according to Eq.(2.2-7). We write s instead of $2s_{\text{p}}$:

$$\mathbf{g}_{\text{m}} + \tau'_{\text{mp}} \frac{\partial \mathbf{g}_{\text{m}}}{\partial t} + \frac{\tau'_{\text{mp}}}{\tau'^2_{\text{p}}} \int \mathbf{g}_{\text{m}} dt = s\mathbf{H} = s \Big(\frac{c}{Z} \operatorname{curl} \mathbf{A}_{\text{m}} - \frac{\partial \mathbf{A}_{\text{e}}}{\partial t} \Big) \tag{6}$$

The parameters τ'_{mp} and τ'_{p} have received a prime in order to distinguish them from τ_{mp} and τ_{p} in Eq.(4).

We substitute $\mathbf{g}_{\text{e}}$ from Eq.(3) into Eq.(4) and $\mathbf{g}_{\text{m}}$ from Eq.(2) into Eq.(6) to obtain the generalization of Eqs.(4.1-6) and (4.1-7):

$$\Big(\nabla^2 \mathbf{A}_{\text{m}} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_{\text{m}}}{\partial t^2} \Big) + \tau_{\text{mp}} \frac{\partial}{\partial t} \Big(\nabla^2 \mathbf{A}_{\text{m}} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_{\text{m}}}{\partial t^2} \Big)$$
$$+ \frac{\tau_{\text{mp}}}{\tau_{\text{p}}^2} \int \Big(\nabla^2 \mathbf{A}_{\text{m}} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_{\text{m}}}{\partial t^2} \Big) dt - \frac{\sigma Z}{c} \Big(Zc \operatorname{curl} \mathbf{A}_{\text{e}} + \frac{\partial \mathbf{A}_{\text{m}}}{\partial t} \Big) = 0 \tag{7}$$

$$\Big(\nabla^2 \mathbf{A}_{\text{e}} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_{\text{e}}}{\partial t^2} \Big) + \tau'_{\text{mp}} \frac{\partial}{\partial t} \Big(\nabla^2 \mathbf{A}_{\text{e}} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_{\text{e}}}{\partial t^2} \Big)$$
$$+ \frac{\tau'_{\text{mp}}}{\tau'^2_{\text{p}}} \int \Big(\nabla^2 \mathbf{A}_{\text{e}} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_{\text{e}}}{\partial t^2} \Big) dt + \frac{s}{Zc} \Big(\frac{c}{Z} \operatorname{curl} \mathbf{A}_{\text{m}} - \frac{\partial \mathbf{A}_{\text{e}}}{\partial t} \Big) = 0 \tag{8}$$

As in Section 4.1 the potentials $\mathbf{A}_{\text{e}}$ and $\mathbf{A}_{\text{m}}$ are connected in these two equations.

In order to write Eqs.(7) and (8) in component form in Cartesian coordinates we differentiate first with respect to t:

$$\left(\tau_{\mathrm{mp}}\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}+\frac{\tau_{\mathrm{mp}}}{\tau_{\mathrm{p}}^2}\right)\left(\nabla^2\mathbf{A}_{\mathrm{m}}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}_{\mathrm{m}}}{\partial t^2}\right)-\frac{\sigma Z}{c}\frac{\partial}{\partial t}\left(Zc\operatorname{curl}\mathbf{A}_{\mathrm{e}}+\frac{\partial\mathbf{A}_{\mathrm{m}}}{\partial t}\right)=0 \quad (9)$$

$$\left(\tau'_{\mathrm{mp}}\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}+\frac{\tau'_{\mathrm{mp}}}{\tau'^2_{\mathrm{p}}}\right)\left(\nabla^2\mathbf{A}_{\mathrm{e}}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}_{\mathrm{e}}}{\partial t^2}\right)+\frac{s}{Zc}\frac{\partial}{\partial t}\left(\frac{c}{Z}\operatorname{curl}\mathbf{A}_{\mathrm{m}}-\frac{\partial\mathbf{A}_{\mathrm{e}}}{\partial t}\right)=0 \quad (10)$$

Using Eqs.(4.1-8)–(4.1-10) we obtain:

$$\left(\tau_{\mathrm{mp}}\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}+\frac{\tau_{\mathrm{mp}}}{\tau_{\mathrm{p}}^2}\right)\left(\frac{\partial^2 A_{\mathrm{m}x}}{\partial x^2}+\frac{\partial^2 A_{\mathrm{m}x}}{\partial y^2}+\frac{\partial^2 A_{\mathrm{m}x}}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{m}x}}{\partial t^2}\right)-\frac{\sigma Z}{c}\frac{\partial}{\partial t}\left[Zc\left(\frac{\partial A_{\mathrm{e}z}}{\partial y}-\frac{\partial A_{\mathrm{e}y}}{\partial z}\right)+\frac{\partial A_{\mathrm{m}x}}{\partial t}\right]=0 \quad (11)$$

$$\left(\tau_{\mathrm{mp}}\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}+\frac{\tau_{\mathrm{mp}}}{\tau_{\mathrm{p}}^2}\right)\left(\frac{\partial^2 A_{\mathrm{m}y}}{\partial x^2}+\frac{\partial^2 A_{\mathrm{m}y}}{\partial y^2}+\frac{\partial^2 A_{\mathrm{m}y}}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{m}y}}{\partial t^2}\right)-\frac{\sigma Z}{c}\frac{\partial}{\partial t}\left[Zc\left(\frac{\partial A_{\mathrm{e}x}}{\partial z}-\frac{\partial A_{\mathrm{e}z}}{\partial x}\right)+\frac{\partial A_{\mathrm{m}y}}{\partial t}\right]=0 \quad (12)$$

$$\left(\tau_{\mathrm{mp}}\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}+\frac{\tau_{\mathrm{mp}}}{\tau_{\mathrm{p}}^2}\right)\left(\frac{\partial^2 A_{\mathrm{m}z}}{\partial x^2}+\frac{\partial^2 A_{\mathrm{m}z}}{\partial y^2}+\frac{\partial^2 A_{\mathrm{m}z}}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{m}z}}{\partial t^2}\right)-\frac{\sigma Z}{c}\frac{\partial}{\partial t}\left[Zc\left(\frac{\partial A_{\mathrm{e}y}}{\partial x}-\frac{\partial A_{\mathrm{e}x}}{\partial y}\right)+\frac{\partial A_{\mathrm{m}z}}{\partial t}\right]=0 \quad (13)$$

$$\left(\tau'_{\mathrm{mp}}\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}+\frac{\tau'_{\mathrm{mp}}}{\tau'^2_{\mathrm{p}}}\right)\left(\frac{\partial^2 A_{\mathrm{e}x}}{\partial x^2}+\frac{\partial^2 A_{\mathrm{e}x}}{\partial y^2}+\frac{\partial^2 A_{\mathrm{e}x}}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{e}x}}{\partial t^2}\right)+\frac{s}{Zc}\frac{\partial}{\partial t}\left[\frac{c}{Z}\left(\frac{\partial A_{\mathrm{m}z}}{\partial y}-\frac{\partial A_{\mathrm{m}y}}{\partial z}\right)-\frac{\partial A_{\mathrm{e}x}}{\partial t}\right]=0 \quad (14)$$

$$\left(\tau'_{\mathrm{mp}}\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}+\frac{\tau'_{\mathrm{mp}}}{\tau'^2_{\mathrm{p}}}\right)\left(\frac{\partial^2 A_{\mathrm{e}y}}{\partial x^2}+\frac{\partial^2 A_{\mathrm{e}y}}{\partial y^2}+\frac{\partial^2 A_{\mathrm{e}y}}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{e}y}}{\partial t^2}\right)+\frac{s}{Zc}\frac{\partial}{\partial t}\left[\frac{c}{Z}\left(\frac{\partial A_{\mathrm{m}x}}{\partial z}-\frac{\partial A_{\mathrm{m}z}}{\partial x}\right)-\frac{\partial A_{\mathrm{e}y}}{\partial t}\right]=0 \quad (15)$$

$$\left(\tau'_{\mathrm{mp}}\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}+\frac{\tau'_{\mathrm{mp}}}{\tau'^2_{\mathrm{p}}}\right)\left(\frac{\partial^2 A_{\mathrm{e}z}}{\partial x^2}+\frac{\partial^2 A_{\mathrm{e}z}}{\partial y^2}+\frac{\partial^2 A_{\mathrm{e}z}}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{e}z}}{\partial t^2}\right)$$
$$+\frac{s}{Zc}\frac{\partial}{\partial t}\left[\frac{c}{Z}\left(\frac{\partial A_{\mathrm{m}y}}{\partial x}-\frac{\partial A_{\mathrm{m}x}}{\partial y}\right)-\frac{\partial A_{\mathrm{e}z}}{\partial t}\right]=0 \quad (16)$$

For planar waves propagating in the direction of y we may use Eqs.(4.1-17) and (4.1-18) to simplify Eqs.(11)–(16):

$$\left(\tau_{\mathrm{mp}}\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}+\frac{\tau_{\mathrm{mp}}}{\tau_{\mathrm{p}}^2}\right)\left(\frac{\partial^2 A_{\mathrm{m}x}}{\partial y^2}-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{m}x}}{\partial t^2}\right)$$
$$-\frac{\sigma Z}{c}\left(Zc\frac{\partial^2 A_{\mathrm{e}z}}{\partial y\partial t}+\frac{\partial^2 A_{\mathrm{m}x}}{\partial t^2}\right)=0 \quad (17)$$

$$\left(\tau_{\mathrm{mp}}\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}+\frac{\tau_{\mathrm{mp}}}{\tau_{\mathrm{p}}^2}\right)\left(\frac{\partial^2 A_{\mathrm{m}y}}{\partial y^2}-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{m}y}}{\partial t^2}\right)-\frac{\sigma Z}{c}\frac{\partial^2 A_{\mathrm{m}y}}{\partial t^2}=0 \quad (18)$$

$$\left(\tau_{\mathrm{mp}}\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}+\frac{\tau_{\mathrm{mp}}}{\tau_{\mathrm{p}}^2}\right)\left(\frac{\partial^2 A_{\mathrm{m}z}}{\partial y^2}-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{m}z}}{\partial t^2}\right)$$
$$+\frac{\sigma Z}{c}\left(Zc\frac{\partial^2 A_{\mathrm{e}x}}{\partial y\partial t}-\frac{\partial^2 A_{\mathrm{m}z}}{\partial t^2}\right)=0 \quad (19)$$

$$\left(\tau'_{\mathrm{mp}}\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}+\frac{\tau'_{\mathrm{mp}}}{\tau'^2_{\mathrm{p}}}\right)\left(\frac{\partial^2 A_{\mathrm{e}x}}{\partial y^2}-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{e}x}}{\partial t^2}\right)$$
$$+\frac{s}{Zc}\left(\frac{c}{Z}\frac{\partial^2 A_{\mathrm{m}z}}{\partial y\partial t}-\frac{\partial^2 A_{\mathrm{e}x}}{\partial t^2}\right)=0 \quad (20)$$

$$\left(\tau'_{\mathrm{mp}}\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}+\frac{\tau'_{\mathrm{mp}}}{\tau'^2_{\mathrm{p}}}\right)\left(\frac{\partial^2 A_{\mathrm{e}y}}{\partial y^2}-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{e}y}}{\partial t^2}\right)-\frac{s}{Zc}\frac{\partial^2 A_{\mathrm{e}y}}{\partial t^2}=0 \quad (21)$$

$$\left(\tau'_{\mathrm{mp}}\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}+\frac{\tau'_{\mathrm{mp}}}{\tau'^2_{\mathrm{p}}}\right)\left(\frac{\partial^2 A_{\mathrm{e}z}}{\partial y^2}-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{e}z}}{\partial t^2}\right)$$
$$-\frac{s}{Zc}\left(\frac{c}{Z}\frac{\partial^2 A_{\mathrm{m}x}}{\partial y\partial t}+\frac{\partial^2 A_{\mathrm{e}z}}{\partial t^2}\right)=0 \quad (22)$$

If we further specialize to a transverse electromagnetic wave we may demand that E_y and H_y are zero. This implies the conditions of Eqs.(4.1-25) and (4.1-26) for $\mathbf{A}_{\mathrm{e}}$ and $\mathbf{A}_{\mathrm{m}}$, which are again reduced to

$$\frac{\partial A_{\mathrm{m}y}}{\partial t}=0, \quad \frac{\partial A_{\mathrm{e}y}}{\partial t}=0 \quad (23)$$

due to the conditions of Eqs.(4.1-17) and (4.1-18) for a planar wave. If we integrate Eqs.(18) and (21) with respect to t we get thus:

$$\left(\tau_{\mathrm{mp}}\frac{\partial}{\partial t}+1+\frac{\tau_{\mathrm{mp}}}{\tau_{\mathrm{p}}^2}\int dt\right)\left(\frac{\partial^2 A_{\mathrm{m}y}}{\partial y^2}-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{m}y}}{\partial t^2}\right)=0 \tag{24}$$

$$\left(\tau'_{\mathrm{mp}}\frac{\partial}{\partial t}+1+\frac{\tau'_{\mathrm{mp}}}{\tau'^2_{\mathrm{p}}}\int dt\right)\left(\frac{\partial^2 A_{\mathrm{e}y}}{\partial y^2}-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{e}y}}{\partial t^2}\right)=0 \tag{25}$$

where the notation

$$\frac{\tau_{\mathrm{mp}}}{\tau_{\mathrm{p}}^2}\int dt\left(\frac{\partial^2 A_{\mathrm{m}y}}{\partial y^2}-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{m}y}}{\partial t^2}\right)=\frac{\tau_{\mathrm{mp}}}{\tau_{\mathrm{p}}^2}\int\left(\frac{\partial^2 A_{\mathrm{m}y}}{\partial y^2}-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{m}y}}{\partial t^2}\right)dt$$

is used. With the substitutions

$$\frac{\partial^2 A_{\mathrm{m}y}}{\partial y^2}-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{m}y}}{\partial t^2}=V_{\mathrm{m}y}(y,t) \tag{26}$$

$$\frac{\partial^2 A_{\mathrm{e}y}}{\partial y^2}-\frac{1}{c^2}\frac{\partial^2 A_{\mathrm{e}y}}{\partial t^2}=V_{\mathrm{e}y}(y,t) \tag{27}$$

one obtains the ordinary differential equations

$$\frac{d^2V_{\mathrm{m}y}}{dt^2}+\frac{1}{\tau_{\mathrm{mp}}}\frac{dV_{\mathrm{m}y}}{dt}+\frac{1}{\tau_{\mathrm{p}}^2}V_{\mathrm{m}y}=0 \tag{28}$$

$$\frac{d^2V_{\mathrm{e}y}}{dt^2}+\frac{1}{\tau'_{\mathrm{mp}}}\frac{dV_{\mathrm{e}y}}{dt}+\frac{1}{\tau'^2_{\mathrm{p}}}V_{\mathrm{e}y}=0 \tag{29}$$

If we solve these equations for $V_{\mathrm{m}y}$ and $V_{\mathrm{e}y}$ we may obtain $A_{\mathrm{m}y}$ and $A_{\mathrm{e}y}$ in analogy to Eqs.(4.1-43) and (4.1-46). This requires the transition to the normalized variables

$$\theta=t/T,\qquad \zeta=y/cT \tag{30}$$

that brings Eqs.(28) and (29) into the following form:

$$\frac{d^2V_{\mathrm{m}y}}{d\theta^2}+\frac{T}{\tau_{\mathrm{mp}}}\frac{dV_{\mathrm{m}y}}{d\theta}+\frac{T^2}{\tau_{\mathrm{mp}}^2}V_{\mathrm{m}y}=0 \tag{31}$$

$$\frac{d^2V_{\mathrm{e}y}}{d\theta^2}+\frac{T}{\tau'_{\mathrm{mp}}}\frac{dV_{\mathrm{e}y}}{d\theta}+\frac{T^2}{\tau'^2_{\mathrm{mp}}}V_{\mathrm{e}y}=0 \tag{32}$$

Equations (26) and (27) become:

$$\frac{\partial^2 A_{my}}{\partial\zeta^2} - \frac{\partial^2 A_{my}}{\partial\theta^2} = c^2T^2V_{my}(\zeta,\theta) \tag{33}$$

$$\frac{\partial^2 A_{ey}}{\partial\zeta^2} - \frac{\partial^2 A_{ey}}{\partial\theta^2} = c^2T^2V_{ey}(\zeta,\theta) \tag{34}$$

With the help of Eqs.(4.1-43) and (4.1-46) we obtain:

$$A_{my}(\zeta,\theta) = -\frac{c^2T^2}{2}\int_0^\theta \left(\int_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} V_{my}(\zeta',\theta')d\zeta'\right)d\theta' \tag{35}$$

$$A_{ey}(\zeta,\theta) = -\frac{c^2T^2}{2}\int_0^\theta \left(\int_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')} V_{ey}(\zeta',\theta')d\zeta'\right)d\theta' \tag{36}$$

The solutions of Eqs.(31) and (32) are given by:

$$\begin{aligned} V_{my}(\zeta,\theta) &= V_{my1}(\zeta)e^{\gamma_{my1}\theta} + V_{my2}(\zeta)e^{\gamma_{my2}\theta} \\ \gamma_{my1,my2} &= -\frac{1}{2}\frac{T}{\tau_{mp}}(1 \mp i\sqrt{3}) \end{aligned} \tag{37}$$

$$\begin{aligned} V_{ey}(\zeta,\theta) &= V_{ey1}(\zeta)e^{\gamma_{ey1}\theta} + V_{ey2}(\zeta)e^{\gamma_{ey2}\theta} \\ \gamma_{ey1,ey2} &= -\frac{1}{2}\frac{T}{\tau'_{mp}}(1 \mp i\sqrt{3}) \end{aligned} \tag{38}$$

When $V_{my}(\zeta,\theta)$ and $V_{ey}(\zeta,\theta)$ are substituted into Eqs.(35) and (36) one must make the substitutions $\zeta \to \zeta'$, $\theta \to \theta'$.

We make again the substitutions of Eq.(4.1-31), with the subscript v alluding to 'variable':

$$A_{ex} = A_{ez} = A_{ev}, \quad A_{mx} = -A_{mz} = A_{mv} \tag{39}$$

Equations (17) and (19) as well as (20) and (22) are reduced to one equation each with the variables A_{mv} and A_{ev}:

$$\left(\tau_{mp}\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial t} + \frac{\tau_{mp}}{\tau_p^2}\right)\left(\frac{\partial^2 A_{mv}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{mv}}{\partial t^2}\right) - \frac{\sigma Z}{c}\left(Zc\frac{\partial^2 A_{ev}}{\partial y\partial t} + \frac{\partial^2 A_{mv}}{\partial t^2}\right) = 0 \tag{40}$$

$$\left(\tau'_{mp}\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial t} + \frac{\tau'_{mp}}{\tau_p'^2}\right)\left(\frac{\partial^2 A_{ev}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{ev}}{\partial t^2}\right) - \frac{s}{Zc}\left(\frac{c}{Z}\frac{\partial^2 A_{mv}}{\partial y\partial t} + \frac{\partial^2 A_{ev}}{\partial t^2}\right) = 0 \tag{41}$$

The comments about polarization made in Section 1.2 following Eq.(1.2-10) apply again if E_x, H_x, E_z, H_z, E, and H are replaced by A_{ex}, A_{mx}, A_{ez}, A_{mz}, A_{ev}, and A_{mv}.

In order to separate A_{ev} and A_{mv} we solve Eq.(40) for $\partial^2 A_{ev}/\partial y \partial t$

$$\frac{\partial^2 A_{ev}}{\partial y \partial t} = \frac{1}{Z^2\sigma}\left[\left(\tau_{mp}\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial t} + \frac{\tau_{mp}}{\tau_p^2}\right)\left(\frac{\partial^2 A_{mv}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{mv}}{\partial t^2}\right) - \frac{Z\sigma}{c}\frac{\partial^2 A_{mv}}{\partial t^2}\right] \quad (42)$$

and differentiate Eq.(41) with respect to y and t:

$$\left(\tau'_{mp}\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial t} + \frac{\tau'_{mp}}{\tau'^2_p}\right)\left(\frac{\partial^4 A_{ev}}{\partial y^3 \partial t} - \frac{1}{c^2}\frac{\partial^4 A_{ev}}{\partial y \partial t^3}\right) - \frac{s}{Zc}\left(\frac{c}{Z}\frac{\partial^4 A_{mv}}{\partial y^2 \partial t^2} + \frac{\partial^4 A_{ev}}{\partial y \partial t^3}\right) = 0 \quad (43)$$

The terms $\partial^4 A_{ev}/\partial y^3 \partial t$ and $\partial^4 A_{ev}/\partial y \partial t^3$ are obtained from Eq.(42):

$$\frac{\partial^4 A_{ev}}{\partial y^3 \partial t} = \frac{1}{Z^2\sigma}\left[\left(\tau_{mp}\frac{\partial^4}{\partial y^2 \partial t^2} + \frac{\partial^3}{\partial y^2 \partial t} + \frac{\tau_{mp}}{\tau_p^2}\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 A_{mv}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{mv}}{\partial t^2}\right) - \frac{Z\sigma}{c}\frac{\partial^4 A_{mv}}{\partial y^2 \partial t^2}\right] \quad (44)$$

$$\frac{\partial^4 A_{ev}}{\partial y \partial t^3} = \frac{1}{Z^2\sigma}\left[\left(\tau_{mp}\frac{\partial^4}{\partial t^4} + \frac{\partial^3}{\partial t^3} + \frac{\tau_{mp}}{\tau_p^2}\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2 A_{mv}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{mv}}{\partial t^2}\right) - \frac{Z\sigma}{c}\frac{\partial^4 A_{mv}}{\partial t^4}\right] \quad (45)$$

Substitution of Eqs.(44) and (45) into Eq.(43) yields:

$$\left(\tau'_{mp}\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial t} + \frac{\tau'_{mp}}{\tau'^2_p}\right)\left(\tau_{mp}\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial t} + \frac{\tau_{mp}}{\tau_p^2}\right)\left(\frac{\partial^2 V_m}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 V_m}{\partial t^2}\right) - \left[\frac{Z\sigma}{c}\left(\tau'_{mp}\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial t} + \frac{\tau'_{mp}}{\tau'^2_p}\right) + \frac{s}{Zc}\left(\tau_{mp}\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial t} + \frac{\tau_{mp}}{\tau_p^2}\right) + s\sigma\right]\frac{\partial^2 V_m}{\partial t^2} = 0 \quad (46)$$

$$V_m(y,t) = \frac{\partial^2 A_{mv}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 A_{mv}}{\partial t^2} \quad (47)$$

These equations should be compared with Eqs.(4.1-38) and (4.1-39). We obtain again a partial differential equation with the wave equation as variable, but it is now of fourth order rather than of second order.

