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basic MATH course for ELECTRONICS

$$Q = \frac{2\pi f L}{R} = \frac{\omega L}{R}$$

by Henry Jacobowitz

Basic Math Course For Electronics

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HENRY JACOBOWITZ

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CONTENTS

CONTENTS

Introduction

IF you are like many of us, you take an instinctive attitude of flight when you hear the word "mathematics". There doesn't seem to be any reason why this suspect term should be coupled in any way with the profession of an electronic technician. As many service technicians have told me, they don't use mathematics—not beyond Ohm's Law anyway—since the equipment manufacturers provide them with all the necessary information. Or at least they think so.

Of course, they ignore the many times they use mathematics in daily practice without being aware of it. Even the usual application of Ohm's Law requires a whole host of mathematical operations. To convert microvolts and milliamperes to volts and amperes, respectively, you may have to use powers of 10, substitute in an algebraic formula, divide or multiply, possibly use a slide rule, and "round off" the arithmetical result to the number of significant figures consistent with the accuracy of the equipment and its components. If you have mastered this kind of essential "test-bench mathematics," you do it instinctively without awareness of the number of mathematical operations involved. It is just like walking or swimming, once you know how. What we are trying to get across, of course, is that learning the language of mathematics—as far as it applies to electronic shop practice—can be not only fairly pleasant and painless, but also highly useful.

The statement that the manufacturers of electronic equipment provide all the information necessary is a half-truth, at best. They publish a variety of specifications, graphs, curves and instructions which they believe will make your work easier and make their equipment operate best. Many times, however, you will have to perform supplementary calculations to utilize the published data and to put together differently formulated specifications for various components. For example, did you recently have to connect a stereo cartridge with an output per coil of 1.5 millivolts to a stereo preamplifier with 70-db gain, which in turn drove another manufacturer's pair of power amplifiers, each of which required 2 volts input for full rated power output? Looking at the specifications alone, you could not possibly tell whether the cartridge had sufficient output to drive the power amplifiers. You may have resolved the problem by trying the equipment out in your client's home, though this might lead to considerable embarrassment if things didn't work out properly. Making several calculations in advance might have spared you this kind of situation. You undoubtedly can add a number of similar examples from your own experience.

What we shall try to do here is to discuss a number of highly useful mathematical tools and techniques as informally as possible. Many times we won't bother to introduce all the formal stipulations, terms, conditions and other verbiage, though the mathematical purists may howl. In brief, we shall learn to manipulate a little arithmetic, algebra, vectors and complex numbers, logarithms and decibels with a minimum of formality.

HENRY JACOBOWITZ

CHAPTER 1

Ohm's Law Arithmetic

LET us begin our electromathematical excursions with Ohm's Law, since this is the most widely known relationship and around it are centered many everyday practical computations. In its most general form, this law—first formulated by Georg Simon Ohm in 1827—simply states that the current flowing in a dc circuit, or portion thereof, is directly proportional to the applied voltage (emf) and inversely proportional to the resistance of the circuit or portion of it. In its mathematical form, Ohm's Law is usually stated thus:

$$\text{Current (I)} = \frac{\text{Voltage (E)}}{\text{Resistance (R)}}$$

$$\text{or briefly, } I = \frac{E}{R} \qquad (1)$$

Note that we have inserted an equal (=) sign between the proportional quantities, rather than a proportional ($\propto$) sign. As we shall see later, this can be done only if a proportionality constant is inserted after the equal sign. In the case of Ohm's Law, the units of current, voltage and resistance have been defined to make the proportionality constant equal to unity (1) and, hence, it disappears altogether. Thus, by definition, a current of 1 ampere is said to flow through a circuit of 1 ohm resistance, if an electromotive force (emf) of 1 volt is applied. Hence,

$$1 \text{ ampere} = \frac{1 \text{ volt}}{1 \text{ ohm}} \quad \text{and in general} \quad \text{amperes} = \frac{\text{volts}}{\text{ohms}}$$

Regardless of the units in which the voltage and resistance are stated in a given problem, you will always have to convert to these or equivalent units, when using Ohm's Law.

Equivalent forms of Ohm's Law

Simple common-sense reasoning shows—if Ohm's Law is true—that the applied voltage E acting in a circuit of resistance R, through which a current I flows, must be equal to the product of the current and the resistance, or

$$E = I \times R \tag{2}$$

This relation applies not only to a complete circuit but also to any portion of it, such as a resistor. Thus, the voltage drop (E) developed by a current (I) flowing through a resistor (R) is equal to the product of the current and the resistance value, or again E = IR (the multiplication sign "×" is understood).

You can also obtain Equation 2 by a simple algebraic manipulation of the originally stated form of Ohm's Law (Equation 1). It is a fundamental rule that you can perform any mathematical operation on an equation, as long as you do it to both sides. Hence, let us multiply both sides of Equation 1 (Ohm's Law) by the resistance, R:

$$I = \frac{E}{R} \qquad I \times R = \frac{E \times R}{R} \quad \text{(multiplying by R)} \tag{1}$$

$$\text{and since} \quad \frac{R}{R} = 1, \quad I \times R = E$$

$$\text{or} \quad E = I \times R \tag{2}$$

which is the relation (Equation 2) we stated before.

From Equation 2 we can easily derive a third, commonly used form of Ohm's Law. Dividing both sides by the current, I:

$$\frac{E}{I} = \frac{I \times R}{I} = R \quad \left(\text{Since, } \frac{I}{I} = 1\right) \quad \text{or} \quad R = \frac{E}{I} \tag{3}$$

Equation 3 states that the resistance (R) of a circuit, or portion thereof, is equal to the voltage (E) applied to the circuit (or portion of it) divided by the current (I) flowing through it.

Using Ohm's Law

Although we have stated the three forms of Ohm's Law in algebraic form, as soon as you start to use it by substituting numbers in the appropriate equation, your calculations become pure arithmetic. This is usually true for most of the simple formulas used in electricity and electronics. To make them generally valid, the

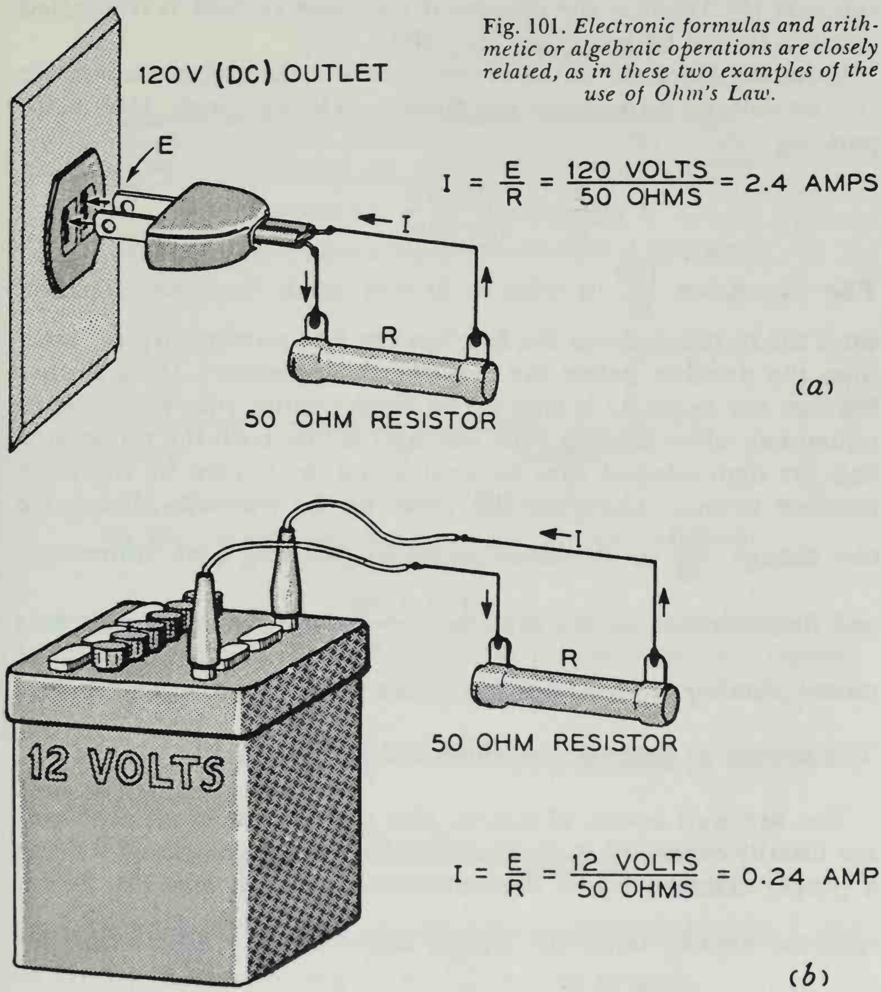


Fig. 101. *Electronic formulas and arithmetic or algebraic operations are closely related, as in these two examples of the use of Ohm's Law.*

formulas are stated in symbol form, such as f for frequency, C for capacitance, L for inductance, etc. but, in any particular case, you will be using specific numerical values in place of the symbols and, hence, the calculations become arithmetical. Your mastery of simple arithmetic, therefore, is the key to solving the majority of ordinary electronics and electrical problems. We shall use Ohm's Law problems to illustrate elementary arithmetical operations, but the examples will apply to any formula, where only simple multiplication and division is indicated.

EXAMPLE 1: (a) A 50-ohm resistor is connected across a 120-volt (dc) line. What is the value of the current flowing through the resistor? (b) What is the current if the same resistor is connected across a 12-volt battery? (See Fig. 101).

Solution: Since the value of the current is desired, we substitute the known quantities in equation 1 (Ohm's Law). Hence for part (a),

$$I = \frac{E}{R} = \frac{120 \text{ volts}}{50 \text{ ohms}} = ? \text{ amperes}$$

The expression $\frac{120}{50}$ is what is known as an improper fraction, since the number above the fraction bar (the numerator) is greater than the number below the bar (the denominator). (In a *proper* fraction the opposite is true.) The fundamental rule you need to remember when dealing with fractions is that both the numerator and the denominator may be multiplied or divided by the same number without changing the value of the fraction. Hence, we can change $\frac{120}{50}$ to its lowest terms by dividing both numerator and denominator by 10: thus, $\frac{120 \div 10 = 12}{50 \div 10 = 5}$, or converting to a mixed number (a whole number plus a proper fraction), $\frac{12}{5} = 2\frac{2}{5}$. The answer to part (a), therefore, is $2\frac{2}{5}$ amperes.

You are well aware, of course, that the answers to all problems are usually expressed in *decimals,* which is a special way of writing a proper fraction whose denominators end with *zero* (0). By inspection we can write the answer above as $2\frac{2}{5} = 2\frac{4}{10}$, which becomes 2.4 since $\frac{4}{10} = 0.4$. (You'll recall that the first place *after* the decimal point are the *tenths,* the second place are *hundredths,* the third place are *thousandths,* and so on.) Knowing that the answer is to be expressed as a decimal, you could have obtained the result more directly by carrying out the division indicated by the fraction bar. Thus, $\frac{120}{50} = \frac{12}{5} = 12 \div 5 = 2.4$

Finally, if you are the proud possessor of a slide rule, you can solve the problem quickly by setting 5 on the C scale above 12 on the D scale. Then set the hairline of the runner (indicator) to 1 on the C scale and read the answer (2.4) immediately below on the D scale (Fig. 102).

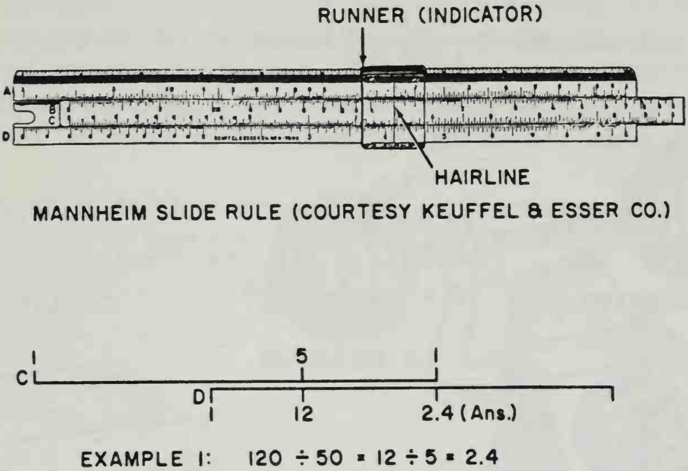


Fig. 102. *The slide rule represents an easy and quick way to solve many problems in electronics.*

We have taken up much time with part (a) of Example 1 to illustrate different ways of solving the problem and the various mathematical operations involved. Now, let us do part (b) in a jiffy.

(b) What is the current if the 50-ohm resistor is connected across a 12-volt battery?

$$I = \frac{E}{R} = \frac{12 \text{ volts}}{50 \text{ ohms}} = ? \text{ amperes}$$

Mentally multiplying both numerator and denominator by 2, you can see by inspection that $\frac{12}{50} = \frac{24}{100} = 0.24$ ampere.

You could have gotten the same result, of course, by carrying out the indicated division $12 \div 50 = 0.24$. The slide rule calculation is the same as in (a), since the numbers are unchanged and the decimal point must be determined by inspection.

EXAMPLE 2: A microammeter inserted into a circuit (Fig. 103) reads a current of 3.5 microamperes (μa) when 7 volts are applied to the circuit. What is the circuit resistance?

Solution: Let's do this problem first in a clumsy way to see how much time we can save by learning a new method. The unit for current in Ohm's Law is the ampere and, hence, we need to convert microamperes to amperes before we can substitute in the formula. We know that 1 microampere equals a millionth of an ampere, or

$$1 \text{ microampere} = \frac{1}{1,000,000} \text{ ampere} = .000001 \text{ ampere}$$

In general, then, we have to move the decimal point six places to the *left,* when converting microamperes to amperes. Therefore,

$$3.5 \text{ microamperes} = \frac{3.5}{1,000,000} \text{ ampere} = .0000035 \text{ ampere}$$

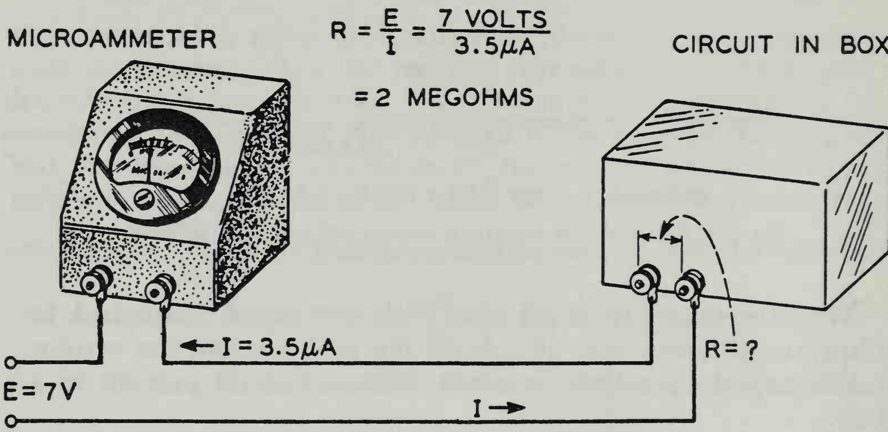


Fig. 103. *Since we know the value of the voltage and the amount of current, we can use Ohm's Law to learn the resistance of the circuit in the box.*

Now we can substitute in the Ohm's Law formula for the resistance,

$$R = \frac{E}{I} = \frac{7 \text{ volts}}{.0000035 \text{ ampere}} = ? \text{ ohms}$$

To get rid of the decimal point, let's multiply both numerator and denominator by 10,000,000, thus obtaining

$$\frac{70,000,000 \text{ (volts)}}{35 \text{ (amperes)}} = 2,000,000 \text{ ohms}$$

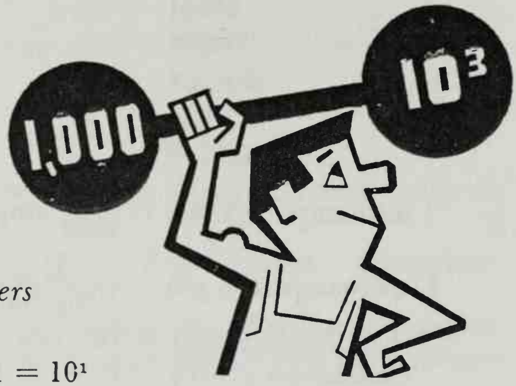
Since the *megohm* is the same as 1 million (1,000,000) ohms, we can write the answer in conventionally used units:

$$2,000,000 \text{ ohms} = 2 \text{ megohms}$$

The power of powers of 10

Now we can tell you that there is a better way of dealing with problems of this kind. This method of notation—called *powers of ten*—is extremely useful whenever you work with very large or very small numbers, where many zeros are involved. The powers-of-10 notation is a short-hand way for indicating the decimal point by raising the number 10 to the appropriate (positive or negative) power. The method is made clear by the following examples for expressing large or small numbers in powers of 10.

