W W L CHEN and X T DUONG

ELEMENTARY MATHEMATICS

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В книзі читач знайде основні відомості з елементарної математики.

Видання може бути використане як посібник з елементарної математики учнями, що вивчають предмети на англійській мові, а також широким колом бажаючих.

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Chapter 1

BASIC ALGEBRA

1.1. The Real Number System

We shall assume that the reader has a knowledge of the real numbers. Some examples of real numbers are 3, $1/2$, π, $\sqrt{23}$ and $-\sqrt[3]{5}$. We shall denote the collection of all real numbers by $\mathbb{R}$, and write $x \in \mathbb{R}$ to denote that x is a real number.

Among the real numbers are the collection $\mathbb{N}$ of all natural numbers and the collection $\mathbb{Z}$ of all integers. These are given by

$$\mathbb{N} = \{1, 2, 3, \ldots\} \qquad \text{and} \qquad \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

Another subcollection of the real numbers is the collection $\mathbb{Q}$ of all rational numbers. To put it simply, this is the collection of all fractions. Clearly we can write any fraction if we allow the numerator to be any integer (positive, negative or zero) and insist that the denominator must be a positive integer. Hence we have

$$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \right\}.$$

It can be shown that the collection $\mathbb{Q}$ contains precisely those numbers which have terminating or repeating decimals in their decimal notations. For example,

$$\frac{3}{4} = 0.75 \qquad \text{and} \qquad -\frac{18}{13} = -1.\overline{384615}$$

are rational numbers. Here the digits with the overline repeat.

It can be shown that the number $\sqrt{2}$ is not a rational number. This is an example of a real number which is not a rational number. Indeed, any real number which is not a rational number is called an irrational number. It can be shown that any irrational number, when expressed in decimal notation, has

† This chapter was written at Macquarie University in 1999.

non-terminating and non-repeating decimals. The number π is an example of an irrational number. It is known that

$$\pi = 3.14159\ldots.$$

The digits do not terminate or repeat. In fact, a book was published some years ago giving the value of the number π to many digits and nothing else – a very uninteresting book indeed! On the other extreme, one of the states in the USA has a law which decrees that $\pi = 3$, no doubt causing a lot of problems for those who have to measure the size of your block of land.

1.2. Arithmetic

In mathematics, we often have to perform some or all of the four major operations of arithmetic on real numbers. These are addition (+), subtraction (−), multiplication (×) and division (÷). There are simple rules and conventions which we need to observe.

SOME RULES OF ARITHMETIC.

(a) *Operations within brackets are performed first.*
(b) *If there are no brackets to indicate priority, then multiplication and division take precedence over addition and subtraction.*
(c) *Addition and subtraction are performed in their order of appearance.*
(d) *Multiplication and division are performed in their order of appearance.*
(e) *A number of additions can be performed in any order. For any real numbers $a, b, c \in \mathbb{R}$, we have $a + (b + c) = (a + b) + c$ and $a + b = b + a$.*
(f) *A number of multiplications can be performed in any order. For any real numbers $a, b, c \in \mathbb{R}$, we have $a \times (b \times c) = (a \times b) \times c$ and $a \times b = b \times a$.*

Example 1.2.1. We have $-3 \times 4 - 5 + (-3) = -(3 \times 4) - 5 + (-3) = -12 - 5 - 3 = -20$. Note that we have recognized that 3×4 takes precedence over the $-$ signs.

Example 1.2.2. We have

$$\begin{aligned} 21 + 32 \div (-4) + (-6) &= 21 + (32 \div (-4)) + (-6) = 21 + (-8) + (-6) \\ &= 21 - 8 - 6 = (21 - 8) - 6 = 13 - 6 = 7. \end{aligned}$$

Note that we have recognized that $32 \div (-4)$ takes precedence over the $+$ signs, and that $21 - 8$ takes precedence over the following $-$ sign.

Example 1.2.3. We have $(366 \div (-6) - (-6)) \div (-11) = ((-61) - (-6)) \div (-11) = (-55) \div (-11) = 5$. Note that the division by -11 is performed last because of brackets.

Example 1.2.4. We have $720 \div (-9) \div 4 \times (-2) = (-80) \div 4 \times (-2) = (-20) \times (-2) = 40$.

Example 1.2.5. Convince yourself that $(76 \div 2 - (-2) \times 9 + 4 \times 8) \div 4 \div 2 - (10 - 3 \times 3) - 6 = 4$.

Another operation on real numbers that we perform frequently is taking square roots. Here we need to exercise great care.

Definition. Suppose that $a \geq 0$. We say that x is a square root of a if $x^2 = a$.

Remarks. (1) If $a > 0$, then there are two square roots of a. We denote by $\sqrt{a}$ the positive square root of a, and by $-\sqrt{a}$ the negative square root of a.

(2) If $a = 0$, then there is only one square root of a. We have $\sqrt{0} = 0$.

(3) Note that square root of a is not defined when $a < 0$. If x is a real number, then $x^2 \geq 0$ and so cannot be equal to any real negative number a.

EXAMPLE 1.2.6. We have $\sqrt{(76 \div 2 - (-2) \times 9 + 4 \times 8) \div 4 \div 2 - (10 - 3 \times 3) - 6} = 2$.

EXAMPLE 1.2.7. We have $\sqrt{27} = 3 \times \sqrt{3}$. To see this, note that

$$(3 \times \sqrt{3})^2 = 3 \times \sqrt{3} \times 3 \times \sqrt{3} = 3 \times 3 \times \sqrt{3} \times \sqrt{3} = 3 \times 3 \times 3 = 27.$$

EXAMPLE 1.2.8. We have $\sqrt{72} = \sqrt{2 \times 2 \times 2 \times 3 \times 3} = 2 \times 3 \times \sqrt{2} = 6 \times \sqrt{2}$.

1.3. Distributive Laws

We now consider the distribution of multiplication inside brackets. For convenience, we usually suppress the multiplication sign $\times$, and write ab to denote the product $a \times b$.

DISTRIBUTIVE LAWS. *For every $a, b, c, d \in \mathbb{R}$, we have*
(a) $a(b + c) = ab + ac$;
(b) $(a + b)c = ac + bc$; *and*
(c) $(a + b)(c + d) = ac + ad + bc + bd$.

Special cases of part (c) above include the following two laws.

LAWS ON SQUARES. *For every $a, b \in \mathbb{R}$, we have*
(a) $(a + b)^2 = a^2 + 2ab + b^2$;
(b) $(a - b)^2 = a^2 - 2ab + b^2$; *and*
(c) $a^2 - b^2 = (a - b)(a + b)$.

PROOF. We have

$$(a + b)^2 = (a + b)(a + b) = a^2 + ab + ba + b^2 = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$$

and

$$(a - b)^2 = (a - b)(a - b) = a^2 - ab - ba + b^2 = a^2 - ab - ab + b^2 = a^2 - 2ab + b^2.$$

Also

$$(a - b)(a + b) = a^2 + ab - ba - b^2 = a^2 + ab - ab - b^2 = a^2 - b^2.$$ ♣

LAWS ON CUBES. *For every $a, b \in \mathbb{R}$, we have*
(a) $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$; *and*
(b) $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

PROOF. We have

$$\begin{aligned}(a - b)(a^2 + ab + b^2) &= a^3 + a^2b + ab^2 - ba^2 - bab - b^3 \\ &= a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3 = a^3 - b^3\end{aligned}$$

and

$$\begin{aligned}(a + b)(a^2 - ab + b^2) &= a^3 - a^2b + ab^2 + ba^2 - bab + b^3 \\ &= a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3 = a^3 + b^3.\end{aligned}$$ ♣

EXAMPLE 1.3.1. Consider the expression $(2x + 5)^2 - (x + 5)^2$. Using part (a) on the Laws on squares, we have $(2x + 5)^2 = 4x^2 + 20x + 25$ and $(x + 5)^2 = x^2 + 10x + 25$. It follows that

$$\begin{aligned}(2x + 5)^2 - (x + 5)^2 &= (4x^2 + 20x + 25) - (x^2 + 10x + 25) \\ &= 4x^2 + 20x + 25 - x^2 - 10x - 25 = 3x^2 + 10x.\end{aligned}$$

EXAMPLE 1.3.2. Consider the expression $(x-y)(x+y-2)+2x$. Using an extended version of part (c) of the Distributive laws, we have

$$(x-y)(x+y-2) = x^2+xy-2x-xy-y^2+2y = x^2-2x-y^2+2y.$$

It follows that

$$(x-y)(x+y-2)+2x = (x^2-2x-y^2+2y)+2x = x^2-2x-y^2+2y+2x = x^2-y^2+2y.$$

Alternatively, we have

$$\begin{aligned}(x-y)(x+y-2)+2x &= (x-y)((x+y)-2)+2x = (x-y)(x+y)-2(x-y)+2x \\ &= x^2-y^2-(2x-2y)+2x = x^2-y^2-2x+2y+2x = x^2-y^2+2y.\end{aligned}$$

EXAMPLE 1.3.3. We have

$$\begin{aligned}(x+1)(x-2)(x+3) &= (x^2-2x+x-2)(x+3) = (x^2-x-2)(x+3) \\ &= x^3+3x^2-x^2-3x-2x-6 = x^3+2x^2-5x-6.\end{aligned}$$

EXAMPLE 1.3.4. We have

$$\begin{aligned}&(5x+3)^2-(2x-3)^2+(3x-2)(3x+2) \\ &\quad = (25x^2+30x+9)-(4x^2-12x+9)+(9x^2-4) \\ &\quad = 25x^2+30x+9-4x^2+12x-9+9x^2-4 \\ &\quad = 30x^2+42x-4.\end{aligned}$$

1.4. Arithmetic of Fractions

In this section, we discuss briefly the arithmetic of fractions. Suppose that we wish to add two fractions and consider

$$\frac{a}{b}+\frac{c}{d},$$

where $a,b,c,d \in \mathbb{Z}$ with $b \neq 0$ and $d \neq 0$. For convenience, we have relaxed the requirement that b and d are positive integers.

ADDITION AND SUBTRACTION OF FRACTIONS. *We have*

$$\frac{a}{b}+\frac{c}{d} = \frac{ad}{bd}+\frac{bc}{bd} = \frac{ad+bc}{bd} \qquad \text{and} \qquad \frac{a}{b}-\frac{c}{d} = \frac{ad}{bd}-\frac{bc}{bd} = \frac{ad-bc}{bd}.$$

In both cases, we first rewrite the fractions with a common denominator, and then perform addition or subtraction on the numerators. Where possible, we may also perform some cancellation to the answer.

EXAMPLE 1.4.1. Following the rules precisely, we have

$$\frac{1}{3}+\frac{1}{6} = \frac{6}{18}+\frac{3}{18} = \frac{6+3}{18} = \frac{9}{18} = \frac{1}{2}.$$

However, we can somewhat simplify the argument by using the lowest common denominator instead of the product of the denominators, and obtain

$$\frac{1}{3}+\frac{1}{6} = \frac{2}{6}+\frac{1}{6} = \frac{2+1}{6} = \frac{3}{6} = \frac{1}{2}.$$

The next few examples may involve ideas discussed in the previous sections. The reader is advised to try to identify the use of the various laws discussed earlier.

EXAMPLE 1.4.2. Consider the expression

$$\frac{(x-4)^2}{(x+4)^2} - \frac{(x+2)^2}{(x+4)^2}.$$

Here the denominators are the same, so we need only perform subtraction on the numerators. We have

$$\begin{aligned}\frac{(x-4)^2}{(x+4)^2} - \frac{(x+2)^2}{(x+4)^2} &= \frac{(x-4)^2-(x+2)^2}{(x+4)^2} = \frac{(x^2-8x+16)-(x^2+4x+4)}{(x+4)^2} \\ &= \frac{x^2-8x+16-x^2-4x-4}{(x+4)^2} = \frac{12-12x}{(x+4)^2}.\end{aligned}$$

EXAMPLE 1.4.3. We have

$$\begin{aligned}\frac{3(x-1)}{x+1} + \frac{2(x+1)}{x-1} &= \frac{3(x-1)^2}{(x+1)(x-1)} + \frac{2(x+1)^2}{(x+1)(x-1)} = \frac{3(x-1)^2+2(x+1)^2}{(x+1)(x-1)} \\ &= \frac{3(x^2-2x+1)+2(x^2+2x+1)}{x^2-1} = \frac{(3x^2-6x+3)+(2x^2+4x+2)}{x^2-1} \\ &= \frac{3x^2-6x+3+2x^2+4x+2}{x^2-1} = \frac{5x^2-2x+5}{x^2-1}.\end{aligned}$$

EXAMPLE 1.4.4. We have

$$\frac{x}{y} - \frac{x}{x+y} = \frac{x(x+y)}{y(x+y)} - \frac{yx}{y(x+y)} = \frac{x(x+y)-yx}{y(x+y)} = \frac{x^2+xy-yx}{y(x+y)} = \frac{x^2+xy-xy}{y(x+y)} = \frac{x^2}{y(x+y)}.$$

EXAMPLE 1.4.5. We have

$$\frac{p}{p-q} + \frac{q}{q-p} = \frac{p}{p-q} + \frac{-q}{p-q} = \frac{p+(-q)}{p-q} = \frac{p-q}{p-q} = 1.$$

Note here that the two denominators are essentially the same, apart from a sign change. Changing the sign of both the numerator and denominator of one of the fractions has the effect of giving two fractions with the same denominator.

EXAMPLE 1.4.6. We have

$$\frac{4}{a} - \frac{2}{a(a+2)} = \frac{4(a+2)}{a(a+2)} - \frac{2}{a(a+2)} = \frac{4(a+2)-2}{a(a+2)} = \frac{(4a+8)-2}{a(a+2)} = \frac{4a+8-2}{a(a+2)} = \frac{4a+6}{a(a+2)}.$$

Note here that the common denominator is not the product of the two denominators, since we have observed the common factor a in the two denominators. If we do not make this observation, then we have

$$\begin{aligned}\frac{4}{a} - \frac{2}{a(a+2)} &= \frac{4a(a+2)}{a^2(a+2)} - \frac{2a}{a^2(a+2)} = \frac{4a(a+2)-2a}{a^2(a+2)} = \frac{(4a^2+8a)-2a}{a^2(a+2)} \\ &= \frac{4a^2+8a-2a}{a^2(a+2)} = \frac{4a^2+6a}{a^2(a+2)} = \frac{a(4a+6)}{a^2(a+2)} = \frac{4a+6}{a(a+2)}.\end{aligned}$$

Note that the common factor a is cancelled from the numerator and denominator in the last step. We still have the same answer, but a little extra work is required.

Suppose next that we wish to multiply two fractions and consider

$$\frac{a}{b} \times \frac{c}{d},$$

where $a, b, c, d \in \mathbb{Z}$ with $b \neq 0$ and $d \neq 0$.

MULTIPLICATION OF FRACTIONS. *We have*

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

We simply multiply the numerators and denominators separately. Where possible, we may also perform some cancellation to the answer.

Example 1.4.7. We have

$$\frac{h}{x^2}\left(1 - \frac{h}{x+h}\right) = \frac{h}{x^2}\left(\frac{x+h}{x+h} - \frac{h}{x+h}\right) = \frac{h}{x^2} \times \frac{x+h-h}{x+h} = \frac{h}{x^2} \times \frac{x}{x+h} = \frac{hx}{x^2(x+h)} = \frac{h}{x(x+h)}.$$

Example 1.4.8. We have

$$\left(\frac{1}{x} + \frac{1}{y}\right)(x+y) = \left(\frac{y}{xy} + \frac{x}{xy}\right)(x+y) = \frac{y+x}{xy} \times (x+y) = \frac{x+y}{xy} \times \frac{x+y}{1} = \frac{(x+y)^2}{xy}.$$

Example 1.4.9. We have

$$\left(\frac{1}{x} - \frac{1}{y}\right)\frac{1}{x-y} = \left(\frac{y}{xy} - \frac{x}{xy}\right)\frac{1}{x-y} = \frac{y-x}{xy} \times \frac{1}{x-y} = \frac{y-x}{xy(x-y)} = \frac{-(x-y)}{xy(x-y)} = -\frac{1}{xy}.$$

Example 1.4.10. We have

$$\frac{b-c}{bc} \times \frac{b^2}{b^2-bc} = \frac{b^2(b-c)}{bc(b^2-bc)} = \frac{b^2(b-c)}{b^2c(b-c)} = \frac{1}{c}.$$

Example 1.4.11. We have

$$\frac{a^2}{a^2-1} \times \frac{a+1}{a} = \frac{a^2(a+1)}{a(a^2-1)} = \frac{a^2(a+1)}{a(a-1)(a+1)} = \frac{a}{a-1}.$$

Example 1.4.12. We have

$$\frac{x+y}{x^2-4y^2} \times \frac{6y-3x}{2x+2y} = \frac{(x+y)(6y-3x)}{(x^2-4y^2)(2x+2y)} = \frac{3(x+y)(2y-x)}{2(x-2y)(x+2y)(x+y)} = -\frac{3}{2(x+2y)}.$$

Suppose finally that we wish to divide one fraction by another and consider

$$\frac{a}{b} \div \frac{c}{d},$$

where $a, b, c, d \in \mathbb{Z}$ with $b \neq 0$, $c \neq 0$ and $d \neq 0$.

DIVISION OF FRACTIONS. *We have*

$$\frac{(a/b)}{(c/d)} = \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}.$$

In other words, we invert the divisor and then perform multiplication instead. Where possible, we may also perform some cancellation to the answer.

REMARK. Note the special cases that

$$\frac{(a/b)}{c} = \frac{a}{bc} \qquad \text{and} \qquad \frac{a}{(c/d)} = \frac{ad}{c}.$$

EXAMPLE 1.4.13. We have

$$\begin{aligned}\left(1+\frac{1}{1+x}\right) \div \frac{4}{5(1+x)} &= \left(\frac{1+x}{1+x}+\frac{1}{1+x}\right) \div \frac{4}{5(1+x)} = \frac{1+x+1}{1+x} \div \frac{4}{5(1+x)} \\ &= \frac{2+x}{1+x} \div \frac{4}{5(1+x)} = \frac{2+x}{1+x} \times \frac{5(1+x)}{4} \\ &= \frac{5(2+x)(1+x)}{4(1+x)} = \frac{5(2+x)}{4}.\end{aligned}$$

EXAMPLE 1.4.14. We have

$$\frac{6a}{a-5} \div \frac{a+5}{a^2-25} = \frac{6a}{a-5} \div \frac{a+5}{(a+5)(a-5)} = \frac{6a}{a-5} \div \frac{1}{a-5} = \frac{6a}{a-5} \times \frac{a-5}{1} = \frac{6a(a-5)}{a-5} = 6a.$$

Alternatively, we have

$$\frac{6a}{a-5} \div \frac{a+5}{a^2-25} = \frac{6a}{a-5} \times \frac{a^2-25}{a+5} = \frac{6a(a^2-25)}{(a-5)(a+5)} = \frac{6a(a^2-25)}{a^2-25} = 6a.$$

EXAMPLE 1.4.15. We have

$$\frac{\frac{1}{x}-\frac{1}{y}}{x-y} = \left(\frac{1}{x}-\frac{1}{y}\right) \div (x-y) = \left(\frac{y}{xy}-\frac{x}{xy}\right) \times \frac{1}{x-y} = \frac{y-x}{xy} \times \frac{1}{x-y} = \frac{y-x}{xy(x-y)} = -\frac{1}{xy}.$$

EXAMPLE 1.4.16. We have

$$\begin{aligned}\frac{\frac{2}{x}+\frac{3}{y}}{\frac{1}{x}-\frac{2}{y}} &= \left(\frac{2}{x}+\frac{3}{y}\right) \div \left(\frac{1}{x}-\frac{2}{y}\right) = \left(\frac{2y}{xy}+\frac{3x}{xy}\right) \div \left(\frac{y}{xy}-\frac{2x}{xy}\right) = \frac{2y+3x}{xy} \div \frac{y-2x}{xy} \\ &= \frac{2y+3x}{xy} \times \frac{xy}{y-2x} = \frac{(2y+3x)xy}{xy(y-2x)} = \frac{2y+3x}{y-2x}.\end{aligned}$$

EXAMPLE 1.4.17. We have

$$\begin{aligned}\frac{x^2-y^2}{\frac{1}{x}+\frac{1}{y}} &= (x^2-y^2) \div \left(\frac{1}{x}+\frac{1}{y}\right) = (x^2-y^2) \div \left(\frac{y}{xy}+\frac{x}{xy}\right) = (x^2-y^2) \div \frac{y+x}{xy} \\ &= (x^2-y^2) \times \frac{xy}{x+y} = \frac{xy(x^2-y^2)}{x+y} = \frac{xy(x-y)(x+y)}{x+y} = xy(x-y).\end{aligned}$$

1.5. Factorization

Very often, we have to handle mathematical expressions that can be simplified. We have seen a few instances of cancellation of common terms in the numerator and denominator of fractions. We now consider the question of factorization. This can be thought of as the reverse process of expansion. It is difficult, if not impossible, to write down rules for factorization. Instead, we shall look at a few examples, and illustrate some of the ideas.

EXAMPLE 1.5.1. Consider the expression $x^3 - x$. First of all, we recognize that both terms x^3 and x have a factor x. Hence we can write $x^3 - x = x(x^2 - 1)$, using one of the Distributive laws. Next, we realize that we can apply one of the Laws on squares on the factor $x^2 - 1$. Hence

$$x^3 - x = x(x^2 - 1) = x(x - 1)(x + 1).$$

EXAMPLE 1.5.2. Consider the expression $a^4 - b^4$. Note that we can apply one of the Laws on squares to obtain $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2)$. We can again apply one of the Laws on squares on the factor $a^2 - b^2$. Hence

$$a^4 - b^4 = (a^2 - b^2)(a^2 + b^2) = (a - b)(a + b)(a^2 + b^2).$$

EXAMPLE 1.5.3. Consider the expression $x^3 - 64$. Note that $64 = 4^3$. Applying one of the Laws on cubes, we obtain $x^3 - 64 = (x - 4)(x^2 + 4x + 16)$.

Sometimes, we can factorize a quadratic polynomial $ax^2 + bx + c$. To do this, we must first study the problem of solving quadratic equations, a problem we shall consider in Section 5.2. We shall consider the problem of factorization further in Section 5.3. Here we confine ourselves to a few more examples.

EXAMPLE 1.5.4. Consider the expression $m^2 - n^2 + 4m + 4n$. We may write

$$m^2 - n^2 + 4m + 4n = m^2 + 4m + 4n - n^2 = m(m + 4) + n(4 - n),$$

and this does not lead anywhere. However, we may recognize that

$$m^2 - n^2 + 4m + 4n = (m - n)(m + n) + 4(m + n) = (m - n + 4)(m + n),$$

and this gives a good factorization.

EXAMPLE 1.5.5. We have

$$\begin{aligned} x^3 - 2x^2 - 4x + 8 &= (x^3 - 2x^2) - (4x - 8) = x^2(x - 2) - 4(x - 2) \\ &= (x^2 - 4)(x - 2) = (x - 2)(x + 2)(x - 2) = (x - 2)^2(x + 2). \end{aligned}$$

EXAMPLE 1.5.6. We have

$$\begin{aligned} \frac{a+1}{a^2-a} - \frac{a-1}{a^2+a} &= \frac{a+1}{a(a-1)} - \frac{a-1}{a(a+1)} = \frac{(a+1)^2}{a(a-1)(a+1)} - \frac{(a-1)^2}{a(a-1)(a+1)} = \frac{(a+1)^2-(a-1)^2}{a(a-1)(a+1)} \\ &= \frac{((a+1)-(a-1))((a+1)+(a-1))}{a(a-1)(a+1)} = \frac{(a+1-a+1)(a+1+a-1)}{a(a-1)(a+1)} \\ &= \frac{4a}{a(a-1)(a+1)} = \frac{4}{(a-1)(a+1)}. \end{aligned}$$

EXAMPLE 1.5.7. We have

$$\begin{aligned} \frac{2}{x^2-1} - \frac{1}{x^2-x} + \frac{x-1}{x^2+x} &= \frac{2}{(x-1)(x+1)} - \frac{1}{x(x-1)} + \frac{x-1}{x(x+1)} \\ &= \frac{2x}{x(x-1)(x+1)} - \frac{x+1}{x(x-1)(x+1)} + \frac{(x-1)^2}{x(x-1)(x+1)} \\ &= \frac{2x-(x+1)+(x-1)^2}{x(x-1)(x+1)} = \frac{2x-x-1+x^2-2x+1}{x(x-1)(x+1)} \\ &= \frac{x^2-x}{x(x-1)(x+1)} = \frac{x(x-1)}{x(x-1)(x+1)} = \frac{1}{x+1}. \end{aligned}$$

Problems for Chapter 1

1. Find the precise value of each of the following expressions:
   a) $5 + 4 \times 3 \div 2 - 1$
   b) $(1 + 2) \times 3 - 4 \div 5$
   c) $(54 \div 3 + 18 \times 2) \div 9$
   d) $4 + 2 - 4 + 5 \times (-2) \times (1 + 3)$
   e) $2 + 5 \times (-1) - (2 + 3) \times 4 \div 10 + 4 - (3 - 5)$
   f) $((4 + 2) \times 3 + 1) \times 5 + 10 \div 2$
   g) $\sqrt{(-4) \times (2 - 11)}$
   h) $-\sqrt{5^2 + 12^2}$
   i) $\sqrt{5 \times 5 - 4 \times 4} - \sqrt{3 \times 3 - 2 \times 2 - (-1) \times (-1)}$

2. Expand each of the following expressions:
   a) $(4x + 3)(5x - 2)$
   b) $(4x + 3)^2 + (5x - 2)^2$
   c) $(4x + 3)^2 - (5x - 2)^2$
   d) $(7x - 2)^2 + (4x + 5)^2$
   e) $(x + y)(x - 2y)$
   f) $(x + 2y + 3)(2x - y - 1)$
   g) $(x + 2y)(x - 2y)^2$
   h) $(x + 2y)^2(x - 2y)^2$

3. Rewrite each of the following expressions, showing all the steps of your argument carefully:
   a) $\dfrac{3}{4} + \dfrac{2}{3}$
   b) $\dfrac{5}{6} - \dfrac{1}{12}$
   c) $\dfrac{5x}{x + 2} + \dfrac{3x}{x + 4}$
   d) $\dfrac{3}{x - 1} - \dfrac{3}{x + 1}$
   e) $\dfrac{5}{x} + \dfrac{3}{x(x + 1)}$
   f) $\dfrac{(x + y)^2}{x^2} - \dfrac{(x - y)^2}{y^2}$

4. Rewrite each of the following expressions, showing all the steps of your argument carefully:
   a) $\dfrac{2 + 3}{4 + 5} \times \dfrac{6 + 7}{8 + 9}$
   b) $\dfrac{2 + 3}{4 + 5} \div \dfrac{6 + 7}{8 + 9}$
   c) $\left(\dfrac{1}{2} + \dfrac{1}{3}\right) \div \left(\dfrac{3}{4} + \dfrac{4}{3}\right) \times \left(\dfrac{5}{14} + \dfrac{3}{2} \times \dfrac{3}{7} + \dfrac{3}{2}\right)$
   d) $\dfrac{x}{y^2} \times \dfrac{xy - yz}{x}$
   e) $\dfrac{x^2 - y^2}{x + y} \div \dfrac{x - y}{x^3 + y^3}$
   f) $\left(\dfrac{4}{x} - \dfrac{3}{y}\right) \div \left(\dfrac{5}{x} + \dfrac{6}{y}\right)$

5. Factorize each of the following expressions, using the laws on squares and cubes as necessary:
   a) $x^4 - x^2$
   b) $x^6 - y^6$
   c) $x^3y - xy^3$
   d) $x^5y^2 + x^2y^5$

6. Simplify each of the following expressions, showing all the steps of your argument carefully:
   a) $\dfrac{3}{x(x + 2)} + \dfrac{1}{x^2 - 2x} - \dfrac{2}{x^2 - 4}$
   b) $\dfrac{1}{x^2 + xy} + \dfrac{1}{y^2 + xy}$
   c) $\dfrac{x}{x - y} - \dfrac{y}{x + y} - \dfrac{2xy}{x^2 - y^2}$
   d) $\left(x + \dfrac{y^2}{x - y}\right) \div \dfrac{x^3 + y^3}{x^2 - y^2}$
   e) $x^3 - y^3 + x^2y - xy^2$

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Chapter 2

INTRODUCTION TO MATRICES

2.1. Linear Equations

EXAMPLE 2.1.1. Consider the two linear equations

$$\begin{aligned} 3x + 4y &= 11, \\ 5x + 7y &= 19. \end{aligned}$$

It is easy to check the two equations are satisfied when $x = 1$ and $y = 2$. We can represent these two linear equations in matrix form as

$$\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ 19 \end{pmatrix},$$

where we adopt the convention that

$$\begin{pmatrix} 3 & 4 \\ \bullet & \bullet \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ \bullet \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \bullet & \bullet \\ 5 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bullet \\ 19 \end{pmatrix}$$

represent respectively the information $3x + 4y = 11$ and $5x + 7y = 19$. Under this convention, it is easy to see that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

for every $x, y \in \mathbb{R}$. Next, observe that

$$\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

† This chapter was written at Macquarie University in 1999.

where, under a convention slightly more general to the one used earlier, we have

$$\begin{pmatrix} 7 & -4 \\ \bullet & \bullet \end{pmatrix}\begin{pmatrix} 3 & \bullet \\ 5 & \bullet \end{pmatrix} = \begin{pmatrix} 1 & \bullet \\ \bullet & \bullet \end{pmatrix}, \qquad \begin{pmatrix} 7 & -4 \\ \bullet & \bullet \end{pmatrix}\begin{pmatrix} \bullet & 4 \\ \bullet & 7 \end{pmatrix} = \begin{pmatrix} \bullet & 0 \\ \bullet & \bullet \end{pmatrix},$$

$$\begin{pmatrix} \bullet & \bullet \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 3 & \bullet \\ 5 & \bullet \end{pmatrix} = \begin{pmatrix} \bullet & \bullet \\ 0 & \bullet \end{pmatrix}, \qquad \begin{pmatrix} \bullet & \bullet \\ -5 & 3 \end{pmatrix}\begin{pmatrix} \bullet & 4 \\ \bullet & 7 \end{pmatrix} = \begin{pmatrix} \bullet & \bullet \\ \bullet & 1 \end{pmatrix},$$

representing respectively $(7 \times 3) + ((-4) \times 5) = 1$, $(7 \times 4) + ((-4) \times 7) = 0$, $((-5) \times 3) + (3 \times 5) = 0$ and $((-5) \times 4) + (3 \times 7) = 1$. It now follows on the one hand that

$$\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix},$$

and on the other hand that

$$\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 11 \\ 19 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The convention mentioned in the example above is simply the rule concerning the multiplication of matrices. The purpose of this chapter is to study the arithmetic in connection with matrices. We shall be concerned primarily with 2×2 real matrices. These are arrays of real numbers of the form

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

consisting of two rows counted from top to bottom, and two columns counted from left to right. An entry a_{ij} thus corresponds to the entry in row i and column j.

2.2. Arithmetic

ADDITION AND SUBTRACTION. *Suppose that*

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

are two 2×2 matrices. Then

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix} \qquad \text{and} \qquad A - B = \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} \\ a_{21} - b_{21} & a_{22} - b_{22} \end{pmatrix}.$$

In other words, we perform addition and subtraction entrywise. The operations addition and subtraction are governed by the following rules:
(a) Operations within brackets are performed first.
(b) Addition and subtraction are performed in their order of appearance.
(c) A number of additions can be performed in any order. For any 2×2 matrices A, B, C, we have $A + (B + C) = (A + B) + C$ and $A + B = B + A$.

EXAMPLE 2.2.1. We have

$$\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} - \left(\begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\right) = \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} - \begin{pmatrix} 3 & 6 \\ 7 & 13 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -2 & -6 \end{pmatrix},$$

and

$$\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 4 & 6 \end{pmatrix}.$$

Example 2.2.2. We have

$$\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} + \left(\begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \right) = \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} + \begin{pmatrix} 3 & 6 \\ 7 & 13 \end{pmatrix} = \begin{pmatrix} 6 & 10 \\ 12 & 20 \end{pmatrix},$$

and

$$\left(\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix} \right) + \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 5 & 8 \\ 9 & 14 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 6 & 10 \\ 12 & 20 \end{pmatrix}.$$

Example 2.2.3. Like real numbers, it is not true in general that $A - (B - C) = (A - B) - C$. Note that

$$\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} - \left(\begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \right) = \begin{pmatrix} 2 & 2 \\ 4 & 6 \end{pmatrix},$$

and

$$\left(\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix} \right) - \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -2 & -6 \end{pmatrix}.$$

Remark. The matrix

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

satisfies $0 + A = A + 0 = A$ for any 2×2 matrix A, and plays a role analogous to the real number 0 in addition of real numbers. This matrix 0 is called the zero matrix.

MULTIPLICATION BY A REAL NUMBER. *Suppose that*

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is a 2×2 matrix, and that r is a real number. Then

$$rA = \begin{pmatrix} ra_{11} & ra_{12} \\ ra_{21} & ra_{22} \end{pmatrix}.$$

In other words, we multiply each entry of A by the same real number r. This operation is governed by the following rules:

(a) Operations within brackets are performed first.
(b) If there are no brackets to indicate priority, then multiplication by a real number takes precedence over addition and subtraction.
(c) A number of multiplications by real numbers can be performed in any order. For any 2×2 matrix A and any real numbers $r, s \in \mathbb{R}$, we have $(rs)A = r(sA)$.
(d) For any 2×2 matrix A and any real numbers $r, s \in \mathbb{R}$, we have $(r + s)A = rA + sA$.
(e) For any 2×2 matrices A, B and any real number $r \in \mathbb{R}$, we have $r(A + B) = rA + rB$.

Example 2.2.4. We have

$$2 \left(\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \right) = 2 \begin{pmatrix} 4 & 6 \\ 8 & 11 \end{pmatrix} = \begin{pmatrix} 8 & 12 \\ 16 & 22 \end{pmatrix}.$$

Example 2.2.5. We have

$$3 \left(\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} + 2 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \right) = 3 \left(\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} \right) = 3 \begin{pmatrix} 5 & 8 \\ 11 & 15 \end{pmatrix} = \begin{pmatrix} 15 & 24 \\ 33 & 45 \end{pmatrix}.$$

EXAMPLE 2.2.6. We have

$$2\left(3\begin{pmatrix}3&4\\5&7\end{pmatrix}\right)=2\begin{pmatrix}9&12\\15&21\end{pmatrix}=\begin{pmatrix}18&24\\30&42\end{pmatrix}=6\begin{pmatrix}3&4\\5&7\end{pmatrix}=(2\times 3)\begin{pmatrix}3&4\\5&7\end{pmatrix}.$$

EXAMPLE 2.2.7. We have

$$5\begin{pmatrix}3&4\\5&7\end{pmatrix}-2\begin{pmatrix}3&4\\5&7\end{pmatrix}=\begin{pmatrix}15&20\\25&35\end{pmatrix}-\begin{pmatrix}6&8\\10&14\end{pmatrix}=\begin{pmatrix}9&12\\15&21\end{pmatrix}=3\begin{pmatrix}3&4\\5&7\end{pmatrix}=(5-2)\begin{pmatrix}3&4\\5&7\end{pmatrix}.$$

EXAMPLE 2.2.8. We have

$$3\left(\begin{pmatrix}3&4\\5&7\end{pmatrix}-\begin{pmatrix}1&2\\3&4\end{pmatrix}\right)=3\begin{pmatrix}2&2\\2&3\end{pmatrix}=\begin{pmatrix}6&6\\6&9\end{pmatrix},$$

and

$$3\begin{pmatrix}3&4\\5&7\end{pmatrix}-3\begin{pmatrix}1&2\\3&4\end{pmatrix}=\begin{pmatrix}9&12\\15&21\end{pmatrix}-\begin{pmatrix}3&6\\9&12\end{pmatrix}=\begin{pmatrix}6&6\\6&9\end{pmatrix}.$$

MULTIPLICATION OF MATRICES. *Suppose that*

$$A=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}\qquad\text{and}\qquad B=\begin{pmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{pmatrix}$$

are 2×2 *matrices. Then*

$$AB=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}\begin{pmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{pmatrix}=\begin{pmatrix}a_{11}b_{11}+a_{12}b_{21}&a_{11}b_{12}+a_{12}b_{22}\\a_{21}b_{11}+a_{22}b_{21}&a_{21}b_{12}+a_{22}b_{22}\end{pmatrix}.$$

This operation is governed by the following rules:
(a) Operations within brackets are performed first.
(b) If there are no brackets to indicate priority, then multiplication takes precedence over addition and subtraction.
(c) For any 2×2 *matrices* A,B,C*, we have* $(AB)C=A(BC)$*.*
(d) For any 2×2 *matrices* A,B *and any real numbers* $r\in\mathbb{R}$*, we have* $r(AB)=(rA)B=A(rB)$*.*
(e) For any 2×2 *matrices* A,B,C*, we have* $A(B+C)=AB+AC$ *and* $(A+B)C=AC+BC$*.*

REMARKS. (1) Note that the definition above agrees with the convention adopted in Example 2.1.1. Observe that we have

$$\begin{pmatrix}a_{11}&a_{12}\\\bullet&\bullet\end{pmatrix}\begin{pmatrix}b_{11}&\bullet\\b_{21}&\bullet\end{pmatrix}=\begin{pmatrix}a_{11}b_{11}+a_{12}b_{21}&\bullet\\\bullet&\bullet\end{pmatrix},$$

$$\begin{pmatrix}a_{11}&a_{12}\\\bullet&\bullet\end{pmatrix}\begin{pmatrix}\bullet&b_{12}\\\bullet&b_{22}\end{pmatrix}=\begin{pmatrix}\bullet&a_{11}b_{12}+a_{12}b_{22}\\\bullet&\bullet\end{pmatrix},$$

$$\begin{pmatrix}\bullet&\bullet\\a_{21}&a_{22}\end{pmatrix}\begin{pmatrix}b_{11}&\bullet\\b_{21}&\bullet\end{pmatrix}=\begin{pmatrix}\bullet&\bullet\\a_{21}b_{11}+a_{22}b_{21}&\bullet\end{pmatrix},$$

$$\begin{pmatrix}\bullet&\bullet\\a_{21}&a_{22}\end{pmatrix}\begin{pmatrix}\bullet&b_{12}\\\bullet&b_{22}\end{pmatrix}=\begin{pmatrix}\bullet&\bullet\\\bullet&a_{21}b_{12}+a_{22}b_{22}\end{pmatrix}.$$

(2) The matrix

$$I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

satisfies $IA=AI=A$ for any 2×2 matrix A, and plays a role analogous to the real number 1 in multiplication of real numbers. This matrix I is called the identity matrix.

(3) Multiplication of matrices is generally not commutative; in other words, given two 2×2 matrices A and B, it is not automatic that $AB = BA$. For example, let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ 20 & 13 \end{pmatrix} \qquad \text{and} \qquad BA = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 13 & 20 \\ 5 & 8 \end{pmatrix}.$$

Example 2.2.9. We have

$$\left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}\right)\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ 20 & 13 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} -4 & 7 \\ -2 & 19 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\left(\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}\right) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -4 & 5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 7 \\ -12 & 19 \end{pmatrix}.$$

Example 2.2.10. We have

$$\left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}\right)\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} -10 & 10 \\ -10 & 10 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} + \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} -6 & 5 \\ -10 & 9 \end{pmatrix} + \begin{pmatrix} -4 & 5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -10 & 10 \\ -10 & 10 \end{pmatrix}.$$

Since I is the identity matrix, we would like to find a technique to obtain, for any given 2×2 matrix A, an inverse 2×2 matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$. This is not always possible, since in the case of real numbers, the number 0 does not have a multiplicative inverse. We therefore need a condition on 2×2 matrices which is equivalent to saying that a real number is non-zero.

MULTIPLICATIVE INVERSE. *For any 2×2 matrix*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

satisfying the condition $ad - bc \neq 0$, the matrix

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

satisfies $AA^{-1} = A^{-1}A = I$. In this case, we say that A is invertible with multiplicative inverse A^{-1}.

Remarks. (1) The quantity $ad - bc$ is known as the determinant of the matrix A. The result above says that any 2×2 matrix is invertible as long as it has non-zero determinant.

(2) If two 2×2 matrices A and B both have non-zero determinants, then it can be shown that the matrix product AB also has non-zero determinant. We also have $(AB)^{-1} = B^{-1}A^{-1}$.

Example 2.2.11. Recall Example 2.1.1. It is easy to check that

$$\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

EXAMPLE 2.2.12. The matrix $\begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$ has determinant 0 and so is not invertible.

EXAMPLE 2.2.13. Consider the matrices

$$A = \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}.$$

Then

$$A^{-1} = \begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix} \qquad \text{and} \qquad B^{-1} = \frac{1}{3}\begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}.$$

Note also that

$$AB = \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 18 & 33 \\ 31 & 57 \end{pmatrix}.$$

We have

$$(AB)^{-1} = \frac{1}{3}\begin{pmatrix} 57 & -33 \\ -31 & 18 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix} = B^{-1}A^{-1}.$$

2.3. Application to Linear Equations

We now return to the problem first discussed in Section 2.1. Consider the two linear equations

$$\begin{aligned} ax + by &= s, \\ cx + dy &= t, \end{aligned}$$

where $a, b, c, d, s, t \in \mathbb{R}$ are given and x and y are the unknowns.

We can represent these two linear equations in matrix form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s \\ t \end{pmatrix}.$$

If $ad - bc \neq 0$, then the 2×2 matrix on the left hand side is invertible. It follows that there exist real numbers $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix},$$

giving the solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix}.$$

EXAMPLE 2.3.1. Suppose that

$$\begin{aligned} x + y &= 32, \\ 3x + 2y &= 70. \end{aligned}$$

The two linear equations can be represented in matrix form

$$\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 32 \\ 70 \end{pmatrix}.$$

Note now that the matrix $\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$ has determinant -1 and multiplicative inverse $\begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$. It follows that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 32 \\ 70 \end{pmatrix} = \begin{pmatrix} 6 \\ 26 \end{pmatrix},$$

giving the solution $x = 6$ and $y = 26$.

We shall discuss a different technique for solving such equations in Section 5.1.

Problems for Chapter 2

1. Write each of the following systems of linear equations in matrix form:
   a) $3x - 8y = 1$, $2x + 3y = 9$
   b) $4x - 3y = 14$, $9x - 4y = 26$
   c) $6x - 2y = 14$, $2x + 3y = 12$
   d) $5x + 2y = 4$, $7x + 3y = 5$

2. Let

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 3 \\ 6 & 7 \end{pmatrix}, \qquad C = \begin{pmatrix} 4 & 3 \\ 5 & 1 \end{pmatrix}.$$

   a) Verify that $(A + B) + C = A + (B + C)$ and $A + B = B + A$.
   b) Find $A + 3B$ and $7A - 2B + 3C$.
   c) Verify that $(AB)C = A(BC)$.
   d) Is it true that $AB = BA$? Comment on the result.
   e) Find A^{-1} and B^{-1}.
   f) Find $(AB)^{-1}$, and verify that $(AB)^{-1} = B^{-1}A^{-1}$.

3. Solve each of the systems of linear equations in Question 1 by using inverse matrices.

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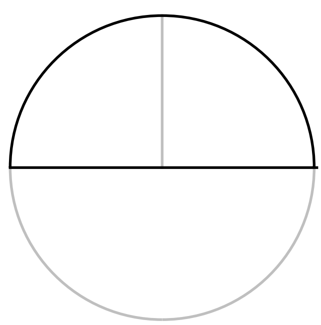
Chapter 3

TRIGONOMETRY

3.1. Radian and Arc Length

The number π plays a central role in the study of trigonometry. We all know that a circle of radius 1 has area π and circumference 2π. It is also very useful in describing angles, as we shall now show.

Let us split a circle of radius 1 along a diameter into two semicircles as shown in the picture below.



The circumference of the circle is now split into two equal parts, each of length π and each suntending an angle $180°$. If we use the convention that $\pi = 180°$, then the arc of the semicircle of radius 1 will be the same as the angle it subtends. If we further split the arc of the semicircle of radius 1 into two equal parts, then each of the two parts forms an arc of length $\pi/2$ and subtends an angle $90° = \pi/2$. In fact, any arc of a circle of radius 1 which subtends an angle θ must have length θ under our convention.

We now formalize our discussion so far.

DEFINITION. An angle of 1 radian is defined to be the angle subtended by an arc of length 1 on a circle of radius 1.

† This chapter was written at Macquarie University in 1999.

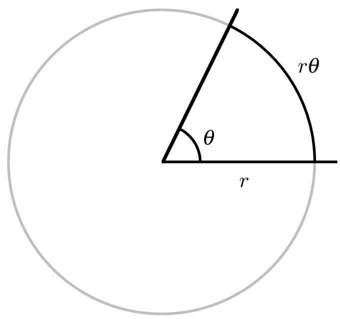
REMARKS. (1) Very often, the term radian is omitted when we discuss angles. We simply refer to an angle 1 or an angle π, rather than an angle of 1 radian or an angle of π radian.

(2) Simple calculation shows that 1 radian is equal to $(180/\pi)^\circ = 57.2957795\ldots^\circ$. Similarly, we can show that 1° is equal to $(\pi/180)$ radian $= 0.01745329\ldots$ radian. In fact, since π is irrational, the digits do not terminate or repeat.

(3) We observe the following special values:

$$\frac{\pi}{6} = 30^\circ, \qquad \frac{\pi}{4} = 45^\circ, \qquad \frac{\pi}{3} = 60^\circ, \qquad \frac{\pi}{2} = 90^\circ, \qquad \pi = 180^\circ, \qquad 2\pi = 360^\circ.$$

Consider now a circle of radius r and an angle θ given in radian, as shown in the picture below.



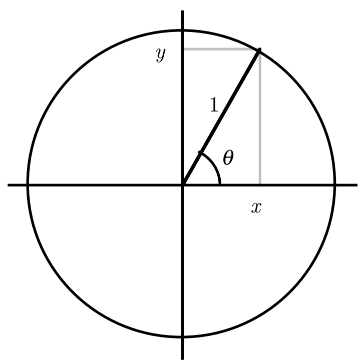
Clearly the length s of the arc which subtends the angle θ satisfies $s = r\theta$, while the area A of the sector satisfies

$$A = \pi r^2 \times \frac{\theta}{2\pi} = \frac{1}{2} r^2 \theta.$$

Note that πr^2 is equal to the area inside the circle, while $\theta/2\pi$ is the proportion of the area in question.

3.2. The Trigonometric Functions

Consider the xy-plane, together with a circle of radius 1 and centred at the origin $(0,0)$. Suppose that θ is an angle measured anticlockwise from the positive x-axis, and the point (x,y) on the circle is as shown in the picture below.



We define

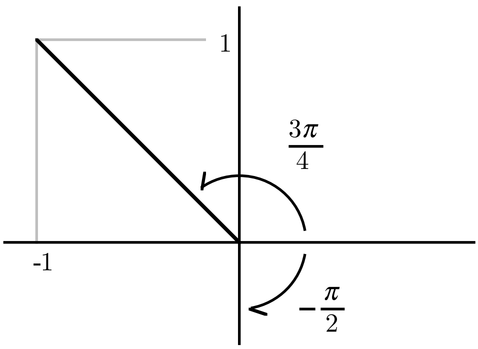
$$\cos\theta = x \qquad \text{and} \qquad \sin\theta = y.$$

Furthermore, we define

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{x}{y}, \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{1}{x} \qquad \text{and} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{1}{y}.$$

REMARKS. (1) Note that $\tan\theta$ and $\sec\theta$ are defined only when $\cos\theta \neq 0$, and that $\cot\theta$ and $\csc\theta$ are defined only when $\sin\theta \neq 0$.

(2) It is a good habit to always measure an angle from the positive x-axis, using the convention that positive angles are measured anticlockwise and negative angles are measured clockwise, as illustrated below.



(3) We then observe that

$$\cos\theta = \frac{\text{horizontal side}}{\text{hypothenuse}} \qquad \text{and} \qquad \sin\theta = \frac{\text{vertical side}}{\text{hypothenuse}},$$

as well as

$$\tan\theta = \frac{\text{vertical side}}{\text{horizontal side}} \qquad \text{and} \qquad \cot\theta = \frac{\text{horizontal side}}{\text{vertical side}},$$

with the convention that

$$\text{horizontal side}\begin{cases} > 0 & \text{(to the right of the (vertical) } y\text{-axis)},\\ < 0 & \text{(to the left of the (vertical) } y\text{-axis)}, \end{cases}$$

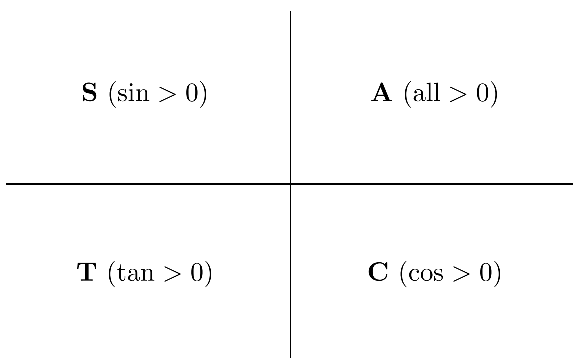
and

$$\text{vertical side}\begin{cases} > 0 & \text{(above the (horizontal) } x\text{-axis)},\\ < 0 & \text{(below the (horizontal) } x\text{-axis)}, \end{cases}$$

while

$$\text{hypothenuse} > 0 \quad \text{(always)}.$$

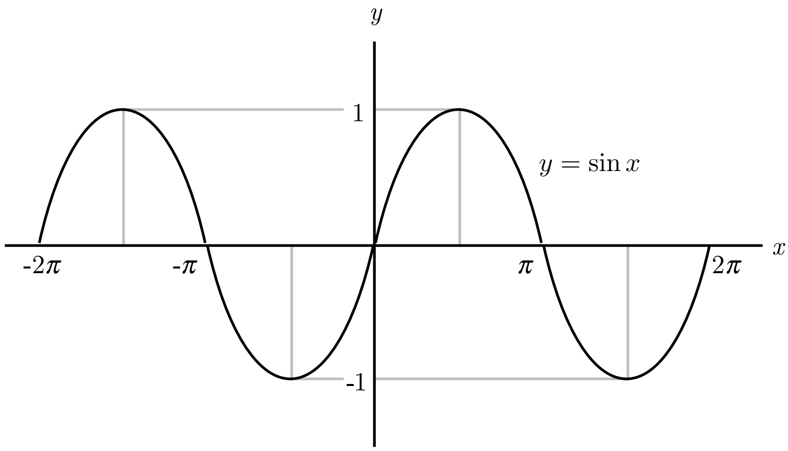
(4) It is also useful to remember the CAST rule concerning sine, cosine and tangent.