Instead of solving Eqs.(40) and (41) for $A_{\rm mv}$ we may solve it for $A_{\rm ev}$. The exchanges

$$\tau_{\rm mp} \leftrightarrow \tau'_{\rm mp},\ \tau_{\rm p} \leftrightarrow \tau'_{\rm p},\ A_{\rm mv} \leftrightarrow A_{\rm ev},\ Z\sigma \leftrightarrow s/Z \tag{48}$$

reproduce Eqs.(40) and (41). Hence, we may make the same exchanges in Eqs.(45) and (40):

$$\begin{aligned}\left(\tau'_{\rm mp}\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}+\frac{\tau'_{\rm mp}}{\tau'^2_{\rm p}}\right)\left(\tau_{\rm mp}\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}+\frac{\tau_{\rm mp}}{\tau^2_{\rm p}}\right)\left(\frac{\partial^2 V_{\rm e}}{\partial y^2}-\frac{1}{c^2}\frac{\partial^2 V_{\rm e}}{\partial t^2}\right)\\ -\left[\frac{Z\sigma}{c}\left(\tau'_{\rm mp}\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}+\frac{\tau'_{\rm mp}}{\tau'^2_{\rm p}}\right)\right.\\ \left.+\frac{s}{Zc}\left(\tau_{\rm mp}\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}+\frac{\tau_{\rm mp}}{\tau^2_{\rm p}}\right)+s\sigma\right]\frac{\partial^2 V_{\rm e}}{\partial t^2}=0\end{aligned} \tag{49}$$

$$V_{\rm e}(y,t)=\frac{\partial^2 A_{\rm ev}}{\partial y^2}-\frac{1}{c^2}\frac{\partial^2 A_{\rm ev}}{\partial t^2} \tag{50}$$

These equations should be compared with Eqs.(4.1-36) and (4.1-37).

Equations (49) and (50) are rewritten into the following form with the help of the notation $\theta = t/T$ and $\zeta = y/cT$:

$$\begin{aligned}\left(\theta_{\rm mp}\frac{\partial^2}{\partial\theta^2}+\frac{\partial}{\partial\theta}+\theta_{\rm p}\right)\left(\theta'_{\rm mp}\frac{\partial^2}{\partial\theta^2}+\frac{\partial}{\partial\theta}+\theta'_{\rm p}\right)\left(\frac{\partial^2 V_{\rm e}}{\partial\zeta^2}-\frac{\partial^2 V_{\rm e}}{\partial\theta^2}\right)\\ -\left[\frac{1}{\rho_{\rm s}}\left(\theta_{\rm mp}\frac{\partial^2}{\partial\theta^2}+\frac{\partial}{\partial\theta}+\theta_{\rm p}\right)+\rho_\sigma\left(\theta'_{\rm mp}\frac{\partial^2}{\partial\theta^2}+\frac{\partial}{\partial\theta}+\theta'_{\rm p}\right)+\rho_2^2\right]\frac{\partial^2 V_{\rm e}}{\partial\theta^2}=0\\ \theta_{\rm mp}=\tau_{\rm mp}/T,\ \theta'_{\rm mp}=\tau'_{\rm mp}/T,\ \theta_{\rm p}=\tau_{\rm mp}T/\tau^2_{\rm p},\ \theta'_{\rm p}=\tau'_{\rm mp}T/\tau'^2_{\rm p}\\ \rho_\sigma=ZTc\sigma,\ \rho_{\rm s}=Z/scT,\ \rho_2^2=c^2T^2\sigma s\end{aligned} \tag{51}$$

$$c^2T^2V_{\rm e}(\zeta,\theta)=\frac{\partial^2 A_{\rm ev}}{\partial\zeta^2}-\frac{\partial^2 A_{\rm ev}}{\partial\theta^2} \tag{52}$$

The solution of Eq.(52) is the same as of Eq.(4.1-42) by Eq.(4.1-43):

$$A_{\rm ev}(\zeta,\theta)=-\frac{c^2T^2}{2}\int_0^\theta\left(\int_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')}V_{\rm e}(\zeta',\theta')d\zeta'\right)d\theta' \tag{53}$$

For the variable $V_{\rm m}(y,t)$ in Eq.(47) we obtain the following three equations in analogy to Eqs.(51)–(53):

$$\left(\theta_{\mathrm{mp}}\frac{\partial^2}{\partial\theta^2}+\frac{\partial}{\partial\theta}+\theta_{\mathrm{p}}\right)\left(\theta'_{\mathrm{m}}\frac{\partial^2}{\partial\theta^2}+\frac{\partial}{\partial\theta}+\theta'_{\mathrm{p}}\right)\left(\frac{\partial^2V_{\mathrm{m}}}{\partial\zeta^2}-\frac{\partial^2V_{\mathrm{m}}}{\partial\theta^2}\right)$$
$$-\left[\frac{1}{\rho_{\mathrm{s}}}\left(\theta_{\mathrm{m}}\frac{\partial^2}{\partial\theta^2}+\frac{\partial}{\partial\theta}+\theta_{\mathrm{p}}\right)+\rho_\sigma\left(\theta'_{\mathrm{mp}}\frac{\partial^2}{\partial\theta^2}+\frac{\partial}{\partial\theta}+\theta'_{\mathrm{p}}\right)+\rho_2^2\right]\frac{\partial^2V_{\mathrm{m}}}{\partial\theta^2}=0 \quad (54)$$

$$c^2T^2V_{\mathrm{m}}(\zeta,\theta)=\frac{\partial^2A_{\mathrm{mv}}}{\partial\zeta^2}-\frac{\partial^2A_{\mathrm{mv}}}{\partial\theta^2} \quad (55)$$

$$A_{\mathrm{mv}}(\zeta,\theta)=-\frac{c^2T^2}{2}\int\limits_0^\theta\left(\int\limits_{\zeta-(\theta-\theta')}^{\zeta+(\theta-\theta')}V_{\mathrm{m}}(\zeta',\theta')d\zeta'\right)d\theta' \quad (56)$$

If $A_{\mathrm{ev}}(\zeta,\theta)$ is obtained from Eq.(53) for certain initial and boundary conditions, one may obtain the component $A_{\mathrm{mv}}(\zeta,\theta)$ of the associated potential from either Eq.(40) or (41). Consider Eq.(41) first:

$$A_{\mathrm{mv}}(\zeta,\theta)=Z\rho_{\mathrm{s}}\int\left\{\int\left[\left(\theta'_{\mathrm{mp}}\frac{\partial^2}{\partial\theta^2}+\frac{\partial}{\partial\theta}+\theta'_{\mathrm{p}}\right)\left(\frac{\partial^2A_{\mathrm{ev}}}{\partial\zeta^2}-\frac{\partial^2A_{\mathrm{ev}}}{\partial\theta^2}\right)\right.\right.$$
$$\left.\left.-\frac{1}{\rho_{\mathrm{s}}}\frac{\partial^2A_{\mathrm{ev}}}{\partial\theta^2}\right]d\theta\right\}d\zeta,\quad \rho_{\mathrm{s}}=\frac{Z}{scT} \quad (57)$$

A second expression for A_{mv} is obtained from Eq.(40) by treating this equation as an inhomogeneous equation for A_{mv} with a known term $\partial^2A_{\mathrm{ev}}/\partial y\partial t$ or $\partial^2A_{\mathrm{ev}}/\partial\zeta\partial\theta$:

$$\left(\theta_{\mathrm{mp}}\frac{\partial^2}{\partial\theta^2}+\frac{\partial}{\partial\theta}+\theta_{\mathrm{p}}\right)\left(\frac{\partial^2A_{\mathrm{mv}}}{\partial\zeta^2}-\frac{\partial^2A_{\mathrm{mv}}}{\partial\theta^2}\right)-\rho_\sigma\frac{\partial^2A_{\mathrm{mv}}}{\partial\theta^2}=Z\rho_\sigma\frac{\partial^2A_{\mathrm{ev}}}{\partial\zeta\partial\theta}$$
$$\rho_\sigma=ZTc\sigma,\ \rho_{\mathrm{s}}=Z/scT,\ \rho_{\mathrm{s}}\rho_\sigma=\sigma\mu/s\epsilon=1/\omega^2 \quad (58)$$

It is generally simpler to integrate Eq.(57) than to solve Eq.(58), but one cannot ignore Eq.(58). Since Eqs.(57) and (58) must yield the same result for A_{mv} we generally need them both to determine integration constants. However, experience teaches that one does not always need all the integration constants.

If $A_{\mathrm{mv}}(\zeta,\theta)$ is found from Eq.(56) for certain initial and boundary conditions one may obtain the associated potential $A_{\mathrm{ev}}(\zeta,\theta)$ from either Eq.(40) or (41). We get from Eq.(40):

$$A_{\mathrm{ev}}(\zeta,\theta)=\frac{1}{Z\rho_\sigma}\int\left\{\int\left[\left(\theta_{\mathrm{mp}}\frac{\partial^2}{\partial\theta^2}+\frac{\partial}{\partial\theta}+\theta_{\mathrm{p}}\right)\left(\frac{\partial^2A_{\mathrm{mv}}}{\partial\zeta^2}-\frac{\partial^2A_{\mathrm{mv}}}{\partial\theta^2}\right)\right.\right.$$
$$\left.\left.-\frac{1}{Z}\frac{\partial^2A_{\mathrm{mv}}}{\partial\theta^2}\right]d\theta\right\}d\zeta \quad (59)$$

The second expression for $A_{\rm ev}$ is obtained from Eq.(41) by treating it as an inhomogeneous equation for $A_{\rm ev}$ with known term $\partial^2 A_{\rm mv}/\partial y \partial t$ or $\partial^2 A_{\rm mv}/\partial\zeta\partial\theta$:

$$\left(\theta'_{\rm mp}\frac{\partial^2}{\partial\theta^2}-\frac{\partial}{\partial\theta}+\theta'_{\rm p}\right)\left(\frac{\partial^2 A_{\rm ev}}{\partial\zeta^2}-\frac{\partial^2 A_{\rm ev}}{\partial\theta^2}\right)-\frac{1}{\rho_{\rm s}}\frac{\partial^2 A_{\rm ev}}{\partial\theta^2}=\frac{1}{Z\rho_{\rm s}}\frac{\partial^2 A_{\rm mv}}{\partial\zeta\partial\theta} \tag{60}$$

Generally one must obtain $A_{\rm ev}(\zeta,\theta)$ from both Eq.(58) and (60) in order to obtain all the integration constants.

We denote the solution of $A_{\rm ev}$ derived from Eq.(53) by $A_{\rm eve}$ and the associated solution obtained from $A_{\rm mv}$ via Eqs.(59) and (60) by $A_{\rm mve}$. The general solution of $A_{\rm ev}$ is the sum

$$A_{\rm ev}(\zeta,\theta)=A_{\rm eve}(\zeta,\theta)+A_{\rm mve}(\zeta,\theta) \tag{61}$$

In analogy we denote the solution of $A_{\rm mv}$ derived from Eq.(56) by $A_{\rm mvm}$ and the associated solution obtained from $A_{\rm ev}$ via Eqs.(57) and (58) by $A_{\rm evm}$ and we obtain the general solution $A_{\rm mv}$ as the sum

$$A_{\rm mv}(\zeta,\theta)=A_{\rm mvm}(\zeta,\theta)+A_{\rm evm}(\zeta,\theta) \tag{62}$$

Hence, one may choose initial and boundary conditions independently for $A_{\rm eve}$ and $A_{\rm mvm}$, but the associated potentials $A_{\rm evm}$ and $A_{\rm mve}$ are always automatically excited with $A_{\rm eve}$ and $A_{\rm mvm}$. One cannot excite $A_{\rm ev}$ without exciting $A_{\rm mv}$ and vice versa.

Let us try to find a solution for $V_{\rm e}(\zeta,\theta)$ of Eq.(51) for a boundary condition at $\zeta=0$ with the time variation of an exponential step function which approaches a step function for $\theta_{\rm s}\to\infty$:

$$\begin{aligned} V_{\rm e}(0,\theta)=V_{\rm e0}e^{-\theta/\theta_{\rm s}}S(\theta)&=0 && \text{for } \theta<0\\ &=V_{\rm e0}e^{-\theta/\theta_{\rm s}} && \text{for } \theta\geq 0 \end{aligned} \tag{63}$$

It is usual to assume a further boundary condition for $\zeta\to\infty$:

$$V_{\rm e}(\infty,\theta)=\text{finite} \tag{64}$$

It was explained in the text following Eq.(4.1-55) why we cannot use it. As initial condition at $\theta=0$ we choose

$$V_{\rm e}(\zeta,0)=0 \quad \text{for } \zeta>0 \tag{65}$$

Differentiation with respect to ζ yields more relations for the derivatives:

$$\frac{\partial^n V_{\rm e}(\zeta,\theta)}{\partial\zeta^n}=0 \quad \text{for } \zeta>0,\ \ n=1,\ 2,\ \ldots \tag{66}$$

Equation (51) may be written in the following form:

$$
\begin{aligned}
&\frac{\partial^2}{\partial\theta^2}\left(\frac{\partial^2 V_e}{\partial\zeta^2} - \frac{\partial^2 V_e}{\partial\theta^2} - \rho_1\frac{\partial V_e}{\partial\theta} - \rho_2^2 V_e\right) \\
&\quad + \frac{\partial}{\partial\theta}\left[\rho_m\frac{\partial^4 V_e}{\partial\theta^2\partial\zeta^2} + \rho_p\frac{\partial^2 V_e}{\partial\zeta^2} - \rho_m\frac{\partial^4 V_e}{\partial\theta^4} - \rho_{em}\frac{\partial^3 V_e}{\partial\theta^3} - (\rho_p + \rho_{ep})\frac{\partial^2 V_e}{\partial\theta^2}\right] \\
&\qquad + \frac{\partial^2}{\partial\zeta^2}\left(\rho_{mm}\frac{\partial^4 V_e}{\partial\theta^4} + \rho_{mp}\frac{\partial^2 V_e}{\partial\theta^2} + \rho_{pp}V_e\right) \\
&\qquad\quad - \frac{\partial^2}{\partial\theta^2}\left(\rho_{mm}\frac{\partial^2 V_e}{\partial\theta^4} + \rho_{mp}\frac{\partial^2 V_e}{\partial\theta^2} + \rho_{pp}V_e\right) = 0
\end{aligned}
$$

$$
\begin{gathered}
\rho_{pm} = \theta_{mp} + \theta'_{mp},\ \rho_p = \theta_p + \theta'_p,\ \rho_{em} = \theta_{mp}/\rho_s + \theta'_{mp}\rho_\sigma \\
\rho_{ep} = \theta_p/\rho_s + \theta'_p\rho_\sigma,\ \rho_{mm} = \theta_{mp}\theta'_{mp},\ \rho_{mp} = \theta_{mp}\theta'_p + \theta_p\theta'_{mp},\ \rho_{pp} = \theta_p\theta'_p
\end{gathered}
\qquad (67)
$$

Some simplification is achieved by integrating twice with respect to θ:

$$
\begin{aligned}
&\frac{\partial^2 V_e}{\partial\zeta^2} - \frac{\partial^2 V_e}{\partial\theta^2} - \rho_1\frac{\partial V_e}{\partial\theta} - \rho_2^2 V_e \\
&\quad + \rho_m\frac{\partial^3 V_e}{\partial\zeta^2\partial\theta} + \rho_p\int\frac{\partial^2 V_e}{\partial\zeta^2}d\theta - \rho_m\frac{\partial^3 V_e}{\partial\theta^3} - \rho_{em}\frac{\partial^2 V_e}{\partial\theta^2} - (\rho_p + \rho_{ep})\frac{\partial V_e}{\partial\theta} \\
&\qquad + \rho_{mm}\frac{\partial^4 V_e}{\partial\zeta^2\partial\theta^2} + \rho_{mp}\frac{\partial^2 V_e}{\partial\zeta^2} + \rho_{pp}\iint\frac{\partial^2 V_e}{\partial\zeta^2}d\theta\,d\theta' \\
&\qquad\quad - \rho_{mm}\frac{\partial^4 V_e}{\partial\theta^4} - \rho_{mp}\frac{\partial^2 V_e}{\partial\theta^2} - \rho_{pp}V_e = 0
\end{aligned}
\qquad (68)
$$

Substitution of Eq.(65) and Eq.(66) for $n = 2$ into Eq.(68) yields the following result for $\theta = 0$, $\zeta > 0$:

$$
\begin{aligned}
&\frac{\partial}{\partial\theta}\left[\frac{\partial V_e}{\partial\theta} + \rho_1 V_e + \rho_m\frac{\partial^2 V_e}{\partial\theta^2} + \rho_m\frac{\partial V_e}{\partial\theta} + (\rho_p + \rho'_{ep})V_e\right. \\
&\quad \left. + \rho_{mm}\frac{\partial^3 V_e}{\partial\theta^3} + \rho_{mp}\frac{\partial V_e}{\partial\theta}\right] - \rho_p\int\frac{\partial^2 V_e}{\partial\zeta^2}d\theta - \rho_{pp}\iint\frac{\partial^2 V_e}{\partial\zeta^2}d\theta\,d\theta' = 0
\end{aligned}
\qquad (69)
$$

We note that $\int(\partial^2 V_e/\partial\zeta^2)d\theta$ can be a function of ζ but its time derivative would still satisfy Eq.(66).

A sufficient condition to satisfy Eq.(69) is that V_e, its time derivatives up to $\partial^3 V_e/\partial\theta^3$, and the two integrals of $\partial^2 V_e/\partial\zeta^2$ are zero. The two terms in Eq.(69) containing V_e yield again Eq.(65), but the derivatives of V_e and the integrals of $\partial^2 V_e/\partial\zeta^2$ yield the following additional conditions:

$$\text{for } \zeta > 0,\ \theta = 0 \qquad \frac{\partial V_e}{\partial \theta} = 0 \tag{70}$$

$$\frac{\partial^2 V_e}{\partial \theta^2} = 0 \tag{71}$$

$$\frac{\partial^3 V_e}{\partial \theta^3} = 0 \tag{72}$$

$$\int \frac{\partial^2 V_e}{\partial \zeta^2} d\theta = 0 \tag{73}$$

$$\iint \frac{\partial^2 V_e}{\partial \zeta^2} d\theta\, d\theta' = 0 \tag{74}$$

These conditions may not be the only possible ones but they are sufficient. Equations (70)–(74) have led previously to useful solutions (Harmuth and Lukin 2000, Section 2.2).