One number just looks larger than the other. They both have the same "weight".



For large numbers

$1 = 10^0$
$10 = 10 \times 1 = 10^1$
$100 = 10 \times 10 = 10^2$
$1{,}000 = 10 \times 10 \times 10 = 10^3$
$10{,}000 = 10 \times 10 \times 10 \times 10 = 10^4$
$100{,}000 = 10^5$
$1{,}000{,}000 = 10^6$
$10{,}000{,}000 = 10^7$ and so forth
$230{,}000 = 2.3 \times 10^5$
$12{,}600{,}000 = 12.6 \times 10^6$ or 1.26×10^7
1 megohm = 1,000,000 ohms = 10^6 ohms
22.5 megohms = 22.5×10^6 ohms or 2.25×10^7 ohms
Speed of light, c = 29,979,000,000 cm/sec. = 2.9979×10^{10} cm/sec.

Evidently, for large numbers, the notation indicates how many times 10 should be multiplied by itself to obtain the required number. For example, 10^5 indicates that 10 should be multiplied 5 times by itself (that is, raised to the fifth power), thus:

$$10 \times 10 \times 10 \times 10 \times 10 = 100{,}000.$$

Small numbers are expressed by taking the reciprocals of powers of 10. This is indicated by placing a small, negative number (exponent) next to 10.

Thus, 10^{-2} is the reciprocal of 10^2 or $\frac{1}{10^2} = \frac{1}{100} = 0.01$.

For small numbers

$$.1 = \frac{1}{10} = 10^{-1}$$

$$.01 = \frac{1}{100} = 10^{-2}$$

$$.001 = \frac{1}{1{,}000} = 10^{-3}$$

$$.0001 = \frac{1}{10{,}000} = 10^{-4}$$

$$.00001 = 10^{-5}$$

$$.000001 = 10^{-6}$$

$$.0000001 = 10^{-7} \text{ and so forth}$$

$$.0036 = 3.6 \times 10^{-3}$$

$$.000000769 = 7.69 \times 10^{-7}$$

$$1 \text{ milliampere } (1 \text{ ma}) = \frac{1}{1000} \text{ ampere} = 10^{-3} \text{ ampere}$$

$$1 \text{ microampere } (1\ \mu\text{a}) = \frac{1}{1{,}000{,}000} \text{ ampere} = 10^{-6} \text{ ampere}$$

$$1 \text{ millivolt } (1 \text{ mV}) = 10^{-3} \text{ volt}$$

$$1 \text{ microvolt } (1\ \mu\text{V}) = 10^{-6} \text{ volt}$$

Wavelength of sodium light, $\lambda = .00005983 \text{ cm} = 5.983 \times 10^{-5} \text{ cm}$

Operations with powers of 10

It is very simple to use numbers expressed in powers of 10 in your calculations. You can add and subtract these numbers just like any others provided they are expressed in the *same* powers of 10. Obviously, you can't add cats and dogs, milliamperes and microamperes, or 10^3 and 10^{-6}, without expressing them first in the same units. For example, let's add $2.345 \times 10^3 + 10.56 \times 10^2 + 8.65 \times 10^{-3}$. Re-expressing the numbers in powers of 10^2:

$$\begin{aligned} 2.345 \times 10^3 &= 23.45 \times 10^2 \\ 10.56 \times 10^2 &= 10.56 \times 10^2 \\ 8.65 \times 10^{-3} &= .0000865 \times 10^2 \\ \text{Adding} \quad & \overline{34.0100865 \times 10^2} = 3{,}401.00865 \end{aligned}$$

As another example, subtract 565 μa from 3.42 milliamperes (ma):

$$\begin{aligned} 3.42 \text{ ma} &= 3.42 \times 10^{-3} \text{ ampere} \\ 565\ \mu\text{a} = 565 \times 10^{-6} \text{ ampere} &= \underline{0.565 \times 10^{-3}} \text{ ampere} \\ \text{Subtracting} \quad & 2.855 \times 10^{-3} \text{ ampere} = 2.855 \text{ ma.} \end{aligned}$$

To multiply or divide numbers expressed in powers of 10, you have to use the law of exponents for powers of the same base (10 in this case). This law states that you simply add the exponents when multiplying powers to the same base (10), and you subtract the exponent of the divisor from that of the dividend when dividing by powers of the same base. For example,

$$(3 \times 10^5) \times (7 \times 10^2) = 3 \times 7 \times 10^5 \times 10^2 = 21 \times 10^{(5+2)} = 21 \times 10^7$$
$$(4 \times 10^7) \times (6 \times 10^{-3}) = 24 \times 10^{[7+(-3)]} = 24 \times 10^4$$
$$(15 \times 10^9) \div (5 \times 10^5) = \frac{15 \times 10^9}{5 \times 10^5} = 3 \times 10^{(9-5)} = 3 \times 10^4$$
$$(12 \times 10^6) \div (2 \times 10^{-6}) = \frac{12 \times 10^6}{2 \times 10^{-6}} = 6 \times 10^{[6-(-6)]} = 6 \times 10^{12}$$

Now that we have mastered operations with powers of 10, let us return to Example 2 and solve it in a much easier way.

SOLUTION OF EXAMPLE 2 (NEW METHOD): $3.5\ \mu a = 3.5 \times 10^{-6}$ ampere;

$$\text{hence, } R = \frac{E}{I} = \frac{7 \text{ volts}}{3.5 \times 10^{-6} \text{ ampere}} = 2 \times 10^6 \text{ ohms} = 2 \text{ megohms}$$

EXAMPLE 3: A pair of push-pull power output tubes have a common 700-ohm cathode resistor[1], through which a total plate current

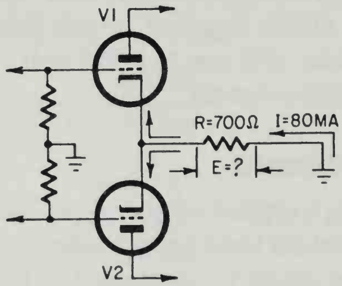


Fig. 104. *Bias voltage can be measured with a voltmeter or, as in this case, it can be calculated using Ohm's Law. Direct measurement is possible only if the equipment exists and is in working order. Calculation is always possible if the required information is available.*

of 80 ma flows (See Fig. 104). What (bias) voltage is developed across the cathode resistor?

Solution: The plate current of 80 ma = 80×10^{-3} ampere = .08 ampere. Hence, the voltage E developed across the resistor

$$E = IR = .08 \text{ ampere} \times 700 \text{ ohms} = 56 \text{ volts}$$
$$\text{Or } E = IR = (80 \times 10^{-3}) \times (0.7 \times 10^3) = 56 \text{ volts.}$$

[1]A value of 700 ohms is used here to make the problem easier. In actual practice we would use a 680-ohm resistor. The difference in the final answer is about a volt and a half.

PRACTICE EXERCISE 1

1. Change 5 microvolts into volts; 15 ma into amperes, and 2.5 megohms into ohms. (Answers: .000005 volt; .015 ampere; 2,500,-000 ohms)

2. A tube filament draws a current of 0.45 ampere when 122 volts is applied. What is the "hot" filament resistance? Answer: (271 ohms)

3. What is the current drawn by a 20,000-ohm bleeder resistor connected across a B-plus supply of 375 volts? (Answer: 18.75 ma)

4. Compute the dc voltage drop across a 200-ohm choke coil rated at 125-ma current flow. (Answer: 25 volts)

5. Express in powers of 10 and as whole numbers or decimals the following prefixes to units: mega, kilo, milli, micro, micromicro.

(Answers: mega = 10^6 = 1,000,000; kilo = 10^3 = 1,000; milli = 10^{-3} = .001; micro = 10^{-6} = .000001; micromicro = 10^{-12} = .000000000001)

6. Convert 250 micromicrofarads ($\mu\mu$f) into microfarads (μf); .00005 farad into μf; 250,000 ohms into kilohms and megohms; 435 millivolts (mv) into volts and microvolts (μv); 896 microamperes (μa) into milliamperes (ma) and amperes. Answers: 250 $\mu\mu$f = .00025 μf; .00005 farad = 50 μf; 250,000 ohms = 250 kilohms = 0.25 megohms; 435 mv = 0.435 volt = 435,000 μv; 896 μa = 0.896 ma = .000896 ampere)

Resistor arithmetic

Fig. 105 is a typical textbook problem in electricity. Five resistors of various values have been hooked in series across a 250-volt supply and you are asked to compute the total series resistance of the circuit and the current flowing in it. To solve this problem in the

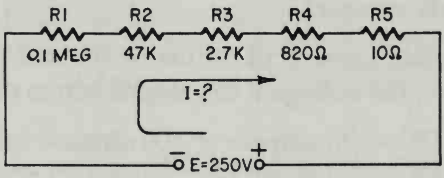


Fig. 105. *Before using Ohm's Law in this circuit, we would need to calculate the total value of the resistors in series.*

conventional manner, you would first use the fact that the total resistance (R_t) in a series circuit is the sum of the individual resistances. This is usually expressed by the formula:

$$R_t = R_1 + R_2 + R_3 + R_4 + R_5 + \ldots R_n$$

You would then apply Ohm's Law to compute the current:

$$I = E/R_t.$$

Let's go through with it to see what we're up against.

$$R_t = 0.1 \text{ megohm} + 47K + 2.7K + 820 + 10$$

Converting to common units (ohms) and adding

$$\begin{array}{r} 100{,}000 \\ 47{,}000 \\ 2{,}700 \\ 820 \\ 10 \\ \hline 150{,}530 \end{array}$$

we obtain a total resistance R_t of 150,530 ohms. Hence, the current

$I = \dfrac{E}{R} = \dfrac{250 \text{ volts}}{150{,}530 \text{ ohms}} = ?$ amperes. Carrying out the indicated division $\quad 250 \div 150{,}530 = .00166$

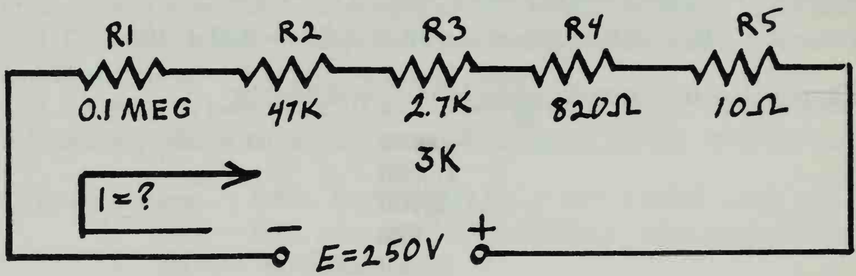
$$\begin{array}{r} .00166 \\ 150{,}530\overline{)\,250.00000} \\ \underline{150\ 530} \\ 99\ 4700 \\ \underline{90\ 3180} \\ 9\ 15200 \\ \underline{9\ 03180} \\ 12020 \end{array}$$

Thus, the final answer is .00166 ampere or 1.66 ma.

Now let's look at the problem from a practical viewpoint. Say you are using 10% commercial resistors to make up the combination. Since "percent" literally means "from hundred," this means that each resistor may differ from its indicated value by as much as 10 parts in 100, or $\frac{10}{100} = 0.1$. This is also true for the entire resistor aggregate, which may be as much as 10% above or below the total computed resistance.

Roughly, therefore, the tolerance in the total resistance is $0.1 \times 150{,}000 = 15{,}000$ ohms, so that the actual total resistance may vary anywhere from 135,000 ohms (15,000 ohms below the computed value) to about 165,000 ohms (15,000 ohms above the computed value).

It would be the height of folly, therefore, to consider the 10-ohm or even the 820-ohm resistor in the total result. The remaining resistance values should be *rounded off* to the nearest 1,000 ohms,



R4 and R5 aren't such important members of this series group.

which is well within the 15,000-ohm total tolerance. Doing this, we can determine by inspection that R1 = 100,000 ohms, R2 = 47,000 ohms and R3 = 3,000 ohms (approximately), while R4 and R5 may be neglected. Mentally adding these three resistance values, we obtain $100{,}000 + 47{,}000 + 3{,}000 = 150{,}000$ ohms total resistance.

Hence the current

$$I = \frac{E}{R} = \frac{250}{150{,}000} = \frac{250}{150 \times 10^3} = \frac{5}{3} \times 10^{-3} \text{ ampere} = 1.67 \text{ ma}$$

Note how close the answer is to our earlier result.

If you used 1% precision resistors, the total resistance might differ by as much as $.01 \times 150{,}000 = 1{,}500$ ohms from the computed value. Again the 10-ohm resistor may be neglected entirely, while the remaining resistance values can be rounded off to the nearest 100 ohms. Doing this, you obtain for the total resistance,

$$R_t = 100{,}000 + 47{,}000 + 2{,}700 + 800 \text{ (approximately)} = 150{,}500 \text{ ohms}$$

Hence, current $I = \dfrac{250 \text{ volts}}{150{,}500 \text{ ohms}} = .00166$ ampere or *1.66 ma.*

Here the division may be performed long-hand or with a slide rule of at least 1% accuracy.

Significant figures and required degree of accuracy

The resistance problem we've just discussed illustrates that in electronics—as anywhere else in real life—we necessarily deal with approximations and never achieve 100% accuracy, in contrast to pure mathematics. An appreciation of the degree of precision of measurement possible in practice and the required accuracy of your calculations can save you countless hours of needless drudgery. For instance, in determining the reactance of a coil by the formula $2\pi fL$, you won't be tempted to use 3.14159 for π, if the inductance (L) of the coil is only within 10% of its nominal value.

Before working with numbers, examine them carefully. Whether you wish to tamper with the numbers depends on how precise your answer must be.

The numbers used in expressing a measured value, called significant figures, generally indicate the precision of measurement. For example, a current value of 25 ma, containing *two* significant figures, indicates a precision of measurement to the nearest (whole) milliampere. If the value is stated as 25.0 ma, it indicates that the measurement has been carried out to the nearest *tenth* of a milliampere (i.e, the current is 25 ma, not 25.3 ma or some other value.)

The zero *after* the decimal point, in this case, is significant and, hence, the number has three significant figures. Similarly, a current value of 100.000 ma has six significant figures, the three zeros after the decimal point indicating that the measurement is precise to the nearest thousandth of a milliampere. As a final illustration, a resistance value of 859.7 ohms has four significant figures, but if the number is rounded off to 860 ohms, it will have only three significant figures.