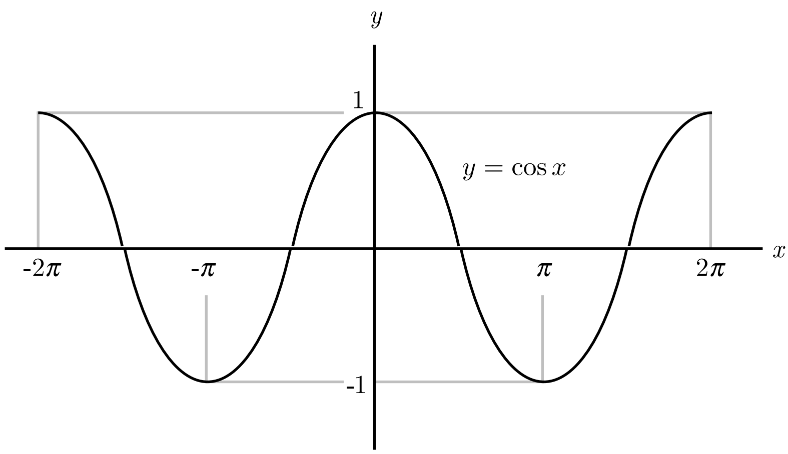


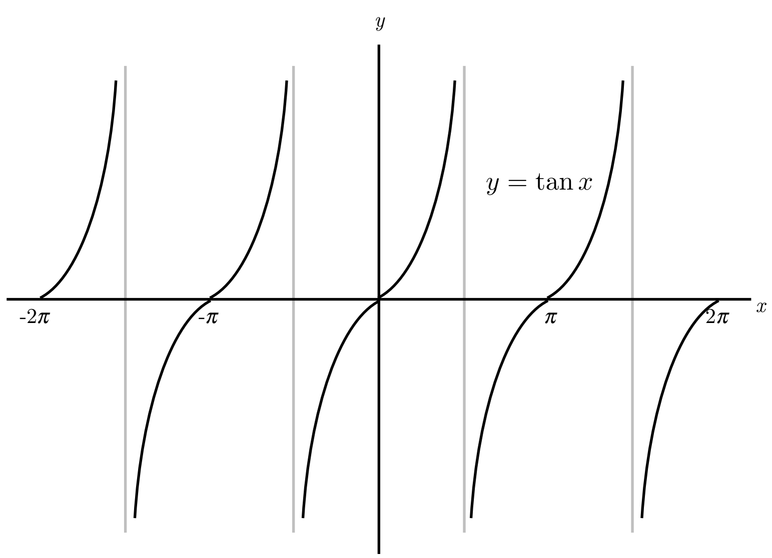
PYTHAGOREAN IDENTITIES. *For every value $\theta \in \mathbb{R}$ for which the trigonometric functions in question are defined, we have*
(a) $\cos^2\theta + \sin^2\theta = 1$*;*
(b) $1 + \tan^2\theta = \sec^2\theta$*; and*
(c) $1 + \cot^2\theta = \csc^2\theta$*.*

PROOF. Note that (a) follows from the classical Pythagoras's theorem. If $\cos\theta \neq 0$, then dividing both sides of (a) by $\cos^2\theta$ gives (b). If $\sin\theta \neq 0$, then dividing both sides of (a) by $\sin^2\theta$ gives (c). ♣

We sketch the graphs of the functions $y = \sin x$, $y = \cos x$ and $y = \tan x$ for $-2\pi \leq x \leq 2\pi$ below:







We shall use these graphs to make some observations about trigonometric functions.

PROPERTIES OF TRIGONOMETRIC FUNCTIONS.

(a) *The functions* $\sin x$ *and* $\cos x$ *are periodic with period* 2π. *More precisely, for every* $x \in \mathbb{R}$, *we have*

$$\sin(x + 2\pi) = \sin x \qquad \textit{and} \qquad \cos(x + 2\pi) = \cos x.$$

(b) *The functions* $\tan x$ *and* $\cot x$ *are periodic with period* π. *More precisely, for every* $x \in \mathbb{R}$ *for which the trigonometric function in question is defined, we have*

$$\tan(x + \pi) = \tan x \qquad \textit{and} \qquad \cot(x + \pi) = \cot x.$$

(c) *The function* $\sin x$ *is an odd function, and the function* $\cos x$ *is an even function. More precisely, for every* $x \in \mathbb{R}$, *we have*

$$\sin(-x) = -\sin x \qquad \textit{and} \qquad \cos(-x) = \cos x.$$

(d) *The functions* $\tan x$ *and* $\cot x$ *are odd functions. More precisely, for every* $x \in \mathbb{R}$ *for which the trigonometric function in question is defined, we have*

$$\tan(-x) = -\tan x \qquad \textit{and} \qquad \cot(-x) = -\cot x.$$

(e) *For every* $x \in \mathbb{R}$, *we have*

$$\sin(x + \pi) = -\sin x \qquad \textit{and} \qquad \cos(x + \pi) = -\cos x.$$

(f) *For every* $x \in \mathbb{R}$, *we have*

$$\sin(\pi - x) = \sin x \qquad \textit{and} \qquad \cos(\pi - x) = -\cos x.$$

(g) *For every* $x \in \mathbb{R}$ *for which the trigonometric function in question is defined, we have*

$$\tan(\pi - x) = -\tan x \qquad \textit{and} \qquad \cot(\pi - x) = -\cot x.$$

(h) *For every* $x \in \mathbb{R}$, *we have*

$$\sin\left(x + \frac{\pi}{2}\right) = \cos x \qquad \textit{and} \qquad \cos\left(x + \frac{\pi}{2}\right) = -\sin x.$$

(i) *For every* $x \in \mathbb{R}$, *we have*

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \qquad \textit{and} \qquad \cos\left(\frac{\pi}{2} - x\right) = \sin x.$$

(j) *For every* $x \in \mathbb{R}$ *for which the trigonometric functions in question are defined, we have*

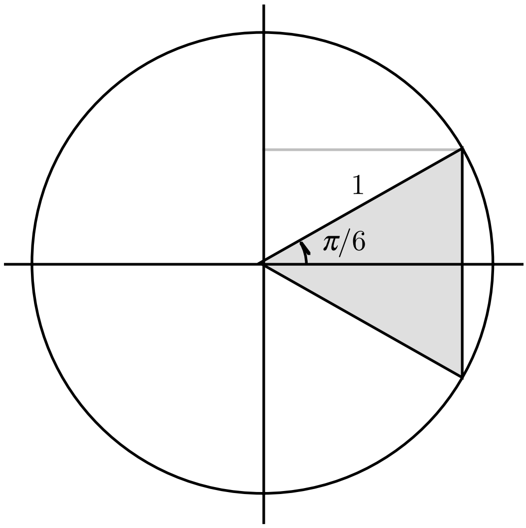
$$\tan\left(x + \frac{\pi}{2}\right) = -\cot x \qquad \textit{and} \qquad \cot\left(x + \frac{\pi}{2}\right) = -\tan x.$$

(k) *For every* $x \in \mathbb{R}$ *for which the trigonometric function in question is defined, we have*

$$\tan\left(\frac{\pi}{2} - x\right) = \cot x \qquad \textit{and} \qquad \cot\left(\frac{\pi}{2} - x\right) = \tan x.$$

Remark. There is absolutely no need to remember any of these properties! We shall discuss later some trigonometric identities which will give all the above as special cases.

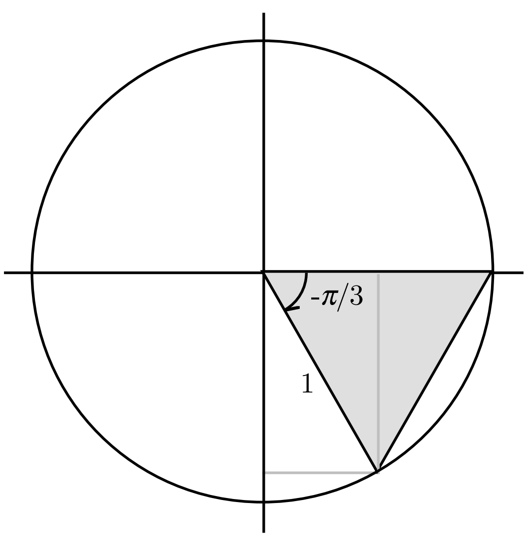
Example 3.2.1. Consider the following picture.



Clearly the triangle shown is an equilateral triangle, with all three sides of equal length. It is also clear that $\sin(\pi/6)$ is half the length of the vertical side. It follows that we must have $\sin(\pi/6) = 1/2$. To find the precise value for $\cos(\pi/6)$, we first observe that $\cos(\pi/6) > 0$. On the other hand, it follows from the first of the Pythagorean identities that $\cos^2(\pi/6) = 3/4$. Hence $\cos(\pi/6) = \sqrt{3}/2$. We can then deduce that $\tan(\pi/6) = 1/\sqrt{3}$.

Example 3.2.2. We have $\tan(13\pi/6) = \tan(7\pi/6) = \tan(\pi/6) = 1/\sqrt{3}$. Note that we have used part (b) of the Properties of trigonometric functions, as well as the result from Example 3.2.1.

Example 3.2.3. Consider the following picture.



Clearly the triangle shown is an equilateral triangle, with all three sides of equal length. It is also clear that $\cos(-\pi/3)$ is half the length the horizontal side. It follows that we must have $\cos(-\pi/3) = 1/2$. To find the precise value for $\sin(-\pi/3)$, we first observe that $\sin(-\pi/3) < 0$. On the other hand, it follows from the first of the Pythagorean identities that $\sin^2(-\pi/3) = 3/4$. Hence $\sin(-\pi/3) = -\sqrt{3}/2$. Alternatively, we can deduce from Example 3.2.1 by using parts (c) and (i) of the Properties of trigonometric functions that

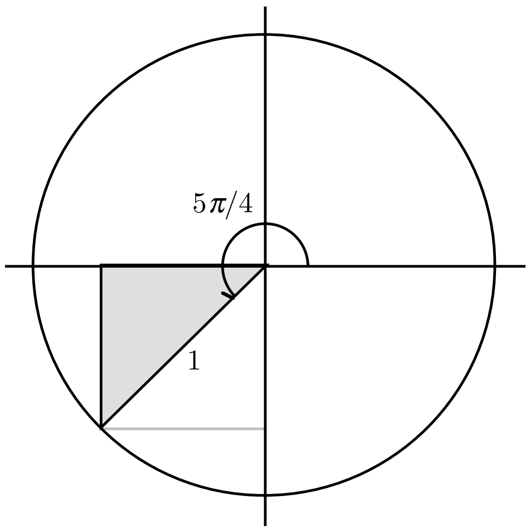
$$\cos\left(-\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$$

and

$$\sin\left(-\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\cos\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}.$$

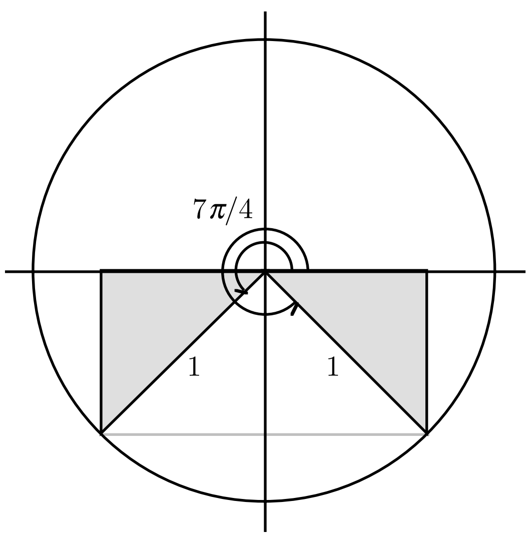
We can then deduce that $\tan(-\pi/3) = -\sqrt{3}$.

Example 3.2.4. Consider the following picture.



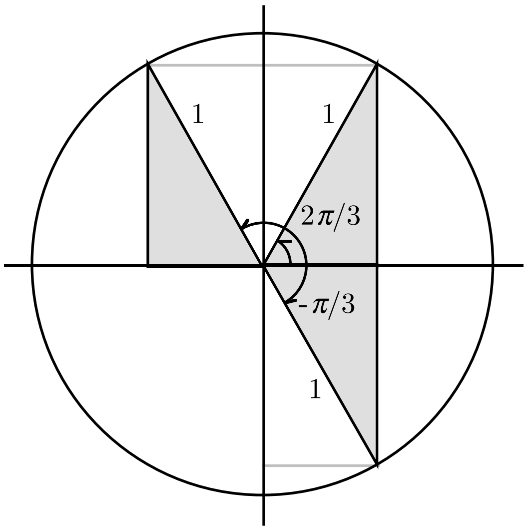
Clearly the triangle shown is a right-angled triangle with the two shorter sides of equal length and hypothenuse of length 1. It is also clear that $\sin(5\pi/4)$ is the length of the vertical side, with a $-$ sign attached as it is below the horizontal axis. On the other hand, it is also clear that $\cos(5\pi/4)$ is the length of the horizontal side, again with a $-$ sign attached as it is left of the vertical axis. It follows from Pythagoras's theorem that $\sin(5\pi/4) = \cos(5\pi/4) = y$, where $y < 0$ and $\sqrt{y^2 + y^2} = 1$. Clearly $y = -1/\sqrt{2}$. We can then deduce that $\tan(5\pi/4) = 1$.

Example 3.2.5. To find all the solutions of the equation $\sin x = -1/\sqrt{2}$ in the interval $0 \le x < 2\pi$, we consider the following picture.



Using Pythagoras's theorem, it is easy to see that the two triangles shown both have horizontal side of length $1/\sqrt{2}$, the same as the length of their vertical sides. Clearly $x = 5\pi/4$ or $x = 7\pi/4$.

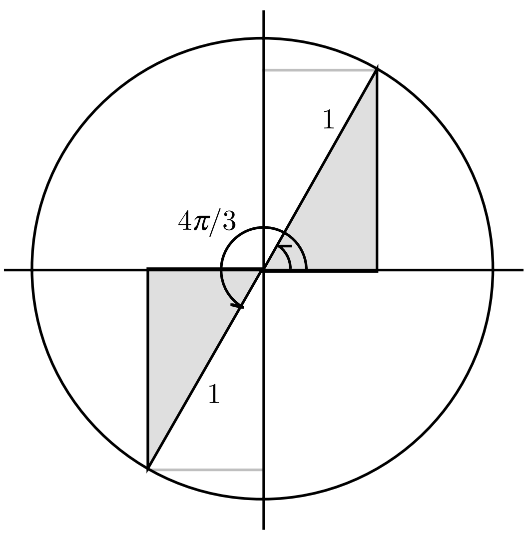
EXAMPLE 3.2.6. Suppose that we wish to find all the solutions of the equation $\cos^2 x = 1/4$ in the interval $-\pi/2 < x \leq \pi$. Observe first of all that either $\cos x = 1/2$ or $\cos x = -1/2$. We consider the following picture.



Using Pythagoras's theorem, it is easy to see that the three triangles shown both have vertical side of length $\sqrt{3}/2$. Clearly $x = -\pi/3$, $x = \pi/3$ or $x = 2\pi/3$.

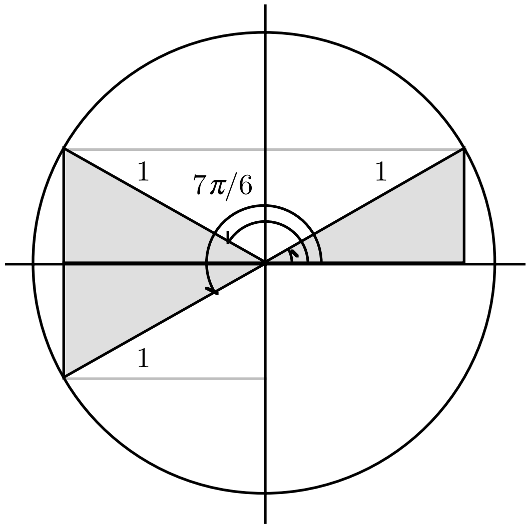
EXAMPLE 3.2.7. Convince yourself that the only two solutions of the equation $\cos^2 x = 1$ in the interval $0 \leq x < 2\pi$ are $x = 0$ and $x = \pi$.

EXAMPLE 3.2.8. To find all the solutions of the equation $\tan x = \sqrt{3}$ in the interval $0 \leq x < 2\pi$, we consider the following picture.



Since $\tan x > 0$, it follows from the CAST rule that we can restrict our attention to the first and third quadrants. It is easy to check that the two triangles shown have horizontal sides of length $1/2$ and vertical sides of length $\sqrt{3}/2$. Clearly $x = \pi/3$ or $x = 4\pi/3$.

EXAMPLE 3.2.9. Suppose that we wish to find all the solutions of the equation $\tan^2 x = 1/3$ in the interval $0 \le x \le 3\pi/2$. Observe first of all that either $\tan x = 1/\sqrt{3}$ or $\tan x = -1/\sqrt{3}$. We consider the following picture.



It is easy to check that the three triangles shown have horizontal sides of length $\sqrt{3}/2$ and vertical sides of length $1/2$. Clearly $x = \pi/6$, $x = 5\pi/6$ or $x = 7\pi/6$.

EXAMPLE 3.2.10. Convince yourself that the only two solutions of the equation $\sec x = \sqrt{2}$ in the interval $0 \le x < 2\pi$ are $x = \pi/4$ and $x = 7\pi/4$.

EXAMPLE 3.2.11. For every $x \in \mathbb{R}$, we have

$$\sin^3 x + \sin x \cos^2 x = \sin x \sin^2 x + \sin x \cos^2 x = (\sin x)(\sin^2 x + \cos^2 x) = \sin x,$$

in view of the first of the Pythagorean identities.

EXAMPLE 3.2.12. For every $x \in \mathbb{R}$ such that $\cos x \neq 0$, we have

$$(\sec x - \tan x)(\sec x + \tan x) = \sec^2 x - \tan^2 x = 1,$$

in view of the second of the Pythagorean identities.

EXAMPLE 3.2.13. For every $x \in \mathbb{R}$ such that $\cos x \neq \pm 1$, we have

$$\frac{1}{1-\cos x} + \frac{1}{1+\cos x} = \frac{(1+\cos x)+(1-\cos x)}{(1-\cos x)(1+\cos x)} = \frac{2}{1-\cos^2 x} = \frac{2}{\sin^2 x} = 2\csc^2 x.$$

EXAMPLE 3.2.14. For every $x \in \mathbb{R}$, we have

$$\begin{aligned}(\cos x + \sin x)^2 + (\cos x - \sin x)^2 &= (\cos^2 x + 2\cos x \sin x + \sin^2 x) + (\cos^2 x - 2\cos x \sin x + \sin^2 x) \\ &= (1 + 2\cos x \sin x) + (1 - 2\cos x \sin x) = 2.\end{aligned}$$

EXAMPLE 3.2.15. For every $x \in \mathbb{R}$ such that the expression on the left hand side makes sense, we have

$$\begin{aligned}\frac{1+\cot x}{\csc x} - \frac{1+\tan x}{\sec x} &= (1+\cot x)\sin x - (1+\tan x)\cos x = \left(1 + \frac{\cos x}{\sin x}\right)\sin x - \left(1 + \frac{\sin x}{\cos x}\right)\cos x \\ &= (\sin x + \cos x) - (\cos x + \sin x) = 0.\end{aligned}$$

EXAMPLE 3.2.16. Let us return to Example 3.2.5 where we showed that the solutions of the equation $\sin x = -1/\sqrt{2}$ in the interval $0 \le x < 2\pi$ are given by $x = 5\pi/4$ and $x = 7\pi/4$. Suppose now that we wish to find all the values $x \in \mathbb{R}$ that satisfy the same equation. To do this, we can use part (a) of the Properties of trigonometric functions, and conclude that the solutions are given by

$$x = \frac{5\pi}{4} + 2k\pi \qquad \text{or} \qquad x = \frac{7\pi}{4} + 2k\pi,$$

where $k \in \mathbb{Z}$.

EXAMPLE 3.2.17. Consider the equation $\sec(x/2) = 2$. Then $\cos(x/2) = 1/2$. If we first restrict our attention to $0 \le x/2 < 2\pi$, then it is not difficult to see that the solutions are given by $x/2 = \pi/3$ and $x/2 = 5\pi/3$. Using part (a) of the Properties of trigonometric functions, we conclude that without the restriction $0 \le x/2 < 2\pi$, the solutions are given by

$$\frac{x}{2} = \frac{\pi}{3} + 2k\pi \qquad \text{or} \qquad \frac{x}{2} = \frac{5\pi}{3} + 2k\pi,$$

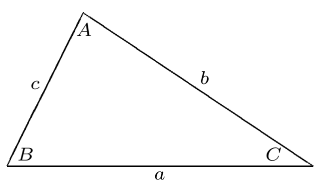
where $k \in \mathbb{Z}$. It follows that

$$x = \frac{2\pi}{3} + 4k\pi \qquad \text{or} \qquad x = \frac{10\pi}{3} + 4k\pi,$$

where $k \in \mathbb{Z}$.

3.3. Some Trigonometric Identities

Consider a triangle with side lengths and angles as shown in the picture below:



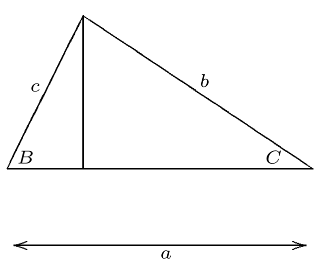
SINE RULE. *We have*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

COSINE RULE. *We have*

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A, \\ b^2 &= a^2 + c^2 - 2ac\cos B, \\ c^2 &= a^2 + b^2 - 2ab\cos C. \end{aligned}$$

SKETCH OF PROOF. Consider the picture below.



Clearly the length of the vertical line segment is given by $c\sin B = b\sin C$, so that

$$\frac{b}{\sin B} = \frac{c}{\sin C}.$$

This gives the sine rule. Next, note that the horizontal side of the the right-angled triangle on the left has length $c\cos B$. It follows that the horizontal side of the right-angled triangle on the right has length $a - c\cos B$. If we now apply Pythagoras's theorem to this latter triangle, then we have

$$(a - c\cos B)^2 + (c\sin B)^2 = b^2,$$

so that

$$a^2 - 2ac\cos B + c^2\cos^2 B + c^2\sin^2 B = b^2,$$

whence

$$b^2 = a^2 + c^2 - 2ac\cos B.$$

This gives the cosine rule. ♣

We mentioned earlier that there is no need to remember any of the Properties of trigonometric functions discussed in the last section. The reason is that they can all be deduced easily from the identities below.

SUM AND DIFFERENCE IDENTITIES. *For every $A, B \in \mathbb{R}$, we have*

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \qquad \textit{and} \qquad \sin(A-B) = \sin A\cos B - \cos A\sin B,$$

as well as

$$\cos(A+B) = \cos A\cos B - \sin A\sin B \qquad \textit{and} \qquad \cos(A-B) = \cos A\cos B + \sin A\sin B.$$

REMARKS. (1) Proofs can be sketched for these identities by drawing suitable pictures, although such pictures are fairly complicated. We omit the proofs here.

(2) It is not difficult to remember these identities. Observe the pattern

$$\sin \pm = \sin\ \cos \pm \cos\ \sin \qquad \text{and} \qquad \cos \pm = \cos\ \cos \mp \sin\ \sin\,.$$

For small positive angles, increasing the angle increases the sine (thus keeping signs) and decreases the cosine (thus reversing signs).

(3) One can also deduce analogous identities for tangent and cotangent. We have

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}.$$

Dividing both the numerator and denominator by $\cos A\cos B$, we obtain

$$\tan(A+B) = \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A\sin B}{\cos A\cos B}} = \frac{\tan A + \tan B}{1 - \tan A\tan B}.$$

Similarly, one can deduce that

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}.$$

EXAMPLE 3.3.1. For every $x \in \mathbb{R}$, we have

$$\sin\left(\frac{\pi}{2} - x\right) = \sin\frac{\pi}{2}\cos x - \cos\frac{\pi}{2}\sin x = \cos x,$$

and

$$\cos\left(\frac{\pi}{2} - x\right) = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x = \sin x.$$

These form part (i) of the Properties of trigonometric functions.

Of particular interest is the special case when $A = B$.

DOUBLE ANGLE IDENTITIES. *For every $x \in \mathbb{R}$, we have*
(a) $\sin 2x = 2\sin x\cos x$; *and*
(b) $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$.

HALF ANGLE IDENTITIES. *For every $y \in \mathbb{R}$, we have*

$$\sin^2\left(\frac{y}{2}\right) = \frac{1 - \cos y}{2} \qquad \text{and} \qquad \cos^2\left(\frac{y}{2}\right) = \frac{1 + \cos y}{2}.$$

PROOF. Let $x = y/2$. Then part (b) of the Double angle identities give

$$\cos y = 1 - 2\sin^2\left(\frac{y}{2}\right) = 2\cos^2\left(\frac{y}{2}\right) - 1.$$

The results follow easily. ♣

EXAMPLE 3.3.2. Suppose that we wish to find the precise values of $\cos(-3\pi/8)$ and $\sin(-3\pi/8)$. We have

$$\cos^2\left(-\frac{3\pi}{8}\right) = \frac{1 + \cos(-3\pi/4)}{2}.$$

It is not difficult to show that $\cos(-3\pi/4) = -1/\sqrt{2}$, so that

$$\cos^2\left(-\frac{3\pi}{8}\right) = \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right) = \frac{\sqrt{2} - 1}{2\sqrt{2}}.$$

It is easy to see that $\cos(-3\pi/8) > 0$, and so

$$\cos\left(-\frac{3\pi}{8}\right) = \sqrt{\frac{\sqrt{2} - 1}{2\sqrt{2}}}.$$

Similarly, we have

$$\sin^2\left(-\frac{3\pi}{8}\right) = \frac{1 - \cos(-3\pi/4)}{2} = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) = \frac{\sqrt{2} + 1}{2\sqrt{2}}.$$

It is easy to see that $\sin(-3\pi/8) < 0$, and so

$$\sin\left(-\frac{3\pi}{8}\right) = -\sqrt{\frac{\sqrt{2} + 1}{2\sqrt{2}}}.$$

EXAMPLE 3.3.3. Suppose that α is an angle in the first quadrant and β is an angle in the third quadrant. Suppose further that $\sin\alpha = 3/5$ and $\cos\beta = -5/13$. Using the first of the Pythagorean identities, we have

$$\cos^2\alpha = 1 - \sin^2\alpha = \frac{16}{25} \qquad \text{and} \qquad \sin^2\beta = 1 - \cos^2\beta = \frac{144}{169}.$$

On the other hand, using the CAST rule, we have $\cos\alpha > 0$ and $\sin\beta < 0$. It follows that $\cos\alpha = 4/5$ and $\sin\beta = -12/13$. Then

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \frac{3}{5}\times\left(-\frac{5}{13}\right) - \frac{4}{5}\times\left(-\frac{12}{13}\right) = \frac{33}{65}$$

and

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \frac{4}{5}\times\left(-\frac{5}{13}\right) + \frac{3}{5}\times\left(-\frac{12}{13}\right) = -\frac{56}{65},$$

so that

$$\tan(\alpha-\beta) = \frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)} = -\frac{33}{56}.$$

EXAMPLE 3.3.4. For every $x \in \mathbb{R}$, we have

$$\sin 6x\cos 2x - \cos 6x\sin 2x = \sin(6x-2x) = \sin 4x = 2\sin 2x\cos 2x.$$

Note that the first step uses a difference identity, while the last step uses a double angle identity.

EXAMPLE 3.3.5. For appropriate values of $\alpha, \beta \in \mathbb{R}$, we have

$$\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\alpha\cos\beta - \cos\alpha\sin\beta} = \frac{1+\cot\alpha\tan\beta}{1-\cot\alpha\tan\beta}.$$

Note that the first step uses sum and difference identities, while the second step involves dividing both the numerator and the denominator by $\sin\alpha\cos\beta$.

EXAMPLE 3.3.6. For every $x \in \mathbb{R}$ for which $\sin 4x \neq 0$, we have

$$\frac{\cos 8x}{\sin^2 4x} = \frac{\cos^2 4x - \sin^2 4x}{\sin^2 4x} = \cot^2 4x - 1.$$

Note that the first step involves a double angle identity.

EXAMPLE 3.3.7. For every $x \in \mathbb{R}$ for which $\cos x \neq 0$, we have

$$\frac{\sin 2x}{1-\sin^2 x} = \frac{2\sin x\cos x}{\cos^2 x} = 2\tan x.$$

Note that the first step involves a double angle identity as well as a Pythagorean identity.

EXAMPLE 3.3.8. For every $x \in \mathbb{R}$ for which $\sin 3x \neq 0$ and $\cos 3x \neq 0$, we have

$$\frac{\cos^2 3x - \sin^2 3x}{2\sin 3x\cos 3x} = \frac{\cos 6x}{\sin 6x} = \cot 6x.$$

Note that the first step involves double angle identities.

EXAMPLE 3.3.9. This example is useful in calculus for finding the derivatives of the sine and cosine functions. For every $x, h \in \mathbb{R}$ with $h \neq 0$, we have

$$\frac{\sin(x+h) - \sin x}{h} = \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} = (\cos x) \times \frac{\sin h}{h} + (\sin x) \times \frac{\cos h - 1}{h}$$

and

$$\frac{\cos(x+h) - \cos x}{h} = \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} = -(\sin x) \times \frac{\sin h}{h} + (\cos x) \times \frac{\cos h - 1}{h}.$$

When h is very close to 0, then

$$\frac{\sin h}{h} \approx 1 \qquad \text{and} \qquad \frac{\cos h - 1}{h} \approx 0,$$

so that

$$\frac{\sin(x+h) - \sin x}{h} \approx \cos x \qquad \text{and} \qquad \frac{\cos(x+h) - \cos x}{h} \approx -\sin x.$$

This is how we show that the derivatives of $\sin x$ and $\cos x$ are respectively $\cos x$ and $-\sin x$.

PROBLEMS FOR CHAPTER 3

1. Find the precise value of each of the following quantities, showing every step of your argument:
   a) $\sin \frac{4\pi}{3}$ b) $\tan \frac{4\pi}{3}$ c) $\cos \frac{3\pi}{4}$ d) $\tan \frac{3\pi}{4}$
   e) $\tan \frac{65\pi}{4}$ f) $\sin \left(-\frac{47\pi}{6}\right)$ g) $\cos \frac{25\pi}{3}$ h) $\cot \left(-\frac{37\pi}{2}\right)$
   i) $\sin \frac{5\pi}{3}$ j) $\tan \frac{5\pi}{3}$ k) $\cos \left(-\frac{53\pi}{6}\right)$ l) $\cot \left(-\frac{53\pi}{6}\right)$

2. Find all solutions for each of the following equations in the intervals given, showing every step of your argument:
   a) $\sin x = -\frac{1}{2}$, $0 \leq x < 2\pi$ b) $\sin x = -\frac{1}{2}$, $0 \leq x < 4\pi$
   c) $\sin x = -\frac{1}{2}$, $-\pi \leq x < \pi$ d) $\sin x = -\frac{1}{2}$, $0 \leq x < 3\pi$
   e) $\cos^2 x = \frac{3}{4}$, $0 \leq x < 2\pi$ f) $\cos^2 x = \frac{3}{4}$, $-\pi \leq x < 3\pi$
   g) $\tan^2 x = 3$, $0 \leq x < 2\pi$ h) $\tan^2 x = 3$, $-2\pi \leq x < \pi$
   i) $\cot^2 x = 3$, $0 \leq x < 2\pi$ j) $\sec^2 x = 2$, $0 \leq x < \frac{5\pi}{2}$
   k) $\cos x = -\frac{1}{2}$, $0 \leq x < 2\pi$ l) $\cos x = -\frac{1}{2}$, $0 \leq x < 4\pi$
   m) $\tan^2 x = 1$, $2\pi \leq x < 4\pi$ n) $\sin^2 x = \frac{1}{4}$, $\frac{\pi}{2} \leq x < 2\pi$

3. Simplify each of the following expressions, showing every step of yor argument:
   a) $\dfrac{\sin x \sec x - \sin^2 x \tan x}{\sin 2x}$ b) $\sin 3x \cos x + \sin x \cos 3x$
   c) $\cos 5x \cos x + \sin x \sin 5x$ d) $\sin 3x - \cos 2x \sin x - 2 \sin x \cos^2 x$
   e) $\dfrac{\sin 4x}{(\cos^2 x - \sin^2 x) \sin x \cos x}$ f) $\cos 4x \cos 3x - 4 \sin x \sin 3x \cos x \cos 2x$
   g) $2 \sin 3x \cos 3x \cos 5x - (\cos^2 3x - \sin^2 3x) \sin 5x$

4. We know that

$$\sin\frac{\pi}{6}=\frac{1}{2},\qquad \cos\frac{\pi}{6}=\frac{\sqrt{3}}{2},\qquad \sin\frac{\pi}{4}=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}\qquad\text{and}\qquad \frac{5\pi}{24}=\frac{1}{2}\left(\frac{\pi}{4}+\frac{\pi}{6}\right).$$

We also know that

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta\qquad\text{and}\qquad \cos 2\theta=2\cos^2\theta-1=1-2\sin^2\theta.$$

Use these to determine the exact values of

$$\cos\frac{5\pi}{24}\qquad\text{and}\qquad\sin\frac{5\pi}{24}.$$

[HINT: Put your calculators away. Your answers will be square roots of expressions involving the numbers $\sqrt{2}$ and $\sqrt{3}$.]

5. Find the precise value of each of the following quantities, showing every step of your argument:
   a) $\cos\left(-\frac{\pi}{8}\right)$ b) $\sin\left(-\frac{\pi}{8}\right)$ c) $\cos\frac{7\pi}{12}$ d) $\tan\frac{7\pi}{12}$

6. Use the sum and difference identities for the sine and cosine functions to deduce each of the following identities:
   a) $\sin(x+\pi)=-\sin x$ b) $\cos(-x)=\cos x$ c) $\tan(\pi-x)=-\tan x$
   d) $\cot\left(\frac{\pi}{2}+x\right)=-\tan x$

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Chapter 4

INDICES AND LOGARITHMS

4.1. Indices

Given any non-zero real number $a \in \mathbb{R}$ and any natural number $k \in \mathbb{N}$, we often write

$$a^k = \underbrace{a \times \ldots \times a}_{k}. \tag{1}$$

This definition can be extended to all integers $k \in \mathbb{Z}$ by writing

$$a^0 = 1 \tag{2}$$

and

$$a^k = \frac{1}{a^{-k}} = \frac{1}{\underbrace{a \times \ldots \times a}_{-k}} \tag{3}$$

whenever k is a negative integer, noting that $-k \in \mathbb{N}$ in this case.

It is not too difficult to establish the following.

LAWS OF INTEGER INDICES. *Suppose that $a, b \in \mathbb{R}$ are non-zero. Then for every $m, n \in \mathbb{Z}$, we have*

(a) $a^m a^n = a^{m+n}$;

(b) $\dfrac{a^m}{a^n} = a^{m-n}$;

(c) $(a^m)^n = a^{mn}$; *and*

(d) $(ab)^m = a^m b^m$.

† This chapter was written at Macquarie University in 1999.

We now further extend the definition of a^k to all rational numbers $k \in \mathbb{Q}$. To do this, we first of all need to discuss the q-th roots of a positive real number a, where $q \in \mathbb{N}$. This is an extension of the idea of square roots discussed in Section 1.2. We recall the following definition, slightly modified here.

DEFINITION. Suppose that $a \in \mathbb{R}$ is positive. We say that $x > 0$ is the positive square root of a if $x^2 = a$. In this case, we write $x = \sqrt{a}$.

We now make the following natural extension.

DEFINITION. Suppose that $a \in \mathbb{R}$ is positive and $q \in \mathbb{N}$. We say that $x > 0$ is the positive q-th root of a if $x^q = a$. In this case, we write $x = \sqrt[q]{a} = a^{1/q}$.

Recall now that every rational number $k \in \mathbb{Q}$ can be written in the form $k = p/q$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. We may, if we wish, assume that p/q is in lowest terms, where p and q have no common factors. For any positive real number $a \in \mathbb{R}$, we can now define a^k by writing

$$a^k = a^{p/q} = (a^{1/q})^p = (a^p)^{1/q}. \tag{4}$$

In other words, we first of all calculate the positive q-th root of a, and then take the p-th power of this q-th root. Alternatively, we can first of all take the p-th power of a, and then calculate the positive q-th root of this p-th power.

We can establish the following generalization of the Laws of integer indices.

LAWS OF INDICES. *Suppose that $a, b \in \mathbb{R}$ are positive. Then for every $m, n \in \mathbb{Q}$, we have*
(a) $a^m a^n = a^{m+n}$;
(b) $\dfrac{a^m}{a^n} = a^{m-n}$;
(c) $(a^m)^n = a^{mn}$; *and*
(d) $(ab)^m = a^m b^m$.

REMARKS. (1) Note that we have to make the restriction that the real numbers a and b are positive. If $a = 0$, then a^k is clearly not defined when k is a negative integer. If $a < 0$, then we will have problems taking square roots.

(2) It is possible to define cube roots of a negative real number a. It is a real number x satisfying the requirement $x^3 = a$. Note that $x < 0$ in this case. A similar argument applies to q-th roots when $q \in \mathbb{N}$ is odd. However, if $q \in \mathbb{N}$ is even, then $x^q \ge 0$ for every $x \in \mathbb{R}$, and so $x^q \ne a$ for any negative $a \in \mathbb{R}$. Hence a negative real number does not have real q-th roots for any even $q \in \mathbb{N}$.

EXAMPLE 4.1.1. We have $2^4 \times 2^3 = 16 \times 8 = 128 = 2^7 = 2^{4+3}$ and $2^{-3} = 1/8$.

EXAMPLE 4.1.2. We have $8^{2/3} = (8^{1/3})^2 = 2^2 = 4$. Alternatively, we have $8^{2/3} = (8^2)^{1/3} = 64^{1/3} = 4$.

EXAMPLE 4.1.3. To show that $\sqrt{5 + 2\sqrt{6}} = \sqrt{2} + \sqrt{3}$, we first of all observe that both sides are positive, and so it suffices to show that the squares of the two sides are equal. Note now that

$$(\sqrt{2} + \sqrt{3})^2 = (\sqrt{2})^2 + 2\sqrt{2}\sqrt{3} + (\sqrt{3})^2 = 5 + 2 \times 2^{1/2}3^{1/2} = 5 + 2(2 \times 3)^{1/2} = 5 + 2\sqrt{6},$$

the square of the left hand side.

EXAMPLE 4.1.4. To show that $\sqrt{8 - 4\sqrt{3}} = \sqrt{6} - \sqrt{2}$, it suffices to show that the squares of the two sides are equal. Note now that

$$\begin{aligned}(\sqrt{6} - \sqrt{2})^2 &= (\sqrt{6})^2 - 2\sqrt{6}\sqrt{2} + (\sqrt{2})^2 = 8 - 2 \times 6^{1/2}2^{1/2} = 8 - 2(6 \times 2)^{1/2} \\ &= 8 - 2\sqrt{12} = 8 - 2 \times 2\sqrt{3} = 8 - 4\sqrt{3},\end{aligned}$$

the square of the left hand side.

In the remaining examples in this section, the variables x and y both represent positive real numbers.

EXAMPLE 4.1.5. We have

$$x^{-1} \times 2x^{1/2} = 2x^{(1/2)-1} = 2x^{-1/2} = \frac{2}{x^{1/2}} = \frac{2}{\sqrt{x}}.$$

EXAMPLE 4.1.6. We have

$$\begin{aligned}(27x^2)^{1/3} \div \frac{1}{3}(x^5)^{1/2} &= 27^{1/3}(x^2)^{1/3} \div \frac{1}{3}x^{5\times(1/2)} = 3x^{2/3} \div \frac{x^{5/2}}{3} = 3x^{2/3} \times \frac{3}{x^{5/2}} \\ &= 3x^{2/3} \times 3x^{-5/2} = 9x^{2/3}x^{-5/2} = 9x^{(2/3)-(5/2)} = 9x^{-11/6} = \frac{9}{x^{11/6}}.\end{aligned}$$

EXAMPLE 4.1.7. We have

$$(9x)^{1/2}(8x^{-1/2})^{1/3} = 9^{1/2}x^{1/2} \times 8^{1/3}x^{(-1/2)\times(1/3)} = 3x^{1/2} \times 2x^{-1/6} = 6x^{(1/2)-(1/6)} = 6x^{1/3}.$$

EXAMPLE 4.1.8. We have

$$\begin{aligned}(8x^{3/4})^{-2} \div \left(\frac{1}{2}x^{-1}\right)^2 &= (8x^{3/4})^{-2} \times \left(\frac{1}{2}x^{-1}\right)^{-2} = 8^{-2}x^{(3/4)\times(-2)} \times \left(\frac{1}{2}\right)^{-2} x^{(-1)\times(-2)} \\ &= \frac{1}{8^2}x^{-3/2} \times 2^2x^2 = \frac{1}{16}x^{(-3/2)+2} = \frac{1}{16}x^{1/2} = \frac{\sqrt{x}}{16}.\end{aligned}$$

EXAMPLE 4.1.9. We have

$$\begin{aligned}\sqrt[3]{8a^2b} \times a^{1/3}b^{5/3} &= (8a^2b)^{1/3}a^{1/3}b^{5/3} = 8^{1/3}(a^2)^{1/3}b^{1/3}a^{1/3}b^{5/3} = 2a^{2/3}b^{1/3}a^{1/3}b^{5/3} \\ &= 2a^{(2/3)+(1/3)}b^{(1/3)+(5/3)} = 2ab^2.\end{aligned}$$

EXAMPLE 4.1.10. We have

$$\begin{aligned}\sqrt[4]{(16x^{1/6}y^2)^3} &= ((16x^{1/6}y^2)^3)^{1/4} = (16x^{1/6}y^2)^{3/4} = 16^{3/4}(x^{1/6})^{3/4}(y^2)^{3/4} \\ &= (16^{1/4})^3x^{(1/6)\times(3/4)}y^{2\times(3/4)} = 8x^{1/8}y^{3/2}.\end{aligned}$$

We now miss out some intermediate steps in the examples below. The reader is advised to fill in all the details in each step.

EXAMPLE 4.1.11. We have

$$\begin{aligned}\frac{16(x^2y^3)^{1/2}}{(2x^{1/2}y)^3} \times \frac{(4x^6y^4)^{1/2}}{(6x^3y^{1/2})^2} &= \frac{16xy^{3/2}}{8x^{3/2}y^3} \times \frac{2x^3y^2}{36x^6y} = \frac{16xy^{3/2} \times 2x^3y^2}{8x^{3/2}y^3 \times 36x^6y} = \frac{32x^4y^{7/2}}{288x^{15/2}y^4} \\ &= \frac{32}{288}x^4y^{7/2}x^{-15/2}y^{-4} = \frac{1}{9}x^{-7/2}y^{-1/2} = \frac{1}{9x^{7/2}y^{1/2}}.\end{aligned}$$

EXAMPLE 4.1.12. We have

$$\begin{aligned}\frac{(3x)^3y^2}{(5xy)^2} \div \frac{(5xy)^4}{(27x^9)^3} &= \frac{(3x)^3y^2}{(5xy)^2} \times \frac{(27x^9)^3}{(5xy)^4} = \frac{(3x)^3y^2 \times (27x^9)^3}{(5xy)^2 \times (5xy)^4} = \frac{3^327^3x^{30}y^2}{5^6x^6y^6} \\ &= \frac{531441}{15625}x^{30}y^2x^{-6}y^{-6} = \frac{531441}{15625}x^{24}y^{-4} = \frac{531441x^{24}}{15625y^4}.\end{aligned}$$

EXAMPLE 4.1.13. We have

$$\begin{aligned}\frac{x^{-1}+y^{-1}}{x+y}-\frac{x^{-1}-y^{-1}}{x-y}&=\frac{(x-y)(x^{-1}+y^{-1})-(x^{-1}-y^{-1})(x+y)}{(x-y)(x+y)}\\&=\frac{(1+xy^{-1}-yx^{-1}-1)-(1+yx^{-1}-xy^{-1}-1)}{(x-y)(x+y)}=\frac{2(xy^{-1}-yx^{-1})}{(x-y)(x+y)}\\&=\frac{2}{(x-y)(x+y)}\times\left(\frac{x}{y}-\frac{y}{x}\right)=\frac{2}{(x-y)(x+y)}\times\frac{x^2-y^2}{xy}=\frac{2}{xy}.\end{aligned}$$

EXAMPLE 4.1.14. We have

$$\begin{aligned}\frac{x^{-1}-y^{-1}}{x^{-2}-y^{-2}}&=(x^{-1}-y^{-1})\div(x^{-2}-y^{-2})=\left(\frac{1}{x}-\frac{1}{y}\right)\div\left(\frac{1}{x^2}-\frac{1}{y^2}\right)=\frac{y-x}{xy}\div\frac{y^2-x^2}{x^2y^2}\\&=\frac{y-x}{xy}\times\frac{x^2y^2}{y^2-x^2}=\frac{x^2y^2(y-x)}{xy(y^2-x^2)}=\frac{x^2y^2(y-x)}{xy(y-x)(y+x)}=\frac{xy}{x+y}.\end{aligned}$$

Alternatively, write $a=x^{-1}$ and $b=y^{-1}$. Then

$$\frac{x^{-1}-y^{-1}}{x^{-2}-y^{-2}}=\frac{a-b}{a^2-b^2}=\frac{1}{a+b}=(a+b)^{-1}=\left(\frac{1}{x}+\frac{1}{y}\right)^{-1}=\left(\frac{x+y}{xy}\right)^{-1}=\frac{xy}{x+y}.$$

EXAMPLE 4.1.15. We have

$$x^2y^{-1}\div(x^{-1}+y^{-1})=\frac{x^2}{y}\div\left(\frac{1}{x}+\frac{1}{y}\right)=\frac{x^2}{y}\div\frac{x+y}{xy}=\frac{x^2}{y}\times\frac{xy}{x+y}=\frac{x^3}{x+y}.$$

EXAMPLE 4.1.16. We have

$$\begin{aligned}xy\div((x^{-1}+y)^{-1})^{-1}=xy\times(x^{-1}+y)^{-1}&=xy\times\left(\frac{1}{x}+y\right)^{-1}\\&=xy\times\left(\frac{1+xy}{x}\right)^{-1}=xy\times\frac{x}{1+xy}=\frac{x^2y}{1+xy}.\end{aligned}$$

4.2. The Exponential Functions

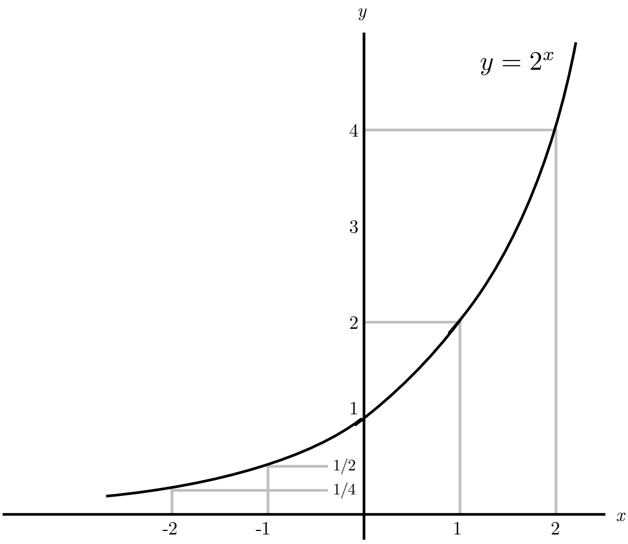
Suppose that $a\in\mathbb{R}$ is a positive real number. We have shown in Section 4.1 that we can use (1)–(4) to define a^k for every rational number $k\in\mathbb{Q}$. Here we shall briefly discuss how we may further extend the definition of a^k to a function a^x defined for every real number $x\in\mathbb{R}$. A thorough treatment of this extension will require the study of the theory of continuous functions as well as the well known result that the rational numbers are "dense" among the real numbers, and is beyond the scope of this set of notes. We shall instead confine our discussion here to a heuristic treatment.

We all know simple functions like $y=x^2$ (a parabola) or $y=2x+3$ (a straight line). It is possible to draw the graph of such a function in one single stroke, without lifting our pen from the paper before completing the drawing. Such functions are called continuous functions.