We assume that the solution of Eq.(68) can be represented as the sum of a steady state solution $F(\zeta)e^{-\theta/\theta_s}$, plus a deviation $w(\zeta,\theta)$ from the steady state solution (Habermann 1983, p. 258):

$$V_e(\zeta,\theta) = V_{e0}[w(\zeta,\theta) + F(\zeta)e^{-\theta/\theta_s}] \tag{75}$$

First we calculate $F(\zeta)$. With

$$\frac{\partial^n}{\partial \theta^n} F(\zeta)e^{-\theta/\theta_s} = (-1)^n \theta_s^{-n} F(\zeta)e^{-\theta/\theta_s} \tag{76}$$

$$\int F\zeta)\mathrm{e}^{-\theta/\theta_s}\mathrm{d}\theta = -\theta_s F(\zeta)e^{-\theta/\theta_s} \tag{77}$$

$$\iint F\zeta)\mathrm{e}^{-\theta/\theta_s}\mathrm{d}\theta\,\mathrm{d}\theta' = \theta_s^2 F(\zeta)e^{-\theta/\theta_s} \tag{78}$$

we obtain from Eq.(68):

$$\begin{aligned}
\Bigg[\frac{d^2F}{d\zeta^2} - \left(\frac{1}{\theta_s^2} - \frac{\rho_1}{\theta_s} + \rho_2^2\right)F \\
- \left(\frac{\rho_m}{\theta_s} + \rho_p\theta_s\right)\frac{d^2F}{d\zeta^2} + \left(\frac{\rho_m}{\theta_s^3} - \frac{\rho_{em}}{\theta_s^2} + \frac{\rho_{p}+\rho_{ep}}{\theta_s}\right)F \\
+ \left(\frac{\rho_{mm}}{\theta_s^2} + \rho_{mp}\right)\frac{d^2F}{d\zeta^2} + \left(\rho_{pp}\theta_s^2 - \frac{\rho_{mm}}{\theta_s^4} - \frac{\rho_{mp}}{\theta_s^2} - \rho_{pp}\right)F\Bigg]e^{-\theta/\theta_s} = 0
\end{aligned} \tag{79}$$

This equation has the trivial solution $\theta_s = 0$ and a non-trivial solution containing the terms $d^2F/d\zeta^2$ and F:

$$\frac{d^2F}{d\zeta^2} - \frac{1}{L^2}F = 0, \quad \frac{1}{L^2} = \frac{L_2}{L_1}, \; L = \sqrt{\frac{L_1}{L_2}}$$

$$L_2 = \rho_2^2 + \rho_{pp} - \rho_{pp}\theta_s^2 - \frac{\rho_1 + \rho_p + \rho_{ep}}{\theta_s} + \frac{1 + \rho_m + \rho_{mp}}{\theta_s^2} - \frac{\rho_m}{\theta_s^3} + \frac{\rho_{mm}}{\theta_s^4}$$

$$L_1 = 1 - \rho_{mp} - \rho_p\theta_s - \frac{\rho_m}{\theta_s} + \frac{\rho_m}{\theta_s^2} \tag{80}$$

Equation (80) has the general solution:

$$F(\zeta) = A_{00}e^{-\zeta/L} + A_{01}e^{\zeta/L} \tag{81}$$

Following Eq.(4.1-60) we choose $A_{00} = 1$ and $A_{01} = 0$:

$$F(\zeta) = e^{-\zeta/L} \tag{82}$$

For the calculation of $w(\zeta, \theta)$ of Eq.(75) we observe that the introduction of the function $F(\zeta)e^{-\theta/\theta_s}$ transforms the boundary condition of Eq.(63) for $V_e(0, \theta)$ into an homogeneous boundary condition for $w(0, \theta)$, which is the purpose of this method of solution:

$$V_e(0,\theta) = V_{e0}[w(0,\theta) + e^{-\theta/\theta_s}] = V_{e0}e^{-\theta/\theta_s}$$

$$w(0,\theta) = 0 \tag{83}$$

The second boundary condition given by Eq.(64) yields:

$$w(\infty, \theta) = \text{finite} \tag{84}$$

The text following Eq.(4.1-63) explains that it cannot be used but it is not required either. The initial conditions of Eqs.(65) and (70)–(74) yield:

$$\text{for } \theta = 0, \; \zeta > 0 \qquad w(\zeta,0) + F(\zeta) = 0, \quad w(\zeta,0) = -e^{-\zeta/L}$$

$$\frac{\partial w(\zeta,0)}{\partial\theta} = 0$$

$$\frac{\partial^2 w(\zeta,0)}{\partial\theta^2} = 0$$

$$\frac{\partial^3 w(\zeta,0)}{\partial\theta^3} = 0$$

$$\int \frac{\partial^2 w(\zeta,0)}{\partial\zeta^2} d\theta = 0$$

$$\iint \frac{\partial^2 w(\zeta,0)}{\partial\zeta^2} d\theta \, d\theta' = 0 \tag{85}$$

Substitution of Eq.(75) into Eq.(68) yields for $w(\zeta,\theta)$ the same equation as for $V_e(\zeta,\theta)$:

$$\frac{\partial^2 w}{\partial\zeta^2} - \frac{\partial^2 w}{\partial\theta^2} - \rho_1\frac{\partial w}{\partial\theta} - \rho_2^2 w + \rho_m\frac{\partial^3 w}{\partial\zeta^2\partial\theta} + \rho_p\int\frac{\partial^2 w}{\partial\zeta^2}d\theta - \rho_m\frac{\partial^3 w}{\partial\theta^3} - \rho_{em}\frac{\partial^2 w}{\partial\theta^2} - (\rho_p+\rho_{ep})\frac{\partial w}{\partial\theta} + \rho_{mm}\frac{\partial^4 w}{\partial\zeta^2\partial\theta^2} + \rho_{mp}\frac{\partial^2 w}{\partial\zeta^2} + \rho_{pp}\iint w\,d\theta\,d\theta' - \rho_{mm}\frac{\partial^4 w}{\partial\theta^4} - \rho_{mp}\frac{\partial^2 w}{\partial\theta^2} - \rho_{pp}w = 0 \quad (86)$$

Using Bernoulli's product method to find a particular solution $w_\kappa(\zeta,\theta)$ by separation of the variables we substitute

$$w_\kappa(\zeta,\theta) = \varphi(\zeta)\psi(\theta) \quad (87)$$

into Eq.(86), divide by $\varphi\psi$, and separate the terms containing φ or ψ:

$$\frac{1}{\varphi}\frac{d^2\varphi}{d\zeta^2} = \left(\rho_{mm}\frac{d^2\psi}{d\theta^2} + \rho_m\frac{d\psi}{d\theta} + \psi + \rho_p\int\psi\,d\theta\right)^{-1} \times\left(\rho_{mm}\frac{d^4\psi}{d\theta^4} + (1+\rho_{em}+\rho_{mp})\frac{d^2\psi}{d\theta^2} + (\rho_1+\rho_p+\rho_{ep})\frac{d\psi}{d\theta} + (\rho_2^2+\rho_{pp})\psi - \rho_{pp}\iint\psi\,d\theta\,d\theta'\right) \quad (88)$$

This equation can only be satisfied if both sides are equal to a constant that we denote $-(2\pi\kappa)^2$:

$$\frac{d^2\varphi}{d\zeta^2} + (2\pi\kappa)^2\varphi = 0 \quad (89)$$

Equation (89) has the solution

$$\varphi(\zeta) = A_{10}\sin 2\pi\kappa\zeta + A_{11}\cos 2\pi\kappa\zeta \quad (90)$$

but the boundary condition of Eq.(83) requires $A_{11} = 0$ and Eq.(90) is reduced to

$$\varphi(\zeta) = A_{10}\sin 2\pi\kappa\zeta \quad (91)$$

The right side of Eq.(88) can be rewritten with the constant $-(2\pi\kappa)^2$ as follows:

$$\rho_{mm}\frac{d^4\psi}{d\theta^4} + \left[(2\pi\kappa)^2\rho_{mm}+1+\rho_{em}+\rho_{mp}\right]\frac{d^2\psi}{d\theta^2} + \left[(2\pi\kappa)^2\rho_m+\rho_1+\rho_p+\rho_{ep}\right]\frac{d\psi}{d\theta} + (\rho_2^2+\rho_{pp})\psi + (2\pi\kappa)^2\rho_p\int\psi\,d\theta - \rho_{pp}\iint\psi\,d\theta\,d\theta' = 0 \quad (92)$$

Equation (92) is solved by the substitution $\psi(\theta) = \exp(\lambda\theta)$ and the following characteristic equation is obtained for the six roots $\lambda = \lambda_1, \ldots, \lambda_6$:

$$\rho_{\rm mm}\lambda^6 + \left[(2\pi\kappa)^2\rho_{\rm mm} + 1 + \rho_{\rm em} + \rho_{\rm mp}\right]\lambda^4 + \left[(2\pi\kappa)^2\rho_{\rm m} + \rho_1 + \rho_{\rm p} + \rho_{\rm ep}\right]\lambda^3 + (\rho_2^2 + \rho_{\rm pp})\lambda^2 + (2\pi\kappa)^2\rho_{\rm p}\lambda - \rho_{\rm pp} = 0 \quad (93)$$

This equation is very similar to a previously derived equation (Harmuth and Lukin 2000, Section 2.2, Eq. 31). We stop here since the evaluation of this equation appears to be as much work as that of the previously derived equation.

For small values of σ and s one may leave out the terms multiplied by σ or s in Eqs.(17)–(22). Six equations of the form of Eqs.(24) and (25) have then to be solved. Using a series expansion one may add correcting terms caused by the terms in Eqs.(17)–(22) multiplied by σ or s.

6.11 Evaluation of Eq.(5.3-4)

For the integration of $(\partial\Psi^*/\partial\theta)(\partial\Psi/\partial\theta)$, $(\partial\Psi^*/\partial\zeta)(\partial\Psi/\partial\zeta)$, and $\Psi^*\Psi$ of Eqs.(5.3-11), (5.3-12), and (5.3-6) with respect to ζ we write the square of the sums in those equations in the following form:

$$\left(\sum_{\kappa=1}^{\infty} B(\kappa)\sin\left\{[(2\pi\kappa)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2]^{1/2}\theta\right\}\sin\left\{[(2\pi\kappa)^2 + \lambda_1^2]^{1/2}\zeta\right\}\right)^2$$
$$= \sum_{\kappa=1}^{\infty} B^2(\kappa)\sin^2\left\{[(2\pi\kappa)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2]^{1/2}\theta\right\}\sin^2\left\{[(2\pi\kappa)^2 + \lambda_1^2]^{1/2}\zeta\right\}$$
$$+ \sum_{\kappa=1}^{\infty,\neq j}\sum_{j=1}^{\infty} B(\kappa)B(j)\sin\left\{[(2\pi\kappa)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2]^{1/2}\theta\right\}\sin\left\{[(2\pi j)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2]^{1/2}\theta\right\}$$
$$\times \sin\left\{[(2\pi\kappa)^2 + \lambda_1^2]^{1/2}\zeta\right\}\sin\left\{[(2\pi j)^2 + \lambda_1^2]^{1/2}\zeta\right\} \quad (1)$$

$$\left(\sum_{\kappa=1}^{\infty} B(\kappa)[(2\pi\kappa)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2]^{1/2}\cos\left\{[(2\pi\kappa)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2]^{1/2}\theta\right\}\right.$$
$$\left.\times \sin\left\{[(2\pi\kappa)^2 + \lambda_1^2]^{1/2}\zeta\right\}\right)^2$$
$$= \sum_{\kappa=1}^{\infty} B^2(\kappa)[(2\pi\kappa)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2]\cos^2\left\{[(2\pi\kappa)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2]^{1/2}\theta\right\}$$
$$\times \sin^2\left\{[(2\pi\kappa)^2 + \lambda_1^2]^{1/2}\zeta\right\}$$
$$+ \sum_{\kappa=1}^{\infty,\neq j}\sum_{j=1}^{\infty} B(\kappa)B(j)[(2\pi\kappa)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2]^{1/2}[(2\pi j)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2]^{1/2}$$
$$\times \cos\left\{[(2\pi\kappa)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2]^{1/2}\theta\right\}\cos\left\{[(2\pi j)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2]^{1/2}\theta\right\}$$
$$\times \sin\left\{[(2\pi\kappa)^2 + \lambda_1^2]^{1/2}\zeta\right\}\sin\left\{[(2\pi j)^2 + \lambda_1^2]^{1/2}\zeta\right\} \quad (2)$$

$$\left(\sum_{\kappa=1}^{\infty} B(\kappa)[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\sin\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}\right.$$
$$\left.\times\cos\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}\right)^2$$
$$=\sum_{\kappa=1}^{\infty} B^2(\kappa)[(2\pi\kappa)^2+\lambda_1^2]\sin^2\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}$$
$$\times\cos^2\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}$$
$$+\sum_{\kappa=1}^{\infty,\neq j}\sum_{j=1}^{\infty} B(\kappa)B(j)[(2\pi\kappa)^2+\lambda_1^2]^{1/2}[(2\pi j)^2+\lambda_1^2]^{1/2}$$
$$\times\sin\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}\sin\left\{[(2\pi j)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}$$
$$\times\cos\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}\cos\left\{[(2\pi j)^2+\lambda_1^2]^{1/2}\zeta\right\} \quad (3)$$

The products of sine and cosine functions of ζ are transformed into sums with the help of the relations:

$$\sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}\sin\left\{[(2\pi j)^2+\lambda_1^2]^{1/2}\zeta\right\}$$
$$=\frac{1}{2}\Big[\cos\left(\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}-[(2\pi j)^2+\lambda_1^2]^{1/2}\right\}\zeta\right)$$
$$-\cos\left(\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}+[(2\pi j)^2+\lambda_1^2]^{1/2}\right\}\zeta\right)\Big] \quad (4)$$

$$\cos\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}\cos\left\{[(2\pi j)^2+\lambda_1^2]^{1/2}\zeta\right\}$$
$$=\frac{1}{2}\Big[\cos\left(\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}-[(2\pi j)^2+\lambda_1^2]^{1/2}\right\}\zeta\right)$$
$$+\cos\left(\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}+[(2\pi j)^2+\lambda_1^2]^{1/2}\right\}\zeta\right)\Big] \quad (5)$$

$$\sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}\cos[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]$$
$$=\frac{1}{2}\Big[\sin\left(\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}-(\lambda_1^2-\lambda_2^2)^{1/2}\right\}\zeta\right)$$
$$+\sin\left(\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}+(\lambda_1^2-\lambda_2^2)^{1/2}\right\}\zeta\right)\Big] \quad (6)$$

$$\sin[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]\sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}$$
$$=\frac{1}{2}\Big[\cos\left(\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}-(\lambda_1^2-\lambda_2^2)^{1/2}\right\}\zeta\right)$$
$$-\cos\left(\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}+(\lambda_1^2-\lambda_2^2)^{1/2}\right\}\zeta\right)\Big] \quad (7)$$

$$\cos[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]\cos\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}$$
$$=\frac{1}{2}\Big[\cos\Big(\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}-(\lambda_1^2-\lambda_2^2)^{1/2}\right\}\zeta\Big)$$
$$+\cos\Big(\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}+(\lambda_1^2-\lambda_2^2)^{1/2}\right\}\zeta\Big)\Big] \quad (8)$$

We shall further need the following integrals:

$$\int_0^1 \sin^2\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}d\zeta=\frac{1}{2}\left(1-\frac{\sin 2[(2\pi\kappa)^2+\lambda_1^2]^{1/2}}{2\,[(2\pi\kappa)^2+\lambda_1^2]^{1/2}}\right) \quad (9)$$

$$\int_0^1 \cos^2\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}d\zeta=\frac{1}{2}\left(1+\frac{\sin 2[(2\pi\kappa)^2+\lambda_1^2]^{1/2}}{2\,[(2\pi\kappa)^2+\lambda_1^2]^{1/2}}\right) \quad (10)$$

$$\int_0^1 \cos\Big(\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\pm[(2\pi j)^2+\lambda_1^2]^{1/2}\right\}\zeta\Big)d\zeta$$
$$=\frac{\sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\pm[(2\pi j)^2+\lambda_1^2]^{1/2}\right\}}{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\pm[(2\pi j)^2+\lambda_1^2]^{1/2}} \quad (11)$$

$$\int_0^1 \sin\Big(\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\pm(\lambda_1^2-\lambda_2^2)^{1/2}\right\}\zeta\Big)d\zeta$$
$$=\frac{1-\cos\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\pm(\lambda_1^2-\lambda_2^2)^{1/2}\right\}}{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\pm(\lambda_1^2-\lambda_2^2)^{1/2}} \quad (12)$$

$$\int_0^1 \cos\Big(\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\pm(\lambda_1^2-\lambda_2^2\right\}\zeta\Big)d\zeta$$
$$=\frac{\sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\pm(\lambda_1^2-\lambda_2^2)^{1/2}\right\}}{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\pm(\lambda_1^2-\lambda_2^2)^{1/2}} \quad (13)$$

$$\int_0^1 \sin^2[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]d\zeta=\frac{1}{2}\left(1-\frac{\sin 2(\lambda_1^2-\lambda_2^2)^{1/2}}{2\,(\lambda_1^2-\lambda_2^2)^{1/2}}\right) \quad (14)$$

$$\int_0^1 \cos^2[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]d\zeta=\frac{1}{2}\left(1+\frac{\sin 2(\lambda_1^2-\lambda_2^2)^{1/2}}{2\,(\lambda_1^2-\lambda_2^2)^{1/2}}\right) \quad (15)$$

For the simplest integral in Eq.(5.3-4) we obtain from Eq.(5.3-6):

$$\int_0^1 \Psi^*\Psi\, d\zeta = \Psi_1^2 \int_0^1 \Bigg[\Bigg(\sum_{\kappa=1}^{\infty} B(\kappa)\sin\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}$$

$$\times \sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}\Bigg)^2 + 2(1-\cos 2\lambda_1\lambda_3\theta)\cos^2[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]\Bigg]\, d\zeta$$

$$= \Psi_1^2\Bigg\{\sum_{\kappa=1}^{\infty} B^2(\kappa)\sin^2\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}$$

$$\times \int_0^1 \sin^2\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\} d\zeta$$

$$+\frac{1}{2}\sum_{\kappa=1}^{\infty,\neq j}\sum_{j=1}^{\infty} B(\kappa)B(j)\sin\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}$$

$$\times \sin[(2\pi j)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\}$$

$$\times\Bigg[\int_0^1 \cos\Big(\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}-[(2\pi j)^2+\lambda_1^2]^{1/2}\right\}\zeta\Big)d\zeta$$

$$-\int_0^1 \cos\Big(\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}+[(2\pi j)^2+\lambda_1^2]^{1/2}\right\}\zeta\Big)d\zeta\Bigg]$$

$$+2(1-\cos 2\lambda_1\lambda_2\theta)\int_0^1 \cos^2[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]d\zeta\Bigg\} \quad (16)$$

Using constants $C_{31}(\kappa)$, $C_{32}(\kappa, j)$, and C_{33} we may rewrite Eq.(16) as follows:

$$\int_0^1 \Psi^*\Psi\, d\zeta = \Psi_1^2\Bigg(\sum_{\kappa=1}^{\infty} C_{31}(\kappa)\sin^2\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}$$

$$+\sum_{\kappa=1}^{\infty,\neq j}\sum_{j=1}^{\infty} C_{32}(\kappa,j)\sin\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}$$

$$\times \sin\left\{[(2\pi j)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\} + C_{33}(1-\cos 2\lambda_1\lambda_3\theta)\Bigg) \quad (17)$$

$$C_{31}(\kappa) = \frac{1}{2}B^2(\kappa)\left(1 - \frac{\sin 2[(2\pi\kappa)^2 + \lambda_1^2]^{1/2}}{2\,[(2\pi\kappa)^2 + \lambda_1^2]^{1/2}}\right) \tag{18}$$

$$C_{32}(\kappa,j) = \frac{1}{2}B(\kappa)B(j)\left(\frac{\sin\left\{[(2\pi\kappa)^2 + \lambda_1^2]^{1/2} - [(2\pi j)^2 + \lambda_1^2]^{1/2}\right\}}{[(2\pi\kappa)^2 + \lambda_1^2]^{1/2} - [(2\pi j)^2 + \lambda_1^2]^{1/2}}\right.$$
$$\left. - \frac{\sin\left\{[(2\pi\kappa)^2 + \lambda_1^2]^{1/2} + [(2\pi j)^2 + \lambda_1^2]^{1/2}\right\}}{[(2\pi\kappa)^2 + \lambda_1^2]^{1/2} + [(2\pi j)^2 + \lambda_1^2]^{1/2}}\right) \tag{19}$$

$$C_{33} = 1 + \frac{\sin 2(\lambda_1^2 - \lambda_2^2)^{1/2}}{2\,(\lambda_1^2 - \lambda_2^2)^{1/2}} \tag{20}$$

Next we integrate Eq.(5.3-11):

$$\int_0^1 \frac{\partial\Psi^*}{\partial\theta}\frac{\partial\Psi}{\partial\theta}\,d\zeta = \Psi_1^2\int_0^1\Bigg[\lambda_1^2\lambda_3^2$$
$$\times\left(\sum_{\kappa=1}^{\infty} B(\kappa)\sin\left\{[(2\pi\kappa)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2]^{1/2}\theta\right\}\sin\left\{[(2\pi\kappa)^2 + \lambda_1^2]^{1/2}\zeta\right\}\right)^2$$
$$+\left(\sum_{\kappa=1}^{\infty} B(\kappa)[(2\pi\kappa)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2]^{1/2}\cos\left\{[(2\pi\kappa)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2]^{1/2}\theta\right\}\right.$$
$$\left.\times\sin\left\{[(2\pi\kappa)^2 + \lambda_1^2]^{1/2}\zeta\right\}\right)^2$$
$$-4\lambda_1\lambda_3\cos[(\lambda_1^2 - \lambda_2^2)^{1/2}\zeta]\sum_{\kappa=1}^{\infty} B(\kappa)\sin\left\{[(2\pi\kappa)^2 + \lambda_1^2]^{1/2}\zeta\right\}$$
$$\times\Big(\lambda_1\lambda_3\cos\lambda_1\lambda_3\theta\sin\left\{[(2\pi\kappa)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2]^{1/2}\theta\right\}$$
$$-[(2\pi\kappa)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2]^{1/2}\sin\lambda_1\lambda_3\theta\cos\left\{[(2\pi\kappa)^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2]^{1/2}\theta\right\}\Big)$$
$$+4\lambda_1^2\lambda_3^2\cos^2[(\lambda_1^2 - \lambda_2^2)^{1/2}\zeta]\Bigg]d\zeta \tag{21}$$

The six integrals over ζ are separated:

$$\int_0^1 \frac{\partial\Psi^*}{\partial\theta}\frac{\partial\Psi}{\partial\theta}\,d\zeta = \Psi_1^2\Bigg[\lambda_1^2\lambda_3^2$$

$$\times \sum_{\kappa=1}^{\infty} B^2(\kappa)\sin^2\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}\int_0^1 \sin^2\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}d\zeta$$

$$+\lambda_1^2\lambda_3^2\sum_{\kappa=1}^{\infty,\neq j}\sum_{j=1}^{\infty} B(\kappa)B(j)\sin\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}$$

$$\times\sin\left\{[(2\pi j)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}$$

$$\times\int_0^1 \sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}\sin\left\{[(2\pi j)^2+\lambda_1^2]^{1/2}\zeta\right\}d\zeta$$

$$+\sum_{\kappa=1}^{\infty} B^2(\kappa)[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]\cos^2\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}$$

$$\times\int_0^1 \sin^2\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}d\zeta$$

$$+\sum_{\kappa=1}^{\infty,\neq j}\sum_{j=1}^{\infty} B(\kappa)B(j)[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}[(2\pi j)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}$$

$$\times\cos\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}\cos\left\{[(2\pi j)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}$$

$$\times\int_0^1 \sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}\sin\left\{[(2\pi j)^2+\lambda_1^2]^{1/2}\zeta\right\}d\zeta$$

$$-4\lambda_1\lambda_3\sum_{\kappa=1}^{\infty} B(\kappa)\Big(\lambda_1\lambda_3\cos\lambda_1\lambda_3\theta\sin\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}$$

$$-[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\sin\lambda_1\lambda_2\theta\cos\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}\Big)$$

$$\times\int_0^1 \sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}\cos[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]d\zeta$$

$$+4\lambda_1^2\lambda_3^2\int_0^1 \cos^2[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]d\zeta\Bigg] \quad (22)$$