Rounding Off: When a number is expressed to a greater degree of accuracy than is necessary, it is usually rounded off by dropping one or more digits at the right. The rule for rounding off a number is very simple: if the last digit (at the right) is less than 5, simply drop it; if the last digit is 5 or more, drop it and increase the preceding digit by 1. For example, in successively rounding off the value for $\pi = 3.1415926536$, the values obtained are 3.141592654, 3.14159265, 3.1415927, 3.141593, 3.14159, 3.1416, 3.142, 3.14, 3.1, and finally 3.

Absolute and Relative Error: If you approximate a value from a more exact known value, then the difference between the two values is known as the absolute error. Thus, if you use an approximate value of 3.7 volts for the exact value of 3.667 volts, the absolute error is $3.7 - 3.667 = 0.033$ volt. Similarly, using an approximate value of 0.5 ampere for the exact one of 0.54 ampere results in an absolute error of $0.5 - 0.54 = -0.04$ ampere. Note that the absolute error is negative when the approximate value is smaller than the exact value, and positive when the approximate value is greater than the exact one.

The ratio of the absolute error to the exact value is known as the relative error. Using the previous examples, the relative error in the first case is $\frac{0.033}{3.667} = 0.009$ (approximately), and in the second case it is $\frac{-0.04}{0.54} = -0.074$ (approximately).

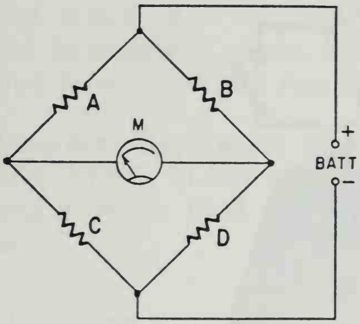
Note that the relative error is a pure number (without units), since the units on top and bottom of the ratio cancel. It is most frequently expressed as percentage error, which is the relative error multiplied by 100. Using the previous examples again, a relative error of 0.009 is equivalent to $0.009 \times 100 = 0.9\%$ and an error of -0.074 is equivalent to $0.074 \times 100 = 7.4\%$. (Usually only the magnitude of the percent error is of interest and the sign is dropped.)

In most practical calculations a one percent (1%) error—the usual slide-rule accuracy—is permissible, since the precision of measurement in electronics rarely equals or exceeds this value. Except for critical circuits, electrical and electronic components, such as resistors, capacitors and coils, have a tolerance of $\pm 10\%$, which means that they may be up to 10% higher or lower than the indicated (nominal) value.

Tubes and transistors usually have an even greater margin of error, and the performance of most electronic circuits appears to be unaffected by as much as 20% tolerance from the stated specifications. In a few sensitive circuits, where such factors as frequency stability or bias voltage may be critical, precision components with ±1% or even ±0.1% tolerance may be used.

Approximating Numbers: Now we have a basis for approximating or rounding off numbers to a required degree of accuracy. You can use the following rule of thumb. To approximate a number within a given relative error (expressed as a decimal), round off the number so that the number of digits retained which are not zero is one greater than the number of decimal places in the relative error. Lest this should sound complicated, let's try a few examples.

EXAMPLE 1: The exact value of a resistance, as determined on a Wheatstone bridge, is 29,735.4 ohms. Round off this value to successive relative errors of .01%, 0.1%, 1% and 10%.



This is the basic circuit of the Wheatstone bridge. The resistances (A, B, C and D) have a special relationship when meter M reads zero. The relationship is:

$$\frac{A}{B} = \frac{C}{D}$$

If B, C and D are 10, 20 and 30 ohms, respectively, and A is unknown, it can be determined by solving the formula for A and substituting the known values. Thus, multiplying both sides of the equation by B:

$$A = \frac{BC}{D} = \frac{10 \times 20}{30} = 6.67 \text{ ohms}$$

Solution: An error of .01% = .0001. Since there are four decimal places in the error, by the rule above we should round off to *five* digits. Hence 29,735.4 ohms becomes 29,735 ohms within .01%.

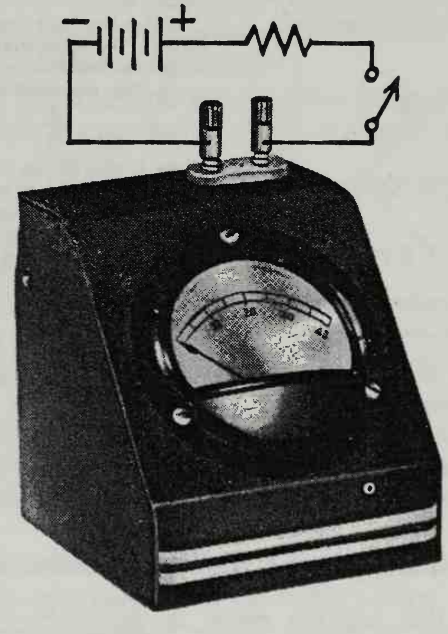
Let's check this answer: the absolute error is 29,735 − 29,735.4 = −0.4; hence, the relative error is $\frac{-0.4}{30{,}000 \text{ (approximately)}}$ = −.000013 or −.0013%. This is much less than the required accuracy of .01%.

An error of 0.1% = .001. Since there are three decimal places, we retain four digits, and the value is rounded off to *29,740* ohms (not counting the zero). Again checking, the absolute error is 29,740 − 29,735.4 = 4.6; hence, the relative error is approximately $\frac{5}{30{,}000}$ = .000167, or 0.0167% (i.e, less than 0.1%).

For an error of 1% or .01 we retain three digits (not counting zeros), and accordingly we round off the exact value to 29,700 ohms. Since the error incurred is about 35 (i.e., 29,735 – 29,700), and 1% of the exact value is approximately 300 (i.e., .01 × 29,735), we are clearly within the required accuracy.

Finally, for a permissible error of 10% or 0.1, we need retain only two digits (ignoring zeros) and, thus, we round off to 30,000 ohms. Again, the error of 265 (approximately) is evidently less than 10% of 29,735, which is approximately 2,974.

EXAMPLE 2: A precision galvanometer reads a current of .0013275 ampere. If a precision (relative error) of 0.1% is required, what value should be used? What is the value for a 1% relative error?

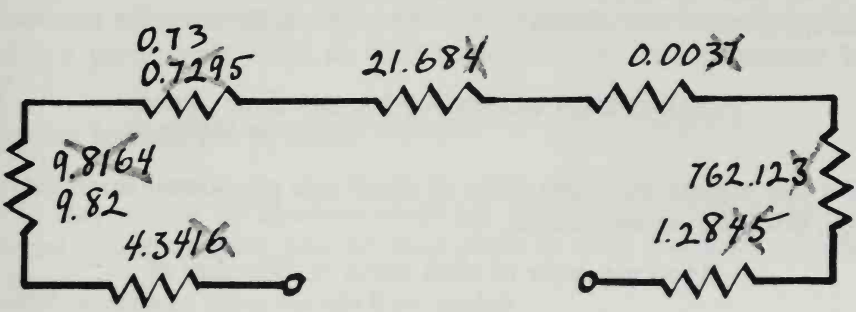


Even with the use of a precision meter, we can round off our reading, depending on the precision we want in our answer.

Solution: For a precision of 0.1% or .001 (three decimal places), we retain four digits. The exact value is then rounded off to .001328 ampere, or 1.328 ma. For a 1% or .01 relative error, we need retain only three digits, hence the value becomes .00133 ampere, or 1.33 ma.

ERROR IN ADDING NUMBERS: We have seen in the earlier resistance example that there is little point in adding numbers to a greater degree of accuracy than the precision of measurement or tolerance of the separate values warrants. In rounding off numbers for addition, we would like to have some idea of the total error incurred, both the absolute value and the percentage. The absolute error in addition is determined very easily: it is simply the *sum* (with proper regard to + or – signs) of the absolute errors of the individual numbers. If you round off to three decimal places, for example, the absolute error of each number cannot be more than one-half of .001, or .0005. Hence, when adding up to 20 numbers, each rounded off to three decimals, the absolute error is less than $20 \times .0005 = .01$. Thus, if the addition of up to 20 numbers is to be correct to *two* decimal places, round off each number to three decimal places. If more than 20 numbers are involved, round off to four decimal places. In general, when adding up to 20 numbers, retain one more decimal place than the required accuracy of the result.

EXAMPLE: Resistances that have measured values of 4.3416, 9.8164, 0.7295, 21.684, .0037, 762.123 and 1.2845 ohms are connected in series. Find the total resistance correct to one decimal place.



By first considering the accuracy we want, we can round off numbers, simplifying our work. In this case one resistor can be eliminated from any consideration since it contributes so little to the final result.

Solution: The total resistance is the sum of the individual resistances. Since the answer is to be correct to one decimal place, we must retain two decimal places for each number during addition.

Hence,

$$
\begin{array}{r}
4.34 \text{ ohms} \\
9.82 \text{ ohms} \\
0.73 \text{ ohms} \\
21.68 \text{ ohms} \\
0.00 \text{ ohms} \\
762.12 \text{ ohms} \\
1.28 \text{ ohms} \\
\hline
799.97 \text{ ohms}
\end{array}
$$

Rounding this result off to one decimal place, we obtain 800.0 ohms for the total resistance. Note that the answer has *four* (not three) significant figures.

How would we find the relative or percentage error incurred in addition. Following the same definition as before, the

$$\text{Relative error of a sum} = \frac{\text{Absolute error of sum}}{\text{Sum}}$$

What we really would like to know, however, is how much each value can be off (that is, its absolute error) for a given permissible percentage or relative error in the sum. This is easily obtained from the definition equation above. Multiplying both sides by the sum, we obtain

$$\text{Absolute error of sum} = \text{Relative error of sum} \times \text{Sum}$$

As we have already seen, the absolute error of the sum is also the product of the absolute error of each value times the number of values:

$$\text{Absolute error of sum} = \text{Absolute error of each value} \times \text{Number of values}$$

We can equate the right sides of these two equations, since their left sides are equal. Hence,

$$\text{Absolute error of each value} \times \text{Number of values} = \text{Relative error of sum} \times \text{Sum}$$

From this equation we obtain the desired result, by dividing both sides by the "Number of values":

$$\text{Absolute (permissible) error of each value} = \frac{\text{Relative error of sum} \times \text{Sum}}{\text{Number of values}}$$

The only trouble with this formula is that we have to know the sum in advance to estimate the permissible error of each value.

For this purpose, however, a very rough mental estimate of the sum will do. An example will clarify the procedure.

EXAMPLE: Find the total series resistance of the resistors listed in the previous example, first to 1% and then to 0.1% accuracy.

Solution: A very rough estimate of only the large numbers shows that the sum is close to 800. Hence, for a permissible error in the sum of 1% or .01 and for seven values,

$$\text{Absolute (permissible) error per value} = \frac{.01 \times 800}{7} = 1.14$$

Since each value can be off by more than 1, we can round off to the units column, thus dropping all decimal places:

$$\begin{array}{r} 4 \\ 10 \\ 1 \\ 22 \\ 0 \\ 762 \\ 1 \\ \hline 800 \end{array} \text{ ohms}$$

The answer, accurate within 1%, therefore is 800 ohms (which is not the same as 800.0 ohms obtained in the previous example). For a permissible error in the sum of 0.1% or .001, the

$$\text{Absolute error per value} = \frac{.001 \times 800}{7} = 0.114$$

To stay within an absolute error per value of less than 0.114, we must retain at least one decimal place in each number. Adding again,

$$\begin{array}{r} 4.3 \\ 9.8 \\ 0.7 \\ 21.7 \\ 0.0 \\ 762.1 \\ 1.3 \\ \hline 799.9 \end{array} \text{ ohms}$$

we obtain an answer of 799.9 ohms, accurate within .001 or 0.1%.

Resistors in parallel

After this digression into error (relative or absolute), let us return to electricity. You will recall that the equivalent resistance (R) of two resistors (R1 and R2) in parallel is the *product of the two resistance values divided by their sum.* This is expressed by the familiar mathematical formula:

$$R = \frac{R1 \times R2}{R1 + R2}$$

Incidentally, this equation also holds for two inductors (L1 and L2) in parallel or for two capacitors (C1 and C2) in series, if you substitute the appropriate symbols.

Although the formula is relatively simple to handle, a lot of time is wasted by some people to get accurate answers, when an approximation will do equally well.

EXAMPLE 1: A phono pickup is connected to the input of an amplifier with a 56,000-ohm input resistance. If the cartridge is shunted by a 0.229-megohm resistor, what is the total load into which the pickup is working? (See Fig. 106.)

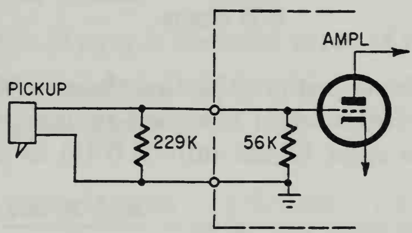


Fig. 106. *Problems in electronics often turn out to be nothing more complicated than calculating the total value of resistors in parallel.*

Solution: Since the cartridge shunt resistor and the amplifier input resistance are in parallel, we can substitute in the formula for two parallel resistors:

$$\text{Load Resistance } R = \frac{56 \times 10^3 \times 229 \times 10^3}{56 \times 10^3 + 229 \times 10^3}$$

$$= \frac{56 \times 229 \times 10^3}{285} \quad \text{(Cancelling } 10^3\text{)}$$

Working this out laboriously by long-hand, we obtain a load resistance of 45,000 ohms, the recommended value for this cartridge.

If we wanted to get an approximate estimate, we might have written

$$R = \frac{56 \times 230 \times 10^3}{285} \cong \frac{230 \times 10^3}{5} \asymp 46{,}000 \text{ ohms*}$$

since 56 goes approximately 5 times into 285. The absolute error incurred by this approximation is 46,000 – 45,000 = 1,000 ohms, and the relative error is $\frac{1{,}000}{45{,}000} = .022$, or 2.2%, which is well within the tolerance of the usual 10% resistors.

RELATIVE ERROR IN MULTIPLICATION: Our intuitive estimate in the last example was good since we were well within the permissible error. We would, however, like to have some rule to determine the relative error in advance, so that our estimates will have the required accuracy. The rule for approximating factors in multiplication is simple: Retain one more digit in each factor than there are decimal places in the permissible relative error.

EXAMPLE: Multiply 3.1415927 (π) by 2.7182818 (ϵ) to an accuracy of 1%.

Percentage of permissible error depends on the application. Sometimes a small error can be a catastrophe.

Solution: For an error of 1% or .01 we should retain *three* digits (one more than in .01) in each factor. Hence,

$$3.14 \times 27.2 = 85.408$$

Compare this to the more exact product of 85.39734.

When several factors are present in combined multiplication and division, the relative error of the result is approximately the difference between the (algebraic) sum of the relative errors of

* $\cong$ represents "approximately equal to"

the factors in the numerator and the sum of the factors in the denominator. Again, retaining one more digit in each factor than decimal places in the permissible relative error is a safe procedure. You can use fewer digits, if you judiciously balance the errors in the factors. Returning to the example of the pickup load resistance:

$$R = \frac{56 \times 229 \times 10^3}{285}$$

Assume we want the result within 10% or 0.1. Retaining alternately one and two digits,

$R = \dfrac{60 \times 230 \times 10^3}{300} = \dfrac{230 \times 10^3}{5} = 46{,}000$ ohms, as before, with a 2.2% error.

Some Multiplication Tricks: Here are a few tricks, based on algebraic formulas, that come in handy when multiplying:

1. $a(b - x) = ab - ax$

This formula is useful when one of the two factors to be multiplied is a little less than an easily multiplied whole number.