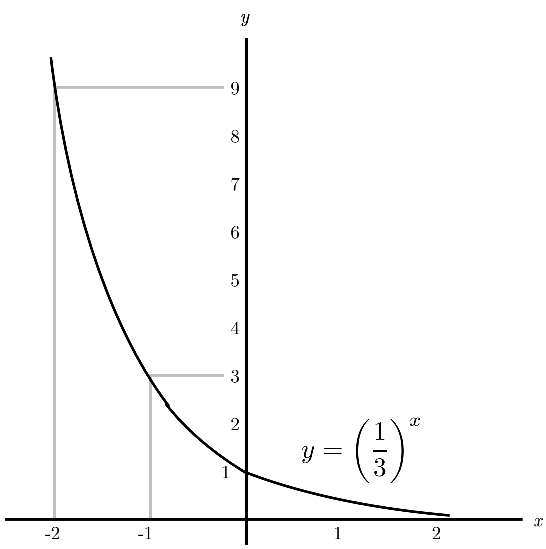
Suppose now that a positive real number $a\in\mathbb{R}$ has been chosen and fixed. Note that we have already defined a^k for every rational number $k\in\mathbb{Q}$. We now draw the graph of a continuous function on the xy-plane which will pass through every point (k,a^k) where $k\in\mathbb{Q}$. It turns out that such a function is unique. In other words, there is one and only one continuous function whose graph on the xy-plane will pass through every point (k,a^k) where $k\in\mathbb{Q}$. We call this function the exponential function corresponding to the positive real number a, and write $f(x)=a^x$.

LAWS FOR EXPONENTIAL FUNCTIONS. *Suppose that $a \in \mathbb{R}$ is positive. Then $a^0 = 1$. For every $x_1, x_2 \in \mathbb{R}$, we have*
(a) $a^{x_1} a^{x_2} = a^{x_1+x_2}$;
(b) $\dfrac{a^{x_1}}{a^{x_2}} = a^{x_1-x_2}$; *and*
(c) $(a^{x_1})^{x_2} = a^{x_1 x_2}$.
(d) Furthermore, if $a \neq 1$, then $a^{x_1} = a^{x_2}$ if and only if $x_1 = x_2$.

Example 4.2.1. The graph of the exponential function $y = 2^x$ is shown below:



Example 4.2.2. The graph of the exponential function $y = (1/3)^x$ is shown below:



Example 4.2.3. We have $8^x \times 2^{4x} = (2^3)^x \times 2^{4x} = 2^{3x}2^{4x} = 2^{7x} = (2^7)^x = 128^x$.

Example 4.2.4. We have $9^{x/2} \times 27^{x/3} = (9^{1/2})^x \times (27^{1/3})^x = 3^x 3^x = 3^{2x} = (3^2)^x = 9^x$.

EXAMPLE 4.2.5. We have

$$32^{x+2} \div 8^{2x-1} = 32^{x+2} \times 8^{1-2x} = (2^5)^{x+2} \times (2^3)^{1-2x} = 2^{5x+10}2^{3-6x} = 2^{13-x}.$$

EXAMPLE 4.2.6. We have

$$64^{2x} \div 16^{2x} = 2^{12x} \div 2^{8x} = \frac{2^{12x}}{2^{8x}} = 2^{4x} = (2^4)^x = 16^x.$$

EXAMPLE 4.2.7. We have

$$\frac{5^{x+1}+5^{x-1}}{5^{x+2}+5^x} = \frac{5^{x+1}+5^{x-1}}{5\times 5^{x+1}+5\times 5^{x-1}} = \frac{5^{x+1}+5^{x-1}}{5(5^{x+1}+5^{x-1})} = \frac{1}{5}.$$

EXAMPLE 4.2.8. We have

$$\frac{4^x-2^{x-1}}{2^x-\frac{1}{2}} = \frac{2^{2x}-2^{-1}2^x}{2^x-\frac{1}{2}} = \frac{2^x2^x-\frac{1}{2}2^x}{2^x-\frac{1}{2}} = \frac{2^x(2^x-\frac{1}{2})}{2^x-\frac{1}{2}} = 2^x.$$

EXAMPLE 4.2.9. Suppose that

$$25^x = \frac{1}{\sqrt{125}}.$$

We can write $25^x = (5^2)^x = 5^{2x}$ and

$$\frac{1}{\sqrt{125}} = (\sqrt{125})^{-1} = ((125)^{1/2})^{-1} = ((5^3)^{1/2})^{-1} = 5^{-3/2}.$$

It follows that we must have $2x = -3/2$, so that $x = -3/4$.

EXAMPLE 4.2.10. Suppose that

$$\left(\frac{1}{9}\right)^{2x-1} = 3(27^{-x}).$$

We can write $3(27^{-x}) = 3((3^3)^{-x}) = 3\times 3^{-3x} = 3^{1-3x}$ and

$$\left(\frac{1}{9}\right)^{2x-1} = \left(\frac{1}{3^2}\right)^{2x-1} = (3^{-2})^{2x-1} = 3^{2-4x}.$$

It follows that we must have $1-3x = 2-4x$, so that $x = 1$.

EXAMPLE 4.2.11. Suppose that $9^x = \sqrt{3}$. We can write $9^x = (3^2)^x = 3^{2x}$ and $\sqrt{3} = 3^{1/2}$. It follows that we must have $2x = 1/2$, so that $x = 1/4$.

EXAMPLE 4.2.12. Suppose that $5^{3x-4} = 1$. We can write $1 = 5^0$. It follows that we must have $3x-4 = 0$, so that $x = 4/3$.

EXAMPLE 4.2.13. Suppose that $(0.125)^x = \sqrt{0.5}$. We can write

$$(0.125)^x = \left(\frac{1}{8}\right)^x = \left(\frac{1}{2^3}\right)^x = (2^{-3})^x = 2^{-3x}$$

and

$$\sqrt{0.5} = (0.5)^{1/2} = \left(\frac{1}{2}\right)^{1/2} = (2^{-1})^{1/2} = 2^{-1/2}.$$

It follows that we must have $-3x = -1/2$, so that $x = 1/6$.

EXAMPLE 4.2.14. Suppose that $8^{1-x} \times 2^{x-3} = 4$. We can write $4 = 2^2$ and

$$8^{1-x} \times 2^{x-3} = (2^3)^{1-x} \times 2^{x-3} = 2^{3-3x} \times 2^{x-3} = 2^{-2x}.$$

It follows that we must have $-2x = 2$, so that $x = -1$.

4.3. The Logarithmic Functions

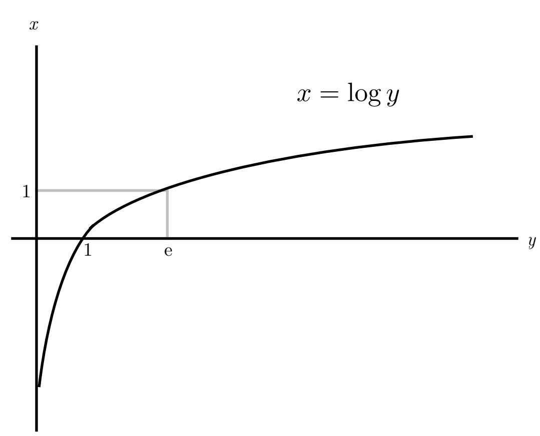
The logarithmic functions are the inverses of the exponential functions. Suppose that $a \in \mathbb{R}$ is a positive real number and $a \neq 1$. If $y = a^x$, then x is called the logarithm of y to the base a, denoted by $x = \log_a y$. In other words, we have

$$y = a^x \qquad \text{if and only if} \qquad x = \log_a y.$$

For the special case when $a = 10$, we have the common logarithm $\log_{10} y$ of y. Another special case is when $a = \mathrm{e} = 2.7182818\ldots$, an irrational number. We have the natural logarithm $\log_{\mathrm{e}} y$ of y, sometimes also denoted by $\log y$ or $\ln y$. Whenever we mention a logarithmic function without specifying its base, we shall assume that it is base e.

REMARK. The choice of the number e is made to ensure that the derivative of the function e^x is also equal to e^x for every $x \in \mathbb{R}$. This is not the case for any other non-zero function, apart from constant multiples of the function e^x.

EXAMPLE 4.3.1. The graph of the logarithmic function $x = \log y$ is shown below:



LAWS FOR LOGARITHMIC FUNCTIONS. *Suppose that $a \in \mathbb{R}$ is positive and $a \neq 1$. Then $\log_a 1 = 0$ and $\log_a a = 1$. For every positive $y_1, y_2 \in \mathbb{R}$, we have*
(a) $\log_a(y_1 y_2) = \log_a y_1 + \log_a y_2$;
(b) $\log_a \left(\dfrac{y_1}{y_2}\right) = \log_a y_1 - \log_a y_2$; *and*
(c) $\log_a(y_1^k) = k \log_a y_1$ *for every* $k \in \mathbb{R}$.
(d) Furthermore, $\log_a y_1 = \log_a y_2$ *if and only if* $y_1 = y_2$.

INVERSE LAWS. *Suppose that $a \in \mathbb{R}$ is positive and $a \neq 1$.*
(a) For every real number $x \in \mathbb{R}$, we have $\log_a(a^x) = x$.
(b) For every positive real number $y \in \mathbb{R}$, we have $a^{\log_a y} = y$.

Example 4.3.2. We have $\log_2 8 = x$ if and only if $2^x = 8$, so that $x = 3$. Alternatively, we can use one of the Inverse laws to obtain $\log_2 8 = \log_2(2^3) = 3$.

Example 4.3.3. We have $\log_5 125 = x$ if and only if $5^x = 125$, so that $x = 3$.

Example 4.3.4. We have $\log_3 81 = x$ if and only if $3^x = 81$, so that $x = 4$.

Example 4.3.5. We have $\log_{10} 1000 = x$ if and only if $10^x = 1000$, so that $x = 3$.

Example 4.3.6. We have

$$\log_4 \frac{1}{8} = x \qquad \text{if and only if} \qquad 4^x = \frac{1}{8}.$$

This means that $2^{2x} = 2^{-3}$, so that $x = -3/2$.

Example 4.3.7. We have

$$\log_2 \frac{\sqrt{2}}{16} = \log_2 \left(\frac{2^{1/2}}{2^4} \right) = \log_2(2^{-7/2}) = -\frac{7}{2}.$$

Example 4.3.8. We have

$$\log_3 \frac{1}{3\sqrt{3}} = \log_3(3^{-3/2}) = -\frac{3}{2}.$$

Example 4.3.9. Use your calculator to confirm that the following are correct to 3 decimal places:

$$\log_{10} 20 \approx 1.301, \quad \log_{10} 7 \approx 0.845, \quad \log_{10} \frac{1}{5} \approx -0.698,$$

$$\log 5 \approx 1.609, \qquad \log 21 \approx 3.044, \qquad \log 0.2 \approx -1.609.$$

Example 4.3.10. We have

$$\log_a \frac{1}{3} = -\frac{1}{3} \qquad \text{if and only if} \qquad a^{-1/3} = \frac{1}{3}.$$

It follows that

$$a = (a^{-1/3})^{-3} = \left(\frac{1}{3}\right)^{-3} = 3^3 = 27.$$

Example 4.3.11. We have

$$\log_a \frac{1}{4} = -\frac{2}{3} \qquad \text{if and only if} \qquad a^{-2/3} = \frac{1}{4}.$$

It follows that

$$a = (a^{-2/3})^{-3/2} = \left(\frac{1}{4}\right)^{-3/2} = 4^{3/2} = 8.$$

Example 4.3.12. We have

$$\log_a 4 = \frac{1}{2} \qquad \text{if and only if} \qquad a^{1/2} = 4.$$

It follows that $a = 16$.

EXAMPLE 4.3.13. Suppose that $\log_5 y = -2$. Then $y = 5^{-2} = 1/25$.

EXAMPLE 4.3.14. Suppose that $\log_a y = \log_a 3 + \log_a 5$. Then since $\log_a 3 + \log_a 5 = \log_a 15$, we must have $y = 15$.

EXAMPLE 4.3.15. Suppose that $\log_a y + 2\log_a 4 = \log_a 20$. Then

$$\log_a y = \log_a 20 - 2\log_a 4 = \log_a 20 - \log_a(4^2) = \log_a 20 - \log_a 16 = \log_a \frac{20}{16},$$

so that $y = 20/16 = 5/4$.

EXAMPLE 4.3.16. Suppose that

$$\frac{1}{2}\log 6 - \log y = \log 12.$$

Then

$$\log y = \frac{1}{2}\log 6 - \log 12 = \log(\sqrt{6}) - \log 12 = \log \frac{\sqrt{6}}{12},$$

so that $y = \sqrt{6}/12$.

For the next four examples, u and v are positive real numbers.

EXAMPLE 4.3.17. We have $\log_a(u^{10}) \div \log_a u = 10\log_a u \div \log_a u = 10$.

EXAMPLE 4.3.18. We have $5\log_a u - \log_a(u^5) = 5\log_a u - 5\log_a u = 0$.

EXAMPLE 4.3.19. We have

$$2\log_a u + 2\log_a v - \log_a((uv)^2) = \log_a(u^2) + \log_a(v^2) - \log_a(u^2v^2) = \log_a(u^2v^2) - \log_a(u^2v^2) = 0.$$

EXAMPLE 4.3.20. We have

$$\begin{aligned}\left(\log_a \frac{u^3}{v} + \log_a \frac{v}{u}\right) \div \log_a(\sqrt{u}) &= \log_a\left(\frac{u^3}{v} \times \frac{v}{u}\right) \div \log_a(u^{1/2}) \\ &= \log_a(u^2) \div \log_a(u^{1/2}) = 2\log_a u \div \frac{1}{2}\log_a u = 4.\end{aligned}$$

EXAMPLE 4.3.21. Suppose that $2\log y = \log(4 - 3y)$. Then since $2\log y = \log(y^2)$, we must have $y^2 = 4 - 3y$, so that $y^2 + 3y - 4 = 0$. This quadratic equation has roots

$$y = \frac{-3 \pm \sqrt{9+16}}{2} = 1 \text{ or } -4.$$

However, we have to discard the solution $y = -4$, since $\log(-4)$ is not defined. The only solution is therefore $y = 1$.

EXAMPLE 4.3.22. Suppose that $\log(\sqrt{y}) = \sqrt{\log y}$. Then

$$\frac{1}{2}\log y = \sqrt{\log y}, \qquad \text{and so} \qquad \log y = 2\sqrt{\log y}.$$

Squaring both sides and letting $x = \log y$, we obtain the quadratic equation $x^2 = 4x$, with solutions $x = 0$ and $x = 4$. The equation $\log y = 0$ corresponds to $y = 1$. The equation $\log y = 4$ corresponds to $y = \mathrm{e}^4$.

Problems for Chapter 4

1. Simplify each of the following expressions:
 a) $216^{2/3}$ b) $32^{3/5}$ c) $64^{-1/6}$ d) $10000^{-3/4}$

2. Simplify each of the following expressions, where x denotes a positive real number:
 a) $(5x^2)^3 \div (25x^{-4})^{1/2}$ b) $(32x^2)^{2/5} \times \left(\dfrac{25}{x^4}\right)^{1/2}$ c) $(16x^{12})^{3/4} \times \left(\dfrac{27}{x^6}\right)^{-1/3}$
 d) $(128x^{14})^{-1/7} \div \left(\dfrac{1}{9}x^{-1}\right)^{-1/2}$ e) $\dfrac{(2x^3)^{1/2}}{(4x)^2} \div \dfrac{(8x)^{1/2}}{(3x^2)^3}$

3. Simplify each of the following expressions, where x and y denote suitable positive real numbers:
 a) $(x^2y^3)^{1/3}(x^3y^2)^{-1/2}$ b) $\dfrac{x^{-1} - y^{-1}}{x^{-3} - y^{-3}}$
 c) $\dfrac{x^{-2} + y^{-2}}{x + y} - \dfrac{x^{-2} - y^{-2}}{x - y}$ d) $xy \div (x^{-1} + y^{-1})^{-1}$
 e) $(x^{-2} - y^{-2})(x - y)^{-1} \left(\dfrac{1}{xy}\right)^{-1} (x^{-1} + y^{-1})^{-1}$ f) $(36x^{1/2}y^2)^{3/2} \div \left(\dfrac{6x^{-2}}{y^{-2/3}}\right)^3$

4. Determine whether each of the following statements is correct:
 a) $\sqrt{8 + 2\sqrt{15}} = \sqrt{3} + \sqrt{5}$ b) $\sqrt{16 - 4\sqrt{15}} = \sqrt{6} - \sqrt{10}$

5. Simplify each of the following expressions, where x denotes a positive real number:
 a) $32^x \times 2^{3x}$ b) $16^{3x/4} \div 4^{2x}$ c) $\dfrac{2^{x+1} + 4^x}{2^{x-1} + 1}$ d) $\dfrac{9^x - 4^x}{3^x + 2^x}$

6. Solve each of the following equations:
 a) $27^{2-x} = 9^{x-2}$ b) $4^x = \dfrac{1}{\sqrt{32}}$ c) $\left(\dfrac{1}{9}\right)^x = 3 \times 81^{-x}$
 d) $2^x \times 16^x = 4 \times 8^x$ e) $4^x = 3^x$ f) $\left(\dfrac{1}{7}\right)^x = 7^{2x}$

7. Find the precise value of each of the following expressions:
 a) $\log_3 \dfrac{1}{27}$ b) $\log_{100} 1000$ c) $\log_5 \sqrt{125}$ d) $\log_4 \dfrac{1}{32}$

8. Solve each of the following equations:
 a) $\log_2 y + 3\log_2(2y) = 3$ b) $\log_3 y = -3$ c) $\log 7 - 2\log y = 2\log 49$
 d) $2\log y = \log(y + 2)$ e) $2\log y = \log(7y - 12)$ f) $\log y = 3\sqrt{\log y}$
 g) $2\log y = \log(y + 6)$ h) $2\log y = \sqrt{\log y}$

9. Simplify each of the following expressions, where u, v and w denote suitable positive real numbers:
 a) $\log \dfrac{u^4}{v^2} - \log \dfrac{v^2}{u}$ b) $\log(u^3v^{-6}) - 3\log \dfrac{u}{v^2}$
 c) $3\log u - \log(uv)^3 + 3\log(vw)$ d) $\log \dfrac{uv}{w} + \log \dfrac{uw}{v} + \log \dfrac{vw}{u} - \log(uvw)$
 e) $\left(\log \dfrac{u^4}{v^2} - \log v + 2\log u\right) \div (2\log u - \log v)$ f) $\log(u^2 - v^2) - \log(u - v) - \log(u + v)$
 g) $\log u^3 + \log v^2 - \log \left(\dfrac{u}{v}\right)^3 - 5\log v$ h) $2\log(\mathrm{e}^u) + \log(\mathrm{e}^v) - \log(\mathrm{e}^{u+v})$

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Chapter 5

POLYNOMIAL EQUATIONS

5.1. Linear Equations

Consider an equation of the type

$$ax + b = 0, \tag{1}$$

where $a, b \in \mathbb{R}$ are constants and $a \neq 0$. To solve such an equation, we first subtract b from both sides of the equation to obtain

$$ax = -b, \tag{2}$$

and then divide both sides of this latter equation by a to obtain

$$x = -\frac{b}{a}.$$

Occasionally a given linear equation may be a little more complicated than (1) or (2). However, with the help of some simple algebra, one can reduce the given equation to one of type (1) or type (2).

EXAMPLE 5.1.1. Suppose that

$$\frac{4x}{x+2} = \frac{18}{5}.$$

Multiplying both sides by $5(x + 2)$, the product of the two denominators, we obtain

$$20x = 18(x + 2) = 18x + 36.$$

Subtracting $18x$ from both sides, we obtain $2x = 36$, and so $x = 36/2 = 18$.

† This chapter was written at Macquarie University in 1999.

EXAMPLE 5.1.2. Suppose that

$$\frac{x-6}{2}+\frac{3x}{4}=x+1.$$

Multiplying both sides by 4, we obtain $2(x-6)+3x=4(x+1)$. Now $2(x-6)+3x=5x-12$ and $4(x+1)=4x+4$. It follows that $5x-12=4x+4$, so that $x-16=0$, giving $x=16$.

EXAMPLE 5.1.3. Suppose that

$$\frac{6x-1}{4x+3}=\frac{3x-7}{2x-5}.$$

Multiplying both sides by $(4x+3)(2x-5)$, the product of the two denominators, we obtain

$$(6x-1)(2x-5)=(3x-7)(4x+3).$$

Now $(6x-1)(2x-5)=12x^2-32x+5$ and $(3x-7)(4x+3)=12x^2-19x-21$. It follows that $12x^2-32x+5=12x^2-19x-21$, so that $-13x+26=0$, whence $x=2$.

EXAMPLE 5.1.4. Suppose that

$$\frac{3x+1}{2x+3}=\frac{3x-3}{2x+5}.$$

Multiplying both sides by $(2x+3)(2x+5)$, the product of the two denominators, we obtain

$$(3x+1)(2x+5)=(3x-3)(2x+3).$$

Now $(3x+1)(2x+5)=6x^2+17x+5$ and $(3x-3)(2x+3)=6x^2+3x-9$. Check that the original equation is equivalent to $14x+14=0$, so that $x=-1$.

We next consider a pair of simultaneous linear equations in two variables, of the type

$$\begin{aligned} a_1x+b_1y&=c_1,\\ a_2x+b_2y&=c_2, \end{aligned} \tag{3}$$

where $a_1,a_2,b_1,b_2,c_1,c_2\in\mathbb{R}$. Multiplying the first equation in (3) by b_2 and multiplying the second equation in (3) by b_1, we obtain

$$\begin{aligned} a_1b_2x+b_1b_2y&=c_1b_2,\\ a_2b_1x+b_1b_2y&=c_2b_1. \end{aligned} \tag{4}$$

Subtracting the second equation in (4) from the first equation, we obtain

$$(a_1b_2x+b_1b_2y)-(a_2b_1x+b_1b_2y)=c_1b_2-c_2b_1.$$

Some simple algebra leads to

$$(a_1b_2-a_2b_1)x=c_1b_2-c_2b_1. \tag{5}$$

On the other hand, multiplying the first equation in (3) by a_2 and multiplying the second equation in (3) by a_1, we obtain

$$\begin{aligned} a_1a_2x+b_1a_2y&=c_1a_2,\\ a_1a_2x+b_2a_1y&=c_2a_1. \end{aligned} \tag{6}$$

Subtracting the second equation in (6) from the first equation, we obtain

$$(a_1a_2x+b_1a_2y)-(a_1a_2x+b_2a_1y)=c_1a_2-c_2a_1.$$

Some simple algebra leads to

$$(b_1a_2-b_2a_1)y=c_1a_2-c_2a_1. \tag{7}$$

Suppose that $a_1b_2-a_2b_1\neq 0$. Then (5) and (7) can be written in the form

$$x=\frac{c_1b_2-c_2b_1}{a_1b_2-a_2b_1}\qquad\text{and}\qquad y=\frac{c_1a_2-c_2a_1}{b_1a_2-b_2a_1}. \tag{8}$$

In practice, we do not need to remember these formulae. It is much easier to do the calculations by using some common sense and cutting a few corners in doing so.

We have the following geometric interpretation. Each of the two linear equations in (3) represents a line on the xy-plane. The condition $a_1b_2 - a_2b_1 \neq 0$ ensures that the two lines are not parallel, so that they intersect at precisely one point, given by (8).

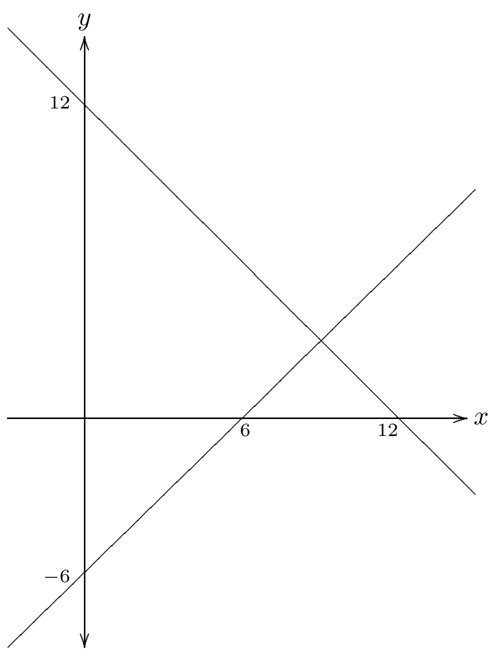
EXAMPLE 5.1.5. Suppose that

$$\begin{aligned} x + y &= 12, \\ x - y &= \ 6. \end{aligned}$$

Note that we can eliminate y by adding the two equations. More precisely, we have

$$(x + y) + (x - y) = 12 + 6.$$

This gives $2x = 18$, so that $x = 9$. We now substitute the information $x = 9$ into one of the two original equations. Simple algebra leads to $y = 3$. We have the following picture.



EXAMPLE 5.1.6. Suppose that

$$\begin{aligned} x + \ \ y &= 32, \\ 3x + 2y &= 70. \end{aligned}$$

We can multiply the first equation by 2 and keep the second equation as it is to obtain

$$\begin{aligned} 2x + 2y &= 64, \\ 3x + 2y &= 70. \end{aligned}$$

The effect of this is that both equations have a term $2y$. We now subtract the first equation from the second equation to eliminate this common term. More precisely, we have

$$(3x + 2y) - (2x + 2y) = 70 - 64.$$

This gives $x = 6$. We now substitute the information $x = 6$ into one of the two original equations. Simple algebra leads to $y = 26$.

EXAMPLE 5.1.7. Suppose that

$$\begin{aligned} 3x + 2y &= 10, \\ 4x - 3y &= 2. \end{aligned}$$

We can multiply the first equation by 4 and the second equation by 3 to obtain

$$\begin{aligned} 12x + 8y &= 40, \\ 12x - 9y &= 6. \end{aligned}$$

The effect of this is that both equations have a term $12x$. We now subtract the second equation from the first equation to eliminate this common term. More precisely, we have

$$(12x + 8y) - (12x - 9y) = 40 - 6.$$

This gives $17y = 34$, so that $y = 2$. We now substitute the information $y = 2$ into one of the two original equations. Simple algebra leads to $x = 2$. The reader may try to eliminate the variable y first and show that we must have $x = 2$.

EXAMPLE 5.1.8. Suppose that

$$\begin{aligned} 7x - 5y &= 16, \\ 2x + 7y &= 13. \end{aligned}$$

We can multiply the first equation by 7 and the second equation by 5 to obtain

$$\begin{aligned} 49x - 35y &= 112, \\ 10x + 35y &= 65. \end{aligned}$$

The effect of this is that both equations have a term $35y$ but with opposite signs. We now add the two equation to eliminate this common term. More precisely, we have

$$(49x - 35y) + (10x + 35y) = 112 + 65.$$

This gives $59x = 177$, so that $x = 3$. We now substitute the information $x = 3$ into one of the two original equations. Simple algebra leads to $y = 1$.

EXAMPLE 5.1.9. Suppose that the difference between two numbers is equal to 11, and that twice the smaller number minus 4 is equal to the larger number. To find the two numbers, let x denote the larger number and y denote the smaller number. Then we have $x - y = 11$ and $2y - 4 = x$, so that

$$\begin{aligned} x - y &= 11, \\ x - 2y &= -4. \end{aligned}$$

We now eliminate the variable x by subtracting the second equation from the first equation. More precisely, we have

$$(x - y) - (x - 2y) = 11 - (-4).$$

This gives $y = 15$. We now substitute the information $y = 15$ into one of the two original equations. Simple algebra leads to $x = 26$.

EXAMPLE 5.1.10. Suppose that a rectangle is 5cm longer than it is wide. Suppose also that if the length and width are both increased by 2cm, then the area of the rectangle increases by 50cm^2. To find the dimension of the rectangle, let x denote its length and y denote its width. Then we have $x = y+5$ and $(x+2)(y+2) - xy = 50$. Simple algebra shows that the second equation is the same as $2x+2y+4 = 50$. We therefore have

$$\begin{aligned} x - y &= 5, \\ 2x + 2y &= 46. \end{aligned}$$

We can multiply the first equation by 2 and keep the second equation as it is to obtain

$$\begin{aligned} 2x - 2y &= 10, \\ 2x + 2y &= 46. \end{aligned}$$

We now eliminate the variable y by adding the two equations. More precisely, we have

$$(2x - 2y) + (2x + 2y) = 10 + 46.$$

This gives $4x = 56$, so that $x = 14$. It follows that $y = 9$.

The idea of eliminating one of the variables can be extended to solve systems of three linear equations. We illustrate the ideas by the following four examples.

EXAMPLE 5.1.11. Suppose that

$$\begin{aligned} x + y + z &= 6, \\ 2x + 3y + z &= 13, \\ x + 2y - z &= 5. \end{aligned}$$

Adding the first equation and the third equation, or adding the second equation and the third equation, we eliminate the variable z on both occasions and obtain respectively

$$\begin{aligned} 2x + 3y &= 11, \\ 3x + 5y &= 18. \end{aligned}$$

Solving this system, the reader can show that $x = 1$ and $y = 3$. Substituting back to one of the original equations, we obtain $z = 2$.

EXAMPLE 5.1.12. Suppose that

$$\begin{aligned} x - y + z &= 10, \\ 4x + 2y - 3z &= 8, \\ 3x - 5y + 2z &= 34. \end{aligned}$$

We can multiply the three equations by 6, 2 and 3 respectively to obtain

$$\begin{aligned} 6x - 6y + 6z &= 60, \\ 8x + 4y - 6z &= 16, \\ 9x - 15y + 6z &= 102. \end{aligned}$$

The reason for the multiplication is to arrange for the term $6z$ to appear in each equation to make the elimination of the variable z easier. Indeed, adding the first equation and the second equation, or adding the second equation and the third equation, we eliminate the variable z on both occasions and obtain respectively

$$\begin{aligned} 14x - 2y &= 76, \\ 17x - 11y &= 118. \end{aligned}$$

Multiplying the first equation by 11 and the second equation by 2, we obtain

$$\begin{aligned} 154x - 22y &= 836, \\ 34x - 22y &= 236. \end{aligned}$$

Eliminating the variable y, we obtain $120x = 600$, so that $x = 5$. It follows that $y = -3$. Using now one of the original equations, we obtain $z = 2$.

EXAMPLE 5.1.13. Suppose that

$$\begin{aligned} 6x + 4y - 2z &= 0, \\ 3x - 2y + 4z &= 3, \\ 5x - 2y + 6z &= 3. \end{aligned}$$

Multiplying the last two equations by 2, we obtain

$$\begin{aligned} 6x + 4y - 2z &= 0, \\ 6x - 4y + 8z &= 6, \\ 10x - 4y + 12z &= 6. \end{aligned}$$

The reason for the multiplication is to arrange for the term $4y$ to appear in each equation to make the elimination of the variable y easier. Indeed, adding the first equation and the second equation, or adding the first equation and the third equation, we eliminate the variable y on both occasions and obtain respectively

$$\begin{aligned} 12x + 6z &= 6, \\ 16x + 10z &= 6. \end{aligned}$$

Solving this system, the reader can show that $x = 1$ and $z = -1$. Substituting back to one of the original equations, we obtain $y = -2$.

EXAMPLE 5.1.14. Suppose that

$$\begin{aligned} 2x + y - z &= 9, \\ 5x + 2z &= -3, \\ 7x - 2y &= 1. \end{aligned}$$

Our strategy here is to eliminate the variable y between the first and third equations. To do this, the first equation can be written in the form $4x + 2y - 2z = 18$. Adding this to the third equation, and also keeping the second equation as it is, we obtain

$$\begin{aligned} 11x - 2z &= 19, \\ 5x + 2z &= -3. \end{aligned}$$

Solving this system, the reader can show that $x = 1$ and $z = -4$. Substituting back to one of the original equations, we obtain $y = 3$. The reader may also wish to first eliminate the variable z between the first two equations and obtain a system of two equations in x and y.

5.2. Quadratic Equations

Consider an equation of the type

$$ax^2 + bx + c = 0, \tag{9}$$

where $a, b, c \in \mathbb{R}$ are constants and $a \neq 0$. To solve such an equation, we observe first of all that

$$\begin{aligned} ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) &= a\left(x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right) \\ &= a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right) = 0 \end{aligned}$$

precisely when

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}. \tag{10}$$

There are three cases:

(1) If $b^2 - 4ac < 0$, then the right hand side of (10) is negative. It follows that (10) is never satisfied for any real number x, so that the equation (9) has no real solution.

(2) If $b^2 - 4ac = 0$, then (10) becomes

$$\left(x + \frac{b}{2a}\right)^2 = 0, \qquad \text{so that} \qquad x = -\frac{b}{2a}.$$

Indeed, this solution occurs twice, as we shall see later.

(3) If $b^2 - 4ac > 0$, then (10) becomes

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}, \qquad \text{so that} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We therefore have two distinct real solutions for the equation (9).

EXAMPLE 5.2.1. For the equation $2x^2+6x+4=0$, we have $(a,b,c)=(2,6,4)$, so that $b^2-4ac=4>0$. It follows that this equation has two distinct real solutions, given by

$$x=\frac{-6\pm\sqrt{4}}{4}=-1 \text{ or } -2.$$

Observe that $2x^2+6x+4=2(x+1)(x+2)$.

EXAMPLE 5.2.2. For the equation $x^2+2x+3=0$, we have $(a,b,c)=(1,2,3)$, so that $b^2-4ac=-8<0$. It follows that this equation has no solution.

EXAMPLE 5.2.3. For the equation $3x^2-12x+12=0$, we have $b^2-4ac=0$. It follows that this equation has one real solution, given by $x=2$. Observe that $3x^2-12x+12=3(x-2)^2$. This is the reason we say that the root occurs twice.

5.3. Factorization Again

Consider equation (9) again. Sometimes we may be able to find a factorization of the form

$$ax^2+bx+c=a(x-\alpha)(x-\beta), \tag{11}$$

where $\alpha,\beta\in\mathbb{R}$. Clearly $x=\alpha$ and $x=\beta$ are solutions of the equation (9).

EXAMPLE 5.3.1. For the equation $x^2-5x=0$, we have the factorization

$$x^2-5x=x(x-5)=(x-0)(x-5).$$

It follows that the two solutions of the equation are $x=0$ and $x=5$.

EXAMPLE 5.3.2. For the equation $x^2-9=0$, we have the factorization $x^2-9=(x-3)(x+3)$. It follows that the two solutions of the equation are $x=\pm3$.

Note that

$$a(x-\alpha)(x-\beta)=a(x^2-(\alpha+\beta)x+\alpha\beta)=ax^2-a(\alpha+\beta)x+a\alpha\beta.$$

It follows from (11) that

$$ax^2+bx+c=ax^2-a(\alpha+\beta)x+a\alpha\beta.$$

Equating corresponding coefficients, we obtain

$$b=-a(\alpha+\beta) \qquad \text{and} \qquad c=a\alpha\beta.$$

We have proved the following result.

SUM AND PRODUCT OF ROOTS OF A QUADRATIC EQUATION. *Suppose that $x=\alpha$ and $x=\beta$ are the two roots of a quadratic equation $ax^2+bx+x=0$. Then*

$$\alpha+\beta=-\frac{b}{a} \qquad \textit{and} \qquad \alpha\beta=\frac{c}{a}.$$

Example 5.3.3. For the equation $x^2 - 5x - 7 = 0$, we have $(a, b, c) = (1, -5, -7)$, and

$$x = \frac{5 \pm \sqrt{25 + 28}}{2} = \frac{5 \pm \sqrt{53}}{2}.$$

Note that

$$\frac{5 + \sqrt{53}}{2} + \frac{5 - \sqrt{53}}{2} = 5 \qquad \text{and} \qquad \frac{5 + \sqrt{53}}{2} \times \frac{5 - \sqrt{53}}{2} = \frac{25 - 53}{4} = -7.$$

Example 5.3.4. For the equation $x^2 - 13x + 4 = 0$, we have $(a, b, c) = (1, -13, 4)$, and

$$x = \frac{13 \pm \sqrt{169 - 16}}{2} = \frac{13 \pm \sqrt{153}}{2}.$$

Note that

$$\frac{13 + \sqrt{153}}{2} + \frac{13 - \sqrt{153}}{2} = 13 \qquad \text{and} \qquad \frac{13 + \sqrt{153}}{2} \times \frac{13 - \sqrt{153}}{2} = \frac{169 - 153}{4} = 4.$$

We conclude this section by studying a few more examples involving factorization of quadratic polynomials. The reader may wish to study Section 1.5 again before proceeding.

Example 5.3.5. Consider the expression $x^2 - 4x + 3$. The roots of the equation $x^2 - 4x + 3 = 0$ are given by

$$\alpha = \frac{4 + \sqrt{16 - 12}}{2} = 3 \qquad \text{and} \qquad \beta = \frac{4 - \sqrt{16 - 12}}{2} = 1.$$

Hence we have $x^2 - 4x + 3 = (x - 3)(x - 1)$.

Example 5.3.6. Consider the expression $2x^2 + 5x + 2$. The roots of the equation $2x^2 + 5x + 2 = 0$ are given by

$$\alpha = \frac{-5 + \sqrt{25 - 16}}{4} = -\frac{1}{2} \qquad \text{and} \qquad \beta = \frac{-5 - \sqrt{25 - 16}}{4} = -2.$$

Hence we have

$$2x^2 + 5x + 2 = 2\left(x + \frac{1}{2}\right)(x + 2) = (2x + 1)(x + 2).$$

Example 5.3.7. Consider the expression $4x^2 - x - 14$. The roots of the equation $4x^2 - x - 14 = 0$ are given by

$$\alpha = \frac{1 + \sqrt{1 + 224}}{8} = 2 \qquad \text{and} \qquad \beta = \frac{1 - \sqrt{1 + 224}}{8} = -\frac{7}{4}.$$

Hence we have

$$4x^2 - x - 14 = 4(x - 2)\left(x + \frac{7}{4}\right) = (x - 2)(4x + 7).$$

Example 5.3.8. We have

$$(x + 2)^2 - (2x - 1)^2 = (x^2 + 4x + 4) - (4x^2 - 4x + 1) = x^2 + 4x + 4 - 4x^2 + 4x - 1 = -3x^2 + 8x + 3.$$

The roots of the equation $-3x^2 + 8x + 3 = 0$ are given by

$$\alpha = \frac{-8 + \sqrt{64 + 36}}{-6} = -\frac{1}{3} \qquad \text{and} \qquad \beta = \frac{-8 - \sqrt{64 + 36}}{-6} = 3.$$

Hence

$$(x+2)^2-(2x-1)^2=-3\left(x+\frac{1}{3}\right)(x-3)=-(3x+1)(x-3)=(3x+1)(3-x).$$

Alternatively, we can use one of the Laws on squares (writing $a=x+2$ and $b=2x-1$). We have

$$\begin{aligned}(x+2)^2-(2x-1)^2&=((x+2)-(2x-1))((x+2)+(2x-1))\\&=(x+2-2x+1)(x+2+2x-1)=(3-x)(3x+1).\end{aligned}$$

EXAMPLE 5.3.9. Consider the expression $6p-17pq+12pq^2$. Taking out a factor p, we have

$$6p-17pq+12pq^2=p(6-17q+12q^2).$$

Consider next the quadratic factor $6-17q+12q^2$. The roots of the equation $6-17q+12q^2=0$ are given by

$$\alpha=\frac{17+\sqrt{289-288}}{24}=\frac{3}{4}\qquad\text{and}\qquad\beta=\frac{17-\sqrt{289-288}}{24}=\frac{2}{3}.$$

Hence

$$6p-17pq+12pq^2=p(6-17q+12q^2)=12p\left(q-\frac{3}{4}\right)\left(q-\frac{2}{3}\right)=p(4q-3)(3q-2).$$

EXAMPLE 5.3.10. We have $10a^2b+11ab-6b=b(10a^2+11a-6)$. Consider the quadratic factor $10a^2+11a-6$. The roots of the equation $10a^2+11a-6=0$ are given by

$$\alpha=\frac{-11+\sqrt{121+240}}{20}=\frac{2}{5}\qquad\text{and}\qquad\beta=\frac{-11-\sqrt{121+240}}{20}=-\frac{3}{2}.$$

Hence

$$10a^2b+11ab-6b=b(10a^2+11a-6)=10b\left(a-\frac{2}{5}\right)\left(a+\frac{3}{2}\right)=b(5a-2)(2a+3).$$

EXAMPLE 5.3.11. Consider the expression

$$\frac{a-3}{a^2-11a+28}-\frac{a+4}{a^2-6a-7}.$$

Note that $a^2-11a+28=(a-7)(a-4)$ and $a^2-6a-7=(a-7)(a+1)$. Hence

$$\begin{aligned}\frac{a-3}{a^2-11a+28}-\frac{a+4}{a^2-6a-7}&=\frac{a-3}{(a-7)(a-4)}-\frac{a+4}{(a-7)(a+1)}\\&=\frac{(a-3)(a+1)}{(a-7)(a-4)(a+1)}-\frac{(a+4)(a-4)}{(a-7)(a-4)(a+1)}\\&=\frac{(a-3)(a+1)-(a+4)(a-4)}{(a-7)(a-4)(a+1)}=\frac{(a^2-2a-3)-(a^2-16)}{(a-7)(a-4)(a+1)}\\&=\frac{13-2a}{(a-7)(a-4)(a+1)}.\end{aligned}$$

EXAMPLE 5.3.12. We have

$$\begin{aligned}&\frac{1}{x+1}+\frac{1}{(x+1)(x+2)}-\frac{4}{(x+1)(x+2)(x+3)}\\&\quad=\frac{(x+2)(x+3)}{(x+1)(x+2)(x+3)}+\frac{(x+3)}{(x+1)(x+2)(x+3)}-\frac{4}{(x+1)(x+2)(x+3)}\\&\quad=\frac{(x+2)(x+3)+(x+3)-4}{(x+1)(x+2)(x+3)}=\frac{(x^2+5x+6)+(x+3)-4}{(x+1)(x+2)(x+3)}=\frac{x^2+6x+5}{(x+1)(x+2)(x+3)}\\&\quad=\frac{(x+1)(x+5)}{(x+1)(x+2)(x+3)}=\frac{x+5}{(x+2)(x+3)}.\end{aligned}$$

5.4. Higher Order Equations

For polynomial equations of degree greater than 2, we do not have general formulae for their solutions. However, we may occasionally be able to find some solutions by inspection. These may help us find other solutions. We shall illustrate the technique here by using three examples.

EXAMPLE 5.4.1. Consider the equation $x^3 - 4x^2 + 2x + 1 = 0$. It is easy to see that $x = 1$ is a solution of this cubic polynomial equation. It follows that $x - 1$ is a factor of the polynomial $x^3 - 4x^2 + 2x + 1$. Using long division, we have the following:

$$
\begin{array}{rrrrr}
 & x^2- & 3x- & 1 & \\ \cline{2-4}
x-\ 1\,) & x^3-4x^2+ & 2x+ & 1 & \\
 & x^3-\ \ x^2 & & & \\ \cline{2-4}
 & -3x^2+ & 2x & & \\
 & -3x^2+ & 3x & & \\ \cline{2-4}
 & - & x+ & 1 & \\
 & - & x+ & 1 & \\ \cline{3-4}
\end{array}
$$

Hence $x^3 - 4x^2 + 2x + 1 = (x - 1)(x^2 - 3x - 1)$. The other two roots of the equation are given by the two roots of $x^2 - 3x - 1 = 0$. These are

$$x = \frac{3 \pm \sqrt{9+4}}{2} = \frac{3 \pm \sqrt{13}}{2}.$$

EXAMPLE 5.4.2. Consider the equation $x^3 + 2x^2 - 5x - 6 = 0$. It is easy to see that $x = -1$ is a solution of this cubic polynomial equation. It follows that $x + 1$ is a factor of the polynomial $x^3 + 2x^2 - 5x - 6$. Using long division, we have the following:

$$
\begin{array}{rrrrr}
 & x^2+ & x- & 6 & \\ \cline{2-4}
x+\ 1\,) & x^3+2x^2- & 5x- & 6 & \\
 & x^3+\ \ x^2 & & & \\ \cline{2-4}
 & x^2- & 5x & & \\
 & x^2+ & x & & \\ \cline{2-4}
 & - & 6x- & 6 & \\
 & - & 6x- & 6 & \\ \cline{3-4}
\end{array}
$$

Hence $x^3 + 2x^2 - 5x - 6 = (x + 1)(x^2 + x - 6)$. The other two roots of the equation are given by the two roots of $x^2 + x - 6 = 0$. These are

$$x = \frac{-1 \pm \sqrt{1+24}}{2} = \frac{-1 \pm 5}{2} = 2 \text{ or } -3.$$

EXAMPLE 5.4.3. Consider the equation $x^4 + 7x^3 - 6x^2 - 2x = 0$. It is easy to see that $x = 0$ and $x = 1$ are solutions of this biquadratic polynomial equation. It follows that $x(x - 1)$ is a factor of the polynomial $x^4 + 7x^3 - 6x^2 - 2x$. Clearly we have $x^4 + 7x^3 - 6x^2 - 2x = x(x^3 + 7x^2 - 6x - 2)$. On the other hand, using long division, we have the following:

$$
\begin{array}{rrrrr}
 & x^2+ & 8x+ & 2 & \\ \cline{2-4}
x-\ 1\,) & x^3+7x^2- & 6x- & 2 & \\
 & x^3-\ \ x^2 & & & \\ \cline{2-4}
 & 8x^2- & 6x & & \\
 & 8x^2- & 8x & & \\ \cline{2-4}
 & & 2x- & 2 & \\
 & & 2x- & 2 & \\ \cline{3-4}
\end{array}
$$

Hence $x^4 + 7x^3 - 6x^2 - 2x = x(x-1)(x^2 + 8x + 2)$. The other two roots of the equation are given by the two roots of $x^2 + 8x + 2 = 0$. These are

$$x = \frac{-8 \pm \sqrt{64-8}}{2} = \frac{-8 \pm \sqrt{56}}{2} = -4 \pm \sqrt{14}.$$

Problems for Chapter 5

1. Solve each of the following equations:
   a) $\dfrac{9x-8}{8x-6} = \dfrac{1}{2}$ b) $\dfrac{5-2x}{3x+7} = 9$ c) $\dfrac{5x+14}{3x+2} = 3$
   d) $\dfrac{2x-2}{3} + \dfrac{10-2x}{6} = 2x-4$ e) $\dfrac{3x-4}{6x-10} = \dfrac{4x+1}{8x-7}$

2. Solve each of the following systems of linear equations:
   a) $3x - 8y = 1$, $2x + 3y = 9$ b) $4x - 3y = 14$, $9x - 4y = 26$ c) $6x - 2y = 14$, $2x + 3y = 12$ d) $5x + 2y = 4$, $7x + 3y = 5$

3. A rectangle is 2 metres longer than it is wide. On the other hand, if each side of the rectangle is increased by 2 metres, then the area increases by 16 square metres. Find the dimension of the rectangle.

4. A rectangle is 10 metres wider than it is long. On the other hand, if the width and length are both decreased by 5 metres, then the area of the rectangle decreases by 125 square metres. Find the dimension of the rectangle.

5. The lengths of the two perpendicular sides of a right-angled triangle differ by 6 centimetres. On the other hand, if the length of the longer of these two sides is increased by 3 centimetres and the length of the shorter of these two sides is decreased by 2 centimetres, then the area of the right-angled triangle formed is decreased by 5 square centimetres. What is the dimension of the original triangle?

6. Solve each of the following systems of linear equations:
   a) $3x - 2y + 4z = 11$, $2x + 3y + 3z = 17$, $4x + 6y - 2z = 10$
   b) $5x + 2y + 4z = 35$, $2x - 3y + 2z = 19$, $3x + 5y + 3z = 19$
   c) $3x + 4y + 2z = 9$, $5x - 2y + 4z = 7$, $2x + 6y - 2z = 6$
   d) $x + 4y - 3z = 2$, $x - 2y + 2z = 1$, $x + 6y - 5z = 2$

7. Determine the number of solutions of each of the following quadratic equations and find the solutions:
   a) $3x^2 - x + 1 = 0$ b) $4x^2 + 12x + 9 = 0$ c) $18x^2 - 84x + 98 = 0$
   d) $6x^2 - 13x + 6 = 0$ e) $5x^2 + 2x + 1 = 0$ f) $x^2 - 2x - 48 = 0$
   g) $12x^2 + 12x + 3 = 0$ h) $2x^2 - 32x + 126 = 0$ i) $3x^2 + 6x + 15 = 0$
   j) $16x^2 + 8x - 3 = 0$ k) $x^2 + 2x + 2 = 0$

8. Factorize each of the following expressions:
   a) $14x^2 + 19x - 3$ b) $6x^2 + x - 12$ c) $(5x+1)^2 - 20x$
   d) $(2x+1)^2 + x(2+4x)$ e) $4x^3 + 9x^2 + 2x$ f) $x^2y - xy - 6y$
   g) $8x - 2xy - xy^2$

9. Simplify each of the following expressions, showing all the steps of your argument carefully:
   a) $\dfrac{2}{x-2} + \dfrac{2}{x-5} - \dfrac{x}{x^2-3x+2} - \dfrac{2}{x-1}$ b) $\dfrac{x-3}{x^2-3x+2} - \dfrac{x-2}{x^2-4x+3}$
   c) $\dfrac{x^4 + x^2y^2 + y^4}{x^3 - y^3} \div \dfrac{x^2 - xy + y^2}{x-y}$ d) $\dfrac{4xy}{(x-y)^2} + \dfrac{x^2-xy}{x^2-y^2}\left(1 + \dfrac{y}{x}\right)$

10. Study each of the following equations for real solutions:
a) $x^3 - 6x^2 + 11x - 6 = 0$
b) $x^3 - 3x^2 + 4 = 0$
c) $x^3 + 2x^2 + 6x + 5 = 0$
d) $x^3 - x^2 - x + 1 = 0$
e) $x^3 + 2x^2 - x - 2 = 0$

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Chapter 6

INEQUALITIES AND ABSOLUTE VALUES

6.1. Some Simple Inequalities

Basic inequalities concerning the real numbers are simple, provided that we exercise due care. We begin by studying the effect of addition and multiplication on inequalities.

