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Contents

CHAPTER 1

Introduction

Quaternionic analysis is the most natural and close generalization of complex analysis that preserves many of its important features. In thirties and forties of the last century in the works of G. Moisil, N. Theodoresco, R. Fueter and other authors this fact started to emerge. Even earlier in [**60**] it was shown that the Dirac equation can be rewritten as a quaternionic equation, and in [**38**] it was observed that the Dirac equation for a massless field can be represented as a condition of analyticity for a function of a quaternionic variable. Later a similar observation was made with respect to Maxwell's equations for a vacuum. An example of a theory developed on the base of this fact can be found in [**37**]. In spite of such clear evidence of the applicability of quaternionic analysis to first order systems of mathematical physics, it was not until quite recently that applications of quaternionic analysis started to gain a wider popularity due to a considerable extent to the book [**31**] as well as to [**6**]. An even wider spectrum of problems from mathematical physics was considered in more recent books [**35, 58, 32**], from the classical equations of elasticity to quark confinement, and from applied electrodynamics to relativistic quantum theory. In principle, the whole building which the equations of mathematical physics inhabit can be erected on the foundations of quaternionic analysis, and this possibility represents some interest due to the lightness and transparency especially of the highest floors of that new building as well as due to high speed horizontal (apart from the vertical) movement allowing an extremely valuable communication between its different parts. Nevertheless the current major interest may be the tools of quaternionic analysis which permit results to be obtained where other "more traditional" methods apparently fail. Examples can be found in the books mentioned above.

The present small book was thought of both as a continuation of the theory developed in [**58**], and as a friendlier introduction to this theory (in fact its writing was inspired by a course given by the author to graduate engineering students). Finally, restricting ourselves only to Maxwell's equations and the Dirac equation we show the progress achieved in applied quaternionic analysis during the last five years, emphasising results which it is not clear how to obtain using other methods. Thus, the main objective of this work is to introduce the reader to some topics of quaternionic analysis whose selection is motivated by particular models from the theory of electromagnetic and spinor fields considered here, and to show the usefulness and necessity of applying the quaternionic analytic tools to these kinds of problems.

Chapter 2 is a really brief familiarization with these topics where together with quite well known facts which can be found for example in [**58**] some very recent results, especially those corresponding to integral representations in unbounded domains, are included.

In Chapter 3 we consider some problems for the Maxwell equations in homogeneous media and for the Dirac equation for a free particle. We start with the well known quaternionic reformulation of the time-dependent Maxwell equations and show how this can be used to obtain the solution for the classical problem of a moving source. Then we consider some boundary value problems for the time-harmonic Maxwell equations as the problem of analytic extension of the electromagnetic field and demonstrate that its complete analysis has become possible due to the quaternionic approach. We show how the same technique works in the case of electromagnetic fields in chiral media, and for the Dirac equation for a free particle. It should be mentioned that the quaternionic reformulation of the Dirac equation used here does not coincide with that proposed in [**60**]. We explain the difference and advantages of our reformulation.

Chapter 4 is dedicated to the Maxwell equations for inhomogeneous media and to the Dirac equation with potentials. In spite of a relatively large number of publications on the quaternionic approach to Maxwell's system, even the question as to how to write the Maxwell equations for arbitrary inhomogeneous media in a compact quaternionic form remained open until recently. In [**49**] such a reformulation was proposed in the case of a time-harmonic electromagnetic field and in [**48**] for the time-dependent case. Here we make one additional step which leads us to Maxwell's system in the form of a single quaternionic equation. As is shown in the same chapter its form is similar to the quaternionic reformulation of the Dirac equation with potentials. For both equations we construct exact solutions using the algebraic advantages of complex quaternions and the relatively simple form of the equations. In the case of a static electromagnetic field in an inhomogeneous medium we show that the solution of the Maxwell equations can be obtained using the fact that the corresponding quaternionic operator factorizes the Schrödinger operator. This factorization is closely related to a nonlinear quaternionic equation which can be considered as a spatial generalization of the ordinary differential Riccati equation. The traditional Riccati equation is distinguished by its numerous applications as well as by some curious and unique properties. If a particular solution of the Riccati equation is known, the equation can be linearized. Given two particular solutions, the general solution can be found in one integration. These two facts were proved by Euler. Other interesting properties were discovered by Weyr and Picard and correspond to three and four particular solutions. We give the generalizations of all these facts for the quaternionic Riccati equation.

Another special case, that of so-called slowly changing media, is discussed separately. We show how in this case using only algebraic properties of complex quaternions explicit solutions can be obtained. The same approach leads us to exact solutions for the Dirac equation with scalar, pseudoscalar or electric potential. In the case of for example electric potentials, this works only for massless fields, which is why in the last section we propose another method for obtaining exact solutions of the Dirac equation with electric potential. This possibility is also based only on algebraic properties of complex quaternions.

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CHAPTER 2

Elements of quaternionic analysis

2.1. Quaternions

An ample class of problems from electrodynamics, wave propagation, quantum mechanics, elasticity theory and many other fields of contemporary science can be reduced to two-dimensional models. This is possible in cases in which we have some additional useful information about geometrical features of the physical process under consideration. For instance, when only radial waves are considered in a propagation phenomenon or when a cylindrical wave guide is studied we can reduce the problem with some well known tricks to a model of fewer dimensions. Why such a reduction is of great importance is quite clear. In numerical analysis the reduction of dimension of the problem can be vital for the precision of its solution. But also for analytical study of the problem the reduction to two dimensions can signify the possibility of its exact solution being found, because of the existence of such a powerful tool as complex analysis which allows us to employ a variety of well developed methods for solving boundary value problems. The root of the power of complex analysis consists in the algebraic properties of complex numbers. Roughly speaking, the introduction of the complex imaginary unit i allows us to multiply any two points in the plane in such a way that multiplication together with addition generates an algebra, that is all axioms of algebra are true: the commutative law, the distributive law, etc.

Nevertheless it is obvious that an overwhelming majority of physically meaningful problems cannot be reduced to two-dimensional models. Usually, in such cases vector calculus is applied, but the difference between the power of complex and vectorial analysis is vast. The vector product does not permit the formation of an algebra. It is not commutative, nor associative. There exists no element which could be called a unit, and it is impossible to divide. In the nineteenth century various attempts were made to construct an algebra in three-dimensional vector space until finally W.R. Hamilton in 1843 found out that it is instead necessary to consider the space of four dimensions where the algebra of quaternions had been waiting for the inquiring researchers. The quaternions enjoy the same properties as the complex numbers with one exception. The commutative law is not valid. This is an important loss which complicates quaternionic arithmetic, but as we will see on the subsequent pages we will actively use the non-commutative peculiarity of quaternions and in some cases it will be the key for obtaining the result.

A quaternion may be regarded as a 4-tuple of real numbers or in other words as an element of $\mathbb{R}^4$ which then is represented as a linear combination of the elements of the standard orthonormal basis:

$$q = q_0 i_0 + q_1 i_1 + q_2 i_2 + q_3 i_3. \tag{2.1.1}$$

Here q_0, q_1, q_2 and q_3 are real numbers and called the components of the quaternion q. We will say that two quaternions are equal if and only if they have exactly the same components:

$$p = q \qquad \text{iff} \qquad p_k = q_k, \quad k = \overline{0,3}.$$

The sum of two quaternions p and q is defined by adding the corresponding components:

$$p+q=\sum_{k=0}^{3}(p_k+q_k)i_k. \tag{2.1.2}$$

Until now we have been considering only the definitions that do not distinguish the quaternions from vectors from $\mathbb{R}^4$. The concept of quaternions properly begins with the definition of quaternionic multiplication. The element i_0 is regarded as the usual unit, that is $i_0 = 1$. Nevertheless sometimes the notation i_0 will be more convenient, e.g., in formulas which include summation such as (2.1.2). The other three basic elements are regarded as imaginary units:

$$i_1^2 = i_2^2 = i_3^2 = -1. \tag{2.1.3}$$

The products of different elements of the basis are defined in the following way

$$i_1 \cdot i_2 = -i_2 \cdot i_1 = i_3, \tag{2.1.4}$$

$$i_2 \cdot i_3 = -i_3 \cdot i_2 = i_1, \tag{2.1.5}$$

$$i_3 \cdot i_1 = -i_1 \cdot i_3 = i_2. \tag{2.1.6}$$

Equalities (2.1.3)-(2.1.6) completely define the multiplication of quaternions. We have

$$\begin{aligned}
p\cdot q &= (p_0i_0+p_1i_1+p_2i_2+p_3i_3)(q_0i_0+q_1i_1+q_2i_2+q_3i_3) = \\
&= (p_0q_0-p_1q_1-p_2q_2-p_3q_3)i_0+ \\
&(p_1q_0+p_0q_1+p_2q_3-p_3q_2)i_1+ \\
&(p_2q_0+p_0q_2+p_3q_1-p_1q_3)i_2+ \\
&(p_3q_0+p_0q_3+p_1q_2-p_2q_1)i_3.
\end{aligned} \tag{2.1.7}$$

Very often the vectorial representation of a quaternion is used. Namely, if $q = \sum_{k=0}^{3} q_k i_k$, then q_0 (which is the same as $q_0 i_0$) is called the scalar part of q and denoted as $\text{Sc}(q)$ and $\sum_{k=1}^{3} q_k i_k$ is called the vector part of q and denoted as $\text{Vec}(q)$ or $\overrightarrow{q}$. Each quaternion q is a sum of a scalar q_0 and of a vector $\overrightarrow{q}$:

$$q = \text{Sc}(q) + \text{Vec}(q) = q_0 + \overrightarrow{q}.$$

If $\text{Sc}(q) = 0$ then $q = \overrightarrow{q}$ is called a purely vectorial quaternion. Let us notice that the basic quaternionic imaginary units i_1, i_2 and i_3 can be identified with the basic coordinate vectors in a three-dimensional space. In this way we identify a vector from $\mathbb{R}^3$ with a purely vectorial quaternion with the same components.

Using the definitions of the scalar and vector products we can rewrite the quaternionic product (2.1.7) in a vector form:

$$p \cdot q = p_0 q_0 - < \vec{p}, \vec{q} > + p_0 \vec{q} + \vec{p} q_0 + [\vec{p} \times \vec{q}]. \tag{2.1.8}$$

Here

$$< \vec{p}, \vec{q} > = p_1 q_1 + p_2 q_2 + p_3 q_3$$

and

$$[\vec{p} \times \vec{q}] = \begin{vmatrix} i_1 & i_2 & i_3 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix}$$

as usual.

Note that

$$\text{Sc}(p \cdot q) = p_0 q_0 - < \vec{p}, \vec{q} >$$

and

$$\text{Vec}(p \cdot q) = p_0 \vec{q} + \vec{p} q_0 + [\vec{p} \times \vec{q}].$$

An important corollary of (2.1.8) is that in general the product of two purely vectorial quaternions is a complete quaternion whose scalar part is not zero but is equal to the scalar product of the two vectors with the sign minus

$$\overrightarrow{p} \cdot \overrightarrow{q} = -<\vec{p},\vec{q}> + [\vec{p} \times \vec{q}].$$

Thus, $\mathrm{Sc}(\overrightarrow{p} \cdot \overrightarrow{q}) = 0$ iff the vectors $\overrightarrow{p}$ and $\overrightarrow{q}$ are orthogonal, and $\mathrm{Vec}(\overrightarrow{p} \cdot \overrightarrow{q}) = 0$ iff they are colinear. Note that

$$\overrightarrow{p}^{\,2} = \overrightarrow{p} \cdot \overrightarrow{p} = -<\vec{p},\vec{p}> = -\left|\vec{p}\right|^2. \tag{2.1.9}$$

Let us define the conjugate of the quaternion $q = q_0 + \overrightarrow{q}$ to be the quaternion $\overline{q} = q_0 - \overrightarrow{q}$. A simple calculation gives us the following important property of the quaternionic conjugation:

$$\overline{p \cdot q} = \overline{q} \cdot \overline{p}.$$

From (2.1.8) we immediately obtain that

$$q \cdot \overline{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2 =: \left|q\right|^2. \tag{2.1.10}$$

This real number represents the square of the distance from the origin of the point with coordinates (q_0, q_1, q_2, q_3) in the Euclidean space of four dimensions. Thus, in order to factorize the squared distance in two dimensions we need the complex imaginary unit i. In three or four dimensions we need three imaginary units i_1, i_2 and i_3.

One more important conclusion which can be reached using (2.1.10) is that each non-zero quaternion q is invertible and its inverse is given by

$$q^{-1} = \frac{\overline{q}}{\left|q\right|^2}. \tag{2.1.11}$$

Note that $|p \cdot q| = |p| \cdot |q|$, as

$$|p \cdot q|^2 = pq \cdot \overline{pq} = pq\overline{q} \cdot \overline{p} = p\,|q|^2\,\overline{p} = p\overline{p} \cdot |q|^2 = |p|^2 \cdot |q|^2 .$$

To finish this introduction to quaternionic arithmetic we should mention that all laws of algebra are valid in the case of quaternions with a unique exception: quaternionic multiplication is not commutative. Thus, in algebraic terms the quaternions form a skew field which we will denote by $\mathbb{H}(\mathbb{R})$. The letter $\mathbb{H}$ is frequently chosen in honour of the inventor of quaternions, and $\mathbb{R}$ here indicates that we consider the real quaternions and not the complex ones, which will be introduced in the next section.

2.2. Complex quaternions

If in the definition of quaternion (2.1.1) we suppose that all components can be complex (instead of real) numbers we arrive at the definition of complex quaternions (which are called also biquaternions). Thus, a complex quaternion q is an object of the form

$$q = q_0 i_0 + q_1 i_1 + q_2 i_2 + q_3 i_3,$$

where q_0, q_1, q_2 and q_3 are complex numbers. In order to complete this definition we must introduce an additional law of multiplication. We establish the commutation rule for the usual complex imaginary unit i with the quaternionic imaginary units i_k, $k = 1, 2, 3$. We define this as follows

$$i \cdot i_k = i_k \cdot i, \quad k = 1, 2, 3.$$

That is, i commutes with the quaternionic imaginary units. Although such a rule seems the most natural, very often i is supposed to anticommute with i_k, $k = 1, 2, 3$. In this way one obtains the algebra of octonions (Cayley numbers). As real vector spaces the set of complex quaternions and the set of octonions are isomorphic, but their algebraic properties are very different. The octonions form a division algebra, - for each non-zero element there exists an inverse. Nevertheless the price for such a wonderful property is the associativity. The algebra of octonions is not associative. The complex quaternions, on the contrary, enjoy the property of associativity but instead, as we will see further on, there exist non-zero elements which do not have inverses. In these lectures we will use the algebra of complex quaternions referring the reader interested in octonions to the books [**73, 39, 25, 35, 83**]. The algebra of complex quaternions will be denoted by $\mathbb{H}(\mathbb{C})$. Note that any $q \in \mathbb{H}(\mathbb{C})$ can be represented as follows $q = \operatorname{Re} q + i \operatorname{Im} q$, where $\operatorname{Re} q = \sum_{k=0}^{3} \operatorname{Re} q_k i_k$ and $\operatorname{Im} q = \sum_{k=0}^{3} \operatorname{Im} q_k i_k$ belong to $\mathbb{H}(\mathbb{R})$.

Let us consider the complex quaternion $q = 1 + ii_1$ and its conjugate $\overline{q} = 1 - ii_1$. Their product gives us

$$q \cdot \overline{q} = (1 + ii_1)(1 - ii_1) = 1 - 1 = 0. \tag{2.2.1}$$

We have two elements different from zero, whose product is equal to zero. In general, if the product of two elements a and b is equal to zero but a and b are not zero, then a and b are called zero divisors. Let us denote the set of all zero divisors in $\mathbb{H}(\mathbb{C})$ by $\mathfrak{S}$:

$$\mathfrak{S} := \{a \in \mathbb{H}(\mathbb{C}) \,|a \neq 0; \exists b \neq 0 : a \cdot b = 0\}\,. \tag{2.2.2}$$

It is important to characterize the subset $\mathfrak{S}$. How one can know a priori if some given quaternion q is a zero divisor? First of all let us note that if $a \in \mathfrak{S}$ then a^{-1} does not exist. The proof of this statement is very simple. We assume that $a \in \mathfrak{S}$, that is, there exists such $b \neq 0$ that $a \cdot b = 0$ and we assume that a^{-1} exists. Then $a^{-1}a = 1$. We multiply this equality by b: $a^{-1}ab = b$, from which we immediately obtain the contradiction $0 = b$. Thus, zero divisors are not invertible. The following lemma gives us a more detailed description of $\mathfrak{S}$.

LEMMA 1. [**58**]*(Structure of the set of zero divisors) Let $a \in \mathbb{H}(\mathbb{C})$ and $a \neq 0$. The following statements are equivalent:*

(1) $a \in \mathfrak{S}$.
(2) $a \cdot \overline{a} = 0$.
(3) $a_0^2 = \overrightarrow{a}^2$.
(4) $a^2 = 2a_0 a = 2\overrightarrow{a}a$.

PROOF. First, let us show the equivalence of 1. and 2. If 2. holds, then we can choose $b = \overline{a}$ and the definition of $\mathfrak{S}$ (2.2.2) is satisfied. If 2. does not hold, i.e., $a \cdot \overline{a} \neq 0$, then there exists a^{-1} defined by $a^{-1} = \overline{a}/|a|^2$ and this means that $a \notin \mathfrak{S}$. Thus, 1. and 2. are equivalent.

The equivalence of 2. and 3. follows immediately from the definition of quaternionic conjugation:

$$a \cdot \overline{a} = 0 \quad \Leftrightarrow \quad (a_0 + \overrightarrow{a})(a_0 - \overrightarrow{a}) = 0 \quad \Leftrightarrow \quad a_0^2 = \overrightarrow{a}^2.$$

To prove the equivalence of 3. and 4. we consider the square of a:

$$a^2 = a_0^2 + 2a_0\overrightarrow{a} + \overrightarrow{a}^2.$$

Then we can see that $a_0^2 = \overrightarrow{a}^2 \quad \Leftrightarrow \quad a^2 = 2\overrightarrow{a}^2 + 2a_0\overrightarrow{a} = 2\overrightarrow{a}a$ and $a_0^2 = \overrightarrow{a}^2 \quad \Leftrightarrow \quad a^2 = 2a_0^2 + 2a_0\overrightarrow{a} = 2a_0 a$. □

REMARK 1. *If $a \in \mathfrak{S}$ and $a_0 \neq 0$ then according to 4. from the preceding lemma the complex quaternion $c := \frac{1}{2a_0}a$ is idempotent, that is $c^2 = c$.*

As we can observe already, the modulus introduced by (2.1.10) in the case of the complex quaternions in general does not give information about the absolute values of their components, see (2.2.1). This is why another kind of modulus is used frequently. We denote by $|q|_c$ the following real number

$$|q|_c := \sqrt{|q_0|^2 + |q_1|^2 + |q_2|^2 + |q_3|^2}, \tag{2.2.3}$$

where $|q_k|^2 = q_k q_k^*$ and "*" stands for the usual complex conjugation. (2.2.3) represents a natural Euclidean metric in $\mathbb{R}^8$ and can be expressed also in the following manner

$$|q|_c^2 = |\operatorname{Re} q|^2 + |\operatorname{Im} q|^2 = \operatorname{Sc}(q \cdot \overline{q}^*) = \operatorname{Sc}(\overline{q}^* \cdot q).$$

Note that $\operatorname{Sc}(p \cdot q) = \operatorname{Sc}(q \cdot p)$ for any $p, q \in \mathbb{H}(\mathbb{C})$.

In general, $|p \cdot q|_c \neq |p|_c \cdot |q|_c$ and even $|p \cdot q|_c$ can be greater than the product $|p|_c \cdot |q|_c$.

Example 1. *Let $p = q = 1 + ii_1$. Then*

$$|p \cdot q|_c = 2\,|1 + ii_1|_c = 2\sqrt{2},$$

but

$$|p|_c \cdot |q|_c = 2.$$

The following simple statement gives us an important estimate.

Lemma 2. *Let p and q be complex quaternions. Then*

$$|p \cdot q|_c \leq \sqrt{2}\,|p|_c \cdot |q|_c\,.$$

Proof. We denote $a := \operatorname{Re} p$, $b := \operatorname{Im} p$, $c := \operatorname{Re} q$, $d := \operatorname{Im} q$. Then

$$|p \cdot q|_c^2 = |(a+ib)(c+id)|_c^2 = |ac - bd + i(bc+ad)|_c^2 =$$
$$= |ac-bd|^2 + |bc+ad|^2 \leq 2(|ac|^2 + |bd|^2 + |bc|^2 + |ad|^2) =$$
$$= 2(|a|^2+|b|^2)(|c|^2+|d|^2) = 2\,|p|_c^2 \cdot |q|_c^2 .$$

□

2.3. Complex quaternionic functions

We will consider functions depending on three or four independent real variables and taking their values in the algebra of complex quaternions, that is functions

$$f : \mathbb{R}^3 \to \mathbb{H}(\mathbb{C})$$

or

$$f : \mathbb{R}^4 \to \mathbb{H}(\mathbb{C}).$$

Such functions will be called complex quaternionic or biquaternionic functions. Let Ω be some domain in $\mathbb{R}^3$ or $\mathbb{R}^4$ and $\mathcal{B}(\Omega)$ some Banach space of complex valued functions defined in Ω, for instance, $C(\Omega)$ or $L_p(\Omega)$. Through the whole book we assume that a complex quaternionic function f belongs to the space $\mathcal{B}(\Omega; \mathbb{H}(\mathbb{C}))$ iff each component f_k of f belongs to $\mathcal{B}(\Omega)$ where the corresponding norm for f is calculated in the following way

$$\|f\|_{\mathcal{B}} = \left(\sum_{k=0}^{3} \|f_k\|_{\mathcal{B}}^2 \right)^{1/2} .$$

If $\mathcal{B}(\Omega)$ is a Banach space then the space $\mathcal{B}(\Omega; \mathbb{H}(\mathbb{C}))$ defined in this manner is also a (complex linear) Banach space.

2.4. The Moisil-Theodoresco differential operator

Let us introduce the following notation which will be used throughout these lectures

$$\partial_k := \frac{\partial}{\partial x_k},$$

and let $f \in C^1(\Omega; \mathbb{H}(\mathbb{C}))$. Then the following operator

$$Df := \sum_{k=1}^{3} i_k \partial_k f$$

is called the Moisil-Theodoresco operator. Let us consider the action of the operator D more explicitly

$$\begin{aligned} Df &= (i_1\partial_1 + i_2\partial_2 + i_3\partial_3)(f_0 + f_1 i_1 + f_2 i_2 + f_3 i_3) = \\ &= (i_1\partial_1 f_0 + i_2\partial_2 f_0 + i_3\partial_3 f_0) - (\partial_1 f_1 + \partial_2 f_2 + \partial_3 f_3) + \\ &+ ((\partial_2 f_3 - \partial_3 f_2)i_1 + (\partial_3 f_1 - \partial_1 f_3)i_2 + (\partial_1 f_2 - \partial_2 f_1)i_3). \end{aligned}$$

The expression in the first brackets is precisely the gradient of the function f_0:

$$\operatorname{grad} f_0 = (i_1\partial_1 + i_2\partial_2 + i_3\partial_3) f_0.$$

The second brackets contain the divergence of the vector $\overrightarrow{f}$:

$$\operatorname{div} \overrightarrow{f} = \partial_1 f_1 + \partial_2 f_2 + \partial_3 f_3.$$

Finally, the third brackets represent the rotational of $\overrightarrow{f}$:

$$\operatorname{rot} \overrightarrow{f} = (\partial_2 f_3 - \partial_3 f_2)i_1 + (\partial_3 f_1 - \partial_1 f_3)i_2 + (\partial_1 f_2 - \partial_2 f_1)i_3.$$

Consequently,

$$Df = -\operatorname{div} \overrightarrow{f} + \operatorname{grad} f_0 + \operatorname{rot} \overrightarrow{f}. \tag{2.4.1}$$

This equality needs some explanation, because if it were written by a student in an exam on vector calculus the result would be deplorable. What does the sum of the vector $\operatorname{grad} f_0$ and the scalar $-\operatorname{div} \overrightarrow{f}$ mean?

What would be impossible in vector calculus, in quaternionic analysis has a very simple and natural meaning. The result of the action of the operator D on the biquaternionic function f is a complex quaternion whose scalar part is equal to $-\operatorname{div}\overrightarrow{f}$ and whose vector part is the sum $\operatorname{grad} f_0 + \operatorname{rot}\overrightarrow{f}$:

$$\operatorname{Sc}(Df) = -\operatorname{div}\overrightarrow{f},$$

$$\operatorname{Vec}(Df) = \operatorname{grad} f_0 + \operatorname{rot}\overrightarrow{f}.$$

We see that all three principal differential operators from vector calculus are contained in the quaternionic operator D. If they are considered separately then many important features are lost, but their combination D allows us to obtain appropriate generalizations of the most of basic facts from complex analysis.

The equation

$$Df = 0 \tag{2.4.2}$$

is equivalent to the system

$$\operatorname{div}\overrightarrow{f} = 0, \tag{2.4.3}$$

$$\operatorname{grad} f_0 + \operatorname{rot}\overrightarrow{f} = 0, \tag{2.4.4}$$

called the Moisil-Theodoresco system. It was studied for the first time in [**65**] which marked the starting point in the development of hypercomplex function theory. Since then the Moisil-Theodoresco system has been considered in hundreds of works (here are only some books [**16, 26, 31, 32, 58, 68**]). Such interest was provoked in some part by some physical applications of (2.4.3), (2.4.4), but mostly because the Moisil-Theodoresco system posessed such important properties that it was regarded as the most natural generalization of the

Cauchy-Riemann system to the space of three dimensions. It became clear what the Cauchy integral theorem, Morera's theorem, Cauchy's integral formula, Plemelj-Sokhotski's formulas and many others are in three dimensions.

Note that using (2.1.9) the following property of the operator D is obtained

$$D^2 = -\Delta, \tag{2.4.5}$$

where $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the usual Laplace operator. This property guarantees that each component of a function f satisfying (2.4.2) is a harmonic function.

The following generalization of Leibniz's rule can be proved by a direct calculation (see [**31**, p. 24]).

THEOREM 1. *(Generalized Leibniz rule) Let* $\{f, g\} \subset C^1(\Omega; \mathbb{H}(\mathbb{C}))$, *where* Ω *is some domain in* $\mathbb{R}^3$. *Then*

$$D[f \cdot g] = D[f] \cdot g + \overline{f} \cdot D[g] + 2(\mathrm{Sc}(fD))[g], \tag{2.4.6}$$

where

$$(\mathrm{Sc}(fD))[g] := -\sum_{k=1}^{3} f_k \partial_k g.$$

An immediate corollary of (2.4.6) is that even if $Df = Dg = 0$ it does not imply that $D[f \cdot g] = 0$ also.

We will actively use the following

REMARK 2. *If in Theorem 1* $\mathrm{Vec}(f) = 0$, *that is* $f = f_0$, *then*

$$D[f_0 \cdot g] = D[f_0] \cdot g + f_0 \cdot D[g]. \tag{2.4.7}$$

From this equality we obtain that the operator $D+\frac{\operatorname{grad} f_0}{f_0}$ *can be factorized as follows*

$$(D+\frac{\operatorname{grad} f_0}{f_0})g = f_0^{-1}D(f_0\cdot g). \tag{2.4.8}$$

Let us note that the Moisil-Theodoresco operator was introduced as acting from the left-hand side. The corresponding operator acting from the right-hand side we will denote D_r:

$$D_r f := \sum_{k=1}^{3} \partial_k f i_k. \tag{2.4.9}$$

In vector form the application of D_r can be represented as follows

$$D_r f = -\operatorname{div}\overrightarrow{f} + \operatorname{grad} f_0 - \operatorname{rot}\overrightarrow{f}$$

(cf. (2.4.1)) and all the corresponding theory can be developed for D_r exactly as for D.

2.5. The operator $D+\alpha I$

In this section we consider the operator $D_\alpha := D+\alpha I$, where α is an arbitrary complex constant and I is the identity operator. The first work in which this operator (for α real) was studied, was by K.Gürlebeck [**30**]. As we will see in the subsequent pages, the addition of α allows us to widen the spectrum of possible applications of the quaternionic analysis techniques under consideration. Having them in mind we will assume that

$$\operatorname{Im}\alpha \geq 0. \tag{2.5.1}$$

As will be clear later, physically α represents the wave number which usually is chosen to satisfy (2.5.1).