Example: Multiply 945×998.

$$945 \times (1{,}000 - 2) = 945{,}000 - 1{,}890 = 943{,}110$$

2. $(a + x)(a - x) = a^2 - x^2$

This result is very handy, when one of the factors is greater and the other is less by the same amount than an easily squared number.

Example 1: Multiply mentally 53×47.

Comparing with Formula 2, above,

$$(50 + 3)(50 - 3) = 50^2 - 3^2 = 2{,}500 - 9 = 2{,}491$$

Example 2: Multiply 195×205.

Since the order of multiplication doesn't matter, $195 \times 205 = 205 \times 195$, or $(200 + 5)(200 - 5) = 200^2 - 5^2 = 40{,}000 - 25 = 39{,}975$.

More Than Two Resistors in Parallel: The equivalent resistance (R) of a number of resistors ($R1, R2, R3 \ldots R_n$) connected in parallel is equal to the reciprocal of the sum of the reciprocals of the individual resistance values. This is expressed mathematically by:

$$R = \frac{1}{\frac{1}{R1} + \frac{1}{R2} + \frac{1}{R3} + \ldots + \frac{1}{R_n}}$$

The proper use of this formula requires skillful and judicious handling of reciprocals, ordinary fractions, decimal fractions, and approximations, wherever possible.

EXAMPLE 1: Resistors of 4, 7 and 14 ohms are connected in parallel across a 6-volt battery. What is the equivalent resistance of the combination and the total current drawn by it? (See Fig. 107.)

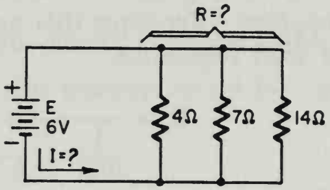


Fig. 107. *When resistors are connected in parallel, the total resistance is always less than the resistor having the smallest value. The total current is the sum of the currents flowing through the individual resistors.*

Solution: Substituting in the formula for the equivalent resistance, $R = \frac{1}{\frac{1}{4} + \frac{1}{7} + \frac{1}{14}} = ?$ ohms.

There are three common proper fractions (under the large fraction bar) which have to be added. To do this, the fractions first have to be converted to equivalent fractions that have the same common denominator. We can find the lowest common denominator (or LCD) by looking for the smallest number that can be divided by all three denominators. In this case, by inspection, the LCD is 28. (In more complicated cases, you'll have to split the denominators into their factors and then find the least common multiple of all numbers.)

Hence,

$$R = \frac{1}{\frac{7}{28} + \frac{4}{28} + \frac{2}{28}} = \frac{1}{\frac{13}{28}} = \frac{28}{13} = 2.154 \text{ ohms.}$$

The current, therefore, is $I = \frac{E}{R} = \frac{6 \text{ volts}}{2.154 \text{ ohms}} = 2.786$ amperes.

It isn't always convenient to solve this type of problem using common fractions, as is shown in the next example.

EXAMPLE 2: A 150-volt plate-voltage supply feeds three electron tubes that have equivalent plate-circuit resistances of 850, 3,900 and 4,700 ohms, respectively, as shown in Fig. 108. Compute the total load resistance and the current drawn.

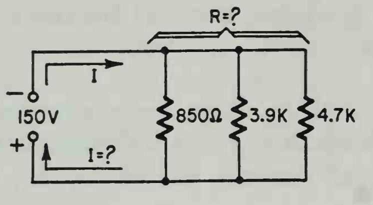


Fig. 108. *This problem can be solved in two ways. You can find the combined value of the resistors and then use Ohm's Law to find the total current. Or, you can find the current through each resistor and then add the individual currents to get the total. You could then use Ohm's Law to find the total resistance.*

Solution: Treating this again as a simple parallel circuit, the total load resistance,

$$R = \frac{1}{\frac{1}{850} + \frac{1}{3{,}900} + \frac{1}{4{,}700}} = ?\text{ ohms.}$$

Obviously, there is no simple way of combining these fractions. We'll therefore have to compute the value of the three reciprocals by changing the common fractions into decimals. This can always be done by dividing the numerator by the denominator. You have several choices in going about this. You can do the division longhand, which is accurate to as many decimal places as you wish, though very time-consuming. You can use a slide rule, obtaining an accuracy of about 1% for a 10-inch rule.

As we shall see in a later chapter, the reciprocals may also be computed rapidly and very accurately by means of a table of five-place logarithms. If you have mathematical tables handy, which you should, you will probably find a table of reciprocals, where you can look up the reciprocal values directly.

Doing it long-hand or by means of mathematical tables, you should have:

$$R = \frac{1}{.0011765 + .0002564 + .0002128}$$

Adding, $R = \frac{1}{.0016457} = 607.64$, or 608 ohms, approximately.

Therefore, the total current $I = \frac{E}{R} = \frac{150\text{ volts}}{608\text{ ohms}} = 0.247$ ampere.

If you have neither tables nor a slide rule and want to save time in computation, you might estimate the result as follows:

$$R = \frac{1}{\frac{1}{850} + \frac{1}{3{,}900} + \frac{1}{4{,}700}} \cong \frac{1}{\frac{1}{900} + \frac{1}{3{,}600} + \frac{1}{4{,}500}}$$

rounding off judiciously, with both positive and negative errors The LCD of the three fractions is 18,000. Hence,

$$R = \frac{1}{\frac{20 + 5 + 4}{18{,}000}} = \frac{1}{\frac{29}{18{,}000}} = \frac{18{,}000}{29} \cong 600 \text{ ohms.}$$

The absolute error in this approximation is roughly 600 – 608 = –8 ohms and the magnitude of the relative error is $\frac{-8}{608} = -.013$, or –1.3%. With the usual 10% resistors, this answer would be quite acceptable.

PRACTICE EXERCISE 2

1. How many significant figures are there in 3.14159; 345,000; 1,000,000; 1.000000; 0.000001; 0.0327850. (Answers: 6, 6, 7, 7, 1, 6).

2. Round off ϵ = 2.7182818285 successively to one decimal place. (Answers: 2.718281829; 2.71828183; 2.7182818; 2.718282; 2.71828; 2.7183; 2.718; 2.72; 2.7).

3. What is the absolute error incurred when a value for ϵ = 2.72 instead of 2.71828 is used in the previous problem? What is the relative and the percentage error? (Answers: .00172, .0006327, or .06327%).

4. A resistance measured with a Wheatstone bridge turns out to be 212,667 ohms. (a) If a value of 212,700 ohms is used for computation, what are the absolute, relative and percentage errors? (b) Will the error be within 1% if a value of 213,000 ohms is used? (Answers: (a) 33; .00011 approximately; .011%. (b) Yes)

5. A Geiger tube and associated counter register 6,789,274 counts per minute (cpm). Round this value off to an accuracy of .01%, 0.1%, 1% and 10%. (Answers: 6,789,300 cpm; 6,789,000 cpm; 6,790,000 cpm; 6,800,000 cpm)

6. Precision resistors with values of 2868.15, 3380.43, 845.31, 27.84 and 343.50 ohms are put in series. What is the total resistance within 1% accuracy? (Answer: 7,470 ohms)

7. (a) What resistance must be placed in parallel with an 8-ohm resistor to make the equivalent resistance of the combination equal to 7 ohms? (b) If an available resistor of 60 ohms is used, what is the approximate relative error of the combination compared to 7 ohms? (Answers: (a) 56 ohms. (b) Less than 1%—actually 0.84%)

8. In an electrical problem the following expression is to be computed: $\frac{3.1416 \times 8 \times 24.44}{10.94 \times 5.22 \times 54.682}$. What is the approximate result and the approximate relative error if, instead, the expression $\frac{3.2 \times 8 \times 24}{10.9 \times 5.2 \times 55}$ is used? (Answers: 0.197; .002 or 0.2% error)

9. Multiply the following numbers rapidly, using the appropriate algebraic formula as shortcut: 1,225 × 99; 198 × 202; 25 × 98 × 35. (Answers: 121,275; 39,996; 85,750)

10. Resistors of 2,500, 10,000 and 50,000 ohms and 100 kilohms are connected in parallel. What is the equivalent resistance? (Answer: 1,886.8 ohms)

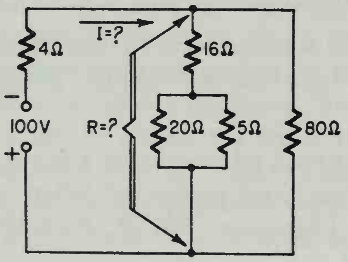


Fig. 109. *With some adjustment, numbers that are awkward to handle become fairly easy. The resulting error is often within limits that can be tolerated.*

11. A 100-volt source is connected into a series-parallel circuit consisting of five resistors, as shown in Fig. 109. What single equivalent resistance can replace the series-parallel combination and what is the total line current flow? (Answers: 20 ohms, 5 amperes)

CHAPTER 2

Network Algebra

ARITHMETIC deals with the rules and laws of manipulating numbers. Algebra has essentially the same rules, but the manipulations are with symbols. By using symbols, such as a, b, c, x, y, z, A, C, L, Z, etc., the rules and operations of algebra assume a universal validity, which is well suited to the problems and formulas of electronics and engineering in general. Of course, in any specific problem you must substitute particular numbers and values in the algebraic formulas and all computations then become purely arithmetical, as you will recall from our "experiments" with Ohm's Law. Thus, perhaps the main advantage of using algebra in electronics is that it allows us to solve most types and classes of problems in advance by manipulating generalized symbols which can stand for the particular values of any specific practical problem.

The symbols, or literal numbers, of algebra may be almost any letter in any alphabet of any language you can think of. Usually, the first few lower-case letters of the alphabet—a, b, c, d, etc.—stand for known numbers (5, 3.14, 659, etc.), while the last few letters—u, v, w, x, y, z—stand for unknown quantities. The following table lists some of the most frequently encountered symbols in electrical work and electronics. A few commonly used Greek symbols and their meaning are also listed.

There just aren't enough letters in the various alphabets to cover all applications of letters as symbols, and so identical letters are often used for several functions. The problem is helped through the use of upper and lower case letters, and also by having superscript and subscript notation. A superscript is a number or letter written slightly above and adjacent to a symbol, such as 10^2 or $(a + b)^2$. In these two examples, the number 2 is a superscript. A subscript is written adjacent to and slightly below the symbol, such as X_L or X_C.

Symbols Frequently Used in Electronics

Symbol	*Meaning*
A	Amplifier gain
C	Capacitance (farads)
E	Emf or voltage (volts)
e	Charge on electron (1.6×10^{-19} coulomb)
F or f	Frequency (cycles per second = cps)
G	Conductance (mhos)
I or i	Current (amperes)
j	Imaginary number or j-operator $= \sqrt{-1}$
k	Dielectric constant, coefficient of coupling
L	Inductance (henrys)
M	Mutual inductance (henrys)
N	Number of turns
Q	Quality factor $= \frac{2\pi fL}{R} = \frac{\omega L}{R}$, Transistor
R	Resistance (ohms)
T	Transformer
V	Electron tube (valve)
X	Reactance (ohms)
X_L	Inductive reactance (ohms) $= 2\pi fL$
X_C	Capacitive reactance (ohms) $= 1/2\pi fC$
Y	Admittance (mhos) $= 1/Z$
Z	Impedance (ohms) $= R + j(X_L - X_C)$
Z	Impedance magnitude $= \sqrt{R^2 + (X_L - X_C)^2}$
α (alpha)	Current gain in transistors $= \frac{i_c}{i_e}\Big\vert V_c$ constant (change in collector current produced by change in emitter current with collector voltage held constant)
α (alpha)	Attenuation constant of rf line
β (beta)	Phase constant of rf line, or feedback factor
ϵ (epsilon)	Natural base of logarithms = 2.71828 (in engineering) dielectric constant
θ (theta)	Angle (degrees)
λ (lambda)	Wavelength $= \frac{\text{velocity of wave}}{\text{frequency}}$
μ (mu)	Tube amplification factor
π (pi)	Ratio of circumference to diameter = 3.14159
Ω, ω (omega)	Symbol for ohms
ω (omega)	Angular velocity $= 2\pi f$

SIGNED QUANTITIES AND GRAPHS: Another distinguishing characteristic of algebra is that all quantities have either plus or minus signs, indicating that they are either greater (+) or less (–) than zero, respectively. Dealing with negative symbols or numbers should be familiar to you from your daily experience with the thermometer or, for that matter, from handling positive (+) or negative (–) errors in the last chapter.

A simple way of looking at signed quantities is to portray them graphically. Fig. 201 illustrates the familiar rectangular coordinate

EQUATION: $y = 4 + 2x$

TABLE OF VALUES

x	y
-5	-6
-4	-4
-3	-2
-2	0
-1	2
0	4
+1	6
+2	8
+3	10
+4	12
+5	14

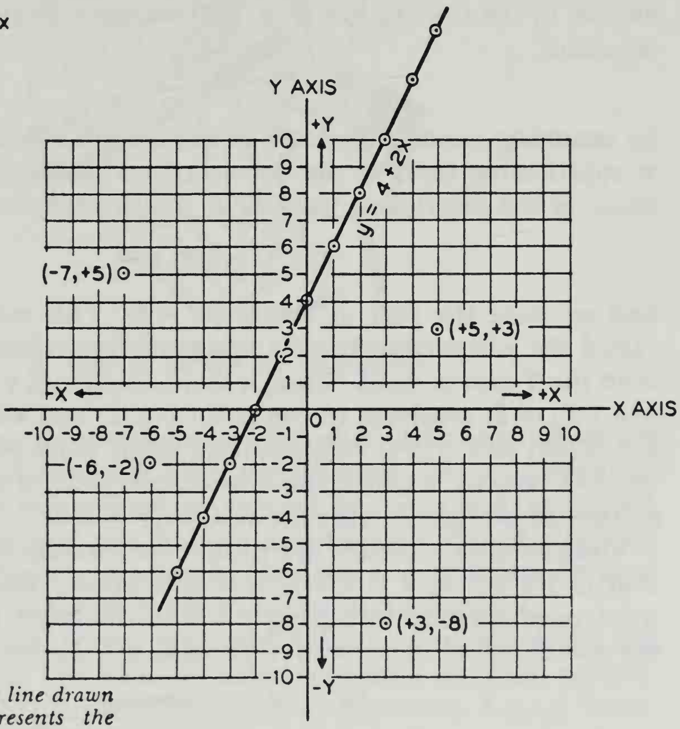


Fig. 201. *The straight line drawn on this graph represents the equation* $y = 4 + 2x$.

system, consisting of a horizontal X-axis and a vertical Y-axis and a series of squares on graph paper. The perpendicular intersection of the two axes marks the zero point (0). Values of x to the right of the Y-axis are considered *positive,* those to the left are negative. Values of y above the X-axis are considered positive, those below the X-axis are negative. It thus becomes possible to assign a unique point on the graph paper for each pair of values, +x or -x and +y or –y. This is written (±x, ±y). For example, the point (0, 0) is the intersection of the X and Y axis; the point (+5, +3) is five squares (units) to the right of zero along the X-axis and three units above zero along the Y-axis. The point (–7, +5) is seven units to the left of zero along the X-axis and five units up along the Y-axis. The point (–6, –2) is six units to left along X and two units down along Y. Finally, the point (+3, –8) is three units to the right along X and eight units down along Y.