ADDITION AND MULTIPLICATION RULES. *Suppose that $a, b \in \mathbb{R}$ and $a < b$. Then*
(a) for every $c \in \mathbb{R}$, we have $a + c < b + c$;
(b) for every $c \in \mathbb{R}$ satisfying $c > 0$, we have $ac < bc$; and
(c) for every $c \in \mathbb{R}$ satisfying $c < 0$, we have $ac > bc$.

In other words, addition by a real number c preserves the inequality. On the other hand, multiplication by a real number c preserves the inequality if $c > 0$ and reverses the inequality if $c < 0$.

REMARK. We can deduce some special rules for positive real numbers. Suppose that $a, b, c, d \in \mathbb{R}$ are all positive. If $a < b$ and $c < d$, then $ac < bd$. To see this, note simply that by part (b) above, we have $ac < bc$ and $bc < bd$.

SQUARE AND RECIPROCAL RULES. *Suppose that $a, b \in \mathbb{R}$ and $0 < a < b$. Then*
(a) $a^2 < b^2$; and
(b) $a^{-1} > b^{-1}$.

PROOF. Part (a) is a special case of our Remark if we take $c = a$ and $d = b$. To show part (b), note that

$$a^{-1} - b^{-1} = \frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab} > 0.$$ ♣

† This chapter was written at Macquarie University in 1999.

CAUCHY'S INEQUALITY. *For every $a, b \in \mathbb{R}$, we have $a^2 + b^2 \geq 2ab$. Furthermore, equality holds precisely when $a = b$.*

PROOF. Simply note that

$$a^2 + b^2 - 2ab = a^2 - 2ab + b^2 = (a - b)^2 \geq 0,$$

and that equality holds precisely when $a - b = 0$. ♣

We now use some of the above rules to solve inequalities. We shall illustrate the ideas by considering a few examples in some detail.

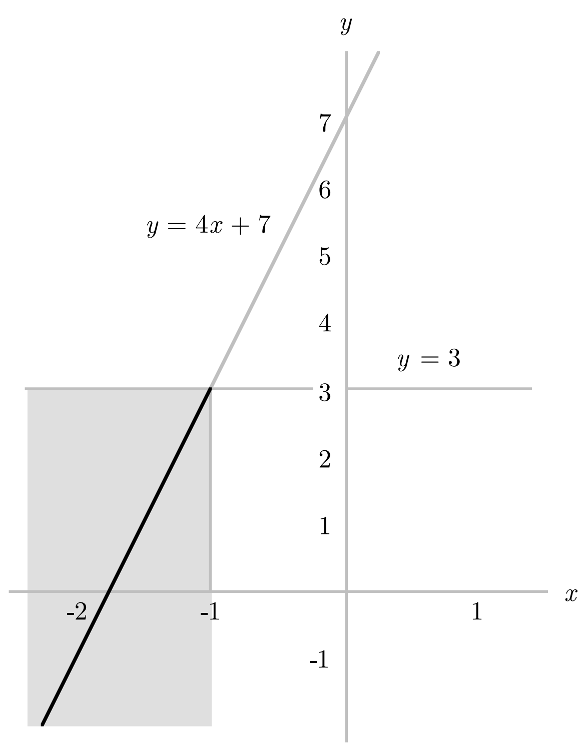
EXAMPLE 6.1.1. Consider the inequality $4x + 7 < 3$. Using the Addition rule and adding -7 to both sides, we obtain $4x < -4$. Using one of the Multiplication rules and multiplying both sides by the positive real number $1/4$, we obtain $x < -1$. We have shown that

$$4x + 7 < 3 \qquad \Longrightarrow \qquad x < -1.$$

Suppose now that $x < -1$. Using one of the Multiplication rules and multiplying both sides by the positive real number 4, we obtain $4x < -4$. Using the Addition rule and adding 7 to both sides, we obtain $4x + 7 < 3$. Combining this with our earlier observation, we have now shown that

$$4x + 7 < 3 \qquad \Longleftrightarrow \qquad x < -1.$$

We can confirm our conclusion by drawing a graph of the line $y = 4x + 7$ and observing that the part of the line below the horizontal line $y = 3$ corresponds to $x < -1$ on the x-axis.



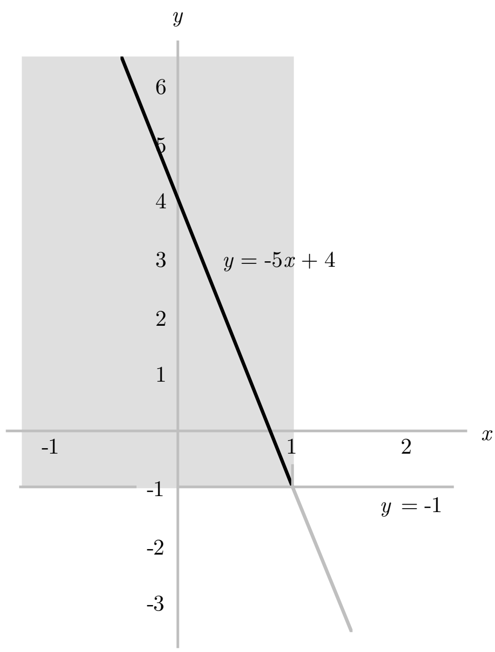
EXAMPLE 6.1.2. Consider the inequality $-5x + 4 > -1$. Using one of the Multiplication rules and multiplying both sides by the negative real number -1, we obtain $5x - 4 < 1$. Using the Addition rule and adding 4 to both sides, we obtain $5x < 5$. Using one of the Multiplication rules and multiplying both sides by the positive real number $1/5$, we obtain $x < 1$. We have shown that

$$-5x + 4 > -1 \qquad \Longrightarrow \qquad x < 1.$$

Suppose now that $x < 1$. Using one of the Multiplication rules and multiplying both sides by the positive real number 5, we obtain $5x < 5$. Using the Addition rule and adding -4 to both sides, we obtain $5x - 4 < 1$. Using one of the Multiplication rules and multiplying both sides by the negative real number -1, we obtain $-5x + 4 > -1$. Combining this with our earlier observation, we have now shown that

$$-5x + 4 > -1 \qquad \Longleftrightarrow \qquad x < 1.$$

We can confirm our conclusion by drawing a graph of the line $y = -5x + 4$ and observing that the part of the line above the horizontal line $y = -1$ corresponds to $x < 1$ on the x-axis.



EXAMPLE 6.1.3. Consider the inequality $x^2 \leq a^2$, where $a > 0$ is fixed. Clearly $x = \pm a$ are the only solutions of the equation $x^2 = a^2$. So let us consider the inequality $x^2 < a^2$. Observe first of all that the inequality is satisfied when $x = 0$. On the other hand, if $0 < x < a$, then the Square rule gives $x^2 < a^2$. However, if $-a < x < 0$, then using one of the Multiplication rules and multiplying all sides by the negative real number -1, we obtain $a > -x > 0$. It follows from the Square rule that $(-x)^2 < a^2$, so that $x^2 < a^2$. We have now shown that

$$-a \leq x \leq a \qquad \Longrightarrow \qquad x^2 \leq a^2.$$

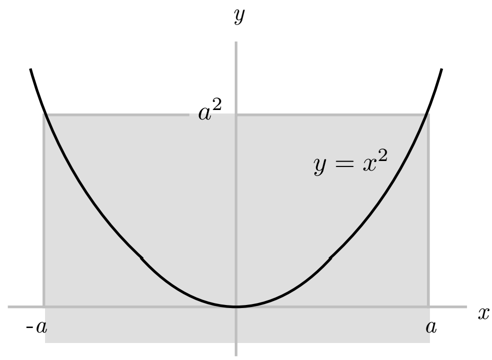
Suppose now that $x > a$. Then it follows from the Square rule that $x^2 > a^2$. On the other hand, suppose that $x < -a$. Using one of the Multiplication rules and multiplying both sides by the negative real number -1, we obtain $-x > a$. It follows from the Square rule that $(-x)^2 > a^2$, so that $x^2 > a^2$. We have now shown that

$$x < -a \quad \text{or} \quad x > a \qquad \Longrightarrow \qquad x^2 > a^2.$$

It now follows that

$$-a \leq x \leq a \qquad \Longleftrightarrow \qquad x^2 \leq a^2.$$

We can confirm our conclusion by drawing a graph of the parabola $y = x^2$ and observing that the part of the parabola on or below the horizontal line $y = a^2$ corresponds to $-a \le x \le a$ on the x-axis.



EXAMPLE 6.1.4. Consider the inequality $x^2 - 4x + 3 \le 0$. We can write

$$x^2 - 4x + 3 = x^2 - 4x + 4 - 1 = (x-2)^2 - 1,$$

so that the inequality is equivalent to $(x-2)^2 - 1 \le 0$, which in turn is equivalent to the inequality $(x-2)^2 \le 1$, in view of the Addition rule. Now write $u = x - 2$. Then it follows from Example 6.1.3 that

$$-1 \le u \le 1 \qquad \Longleftrightarrow \qquad u^2 \le 1.$$

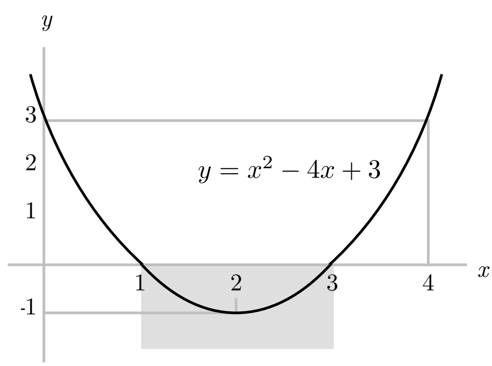
Hence

$$-1 \le x - 2 \le 1 \qquad \Longleftrightarrow \qquad (x-2)^2 \le 1.$$

Using the addition rule on the inequalities on the left hand side, and using our earlier observation, we conclude that

$$1 \le x \le 3 \qquad \Longleftrightarrow \qquad x^2 - 4x + 3 \le 0.$$

We can confirm our conclusion by drawing a graph of the parabola $y = x^2 - 4x + 3$ and observing that the part of the parabola on or below the horizontal line $y = 0$ corresponds to $1 \le x \le 3$ on the x-axis.



EXAMPLE 6.1.5. Consider the inequality

$$\frac{1}{x} < 2.$$

Clearly we cannot have $x = 0$, as $1/0$ is meaningless. We have two cases:

(1) Suppose that $x > 0$. Using one of the Multiplication rules and multiplying both sides by the positive real number x, we obtain the inequality $1 < 2x$. Multiplying both sides by the positive real number $1/2$, we obtain $1/2 < x$. Suppose now that $1/2 < x$. Using one of the Multiplication rules

and multiplying both sides by the positive real number $2/x$, we obtain the original inequality. We have therefore shown that for $x > 0$, we have

$$x > \frac{1}{2} \qquad \Longleftrightarrow \qquad \frac{1}{x} < 2.$$

(2) Try to use one of the Multiplication rules to show that

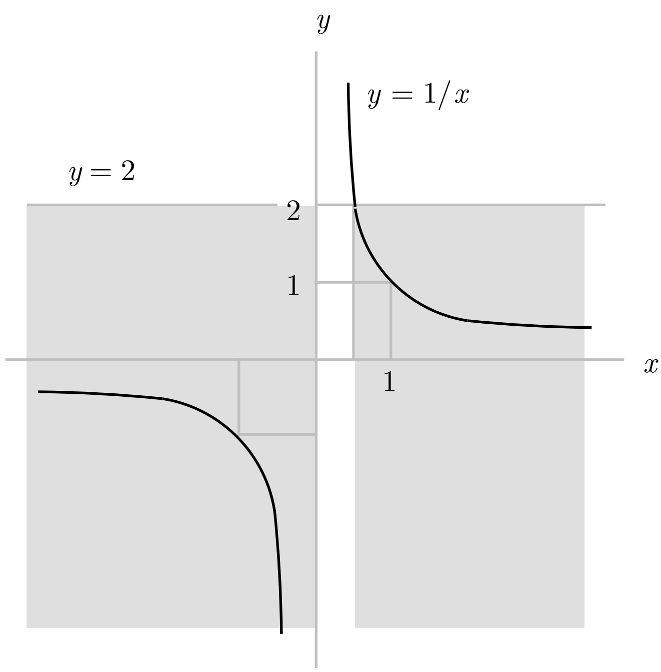
$$x < 0 \qquad \Longleftrightarrow \qquad \frac{1}{x} < 0.$$

The result is obvious, but the proof is slightly tricky.

Combining the two parts, we conclude that

$$x < 0 \quad \text{or} \quad x > \frac{1}{2} \qquad \Longleftrightarrow \qquad \frac{1}{x} < 2.$$

We can confirm our conclusion by drawing a graph of the hyperbola $y = 1/x$ and observing that the part of the hyperbola below the horizontal line $y = 2$ corresponds to $x < 0$ together with $x > 1/2$ on the x-axis.



For the remaining examples in this section, we shall use M to denote an application of one of the Multiplication rules, A to denote an application of the Addition rule and S to denote an application of the Square rule.

Example 6.1.6. Consider the inequality

$$\frac{x+4}{2x} < 3.$$

Clearly we cannot have $x = 0$. We therefore have two cases:

(1) Suppose that $x > 0$. We have

$$\frac{x+4}{2x} < 3 \quad \overset{\text{M}}{\Longleftrightarrow} \quad x+4 < 6x \quad \overset{\text{A}}{\Longleftrightarrow} \quad 4 < 5x \quad \overset{\text{M}}{\Longleftrightarrow} \quad \frac{4}{5} < x.$$

(2) Suppose that $x < 0$. We have

$$\frac{x+4}{2x} < 3 \quad \overset{\text{M}}{\Longleftrightarrow} \quad x+4 > 6x \quad \overset{\text{A}}{\Longleftrightarrow} \quad 4 > 5x \quad \overset{\text{M}}{\Longleftrightarrow} \quad \frac{4}{5} > x.$$

Since the rightmost inequality is always satisfied when $x < 0$, it follows that the leftmost inequality is always satisfied when $x < 0$.

Combining the two parts, we conclude that

$$x < 0 \quad \text{or} \quad x > \frac{4}{5} \quad \Longleftrightarrow \quad \frac{x+4}{2x} < 3.$$

EXAMPLE 6.1.7. Consider the inequality

$$\frac{x^2-3x+4}{x+1} \le 1.$$

Clearly we cannot have $x = -1$. We therefore have two cases:

(1) Suppose that $x > -1$, so that $x+1 > 0$. Then

$$\frac{x^2-3x+4}{x+1} \le 1 \quad \overset{\text{M}}{\Longleftrightarrow} \quad x^2-3x+4 \le x+1 \quad \overset{\text{A}}{\Longleftrightarrow} \quad x^2-4x+3 \le 0.$$

Recall from Example 6.1.4 that

$$1 \le x \le 3 \quad \Longleftrightarrow \quad x^2-4x+3 \le 0.$$

(2) Suppose that $x < -1$, so that $x+1 < 0$. Then

$$\frac{x^2-3x+4}{x+1} \le 1 \quad \overset{\text{M}}{\Longleftrightarrow} \quad x^2-3x+4 \ge x+1 \quad \overset{\text{A}}{\Longleftrightarrow} \quad x^2-4x+3 \ge 0.$$

It can be deduced from Example 6.1.4 that

$$x \le 1 \quad \text{or} \quad x \ge 3 \quad \Longleftrightarrow \quad x^2-4x+3 \ge 0.$$

Combining the two parts, we conclude that

$$x < -1 \quad \text{or} \quad 1 \le x \le 3 \quad \Longleftrightarrow \quad \frac{x^2-3x+4}{x+1} \le 1.$$

EXAMPLE 6.1.8. Suppose that $a, b \in \mathbb{R}$ are non-negative. We shall prove that

$$\frac{a+b}{2} \ge \sqrt{ab}.$$

Suppose on the contrary that this is not true. Then

$$\begin{aligned} \frac{a+b}{2} < \sqrt{ab} \quad &\overset{\text{S}}{\Longrightarrow} \quad \frac{(a+b)^2}{4} < ab \quad \overset{\text{M}}{\Longrightarrow} \quad (a+b)^2 < 4ab \\ &\overset{\text{A}}{\Longrightarrow} \quad (a-b)^2 = (a+b)^2 - 4ab < 0. \end{aligned}$$

But the last inequality is absurd.

EXAMPLE 6.1.9. Suppose that $x \in \mathbb{R}$ and $x > 0$. We shall prove that $x + x^{-1} \ge 2$. Suppose on the contrary that this is not true. Then

$$x + x^{-1} < 2 \quad \overset{\text{M}}{\Longrightarrow} \quad x^2+1 < 2x \quad \overset{\text{A}}{\Longrightarrow} \quad (x-1)^2 = x^2-2x+1 < 0.$$

But the last inequality is absurd.

EXAMPLE 6.1.10. Suppose that $x \in \mathbb{R}$. We shall prove that

$$x^2 - 4x + 3 \geq -1.$$

Suppose on the contrary that this is not true. Then

$$x^2 - 4x + 3 < -1 \quad \overset{\text{A}}{\Longrightarrow} \quad (x-2)^2 = x^2 - 4x + 4 < 0.$$

But the last inequality is absurd.

6.2. Absolute Values

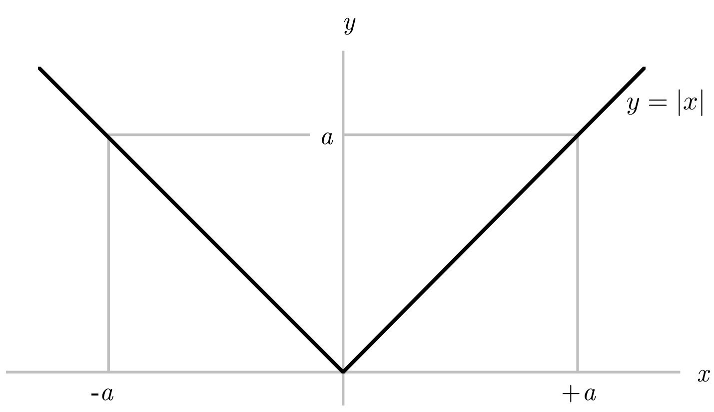
DEFINITION. For every $a \in \mathbb{R}$, the absolute value $|a|$ of a is a non-negative real number satisfying

$$|a| = \begin{cases} a & \text{if } a \geq 0; \\ -a & \text{if } a < 0. \end{cases}$$

REMARK. If we place the number a on the real number line, then the absolute value $|a|$ represents the distance of a from the origin 0.

PROPERTIES OF ABSOLUTE VALUES. *For every $a, b \in \mathbb{R}$, we have*
(a) $|a| \geq 0$;
(b) $|a| \geq a$;
(c) $|a|^2 = a^2$;
(d) $|ab| = |a||b|$; *and*
(e) $|a+b| \leq |a| + |b|$.

The graph of the function $y = |x|$ is given below, where $a \in \mathbb{R}$ is a non-negative real number.



The following is easily seen from the graph.

RULE CONCERNING REMOVAL OF ABSOLUTE VALUES. *Suppose that $a \in \mathbb{R}$ is non-negative. Then*

$$|x| \leq a \quad \textit{if and only if} \quad -a \leq x \leq +a,$$

and

$$|x| < a \quad \textit{if and only if} \quad -a < x < +a.$$

EXAMPLE 6.2.1. The equation $|x| = 4$ has two solutions $x = \pm 4$.

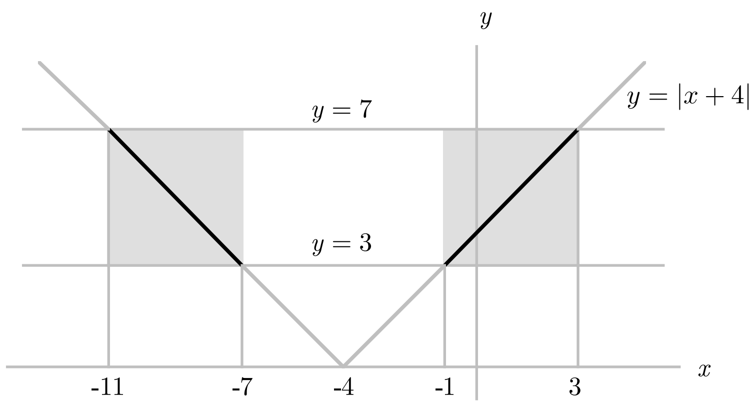
EXAMPLE 6.2.2. The equation $|2x + 1| = 5$ has two solutions, one satisfying $2x + 1 = 5$ and the other satisfying $2x + 1 = -5$. Hence $x = 2$ or $x = -3$.

EXAMPLE 6.2.3. The inequality $|x| < 5$ is satisfied precisely when $-5 < x < 5$.

EXAMPLE 6.2.4. The inequality $|2x + 1| \leq 9$ is satisfied precisely when $-9 \leq 2x + 1 \leq 9$; in other words, when $-5 \leq x \leq 4$.

EXAMPLE 6.2.5. The equation $\sqrt{x^2 + 4x + 13} = x - 1$ is satisfied only if the right hand side is non-negative, so that we must have $x \geq 1$. Squaring both sides, we have $x^2+4x+13 = (x-1)^2 = x^2-2x+1$, so that $6x + 12 = 0$, giving $x = -2$. Hence the equation has no real solution x.

EXAMPLE 6.2.6. Consider the inequalities $3 < |x+4| < 7$. Note first of all that the inequality $|x+4| < 7$ holds precisely when $-7 < x + 4 < 7$; in other words, when $-11 < x < 3$. On the other hand, the inequality $|x+4| \leq 3$ holds precisely when $-3 \leq x+4 \leq 3$; in other words, precisely when $-7 \leq x \leq -1$. Hence the inequality $|x + 4| > 3$ holds precisely when $x < -7$ or $x > -1$. It follows that the original inequalities hold precisely when $-11 < x < -7$ or $-1 < x < 3$. We can confirm our conclusion by drawing a graph of the function $y = |x + 4|$ and observing that the part of the graph between the horizontal lines $y = 3$ and $y = 7$ corresponds to $-11 < x < -7$ together with $-1 < x < 3$ on the x-axis.



PROBLEMS FOR CHAPTER 6

1. Suppose that α and β are two positive real numbers. The number $\frac{1}{2}(\alpha+\beta)$ is called the arithmetic mean of α and β, while the number $\sqrt{\alpha\beta}$ is called the geometric mean of α and β.
   a) Prove that $\sqrt{\alpha\beta} \leq \frac{1}{2}(\alpha + \beta)$; in other words, the geometric mean never exceeds the arithmetic mean.
   b) Show that equality holds in part (a) precisely when $\alpha = \beta$.

   [REMARK: The famous arithmetic-mean-geometric-mean theorem states that for any k positive integers $\alpha_1, \ldots, \alpha_k$, we have $\sqrt[k]{\alpha_1 \ldots \alpha_k} \leq \frac{1}{k}(\alpha_1 + \ldots + \alpha_k)$, and that equality holds precisely when $\alpha_1 = \ldots = \alpha_k$. The proof is rather complicated when $k > 2$, and is beyond the scope of our present discussion.]

2. For each of the following inequalities, find all real values of x satisfying the inequality:
   a) $2x + 4 < 6$ b) $5 - 3x > 11$ c) $7x + 9 > -5$ d) $4x + 4 < 28$
   e) $2x + 5 < 3$ f) $4 - 6x \geq 10$

3. Determine all real values of x for which the inequalities $5 < 2x + 7 \leq 13$ hold.

4. For each of the following inequalities, determine all real values of x for which the inequality holds:
   a) $6 + x - x^2 \geq 0$ b) $x^2 - 1 > 0$ c) $x^2 - 4 \leq 0$ d) $2 - x - x^2 \geq 0$
   e) $x^2 + 2x + 1 \leq 0$ f) $(2x + 3)^2 \leq 4$ g) $(3x - 1)^2 > 9$

5. For each of the following inequalities, determine all real values of x for which the inequality holds, taking care to distinguish the two cases $x > 0$ and $x < 0$, and explain each step of your argument by quoting the relevant rules concerning inequalities:
   a) $\dfrac{x+4}{2x} < 3$ b) $\dfrac{1}{x} < 3$ c) $-2 < \dfrac{1}{x} < 3$

6. For each of the following inequalities, determine all real values of x for which the inequality holds, taking care to distinguish two cases, and explain each step of your argument by quoting the relevant rules concerning inequalities:
   a) $\dfrac{2x+3}{3x+1} < 1$ b) $\dfrac{4x-2}{x+4} \geq 2$ c) $2 \leq \dfrac{4x-2}{x+4} < 3$

7. Show that $\dfrac{x^2 + 4x - 7}{x+4} \geq 2$ precisely when $-5 \leq x < -4$ or $x \geq 3$.

8. For each of the following inequalities, determine all real values of x for which the inequality holds:
   a) $\dfrac{x^2 - 5x + 3}{x - 2} \leq 1$ b) $\dfrac{x^2 - 4x + 1}{x - 3} \leq 1$

9. Find all solutions of the inequality $|x + 2| < 6$, and confirm your answer by drawing a suitable picture.

10. For each of the following inequalities, determine all real values of x for which the inequality holds:
   a) $1 < |3x - 5| \leq 7$ b) $1 \leq |(x - 1)^3| \leq 8$

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Chapter 7

PROGRESSIONS

7.1. Arithmetic Progressions

EXAMPLE 7.1.1. Consider the finite sequence of numbers

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31.$$

This sequence has the property that the difference between successive terms is constant and equal to 2. It follows that the k-th term is obtained from the first term by adding $(k-1) \times 2$, and is therefore equal to $1 + 2(k-1)$. On the other hand, if we want to add all the numbers together, then we observe that

$$1 + 31 = 3 + 29 = 5 + 27 = 7 + 25 = 9 + 23 = 11 + 21 = 13 + 19 = 15 + 17,$$

so that the numbers can be paired off in such a way that the sum of the pair is always the same and equal to 32. Note now that there are 16 numbers which form 8 pairs. It follows that

$$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 + 23 + 25 + 27 + 29 + 31 = 8 \times 32 = 256.$$

EXAMPLE 7.1.2. Consider the finite sequence of numbers

$$2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32.$$

This sequence has the property that the difference between successive terms is constant and equal to 3. If we want to add all the numbers together, then we observe that

$$2 + 32 = 5 + 29 = 8 + 26 = 11 + 23 = 14 + 20 = 2 \times 17,$$

† This chapter was written at Macquarie University in 1999.

so that the numbers other than the middle one can be paired off in such a way that the sum of the pair is always the same and equal to 34. Note now that there are 11 numbers which form 5 pairs, as well as the number 17 which is equal to half the sum of a pair. We can therefore pretend that there are $5\frac{1}{2}$ pairs, each adding to 34. It follows that

$$2+5+8+11+14+17+20+23+26+29+32 = \frac{11}{2} \times 34 = 187.$$

DEFINITION. By an arithmetic progression of m terms, we mean a finite sequence of the form

$$a, a+d, a+2d, a+3d, \ldots, a+(m-1)d. \tag{1}$$

The real number a is called the first term of the arithmetic progression, and the real number d is called the difference of the arithmetic progression. The term $a+(k-1)d$ is called the k-th term of the arithmetic progression.

SUM OF AN ARITHMETIC PROGRESSION. *The sum of the m terms of an arithmetic progression of the type* (1) *is equal to*

$$\frac{m}{2} \times (2a + (m-1)d).$$

REMARK. Note that the sum of an arithmetic progression is equal to

$$\frac{\text{number of terms}}{2} \times (\text{first term} + \text{last term}).$$

EXAMPLE 7.1.3. Suppose that the 4-th and 7-th terms of an arithmetic progression are equal to 9 and -15 respectively. Then we have

$$\begin{aligned} 9 &= a + 3d, \\ -15 &= a + 6d, \end{aligned}$$

so that $3d = -24$. It follows that $d = -8$ and $a = 33$. The arithmetic progression is given by

$$33, 25, 17, 9, 1, -7, -15, \ldots.$$

The 10-th term is given by $a + 9d = 33 - 72 = -39$. The sum of the first 10 terms is equal to

$$\frac{10}{2} \times (33 - 39) = -30.$$

EXAMPLE 7.1.4. We have

$$1+3+5+\ldots+(2n-1) = \frac{n}{2} \times (1+2n-1) = n^2$$

and

$$2+4+6+\ldots+2n = \frac{n}{2} \times (2+2n) = n+n^2.$$

Note also that

$$1+2+3+\ldots+2n = \frac{2n}{2} \times (1+2n) = n+2n^2.$$

7.2. Geometric Progressions

EXAMPLE 7.2.1. Consider the finite sequence of numbers

$$4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192.$$

This sequence has the property that the ratio between successive terms is constant and equal to 2. It follows that the k-th term is obtained from the first term by multiplying 2^{k-1}, and is therefore equal to $4 \times 2^{k-1}$.

DEFINITION. By a geometric progression of m terms, we mean a finite sequence of the form

$$a, ar, ar^2, ar^3, \ldots, ar^{m-1}. \tag{2}$$

The real number a is called the first term of the geometric progression, and the real number r is called the ratio of the geometric progression. The term ar^{k-1} is called the k-th term of the geometric progression.

Suppose now that we wish to add the numbers in (2). Write

$$S = a + ar + ar^2 + ar^3 + \ldots + ar^{m-1}.$$

Then

$$rS = \quad ar + ar^2 + ar^3 + \ldots + ar^{m-1} + ar^m.$$

It follows that

$$S - rS = a \qquad\qquad\qquad\qquad - ar^m.$$

Hence

$$S = \frac{a - ar^m}{1 - r},$$

provided that $r \neq 1$. On the other hand, if $r = 1$, then $S = am$.

We have proved the following result.

SUM OF A GEOMETRIC PROGRESSION. *The sum of the m terms of a geometric progression of the type* (2) *is equal to am if $r = 1$, and equal to*

$$\frac{a - ar^m}{1 - r}$$

if $r \neq 1$.

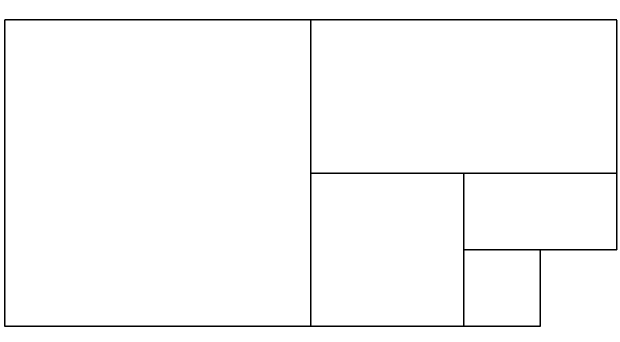
EXAMPLE 7.2.2. Consider the geometric sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots.$$

The sum of the first m terms is equal to

$$\frac{1 - 2^{-m}}{1 - 2^{-1}} = 2 - \frac{1}{2^{m-1}},$$

very close to 2 when m is very large. We can explain this geometrically by the picture below:



Suppose that the square on the left has area 1. Then the rectangle on the top right has area 1/2. The square the next size down has area 1/4. The rectangle the next size down has area 1/8. The square the next size down has area 1/16. The next term 1/32 will fill half of the missing square on the bottom right. The term 1/64 will fill half of what is still missing on the bottom right. If m is very large, then we account for nearly all of the missing piece and so get a big rectangle of area 2.

Problems for Chapter 7

1. Find each of the following sums without using your calculators, taking care to explain each step of your argument:
   a) $1+3+5+7+\ldots+999$ (sum of arithmetic progression).
   b) $2+4+8+16+\ldots+2^n$ (sum of geometric progression), where $n \in \mathbb{N}$.

2. Using the idea of arithmetic progressions and geometric progressions, without the help of calculators to find the values of the individual terms or to add them together, find the sum

$$5+12+21+\ldots+1048674$$

   of 20 terms, where the k-th term of the sum is given by $2^k+3+5(k-1)$.
   [Hint: Consider the sum of the terms 2^k separately from the sum of the terms $3+5(k-1)$.]

3. Using the idea of arithmetic progressions and geometric progressions, without the help of calculators to find the values of the individual terms or to add them together, find the sum

$$3+10+25+\ldots+39394$$

   of 10 terms, where the k-th term of the sum is given by $2\times 3^{k-1}+1+3(k-1)$.

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Chapter 8

ELEMENTARY COUNTING TECHNIQUES

8.1. The Fundemental Principle of Counting

We begin by studying two very simple examples.

EXAMPLE 8.1.1. Consider the collection of all 2-digit numbers where the first digit is either 1 or 2, and where the second digit is either 6, 7 or 8. Clearly there are 6 such numbers and they are listed below:

16	17	18
26	27	28

Arranged this way, we note that each row corresponds to a choice for the first digit and each column corresponds to a choice for the second digit. We have 2 rows and 3 columns, and hence $2 \times 3 = 6$ possibilities.

EXAMPLE 8.1.2. Consider the collection of all 3-digit numbers where the first digit is either 1, 2, 3 or 4, where the second digit is either 5 or 6, and where the third digit is either 7, 8 or 9. The candidates are listed below:

157	158	159	257	258	259	357	358	359	457	458	459
167	168	169	267	268	269	367	368	369	467	468	469

Arranged this way, we note that each block corresponds to a choice for the first digit. Within each block, each row corresponds to a choice for the second digit and each column corresponds to a choice for the third digit. We have 4 blocks, each with 2 rows and 3 columns, and hence $4 \times 2 \times 3 = 24$ possibilities.

These two examples are instances of a simple but very useful principle.

† This chapter was written at Macquarie University in 1999.

FUNDAMENTAL PRINCIPLE OF COUNTING. *Suppose that a first event can occur in n_1 different ways, a second event can occur in n_2 different ways, and so on, and a k-th event can occur in n_k different ways. Then the number of different ways for these k events to occur in succession is given by $n_1 \times n_2 \times \ldots \times n_k$.*

EXAMPLE 8.1.3. Consider motor vehicle licence plates made up of 3 letters followed by 3 digits, such as *ABC012*. To determine the total number of possible different licence plates, note that there are 26 choices for each letter and 10 choices for each digit. Hence the total number is $26 \times 26 \times 26 \times 10 \times 10 \times 10$. On the other hand, if the first digit is restricted to be non-zero, then the total number is only $26 \times 26 \times 26 \times 9 \times 10 \times 10$. Furthermore, if the letters are required to be distinct and the first digit is restricted to be non-zero, then the total number is only $26 \times 25 \times 24 \times 9 \times 10 \times 10$. Finally, if the letters and digits are required to be distinct and the first digit is restricted to be non-zero, then the total number is only $26 \times 25 \times 24 \times 9 \times 9 \times 8$.

8.2. Permutation

Again, we begin by studying a simple example.

EXAMPLE 8.2.1. Suppose that we are required to choose 6 people out of 10 and seat them in a row from the left to the right. To do this, let us take a sensible approach and choose them one at a time. We can fill the leftmost seat by any one of the 10 people, so we clearly have 10 choices here. Having made a choice, we can fill the second seat by any one of the 9 remaining people, so we clearly have 9 choices here. Having made a choice, we can fill the third seat by any one of the 8 remaining people, so we clearly have 8 choices here, and so on. Clearly the total number of ways is equal to

$$P(10,6) = \underbrace{10 \times 9 \times 8 \times 7 \times 6 \times 5}_{6}.$$

PERMUTATION. *The number of ways of choosing k objects from n objects and arranging them in order is equal to*

$$P(n,k) = \underbrace{n \times (n-1) \times \ldots \times (n-k+1)}_{k} = \frac{n!}{(n-k)!}.$$

Here, for every positive integer m, the number $m!$ denotes the product of the first m positive integers. We shall also use the convention that $0! = 1$.

EXAMPLE 8.2.2. We wish to determine the number of 5-digit numbers where all the digits are non-zero and where at least one digit is used more than once. To do so, we formulate the following strategy. Our desired number is equal to $N_1 - N_2$, where N_1 denotes the total number of 5-digit numbers where all the digits are non-zero, and where N_2 denotes the total number of 5-digit numbers where all the digits are non-zero and where no digit is used more than once. Then

$$N_1 = 9 \times 9 \times 9 \times 9 \times 9 = 59049 \qquad \text{and} \qquad N_2 = P(9,5) = 9 \times 8 \times 7 \times 6 \times 5 = 15120,$$

so that $N_1 - N_2 = 43929$.

EXAMPLE 8.2.3. We wish to determine the number of ways of arranging 6 people in a circle. To do this, we put 6 chairs in a circle and ask person A to choose a chair. Clearly it does not matter which chair he chooses. It is only a question of which chairs the others choose relative to him. So he really has only one choice. Now person B has 5 chairs to choose from, and person C has 4 chairs to choose from, and so on, while person F takes whatever is left. The total number of ways is therefore equal to $P(5,5) = 120$. Alternatively, we nominate one of these chairs and call it the first chair. Then we fill the chairs in clockwise order. We can put any one of the 6 people in the first chair, any one of the remaining 5 in the second chair, and so on. Hence there are $P(6,6)$ ways of achieving these. Now note that the following two arrangements are the same:

$$\begin{array}{ccccc} & A & B & & \\ F & & & C \\ & E & D & & \end{array} \qquad\qquad \begin{array}{ccccc} & F & A & & \\ E & & & B \\ & D & C & & \end{array}$$

Indeed, there are 4 more arrangements which are the same as these two. Hence we have overcounted by a multiple of 6, and so the desired number is

$$\frac{1}{6} \times P(6,6) = 120$$

as before.

EXAMPLE 8.2.4. We wish to determine the number of ways of arranging the letters of

$$WOOLLOOMOOLOO.$$

Here there are 8 O's, 3 L's and one each of W and M, making a total of 13 letters. First of all, let us label the O's and L's to make them different, so that

$$MO_1O_2L_aL_bO_3O_4WO_5O_6L_cO_7O_8 \qquad \text{and} \qquad MO_2O_1L_aL_bO_3O_4WO_5O_6L_cO_7O_8$$

are considered to be different arrangements. In this case, there are clearly $P(13,13) = 13!$ arrangements. On the other hand, the 8 different O's are having a private contest among themselves to see who is used first, second, and so on, and they have $P(8,8) = 8!$ ways of resolving this. Similarly, the 3 different L's have $P(3,3) = 3!$ ways of resolving their own little dispute. Unlabelling the O's and L's, we see that we have overcounted, and the correct number is really only

$$\frac{13!}{8!3!} = 25740.$$

EXAMPLE 8.2.5. We shall make 5-digit numbers from the digits $1, 2, 3, 4, 5, 6, 7, 8$, where no digit is used more than once.

- If there are no further restrictions, then clearly there are $P(8,5)$ ways.
- Suppose that the 5-digit number must begin with the digit 1. Then there are clearly $P(7,4)$ ways of choosing the remaining 4 digits.
- Suppose that the 5-digit number must contain the digit 1. Then there are $P(5,1)$ choices for placing the digit 1. Having made this choice, there are $P(7,4)$ ways of choosing the remaining 4 digits. Hence the total number of ways is equal to $P(5,1) \times P(7,4)$.
- Suppose that the 5-digit number must begin with the digit 1 and contain the digit 2. Then there are $P(4,1)$ choices for placing the digit 2. Having made this choice, there are $P(6,3)$ ways of choosing the remaining 3 digits. Hence the total number of ways is equal to $P(4,1) \times P(6,3)$.
- Suppose that the 5-digit number must contain the digits 1 and 2. Then there are $P(5,2)$ choices for placing the digits 1 and 2. Having made this choice, there are $P(6,3)$ ways of choosing the remaining 3 digits. Hence the total number of ways is equal to $P(5,2) \times P(6,3)$.

8.3. Combination

Yet again, we begin by studying a simple example.

EXAMPLE 8.3.1. Suppose that we are required to choose 6 people out of 10 but not in any particular order. To do this, let us first return to Example 8.2.1 where 6 people are chosen in order and seated from left to right. In this case, there are $P(10,6)$ choices. However, if the order is no longer important, the 6 chosen people can resolve their differences, and the number of duplications which end up with the same 6 people is the number of ways of arranging 6 people in order, equal to $P(6,6)$. It follows that the number of ways of choosing 6 people out of 10 but not in any particular order is equal to

$$\frac{P(10,6)}{P(6,6)}.$$

COMBINATION. *The number of ways of choosing k objects from n objects not in any particular order is equal to*

$$C(n,k) = \binom{n}{k} = \frac{P(n,k)}{P(k,k)} = \frac{n!}{k!(n-k)!}.$$

EXAMPLE 8.3.2. Consider the set $S = \{a,b,c,d,e,f,g,h,i\}$ of 9 letters. Suppose that we wish to choose 4 letters from S.

- If there are no restrictions, then there are clearly $C(9,4)$ ways.
- Suppose that we must choose 2 vowels and 2 consonants. Then there are $C(3,2)$ ways of choosing the vowels and $C(6,2)$ ways of choosing the consonants. Hence the total number of ways is equal to $C(3,2) \times C(6,2)$.
- Suppose that we must choose at least one vowel. Then we note that there are $C(6,4)$ ways of choosing all consonants. Hence the desired number of ways is equal to $C(9,4) - C(6,4)$.
- Suppose that we must choose more consonants than vowels. Then we can either choose no vowels or one vowel. If we choose no vowels, then there are $C(6,4)$ possibilities. If we choose one vowel, then there are $C(3,1) \times C(6,3)$ possibilities. Hence the desired number of ways is equal to

$$C(6,4) + C(3,1) \times C(6,3).$$

EXAMPLE 8.3.3. Some crazy professor has set an examination paper which contains 5 history questions and 6 engineering questions. An unfortunate candidate is required to choose exactly 8 questions.

- If there are no further restrictions, then the number of ways is $C(11,8)$.
- Suppose that the unfortunate candidate is required to choose exactly 4 history questions and 4 engineering questions. Then there are $C(5,4)$ ways of choosing the history questions and $C(6,4)$ ways of choosing the engineering questions. Hence the total number of ways is equal to

$$C(5,4) \times C(6,4).$$

- Suppose that the unfortunate candidate is required to choose at least 4 history questions. Then apart from the $C(5,4) \times C(6,4)$ ways of choosing exactly 4 history questions and 4 engineering questions, there are also $C(5,5) \times C(6,3)$ ways of choosing exactly 5 history questions and 3 engineering questions. Hence the total number of ways is equal to

$$C(5,4) \times C(6,4) + C(5,5) \times C(6,3).$$

- Suppose that the unfortunate candidate is required to choose at most 4 history questions. Then apart from the $C(5,4) \times C(6,4)$ ways of choosing exactly 4 history questions and 4 engineering questions, there are also $C(5,3) \times C(6,5)$ ways of choosing exactly 3 history questions and 5 engineering questions, as well as $C(5,2) \times C(6,6)$ ways of choosing exactly 2 history questions and 6 engineering questions. Hence the total number of ways is equal to

$$C(5,4) \times C(6,4) + C(5,3) \times C(6,5) + C(5,2) \times C(6,6).$$

Alternatively, the only way of choosing 8 questions and falling foul of the crazy professor is to choose all 5 history questions and 3 engineering questions. There are $C(5,5) \times C(6,3)$ ways of doing so. Hence the total number of ways is also equal to $C(11,8) - C(5,5) \times C(6,3)$.

EXAMPLE 8.3.4. Consider a usual deck of 52 playing cards, without jokers. We wish to choose 5 cards from this deck.

- Without any restrictions, the number of ways is equal to $C(52,5)$.
- Suppose that we wish to choose exactly two A's and three K's. Then the total number of ways is equal to $C(4,2) \times C(4,3)$.
- Suppose that we wish to have four cards of the same suit and a fifth card from another suit. There are $C(4,1)$ ways of choosing a suit, and then $C(13,4)$ ways of choosing four cards from it. There are then $C(39,1)$ ways of choosing the fifth card. Hence the total number of ways is equal to $C(4,1) \times C(13,4) \times C(39,1)$.
- Suppose that we wish to choose four cards with the same value and another card. There are $C(13,1)$ ways of choosing a value and then $C(4,4)$ ways of choosing the four cards with this value. There are then $C(48,1)$ ways of choosing the remaining card. Hence the total number of ways is equal to $C(13,1) \times C(4,4) \times C(48,1)$.
- Suppose that the five cards we choose must have total value equal to 7. Here let us assume that A's carry the value 1. Then the only possibilities are four A's and one 3, or three A's and two 2's. The number of choices for the former is $C(4,4) \times C(4,1)$, while the number of choices for the latter is $C(4,3) \times C(4,2)$. Hence the total number is equal to $C(4,4) \times C(4,1) + C(4,3) \times C(4,2)$.

8.4. The Binomial Theorem

Consider the array of numbers given below:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
\cdot^{\cdot^{\cdot}} & & & & & & & & \ddots
\end{array}
$$

This is known as Pascal's triangle. It has the property that every entry is equal to the sum of the two immediate entries on the row above. In fact, Pascal's triangle can be written in the following form:

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\
\cdot^{\cdot^{\cdot}} & & & & & & & & \ddots
\end{array}
$$

In this case, the rule concerning an entry being equal to the sum of the two immediate entries on the row above translates to the observation that

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

which can easily be established from the definition. Indeed, we have

$$\binom{n}{k-1}+\binom{n}{k}=\frac{n!}{(k-1)!(n-k+1)!}+\frac{n!}{k!(n-k)!}=\frac{n!}{(k-1)!(n-k)!}\left(\frac{1}{n-k+1}+\frac{1}{k}\right)$$
$$=\frac{n!}{(k-1)!(n-k)!}\times\frac{n+1}{(n-k+1)k}=\frac{(n+1)!}{k!(n+1-k)!}=\binom{n+1}{k}.$$

An important deduction from this observation is the following result.

BINOMIAL THEOREM. *Suppose that $n \in \mathbb{N}$. Then for any real numbers $a, b \in \mathbb{R}$, we have*

$$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k.$$

REMARKS. (1) Note the special case $n = 2$, where we obtain

$$(a+b)^2=a^2+2ab+b^2.$$

(2) Note also the special case when $a = 1$ and $b = x$. We have

$$(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k.$$

In other words, the binomial coefficient

$$\binom{n}{k}$$

is equal to the coefficient of x^k in the expansion of $(1 + x)^n$.

EXAMPLE 8.4.1. Let $a = b = 1$. Then the binomial theorem gives

$$2^n=\sum_{k=0}^{n}\binom{n}{k}.$$

In other words, the entries in the n-th row of Pascal's triangle have a sum equal to 2^n. Here, we use for convenience the convention that the top entry in Pascal's triangle represents the 0-th row.

EXAMPLE 8.4.2. Suppose that in the expansion of $(1+x)^n$, the coefficient of x^2 is equal to 28. Clearly the term corresponding to x^2 is equal to

$$\binom{n}{2}x^2,$$

so that

$$\frac{n!}{2!(n-2)!}=\frac{n(n-1)}{2}=28.$$

Hence $n = 8$. In fact, the first four terms of the expansion of $(1 + x)^8$ sum to

$$\binom{8}{0}+\binom{8}{1}x+\binom{8}{2}x^2+\binom{8}{3}x^3=1+8x+28x^2+56x^3.$$

EXAMPLE 8.4.3. We have

$$\left(3x^2-\frac{1}{2x}\right)^9=\sum_{k=0}^{9}\binom{9}{k}(3x^2)^{9-k}\left(-\frac{1}{2x}\right)^k=\sum_{k=0}^{9}\binom{9}{k}(-1)^k\frac{3^{9-k}}{2^k}x^{18-3k}.$$

It follows that the term independent of x in this expansion is given by $k = 6$. This is equal to

$$\binom{9}{6}(-1)^6 \frac{3^3}{2^6} = \frac{567}{16}.$$

EXAMPLE 8.4.4. We have

$$\left(x + \frac{1}{x^2}\right)^{12} = \sum_{k=0}^{12} \binom{12}{k} x^{12-k} \left(\frac{1}{x^2}\right)^k = \sum_{k=0}^{12} \binom{12}{k} x^{12-3k}.$$

To obtain the coefficient of x^3, we take $k = 3$. This is given by $\binom{12}{3} = 220$. To obtain the coefficient of x^{-9}, we take $k = 7$. This is given by $\binom{12}{7} = 792$. To obtain the coefficient of the constant term, we take $k = 4$. This is given by $\binom{12}{4} = 495$.

8.5. Application to Probability Theory

We begin by studying a very simple example.

EXAMPLE 8.5.1. Suppose that two coins are tossed, and that each is either heads (h) or tails (t) equally likely. Then the following four outcomes are equally likely:

$$\begin{array}{cc} (h,h) & (h,t) \\ (t,h) & (t,t) \end{array}$$

- The probability of a pair of heads is $1/4$, representing one of the four possible outcomes.
- The probability of both coins landing the same way is $1/2$, representing two of the four possible outcomes.
- The probability of getting at least one head is $3/4$, representing three of the four possible outcomes.

In general, if there are several equally likely, mutually exclusive and collectively exhaustive outcomes of an experiment, then the probability of an event $\mathcal{E}$ is given by

$$p(\mathcal{E}) = \frac{\text{number of outcomes favourable to } \mathcal{E}}{\text{total number of outcomes}}.$$

EXAMPLE 8.5.2. Suppose that two dice are thrown. What is the probability that the sum of the two numbers is equal to 7? To answer this question, let us look at all the possible and equally likely outcomes:

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \underline{(1,6)} \\
(2,1) & (2,2) & (2,3) & (2,4) & \underline{(2,5)} & (2,6) \\
(3,1) & (3,2) & (3,3) & \underline{(3,4)} & (3,5) & (3,6) \\
(4,1) & (4,2) & \underline{(4,3)} & (4,4) & (4,5) & (4,6) \\
(5,1) & \underline{(5,2)} & (5,3) & (5,4) & (5,5) & (5,6) \\
\underline{(6,1)} & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)
\end{array}$$

The cases where the sum of the two numbers shown is equal to 7 has been underlined. These represent 6 out of 36 equally likely outcomes. It follows that the probability is equal to $6/36 = 1/6$.