2.5.1. Factorization of the Helmholtz operator and fundamental solutions. The operator D_α is closely related to the Helmholtz operator $\Delta + \alpha^2 I$ because of the following factorization

$$\Delta + \alpha^2 = -(D + \alpha)(D - \alpha) = -D_\alpha D_{-\alpha} \tag{2.5.2}$$

which is a corollary of (2.4.5). The equality (2.5.2) means that any function satisfying the equation

$$D_\alpha f = 0 \tag{2.5.3}$$

or

$$D_{-\alpha} f = 0 \tag{2.5.4}$$

also satisfies the Helmholtz equation

$$(\Delta + \alpha^2) f = 0. \tag{2.5.5}$$

In other words, each component of the quaternionic function f satisfying (2.5.3) or (2.5.4) is also a solution of the Helmholtz equation.

Another important corollary of (2.5.2) is the possibility of the calculation of the fundamental solutions of the operators D_α and $D_{-\alpha}$. Suppose that ϑ is a fundamental solution of the Helmholtz operator:

$$(\Delta + \alpha^2)\vartheta = \delta.$$

Then using (2.5.2) we obtain that the function

$$\mathcal{K}_\alpha := -(D - \alpha)\vartheta \tag{2.5.6}$$

is a fundamental solution of D_α and the function

$$\mathcal{K}_{-\alpha} := -(D + \alpha)\vartheta \tag{2.5.7}$$

is a fundamental solution of $D_{-\alpha}$, that is

$$D_{\pm\alpha} \mathcal{K}_{\pm\alpha} = \delta.$$

Normally, the election of a unique fundamental solution is related to its physical meaning. In the case of the Helmholtz operator the additional assumption (2.5.1) leaves no choice (see the discussion on p. 27). The fundamental solution

$$\vartheta(x) = -\frac{e^{i\alpha|x|}}{4\pi\,|x|} \tag{2.5.8}$$

chosen in this case represents an outgoing wave (decreasing at infinity) generated by a point source situated at the origin. Another possible candidate, the distribution $-e^{-i\alpha|x|}/(4\pi\,|x|)$, if $\operatorname{Im}\alpha > 0$, increases exponentially at infinity and for this reason does not serve for describing fields produced by sources in a finite part of space. As will be seen later, the problem is to "distinguish" the behavior at infinity of these two fundamental solutions in the case when $\operatorname{Im}\alpha = 0$. This difficulty is overcome with the aid of the so-called radiation condition. But everything is good in its season, and this discussion will be continued on p. 27. At this moment due to the physical substantiality of the solution (2.5.8) we choose it for constructing the fundamental solutions for the operators D_α and $D_{-\alpha}$. Substituting the function (2.5.8) into the equality (2.5.6) we obtain that

$$\mathcal{K}_\alpha(x) = -\operatorname{grad}\vartheta(x) + \alpha\vartheta(x) = (\alpha + \frac{x}{|x|^2} - i\alpha\frac{x}{|x|})\cdot\vartheta(x), \tag{2.5.9}$$

where $x := \sum_{k=1}^{3} x_k i_k$. Substituting (2.5.8) in equality (2.5.7) we find the fundamental solution of the operator $D_{-\alpha}$:

$$\mathcal{K}_{-\alpha}(x) = -\operatorname{grad}\vartheta(x) - \alpha\vartheta(x) = (-\alpha + \frac{x}{|x|^2} - i\alpha\frac{x}{|x|})\cdot\vartheta(x). \tag{2.5.10}$$

The functions (2.5.9) and (2.5.10) will play a crucial role in what follows. They were obtained in [**44**], see also [**58**, Section 3].

2.5.2. Integral representations in bounded domains. Now we will prove some important facts related with the integral representations of solutions of (2.5.3). Let us start with the following auxiliary theorem which is nothing but a quaternionic version of Stokes' formula. We assume that Ω is a bounded domain in $\mathbb{R}^3$ with a piecewise smooth boundary $\Gamma := \partial\Omega$.

THEOREM 2. *(Quaternionic Stokes' formula) Let f and g belong to $C^1(\Omega; \mathbb{H}(\mathbb{C})) \cap C(\overline{\Omega}; \mathbb{H}(\mathbb{C}))$. Then*

$$\int_\Omega (D_r f(y))g(y) + f(y)(Dg(y)))dy = \int_\Gamma f(y)\overrightarrow{n}(y)g(y)d\Gamma_y, \tag{2.5.11}$$

where $\overrightarrow{n}$ denotes the outward unitary normal on Γ in quaternionic form: $\overrightarrow{n} := \sum_{k=1}^3 n_k i_k$.

The proof of this fact can be found in [**58**, Chapter 4] or [**32**, p. 86].

COROLLARY 1. *Let $g \in C^1(\Omega; \mathbb{H}(\mathbb{C})) \cap C(\overline{\Omega}; \mathbb{H}(\mathbb{C}))$. Then*

$$\int_\Omega Dg(y)dy = \int_\Gamma \overrightarrow{n}(y)g(y)d\Gamma_y.$$

PROOF. In (2.5.11) f was chosen equal to 1. □

COROLLARY 2. *(Quaternionic Cauchy's integral theorem) Let $g \in C^1(\Omega; \mathbb{H}(\mathbb{C})) \cap C(\overline{\Omega}; \mathbb{H}(\mathbb{C}))$ and satisfy (2.5.3) in Ω. Then*

$$\int_\Gamma \overrightarrow{n}(y)g(y)d\Gamma_y = -\alpha \int_\Omega g(y)dy.$$

There is also an inverse result, the proof of which the reader can find in [**58**, p. 70]:

THEOREM 3. *(Quaternionic Morera theorem)*

Let $\alpha \in \mathbb{C}$, $g \in C^1(\Omega; \mathbb{H}(\mathbb{C}))$, $D_\alpha g \in L_p(\Omega; \mathbb{H}(\mathbb{C}))$ for some $p > 1$. If for any Liapunov manifold without boundary $\widehat{\Gamma}$ ($\widehat{\Gamma} = \partial\widehat{\Omega}$, $\widehat{\Omega} \subset \Omega$, $\widehat{\Gamma} \subset \Omega$) the following equality holds

$$\int_{\widehat{\Gamma}} \overrightarrow{n}(y) g(y) d\widehat{\Gamma}_y = -\alpha \int_{\widehat{\Omega}} g(y) dy,$$

then g is a solution of (2.5.3) in Ω.

The reader is reminded that a closed bounded surface Γ is called a Liapunov surface if in each point $x \in \Gamma$ there exists a normal $\overrightarrow{n}(x)$ satisfying the Hölder condition on Γ, that is there exist numbers $C > 0$ and $\epsilon > 0$, $\epsilon \leq 1$ such that

$$\left|\overrightarrow{n}(x) - \overrightarrow{n}(y)\right| \leq C \left|x - y\right|^{\epsilon}$$

for arbitrary $x, y \in \Gamma$. From this definition it follows that the class of Liapunov surfaces is contained in the set of C^1-surfaces, and each C^2-surface is necessarily a Liapunov surface. More information about Liapunov surfaces can be found, for instance, in [**82**, Section 27].

In what follows, unless stated otherwise, we assume that Γ is a Liapunov surface. Practically all results presented here for Liapunov boundaries can be generalized to the case of Lipshitz boundaries (see, e.g., [**64**] and the bibliography there). However this requires some technical complications unnecessary for explaining the main ideas in these lectures.

Now we use Theorem 2 in order to prove a generalization of Borel-Pompeiu's formula. Let us introduce the main integral operators which in their properties are very similar to their famous complex prototypes: the T-operator, Cauchy's integral operator and the operator of singular integration, which guarantee an efficient solution of different kinds of boundary value problems:

$$T_\alpha[f](x) := \int_\Omega \mathcal{K}_\alpha(x-y)f(y)dy, \quad x \in \mathbb{R}^3, \tag{2.5.12}$$

$$K_\alpha[f](x) := -\int_\Gamma \mathcal{K}_\alpha(x-y)\overrightarrow{n}(y)f(y)d\Gamma_y, \quad x \in \mathbb{R}^3 \setminus \Gamma, \tag{2.5.13}$$

$$S_\alpha[f](x) := -2\int_\Gamma \mathcal{K}_\alpha(x-y)\overrightarrow{n}(y)f(y)d\Gamma_y, \quad x \in \Gamma. \tag{2.5.14}$$

Note that the integral in (2.5.14) is considered in the sense of the Cauchy principal value.

As usual, the operator of singular integration generates two important operators

$$P_\alpha := \frac{1}{2}(I + S_\alpha) \quad \text{and} \quad Q_\alpha := \frac{1}{2}(I - S_\alpha). \tag{2.5.15}$$

In what follows we assume that Ω is an open bounded domain in $\mathbb{R}^3$ with a Liapunov boundary $\Gamma := \partial\Omega$.

THEOREM 4. *(Quaternionic Borel-Pompeiu formula)*
Let $f \in C^1(\Omega;\mathbb{H}(\mathbb{C})) \cap C(\overline{\Omega};\mathbb{H}(\mathbb{C}))$. *Then*

$$K_\alpha[f](x) + T_\alpha D_\alpha[f](x) = f(x), \quad \forall x \in \Omega.$$

PROOF. Let us consider the integral

$$T_\alpha D_\alpha[f](x) = \int_\Omega \mathcal{K}_\alpha(x-y)D_{\alpha,y}f(y)dy, \tag{2.5.16}$$

where the index y in $D_{\alpha,y}$ means differentiation with respect to y. The main idea of the proof, as will be seen later, is to apply Theorem 2 to the volume integral (2.5.16) in order to reduce it to a surface integral, but the problem here is that the function $\mathcal{K}_\alpha(x-y)$ does not match the conditions of Theorem 2. Namely, $\mathcal{K}_\alpha(x-y)$ is a C^1-function in the whole domain Ω excepted point $y = x$. Thus the first

Figure 1. Domain Ω_ϵ.

step of the proof is to cut out the point x together with a small ball $B_\epsilon := \{y \mid \ |x-y| \le \epsilon\}$ from Ω and consider the integral (2.5.16) as the following limit

$$\int_\Omega \mathcal{K}_\alpha(x-y) D_{\alpha,y} f(y) dy = \lim_{\epsilon \to 0} \int_{\Omega_\epsilon} \mathcal{K}_\alpha(x-y) D_{\alpha,y} f(y) dy, \tag{2.5.17}$$

where $\Omega_\epsilon := \Omega \diagdown B_\epsilon$ as shown in Fig. 1. Now the integral

$$\int_{\Omega_\epsilon} \mathcal{K}_\alpha(x-y) D_{\alpha,y} f(y) dy$$

is almost ready for the application of Theorem 2. We only need to transform it to some more suitable form:

$$\begin{aligned}
&\int_{\Omega_\epsilon} \mathcal{K}_\alpha(x-y) D_{\alpha,y} f(y) dy = \\
&= \int_{\Omega_\epsilon} \mathcal{K}_\alpha(x-y) D_y f(y) dy + \alpha \int_{\Omega_\epsilon} \mathcal{K}_\alpha(x-y) f(y) dy = \\
&= \int_{\Omega_\epsilon} \mathcal{K}_\alpha(x-y) D_y f(y) dy + \alpha \int_{\Omega_\epsilon} \mathcal{K}_\alpha(x-y) f(y) dy - \\
&- \int_{\Omega_\epsilon} D_{r,y}[\mathcal{K}_\alpha(x-y)] f(y) dy + \int_{\Omega_\epsilon} D_{r,y}[\mathcal{K}_\alpha(x-y)] f(y) dy,
\end{aligned}$$

where $D_{r,y}$ is the right Moisil-Theodoresco operator (formula (2.4.9)) with respect to the variable y. Let us note that by definition of $\mathcal{K}_\alpha$ we have

$$-D_{r,y}[\mathcal{K}_\alpha(x-y)] = -D_y[\mathcal{K}_\alpha(x-y)] = D_x[\mathcal{K}_\alpha(x-y)].$$

Thus, continuing the last equality we obtain

$$\int_{\Omega_\epsilon} \mathcal{K}_\alpha(x-y) D_{\alpha,y} f(y) dy =$$

$$= \int_{\Omega_\epsilon} (\mathcal{K}_\alpha(x-y) D_y f(y) + D_{r,y}[\mathcal{K}_\alpha(x-y)] f(y)) dy +$$

$$+ \int_{\Omega_\epsilon} (\alpha \mathcal{K}_\alpha(x-y) + D_x[\mathcal{K}_\alpha(x-y)]) f(y) dy.$$

The first integral in the right-hand side falls victim to Theorem 2:

$$\int_{\Omega_\epsilon} (\mathcal{K}_\alpha(x-y) D_y f(y) + D_{r,y}[\mathcal{K}_\alpha(x-y)] f(y)) dy =$$

$$= \int_{\Gamma_\epsilon} \mathcal{K}_\alpha(x-y) \overrightarrow{n}(y) f(y) d\Gamma_{\epsilon,y},$$

where $\Gamma_\epsilon := \partial\Omega_\epsilon$. Turning back to (2.5.17) we see that

$$T_\alpha D_\alpha[f](x) = \lim_{\epsilon\to 0} \int_{\Gamma_\epsilon} \mathcal{K}_\alpha(x-y) \overrightarrow{n}(y) f(y) d\Gamma_{\epsilon,y}$$

$$+ \lim_{\epsilon\to 0} \int_{\Omega_\epsilon} D_{\alpha,x}[\mathcal{K}_\alpha(x-y)] f(y) dy.$$

The first limit gives us $-K_\alpha[f](x)$ while the second is equal to

$$\int_\Omega \delta(x-y) f(y) dy$$

and consequently to $f(x)$. This completes the proof. □

This theorem immediately implies the following analogue of the Cauchy integral formula.

THEOREM 5. *(Quaternionic Cauchy integral formula)*
Let $f \in C^1(\Omega; \mathbb{H}(\mathbb{C})) \cap C(\overline{\Omega}; \mathbb{H}(\mathbb{C}))$ *and* $f \in \ker D_\alpha(\Omega)$. *Then*

$$f(x) = K_\alpha[f](x), \quad \forall x \in \Omega. \tag{2.5.18}$$

(Recall that $f \in \ker A$ *in some domain* Ω *iff* $Af = 0$, $\forall x \in \Omega$*).*

2.5.3. The radiation condition and integral representations in unbounded domains. The next step is to obtain the Cauchy integral formula for null-solutions of the operator D_α in the exterior domain $\mathbb{R}^3 \setminus \overline{\Omega}$. Let us first discuss some heuristic arguments leading to the notion of a radiation condition at the infinity.

Consider the Helmholtz equation

$$(\Delta + \alpha^2)u(x) = \delta(x), \quad x \in \mathbb{R}^3, \tag{2.5.19}$$

which mathematically defines the fundamental solution of the Helmholtz operator. In order that the solution of (2.5.19) has a physical sense we must remember that it describes a monochromatic wave generated by a point source situated in the origin. It is physically reasonable to require that u decrease at infinity, which ensures finite energy of the propagation process. Supposing that $\operatorname{Im}\alpha > 0$ it is not difficult to see (applying the Fourier transform to (2.5.19)) that the only solution of (2.5.19) satisfying this requirement is the (generalized) function $u = \vartheta$ defined by (2.5.8). The situation changes dramatically when we suppose that $\operatorname{Im}\alpha = 0$. In this case there are two solutions of (2.5.19) decreasing at infinity,

$$u^+(x) := -\frac{e^{i\alpha|x|}}{4\pi\,|x|} \quad \text{and} \quad u^-(x) := -\frac{e^{-i\alpha|x|}}{4\pi\,|x|}.$$

Nevertheless the uniqueness of the fundamental solution is crucial in order to ensure the uniqueness of integral representations and of the solutions of physically meaningful boundary value problems. There are two ways to resolve this difficulty. The first is related to the so-called limiting absorption principle which applied to (2.5.19) consists in the following. We assume that there is some small absorption in the medium characterized by a small parameter ϵ. Then the corresponding equation becomes

$$(\Delta + (\alpha + i\epsilon)^2)u_\epsilon(x) = \delta(x), \quad x \in \mathbb{R}^3. \tag{2.5.20}$$

As we have seen, for $\epsilon > 0$ the solution of (2.5.20) decreasing at infinity is unique:

$$u_\epsilon(x) = -\frac{e^{i(\alpha+i\epsilon)|x|}}{4\pi\,|x|}.$$

Considering the limit when $\epsilon \to 0$ we arrive at the solution u^+ of (2.5.19).

Another way to obtain a unique solution of (2.5.19) in the case $\operatorname{Im}\alpha = 0$ is to impose a radiation condition at infinity. For the Helmholtz equation, this was proposed by Sommerfeld and has the following form. It is required that u satisfy the asymptotic equality

$$\frac{\partial u(x)}{\partial\,|x|} - i\alpha u(x) = o(\frac{1}{|x|}), \quad \text{when} \quad |x| \to \infty. \tag{2.5.21}$$

It can be verified immediately that this condition is fulfilled by u^+ but not by u^-.

We observe a similar situation in the case of the operator D_α. When $\operatorname{Im}\alpha = 0$ the equation

$$D_\alpha \mathcal{K}(x) = \delta(x), \quad x \in \mathbb{R}^3$$

admits two solutions obtained (according to (2.5.6)) by application of the operator $-D_{-\alpha}$ to the fundamental solutions of the Helmholtz operator u^+ and u^- respectively:

$$\mathcal{K}^\pm(x) = (\alpha + \frac{x}{|x|^2} \mp i\alpha\frac{x}{|x|}) \cdot u^\pm(x).$$

In order to omit one of these possibilities we impose the following radiation condition

$$(\alpha - \frac{x}{|x|^2} + i\alpha\frac{x}{|x|}) \cdot \mathcal{K}(x) = o(\frac{1}{|x|}), \quad \text{when} \quad |x| \to \infty. \tag{2.5.22}$$

Let us see what happens with the function $\mathcal{K}^+$. Consider

$$-(\alpha - \frac{x}{|x|^2} + i\alpha\frac{x}{|x|}) \cdot (\alpha + \frac{x}{|x|^2} - i\alpha\frac{x}{|x|})\frac{e^{i\alpha|x|}}{4\pi|x|} =$$

$$= -(\alpha - \frac{x}{|x|}(\frac{1 - i\alpha|x|}{|x|})) \cdot (\alpha + \frac{x}{|x|}(\frac{1 - i\alpha|x|}{|x|}))\frac{e^{i\alpha|x|}}{4\pi|x|} =$$

$$= -(\alpha^2 + \frac{(1 - i\alpha|x|)^2}{|x|^2})\frac{e^{i\alpha|x|}}{4\pi|x|} = (-\frac{1}{|x|^2} + \frac{2i\alpha}{|x|})\frac{e^{i\alpha|x|}}{4\pi|x|}.$$

This function decreases at the infinity as $O(1/|x|^2)$ and thus (2.5.22) is fulfilled by $\mathcal{K}^+$. The same procedure shows us that $\mathcal{K}^-$ does not satisfy (2.5.22). Note that $\mathcal{K}^+$ is precisely the fundamental solution $\mathcal{K}_\alpha$ used already on the preceding pages.

Now we are ready to prove the Cauchy integral formula for the exterior domain.

THEOREM 6. *(Quaternionic Cauchy integral formula for the exterior domain) Let*

$$f \in C^1(\mathbb{R}^3 \backslash \overline{\Omega}; \mathbb{H}(\mathbb{C})) \cap C(\mathbb{R}^3 \backslash \Omega; \mathbb{H}(\mathbb{C})), \quad f \in \ker D_\alpha(\mathbb{R}^3 \backslash \overline{\Omega}), \quad \operatorname{Im}\alpha \geq 0$$

and satisfy the radiation condition

$$(\alpha - \frac{x}{|x|^2} + i\alpha\frac{x}{|x|}) \cdot f(x) = o(\frac{1}{|x|}), \quad \textit{when} \quad |x| \to \infty.$$

Then

$$f(x) = -K_\alpha[f](x), \quad \forall x \in \mathbb{R}^3 \setminus \overline{\Omega}. \tag{2.5.23}$$

PROOF. Let Γ^R be a sphere with center in the origin and with radius R sufficiently large so that Ω is contained in the ball B^R with the boundary $\partial B^R = \Gamma^R$ (Fig. 2).

FIGURE 2. Domain Ω^R.

According to Theorem 5 in each point x of the domain $\Omega^R := B^R \backslash \overline{\Omega}$ we have the equality

$$f(x) = \int_\Gamma \mathcal{K}_\alpha(x-y)\overrightarrow{n}(y)f(y)d\Gamma_y - \int_{\Gamma^R} \mathcal{K}_\alpha(x-y)\frac{y}{|y|}f(y)d\Gamma_y^R.$$

Now let us consider the limit of this equality when $R \to \infty$. We have the following asymptotic relation

$$\int_{\Gamma^R} \mathcal{K}_\alpha(x-y)\frac{y}{|y|}f(y)d\Gamma_y^R \sim \int_{\Gamma^R} \vartheta(y)(\alpha - \frac{y}{|y|^2} + i\alpha\frac{y}{|y|})f(y)d\Gamma_y^R, \quad R \to \infty.$$

Using the radiation condition we obtain that this integral tends to zero when $R \to \infty$. Thus,

$$f(x) = \int_\Gamma \mathcal{K}_\alpha(x-y)\overrightarrow{n}(y)f(y)d\Gamma_y$$

which gives us the statement of the theorem. □

The introduction of the radiation condition in the form (2.5.22) allowed us to obtain a very simple proof of the Cauchy integral formula for the exterior domain. Nevertheless we should mention that A.McIntosh and M.Mitrea in [**64**] proposed a radiation condition for the operator D_α in a more elegant form. Let us note that the expression which appears in (2.5.22) can be represented in the following form

$$(\alpha - \frac{x}{|x|^2} + i\alpha\frac{x}{|x|}) = \alpha(1 + \frac{ix}{|x|}) + O(\frac{1}{|x|}), \quad |x| \to \infty.$$

Thus, a natural idea is to introduce the radiation condition as

$$(1 + \frac{ix}{|x|})f(x) = o(\frac{1}{|x|}), \tag{2.5.24}$$

because the term $x/|x|^2$ apparently gives a faster decay. The problem here is that (2.5.24) does not imply the decay of the product $\frac{x}{|x|^2} \cdot f(x)$ when $|x| \to \infty$ due to the fact that $(1 + \frac{ix}{|x|})$ is a zero divisor. The following simple example makes this clear. Consider $f(x) := (1 - \frac{ix}{|x|}) \cdot |x|^2$. This function obviously satisfies (2.5.24) but $\frac{x}{|x|^2} \cdot f(x) = O(|x|^2)$, $|x| \to \infty$. Thus, in order to be able to apply (2.5.24) instead of (2.5.22), we should prove that if f belongs to $\ker D_\alpha(\mathbb{R}^3 \backslash \overline{\Omega})$ and satisfies (2.5.24) then it decreases at infinity. Following arguments from [**64**] we prove the necessary fact.

THEOREM 7. *Let* $f \in C^1(\mathbb{R}^3 \setminus \overline{\Omega}; \mathbb{H}(\mathbb{C})) \cap C(\mathbb{R}^3 \setminus \Omega; \mathbb{H}(\mathbb{C}))$, $f \in \ker D_\alpha(\mathbb{R}^3 \setminus \overline{\Omega})$, $\quad \operatorname{Im} \alpha \geq 0$ *and satisfy (2.5.24). Then*

$$\int_{|x|=R} |f(x)|_c^2 \, d\Gamma_x^R = O(1) \quad as \quad R \to \infty. \tag{2.5.25}$$

PROOF. Let us consider the expression

$$\left|(1 + \frac{ix}{|x|})f\right|_c^2 = \operatorname{Sc}((1 + \frac{ix}{|x|})f\overline{f}^*(1 + \frac{ix}{|x|})) = 2\operatorname{Sc}((1 + \frac{ix}{|x|})f\overline{f}^*) =$$

$$= 2(|f|_c^2 - \operatorname{Im}\operatorname{Sc}(\overline{f}^* \frac{x}{|x|} f)).$$

Consequently,

$$\int_{|x|=R} \left|(1 + \frac{ix}{|x|})f(x)\right|_c^2 d\Gamma_x^R = 2(\int_{|x|=R} |f(x)|_c^2 \, d\Gamma_x^R - \tag{2.5.26}$$

$$\operatorname{Im}\operatorname{Sc}(\int_{|x|=R} \overline{f}^*(x) \frac{x}{|x|} f(x) d\Gamma_x^R)).$$

Due to Theorem 2 we have (using the notation of Theorem 6)

$$\int_{|x|=R} \overline{f}^*(x)\frac{x}{|x|}f(x)d\Gamma_x^R = \int_\Gamma \overline{f}^*(x)\overrightarrow{n}(x)f(x)d\Gamma_x +$$

$$\int_{\Omega^R}((D_r(\overline{f}^*(x))\cdot f(x)+\overline{f}^*(x)\cdot Df(x))dx.$$

The first integral on the right-hand side is some constant C. In order to simplify the second one we remember that $Df=-\alpha f$ and observe that

$$D_r\overline{f}^* = -\overline{Df^*} = \overline{\alpha^* f^*} = \alpha^*\overline{f}^*.$$

Then

$$\begin{aligned}\int_{\Omega^R}((D_r(\overline{f}^*(x))\cdot f(x)+\overline{f}^*(x)\cdot Df(x))dx &= \\ =\int_{\Omega^R}(\alpha^*\overline{f}^*(x)\cdot f(x)-\alpha\overline{f}^*(x)\cdot f(x))dx& \\ =-2i\operatorname{Im}\alpha\int_{\Omega^R}\overline{f}^*(x)\cdot f(x)dx.&\end{aligned}$$

We obtain that

$$\operatorname{Sc}(\int_{|x|=R}\overline{f}^*(x)\frac{x}{|x|}f(x)d\Gamma_x^R) = C_0 - 2i\operatorname{Im}\alpha\int_{\Omega^R}|f(x)|_c^2\,dx.$$

Substituting this expression in (2.5.26) and using the radiation condition (2.5.24) we have

$$\int_{|x|=R}|f(x)|_c^2\,d\Gamma_x^R = \operatorname{Im}C_0 - 2\operatorname{Im}\alpha\int_{\Omega^R}|f(x)|_c^2\,dx, \quad R\to\infty.$$

The last term above vanishes when $\operatorname{Im}\alpha=0$ and is negative when $\operatorname{Im}\alpha>0$. In both cases (2.5.25) is proved. □

This theorem establishes the equivalence of the radiation conditions (2.5.22) and (2.5.24) for functions from $\ker D_\alpha$. In what follows we will use the simpler version (2.5.24) of the radiation condition.

Remark 3. *It is easy to see that for functions from* $\ker D_{-\alpha}$ *the corresponding radiation condition at infinity has the form*

$$(1 - \frac{ix}{|x|})f(x) = o(\frac{1}{|x|}). \tag{2.5.27}$$

Due to a close relation between the operators $D_{\pm\alpha}$ and the Helmholtz operator one might expect that conditions (2.5.24) and (2.5.27) must be connected with the Sommerfeld radiation condition (2.5.21), and this is in fact the case. We explain this connection in the following remark.