Carrying out the integrations is lengthy but straight forward:

$$\int_0^1 \frac{\partial \Psi^*}{\partial \theta}\frac{\partial \Psi}{\partial \theta}\, d\zeta = \Psi_1^2\Bigg(\sum_{\kappa=1}^{\infty} C_{11}(\kappa)\sin^2\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}$$

$$+\sum_{\kappa=1}^{\infty,\neq j}\sum_{j=1}^{\infty} C_{12}(\kappa,j)\sin\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}\sin\left\{[(2\pi j)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}$$

$$+\sum_{\kappa=1}^{\infty} C_{13}(\kappa)\cos^2\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}$$

$$+\sum_{\kappa=1}^{\infty,\neq j}\sum_{j=1}^{\infty} C_{14}(\kappa,j)\cos\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}\cos\left\{[(2\pi j)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}$$

$$-\sum_{\kappa=1}^{\infty} C_{15}(\kappa)\cos\lambda_1\lambda_3\theta\sin\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}$$

$$+\sum_{\kappa=1}^{\infty} C_{16}(\kappa)\sin\lambda_1\lambda_3\theta\cos\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}+C_{17}\Bigg) \quad (23)$$

$$C_{11}(\kappa) = \frac{1}{2}\lambda_1^2\lambda_3^2 B^2(\kappa)\left(1-\frac{\sin 2[(2\pi\kappa)^2+\lambda_1^2]^{1/2}}{2\,[(2\pi\kappa)^2+\lambda_1^2]^{1/2}}\right)^2 = \lambda_1^2\lambda_3^2 C_{31}(\kappa) \quad (24)$$

$$C_{12}(\kappa,j) = \frac{1}{2}\lambda_1^2\lambda_3^2 B(\kappa)B(j)\left(\frac{\sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}-[(2\pi j)^2+\lambda_1^2]^{1/2}\right\}}{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}-[(2\pi j)^2+\lambda_1^2]^{1/2}}\right.$$
$$\left.-\frac{\sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}+[(2\pi j)^2+\lambda_1^2]^{1/2}\right\}}{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}+[(2\pi j)^2+\lambda_1^2]^{1/2}}\right) \quad (25)$$

$$C_{13}(\kappa) = \frac{1}{2}B^2(\kappa)[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]\left(1-\frac{\sin 2[(2\pi\kappa)^2+\lambda_1^2]^{1/2}}{2\,[(2\pi\kappa)^2+\lambda_1^2]^{1/2}}\right)$$
$$= [(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]C_{31}(\kappa) \quad (26)$$

$$C_{14}(\kappa,j) = \frac{1}{2}B(\kappa)B(j)[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}[(2\pi j)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}$$
$$\times\left(\frac{\sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}-[(2\pi j)^2+\lambda_1^2]^{1/2}\right\}}{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}-[(2\pi j)^2+\lambda_1^2]^{1/2}}\right.$$
$$\left.-\frac{\sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}+[(2\pi j)^2+\lambda_1^2]^{1/2}\right\}}{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}+[(2\pi j)^2+\lambda_1^2]^{1/2}}\right) \quad (27)$$

$$C_{15}(\kappa) = 2\lambda_1^2\lambda_3^2 B(\kappa)C_{156}(\kappa) \quad (28)$$

$$C_{16}(\kappa) = 2\lambda_1\lambda_3[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}B(\kappa)C_{156}(\kappa) \quad (29)$$

$$C_{156}(\kappa) = \frac{1-\cos\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}-(\lambda_1^2-\lambda_2^2)^{1/2}\right\}}{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}-(\lambda_1^2-\lambda_2^2)^{1/2}} + \frac{1-\cos\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}+(\lambda_1^2-\lambda_2^2)^{1/2}\right\}}{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}+(\lambda_1^2-\lambda_2^2)^{1/2}}$$

$$C_{17} = 2\lambda_1^2\lambda_3^2\left(1+\frac{\sin 2(\lambda_1^2-\lambda_2^2)^{1/2}}{2\,(\lambda_1^2-\lambda_2^2)^{1/2}}\right) = 2\lambda_1^2\lambda_3^2 C_{33} \tag{30}$$

Finally, Eq.(5.3-12) yields:

$$\begin{aligned}\int_0^1 \frac{\partial\Psi^*}{\partial\zeta}\frac{\partial\Psi}{\partial\zeta}\,d\zeta = \Psi_1^2\int_0^1 \Bigg[\Bigg(&\sum_{\kappa=1}^{\infty} B(\kappa)\lambda_1\sin\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\} \\ &\times\sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}\Bigg)^2 \\ +\Bigg(&\sum_{\kappa=1}^{\infty} B(\kappa)[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\sin\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\} \\ &\times\cos\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}\Bigg)^2 \\ &+4\lambda_1\sin\lambda_1\lambda_3\theta\sum_{\kappa=1}^{\infty}B(\kappa)\sin\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\} \\ \times\Big(&(\lambda_1^2-\lambda_2^2)^{1/2}\sin[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]\sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\} \\ &+[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\cos[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]\cos\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}\Big) \\ &+2(1-\cos 2\lambda_1\lambda_3\theta)\left\{\lambda_1^2-\lambda_2^2\sin^2[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]\right\}\Bigg]d\zeta\end{aligned} \tag{31}$$

There are eight integrals over ζ that need to be separated:

$$\begin{aligned}\int_0^1 \frac{\partial\Psi^*}{\partial\zeta}\frac{\partial\Psi}{\partial\zeta}\,d\zeta = \Psi_1^2\Bigg[&\lambda_1^2\sum_{\kappa=1}^{\infty}B^2(\kappa)\sin^2\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\} \\ &\times\int_0^1\sin^2\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}d\zeta \\ &+\lambda_1^2\sum_{\kappa=1}^{\infty,\neq j}\sum_{j=1}^{\infty}B(\kappa)B(j)\sin\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\} \\ &\times\sin\left\{[(2\pi j)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}\end{aligned}$$

$$\times \int_0^1 \sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}\sin\left\{[(2\pi j)^2+\lambda_1^2]^{1/2}\zeta\right\}d\zeta$$

$$+\sum_{\kappa=1}^{\infty} B^2(\kappa)[(2\pi\kappa)^2+\lambda_1^2]\sin^2\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}$$

$$\times \int_0^1 \cos^2\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}d\zeta$$

$$+\sum_{\kappa=1}^{\infty,\neq j}\sum_{j=1}^{\infty} B(\kappa)B(j)[(2\pi\kappa)^2+\lambda_1^2]^{1/2}[(2\pi j)^2+\lambda_1^2]^{1/2}$$

$$\times\sin\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}\sin\left\{[(2\pi j)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}$$

$$\times \int_0^1 \cos\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}\cos\left\{[(2\pi j)^2+\lambda_1^2]^{1/2}\zeta\right\}d\zeta$$

$$+4\lambda_1\sin\lambda_1\lambda_3\theta\sum_{\kappa=1}^{\infty}B(\kappa)\sin\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}$$

$$\times\Bigg((\lambda_1^2-\lambda_2^2)^{1/2}\int_0^1\sin[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]\sin\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}d\zeta$$

$$+[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\int_0^1\cos[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]\cos\left\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}\zeta\right\}d\zeta\Bigg)$$

$$+2(1-\cos 2\lambda_1\lambda_3\theta)\Bigg(\lambda_1^2\int_0^1 d\zeta-\lambda_2^2\int_0^1\sin^2[(\lambda_1^2-\lambda_2^2)^{1/2}\zeta]d\zeta\Bigg)\Bigg] \quad (32)$$

The integrations are cumbersome but not scientifically challenging:

$$\int_0^1\frac{\partial\Psi^*}{\partial\zeta}\frac{\partial\Psi}{\partial\zeta}d\zeta=\Psi_1^2\Bigg(\sum_{\kappa=1}^{\infty}C_{21}(\kappa)\sin^2\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}$$

$$+\sum_{\kappa=1}^{\infty,\neq j}\sum_{j=1}^{\infty}C_{22}(\kappa,j)\sin\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}$$

$$\times\sin\left\{[(2\pi j)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}$$

$$+\sum_{\kappa=1}^{\infty}C_{23}(\kappa)\sin^2\left\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\right\}$$ continued

$$+\sum_{\kappa=1}^{\infty,\neq j}\sum_{j=1}^{\infty} C_{24}(\kappa,j)\sin\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\}$$
$$\times\sin\{[(2\pi j)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\}$$
$$+\sum_{\kappa=1}^{\infty} C_{25}(\kappa)\sin\lambda_1\lambda_3\theta\sin\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\}$$
$$+C_{26}(1-\cos 2\lambda_1\lambda_3\theta)\Big) \quad (33)$$

$$C_{21}(\kappa)=\frac{1}{2}\lambda_1^2B^2(\kappa)\left(1-\frac{\sin 2[(2\pi\kappa)^2+\lambda_1^2]^{1/2}}{2\,[(2\pi\kappa)^2+\lambda_1^2]^{1/2}}\right)=\lambda_1^2C_{31}(\kappa) \quad (34)$$

$$C_{22}(\kappa,j)=\frac{1}{2}\lambda_1^2B(\kappa)B(j)\left(\frac{\sin\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}-[(2\pi j)^2+\lambda_1^2]^{1/2}\}}{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}-[(2\pi j)^2+\lambda_1^2]^{1/2}}\right.$$
$$\left.-\frac{\sin\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}+[(2\pi j)^2+\lambda_1^2]^{1/2}\}}{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}+[(2\pi j)^2+\lambda_1^2]^{1/2}}\right) \quad (35)$$

$$C_{23}(\kappa)=\frac{1}{2}B^2(\kappa)[(2\pi\kappa)^2+\lambda_1^2]\left(1+\frac{\sin 2[(2\pi\kappa)^2+\lambda_1^2]^{1/2}}{2\,[(2\pi\kappa)^2+\lambda_1^2]^{1/2}}\right) \quad (36)$$

$$C_{24}(\kappa,j)=\frac{1}{2}B(\kappa)B(j)[(2\pi\kappa)^2+\lambda_1^2]^{1/2}[(2\pi j)^2+\lambda_1^2]^{1/2}$$
$$\times\left(\frac{\sin\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}-[(2\pi j)^2+\lambda_1^2]^{1/2}\}}{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}-[(2\pi j)^2+\lambda_1^2]^{1/2}}\right.$$
$$\left.+\frac{\sin\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}+[(2\pi j)^2+\lambda_1^2]^{1/2}\}}{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}+[(2\pi j)^2+\lambda_1^2]^{1/2}}\right) \quad (37)$$

$$C_{25}(\kappa)=2\lambda_1B(\kappa)\{(\lambda_1^2-\lambda_2^2)[C_{251}(\kappa)-C_{252}(\kappa)]$$
$$+[(2\pi\kappa)^2+\lambda_1^2]^{1/2}[C_{251}(\kappa)+C_{252}(\kappa)]\}$$
$$C_{251}(\kappa)=\frac{\sin\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}-(\lambda_1^2-\lambda_2^2)^{1/2}\}}{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}-(\lambda_1^2-\lambda_2^2)^{1/2}}$$
$$C_{252}(\kappa)=\frac{\sin\{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}+(\lambda_1^2-\lambda_2^2)^{1/2}\}}{[(2\pi\kappa)^2+\lambda_1^2]^{1/2}+(\lambda_1^2-\lambda_2^2)^{1/2}} \quad (38)$$

$$C_{26}=2\lambda_1^2-\lambda_2^2\left(1-\frac{\sin 2(\lambda_1^2-\lambda_2^2)^{1/2}}{2\,(\lambda_1^2-\lambda_2^2)^{1/2}}\right) \quad (39)$$

Equation (5.3-4) is rewritten:

$$U=\frac{L^2}{cT}\int_0^1\left(\frac{\partial\Psi^*}{\partial\theta}\frac{\partial\Psi}{\partial\theta}+\frac{\partial\Psi^*}{\partial\zeta}\frac{\partial\Psi}{\partial\zeta}+\frac{m_0^2c^4T^2}{\hbar^2}\Psi^*\Psi\right)d\zeta \quad (40)$$

In order for U to have the dimension or energy, J=VAs, the constant Ψ_1^2 in Eqs.(17), (23), and (33) must have the dimension J/m=VAs/m. Equations (17), (23), and (33) are substituted into Eq.(40). The resulting equation can be rewritten with the help of the relations

$$\cos^2 x = \frac{1}{2}(1+\cos 2x), \quad \sin^2 x = \frac{1}{2}(1-\cos 2x), \quad x = \kappa_0 \text{ or } j_0 \tag{41}$$

to yield:

$$\begin{aligned}
U = \frac{L^2\Psi_1^2}{cT}\Bigg\{ & \frac{1}{2}\sum_{\kappa=1}^{\infty}\left(C_{11}(\kappa)+C_{13}(\kappa)+C_{21}(\kappa)+C_{23}(\kappa)+\frac{m_0^2c^4T^2}{\hbar^2}C_{31}(\kappa)\right)\\
& -\frac{1}{2}\sum_{\kappa=1}^{\infty}\left(C_{11}(\kappa)-C_{13}(\kappa)+C_{21}(\kappa)+C_{23}(\kappa)+\frac{m_0^2c^4T^2}{\hbar^2}C_{31}(\kappa)\right)\cos 2\kappa_0\theta\\
& +\sum_{\kappa=1}^{\infty,\neq j}\sum_{j=1}^{\infty}\Bigg[\left(C_{12}(\kappa,j)+C_{22}(\kappa,j)+C_{24}(\kappa,j)+\frac{m_0^2c^4T^2}{\hbar^2}C_{32}(\kappa,j)\right)\\
& \qquad \times \sin\kappa_0\theta\sin j_0\theta + C_{14}(\kappa,j)\cos\kappa_0\theta\cos j_0\theta\Bigg]\\
& -\sum_{\kappa=1}^{\infty}[C_{15}(\kappa)\cos\lambda_1\lambda_3\theta\sin\kappa_0\theta + C_{16}(\kappa)\sin\lambda_1\lambda_3\theta\cos\kappa_0\theta\\
& \qquad + C_{25}(\kappa)\sin\lambda_1\lambda_3\theta\sin\kappa_0\theta]\\
& + C_{17} + C_{26}(1-\cos 2\lambda_1\lambda_3\theta) + \frac{m_0^2c^4T^2}{\hbar^2}C_{33}(1-\cos 2\lambda_1\lambda_3\theta)\Bigg\}
\end{aligned}$$

$$\kappa_0 = [(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}, \quad j_0 = [(2\pi j)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2} \tag{42}$$

The energy U consists of a constant part U_c and a time-variable part U_v with time-average equal to zero. The part U_c is readily copied from Eq.(42):

$$U = U_c + U_v \tag{43}$$

Note that in the following equation the second line becomes the term $\kappa = 0$ of the sum in the third line:

$$\begin{aligned}
U_c = \frac{L^2\Psi_1^2}{2cT}\Bigg[& \sum_{\kappa=1}^{\infty}\left(C_{11}(\kappa)+C_{13}(\kappa)+C_{21}(\kappa)+C_{23}(\kappa)+\frac{m_0^2c^4T^2}{\hbar^2}C_{31}(\kappa)\right)\\
& +2\left(C_{17}+C_{26}+\frac{m_0^2c^4T^2}{\hbar^2}C_{33}\right)\Bigg]
\end{aligned}$$

continued on following page

$$= \frac{L^2\Psi_1^2}{2cT}\sum_{\kappa=0}^{\infty}\left(C_{11}(\kappa)+C_{13}(\kappa)+C_{21}(\kappa)+C_{25}(\kappa)+\frac{m_0^2c^4T^2}{\hbar^2}C_{31}(\kappa)\right)$$

$$= \frac{L^2\Psi_1^2}{2cT}\sum_{\kappa=0}^{\infty}D(\kappa)$$

$$C_{11}(0)=2(C_{17}+C_{26}),\ C_{13}(0)=C_{21}(0)=C_{25}(0)=0,\ C_{31}(0)=2C_{33} \quad (44)$$

A few more transformations are required to write the variable part U_v of the energy U in a practical form:

$$\sin 2\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\}\sin 2\{[(2\pi j)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\} = \frac{1}{2}(\cos\kappa_1\theta-\cos\kappa_2\theta) \quad (45)$$

$$\cos 2\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\}\cos 2\{[(2\pi j)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\theta\} = \frac{1}{2}(\cos\kappa_1\theta+\cos\kappa_2\theta) \quad (46)$$

$$\kappa_1 = 2\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}-[(2\pi j)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\} \quad (47)$$

$$\kappa_2 = 2\{[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}+[(2\pi j)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}\} \quad (48)$$

$$\cos\lambda_1\lambda_3\theta\sin 2\kappa_0\theta = \frac{1}{2}(\sin\kappa_3\theta+\sin\kappa_4\theta) \quad (49)$$

$$\sin\lambda_1\lambda_3\theta\cos 2\kappa_0\theta = \frac{1}{2}(\sin\kappa_3\theta-\sin\kappa_4\theta) \quad (50)$$

$$\sin\lambda_1\lambda_3\theta\sin 2\kappa_0\theta = \frac{1}{2}(\cos\kappa_3\theta-\cos\kappa_4\theta) \quad (51)$$

$$\kappa_3 = 2[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^{1/2}+\lambda_1\lambda_3 \quad (52)$$

$$\kappa_4 = 2[(2\pi\kappa)^2+\lambda_1^2\lambda_3^2+\lambda_2^2]^1/2-\lambda_1\lambda_3 \quad (53)$$

The time-variable part U_v of Eq.(42) may now be written in the following form:

$$U_v = \frac{L^2\Psi_1^2}{2cT}\left\{\sum_{\kappa=1}^{\infty}\left(C_{13}(\kappa)-C_{11}(\kappa)-C_{21}(\kappa)-C_{23}(\kappa) - \frac{m_0^2c^4T^2}{\hbar^2}C_{31}(\kappa)\right)\cos 2\kappa_0\theta\right.$$

$$+\sum_{\kappa=1}^{\infty,\neq j}\sum_{j=1}^{\infty}\Bigg[\bigg(C_{12}(\kappa,j)+C_{14}(\kappa,j)+C_{22}(\kappa,j)+C_{24}(\kappa,j)$$
$$+\frac{m_0^2c^4T^2}{\hbar^2}C_{32}(\kappa,j)\bigg)\cos\kappa_1\theta$$
$$+\bigg(C_{14}(\kappa,j)-C_{12}(\kappa,j)-C_{22}(\kappa,j)-C_{24}(\kappa,j)$$
$$-\frac{m_0^2c^4T^2}{\hbar^2}C_{32}(\kappa,j)\bigg)\cos\kappa_2\theta\Bigg]$$
$$+\sum_{\kappa=1}^{\infty}[-C_{15}(\kappa)(\sin\kappa_3\theta+\sin\kappa_4\theta)+C_{16}(\kappa)(\sin\kappa_3\theta-\sin\kappa_4\theta)$$
$$+C_{25}(\kappa)(\cos\kappa_4\theta-\cos\kappa_3\theta)]$$
$$-\bigg(C_{26}+\frac{m_0^2c^4T^2}{\hbar^2}C_{33}\bigg)\cos\lambda_1\lambda_3\theta\bigg\} \quad (54)$$

The term $D(\kappa)$ in Eq.(44) will be needed explicitly and we write it in more detail:

$$D(\kappa)=\lambda_1^2(\lambda_3^2+1)C_{31}(\kappa)+C_{13}(\kappa)+C_{23}(\kappa)+\frac{m_0^2c^4T^2}{\hbar^2}C_{31}(\kappa)$$
$$=\frac{1}{2}B^2(\kappa)\Bigg[2(2\pi\kappa)^2+2\lambda_1^2\lambda_3^2+2\lambda_1^2+\lambda_2^2+\frac{m_0^2c^4T^2}{\hbar^2}$$
$$-\bigg(2\lambda_1^2\lambda_3^2+\lambda_2^2+\frac{m_0^2c^4T^2}{\hbar^2}\bigg)\frac{\sin 2[(2\pi\kappa)^2+\lambda_1^2]^{1/2}}{2\,[(2\pi\kappa)^2+\lambda_1^2]^{1/2}}\Bigg]$$
$$\text{for } \kappa=1,\ 2,\ldots \quad (55)$$

$$D(0)=2\bigg(C_{17}+C_{26}+\frac{m_0^2c^4T^2}{\hbar^2}C_{33}\bigg)$$
$$=2\Bigg[2\lambda_1^2+2\lambda_1^2\lambda_3^2-\lambda_2^2+\frac{m_0^2c^4T^2}{\hbar^2}$$
$$+\bigg(2\lambda_1^2\lambda_3^2+\lambda_2^2+\frac{m_0^2c^4T^2}{\hbar^2}\bigg)\frac{\sin 2(\lambda_1^2-\lambda_2^2)^{1/2}}{2\,(\lambda_1^2-\lambda_2^2)^{1/2}}\Bigg] \quad \text{for } \kappa=0 \quad (56)$$

The term $B(\kappa)$ is defined in Eq.(5.3-5), the constants λ_1, λ_2, and λ_3 are found in Eq.(5.2-8).