Since every pair of values (x, y) defines a point, an algebraic equation expressing a relation between the independent variable x and the dependent variable y may be plotted by assuming various values for x and computing the value of y in each case. For example, in the illustration (Fig. 201) we have plotted the algebraic equation

$$y = 4 + 2x$$

by assuming various arbitrary values for the independent variable x, substituting them in the equation and computing the resultant value of the dependent variable y. Thus, when $x = 0$,

$$y = 4 + 2(0) = 4$$

and we have the pair of values (0, +4). This value of y (+4) is called the y-intercept, since it marks the intersection of the graph with the Y-axis ($x = 0$). When $x = -1$, we obtain $y = 4 + 2(-1) = 4 + (-2) = 2$, resulting in the point (–1, 2) and when $x = -2$, $y = 4 + 2(-2) = 4 + (-4) = 0$, resulting in the point (–2, 0). This point ($x = -2$, $y = 0$) is called the x-intercept, since it marks the intersection of the graph with the X-axis ($y = 0$).

You can always obtain the y-intercept directly by setting $x = 0$; that is, for $x = 0$, $y = 4 + 2(0) = 4$, as above. You can obtain the x-intercept directly by setting $y = 0$ in the given equation. Thus, for $y = 0$, we obtain $0 = 4 + 2x$, and solving for x:

$$2x = -4 \text{ (transposing)}$$
$$x = -4/2 = -2 \text{ (dividing by 2)}$$

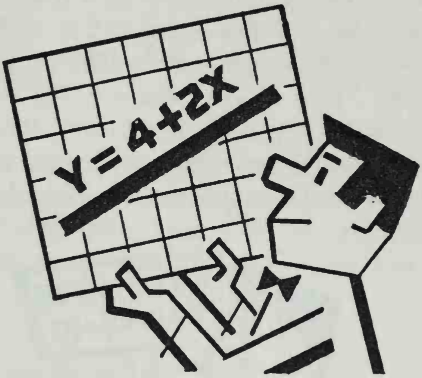
Hence, the x-intercept is –2, as was obtained previously.

By continuing the process of assuming values for x and computing y from the equation, you can make up a table of values, listing pairs of corresponding x and y values that signify points of the graph. Thus, in the present example (Fig. 201), the table of values looks like this:

X . . .	–5	–4	–3	–2	–1	0	+1	+2	+3	+4	+5 . . .
Y . . .	–6	–4	–2	0	+2	+4	+6	+8	+10	+12	+14 . . .

Since the graph of the equation $y = 4 + 2x$ is obviously a straight line, it would have been sufficient in this case to compute the x- and y-intercepts and possibly one additional pair of values, to make sure the graph is a straight line.

After obtaining three pairs of values and marking three points of the graph, we can easily draw a straight line of indefinite length through the points, as shown in Fig. 201. This line represents the equation $y = 4 + 2x$ at every point and is indefinite in length,



When we plot the graph of an equation, we begin to see that the equation has meaning and isn't just a group of symbols.

since there are an infinite number of positive and negative values for x and y. It is apparent from the example that any algebraic equation can be represented by a graph or curve and, moreover, any curve or line you can draw represents some sort of algebraic equation. The branch of analytic geometry is exclusively devoted to exploring the relations between curves and equations.

RULES OF SIGNS: If you had any difficulties in following the previous example, you have probably forgotten the rules governing signs and other elementary algebraic operations. Let us, therefore, briefly review these before getting into electronic applications.

RULE 1: To add numbers of like signs, assign the common sign to the result.

EXAMPLES: $+5 +7 +12 +3 = +27$, or 27 (the + sign may be omitted)

$$5x + 7x + 12x + 3x = 27x$$
$$-3 -8 -12 -5 -4 = -32$$

RULE 2: To add numbers of unlike signs, first add all positive and negative quantities, subtract the smaller from the larger, and place in front of your answer the sign of the larger combination.

EXAMPLE 1: Find the algebraic sum of –4, –8, +2, +6 and +10

Solution: $-4 + (-8) = -12;\ 2 + 6 + 10 = 18;\ 18 - 12 = 6$

EXAMPLE 2: Add $3y + 19y + 4y - 45y$

Solution: $3y + 19y + 4y = 26y;$

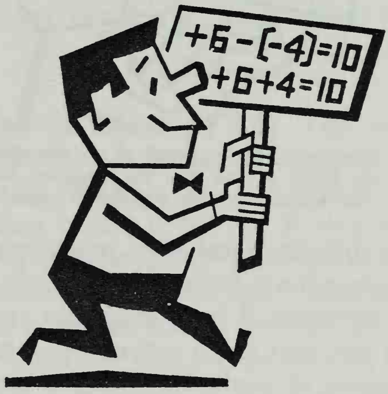
$26y - 45y = -(45y - 26y) = -19y$

EXAMPLE 3: Find the algebraic sum of

$18a + 23b - 12a + 6b + 4a - 16a - 39b + 3b$

Solution: $(18a + 4a) + (-12a - 16a) = 22a - 28a = -(28a - 22a)$
$= -6a; (23b + 6b + 3b) - 39b = 32b - 39b =$
$-(39b - 32b) = -7b;\ -6a + (-7b) = -6a - 7b$

RULE 3: To subtract signed quantities (i.e., find the algebraic difference) change the sign of the quantity to be subtracted and add



Learn the algebraic rules and you won't have a "problem".

EXAMPLE 1: $12 - (-16) = 12 + (+16) = 12 + 16 = 28$

EXAMPLE 2: $4c - (-5c) = 4c + (+5c) = 4c + 5c = 9c$

EXAMPLE 3: Find the algebraic difference between

$(12x - 22x + 3x - 2x)$ and $(-44x + 23x - 8x + 13x)$

Solution: $12x + 3x - 22x - 2x = 15x - 24x = -9x$

$23x + 13x - 44x - 8x = 36x - 52x = -16x$

$-9x - (-16x) = -9x + (+16x) = 16x - 9x = 7x$

OR: $(12x - 22x + 3x - 2x) - (-44x + 23x - 8x + 13x)$
$= (-9x) - (-16x) = -9x + 16x = 7x$

RULE 4: The product of any two numbers that have like signs (either "+" or "–") is positive (+), and the product of a positive and a negative quantity is negative (–).

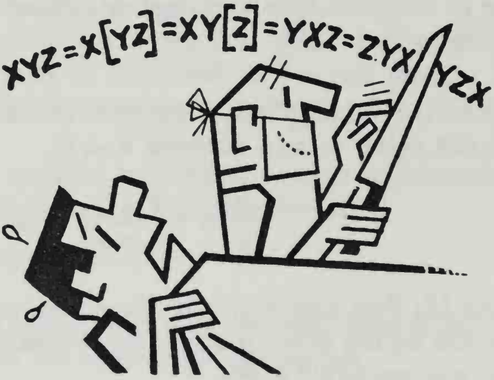
EXAMPLES: $(-7) \times (-3) = +21$; $(-3y) \times (-9z) = +27yz$
$(4u) \times (6v) \times (9w) = 216uvw$
$(-8y) \times (6z) = -48yz$; $(-3) \times (8) \times (6) = -144$
$(5u)(3v)(-4w)(6x)(-2y)(10z) = +7{,}2000uvwxyz$

RULE 5: The quotient of two quantities of like sign ("+" or "–") is positive, while the quotient of a positive and a negative quantity is negative.

EXAMPLES: $\frac{-32}{-16} = +2$; $\frac{-39}{3} = -13$; $\frac{-44 \times 12}{4 \times -11} = +12$

These are the basic rules for manipulating signed quantities. There are some additional fundamental algebraic operations which you should review at this time if you are not sure you can do them without difficulty. You will be reviewing arithmetic at the same time, since a substitution of numbers in the formulas results in purely arithmetical operations.

Fundamental operations



Algebraic symbols can be written in different ways—all having the same value.

RULE 6: Quantities may be added in any order or grouping.

$$x + y + z = y + x + z = z + y + x$$
$$(x + y) + z = x + (y + z) = x + y + z$$

RULE 7: Quantities may be multiplied in any order or grouping.

$$xyz = yxz = zyx = xzy = yzx = zxy$$
$$x(yz) = (xy)z = xyz$$

RULE 8: A coefficient (known quantity) outside a parenthesis means that all terms within the parenthesis are to be multiplied by it;

conversely, a factor common to all terms may be placed outside a parenthesis.

$$a(x + y) = ax + ay$$
$$ax + ay + az = a(x + y + z)$$

RULE 9: A minus (–) sign preceding a parenthesis applies to all terms within the parenthesis.

$$-(x - y + z) = -x - (-y) - (+z) = -x + y - z$$
$$-(-x + y - z) = x - (+y) + z = x - y + z$$

RULE 10: You may remove sets of parentheses or brackets "from the inside out" or "from the outside in."

$$x - b\,[(y - z)] = x - [by - bz] = x - by + bz$$
or
$$x - b\,[(y - z)] = x - b(y - z) = x - by + bz$$

Operations with fractions

The following rules review operations with fractions, irrespective of whether they are algebraic (literal numbers) or arithmetic (ordinary numbers).

RULE 11: Dividing by x is the same as multiplying by 1/x, provided x is not equal to zero (written $x \neq 0$).

$$\frac{y}{x} = y\left(\frac{1}{x}\right), \quad (x \neq 0)$$

RULE 12: You can multiply the numerator and denominator of a fraction by the same quantity (k) without changing the value of the fraction, provided $k \neq 0$.

$$\frac{y}{x} = \frac{ky}{kx} = \frac{\frac{y}{k}}{\frac{x}{k}}, \quad (k \neq 0)$$

RULE 13: Dividing by a fraction is equivalent to multiplying by the reciprocal of the fraction.

$$\frac{a}{\frac{y}{x}} = \frac{ax}{y}$$

RULE 14: Fractions with different denominators must be reduced to their lowest common denominator (LCD) before they can be added or subtracted.

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \frac{bcx}{abc} + \frac{acy}{abc} + \frac{abz}{abc} = \frac{bcx + acy + abz}{abc}$$

RULE 15: The product of two fractions is the product of their numerators divided by the product of their denominators.

$$\left(\frac{y}{a}\right)\left(\frac{x}{b}\right) = \frac{yx}{ab}$$

$$\left(\frac{ac}{b}\right)\left(\frac{x}{yz}\right) = \frac{acx}{byz}$$

Operations with exponents (powers and roots)

The power of a quantity is the product obtained by multiplying the quantity by itself a given number of times. The exponent indicates the power to which the quantity is to be raised. Thus, x^5 means that x is to be raised to the fifth power, or $x \bullet x \bullet x \bullet x \bullet x = x^5$ The quantity that is to be raised to a power is called the base. Thus, y^n means that the base, y, is to be raised to the nth power. The following rules review operations with exponents.

RULE 16: An exponent outside a product within a parenthesis applies to each of the factors within the parenthesis.

$$(xyz)^m = x^m y^m z^m$$

RULE 17: The product of two powers with the same base is the base raised to a power equal to the sum of the exponents.

$$y^m y^n = y^{m+n}$$

RULE 18: The quotient of two powers with the same base is the base raised to a power equal to the exponent of the numerator minus the exponent of the denominator.

$$\frac{y^m}{y^n} = y^{m-n}$$

RULE 19: Any quantity, except 0, with the exponent 0 is equal to 1.

$$y^0 = 1 \quad (y \neq 0)$$

RULE 20: A base raised to a negative power is equal to the reciprocal of the base raised to the same positive power (that is, take the reciprocal and change the sign of the exponent).

$$z^{-n} = \frac{1}{z^n}$$

RULE 21: The nth power of the mth power of a quantity is the same as the mth power of the nth power of that quantity, or the mnth power of that quantity.

$$(y^m)^n = (y^n)^m = y^{mn}$$

RULE 22: The numerator of a fractional exponent indicates a power and the denominator a root, of the base quantity.

$$y^{\frac{1}{m}} = \sqrt[m]{y^1} = \sqrt[m]{y} \quad \text{(i.e., the mth root of y)}.$$

(Note: The exponent 1 is understood and hence may be omitted.)

$$y^{\frac{m}{n}} = \left(y^{\frac{1}{n}}\right)^m = (\sqrt[n]{y})^m$$

$$\text{OR} \quad y^{\frac{m}{n}} = (y^m)^{\frac{1}{n}} = \sqrt[n]{y^m} \quad \left(\text{hence: } (\sqrt[n]{y})^m = \sqrt[n]{y^m}\right)$$

Special factors and expansions

The following special factors and their expansions (when multiplied) are frequently used in algebraic processes and you might do well to memorize them.

To manipulate an algebraic quantity you may need to expand it.

RULE 23: $(a + b)^2 = a^2 + 2ab + b^2$

RULE 24: $(a - b)^2 = a^2 - 2ab + b^2$

RULE 25: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

RULE 26: $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

RULE 27: $a^2 - b^2 = (a - b)(a + b)$

RULE 28: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

RULE 29: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Mathematics is like a game; you must know the rules before you can play.

Operations with equations (axioms of equality)

The rules listed below govern operations with equations and are used in their solution.

RULE 30: Equal quantities may be added to, subtracted from, multiplied by or divided into both sides of an equation without destroying the equality. Division by zero (0) is not possible.

If $y = z$ and $a = b$, then:

1: $y + a = z + b$ (adding equal quantities)

2: $y - a = z - b$ (subtracting equal quantities)

3: $ay = bz$ (multiplying by equal quantities)

4: $\frac{y}{a} = \frac{z}{b}$ (dividing by equal quantities)
$(a = b \neq 0)$

RULE 31: Raising both sides of an equation to the same power, or taking the same root, does not affect the equality.

$y^n = z^n$ (raising to the same power)

$\sqrt[n]{y} = \sqrt[n]{z}$ (taking the same root)

PRACTICE EXERCISE 3

1. Compute the value of each side of the equation in Rules 6 through 18 and in Rule 21, when $x = 2, y = 3, z = 4, a = 5, b = 6, c = 8, n = 2, m = 3$, and $K = 7$.

(Answers: Rule 6: 9. Rule 7: 24. Rule 8: 25 and 45. Rule 9: –3.

Rule 10: 8. Rule 11: 3/2 = 1.5. Rule 12: 3/2 = 21/14 = 1.5.

Rule 13: $\frac{5}{3/2} = \frac{5 \times 2}{3} = \frac{10}{3} = 3\frac{1}{3}$

Rule 14: $\frac{2}{5} + \frac{3}{6} + \frac{4}{8} = \frac{48 + 60 + 60}{120} = \frac{14}{10} = 1.4.$

Rule 15: $\frac{6}{30} = \frac{1}{5}$. Rule 16: 13,824. Rule 17: 243. Rule 18: 3.

Rule 21: 729.

2. Derive Rule 19 by letting $n = m$ in Rule 18.

3. Derive Rule 20 by letting $m = 0$ and $y = z$ in Rule 18, and applying Rule 19.

4. Find the value of each side of the equation in Rule 22 when $y = 4, m = 3$ and $n = 2$. [Answer: $4^{1/3} = \sqrt[3]{4} = 1.587$

$4^{3/2} = (\sqrt{4})^3 = 8$)]

5. Continue problem 1, above for Rules 23 through 29.

(Answers: Rule 23: 121. Rule 24: 1. Rule 25: 1,331. Rule 26: –1. Rule 27: –11. Rule 28: –91. Rule 29: 341).

6. What's wrong with this demonstration: Let $a = b$; hence

1: $a^2 = ab$ (multiplying by a).
2: $a^2 - b^2 = ab - b^2$ (subtracting b^2).
3: $(a + b)(a - b) = b(a - b)$ (factoring).
4: $a + b = b$ (dividing by $a - b$).
5: $2b = b$ (substituting $a = b$).
6: $2 = 1$ (dividing by b).