Remark 4. *Let us make use of the following fact proved in* [**57**]*:*

$$\ker(\Delta + \alpha^2) = \ker D_\alpha \oplus \ker D_{-\alpha}.$$

Here the Helmholtz operator is considered in general on $\mathbb{H}(\mathbb{C})$*-valued functions. In other words each function* $u \in \ker(\Delta + \alpha^2)$ *can be represented in a unique way as the following sum*

$$u = f + g,$$

where $f \in \ker D_\alpha$ *and* $g \in \ker D_{-\alpha}$*. For a given* u *the corresponding functions* f *and* g *are found easily. Namely,*

$$f = -\frac{1}{2\alpha} D_{-\alpha} u \qquad \text{and} \qquad g = \frac{1}{2\alpha} D_\alpha u.$$

This fact is true also for scalar solutions u_0 *of the Helmholtz equation. In this case*

$$f = -\frac{1}{2\alpha} \operatorname{grad} u_0 + \frac{1}{2} u_0$$

and

$$g = \frac{1}{2\alpha} \operatorname{grad} u_0 + \frac{1}{2} u_0.$$

If now f fulfills (2.5.24) and g fulfills (2.5.27), we obtain the following asymptotic equality for u_0:

$$u_0 = -\frac{ix}{|x|} \cdot f + \frac{ix}{|x|} \cdot g + o\left(\frac{1}{|x|}\right) =$$

$$= \frac{ix}{|x|}\left(\frac{1}{2\alpha}\operatorname{grad} u_0 - \frac{1}{2}u_0\right) +$$

$$+\frac{ix}{|x|}\left(\frac{1}{2\alpha}\operatorname{grad} u_0 + \frac{1}{2}u_0\right) + o\left(\frac{1}{|x|}\right) =$$

$$= \frac{i}{\alpha}\frac{x}{|x|} \cdot \operatorname{grad} u_0 + o\left(\frac{1}{|x|}\right),$$

which gives us the Sommerfeld radiation condition:

$$i\alpha u_0(x) - \left\langle \frac{x}{|x|}, \operatorname{grad} u_0(x)\right\rangle = o\left(\frac{1}{|x|}\right), \qquad \textit{when } |x| \to \infty.$$

2.5.4. The quaternionic Plemelj-Sokhotski formulas and some boundary value problems. Theorems 5 and 6 allow reconstruction of solutions of the equation $D_\alpha f = 0$ both in the interior of the domain Ω and in its exterior when the values of the function f are given in all points of the surface $\Gamma = \partial\Omega$. The solution is then represented in the form of the Cauchy integral $K_\alpha f$. Now the question arises as to what happens to this integral representation near the boundary Γ. It is clear that for “reasonably good” functions (such as those satisfying the conditions of the above mentioned theorems) the integral $K_\alpha f$ is a continuous function (even more, its derivatives of any order exist), but on the boundary Γ the kernel of the integral, the function $\mathcal{K}_\alpha(x-y)$, has a singularity which leaves no hope for the continuity of $K_\alpha f$ in the points of Γ. The following fundamental fact, which is an analogue of the Plemelj-Sokhotski formulas from complex analysis, makes this completely clear.

Let us denote $\Omega^+ := \Omega$, $\Omega^- := \mathbb{R}^3\backslash\overline{\Omega}$. The set of all complex quaternionic functions f whose components satisfy the Hölder condition on Γ:

$$|f_k(x) - f_k(y)| \le C\,|x-y|^{\epsilon}, \quad 0 < \epsilon \le 1$$

for arbitrary $x, y \in \Gamma$, we will denote $C^{0,\epsilon}(\Gamma; \mathbb{H}(\mathbb{C}))$.

THEOREM 8. *(Quaternionic Plemelj-Sokhotski formulas) Let Γ be a closed Liapunov surface, $f \in C^{0,\epsilon}(\Gamma; \mathbb{H}(\mathbb{C}))$, $0 < \epsilon \le 1$. Then everywhere on Γ the following limits exist*

$$\lim_{\Omega^{\pm}\ni x \to \tau \in \Gamma} K_\alpha[f](x) =: K_\alpha[f]^{\pm}(\tau),$$

and the following formulas hold

$$K_\alpha[f]^+(\tau) = P_\alpha[f](\tau), \qquad K_\alpha[f]^-(\tau) = -Q_\alpha[f](\tau), \tag{2.5.28}$$

where the operators P_α and Q_α are those introduced on p. 23.

PROOF. For $\alpha = 0$ this theorem is very well known and its proof can be found, for instance, in [**16**, p.177], [**31**, p.59] or [**32**, p.105]. Thus we need prove it only for $\alpha \neq 0$.

Let us rewrite $\mathcal{K}_\alpha$ in the following form

$$\mathcal{K}_\alpha(x) = -\frac{\alpha e^{i\alpha|x|}}{4\pi\,|x|} + \mathcal{K}_0(x)\cdot(e^{i\alpha|x|} - i\alpha\,|x|\,e^{i\alpha|x|}).$$

Having expanded the function $e^{i\alpha|x|}$ into its Taylor series we obtain:

$$\lim_{\Omega^{\pm}\ni x\to\tau\in\Gamma} K_\alpha[f](x) = -\lim_{\Omega^{\pm}\ni x\to\tau\in\Gamma}\int_\Gamma (-\frac{\alpha e^{i\alpha|x-y|}}{4\pi\,|x-y|}$$

$$-i\alpha\mathcal{K}_0(x-y)\,|x-y|\,e^{i\alpha|x-y|} + \mathcal{K}_0(x-y)\sum_{k=1}^{\infty}\frac{(i\alpha\,|x-y|)^k}{k!})\overrightarrow{n}(y)f(y)d\Gamma_y$$

$$-\lim_{\Omega^{\pm}\ni x\to\tau\in\Gamma}\int_\Gamma \mathcal{K}_0(x-y)\overrightarrow{n}(y)f(y)d\Gamma_y.$$

The kernel of the first integral has a weak singularity and hence it is a continuous function. The second integral is nothing but $K_0[f](x)$, for which as was mentioned above the theorem is valid. Thus we have

$$\lim_{\Omega^{+}\ni x\to\tau\in\Gamma} K_0[f](x) = P_0[f](\tau),$$

$$\lim_{\Omega^{-}\ni x\to\tau\in\Gamma} K_0[f](x) = -Q_0[f](\tau).$$

Substituting this into the preceding equality we complete the proof. $\square$

Theorem 8 implies some very nice properties of the operators P_α, Q_α and S_α. First of all we prove the following

THEOREM 9. *The operator S_α is an involution on the space $C^{0,\epsilon}(\Gamma;\mathbb{H}(\mathbb{C}))$, $0<\epsilon\le 1$ and hence P_α and Q_α are mutually complementary projection operators on the same space:*

$$S_\alpha^2 = I, \tag{2.5.29}$$

$$P_\alpha^2 = P_\alpha,\quad Q_\alpha^2 = Q_\alpha;\quad P_\alpha Q_\alpha = Q_\alpha P_\alpha = 0. \tag{2.5.30}$$

PROOF. A simple calculation using the definition of these operators shows us that (2.5.29) and (2.5.30) are equivalent, so that it is enough to prove (2.5.30). Let $f \in C^{0,\epsilon}(\Gamma; \mathbb{H}(\mathbb{C}))$. $K_\alpha[f] \in \ker D_\alpha(\Omega)$ and due to the Cauchy integral formula we obtain that

$$K_\alpha[f](x) = K_\alpha[K_\alpha[f]](x).$$

Now letting $x \to \tau \in \Gamma$ and using (2.5.28) we obtain the first of the equalities (2.5.30). The second is proved in a similar way considering the exterior of Ω. Thus, P_α and Q_α are projection operators. By their definition they are mutually complementary, that is, $Q_\alpha = I - P_\alpha$. Then $P_\alpha Q_\alpha = P_\alpha(I - P_\alpha) = P_\alpha - P_\alpha^2 = 0$ and we obtain the necessary result. □

We proved that P_α and Q_α are projection operators on the space $C^{0,\epsilon}(\Gamma; \mathbb{H}(\mathbb{C}))$, but where do they project the functions from this space? In other words, any $f \in C^{0,\epsilon}(\Gamma; \mathbb{H}(\mathbb{C}))$ can be represented in a unique way as the sum $f = P_\alpha f + Q_\alpha f$. What then are these "parts" of f, $P_\alpha f$ and $Q_\alpha f$? The answer is given in the following statement.

THEOREM 10. *Let Γ be a closed Liapunov surface which is the boundary of a finite domain Ω^+ and of an infinite domain Ω^-. Let $f \in C^{0,\epsilon}(\Gamma; \mathbb{H}(\mathbb{C}))$, $0 < \epsilon \leq 1$.*

(1) *In order for f to be the boundary value of a function F from $\ker D_\alpha(\Omega^+)$, the following condition is necessary and sufficient:*

$$f \in \operatorname{im} P_\alpha \tag{2.5.31}$$

(that is, there exists a function $g \in C^{0,\epsilon}(\Gamma; \mathbb{H}(\mathbb{C}))$ such that $f = P_\alpha g$).

(2) *In order for f to be the boundary value of a function F from* $\ker D_\alpha(\Omega^-)$*, satisfying (2.5.24) at infinity, the following condition is necessary and sufficient:*

$$f \in \operatorname{im} Q_\alpha. \tag{2.5.32}$$

PROOF. First, let $f \in C^{0,\epsilon}(\Gamma; \mathbb{H}(\mathbb{C}))$ be the boundary value of $F \in \ker D_\alpha(\Omega^+)$. Then F is representable by its Cauchy integral:

$$F(x) = K_\alpha[f](x), \quad \forall x \in \Omega^+.$$

Now let $\tau \in \Gamma$ and $\Omega^+ \ni x \to \tau$. Then $F(x) \to f(\tau)$ and according to the Plemelj-Sokhotski formulas $K_\alpha[f](x) \to P_\alpha[f](\tau)$. Thus, $f(\tau) = P_\alpha[f](\tau)$ which gives us (2.5.31).

Now, on the contrary, let (2.5.31) hold. Let us consider $F(x) := K_\alpha[f](x)$, $x \in \Omega^+$. Then $F \in \ker D_\alpha(\Omega^+)$ and again by the Plemelj-Sokhotski formulas, $F|_\Gamma = P_\alpha[f] = f$. The first part of the theorem is proved. The proof of the second part is analogous. □

REMARK 5. *The condition (2.5.31) can be rewritten as follows*

$$f(\tau) = S_\alpha[f](\tau), \quad \forall \tau \in \Gamma, \tag{2.5.33}$$

and (2.5.32) as

$$f(\tau) = -S_\alpha[f](\tau), \quad \forall \tau \in \Gamma. \tag{2.5.34}$$

Theorem 10, in particular, signifies that for any $f \in C^{0,\epsilon}(\Gamma; \mathbb{H}(\mathbb{C}))$ its "part" $P_\alpha[f]$ is extendable into the domain Ω^+ in such a way that the extension belongs to $\ker D_\alpha(\Omega^+)$, and the other "part" $Q_\alpha[f]$ is extendable into Ω^- in such a way that the extension belongs to $\ker D_\alpha(\Omega^-)$ and satisfies the radiation condition (2.5.24). The function f itself is extendable in this sense into Ω^+ or Ω^- iff $Q_\alpha[f] \equiv 0$ or $P_\alpha[f] \equiv 0$ on Γ

respectively. In these cases we call the function α-extendable into Ω^+ or Ω^- respectively.

Another possible interpretation of Theorem 10 consists in considering the following boundary value problems for the operator D_α.

PROBLEM 1. *(The interior Dirichlet problem for the operator D_α) Given a complex quaternionic function $g \in C^{0,\epsilon}(\Gamma; \mathbb{H}(\mathbb{C}))$, find a function f such that*

$$D_\alpha f(x) = 0, \quad x \in \Omega^+$$

and

$$f(x) = g(x), \quad x \in \Gamma.$$

PROBLEM 2. *(The exterior Dirichlet problem for the operator D_α) Given a complex quaternionic function $g \in C^{0,\epsilon}(\Gamma; \mathbb{H}(\mathbb{C}))$, find a function f such that*

$$D_\alpha f(x) = 0, \quad x \in \Omega^-,$$

$$f(x) = g(x), \quad x \in \Gamma,$$

and f satisfies (2.5.24) at infinity.

Let us analyse Problem 1 (Problem 2 can be analysed in a similar way). From Theorem 10 we see immediately that the solution of Problem 1 does not always exist because not all functions g are α-extendable into Ω^+. They must satisfy the condition (2.5.31) or equivalently, the condition (2.5.33). If this is the case then the solution of Problem 1, according to the Cauchy integral formula, is obtained from the Cauchy integral of g: $f = K_\alpha g$.

Let us consider another boundary value problem for the operator D_α, the so-called jump problem.

PROBLEM 3. *(The jump problem for the operator D_α)*

Given $g \in C^{0,\epsilon}(\Gamma; \mathbb{H}(\mathbb{C}))$, find a pair of functions f^+ and f^- such that $f^\pm \in \ker D_\alpha(\Omega^\pm)$, f^- satisfies (2.5.24) and

$$f^+(x) - f^-(x) = g(x), \quad x \in \Gamma. \tag{2.5.35}$$

The solution of this problem always exists and is obtained in the following form

$$f^+(x) = K_\alpha g(x), \quad x \in \Omega^+,$$

$$f^-(x) = K_\alpha g(x), \quad x \in \Omega^-.$$

We can see that due to the Plemelj-Sokhotski formulas the condition (2.5.35) is fulfilled:

$$f^+ - f^- = K_\alpha[g]^+ - K_\alpha[g]^- = P_\alpha[g] + Q_\alpha[g] = g.$$

A much more difficult problem is the analogue of the famous Riemann boundary value problem.

PROBLEM 4. *(The Riemann boundary value problem) Given two functions $g, G \in C^{0,\epsilon}(\Gamma; \mathbb{H}(\mathbb{C}))$, find a pair of functions f^+ and f^- such that $f^\pm \in \ker D_\alpha(\Omega^\pm)$, f^- satisfies (2.5.24) and*

$$f^+(x) = f^-(x)G(x) + g(x), \quad x \in \Gamma. \tag{2.5.36}$$

Note that in the case when G is constant, the condition (2.5.36) is reduced to (2.5.35) by denoting $f^-(x)G =: \widetilde{f}^-$ (the operator D_α is right-linear). In the general case when G is a complex quaternionic function, only results about the Fredholmness of the problem have been obtained (see [**74, 75, 54, 10, 11**]).

2.6. The operator $D + M^\alpha$

In 1975, even before the operator D_α, $\alpha \in \mathbb{R}$ was considered by K. Gürlebeck, there appeared an article by E. Obolashvili [**67**] in which the operator acting on quaternionic functions in the following way

$$D_{\overrightarrow{\alpha}} f := Df + f\overrightarrow{\alpha}$$

was studied, where $\overrightarrow{\alpha}$ is a purely vectorial real quaternion. This work was written in matrix terms without using the notion of quaternions. The development of both theories (for $\alpha = \mathrm{Sc}(\alpha)$ and for $\alpha = \mathrm{Vec}(\alpha)$) led to principally similar results, but they existed separately one from another. The natural desire to construct a theory combining them and including the case when the components of α are complex numbers led to a series of works [**55, 56, 57**] in which the operator $D + M^\alpha$ was studied. Here M^α denotes the operator of multiplication by $\alpha \in \mathbb{H}(\mathbb{C})$ from the right-hand side:

$$M^\alpha f := f \cdot \alpha.$$

As we will see in the next chapter, apart from this natural "mathematical" desire there exists an important "physical" reason for studying the operator $D + M^\alpha$: it is closely related to the classical Dirac operator from quantum mechanics.

The theory of the operator $D + M^\alpha$ was expounded in detail in the book [**58**]. Here we only outline some results necessary for these lectures, giving them without proof. However we should answer first the following obvious question which no doubt the observant reader is already asking at this point. Why do we consider the multiplication by α from the right-hand side and not the operator $D+\alpha I$, $\alpha \in \mathbb{H}(\mathbb{C})$? The explanation is that the study of this operator reduces to the study of the operator $D+\alpha_0 I$, where $\alpha_0 = \mathrm{Sc}(\alpha)$. This stems from the following fact [**45**], [**58**, p. 64]. A complex quaternionic function f belongs to $\ker(D+\alpha I)(\Omega)$ if and only if the function $g(x) := e^{\langle \overrightarrow{\alpha}, \overrightarrow{x} \rangle} f(x)$ belongs to $\ker(D+\alpha_0 I)(\Omega)$. The proof consists in the application of the operator D to g. Let $f \in \ker(D+\alpha I)(\Omega)$. Then using Remark 2 we obtain

$$\begin{aligned} Dg(x) &= D[e^{\langle \overrightarrow{\alpha}, \overrightarrow{x} \rangle}] \cdot f(x) + e^{\langle \overrightarrow{\alpha}, \overrightarrow{x} \rangle} \cdot D[f](x) \\ &= \overrightarrow{\alpha} e^{\langle \overrightarrow{\alpha}, \overrightarrow{x} \rangle} f(x) - \alpha e^{\langle \overrightarrow{\alpha}, \overrightarrow{x} \rangle} f(x) = -\alpha_0 g(x). \end{aligned}$$

Of course, in the opposite direction the proof is similar. More details can be found in [**8**].

Thus the operator $D+\alpha I$, when α is a constant complex quaternion, being reduced to $D+\alpha_0 I$ represents less interest compared to the operator $D+M^\alpha$. We will use the notation D_α for this operator:

$$D_\alpha := D + M^\alpha, \tag{2.6.1}$$

where $\alpha \in \mathbb{H}(\mathbb{C})$. In order to introduce the integral operators K_α, T_α and S_α corresponding to (2.6.1) we have to distinguish different cases depending on the algebraic properties of α. The following observations will help us to understand the structure of the integral operators.

(1) Let $\alpha \notin \mathfrak{S}$ and $\overrightarrow{\alpha}^2 \neq 0$. We introduce the following auxiliary complex numbers: $\lambda := \sqrt{\overrightarrow{\alpha}^2}$ and $\xi_\pm := \alpha_0 \pm \lambda$. λ is chosen so that $\operatorname{Im}\lambda \geq 0$. The complex quaternions $\lambda + \overrightarrow{\alpha}$ and $\lambda - \overrightarrow{\alpha}$ are zero divisors. Hence the operators

$$P^+ := \frac{1}{2\lambda} M^{(\lambda+\overrightarrow{\alpha})} \quad \text{and} \quad P^- := \frac{1}{2\lambda} M^{(\lambda-\overrightarrow{\alpha})} \tag{2.6.2}$$

are mutually complementary projection operators. Let us consider the product

$$\alpha \cdot (\lambda \pm \overrightarrow{\alpha}) = (\alpha_0 + \overrightarrow{\alpha})(\lambda \pm \overrightarrow{\alpha}) = \alpha_0\lambda + \lambda\overrightarrow{\alpha} \pm \lambda^2 \pm \alpha_0\overrightarrow{\alpha}$$
$$= \alpha_0(\lambda \pm \overrightarrow{\alpha}) + \lambda(\pm\lambda + \overrightarrow{\alpha}) = \xi_\pm \cdot (\lambda \pm \overrightarrow{\alpha}).$$

Consequently,

$$P^+(D + M^\alpha) = P^+(D + \xi_+ I)$$

and

$$P^-(D + M^\alpha) = P^-(D + \xi_- I).$$

Thus, the operator $D + M^\alpha$ can be rewritten in the following form

$$D_\alpha = P^+ D_{\xi_+} + P^- D_{\xi_-}. \tag{2.6.3}$$

Moreover, the operators P^+ and P^- commute with the operators D_{ξ_+} and D_{ξ_-}. This implies, for instance, that

$$\ker D_\alpha = P^+(\ker D_{\xi_+}) \oplus P^-(\ker D_{\xi_-}). \tag{2.6.4}$$

The right inverse operator T_α for D_α can be written as follows

$$T_\alpha = P^+ T_{\xi_+} + P^- T_{\xi_-},$$

where the $T_{\xi_\pm}$ are defined by (2.5.12). All other integral operators in this case can be constructed by analogy.

(2) Let $\alpha \notin \mathfrak{S}$ and $\overrightarrow{\alpha}^2 = 0$. Then the following trick helps us to find a convenient form for the operator D_α. We have the equality

$$f = \frac{\partial}{\partial \alpha_0}(D_{\alpha_0} f).$$

Then

$$D_\alpha f = D_{\alpha_0} f + M^{\vec{\alpha}} f = D_{\alpha_0} f + M^{\vec{\alpha}} \frac{\partial}{\partial \alpha_0}(D_{\alpha_0} f) = (I + M^{\vec{\alpha}} \frac{\partial}{\partial \alpha_0}) D_{\alpha_0} f.$$

We have managed to again reduce the operator D_α with $\alpha \in \mathbb{H}(\mathbb{C})$ to the operator with a scalar parameter. Thus for example, the right-inverse operator T_α is obtained in the form

$$T_\alpha = (I + M^{\vec{\alpha}} \frac{\partial}{\partial \alpha_0}) T_{\alpha_0}.$$

(3) Let $\alpha \in \mathfrak{S}$ and $\alpha_0 \neq 0$. In this case, as earlier, we denote

$$P^\pm := \frac{1}{2\lambda} M^{(\lambda \pm \vec{\alpha})},$$

and note that $\lambda = \alpha_0$ or $\lambda = -\alpha_0$. The sign is chosen so that $\operatorname{Im} \lambda \geq 0$. For the sake of simplicity we suppose that $\operatorname{Im} \alpha_0 \geq 0$ and $\lambda = \alpha_0$. Then

$$D_\alpha = P^+(D + M^\alpha) + P^-(D + M^\alpha) = P^+ D_{2\alpha_0} + P^- D.$$

Hence,

$$T_\alpha = P^+ T_{2\alpha_0} + P^- T_0.$$

(4) Let $\alpha \in \mathfrak{S}$ and $\alpha_0 = 0$. Then all necessary results for the operator D_α can be obtained using the following observation

$$-D_\alpha D_{-\alpha} = \Delta.$$

If we denote the right-inverse operator for Δ as

$$W f(x) := \int_\Omega \vartheta_0(x - y) f(y) dy,$$

where $\vartheta_0(x) := -1/(4\pi\,|x|)$, then

$$T_\alpha = (-D + M^\alpha) W = T_0 + M^\alpha W.$$

We have now considered all possible cases depending on algebraic properties of α, and explained the construction of the right-inverse operator T_α, which we can resume in the following equality

$$T_\alpha := \begin{cases} P^+T_{\xi_+} + P^-T_{\xi_-}, & \alpha \notin \mathfrak{S} \quad \text{and} \quad \overrightarrow{\alpha}^2 \neq 0, \\ (I + M^{\overrightarrow{\alpha}}\frac{\partial}{\partial\alpha_0})T_{\alpha_0}, & \alpha \notin \mathfrak{S} \quad \text{and} \quad \overrightarrow{\alpha}^2 = 0, \\ P^+T_{2\alpha_0} + P^-T_0, & \alpha \in \mathfrak{S} \quad \text{and} \quad \alpha_0 \neq 0, \\ T_0 + M^\alpha W, & \alpha \in \mathfrak{S} \quad \text{and} \quad \alpha_0 = 0. \end{cases} \tag{2.6.5}$$

In a similar way we construct the two other integral operators:

$$K_\alpha := \begin{cases} P^+K_{\xi_+} + P^-K_{\xi_-}, & \alpha \notin \mathfrak{S} \quad \text{and} \quad \overrightarrow{\alpha}^2 \neq 0, \\ (I + M^{\overrightarrow{\alpha}}\frac{\partial}{\partial\alpha_0})K_{\alpha_0}, & \alpha \notin \mathfrak{S} \quad \text{and} \quad \overrightarrow{\alpha}^2 = 0, \\ P^+K_{2\alpha_0} + P^-K_0, & \alpha \in \mathfrak{S} \quad \text{and} \quad \alpha_0 \neq 0, \\ K_0 - M^\alpha V, & \alpha \in \mathfrak{S} \quad \text{and} \quad \alpha_0 = 0, \end{cases} \tag{2.6.6}$$

where

$$Vf(x) := \int_\Gamma \vartheta_0(x-y)\overrightarrow{n}(y)f(y)d\Gamma_y, \quad x \in \mathbb{R}^3 \setminus \Gamma, \tag{2.6.7}$$

and

$$S_\alpha := \begin{cases} P^+S_{\xi_+} + P^-S_{\xi_-}, & \alpha \notin \mathfrak{S} \quad \text{and} \quad \overrightarrow{\alpha}^2 \neq 0, \\ (I + M^{\overrightarrow{\alpha}}\frac{\partial}{\partial\alpha_0})S_{\alpha_0}, & \alpha \notin \mathfrak{S} \quad \text{and} \quad \overrightarrow{\alpha}^2 = 0, \\ P^+S_{2\alpha_0} + P^-S_0, & \alpha \in \mathfrak{S} \quad \text{and} \quad \alpha_0 \neq 0, \\ S_0 - M^\alpha \widehat{V}, & \alpha \in \mathfrak{S} \quad \text{and} \quad \alpha_0 = 0, \end{cases} \tag{2.6.8}$$

where

$$\widehat{V}f(x) := 2\int_\Gamma \vartheta_0(x-y)\overrightarrow{n}(y)f(y)d\Gamma_y, \quad x \in \Gamma.$$

As before, we define the operators P_α and Q_α by (2.5.15), where S_α is the operator (2.6.8).

The integral operators introduced in the way described above enjoy the same properties as those which were obtained earlier in the case of a scalar α. We give these without proof (which can be found in [**58**, p. 75]) as the following theorem.

THEOREM 11. *(Main integral theorems in the case of an arbitrary biquaternionic parameter α) Let α be an arbitrary complex quaternion, Ω be a bounded domain in $\mathbb{R}^3$ with the Liapunov boundary $\Gamma := \partial\Omega$, and let T_α, K_α and S_α be as in formulas (2.6.5)-(2.6.8). Then the following assertions are true:*

(1) *(Borel-Pompeiu's formula for a biquaternionic parameter α) Let*

$$f \in C^1(\Omega; \mathbb{H}(\mathbb{C})) \cap C(\overline{\Omega}; \mathbb{H}(\mathbb{C})).$$

Then

$$K_\alpha[f](x) + T_\alpha D_\alpha[f](x) = f(x), \quad \forall x \in \Omega.$$

(2) *(Cauchy's integral formula for a biquaternionic parameter α) Let $f \in C^1(\Omega; \mathbb{H}(\mathbb{C})) \cap C(\overline{\Omega}; \mathbb{H}(\mathbb{C}))$ and $f \in \ker D_\alpha(\Omega)$. Then*

$$f(x) = K_\alpha[f](x), \quad \forall x \in \Omega.$$

(3) *(Right-inverse operator for D_α, $\alpha \in \mathbb{H}(\mathbb{C})$) Let*

$$f \in C^1(\Omega; \mathbb{H}(\mathbb{C})) \cap C(\overline{\Omega}; \mathbb{H}(\mathbb{C})).$$

Then

$$D_\alpha T_\alpha[f](x) = f(x), \quad \forall x \in \Omega.$$

The Cauchy integral formula for the exterior domain requires more detailed analysis. We have to obtain the radiation conditions corresponding to the different types of α. In other words we have to introduce some appropriate conditions which guarantee the decay of the integral $K_\alpha[f]$ taken over a sphere whose radius R tends to infinity (see the proof of Theorem 6).

(1) Let $\alpha \notin \mathfrak{S}$ and $\overrightarrow{\alpha}^2 \neq 0$. Then as

$$K_\alpha f = P^+ K_{\xi_+} f + P^- K_{\xi_-} f,$$

we obtain that the following independent conditions must be fulfilled:

$$(1 + \frac{ix}{|x|}) \cdot f(x) \cdot (\lambda + \overrightarrow{\alpha}) = o(\frac{1}{|x|}), \quad |x| \to \infty$$

and

$$(1 - \frac{ix}{|x|}) \cdot f(x) \cdot (\lambda - \overrightarrow{\alpha}) = o(\frac{1}{|x|}), \quad |x| \to \infty$$

or in another form:

$$(1 + \frac{ix}{|x|}) \cdot P^+ f(x) + (1 - \frac{ix}{|x|}) \cdot P^- f(x) = o(\frac{1}{|x|}), \quad |x| \to \infty.$$

(2) Let $\alpha \notin \mathfrak{S}$ and $\overrightarrow{\alpha}^2 = 0$. Then $K_\alpha = (I + M^{\overrightarrow{\alpha}} \frac{\partial}{\partial \alpha_0}) K_{\alpha_0}$. Consider

$$\frac{\partial}{\partial \alpha_0} \mathcal{K}_{\alpha_0} = \frac{\partial}{\partial \alpha_0} \left[-(\alpha_0 + \frac{x}{|x|^2} - i\alpha_0 \frac{x}{|x|}) \cdot \frac{e^{i\alpha_0 |x|}}{4\pi |x|} \right]$$

$$= -(1 + i\alpha_0 |x| + \alpha_0 x) \cdot \frac{e^{i\alpha_0 |x|}}{4\pi |x|}$$

$$= i\alpha_0 |x| (1 - \frac{ix}{|x|}) \cdot \vartheta(x) + O(\frac{1}{|x|}).$$

Thus, in order that the integral $K_\alpha f$, taken over a sphere whose radius R tends to infinity, decrease the radiation conditions

$$(1 + \frac{ix}{|x|}) \cdot f(x) = o(\frac{1}{|x|}), \quad |x| \to \infty$$

and

$$(1 + \frac{ix}{|x|}) \cdot f(x) \cdot \overrightarrow{\alpha} = o(\frac{1}{|x|^2}), \quad |x| \to \infty$$

are necessary.