6.12 Calculations for Sections 4.2 and 4.3

The auxiliary variables d^2, q_1, q_2, q_3 and q_4 are frequently needed. We list them here together with a number of relations derived from them to help with the equations of this section:

$$d^2 = 4[(2\pi\kappa)^2 + \rho_2^2] \tag{1}$$

$$q_1 = +\frac{1}{2}(d^2 - \rho_1^2)^{1/2} + 2\pi\kappa, \quad d^2 > \rho_1^2 \tag{2}$$

$$q_2 = +\frac{1}{2}(d^2 - \rho_1^2)^{1/2} - 2\pi\kappa, \quad d^2 > \rho_1^2 \tag{3}$$

$$q_3 = +\frac{1}{2}(\rho_1^2 - d^2)^{1/2} - \frac{1}{2}\rho_1, \quad d^2 < \rho_1^2 \tag{4}$$

$$q_4 = +\frac{1}{2}(\rho_1^2 - d^2)^{1/2} + \frac{1}{2}\rho_1, \quad d^2 < \rho_1^2 \tag{5}$$

$$q_2 = q_1 - 4\pi\kappa, \quad q_4 = q_3 + \rho_1 \tag{6}$$

$$q_1 = 2\pi\kappa + [(2\pi\kappa)^2 + \rho_2^2 - \rho_1^2/4]^{1/2} \tag{7}$$

$$q_2 = -2\pi\kappa + [(2\pi\kappa)^2 + \rho_2^2 - \rho_1^2/4)^{1/2} \tag{8}$$

$$(d^2 - \rho_1^2)^{1/2} = 2(q_1 - 2\pi\kappa) = 2(q_2 + 2\pi\kappa), \quad d^2 > \rho_1^2 \tag{9}$$

$$(\rho_1^2 - d^2)^{1/2} = 2q_3 + \rho_1 = 2q_4 - \rho_1, \quad d^2 < \rho_1^2 \tag{10}$$

$$q_1^2 + \left(\frac{\rho_1}{2}\right)^2 = 2(2\pi\kappa)^2 + \rho_2^2 + 2(2\pi\kappa)^2\left(1 + \frac{\rho_2^2 - \rho_1^2/4}{(2\pi\kappa)^2}\right)^{1/2}, \; d^2 > \rho_1^2 \tag{11}$$

$$q_2^2 + \left(\frac{\rho_1}{2}\right)^2 = 2(2\pi\kappa)^2 + \rho_2^2 - 2(2\pi\kappa)^2\left(1 + \frac{\rho_2^2 - \rho_1^2/4}{(2\pi\kappa)^2}\right)^{1/2}, \; d^2 > \rho_1^2 \tag{12}$$

We start with the second sum holding for $\kappa > K$ in Eq.(4.1-103) which is the same as Eq.(4.1-105) for $K < 1$. Later we shall extend the investigation to the whole of Eq.(4.1-103). Equation (4.1-105) is shown once more in Eq.(4.2-11) with the terms $L_{15}(\theta,\kappa)$ to $L_{18}(\theta,\kappa)$ broken up into $L_{15\mathrm{A}}(\kappa)$, $L_{15\mathrm{B}}(\theta,\kappa)$, ..., $L_{18\mathrm{B}}(\theta,\kappa)$. We denote the two terms of the second sum of Eq.(4.2-11) by $A_{\mathrm{es}}(\kappa)$ and $A_{\mathrm{ec}}(\kappa)$:

$$\begin{aligned} A_{\mathrm{es}}(\kappa) &= \frac{1}{(2\pi\kappa)^2 + \rho_2^2}\left(L_{17\mathrm{A}}(\kappa) + \frac{\rho_1 L_{15\mathrm{A}}(\kappa)}{(d^2 - \rho_1^2)^{1/2}}\right) \\ &= \frac{\rho_1}{(2\pi\kappa)^2 + \rho_2^2}\,\frac{1}{2(q_1 - 2\pi\kappa)} \\ &\qquad \times \left(\frac{q_1 - \pi\kappa}{q_1^2 + (\rho_1/2)^2} + \frac{q_2 + \pi\kappa}{q_2^2 + (\rho_1/2)^2}\right) \end{aligned} \tag{13}$$

$$A_{\mathrm{ec}}(\kappa) = -\frac{1}{(2\pi\kappa)^2 + \rho_2^2}\left(L_{18\mathrm{A}}(\kappa) + \frac{\rho_1 L_{16\mathrm{A}}(\kappa)}{(d^2 - \rho_1^2)^{1/2}}\right)$$

$$= -\frac{1}{(2\pi\kappa)^2 + \rho_2^2} \frac{1}{2(q_1 - 2\pi\kappa)}$$
$$\times \left(\frac{q_1(q_1 - 2\pi\kappa) - \rho_1^2/4}{q_1^2 + (\rho_1/2)^2} - \frac{q_2(q_2 + 2\pi\kappa) - \rho_1^2/4}{q_2^2 + (\rho_1/2)^2} \right)$$
$$\kappa > K = cT|(\sigma Z - s/Z)|/4\pi \text{ if used for Eq.(4.1-103)} \quad (14)$$

For the third term of the second sum in Eq.(4.2-11) we obtain with Eqs.(4.2-2)–(4.2-5):

$$\frac{(d^2 - \rho_1^2)^{1/2} L_{17\mathrm{B}}(\theta,\kappa) + \rho_1 L_{15\mathrm{B}}(\theta,\kappa)}{(d^2 - \rho_1^2)^{1/2} \left[(2\pi\kappa)^2 + \rho_2^2\right]} = -\frac{1}{2\,(2q_1 - 2\pi\kappa)\left[(2\pi\kappa)^2 + \rho_2^2\right]}$$
$$\times \left(\frac{[\rho_1(q_1 - 2\pi\kappa) + \rho_1 q_1]\cos q_1\theta - [2q_1(q_1 - 2\pi\kappa) - \rho_1^2/2]\sin q_1\theta}{2\left[q_1^2 + (\rho_1/2)^2\right]} \right.$$
$$\left. + \frac{[\rho_1(q_1 - 2\pi\kappa) + \rho_1 q_2]\cos q_2\theta - [2q_2(q_1 - 2\pi\kappa) - \rho_1^2/2]\sin q_2\theta}{2\left[q_2^2 + (\rho_1/2)^2\right]} \right)$$
$$= A_{57}(\kappa)\cos q_1\theta + B_{57}(\kappa)\sin q_1\theta + C_{57}(\kappa)\cos q_2\theta + D_{57}(\kappa)\sin q_2\theta \quad (15)$$

$$A_{57}(\kappa) = -\frac{\rho_1(q_1 - 2\pi\kappa) + q_1\rho_1}{4(q_1 - 2\pi\kappa)\left[(2\pi\kappa)^2 + \rho_2^2\right]\left[q_1^2 + (\rho_1/2)^2\right]} \quad (16)$$

$$B_{57}(\kappa) = +\frac{2q_1(q_1 - 2\pi\kappa) - \rho_1^2/2}{4\,(q_1 - 2\pi\kappa)\left[(2\pi\kappa)^2 + \rho_2^2\right]\left[q_1^2 + (\rho_1/2)^2\right]} \quad (17)$$

$$C_{57}(\kappa) = -\frac{\rho_1(q_1 - 2\pi\kappa) + \rho_1 q_2}{4\,(q_1 - 2\pi\kappa)\left[(2\pi\kappa)^2 + \rho_2^2\right]\left[q_2^2 + (\rho_1/2)^2\right]} \quad (18)$$

$$D_{57}(\kappa) = +\frac{2q_2(q_1 - 2\pi\kappa) - \rho_1^2/2}{4\,(q_1 - 2\pi\kappa)\left[(2\pi\kappa)^2 + \rho_2^2\right]\left[q_2^2 + (\rho_1/2)^2\right]} \quad (19)$$

The third term of the second sum in Eq.(4.2-11) may now be written as follows:

$$\left(L_{17\mathrm{B}}(\theta,\kappa) + \frac{\rho_1 L_{15\mathrm{B}}(\theta,\kappa)}{(d^2 - \rho_1^2)^{1/2}} \right) \frac{\sin 2\pi\kappa\theta}{(2\pi\kappa)^2 + \rho_2^2}$$
$$= A_{57}(\kappa)\cos q_1 \sin 2\pi\kappa\theta + B_{57}(\kappa)\sin q_1 \sin 2\pi\kappa\theta$$
$$+ C_{57}(\kappa)\cos q_2\theta \sin 2\pi\kappa\theta + D_{57}(\kappa)\sin q_2\theta \sin 2\pi\kappa\theta \quad (20)$$

Using the changed wavenumbers $2\pi\kappa \pm q_1$ and $2\pi\kappa \pm q_2$

$$
\begin{aligned}
2\pi\kappa - q_1 &= \ \ \pi\kappa - (\pi^2\kappa^2 + \rho_2^2 - \rho_1^2/4)^{1/2} && (21)\\
2\pi\kappa + q_1 &= 3\pi\kappa + (\pi^2\kappa^2 + \rho_2^2 - \rho_1^2/4)^{1/2} && (22)\\
2\pi\kappa - q_2 &= 3\pi\kappa - (\pi^2\kappa^2 + \rho_2^2 - \rho_1^2/4)^{1/2} && (23)\\
2\pi\kappa + q_2 &= \ \ \pi\kappa + (\pi^2\kappa^2 + \rho_2^2 - \rho_1^2/4)^{1/2} && (24)
\end{aligned}
$$

we obtain for the products of sine and cosine functions in Eq.(20):

$$
\begin{aligned}
\sin 2\pi\kappa\theta \cos q_1\theta &= \frac{1}{2}[\sin(2\pi\kappa - q_1)\theta + \sin(2\pi\kappa + q_1)\theta] && (25)\\
\sin 2\pi\kappa\theta \sin q_1\theta &= \frac{1}{2}[\cos(2\pi\kappa - q_1)\theta - \cos(2\pi\kappa + q_1)\theta] && (26)\\
\sin 2\pi\kappa\theta \cos q_2\theta &= \frac{1}{2}[\sin(2\pi\kappa - q_2)\theta + \sin(2\pi\kappa + q_2)\theta] && (27)\\
\sin 2\pi\kappa\theta \sin q_2\theta &= \frac{1}{2}[\cos(2\pi\kappa - q_2)\theta - \cos(2\pi\kappa + q_2)\theta] && (28)
\end{aligned}
$$

We turn to the fourth term of the second sum in Eq.(4.2-11). Instead of Eqs.(15)–(19) one obtains:

$$
\begin{aligned}
&-\frac{(d^2-\rho_1^2)^{1/2}L_{18\mathrm{B}}(\theta,\kappa) + \rho_1 L_{16\mathrm{B}}(\theta,\kappa)}{(d^2-\rho_1^2)^{1/2}\left[(2\pi\kappa)^2+\rho_2^2\right]} = +\frac{1}{2\,(q_1-2\pi\kappa))\left[(2\pi\kappa)^2+\rho_2^2\right]}\\
&\times\left(\frac{[\rho_1(q_1-2\pi\kappa)+\rho_1 q_1]\sin q_1\theta + [2q_1(q_1-2\pi\kappa)-\rho_1^2/2]\cos q_1\theta}{2\left[q_1^2+(\rho_1/2)^2\right]}\right.\\
&\left.-\frac{[\rho_1(q_1-2\pi\kappa)-\rho_1 q_2]\sin q_2\theta + [2q_2(q_1-2\pi\kappa)+\rho_1^2/2]\cos q_2\theta}{2\left[q_2^2+(\rho_1/2)^2\right]}\right)\\
&= A_{68}(\kappa)\sin q_1\theta + B_{68}(\kappa)\cos q_1\theta + C_{68}(\kappa)\sin q_2\theta + D_{68}(\kappa)\cos q_2\theta \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
A_{68}(\kappa) &= +\frac{\rho_1(q_1-2\pi\kappa)+\rho_1 q_1}{4\,(q_1-2\pi\kappa)\left[(2\pi\kappa)^2+\rho_2^2\right]\left[q_1^2+(\rho_1/2)^2\right]} = -A_{57}(\kappa) && (30)\\
B_{68}(\kappa) &= +\frac{2q_1(q_1-2\pi\kappa)-\rho_1^2/2}{4\,(q_1-2\pi\kappa)\left[(2\pi\kappa)^2+\rho_2^2\right]\left[q_1^2+(\rho_1/2)^2\right]} = +B_{57}(\kappa) && (31)\\
C_{68}(\kappa) &= -\frac{\rho_1(q_1-2\pi\kappa)-\rho_1 q_2}{4\,(q_1-2\pi\kappa)\left[(2\pi\kappa)^2+\rho_2^2\right]\left[q_2^2+(\rho_1/2)^2\right]} && (32)\\
D_{68}(\kappa) &= -\frac{2q_2(q_1-2\pi\kappa)+\rho_1^2/2}{4\,(q_1^2-2\pi\kappa)\left[(2\pi\kappa)^2+\rho_2^2\right]\left[q_2^2+(\rho_1/2)^2\right]} && (33)
\end{aligned}
$$

The fourth term of the second sum in Eq.(4.2-11) may be written as follows:

$$
\begin{aligned}
-\left(L_{18\mathrm{B}}(\theta,\kappa)+\frac{\rho_1 L_{16\mathrm{B}}(\theta,\kappa)}{(d^2-\rho_1^2)^{1/2}}\right)\frac{\cos 2\pi\kappa\theta}{(2\pi\kappa)^2+\rho_2^2} \\
= A_{68}(\kappa)\sin q_1\theta\cos 2\pi\kappa\theta + B_{68}(\kappa)\cos q_1\theta\cos 2\pi\kappa\theta \\
+ C_{68}(\kappa)\sin q_2\theta\cos 2\pi\kappa\theta + D_{68}(\kappa)\cos q_2\theta\cos 2\pi\kappa\theta \quad (34)
\end{aligned}
$$

Using the changed wavenumbers of Eqs.(21)–(24) we obtain for the products of sine and cosine functions in Eq.(34):

$$\cos 2\pi\kappa\theta\sin q_1\theta = \frac{1}{2}[-\sin(2\pi\kappa-q_1)\theta+\sin(2\pi\kappa+q_1)\theta] \quad (35)$$

$$\cos 2\pi\kappa\theta\cos q_1\theta = \frac{1}{2}[+\cos(2\pi\kappa-q_1)\theta+\cos(2\pi\kappa+q_1)\theta] \quad (36)$$

$$\cos 2\pi\kappa\theta\sin q_2\theta = \frac{1}{2}[-\sin(2\pi\kappa-q_2)\theta+\sin(2\pi\kappa+q_2)\theta] \quad (37)$$

$$\cos 2\pi\kappa\theta\cos q_2\theta = \frac{1}{2}[+\cos(2\pi\kappa-q_2)\theta+\cos(2\pi\kappa+q_2)\theta] \quad (38)$$

We introduce four more auxiliary variables:

$$
\begin{aligned}
E_{57}(\kappa) &= \frac{1}{2}[C_{57}(\kappa)-C_{68}(\kappa)] \\
&= -\frac{\rho_1 q_1}{4(q_1-2\pi\kappa)\left[(2\pi\kappa)^2+\rho_2^2\right]\left[q_2^2+(\rho_1/2)^2\right]} \quad (39)
\end{aligned}
$$

$$
\begin{aligned}
E_{68}(\kappa) &= \frac{1}{2}[C_{57}(\kappa)+C_{68}(\kappa)] \\
&= -\frac{\rho_1(2q_1-2\pi\kappa)}{4(q_1-2\pi\kappa)\left[(2\pi\kappa)^2+\rho_2^2\right]\left[q_2^2+(\rho_1/2)^2\right]} \quad (40)
\end{aligned}
$$

$$
\begin{aligned}
F_{57}(\kappa) &= \frac{1}{2}[D_{57}(\kappa)+D_{68}(\kappa)] \\
&= -\frac{\rho_1^2/2}{4(q_1-2\pi\kappa)\left[(2\pi\kappa)^2+\rho_2^2\right]\left[q_2^2+(\rho_1/2)^2\right]} \quad (41)
\end{aligned}
$$

$$
\begin{aligned}
F_{68}(\kappa) &= -\frac{1}{2}[D_{57}(\kappa)-D_{68}(\kappa)] \\
&= -\frac{2q_2(q_1-2\pi\kappa)}{4(q_1-2\pi\kappa)\left[(2\pi\kappa)^2+\rho_2^2\right]\left[q_2^2+(\rho_1/2)^2\right]} \quad (42)
\end{aligned}
$$

The third and fourth terms of the second sum of Eq.(4.2-11) become:

$$\begin{aligned}
&\left(L_{17\mathrm{B}}(\theta,\kappa)+\frac{\rho_1 L_{15\mathrm{B}}(\theta,\kappa)}{(d^2-\rho_1^2)^{1/2}}\right)\frac{\sin 2\pi\kappa\theta}{(2\pi\kappa)^2+\rho_2^2}\\
&\qquad\qquad-\left(L_{18\mathrm{B}}(\theta,\kappa)+\frac{\rho_1 L_{11\mathrm{B}}(\theta,\kappa)}{(d^2-\rho_1^2)^{1/2}}\right)\frac{\cos 2\pi\kappa\theta}{(2\pi\kappa)^2+\rho_2^2}\\
&= A_{57}(\kappa)\sin(2\pi\kappa-q_1)\theta+B_{68}(\kappa)\cos(2\pi\kappa-q_1)\theta\\
&\quad+E_{57}(\kappa)\sin(2\pi\kappa-q_2)\theta+E_{68}(\kappa)\sin(2\pi\kappa+q_2)\theta\\
&\quad+F_{57}(\kappa)\cos(2\pi\kappa-q_2)\theta+F_{68}(\kappa)\cos(2\pi\kappa+q_2)\theta \qquad (43)
\end{aligned}$$

We turn to the first sum in Eq.(4.1-103) that contains more complicated terms. With the help of Eqs.(4.2-7)–(4.2-10) we obtain:

$$\begin{aligned}
A_{\mathrm{es}}(\kappa)&=\frac{(\rho_1^2-d^2)^{1/2}L_{13\mathrm{A}}(\kappa)+\rho_1 L_{11\mathrm{A}}(\kappa)}{(\rho_1^2-d^2)^{1/2}\left[(2\pi\kappa)^2+\rho_2^2\right]}\\
&=-\frac{q_3q_4}{(2q_3+\rho_1)\left[(2\pi\kappa)^2+\rho_2^2\right]}\left(\frac{1}{(2\pi\kappa)^2+q_3^2}-\frac{1}{(2\pi\kappa)^2+q_4^2}\right) \qquad (44)\\
A_{\mathrm{ec}}(\kappa)&=-\frac{(\rho_1^2-d^2)^{1/2}L_{14\mathrm{A}}(\kappa)+\rho_1 L_{12\mathrm{A}}(\kappa)}{(\rho_1^2-d^2)^{1/2}\left[(2\pi\kappa)^2+\rho_2^2\right]}\\
&=-\frac{2\pi\kappa}{(2q_3+\rho_1)\left[(2\pi\kappa)^2+\rho_2^2\right]}\left(\frac{q_4}{(2\pi\kappa)^2+q_3^2}+\frac{q_3}{(2\pi\kappa)^2+q_4^2}\right)\\
&\kappa<K=cT|(\sigma Z-s/Z)| \text{ if used for Eq.(4.1-103)} \qquad (45)
\end{aligned}$$

The terms with the subscript B rather than A are more complicated:

$$\begin{aligned}
&\frac{(\rho_1^2-d^2)^{1/2}L_{13\mathrm{B}}(\theta,\kappa)+\rho_1 L_{11\mathrm{B}}(\theta,\kappa)}{(\rho_1^2-d^2)^{1/2}\left[(2\pi\kappa)^2+\rho_2^2\right]}=\frac{1}{(2q_3+\rho_1)\left[(2\pi\kappa)^2+\rho_2^2\right]}\\
&\qquad\times\left(q_4e^{(2q_3+\rho_1)\theta/2}\,\frac{q_3\cos 2\pi\kappa\theta-2\pi\kappa\sin 2\pi\kappa\theta}{(2\pi\kappa)^2+q_3^2}\right.\\
&\qquad\qquad\left.-\,q_3e^{-(2q_3+\rho_1)\theta/2}\,\frac{q_4\cos 2\pi\kappa\theta-2\pi\kappa\sin 2\pi\kappa\theta}{(2\pi\kappa)^2+q_4^2}\right)\\
&=e^{(2q_3+\rho_1)\theta/2}[A_{13}(\kappa)\cos 2\pi\kappa\theta+B_{13}(\kappa)\sin 2\pi\kappa\theta]\\
&\qquad+e^{-(2q_3+\rho_1)\theta/2}[C_{13}(\kappa)\cos 2\pi\kappa\theta+D_{13}(\kappa)\sin 2\pi\kappa\theta] \qquad (46)
\end{aligned}$$

$$A_{13}(\kappa)=+\frac{q_3q_4}{(2q_3+\rho_1)\left[(2\pi\kappa)^2+\rho_2^2\right]\left[(2\pi\kappa)^2+q_3^2\right]} \qquad (47)$$

$$B_{13}(\kappa) = -\frac{2\pi\kappa q_4}{(2q_3+\rho_1)\left[(2\pi\kappa)^2+\rho_2^2\right]\left[(2\pi\kappa)^2+q_3^2\right]} \tag{48}$$

$$C_{13}(\kappa) = -\frac{q_3 q_4}{(2q_3+\rho_1)\left[(2\pi\kappa)^2+\rho_2^2\right]\left[(2\pi\kappa)^2+q_4^2\right]} \tag{49}$$

$$D_{13}(\kappa) = +\frac{2\pi\kappa q_3}{(2q_3+\rho_1)\left[(2\pi\kappa)^2+\rho_2^2\right]\left[(2\pi\kappa)^2+q_4^2\right]} \tag{50}$$