[Answer: dividing by $(a - b) = 0$]

7. If $a = 2$ and $b = 3$, what is the value of $(a + b)^2$?
[Answer: 25]

8. If $a = 2$ and $b = 3$, what is the value of $a^2 + b^2$?
[Answer: 13]

9. Multiply $(a + b)(a^2 - ab + b^2)$ to show that it is equal to $a^3 + b^3$.

10. What is the algebraic sum of $(4a - 5b) - (-3a - 4b)$?
[Answer: 7a–b]

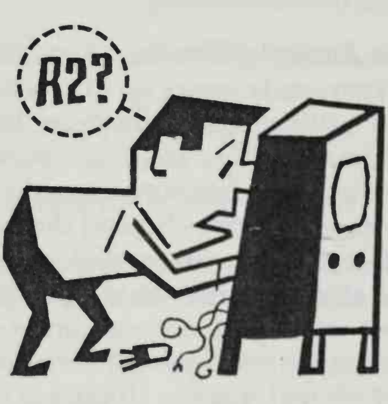
Using algebra in electronics

Knowing a little algebra can be a very useful tool for daily work in electronics. Though the rules and operations of the "dismal science"—as mathematics has been called—may be dry as dust, solving problems arising in practice or deriving your own formulas for specific situations can be enjoyable.

What to put in parallel: In Chapter 1 we gave the formula for the resistance of two resistors in parallel:

$$R = \frac{R1 \times R2}{R1 + R2}$$

and we cited as example (Fig. 106) a phono pickup, shunted by a 0.229-megohm resistor, which was connected to the 56,000-ohm input termination of an amplifier. The question was to determine the value of the total load resistance. This (by an application of the formula) turned out to be 45,000 ohms. It would be more likely, in practice, that you would have to determine what value resistor you should place in parallel with the pickup and amplifier input to obtain a recommended load of 45,000 ohms, for example. Of course, you could do this easily for this particular case by letting R = 45,000, R1 = 56,000 and solving for R2 (the value of the shunt resistor) in the formula above. But if this problem occurs frequently in your work, you'll want a general formula for the shunt resistor, R2, which applies to any particular case. We can derive this easily.



Servicing television sets? Finding the value of resistors in series or parallel can be a problem.

1: $\dfrac{R1 \times R2}{R1 + R2} = R$

2: $R1\ R2 = R(R1 + R2)$ Multiplying by (R1 + R2)

3: $R1\ R2 = R\ R1 + R\ R2$ Multiplying

4: $R1\ R2 - R\ R2 = R\ R1$. Subtracting R R2; also called "transposing"

5: $R2(R1 - R) = R\ R1$ Factoring R2

6: $R2 = \dfrac{R\ R1}{R1 - R}$ Dividing by R1 − R

Step 6 is, of course, the desired formula for the shunt resistance, R2, to be placed across the pickup. We can now substitute the specific values of our example (R = 45,000 ohms and R1 = 56,000 ohms) and obtain

$$R2 = \frac{R\ R1}{R1 - R} = \frac{(45 \times 10^3) \times (56 \times 10^3)}{(56 \times 10^3) - (45 \times 10^3)}$$

$$= \frac{45 \times 56 \times 10^3 \times 10^3}{56 - 45 \times 10^3} = 229 \times 10^3 = 229{,}000 \text{ ohms}$$

Hence, the shunt resistor should be 229,000 ohms, or 0.229 megohm. As another example of the use of the formula we just derived, let us solve problem 7a of Practice Exercise 2:

Solution: Let R1 = 8 ohms and R = 7 ohms. Hence,

$$R2 = \frac{R\ R1}{R1 - R} = \frac{7 \times 8}{8 - 7} = \frac{7 \times 8}{1} = 56 \text{ ohms}$$

Doing this or most other problems by "trial and error", is obviously more inefficient and time-consuming.

How to Determine Ammeter Shunts

A problem that frequently comes up in practice is to extend the range of a sensitive milliammeter to measure larger currents, possibly up to several amperes. You know, of course, that you can do this readily by placing a shunt resistor across the meter movement, through which the excess current—beyond the full-scale meter current—is made to flow. Such a current-divider arrangement works very well, provided the shunt resistor is a precision component of exactly the required value, so that the accuracy of your meter will be maintained. Let us derive a general formula for the required shunt resistance, which will apply to all cases of this type.

In Fig. 202, the schematic circuit of such an ammeter shunt arrangement, the symbols stand for:

R_m = the internal resistance of the meter movement
R_s = the shunt resistance
E = the voltage applied across the meter and shunt (for full-scale deflection)
I_m = the current through R_m for full-scale meter deflection
I_s = the shunt current through shunt resistor R_s
I_t = the total (meter + shunt) current = $I_m + I_s$

Since the voltage, E, applied across the meter is also across the shunt, we can write:

$$E = I_m R_m = I_s R_s \quad (1)$$

Hence, $$I_m = \frac{I_s R_s}{R_m} \quad \text{(dividing by } R_m\text{)} \quad (2)$$

and $$\frac{I_m}{I_s} = \frac{R_s}{R_m} \quad \text{(dividing by } I_s\text{)} \quad (3)$$

But since $$I_t = I_m + I_s,\ I_s = I_t - I_m \quad \text{(subtracting } I_m\text{)} \quad (4)$$

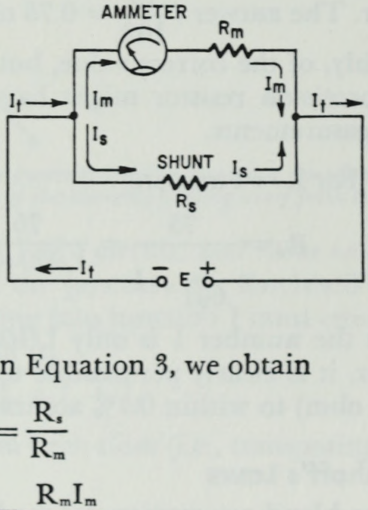


Fig. 202. *Basically, an ammeter and its shunt represent a problem involving resistors in parallel.*

Substituting Equation 4 for I_s in Equation 3, we obtain

$$\frac{I_m}{I_t - I_m} = \frac{R_s}{R_m} \quad (5)$$

And solving for $$R_s = \frac{R_m I_m}{I_t - I_m} \quad (6)$$

It will be convenient to divide the numerator and denominator of this expression by I_m and obtain the solution in the form

$$R_s = \frac{R_m}{\frac{I_t}{I_m} - 1} \quad \text{(dividing by } I_m\text{)} \quad (7)$$

Equation 7 is the desired expression for the shunt resistance R_s.

EXAMPLE: Extend the range of a 0–1-ma (full-scale) milliammeter with 75 ohms internal resistance to 10 ma, 0.1 ampere and 1 ampere. What should be the values of the three shunt resistors?

Solution: Letting $R_m = 75$, $I_m = .001$ and substituting in the formula, we obtain:

1: For $I_t = 10$ ma, or .01 ampere

$$R_s = \frac{75}{\frac{.01}{.001} - 1} = \frac{75}{10 - 1} = \frac{75}{9} = 8.33 \text{ ohms}$$

2: For $I_t = 0.1$ ampere

$$R_s = \frac{75}{\frac{0.1}{.001} - 1} = \frac{75}{100 - 1} = \frac{75}{99} = 0.758 \text{ ohm}$$

In this case we might have been tempted to ignore the number 1 in the denominator, since it is only 1/100 or 1% of the entire denominator. The answer ($\frac{75}{100} = 0.75$ ohm) would have been within 1%, roughly, of the correct value, but adding to this the tolerance of a 1% precision resistor might have seriously affected the accuracy of measurements.

3: For $I_t = 1$ ampere

$$R_s = \frac{75}{\frac{1}{.001} - 1} = \frac{75}{1{,}000 - 1} \cong \frac{75}{1{,}000} = 0.075 \text{ ohm}$$

Since the number 1 is only 1/1000 or 0.1% of the entire denominator, it is clearly permissible to ignore it and obtain the answer (.075 ohm) to within 0.1% accuracy.

Kirchhoff's Laws

The bland assumption we made in the computation of the ammeter shunts—that the sum of the meter and shunt currents equals the total current—is actually not as self-evident as it may appear at first. The example illustrates one of two network laws, first formulated by the German physicist Gustav Robert Kirchhoff (1824–1887). Kirchhoff's Laws, rather than Ohm's Law (on which they are based), are used constantly for determining the currents in the intricate combinations of resistances and voltages called networks. In brief, Kirchhoff's Laws state:

1. The sum of the currents flowing into a junction of a circuit equals the sum of the currents flowing out of the junction.

2. The sum of the electromotive forces (emf) (battery or generator voltages) around any closed loop of a circuit equals the sum of the voltage drops across the resistances (or impedances) in that loop.

You can see from Fig. 203 how you would go about using Kirchhoff's first law in a practical case. Here two branch currents (I_1 and

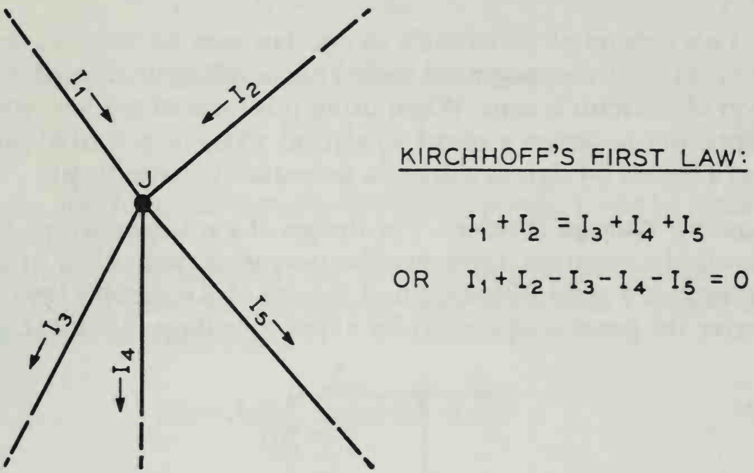


Fig. 203. *The algebraic sum of the currents flowing toward a junction must be equal to the algebraic sum of the currents flowing away from it.*

I_2) are flowing into the junction J of a circuit, and three currents (I_3, I_4 and I_5) are flowing out of the junction. By Kirchhoff's first law, the sum of the currents flowing into junction J must equal the sum of the currents flowing out of it:

$$I_1 + I_2 = I_3 + I_4 + I_5$$

By subtracting ($I_3 + I_4 + I_5$) from both sides (i.e., transposing), we may also write this equation:

$$I_1 + I_2 - I_3 - I_4 - I_5 = 0$$

This suggests that we can simplify Kirchhoff's first law somewhat by assigning a plus (+) sign to all currents flowing toward a junction and a minus (–) sign to currents flowing away from the junction. We may then rephrase the first law:

The algebraic sum of the currents at a junction is zero, or in concise mathematical form:

$$\text{Sum } I = 0$$

(This is sometimes written Σ I = 0, where the symbol Σ (sigma) stands for "the sum of.")

We can similarly simplify Kirchhoff's second law:

Sum of the emf's = sum of the voltage drops (around a closed loop) or sum E = sum of the IR drops (around a closed loop) (Recall that a voltage drop = I × resistance, or IR.) And, finally, in concise mathematical form:

$$\Sigma E - \Sigma IR = 0 \text{ (around closed loop)}$$

This version of Kirchhoff's second law may be worded: the algebraic sum of the potential differences (voltages) around a closed loop of a circuit is zero. When using this form of the law, you must remember to assign a plus (+) sign to a rise in potential (an emf) and a minus (–) sign to a drop in potential (voltage drop).

Figuring Voltage Dividers: The design of a voltage divider–bleeder to supply required vacuum-tube operating potentials at certain currents is a good illustration of the use of Kirchhoff's laws. Let us derive the general equations for a typical voltage divider (Fig. 204),

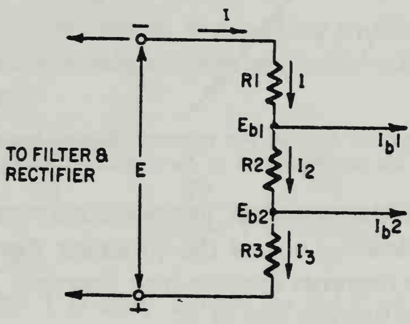


Fig. 204. *This is a series voltage divider of the type found across the output of power supplies. It is sometimes also called a bleeder since it puts a constant load on the filter capacitors.*

though in a particular case you would, of course, substitute the values called for in the problem. Assume, an output voltage, E, is available from the rectifier-filter of the power supply, and plate voltages E_{b1}, with a plate current I_{b1}, and E_{b2}, with a plate current I_{b2}, are to be supplied by the voltage divider. In addition, a minimum bleeder current, I_3, is to flow at all times for adequate voltage regulation. What are the values needed for resistors R1, R2 and R3 to provide these voltages and currents, and what is the required total current, I. We can find the required current easily by an application of Kirchhoff's first law:

At the junction of R1 and R2:

$$I = I_2 + I_{b1} \tag{1}$$

At the junction of R2 and R3:

$$I_2 = I_3 + I_{b2} \tag{2}$$

Substituting for I_2 in Equation 1, the total current:

$$I = I_3 + I_{b1} + I_{b2} \tag{3}$$

Since I_3, I_{b1} and I_{b2} are known, the required current, I, can be found from Equation 3.

The values of resistors R1, R2 and R3 may be determined by applying Ohm's Law, in conjunction with Equations 1, 2 and 3. At the junction of R2 and R3,

$$E_{b2} = I_3 R3; \text{ hence } R3 = \frac{E_{b2}}{I_3} \tag{4}$$

also, $E_{b1} - E_{b2} = I_2 R2$. But from Equation 2, $I_2 = I_3 + I_{b2}$, hence, $E_{b1} - E_{b2} = (I_3 + I_{b2})R2$ and dividing by $(I_3 + I_{b2})$:

$$R2 = \frac{E_{b1} - E_{b2}}{I_3 + I_{b2}} \tag{5}$$

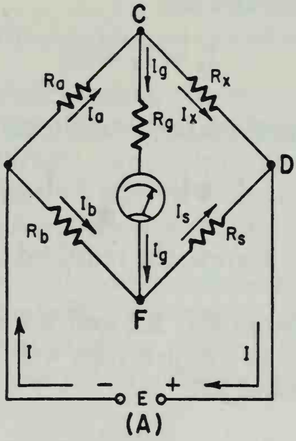
Finally, $E - E_{b1} = IR1 = (I_3 + I_{b1} + I_{b2})\, R1$ (from Equation 3). Solving for R_1 by dividing through by $I = I_3 + I_{b1} + I_{b2}$,

$$R1 = \frac{E - E_{b1}}{I_3 + I_{b1} + I_{b2}} = \frac{E - E_{b1}}{I} \tag{6}$$

Equations 3, 6, 5 and 4 determine the required values of I, R1, R2 and R3 for the given voltages and currents. If you have to design a voltage divider with two taps, therefore, you can determine the settings of the resistance taps either by substituting the given values directly in these equations, or by working out the problem fresh from start with specific values.

Wheatstone bridge

As another application of Kirchhoff's laws, let us analyze the Wheatstone bridge (Fig. 205), which, as you know, is used for precision resistance measurements. In practice, resistors R_a and R_b are given suitable, fixed values and a standard resistance, R_s, is adjusted in value until a sensitive galvanometer, connected across the resistance junction, shows zero current deflection. The bridge is



AT BALANCE-$R_x = \frac{R_a}{R_b} R_s$

Fig. 205. *The Wheatstone bridge can be analyzed by using Kirchhoff's laws.*

then said to be balanced and the value of the unknown resistance, R_x, can be determined very accurately from a knowledge of resistors R_a, R_b, and R_s. (There are many types of bridges, but this is one of the most common.)

Let us derive the equation for the galvanometer current, I_g, in terms of the known voltage, E, and the resistances. We can then determine the condition for bridge balance and the relations resulting between the four resistance arms.