EXAMPLE 8.5.3. Let us return to Example 8.3.4 concerning drawing 5 cards out of a deck of 52 playing cards.

- The probability of drawing exactly two A's and three K's is equal to

$$\frac{C(4,2) \times C(4,3)}{C(52,5)} = \frac{1}{108290}.$$

- The probability of drawing 5 cards with a total value of 7 is equal to

$$\frac{C(4,4) \times C(4,1) + C(4,3) \times C(4,2)}{C(52,5)} = \frac{1}{92820}.$$

In probability theory, we often have to deal with multiple events. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two events.

- By the event $\mathcal{A} \cap \mathcal{B}$, we mean the event that both $\mathcal{A}$ and $\mathcal{B}$ occur.
- By the event $\mathcal{A} + \mathcal{B}$, we mean the event that at least one of $\mathcal{A}$ or $\mathcal{B}$ occurs.

These are related to events $\mathcal{A}$ and $\mathcal{B}$ in probability as follows.

PROBABILITY OF DOUBLE EVENTS. *For any two events $\mathcal{A}$ and $\mathcal{B}$, we have*

$$p(\mathcal{A} + \mathcal{B}) = p(\mathcal{A}) + p(\mathcal{B}) - p(\mathcal{A} \cap \mathcal{B}).$$

An important result in probability theory concerns independent events. Roughly speaking, two events $\mathcal{A}$ and $\mathcal{B}$ are independent if each does not affect the outcome of the other.

PROBABILITY OF INDEPENDENT EVENTS. *For any two independent events $\mathcal{A}$ and $\mathcal{B}$, we have*

$$p(\mathcal{A} \cap \mathcal{B}) = p(\mathcal{A}) \times p(\mathcal{B}).$$

EXAMPLE 8.5.4. Suppose that 6 people out of 10 are to be chosen not in any particular order, and that A and B are two of these 10 people. Assume that each of the 10 people is equally likely to be chosen. Let $\mathcal{A}$ denote the event that A is chosen, and let $\mathcal{B}$ denote the event that B is chosen.

- What is the probability $p(\mathcal{A})$ that A is chosen? To answer this question, let us consider the number of ways of choosing 6 people out of 10 to include A. Clearly we choose A and then choose 5 from the remaining 9 people. Hence the number of ways of choosing 6 people out of 10 to include A is equal to $C(9,5)$. On the other hand, the number of ways of choosing 6 people out of 10 is equal to $C(10,6)$. It follows that the probability that A is chosen is also equal to

$$p(\mathcal{A}) = \frac{C(9,5)}{C(10,6)} = \frac{3}{5}.$$

- Similarly, the probability that B is chosen is also equal to

$$p(\mathcal{B}) = \frac{C(9,5)}{C(10,6)} = \frac{3}{5}.$$

- What is the probability $p(\mathcal{A} \cap \mathcal{B})$ that both A and B are chosen? To answer this question, let us consider the number of ways of choosing 6 people out of 10 to include both A and B. Clearly we choose both A and B and then choose 4 from the remaining 8 people. Hence the number of ways of choosing 6 people out of 10 to include both A and B is equal to $C(8,4)$. On the other hand, the number of ways of choosing 6 people out of 10 is equal to $C(10,6)$. It follows that the probability that both A and B are chosen is equal to

$$p(\mathcal{A} \cap \mathcal{B}) = \frac{C(8,4)}{C(10,6)} = \frac{1}{3}.$$

- What is the probability $p(\mathcal{A}+\mathcal{B})$ that at least one of A or B is chosen? Let us consider instead the complementary event that neither A nor B is chosen. To do so, we simply choose all 6 people from the remaining 8, and the number of ways of doing this is equal to $C(8,6)$. It follows that the probability that neither A nor B is chosen is equal to

$$\frac{C(8,6)}{C(10,6)} = \frac{2}{15}.$$

  Hence the probability that at least one of A or B is chosen is equal to

$$p(\mathcal{A}+\mathcal{B}) = \frac{13}{15}.$$

- Note that $p(\mathcal{A}+\mathcal{B}) = p(\mathcal{A}) + p(\mathcal{B}) - p(\mathcal{A}\cap\mathcal{B})$.
- Note that $p(\mathcal{A}\cap\mathcal{B}) \neq p(\mathcal{A}) \times p(\mathcal{B})$, so that the events $\mathcal{A}$ and $\mathcal{B}$ are not independent. It is clear that A not being chosen enhances the chances of B being chosen.

Problems for Chapter 8

1. Consider the collection $\mathcal{S}$ of all 7-digit numbers where each digit is non-zero.
   a) How many numbers are there in the collection $\mathcal{S}$?
   b) How many numbers in $\mathcal{S}$ have distinct digits?
   c) How many numbers in $\mathcal{S}$ have 1 as its first digit?
   d) How many numbers in $\mathcal{S}$ have distinct digits as well as 2 as its first digit and 4 as its last digit?

2. Consider the collection $\mathcal{S}$ of all 5-digit numbers where each digit is odd.
   a) How many numbers are there in the collection $\mathcal{S}$?
   b) How many numbers in $\mathcal{S}$ have distinct digits?
   c) How many numbers in $\mathcal{S}$ have 1 as its first digit?
   d) How many numbers in $\mathcal{S}$ have distinct digits as well as 1 as its first digit and 3 as its last digit?

3. Determine the number of ways of arranging the letters of *MACQUARIEUNIVERSITYSYDNEY* and leave your answer in terms of factorials.

4. Determine the number of ways of arranging the letters of *SYDNEYOLYMPICGAMES* and leave your answer in terms of factorials.

5. In how many ways can one choose 12 people from 20 people and seat them
   a) in a row from left to right?
   b) in a circle?
   c) in a square with 3 on each side?
   d) in a triangle with 4 on each side?
   e) in two rows of 6 facing each other?

6. In an examination paper, there are 8 questions on modern history and 7 questions on ancient history. A candidate is required to choose exactly 10 questions.
   a) In how many ways can this be achieved?
   b) In how many ways can this be achieved if the candidate is required additionally to choose exactly 5 questions in modern history?
   c) In how many ways can this be achieved if the candidate is required additionally to choose at least 5 questions in modern history?

7. There are 20 men and 20 women in a class. A mad professor decides to choose 10 students and give them extra assignments.
   a) In how many ways can the professor choose his victims?
   b) In how many ways can the professor choose his victims if he decides to choose exactly 5 men and 5 women?
   c) In how many ways can the professor choose his victims if he decides to choose at least 5 men?
   d) In how many ways can the professor choose his victims if he decides to choose at most 6 women?

8. We wish to choose 10 cards from a usual deck of 52 playing cards.
   a) In how many ways can be achieve this?
   b) In how many ways can we achieve this if we are required to choose all cards from the same suit?
   c) In how many ways can we achieve this if we are required to choose exactly 3 aces and 3 kings?
   d) In how many ways can we achieve this if we are required to choose cards of different values (assuming that the 13 cards in each suit are of different values)?

9. If we roll two dice, what is the probability that the sum of the two numbers is a multiple of 3?

10. Suppose that you are one of 10 candidates for election to a small committee of 3 people. Suppose further that each candidate is equally likely to be elected.
    a) What is the probability that you will be successful?
    b) Your best friend is also one of the candidates. What is the probability that both of you are successful?

11. We wish to elect 10 members to a committee from 100 candidates, and you are one of the candidates.
    a) What is the probability that you are elected?
    b) Two of your friends are also among the candidates.
       (i) What is the probability that you and both your friends are elected?
       (ii) What is the probability that you and exactly one of your two two friends are elected?
       (iii) What is the probability that you and at least one of your two friends are elected?
       (iv) What is the probability that both your friends are elected but you are not?

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Chapter 9

COMPLEX NUMBERS

9.1. Introduction

It is easy to see that the equation $x^2 + 1 = 0$ has no solution $x \in \mathbb{R}$. In order to "solve" this equation, we have to introduce extra numbers into our number system.

Define the number i by writing $\mathrm{i}^2 + 1 = 0$. We then extend the collection of all real numbers by adjoining the number i, which is then combined with the real numbers by the operations addition and multiplication in accordance with the rules of arithmetic for real numbers. The numbers $a + b\mathrm{i}$, where $a, b \in \mathbb{R}$, of the extended collection are then added and multiplied in accordance with the rules of arithmetic for real numbers, suitably extended, and the restriction $\mathrm{i}^2 + 1 = 0$. Note that the number $a + 0\mathrm{i}$, where $a \in \mathbb{R}$, behaves like the real number a.

Definition. We denote by $\mathbb{C}$ the collection of all complex numbers; in other words, the collection of all numbers of the form $a + b\mathrm{i}$, where $a, b \in \mathbb{R}$.

ARITHMETIC OF COMPLEX NUMBERS. *Consider two complex numbers $a + b\mathrm{i}$ and $c + d\mathrm{i}$, where $a, b, c, d \in \mathbb{R}$. We have the addition rule*

$$(a + b\mathrm{i}) + (c + d\mathrm{i}) = (a + c) + (b + d)\mathrm{i},$$

and the multiplication rule

$$(a + b\mathrm{i})(c + d\mathrm{i}) = (ac - bd) + (ad + bc)\mathrm{i}.$$

A simple consequence is the subtraction rule

$$(a + b\mathrm{i}) - (c + d\mathrm{i}) = (a - c) + (b - d)\mathrm{i}.$$

† This chapter was written at Macquarie University in 1999.

For the division rule, suppose that $c + d\mathrm{i} \neq 0$. Then $c \neq 0$ or $d \neq 0$, so that $c^2 + d^2 \neq 0$. We have

$$\frac{a+b\mathrm{i}}{c+d\mathrm{i}} = \frac{(a+b\mathrm{i})(c-d\mathrm{i})}{(c+d\mathrm{i})(c-d\mathrm{i})} = \frac{(ac+bd)+(bc-ad)\mathrm{i}}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\mathrm{i}.$$

Alternatively, write

$$\frac{a+b\mathrm{i}}{c+d\mathrm{i}} = x + y\mathrm{i},$$

where $x, y \in \mathbb{R}$. Then

$$a + b\mathrm{i} = (c+d\mathrm{i})(x+y\mathrm{i}) = (cx - dy) + (cy + dx)\mathrm{i}.$$

It follows that

$$a = cx - dy,$$
$$b = cy + dx.$$

This system of simultaneous linear equations has the unique solution

$$x = \frac{ac+bd}{c^2+d^2} \qquad \text{and} \qquad y = \frac{bc-ad}{c^2+d^2},$$

so that

$$\frac{a+b\mathrm{i}}{c+d\mathrm{i}} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\mathrm{i}.$$

However, there is absolutely no need to commit either of these two techniques to memory. For the special case $a = 1$ and $b = 0$ gives

$$\frac{1}{c+d\mathrm{i}} = \frac{c-d\mathrm{i}}{c^2+d^2}.$$

This can be obtained by noting that $(c+d\mathrm{i})(c-d\mathrm{i}) = c^2 + d^2$, so that

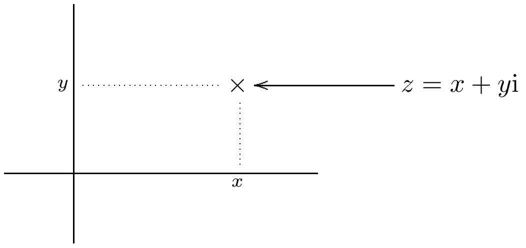
$$\frac{1}{c+d\mathrm{i}} = \frac{c-d\mathrm{i}}{(c+d\mathrm{i})(c-d\mathrm{i})} = \frac{c-d\mathrm{i}}{c^2+d^2}.$$

It is also useful to note that i^n has exactly four possible values, with $\mathrm{i}^2 = -1$, $\mathrm{i}^3 = -\mathrm{i}$ and $\mathrm{i}^4 = 1$.

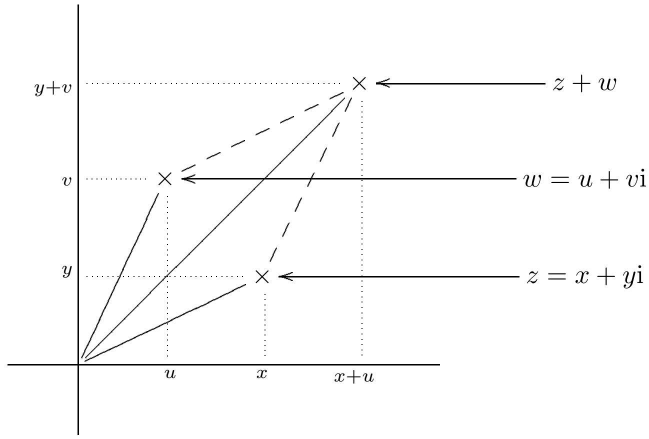
9.2. The Complex Plane

Definition. Suppose that $z = x + y\mathrm{i}$, where $x, y \in \mathbb{R}$. The real number x is called the real part of z, and denoted by $x = \mathfrak{Re}\, z$. The real number y is called the imaginary part of z, and denoted by $y = \mathfrak{Im}\, z$.

A useful way of representing complex numbers is to use the Argand diagram. This is made up of the plane together with two axes. The horizontal axis, usually called the real axis, is used to denote the real part of the complex number, while the vertical axis, usually called the imaginary axis, is used to denote the imaginary part of the complex number.



Addition of two complex numbers is then represented by the Argand diagram below.



Example 9.2.1. We have $(2 + 3\mathrm{i}) + (4 - 5\mathrm{i}) = (2 + 4) + (3 - 5)\mathrm{i} = 2 - 2\mathrm{i} = 2(1 - \mathrm{i})$.

Example 9.2.2. We have $(2 + 3\mathrm{i})(4 - 5\mathrm{i}) = (2 \times 4 - 3 \times (-5)) + (2 \times (-5) + 3 \times 4)\mathrm{i} = 23 + 2\mathrm{i}$.

Example 9.2.3. We have

$$\frac{2+3\mathrm{i}}{4-5\mathrm{i}} = \frac{(2+3\mathrm{i})(4+5\mathrm{i})}{(4-5\mathrm{i})(4+5\mathrm{i})} = \frac{-7+22\mathrm{i}}{41} = -\frac{7}{41} + \frac{22}{41}\mathrm{i}.$$

Hence

$$\mathfrak{Re}\frac{2+3\mathrm{i}}{4-5\mathrm{i}} = -\frac{7}{41} \qquad \text{and} \qquad \mathfrak{Im}\frac{2+3\mathrm{i}}{4-5\mathrm{i}} = \frac{22}{41}.$$

Example 9.2.4. We have

$$\frac{(1+2\mathrm{i})^2}{1-\mathrm{i}} = \frac{-3+4\mathrm{i}}{1-\mathrm{i}} = \frac{(-3+4\mathrm{i})(1+\mathrm{i})}{(1-\mathrm{i})(1+\mathrm{i})} = \frac{-7+\mathrm{i}}{2} = -\frac{7}{2} + \frac{1}{2}\mathrm{i}.$$

Hence

$$\mathfrak{Re}\frac{(1+2\mathrm{i})^2}{1-\mathrm{i}} = -\frac{7}{2} \qquad \text{and} \qquad \mathfrak{Im}\frac{(1+2\mathrm{i})^2}{1-\mathrm{i}} = \frac{1}{2}.$$

Example 9.2.5. We have $1 + \mathrm{i} + \mathrm{i}^2 + \mathrm{i}^3 = 0$ and $5 + 7\mathrm{i}^{2000} = 12$.

Many equations over a complex variable can be studied by similar techniques to equations over a real variable. Here we give two examples to illustrate this point.

Example 9.2.6. Suppose that $(1 + 3\mathrm{i})z + 2 + 2\mathrm{i} = 3 + \mathrm{i}$. Then subtracting $2 + 2\mathrm{i}$ from both sides, we obtain $(1 + 3\mathrm{i})z = 1 - \mathrm{i}$, and so

$$z = \frac{1-\mathrm{i}}{1+3\mathrm{i}} = \frac{(1-\mathrm{i})(1-3\mathrm{i})}{(1+3\mathrm{i})(1-3\mathrm{i})} = \frac{-2-4\mathrm{i}}{10} = -\frac{1}{5} - \frac{2}{5}\mathrm{i}.$$

Example 9.2.7. Suppose that $z^2 + 2z + 10 = 0$. Then following the technique for solving quadratic equations, we have

$$z = \frac{-2 \pm \sqrt{4-40}}{2} = -1 \pm 3\mathrm{i}.$$

Alternatively, we observe that

$$z^2 + 2z + 10 = z^2 + 2z + 1 + 9 = (z + 1)^2 + 9 = 0$$

precisely when $(z + 1)^2 = -9$; in other words, precisely when $z + 1 = \pm 3\mathrm{i}$, so that $z = -1 \pm 3\mathrm{i}$.

Problems for Chapter 9

1. Find each of the following:

   a) $(1 - 2\mathrm{i})^2$ b) $(1 + 2\mathrm{i})(3 + 4\mathrm{i})$ c) $\dfrac{1}{3 + 2\mathrm{i}}$ d) $\dfrac{(1 - 2\mathrm{i})^2}{1 - \mathrm{i}}$

   e) $\dfrac{1}{1 + \mathrm{i}} + \dfrac{1}{1 - 2\mathrm{i}}$ f) $\mathrm{i}^3 + \mathrm{i}^4 + \mathrm{i}^5 + \mathrm{i}^6$ g) $\dfrac{1}{(4 + 2\mathrm{i})(2 - 3\mathrm{i})}$ h) $\dfrac{3 + 4\mathrm{i}}{1 + 2\mathrm{i}}$

2. Find the real and imaginary parts of each of the complex numbers in Question 1.

3. Solve each of the following equations:

   a) $(2 + \mathrm{i})z + \mathrm{i} = 3$ b) $\dfrac{z - 1}{z - \mathrm{i}} = \dfrac{2}{3}$ c) $\dfrac{z - 3\mathrm{i}}{z + 4} = \dfrac{1}{5}$ d) $\dfrac{1}{z + \mathrm{i}} = \dfrac{3}{2 - z}$

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Chapter 10

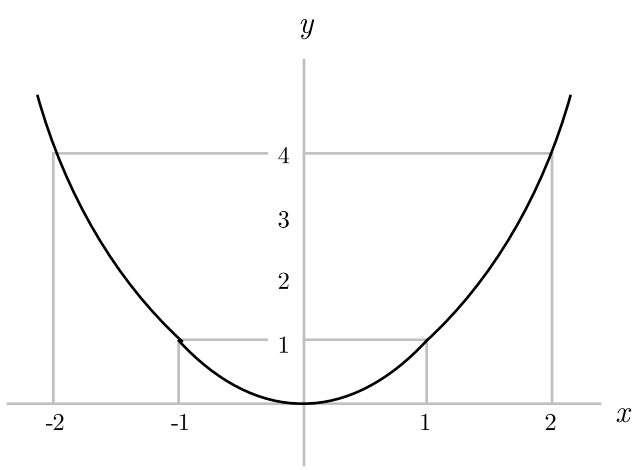
FUNCTIONS AND LINES

10.1. Functions and Graphs

We shall be concerned with real valued functions of a real variable x. In other words, we study functions of the form $f(x)$, where $x \in \mathbb{R}$ and $f(x)$ is real valued.

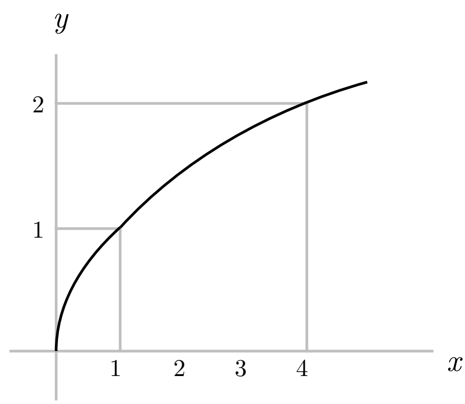
A convenient way to study and understand the properties of a function is to draw its graph. To do so, we make use of the xy-plane, and denote the values $f(x)$ by using the y-axis. Then the graph of the function consists of all points $(x, f(x))$ for which the function is defined.

EXAMPLE 10.1.1. Consider the function $f(x) = x^2$. This function is defined for all real values of x. For every $x \in \mathbb{R}$, the value $f(x)$ is real. We have $f(0) = 0$, $f(-1) = f(1) = 1$ and $f(-2) = f(2) = 4$. The graph of this function is given below.



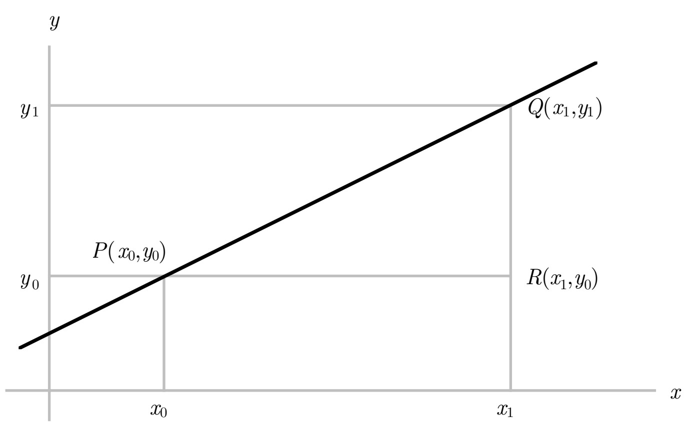
† This chapter was written at Macquarie University in 1999.

EXAMPLE 10.1.2. Consider the function $f(x) = \sqrt{x}$. This function is defined for all non-negative real values of x but not defined for any negative values of x. For every non-negative $x \in \mathbb{R}$, the value $f(x)$ is real. We have $f(0) = 0$, $f(1) = 1$, $f(2) = \sqrt{2}$, $f(3) = \sqrt{3}$ and $f(4) = 2$. The graph of this function is given below.



10.2. Lines on the Plane

In this section, we shall study the problem of lines and their graphs. Recall first of all that a line is determined if we know two of its points. Suppose that $P(x_0, y_0)$ and $Q(x_1, y_1)$ are two points on the xy-plane as shown in the picture below.



We can consider the triangle PQR formed by P and Q as well as the point $R(x_1, y_0)$. The length of the vertical side of this triangle is given by $y_1 - y_0$, while the length of the horizontal side of this triangle is given by $x_1 - x_0$. The ratio

$$\frac{y_1 - y_0}{x_1 - x_0}$$

is called the slope of the line through the points P and Q.

EXAMPLE 10.2.1. The line through the points $P(1, 2)$ and $Q(3, 6)$ has slope

$$\frac{6 - 2}{3 - 1} = 2.$$

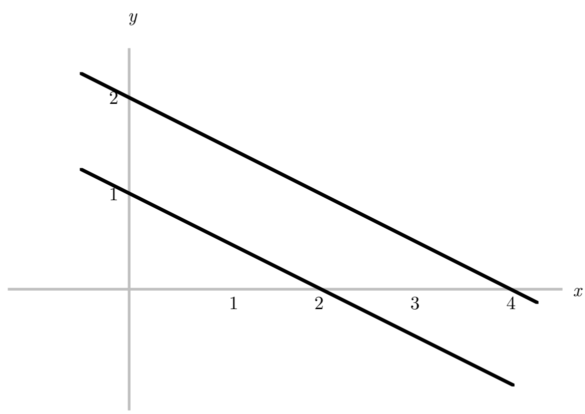
EXAMPLE 10.2.2. The line through the points $(4,0)$ and $(0,2)$ has slope

$$\frac{2-0}{0-4} = -\frac{1}{2}.$$

The line through the points $(2,0)$ and $(0,1)$ has slope

$$\frac{1-0}{0-2} = -\frac{1}{2}.$$

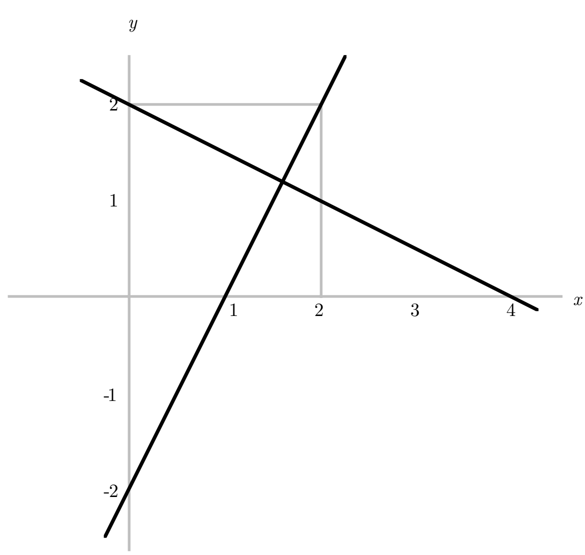
Note that the slope is negative in both cases; see the picture below.



It is clear that the two lines are parallel to each other. Note that their slopes are equal.

PARALLEL LINES. *Two lines on the plane are parallel to each other if and only if their slopes are equal.*

EXAMPLE 10.2.3. The line through the points $(4,0)$ and $(0,2)$ has slope $-1/2$. It is easy to check that the line through $(1,0)$ and $(2,2)$ has slope 2. We have the following picture.



It is clear that the two lines are perpendicular to each other. Note that the product of their slopes is equal to -1.

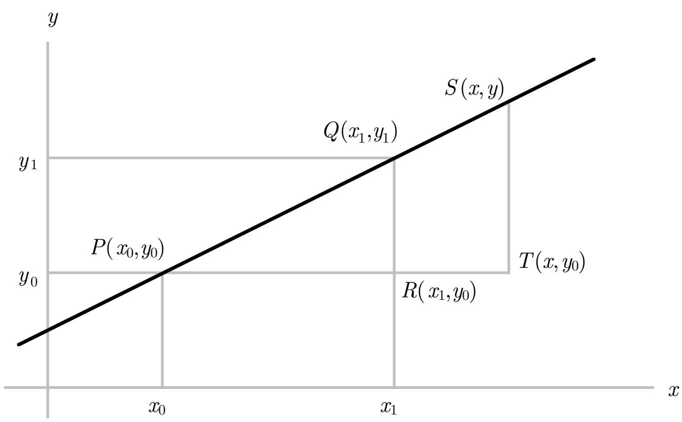
PERPENDICULAR LINES. *Two lines on the plane are perpendicular to each other if and only if the product of their slopes is equal to* -1.

REMARK. The two results above are in fact not quite correct. For example, according to our definition, the line through the points $(1, 0)$ and $(1, 1)$ has slope

$$\frac{1-0}{1-1} = \frac{1}{0}$$

which is undefined. This represents a vertical line, parallel to the y-axis. On the other hand, a horizontal line, parallel to the x-axis, has slope 0 (why?). However, in these special cases, we do not need to use the two results above.

We now consider the problem of finding the equation of the line through two points $P(x_0, y_0)$ and $Q(x_1, y_1)$. Consider the picture below.



Suppose that (x, y) is a typical point on the line joining (x_0, y_0) and (x_1, y_1). Note that the triangles PQR and PST are similar, so that

$$\frac{TS}{PT} = \frac{RQ}{PR}.$$

In other words, we must have

$$\frac{y-y_0}{x-x_0} = \frac{y_1-y_0}{x_1-x_0} = m,$$

where

$$m = \frac{y_1-y_0}{x_1-x_0}$$

is the slope of the line joining (x_0, y_0) and (x_1, y_1). It follows that the equation of the line is given by

$$\frac{y-y_0}{x-x_0} = m.$$

This can be written in the form

$$y = mx + c, \tag{1}$$

where

$$c = y_0 - mx_0.$$

Note that the y-axis is the line $x = 0$. If we substitute this into (1), then we ontain $y = c$. This shows that the line (1) intersects the y-axis at the point $(0, c)$. This point is called the y-intercept of the line. On the other hand, the x-intercept of the line is the point of intersection with the x-axis. Note that the x-axis is the line $y = 0$. If we substitute this into (1), then we obtain $x = -c/m$. This shows that the line (1) intersects the x-axis at the point $(-c/m, 0)$.

Remark. Again, our argument here fails if the line in question is a vertical line, parallel to the y-axis, or a horizontal line, parallel to the x-axis.

Example 10.2.4. The line joining the points $(1,0)$ and $(2,2)$ is given by

$$\frac{y-0}{x-1} = \frac{2-0}{2-1} = 2.$$

This can be written in the form $y = 2x - 2$. Note that this line intersects the y-axis at the point $(0,-2)$ and the x-axis at the point $(1,0)$. See the picture in Example 10.2.3.

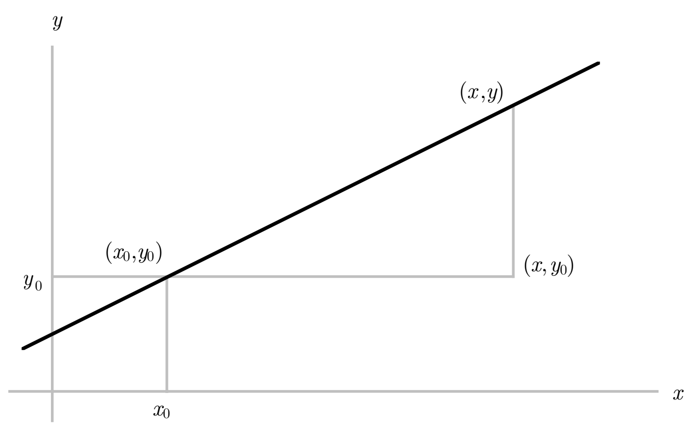
Example 10.2.5. The line joining the points $(1,1)$ and $(3,1)$ is given by

$$\frac{y-1}{x-1} = \frac{1-1}{3-1} = 0.$$

This can be written in the form $y = 1$, with $m = 0$ and $c = 1$. Note that this line clearly intersects the y-axis at the point $(0,1)$ but does not intersect the x-axis. Try to draw the graph of this line.

Example 10.2.6. The line joining the points $(1,1)$ and $(1,3)$ is given by $x = 1$. This is an example of the exception to our technique. It is easy to see that the line is parallel to the y-axis. It intersects the x-axis at the point $(1,0)$ but does not intersect the y-axis. Try to draw the graph of this line.

A line can also be determined if we know its slope and one of its points. Suppose that (x_0, y_0) is a given point, and that (x, y) is a typical point on the line through (x_0, y_0) with slope m. Consider the following picture.



Clearly we must have

$$\frac{y-y_0}{x-x_0} = m.$$

This again can be rewritten in the form $y = mx + c$, where $c = y_0 - mx_0$.

Example 10.2.7. The line through the point $(1,2)$ with slope 3 is given by

$$\frac{y-2}{x-1} = 3.$$

This can be rewritten in the form $y = 3x - 1$. Note that this line intersects the y-axis at the point $(0,-1)$ and the x-axis at the point $(1/3, 0)$.

EXAMPLE 10.2.8. The line $y = 4x - 2$ has slope 4. We now consider a line parallel to this and passing through the point $(1, 3)$. This parallel line must also have slope 4, and therefore has equation

$$\frac{y-3}{x-1} = 4.$$

This can be rewritten in the form $y = 4x - 1$. We consider next a line perpendicular to these two lines and passing through the point $(2, 1)$. This new line must have slope $-1/4$, and therefore has equation

$$\frac{y-1}{x-2} = -\frac{1}{4}.$$

This can be rewritten in the form $y = -\frac{1}{4}x + \frac{3}{2}$. Try to draw the graphs of these three lines, and find their x-intercepts and y-intercepts.

PROBLEMS FOR CHAPTER 10

1. For each of the following functions, determine the value $f(x)$ for every integer x satisfying the inequalities $-3 \le x \le 3$, and draw a rough sketch of the graph of the function:
   a) $f(x) = x^2 + 2x - 1$ b) $f(x) = 4 - x^2$ c) $f(x) = \sqrt{9 - x^2}$

2. For each of the following, find the equation of the line through the given points:
   a) $(1, 3)$ and $(2, 5)$ b) $(-1, 3)$ and $(2, 5)$ c) $(-3, 5)$ and $(2, 6)$

3. For each of the following, find the equation of the line through the given points, and determine the x-intercept and y-intercept of this line:
   a) $(2, 0)$ and $(3, 5)$ b) $(3, -2)$ and $(3, 1)$ c) $(5, -1)$ and $(-1, -1)$

4. For each of the following, find the equation of the line with the given slope and passing through the given point, and determine the x-intercept and y-intercept of this line:
   a) slope 3 and point $(3, 1)$ b) slope -2 and point $(-1, 4)$ c) slope 4 and point $(4, 2)$

5. For each of the three lines in Question 3, find the equation of the perpendicular line which passes through the point $(-4, -4)$, as well as the intersection of the perpendicular line with the given line.

6. For each of the three lines in Question 4, find the equation of the perpendicular line which passes through the point $(-1, -1)$, determine its point of intersection with the original line, and then find the equation of the line which passes through this intersection point and the origin $(0, 0)$.

7. Consider the parabola $y = f(x)$ where $f(x) = x^2 + 3$.
   a) Find the slope of the line joining the points (x_1, y_1) and (x_2, y_2) on this parabola, where $x_1 = 1$ and $x_2 = 3$.
   b) Find the slope of the line joining the points (x_3, y_3) and (x_4, y_4) on this parabola, where $y_3 = 7$ and $y_4 = 19$ and where both x_3 and x_4 are negative.
   c) Confirm that the slope of the line in part (a) is equal to $x_1 + x_2$, and that the slope of the line in part (b) is equal to $x_3 + x_4$.
   d) Can you obtain the conclusion of part (c) without using the information about the specific values of x_1, x_2, y_3 and y_4?

8. Let m_1 be the slope of some line, and let m_2 be the slope of a perpendicular line. Use trigonometry to explain why the product $m_1 m_2$ is equal to -1, except when one line is horizontal and the other line is vertical.

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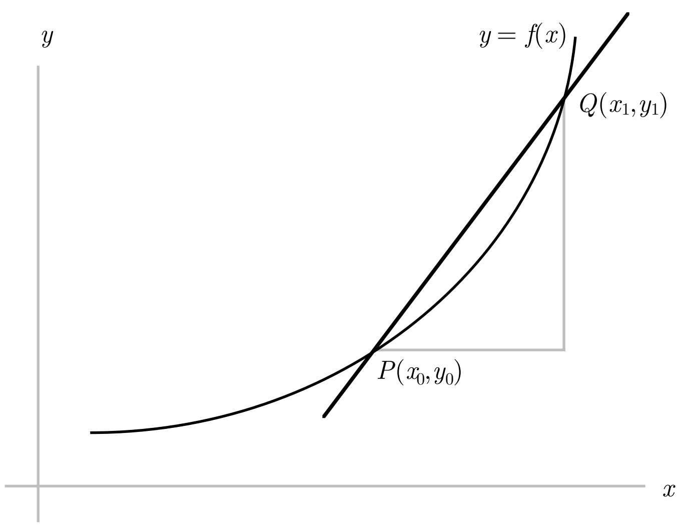
Chapter 11

INTRODUCTION TO DIFFERENTIATION

11.1. Tangent to a Curve

Consider the graph of a function $y = f(x)$. Suppose that $P(x_0, y_0)$ is a point on the curve $y = f(x)$. Consider now another point $Q(x_1, y_1)$ on the curve close to the point $P(x_0, y_0)$. We draw the line joining the points $P(x_0, y_0)$ and $Q(x_1, y_1)$, and obtain the picture below.

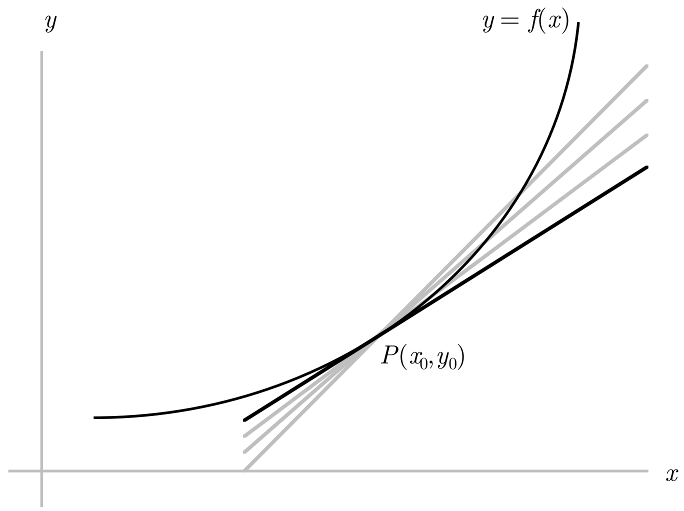


Clearly the slope of this line is equal to

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

† This chapter was written at Macquarie University in 1999.

Now let us keep the point $P(x_0, y_0)$ fixed, and move the point $Q(x_1, y_1)$ along the curve towards the point P. Eventually the line PQ becomes the tangent to the curve $y = f(x)$ at the point $P(x_0, y_0)$, as shown in the picture below.

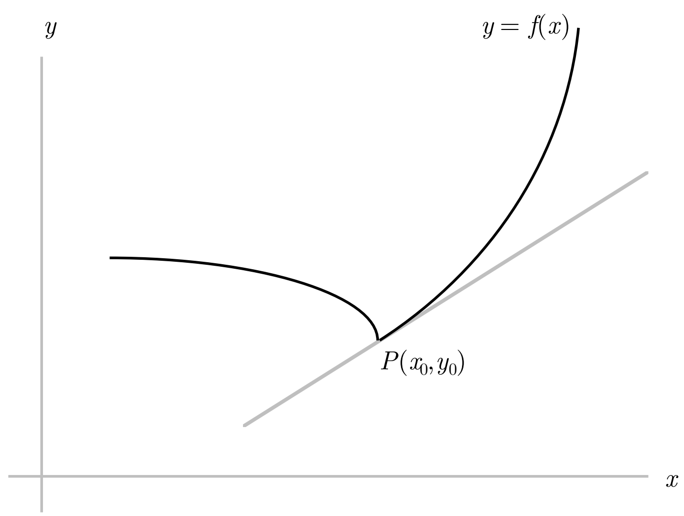


We are interested in the slope of this tangent line. Its value is called the derivative of the function $y = f(x)$ at the point $x = x_0$, and denoted by

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0} \qquad \text{or} \qquad f'(x_0).$$

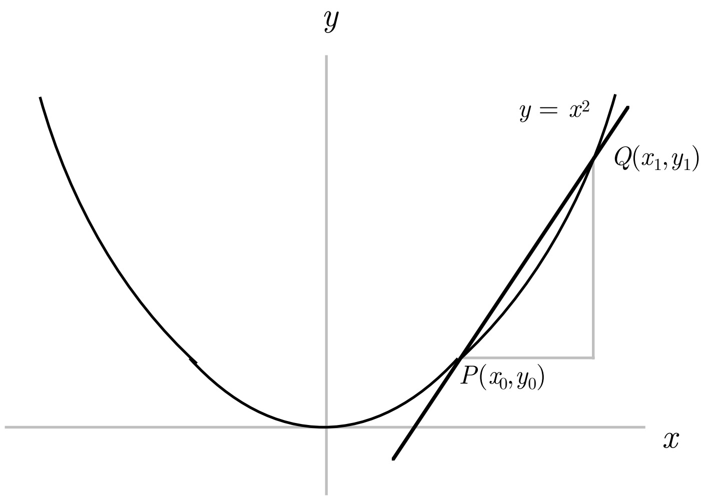
In this case, we say that the function $y = f(x)$ is differentiable at the point $x = x_0$.

Remark. Sometimes, when we move the point $Q(x_1, y_1)$ along the curve $y = f(x)$ towards the point $P(x_0, y_0)$, the line PQ does not become the tangent to the curve $y = f(x)$ at the point $P(x_0, y_0)$. In this case, we say that the function $y = f(x)$ is not differentiable at the point $x = x_0$. An example of such a situation is given in the picture below.



Note that the curve $y = f(x)$ makes an abrupt turn at the point $P(x_0, y_0)$.

EXAMPLE 11.1.1. Consider the graph of the function $y = f(x) = x^2$.



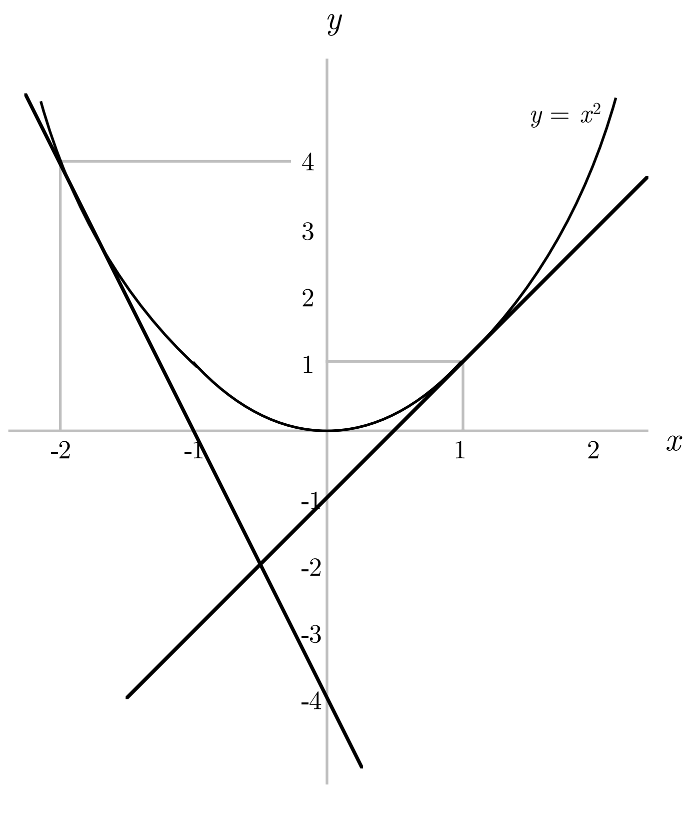
Here the slope of the line joining the points $P(x_0, y_0)$ and $Q(x_1, y_1)$ is equal to

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{x_1^2 - x_0^2}{x_1 - x_0} = x_1 + x_0.$$

It follows that if we move the point $Q(x_1, y_1)$ along the curve towards the point $P(x_0, y_0)$, then the slope of this line will eventually be equal to $x_0 + x_0 = 2x_0$. Hence for the function $y = f(x) = x^2$, we have

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0} = f'(x_0) = 2x_0.$$

In particular, the tangent to the curve at the point $(1, 1)$ has slope 2 and so has equation $y = 2x - 1$, whereas the tangent to the curve at the point $(-2, 4)$ has slope -4 and so has equation $y = -4x - 4$.



EXAMPLE 11.1.2. Consider the graph of the function $y = f(x) = x^3$. Here the slope of the line joining the points $P(x_0, y_0)$ and $Q(x_1, y_1)$ is equal to

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{x_1^3 - x_0^3}{x_1 - x_0} = x_1^2 + x_1 x_0 + x_0^2.$$

It follows that if we move the point $Q(x_1, y_1)$ along the curve towards the point $P(x_0, y_0)$, then the slope of this line will eventually be equal to $x_0^2 + x_0 x_0 + x_0^2 = 3x_0^2$. Hence for the function $y = f(x) = x^3$, we have

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0} = f'(x_0) = 3x_0^2.$$

In particular, the tangent to the curve at the point $(0, 0)$ has slope 0 and so has equation $y = 0$, whereas the tangent to the curve at the point $(2, 8)$ has slope 12 and so has equation $y = 12x - 16$.

EXAMPLE 11.1.3. Consider the graph of the function $y = f(x) = x$. Here the slope of the line joining the points $P(x_0, y_0)$ and $Q(x_1, y_1)$ is equal to

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{x_1 - x_0}{x_1 - x_0} = 1.$$

It follows that if we move the point $Q(x_1, y_1)$ along the curve towards the point $P(x_0, y_0)$, then the slope of this line will remain equal to 1. Hence for the function $y = f(x) = x$, we have

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0} = f'(x_0) = 1.$$

EXAMPLE 11.1.4. Consider the graph of the function $y = f(x) = x^{1/2}$, defined for all real numbers $x \geq 0$. Suppose that $x_0 > 0$ and $x_1 > 0$. Then the slope of the line joining the points $P(x_0, y_0)$ and $Q(x_1, y_1)$ is equal to

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{x_1^{1/2} - x_0^{1/2}}{x_1 - x_0} = \frac{1}{x_1^{1/2} + x_0^{1/2}}.$$

It follows that if we move the point $Q(x_1, y_1)$ along the curve towards the point $P(x_0, y_0)$, then the slope of this line will eventually be equal to

$$\frac{1}{x_0^{1/2} + x_0^{1/2}} = \frac{1}{2x_0^{1/2}} = \frac{1}{2}x_0^{-1/2}.$$

Hence for the function $y = f(x) = x^{1/2}$, we have

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0} = f'(x_0) = \frac{1}{2}x_0^{-1/2}.$$

The above four examples are special cases of the following result.

DERIVATIVES OF POWERS. *Suppose that n is a fixed non-zero real number. Then for the function $y = f(x) = x^n$, we have*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x) = nx^{n-1}$$

for every real number x for which x^{n-1} is defined.

Here and henceforth, we shall slightly abuse our notation and refer to $f'(x)$ as the derivative of the function $y = f(x)$, and write

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x).$$

EXAMPLE 11.1.5. For the function $y = f(x) = x^{1/4}$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x) = \frac{1}{4}x^{-3/4}$$

for every positive real number x.

The rule concerning derivatives of powers does not apply in the case $n = 0$.

DERIVATIVES OF CONSTANTS. *Suppose that $f(x) = c$, where c is a fixed real number. Then $f'(x) = 0$ for every real number x.*

11.2. Arithmetic of Derivatives

Very often, we need to find the derivatives of complicated functions which are constant multiples, sums, products and/or quotients of much simpler functions. To achieve this, we can make use of our knowledge concerning the derivatives of these simpler functions. We have four extremely useful results.

CONSTANT MULTIPLE RULE. *Suppose that $m(x) = cf(x)$, where c is a fixed real number. Then*

$$m'(x) = cf'(x)$$

for every real number x for which $f'(x)$ exists.

SUM RULE. *Suppose that $s(x) = f(x) + g(x)$ and $d(x) = f(x) - g(x)$. Then*

$$s'(x) = f'(x) + g'(x) \qquad \text{and} \qquad d'(x) = f'(x) - g'(x)$$

for every real number x for which $f'(x)$ and $g'(x)$ exist.

EXAMPLE 11.2.1. Consider the function $h(x) = 5x^2 + 3x^5$. We can write

$$h(x) = f(x) + g(x),$$

where $f(x) = 5x^2$ and $g(x) = 3x^5$. It follows from the sum rule that

$$h'(x) = f'(x) + g'(x).$$

Next, the function $f(x) = 5x^2$ is a constant (5) multiple of the function x^2, and so it follows from the constant multiple rule and the rule on the derivatives of powers that

$$f'(x) = 5(x^2)' = 5(2x) = 10x.$$

Similarly, the function $g(x) = 3x^5$ is a constant (3) multiple of the function x^5, and so it follows from the constant multiple rule and the rule on the derivatives of powers that

$$g'(x) = 3(x^5)' = 3(5x^4) = 15x^4.$$

Hence $h'(x) = 10x + 15x^4$.

EXAMPLE 11.2.2. Consider the function $h(x) = (3x)^4 - (2x)^6$. We can write

$$h(x) = f(x) - g(x),$$

where $f(x) = 81x^4$ and $g(x) = 64x^6$. It follows from the sum rule that

$$h'(x) = f'(x) - g'(x).$$

Applying the constant multiple rule and the rule on the derivatives of powers, we obtain $f'(x) = 324x^3$ and $g'(x) = 384x^5$. Hence $h'(x) = 324x^3 - 384x^5$.

The sum rule can be extended to the sum or difference of more than two functions in the natural way. We illustrate the technique in the following three examples.

EXAMPLE 11.2.3. Consider the function $h(x) = 4x^3 - 15x^2 + 4x - 1$. We can write

$$h(x) = f(x) - g(x) + k(x) - t(x),$$

where $f(x) = 4x^3$, $g(x) = 15x^2$, $k(x) = 4x$ and $t(x) = 1$. It follows from the sum rule that

$$h'(x) = f'(x) - g'(x) + k'(x) - t'(x).$$

Applying the constant multiple rule and the rule on the derivatives of powers, we obtain $f'(x) = 12x^2$, $g'(x) = 30x$ and $k'(x) = 4$. Applying the rule on the derivatives of constants, we obtain $t'(x) = 0$. Hence $h'(x) = 12x^2 - 30x + 4$.

EXAMPLE 11.2.4. Consider the function $h(x) = 8x^3 - 2(x+2)^2 + 3$. Then $h(x) = 8x^3 - 2x^2 - 8x - 5$, and so we can write

$$h(x) = f(x) - g(x) - k(x) - t(x),$$

where $f(x) = 8x^3$, $g(x) = 2x^2$, $k(x) = 8x$ and $t(x) = 5$. It follows from the sum rule that

$$h'(x) = f'(x) - g'(x) - k'(x) - t'(x).$$

Applying the constant multiple rule and the rule on the derivatives of powers, we obtain $f'(x) = 24x^2$, $g'(x) = 4x$ and $k'(x) = 8$. Applying the rule on the derivatives of constants, we obtain $t'(x) = 0$. Hence $h'(x) = 24x^2 - 4x - 8$.