We still need the terms with subscripts 14B and 12B:

$$\begin{aligned}
-\frac{(\rho_1^2-d^2)^{1/2}L_{14\mathrm{B}}(\theta,\kappa)+\rho_1 L_{12\mathrm{B}}(\theta,\kappa)}{(\rho_1^2-d^2)^{1/2}\left[(2\pi\kappa)^2+\rho_2^2\right]} &= -\frac{1}{(2q_3+\rho_1)\left[(2\pi\kappa)^2+\rho_2^2\right]} \\
&\times\left(q_4 e^{(2q_3+\rho_1)\theta/2}\,\frac{q_3\sin 2\pi\kappa\theta-2\pi\kappa\cos 2\pi\kappa\theta}{(2\pi\kappa)^2+q_3^2}\right. \\
&\left.\quad - q_3 e^{-(2q_3+\rho_1)\theta/2}\,\frac{q_4\sin 2\pi\kappa\theta+2\pi\kappa\cos 2\pi\kappa\theta}{(2\pi\kappa)^2+q_4^2}\right) \\
= e^{(2q_3+\rho_1)\theta/2}[A_{24}(\kappa)\sin 2\pi\kappa\theta &+ B_{24}(\kappa)\cos 2\pi\kappa\theta] \\
&+ e^{-(2q_3+\rho_1)\theta/2}[C_{24}(\kappa)\sin 2\pi\kappa\theta + D_{24}(\kappa)\cos 2\pi\kappa\theta]
\end{aligned} \tag{51}$$

$$A_{24}(\kappa) = -\frac{q_3 q_4}{(2q_3+\rho_1)\left[(2\pi\kappa)^2+\rho_2^2\right]\left[(2\pi\kappa)^2+q_3^2\right]} \tag{52}$$

$$B_{24}(\kappa) = +\frac{2\pi\kappa q_4}{(2q_3+\rho_1)\left[(2\pi\kappa)^2+\rho_2^2\right]\left[(2\pi\kappa)^2+q_3^2\right]} \tag{53}$$

$$C_{24}(\kappa) = +\frac{q_3 q_4}{(2q_3+\rho_1)\left[(2\pi\kappa)^2+\rho_2^2\right]\left[(2\pi\kappa)^2+q_4^2\right]} \tag{54}$$

$$D_{24}(\kappa) = +\frac{2\pi\kappa q_3}{(2q_3+\rho_1)\left[(2\pi\kappa)^2+\rho_2^2\right]\left[(2\pi\kappa)^2+q_4^2\right]} \tag{55}$$

The comparison of Eqs.(52) to (55) with Eqs.(47) to (50) provides the following relations:

$$A_{24}(\kappa) = -A_{13}(\kappa) \tag{56}$$
$$B_{24}(\kappa) = -B_{13}(\kappa) \tag{57}$$
$$C_{24}(\kappa) = +C_{13}(\kappa) \tag{58}$$
$$D_{24}(\kappa) = +D_{13}(\kappa) \tag{59}$$

The sum of Eq.(46) multiplied by $\sin 2\pi\kappa\theta$ and of Eq.(51) multiplied by $\cos 2\pi\kappa\theta$ yields:

$$\left(L_{13\mathrm{B}}(\theta,\kappa)+\frac{\rho_1 L_{11\mathrm{B}}(\theta,\kappa)}{(\rho_1^2-d^2)^{1/2}}\right)\frac{\sin 2\pi\kappa\theta}{(2\pi\kappa)^2+\rho_2^2}$$
$$-\left(L_{14\mathrm{B}}(\theta,\kappa)+\frac{\rho_1 L_{12\mathrm{B}}(\theta,\kappa)}{(\rho_1^2-d^2)^{1/2}}\right)\frac{\cos 2\pi\kappa\theta}{(2\pi\kappa)^2+\rho_2^2}$$
$$=-e^{(q_3+\rho_1)\theta/2}B_{13}(\kappa)\cos 4\pi\kappa\theta$$
$$+e^{-(2q_3+\rho_1)\theta/2}[C_{13}(\kappa)\sin 4\pi\kappa\theta+D_{13}(\kappa)] \quad (60)$$

Substitution of Eqs.(13), (14), (43)–(45), and (60) into Eq.(4.2-7) yields:

$$A_{\mathrm{ev}}(\zeta,\theta)=c^2T^2V_{\mathrm{e0}}\Bigg\{\frac{1}{\rho_2^2}e^{-\rho_2\zeta}(1-\mathrm{ch}\,\rho_2\theta)+2(1-e^{-\rho_2})$$
$$\times\Bigg[\sum_{\kappa=1}^{<K}[A_{\mathrm{es}}(\kappa)\sin 2\pi\kappa\theta+A_{\mathrm{ec}}(\kappa)\cos 2\pi\kappa\theta]\sin 2\pi\kappa\zeta$$
$$+e^{-\rho_1\theta/2}$$
$$+\Bigg(\sum_{\kappa=1}^{<K}-e^{(2q_3+\rho_1)\theta/2}B_{13}(\kappa)\cos 4\pi\kappa\theta\sin 2\pi\kappa\zeta$$
$$+\sum_{\kappa=1}^{<K}e^{-(2q_3+\rho_1)\theta/2}[C_{13}(\kappa)\sin 4\pi\kappa\theta+D_{13}]\sin 2\pi\kappa\zeta\Bigg)$$
$$+\sum_{\kappa>K}^{\infty}[A_{\mathrm{es}}(\kappa)\sin 2\pi\kappa\theta+A_{\mathrm{ec}}(\kappa)\cos 2\pi\kappa\theta]\sin 2\pi\kappa\zeta$$
$$+e^{-\rho_1\theta/2}$$
$$\times\Bigg(\sum_{\kappa>K}^{\infty}[A_{57}(\kappa)\sin(2\pi\kappa-q_1)\theta+B_{68}(\kappa)\cos(2\pi\kappa-q_1)\theta]\sin 2\pi\kappa\zeta$$
$$+\sum_{\kappa>K}^{\infty}[E_{57}(\kappa)\sin(2\pi\kappa-q_2)\theta+F_{57}(\kappa)\cos(2\pi\kappa-q_2)\theta]\sin 2\pi\kappa\zeta$$
$$+\sum_{\kappa>K}^{\infty}[E_{68}(\kappa)\sin(2\pi\kappa+q_2)\theta+F_{68}(\kappa)\cos(2\pi\kappa+q_2)\theta]\sin 2\pi\kappa\zeta\Bigg)\Bigg]\Bigg\} \quad (61)$$

The term $e^{(2q_3+\rho_1)\theta/2}$ may cause concern but it is multiplied with $e^{-\rho_1\theta/2}$. The exponent of the product equals:

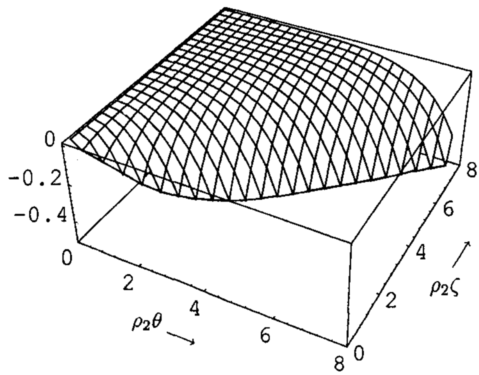

FIG.6.12-1. Three-dimensional plot of the function $\exp(-\rho_2\zeta)(1-\operatorname{ch}\rho_2\theta)$ for $\zeta-\theta\geq 0$ in the interval $0\leq\rho_2\zeta\leq 8$, $0\leq\rho_2\theta\leq 8$.

$$\begin{aligned}[(2q_3+\rho_1)-\rho_1]\theta/2 &= [(\rho_1^2-d^2)^{1/2}-\rho_1]\theta/2\\ &= \{[\rho_1^2-4(2\pi\kappa)^2-4\rho_2^2]^{1/2}-\rho_1\}\theta/2\end{aligned} \tag{62}$$

The relation

$$(\rho_1^2-d^2)^{1/2}=[\rho_1^2-4(2\pi\kappa)^2-4\rho_2^2]^{1/2}>0$$

implies

$$[\rho_1^2-4(2\pi\kappa)^2-4\rho_2^2]-\rho_1<0 \tag{63}$$

and the terms in Eq.(61) multiplied with $e^{-\rho_1\theta/2}e^{(2q_3+\rho_1)\theta/2}$ become very small for large values of θ.

Consider the very first term in Eq.(57). To recognize what values it may assume we rewrite it as follows:

$$\begin{aligned}\frac{1}{\rho_2^2}e^{-\rho_2\zeta}(1-\operatorname{ch}\rho_2\theta) &= \frac{1}{\rho_2^2}\left(e^{-\rho_2\zeta}-\frac{1}{2}e^{-\rho_2(\zeta-\theta)}-\frac{1}{2}e^{-\rho_2(\zeta+\theta)}\right)\\ &\doteq -\frac{1}{2\rho_2^2}e^{-\rho_2(\zeta-\theta)}\quad\text{for } \zeta,\ \theta\gg 1\end{aligned} \tag{64}$$

The constraint $\zeta-\theta\geq 0$ assures that this term will vary only in the interval from 0 to $-1/2\rho_2^2$. Figure 6.12-1 shows this variation in detail.

Equation (61) contains functions like $e^{-\rho_1\theta/2}$ and $e^{(2q_3+\rho_1)\theta/2}$ that represents attenuation due to losses. Such terms have no meaning in quantum mechanics since photons are never attenuated. To eliminate these attenuation terms, as well as the phase shifted arguments $2\pi\kappa-q_1$ and $2\pi\kappa\pm q_2$ of some of the sine and cosine functions we resort to the Fourier series. We write it in the following form:

$$f_\kappa(\theta) = g_0 + \sum_{\nu=1}^{\infty}[g_{s\kappa}(\nu)\sin 2\pi\nu\theta + g_{c\kappa}(\nu)\cos 2\pi\nu\theta]$$

$$g_{s\kappa}(\nu) = 2\int_0^1 f_\kappa(\theta)\sin 2\pi\nu\theta\, d\theta,\ \ g_{c\kappa}(\nu) = 2\int_0^1 f_\kappa(\theta)\cos 2\pi\nu\theta\, d\theta$$

$$g_0 = \int_0^1 f_\kappa(\theta)\, d\theta, \quad 0 \le \theta = t/T \le 1 \tag{65}$$

We apply this series expansion to the second sum in Eq.(61):

$$f_{\kappa 1}(\theta) = e^{q_3\theta}B_{13}(\kappa)\cos 4\pi\kappa\theta \tag{66}$$

$$g_{s\kappa 1}(\nu) = 2\int_0^1 e^{q_3\theta}B_{13}(\kappa)\cos 4\pi\kappa\theta\sin 2\pi\nu\theta\, d\theta \tag{67}$$

$$g_{c\kappa 1}(\nu) = \int_0^1 e^{q_3\theta}B_{13}(\kappa)\cos 4\pi\kappa\theta\cos 2\pi\nu\theta\, d\theta \tag{68}$$

$$g_0 = 0 \tag{69}$$

The following integrals are required to evaluate $g_{s\kappa 1}(\nu)$ and $g_{c\kappa 1}(\nu)$:

$$I_1(\kappa,\nu) = 2\int_0^1 e^{q_3\theta}\cos 4\pi\kappa\theta\sin 2\pi\nu\theta\, d\theta = I_{31}(\kappa,\nu) + I_{32}(\kappa,\nu) \tag{70}$$

$$I_2(\kappa,\nu) = 2\int_0^1 e^{q_3\theta}\cos 4\pi\kappa\theta\cos 2\pi\nu\theta\, d\theta = I_{41}(\kappa,\nu) + I_{42}(\kappa,\nu) \tag{71}$$

$$I_{31}(\kappa,\nu) = \int_0^1 e^{q_3\theta}\sin[2\pi(2\kappa+\nu)]d\theta = +\frac{2\pi(2\kappa+\nu)(1-e^{q_3})}{q_3^2 + [2\pi(2\kappa+\nu)]^2} \tag{72}$$

$$I_{32}(\kappa,\nu) = \int_0^1 e^{q_3\theta}\sin[2\pi(2\kappa-\nu)]d\theta = -\frac{2\pi(2\kappa-\nu)(1-e^{q_3})}{q_3^2 + [2\pi(2\kappa-\nu)]^2} \tag{73}$$

$$I_{41}(\kappa,\nu) = \int_0^1 e^{q_3\theta}\cos[2\pi(2\kappa+\nu)]d\theta = -\frac{q_3(1-e^{q_3})}{q_3^2 + [2\pi(2\kappa+\nu)]^2} \tag{74}$$

$$I_{42}(\kappa,\nu) = \int_0^1 e^{q_3\theta}\cos[2\pi(2\kappa-\nu)]d\theta = -\frac{q_3(1-e^{q_3})}{q_3^2 + [2\pi(2\kappa-\nu)]^2} \tag{75}$$

Substitution of Eqs.(70) to (75) into Eqs.(66) to (68) yields for $g_{s\kappa 1}(\nu)$, $g_{c\kappa 1}(\nu)$, and $f_{\kappa 1}(\theta)$ the following relations:

$$g_{s\kappa 1}(\nu) = B_{13}(\kappa)[I_{31}(\kappa,\nu) + I_{32}(\kappa,\nu)] \tag{76}$$

$$g_{c\kappa 1}(\nu) = B_{13}(\kappa)[I_{41}(\kappa,\nu) + I_{42}(\kappa,\nu)] \tag{77}$$

$$\begin{aligned} f_{\kappa 1}(\theta) = \sum_{\nu=1}^{\infty} B_{13}(\kappa)\{&[I_{31}(\kappa,\nu) + I_{32}(\kappa,\nu)]\sin 2\pi\nu\theta \\ &+ [I_{41}(\kappa,\nu) + I_{42}(\kappa,\nu)]\cos 2\pi\nu\theta\} \end{aligned} \tag{78}$$

Let us turn to Eq.(61) for $A_{\mathrm{ev}}(\zeta,\theta)$. The second sum, multiplied by $e^{-\rho_1}$ in Eq.(61) becomes:

$$\begin{aligned} &\sum_{\kappa=1}^{<K} -e^{q_3\theta} B_{13}(\kappa)\cos 4\pi\kappa\theta \sin 2\pi\kappa\zeta \\ &\quad = -\sum_{\kappa=1}^{<K}\sum_{\nu=1}^{\infty} B_{13}(\kappa)[I_1(\kappa,\nu)\sin 2\pi\nu\theta + I_2(\kappa,\nu)\cos 2\pi\nu\theta]\sin 2\pi\kappa\zeta \end{aligned} \tag{79}$$

$$I_1(\kappa,\nu) = (1-e^{q_3})\left(\frac{2\pi(2\kappa+\nu)}{q_3^2 + [2\pi(2\kappa+\nu)]^2} - \frac{2\pi(2\kappa-\nu)}{q_3^2 + [2\pi(2\kappa-\nu)]^2}\right) \tag{80}$$

$$I_2(\kappa,\nu) = -(1-e^{q_3})\left(\frac{q_3}{q_3^2 + [(2\pi(2\kappa+\nu)]^2} + \frac{q_3}{q_3^2 + [2\pi(2\kappa-\nu)]^2}\right) \tag{81}$$

We turn to the third sum in Eq.(61). The term $e^{-\rho_1\theta/2}e^{-(2q_3+\rho_1)\theta/2}$ is also eliminated by means of a Fourier expansion:

$$f_{\kappa 2}(\theta) = e^{-(q_3+\rho_1)\theta}[C_{13}(\kappa)\sin 4\pi\kappa\theta + D_{13}(\kappa)] \tag{82}$$

Following the steps from Eq.(66) to (79) we get a similar but considerably more complicated result:

$$\begin{aligned} &\sum_{\kappa=1}^{<K} e^{-(q_3+\rho_1)\theta}[C_{13}(\kappa)\sin 4\pi\kappa\theta + D_{13}(\kappa)\cos 2\pi\kappa\theta]\sin 2\pi\kappa\zeta \\ &\quad = \sum_{\kappa=1}^{<K}\sum_{\nu=1}^{\infty}\{[C_{13}(\kappa)I_5(\kappa,\nu) + D_{13}(\kappa)I_6(\kappa,\nu)]\sin 2\pi\nu\theta \\ &\qquad + [C_{13}(\kappa)I_7(\kappa,\nu) + D_{13}(\kappa)I_8(\kappa,\nu)]\cos 2\pi\nu\theta\}\sin 2\pi\kappa\zeta \end{aligned} \tag{83}$$

The four integrals $I_5(\kappa,\nu)$ to $I_8(\kappa,\nu)$ are defined as follows:

$$I_5(\kappa,\nu) = 2\int_0^1 e^{-(q_3+\rho_1)\theta}\sin 4\pi\kappa\theta \sin 2\pi\nu\theta\, d\theta = -(q_3+\rho_1)\left(1-e^{-(q_3+\rho_1)}\right)$$
$$\times\left(\frac{1}{(q_3+\rho_1)^2+[2\pi(2\kappa+\nu)]^2}-\frac{1}{(q_3+\rho_1)^2+[2\pi(2\kappa-\nu)]^2}\right) \tag{84}$$

$$I_6(\kappa,\nu) = 2\int_0^1 e^{-(q_3+\rho_1)\theta}\sin 2\pi\nu\theta\, d\theta = 2\left(1-e^{-(q_3+\rho_1)}\right)$$
$$\times\frac{2\pi\nu}{(q_3^2+\rho_1)^2+(2\pi\nu)^2} \tag{85}$$

$$I_7(\kappa,\nu) = 2\int_0^1 e^{-(q_3+\rho_1)\theta}\sin 4\pi\kappa\theta \cos 2\pi\nu\theta\, d\theta = \left(1-e^{-(q_3+\rho_1)}\right)$$
$$\times\left(\frac{2\pi(2\kappa+\nu)}{(q_3+\rho_1)^2+[2\pi(2\kappa+\nu)]^2}+\frac{2\pi(2\kappa-\nu)}{(q_3+\rho_1)^2+[2\pi(2\kappa-\nu)]^2}\right) \tag{86}$$

$$I_8(\kappa,\nu) = 2\int_0^1 e^{-(q_3+\rho_1)\theta}\cos 2\pi\nu\theta\, d\theta = 2(q_3+\rho_1)\left(1-e^{-(q_3+\rho_1)}\right)$$
$$\times\frac{q_3+\rho_1}{(q_3+\rho_1)^2+(2\pi\nu)^2} \tag{87}$$

The first three sums in Eq.(61) may now be combined:

$$\sum_{\kappa=1}^{<K}\{A_{\mathrm{es}}(\kappa)\sin 2\pi\kappa\theta + A_{\mathrm{ec}}(\kappa)\cos 2\pi\kappa\theta$$
$$- e^{q_3\theta}B_{13}(\kappa)\cos 4\pi\kappa\theta$$
$$+ e^{-(q_3+\rho_1)\theta}[C_{13}(\kappa)\sin 4\pi\kappa\theta + D_{13}(\kappa)]\}\sin 2\pi\kappa\zeta$$
$$= \sum_{\kappa=1}^{<K}\{A_{\mathrm{es}}(\kappa)\sin 2\pi\kappa\theta + A_{\mathrm{ec}}(\kappa)\cos 2\pi\kappa\theta$$
$$+\sum_{\nu=1}^{\infty}[B_{\mathrm{es}}(\kappa,\nu)\sin 2\pi\nu\theta + B_{\mathrm{ec}}(\kappa,\nu)\cos 2\pi\nu\theta]\}\sin 2\pi\kappa\zeta$$

$$B_{\mathrm{es}}(\kappa,\nu) = B_{13}(\kappa)I_1(\kappa,\nu) + C_{13}(\kappa)I_5(\kappa,\nu) + D_{13}(\kappa)I_6(\kappa,\nu)$$
$$B_{\mathrm{ec}}(\kappa,\nu) = B_{13}(\kappa)I_2(\kappa,\nu) + C_{13}(\kappa)I_7(\kappa,\nu) + D_{13}(\kappa)I_8(\kappa,\nu) \tag{88}$$

We turn to the last three sums in Eq.(61). They are all multiplied by $e^{-\rho_1\theta/2}$ and they all have a shift $-q_1$ or $\pm q_2$ in the arguments of the sine and cosine functions. Following Eq.(82) we write:

$$f_{\kappa 3}(\theta) = e^{-\rho_1\theta/2}[A_{57}(\kappa)\sin(2\pi\kappa - q_1)\theta + B_{68}(\kappa)\cos(2\pi\kappa - q_1)\theta] \tag{89}$$

Following the steps from Eq.(66) to (77) we get:

$$\begin{aligned} e^{-\rho_1\theta/2}\sum_{\kappa>K}^{\infty}&[A_{57}(\kappa)\sin(2\pi\kappa - q_1)\theta + B_{68}(\kappa)\cos(2\pi\kappa - q_1)\theta]\sin 2\pi\kappa\zeta \\ &= \sum_{\kappa>K}^{\infty}\sum_{\nu=1}^{\infty}\{[A_{57}(\kappa)I_9(\kappa,\nu) + B_{68}(\kappa)I_{10}(\kappa,\nu)]\sin 2\pi\nu\theta \\ &\quad + [A_{57}(\kappa)I_{11}(\kappa,\nu) + B_{68}(\kappa)I_{12}(\kappa,\nu)]\cos 2\pi\nu\theta\}\sin 2\pi\kappa\zeta \end{aligned} \tag{90}$$