In analyzing such a circuit using Kirchhoff's laws, we assume an arbitrary current direction in each branch and write the laws in as many independent equations as there are unknown currents. (Independent equations will not reduce to identical forms by algebraic substitution.) We then solve the equations simultaneously to obtain the unknown currents. If any current values come out negative (–), this simply means that we have assumed the wrong current direction in the particular branch and that it should be reversed. Applying Kirchhoff's first law, for the assumed current directions shown in the diagram (Fig. 205), we can write at junction C:

$$I_a = I_g + I_x \quad (1)$$

and at junction F:

$$I_b + I_g = I_s \quad (2)$$

Then, applying Kirchhoff's second law for the voltage drops around the loops of the circuit, and recalling that IR drops are negative (–), we obtain around loop ABFDA:

$$+E - I_bR_b - I_sR_s = 0, \quad (3)$$

around loop ABCDA:

$$+E - I_aR_a - I_xR_x = 0, \quad (4)$$

and around loop BCFB:

$$-I_aR_a - I_gR_g + I_bR_b = 0. \quad (5)$$

(Here the voltage drop I_bR_b is plus, since it is opposed to the assumed current direction.)

Voltage drops are quite common.

To solve for the galvanometer current, I_g, we progressively reduce the five simultaneous equations in number, eliminating one of the unknown currents each time. Substituting for $I_b = I_s - I_g$ from Equation 2 and for $I_a = I_g + I_x$ from Equation 1 in Equations 3, 4 and 5 we obtain:

In Equation 3: $+E - (I_s - I_g)R_b - I_sR_s = 0$, and simplifying

$$+E + I_gR_b - I_s(R_b + R_s) = 0 \qquad (6)$$

In Equation 4: $+E - (I_g + I_x)R_a - I_xR_x = 0$, and again simplifying:

$$+E - I_gR_a - I_x(R_a + R_x) = 0 \qquad (7)$$

In Equation 5: $-(I_g + I_x)R_a - I_gR_g + (I_s - I_g)R_b = 0$ simplifying:

$$-I_g(R_a + R_g + R_b) + I_sR_b - I_xR_a = 0 \qquad (8)$$

By solving Equation 6 for I_s and Equation 7 for I_x, and substituting the results in Equation 8, we obtain from Equation 6:

Sometimes, as in manufacturing and in mathematics, changing the original form is both helpful and necessary.

$$I_s = \frac{E + I_gR_b}{R_b + R_s}\text{; from Equation 7: } I_x = \frac{E - I_gR_a}{R_a + R_x}$$

and, hence, Equation 8 becomes

$$-I_g(R_a + R_g + R_b) + \frac{(E + I_gR_b)R_b}{R_b + R_s} - \frac{(E - I_gR_a)R_a}{R_a + R_x} = 0 \qquad (9)$$

Transposing the first term to the right and placing the fractions over the common denominator $(R_b + R_s)(R_a + R_x)$:

$$\frac{(E + I_gR_b)R_b(R_a + R_x) - (E - I_gR_a)R_a(R_b + R_s)}{(R_b + R_s)(R_a + R_x)} = I_g(R_a + R_b + R_g)$$

Multiplying by $(R_b + R_s)(R_a + R_x)$ to get rid of the fractions:

$$(E + I_gR_b)R_b(R_a + R_x) - (E - I_gR_a)R_a(R_b + R_s) = I_g(R_a + R_b + R_g)(R_b + R_s)(R_a + R_x).$$

Finally, multiplying the parentheses, cancelling equal terms on both sides of the equation and collecting all terms with I_g:

$$E\,R_bR_x - E\,R_aR_s = I_g(R_aR_bR_g + R_aR_gR_s + R_aR_bR_s + R_aR_bR_x + R_bR_gR_x + R_aR_sR_x + R_gR_sR_x + R_bR_sR_x)$$

Solving for I_g by dividing through by the parenthesis at the right and simplifying, we obtain the desired galvanometer current:

$$I_g = E\frac{R_bR_x - R_aR_s}{(R_a + R_x)(R_bR_g + R_bR_s + R_sR_g) + R_aR_x(R_b + R_s)} \qquad (10)$$

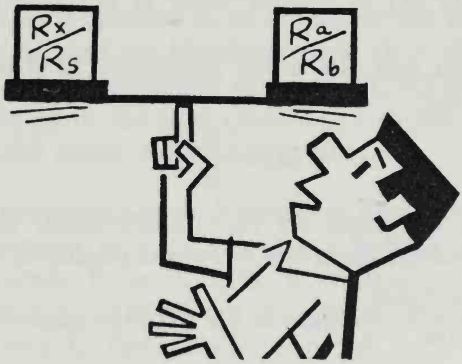
Although this expression looks very complex, substitution of actual values results in a relatively simple computation. We see at once that the numerator of Equation 10 must equal zero (0) for bridge balance, since by definition the galvanometer current must be zero. Hence, for $I_g = 0$, we let

$$R_bR_x - R_aR_s = 0 \qquad (11)$$

and, thus, obtain the desired condition for bridge balance. Dividing Equation 11 through by R_bR_s and transposing, we may write

$$\frac{R_x}{R_s} = \frac{R_a}{R_b} \qquad (12)$$

This shows that the ratio of the unknown resistance, R_x, to the standard resistance, R_s, equals the ratio of the bridge arm resistances, R_a to R_b. The statement that two ratios are equal is called a proportion, and all kinds of interesting things can be proved



If the equation doesn't balance, it isn't an equation.

about proportions. For instance, you can see (by multiplying) that for $R_x : R_s = R_a : R_b$, or $\frac{R_x}{R_s} = \frac{R_a}{R_b}$, we get $\frac{R_x + R_s}{R_x - R_s} = \frac{R_a + R_b}{R_a - R_b}$ which may come in handy occasionally during numerical computation.

Proportions are particularly easy to handle on a slide rule. For example, if a Wheatstone bridge is balanced with $R_a = 3{,}000$ ohms, $R_b = 2{,}000$ ohms and $R_s = 6{,}445$ ohms, you can find unknown resistance R_x easily by setting up the proportion:

$$R_a : R_b = R_x : R_s \text{ or } 3{,}000 : 2{,}000 = R_x : 6{,}445$$

To do this on any slide rule, simply set the C and D scales to the same proportions; (Fig. 206):

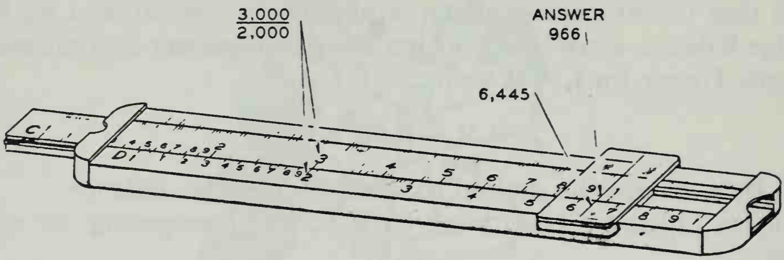


Fig. 206. *The slide rule is ideal for providing quick answers to problems in proportion.*

$$\begin{array}{llll} C\ D & C & D \\ 3:2 = & x & : 6{,}445 \\ & (9{,}660) & \end{array}$$

As shown in the illustration, opposite 2 on D set 3 on C. Then, opposite 6,445 (approximately) on D read the answer 966 (approximately). Since C (3) is greater than D (2), the result must be 9,660 ohms.

Of course, you can also solve Equation 12 directly for R_x by multiplying by R_s:

$$R_x = \frac{R_a}{R_b} R_s \qquad (13)$$

Substituting in Equation 13 $R_a = 3{,}000$ ohms, $R_b = 2{,}000$ ohms and $R_s = 6{,}445$ ohms from the previous example, we obtain:

$$R_x = \frac{3{,}000 \times 6{,}445}{2{,}000} \text{ ohms} = 1.5 \text{ x } 6{,}445 \text{ ohms} = 9{,}668 \text{ ohms}$$

(The arithmetical answer, naturally, is more exact than the slide rule answer.)

PRACTICE EXERCISE 4

1. Derive a formula for the equivalent parallel resistance of n equal resistors, of value R each. (Answer: $\frac{R}{n}$)

2. Resistance R1 is inserted in parallel with two equal parallel resistors of value R2 each. (a) What is the general formula for the total combined resistance, and (b) what is its value, if R1 = 3,000 ohms and R2 = 4,000 ohms (each).

[Answer: (a) $\frac{R1R2}{2R1 + R2}$; (b) 1,200 ohms]

3. An amplifier input termination of 100,000 ohms is in parallel with an input tube grid resistor of 0.5 megohm. (a) What value of resistance should be placed in shunt with a phono pickup connected to the input to comply with the manufacturer's recommended input load resistance of 20,000 ohms? (b) If the shunt resistor = R1, the grid resistor = R2, the input termination = R3 and the recommended load resistance = R, find a formula for R1.

[Answer: (a) 26,300 ohms; (b) $R1 = \frac{RR2R3}{R2R3 - RR3 - RR2}$]

4. A 0–1-ma meter with an internal resistance of 100 ohms is to be extended to a range of (a) 0.1 ampere and (b) 1 ampere. Determine the value of the shunt resistor within 0.1% for each case.

[Answer: (a) 1.010 ohms; (b) 0.100 ohm]

5. The voltage divider of Fig. 204 is to deliver a plate voltage E_{b1} = 300 volts, at a current I_{b1} = 100 ma, and a plate voltage E_{b2} = 150 volts, at a current I_{b2} = 30 ma. If the filter output voltage E = 450 volts, and the bleeder curent I3 = 20 ma, determine total current I and the values of resistors R1, R2 and R3, using the relations developed in the text. (Answer: I = 150 ma; R1 = 1,000 ohms; R2 = 3,000 ohms; R3 = 7,500 ohms.)

6. A known resistance R_x = 50 ohms is inserted into the Wheatstone-bridge network of Fig. 205. If the battery voltage E = 100 volts, R_a = 30 ohms, R_b = 60 ohms, R_s =40 ohms, and the galvanometer resistance R_g = 20 ohms, determine the unbalanced galvanometer current I_g, the currents through each of the resistance arms and the voltage drop across each, and also total current I and equivalent resistance R of the network.

[Hint: Determine the currents by substituting the resistance and voltage values directly into the equations developed in the text, solve the simultaneous equations for the currents I_a, I_b, I_s, I_x and I_g. Then determine the voltage (IR) drops across each resistor and, finally, determine the total current at point B ($I = I_a + I_b$) and find the equivalent resistance, $R = \frac{E}{I}$.]

[Answer: $I_g = 0.36$ amp. $I_a = 1.475$ amp. $I_b = 0.86$ amp. $I_x = 1.115$ amp. $I_s = 1.22$ amp. $I = 2.335$ amp. $R = 42.8$ ohms. Voltage across $R_a = 44.25$; across $R_b = 51.6$; across $R_x = 55.75$; across $R_s = 48.8$. Voltage across the meter $(R_g) = 7.2$.]

7. If resistor R_x in problem 6 is removed from the bridge, what value should be substituted to obtain a balanced bridge? (Answer: 20 ohms.)

CHAPTER 3

From AC to Complex Numbers

THE signals of electronics are alternating currents and voltages, which continuously rise and fall in magnitude and periodically reverse their polarity. Some circuit components slow down alternating currents, while others speed them up, so that the current and applied voltage do not always rise and fall together. Because of these added complexities, alternating currents are not successfully handled by the limited mathematical tools we have developed for direct currents, and hence, we must expand our mathematical horizon somewhat. An understanding of vectors and phase relations, and the correct mathematical manipulation of these by elementary trigonometry and complex numbers, will enable us to solve most ac problems rapidly and efficiently. Moreover, the mastery of these relatively simple methods will lend your calculations an aura of elegance, which other, more clumsy methods cannot attain.

Elementary ac generator

Let us investigate briefly the generation of an ac voltage by an armature coil rotating in a uniform magnetic field (Fig. 301). A coil (or conductors) cutting lines of magnetic flux will have an electromotive force (emf) induced in it proportional to the flux density of the field, the length of the conductor and the velocity of rotation. As the coil rotates between the poles of a magnet, its long sides cut the flux first in one direction, then in the other, thus producing an emf (and current) that alternates continuously in polarity (plus during one half-cycle, minus during the other).

When the coil is in approximately the position illustrated (Fig. 301), its sides cut a maximum number of lines of force at right angles (perpendicularly) to the field. In this position, the emf induced in the coil reaches a maximum value. A quarter revolution or 90° later, the two long conductors of the coil move parallel to the lines of force and do not cut any of them. The voltage induced in the coil, consequently, is zero. As the coil continues to rotate (counterclockwise in the illustration), the direction of the induced emf and current reverses and the voltage begins to build up in the opposite direction. As the armature coil completes one-half revolution (180°), its sides move once again perpendicularly to the field

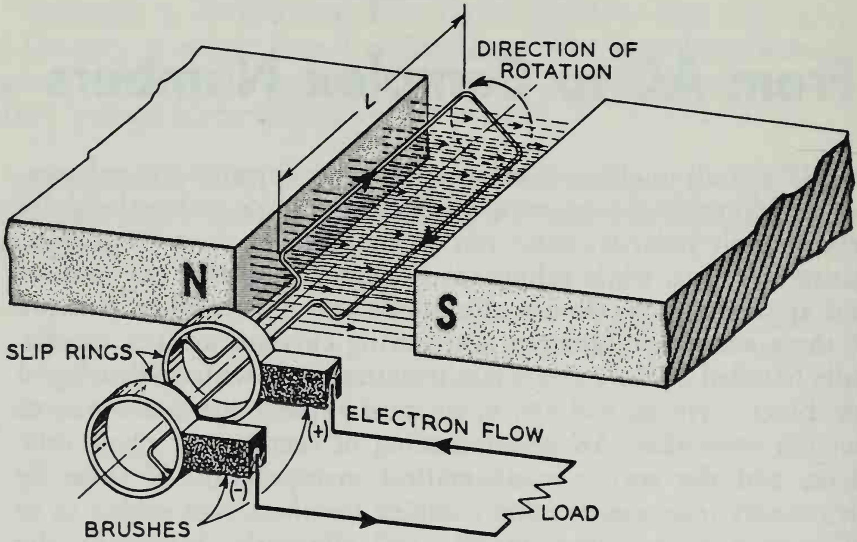


Fig. 301. *This is the basic arrangement of an alternating-current generator. When the coil rotates, a voltage is induced across those portions of the coil that move at angles to the magnetic field. One of these is marked L. With the help of slip rings and carbon brushes, this induced voltage is impressed across the resistive load, causing a current to flow through it.*

and the induced voltage is again at a maximum, though in the opposite direction. If the coil voltage is connected to an external load circuit through two continuously contacting slip rings, an alternating current will flow through the load circuit. Fig. 301 shows the direction of (electron) current flow for the field direction and coil position illustrated. The current direction reverses during each half-revolution.

What is a sine wave?

Now let us investigate how the induced emf or current varies from moment to moment, between its maximum and zero values. Fig. 302 illustrates the magnetic field and coil in cross-section, with one of the long coil sides facing the reader. For simplicity, assume that the radius of the circle of rotation represents one side of the armature coil and that this same length also represents the maximum value, E_m, of the induced alternating voltage. We want to find out how this induced voltage varies in magnitude (with time or angle of rotation, θ) as the radius turns counterclockwise in the magnetic field to simulate the actual rotation of the armature in a generator.