(3) When $\alpha \in \mathfrak{S}$ and $\alpha_0 \neq 0$, by analogy with the first case, we obtain the radiation conditions:

$$(1 + \frac{ix}{|x|}) \cdot P^+ f(x) = o(\frac{1}{|x|}), \quad |x| \to \infty$$

and

$$P^- f(x) = o(1), \quad |x| \to \infty.$$

(4) If $\alpha \in \mathfrak{S}$ and $\alpha_0 = 0$, then we have the conditions in the form

$$f(x) = o(1), \quad |x| \to \infty$$

and

$$f(x)\alpha = o(\frac{1}{|x|}), \quad |x| \to \infty.$$

Under the obtained radiation conditions the Cauchy integral formula (2.5.23) for the functions from $\ker D_\alpha(\mathbb{R}^3 \setminus \overline{\Omega})$, $\alpha \in \mathbb{H}(\mathbb{C})$ is valid.

In what follows we will be primarily interested in two cases $\alpha = \alpha_0$ (considered in Section 2.5) and $\alpha = \overrightarrow{\alpha}$ (that is $\alpha_0 = 0$). In this second case we can consider both situations $\overrightarrow{\alpha} \in \mathfrak{S}$ and $\overrightarrow{\alpha} \notin \mathfrak{S}$ together. First of all let us notice that in both cases the operator $K_{\overrightarrow{\alpha}}$ admits the following representation

$$K_{\overrightarrow{\alpha}}[f](x) = -\int_\Gamma \{\vartheta(x-y)\left(\frac{x-y}{|x-y|^2} - \frac{i\lambda(x-y)}{|x-y|}\right)\overrightarrow{n}(y)f(y)+$$

$$+\vartheta(x-y)\overrightarrow{n}(y)f(y)\overrightarrow{\alpha}\}d\Gamma_y, \qquad x \in \mathbb{R}^3 \setminus \Gamma. \tag{2.6.9}$$

Here

$$\vartheta(x) = -\frac{\exp(i\lambda\,|x|)}{4\pi\,|x|}.$$

Moreover, the radiation condition for any $\alpha = \overrightarrow{\alpha} \in \mathbb{H}(\mathbb{C})$ can be written also in a unified form as can be seen from the following statement, which is nothing but the Cauchy integral formula for the operator $D_{\overrightarrow{\alpha}}$ in an exterior domain.

PROPOSITION 1. *Let* $f \in C^1(\mathbb{R}^3 \setminus \overline{\Omega}; \mathbb{H}(\mathbb{C})) \cap C(\mathbb{R}^3 \setminus \Omega; \mathbb{H}(\mathbb{C}))$, $f \in \ker D_\alpha(\mathbb{R}^3 \setminus \overline{\Omega})$, $\alpha \in \mathbb{H}(\mathbb{C})$, $\operatorname{Sc}\alpha = 0$ *and let* f *satisfy the radiation condition*

$$\lambda f(x) + \frac{ix}{|x|} f(x)\overrightarrow{\alpha} = o\left(\frac{1}{|x|}\right), \qquad \textit{when } |x| \to \infty, \tag{2.6.10}$$

where $\lambda := \sqrt{\overrightarrow{\alpha}^2} \in \mathbb{C}$ *and* $\operatorname{Im}\lambda \geq 0$. *If* $\alpha \in \mathfrak{S}$ *then we suppose additionally that* $f(x) = o(1)$. *Then*

$$f(x) = -K_{\overrightarrow{\alpha}}[f](x), \qquad \forall x \in \mathbb{R}^3 \setminus \overline{\Omega}.$$

PROOF. First we suppose that $\overrightarrow{\alpha} \notin \mathfrak{S}$. Multiplying (2.6.10) by $\overrightarrow{\alpha}/\lambda$ from the right-hand side we obtain

$$f(x)\overrightarrow{\alpha} + \lambda\frac{ix}{|x|} f(x) = o\left(\frac{1}{|x|}\right). \tag{2.6.11}$$

Adding and subtracting (2.6.10) and (2.6.11) we obtain

$$\left(\lambda f(x) + \frac{ix}{|x|} f(x)\overrightarrow{\alpha}\right) \pm \left(f(x)\overrightarrow{\alpha} + \lambda\frac{ix}{|x|} f(x)\right) =$$

$$= \left(1 \pm \frac{ix}{|x|}\right) f(x)\lambda \pm \left(1 \pm \frac{ix}{|x|}\right) f(x)\overrightarrow{\alpha} =$$

$$= \left(1 \pm \frac{ix}{|x|}\right) f(x)\,(\lambda \pm \overrightarrow{\alpha}) = o\left(\frac{1}{|x|}\right), \tag{2.6.12}$$

which can be written as follows

$$P^+\left(\left(1+\frac{ix}{|x|}\right)f(x)\right)+P^-\left(\left(1-\frac{ix}{|x|}\right)f(x)\right)=o\left(\frac{1}{|x|}\right) \tag{2.6.13}$$

when $|x| \to \infty$. Thus, (2.6.10) is equivalent to (2.6.13), from which it can be seen that P^+f fulfills the radiation condition (2.5.24) and P^-f fulfills (2.5.27). Consequently the integrals $K_{\pm\lambda}P^{\pm}f$ taken over the sphere Γ^R (see the proof of Theorem 6) decrease when $R \to \infty$.

Since $K_{\pm\lambda}$ commute with $P^{\pm}$ we obtain that the integral $K_{\overrightarrow{\alpha}}f(x)$ taken over the sphere Γ^R also decreases when $R \to \infty$.

In the case when α is a zero divisor the radiation condition (2.6.10) becomes

$$\frac{ix}{|x|}f\overrightarrow{\alpha}=o\left(\frac{1}{|x|}\right), \qquad \text{when } |x| \to \infty. \tag{2.6.14}$$

Since the behavior of ϑ_0 in (2.6.7) is of the type $O\left(\frac{1}{|x|}\right)$ when $|x| \to \infty$ and since the expression $M^{\overrightarrow{\alpha}}V$ contains the multiplication from the right-hand side by $\overrightarrow{\alpha}$, and in this case $f\overrightarrow{\alpha} = o\left(\frac{1}{|x|}\right)$, it can be seen that the integral (2.6.9) taken over the sphere Γ^R decreases at infinity also. □

CHAPTER 3

Physical models reducing to the operator D_α

3.1. Maxwell's equations

The impressive diversity of electromagnetic phenomena reduce to the four Maxwell's equations which are axioms of electromagnetic theory:

$$\operatorname{rot}\mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}, \tag{3.1.1}$$

$$\operatorname{rot}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{3.1.2}$$

$$\operatorname{div}\mathbf{D} = \rho, \tag{3.1.3}$$

$$\operatorname{div}\mathbf{B} = 0. \tag{3.1.4}$$

The notation used in these equations is explained in the following table.

Notation	Physical quantity	Units in the mks system
ρ	volume charge density	coulomb/m^3
$\mathbf{j}$	current density	amper/m^2
$\mathbf{E}$	electric field intensity	volt/m
$\mathbf{H}$	magnetic field intensity	amper/m
$\mathbf{D}$	electric induction vector	coulomb/m^2
$\mathbf{B}$	magnetic induction vector	tesla=volt·sec/m^2

The Maxwell equations are considered together with the so-called constitutive relations which describe the relations between the induction vectors and the field vectors. In general they can be written as follows

$$\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{H}), \qquad \mathbf{B} = \mathbf{B}(\mathbf{E}, \mathbf{H}).$$

The simplest interpretation of these relations is that, for instance, the electric induction $\mathbf{D}(\overrightarrow{x}, t)$ is completely determined by the intensity $\mathbf{E}(\overrightarrow{x}, t)$ in the same point $\overrightarrow{x}$ and at the same instant t ($\mathbf{B}$ and $\mathbf{H}$ are considered in a similar way). In other words, electromagnetic phenomena in the medium are considered to be local and non-inertial: in each point the state does not depend on the surrounding medium and at each moment of time the state does not depend on its "history". Although such an interpretation is quite idealized, it is applicable in many practical cases. Then

$$\mathbf{D} = \varepsilon_0 \varepsilon_r \mathbf{E} \tag{3.1.5}$$

and

$$\mathbf{B} = \mu_0 \mu_r \mathbf{H}, \tag{3.1.6}$$

where ε_0 is the free-space permittivity measured in farad/meter and μ_0 is the free-space permeability measured in henry/meter; the adimensional quantities ε_r and μ_r are called relative permittivity and permeability respectively. The coefficients $\varepsilon := \varepsilon_0 \varepsilon_r$ and $\mu := \mu_0 \mu_r$ are the absolute permittivity and permeability respectively.

The constitutive relations (3.1.5) and (3.1.6) describe the rich variety of physical phenomena which represent the response of the medium to the application of the electromagnetic field. We will assume that the electromagnetic characteristics of the medium ε and μ do not change in time. If in addition they have the same values in each point of some volume $\Omega \subset \mathbb{R}^3$ then the medium which fills the volume is called homogeneous. If on the other hand $\varepsilon = \varepsilon(\overrightarrow{x})$ and/or $\mu = \mu(\overrightarrow{x})$, the medium is inhomogeneous. We will also suppose that the pairs of vectors **D**, **E** and **B**, **H** are colinear. In this case the medium is called isotropic (otherwise, anisotropic).

3.1.1. Maxwell´s equations for homogeneous media in quaternionic form and the wave operator. The main part of the material of these lectures corresponding to the electromagnetic models is dedicated to the treatment of time-harmonic (monochromatic) electromagnetic fields. Nevertheless, methods of quaternionic analysis can also be very useful in the general situation. Let us suppose that the medium is homogeneous (and isotropic). Consider the function

$$\mathbf{V} := \sqrt{\varepsilon}\mathbf{E} + i\sqrt{\mu}\mathbf{H}. \tag{3.1.7}$$

It is easy to verify that the Maxwell equations (3.1.1)-(3.1.4) can be rewritten in the following form

$$(\frac{1}{c}\partial_t + iD)\mathbf{V} = -(\sqrt{\mu}\mathbf{j} + \frac{i\rho}{\sqrt{\varepsilon}}), \tag{3.1.8}$$

where c is the speed of propagation of electromagnetic waves in the medium:

$$c = \frac{1}{\sqrt{\varepsilon\mu}}$$

(in particular, in the case of a vacuum we have that $c = c_0 = 1/\sqrt{\varepsilon_0\mu_0}$ is the speed of light).

We note an important property of the quaternionic Maxwell operator $\frac{1}{c}\partial_t + iD$. That is, it factorizes the wave operator:

$$\frac{1}{c^2}\partial_t^2 - \Delta = (\frac{1}{c}\partial_t + iD)(\frac{1}{c}\partial_t - iD). \tag{3.1.9}$$

Moreover, each solution (scalar or biquaternionic) of the wave equation

$$(\frac{1}{c^2}\partial_t^2 - \Delta)u = 0$$

can be written in a unique way as the sum of two functions f and g, where f satisfies the equation

$$(\frac{1}{c}\partial_t + iD)f = 0$$

and g satisfies the conjugate equation

$$(\frac{1}{c}\partial_t - iD)g = 0.$$

In other words,

$$\ker(\frac{1}{c^2}\partial_t^2 - \Delta) = \ker(\frac{1}{c}\partial_t + iD) \oplus \ker(\frac{1}{c}\partial_t - iD).$$

The proof of this result can be found in [**20**] and [**50**].

3.1.2. Moving charge. As an example of the application of (3.1.9), let us consider the well known problem of a moving charge. We follow here the reasoning from [**40**]. It is required to find the electromagnetic field generated by the charge q moving with velocity $\overrightarrow{v}(t)$. In this case the charge density ρ has the form

$$\rho(t, x) = q\delta(x - s(t)),$$

where s is the trajectory of the charge, and the vector $\mathbf{j}$ is written as follows

$$\mathbf{j}(t, x) = \overrightarrow{v}(t)\rho(t, x).$$

We will use equality (3.1.9) and the known solution (see, e.g., [**19**]) of the equation

$$\left(\frac{1}{c^2}\partial_t^2 - \Delta\right) u(t, x) = A(t)\delta(x - s(t)), \tag{3.1.10}$$

which is given by the formula

$$u(t,x) = \frac{A(\tau(t))}{4\pi\,|x-s(\tau(t))|\,(1-M(\tau(t)))}, \tag{3.1.11}$$

where

$$M(\tau) := \frac{\langle \overrightarrow{v}(\tau), x-s(\tau)\rangle}{c\,|x-s(\tau)|}$$

is called the Mach number and τ satisfies the equation

$$\frac{|x-s(\tau)|}{c} - (t-\tau) = 0.$$

Taking into account the explicit form of ρ and $\mathbf{j}$, equation (3.1.8) is rewritten as follows

$$(\frac{1}{c}\partial_t + iD)\mathbf{V} = -(\sqrt{\mu}\,\overrightarrow{v}(t)q + \frac{iq}{\sqrt{\varepsilon}})\delta(x-s(t)). \tag{3.1.12}$$

Thus, the purely vectorial biquaternion

$$\mathbf{V}(t,x) = (\frac{1}{c}\partial_t - iD)u(t,x)$$

is a solution of (3.1.12) if u is a solution of (3.1.10) with

$$A(t) = -(\sqrt{\mu}\,\overrightarrow{v}(t)q + \frac{iq}{\sqrt{\varepsilon}}).$$

Note that both A and u are complex quaternionic functions.

Using (3.1.11) we obtain that

$$u_0(t,x) = -\frac{iq}{4\pi\sqrt{\varepsilon}\,|x-s(\tau(t))|\,(1-M(\tau(t)))},$$

and

$$\overrightarrow{u}(t,x) = -\frac{\sqrt{\mu}q\,\overrightarrow{v}(\tau(t))}{4\pi\,|x-s(\tau(t))|\,(1-M(\tau(t)))}.$$

Thus, solution of (3.1.12) reduces to a simple differentiation. That is, we have to calculate the following expression

$$\mathbf{V} = -(\frac{1}{c}\partial_t - iD)\left(\left(\frac{iq}{4\pi\sqrt{\varepsilon}} + \frac{\sqrt{\mu}q\,\overrightarrow{v}(\tau(t))}{4\pi}\right)\frac{1}{|x-s(\tau(t))|\,(1-M(\tau(t)))}\right).$$

We introduce the auxiliary functions

$$\varphi = \frac{1}{|x - s(\tau(t))|\,(1 - M(\tau(t)))},$$

$$f = \frac{iq}{4\pi\sqrt{\varepsilon}} + \frac{\sqrt{\mu} q \overrightarrow{v}(\tau(t))}{4\pi}.$$

Then

$$\mathbf{V} = -(\frac{1}{c}\partial_t - iD)\,[f]\cdot\varphi - (\frac{1}{c}\partial_t - iD)\,[\varphi]\cdot f. \tag{3.1.13}$$

It is easy to see that the scalar part of the expression on the right-hand side is zero. For this purpose we give the following formulas, which can be verified by a straightforward computation.

$$\tau\prime = \frac{1}{1 - M},$$

$$\tau\prime\prime(t) = \frac{1}{c\,|x - s|\,(1 - M)^3}(c^2M^2 + \langle s\prime\prime, x - s\rangle - |s\prime|^2),$$

$$\operatorname{grad}\tau = -\frac{1}{c(1 - M)}\frac{x - s}{|x - s|},$$

$$\partial_t\varphi = \frac{\tau\prime\prime(t)}{|x - s|} + \frac{cM}{|x - s|^2\,(1 - M)^2},$$

$$\operatorname{grad}\varphi = -\frac{x - s}{|x - s|^3}\frac{M + 1}{(1 - M)^2} + \frac{1}{c\,|x - s|^2}\left(\frac{\overrightarrow{v}(\tau)}{(1 - M)^2} - (x - s)\tau\prime\prime\right).$$

We are principally interested in the vector part of (3.1.13), which contains the solution of Maxwell's equations (3.1.1)-(3.1.4). Rewriting it in vector form we obtain the following

$$\mathbf{V} = -\frac{1}{c}(\partial_t\overrightarrow{f}\cdot\varphi + \partial_t\varphi\cdot\overrightarrow{f}) + i\varphi\operatorname{rot}\overrightarrow{f} + if_0\operatorname{grad}\varphi + i\left[\operatorname{grad}\varphi\times\overrightarrow{f}\right].$$

By the definition of $\mathbf{V}$ (3.1.7) we have:

$$\sqrt{\varepsilon}\mathbf{E} = -\frac{1}{c}(\partial_t\overrightarrow{f}\cdot\varphi + \partial_t\varphi\cdot\overrightarrow{f}) - \frac{q}{4\pi\sqrt{\varepsilon}}\operatorname{grad}\varphi$$

and

$$\sqrt{\mu}\mathbf{H} = \varphi\operatorname{rot}\overrightarrow{f} + \left[\operatorname{grad}\varphi\times\overrightarrow{f}\right].$$

To obtain the vectors **E** and **H** in explicit form we use the following intermediate equalities

$$\partial_t \overrightarrow{f} = \frac{\sqrt{\mu}q}{4\pi}\frac{\overrightarrow{v}\prime(\tau)}{1-M(\tau)},$$

$$\operatorname{rot}\overrightarrow{f} = \frac{\sqrt{\mu}q}{4\pi}\left[\operatorname{grad}\tau\times\overrightarrow{v}\prime(\tau)\right],$$

$$\left[\operatorname{grad}\varphi\times\overrightarrow{f}\right] = -\frac{\sqrt{\mu}q}{4\pi}\frac{[(x-s)\times\overrightarrow{v}]}{c^2\left|x-s\right|^3(1-M)^3}(c^2+\langle\overrightarrow{v}\prime,x-s\rangle-|\overrightarrow{v}|^2).$$

Finally, the solution of the problem of a moving charge is obtained in the form

$$\mathbf{E} = -\frac{\mu q}{4\pi}(\frac{\overrightarrow{v}\prime}{|x-s|\,(1-M)^2}+$$
$$(\frac{\overrightarrow{v}\,|x-s|-c(x-s)}{c\left|x-s\right|^3(1-M)^3})(c^2+\langle\overrightarrow{v}\prime,x-s\rangle-|\overrightarrow{v}|^2)),$$

$$\mathbf{H} = -\frac{q}{4\pi}(\frac{[(x-s)\times\overrightarrow{v}\prime]}{c\left|x-s\right|^2(1-M)^2}+$$
$$\frac{[(x-s)\times\overrightarrow{v}]}{c^2\left|x-s\right|^3(1-M)^3}(c^2+\langle\overrightarrow{v}\prime,x-s\rangle-|\overrightarrow{v}|^2)).$$

3.2. Time-harmonic Maxwell's equations

Using the Fourier transform any electromagnetic field can be represented as an infinite superposition of time-harmonic (=monochromatic) fields. These fields are normally the main object of study in radioelectronics, wave propagation theory and many other branches of physics and engineering. A time-harmonic electromagnetic field has the following form

$$\mathbf{E}(x,t) = \operatorname{Re}(\overrightarrow{E}(x)e^{-i\omega t}) \tag{3.2.1}$$

and

$$\mathbf{H}(x,t) = \operatorname{Re}(\overrightarrow{H}(x)e^{-i\omega t}), \tag{3.2.2}$$

where the vectors $\overrightarrow{E}$ and $\overrightarrow{H}$ depend only on the spatial variables $x = (x_1, x_2, x_3)$ and all dependence on time is contained in the factor $e^{-i\omega t}$. $\overrightarrow{E}$ and $\overrightarrow{H}$ are complex vectors called the complex amplitudes of the electromagnetic field; ω is the frequency of oscillations.

Substituting (3.2.1) and (3.2.2) into the Maxwell equations (3.1.1)-(3.1.4) we obtain the equations for the complex amplitudes $\overrightarrow{E}$ and $\overrightarrow{H}$ (the quantities ρ and $\mathbf{j}$ characterizing the sources are also assumed to be time-harmonic $\rho(x,t) = \mathrm{Re}(\rho(x)e^{-i\omega t})$, $\mathbf{j}(x,t) = \mathrm{Re}(\overrightarrow{j}(x)e^{-i\omega t})$):

$$\mathrm{rot}\,\overrightarrow{H} = -i\omega\varepsilon\overrightarrow{E} + \overrightarrow{j}, \tag{3.2.3}$$

$$\mathrm{rot}\,\overrightarrow{E} = i\omega\mu\overrightarrow{H}, \tag{3.2.4}$$

$$\mathrm{div}\,\overrightarrow{E} = \frac{\rho}{\varepsilon}, \tag{3.2.5}$$

$$\mathrm{div}\,\overrightarrow{H} = 0. \tag{3.2.6}$$

Applying the divergence to (3.2.3) we find the relation between ρ and $\overrightarrow{j}$:

$$\mathrm{div}\,\overrightarrow{j} - i\omega\rho = 0. \tag{3.2.7}$$

Note that until Section 4.1.1 the medium is supposed to be homogeneous. Very often ε and μ are considered to be complex quantities. This allows us to describe additional physical phenomena such as, for example, the inertia of the polarization processes (see, e.g., [**66**, Section 3.2]).

Let us denote $\alpha := \omega\sqrt{\varepsilon\mu}$, where the square root is chosen so that $\mathrm{Im}\,\alpha \geq 0$. The quantity α is called the wave number. Note that equations (3.2.3)-(3.2.6) can be rewritten as two quaternionic equations

$$D\overrightarrow{E} = i\omega\mu\overrightarrow{H} - \frac{\rho}{\varepsilon}, \tag{3.2.8}$$

$$D\overrightarrow{H} = -i\omega\varepsilon\overrightarrow{E} + \overrightarrow{j}. \tag{3.2.9}$$

We now explain a simple procedure for the diagonalization of Maxwell's equations which was proposed for the first time in [**44**] (see also [**58**]). We introduce the following pair of purely vectorial biquaternionic functions

$$\overrightarrow{\varphi} := -i\omega\varepsilon\overrightarrow{E} + \alpha\overrightarrow{H}$$

and

$$\overrightarrow{\psi} := i\omega\varepsilon\overrightarrow{E} + \alpha\overrightarrow{H}.$$

We apply the operator D to $\overrightarrow{\varphi}$ and use (3.2.8), (3.2.9):

$$D\overrightarrow{\varphi} = -i\omega\varepsilon D\overrightarrow{E} + \alpha D\overrightarrow{H} = \alpha^2\overrightarrow{H} + i\omega\rho - i\omega\varepsilon\alpha\overrightarrow{E} + \alpha\overrightarrow{j} = \alpha\overrightarrow{\varphi} + i\omega\rho + \alpha\overrightarrow{j}.$$

Thus, $\overrightarrow{\varphi}$ satisfies the equation

$$(D-\alpha)\overrightarrow{\varphi} = \operatorname{div}\vec{j} + \alpha\vec{j}. \tag{3.2.10}$$

Analogously, $\overrightarrow{\psi}$ satisfies the equation

$$(D+\alpha)\overrightarrow{\psi} = -\operatorname{div}\vec{j} + \alpha\vec{j}. \tag{3.2.11}$$

Equations (3.2.10) and (3.2.11) are equivalent to the Maxwell system (3.2.3)-(3.2.6). This procedure of diagonalization can be written in matrix form

$$\begin{pmatrix} D & -i\omega\mu \\ i\omega\varepsilon & D \end{pmatrix} \begin{pmatrix} \overrightarrow{E} \\ \overrightarrow{H} \end{pmatrix} = B_\alpha^{-1} \begin{pmatrix} D-\alpha & 0 \\ 0 & D+\alpha \end{pmatrix} B_\alpha \begin{pmatrix} \overrightarrow{E} \\ \overrightarrow{H} \end{pmatrix},$$

where

$$B_\alpha := \begin{pmatrix} -i\omega\varepsilon & \alpha \\ i\omega\varepsilon & \alpha \end{pmatrix} \quad \text{and} \quad B_\alpha^{-1} = \frac{1}{2}\begin{pmatrix} -1/i\omega\varepsilon & 1/i\omega\varepsilon \\ 1/\alpha & 1/\alpha \end{pmatrix}.$$

In this way we obtain two decoupled equations for the unknown vectors $\overrightarrow{\varphi}$ and $\overrightarrow{\psi}$, which substantially simplifies the analysis of the Maxwell equations. Now we use this fact in order to obtain integral representations for vectors of the electromagnetic field.