The integrals $I_9(\kappa,\nu)$ to $I_{12}(\kappa,\nu)$ are more complicated than the previous integrals $I_1(\kappa,\nu)$ to $I_8(\kappa,\nu)$:

$$\begin{aligned} I_9(\kappa,\nu) &= 2\int_0^1 e^{-\rho_1\theta/2}\sin(2\pi\kappa - q_1)\theta\sin 2\pi\nu\theta\,d\theta \\ &= \frac{e^{-\rho_1/2}\{(\rho_1/2)\cos q_1 + [2\pi(\kappa+\nu) - q_1]\sin q_1\} - \rho_1/2}{(\rho_1/2)^2 + [2\pi(\kappa+\nu) - q_1]^2} \\ &- \frac{e^{-\rho_1/2}\{(\rho_1/2)\cos q_1 + [2\pi(\kappa-\nu) - q_1]\sin q_1\} - \rho_1/2}{(\rho_1/2)^2 + [2\pi(\kappa-\nu) - q_1]^2} \end{aligned} \tag{91}$$

$$\begin{aligned} I_{10}(\kappa,\nu) &= 2\int_0^1 e^{-\rho_1\theta/2}\cos(2\pi\kappa - q_1)\theta\sin 2\pi\nu\theta\,d\theta \\ &= \frac{e^{-\rho_1/2}\{(\rho_1/2)\sin q_1 - [2\pi(\kappa+\nu) - q_1]\cos q_1\} + 2\pi(\kappa+\nu) - q_1}{(\rho_1/2)^2 + [2\pi(\kappa+\nu) - q_1]^2} \\ &- \frac{e^{-\rho_1/2}\{(\rho_1/2)\sin q_1 - [2\pi(\kappa-\nu) - q_1]\cos q_1\} + 2\pi(\kappa-\nu) - q_1}{(\rho_1/2)^2 + [2\pi(\kappa-\nu) - q_1]^2} \end{aligned} \tag{92}$$

$$\begin{aligned} I_{11}(\kappa,\nu) &= 2\int_0^1 e^{-\rho_1\theta/2}\sin(2\pi\kappa - q_1)\theta\cos 2\pi\nu\theta\,d\theta \\ &= \frac{e^{-\rho_1/2}\{(\rho_1/2)\sin q_1 - [2\pi(\kappa+\nu) - q_1]\cos q_1\} + 2\pi(\kappa+\nu) - q_1}{(\rho_1/2)^2 + [2\pi(\kappa+\nu) - q_1]^2} \\ &+ \frac{e^{-\rho_1/2}\{(\rho_1/2)\sin q_1 - [2\pi(\kappa-\nu) - q_1]\cos q_1\} + 2\pi(\kappa-\nu) - q_1}{(\rho_1/2)^2 + [2\pi(\kappa-\nu) - q_1]^2} \end{aligned} \tag{93}$$

$$I_{12}(\kappa,\nu) = 2\int_0^1 e^{-\rho_1\theta/2}\cos(2\pi\kappa - q_1)\theta\cos 2\pi\nu\theta\,d\theta$$
$$= -\frac{e^{-\rho_1/2}\{(\rho_1/2)\cos q_1 + [2\pi(\kappa+\nu) - q_1]\sin q_1\} - \rho_1/2}{(\rho_1/2)^2 + [2\pi(\kappa+\nu) - q_1]^2}$$
$$- \frac{e^{-n\rho_1/2}\{(\rho_1/2)\cos q_1 + [2\pi(\kappa-\nu) - q_1]\sin q_1\} - \rho_1/2}{(\rho_1/2)^2 + [2\pi(\kappa-\nu) - q_1]^2} \quad (94)$$

The second sum from the end in Eq.(61) is written in analogy to Eq.(82) as follows:

$$f_{\kappa 4}(\theta) = e^{-\rho_1\theta/2}[E_{57}(\kappa)\sin(2\pi\kappa - q_1)\theta + F_{57}(\kappa)\cos(2\pi\kappa - q_2)\theta] \quad (95)$$

Again we follow the steps from Eq.(66) to (77) and obtain:

$$e^{-\rho_1\theta/2}\sum_{\kappa>K}^{\infty}[E_{57}(\kappa)\sin(2\pi\kappa - q_2)\theta + F_{57}\cos(2\pi\kappa - q_2)\theta]\sin 2\pi\kappa\zeta$$
$$= \sum_{\kappa>K}^{\infty}\sum_{\nu=1}^{\infty}\{[E_{57}(\kappa)I_{13}(\kappa,\nu) + F_{57}(\kappa)I_{14}(\kappa,\nu)]\sin 2\pi\nu\theta$$
$$+ E_{57}(\kappa)I_{15}(\kappa,\nu) + F_{57}(\kappa)I_{16}(\kappa,\nu)]\cos 2\pi\nu\theta\}\sin 2\pi\kappa\zeta \quad (96)$$

The integrals $I_{13}(\kappa,\nu)$ to $I_{16}(\kappa,\nu)$ can be written with the help of the integrals $I_9(\kappa,\nu)$ to $I_{12}(\kappa,\nu)$:

$$I_{13}(\kappa,\nu) = 2\int_0^1 e^{-\rho_1\theta/2}\sin(2\pi\kappa - q_2)\theta\sin 2\pi\nu\theta\,d\theta$$
$$= I_9(\kappa,\nu) \text{ with } q_1 \text{ replaced by } q_2 \quad (97)$$
$$I_{14}(\kappa,\nu) = 2\int_0^1 e^{-\rho_1\theta/2}\cos(2\pi\kappa - q_2)\theta\sin 2\pi\nu\theta\,d\theta$$
$$= I_{10}(\kappa,\nu) \text{ with } q_1 \text{ replaced by } q_2 \quad (98)$$
$$I_{15}(\kappa,\nu) = 2\int_0^1 e^{-\rho_1\theta/2}\sin(2\pi\kappa - q_2)\theta\cos 2\pi\nu\theta\,d\theta$$
$$= I_{11}(\kappa,\nu) \text{ with } q_1 \text{ replaced by } q_2 \quad (99)$$
$$I_{16}(\kappa,\nu) = 2\int_0^1 e^{-\rho_1\theta/2}\cos(2\pi\kappa - q_2)\theta\cos 2\pi\nu\theta\,d\theta$$
$$= I_{12}(\kappa,\nu) \text{ with } q_1 \text{ replaced by } q_2 \quad (100)$$

We turn to the last sum in Eq.(61). In analogy to Eq.(82) we write it in the following form:

$$f_{\kappa 5}(\theta) = e^{-\rho_1\theta/2}[E_{68}(\kappa)\sin(2\pi\kappa + q_2)\theta + F_{68}(\kappa)\cos(2\pi\kappa + q_2)\theta] \qquad (101)$$

Following the steps from Eq.(66) to (77) a final time we obtain:

$$\begin{aligned} e^{-\rho_1\theta/2}\sum_{\kappa>K}^{\infty}\{[E_{68}(\kappa)\sin(2\pi\kappa+q_2)\theta + F_{68}(\kappa)(2\pi\kappa+q_2)\theta]\sin 2\pi\kappa\zeta \\ = \sum_{\kappa>K}^{\infty}\sum_{\nu=1}^{\infty}\{[E_{68}(\kappa)I_{17}(\kappa,\nu) + F_{68}(\kappa)I_{18}(\kappa,\nu)]\sin 2\pi\nu\theta \\ + [E_{68}(\kappa)I_{19}(\kappa,\nu) + F_{68}(\kappa)I_{20}(\kappa,\nu)]\cos 2\pi\nu\theta\}\sin 2\pi\kappa\zeta \end{aligned} \qquad (102)$$

The integrals $I_{17}(\kappa,\nu)$ to $I_{20}(\kappa,\nu)$ can be written with the help of the integrals $I_9(\kappa,\nu)$ to $I_{12}(\kappa,\nu)$:

$$\begin{aligned} I_{17}(\kappa,\nu) &= 2\int_0^1 e^{-\rho_1\theta/2}\sin(2\pi\kappa+q_2)\theta\sin 2\pi\nu\theta\,d\theta \\ &= I_9(\kappa,\nu) \text{ with } q_1 \text{ replaced by } -q_2 \end{aligned} \qquad (103)$$

$$\begin{aligned} I_{18}(\kappa,\nu) &= 2\int_0^1 e^{-\rho_1\theta/2}\cos(2\pi\kappa+q_2)\theta\sin 2\pi\nu\theta\,d\theta \\ &= I_{10}(\kappa,\nu) \text{ with } q_1 \text{ replaced by } -q_2 \end{aligned} \qquad (104)$$

$$\begin{aligned} I_{19}(\kappa,\nu) &= 2\int_0^1 e^{-\rho_1\theta/2}\sin(2\pi\kappa+q_2)\theta\cos 2\pi\nu\theta\,d\theta \\ &= I_{11}(\kappa,\nu) \text{ with } q_1 \text{ replaced by } -q_2 \end{aligned} \qquad (105)$$

$$\begin{aligned} I_{20}(\kappa,\nu) &= 2\int_0^1 e^{-\rho_1\theta/2}\cos(2\pi\kappa+q_2)\theta\cos 2\pi\nu\theta\,d\theta \\ &= I_{12}(\kappa,\nu) \text{ with } q_1 \text{ replaced by } -q_2 \end{aligned} \qquad (106)$$

The sums 4 to 7 of Eq.(61) may be combined:

$$\sum_{\kappa>K}^{\infty}\{A_{\rm es}(\kappa)\sin 2\pi\kappa\theta+A_{\rm ec}(\kappa)\cos 2\pi\kappa\theta$$
$$+e^{-\rho_1\theta/2}[A_{57}(\kappa)\sin(2\pi\kappa-q_1)\theta+B_{68}(\kappa)\cos(2\pi\kappa-q_1)\theta$$
$$+E_{57}(\kappa)\sin(2\pi\kappa-q_2)\theta+F_{57}\cos(2\pi\kappa-q_2)\theta$$
$$+E_{68}(\kappa)\sin(2\pi\kappa+q_2)\theta+F_{68}(\kappa)\cos(2\pi\kappa+q_2)\theta]\}\sin 2\pi\kappa\zeta$$
$$=\sum_{\kappa>K}^{\infty}\{A_{\rm es}(\kappa)\sin 2\pi\kappa\theta+A_{\rm ec}(\kappa)\cos 2\pi\kappa\theta$$
$$+\sum_{\nu=1}^{\infty}[B_{\rm es}(\kappa,\nu)\sin 2\pi\nu\theta+B_{\rm ec}(\kappa,\nu)\cos 2\pi\nu\theta]\}\sin 2\pi\kappa\zeta$$
$$B_{\rm es}(\kappa,\nu)=A_{57}(\kappa)I_9(\kappa,\nu)+B_{68}(\kappa)I_{10}(\kappa,\nu)+E_{57}(\kappa)I_{13}(\kappa,\nu)$$
$$+F_{57}(\kappa)I_{14}(\kappa,\nu)+E_{68}(\kappa)I_{17}(\kappa,\nu)+F_{68}(\kappa)I_{18}(\kappa,\nu)$$
$$B_{\rm ec}(\kappa,\nu)=A_{57}(\kappa)I_{11}(\kappa,\nu)+B_{68}(\kappa)I_{12}(\kappa,\nu)+E_{57}(\kappa)I_{15}(\kappa,\nu)$$
$$+F_{57}(\kappa)I_{16}(\kappa,\nu)+E_{68}(\kappa)I_{19}(\kappa,\nu)$$
$$+F_{68}(\kappa)I_{20}(\kappa,\nu)\qquad(107)$$

The very first term $e^{-\rho_2\zeta}(1-\operatorname{ch}\rho_2\theta)$ in Eq.(61) can be represented by a product of Fourier series with ζ and θ as the transformed variables. We shall not do so now in order to preserve the compactness and clarity of the notation $e^{-\rho_2\zeta}(1-\operatorname{ch}\rho_2\theta)$ compared with its Fourier representation. Equation (61) is written in the following form with the help of Eqs.(88) and (107):

$$A_{\rm ev}(\zeta,\theta)=c^2T^2V_{\rm e0}\left[\frac{1}{\rho_2^2}e^{-\rho_2\zeta}(1-\operatorname{ch}\rho_2\theta)+2(1-e^{-\rho_2})\right.$$
$$\times\left(\sum_{\kappa=1}^{<K}\{A_{\rm es}(\kappa)\sin 2\pi\kappa\theta+A_{\rm ec}(\kappa)\cos 2\pi\kappa\theta\right.$$
$$+\sum_{\nu=1}^{\infty}[B_{\rm es}(\kappa,\nu)\sin 2\pi\nu\theta+B_{\rm ec}(\kappa,\nu)\cos 2\pi\nu\theta]\}\sin 2\pi\kappa\zeta$$
$$+\sum_{\kappa>K}^{\infty}\{A_{\rm es}(\kappa)\sin 2\pi\kappa\theta+A_{\rm ec}(\kappa)\cos 2\pi\kappa\theta$$
$$\left.\left.+\sum_{\nu=1}^{\infty}[B_{\rm es}(\kappa,\nu)\sin 2\pi\nu\theta+B_{\rm ec}(\kappa,\nu)\cos 2\pi\nu\theta]\}\sin 2\pi\kappa\zeta\right)\right]\qquad(108)$$

A radical simplification of the writing of this equation is necessary to make it usable:

$$A_{\text{ev}}(\zeta,\theta) = c^2T^2V_{\text{e0}}\left(A_{\text{e0}}(\zeta,\theta) + \sum_{\kappa=1}^{\infty} C_{\text{e}\kappa}(\theta)\sin 2\pi\kappa\zeta\right)$$

$$A_{\text{e0}}(\zeta,\theta) = \frac{1}{\rho_2^2}e^{-\rho_2\zeta}(1 - \operatorname{ch}\rho_2\theta)$$

$$C_{\text{e}\kappa}(\theta) = 2(1 - e^{-\rho_2})\Big(A_{\text{es}}(\kappa)\sin 2\pi\kappa\theta + A_{\text{ec}}(\kappa)\cos 2\pi\kappa\theta$$

$$+\sum_{\nu=1}^{\infty}[B_{\text{es}}(\kappa,\nu)\sin 2\pi\nu\theta + B_{\text{ec}}(\kappa,\nu)\cos 2\pi\nu\theta]\Big) \tag{109}$$

We may make two simplifications that hold generally by using the relation $\rho_2 = cT\sqrt{\sigma s} \gg 1$ of Eq.(2.2-9). First the factor $1 - \exp(-\rho_2)$ is essentially equal to 1. Second, the absolute value of $A_{\text{e0}}(\zeta,\theta)$ is less than $0.5/\rho_2^2$ according to Fig.6.12-1. Hence, we write:

$$A_{\text{ev}}(\zeta,\theta) = c^2T^2V_{\text{e0}}\sum_{\kappa=1}^{\infty} C_{\text{e}\kappa}(\theta)\sin 2\pi\kappa\zeta$$

$$C_{\text{e}\kappa}(\theta) = 2\Big(A_{\text{es}}(\kappa)\sin 2\pi\kappa\theta + A_{\text{ec}}(\kappa)\cos 2\pi\kappa\theta$$

$$+\sum_{\nu=1}^{\infty}[B_{\text{es}}(\kappa,\nu)\sin 2\pi\kappa\nu\theta + B_{\text{ec}}(\kappa,\nu)\cos 2\pi\nu\theta]\Big) \tag{110}$$

We need the limits for $\kappa \to \infty$ of most of the expressions derived in this section. First we write the auxiliary expressions of Eqs.(1) to (12) as well as a few more frequently required expressions:

$$\text{for } d^2 = 4[(2\pi\kappa)^2 + \rho_2^2] > \rho_1^2, \quad \kappa \to \infty$$

$$d^2 \approx 4(2\pi\kappa)^2 \tag{111}$$

$$q_1 \approx 4\pi\kappa + \frac{\rho_2^2 - \rho_1^2/4}{4\pi\kappa}, \qquad q_2 \approx \frac{\rho_2^2 - \rho_1^2/4}{4\pi\kappa} \tag{112}$$

$$(d^2 - \rho_1^2)^{1/2} \approx 4\pi\kappa \tag{113}$$

$$q_1^2 + \left(\frac{\rho_1}{2}\right)^2 \approx 4(2\pi\kappa)^2 + 2\rho_2^2 - \frac{1}{4}\rho_1^2 \tag{114}$$

$$q_2^2 + \left(\frac{\rho_1}{2}\right)^2 \approx \frac{1}{4}\rho_1^2 \tag{115}$$

$$q_1 - 2\pi\kappa \approx 2\pi\kappa + \frac{\rho_2^2 - \rho_1^2/4}{4\pi\kappa} \tag{116}$$

$$q_2 + 2\pi\kappa \approx 2\pi\kappa + \frac{\rho_2^2 - \rho_1^2/4}{4\pi\kappa} \tag{117}$$

Starting with $A_{es}(\kappa)$ of Eq.(13) we follow the listing and obtain for the limit $\kappa \to \infty$ the following results:

$$A_{es}(\kappa) \approx \frac{1}{(2\pi\kappa)^2\rho_1} \qquad A_{ec}(\kappa) \approx \frac{1}{(2\pi\kappa)^3}\left(1 - \frac{\rho_2^2}{\rho_1^2}\right) \tag{118}$$

$$A_{57}(\kappa) \approx -\frac{3\rho_1}{16(2\pi\kappa)^4} \qquad B_{57}(\kappa) \approx \frac{1}{4(2\pi\kappa)^3} \tag{119}$$

$$C_{57}(\kappa) \approx -\frac{3}{(2\pi\kappa)^2\rho_1} \qquad D_{57}(\kappa) \approx -\frac{3}{4(2\pi\kappa)^3}\left(1 - \frac{4\rho_2^2}{3\rho_1^2}\right) \tag{120}$$

$$A_{68}(\kappa) \approx \frac{3\rho_1}{16(2\pi\kappa)^4} \approx -A_{57}(\kappa) \quad B_{68}(\kappa) \approx \frac{1}{4(2\pi\kappa)^3} \approx B_{57}(\kappa) \tag{121}$$

$$C_{68}(\kappa) \approx -\frac{1}{(2\pi\kappa)^2\rho_1} \qquad D_{68}(\kappa) \approx -\frac{\rho_2^2 + \rho_1^2/4}{(2\pi\kappa)^3\rho_1^2} \tag{122}$$

$$E_{57}(\kappa) \approx -\frac{1}{(2\pi\kappa)^2\rho_1} \qquad E_{68}(\kappa) \approx -\frac{2}{(2\pi\kappa)^2\rho_1} \tag{123}$$

$$F_{57}(\kappa) \approx -\frac{1}{2(2\pi\kappa)^3} \qquad F_{68}(\kappa) \approx \frac{1}{4(2\pi\kappa)^3}\left(1 - 4\frac{\rho_2^2}{\rho_1^2}\right) \tag{124}$$

The integrals $I_1(\kappa,\nu)$ to $I_8(\kappa,\nu)$ are not needed for the limits $\kappa \to \infty$, but the integrals $I_9(\kappa,\nu)$ to $I_{20}(\kappa,\nu$ are:

$$I_9(\kappa,\nu) \approx -\frac{4\rho_1\nu}{(2\pi\kappa)^3} \qquad I_{10}(\kappa,\nu) \approx -\frac{4\pi\nu}{(2\pi\kappa)^2} \tag{125}$$

$$I_{11}(\kappa,\nu) \approx -\frac{2}{2\pi\kappa} \qquad I_{12}(\kappa,\nu) \approx \frac{\rho_1}{(2\pi\kappa)^2} \tag{126}$$

$$I_{13}(\kappa,\nu) \approx 2\rho_1\frac{2\pi\nu}{(2\pi\kappa)^3} \qquad I_{14}(\kappa,\nu) \approx -\frac{4\pi\nu}{(2\pi\kappa)^2} \tag{127}$$

$$I_{15}(\kappa,\nu) \approx \frac{2}{2\pi\kappa} \qquad I_{16}(\kappa,\nu) \approx \frac{\rho_1}{2(2\pi\kappa)^2} \tag{128}$$

$$I_{17}(\kappa,\nu) \approx 2\rho_1\frac{2\pi\nu}{(2\pi\kappa)^3} \qquad I_{18}(\kappa,\nu) \approx -\frac{2\pi\nu}{(2\pi\kappa)^2} \tag{129}$$

$$I_{19}(\kappa,\nu) \approx \frac{2}{2\pi\kappa} \qquad I_{20}(\kappa,\nu) \approx \frac{\rho_1}{2(2\pi\kappa)^2} \tag{130}$$

We may now produce the functions $B_{es}(\kappa,\nu)$ and $B_{ec}(\kappa,\nu)$ of Eq.(107). Since we do not want to evaluate the sums over ν we cannot write $\approx$ (approximately) but must write $\propto$ (proportionate):

$$B_{es}(\kappa,\nu) \propto \frac{1}{(2\pi\kappa)^5} \qquad B_{ec}(\kappa,\nu) \propto \frac{1}{(2\pi\kappa)^3} \tag{131}$$

The functions $C_{es}(\kappa,\nu)$ and $C_{ec}(\kappa,\nu)$ of Eq.(4.2-16) vary for $\kappa \to \infty$ as follows:

$$C_{\rm es}(\kappa,\nu) \approx 2\pi\kappa\rho_{\rm s}B_{\rm es}(\kappa,\nu) - \frac{2\pi\nu}{2\pi\kappa}B_{\rm ec}(\kappa,\nu) \propto \frac{1}{(2\pi\kappa)^4} \tag{132}$$

$$C_{\rm ec}(\kappa,\nu) \approx 2\pi\kappa\rho_{\rm s}B_{\rm ec}(\kappa,\nu) + \frac{2\pi\nu}{2\pi\kappa}B_{\rm es}(\kappa,\nu) \propto \frac{1}{(2\pi\kappa)^2} \tag{133}$$

For $U^2_{\rm cs}(\kappa)$ and $U^2_{\rm cc}(\kappa)$ in Eqs.(4.3-30) and (4.3-31) we get:

$$U^2_{\rm cs}(\kappa) \propto \left(B_{\rm ec}(\kappa,\nu) + \frac{\nu}{\kappa}C_{\rm es}(\kappa,\nu)\right)^2 + \left(B_{\rm es}(\kappa,\nu) - \frac{\nu}{\kappa}C_{\rm ec}(\kappa,\nu)\right)^2$$

$$\propto \frac{1}{(2\pi\kappa)^6} \tag{134}$$

$$U^2_{\rm cc}(\kappa) \propto \left(\frac{\nu}{\kappa}B_{\rm ec}(\kappa,\nu) + C_{\rm es}(\kappa,\nu)\right)^2 + \left(\frac{\nu}{\kappa}B_{\rm es}(\kappa,\nu) - C_{\rm ec}(\kappa,\nu)\right)^2$$

$$\propto \frac{1}{(2\pi\kappa)^4} \tag{135}$$

The normalized energy of the component of the wave represented by the sinusoidal pulse with κ cycles in the interval $0 \leq y \leq cT$ or by all the photons with the period number κ varies for $\kappa \to \infty$ as follows:

$$\mathcal{H}_\kappa = (2\pi\kappa)^2[U^2_{\rm cs}(\kappa) + U^2_{\rm cc}(\kappa)] \propto \frac{1}{(2\pi\kappa)^2} \qquad \text{for } \kappa \to \infty \tag{136}$$

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7 References and Bibliography

Abraham, M. and Becker, R. (1932). *The Classical Theory of Electricity and Magnetism* (transl. by J.Dougall of 8th ed. of *Theorie der Elektrizität*). Hafner, New York.