EXAMPLE 11.2.5. Consider the function $h(x) = (x^2 + 2x)^2$. Then $h(x) = x^4 + 4x^3 + 4x^2$, and so we can write

$$h(x) = f(x) + g(x) + k(x),$$

where $f(x) = x^4$, $g(x) = 4x^3$ and $k(x) = 4x^2$. It follows from the sum rule that

$$h'(x) = f'(x) + g'(x) + k'(x).$$

Applying the constant multiple rule and the rule on the derivatives of powers, we obtain $f'(x) = 4x^3$, $g'(x) = 12x^2$ and $k'(x) = 8x$. Hence $h'(x) = 4x^3 + 12x^2 + 8x$.

EXAMPLE 11.2.6. Consider the function

$$h(x) = \frac{3}{x} + 2x.$$

We can write

$$h(x) = f(x) + g(x),$$

where $f(x) = 3x^{-1}$ and $g(x) = 2x$. It follows from the sum rule that

$$h'(x) = f'(x) + g'(x).$$

Applying the constant multiple rule and the rule on the derivatives of powers, we obtain $f'(x) = -3x^{-2}$ and $g'(x) = 2$. Hence $h'(x) = 2 - 3x^{-2}$.

EXAMPLE 11.2.7. Consider the function

$$h(x) = 6x^2\sqrt{x} - \frac{4}{\sqrt{x}} + 3x^{1/3}.$$

We can write

$$h(x) = f(x) - g(x) + k(x),$$

where $f(x) = 6x^{5/2}$, $g(x) = 4x^{-1/2}$ and $k(x) = 3x^{1/3}$. It follows from the sum rule that

$$h'(x) = f'(x) - g'(x) + k'(x).$$

Applying the constant multiple rule and the rule on the derivatives of powers, we obtain $f'(x) = 15x^{3/2}$, $g'(x) = -2x^{-3/2}$ and $k'(x) = x^{-2/3}$. Hence $h'(x) = 15x^{3/2} + 2x^{-3/2} + x^{-2/3}$.

EXAMPLE 11.2.8. Consider the function $h(x) = \sqrt{3x} + \sqrt[3]{2x}$. We can write

$$h(x) = f(x) + g(x),$$

where $f(x) = \sqrt{3}x^{1/2}$ and $g(x) = \sqrt[3]{2}x^{1/3}$. It follows from the sum rule that

$$h'(x) = f'(x) + g'(x).$$

Applying the constant multiple rule and the rule on the derivatives of powers, we obtain

$$f'(x) = \frac{\sqrt{3}}{2}x^{-1/2} \qquad \text{and} \qquad g'(x) = \frac{\sqrt[3]{2}}{3}x^{-2/3}.$$

Hence

$$h'(x) = \frac{\sqrt{3}}{2}x^{-1/2} + \frac{\sqrt[3]{2}}{3}x^{-2/3} = \sqrt{\frac{3}{4x}} + \sqrt[3]{\frac{2}{27x^2}}.$$

PRODUCT RULE. *Suppose that $p(x) = f(x)g(x)$. Then*

$$p'(x) = f'(x)g(x) + f(x)g'(x)$$

for every real number x for which $f'(x)$ and $g'(x)$ exist.

EXAMPLE 11.2.9. Consider the function $h(x) = (x^3 - x^5)(x^2 + x^4)$. We can write

$$h(x) = f(x)g(x),$$

where $f(x) = x^3 - x^5$ and $g(x) = x^2 + x^4$. It follows from the product rule that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

Applying the sum rule and the rule on the derivatives of powers, we obtain $f'(x) = 3x^2 - 5x^4$ and $g'(x) = 2x + 4x^3$. Hence

$$h'(x) = (3x^2 - 5x^4)(x^2 + x^4) + (x^3 - x^5)(2x + 4x^3) = 5x^4 - 9x^8.$$

Alternatively, we observe that $h(x) = (x^3 - x^5)(x^2 + x^4) = x^5 - x^9$. Applying the sum rule and the rule on the derivatives of powers, we obtain $h'(x) = 5x^4 - 9x^8$ as before.

EXAMPLE 11.2.10. Let us return to Example 11.2.5 and consider again the function $h(x) = (x^2+2x)^2$. We can write

$$h(x) = f(x)g(x),$$

where $f(x) = g(x) = x^2 + 2x$. It follows from the product rule that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

Applying the sum rule, the constant multiple rule and the rule on the derivatives of powers, we obtain $f'(x) = g'(x) = 2x + 2$. Hence

$$h'(x) = (2x + 2)(x^2 + 2x) + (x^2 + 2x)(2x + 2) = 2(2x + 2)(x^2 + 2x) = 4x^3 + 12x^2 + 8x$$

as before. We shall return to example again in Section 12.1.

EXAMPLE 11.2.11. Consider the function $h(x) = (x^2 + x)(x^3 - 6x^2 + 2x)$. We can write

$$h(x) = f(x)g(x),$$

where $f(x) = x^2 + x$ and $g(x) = x^3 - 6x^2 + 2x$. It follows from the product rule that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

Applying the sum rule, the constant multiple rule and the rule on the derivatives of powers, we obtain $f'(x) = 2x + 1$ and $g'(x) = 3x^2 - 12x + 2$. Hence

$$h'(x) = (2x + 1)(x^3 - 6x^2 + 2x) + (x^2 + x)(3x^2 - 12x + 2).$$

EXAMPLE 11.2.12. Consider the function

$$h(x) = (x + \sqrt{x})\left(x - \frac{1}{\sqrt{x}}\right).$$

We can write

$$h(x) = f(x)g(x),$$

where $f(x) = x + x^{1/2}$ and $g(x) = x - x^{-1/2}$. It follows from the product rule that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

Applying the sum rule and the rule on the derivatives of powers, we obtain

$$f'(x) = 1 + \frac{1}{2}x^{-1/2} \qquad \text{and} \qquad g'(x) = 1 + \frac{1}{2}x^{-3/2}.$$

Hence

$$\begin{aligned} h'(x) &= \left(1 + \frac{1}{2}x^{-1/2}\right)(x - x^{-1/2}) + (x + x^{1/2})\left(1 + \frac{1}{2}x^{-3/2}\right) \\ &= \left(x - x^{-1/2} + \frac{1}{2}x^{1/2} - \frac{1}{2}x^{-1}\right) + \left(x + \frac{1}{2}x^{-1/2} + x^{1/2} + \frac{1}{2}x^{-1}\right) \\ &= 2x + \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-1/2}. \end{aligned}$$

Alternatively, we observe that $h(x) = (x + x^{1/2})(x - x^{-1/2}) = x^2 - x^{1/2} + x^{3/2} - 1$. Applying the sum rule and the rules on the derivatives of powers and constants, we obtain

$$h'(x) = 2x - \frac{1}{2}x^{-1/2} + \frac{3}{2}x^{1/2}$$

as before.

The product rule can be extended to the product of more than two functions. The extension is at first sight somewhat less obvious than in the case of the sum rule. However, with a bit of care, it is in fact rather straightforward.

EXAMPLE 11.2.13. Consider the function $h(x) = (x^2 + 4x)(2x + 1)(6 - 2x^2)$. We can write

$$h(x) = f(x)r(x),$$

where $f(x) = x^2 + 4x$ and $r(x) = (2x + 1)(6 - 2x^2)$. It follows from the product rule that

$$h'(x) = f'(x)r(x) + f(x)r'(x).$$

We can now write

$$r(x) = g(x)k(x),$$

where $g(x) = 2x + 1$ and $k(x) = 6 - 2x^2$. It follows from the product rule that

$$r'(x) = g'(x)k(x) + g(x)k'(x).$$

Hence $h(x) = f(x)g(x)k(x)$, and

$$h'(x) = f'(x)g(x)k(x) + f(x)g'(x)k(x) + f(x)g(x)k'(x).$$

Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain $f'(x) = 2x + 4$, $g'(x) = 2$ and $k'(x) = -4x$. Hence

$$h'(x) = (2x + 4)(2x + 1)(6 - 2x^2) + 2(x^2 + 4x)(6 - 2x^2) - 4x(x^2 + 4x)(2x + 1).$$

REMARK. The interested reader is challenged to show that if $p(x) = f(x)g(x)k(x)t(x)$, then

$$p'(x) = f'(x)g(x)k(x)t(x) + f(x)g'(x)k(x)t(x) + f(x)g(x)k'(x)t(x) + f(x)g(x)k(x)t'(x).$$

QUOTIENT RULE. *Suppose that $q(x) = f(x)/g(x)$. Then*

$$q'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

for every real number x for which $f'(x)$ and $g'(x)$ exist, and for which $g(x) \neq 0$.

EXAMPLE 11.2.14. Consider the function

$$h(x) = \frac{x^2 - 1}{x^3 + 2x}.$$

We can write

$$h(x) = \frac{f(x)}{g(x)},$$

where $f(x) = x^2 - 1$ and $g(x) = x^3 + 2x$. It follows from the quotient rule that

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain $f'(x) = 2x$ and $g'(x) = 3x^2 + 2$. Hence

$$h'(x) = \frac{2x(x^3 + 2x) - (x^2 - 1)(3x^2 + 2)}{(x^3 + 2x)^2}.$$

EXAMPLE 11.2.15. Consider the function

$$h(x) = \frac{4x^2+1}{3x}.$$

We can write

$$h(x) = \frac{f(x)}{g(x)},$$

where $f(x) = 4x^2 + 1$ and $g(x) = 3x$. It follows from the quotient rule that

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain $f'(x) = 8x$ and $g'(x) = 3$. Hence

$$h'(x) = \frac{24x^2 - 3(4x^2+1)}{9x^2}.$$

EXAMPLE 11.2.16. Consider the function

$$h(x) = \frac{3x^2+4x^7}{5x^{-2}+3}.$$

We can write

$$h(x) = \frac{f(x)}{g(x)},$$

where $f(x) = 3x^2 + 4x^7$ and $g(x) = 5x^{-2} + 3$. It follows from the quotient rule that

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain $f'(x) = 6x + 28x^6$ and $g'(x) = -10x^{-3}$. Hence

$$\begin{aligned} h'(x) &= \frac{(5x^{-2}+3)(6x+28x^6) + 10x^{-3}(3x^2+4x^7)}{(5x^{-2}+3)^2} = \frac{30x^{-1}+140x^4+18x+84x^6+30x^{-1}+40x^4}{25x^{-4}+30x^{-2}+9} \\ &= \frac{60x^{-1}+18x+180x^4+84x^6}{25x^{-4}+30x^{-2}+9} \times \frac{x^4}{x^4} = \frac{60x^3+18x^5+180x^8+84x^{10}}{25+30x^2+9x^4}. \end{aligned}$$

Alternatively, we observe that

$$h(x) = \frac{3x^2+4x^7}{5x^{-2}+3} \times \frac{x^2}{x^2} = \frac{3x^4+4x^9}{5+3x^2}.$$

We can write

$$h(x) = \frac{k(x)}{t(x)},$$

where $k(x) = 3x^4 + 4x^9$ and $t(x) = 5 + 3x^2$. It follows from the quotient rule that

$$h'(x) = \frac{t(x)k'(x) - k(x)t'(x)}{t^2(x)}.$$

Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain $k'(x) = 12x^3 + 36x^8$ and $g'(x) = 6x$. Hence

$$\begin{aligned} h'(x) &= \frac{(5+3x^2)(12x^3+36x^8) - 6x(3x^4+4x^9)}{(5+3x^2)^2} = \frac{60x^3+36x^5+180x^8+108x^{10}-18x^5-24x^{10}}{25+30x^2+9x^4} \\ &= \frac{60x^3+18x^5+180x^8+84x^{10}}{25+30x^2+9x^4} \end{aligned}$$

as before.

EXAMPLE 11.2.17. Consider the function

$$h(x) = \frac{(x^2+4)(x-2)}{x^2+2}.$$

We can write

$$h(x) = \frac{f(x)}{g(x)},$$

where $f(x) = (x^2+4)(x-2)$ and $g(x) = x^2+2$. It follows from the quotient rule that

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

We can now write

$$f(x) = k(x)t(x),$$

where $k(x) = x^2+4$ and $t(x) = x-2$. It follows from the product rule that

$$f'(x) = k'(x)t(x) + k(x)t'(x).$$

Hence

$$h(x) = \frac{k(x)t(x)}{g(x)},$$

and

$$h'(x) = \frac{g(x)k'(x)t(x) + g(x)k(x)t'(x) - k(x)t(x)g'(x)}{g^2(x)}.$$

Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain $k'(x) = 2x$, $t'(x) = 1$ and $g'(x) = 2x$. Hence

$$h'(x) = \frac{2x(x^2+2)(x-2) + (x^2+2)(x^2+4) - 2x(x^2+4)(x-2)}{(x^2+2)^2} = \frac{x^4+2x^2+8x+8}{(x^2+2)^2}.$$

Alternatively, we observe that

$$f(x) = (x^2+4)(x-2) = x^3 - 2x^2 + 4x - 8.$$

Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain $f'(x) = 3x^2 - 4x + 4$. Hence

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)} = \frac{(x^2+2)(3x^2-4x+4) - 2x(x^3-2x^2+4x-8)}{(x^2+2)^2} = \frac{x^4+2x^2+8x+8}{(x^2+2)^2}$$

as before.

For those who want a small challenge, here is one more example.

EXAMPLE 11.2.18. Consider the function

$$h(x) = \frac{(4x-3)(2x^2-3x)}{(2x+2)(x^3+6)}.$$

We can write

$$h(x) = \frac{f(x)}{g(x)},$$

where $f(x) = (4x-3)(2x^2-3x)$ and $g(x) = (2x+2)(x^3+6)$. It follows from the quotient rule that

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

We can now write
$$f(x)=k(x)t(x)\qquad\text{and}\qquad g(x)=u(x)v(x),$$
where $k(x)=4x-3$, $t(x)=2x^2-3x$, $u(x)=2x+2$ and $v(x)=x^3+6$. It follows from the product rule that
$$f'(x)=k'(x)t(x)+k(x)t'(x)\qquad\text{and}\qquad g'(x)=u'(x)v(x)+u(x)v'(x).$$
Hence
$$h(x)=\frac{k(x)t(x)}{u(x)v(x)},$$
and
$$\begin{aligned}h'(x)&=\frac{u(x)v(x)k'(x)t(x)+u(x)v(x)k(x)t'(x)-k(x)t(x)u'(x)v(x)-k(x)t(x)u(x)v'(x)}{u^2(x)v^2(x)}\\&=\frac{u(x)v(x)k'(x)t(x)+u(x)v(x)k(x)t'(x)}{u^2(x)v^2(x)}-\frac{k(x)t(x)u'(x)v(x)+k(x)t(x)u(x)v'(x)}{u^2(x)v^2(x)}.\end{aligned}$$
Applying the sum rule, the constant multiple rule and the rules on the derivatives of powers and constants, we obtain $k'(x)=4$, $t'(x)=4x-3$, $u'(x)=2$ and $v'(x)=3x^2$. Hence
$$\begin{aligned}h'(x)=&\frac{4(2x+2)(x^3+6)(2x^2-3x)+(2x+2)(x^3+6)(4x-3)^2}{(2x+2)^2(x^3+6)^2}\\&-\frac{2(4x-3)(2x^2-3x)(x^3+6)+3x^2(4x-3)(2x^2-3x)(2x+2)}{(2x+2)^2(x^3+6)^2}.\end{aligned}$$

11.3. Derivatives of the Trigonometric Functions

Consider the curve $y=f(x)=\sin x$. Suppose that $P(x,f(x))$ is a point on this curve. Consider another point $Q(x+h,f(x+h))$, where $h\neq 0$, which also lies on this curve. Clearly the slope of the line joining the two points P and Q is equal to
$$\frac{f(x+h)-f(x)}{(x+h)-x}=\frac{\sin(x+h)-\sin x}{h}.$$
Consider the curve $y=g(x)=\cos x$. Suppose that $R(x,g(x))$ is a point on this curve. Consider another point $S(x+h,g(x+h))$, where $h\neq 0$, which also lies on this curve. Clearly the slope of the line joining the two points R and S is equal to
$$\frac{g(x+h)-g(x)}{(x+h)-x}=\frac{\cos(x+h)-\cos x}{h}.$$
We now move the point Q along the curve $y=f(x)=\sin x$ towards the point P, and move the point S along the curve $y=g(x)=\cos x$ towards the point R. Recall Example 3.3.9, that when h is very close to 0, we have
$$\frac{\sin(x+h)-\sin x}{h}\approx\cos x\qquad\text{and}\qquad\frac{\cos(x+h)-\cos x}{h}\approx-\sin x.$$
We have established the first two parts of the result below.

DERIVATIVES OF THE TRIGONOMETRIC FUNCTIONS.
(a) *If* $f(x)=\sin x$, *then* $f'(x)=\cos x$.
(b) *If* $g(x)=\cos x$, *then* $g'(x)=-\sin x$.
(c) *If* $t(x)=\tan x$, *then* $t'(x)=\sec^2 x$.
(d) *If* $t(x)=\cot x$, *then* $t'(x)=-\csc^2 x$.
(e) *If* $t(x)=\sec x$, *then* $t'(x)=\tan x\sec x$.
(f) *If* $t(x)=\csc x$, *then* $t'(x)=-\cot x\csc x$.

PROOF. The proofs of parts (c)–(f) depend on the quotient rule as well as parts (a) and (b). For the sake of convenience, we use the functions $f(x) = \sin x$ and $g(x) = \cos x$ throughout this proof, as well as the function $c(x) = 1$, with $c'(x) = 0$.

(c) Suppose that $t(x) = \tan x$. Then $t(x) = f(x)/g(x)$. It follows from the quotient rule that

$$t'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

(d) Suppose that $t(x) = \cot x$. Then $t(x) = g(x)/f(x)$. It follows from the quotient rule that

$$t'(x) = \frac{f(x)g'(x) - g(x)f'(x)}{f^2(x)} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x.$$

(e) Suppose that $t(x) = \sec x$. Then $t(x) = c(x)/g(x)$. It follows from the quotient rule that

$$t'(x) = \frac{g(x)c'(x) - c(x)g'(x)}{g^2(x)} = \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x} \times \frac{1}{\cos x} = \tan x \sec x.$$

(f) Suppose that $t(x) = \csc x$. Then $t(x) = c(x)/f(x)$. It follows from the quotient rule that

$$t'(x) = \frac{f(x)c'(x) - c(x)f'(x)}{f^2(x)} = -\frac{\cos x}{\sin^2 x} = -\frac{\cos x}{\sin x} \times \frac{1}{\sin x} = -\cot x \csc x.$$ ♣

We next combine our knowledge on trigonometric functions with the arithmetic of derivatives. The reader is advised to identify the rules used at each step in the following examples.

EXAMPLE 11.3.1. Consider the function $h(x) = (x^3 - 2)(\sin x + \cos x)$. We can write

$$h(x) = f(x)g(x),$$

where $f(x) = x^3 - 2$ and $g(x) = \sin x + \cos x$. It follows that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

Observe next that $f'(x) = 3x^2$ and $g'(x) = \cos x - \sin x$. Hence

$$h'(x) = 3x^2(\sin x + \cos x) + (x^3 - 2)(\cos x - \sin x).$$

EXAMPLE 11.3.2. Consider the function

$$h(x) = \frac{\sin x}{x}.$$

We can write

$$h(x) = \frac{f(x)}{g(x)},$$

where $f(x) = \sin x$ and $g(x) = x$. It follows that

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

Observe next that $f'(x) = \cos x$ and $g'(x) = 1$. Hence

$$h'(x) = \frac{x \cos x - \sin x}{x^2}.$$

EXAMPLE 11.3.3. Consider the function $h(x) = \sin^2 x$. We can write

$$h(x) = f(x)g(x),$$

where $f(x) = g(x) = \sin x$. It follows that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

Observe next that $f'(x) = g'(x) = \cos x$. Hence

$$h'(x) = \cos x \sin x + \sin x \cos x = 2 \sin x \cos x.$$

EXAMPLE 11.3.4. Consider the function $y = \sin 2x$. We can write

$$h(x) = 2f(x)g(x),$$

where $f(x) = \sin x$ and $g(x) = \cos x$. It follows that

$$h'(x) = 2(f'(x)g(x) + f(x)g'(x)).$$

Observe next that $f'(x) = \cos x$ and $g'(x) = -\sin x$. Hence

$$h'(x) = 2(\cos^2 x - \sin^2 x) = 2 \cos 2x.$$

We shall return to Examples 11.3.3 and 11.3.4 in Section 12.1.

EXAMPLE 11.3.5. Consider the function

$$h(x) = \frac{\cos x}{x^2 - x}.$$

We can write

$$h(x) = \frac{f(x)}{g(x)},$$

where $f(x) = \cos x$ and $g(x) = x^2 - x$. It follows that

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

Observe next that $f'(x) = -\sin x$ and $g'(x) = 2x - 1$. Hence

$$h'(x) = \frac{(x - x^2) \sin x - (2x - 1) \cos x}{(x^2 - x)^2}.$$

EXAMPLE 11.3.6. Consider the function

$$h(x) = \frac{\sin x + \cos x}{1 - x^4}.$$

We can write

$$h(x) = \frac{f(x)}{g(x)},$$

where $f(x) = \sin x + \cos x$ and $g(x) = 1 - x^4$. It follows that

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

Observe next that $f'(x) = \cos x - \sin x$ and $g'(x) = -4x^3$. Hence

$$h'(x) = \frac{(1-x^4)(\cos x - \sin x) + 4x^3(\sin x + \cos x)}{(1-x^4)^2}.$$

Example 11.3.7. Consider the function

$$h(x) = \left(\frac{x^2+1}{\cos x}\right)\sin x.$$

We can write

$$h(x) = f(x)g(x),$$

where

$$f(x) = \frac{x^2+1}{\cos x} \qquad \text{and} \qquad g(x) = \sin x.$$

It follows that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

We can also write

$$f(x) = \frac{k(x)}{t(x)},$$

where $k(x) = x^2 + 1$ and $t(x) = \cos x$. It follows that

$$f'(x) = \frac{t(x)k'(x) - k(x)t'(x)}{t^2(x)},$$

and so

$$h'(x) = \frac{t(x)k'(x) - k(x)t'(x)}{t^2(x)}g(x) + \frac{k(x)}{t(x)}g'(x).$$

Observe next that $k'(x) = 2x$, $t'(x) = -\sin x$ and $g'(x) = \cos x$. Hence

$$\begin{aligned} h'(x) &= \left(\frac{2x\cos x + (x^2+1)\sin x}{\cos^2 x}\right)\sin x + \left(\frac{x^2+1}{\cos x}\right)\cos x \\ &= 2x\tan x + (x^2+1)\tan^2 x + (x^2+1) = 2x\tan x + (x^2+1)\sec^2 x. \end{aligned}$$

Alternatively, we observe that $h(x) = (x^2+1)\tan x$. We can write

$$h(x) = u(x)v(x),$$

where $u(x) = x^2 + 1$ and $v(x) = \tan x$. It follows that

$$h'(x) = u'(x)v(x) + u(x)v'(x).$$

Observe next that $u'(x) = 2x$ and $v'(x) = \sec^2 x$. Hence $h'(x) = 2x\tan x + (x^2+1)\sec^2 x$ as before.

Example 11.3.8. Consider the function $h(x) = \sin^2 x + \cos^2 x$. We can write

$$h(x) = f(x)g(x) + k(x)t(x),$$

where $f(x) = g(x) = \sin x$ and $k(x) = t(x) = \cos x$. It follows that

$$h'(x) = f'(x)g(x) + f(x)g'(x) + u'(x)v(x) + u(x)v'(x).$$

Observe next that $f'(x) = g'(x) = \cos x$ and $k'(x) = t'(x) = -\sin x$. Hence

$$h'(x) = \cos x \sin x + \sin x \cos x - \sin x \cos x - \cos x \sin x = 0.$$

A far simpler way to obtain the same result is to merely observe that $h(x) = 1$.

Problems for Chapter 11

1. For each of the following functions $f(x)$, write down the derivative $f'(x)$ as a function of x, and find the slope of the tangent at the point $P(1, f(1))$:
   a) $f(x) = x^4$ b) $f(x) = 5x^2$ c) $f(x) = \dfrac{1}{6}x^{-3}$ d) $f(x) = \pi x^{1.5}$

2. Find the derivative of each of the following functions, using the rules concerning the derivatives of powers, constants and sums:
   a) $h(x) = 6x^3$ b) $h(x) = 5x^{-7}$
   c) $h(x) = 12x - 3x^2$ d) $h(x) = x^3 + 4x$
   e) $h(x) = 6x^2 - 40x$ f) $h(x) = x^7 + 6x^5 - 8x^2 + 3x$
   g) $h(x) = -\dfrac{3}{x}$ h) $h(x) = \dfrac{7}{x^6}$
   i) $h(x) = \dfrac{6}{x^2}$ j) $h(x) = x^3 + 3x - \dfrac{5}{x^3}$
   k) $h(x) = x^2 - 10x + 100 + \dfrac{4}{x}$ l) $h(x) = x^{100} + 50x + 1 - 2x^{-3} + 7x^{-6}$
   m) $h(x) = \pi x^3 - \dfrac{\pi^2}{x^6}$ n) $h(x) = x^2(x^3 + 3x)$
   o) $h(x) = (x^2 + 3)(2x - 5)$ p) $h(x) = -5\sqrt{x}$
   q) $h(x) = \dfrac{\sqrt{x}}{x^3}$ r) $h(x) = \sqrt{3x}$
   s) $h(x) = \sqrt{4x} + \sqrt{\dfrac{4}{x}}$ t) $h(x) = x^5 + 6x^{-3/2}$

3. Find the derivative of each of the following functions, using the rules concerning the derivatives of powers, constants, sums and products as appropriate:
   a) $h(x) = (x^2 + 3)(2x - 5)$ b) $h(x) = (x^2 - x + 2)(x^2 - 2)$
   c) $h(x) = (x^2 + 5)(x^3 - 4x^2)$ d) $h(x) = (x^4 - 3x^3 + 2x)(3x^2 + 4x)$
   e) $h(x) = (x^9 + 2x^3)x^{-4}$ f) $h(x) = (x^4 - 2x^3 + 7x + 8)^2$
   g) $h(x) = x^{2/3}(x + 2)$ h) $h(x) = (x + 3)(x - 5)(x^2 - 4)$
   i) $h(x) = x^{1/2}(x^3 + x - 2)(3x + 1)$ j) $h(x) = x(x - 1)(x - 2)$

4. Find the derivative of each of the following functions, using the rules concerning the derivatives of powers, constants, sums, products and quotients as appropriate:
   a) $h(x) = \dfrac{1}{x^4 + x^3 + 1}$ b) $h(x) = 1 + \dfrac{3}{x} - \dfrac{2}{x^2}$ c) $h(x) = \dfrac{x - 2}{x + 1}$
   d) $h(x) = \dfrac{1 + x^2}{1 - x^2}$ e) $h(x) = \dfrac{\sqrt{x} - 1}{\sqrt{x} + 1}$ f) $h(x) = \dfrac{x}{x + x^{-1}}$
   g) $h(x) = \dfrac{2x + 3}{3x + 2}$ h) $h(x) = \dfrac{2x + 1}{x - 1}$

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Chapter 12

FURTHER TECHNIQUES OF DIFFERENTIATION

12.1. The Chain Rule

We begin by re-examining a few examples discussed in the previous chapter.

EXAMPLE 12.1.1. Recall Examples 11.2.5 and 11.2.10, that for the function

$$y = h(x) = (x^2 + 2x)^2,$$

we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = h'(x) = 4x^3 + 12x^2 + 8x.$$

On the other hand, we can build a chain and describe the function $y = h(x)$ by writing

$$y = g(u) = u^2 \qquad \text{and} \qquad u = f(x) = x^2 + 2x.$$

Note that

$$\frac{\mathrm{d}y}{\mathrm{d}u} = 2u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x + 2,$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 2u(2x + 2) = 2(x^2 + 2x)(2x + 2) = 4x^3 + 12x^2 + 8x.$$

† This chapter was written at Macquarie University in 1999.

EXAMPLE 12.1.2. Recall Example 11.3.3, that for the function

$$y = h(x) = \sin^2 x,$$

we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = h'(x) = 2\sin x \cos x.$$

On the other hand, we can build a chain and describe the function $y = h(x)$ by writing

$$y = g(u) = u^2 \qquad \text{and} \qquad u = f(x) = \sin x.$$

Note that

$$\frac{\mathrm{d}y}{\mathrm{d}u} = 2u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x,$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 2u\cos x = 2\sin x \cos x.$$

EXAMPLE 12.1.3. Recall Example 11.3.4, that for the function

$$y = h(x) = \sin 2x,$$

we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = h'(x) = 2\cos 2x.$$

On the other hand, we can build a chain and describe the function $y = h(x)$ by writing

$$y = g(u) = \sin u \qquad \text{and} \qquad u = f(x) = 2x.$$

Note that

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \cos u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2,$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 2\cos u = 2\cos 2x.$$

In these three examples, we consider functions of the form $y = h(x)$ which can be described in a chain by $y = g(u)$ and $u = f(x)$, where u is some intermediate variable. Suppose that $x_0, x_1 \in \mathbb{R}$. Write $u_0 = f(x_0)$ and $u_1 = f(x_1)$, and write $y_0 = g(u_0)$ and $y_1 = g(u_1)$. Then clearly $h(x_0) = g(f(x_0))$ and $h(x_1) = g(f(x_1))$. Heuristically, we have

$$\frac{h(x_1) - h(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1 - y_0}{u_1 - u_0} \times \frac{u_1 - u_0}{x_1 - x_0} = \frac{g(u_1) - g(u_0)}{u_1 - u_0} \times \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

If x_1 is close to x_0, then we expect that u_1 is close to u_0, and so the product

$$\frac{g(u_1) - g(u_0)}{u_1 - u_0} \times \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

is close to $g'(u_0)f'(x_0)$, while the product

$$\frac{h(x_1) - h(x_0)}{x_1 - x_0}$$

is close to $h'(x_0)$. It is therefore not unreasonable to expect the following result, although a formal proof is somewhat more complicated.

CHAIN RULE. *Suppose that $y = g(u)$ and $u = f(x)$. Then*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x},$$

provided that the two derivatives on the right hand side exist.

We can interpret the rule in the following way. As we vary x, the value $u = f(x)$ changes at the rate of $\mathrm{d}u/\mathrm{d}x$. This change in the value of $u = f(x)$ in turn causes a change in the value of $y = g(u)$ at the rate of $\mathrm{d}y/\mathrm{d}u$.

EXAMPLE 12.1.4. Consider the function $y = h(x) = (x^2 - 6x + 5)^3$. We can set up a chain by writing

$$y = g(u) = u^3 \qquad \text{and} \qquad u = f(x) = x^2 - 6x + 5.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = 3u^2 \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x - 6,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 3u^2(2x-6) = 6(x^2 - 6x + 5)^2(x-3).$$

EXAMPLE 12.1.5. Consider the function $y = h(x) = \sin^4 x$. We can set up a chain by writing

$$y = g(u) = u^4 \qquad \text{and} \qquad u = f(x) = \sin x.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = 4u^3 \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 4u^3 \cos x = 4 \sin^3 x \cos x.$$

EXAMPLE 12.1.6. Consider the function $y = h(x) = \sec(x^4)$. We can set up a chain by writing

$$y = g(u) = \sec u \qquad \text{and} \qquad u = f(x) = x^4.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \tan u \sec u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 4x^3,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 4x^3 \tan u \sec u = 4x^3 \tan(x^4) \sec(x^4).$$

EXAMPLE 12.1.7. Consider the function $y = h(x) = \tan(x^2 - 3x + 4)$. We can set up a chain by writing

$$y = g(u) = \tan u \qquad \text{and} \qquad u = f(x) = x^2 - 3x + 4.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \sec^2 u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x - 3,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}=(2x-3)\sec^2 u=(2x-3)\sec^2(x^2-3x+4).$$

EXAMPLE 12.1.8. Consider the function $y=h(x)=(x^2+5x-1)^{2/3}$. We can set up a chain by writing

$$y=g(u)=u^{2/3}\qquad\text{and}\qquad u=f(x)=x^2+5x-1.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u}=\frac{2}{3}u^{-1/3}\qquad\text{and}\qquad\frac{\mathrm{d}u}{\mathrm{d}x}=2x+5,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{2}{3}u^{-1/3}(2x+5)=\frac{2}{3}(x^2+5x-1)^{-1/3}(2x+5).$$

EXAMPLE 12.1.9. Consider the function

$$y=h(x)=\frac{1}{\cos^3 x}.$$

We can set up a chain by writing

$$y=g(u)=\frac{1}{u^3}=u^{-3}\qquad\text{and}\qquad u=f(x)=\cos x.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u}=-3u^{-4}\qquad\text{and}\qquad\frac{\mathrm{d}u}{\mathrm{d}x}=-\sin x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}=3u^{-4}\sin x=\frac{3\sin x}{u^4}=\frac{3\sin x}{\cos^4 x}.$$

EXAMPLE 12.1.10. Consider the function

$$y=h(x)=\frac{1}{(2x^3-5x+1)^4}.$$

We can set up a chain by writing

$$y=g(u)=\frac{1}{u^4}=u^{-4}\qquad\text{and}\qquad u=f(x)=2x^3-5x+1.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u}=-4u^{-5}\qquad\text{and}\qquad\frac{\mathrm{d}u}{\mathrm{d}x}=6x^2-5,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}=4u^{-5}(5-6x^2)=\frac{4(5-6x^2)}{u^5}=\frac{4(5-6x^2)}{(2x^3-5x+1)^5}.$$

Example 12.1.11. Consider the function $y = h(x) = \sin(\cos x)$. We can set up a chain by writing

$$y = g(u) = \sin u \qquad \text{and} \qquad u = f(x) = \cos x.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \cos u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = -\cos u \sin x = -\cos(\cos x)\sin x.$$

Example 12.1.12. Consider the function

$$y = h(x) = \frac{1}{2(x+1)} + \frac{1}{4(x+1)^2}.$$

We can set up a chain by writing

$$y = g(u) = \frac{1}{2u} + \frac{1}{4u^2} = \frac{1}{2}u^{-1} + \frac{1}{4}u^{-2} \qquad \text{and} \qquad u = f(x) = x + 1.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = -\frac{1}{2}u^{-2} - \frac{1}{2}u^{-3} \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 1,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{1}{2}u^{-2} - \frac{1}{2}u^{-3} = -\frac{1}{2(x+1)^2} - \frac{1}{2(x+1)^3}.$$

Example 12.1.13. Consider the function

$$y = h(x) = \left(\frac{x-1}{x+1}\right)^3.$$

We can set up a chain by writing

$$y = g(u) = u^3 \qquad \text{and} \qquad u = f(x) = \frac{x-1}{x+1}.$$

Clearly we have (using the quotient rule for the latter)

$$\frac{\mathrm{d}y}{\mathrm{d}u} = 3u^2 \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2}{(x+1)^2},$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 3u^2 \times \frac{2}{(x+1)^2} = 3\left(\frac{x-1}{x+1}\right)^2 \times \frac{2}{(x+1)^2} = \frac{6(x-1)^2}{(x+1)^4}.$$

The chain rule can be extended to chains of more than two functions. We illustrate the ideas by considering the next four examples.

Example 12.1.14. Consider the function $y = h(x) = \sin^3(x^2+2)$. We can set up a chain by writing

$$y = k(v) = v^3, \qquad v = g(u) = \sin u \qquad \text{and} \qquad u = f(x) = x^2 + 2.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}v} = 3v^2, \qquad \frac{\mathrm{d}v}{\mathrm{d}u} = \cos u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}v} \times \frac{\mathrm{d}v}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 6xv^2 \cos u = 6x \sin^2 u \cos u = 6x \sin^2(x^2+2)\cos(x^2+2).$$

EXAMPLE 12.1.15. Consider the function $y = h(x) = (1 + (1 + x)^{1/2})^5$. We can set up a chain by writing

$$y = k(v) = v^5, \qquad v = g(u) = 1 + u^{1/2} \qquad \text{and} \qquad u = f(x) = 1 + x.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}v} = 5v^4, \qquad \frac{\mathrm{d}v}{\mathrm{d}u} = \frac{1}{2}u^{-1/2} \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 1,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}v} \times \frac{\mathrm{d}v}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{5}{2}v^4u^{-1/2} = \frac{5}{2}(1 + u^{1/2})^4u^{-1/2} = \frac{5(1 + (1 + x)^{1/2})^4}{2(1 + x)^{1/2}}.$$

EXAMPLE 12.1.16. Consider the function $y = h(x) = \tan((x^4 - 3x)^3)$. We can set up a chain by writing

$$y = k(v) = \tan v, \qquad v = g(u) = u^3 \qquad \text{and} \qquad u = f(x) = x^4 - 3x.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}v} = \sec^2 v, \qquad \frac{\mathrm{d}v}{\mathrm{d}u} = 3u^2 \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 4x^3 - 3,$$

so it follows from the chain rule that

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}v} \times \frac{\mathrm{d}v}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} &= 3u^2(4x^3 - 3)\sec^2 v = 3u^2(4x^3 - 3)\sec^2(u^3) \\ &= 3(x^4 - 3x)^2(4x^3 - 3)\sec^2((x^4 - 3x)^3).\end{aligned}$$

EXAMPLE 12.1.17. Consider the function $y = h(x) = \sqrt{x^2 + \sin(x^2)}$. We can set up a chain by writing

$$y = k(v) = v^{1/2}, \qquad v = g(u) = u + \sin u \qquad \text{and} \qquad u = f(x) = x^2.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}v} = \frac{1}{2v^{1/2}}, \qquad \frac{\mathrm{d}v}{\mathrm{d}u} = 1 + \cos u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}v} \times \frac{\mathrm{d}v}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{x(1 + \cos u)}{v^{1/2}} = \frac{x(1 + \cos u)}{(u + \sin u)^{1/2}} = \frac{x(1 + \cos(x^2))}{\sqrt{x^2 + \sin(x^2)}}.$$

We conclude this section by studying three examples where the chain rule is used only in part of the argument. These examples are rather hard, and the reader is advised to concentrate on the ideas and not to get overly worried about the arithmetic details. For accuracy, it is absolutely crucial that we exercise great care.

EXAMPLE 12.1.18. Consider the function $y = h(x) = (x^2 - 1)^{1/2}(x^2 + 4x + 3)$. We can write

$$h(x) = f(x)g(x),$$

where

$$f(x) = (x^2 - 1)^{1/2} \qquad \text{and} \qquad g(x) = x^2 + 4x + 3.$$

It follows from the product rule that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

It is easy to see that $g'(x) = 2x + 4$. To find $f'(x)$, we shall use the chain rule. Let

$$z = f(x) = (x^2 - 1)^{1/2}.$$

We can set up a chain by writing

$$z = u^{1/2} \qquad \text{and} \qquad u = x^2 - 1.$$

Clearly we have

$$\frac{\mathrm{d}z}{\mathrm{d}u} = \frac{1}{2u^{1/2}} \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x,$$

so it follows from the chain rule that

$$f'(x) = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{x}{u^{1/2}} = \frac{x}{(x^2 - 1)^{1/2}}.$$

Hence

$$h'(x) = \frac{x(x^2 + 4x + 3)}{(x^2 - 1)^{1/2}} + (x^2 - 1)^{1/2}(2x + 4).$$

Example 12.1.19. Consider the function

$$y = h(x) = \frac{(1 - x^3)^2}{(1 + 2x + 3x^2)^2}.$$

We can write

$$h(x) = \frac{f(x)}{g(x)},$$

where

$$f(x) = (1 - x^3)^2 \qquad \text{and} \qquad g(x) = (1 + 2x + 3x^2)^2.$$

It follows from the quotient rule that

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

To find $f'(x)$ and $g'(x)$, we shall use the chain rule. Let

$$z = f(x) = (1 - x^3)^2 \qquad \text{and} \qquad w = g(x) = (1 + 2x + 3x^2)^2.$$

We can set up a chain by writing

$$z = u^2 \qquad \text{and} \qquad u = 1 - x^3.$$

Then

$$f'(x) = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 2u \times (-3x^2) = -6ux^2 = -6(1 - x^3)x^2.$$

Similarly, we can set up a chain by writing

$$w = v^2 \qquad \text{and} \qquad v = 1 + 2x + 3x^2.$$

Then

$$g'(x) = \frac{\mathrm{d}w}{\mathrm{d}x} = \frac{\mathrm{d}w}{\mathrm{d}v} \times \frac{\mathrm{d}v}{\mathrm{d}x} = 2v \times (2 + 6x) = 4v(1 + 3x) = 4(1 + 2x + 3x^2)(1 + 3x).$$

Hence

$$h'(x) = \frac{-6(1+2x+3x^2)^2(1-x^3)x^2 - 4(1-x^3)^2(1+2x+3x^2)(1+3x)}{(1+2x+3x^2)^4}$$
$$= -\frac{6(1+2x+3x^2)(1-x^3)x^2 + 4(1-x^3)^2(1+3x)}{(1+2x+3x^2)^3}.$$

Alternatively, observe that we can set up a chain by writing

$$y = s^2 \qquad \text{and} \qquad s = \frac{1-x^3}{1+2x+3x^2}.$$

Clearly we have (using the quotient rule for the latter)

$$\frac{\mathrm{d}y}{\mathrm{d}s} = 2s \qquad \text{and} \qquad \frac{\mathrm{d}s}{\mathrm{d}x} = \frac{-3(1+2x+3x^2)x^2 - (1-x^3)(2+6x)}{(1+2x+3x^2)^2},$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}s} \times \frac{\mathrm{d}s}{\mathrm{d}x} = -2s \times \frac{3(1+2x+3x^2)x^2 + (1-x^3)(2+6x)}{(1+2x+3x^2)^2}$$
$$= -\frac{2(1-x^3)}{1+2x+3x^2} \times \frac{3(1+2x+3x^2)x^2 + 2(1-x^3)(1+3x)}{(1+2x+3x^2)^2}.$$

It can be easily checked that the answer is the same as before.

EXAMPLE 12.1.20. Consider the function $y = h(x) = (x^2 + (x^3 + x^5)^7)^{11}$. We can set up a chain by writing

$$y = g(u) = u^{11} \qquad \text{and} \qquad u = f(x) = x^2 + (x^3 + x^5)^7.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = 11u^{10}.$$

On the other hand, we have $f(x) = k(x) + t(x)$, where $k(x) = x^2$ and $t(x) = (x^3 + x^5)^7$. It follows that $f'(x) = k'(x) + t'(x)$. Note that $k'(x) = 2x$. To find $t'(x)$, we shall use the chain rule. Let

$$z = t(x) = (x^3 + x^5)^7.$$

We can set up a chain by writing

$$z = v^7 \qquad \text{and} \qquad v = x^3 + x^5.$$

Clearly we have

$$\frac{\mathrm{d}z}{\mathrm{d}v} = 7v^6 \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = 3x^2 + 5x^4,$$

so it follows from the chain rule that

$$t'(x) = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}v} \times \frac{\mathrm{d}v}{\mathrm{d}x} = 7v^6(3x^2 + 5x^4) = 7(x^3 + x^5)^6(3x^2 + 5x^4),$$

and so

$$f'(x) = \frac{\mathrm{d}u}{\mathrm{d}x} = 2x + 7(x^3 + x^5)^6(3x^2 + 5x^4).$$

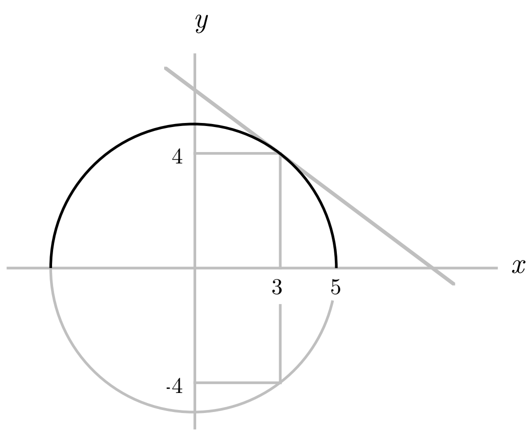
It then follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 11u^{10}(2x + 7(x^3 + x^5)^6(3x^2 + 5x^4))$$
$$= 11(x^2 + (x^3 + x^5)^7)^{10}(2x + 7(x^3 + x^5)^6(3x^2 + 5x^4)).$$

12.2. Implicit Differentiation

A function $y = f(x)$ can usually be viewed as a curve on the xy-plane, and gives a relationship between the (independent) variable x and the (dependent) variable y by describing y explicitly in terms of x. However, a relationship between two variables x and y cannot always be expressed as a function $y = f(x)$. Moreover, we may even choose to describe a function $y = f(x)$ implicitly by simply giving some relationship between the variables x and y, and not describing y explicitly in terms of x.

EXAMPLE 12.2.1. Consider the equation $x^2 + y^2 = 25$, representing a circle of radius 5 and centred at the origin $(0, 0)$. This equation expresses a relationship between the two variables x and y, but y is not given explicitly in terms of x. Indeed, it is not possible to give y explicitly in terms of x, as this equation does not represent a function $y = f(x)$. To see this, note that if $x = 3$, then both $y = 4$ and $y = -4$ will satisfy the equation, so it is meaningless to talk of $f(3)$. On the other hand, we see that the point $(3, 4)$ is on the circle, and clearly there is a tangent line to the circle at the point $(3, 4)$, as shown in the picture below.



If we restrict our attention to the upper semicircle, then we can express the variable y explicitly as a function of the variable x by writing

$$y = (25 - x^2)^{1/2}.$$

We can set up a chain by writing

$$y = u^{1/2} \qquad \text{and} \qquad u = 25 - x^2.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{1}{2}u^{-1/2} \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = -2x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = -xu^{-1/2} = -x(25 - x^2)^{-1/2}.$$

Hence

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(3,4)} = -\frac{3}{4}.$$

Note that our argument here involves obtaining an explicit expression for the variable y in terms of the variable x from similar information given implicitly by the equation $x^2 + y^2 = 25$. Now let us see whether we can obtain a similar conclusion concerning the slope of the tangent line at the point $(3, 4)$ without first having to obtain the explicit expression $y = (25 - x^2)^{1/2}$ of the upper semicircle. Let us start from the equation $x^2 + y^2 = 25$ of the circle. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + y^2) = \frac{\mathrm{d}}{\mathrm{d}x}(25).$$

Using the rule on the derivatives of constants, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(25) = 0.$$

Using the sum rule and the rule on the derivatives of powers, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + y^2) = \frac{\mathrm{d}}{\mathrm{d}x}(x^2) + \frac{\mathrm{d}}{\mathrm{d}x}(y^2) = 2x + \frac{\mathrm{d}}{\mathrm{d}x}(y^2).$$

We next set up a chain by writing

$$z = y^2 \qquad \text{and} \qquad y = f(x),$$

where there is no need to know precisely what $f(x)$ is. Then using the chain rule and the rule on the derivatives of powers, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^2) = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y} \times \frac{\mathrm{d}y}{\mathrm{d}x} = 2y\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we obtain

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}.$$

Hence

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(3,4)} = -\frac{3}{4}$$

as before.

The second part of the example above is a case of using implicit differentiation, where we find the derivative of a function $y = f(x)$ without knowing any explicit expression for the variable y in terms of the variable x. We shall describe this technique further by discussing a few more examples. In some of these examples, it may be very difficult, if not impossible, to find any explicit expression for the variable y in terms of the variable x.