Using the Borel–Pompeiu formula we can rewrite (3.2.10) and (3.2.11) in the following form

$$\overrightarrow{\varphi} = T_{-\alpha}(\operatorname{div}\vec{j} + \alpha\vec{j}) + K_{-\alpha}\overrightarrow{\varphi}, \tag{3.2.12}$$

$$\overrightarrow{\psi} = T_{\alpha}(-\operatorname{div}\vec{j} + \alpha\vec{j}) + K_{\alpha}\overrightarrow{\psi}. \tag{3.2.13}$$

Rewriting (3.2.12), (3.2.13) for the electromagnetic field we obtain:

$$-i\omega\varepsilon\vec{E} + \alpha\vec{H} = T_{-\alpha}(\operatorname{div}\vec{j} + \alpha\vec{j}) + K_{-\alpha}(-i\omega\varepsilon\vec{E} + \alpha\vec{H}),$$

$$i\omega\varepsilon\vec{E} + \alpha\vec{H} = T_{\alpha}(-\operatorname{div}\vec{j} + \alpha\vec{j}) + K_{\alpha}(i\omega\varepsilon\vec{E} + \alpha\vec{H}),$$

or equivalently:

$$\begin{aligned}\vec{E} = \frac{1}{2i\omega\varepsilon}&(i\omega\varepsilon(K_\alpha + K_{-\alpha})\vec{E} + \alpha(K_\alpha - K_{-\alpha})\vec{H} \\ &- (T_\alpha + T_{-\alpha})\operatorname{div}\vec{j} + \alpha(T_\alpha - T_{-\alpha})\vec{j}),\end{aligned} \tag{3.2.14}$$

$$\begin{aligned}\vec{H} = \frac{1}{2\alpha}&(\alpha(K_\alpha + K_{-\alpha})\vec{H} + i\omega\varepsilon(K_\alpha - K_{-\alpha})\vec{E} \\ &- (T_\alpha - T_{-\alpha})\operatorname{div}\vec{j} + \alpha(T_\alpha + T_{-\alpha})\vec{j}).\end{aligned} \tag{3.2.15}$$

To compare the last two equalities with some known formulas from electromagnetic theory (3.2.14), (3.2.15) must be rewritten in vector terms. We remind readers that

$$K_{\pm\alpha}\vec{f}(x) = \int\limits_{\Gamma} \{\pm\alpha\vartheta(x-y)\langle\vec{n}(y), \vec{f}(y)\rangle - \langle[\operatorname{grad}_x \vartheta(x-y) \times \vec{n}(y)], \vec{f}(y)\rangle \mp$$

$$\mp \alpha\vartheta(x-y)[\vec{n}(y) \times \vec{f}(y)] - \langle \operatorname{grad}_x \vartheta(x-y), \vec{n}(y)\rangle \vec{f}(y) +$$

$$+ [[\operatorname{grad}_x \vartheta(x-y) \times \vec{n}(y)] \times \vec{f}(y)]\} d\Gamma_y.$$

From this explicit form of the quaternionic integral we obtain the following necessary combinations

$$(K_\alpha + K_{-\alpha})\vec{f}(x) = 2\int\limits_{\Gamma} \{-\langle[\operatorname{grad}_x \vartheta(x-y) \times \vec{n}(y)], \vec{f}(y)\rangle$$

$$- \langle \operatorname{grad}_x \vartheta(x-y), \vec{n}(y)\rangle \vec{f}(y)$$

$$+ [[\operatorname{grad}_x \vartheta(x-y) \times \vec{n}(y)] \times \vec{f}(y)]\} d\Gamma_y,$$

$$(K_\alpha - K_{-\alpha})\vec{f}(x) = 2\alpha\int\limits_{\Gamma} \{\vartheta(x-y)\langle\vec{n}(y), \vec{f}(y)\rangle$$

$$- \vartheta(x-y)[\vec{n}(y) \times \vec{f}(y)]\} d\Gamma_y.$$

Reasoning along similar lines we obtain explicit representations for the necessary combinations of the operators T_α and $T_{-\alpha}$:

$$(T_\alpha + T_{-\alpha})f_0(x) = -2\int\limits_{\Omega} \operatorname{grad}_x \vartheta(x-y) f_0(y) dy,$$

$$(T_\alpha - T_{-\alpha})f_0(x) = 2\alpha\int\limits_{\Omega} \vartheta(x-y) f_0(y) dy,$$

$$(T_\alpha+T_{-\alpha})\vec{f}(x) = 2\int\limits_\Omega \{\langle \operatorname{grad}_x \vartheta(x-y), \vec{f}(y)\rangle - [\operatorname{grad}_x \vartheta(x-y)\times \vec{f}(y)]\}dy,$$

$$(T_\alpha - T_{-\alpha})\vec{f}(x) = 2\alpha \int\limits_\Omega \vartheta(x-y)\vec{f}(y)dy.$$

Using all of these combinations we obtain the following integral representations for the electromagnetic field

$$\vec{E}(x) = \int\limits_\Gamma \{-\langle[\operatorname{grad}_x \vartheta(x-y)\times \vec{n}(y)], \vec{E}(y)\rangle -$$

$$-\langle \operatorname{grad}_x \vartheta(x-y), \vec{n}(y)\rangle \vec{E}(y) + [[\operatorname{grad}_x \vartheta(x-y)\times \vec{n}(y)]\times \vec{E}(y)]\}d\Gamma_y +$$

$$+i\omega\mu \int\limits_\Gamma \{-\vartheta(x-y)\langle \vec{n}(y), \vec{H}(y)\rangle + \vartheta(x-y)[\vec{n}(y)\times \vec{H}(y)]\}d\Gamma_y +$$

$$+\int\limits_\Omega \{\frac{1}{i\omega\varepsilon} \operatorname{grad}_x \vartheta(x-y) \operatorname{div} \vec{j}(y) - i\omega\mu\vartheta(x-y)\vec{j}(y)\}dy,$$

$$\vec{H}(x) = \int\limits_\Gamma \{-\langle[\operatorname{grad}_x \vartheta(x-y)\times \vec{n}(y)], \vec{H}(y)\rangle -$$

$$-\langle \operatorname{grad}_x \vartheta(x-y), \vec{n}(y)\rangle \vec{H}(y) + [[\operatorname{grad}_x \vartheta(x-y)\times \vec{n}(y)]\times \vec{H}(y)]\}d\Gamma_y -$$

$$-i\omega\varepsilon \int\limits_\Gamma \{-\vartheta(x-y)\langle \vec{n}(y), \vec{E}(y)\rangle + \vartheta(x-y)[\vec{n}(y)\times \vec{E}(y)]\}d\Gamma_y -$$

$$-\int\limits_\Omega \{\vartheta(x-y) \operatorname{div} \vec{j}(y) - \langle \operatorname{grad}_x \vartheta(x-y), \vec{j}(y)\rangle + [\operatorname{grad}_x \vartheta(x-y)\times \vec{j}(y)]\}dy.$$

Separating the scalar and the vector parts in the last two equalities we obtain the following four equations

(3.2.16)
$$\int_\Gamma \{\langle[\operatorname{grad}_x \vartheta(x-y)\times\vec{n}(y)],\vec{E}(y)\rangle + i\omega\mu\vartheta(x-y)\langle\vec{n}(y),\vec{H}(y)\rangle\}d\Gamma_y = 0,$$

$$\int_\Gamma \{\langle[\operatorname{grad}_x \vartheta(x-y)\times\vec{n}(y)],\vec{H}(y)\rangle - i\omega\varepsilon\vartheta(x-y)\langle\vec{n}(y),\vec{E}(y)\rangle\}d\Gamma_y =$$

$$= -\int_\Omega \{\vartheta(x-y)\operatorname{div}\vec{j}(y) - \langle\operatorname{grad}_x \vartheta(x-y),\vec{j}(y)\rangle\}dy, \tag{3.2.17}$$

(3.2.18)
$$\begin{aligned}\vec{E}(x) \quad = \quad & \int_\Gamma \{-\langle\operatorname{grad}_x \vartheta(x-y),\vec{n}(y)\rangle\vec{E}(y)+ \\ & [[\operatorname{grad}_x \vartheta(x-y)\times\vec{n}(y)]\times\vec{E}(y)] + i\omega\mu\vartheta(x-y)[\vec{n}(y)\times\vec{H}(y)]\}d\Gamma_y+ \\ & \int_\Omega \{\frac{1}{i\omega\varepsilon}\operatorname{grad}_x \vartheta(x-y)\operatorname{div}\vec{j}(y) - i\omega\mu\vartheta(x-y)\vec{j}(y)\}dy,\end{aligned}$$

$$\begin{aligned}\vec{H}(x) \quad = \quad & \int_\Gamma \{-\langle\operatorname{grad}_x \vartheta(x-y),\vec{n}(y)\rangle\vec{H}(y)+ \\ & [[\operatorname{grad}_x \vartheta(x-y)\times\vec{n}(y)]\times\vec{H}(y)]- \\ & i\omega\varepsilon\vartheta(x-y)[\vec{n}(y)\times\vec{E}(y)]\}d\Gamma_y- \\ & \int_\Omega [\operatorname{grad}_x \vartheta(x-y)\times\overrightarrow{j}(y)]dy.\end{aligned} \tag{3.2.19}$$

The last two equalities represent one of the central theoretical facts of the modern electrodynamics, the Stratton–Chu formulas, which have many useful applications in different types of boundary value problems for the Maxwell equations (see, e.g., [**22, 23, 86**]).

To understand what the integral equalities (3.2.16), (3.2.17) are, we remind readers of a simple auxiliary vectorial identity (see [**44**] and also [**58**, p.120])

$$\int_\Gamma \{\langle \operatorname{grad}_x \vartheta(x-y), [\vec{n}(y) \times \vec{f}(y)]\rangle + \vartheta(x-y)\langle \vec{n}(y), \operatorname{rot} \vec{f}(y)\rangle\} d\Gamma_y = 0, \tag{3.2.20}$$

$x \in \Omega$, which is valid for any $\vec{f} \in C^1(\overline{\Omega})$ and in general does not make sense for $x \in \Gamma$. Then to obtain (3.2.16) from (3.2.20) we have to take $\vec{E}$ as $\vec{f}$ and use one of the Maxwell equations: $\operatorname{rot} \vec{E} = i\omega\mu\vec{H}$. In order to obtain (3.2.17) it is necessary to use the following chain of equalities:

$$\int_\Omega \{\vartheta(x-y) \operatorname{div}_y \vec{j}(y) + \langle \operatorname{grad}_y \vartheta(x-y), \vec{j}(y)\rangle\} dy =$$

$$= \int_\Omega \operatorname{div}_y[\vartheta(x-y)\vec{j}(y)] dy = \int_\Gamma \vartheta(x-y)\langle \vec{n}(y), \vec{j}(y)\rangle d\Gamma_y,$$

where Gauss' theorem was used as well as the following equality:

$$\operatorname{grad}_x \vartheta(x-y) = -\operatorname{grad}_y \vartheta(x-y).$$

Thus, (3.2.17) takes the form:

$$\int_\Gamma \{\langle [\operatorname{grad}_x \vartheta(x-y) \times \vec{n}(y)], \vec{H}(y)\rangle - i\omega\varepsilon\vartheta(x-y)\langle \vec{n}(y), \vec{E}(y)\rangle\} d\Gamma_y =$$

$$= -\int_\Gamma \vartheta(x-y)\langle \vec{n}(y), \vec{j}(y)\rangle d\Gamma_y, \tag{3.2.21}$$

which equals (3.2.20) if we take $\vec{H}$ as $\vec{f}$ and use the Maxwell equation: $\operatorname{rot} \vec{H} = -i\omega\varepsilon\vec{E} + \vec{j}$.

3.3. Boundary value problems for electromagnetic fields

3.3.1. The Dirichlet boundary value problem. Let $g \in C(\Omega; \mathbb{H}(\mathbb{C}))$ and $v \in C^{0,\epsilon}(\Gamma; \mathbb{H}(\mathbb{C}))$. Consider the following boundary value problem.

PROBLEM 5.

$$(D + \alpha)f = g \qquad \textit{in} \;\; \Omega \tag{3.3.1}$$

and

$$f|_\Gamma = v. \tag{3.3.2}$$

We introduce a new function

$$u := f - T_\alpha g.$$

If f solves the problem (3.3.1), (3.3.2), u is a solution of the boundary value problem

$$D_\alpha u = 0 \qquad \text{in} \;\; \Omega,$$

$$u|_\Gamma = w,$$

where

$$w(x) := v(x) - T_\alpha g(x), \qquad x \in \Gamma.$$

Thus we obtain the Dirichlet problem considered on p. 38. The solution of this problem exists if and only if the function w fulfills condition (2.5.33)

$$w = S_\alpha w \qquad \text{on} \;\; \Gamma.$$

In other words,

$$v - T_\alpha g = S_\alpha v - S_\alpha T_\alpha g \qquad \text{on} \;\; \Gamma. \tag{3.3.3}$$

If this condition is fulfilled then the solution is

$$u = K_\alpha w = K_\alpha (v - T_\alpha g).$$

Consider the expression $K_\alpha T_\alpha g$. From Borel-Pompeiu's formula we have that

$$K_\alpha T_\alpha g = (I - T_\alpha D_\alpha) T_\alpha g = 0. \tag{3.3.4}$$

Thus

$$u = K_\alpha v. \tag{3.3.5}$$

Moreover, (3.3.4) gives us that $P_\alpha T_\alpha g = 0$ on Γ, that is $T_\alpha g = -S_\alpha T_\alpha g$, and condition (3.3.3) can be rewritten as follows

$$v - S_\alpha v = 2T_\alpha g \qquad \text{on} \ \ \Gamma,$$

or

$$Q_\alpha v = T_\alpha g \qquad \text{on} \ \ \Gamma. \tag{3.3.6}$$

Now, returning to the boundary value problem 5, we obtain its solution in the form

$$f = K_\alpha v + T_\alpha g$$

under the necessary and sufficient condition (3.3.6). In fact such a function obviously satisfies (3.3.1), and due to the Plemelj-Sokhotski formulas and (3.3.6), we obtain on the boundary the required equality (3.3.2)

$$f|_\Gamma = P_\alpha v + (T_\alpha g)|_\Gamma = P_\alpha v + Q_\alpha v = v.$$

We now consider the inhomogeneous Dirichlet boundary value problem for the Maxwell equations.

PROBLEM 6. *Find a pair of vectors $\overrightarrow{E}$ and $\overrightarrow{H}$ satisfying (3.2.3)-(3.2.6) in Ω as well as the following conditions on the boundary:*

$$\overrightarrow{E}|_\Gamma = \overrightarrow{e} \qquad \text{and} \qquad \overrightarrow{H}|_\Gamma = \overrightarrow{h},$$

where $\overrightarrow{e}$ and $\overrightarrow{h}$ are given complex vector functions defined on Γ and satisfying the Hölder condition there.

This problem is equivalent to the following boundary value problem for a pair of complex quaternionic functions.

PROBLEM 7. *Find two purely vectorial biquaternionic functions $\overrightarrow{\varphi}$ and $\overrightarrow{\psi}$ satisfying (3.2.10), (3.2.11) and the boundary conditions*

$$\overrightarrow{\varphi}|_\Gamma = -i\omega\varepsilon\overrightarrow{e} + \alpha\overrightarrow{h} \qquad \text{and} \qquad \overrightarrow{\psi}|_\Gamma = i\omega\varepsilon\overrightarrow{e} + \alpha\overrightarrow{h}.$$

Necessary and sufficient conditions for the existence of the solution of this problem are the equalities

$$Q_{-\alpha}(-i\omega\varepsilon\overrightarrow{e} + \alpha\overrightarrow{h}) = T_{-\alpha}(\operatorname{div}\overrightarrow{j} + \alpha\overrightarrow{j}) \qquad \text{on} \quad \Gamma \tag{3.3.7}$$

and

$$Q_{\alpha}(i\omega\varepsilon\overrightarrow{e} + \alpha\overrightarrow{h}) = T_{\alpha}(-\operatorname{div}\overrightarrow{j} + \alpha\overrightarrow{j}) \qquad \text{on} \quad \Gamma. \tag{3.3.8}$$

Thus, Problem 6 is solvable if and only if (3.3.7) and (3.3.8) are fulfilled and if this is the case, the solution is given by the equalities

$$\overrightarrow{E} = \frac{1}{2i\omega\varepsilon}(i\omega\varepsilon(K_\alpha + K_{-\alpha})\overrightarrow{e} + \alpha(K_\alpha - K_{-\alpha})\overrightarrow{h}$$

$$-(T_\alpha + T_{-\alpha})\operatorname{div}\overrightarrow{j} + \alpha(T_\alpha - T_{-\alpha})\overrightarrow{j})$$

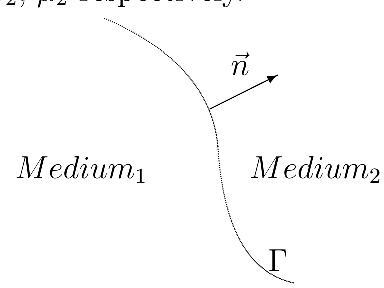
and

$$\overrightarrow{H}=\frac{1}{2\alpha}(\alpha(K_\alpha+K_{-\alpha})\overrightarrow{h}+i\omega\varepsilon(K_\alpha-K_{-\alpha})\overrightarrow{e}$$

$$-(T_\alpha-T_{-\alpha})\operatorname{div}\overrightarrow{j}+\alpha(T_\alpha+T_{-\alpha})\overrightarrow{j}),$$

or in vectorial form by the Stratton-Chu formulas on p. 63, where the vectors $\overrightarrow{E}$ and $\overrightarrow{H}$ on the right-hand side must be replaced by their boundary values $\overrightarrow{e}$ and $\overrightarrow{h}$.

3.3.2. Charged boundary. Let Γ be a boundary between two different physical media, as shown in the next figure, with electromagnetic characteristics ε_1, μ_1 and ε_2, μ_2 respectively.



In general Γ can be charged. The corresponding surface charge density we denote by ρ_s and the surface current density by $\overrightarrow{j}_s$. In this case the set of the so-called natural boundary conditions for an electromagnetic field has the form

$$\varepsilon_1\left\langle\overrightarrow{E}^{(1)},\overrightarrow{n}\right\rangle-\varepsilon_2\left\langle\overrightarrow{E}^{(2)},\overrightarrow{n}\right\rangle=\rho_s,$$

$$\mu_1\left\langle\overrightarrow{H}^{(1)},\overrightarrow{n}\right\rangle-\mu_2\left\langle\overrightarrow{H}^{(2)},\overrightarrow{n}\right\rangle=0,$$

$$\left[\overrightarrow{n}\times\overrightarrow{E}^{(1)}\right]-\left[\overrightarrow{n}\times\overrightarrow{E}^{(2)}\right]=0,$$

$$\left[\overrightarrow{n} \times \overrightarrow{H}^{(1)}\right] - \left[\overrightarrow{n} \times \overrightarrow{H}^{(2)}\right] = \overrightarrow{j}_s.$$

Here $\overrightarrow{E}^{(1)}$ and $\overrightarrow{H}^{(1)}$ represent the boundary values of the electromagnetic field from Medium_1 and $\overrightarrow{E}^{(2)}$, $\overrightarrow{H}^{(2)}$ those from Medium_2. The scalar products $\langle \cdot, \overrightarrow{n} \rangle$ are called the normal components of the field vectors which stand for the point in the brackets. The vector products $[\overrightarrow{n} \times \cdot]$ are called the tangential components of the corresponding vectors. If the boundary is not charged, then we see that the tangential components of the electromagnetic field are continuous and only the normal components "suffer" because of a sharp change of the electromagnetic characteristics of the medium.

Let us consider now the situation in which the charged surface Γ is submerged in a homogeneous medium with characteristics ε and μ. In other words, Medium_1 and Medium_2 are the same but ρ_s and $\overrightarrow{j}_s$ are not zero. We suppose that Γ is a closed Liapunov surface which divides the whole space $\mathbb{R}^3$ into two domains. We denote the bounded domain Ω^+ and its complement Ω^-. We suppose also that there are no field sources in $\Omega^\pm$. We find an electromagnetic field $\overrightarrow{E}$ and $\overrightarrow{H}$ in the whole space $\mathbb{R}^3$ which satisfies the Maxwell equations

$$\text{rot}\,\overrightarrow{H} = -i\omega\varepsilon\overrightarrow{E}, \tag{3.3.9}$$

$$\text{rot}\,\overrightarrow{E} = i\omega\mu\overrightarrow{H}, \tag{3.3.10}$$

$$\text{div}\,\overrightarrow{E} = 0, \tag{3.3.11}$$

$$\text{div}\,\overrightarrow{H} = 0 \tag{3.3.12}$$

for any $x \in \mathbb{R}^3 \setminus \Gamma$, as well as the boundary conditions on Γ

$$\left\langle \overrightarrow{E}^+, \overrightarrow{n} \right\rangle - \left\langle \overrightarrow{E}^-, \overrightarrow{n} \right\rangle = \frac{\rho_s}{\varepsilon}, \tag{3.3.13}$$

$$\left\langle \overrightarrow{H}^{+}, \overrightarrow{n} \right\rangle - \left\langle \overrightarrow{H}^{-}, \overrightarrow{n} \right\rangle = 0, \tag{3.3.14}$$

$$\left[\overrightarrow{n} \times \overrightarrow{E}^{+} \right] - \left[\overrightarrow{n} \times \overrightarrow{E}^{-} \right] = 0, \tag{3.3.15}$$

$$\left[\overrightarrow{n} \times \overrightarrow{H}^{+} \right] - \left[\overrightarrow{n} \times \overrightarrow{H}^{-} \right] = \overrightarrow{j}_s. \tag{3.3.16}$$

and the Silver-Müller condition at infinity:

$$\left[\overrightarrow{E}^{-} \times \frac{x}{|x|} \right] + W\overrightarrow{H}^{-} = o(\frac{1}{|x|}), \qquad |x| \to \infty, \tag{3.3.17}$$

which represents the radiation condition for the Maxwell equations. Here W is the intrinsic wave impedance of the medium defined as $W = \sqrt{\mu}/\sqrt{\varepsilon}$ and measured in ohms. It is known (see, e.g., [**22**]) that if the field $\overrightarrow{E}^{-}$, $\overrightarrow{H}^{-}$ fulfills (3.3.17) then another version of the Silver-Müller condition,

$$\left[\overrightarrow{H}^{-} \times \frac{x}{|x|} \right] - \frac{1}{W}\overrightarrow{E}^{-} = o(\frac{1}{|x|}), \qquad |x| \to \infty \tag{3.3.18}$$

also holds.

We rewrite the boundary value problem (3.3.9)-(3.3.17) in quaternionic form using the auxiliary biquaternions $\overrightarrow{\varphi}$ and $\overrightarrow{\psi}$. The Maxwell equations (3.3.9)-(3.3.12) take the form

$$D_{-\alpha}\overrightarrow{\varphi} = 0, \tag{3.3.19}$$

$$D_{\alpha}\overrightarrow{\psi} = 0 \tag{3.3.20}$$

for any $x \in \mathbb{R}^3 \backslash \Gamma$. In order to rewrite the boundary conditions (3.3.13)-(3.3.16) we consider the expression

$$\begin{aligned} \overrightarrow{n}(\overrightarrow{\varphi}^{+} - \overrightarrow{\varphi}^{-}) &= -i\omega\varepsilon\overrightarrow{n}(\overrightarrow{E}^{+} - \overrightarrow{E}^{-}) + \alpha\overrightarrow{n}(\overrightarrow{H}^{+} - \overrightarrow{H}^{-}) \\ &= i\omega\rho_s + \alpha\overrightarrow{j}_s. \end{aligned}$$

That is, the boundary condition for $\overrightarrow{\varphi}$ is obtained in the form

$$\overrightarrow{\varphi}^{+} - \overrightarrow{\varphi}^{-} = -\overrightarrow{n}(i\omega\rho_s + \alpha\overrightarrow{j}_s). \tag{3.3.21}$$

In the same way we obtain the boundary condition for $\overrightarrow{\psi}$:

$$\overrightarrow{\psi}^{+} - \overrightarrow{\psi}^{-} = \overrightarrow{n}(i\omega\rho_s - \alpha\overrightarrow{j}_s). \tag{3.3.22}$$

Now let us consider the radiation condition (2.5.24) applied to $\overrightarrow{\varphi}$ and $\overrightarrow{\psi}$. We have

$$(1 - \frac{ix}{|x|})\overrightarrow{\varphi}(x) = o(\frac{1}{|x|}), \qquad |x| \to \infty, \tag{3.3.23}$$

and

$$(1 + \frac{ix}{|x|})\overrightarrow{\psi}(x) = o(\frac{1}{|x|}), \qquad |x| \to \infty. \tag{3.3.24}$$

For

$$\overrightarrow{E} = \frac{1}{2i\omega\varepsilon}(\overrightarrow{\psi} - \overrightarrow{\varphi}) \qquad \text{and} \qquad \overrightarrow{H} = \frac{1}{2\alpha}(\overrightarrow{\varphi} + \overrightarrow{\psi})$$

we obtain

$$\overrightarrow{E} = -\frac{1}{2\omega\varepsilon}\frac{x}{|x|}(\overrightarrow{\varphi} + \overrightarrow{\psi}) + o(\frac{1}{|x|}) = -\frac{\alpha}{\omega\varepsilon}\frac{x}{|x|}\frac{1}{2\alpha}(\overrightarrow{\varphi} + \overrightarrow{\psi}) + o(\frac{1}{|x|})$$

$$= -W\frac{x}{|x|}\overrightarrow{H} + o(\frac{1}{|x|}),$$

from which we have (3.3.18). Starting from $\overrightarrow{H}$ we obtain (3.3.17):

$$\overrightarrow{H} = \frac{1}{2\alpha}\frac{ix}{|x|}(\overrightarrow{\varphi} - \overrightarrow{\psi}) + o(\frac{1}{|x|}) = \frac{\omega\varepsilon}{\alpha}\frac{1}{2i\omega\varepsilon}\frac{x}{|x|}(\overrightarrow{\psi} - \overrightarrow{\varphi}) + o(\frac{1}{|x|})$$

$$= \frac{1}{W}\frac{x}{|x|}\overrightarrow{E} + o(\frac{1}{|x|}).$$

Thus, the boundary value problem for Maxwell's equations (3.3.9)-(3.3.17) in terms of complex quaternionic functions $\overrightarrow{\varphi}$ and $\overrightarrow{\psi}$ has the form (3.3.19)-(3.3.24). It is easy to see that for each of the functions $\overrightarrow{\varphi}$

and $\overrightarrow{\psi}$ we now have a jump problem with the corresponding radiation condition at infinity (see Problem 3 on p. 39). The solution can be found immediately:

$$\overrightarrow{\varphi} = -K_{-\alpha}[\overrightarrow{n} \cdot (i\omega\rho_s + \alpha \overrightarrow{j}_s)],$$

$$\overrightarrow{\psi} = K_{\alpha}[\overrightarrow{n} \cdot (i\omega\rho_s - \alpha \overrightarrow{j}_s)].$$

For the electromagnetic field vectors we obtain the following representations

$$\overrightarrow{E} = \frac{1}{2i\omega\varepsilon}(\overrightarrow{\psi} - \overrightarrow{\varphi})$$

$$= \frac{1}{2i\omega\varepsilon}\left(K_{-\alpha}[\overrightarrow{n} \cdot (i\omega\rho_s + \alpha \overrightarrow{j}_s)] + K_{\alpha}[\overrightarrow{n} \cdot (i\omega\rho_s - \alpha \overrightarrow{j}_s)]\right)$$

$$= \frac{1}{2}\left(\frac{1}{\varepsilon}(K_{-\alpha} + K_{\alpha})[\overrightarrow{n} \cdot \rho_s] - iW(K_{-\alpha} - K_{\alpha})[\overrightarrow{n} \cdot \overrightarrow{j}_s]\right)$$

and

$$\overrightarrow{H} = \frac{1}{2\alpha}(\overrightarrow{\varphi} + \overrightarrow{\psi})$$

$$= \frac{1}{2\alpha}\left(-K_{-\alpha}[\overrightarrow{n} \cdot (i\omega\rho_s + \alpha \overrightarrow{j}_s)] + K_{\alpha}[\overrightarrow{n} \cdot (i\omega\rho_s - \alpha \overrightarrow{j}_s)]\right)$$

$$= \frac{i}{2W}\left(\frac{1}{\varepsilon}(K_{\alpha} - K_{-\alpha})[\overrightarrow{n} \cdot \rho_s] + iW(K_{\alpha} + K_{-\alpha})[\overrightarrow{n} \cdot \overrightarrow{j}_s]\right),$$

which give us the solution of (3.3.9)-(3.3.17).

3.4. Chiral media

In this section we show that the same technique can be applied also to the Maxwell equations in chiral media. The analysis of Maxwell's equations for chiral media is of great importance due to the numerous applications of such materials in telecommunications engineering, chemistry, physics and other fields (see, e.g., [**59**]). Here we use the results of [**41**].

3.4.1. Electromagnetic fields in chiral media. Let us consider Maxwell's equations

$$\operatorname{div} \widetilde{E}\,(x) = \frac{\rho(x)}{\varepsilon}, \tag{3.4.1}$$

$$\operatorname{rot} \widetilde{E}\,(x) = i\omega \widetilde{B}\,(x)\,, \tag{3.4.2}$$

$$\operatorname{rot} \widetilde{H}\,(x) = -i\omega \widetilde{D}\,(x) + \widetilde{j}\,(x)\,, \tag{3.4.3}$$

$$\operatorname{div} \widetilde{H}\,(x) = 0 \tag{3.4.4}$$

with the constitutive relations [**59**]

$$\widetilde{D} = \varepsilon\left(\widetilde{E}\,(x) + \beta \operatorname{rot} \widetilde{E}\,(x)\right), \tag{3.4.5}$$

$$\widetilde{B} = \mu\left(\widetilde{H}\,(x) + \beta \operatorname{rot} \widetilde{H}\,(x)\right), \tag{3.4.6}$$

where β is the chirality measure of the medium. The Maxwell equations (3.4.1)-(3.4.4) can be also written as follows

$$\operatorname{rot} \widetilde{E}(x) = i\omega\mu \left(\widetilde{H}(x) + \beta \operatorname{rot} \widetilde{H}(x) \right), \tag{3.4.7}$$

$$\operatorname{rot} \widetilde{H}(x) = -i\omega\varepsilon \left(\widetilde{E}(x) + \beta \operatorname{rot} \widetilde{E}(x) \right) + \widetilde{j}(x). \tag{3.4.8}$$

Introducing the notations

$$\widetilde{E}(x) = -\sqrt{\mu} \cdot \overrightarrow{E}(x), \tag{3.4.9}$$

$$\widetilde{H}(x) = \sqrt{\varepsilon} \cdot \overrightarrow{H}(x), \tag{3.4.10}$$

$$\widetilde{j}(x) = \sqrt{\varepsilon} \cdot \overrightarrow{j}(x), \tag{3.4.11}$$

we obtain the equations

$$\operatorname{rot} \overrightarrow{E}(x) = -i\alpha \left(\overrightarrow{H}(x) + \beta \operatorname{rot} \overrightarrow{H}(x) \right) \tag{3.4.12}$$

and

$$\operatorname{rot} \overrightarrow{H}(x) = i\alpha \left(\overrightarrow{E}(x) + \beta \operatorname{rot} \overrightarrow{E}(x) \right) + \overrightarrow{j}(x), \tag{3.4.13}$$

where as before $\alpha = \omega\sqrt{\varepsilon\mu}$ is the wave number.