Abraham, M. and Becker, R. (1950). *The Classical Theory of Electricity and Magnetism* (transl. by J.Dougall of 14th ed. of *Theorie der Elektrizität*). Hafner, New York.

Abramowitz, M. and Stegun, I.A. (1964). Handbook of Mathematical Functions. National Bureau of Standards, Government Printing Office, Washington DC.

Akhiezer, A.I. and Berestecki, V.B. (1981). *Quantum Electrodynamics*, (in Russian). Nauka, Moscow.

Apostle, H.G. (1969). *Aristotle's Physics, Translated with Commentaries and Glossary.* Indiana University Press, Bloomington.

Aristotle (1930). *The Works of Aristotle, vol. II. Physics*, R.R. Hardie and R.K. Gaye transl. Clarendon Press, Oxford.

Barrett, T.W. (1988). Comments on the Harmuth ansatz: Use of magnetic current density in the calculation of the propagation velocity of signals by amended Maxwell theory. *IEEE Trans. Electromagn. Compat.*, vol. EMC-30, 419–420.

Barrett, T.W. (1989a). Comments on 'Solution of Maxwell's equations for general nonperiodic waves in lossy media'. *IEEE Trans. Electromagn. Compat.*, vol. EMC-31, 197–199.

Barrett, T.W. (1989b). Comments on 'Some comments on Harmuth and his critics'. *IEEE Trans. Electromagn. Compat.*, vol. EMC-31, 201–202.

Barrett, T.W. (1989c). On the distinction between fields and their metric: the fundamental difference between specifications concerning medium-independent fields and constitutive specifications concerning relations to the medium in which they exist. *Annales de la Fondation Louis de Broglie*, vol. 14, 37–75.

Barrett, T.W. (1990a). Maxwell's theory extended. Part 1. Empirical reasons for questioning the completeness of Maxwell's theory—effects demonstrating the physical significance of the A potentials. *Annales de la Fondation Louis de Broglie*, vol. 15, 143–183.

Barrett, T.W. (1990b). Maxwell's theory extended. Part 2. Theoretical and pragmatic reasons for questioning the completeness of Maxwell's theory. *Annales de la Fondation* Louis de Broglie, vol. 15, 253–283.

Barrett, T.W. (1991). Energy transfer and propagation and the dielectrics of materials: transient versus steady state effects. *Ultra-Wideband Radar: Proc. First Los Alamos Symposium* (B.Noel ed.). CRC Press, Boca Raton, Florida.

Barrett, T.W. (1993). Electromagnetic phenomena not explained by Maxwell's equations. In *Essays on the Formal Aspects of Electromagnetic Theory* (A.Lakhtakia ed.), 6–86. World Scientific Publishing Co., Singapore.

Barrett, T.W. (1994). The Ehrenhaft-Mikhailov effect described as the behavior of a low energy density magnetic monopole instanton. *Annales de la Fondation Louis de Broglie*, vol. 19, 291–301.

Barrett, T.W. (1995a). Energy transfer through media and sensing of the media. In *Introduction to Ultrawideband Radar Systems* (J.D.Taylor ed.), 365–434. CRC Press, Boca Raton, Florida.

Barrett, T.W. (1995b). Sagnac effect: A consequence of conservation of action due to gauge field global conformal invariance in multiply-joined topology of coherent fields. In *Advanced Electromagnetism: Foundations, Theory, Applications* (T.W.Barrett and D.M.Grimes eds.), 278–313. World Scientific Publishing Co., Singapore.

Barrett, T.W. (1998). The impact of topology and group theory on future progress in electromagnetics. *Progress in Electromagnetics Research Symposium, PIERS'98*, Nantes, France.

Becker, R. (1957), *Theorie der Elektrizität*, vol. 1 (revised by F.Sauter), 16th ed. Teubner, Stuttgart.

Becker, R. (1963). *Theorie der Elektrizität*, vol. 2 (revised by G.Leibfried and W.Breuig), 9th ed. Teubner, Stuttgart.

Becker, R. (1964a). *Electromagnetic Fields and Interactions* (transl. by A.W.Knudsen of vol. 1, 16th ed. and by I. de Teissier of vol. 2, 8th ed. of *Theorie der Elektrizität*). Blaisdell, New York. Reprinted 1982 by Dover, New York.

Becker, R. (1964b). *Theorie der Elektrizität*, vol. 1 (revised by F.Sauter), 18th ed. Teubner, Stuttgart.

Berestetzki, W.B., Lifschitz, E.M. and Pitajewski, L.P. (1970). *Relativistische Quantentheorie*, vol. IVa of *Lehrbuch der Theoretischen Physik*, L.D.Landau and E.M.Lifschitz eds.; transl. from Russian. Akademie Verlag, Berlin.

Berestetskii, V.B., Lifshitz, E.M. and Pitajevskii (1982). *Quantum Electrodynamics*, transl. from Russian. Pergamon Press, New York.

Berry, M.V. (1980). Exact Aharonov-Bohm wavefunction obtained by applying Dirac's magnetic phase factor. *Eur. J. Phys.*, vol. 1, 240–244.

Bjorken, J.D. and Drell, S.D. (1964). *Relativistic Quantum Mechanics.* McGraw-Hill, New York.

Bjorken, J.D. and Drell, S.D. (1965). *Relativistic Quantum Fields.* McGraw-Hill, New York.

Bogoliubov, N.N. and Shirkov, D.V. (1980). *Introduction to the Theory of Quantized Fields*, 3rd ed. Wiley, New York.

Bolyai, J. (1832). *The Science of Absolute Space* (original in Latin). See appendix of Bonola (1955) for English translation.

Bonola, R. (1955). *Non-Euclidean Geometry.* Dover, New York (Originally published in Italian in 1906).

Börsch, H., Hamisch, H., Wohlleben, D. and Grohmann, K. (1960). Antiparallele Weissche Bereiche als Biprisma für Elektroneninterferenzen. *Zeitschrift für Physik.* vol. 159, 397–404.

Boules, R.N. (1989). *Propagation Velocity of Electromagnetic Signals in Lossy Media in the Presence of Noise.* PhD Thesis, Dept. of Electrical Engineering, The Catholic University of America, Washington, DC.

Brandt, S. (1997). *Electrodynamik: Eine Einführung in Experiment und Theorie.* Springer-Verlag, Berlin.

Bub, J. (1999). Interpreting the Quantum World. Cambridge University Press, Cambridge.

Callender, C. and Hugget, N. (2000). *Physics Meets Philosophy at the Planck Scale.* Cambridge University Press, Cambridge.

Chambers, R.G. (1960). Shift of an electron interference pattern by enclosed magnetic flux. *Phys. Rev. Let.*, vol. 5, July, 3–5.

Chuang, I.L. and Nielsen, M.A. (2000).*Quantum Computation and Information.* Cambridge University Press, Cambridge.

Collins, J. (1984). *Renormalization.* Cambridge University Press, Cambridge.

Dickson, W.M. (1998). *Quantum Chance and Nonlocality.* Cambridge University Press, Cambridge.

Dittrich, W. and Gies, H. (2000). *Probing the Quantum Vacuum / Perturbative Effective Action Approach in Quantum Electrodynamics and its Applications.* Springer-Verlag, Berlin.

Feynman, R.P. (1961). *Quantum Electrodynamics.* Benjamin, New York.

Fushich, W.I. and Nikitin, A.G. (1987). *Symmetries of Maxwell's Equations*; transl. from Russian. Reidel Publishing Co., Dordrecht, Netherlands.

Gasiorowitcz, S. (1966). *Elementary Particle Physics.* Wiley, New York.

Ghose, P. (1999). *Testing Quantum Mechanics on New Ground.* Cambridge University Press, Cambridge.

Gradshteyn, I.S. and Ryzhik, I.M. (1980). *Tables of Integrals, Series, and Products* (English edition of Russian original by A.Jeffrey). Academic Press, New York.

Greiner, W. (1998). *Classical Electrodynamics.* Springer-Verlag, Berlin.

Habermann, R. (1983). *Elementary Applied Partial Differential Equations with Fourier Series and Boundary Value Problems.* Prentice Hall, Englewood Cliffs, New Jersey.

Haken, H. and Wolf, H.C. (2000). *Atom und Quantenphysik: Einführung in die experimentellen und theoretischen Grundlagen.* Springer-Verlag, Berlin.

Hamermesh, M. (1962). *Group Theory and Its Applications to Physical Problems.* Addison-Wesley Publ. Co., Reading, MA. Reprinted: Dover, New York 1989.

Harman, P.M. (1998). *The Natural Philosophy of James Clerk Maxwell.* Cambridge University Press, Cambridge.

Harmuth, H.F. (1986a). *Propagation of Nonsinusoidal Electromagnetic Waves.* Academic Press, New York.

Harmuth, H.F. (1986b, c). Correction of Maxwell's equations for signals I, II. *IEEE Trans. Electromagn. Compat.*, vol. EMC-28, 250–258, 259–266.

Harmuth, H.F. (1989). *Information Theory Applied to Space-Time Physics* (in Russian). Mir, Moscow.

Harmuth, H.F. (1992). *Information Theory Applied to Space-Time Physics* (English edition). World Scientific Publishing Co., Singapore.

Harmuth, H.F. and Hussain, M.G.M. (1994). *Propagation of Electromagnetic Signals.* World Scientific Publishing Co., Singapore.

Harmuth, H.F., Boules, R.N. and Hussain, M.G.M. (1999). *Electromagnetic Signals—Reflection, Focusing, Distortions, and Their Application.* Kluwer Academic/Plenum Publishers, New York.

Harmuth, H.F. and Lukin, K.A. (2000). Interstellar Propagation of Electromagnetic Signals. Kluwer Academic/Plenum Publishers, New York. The empty page 224 of this book was censored. Uncensored copies are available at libraries of universities in Eastern Europe, Asia, North Africa, and Latin America.

Heine, V. (1960). *Group Theory in Quantum Mechanics: An Introduction to Its Present Usage.* Pergamon Press, New York (Reprinted: Dover, New York 1993).

Heisenberg, W. and Pauli, W. (1929). Zur Quantendynamik der Wellenfelder. *Zeitschrift für Physik*, vol. 56, 1-61.

Heisenberg, W. and Pauli, W. (1930). Zur Quantentheorie der Wellenfelder II. *Zeittschrift für Physik*, vol. 59, 168-190.

Heitler, W. (1954). *The Quantum Theory of Radiation*, 3rd ed. Oxford University Press, London.

Hillion, P. (1990). Boundary value problems for the wave equation. *Rev. math. Phys.*, vol. 2, 177–191.

Hillion, P. (1991). Remarks on Harmuth's 'Correction of Maxwell's equations for signals I'. *IEEE Trans. Electromagn. Compat.*, vol. EMC-33, 144.

Hillion, P. (1992a). Response to 'The magnetic conductivity and wave propagation'. *IEEE Trans. Electromagn. Compat.*, vol. EMC-34, 376–377.

Hillion, P. (1992b). A further remark on Harmuth's problem. *IEEE Trans. Electromagn. Compat.*, vol. EMC-34, 377–378.

Hillion, P. (1993). Some comments on electromagnetic signals; in *Essays on the Formal Aspects of Electromagnetic Theory.* A.Lakhtakia ed., 127–137. World Scientific Publishing Co., Singapore.

Hillion, P. and Lakhtakia, A. (1993). On an initial-boundary value problem involving Beltram-Moses fields in electromagnetic theory. *Phil. Trans. R. Soc. London*, vol. A344, 235–248.

Horvathy, P.A. (1986). The Wu-Yang factor and the non-Abelian Aharonov-Bohm experiment. *Phys. Rev.*, vol. D33, no. 2, 407–414.

Hussain, M.G.M. (1992). A comparison of transient solutions of Maxwell's equations to that of the modified Maxwell's equations. *IEEE Trans. Electromagn. Compat.*, vol. EMC-34, 482–486.

Icke, V. (1995). *The Force of Symmetry.* Cambridge University Press, Cambridge.

Itzykson, C. and Zuber, J.B. (1980). *Quantum Field Theory.* McGraw-Hill, New York.

Jackson, J.D. (1975). *Classical Electrodynamics*, 2nd ed., p.251. Wiley, New York.

Jordan, P. and Pauli, W. (1928). Zur Quantenelektrodynamik ladungsfreier Felder. *Zeitschrift für Physik*, vol. 47, 151-173.

Källén, G. (1972). *Quantum Electrodynamics.* Springer-Verlag, Berlin.

King, R.W. (1993). The propagation of a Gaussian pulse in seawater and its application to remote sensing. *IEEE Trans. Geoscience, Remote Sensing*, vol. 31, 595–605.

King. R.W. and Harrison, C.W. (1968). The transmission of electromagnetic waves and pulses into the Earth. *J. Appl. Phys.*, vol. 39, 4444–4452.

Kinoshita, T. (ed.) (1990). *Quantum Electrodynamics.* World Scientific Publishing Co., Singapore.

Krylov, A.N. (1929). On the transmission of currents in cables (in Russian). *J. Appl. Phys. (Moscow)* VI(2), 66.

Kuester, E.F. and Harmuth, H.F. (1987). Comments on 'Correction of Maxwell's equations for signals I', 'Correction of Maxwell's equations for signals II', and 'Propagation velocity of electromagnetic signals'. *IEEE Trans. Electromagn. Compat.*, vol. EMC-29, 187–191.

Landau, L.D. and Lifschitz, E.M. (1966). *Lehrbuch der Theoretischen Physik* (German edition of Russian original by G.Heber). Akademie-Verlag, Berlin.

Leader, E. and Predazzi, E. (1996). *An Introduction to Gauge Theories and Modern Particle Physics.* Cambridge University Press, Cambridge.

Lifschitz, E.M. and Pitajewski, L.P. (1973). *Relativistische Quantentheorie*, vol. IVb of *Lehrbuch der Theoretischen Physik* (L.D.Landau and E.M.Lifschitz eds.); transl. from Russian. Akademie Verlag, Berlin.

Lobachevskii, N.I. (1840). *Geometrische Untersuchungen zur Theorie der Parallellinien.* G.Finke, Berlin. Reprinted Mayer und Müller, Berlin 1887. English translation in the appendix of Bonola (1955).

Lobachevskii, N.I. (1856). *Pangéometrie; ou, Précis de Géométrie Fondée sur une Théorie Générale et Rigoureuse des Parallèls.* Kasan University (Reprinted by Herman, Paris 1905).

Lurié, D. (1968). *Particles and Fields.* Interscience Publishers, New York.

Matteucci, G. and Pozzi, G. (1985). New diffraction experiment on the electrostatic Aharonov-Bohm effect. *Phys. Rev. Lett.*, vol. 54, 2469–2472.

Messiah, A. (1970). *Quantum Mechanics*, 2 vols. (transl. by G.M.Temmer of *Méchanique Quantique*, Dunod, Paris), North Holland Publishing Co., Amsterdam.

Milonni, P.W. (1994). *The Quantum Vacuum.* Academic Press, New York.

Möllenstedt, G. and Bayh, W. (1962). Messung der kontinuierlichen Phasenverschiebung von Elektronenwellen im kraftfreien Raum durch das magnetische Vektorpotential einer Luftspule. *Naturwissenschaften*, vol. 49, 81–82.

Morgenthau, H. (1945). *Germany Is Our Problem.* Harper & Brothers, New York and London.

Olariu, S. and Popescu, I.I. (1985). The quantum effect of electromagnetic fluxes. *Rev. Mod. Phys.*, vol. 157, 349–436.

Ottesen, J.T. (1995). *Infinite Dimensional Groups and Algebras in Quantum Physics.* Springer Verlag, Berlin.

Pearl, J. (2000). *Causality—Models, Reasoning, and Inference.* Cambridge University Press, Cambridge.

Peshkin, M. (1981). The Aharonov-Bohm effect: why it cannot be eliminated from quantum mechanics. *Physics Reports*, vol. 80, 375–386.

Peshkin, M. and Tonomura, A. (1989). *The Aharonov-Bohm Effect.* Springer-Verlag, New York.

Peskin, M.E. and Schroeder, D.V. (1995). *An Introduction to Quantum Field Theory.* Addison Wesley Publishers, Reading, Massachusetts.

Ramond, P. (1981). *Field Theory: A Modern Primer.* Benjamin, New York.

Reitz, J.R., Milford, F.J. and Christy, R.W. (1979). *Foundations of Electromagnetic Theory*, 3rd ed. Addison Wesley Publishing Co., Reading, Massachusetts.

Riemann, B. (1854). Über die Hypothesen, welche der Geometrie zu Grunde liegen. In *Gesammelte Mathematische Werke*, H.Weber ed., 272–287. Teubner, Leipzig 1892.

Roepstorff, G. (1994). *Pfadintegrale in der Quantenphysik.* Springer-Verlag, Berlin.

Ryder, L.H. (1996). *Quantum Field Theory*, 2nd ed. Cambridge University Press, Cambridge.

Salmhofer, M. (1999). *Renormalization: An Introduction.* Springer-Verlag, Berlin.

Scharf, G. (1995). *Finite Quantum Electrodynamics: The Causal Approach*, 2nd ed. Springer-Verlag, Berlin.

Smirnov, W.I. (1961). *Lehrgang der Höheren Mathematik*, 4th ed. (transl. of the 12th Russian edition). Deutscher Verlag der Wissenschaften, Berlin. English edition: *A Course in Higher Mathematics.* Pergamon Press, Oxford 1964.

Steinmann, O. (2000). *Perturbation Quantum Electrodynamics and Axiomatic Field Theory.* Springer-Verlag, Berlin.

Stratton, J.A. (1941). *Electromagnetic Theory.* McGraw-Hill, New York.

Tian, Y.C. (1999). *Conceptual Foundations of Quantum Field Theory.* Cambridge University Press, Cambridge.

Toll, J.S. (1956). Causality and the dispersion relation: Logical foundations. *Phys. Rev.*, vol. 104, No. 6, 1760–1770.

Tonomura, A., Matsuda, T., Suzuki, R., Fukuhara, A., Osakabe, N., Umezaki, H., Endo, J., Shinagawa, K., Sugita, Y., and Fujiwara, H. (1982). Observation of Aharonov-Bohm effect by electron microscopy. *Phys. Rev. Lett.*, vol. 48, 1443–1446.

Tonomura, A., Umezaki, H., Matsuda, T., Osakabe, N., Endo, J., and Sugita, Y. (1983). Is magnetic flux quantized in a torroidal ferromagnet? *Phys. Rev. Lett.*, vol. 51, 331–334.

Tonomura, A., Osakabe, N., Matsuda, T., Kawasaki, T., Endo, J., Yano, S., and Yamada, H. (1986). Evidence for Aharonov-Bohm effect with magnetic field completely shielded from electron wave. *Phys. Rev. Let.*, vol. 56, 792–795.

Tonomura, A. and Callen, E. (1987). Phase, electron holography, and a conclusive demonstration of the Aharonov-Bohm effect. *ONRE Sci. Bul.*, vol. 12, 93–104.

Tsvelik, A.M. (1996). *Quantum Field Theory in Condensed Matter Physics.* Cambridge University Press, Cambridge.

Weinberg, S. (1995). *The Quantum Theory of Fields: I. Foundations.* Cambridge University Press, Cambridge.

Weinberg, S. (1996). *The Quantum Theory of Fields: II. Modern Applications.* Cambridge University Press, Cambridge.

Weinberg, S. (2000). *The Quantum Theory of Fields: III. Supersymmetry.* Cambridge University Press, Cambridge.

Weisskopf, V. and Wigner, E. (1930a). Berechnung der natürlichen Linienbreite auf Grund der Diracschen Lichttheorie. *Zeitschrift für Physik* **63**, 54–73.

Weisskopf, V. and Wigner, E. (1930b). Über die natürliche Linienbreite in der Strahlung des harmonischen Oszillators. *Zeitschrift für Physik* **65**, 18–29.

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MODIFIED MAXWELL EQUATIONS IN QUANTUM ELECTRODYNAMICS

Divergencies in quantum field theory referred to as "infinite zero-point energy" have been a problem for 70 years. Renormalization has always been considered an unsatisfactory remedy.

In 1985 it was found that Maxwell's equations generally do not have solutions that satisfy the causality law. An additional term for magnetic dipole currents corrected this shortcoming. Rotating magnetic dipoles produce magnetic dipole currents, just as rotating electric dipoles in a material like barium titanate produce electric dipole currents. Electric dipole currents were always part of Maxwell's equations.

This book shows that the correction of Maxwell's equations eliminates the infinite zero-point energy in quantum electrodynamics. In addition, it presents many more new results.

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