Example 12.2.2. Suppose that $y^2 - x^2 = 4$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^2 - x^2) = \frac{\mathrm{d}}{\mathrm{d}x}(4) = 0.$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^2 - x^2) = \frac{\mathrm{d}}{\mathrm{d}x}(y^2) - \frac{\mathrm{d}}{\mathrm{d}x}(x^2) = \frac{\mathrm{d}}{\mathrm{d}x}(y^2) - 2x = 0.$$

We next set up a chain by writing

$$z = y^2 \qquad \text{and} \qquad y = f(x),$$

where there is no need to know precisely what $f(x)$ is. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^2) = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y} \times \frac{\mathrm{d}y}{\mathrm{d}x} = 2y\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we obtain

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} - 2x = 0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}.$$

EXAMPLE 12.2.3. Suppose that $y^3 + \sin x = 3$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^3 + \sin x) = \frac{\mathrm{d}}{\mathrm{d}x}(3) = 0.$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^3 + \sin x) = \frac{\mathrm{d}}{\mathrm{d}x}(y^3) + \frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \frac{\mathrm{d}}{\mathrm{d}x}(y^3) + \cos x = 0.$$

We next set up a chain by writing

$$z = y^3 \qquad \text{and} \qquad y = f(x),$$

where there is no need to know precisely what $f(x)$ is. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^3) = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y} \times \frac{\mathrm{d}y}{\mathrm{d}x} = 3y^2\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we obtain

$$3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + \cos x = 0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\cos x}{3y^2}.$$

EXAMPLE 12.2.4. Suppose that $y^5 + 3y^2 - 2x^2 + 4 = 0$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^5 + 3y^2 - 2x^2 + 4) = \frac{\mathrm{d}}{\mathrm{d}x}(0) = 0.$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^5 + 3y^2 - 2x^2 + 4) = \frac{\mathrm{d}}{\mathrm{d}x}(y^5) + 3\frac{\mathrm{d}}{\mathrm{d}x}(y^2) - 2\frac{\mathrm{d}}{\mathrm{d}x}(x^2) + \frac{\mathrm{d}}{\mathrm{d}x}(4) = \frac{\mathrm{d}}{\mathrm{d}x}(y^5) + 3\frac{\mathrm{d}}{\mathrm{d}x}(y^2) - 4x = 0.$$

Using the chain rule, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^5) = \frac{\mathrm{d}}{\mathrm{d}y}(y^5) \times \frac{\mathrm{d}y}{\mathrm{d}x} = 5y^4\frac{\mathrm{d}y}{\mathrm{d}x} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(y^2) = \frac{\mathrm{d}}{\mathrm{d}y}(y^2) \times \frac{\mathrm{d}y}{\mathrm{d}x} = 2y\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we obtain

$$(5y^4 + 6y)\frac{\mathrm{d}y}{\mathrm{d}x} - 4x = 0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x}{5y^4 + 6y}.$$

EXAMPLE 12.2.5. Suppose that $xy = 6$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = \frac{\mathrm{d}}{\mathrm{d}x}(6) = 0.$$

It follows from the product rule that

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = \frac{\mathrm{d}}{\mathrm{d}x}(x) \times y + x \times \frac{\mathrm{d}}{\mathrm{d}x}(y) = y + x\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x}.$$

EXAMPLE 12.2.6. Suppose that $x^3+2x^2y^3+3y^4=6$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^3+2x^2y^3+3y^4)=\frac{\mathrm{d}}{\mathrm{d}x}(6)=0.$$

It follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(x^3+2x^2y^3+3y^4)&=\frac{\mathrm{d}}{\mathrm{d}x}(x^3)+2\left(\frac{\mathrm{d}}{\mathrm{d}x}(x^2)\times y^3+x^2\times\frac{\mathrm{d}}{\mathrm{d}x}(y^3)\right)+3\frac{\mathrm{d}}{\mathrm{d}x}(y^4)\\&=3x^2+4xy^3+2x^2\frac{\mathrm{d}}{\mathrm{d}x}(y^3)+3\frac{\mathrm{d}}{\mathrm{d}x}(y^4)=0.\end{aligned}$$

Using the chain rule, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^3)=\frac{\mathrm{d}}{\mathrm{d}y}(y^3)\times\frac{\mathrm{d}y}{\mathrm{d}x}=3y^2\frac{\mathrm{d}y}{\mathrm{d}x}\qquad\text{and}\qquad\frac{\mathrm{d}}{\mathrm{d}x}(y^4)=\frac{\mathrm{d}}{\mathrm{d}y}(y^4)\times\frac{\mathrm{d}y}{\mathrm{d}x}=4y^3\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we obtain

$$3x^2+4xy^3+(6x^2y^2+12y^3)\frac{\mathrm{d}y}{\mathrm{d}x}=0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{3x^2+4xy^3}{6x^2y^2+12y^3}.$$

Note next that the point $(1,1)$ satisfies the equation. It follows that

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(1,1)}=-\frac{7}{18}.$$

Check that the equation of the tangent line at this point is given by $7x+18y=25$.

EXAMPLE 12.2.7. Suppose that $(x^2+y^3)^2=9$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}((x^2+y^3)^2)=\frac{\mathrm{d}}{\mathrm{d}x}(9)=0.$$

Let $w=(x^2+y^3)^2$. We can set up a chain by writing

$$w=z^2\qquad\text{and}\qquad z=x^2+y^3,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}}{\mathrm{d}x}((x^2+y^3)^2)=\frac{\mathrm{d}w}{\mathrm{d}x}=\frac{\mathrm{d}w}{\mathrm{d}z}\times\frac{\mathrm{d}z}{\mathrm{d}x}=2z\frac{\mathrm{d}z}{\mathrm{d}x}=2(x^2+y^3)\frac{\mathrm{d}}{\mathrm{d}x}(x^2+y^3).$$

Hence

$$(x^2+y^3)\frac{\mathrm{d}}{\mathrm{d}x}(x^2+y^3)=0.$$

On the other hand,

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2+y^3)=\frac{\mathrm{d}}{\mathrm{d}x}(x^2)+\frac{\mathrm{d}}{\mathrm{d}x}(y^3)=2x+3y^2\frac{\mathrm{d}y}{\mathrm{d}x},$$

where we have used the chain rule at the last step. Summarizing, we obtain

$$(x^2+y^3)\left(2x+3y^2\frac{\mathrm{d}y}{\mathrm{d}x}\right)=0.$$

It is clear that $x^2+y^3\neq 0$ for any point (x,y) satisfying the equation. It follows that

$$2x+3y^2\frac{\mathrm{d}y}{\mathrm{d}x}=0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x}{3y^2}.$$

Note next that the point $(2,-1)$ satisfies the equation. It follows that

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(2,-1)} = -\frac{4}{3}.$$

Check that the equation of the tangent line at this point is given by $4x+3y=5$.

EXAMPLE 12.2.8. The point $(1,1)$ is one of the intersection points of the parabola $y-x^2=0$ and the ellipse $x^2+2y^2=3$. We shall show that the two tangents at $(1,1)$ are perpendicular to each other. Consider first of all the parabola $y-x^2=0$. Here we can write $y=x^2$, so that $\mathrm{d}y/\mathrm{d}x=2x$. Hence

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(1,1)} = 2.$$

Consider next the ellipse $x^2+2y^2=3$. Using implicit differentiation, it is not difficult to show that

$$2x+4y\frac{\mathrm{d}y}{\mathrm{d}x}=0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{2y}.$$

Hence

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(1,1)} = -\frac{1}{2}.$$

Since the product of the two derivatives is equal to -1, it follows that the two tangents are perpendicular to each other.

12.3. Derivatives of the Exponential and Logarithmic Functions

We shall state without proof the following result.

DERIVATIVE OF THE EXPONENTIAL FUNCTION. *If $f(x)=\mathrm{e}^x$, then $f'(x)=\mathrm{e}^x$.*

EXAMPLE 12.3.1. Consider the function $y=h(x)=\mathrm{e}^x(\sin x+2\cos x)$. We can write

$$h(x)=f(x)g(x),$$

where $f(x)=\mathrm{e}^x$ and $g(x)=\sin x+2\cos x$. It follows from the product rule that

$$h'(x)=f'(x)g(x)+f(x)g'(x).$$

Clearly

$$f'(x)=\mathrm{e}^x \qquad \text{and} \qquad g'(x)=\cos x-2\sin x.$$

Hence

$$h'(x)=\mathrm{e}^x(\sin x+2\cos x)+\mathrm{e}^x(\cos x-2\sin x)=\mathrm{e}^x(3\cos x-\sin x).$$

EXAMPLE 12.3.2. Consider the function $y = h(x) = \mathrm{e}^x(x^2 + x + 2)$. We can write

$$h(x) = f(x)g(x),$$

where $f(x) = \mathrm{e}^x$ and $g(x) = x^2 + x + 2$. It follows from the product rule that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

Clearly

$$f'(x) = \mathrm{e}^x \qquad \text{and} \qquad g'(x) = 2x + 1.$$

Hence

$$h'(x) = \mathrm{e}^x(x^2 + x + 2) + \mathrm{e}^x(2x + 1) = \mathrm{e}^x(x^2 + 3x + 3).$$

EXAMPLE 12.3.3. Consider the function $y = h(x) = \mathrm{e}^{2x}$. We can set up a chain by writing

$$y = g(u) = \mathrm{e}^u \qquad \text{and} \qquad u = f(x) = 2x.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \mathrm{e}^u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 2\mathrm{e}^u = 2\mathrm{e}^{2x}.$$

Alternatively, we can set up a chain by writing

$$y = t(v) = v^2 \qquad \text{and} \qquad v = k(x) = \mathrm{e}^x.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}v} = 2v \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}v} \times \frac{\mathrm{d}v}{\mathrm{d}x} = 2v\mathrm{e}^x = 2\mathrm{e}^x\mathrm{e}^x = 2\mathrm{e}^{2x}.$$

Yet another alternative is to observe that $h(x) = \mathrm{e}^x\mathrm{e}^x$. It follows that we can use the product rule instead of the chain rule. Try it!

EXAMPLE 12.3.4. Consider the function $y = h(x) = \mathrm{e}^{x^3}$. We can set up a chain by writing

$$y = g(u) = \mathrm{e}^u \qquad \text{and} \qquad u = f(x) = x^3.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \mathrm{e}^u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2\mathrm{e}^u = 3x^2\mathrm{e}^{x^3}.$$

EXAMPLE 12.3.5. Consider the function $y = h(x) = \mathrm{e}^{\sin x + 4\cos x}$. We can set up a chain by writing

$$y = g(u) = \mathrm{e}^u \qquad \text{and} \qquad u = f(x) = \sin x + 4\cos x.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \mathrm{e}^u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x - 4\sin x,$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^u(\cos x - 4\sin x) = \mathrm{e}^{\sin x + 4\cos x}(\cos x - 4\sin x).$$

EXAMPLE 12.3.6. Consider the function

$$y = h(x) = \sin^3\left(\mathrm{e}^{4x^2}\right).$$

We can set up a chain by writing

$$y = t(w) = w^3, \qquad w = k(v) = \sin v, \qquad v = g(u) = \mathrm{e}^u \qquad \text{and} \qquad u = f(x) = 4x^2.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}w} = 3w^2, \qquad \frac{\mathrm{d}w}{\mathrm{d}v} = \cos v, \qquad \frac{\mathrm{d}v}{\mathrm{d}u} = \mathrm{e}^u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 8x,$$

so it follows from the chain rule that

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}w} \times \frac{\mathrm{d}w}{\mathrm{d}v} \times \frac{\mathrm{d}v}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} &= 24x\mathrm{e}^u w^2 \cos v = 24x\mathrm{e}^u \sin^2 v \cos v = 24x\mathrm{e}^u \sin^2(\mathrm{e}^u)\cos(\mathrm{e}^u) \\ &= 24x\mathrm{e}^{4x^2}\sin^2(\mathrm{e}^{4x^2})\cos(\mathrm{e}^{4x^2}).\end{aligned}$$

EXAMPLE 12.3.7. Suppose that $\mathrm{e}^{2x} + y^2 = 5$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x} + y^2) = \frac{\mathrm{d}}{\mathrm{d}x}(5) = 0.$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x} + y^2) = \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x}) + \frac{\mathrm{d}}{\mathrm{d}x}(y^2) = 2\mathrm{e}^{2x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

using Example 12.3.3 and the chain rule. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\mathrm{e}^{2x}}{y}.$$

The next example is rather complicated, and the reader is advised to concentrate on the ideas and not to get overly worried about the arithmetic details. For accuracy, it is absolutely crucial that we exercise great care.

EXAMPLE 12.3.8. Suppose that $\mathrm{e}^{2x}\sin 3y + x^2y^3 = 3$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x}\sin 3y + x^2y^3) = \frac{\mathrm{d}}{\mathrm{d}x}(3) = 0.$$

Using the sum and product rules, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x}\sin 3y + x^2y^3) = \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x}) \times \sin 3y + \mathrm{e}^{2x} \times \frac{\mathrm{d}}{\mathrm{d}x}(\sin 3y) + \frac{\mathrm{d}}{\mathrm{d}x}(x^2) \times y^3 + x^2 \times \frac{\mathrm{d}}{\mathrm{d}x}(y^3).$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x}) = 2\mathrm{e}^{2x} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x.$$

Writing $z = 3y$ and using the chain rule, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin 3y) = \frac{\mathrm{d}}{\mathrm{d}y}(\sin 3y) \times \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}y}(\sin z) \times \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}z}(\sin z) \times \frac{\mathrm{d}z}{\mathrm{d}y} \times \frac{\mathrm{d}y}{\mathrm{d}x} = 3\cos z\frac{\mathrm{d}y}{\mathrm{d}x} = 3\cos 3y\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Using the chain rule, we also obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^3) = \frac{\mathrm{d}}{\mathrm{d}y}(y^3) \times \frac{\mathrm{d}y}{\mathrm{d}x} = 3y^2\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we have

$$2\mathrm{e}^{2x}\sin 3y + 3\mathrm{e}^{2x}\cos 3y\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy^3 + 3x^2y^2\frac{\mathrm{d}y}{\mathrm{d}x} = 2(\mathrm{e}^{2x}\sin 3y + xy^3) + 3(\mathrm{e}^{2x}\cos 3y + x^2y^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

and so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2(\mathrm{e}^{2x}\sin 3y + xy^3)}{3(\mathrm{e}^{2x}\cos 3y + x^2y^2)}.$$

Next, we turn to the logarithmic function. Using implicit differentiation, we can establish the following result.

DERIVATIVE OF THE LOGARITHMIC FUNCTION. *If $f(x) = \log x$, then $f'(x) = 1/x$.*

PROOF. Suppose that $y = \log x$. Then $\mathrm{e}^y = x$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^y) = \frac{\mathrm{d}}{\mathrm{d}x}(x) = 1.$$

Using the chain rule and the rule on the derivative of the exponential function, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^y) = \frac{\mathrm{d}}{\mathrm{d}y}(\mathrm{e}^y) \times \frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we have

$$\mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x} = 1,$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{e}^y} = \frac{1}{x}.$$

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EXAMPLE 12.3.9. Consider the function $y = h(x) = x\log x$. We can write

$$h(x) = f(x)g(x),$$

where $f(x) = x$ and $g(x) = \log x$. It follows from the product rule that

$$h'(x) = f'(x)g(x) + f(x)g'(x).$$

Clearly $f'(x) = 1$ and $g'(x) = 1/x$. Hence $h'(x) = \log x + 1$.

EXAMPLE 12.3.10. Consider the function

$$y = h(x) = \frac{x \log x + \sin x}{e^x}.$$

We can write

$$h(x) = \frac{f(x) + k(x)}{g(x)},$$

where $f(x) = x \log x$, $k(x) = \sin x$ and $g(x) = e^x$. It follows from the sum and quotient rules that

$$h'(x) = \frac{g(x)(f'(x) + k'(x)) - (f(x) + k(x))g'(x)}{g^2(x)}.$$

Clearly $k'(x) = \cos x$ and $g'(x) = e^x$. Observe also from Example 12.3.9 that $f'(x) = \log x + 1$. Hence

$$h'(x) = \frac{e^x(\log x + 1 + \cos x) - (x \log x + \sin x)e^x}{e^{2x}} = \frac{(1-x)\log x + 1 + \cos x - \sin x}{e^x}.$$

EXAMPLE 12.3.11. Consider the function $y = h(x) = \log(5x^2 + 3)$. We can set up a chain by writing

$$y = g(u) = \log u \qquad \text{and} \qquad u = f(x) = 5x^2 + 3.$$

Clearly we have

$$\frac{dy}{du} = \frac{1}{u} \qquad \text{and} \qquad \frac{du}{dx} = 10x,$$

so it follows from the chain rule that

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{10x}{u} = \frac{10x}{5x^2 + 3}.$$

EXAMPLE 12.3.12. Consider the function $y = h(x) = \log(\tan x + \sec x)$. We can set up a chain by writing

$$y = g(u) = \log u \qquad \text{and} \qquad u = f(x) = \tan x + \sec x.$$

Clearly we have

$$\frac{dy}{du} = \frac{1}{u} \qquad \text{and} \qquad \frac{du}{dx} = \sec^2 x + \tan x \sec x,$$

so it follows from the chain rule that

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{\sec^2 x + \tan x \sec x}{u} = \frac{\sec^2 x + \tan x \sec x}{\tan x + \sec x} = \sec x.$$

EXAMPLE 12.3.13. Consider the function $y = h(x) = \log(\cot x + \csc x)$. We can set up a chain by writing

$$y = g(u) = \log u \qquad \text{and} \qquad u = f(x) = \cot x + \csc x.$$

Clearly we have

$$\frac{dy}{du} = \frac{1}{u} \qquad \text{and} \qquad \frac{du}{dx} = -\csc^2 x - \cot x \csc x,$$

so it follows from the chain rule that

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{-\csc^2 x - \cot x \csc x}{u} = -\frac{\csc^2 x + \cot x \csc x}{\cot x + \csc x} = -\csc x.$$

EXAMPLE 12.3.14. Consider the function $y = h(x) = \log(\sin(x^{1/2}))$. We can set up a chain by writing

$$y = k(v) = \log v, \qquad v = g(u) = \sin u \qquad \text{and} \qquad u = f(x) = x^{1/2}.$$

Clearly we have

$$\frac{\mathrm{d}y}{\mathrm{d}v} = \frac{1}{v}, \qquad \frac{\mathrm{d}v}{\mathrm{d}u} = \cos u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2x^{1/2}},$$

so it follows from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}v} \times \frac{\mathrm{d}v}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\cos u}{2x^{1/2}v} = \frac{\cos u}{2x^{1/2}\sin u} = \frac{\cos(x^{1/2})}{2x^{1/2}\sin(x^{1/2})}.$$

EXAMPLE 12.3.15. Suppose that $x\log y + y^2 = 4$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(x\log y + y^2) = \frac{\mathrm{d}}{\mathrm{d}x}(4) = 0.$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}(x\log y + y^2) = \frac{\mathrm{d}}{\mathrm{d}x}(x) \times \log y + x \times \frac{\mathrm{d}}{\mathrm{d}x}(\log y) + \frac{\mathrm{d}}{\mathrm{d}x}(y^2) = \log y + x\frac{\mathrm{d}}{\mathrm{d}x}(\log y) + \frac{\mathrm{d}}{\mathrm{d}x}(y^2).$$

By the chain rule, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log y) = \frac{\mathrm{d}}{\mathrm{d}y}(\log y) \times \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}(y^2) = \frac{\mathrm{d}}{\mathrm{d}y}(y^2) \times \frac{\mathrm{d}y}{\mathrm{d}x} = 2y\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we have

$$\log y + \left(\frac{x}{y} + 2y\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y\log y}{x + 2y^2}.$$

EXAMPLE 12.3.16. Suppose that $\log(xy^2) = 2x^2$. Differentiating both sides with respect to x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log(xy^2)) = \frac{\mathrm{d}}{\mathrm{d}x}(2x^2) = 4x.$$

Let $z = \log(xy^2)$. We can set up a chain by writing

$$z = \log u \qquad \text{and} \qquad u = xy^2.$$

Then it follows from the chain rule that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log(xy^2)) = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{xy^2} \times \frac{\mathrm{d}}{\mathrm{d}x}(xy^2).$$

Next, we observe that

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy^2) = \frac{\mathrm{d}}{\mathrm{d}x}(x) \times y^2 + x \times \frac{\mathrm{d}}{\mathrm{d}x}(y^2) = y^2 + x \times \frac{\mathrm{d}}{\mathrm{d}y}(y^2) \times \frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + 2xy\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Summarizing, we have

$$y^2 + 2xy\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^2y^2,$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(4x^2 - 1)y^2}{2xy}.$$

12.4. Derivatives of the Inverse Trigonometric Functions

The purpose of this last section is to determine the derivatives of the inverse trigonometric functions by using implicit differentiation and our knowledge on the derivatives of the trigonometric functions.

For notational purposes, we shall write

$$y = \sin^{-1} x \qquad \text{if and only if} \qquad x = \sin y,$$

and similarly for the other trigonometric functions. These inverse trigonometric functions are well defined, provided that we restrict the values for x to suitable intervals of real numbers.

DERIVATIVES OF THE INVERSE TRIGONOMETRIC FUNCTIONS.

(a) If $y = \sin^{-1} x$, *then* $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$.

(b) If $y = \cos^{-1} x$, *then* $\dfrac{dy}{dx} = -\dfrac{1}{\sqrt{1-x^2}}$.

(c) If $y = \tan^{-1} x$, *then* $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$.

(d) If $y = \cot^{-1} x$, *then* $\dfrac{dy}{dx} = -\dfrac{1}{1+x^2}$.

(e) If $y = \sec^{-1} x$, *then* $\dfrac{dy}{dx} = \dfrac{1}{x\sqrt{x^2-1}}$.

(f) If $y = \csc^{-1} x$, *then* $\dfrac{dy}{dx} = -\dfrac{1}{x\sqrt{x^2-1}}$.

SKETCH OF PROOF. For simplicity, we shall assume that $0 < y < \pi/2$, so that y is in the first quadrant, and so all the trigonometric functions have positive values.

(a) If $y = \sin^{-1} x$, then $x = \sin y$. Differentiating with respect to x, we obtain

$$1 = \cos y \frac{dy}{dx},$$

so that

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

(b) If $y = \cos^{-1} x$, then $x = \cos y$. Differentiating with respect to x, we obtain

$$1 = -\sin y \frac{dy}{dx},$$

so that

$$\frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}}.$$

(c) If $y = \tan^{-1} x$, then $x = \tan y$. Differentiating with respect to x, we obtain

$$1 = \sec^2 y \frac{dy}{dx},$$

so that

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

(d) If $y = \cot^{-1} x$, then $x = \cot y$. Differentiating with respect to x, we obtain

$$1 = -\csc^2 y \frac{dy}{dx},$$

so that

$$\frac{dy}{dx} = -\frac{1}{\csc^2 y} = -\frac{1}{1+\cot^2 y} = -\frac{1}{1+x^2}.$$

(e) If $y = \sec^{-1} x$, then $x = \sec y$. Differentiating with respect to x, we obtain

$$1 = \tan y \sec y \frac{\mathrm{d}y}{\mathrm{d}x},$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\tan y \sec y} = \frac{1}{(\sec^2 y - 1)^{1/2} \sec y} = \frac{1}{x\sqrt{x^2-1}}.$$

(f) If $y = \csc^{-1} x$, then $x = \csc y$. Differentiating with respect to x, we obtain

$$1 = -\cot y \csc y \frac{\mathrm{d}y}{\mathrm{d}x},$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\cot y \csc y} = -\frac{1}{(\csc^2 y - 1)^{1/2} \csc y} = -\frac{1}{x\sqrt{x^2-1}}.$$ ♣

There is no need to remember the derivatives of any of these inverse trigonometric functions.

Problems for Chapter 12

1. By making suitable use of the chain rule and other rules as appropriate, find the derivative of each of the following functions:
   a) $h(x) = \sqrt{1-\cos x}$ b) $h(x) = \sin(3x)$ c) $h(x) = \cos(\sin x)$
   d) $h(x) = x^2 \cos x$ e) $h(x) = \sin(2x)\sin(3x)$ f) $h(x) = 2x\sin(3x)$
   g) $h(x) = \tan(3x)$ h) $h(x) = 4\sec(5x)$ i) $h(x) = \cos(x^3)$
   j) $h(x) = \cos^3 x$ k) $h(x) = (1+\cos^2 x)^6$ l) $h(x) = \tan(x^2) + \tan^2 x$
   m) $h(x) = \cos(\tan x)$ n) $h(x) = \sin(\sin x)$

2. Find the derivative of each of the following functions:
   a) $h(x) = \sin(\mathrm{e}^x)$ b) $h(x) = \mathrm{e}^{\sin x}$ c) $h(x) = \mathrm{e}^{-2x}\sin x$
   d) $h(x) = \mathrm{e}^x \sin(2x)$ e) $h(x) = \tan(\mathrm{e}^{-3x})$ f) $h(x) = \tan(\mathrm{e}^x)$

3. a) For each $k = 0, 1, 2, 3, \ldots$, find a function $f_k(x)$ such that $f_k'(x) = x^k$.
   b) For each $k = -2, -3, -4, \ldots$, find a function $f_k(x)$ such that $f_k'(x) = x^k$.
   c) Find a function $f_{-1}(x)$ such that $f_{-1}'(x) = x^{-1}$.

4. Find the derivative of each of the following functions:
   a) $h(x) = (3x^2+\pi)(\mathrm{e}^x - 4)$ b) $h(x) = x^5 + 3x^2 + \dfrac{2}{x^4} + 1$ c) $h(x) = 2x - \dfrac{1}{\sqrt[3]{x}} + \mathrm{e}^{2x}$
   d) $h(x) = 2\mathrm{e}^x + x\mathrm{e}^{3x}$ e) $h(x) = \mathrm{e}^{\tan x}$ f) $h(x) = 2x\mathrm{e}^x - x^{-2}$
   g) $h(x) = \log(\log(2x^3))$ h) $h(x) = \sqrt{x+5}$ i) $h(x) = \dfrac{x+2}{x^2+1}$
   j) $h(x) = \sin(2x+3)$ k) $h(x) = \cos^2(2x)$ l) $h(x) = \log(\mathrm{e}^{-x} - 1)$
   m) $h(x) = \mathrm{e}^{\mathrm{e}^x + \mathrm{e}^{-x}}$ n) $h(x) = \dfrac{x^2+1}{\sqrt{x}}$ o) $h(x) = (x+3)^2$
   p) $h(x) = \log(2x+3)$ q) $h(x) = \tan(3x+2x^2)$ r) $h(x) = \cos(\mathrm{e}^{2x})$

5. Use implicit differentiation to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for each of the following relations:
   a) $x^2 + xy - y^3 = xy^2$ b) $x^2 + y^2 = \sqrt{7}$ c) $\sqrt{x} + \sqrt{y} = 25$
   d) $\sin(xy) = 2x + 5$ e) $x\log y + y^3 = \log x$ f) $y^3 - xy = -6$
   g) $x^2 - xy + y^4 = x^2 y$ h) $\sin(xy) = 3x^2 - 2$

6. For each of the following, verify first that the given point satisfies the relation defining the curve, then find the equation of the tangent line to the curve at the point:

   a) $xy^2 = 1$ at $(1, 1)$

   b) $y^2 = \dfrac{x^2}{xy - 4}$ at $(4, 2)$

   c) $y + \sin y + x^2 = 9$ at $(3, 0)$

   d) $x^{2/3} + y^{2/3} = a^{2/3}$ at $(a, 0)$

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Chapter 13

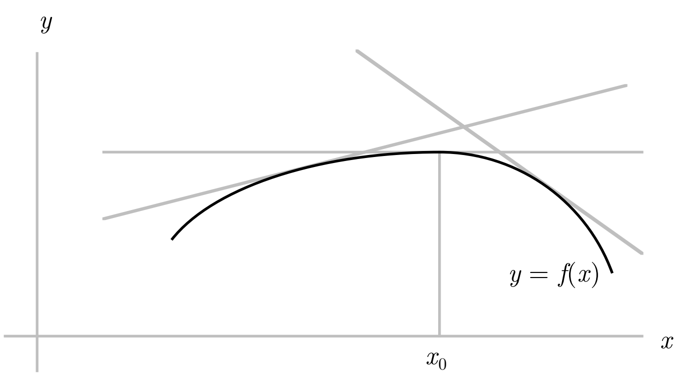
APPLICATIONS OF DIFFERENTIATION

13.1. Second Derivatives

Recall that for a function $y = f(x)$, the derivative $f'(x)$ represents the slope of the tangent. It is easy to see from a picture that if the derivative $f'(x) > 0$, then the function $f(x)$ is increasing; in other words, $f(x)$ increases in value as x increases. On the other hand, if the derivative $f'(x) < 0$, then the function $f(x)$ is decreasing; in other words, $f(x)$ decreases in value as x increases. We are interested in the case when the derivative $f'(x) = 0$. Values $x = x_0$ such that $f'(x_0) = 0$ are called stationary points.

Let us introduce the second derivative $f''(x)$ of the function $f(x)$. This is defined to be the derivative of the derivative $f'(x)$. With the same reasoning as before but applied to the function $f'(x)$ instead of the function $f(x)$, we conclude that if the second derivative $f''(x) > 0$, then the derivative $f'(x)$ is increasing. Similarly, if the second derivative $f''(x) < 0$, then the derivative $f'(x)$ is decreasing.

Suppose that $f'(x_0) = 0$ and $f''(x_0) < 0$. The condition $f''(x_0) < 0$ tells us that the derivative $f'(x)$ is decreasing near the point $x = x_0$. Since $f'(x_0) = 0$, this suggests that $f'(x) > 0$ when x is a little smaller than x_0, and that $f'(x) < 0$ when x is a little greater than x_0, as indicated in the picture below.

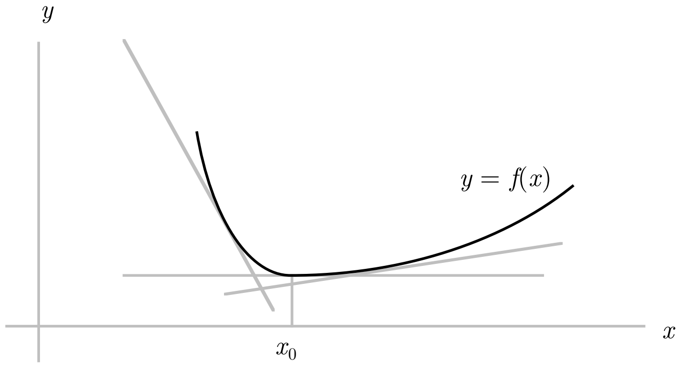


† This chapter was written at Macquarie University in 1999.

In this case, we say that the function has a local maximum at the point $x = x_0$. This means that if we restrict our attention to real values x near enough to the point $x = x_0$, then $f(x) \leq f(x_0)$ for all such real values x.

LOCAL MAXIMUM. *Suppose that $f'(x_0) = 0$ and $f''(x_0) < 0$. Then the function $f(x)$ has a local maximum at the point $x = x_0$.*

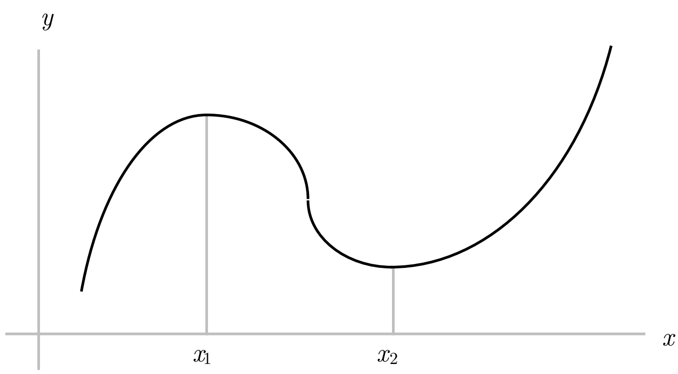
Suppose next that $f'(x_0) = 0$ and $f''(x_0) > 0$. The condition $f''(x_0) > 0$ tells us that the derivative $f'(x)$ is increasing near the point $x = x_0$. Since $f'(x_0) = 0$, this suggests that $f'(x) < 0$ when x is a little smaller than x_0, and that $f'(x) > 0$ when x is a little greater than x_0, as indicated in the picture below.



In this case, we say that the function has a local minimum at the point $x = x_0$. This means that if we restrict our attention to real values x near enough to the point $x = x_0$, then $f(x) \geq f(x_0)$ for all such real values x.

LOCAL MINIMUM. *Suppose that $f'(x_0) = 0$ and $f''(x_0) > 0$. Then the function $f(x)$ has a local minimum at the point $x = x_0$.*

Remark. These stationary points are called local maxima or local minima because such points may not maximize or minimize the functions in question. Consider the picture below, with a local maximum at $x = x_1$ and a local minumum at $x = x_2$.



We also say that a point $x = x_0$ is a point of inflection if $f''(x_0) = 0$, irrespective of whether $f'(x_0) = 0$ or not. A simple way of visualizing the graph of a function at a point of inflection is to imagine that one is steering a car along the curve. A point of inflection then corresponds to the place on the curve where the steering wheel of the car is momentarily straight while being turned from a little left to a little right, or while being turned from a little right to a little left.

EXAMPLE 13.1.1. Consider the function $f(x) = \cos x$. Since $f'(x) = -\sin x = 0$ whenever $x = k\pi$, where $k \in \mathbb{Z}$, it follows that the function $f(x) = \cos x$ has a stationary point at $x = k\pi$ for every $k \in \mathbb{Z}$. Next, note that $f''(x) = -\cos x$. If k is even, then $f''(k\pi) = -1$, so that $f(x)$ has a local maximum at $x = k\pi$. If k is odd, then $f''(k\pi) = 1$, so that $f(x)$ has a local minimum at $x = k\pi$. See the graph of this function in Chapter 3.

EXAMPLE 13.1.2. Consider the function $f(x) = 3x^4 + 4x^3 - 12x^2 + 5$. Since

$$f'(x) = 12x^3 + 12x^2 - 24x = 12x(x^2 + x - 2) = 12x(x-1)(x+2),$$

it follows that the function $f(x)$ has stationary points at $x = 0$, $x = 1$ and $x = -2$. On the other hand, we have $f''(x) = 36x^2 + 24x - 24$. Since $f''(0) = -24$, $f''(1) = 36$ and $f''(-2) = 72$, it follows that $f(x)$ has a local maximum at $x = 0$ and local minima at $x = 1$ and $x = -2$.

EXAMPLE 13.1.3. Consider the function $f(x) = x^3 - 3x^2 + 2$. Since $f'(x) = 3x^2 - 6x = 3x(x-2)$, it follows that the function $f(x)$ has stationary points at $x = 0$ and $x = 2$. On the other hand, we have $f''(x) = 6x - 6$. Since $f''(0) = -6$ and $f''(2) = 6$, it follows that $f(x)$ has a local maximum at $x = 0$ and a local minimum at $x = 2$. Observe also that there is a point of inflection at $x = 1$.

EXAMPLE 13.1.4. Consider the function $f(x) = x^4 - 2x^2 + 7$. Since

$$f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1),$$

it follows that the function $f(x)$ has stationary points at $x = 0$, $x = 1$ and $x = -1$. On the other hand, we have $f''(x) = 12x^2 - 4$. Since $f''(0) = -4$, $f''(1) = 8$ and $f''(-1) = 8$, it follows that $f(x)$ has a local maximum at $x = 0$ and local minima at $x = 1$ and $x = -1$. Note also that $f''(x) = 0$ if $x = \pm\sqrt{1/3}$, so there are points of inflection at $x = \pm\sqrt{1/3}$.

EXAMPLE 13.1.5. Consider the function $f(x) = 3x^4 - 16x^3 + 24x^2 - 1$. Since

$$f'(x) = 12x^3 - 48x^2 + 48x = 12x(x^2 - 4x + 4) = 12x(x-2)^2,$$

it follows that the function $f(x)$ has stationary points at $x = 0$ and $x = 2$. On the other hand, we have $f''(x) = 36x^2 - 96x + 48$. Since $f''(0) = 48$ and $f''(2) = 0$, it follows that $f(x)$ has a local minimum at $x = 0$ and a point of inflection at $x = 2$. Note also that $36x^2 - 96x + 48 = 12(x-2)(3x-2)$, so there is another point of inflection at $x = 2/3$.

EXAMPLE 13.1.6. Consider the function

$$f(x) = \frac{1}{x^2+1}, \qquad \text{with} \qquad f'(x) = -\frac{2x}{(x^2+1)^2}.$$

Clearly $f(x)$ has a stationary point at $x = 0$. On the other hand, it is easy to check that

$$f''(x) = \frac{6x^2 - 2}{(x^2+1)^3}.$$

Since $f''(0) = -2$, it follows that $f(x)$ has a local maximum at $x = 0$. We also have points of inflection when $6x^2 - 2 = 0$; in other words, when $x = \pm\sqrt{1/3}$.

EXAMPLE 13.1.7. Consider the function

$$f(x) = \frac{x}{x^2+1}, \qquad \text{with} \qquad f'(x) = \frac{1 - x^2}{(x^2+1)^2}.$$

Clearly $f(x)$ has stationary points at $x = 1$ and $x = -1$. On the other hand, it is easy to check that

$$f''(x) = -\frac{2x(x^2+1) + 4x(1-x^2)}{(x^2+1)^3} = \frac{2x(x^2-3)}{(x^2+1)^3}.$$

Since $f''(1) = -1/2$ and $f''(-1) = 1/2$, it follows that $f(x)$ has a local maximum at $x = 1$ and a local minimum at $x = -1$. We also have points of inflection when $2x(x^2 - 3) = 0$; in other words, when $x = 0$ or $x = \pm\sqrt{3}$.

EXAMPLE 13.1.8. Consider the function $f(x) = \mathrm{e}^x + \mathrm{e}^{-x}$. Since $f'(x) = \mathrm{e}^x - \mathrm{e}^{-x}$, it follows that the function $f(x)$ has a stationary point at $x = 0$. On the other hand, we have $f''(x) = \mathrm{e}^x + \mathrm{e}^{-x}$. Since $f''(0) = 2$, it follows that $f(x)$ has a local minimum at $x = 0$.

EXAMPLE 13.1.9. Consider the function $f(x) = \sin x - \cos^2 x$, restricted to the interval $0 \le x \le 2\pi$. It is easy to see that

$$f'(x) = \cos x + 2\cos x \sin x = (1 + 2\sin x)\cos x.$$

We therefore have stationary points when $\cos x = 0$ or $\sin x = -1/2$. There are four stationary points in the interval $0 \le x \le 2\pi$, namely

$$x = \frac{\pi}{2}, \qquad x = \frac{3\pi}{2}, \qquad x = \frac{7\pi}{6}, \qquad x = \frac{11\pi}{6}.$$

Next, note that we can write $f'(x) = \cos x + \sin 2x$, so that $f''(x) = 2\cos 2x - \sin x$. It is easy to check that

$$f''\left(\frac{\pi}{2}\right) = -3, \qquad f''\left(\frac{3\pi}{2}\right) = -1, \qquad f''\left(\frac{7\pi}{6}\right) = \frac{3}{2}, \qquad f''\left(\frac{11\pi}{6}\right) = \frac{3}{2}.$$

Hence $f(x)$ has local maxima at $x = \pi/2$ and $x = 3\pi/2$, and local minima at $x = 7\pi/6$ and $x = 11\pi/6$.

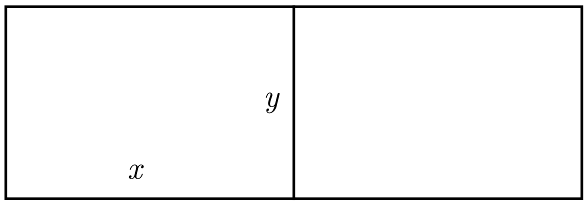
13.2. Applications to Problem Solving

In this section, we discuss how we can apply ideas in differentiation to solve various problems. We shall illustrate the techniques by discussing a few examples. Central to all of these is the crucial step where we set up the problems mathematically and in a suitable way.

EXAMPLE 13.2.1. We wish to find positive real numbers x and y such that $x + y = 6$ and the quantity xy^2 is as large as possible. In view of the restriction $x + y = 6$, the quantity $xy^2 = x(6 - x)^2$. We can therefore try to find a real number x which makes the quantity $x(6 - x)^2$ as large as possible. The idea here is to consider the function $f(x) = x(6 - x)^2$ and hope to find a local maximum. We can write $f(x) = 36x - 12x^2 + x^3$, and so $f'(x) = 36 - 24x + 3x^2 = 3(x^2 - 8x + 12) = 3(x - 2)(x - 6)$. Hence $x = 2$ and $x = 6$ are stationary points. Next, note that $f''(x) = 6x - 24$. Hence $f''(2) = -12$ and $f''(6) = 12$. It follows that the function $f(x)$ has a local maximum at the point $x = 2$. Then $y = 6 - x = 4$, with $f(2) = 32$. This choice of x and y makes xy^2 as large as possible, with value $f(2) = 32$.

EXAMPLE 13.2.2. We have 20 metres of fencing material, and wish to find the largest rectangular area that we can enclose. Suppose that the rectangular area has sides x and y in metres. Then the area is equal to xy, while the perimeter is equal to $2x + 2y$. Hence we wish to maximize the quantity xy subject to the restriction $2x + 2y = 20$. Under the restriction $2x + 2y = 20$, the quantity $xy = x(10 - x)$. We can therefore try to find a real number x which makes the quantity $x(10 - x)$ as large as possible. Consider the function $f(x) = x(10 - x) = 10x - x^2$. Then $f'(x) = 10 - 2x$, and so $x = 5$ is a stationary point. Since $f''(x) = -2$, the point $x = 5$ is a local maximum. Then $y = 10 - x = 5$, with $f(5) = 25$. This choice of x and y makes xy as large as possible, with area 25 square metres.

EXAMPLE 13.2.3. We have 1200 metres of fencing material, and wish to enclose a double paddock with two equal rectangular areas as shown in the diagram below.



Suppose that each of the two rectangular areas has sides x and y in metres, as shown in the picture. Then the total area is equal to $2xy$, while the total perimeter is equal to $4x + 3y$. Hence we wish to maximize the quantity $2xy$ subject to the restriction $4x + 3y = 1200$. Under the restriction $4x + 3y = 1200$, the quantity

$$2xy = 2x\left(400 - \frac{4x}{3}\right).$$

We can therefore try to find a real number x which makes this quantity as large as possible. Consider the function

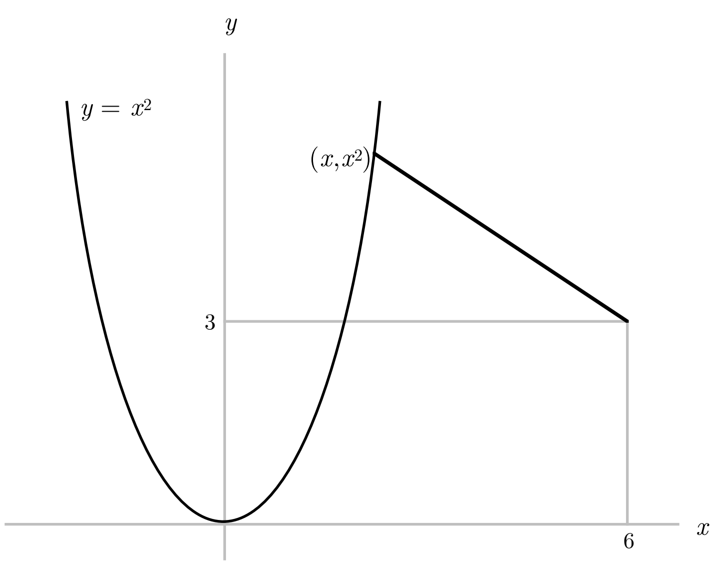
$$f(x) = 2x\left(400 - \frac{4x}{3}\right) = 800x - \frac{8x^2}{3}.$$

Then

$$f'(x) = 800 - \frac{16x}{3},$$

and so $x = 150$ is a stationary point. Since $f''(x) = -16/3$, the point $x = 150$ is a local maximum. Then $y = 200$, with $f(150) = 60000$. This choice of x and y makes $2xy$ as large as possible, with total area 60000 square metres.

EXAMPLE 13.2.4. We wish to find the point on the parabola $y = x^2$ which is closest to the point $(6, 3)$. We begin by drawing a picture.



Note that a typical point on the parabola $y = x^2$ is given by $(x, y) = (x, x^2)$. The distance between this point and the point $(6, 3)$ is given by

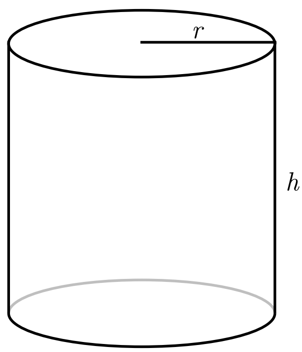
$$\sqrt{(x-6)^2 + (x^2-3)^2},$$

in view of the theorem of Pythagoras. Now let $f(x) = (x-6)^2 + (x^2-3)^2$. Then $f(x)$ represents the square of this distance. We now need to find a local minimum for the function $f(x)$. Differentiating, we obtain $f'(x) = 2(x-6) + 4x(x^2-3) = 4x^3 - 10x - 12$. We observe that $x = 2$ is a root of the polynomial $4x^3 - 10x - 12$. Hence we have

$$4x^3 - 10x - 12 = (x-2)(4x^2+8x+6) = (x-2)(4(x^2+2x+1)+2) = (x-2)(4(x+1)^2+2).$$

It follows that there is only one stationary point $x = 2$. Note next that $f''(x) = 12x^2 - 10$, so that $f''(2) > 0$. Hence $x = 2$ is a local minimum. It follows that the point $(2,4)$ on the parabola is closest to the point $(6,3)$, with distance $\sqrt{(2-6)^2 + (4-3)^2} = \sqrt{17}$.

EXAMPLE 13.2.5. A manufacturer wishes to maximize the volume of cylindrical metal cans made out of a fixed quantity of metal. To understand this problem, suppose that a typical can has radius r and height h as shown in the picture below:



Then the total surface area is equal to $2\pi r^2 + 2\pi rh = S$, where S is fixed, so that

$$h = \frac{S}{2\pi r} - r. \tag{1}$$

On the other hand, the volume of such a can is equal to $V = \pi r^2 h$. Under the restriction (1), we have

$$V = \pi r^2 h = \frac{Sr}{2} - \pi r^3.$$

Consider now the function

$$V(r) = \frac{Sr}{2} - \pi r^3.$$

Differentiating, we have

$$V'(r) = \frac{S}{2} - 3\pi r^2,$$

so that $r = \sqrt{S/6\pi}$ is the only stationary point, since negative values of r are meaningless. Furthermore, we have $V''(r) = -6\pi r$, and so this stationary point is a local maximum. For this value of r, we have

$$h = \frac{S}{2\pi r} - r = \sqrt{\frac{3S}{2\pi}} - \sqrt{\frac{S}{6\pi}} = \sqrt{\frac{9S}{6\pi}} - \sqrt{\frac{S}{6\pi}} = 3\sqrt{\frac{S}{6\pi}} - \sqrt{\frac{S}{6\pi}} = 2\sqrt{\frac{S}{6\pi}} = 2r.$$

This means that the most economical shape of a cylindrical can is when the height is twice the radius.

EXAMPLE 13.2.6. A steamer travelling at constant speed due east passes a buoy at 9 am. A hydrofoil travelling at twice this speed due north passes the same buoy at 11 am. We would like to determine the time when the distance between the two vessels is smallest. To set up the problem mathematically, we consider the xy-plane, and assume that the position of the buoy is at the origin $(0,0)$. Let s be the speed of the steamer. Then at t am, the position of the steamer is given by the point $(s(t-9), 0)$ if we relate due east to the positive horizontal axis. Furthermore, the position of the hydrofoil is given by the point $(0, 2s(t-11))$ if we relate due north to the positive vertical axis. By the theorem of Pythagoras, the distance between the two vessels at t am is given by $\sqrt{s^2(t-9)^2 + 4s^2(t-11)^2}$. Consider now the function $f(t) = s^2(t-9)^2 + 4s^2(t-11)^2$. Clearly this represents the square of the distance between the two vessels, so we need to find a local minimum for this function. Differentiating, we obtain

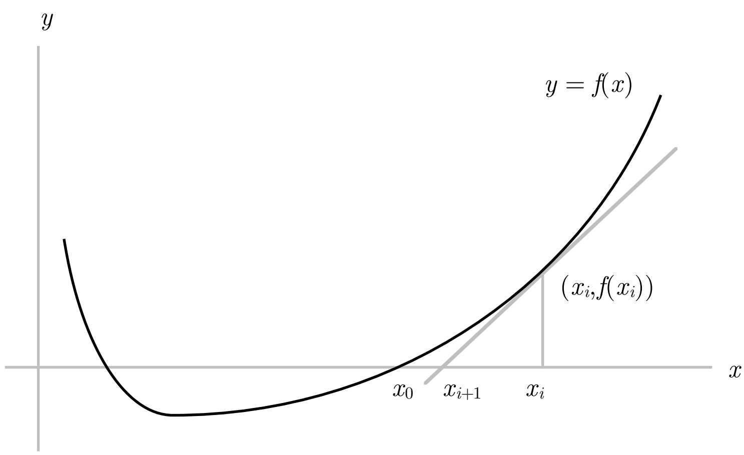
$$f'(t) = 2s^2(t-9) + 8s^2(t-11) = s^2(10t - 106),$$

with stationary point $t = 10.6$, representing the time 10:36 am. Can you convince yourself that this is a local minimum?

13.3. Newton's Method

In this section, we briefly describe a numerical technique which allows us to obtain approximations to solutions of some problems where exact answers may be hard or even impossible to calculate.

Let us consider an equation of the form $f(x) = 0$. Suppose that we wish to find some real number x for which the equation is satisfied. Consider the picture below:



Here x_0 represents a solution of the equation $f(x) = 0$. Unfortunately, we are unable to calculate the value of x_0 precisely. We now take some number x_i close to x_0, and consider the tangent to the curve $y = f(x)$ at the point $(x_i, f(x_i))$. Clearly the tangent has slope $f'(x_i)$. It follows that the equation of the tangent is given by

$$\frac{y - f(x_i)}{x - x_i} = f'(x_i).$$

Let x_{i+1} be the x-intercept of this tangent line. Then it is easy to see that

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}. \qquad (2)$$

From the picture, this new point x_{i+1} gives a better approximation to x_0 than the point x_i does.

Newton's method is now to start with some point x_1 close to a solution x_0 of $f(x) = 0$, and then obtain a sequence of successive approximations $x_2, x_3, x_4, \ldots$ by using the formula (2).

EXAMPLE 13.3.1. We shall try to obtain some approximation for $\sqrt{2}$. To do so, we consider the equation $f(x) = 0$, where $f(x) = x^2 - 2$. Then $f'(x) = 2x$, and so equation (2) becomes

$$x_{i+1} = x_i - \frac{x_i^2 - 2}{2x_i}.$$

Using this, and taking $x_1 = 2$, we obtain $x_2 = 1.5$, $x_3 = 1.41667$, $x_4 = 1.41422$, $x_5 = 1.41421$, and so on, all to 5 decimal places. On the other hand, if we take $x_1 = -2$, then we obtain $x_2 = -1.5$, $x_3 = -1.41667$, $x_4 = -1.41422$, $x_5 = -1.41421$, and so on, all to 5 decimal places.

EXAMPLE 13.3.2. We shall try to obtain some approximation to a solution of the equation $x^3+x-1 = 0$. To do so, we consider the function $f(x) = x^3 + x - 1$. Then $f'(x) = 3x^2 + 1$, and so equation (2) becomes

$$x_{i+1} = x_i - \frac{x_i^3 + x_i - 1}{3x_i^2 + 1}.$$

Using this, and taking $x_1 = 1$, we obtain $x_2 = 0.75$, $x_3 = 0.68605$, $x_4 = 0.68234$, $x_5 = 0.68233$, and so on, all to 5 decimal places.