3.4.2. Maxwell's equations for chiral media in quaternionic form. Let us consider the following purely vectorial biquaternionic functions:

$$\overrightarrow{\varphi}(x) = \overrightarrow{E}(x) + i\overrightarrow{H}(x) \tag{3.4.14}$$

and

$$\overrightarrow{\psi}(x) = \overrightarrow{E}(x) - i\overrightarrow{H}(x). \tag{3.4.15}$$

We have that

$$D\overrightarrow{\varphi}(x) = \frac{\rho(x)}{\varepsilon\sqrt{\mu}} + \operatorname{rot}\overrightarrow{E}(x) + i\operatorname{rot}\overrightarrow{H}(x). \tag{3.4.16}$$

Using (3.4.12) and (3.4.13) we obtain

$$D\overrightarrow{\varphi}(x) = \frac{\rho(x)}{\varepsilon\sqrt{\mu}} - i\alpha\left(\overrightarrow{H}(x) + \beta\operatorname{rot}\overrightarrow{H}(x)\right) +$$

$$i\left(i\alpha\left(\overrightarrow{E}(x) + \beta\operatorname{rot}\overrightarrow{E}(x)\right) + \overrightarrow{j}(x)\right)$$

$$= \frac{\rho(x)}{\varepsilon\sqrt{\mu}}(1+\alpha\beta) - (i\alpha\overrightarrow{H}(x) + \alpha\overrightarrow{E}(x)) -$$

$$\alpha\beta\left(D\overrightarrow{E}(x) + iD\overrightarrow{H}(x)\right) + i\overrightarrow{j}(x).$$

That is,

$$D\overrightarrow{\varphi}(x) = \frac{\rho(x)}{\varepsilon\sqrt{\mu}}(1+\alpha\beta) - \alpha\overrightarrow{\varphi}(x) - \alpha\beta D\overrightarrow{\varphi}(x) + i\overrightarrow{j}(x).$$

Thus the complex quaternionic function $\overrightarrow{\varphi}$ satisfies the following equation

$$\left(D+\frac{\alpha}{(1+\alpha\beta)}\right)\overrightarrow{\varphi}(x)=\frac{\rho(x)}{\varepsilon\sqrt{\mu}}+i\frac{\overrightarrow{j}(x)}{(1+\alpha\beta)}. \tag{3.4.17}$$

By analogy we obtain the equation for $\overrightarrow{\psi}$

$$\left(D-\frac{\alpha}{(1-\alpha\beta)}\right)\overrightarrow{\psi}(x)=\frac{\rho(x)}{\varepsilon\sqrt{\mu}}-i\frac{\overrightarrow{j}(x)}{(1-\alpha\beta)}. \tag{3.4.18}$$

Note that from (3.4.1) and (3.4.13) the continuity equation follows:

$$\frac{\rho}{\varepsilon\sqrt{\mu}}=-\frac{i}{\alpha}\operatorname{div}\overrightarrow{j}. \tag{3.4.19}$$

Introducing the notations

$$\alpha_1=\frac{\alpha}{(1+\alpha\beta)},\qquad \alpha_2=\frac{\alpha}{(1-\alpha\beta)}$$

and using (3.4.19) we rewrite the equations (3.4.17) and (3.4.18) in the form

$$(D+\alpha_1)\overrightarrow{\varphi}(x)=\frac{i}{\alpha}(\alpha_1\overrightarrow{j}(x)-\operatorname{div}\overrightarrow{j}(x)) \tag{3.4.20}$$

and

$$(D-\alpha_2)\overrightarrow{\psi}(x)=-\frac{i}{\alpha}(\alpha_2\overrightarrow{j}(x)+\operatorname{div}\overrightarrow{j}(x)). \tag{3.4.21}$$

When $\beta=0$ we arrive at the quaternionic form of the Maxwell equations in the nonchiral case, but in general the wave numbers α_1 and α_2 are different and physically characterize the propagation of waves of opposing circular polarizations.

In any case the analysis of Maxwell's equations reduces again to the analysis of the operators $D_{\pm\alpha}$ and results such as those discussed in the preceding section can be obtained also for chiral media (see [**41**]).

3.5. The Dirac equation for a free particle

We will consider the Dirac operator in its covariant form

$$\mathbb{D}[\Phi] := \left(\gamma_0 \partial_t - \sum_{k=1}^{3} \gamma_k \partial_k + im\right)[\Phi],$$

where Φ is a $\mathbb{C}^4$-valued function $\Phi = (\Phi_0, \Phi_1, \Phi_2, \Phi_3)^T$; $m \in \mathbb{R}$ represents the mass of a particle and the Dirac γ-matrices are defined as follows (see, e.g., [**18, 79**])

$$\gamma_0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma_1 := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_2 := \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma_3 := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

We will need also the following matrix, usually denoted as γ_5

$$\gamma_5 := i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Recall some important properties of the γ-matrices:

(1) $\gamma_0^2 = \gamma_5^2 = E_4$, the identity matrix.

(2) $\gamma_k^2 = -E_4$, $k = 1, 2, 3$.

(3) $\gamma_j\gamma_k = -\gamma_k\gamma_j$, $j, k = 0, 1, 2, 3, 5$, $j \neq k$.

3.5.1. Historical remarks. From the very appearance of Dirac's equation (introduced by P. Dirac in 1928) in quantum mechanics many researchers considered that the form in which it was written is unsatisfactory. Their reason is that the algebra of matrices used for this purpose has dimension 16, and there does not exist any physically meaningful reason for using an algebra of 16 dimensions if the number of equations under consideration is four. This discrepancy, which sometimes makes work with the Dirac equation more difficult that it could be, motivates the posing by A. Sommerfeld of the following problem: to rewrite the Dirac equation in a form in which the rank of the algebra of the matrices involved coincides with the number of components of the wavefunction. Apparently he made a first attempt to solve it (see [**76**]). As early as 1929 appeared an article by C. Lanczos [**60**] in which the Dirac equation

$$\mathbb{D}[\Phi] = 0 \tag{3.5.1}$$

was rewritten in the following form

$$(i\partial_t + D - imZ_c M^{i_3})F = 0, \tag{3.5.2}$$

where F is an $\mathbb{H}(\mathbb{C})$-valued function and Z_c is the operator of complex conjugation $Z_c F = F^* = \sum_{k=0}^{3} F_k^* i_k$. This equation is equivalent to the Dirac equation (see [**58**, Section 12]), and after C. Lanczos it was rediscovered (see [**29**]) by a number of people, e.g., [**33**] and [**36**]. Equation (3.5.2) played an important role in the development of the Clifford algebra approach to the Dirac equation, but it is not free of some important disadvantages which limited its possible applications. First of all, the simple map transforming the classical Dirac equation into

(3.5.2) is not even $\mathbb{C}$-linear. However the most serious disadvantage is the presence of the conjugation operator Z_c which impedes application of quaternionic analysis methods to (3.5.2). Obviously, the simplest way to eliminate this difficulty is annihilating the corresponding term, making m equal to zero. In 1930, one year after the publication of Lanczos' work, there appeared an article by D. D. Ivanenko and K. V. Nikolski [**38**] in which the Dirac equation for a massless field was written as an analyticity condition for a function of a quaternionic variable. This observation was used in a number of works (see, e.g., [**77**]). The problem remained, as to how to rewrite the massive Dirac equation in quaternionic terms without undesirable presence of the complex conjugation operator. An important contribution was made in [**7**], where the symmetrical analysis of the Dirac equation for a free particle with spin 1/2 and non-zero rest mass was essentially simplified after using a quaternionic analogue which fulfilled this requirement (see also [**6**]). Nevertheless, there remained a very natural problem: to introduce a transformation which would transform the Dirac equation from its traditional form to an appropriate quaternionic one. Not to introduce one more analogue of the classical Dirac equation, but to establish an equivalence between this and a quaternionic equation. A solution of this problem was proposed in [**46**] where a simple matrix transformation which allows us to rewrite the classical Dirac operator (given in its "traditional" form using γ-matrices) applied to $\mathbb{C}^4$-vectors as a quaternionic operator applied to $\mathbb{H}(\mathbb{C})$-valued functions (and not containing the conjugation operator) was obtained. We will not discuss here a quite lengthy procedure which led to the matrix transformation, referring the reader to [**46**] or [**58**, Section 12]. Its final form is simple and easy to use. It is $\mathbb{C}$-linear, as opposed to the Lanczos transformation.

We will introduce only some necessary properties of this transformation and with their aid apply the results of Chapter 2 to the Dirac operator.

3.5.2. Dirac's operator in a quaternionic form. Let us introduce an auxiliary notation $\widetilde{f} := f(t, x_1, x_2, -x_3)$. The domain $\widetilde{G}$ is assumed to be obtained from the domain $G \subset \mathbb{R}^4$ by the reflection $x_3 \to -x_3$. The transformation described above we denote as $\mathcal{A}$ and define it in the following way. A function $\Phi : G \subset \mathbb{R}^4 \to \mathbb{C}^4$ is transformed into a function $F : \widetilde{G} \subset \mathbb{R}^4 \to \mathbb{H}(\mathbb{C})$ by the rule

$$F = \mathcal{A}[\Phi] := \frac{1}{2}\left(-(\widetilde{\Phi}_1 - \widetilde{\Phi}_2)i_0 + i(\widetilde{\Phi}_0 - \widetilde{\Phi}_3)i_1 - (\widetilde{\Phi}_0 + \widetilde{\Phi}_3)i_2 + i(\widetilde{\Phi}_1 + \widetilde{\Phi}_2)i_3\right).$$

The inverse transformation $\mathcal{A}^{-1}$ is defined as follows

$$\Phi = \mathcal{A}^{-1}[F] = (-i\widetilde{F}_1 - \widetilde{F}_2, -\widetilde{F}_0 - i\widetilde{F}_3, \widetilde{F}_0 - i\widetilde{F}_3, i\widetilde{F}_1 - \widetilde{F}_2).$$

Let us present the introduced transformations in a more explicit matrix form which relates the components of a $\mathbb{C}^4$-valued function Φ with the components of an $\mathbb{H}(\mathbb{C})$-valued function F:

$$F = \mathcal{A}[\Phi] = \frac{1}{2}\begin{pmatrix} 0 & -1 & 1 & 0 \\ i & 0 & 0 & -i \\ -1 & 0 & 0 & -1 \\ 0 & i & i & 0 \end{pmatrix}\begin{pmatrix} \widetilde{\Phi}_0 \\ \widetilde{\Phi}_1 \\ \widetilde{\Phi}_2 \\ \widetilde{\Phi}_3 \end{pmatrix}$$

and

$$\Phi = \mathcal{A}^{-1}[F] = \begin{pmatrix} 0 & -i & -1 & 0 \\ -1 & 0 & 0 & -i \\ 1 & 0 & 0 & -i \\ 0 & i & -1 & 0 \end{pmatrix}\begin{pmatrix} \widetilde{F}_0 \\ \widetilde{F}_1 \\ \widetilde{F}_2 \\ \widetilde{F}_3 \end{pmatrix}.$$

We will need the following properties of the transformation $\mathcal{A}$.

LEMMA 3. *(Algebraic properties of the transformation $\mathcal{A}$)*

(1) $\mathcal{A}\gamma_1\gamma_2\gamma_3\gamma_1\mathcal{A}^{-1}[F] = -\mathcal{A}\gamma_2\gamma_3\mathcal{A}^{-1}[F] = i_1 F$;

(2) $\mathcal{A}\gamma_1\gamma_2\gamma_3\gamma_2\mathcal{A}^{-1}[F] = \mathcal{A}\gamma_1\gamma_3\mathcal{A}^{-1}[F] = i_2 F$;

(3) $\mathcal{A}\gamma_1\gamma_2\gamma_3\gamma_3\mathcal{A}^{-1}[F] = -\mathcal{A}\gamma_1\gamma_2\mathcal{A}^{-1}[F] = -i_3 F$;

(4) $\mathcal{A}\gamma_1\gamma_2\gamma_3\gamma_0\mathcal{A}^{-1}[F] = -\mathcal{A}\gamma_0\gamma_1\gamma_2\gamma_3\mathcal{A}^{-1}[F] = F i_1$;

(5) $\mathcal{A}\gamma_1\gamma_2\gamma_3\mathcal{A}^{-1}[F] = -iFi_2$.

PROOF. First of all let us note that $\mathcal{A}$ is a $\mathbb{C}$–linear transform. We prove first statement 1:

$$\mathcal{A}\gamma_2\gamma_3\mathcal{A}^{-1}[F] = -i_1 F.$$

We have

$$\gamma_2\gamma_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix}.$$

By the definition of $\mathcal{A}^{-1}$, we have

$$\mathcal{A}^{-1}[F] = \begin{pmatrix} -iF_1 - F_2 \\ -F_0 - iF_3 \\ F_0 - iF_3 \\ iF_1 - F_2 \end{pmatrix}.$$

Thus,

$$\gamma_2\gamma_3\mathcal{A}^{-1}[F] = -i\begin{pmatrix} -F_0 - iF_3 \\ -iF_1 - F_2 \\ iF_1 - F_2 \\ F_0 - iF_3 \end{pmatrix}.$$

Then

$$\mathcal{A}\gamma_2\gamma_3\mathcal{A}^{-1}[F] =$$

$$= -\frac{i}{2}[-(-iF_1 - F_2 - iF_1 + F_2)i_0 + i(-F_0 - iF_3 - F_0 + iF_3)i_1 -$$

$$-(-F_0 - iF_3 + F_0 - iF_3)i_2 + i(-iF_1 - F_2 + iF_1 - F_2)i_3] =$$

$$= -i[iF_1 i_0 - iF_0 i_1 + iF_3 i_2 - iF_2 i_3] = F_1 i_0 - F_0 i_1 + F_3 i_2 - F_2 i_3 = -i_1 F.$$

By analogy one can prove 2.–5. □

We introduce the following quaternionic operator

$$R := P_1^+(i\partial_t + D) + P_1^-(-i\partial_t + D) - mM^{i_2},$$

where

$$P_k^{\pm} := \frac{1}{2} M^{(1 \pm i i_k)}.$$

Using the algebraic properties of $\mathcal{A}$ and $\mathcal{A}^{-1}$ we obtain the following equality

$$R = -\mathcal{A}\gamma_1\gamma_2\gamma_3 \mathbb{D}\mathcal{A}^{-1}. \tag{3.5.3}$$

Thus, the operator R is the Dirac operator for a free massive particle of spin 1/2 in quaternionic form.

3.5.3. Solutions with given energy. We will consider time-harmonic solutions of the Dirac equation (3.5.1) or in other words solutions with a specified energy ω. In this case the solution Φ of (3.5.1) has the form

$$\Phi(t, x) = q(x)e^{i\omega t},$$

where $\omega \in \mathbb{R}$ and q is a $\mathbb{C}^4$-valued function depending only on $x = (x_1, x_2, x_3)$. From (3.5.1) we obtain the Dirac equation for q

$$\mathbb{D}_{\omega,m} q = 0, \tag{3.5.4}$$

where

$$\mathbb{D}_{\omega,m} := i\omega\gamma_0 - \sum_{k=1}^{3} \gamma_k \partial_k + imI.$$

The corresponding quaternionic reformulation of $\mathbb{D}_{\omega,m}$ will be the operator $D_{\overrightarrow{\alpha}}$, where $\overrightarrow{\alpha} := -(i\omega i_1 + m i_2)$. We have the following equality

$$D_{\overrightarrow{\alpha}} = -\mathcal{A}\gamma_1\gamma_2\gamma_3\mathbb{D}_{\omega,m}\mathcal{A}^{-1}. \tag{3.5.5}$$

Thus, q is a solution of (3.5.4) if and only if $f := \mathcal{A}q$ is a solution of the equation

$$D_{\overrightarrow{\alpha}} f = 0. \tag{3.5.6}$$

All the results obtained for solutions of this equation can be immediately transformed into results for solutions of the Dirac equation (3.5.4). For example, introducing the operator

$$\mathbb{K}_{\omega,m} := \mathcal{A}^{-1}K_{\overrightarrow{\alpha}}\mathcal{A},$$

where $K_{\overrightarrow{\alpha}}$ is defined by (2.6.9), we obtain the following statement.

THEOREM 12. [**46**] *Let* $q \in C^1(\Omega;\mathbb{C}^4)\cap C(\overline{\Omega};\mathbb{C}^4)$ *and* $q \in \ker \mathbb{D}_{\omega,m}(\Omega)$. *Then*

$$q(x) = \mathbb{K}_{\omega,m}[q](x), \qquad x \in \Omega.$$

This equality, which can be considered as a Cauchy integral formula for spinor fields, together with the Plemelj-Sokhotski formulas (Subsection 2.5.4) allows us to solve boundary value problems for the operator $\mathbb{D}_{\omega,m}$. In particular, the boundary value problem corresponding to the MIT bag model, well known in the theory of the quark confinement, was completely solved [**46**]. Here we will not reproduce these results instead referring the reader to [**58**, Section 14].

In order to obtain a result similar to that of Theorem 12 for the exterior domain $\mathbb{R}^3\backslash\overline{\Omega}$, we have to rewrite the radiation condition (2.6.10) for functions from $\ker \mathbb{D}_{\omega,m}$. Here we use results from [**51**].

Let us denote

$$q := \mathcal{A}^{-1}f. \tag{3.5.7}$$

According to (3.5.5), if $f \in \ker D_{\overrightarrow{\alpha}}$ then $q \in \ker \mathbb{D}_{\omega,m}$. Suppose that f fulfills the radiation condition (2.6.10). Then we have

$$\lambda\mathcal{A}\gamma_1\gamma_2\gamma_3\mathcal{A}^{-1}f + \mathcal{A}\gamma_1\gamma_2\gamma_3\mathcal{A}^{-1}\frac{ix}{|x|}f\overrightarrow{\alpha} = o\left(\frac{1}{|x|}\right).$$

Let us denote

$$f\overrightarrow{\alpha} =: F$$

and consider the product

$$xF = (x_1i_1 + x_2i_2 + x_3i_3)F.$$

Using properties 1.-3. of the transformations $\mathcal{A}$ and $\mathcal{A}^{-1}$ (Lemma 3), we obtain

$$xF = \mathcal{A}\gamma_1\gamma_2\gamma_3\left(\sum_{k=1}^{3} x_k\gamma_k\right)\mathcal{A}^{-1}[F].$$

Consequently, using the properties 4. and 5., we have

$$xF(x) = \mathcal{A}\gamma_1\gamma_2\gamma_3\left(\sum_{k=1}^{3} x_k\gamma_k\right)\mathcal{A}^{-1}[f(x)\cdot(-i\omega i_1 - mi_2)] =$$

$$= \mathcal{A}\gamma_1\gamma_2\gamma_3\left(\sum_{k=1}^{3} x_k\gamma_k\right)\mathcal{A}^{-1}\mathcal{A}\gamma_1\gamma_2\gamma_3\gamma_0\mathcal{A}^{-1}f(x)\cdot(-i\omega) +$$

$$+\mathcal{A}\gamma_1\gamma_2\gamma_3\left(\sum_{k=1}^{3} x_k\gamma_k\right)\mathcal{A}^{-1}\mathcal{A}\gamma_1\gamma_2\gamma_3\mathcal{A}^{-1}f(x)\cdot(-im) =$$

$$= -i\omega\mathcal{A}\left(\sum_{k=1}^{3} x_k\gamma_k\right)\gamma_0\mathcal{A}^{-1}f(x) - im\mathcal{A}\left(\sum_{k=1}^{3} x_k\gamma_k\right)\mathcal{A}^{-1}f(x).$$

Using (3.5.7) we can rewrite the previous equation as

$$xF \ = \mathcal{A}\left(\sum_{k=1}^{3} x_k\gamma_k\right)(-i\omega\gamma_0 - im)q.$$

Thus (2.6.10) can be rewritten as follows

$$\lambda f + \frac{ix}{|x|} f\overrightarrow{\alpha} = \lambda\mathcal{A}q + \mathcal{A}\frac{1}{|x|}\sum_{k=1}^{3} x_k\gamma_k(\omega\gamma_0 + m)q = o\left(\frac{1}{|x|}\right),$$

and hence the radiation condition for the Dirac operator in its traditional form is

$$\lambda q(x) - (\omega\gamma_0 - m)\frac{\overrightarrow{x_\gamma}}{|x|}q(x) = o\left(\frac{1}{|x|}\right), \qquad |x| \to \infty \tag{3.5.8}$$

where $\overrightarrow{x_\gamma} := \sum_{k=1}^{3} x_k\gamma_k$.

With the aid of this radiation condition we immediately obtain the following statement.

THEOREM 13. *(Cauchy integral formula for $\mathbb{D}_{\omega,m}$ in the exterior domain $\mathbb{R}^3\backslash\overline{\Omega}$) Let $q \in C^1(\mathbb{R}^3\backslash\overline{\Omega};\mathbb{C}^4)\cap C(\mathbb{R}^3\backslash\Omega;\mathbb{C}^4)$, $q \in \ker \mathbb{D}_{\omega,m}(\mathbb{R}^3\backslash\overline{\Omega})$, and let q satisfy the radiation condition (3.5.8). Then*

$$q(x) = -\mathbb{K}_{\omega,m}[q](x), \qquad x \in (\mathbb{R}^3\backslash\overline{\Omega}). \tag{3.5.9}$$

The proof consists in the application of (3.5.5) and Proposition 1.

This theorem together with the radiation condition for the classical Dirac operator allows us to treat the extension problem corresponding to this operator for the exterior domain (the procedure is considered in [**58**] for the interior domain).

Let us introduce an integral operator which by its properties is very similar to its famous complex prototype, the operator of singular integration

$$S_{\overrightarrow{\alpha}}[f](x) = -2\int_\Gamma \{\vartheta(x-y)\left(\frac{x-y}{|x-y|^2} - \frac{i\lambda(x-y)}{|x-y|}\right)\overrightarrow{n}(y)f(y)+$$

$$+\vartheta(x-y)\overrightarrow{n}(y)f(y)\overrightarrow{\alpha}\}d\Gamma_y, \qquad x \in \Gamma.$$

Corollary 3. *(On the spinor extension from the boundary) Suppose $p : \Gamma \to \mathbb{C}^4$ satisfies the Hölder condition. In order for this function to be a boundary value of a solution q to (3.5.4) in $\mathbb{R}^3 \setminus \overline{\Omega}$ satisfying the radiation condition (3.5.8), it is necessary and sufficient that*

$$p = -\mathbb{S}_{\omega,m}p \qquad \text{on } \Gamma, \tag{3.5.10}$$

where $\mathbb{S}_{\omega,m} := \mathcal{A}^{-1}S_{\overrightarrow{\alpha}}\mathcal{A}$. Moreover, if (3.5.10) is satisfied, then $q := -\mathbb{K}_{\omega,m}[p]$ is the above-mentioned solution to (3.5.4).

CHAPTER 4

Fields in inhomogeneous media

4.1. Electromagnetic fields

4.1.1. Maxwell's equations for inhomogeneous media in quaternionic form. Let us return to the Maxwell equations (3.1.1)-(3.1.4) assuming now that ε and μ are functions of coordinates:

$$\varepsilon = \varepsilon(x) \quad \text{and} \quad \mu = \mu(x).$$

Using the constitutive relations (3.1.5) and (3.1.6) we rewrite the Maxwell system in the following form

$$\operatorname{rot} \mathbf{H} = \varepsilon \partial_t \mathbf{E} + \mathbf{j}, \tag{4.1.1}$$

$$\operatorname{rot} \mathbf{E} = -\mu \partial_t \mathbf{H}, \tag{4.1.2}$$

$$\operatorname{div}(\varepsilon \mathbf{E}) = \rho, \tag{4.1.3}$$

$$\operatorname{div}(\mu \mathbf{H}) = 0. \tag{4.1.4}$$

Equations (4.1.3) and (4.1.4) can be written as follows

$$\operatorname{div} \mathbf{E} + < \frac{\operatorname{grad} \varepsilon}{\varepsilon}, \mathbf{E} >= \frac{\rho}{\varepsilon}$$

and

$$\operatorname{div} \mathbf{H} + < \frac{\operatorname{grad} \mu}{\mu}, \mathbf{H} >= 0.$$

Combining these equations with (4.1.1) and (4.1.2) we obtain the Maxwell system in the form

$$DE =< \frac{\operatorname{grad}\varepsilon}{\varepsilon}, \mathbf{E} > -\mu\partial_t \mathbf{H} - \frac{\rho}{\varepsilon} \tag{4.1.5}$$

and

$$D\mathbf{H} =< \frac{\operatorname{grad}\mu}{\mu}, \mathbf{H} > +\varepsilon\partial_t \mathbf{E} + \mathbf{j}. \tag{4.1.6}$$

Let us make a simple observation: the scalar product of two vectors $\overrightarrow{p}$ and $\overrightarrow{q}$ can be written as follows

$$< \overrightarrow{p}, \overrightarrow{q} >= -\frac{1}{2}(^{\overrightarrow{p}}M + M^{\overrightarrow{p}})\overrightarrow{q}.$$

Using this fact, from (4.1.5) and (4.1.6) we obtain the pair of equations

$$(D + \frac{1}{2}\frac{\operatorname{grad}\varepsilon}{\varepsilon})\mathbf{E} = -\frac{1}{2}M^{\frac{\operatorname{grad}\varepsilon}{\varepsilon}}\mathbf{E} - \mu\partial_t \mathbf{H} - \frac{\rho}{\varepsilon} \tag{4.1.7}$$

and

$$(D + \frac{1}{2}\frac{\operatorname{grad}\mu}{\mu})\mathbf{H} = -\frac{1}{2}M^{\frac{\operatorname{grad}\mu}{\mu}}\mathbf{H} + \varepsilon\partial_t \mathbf{E} + \mathbf{j}. \tag{4.1.8}$$

Note that

$$\frac{1}{2}\frac{\operatorname{grad}\varepsilon}{\varepsilon} = \frac{\operatorname{grad}\sqrt{\varepsilon}}{\sqrt{\varepsilon}}.$$

Then using Remark 2, (4.1.7) can be rewritten in the following form

$$\frac{1}{\sqrt{\varepsilon}}D(\sqrt{\varepsilon}\cdot\mathbf{E}) + \mathbf{E}\cdot\overrightarrow{\varepsilon} = -\mu\partial_t \mathbf{H} - \frac{\rho}{\varepsilon}, \tag{4.1.9}$$

where

$$\overrightarrow{\varepsilon} := \frac{\operatorname{grad}\sqrt{\varepsilon}}{\sqrt{\varepsilon}}.$$

Analogously, (4.1.8) takes the form

$$\frac{1}{\sqrt{\mu}}D(\sqrt{\mu}\cdot\mathbf{H}) + \mathbf{H}\cdot\overrightarrow{\mu} = \varepsilon\partial_t \mathbf{E} + \mathbf{j}, \tag{4.1.10}$$

where

$$\overrightarrow{\mu} := \frac{\operatorname{grad}\sqrt{\mu}}{\sqrt{\mu}}.$$

Introducing the notations

$$\overrightarrow{\mathcal{E}} := \sqrt{\varepsilon}\mathbf{E}, \qquad \overrightarrow{\mathcal{H}} := \sqrt{\mu}\mathbf{H},$$

multiplying (4.1.9) by $\sqrt{\varepsilon}$ and (4.1.10) by $\sqrt{\mu}$ we arrive at the equations

$$(D + M^{\overrightarrow{\varepsilon}})\overrightarrow{\mathcal{E}} = -\frac{1}{c}\partial_t\overrightarrow{\mathcal{H}} - \frac{\rho}{\sqrt{\varepsilon}}, \tag{4.1.11}$$

and

$$(D + M^{\overrightarrow{\mu}})\overrightarrow{\mathcal{H}} = \frac{1}{c}\partial_t\overrightarrow{\mathcal{E}} + \sqrt{\mu}\mathbf{j}, \tag{4.1.12}$$

where as before $c = 1/\sqrt{\varepsilon\mu}$ is the speed of propagation of electromagnetic waves in the medium.

Equations (4.1.11) and (4.1.12) can be rewritten even in a more elegant form. Consider the function

$$\mathbf{V} := \overrightarrow{\mathcal{E}} + i\overrightarrow{\mathcal{H}}$$

(this notation is in full agreement with (3.1.7)). Let us apply to it the quaternionic Maxwell operator

$$\frac{1}{c}\partial_t + iD$$

introduced in Subsection 3.1.1. We obtain

$$(\frac{1}{c}\partial_t + iD)\mathbf{V} = \frac{1}{c}\partial_t\overrightarrow{\mathcal{E}} - D\overrightarrow{\mathcal{H}} + i(\frac{1}{c}\partial_t\overrightarrow{\mathcal{H}} + D\overrightarrow{\mathcal{E}}).$$

Applying (4.1.12) and (4.1.11) to the real and imaginary parts of this equation gives

$$(\frac{1}{c}\partial_t + iD)\mathbf{V} = -i(M^{\overrightarrow{\varepsilon}}\overrightarrow{\mathcal{E}} + iM^{\overrightarrow{\mu}}\overrightarrow{\mathcal{H}}) - \sqrt{\mu}\mathbf{j} - \frac{i\rho}{\sqrt{\varepsilon}}. \tag{4.1.13}$$

Note that

$$\overrightarrow{\mathcal{E}} = \frac{1}{2}(\mathbf{V} + \mathbf{V}^*) \qquad \text{and} \qquad \overrightarrow{\mathcal{H}} = \frac{1}{2i}(\mathbf{V} - \mathbf{V}^*).$$

Hence

$$M^{\overrightarrow{\varepsilon}}\overrightarrow{\mathcal{E}}+iM^{\overrightarrow{\mu}}\overrightarrow{\mathcal{H}}=\frac{1}{2}(M^{(\overrightarrow{\varepsilon}+\overrightarrow{\mu})}\mathbf{V}+M^{(\overrightarrow{\varepsilon}-\overrightarrow{\mu})}\mathbf{V}^*).$$

Let us notice that

$$\overrightarrow{\varepsilon}+\overrightarrow{\mu}=-\frac{\operatorname{grad} c}{c} \qquad \text{and} \qquad \overrightarrow{\varepsilon}-\overrightarrow{\mu}=-\frac{\operatorname{grad} W}{W},$$

where W is the intrinsic wave impedance of the medium (p. 70). Denote

$$\overrightarrow{c}:=\frac{\operatorname{grad}\sqrt{c}}{\sqrt{c}} \qquad \text{and} \qquad \overrightarrow{W}:=\frac{\operatorname{grad}\sqrt{W}}{\sqrt{W}}.$$

Then

$$M^{\overrightarrow{\varepsilon}}\overrightarrow{\mathcal{E}}+iM^{\overrightarrow{\mu}}\overrightarrow{\mathcal{H}}=-(M^{\overrightarrow{c}}\mathbf{V}+M^{\overrightarrow{W}}\mathbf{V}^*).$$

From (4.1.13) we obtain the Maxwell equations for an inhomogeneous medium in the following form

$$(\frac{1}{c}\partial_t+iD)\mathbf{V}-M^{i\overrightarrow{c}}\mathbf{V}-M^{i\overrightarrow{W}}\mathbf{V}^*=-(\sqrt{\mu}\mathbf{j}+\frac{i\rho}{\sqrt{\varepsilon}}) \tag{4.1.14}$$

(compare with (3.1.8)). This equation is completely equivalent to the Maxwell system (4.1.1)-(4.1.4) and represents Maxwell's equation for inhomogeneous media in a quaternionic form.

REMARK 6. *Equation (4.1.14) can be considered as a generalization of the Vekua equation, well known in complex analysis, that describes generalized analytic functions* [**80**]. *Recently in* [**62**] *using the L. Bers approach* [**13, 14**] *another generalization of the Vekua equation was considered. Most likely some of the interesting results discussed in* [**62**] *can be obtained for (4.1.14) also. Their physical meaning would be of great interest.*

4.1.2. Static case and factorization of the Schrödinger operator. When the vectors of the electromagnetic field do not depend on time from (4.1.11) and (4.1.12) we obtain two independent equations

$$(D + M^{\overrightarrow{\varepsilon}})\overrightarrow{\mathcal{E}} = -\frac{\rho}{\sqrt{\varepsilon}}$$

and

$$(D + M^{\overrightarrow{\mu}})\overrightarrow{\mathcal{H}} = \sqrt{\mu}\mathbf{j}.$$

Let us consider the sourceless situation, that is we are interested in the solutions for the operator $D + M^{\overrightarrow{\alpha}}$, where the complex quaternion $\overrightarrow{\alpha}$ represents $\overrightarrow{\varepsilon}$ or $\overrightarrow{\mu}$ and has the form

$$\overrightarrow{\alpha} = \frac{\operatorname{grad}\varphi}{\varphi}.$$

The function φ is different from zero.

Note that due to (2.4.8) the study of the operator $D +^{\overrightarrow{\alpha}} M$ practically reduces to that of D, as shown in [**78**]. In the case of the operator $D + M^{\overrightarrow{\alpha}}$ the situation is quite different.

Consider the equation

$$(D + M^{\overrightarrow{\alpha}})\overrightarrow{f} = 0. \tag{4.1.15}$$

Denote

$$v = \frac{\Delta\varphi}{\varphi}.$$

In other words φ is a solution of the Schrödinger equation

$$-\Delta\varphi + v\varphi = 0. \tag{4.1.16}$$

Proposition 2. *Let ψ be another solution of (4.1.16). Then the function*

$$\overrightarrow{f} = (D - \overrightarrow{\alpha})\psi \tag{4.1.17}$$

is a solution of (4.1.15).

PROOF. The proof consists of a simple calculation. Consider

$$\begin{aligned} D\overrightarrow{f} &= -\Delta\psi - D\psi \cdot \frac{D\varphi}{\varphi} - \psi \cdot D(\frac{D\varphi}{\varphi}) \\ &= -v\psi - D\psi \cdot \frac{D\varphi}{\varphi} + \psi \cdot \left(\frac{D\varphi}{\varphi}\right)^2 + \psi\frac{\Delta\varphi}{\varphi} \\ &= -(D\psi - \frac{D\varphi}{\varphi} \cdot \psi)\frac{D\varphi}{\varphi} = -\overrightarrow{f} \cdot \overrightarrow{\alpha}. \end{aligned}$$

□

This proposition gives us the possibility to reduce the solution of (4.1.15) to that of the Schrödinger equation (4.1.16). Here an important detail is that we arrive at the Schrödinger equation already having one particular solution. Moreover, if ψ is a fundamental solution of the Schrödinger operator:

$$(-\Delta + v)\psi = \delta,$$

then the function $\overrightarrow{f}$ defined by (4.1.17) is a fundamental solution of the operator $D + M^{\overrightarrow{\alpha}}$ as can be seen following the proof of Proposition 2.

Let us consider the following simple example of application of the Proposition 2.

EXAMPLE 2. *Consider equation (4.1.15) in some domain* $\Omega \subset \mathbb{R}^3$ *and let* $\Delta\varphi/\varphi = -c^2$ *in* Ω*, where* c *is a complex constant. In this case we are able to construct a fundamental solution for the operator* $D + M^{\overrightarrow{\alpha}}$*. Denote*

$$\psi(x) = \frac{e^{ic|x|}}{4\pi|x|}.$$

This is a fundamental solution of the operator $-\Delta - c^2$*. Then the fundamental solution of* $D + M^{\overrightarrow{\alpha}}$ *is constructed as follows*

$$\overrightarrow{f}(x) = (D - \frac{\operatorname{grad}\varphi(x)}{\varphi(x)})\frac{e^{ic|x|}}{4\pi|x|} = (-\frac{x}{|x|^2} + ic\frac{x}{|x|} - \frac{\operatorname{grad}\varphi(x)}{\varphi(x)}) \cdot \frac{e^{ic|x|}}{4\pi|x|},$$

where $x = \sum_{k=1}^{3} x_k i_k$.

Note that it is not clear how to obtain this result for the Maxwell operators $D + M^{\overrightarrow{\varepsilon}}$ and $D + M^{\overrightarrow{\mu}}$ using other known methods.

REMARK 7. *Proposition 2 is closely related to the following factorization of the Schrödinger operator proposed in* [**9**], [**12**]. *For a scalar function* u *we have the equality*

$$(D + M^{\alpha})(D - M^{\alpha})u = (-\Delta + v)u, \tag{4.1.18}$$

if the complex quaternionic function α *satisfies the equation*

$$D\alpha + \alpha^2 = -v. \tag{4.1.19}$$

It is easy to check that for $\alpha = (\operatorname{grad}\varphi)/\varphi$ *the equation (4.1.19) is equivalent to (4.1.16). Equation (4.1.19) can be considered as a natural generalization of the ordinary differential Riccati equation. In* [**43**] *and* [**85**] *the corresponding generalizations of the well known Euler theorems for the Riccati equation were obtained. We consider these results in the next subsection.*

4.1.3. A quaternionic generalization of the Riccati differential equation. The Riccati equation

$$\partial u = pu^2 + qu + r,$$

where ∂ is the operator of differentiation with respect to the independent variable and $p,\ q$ and r are real valued functions, has received a great deal of attention since a particular version was first studied by Count Riccati in 1724, owing to both its peculiar properties and the wide range of applications in which it appears. For a survey of the history and classical results on this equation, see for example [**84**], [**24**] and [**72**]. This equation can be always reduced to its canonical form (see, e.g., [**17**]),

$$\partial y + y^2 = -v, \tag{4.1.20}$$

and this is the form that we will consider.

One of the reasons for which the Riccati equation has so many applications is that it is related to the general second order homogeneous differential equation. In particular, the one-dimensional Schrödinger equation

$$-\partial^2 u - vu = 0 \tag{4.1.21}$$

where v is a function, is related to the (4.1.20) by the easily inverted substitution

$$y = \frac{\partial u}{u}.$$

This substitution, which as its most spectacular application reduces Burger's equation to the standard one-dimensional heat equation, is the basis of the well-developed theory of logarithmic derivatives for the integration of nonlinear differential equations [**63**]. A generalization of this substitution will be used in this work.

A second relation between the one-dimensional Schrödinger equation and the Riccati equation is as follows. The one-dimensional Schrödinger operator can be factorized in the form

$$-\partial^2 - v(x) = -(\partial + y(x))(\partial - y(x))$$

if and only if (4.1.20) holds.

Among the peculiar properties of the Riccati equation stand out two theorems of Euler, dating from 1760. The first of these states that if a particular solution y_0 of the Riccati equation is known, the substitution

$y = y_0 + z$ reduces (4.1.20) to a Bernoulli equation which in turn is reduced by the substitution $z = \frac{1}{u}$ to a first order linear equation. Thus given a particular solution of the Riccati equation, it can be linearized and the general solution can be found in two integrations. The second of these theorems states that given two particular solutions $y_0,\ y_1$ of the Riccati equation, the general solution can be found in the form

$$y = \frac{ky_0 \exp(\int y_0 - y_1) - y_1}{k \exp(\int y_0 - y_1) - 1} \tag{4.1.22}$$

where k is a constant. That is, given two particular solutions of (4.1.20), the general solution can be found in one integration.

Other interesting properties are those discovered by Weyr and Picard ([**84**],[**24**]). The first is that given a third particular solution y_3, the general solution can be found without integrating. That is, an explicit combination of three particular solutions gives the general solution. The second is that given a fourth particular solution y_4, the cross ratio

$$\frac{(y_1 - y_2)(y_3 - y_4)}{(y_1 - y_4)(y_3 - y_2)}$$

is a constant. Thus the derivative of this ratio is zero, which holds if and only if the numerator of the derivative is zero. That is iff

$$(y_1-y_4)(y_3-y_2)\partial((y_1-y_2)(y_3-y_4))-(y_1-y_2)(y_3-y_4)\partial((y_1-y_4)(y_3-y_2)) = 0.$$

Dividing by $(y_1 - y_2)(y_3 - y_4)(y_1 - y_4)(y_3 - y_2)$, we see that Picard's theorem is equivalent to the statement

$$\frac{\partial(y_1 - y_2)}{y_1 - y_2} + \frac{\partial(y_3 - y_4)}{y_3 - y_4} - \frac{\partial(y_1 - y_4)}{y_1 - y_4} - \frac{\partial(y_3 - y_2)}{y_3 - y_2} = 0. \tag{4.1.23}$$

In Remark 7 we observed that equation (4.1.19) is related to the three-dimensional Schrödinger equation (4.1.16) in the same way as in the one-dimensional situation. That is, if φ is a solution of (4.1.16) then its logarithmic derivative $\overrightarrow{\alpha} = (\operatorname{grad}\varphi)/\varphi$ is a solution of (4.1.19) and if a complex quaternionic function α satisfies (4.1.19) then the Schrödinger operator with the potential v can be factorized (4.1.18). These relationships suggest that (4.1.19) can be considered a natural generalization of (4.1.20).

For our purposes it is sufficient to consider purely vectorial solutions of (4.1.19). In this case the equation which will be refered to as the Riccati partial differential equation

$$D\overrightarrow{\alpha} + \overrightarrow{\alpha}^2 = -v \tag{4.1.24}$$

is equivalent to the following system

$$-\operatorname{div}\overrightarrow{\alpha} + \overrightarrow{\alpha}^2 = -v,$$

$$\operatorname{rot}\overrightarrow{\alpha} = 0.$$

The second equation implies that for a simply-connected domain Ω, there exists a scalar function η such that $\overrightarrow{\alpha} = \operatorname{grad}\eta$. Substituting this in the first equation gives

$$\triangle\eta + \langle\nabla\eta, \nabla\eta\rangle = v. \tag{4.1.25}$$

THEOREM 14. *(Generalization of Euler's first theorem) Let* $\overrightarrow{h} = \operatorname{grad}\xi$ *be an arbitrary particular solution of (4.1.24). Then the general solution of (4.1.24) has the form*

$$\overrightarrow{\alpha} = \overrightarrow{g} + \overrightarrow{h}, \tag{4.1.26}$$

where $\overrightarrow{g} = (\operatorname{grad}\Psi)/\Psi$ *and* Ψ *is a general solution of the equation*

$$\triangle\Psi + 2\langle\operatorname{grad}\xi, \operatorname{grad}\Psi\rangle = 0, \tag{4.1.27}$$

or equivalently of

$$\operatorname{div}(e^{2\xi}\operatorname{grad}\Psi) = 0. \tag{4.1.28}$$

PROOF. Substituting (4.1.26) in (4.1.24) gives

$$D\overrightarrow{g} - 2\left\langle \overrightarrow{h}, \overrightarrow{g} \right\rangle + \overrightarrow{g}^2 = 0. \tag{4.1.29}$$

Note that the vector part of (4.1.29) is rot $\overrightarrow{g} = 0$, so that

$$\overrightarrow{g} = \operatorname{grad}\Phi$$

for some function Φ. If $\Psi = e^{\Phi}$, this is equivalent to

$$\overrightarrow{g} = (\operatorname{grad}\Psi)/\Psi.$$

Equation (4.1.29), written in terms of Ψ, is

$$-\frac{1}{\Psi^2}(\nabla\Psi)^2 - \frac{1}{\Psi}\triangle\Psi - \frac{2}{\Psi}\langle\nabla\xi, \nabla\Psi\rangle + \frac{1}{\Psi^2}(\nabla\Psi)^2 = 0,$$

so that (4.1.29) is equivalent to

$$\triangle\Psi + 2\langle\nabla\xi, \nabla\Psi\rangle = 0.$$

Noting that

$$\operatorname{div}(e^{2\xi}\nabla\Psi) = 2e^{2\xi}\langle\nabla\xi, \nabla\Psi\rangle + e^{2\xi}\triangle\Psi = e^{2\xi}(\triangle\Psi + 2\langle\nabla\xi, \nabla\Psi\rangle),$$

this equation can be rewritten in the form

$$\operatorname{div}(e^{2\xi}\nabla\Psi) = 0.$$

□

Thus, as in the one-dimensional case, having one particular solution of the Riccati equation allows us to linearize it. In the three-dimensional case the corresponding linear equation (4.1.28) is quite well known and appears in various applications, for example in electrostatics [**61**], where $e^{2\xi}$ is the dielectric permeability and Ψ is the electric field potential, and as the continuity equation of hydromechanics in the case of an irrotational steady flow, where $e^{2\xi}$ is the density of the medium [**27**].

Equation (4.1.27) is the well-known transport equation, which appears for example in the ray method of approximations of solutions to the wave equation, coupled with the eikonal equation [**4**].

The second Euler theorem and Weyr's property mentioned above admit the following simple generalization.

PROPOSITION 3. [**85**] *Let* $\overrightarrow{\alpha}_k = \operatorname{grad}\eta_k$*,* $k = 1, \ldots, n$ *be* n *particular solutions of the Riccati partial differential equation (4.1.24). Then*

$$\overrightarrow{\alpha} = \frac{1}{e^{\eta_1} + \sum_{k=2}^{n} a_k e^{\eta_k}} (\overrightarrow{\alpha}_1 e^{\eta_1} + \sum_{k=2}^{n} a_k \overrightarrow{\alpha}_k e^{\eta_k}),$$

where a_k *are arbitrary complex constants, is a solution of (4.1.24).*

PROOF. We have that e^{η_k} are solutions of the Schrödinger equation (4.1.16) which is linear. This implies that

$$\varphi = e^{\eta_1} + \sum_{k=2}^{n} a_k e^{\eta_k}$$

is also its solution. Consequently, $\overrightarrow{\alpha} = (\operatorname{grad}\varphi)/\varphi$ is a solution of (4.1.24). □

Finally, we give a generalisation of the expression (4.1.23) shown earlier to be equivalent, in one dimension, to Picard's theorem.

PROPOSITION 4. [**85**] *(Generalization of Picard's theorem) Let* $\overrightarrow{\alpha}_k$*,* $k = 1, .., 4$ *be four solutions of the Riccati PDE. Then*

$$(D(\overrightarrow{\alpha}_1 - \overrightarrow{\alpha}_2) + [\overrightarrow{\alpha}_1, \overrightarrow{\alpha}_2])(\overrightarrow{\alpha}_1 - \overrightarrow{\alpha}_2)^{-1}$$

$$+(D(\overrightarrow{\alpha}_3 - \overrightarrow{\alpha}_4) + [\overrightarrow{\alpha}_3, \overrightarrow{\alpha}_4])(\overrightarrow{\alpha}_3 - \overrightarrow{\alpha}_4)^{-1}$$

$$-(D(\overrightarrow{\alpha}_1 - \overrightarrow{\alpha}_4) + [\overrightarrow{\alpha}_1, \overrightarrow{\alpha}_4])(\overrightarrow{\alpha}_1 - \overrightarrow{\alpha}_4)^{-1}$$

$$-(D(\overrightarrow{\alpha}_3 - \overrightarrow{\alpha}_2) + [\overrightarrow{\alpha}_3, \overrightarrow{\alpha}_2])(\overrightarrow{\alpha}_3 - \overrightarrow{\alpha}_2)^{-1} = 0.$$

where $[a,b] = ab - ba$ *is the standard anticommutator, assuming that the inverses exist.*

PROOF. Obviously

$$(\overrightarrow{\alpha}_1 + \overrightarrow{\alpha}_4) + (\overrightarrow{\alpha}_3 + \overrightarrow{\alpha}_2) - (\overrightarrow{\alpha}_1 + \overrightarrow{\alpha}_2) - (\overrightarrow{\alpha}_3 + \overrightarrow{\alpha}_4) = 0.$$

Multiplying each parenthesis on the right by $1 = (\overrightarrow{\alpha}_i - \overrightarrow{\alpha}_j)(\overrightarrow{\alpha}_i - \overrightarrow{\alpha}_j)^{-1}$ we obtain the equality

$$(\overrightarrow{\alpha}_1 + \overrightarrow{\alpha}_4)(\overrightarrow{\alpha}_1 - \overrightarrow{\alpha}_4)(\overrightarrow{\alpha}_1 - \overrightarrow{\alpha}_4)^{-1} + (\overrightarrow{\alpha}_3 + \overrightarrow{\alpha}_2)(\overrightarrow{\alpha}_3 - \overrightarrow{\alpha}_2)(\overrightarrow{\alpha}_3 - \overrightarrow{\alpha}_2)^{-1}$$

$$-(\overrightarrow{\alpha}_1 + \overrightarrow{\alpha}_2)(\overrightarrow{\alpha}_1 - \overrightarrow{\alpha}_2)(\overrightarrow{\alpha}_1 - \overrightarrow{\alpha}_2)^{-1} - (\overrightarrow{\alpha}_3 + \overrightarrow{\alpha}_4)(\overrightarrow{\alpha}_3 - \overrightarrow{\alpha}_4)(\overrightarrow{\alpha}_3 - \overrightarrow{\alpha}_4)^{-1} = 0.$$

Using

$$(\overrightarrow{\alpha}_i + \overrightarrow{\alpha}_j)(\overrightarrow{\alpha}_i - \overrightarrow{\alpha}_j) = \overrightarrow{\alpha}_i^2 - \overrightarrow{\alpha}_j^2 - \overrightarrow{\alpha}_i \overrightarrow{\alpha}_j + \overrightarrow{\alpha}_j \overrightarrow{\alpha}_i = D(\overrightarrow{\alpha}_j - \overrightarrow{\alpha}_i) + [\overrightarrow{\alpha}_j, \overrightarrow{\alpha}_i],$$

the result is obtained. □

4.1.4. Monochromatic fields in slowly changing media. When the electromagnetic field is monochromatic (see Section 3.2) from (4.1.11), (4.1.12) we obtain the following equations for the complex amplitudes $\overrightarrow{E}$ and $\overrightarrow{H}$ of the vectors $\overrightarrow{\mathcal{E}}$ and $\overrightarrow{\mathcal{H}}$:

$$D_{\overrightarrow{\varepsilon}} \overrightarrow{E} = i\alpha \overrightarrow{H} - \frac{\rho}{\sqrt{\varepsilon}},$$

$$D_{\overrightarrow{\mu}} \overrightarrow{H} = -i\alpha \overrightarrow{E} + \sqrt{\mu}\, \overrightarrow{j}\,.$$

Here, as before, $\alpha = \omega/c$ is the wave number. In what follows we consider the sourceless situation, that is the following pair of equations

$$D_{\overrightarrow{\varepsilon}} \overrightarrow{E} = i\alpha \overrightarrow{H}, \tag{4.1.30}$$

$$D_{\overrightarrow{\mu}} \overrightarrow{H} = -i\alpha \overrightarrow{E}. \tag{4.1.31}$$

The medium is said to be slowly changing when its properties change appreciably over distances much greater than the wavelength [**4, 81**]. Usually this is associated with the possibility of reducing the Maxwell equations (4.1.30) and (4.1.31) to the Helmholtz equations

$$(\Delta+\alpha^2)\overrightarrow{E}=0 \qquad \text{and} \qquad (\Delta+\alpha^2)\overrightarrow{H}=0.$$

It is easy to check that such a reduction is possible if only $|\overrightarrow{\varepsilon}|$ and $|\overrightarrow{\mu}|$ are considered as relatively very small and the terms containing the vectors $\overrightarrow{\varepsilon}$ and $\overrightarrow{\mu}$ are supposed to be negligible. Then (4.1.30) and (4.1.31) take the form

$$D\overrightarrow{E}=i\alpha\overrightarrow{H},$$

$$D\overrightarrow{H}=-i\alpha\overrightarrow{E}$$

and can be diagonalized. For the functions $\overrightarrow{\varphi}:=\overrightarrow{E}+i\overrightarrow{H}$ and $\overrightarrow{\psi}:=\overrightarrow{E}-i\overrightarrow{H}$ we obtain the equations

$$D_{-\alpha}\overrightarrow{\varphi}=0 \qquad \text{and} \qquad D_{\alpha}\overrightarrow{\psi}=0.$$

Let us analyze some possiblities for obtaining analytic solutions of these equations.

4.1.5. The equation $(D+\alpha(x))u(x)=0$**.** Let α be a complex valued scalar function defined in $\Omega\subset\mathbb{R}^3$. We consider the following equation

$$(D+\alpha(x))u(x)=0 \quad \text{in } \Omega. \tag{4.1.32}$$

Suppose that the function ϕ is some solution of the eikonal equation

$$(\nabla\phi)^2=\alpha^2 \quad \text{in } \Omega. \tag{4.1.33}$$

Then note that the $\mathbb{H}(\mathbb{C})$-valued functions $Q^{\pm} := \alpha \pm \nabla\phi$ are zero divisors in Ω. Let $\eta = e^{\phi}$. Then $\nabla\phi = \nabla\eta/\eta$, and the operator $D + \alpha I$ can be rewritten in the following manner

$$\begin{aligned} D + \alpha I &= D + (\alpha + \nabla\phi - \nabla\phi)I = \\ &= D + (\alpha + \nabla\phi - \frac{\nabla\eta}{\eta})I = \\ &= \eta(D + Q^{+})\eta^{-1}I. \end{aligned}$$

Consequently, (4.1.32) is reduced to the following equation

$$(D + Q^{+}(x))v(x) = 0 \quad \text{in } \Omega, \tag{4.1.34}$$

where $v = u/\eta$. Note that (4.1.34) is equivalent to (4.1.32).

Let us look for the solution of (4.1.34) in the form

$$v = Q^{-}s, \tag{4.1.35}$$

where s is an $\mathbb{H}(\mathbb{C})$-valued function. Substituting (4.1.35) into (4.1.34) we obtain an equation for s:

$$DQ^{-}s = 0. \tag{4.1.36}$$

Thus, we should describe the set $\ker D \cap \operatorname{im}(Q^{-}I)$ which would be a simple task if Q^{-} were not a zero divisor. Let us consider the following important cases in which it is possible to solve (4.1.36).

4.1.5.1. *α is a function of one variable.* Assume that α depends only on one variable: $\alpha = \alpha(x_1)$. Then, e.g., the functions

$$\phi_1 = i\Theta, \quad \phi_2 = -i\Theta$$

are solutions of (4.1.33). Here Θ is an antiderivative of α. Note that any function of the form

$$\phi = \pm i\Theta + \zeta, \tag{4.1.37}$$

where ζ is an arbitrary analytic or antianalytic function of complex variable $z = x_2 + ix_3$, will be also a solution of (4.1.33). Here the trick consists in the fact that (e.g., for an analytic function ζ):

$$\nabla\zeta = \frac{\partial\zeta}{\partial z}(i_2 + ii_3).$$

The complex quaternion $i_2 + ii_3$ is a purely vectorial zero divisor which being squared gives us zero.

Let us consider first the function ϕ_1. We have

$$\eta_1(x_1) = e^{\phi_1(x_1)} = e^{i\Theta(x_1)}, \quad Q_1^{\pm}(x_1) = \alpha(x_1) \pm ii_1\alpha(x_1) = \alpha(x_1)(1 \pm ii_1).$$

The function v is related with u by $v = e^{-\phi_1}u$, and we are looking for it in the form

$$v(x) = \alpha(x_1)(1 - ii_1)s(x).$$

For the function s we obtain the following equation

$$D(\alpha(x_1)(1 - ii_1)s(x)) = 0. \tag{4.1.38}$$

Let us denote $f = \alpha \cdot s$ and use the following quaternionic representation of f:

$$f = F_1 + F_2 i_2,$$

where $F_1 = f_0 + f_1 i_1$, $F_2 = f_2 + f_3 i_1$. We note that F_1 and F_2 commute with $(1 - ii_1)$, and $(1 - ii_1)i_2 = i_2(1 + ii_1)$. Then (4.1.38) can be rewritten as follows

$$D(F_1(1 - ii_1) + F_2 i_2(1 + ii_1)) = 0. \tag{4.1.39}$$

Note that $DM^{(1\pm ii_1)} = M^{(1\pm ii_1)}D$. Multiplying (4.1.39) from the right-hand side first by $(1 - ii_1)$ and then by $(1 + ii_1)$ we obtain that (4.1.39) is equivalent to the system

$$D(F_1)(1 - ii_1) = 0, \tag{4.1.40}$$

$$D(F_2 i_2)(1 + i i_1) = 0.$$

The last equation can be rewritten in the form

$$D(F_2)(1 - i i_1) = 0. \tag{4.1.41}$$

Thus, F_1 and F_2 must satisfy the same equation. Let us consider the equation (4.1.40). Its solution obviously has the form

$$F_1(x) = H_1(x) + S_1(x)(1 + i i_1), \tag{4.1.42}$$

where S_1 is an arbitrary two-component function, and $H_1 = h_0 + h_1 i_1$ satisfies the equation

$$DH_1 = 0. \tag{4.1.43}$$

We note that the last term in (4.1.42) does not contribute in the final solution of (4.1.32) because of multiplication by Q^- (4.1.35).

In order to solve (4.1.43) we rewrite it in explicit form $(i_1\partial_1 + i_2\partial_2 + i_3\partial_3)(h_0 + h_1 i_1) = 0$ and obtain that it is equivalent to the following system

$$\partial_1 h_0 = \partial_1 h_1 = 0, \tag{4.1.44}$$

$$\partial_2 h_0 + \partial_3 h_1 = 0, \tag{4.1.45}$$

$$\partial_3 h_0 - \partial_2 h_1 = 0. \tag{4.1.46}$$

From (4.1.44)-(4.1.46) we have that H_1 is independent of the variable x_1 and is analytic in the usual complex sense with respect to the complex variable $z = x_3 + i_1 x_2$ as (4.1.45) and (4.1.46) represent the corresponding Cauchy-Riemann conditions. More precisely, both $\operatorname{Re} H_1 = \operatorname{Re} h_0 + i_1 \operatorname{Re} h_1$, and $\operatorname{Im} H_1 = \operatorname{Im} h_0 + i_1 \operatorname{Im} h_1$ are analytic with respect to z.