REMARK. As is the case for much of numerical mathematics, Newton's method is imprecise. It may fail to work in some instances. If there are many possible solutions, then it is sometimes unclear which solution the method will give.

EXAMPLE 13.3.3. Let us return to Example 13.1.7, and consider the function

$$f(x) = \frac{x}{x^2+1}, \qquad \text{with} \qquad f'(x) = \frac{1-x^2}{(x^2+1)^2}.$$

It is easy to see that the equation $f(x) = 0$ has precisely one solution, namely $x = 0$. Nevertheless, let us apply Newton's method to this function. Then equation (2) becomes

$$x_{i+1} = x_i - \frac{x_i(x_i^2+1)}{1-x_i^2}.$$

Using this, and taking $x_1 = 0.5$, we obtain $x_2 = -0.33333$, $x_3 = 0.08333$, $x_4 = -0.00117$, $x_5 = 0.00000$, and so on, all to 5 decimal places. On the other hand, if we take $x_1 = 2$, then we obtain $x_2 = 5.33333$, $x_3 = 11.05533$, $x_4 = 22.29306$, $x_5 = 44.67602$, and so on, all to 5 decimal places. The method clearly fails in this second case.

PROBLEMS FOR CHAPTER 13

1. For each of the following functions, find all of the stationary points. For each such stationary point, determine whether it is a local maximum, a local minimum or another type of stationary point:
   a) $f(x) = 3x^2 + 6x + 9$ b) $f(x) = 6x - x^2$ c) $f(x) = 2 - 3x - 3x^2$
   d) $f(x) = 6 + 9x - 3x^2 - x^3$ e) $f(x) = x + \dfrac{4}{x+1}$ f) $f(x) = 4x - 1 + \dfrac{36}{x-1}$
   g) $f(x) = (x+1)^2 - (x-1)^2$ h) $f(x) = 6 - \dfrac{2}{x} - x^2$
2. A bullet is shot upwards at time $t = 0$ from the top of a building 176 metres tall, with an initial speed of 160 metres per second. The height of the bullet is given by $h(t) = -16t^2 + 160t + 176$ after t seconds. At what time is the bullet at maximum height above the ground? What is this height?
3. What number, when squared and added to 16 times its reciprocal, gives a minimum value for this sum?

4. A piece of wire is to be cut into two pieces to form a circle and a square. How should the wire be cut to minimize the total area of the two pieces?

5. Find two real numbers whose sum is 16 and whose product is a maximum.

6. Find two positive real numbers whose product is 81 and the sum of whose squares is a minimum.

7. What positive real number is exceeded by its square root by the greatest amount?

8. Find the dimension of a right circular cylinder of volume 1 cubic metre and having the minimum surface area.

9. A closed box is to be constructed with a square base and volume of 1500 cubic metres. The material used for the base costs twice as much as for the top and sides. What dimension should the box have to keep the cost to a minimum?

10. Find the maximum area of a rectangle inscribed in a semicircle of radius 12 metres.

11. A rectangular beam, of width w and depth d, is cut from a circular log of diameter $a = 25$ centimetres. The beam has strength S given by $S = 2wd^2$. Find the dimension that will give the strongest beam.
[HINT: Use $d^2 + w^2 = a^2$ to relate the variables d and w.]

12. For each of the following, use a calculator and Newton's method to obtain estimates for the desired quantity to at least 4 decimal places:
   a) The number $\sqrt{5}$, starting with an initial estimate of $x_0 = 2$.
   b) The number $\sqrt{7}$, starting with an initial estimate of $x_0 = 3$.
   c) The number $\sqrt[3]{2}$.
   d) A solution to the equation $x^3 - 2x - 5 = 0$.
   e) A solution to the equation $\cos x = x$.
   f) The largest real root of the polynomial $x^3 + x - 1$.
   g) The largest solution to the equation $\sin x = e^x$.

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Chapter 14

INTRODUCTION TO INTEGRATION

14.1. Antiderivatives

In this chapter, we discuss the inverse process of differentiation. In other words, given a function $f(x)$, we wish to find a function $F(x)$ such that $F'(x) = f(x)$. Any such function $F(x)$ is called an antiderivative, or indefinite integral, of the function $f(x)$, and we write

$$F(x) = \int f(x)\,\mathrm{d}x.$$

A first observation is that the antiderivative, if it exists, is not unique. Suppose that the function $F(x)$ is an antiderivative of the function $f(x)$, so that $F'(x) = f(x)$. Let $G(x) = F(x) + C$, where C is any fixed real number. Then it is easy to see that $G'(x) = F'(x) = f(x)$, so that $G(x)$ is also an antiderivative of $f(x)$. A second observation, somewhat less obvious, is that for any given function $f(x)$, any two distinct antiderivatives of $f(x)$ must differ only by a constant. In other words, if $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$, then $F(x) - G(x)$ is a constant. In this chapter, we shall denote any such constant by C, with or without subscripts.

An immediate consequence of this second observation is the following simple result related to the derivatives of constants in Section 11.1.

ANTIDERIVATIVES OF ZERO. *We have*

$$\int 0\,\mathrm{d}x = C.$$

In other words, the antiderivatives of the zero function are precisely all the constant functions.

Indeed, many antiderivatives can be obtained simply by referring to various rules concerning derivatives. We list here a number of such results. The first of these is related to the constant multiple rule for differentiation in Section 11.2.

† This chapter was written at Macquarie University in 1999.

CONSTANT MULTIPLE RULE. *Suppose that a function $f(x)$ has antiderivatives. Then for any fixed real number c, we have*

$$\int cf(x)\,\mathrm{d}x = c\int f(x)\,\mathrm{d}x.$$

ANTIDERIVATIVES OF POWERS.

(a) Suppose that n is a fixed real number such that $n \neq -1$. Then

$$\int x^n\,\mathrm{d}x = \frac{1}{n+1}x^{n+1} + C.$$

(b) We have

$$\int x^{-1}\,\mathrm{d}x = \log|x| + C.$$

Proof. Part (a) is a consequence of the rule concerning derivatives of powers in Section 11.1. If $x > 0$, then part (b) is a consequence of the rule concerning the derivative of the logarithmic function in Section 12.3. If $x < 0$, we can write $|x| = u$, where $u = -x$. It then follows from the Chain rule that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log|x|) = \frac{\mathrm{d}u}{\mathrm{d}x} \times \frac{\mathrm{d}}{\mathrm{d}u}(\log u) = -\frac{1}{u} = \frac{1}{x} \qquad (1)$$

again. ♣

Corresponding to the sum rule for differentiation in Section 11.2, we have the following.

SUM RULE. *Suppose that functions $f(x)$ and $g(x)$ have antiderivatives. Then*

$$\int (f(x) + g(x))\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x.$$

We next consider trigonometric functions.

ANTIDERIVATIVES OF TRIGONOMETRIC FUNCTIONS.

(a) We have

$$\int \cos x\,\mathrm{d}x = \sin x + C \qquad \text{and} \qquad \int \sin x\,\mathrm{d}x = -\cos x + C.$$

(b) We have

$$\int \sec^2 x\,\mathrm{d}x = \tan x + C \qquad \text{and} \qquad \int \csc^2 x\,\mathrm{d}x = -\cot x + C.$$

(c) We have

$$\int \tan x \sec x\,\mathrm{d}x = \sec x + C \qquad \text{and} \qquad \int \cot x \csc x\,\mathrm{d}x = -\csc x + C.$$

(d) We have

$$\int \sec x\,\mathrm{d}x = \log|\tan x + \sec x| + C \qquad \text{and} \qquad \int \csc x\,\mathrm{d}x = -\log|\cot x + \csc x| + C.$$

Proof. Parts (a)–(c) follow immediately from the rules concerning derivatives of the trigonometric functions in Section 11.3. Part (d) follows from Example 12.3.12 and Example 12.3.13 if we note (1). ♣

Corresponding to the rule concerning the derivative of the exponential function in Section 12.3, we have the following.

ANTIDERIVATIVES OF THE EXPONENTIAL FUNCTION. *We have*

$$\int \mathrm{e}^x \,\mathrm{d}x = \mathrm{e}^x + C.$$

EXAMPLE 14.1.1. Using the sum rule, the constant multiple rule and the rule concerning antiderivatives of powers, we have

$$\int (x^2 + 3x + 1)\,\mathrm{d}x = \int x^2 \,\mathrm{d}x + 3\int x \,\mathrm{d}x + \int x^0 \,\mathrm{d}x = \frac{1}{3}x^3 + \frac{3}{2}x^2 + x + C.$$

EXAMPLE 14.1.2. Using the sum rule and the rules concerning antiderivatives of powers and of trigonometric functions, we have

$$\int (x^3 + \sin x)\,\mathrm{d}x = \int x^3 \,\mathrm{d}x + \int \sin x \,\mathrm{d}x = \frac{1}{4}x^4 - \cos x + C.$$

EXAMPLE 14.1.3. We have

$$\int (\sin x + \sec x)\,\mathrm{d}x = \int \sin x \,\mathrm{d}x + \int \sec x \,\mathrm{d}x = -\cos x + \log|\tan x + \sec x| + C.$$

EXAMPLE 14.1.4. We have

$$\int (\mathrm{e}^x + 3\cos x)\,\mathrm{d}x = \int \mathrm{e}^x \,\mathrm{d}x + 3\int \cos x \,\mathrm{d}x = \mathrm{e}^x + 3\sin x + C.$$

EXAMPLE 14.1.5. To find

$$\int \frac{1-\sin x}{1+\sin x}\,\mathrm{d}x,$$

note first of all that

$$\begin{aligned}\frac{1-\sin x}{1+\sin x} &= \frac{(1-\sin x)(1-\sin x)}{(1+\sin x)(1-\sin x)} = \frac{1-2\sin x+\sin^2 x}{1-\sin^2 x} = \frac{1-2\sin x+\sin^2 x}{\cos^2 x}\\ &= \sec^2 x - 2\tan x \sec x + \tan^2 x = 2\sec^2 x - 2\tan x\sec x - 1.\end{aligned}$$

It follows that

$$\int \frac{1-\sin x}{1+\sin x}\,\mathrm{d}x = 2\int \sec^2 x \,\mathrm{d}x - 2\int \tan x \sec x \,\mathrm{d}x - \int \mathrm{d}x = 2\tan x - 2\sec x - x + C.$$

14.2. Integration by Substitution

We now discuss how we can use the chain rule in differentiation to help solve problems in integration. This technique is usually called integration by substitution. As we shall not prove any result here, our discussion will be only heuristic.

We emphasize that the technique does not always work. First of all, we have little or no knowledge of the antiderivatives of many functions. Secondly, there is no simple routine that we can describe to help us find a suitable substitution even in the cases where the technique works. On the other hand, when the technique does work, there may well be more than one suitable substitution!

REMARK. It is imperative that one does not give up when one's effort does not seem to yield results. We learn far more from indefinite integrals that we cannot find than from those than we can.

INTEGRATION BY SUBSTITUTION – VERSION 1. *If we make a substitution $x = g(u)$, then $\mathrm{d}x = g'(u)\,\mathrm{d}u$, and*

$$\int f(x)\,\mathrm{d}x = \int f(g(u))g'(u)\,\mathrm{d}u.$$

EXAMPLE 14.2.1. Consider the indefinite integral

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x.$$

If we make a substitution $x = \sin u$, then $\mathrm{d}x = \cos u\,\mathrm{d}u$, and

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \int \frac{\cos u}{\sqrt{1-\sin^2 u}}\,\mathrm{d}u = \int \mathrm{d}u = u + C = \sin^{-1} x + C.$$

On the other hand, if we make a substitution $x = \cos v$, then $\mathrm{d}x = -\sin v\,\mathrm{d}v$, and

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = -\int \frac{\sin v}{\sqrt{1-\cos^2 v}}\,\mathrm{d}v = -\int \mathrm{d}v = -v + C = -\cos^{-1} x + C.$$

See Section 12.4 concerning derivatives of inverse trigonometric functions.

EXAMPLE 14.2.2. Consider the indefinite integral

$$\int \frac{1}{1+x^2}\,\mathrm{d}x.$$

If we make a substitution $x = \tan u$, then $\mathrm{d}x = \sec^2 u\,\mathrm{d}u$, and

$$\int \frac{1}{1+x^2}\,\mathrm{d}x = \int \frac{\sec^2 u}{1+\tan^2 u}\,\mathrm{d}u = \int \mathrm{d}u = u + C = \tan^{-1} x + C.$$

On the other hand, if we make a substitution $x = \cot v$, then $\mathrm{d}x = -\csc^2 v\,\mathrm{d}v$, and

$$\int \frac{1}{1+x^2}\,\mathrm{d}x = -\int \frac{\csc^2 v}{1+\cot^2 v}\,\mathrm{d}v = -\int \mathrm{d}v = -v + C = -\cot^{-1} x + C.$$

EXAMPLE 14.2.3. Consider the indefinite integral

$$\int x\sqrt{x+1}\,\mathrm{d}x.$$

If we make a substitution $x = u^2 - 1$, then $\mathrm{d}x = 2u\,\mathrm{d}u$, and

$$\begin{aligned}\int x\sqrt{x+1}\,\mathrm{d}x &= \int 2(u^2-1)u^2\,\mathrm{d}u = 2\int u^4\,\mathrm{d}u - 2\int u^2\,\mathrm{d}u \\ &= \frac{2}{5}u^5 - \frac{2}{3}u^3 + C = \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C.\end{aligned}$$

On the other hand, if we make a substitution $x = v - 1$, then $\mathrm{d}x = \mathrm{d}v$, and

$$\begin{aligned}\int x\sqrt{x+1}\,\mathrm{d}x &= \int (v-1)v^{1/2}\,\mathrm{d}v = \int v^{3/2}\,\mathrm{d}v - \int v^{1/2}\,\mathrm{d}v \\ &= \frac{2}{5}v^{5/2} - \frac{2}{3}v^{3/2} + C = \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C.\end{aligned}$$

We can confirm that the indefinite integral is correct by checking that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C\right) = x\sqrt{x+1}.$$

INTEGRATION BY SUBSTITUTION – VERSION 2. *Suppose that a function $f(x)$ can be written in the form $f(x) = g(h(x))h'(x)$. If we make a substitution $u = h(x)$, then $\mathrm{d}u = h'(x)\,\mathrm{d}x$, and*

$$\int f(x)\,\mathrm{d}x = \int g(h(x))h'(x)\,\mathrm{d}x = \int g(u)\,\mathrm{d}u.$$

Remark. Note that in Version 1, the variable x is initially written as a function of the new variable u, whereas in Version 2, the new variable u is written as a function of x. The difference, however, is minimal, as the substitution $x = g(u)$ in Version 1 has to be invertible to enable us to return from the new variable u to the original variable x at the end of the process.

Example 14.2.4. Consider the indefinite integral

$$\int x(x^2+3)^4\,\mathrm{d}x.$$

Note first of all that the derivative of the function x^2+3 is equal to $2x$, so it is convenient to make the substitution $u = x^2+3$. Then $\mathrm{d}u = 2x\,\mathrm{d}x$, and

$$\int x(x^2+3)^4\,\mathrm{d}x = \frac{1}{2}\int 2x(x^2+3)^4\,\mathrm{d}x = \frac{1}{2}\int u^4\,\mathrm{d}u = \frac{1}{10}u^5 + C = \frac{1}{10}(x^2+3)^5 + C.$$

Example 14.2.5. Consider the indefinite integral

$$\int \frac{1}{x\log x}\,\mathrm{d}x.$$

Note first of all that the derivative of the function $\log x$ is equal to $1/x$, so it is convenient to make the substitution $u = \log x$. Then $\mathrm{d}u = (1/x)\,\mathrm{d}x$, and

$$\int \frac{1}{x\log x}\,\mathrm{d}x = \int \frac{1}{u}\,\mathrm{d}u = \log|u| + C = \log|\log x| + C.$$

Example 14.2.6. Consider the indefinite integral

$$\int x^2\mathrm{e}^{x^3}\,\mathrm{d}x.$$

Note first of all that the derivative of the function x^3 is equal to $3x^2$, so it is convenient to make the substitution $u = x^3$. Then $\mathrm{d}u = 3x^2\,\mathrm{d}x$, and

$$\int x^2\mathrm{e}^{x^3}\,\mathrm{d}x = \frac{1}{3}\int 3x^2\mathrm{e}^{x^3}\,\mathrm{d}x = \frac{1}{3}\int \mathrm{e}^u\,\mathrm{d}u = \frac{1}{3}\mathrm{e}^u + C = \frac{1}{3}\mathrm{e}^{x^3} + C.$$

A somewhat more complicated alternative is to note that the derivative of the function e^{x^3} is equal to $3x^2\mathrm{e}^{x^3}$, so it is convenient to make the substitution $v = \mathrm{e}^{x^3}$. Then $\mathrm{d}v = 3x^2\mathrm{e}^{x^3}\,\mathrm{d}x$, and

$$\int x^2\mathrm{e}^{x^3}\,\mathrm{d}x = \frac{1}{3}\int 3x^2\mathrm{e}^{x^3}\,\mathrm{d}x = \frac{1}{3}\int \mathrm{d}v = \frac{1}{3}v + C = \frac{1}{3}\mathrm{e}^{x^3} + C.$$

Example 14.2.7. Consider the indefinite integral

$$\int \tan^3 x\sec^2 x\,\mathrm{d}x.$$

Note first of all that the derivative of the function $\tan x$ is equal to $\sec^2 x$, so it is convenient to make the substitution $u = \tan x$. Then $\mathrm{d}u = \sec^2 x\,\mathrm{d}x$, and

$$\int \tan^3 x\sec^2 x\,\mathrm{d}x = \int u^3\,\mathrm{d}u = \frac{1}{4}u^4 + C = \frac{1}{4}\tan^4 x + C.$$

Occasionally, the possibility of substitution may not be immediately obvious, and a certain amount of trial and error does occur. The fact that one substitution does not appear to work does not mean that the method fails. It may very well be the case that we have used a bad substitution. Or perhaps we may slightly modify the problem first. We illustrate this point by looking at two more examples.

Example 14.2.8. Consider the indefinite integral

$$\int \tan x \,\mathrm{d}x.$$

Here it does not appear that any substitution will work. However, if we write

$$\int \tan x \,\mathrm{d}x = \int \frac{\sin x}{\cos x}\,\mathrm{d}x,$$

then we observe that the derivative of the function $\cos x$ is equal to $-\sin x$, so it is convenient to make the substitution $u = \cos x$. Then $\mathrm{d}u = -\sin x \,\mathrm{d}x$, and

$$\int \tan x \,\mathrm{d}x = -\int \frac{-\sin x}{\cos x}\,\mathrm{d}x = -\int \frac{1}{u}\,\mathrm{d}u = -\log|u| + C = -\log|\cos x| + C.$$

Example 14.2.9. The indefinite integral

$$\int \frac{9+6x+2x^2+x^3}{4+x^2}\,\mathrm{d}x$$

is rather daunting at first sight, but we have enough technique to study it. Note first of all that

$$\begin{aligned}9+6x+2x^2+x^3 &= 9+2x+2x^2+4x+x^3 = 9+2x+2x^2+x(4+x^2)\\ &= 1+2x+8+2x^2+x(4+x^2) = 1+2x+2(4+x^2)+x(4+x^2).\end{aligned}$$

It follows that

$$\int \frac{9+6x+2x^2+x^3}{4+x^2}\,\mathrm{d}x = \int \frac{1}{4+x^2}\,\mathrm{d}x + \int \frac{2x}{4+x^2}\,\mathrm{d}x + \int (2+x)\,\mathrm{d}x. \tag{2}$$

To study the first integral on the right hand side of (2), we can make a substitution $x = 2\tan u$. Then $\mathrm{d}x = 2\sec^2 u \,\mathrm{d}u$, and

$$\int \frac{1}{4+x^2}\,\mathrm{d}x = \int \frac{2\sec^2 u}{4+4\tan^2 u}\,\mathrm{d}u = \frac{1}{2}\int \mathrm{d}u = \frac{1}{2}u + C_1 = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C_1. \tag{3}$$

To study the second integral on the right hand side of (2), we note that the derivative of the function $4+x^2$ is equal to $2x$. If we make a substitution $v = 4+x^2$, then $\mathrm{d}v = 2x\,\mathrm{d}x$, and

$$\int \frac{2x}{4+x^2}\,\mathrm{d}x = \int \frac{1}{v}\,\mathrm{d}v = \log|v| + C_2 = \log(4+x^2) + C_2. \tag{4}$$

The third integral on the right hand side of (2) is easy to evaluate. We have

$$\int (2+x)\,\mathrm{d}x = 2x + \frac{1}{2}x^2 + C_3. \tag{5}$$

Substituting (3)–(5) into (2) and writing $C = C_1 + C_2 + C_3$, we obtain

$$\int \frac{9+6x+2x^2+x^3}{4+x^2}\,\mathrm{d}x = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + \log(4+x^2) + 2x + \frac{1}{2}x^2 + C.$$

It may be worth checking that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + \log(4+x^2) + 2x + \frac{1}{2}x^2 + C\right) = \frac{9+6x+2x^2+x^3}{4+x^2}.$$

14.3. Definite Integrals

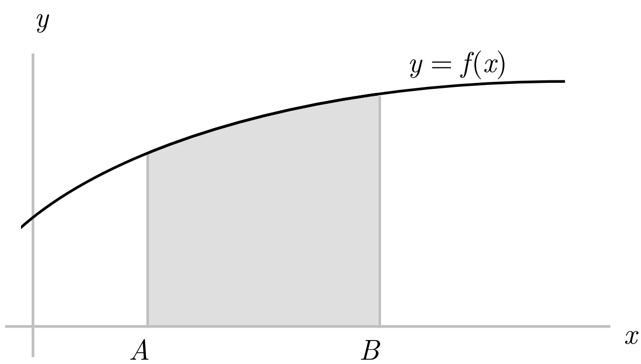
The formal definition of a definite integral is rather complicated, and we do not propose to discuss it here. Instead, we shall only give some geometric motivation, and then relate the definite integral to indefinite integrals we have discussed earlier.

Suppose that $f(x)$ is a real valued function, defined on an interval $[A, B] = \{x \in \mathbb{R} : A \le x \le B\}$. We shall suppose also that $f(x)$ has an antiderivative $F(x)$ for every $x \in [A, B]$.

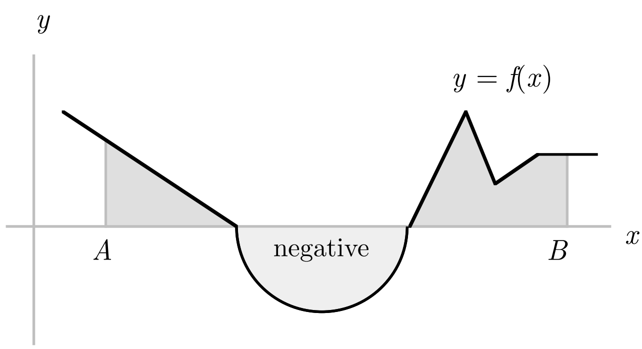
Consider first of all the special case that $f(x) \ge 0$ for every $x \in [A, B]$. By the definite integral

$$\int_A^B f(x)\,\mathrm{d}x,$$

we mean the area below the curve $y = f(x)$ and above the horizontal axis $y = 0$, bounded between the vertical lines $x = A$ and $x = B$, as shown in the picture below.



In general, we take the area between the curve $y = f(x)$ and the horizontal axis $y = 0$, bounded between the vertical lines $x = A$ and $x = B$, with the convention that the area below the horizontal axis $y = 0$ is taken to be negative, as shown in the picture below.



We now need a way of calculating this area. In some very special cases, this is very simple.

Example 14.3.1. If we examine the graph of the trigonometric functions in Chapter 3, then it is easy to see that

$$\int_0^{2\pi} \sin x\,\mathrm{d}x = 0 \qquad \text{and} \qquad \int_0^{\pi} \cos x\,\mathrm{d}x = 0.$$

In each case, it is easy to see that the area in question above the horizontal axis $y = 0$ is equal to the area in question below this axis.

Example 14.3.2. It is easy to see that the area between the line $y = x$ and the horizontal axis $y = 0$, bounded between the vertical lines $x = 0$ and $x = 1$, is the area of a triangle with base 1 and height 1. Hence

$$\int_0^1 x \, \mathrm{d}x = \frac{1}{2}.$$

In many instances, we do not have such geometric information to help us calculate the area in question. Instead, we can use the indefinite integral.

FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS. *Suppose that a function $F(x)$ satisfies $F'(x) = f(x)$ for every $x \in [A, B]$. Then*

$$\int_A^B f(x) \, \mathrm{d}x = \Big[F(x) \Big]_A^B = F(B) - F(A).$$

Remark. A simple consequence of the above is that the constant multiple rule and sum rule for indefinite integrals extend to definite integrals. For any fixed real number c, we have

$$\int_A^B c f(x) \, \mathrm{d}x = c \int_A^B f(x) \, \mathrm{d}x.$$

We also have

$$\int_A^B (f(x) + g(x)) \, \mathrm{d}x = \int_A^B f(x) \, \mathrm{d}x + \int_A^B g(x) \, \mathrm{d}x.$$

A further consequence of the Fundamental theorem of integral calculus is a rule concerning splitting up an interval $[A, B]$ into two. Suppose that $A < A^* < B$. Then

$$\int_A^B f(x) \, \mathrm{d}x = \int_A^{A^*} f(x) \, \mathrm{d}x + \int_{A^*}^B f(x) \, \mathrm{d}x.$$

Example 14.3.3. Returning to Example 14.3.1, we have

$$\int_0^{2\pi} \sin x \, \mathrm{d}x = \Big[-\cos x \Big]_0^{2\pi} = -\cos 2\pi + \cos 0 = 0$$

and

$$\int_0^{\pi} \cos x \, \mathrm{d}x = \Big[\sin x \Big]_0^{\pi} = \sin \pi - \sin 0 = 0.$$

Example 14.3.4. Returning to Example 14.3.2, we have

$$\int_0^1 x \, \mathrm{d}x = \left[\frac{1}{2} x \right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}.$$

Example 14.3.5. We have

$$\int_0^{\pi} \sin x \, \mathrm{d}x = \Big[-\cos x \Big]_0^{\pi} = -\cos \pi + \cos 0 = 2.$$

Example 14.3.6. We have

$$\int_1^2 \frac{1}{x} \, \mathrm{d}x = \Big[\log |x| \Big]_1^2 = \log 2 - \log 1 = \log 2.$$

EXAMPLE 14.3.7. We have

$$\int_0^1 \mathrm{e}^x \,\mathrm{d}x = \Big[\mathrm{e}^x\Big]_0^1 = \mathrm{e}^1 - \mathrm{e}^0 = \mathrm{e} - 1.$$

EXAMPLE 14.3.8. We have

$$\int_0^{\pi/4} \sec^2 x \,\mathrm{d}x = \Big[\tan x\Big]_0^{\pi/4} = \tan\frac{\pi}{4} - \tan 0 = 1.$$

EXAMPLE 14.3.9. We have

$$\int_{-1}^1 (x^3 + x^2) \,\mathrm{d}x = \left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^1 = \left(\frac{1}{4} + \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{3}\right) = \frac{2}{3}.$$

EXAMPLE 14.3.10. Recall Example 14.2.1. Since

$$\int \frac{1}{\sqrt{1 - x^2}} \,\mathrm{d}x = \sin^{-1} x + C, \tag{6}$$

we have

$$\int_0^{1/2} \frac{1}{\sqrt{1 - x^2}} \,\mathrm{d}x = \Big[\sin^{-1} x\Big]_0^{1/2} = \sin^{-1}\frac{1}{2} - \sin^{-1} 0 = \frac{\pi}{6}.$$

To obtain (6), recall that we can use the substitution $x = \sin u$ to show that

$$\int \frac{1}{\sqrt{1 - x^2}} \,\mathrm{d}x = \int \mathrm{d}u = u + C,$$

followed by an inverse substitution $u = \sin^{-1} x$. Here, we need to make the extra step of substituting the values $x = 0$ and $x = 1/2$ to the indefinite integral $\sin^{-1} x$. Observe, however, that with the substitution $x = \sin u$, the variable x increases from 0 to $1/2$ as the variable u increases from 0 to $\pi/6$. But then

$$\int_0^{\pi/6} \mathrm{d}u = \Big[u\Big]_0^{\pi/6} = \frac{\pi}{6} = \int_0^{1/2} \frac{1}{\sqrt{1 - x^2}} \,\mathrm{d}x,$$

so it appears that we do not need the inverse substitution $u = \sin^{-1} x$. Perhaps we can directly substitute $u = 0$ and $u = \pi/6$ to the indefinite integral u.

DEFINITE INTEGRAL BY SUBSTITUTION – VERSION 1. *Suppose that a substitution $x = g(u)$ satisfies the following conditions:*
(a) There exist $\alpha, \beta \in \mathbb{R}$ such that $g(\alpha) = A$ and $g(\beta) = B$.
(b) The derivative $g'(u) > 0$ for every u satisfying $\alpha < u < \beta$.
Then $\mathrm{d}x = g'(u)\,\mathrm{d}u$, and

$$\int_A^B f(x) \,\mathrm{d}x = \int_\alpha^\beta f(g(u))g'(u) \,\mathrm{d}u.$$

REMARK. If condition (b) above is replaced by the condition that the derivative $g'(u) < 0$ for every u satisfying $\beta < u < \alpha$, then the same conclusion holds if we adopt the convention that

$$\int_\alpha^\beta f(g(u))g'(u) \,\mathrm{d}u = -\int_\beta^\alpha f(g(u))g'(u) \,\mathrm{d}u.$$

EXAMPLE 14.3.11. To calculate the definite integral

$$\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x,$$

we can use the substitution $x = \tan u$, so that $\mathrm{d}x = \sec^2 u\,\mathrm{d}u$. Note that $\tan 0 = 0$ and $\tan(\pi/4) = 1$, and that $\sec^2 u > 0$ whenever $0 < u < \pi/4$. It follows that

$$\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = \int_0^{\pi/4} \frac{\sec^2 u}{1+\tan^2 u}\,\mathrm{d}u = \int_0^{\pi/4} \mathrm{d}u = \Big[u\Big]_0^{\pi/4} = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

We can compare this to first observing Example 14.2.2, so that

$$\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = \Big[\tan^{-1} x\Big]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

EXAMPLE 14.3.12. To calculate the definite integral

$$\int_0^3 x\sqrt{x+1}\,\mathrm{d}x,$$

we can use the substitution $x = g(u) = u^2 - 1$, so that $\mathrm{d}x = 2u\,\mathrm{d}u$. Note that $g(1) = 0$ and $g(2) = 3$, and that $g'(u) = 2u > 0$ whenever $1 < u < 2$. It follows that

$$\int_0^3 x\sqrt{x+1}\,\mathrm{d}x = \int_1^2 2(u^2-1)u^2\,\mathrm{d}u = \left[\frac{2}{5}u^5 - \frac{2}{3}u^3\right]_1^2 = \left(\frac{64}{5} - \frac{16}{3}\right) - \left(\frac{2}{5} - \frac{2}{3}\right) = \frac{62}{5} - \frac{14}{3} = \frac{116}{15}.$$

DEFINITE INTEGRAL BY SUBSTITUTION – VERSION 2. *Suppose that a substitution $u = h(x)$ satisfies the following conditions:*
(a) There exists a function $g(u)$ such that $f(x) = g(h(x))h'(x)$ for every $x \in [A, B]$.
(b) The derivative $h'(x) > 0$ for every x satisfying $A < x < B$.
Then $\mathrm{d}u = h'(x)\,\mathrm{d}x$, and

$$\int_A^B f(x)\,\mathrm{d}x = \int_A^B g(h(x))h'(x)\,\mathrm{d}x = \int_{h(A)}^{h(B)} g(u)\,\mathrm{d}u.$$

REMARK. If condition (b) above is replaced by the condition that the derivative $h'(x) < 0$ for every x satisfying $A < x < B$, then the same conclusion holds if we adopt the convention that

$$\int_{h(A)}^{h(B)} g(u)\,\mathrm{d}u = -\int_{h(B)}^{h(A)} g(u)\,\mathrm{d}u.$$

EXAMPLE 14.3.13. To calculate the definite integral

$$\int_0^1 x(x^2+3)^4\,\mathrm{d}x,$$

we can use the substitution $u = h(x) = x^2 + 3$, so that $\mathrm{d}u = 2x\,\mathrm{d}x$. Note that $h(0) = 3$ and $h(1) = 4$, and that $h'(x) = 2x > 0$ whenever $0 < x < 1$. It follows that

$$\int_0^1 x(x^2+3)^4\,\mathrm{d}x = \frac{1}{2}\int_3^4 u^4\,\mathrm{d}x = \frac{1}{2}\left[\frac{u^5}{5}\right]_3^4 = \frac{1}{2}\left(\frac{1024}{5} - \frac{243}{5}\right) = \frac{781}{10}.$$

EXAMPLE 14.3.14. To calculate the definite integral

$$\int_2^4 \frac{1}{x\log x}\,\mathrm{d}x,$$

we can use the substitution $u = h(x) = \log x$, so that $\mathrm{d}u = h'(x)\,\mathrm{d}x$, where $h'(x) = 1/x > 0$ whenever $2 < x < 4$. Note also that $h(2) = \log 2$ and $h(4) = \log 4$. It follows that

$$\int_2^4 \frac{1}{x\log x}\,\mathrm{d}x = \int_{\log 2}^{\log 4} \frac{1}{u}\,\mathrm{d}u = \Big[\log|u|\Big]_{\log 2}^{\log 4} = \log\log 4 - \log\log 2 = \log\left(\frac{\log 4}{\log 2}\right) = \log 2.$$

EXAMPLE 14.3.15. To calculate the definite integral

$$\int_0^\pi \sin^2 x\cos x\,\mathrm{d}x,$$

we can use the substitution $u = h(x) = \sin x$, so that $\mathrm{d}u = \cos x\,\mathrm{d}x$. Now $h(0) = 0$ and $h(\pi) = 0$, so something is funny here! The problem is that

$$h'(x) = \cos x \begin{cases} > 0 & \left(0 < x < \frac{\pi}{2}\right), \\ < 0 & \left(\frac{\pi}{2} < x < \pi\right). \end{cases}$$

It follows that we must first write

$$\int_0^\pi \sin^2 x\cos x\,\mathrm{d}x = \int_0^{\pi/2} \sin^2 x\cos x\,\mathrm{d}x + \int_{\pi/2}^{\pi} \sin^2 x\cos x\,\mathrm{d}x \tag{7}$$

before we can make any substitution. Consider now the first integral on the right hand side of (7). Using the substitution $u = h(x) = \sin x$, we note that $h(0) = 0$ and $h(\pi/2) = 1$, and that $h'(x) > 0$ whenever $0 < x < \pi/2$. Hence

$$\int_0^{\pi/2} \sin^2 x\cos x\,\mathrm{d}x = \int_0^1 u^2\,\mathrm{d}u = \frac{1}{3}.$$

Consider next the second integral on the right hand side of (7). Using the substitution $u = h(x) = \sin x$, we note that $h(\pi/2) = 1$ and $h(\pi) = 0$, and that $h'(x) < 0$ whenever $\pi/2 < x < \pi$. Hence

$$\int_{\pi/2}^{\pi} \sin^2 x\cos x\,\mathrm{d}x = \int_1^0 u^2\,\mathrm{d}u = -\int_0^1 u^2\,\mathrm{d}u = -\frac{1}{3}.$$

Combining the two parts, we conclude that

$$\int_0^\pi \sin^2 x\cos x\,\mathrm{d}x = 0.$$

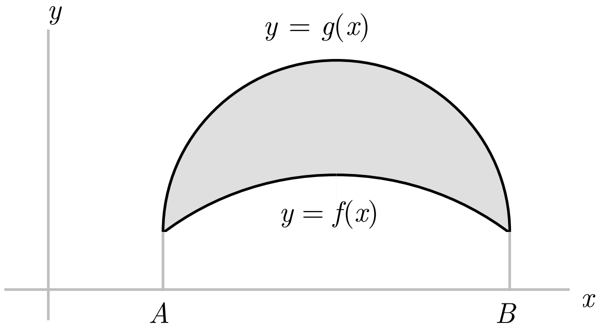
Alternatively, we can make the substitution $x = g(v) = \pi - v$ to the second integral on the right hand side of (7). Then $g(\pi/2) = \pi/2$ and $g(0) = \pi$, and $g'(v) = -1 < 0$ for every v satisfying $0 < v < \pi/2$. It follows that

$$\int_{\pi/2}^{\pi} \sin^2 x\cos x\,\mathrm{d}x = \int_{\pi/2}^{0} \sin^2 v\cos v\,\mathrm{d}v = -\int_0^{\pi/2} \sin^2 v\cos v\,\mathrm{d}v.$$

This, combined with (7), gives the same conclusion.

14.4. Areas

We conclude this chapter by describing how we may use definite integrals to evaluate areas. Suppose that the boundary of a region on the xy-plane can be described by a top edge $y = g(x)$ and a bottom edge $y = f(x)$ bounded between two vertical lines $x = A$ and $x = B$, as shown in the picture below.



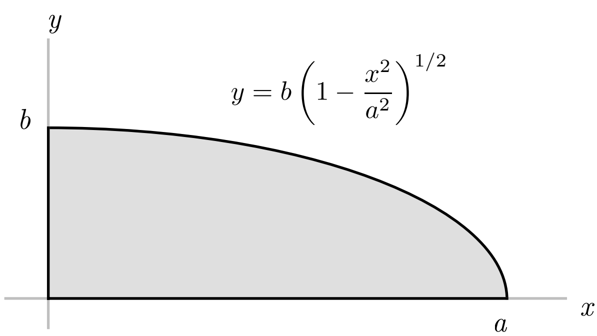
Then the area of the region is given by the definite integral

$$\int_A^B (g(x) - f(x))\,\mathrm{d}x.$$

EXAMPLE 14.4.1. We wish to show that the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a, b \in \mathbb{R}$ are positive, is equal to πab. To do this, we may consider the quarter of the ellipse in the first quadrant, as shown in the picture below.



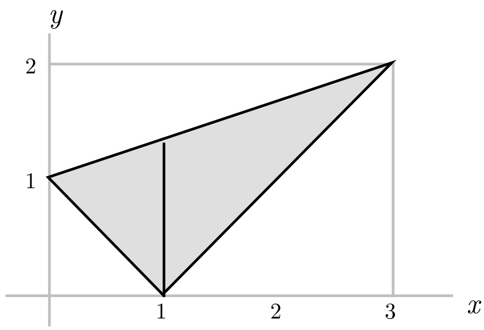
It follows that the shaded region has area

$$\int_0^a b\left(1 - \frac{x^2}{a^2}\right)^{1/2} \mathrm{d}x.$$

We can use the substitution $x = g(u) = a \sin u$. Then $g(0) = 0$ and $g(\pi/2) = a$. Furthermore, we have $\mathrm{d}x = g'(u)\,\mathrm{d}u$, where $g'(u) = a\cos u > 0$ whenever $0 < u < \pi/2$. It follows that

$$\begin{aligned}\int_0^a b\left(1 - \frac{x^2}{a^2}\right)^{1/2} \mathrm{d}x &= \int_0^{\pi/2} ab(1 - \sin^2 u)^{1/2} \cos u\,\mathrm{d}u = ab\int_0^{\pi/2} \cos^2 u\,\mathrm{d}u \\ &= ab\int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos 2u\right) \mathrm{d}u = ab\left[\frac{1}{2}u + \frac{1}{4}\sin 2u\right]_0^{\pi/2} = \frac{\pi ab}{4}.\end{aligned}$$

EXAMPLE 14.4.2. We wish to evaluate the area of the triangle with vertices $(0,1)$, $(1,0)$ and $(3,2)$. To do this, we split the triangle into two regions as shown in the picture below.



The triangle on the left is bounded between the vertical lines $x=0$ and $x=1$, and the top edge and the bottom edge are given respectively by

$$y=\frac{1}{3}x+1 \qquad \text{and} \qquad y=1-x.$$

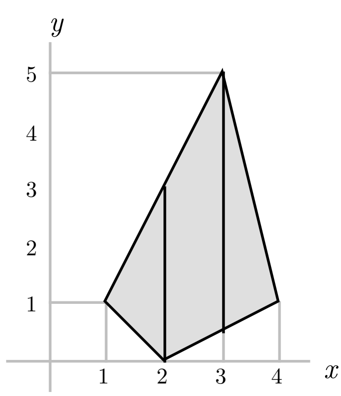
The triangle on the right is bounded between the vertical lines $x=1$ and $x=3$, and the top edge and the bottom edge are given respectively by

$$y=\frac{1}{3}x+1 \qquad \text{and} \qquad y=x-1.$$

It follows that the area of the original triangle is given by

$$\int_0^1 \left(\left(\frac{1}{3}x+1\right)-(1-x)\right)\,\mathrm{d}x+\int_1^3 \left(\left(\frac{1}{3}x+1\right)-(x-1)\right)\,\mathrm{d}x$$
$$=\int_0^1 \frac{4}{3}x\,\mathrm{d}x+\int_1^3 \left(2-\frac{2}{3}x\right)\,\mathrm{d}x=\left[\frac{2}{3}x^2\right]_0^1+\left[2x-\frac{1}{3}x^2\right]_1^3=2.$$

EXAMPLE 14.4.3. We wish to evaluate the area of the quadrilateral with vertices $(1,1)$, $(2,0)$, $(4,1)$ and $(3,5)$. To do this, we split the quadrilateral into three regions as shown in the picture below.



The triangle on the left is bounded between the vertical lines $x=1$ and $x=2$, and the top edge and the bottom edge are given respectively by

$$y=2x-1 \qquad \text{and} \qquad y=2-x.$$

The quadrilateral in the middle is bounded between the vertical lines $x = 2$ and $x = 3$, and the top edge and the bottom edge are given respectively by

$$y = 2x - 1 \qquad \text{and} \qquad y = \frac{1}{2}x - 1.$$

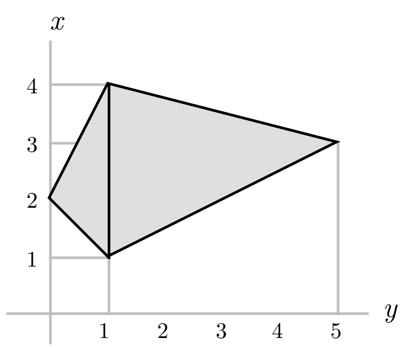
The triangle on the right is bounded between the vertical lines $x = 3$ and $x = 4$, and the top edge and the bottom edge are given respectively by

$$y = 17 - 4x \qquad \text{and} \qquad y = \frac{1}{2}x - 1.$$

It follows that the area of the original quadrilateral is given by

$$\begin{aligned}
&\int_1^2 ((2x-1)-(2-x))\,\mathrm{d}x + \int_2^3 \left((2x-1) - \left(\frac{1}{2}x - 1\right)\right)\mathrm{d}x + \int_3^4 \left((17-4x) - \left(\frac{1}{2}x - 1\right)\right)\mathrm{d}x \\
&\quad = \int_1^2 (3x-3)\,\mathrm{d}x + \int_2^3 \frac{3}{2}x\,\mathrm{d}x + \int_3^4 \left(18 - \frac{9}{2}x\right)\mathrm{d}x \\
&\quad = \left[\frac{3}{2}x^2 - 3x\right]_1^2 + \left[\frac{3}{4}x^2\right]_2^3 + \left[18x - \frac{9}{4}x^2\right]_3^4 = \frac{15}{2}.
\end{aligned}$$

Alternatively, we can transpose the picture above and split the quadrilateral into two regions as shown in the picture below:



Note that the roles of x and y are now interchanged. The triangle on the left is bounded between the vertical lines $y = 0$ and $y = 1$, and the top edge and the bottom edge are given respectively by

$$x = 2y + 2 \qquad \text{and} \qquad x = 2 - y.$$

The triangle on the right is bounded between the vertical lines $y = 1$ and $y = 5$, and the top edge and the bottom edge are given respectively by

$$x = \frac{17}{4} - \frac{1}{4}y \qquad \text{and} \qquad x = \frac{1}{2}y + \frac{1}{2}.$$

It follows that the area of the original quadrilateral is given by

$$\begin{aligned}
&\int_0^1 ((2y+2)-(2-y))\,\mathrm{d}y + \int_1^5 \left(\left(\frac{17}{4} - \frac{1}{4}y\right) - \left(\frac{1}{2}y + \frac{1}{2}\right)\right)\mathrm{d}y \\
&\quad = \int_0^1 3y\,\mathrm{d}y + \int_1^5 \left(\frac{15}{4} - \frac{3}{4}y\right)\mathrm{d}y = \left[\frac{3}{2}y^2\right]_0^1 + \left[\frac{15}{4}y - \frac{3}{8}y^2\right]_1^5 = \frac{15}{2}
\end{aligned}$$

as before.

Problems for Chapter 14

1. Find each of the following indefinite integrals:

a) $\int \sqrt{3}\,\mathrm{d}x$ b) $\int (5x+3)\,\mathrm{d}x$ c) $\int (2x^2-3x+1)\,\mathrm{d}x$

d) $\int x^3\,\mathrm{d}x$ e) $\int (x-2)(x+3)\,\mathrm{d}x$ f) $\int (1-2\cos x)\,\mathrm{d}x$

g) $\int (5\cos x+4x)\,\mathrm{d}x$ h) $\int 8\mathrm{e}^x\,\mathrm{d}x$ i) $\int \frac{1}{x}\,\mathrm{d}x$

2. Evaluate each of the following indefinite integrals using the given substitution:

a) $\int x(x^2-1)^{99}\,\mathrm{d}x$ (use the substitution $u=x^2-1$)

b) $\int \frac{x^2}{\sqrt{2+x^3}}\,\mathrm{d}x$ (use the substitution $u=x^3+2$)

c) $\int \sin 4x\,\mathrm{d}x$ (use the substitution $u=4x$)

d) $\int \frac{\mathrm{d}x}{(2x+1)^2}$ (use the substitution $u=2x+1$)

e) $\int \frac{x+3}{(x^2+6x)^2}\,\mathrm{d}x$ (use the substitution $u=x^2+6x$)

f) $\int \sec ax\,\tan ax\,\mathrm{d}x$ (use the substitution $u=ax$)

3. Evaluate each of the following indefinite integrals:

a) $\int \cos 2x\,\mathrm{d}x$ b) $\int \sqrt{x-1}\,\mathrm{d}x$ c) $\int x^2\cos(1-x^3)\,\mathrm{d}x$

d) $\int x\sin(x^2)\,\mathrm{d}x$ e) $\int \frac{1}{(1-3x)^4}\,\mathrm{d}x$ f) $\int \frac{x}{\sqrt{x^2+1}}\,\mathrm{d}x$

g) $\int \sec^2(3x)\,\mathrm{d}x$ h) $\int \sin^3 x\cos x\,\mathrm{d}x$ i) $\int x(x^2+16)^2\,\mathrm{d}x$

j) $\int x^2\sqrt{x^3+8}\,\mathrm{d}x$ k) $\int \frac{1}{\sqrt{2x+5}}\,\mathrm{d}x$ l) $\int \left(x-\frac{1}{x}\right)\,\mathrm{d}x$

m) $\int \frac{2x+1}{x^2+x+3}\,\mathrm{d}x$ n) $\int \frac{1}{x^2-4x+4}\,\mathrm{d}x$ o) $\int \frac{\log x}{x}\,\mathrm{d}x$

p) $\int \frac{\mathrm{e}^x}{1+\mathrm{e}^x}\,\mathrm{d}x$ q) $\int x\mathrm{e}^{x^2}\,\mathrm{d}x$ r) $\int \mathrm{e}^{2x-1}\,\mathrm{d}x$

s) $\int \sec(4x)\tan(4x)\,\mathrm{d}x$ t) $\int x^3\cos(5x^4)\,\mathrm{d}x$ u) $\int \sec^2(2x+1)\,\mathrm{d}x$

v) $\int \mathrm{e}^x\cos(\mathrm{e}^x)\,\mathrm{d}x$ w) $\int \frac{(\log x)^2}{x}\,\mathrm{d}x$ x) $\int \tan x\sec^3 x\,\mathrm{d}x$

4. Evaluate each of the following definite integrals:

a) $\int_1^2 2x\,\mathrm{d}x$ b) $\int_1^3 \frac{1}{x}\,\mathrm{d}x$ c) $\int_0^2 \mathrm{e}^{-x}\,\mathrm{d}x$

d) $\int_2^3 (3x+1)\,\mathrm{d}x$ e) $\int_0^\pi \sin x\,\mathrm{d}x$ f) $\int_3^6 (x-3)^2\,\mathrm{d}x$

g) $\int_0^2 \frac{1}{4+x^2}\,\mathrm{d}x$ h) $\int_0^1 x\mathrm{e}^{x^2}\,\mathrm{d}x$ i) $\int_0^a (x^2+a^2)\,\mathrm{d}x$

j) $\int_0^1 (1+x+3x^2)\,\mathrm{d}x$ k) $\int_0^1 \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x$ l) $\int_{-\pi/2}^{\pi/2} \sin x\,\mathrm{d}x$

5. a) Draw the graphs of the line $y = x$ and the parabola $y = x^2$.
   b) Find the two points of intersection of the two curves.
   c) Use definite integrals to find the area bounded between the two curves.

6. Use definite integrals to find the area between the curves $y = e^x$ and $y = e^{2x}$, bounded between the lines $x = 0$ and $x = 1$.

7. Find the area of the triangle with vertices $(0, 0)$, $(4, 3)$ and $(1, 5)$.

8. Find the area of the quadrilateral with vertices $(1, 1)$, $(5, 2)$, $(2, 3)$ and $(4, 3)$.

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