In a similar way, $F_2 = H_2(x_2, x_3)$ is an analytic function with respect to z. Thus,

$$s(x) = \frac{1}{\alpha(x_1)}(H_1(x_2, x_3) + H_2(x_2, x_3)i_2),$$

and the function

$$\begin{aligned}\widehat{u}_1(x) &= e^{\phi_1(x_1)}\alpha(x_1)(1 - ii_1)s(x) = \\ &= e^{i\Theta(x_1)}(H_1(x_2, x_3)(1 - ii_1) + H_2(x_2, x_3)i_2(1 + ii_1))\end{aligned}$$

is a solution of (4.1.32). Moreover, due to the right $\mathbb{H}(\mathbb{C})$-linearity of (4.1.32), the following function is also a solution

(4.1.47)
$$u_1(x) = e^{i\Theta(x_1)}(H_1(x_2, x_3)(1 - ii_1)A_1 + H_2(x_2, x_3)i_2(1 + ii_1)A_2),$$

where A_1 and A_2 are arbitrary constant complex quaternions.

Taking the function ϕ_2 as a solution of the eikonal equation (4.1.33) and repeating the procedure described above we arrive at another solution of (4.1.32):

(4.1.48)
$$u_2(x) = e^{-i\Theta(x_1)}(G_1(x_2, x_3)(1 + ii_1)B_1 + G_2(x_2, x_3)i_2(1 - ii_1)B_2),$$

where G_1 and G_2, similarly to H_1 and H_2, are analytic functions with respect to z, and B_1, B_2 are arbitrary constant complex quaternions. Thus, we obtain the following proposition.

PROPOSITION 5. *Let $\Theta(x_1)$ be an antiderivative of the function $\alpha(x_1)$; H_1, H_2, G_1 and G_2 satisfy the Cauchy-Riemann conditions (4.1.45), (4.1.46), and A_1, A_2, B_1 and B_2 be arbitrary constant complex quaternions. Then the functions (4.1.47) and (4.1.48) are solutions of the equation*

$$(D + \alpha(x_1))u(x) = 0.$$

4.1.5.2. *α is an analytic function.* Let us return to the equation (4.1.36). We are interested in describing all complex quaternionic solutions of the Moisil-Theodoresco equation

$$Df = 0 \tag{4.1.49}$$

with the following additional condition

$$f \cdot \overline{f} = 0. \tag{4.1.50}$$

In [**42**] the equation

$$\partial_t f + Df + f \cdot \overline{f} = 0 \tag{4.1.51}$$

was studied and its solutions obtained. Equation (4.1.51) is obtained [**34**, p. 99] from the general self-duality equation taking the Jackiw-Nohl-Rebbi-t' Hooft ansatz for the gauge potential. The main result of [**42**] reads as follows. For any $u \in \ker(\partial_t + D)(G)$, $u_0 \neq 0$, the function

$$f = -\frac{1}{2u_0}(Du + D_r u)$$

is a solution of (4.1.51). Here G is some domain in $\mathbb{R}^4$ and D_r denotes the right Moisil-Theodoresco operator $D_r u = \sum_{k=1}^{3} \partial_k u i_k$. Note that

$$D_r u = -\operatorname{div} \overrightarrow{u} + \operatorname{grad} u_0 - \operatorname{rot} \overrightarrow{u}$$

(compare with (2.4.1)). For the purposes of the present work it is sufficient to limit ourselves with the case $\partial_t f \equiv 0$. Then $Du = 0$ and from (2.4.3), (2.4.4) we obtain that $D_r u = 2 \operatorname{grad} u_0 = -2 \operatorname{rot} \overrightarrow{u}$. Thus, considering the first of these equalities we arrive at the following

Proposition 6. *Let $u_0 \in \ker \Delta(\Omega)$, $\Omega \subset \mathbb{R}^3$. Then $f = \nabla u_0 / u_0$ satisfies the equation*

$$Df + f \cdot \overline{f} = 0. \tag{4.1.52}$$

REMARK 8. *Equation (4.1.52) is a homogeneous Riccati partial differential equation discussed in Subsection 4.1.3.*

Then as a class of solutions of (4.1.49) and (4.1.50) we can consider $f = \nabla u_0$, where $u_0 \in \ker \Delta(\Omega)$ is such that $\langle \nabla u_0, \nabla u_0 \rangle = 0$. For instance, u_0 can be an arbitrary analytic function of the complex variable $z_1 = x_1 + ix_2$ (and independent of x_3). In this case

$$\nabla u_0 = \frac{\partial u_0}{\partial z_1}(i_1 + ii_2)$$

and $f = \nabla u_0$ satisfies (4.1.49) and (4.1.50).

Let us consider the following question. Under which conditions can the expression $Q^- s$ from (4.1.36) be equal to such a purely vectorial f? In other words, when do we have the equality

$$Q^- = \frac{\partial u_0}{\partial z_1}(i_1 + ii_2)s^{-1} \ \ ? \tag{4.1.53}$$

Denote $\nu = s^{-1}$ and recall that $Q^- = \alpha - \nabla \phi$. Then (4.1.53) is equivalent to the system

$$\alpha = -\frac{\partial u_0}{\partial z_1}(\nu_1 + i\nu_2), \tag{4.1.54}$$

$$\partial_1 \phi = -\frac{\partial u_0}{\partial z_1}(\nu_0 + i\nu_3), \tag{4.1.55}$$

$$\partial_2 \phi = -i\frac{\partial u_0}{\partial z_1}(\nu_0 + i\nu_3), \tag{4.1.56}$$

$$\partial_3 \phi = i\frac{\partial u_0}{\partial z_1}(\nu_1 + i\nu_2). \tag{4.1.57}$$

From (4.1.54) and (4.1.57) we obtain that

$$\phi = -i\Theta_3 + \zeta, \tag{4.1.58}$$

where Θ_3 is an antiderivative of α with respect to x_3 and ζ is an arbitrary function of x_1 and x_2. Then from (4.1.55) and (4.1.56) we obtain the following equality

$$i(\partial_1 + i\partial_2)\Theta_3 = (\partial_1 + i\partial_2)\zeta. \tag{4.1.59}$$

Taking into account that ζ must not depend on x_3 we see that (4.1.59) is possible only if

$$(\partial_1 + i\partial_2)\alpha = \frac{\partial\alpha}{\partial\overline{z}_1} = 0$$

and

$$\frac{\partial\zeta}{\partial\overline{z}_1} = 0.$$

Thus, α and ζ are analytic functions with respect to $z_1 = x_1 + ix_2$. Then for $\partial_1\phi$ and $\partial_2\phi$ we obtain

$$\partial_1\phi = -i\partial_2\phi = -i\frac{\partial\Theta_3}{\partial z_1} + \frac{\partial\zeta}{\partial z_1} = \partial_{z_1}\zeta(1 - i\frac{\partial_{z_1}\Theta_3}{\partial_{z_1}\zeta}).$$

Consequently, comparing the last equality with (4.1.54)-(4.1.57) we can see that $\zeta = u_0$. Then the following election of the functions ν_k, $k = \overline{0,3}$ is possible

$$\nu_0 = -1, \qquad \nu_1 = -\frac{\alpha}{\partial_{z_1}\zeta}, \qquad \nu_2 = 0, \qquad \nu_3 = \frac{\partial_{z_1}\Theta_3}{\partial_{z_1}\zeta}.$$

Then

$$s = \nu^{-1} = \frac{\partial_{z_1}\zeta}{(\partial_{z_1}\zeta)^2 + \alpha^2 + (\partial_{z_1}\Theta_3)^2}(-\partial_{z_1}\zeta + \alpha i_1 - \partial_{z_1}\Theta_3 i_3).$$

For Q^- we have the expression

$$Q^- = \alpha - \nabla\phi = \alpha + (i\frac{\partial\Theta_3}{\partial z_1} - \frac{\partial\zeta}{\partial z_1})(i_1 + ii_2) + i\alpha i_3.$$

It is easy to see that

$$Q^- \cdot s = \frac{\partial\zeta}{\partial z_1}(i_1 + ii_2) = \nabla\zeta.$$

Thus, the function v (4.1.35) is found and to obtain a solution $\widehat{u}$ of (4.1.32) we merely have to consider the product

$$\widehat{u} = \eta \cdot v = e^{\phi} \cdot v = e^{-i\Theta_3 + \zeta} \cdot \frac{\partial \zeta}{\partial z_1} \cdot (i_1 + ii_2).$$

Finally, the expression $e^{\zeta} \frac{\partial \zeta}{\partial z_1}$, where ζ is an arbitrary analytic function, can be substituted by an arbitrary analytic function h, related by $\zeta(z_1) = \ln H(z_1)$, where H is an antiderivative of h with respect to z_1. We obtain $\widehat{u}$ in the form

$$\widehat{u} = e^{-i\Theta_3(z_1, x_3)} \cdot h(z_1) \cdot (i_1 + ii_2).$$

Due to the right-linearity of the operator $D + \alpha(x)I$, the solutions of (4.1.32) are invariant under multiplication by an arbitrary constant complex quaternion from the right-hand side. We summarize the results of this subsection in the following

Proposition 7. *Let α be an analytic function with respect to the complex variable $z_1 = x_1 + ix_2$ and Θ_3 be an antiderivative of α with respect to x_3. Let h be an arbitrary analytic function of z_1 and independent of x_3, A be an arbitrary constant complex quaternion. Then the function*

$$u(z_1, x_3) = e^{-i\Theta_3(z_1, x_3)} \cdot h(z_1) \cdot (i_1 + ii_2)A \qquad (4.1.60)$$

is a solution of (4.1.32).

Remark 9. *We note that the same can be done for an antianalytic function α. In this case the solution is*

$$u(z_1, x_3) = e^{i\Theta_3(z_1, x_3)} \cdot h^*(z_1) \cdot (i_1 - ii_2)A, \qquad (4.1.61)$$

where h^* is an arbitrary antianalytic function with respect to z_1.

REMARK 10. *Note that α can be independent of x_1 and x_2. Then we return to the case considered in 4.1.1. The solution (4.1.48) is obtained from (4.1.60) and (4.1.47) from (4.1.61).*

4.2. Spinor fields

4.2.1. The Dirac operator with potentials in quaternionic form. In Subsection 3.5.2 we obtained a simple relation (3.5.3) between the classic Dirac operator for a free particle

$$\mathbb{D} = \gamma_0 \partial_t - \sum_{k=1}^{3} \gamma_k \partial_k + im$$

and the quaternionic operator

$$R = P_1^+(i\partial_t + D) + P_1^-(-i\partial_t + D) - mM^{i_2}.$$

These two operators are related by the transformation $\mathcal{A}$. Now we will see the result of transformation of the most frequently studied potentials. The Dirac operator with electric potential $g_{el}(x)$ has the form (see, e.g., [**79**])

$$\mathbb{D}^{el} := \mathbb{D} + ig_{el}\gamma_0.$$

Making use of the property 4 from Lemma 3 we see that

$$M^{i\widetilde{g}_{el}i_1} = \mathcal{A}\gamma_1\gamma_2\gamma_3(ig_{el}\gamma_0)\mathcal{A}^{-1}$$

and hence

(4.2.1) $$R^{el} = -\mathcal{A}\gamma_1\gamma_2\gamma_3\mathbb{D}^{el}\mathcal{A}^{-1},$$

where

$$R^{el} := R - M^{i\widetilde{g}_{el}i_1}.$$

Thus, R^{el} is a quaternionic form of the Dirac operator with electric potential. In the same way we obtain quaternionic reformulations similar to (4.2.1) for the Dirac operator:

a): with scalar potential

$$\mathbb{D}^{sc} := \mathbb{D} + g_{sc} I,$$

b): with pseudoscalar potential

$$\mathbb{D}^{ps} := \mathbb{D} + g_{ps}\gamma_0\gamma_5,$$

c): with magnetic potential

$$\mathbb{D}^{m} := \mathbb{D} + \sum_{k=1}^{3} A_k\gamma_k.$$

Namely,

$$R^{sc} := R + M^{i\widetilde{g}_{sc} i_2},$$

$$R^{ps} := R - i\widetilde{g}_{ps} I,$$

$$R^{m} := R - \overrightarrow{B} I,$$

where the purely vectorial quaternionic function $\overrightarrow{B}$ has the form

$$\overrightarrow{B} = \widetilde{A}_1 i_1 + \widetilde{A}_2 i_2 - \widetilde{A}_3 i_3.$$

In all these cases we have equalities similar to (4.2.1), where the operators R and $\mathbb{D}$ have the corresponding superscripts. For example,

$$R^{m} = -\mathcal{A}\gamma_1\gamma_2\gamma_3\mathbb{D}^{m}\mathcal{A}^{-1}.$$

Thus, the Dirac operator with different potentials can be studied not only in its traditional form using Dirac matrices but in a quaternionic form which provides the possibility of utilizing algebraic advantages of quaternions.

4.2.2. Exact solutions. Exact solutions of the Dirac equation are of special interest, since in many cases, due to general facts from functional analysis and the theory of partial differential equations, it is possible to make some qualitative conclusions from them about the behaviour of the quantum system or even solve the corresponding Cauchy or boundary value problem. In particular, all corresponding Green's functions have been constructed using exact solutions of the Dirac equation. Of course, there exist dozens of works on the topic. The reader is referred to the encyclopaedic monograph [**5**] for the corresponding bibliography and review of known exact solutions of the Dirac equation up to the late 1980's.

In this subsection we explain how the results of Subsection 4.1.5 can be used to obtain exact solutions of the Dirac equation with different potentials. In several cases inaccessible to the application of these results we demonstrate another possibility leading to a quite general class of exact solutions.

4.2.2.1. *Pseudoscalar potential.* We begin with the pseudoscalar potential because, as we will see, the corresponding quaternionic equation can be reduced to that considered in Subsection 4.1.5. We will look for solutions with a specified energy $\omega \in \mathbb{R}$, that is for $\mathbb{H}(\mathbb{C})$-valued functions of the form $F(t,x) = f(x) \cdot e^{i\omega t}$ satisfying the equation

$$R^{ps}F = 0,$$

from which we obtain the following equation for f

$$(D + \nu(x)I + M^{\overrightarrow{\alpha}})f(x) = 0. \tag{4.2.2}$$

Here $\nu := -i\widetilde{g}_{ps}$ and as in Subsection 3.5.3, $\overrightarrow{\alpha} = -(i\omega i_1 + m i_2)$. Denote

$$R_{\overrightarrow{\alpha}} := D + \nu(x)I + M^{\overrightarrow{\alpha}}.$$

First we suppose that $\overrightarrow{\alpha} \notin \mathfrak{S}$. Using the notations of Section 2.6 we obtain the following statement.

THEOREM 15. *Let $\overrightarrow{\alpha}$ be an arbitrary constant purely vectorial complex quaternion and $\overrightarrow{\alpha} \notin \mathfrak{S}$. Let Ω be some domain in $\mathbb{R}^3$ and ν be an arbitrary complex-valued function defined in Ω. Then*

$$\ker R_{\overrightarrow{\alpha}} = P^+ \ker(D + (\nu(x) + \lambda)I) \oplus P^- \ker(D + (\nu(x) - \lambda)I) \qquad in \quad \Omega,$$

where $\lambda^2 = \overrightarrow{\alpha}^2$ and $P^\pm$ are defined by (2.6.2).

In other words f satisfies (4.2.2) if and only if the functions $f^+ := P^+ f$ and $f^- := P^- f$ are respectively solutions of the equations

$$(D + (\nu(x) + \lambda)I)f^+(x) = 0$$

and

$$(D + (\nu(x) - \lambda)I)f^-(x) = 0.$$

Thus, in the case $\overrightarrow{\alpha} \notin \mathfrak{S}$ we reduce equation (4.2.2) to an equivalent pair of equations with scalar coefficients as considered in Subsection 4.1.5.

Now let us consider the case $\overrightarrow{\alpha} \in \mathfrak{S}$. We look for solutions f of (4.2.2) that satisfy the equations

$$(D + \nu)f = 0 \tag{4.2.3}$$

and

$$f \cdot \overrightarrow{\alpha} = 0$$

simultaneously. Such solutions have the form

$$f = u \cdot \overrightarrow{\alpha},$$

where u is a solution of (4.2.3). In Subsection 4.1.5 we constructed solutions of equation (4.2.3) for the case in which the coefficient ν is a function of one variable. Using the results gathered in Proposition 5 we obtain solutions for equation (4.2.2).

PROPOSITION 8. *Let* $\Theta(x_1)$ *be an antiderivative of the function* $\nu(x_1)$*;* $H_1^{\pm}$, $H_2^{\pm}$, $G_1^{\pm}$ *and* $G_2^{\pm}$ *satisfy the Cauchy-Riemann conditions (4.1.45), (4.1.46) and* $A_1^{\pm}$, $A_2^{\pm}$, $B_1^{\pm}$ *and* $B_2^{\pm}$ *be arbitrary constant complex quaternions. Then*

1) if $\overrightarrow{\alpha} \notin \mathfrak{S}$*, the function*

(4.2.4)

$$f = \frac{1}{2\lambda}(e^{i(\Theta(x_1)+\lambda x_1)}(H_1^+(x_2,x_3)(1-ii_1)A_1^+ + H_2^+(x_2,x_3)i_2(1+ii_1)A_2^+)+$$
$$+ e^{-i(\Theta(x_1)+\lambda x_1)}(G_1^+(x_2,x_3)(1+ii_1)B_1^+ + G_2^+(x_2,x_3)i_2(1-ii_1)B_2^+)(\lambda+\overrightarrow{\alpha})+$$
$$+ \frac{1}{2\lambda}(e^{i(\Theta(x_1)-\lambda x_1)}(H_1^-(x_2,x_3)(1-ii_1)A_1^- + H_2^-(x_2,x_3)i_2(1+ii_1)A_2^-)+$$
$$+ e^{-i(\Theta(x_1)-\lambda x_1)}(G_1^-(x_2,x_3)(1+ii_1)B_1^- + G_2^-(x_2,x_3)i_2(1-ii_1)B_2^-)(\lambda-\overrightarrow{\alpha}))$$

is a solution of (4.2.2),

2) if $\overrightarrow{\alpha} \in \mathfrak{S}$*, the function*

(4.2.5)

$$f = (e^{i\Theta(x_1)}(H_1^+(x_2,x_3)(1-ii_1)A_1^+ + H_2^+(x_2,x_3)i_2(1+ii_1)A_2^+)+$$
$$+ e^{-i\Theta(x_1)}(G_1^+(x_2,x_3)(1+ii_1)B_1^+ + G_2^+(x_2,x_3)i_2(1-ii_1)B_2^+))\overrightarrow{\alpha}$$

is a solution of (4.2.2).

REMARK 11. *With the aid of the transformation* $\mathcal{A}^{-1}$ *these solutions can be rewritten in a "traditional" form and it can be checked that they really satisfy the equation* $\mathbb{D}^{ps}\Phi = 0$*. This is an easy excercise executed in* [**47**]*, but in fact unnecessary. We showed that equation (4.2.2) is the same Dirac equation with pseudoscalar potential and all the interesting physical information contained in the Dirac equation in its "traditional" form is contained also in this quaternionic equation.*

4.2.2.2. *Electric potential.* We now consider the Dirac operator with electric potential R^{el}. As before we look for the solutions of the form $F(t,x) = f(x) \cdot e^{i\omega t}$. Then from the equation

$$R^{el} F = 0$$

we obtain the following equation for f:

$$(D - M^{(i(\widetilde{g}_{el}(x)+\omega)i_1 + mi_2)}) f(x) = 0. \tag{4.2.6}$$

Denote

$$R^{el}_{\omega,m} := D - M^{(i(\widetilde{g}_{el}(x)+\omega)i_1 + mi_2)}.$$

Equation (4.2.6) in general cannot be reduced to the equation with a scalar coefficient studied in Subsection 4.1.5. - it is possible in some special cases only. For example, assuming that $m = 0$ we have the operator $R^{el}_{\omega,0}$ which can be represented in the following form

$$R^{el}_{\omega,0} = P_1^-(D + (\widetilde{g}_{el}(x) + \omega)I) + P_1^+(D - (\widetilde{g}_{el}(x) + \omega)I),$$

where the operators

$$P_1^\pm := \frac{1}{2} M^{(1 \pm i i_1)}$$

commute with the operators in parentheses, and therefore we obtain that any solution f of the equation

$$R^{el}_{\omega,0} f = 0 \tag{4.2.7}$$

has the form

$$f = P_1^- f^+ + P_1^+ f^-,$$

where

$$(D \pm (\widetilde{g}_{el}(x) + \omega)) f^\pm(x) = 0. \tag{4.2.8}$$

Thus, when $m = 0$ we reduce equation (4.2.6) to a pair of equations (4.2.8) with scalar coefficients, and the results of Subsection 4.1.5 can be used again to obtain solutions of (4.2.7) (see [**47**]).

Of course, the requirement $m = 0$ is very restrictive. Let us analyse another approach proposed in [**52**] which allows us to obtain a wide class of solutions of (4.2.6) without this requirement.

We assume that the potential g_{el} is a function of one variable x_1. Denote

$$h(x_1) := i(g_{el}(x_1) + \omega).$$

Then we are looking for solutions of the equation

$$(D - M^{(h(x_1)i_1 + mi_2)})f(x) = 0. \tag{4.2.9}$$

Obviously any solution of (4.2.9) can be written in the form

$$f = \varphi \cdot Q,$$

where φ is a scalar function and Q is a complex quaternionic function. From (4.2.9) we obtain

$$D(\varphi Q) - \varphi Q \cdot (hi_1 + mi_2) = 0.$$

According to Remark 6 this is

$$\varphi \cdot DQ + \operatorname{grad} \varphi \cdot Q - \varphi Q \cdot (hi_1 + mi_2) = 0. \tag{4.2.10}$$

Let us require the function Q to satisfy the following equation

$$DQ = Q \cdot mi_2. \tag{4.2.11}$$

Substituting this condition in (4.2.10), we obtain the equality

$$\operatorname{grad} \varphi \cdot Q = \varphi Q \cdot hi_1. \tag{4.2.12}$$

We consider first the case when the function Q has the form

$$Q = Q_1 = q_0 + q_1 i_1.$$

Thus we are looking for $f_1 = \varphi_1 Q_1$. Then (4.2.12) takes the form

$$\partial_1 \varphi_1 = h\varphi_1, \qquad \partial_2 \varphi_1 = \partial_3 \varphi_1 = 0$$

(we assume that Q_1 is not a zero divisor). Consequently,

$$\varphi_1 = a \cdot e^{H(x_1)}, \tag{4.2.13}$$

where H is the antiderivative of h and a is a complex constant.

Equation (4.2.11) takes the form

$$(i_1\partial_1 + i_2\partial_2 + i_3\partial_3)(q_0 + q_1 i_1) = (q_0 + q_1 i_1) \cdot m i_2$$

which is equivalent to the system

$$\begin{aligned} \partial_2 q_0 + \partial_3 q_1 &= m q_0, \\ \partial_3 q_0 - \partial_2 q_1 &= m q_1 \end{aligned} \tag{4.2.14}$$

with the additional condition

$$\partial_1 q_0 = \partial_1 q_1 = 0.$$

Let us introduce the following notation

$$\partial_{\overline{z}} := \frac{1}{2}(\partial_3 + i_1\partial_2).$$

Then the system (4.2.14) can be rewritten as follows

$$\partial_{-} Q_1 := (\partial_{\overline{z}} - \frac{m i_1}{2} C_1) Q_1 = 0, \tag{4.2.15}$$

where C_1 is the operator of conjugation with respect to i_1:

$$C_1 Q_1 = q_0 - i_1 q_1.$$

We can construct a conjugate operator to the operator ∂_-:

$$\partial_+ := \partial_z + \frac{m i_1}{2} C_1,$$

where

$$\partial_z := \frac{1}{2}(\partial_3 - i_1 \partial_2).$$

Note that the operators ∂_- and ∂_+ factorize the Helmholtz operator

$$\partial_- \partial_+ = \partial_+ \partial_- = \frac{1}{4}(\Delta_2 - m^2),$$

where $\Delta_2 := \partial_2^2 + \partial_3^2$. Thus we can obtain solutions of (4.2.14) from solutions of the two-dimensional Helmholtz equation by applying to them the operator ∂_+. We will restrict our consideration to the simplest class of solutions of the Helmholtz equation, namely, to plane waves

$$\exp(-(m_a x_2 + m_b x_3)), \tag{4.2.16}$$

where m_a and m_b are arbitrary numbers with the property

$$m_a^2 + m_b^2 = m^2.$$

Applying the operator ∂_+ to (4.2.16), we obtain a solution to (4.2.14) or equivalently to equation (4.2.15):

$$Q_1 = \partial_+ e^{-(m_a x_2 + m_b x_3)} = -\frac{1}{2}(m_b - (m + m_a) i_1) \cdot e^{-(m_a x_2 + m_b x_3)}. \tag{4.2.17}$$

From (4.2.13) and (4.2.17) we obtain a particular solution to equation (4.2.9) in the form

$$f_1 = \varphi_1 Q_1 = c \cdot (m_b - (m + m_a) i_1) \cdot \exp(H(x_1) - (m_a x_2 + m_b x_3)),$$

where c is a complex constant.

We consider now the quaternionic function $Q_2 = q_2 i_2 + q_3 i_3$, and the corresponding solution $f_2 = \varphi_2 Q_2$. Note that $Q_2 = \theta \cdot i_2$, where $\theta = q_2 + q_3 i_1$. Equation (4.2.12) takes the form

$$\partial_1 \varphi_2 = -h \cdot \varphi_2, \qquad \partial_2 \varphi_2 = \partial_3 \varphi_2 = 0.$$

Thus $\varphi_2 = b \cdot e^{-H(x_1)}$, where b is a complex constant.

From (4.2.11) we obtain that θ is a solution of (4.2.15). Then

$$Q_2 = \theta \cdot i_2 = -\frac{1}{2}(m_d - (m + m_c) i_1) \cdot e^{-(m_c x_2 + m_d x_3)} \cdot i_2$$

$$= -\frac{1}{2}(m_d i_2 - (m + m_c) i_3) \cdot e^{-(m_c x_2 + m_d x_3)},$$

where $m_c^2 + m_d^2 = m^2$. Thus, another particular solution of (4.2.9) has the form

$$f_2 = \varphi_2 Q_2 = d \cdot (m_d i_2 - (m + m_c) i_3) \cdot \exp(-H(x_1) - (m_c x_2 + m_d x_3)),$$

where d is an arbitrary complex constant. Combining f_1 and f_2 we obtain the solution of (4.2.9)

$$\begin{aligned} f &= f_1 + f_2 \\ &= c \cdot (m_b - (m + m_a) i_1) \cdot \exp(H(x_1) - (m_a x_2 + m_b x_3)) \\ &\quad + d \cdot (m_d i_2 - (m + m_c) i_3) \cdot \exp(-H(x_1) - (m_c x_2 + m_d x_3)). \end{aligned} \tag{4.2.18}$$

Remark 12. *Using algebraic properties of quaternions we obtained exact solutions for the Dirac equation with electric potential, but the same procedure can be applied also to the Dirac equation with scalar potential* [**52**].

Remark 13. *The operator $D - M^{(h(x_1) i_1 + m i_2)}$ considered in this subsection can be written in the form $D + M^{\overrightarrow{\alpha}}$ with $\overrightarrow{\alpha} = (\operatorname{grad} \varphi)/\varphi$*

and $\varphi = \exp(-(H(x_1) + mx_2))$. *Thus the exact solutions obtained here also give us expressions for static electric or magnetic fields (see Subsection 4.1.2) in a medium whose permittivity or permeability is* φ^2, *that is when* ε *or* μ *are functions of the form* $\exp(\eta(x_1) + cx_2)$, *where* η *is an arbitrary function and* c *is an arbitrary constant. In order to obtain from (4.2.18) an expression for the electric or magnetic field we must ensure that the scalar part of (4.2.18) be zero. Hence under the condition* $m_b = 0$ *the function (4.2.18) can be useful for obtaining exact expressions for static electric or magnetic fields in this special class of physical media. More details can be found in* [**53**] *and* [**71**].

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Quaternionic analysis is the most natural and close generalization of complex analysis that preserves many of its important features. The present book is meant as an introduction and invitation to this theory and its applications (in fact it was inspired by a course given by the author to graduate engineering students). Restricting ourselves to Maxwell's equations and the Dirac equation only we show the progress achieved in applied quaternionic analysis during the last five years, emphasising results which can not so easily be obtained by other methods. Thus, the main objective of this work is to introduce the reader to some topics of quaternionic analysis whose selection is motivated by particular models from the theory of electromagnetic and spinor fields, and to show the usefulness and necessity of applying the tools of quaternionic analysis to these kinds of problems.

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