Let us start the rotation with the radius in the horizontal position and its end point, A, moving parallel with the field lines (point 1). At this instant (0° rotation), the induced voltage is zero, as shown by point 1 at 0° for the plot of the voltage at right. As the radius rotates, only the portion perpendicular to the flux (i.e., cuts it at right angles) will have a voltage induced in it. We can obtain the effective perpendicular portion at any time by drawing a line from the end of the radius perpendicular to the horizontal diameter

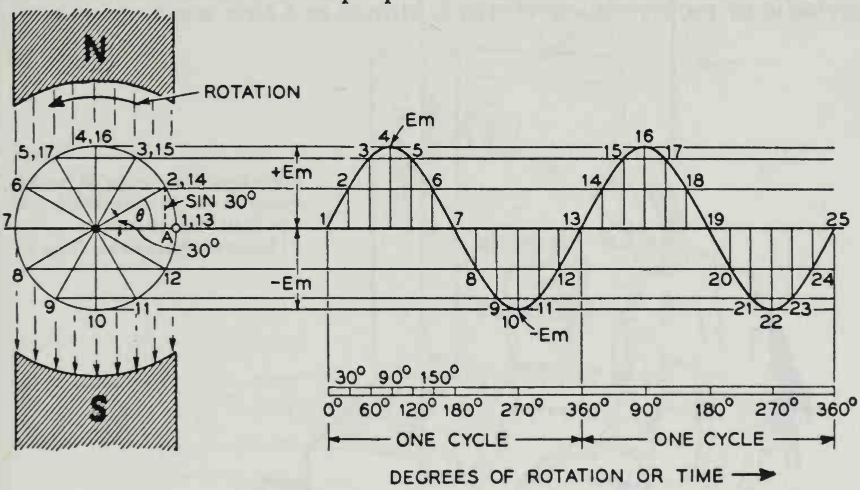


Fig. 302. *Generation of a sine wave by a rotating coil in a magnetic field.*

of the circle. The length of this perpendicular line at any instant will then represent the magnitude of the voltage induced in the armature coil at that instant. Hence, by dropping perpendiculars from the end point of the radius to the horizontal at regular intervals, we can determine the variation of the instantaneous induced voltage with the amount of angular rotation or with time, if the speed of rotation is uniform.

To the right of the rotating radius in Fig. 302 we have plotted the length of the perpendicular (also called vertical projection of the radius) against the counterclockwise angle that the radius forms with the horizontal diameter. For example, when the rotating radius makes an angle of 30° with the horizontal (point 2), a horizontal line drawn from the end point of the radius to the 30° ordinate of the waveform plot at the right determines the height of the perpendicular at 30° rotation and, hence, the magnitude of the voltage induced at that instant.

Similarly, when the radius makes an angle of 90° (point 4), the horizontal line drawn to the 90° ordinate of the voltage waveform determines the value of the induced voltage at that instant. This turns out to be the maximum value, E_m. As we continue to plot the length of the perpendicular (or vertical projection) against the angular rotation, or time, we obtain the voltage waveform shown at the right of Fig. 302. Evidently, as the radius rotates, its vertical projection varies between maximum values of $+E_m$ and $-E_m$, and generates the smoothly varying waveform shown. After one complete rotation cycle, the waveform repeats itself. This type of periodic or recurrent waveform is known as a sine wave.

A working knowledge of mathematics is an indispensable tool in the laboratory. [Spectran Electronics Corp.]

Sines, cosines and tangents

You can see in Fig. 302 that the rotating radius, its vertical projection and the horizontal diameter always form a *right triangle,* regardless of the angular position (θ) of the radius. Certain special relationships, known as trigonometric functions, hold for any right triangle. Figs. 303 through 306 review these relationships. In Fig. 303 we have drawn a line of length r to a point P, with rectangular

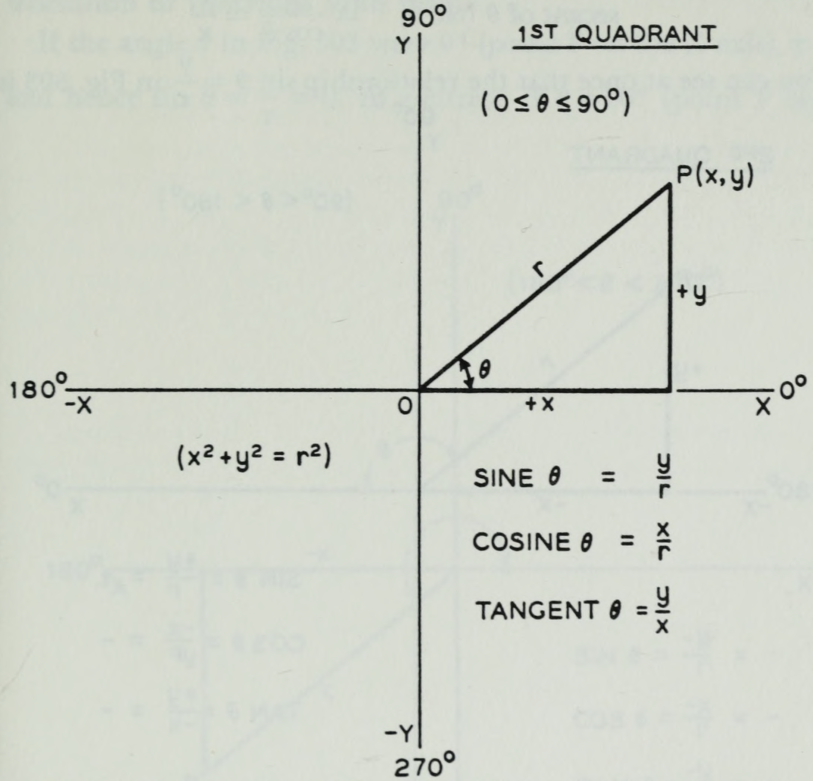


Fig. 303. *Trigonometric functions in the first quadrant. In this quadrant the sine, cosine and tangent all have positive values.*

coordinates x and y. The line forms an arbitrary angle, θ, with the X-axis, which is somewhere between 0° and 90°. The line r to point P, together with the x and y coordinates of P, forms a right triangle, in which the following trigonometric functions are defined:

$$\text{sine of } \theta \ (\sin \theta) = \frac{y}{r}$$

$$\text{cosine of } \theta \ (\cos \theta) = \frac{x}{r}$$

$$\text{tangent of } \theta \ (\tan \theta) = \frac{y}{x}$$

While these three are the most frequently used trigonometric functions, the following reciprocal functions also hold:

$$\text{cotangent of } \theta \ (\cot \theta) = \frac{1}{\tan \theta} = \frac{x}{y}$$

$$\text{cosecant of } \theta \ (\text{cosec } \theta) = \frac{1}{\sin \theta} = \frac{r}{y}$$

$$\text{secant of } \theta \ (\sec \theta) = \frac{1}{\cos \theta} = \frac{r}{x}$$

You can see at once that the relationship $\sin \theta = \frac{y}{r}$ in Fig. 303 is

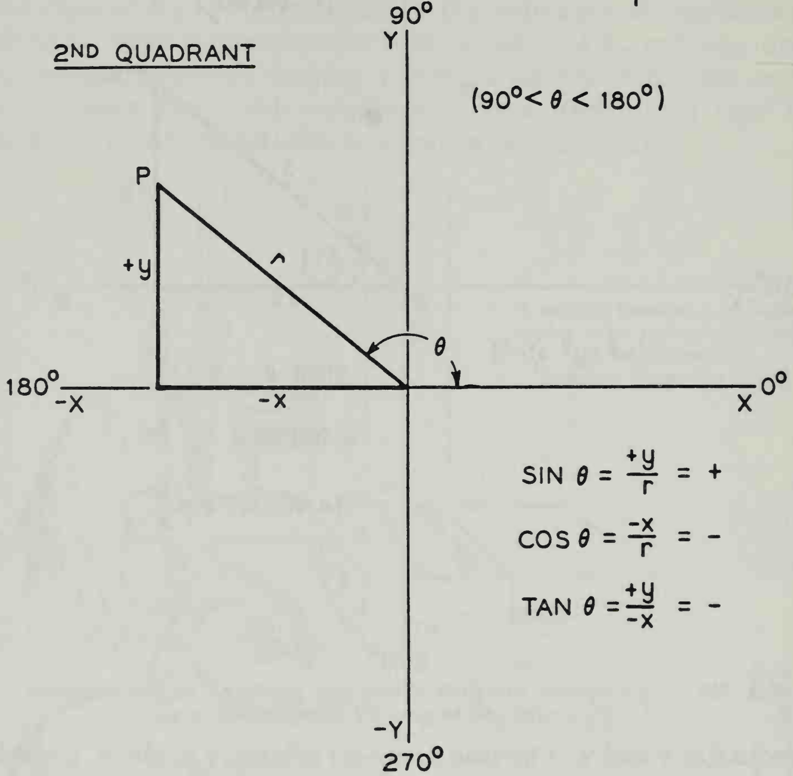


Fig. 304. *Trigonometric functions in the second quadrant. The angle, theta, is greater than 90° but less than 180°.*

identical with the relationship between the rotating radius and its vertical projection in Fig. 302. If we let the radius r = 1 (unit length), then $\sin \theta = y/1 = y$, and hence, the vertical projection (y) of the radius in Fig. 302 at any instant equals the sine of the angle of rotation. Therefore, the voltage induced in the armature coil (represented by the radius) varies as the sine of the angle of rotation (θ).

Moreover, the plot of the vertical projection of the radius against the angle of rotation (at the right in Fig. 302) is the graph of the sine function against θ, and is known as a sine wave. The ordinary ac generator, thus, automatically generates a sine wave. Let us see how such a sine wave varies in magnitude and sign ("+" or "–") with increasing angle θ.

Variation of functions with angle

If the angle θ in Fig. 303 were 0° (point P on the X-axis), $y = 0$, and hence $\sin\theta = \frac{y}{r} = 0$. In contrast, if $\theta = 90°$ (point P on the

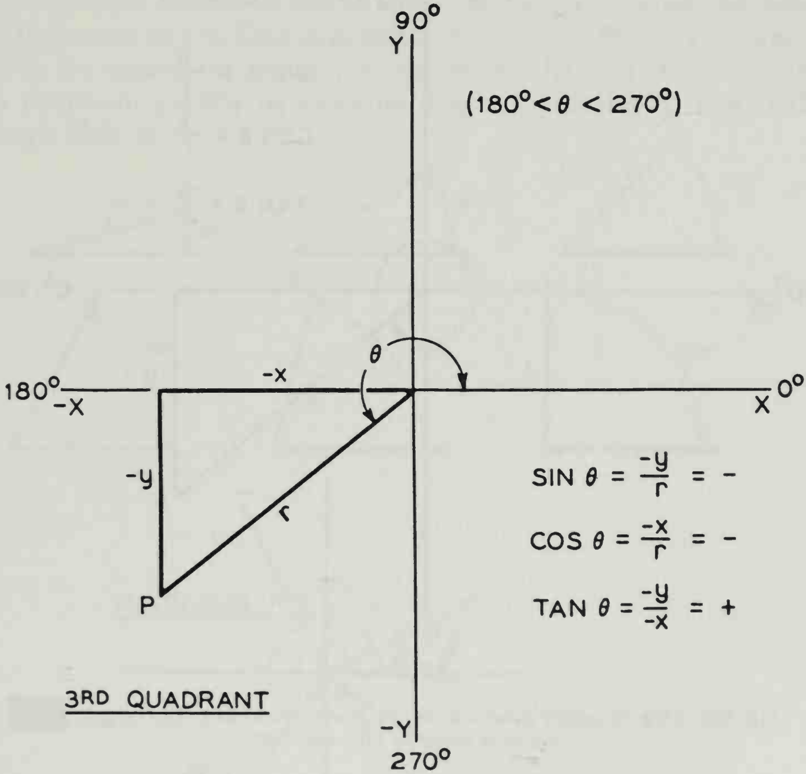


Fig. 305. *Trigonometric functions in the third quadrant. The sine and cosine are both negative, the tangent positive.*

Y-axis), $y = r$, and hence $\sin\theta = \frac{y}{r} = \frac{r}{r} = 1$. Thus, for angles ($\theta$) between 0° and 90° (called the first quadrant) the sine of θ varies between 0 and 1. Moreover, since the length r is always taken as positive and the y-coordinate is positive in the entire first quadrant, the ratio $\frac{y}{r}$ = sin θ is positive (+) in the first quadrant. Similar reasoning shows that the cosine of θ varies between 1 (for $\theta = 0°$) and 0 (for $\theta = 90°$) and is also positive (+) in the first quadrant.

To determine the limits of the tangent function, first let $\theta = 0°$ and then set $\theta = 90°$. For $\theta = 0°$, y = 0, and tan $\theta = \frac{y}{x} = \frac{0}{x} = 0$.

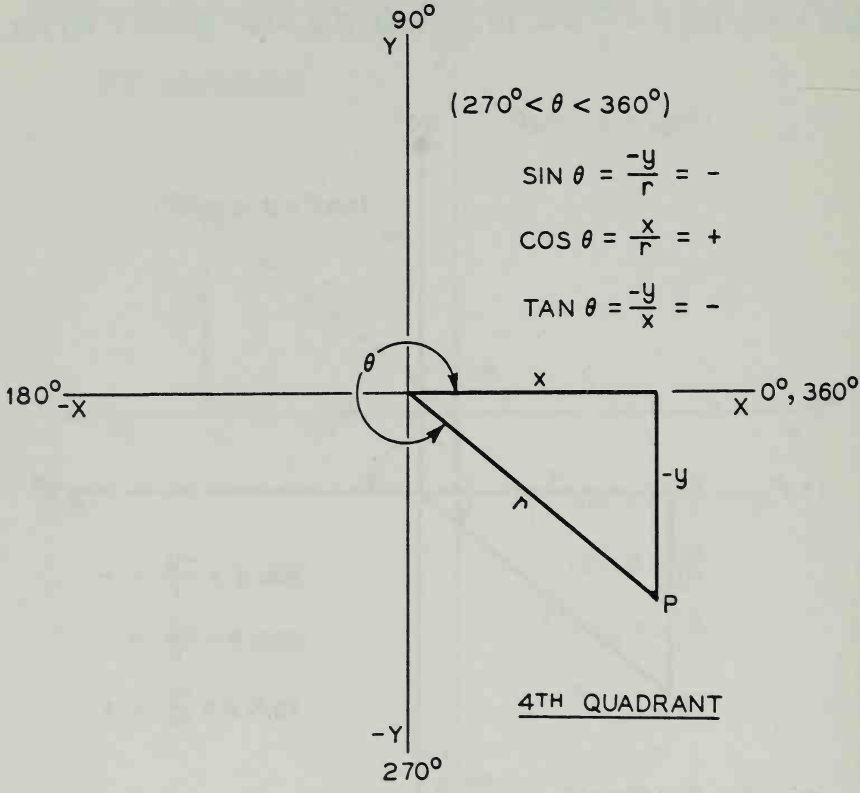
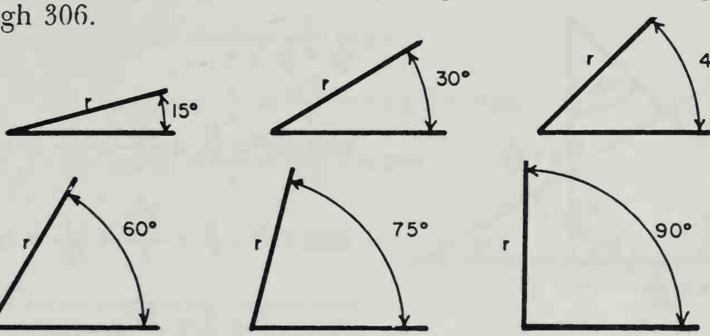


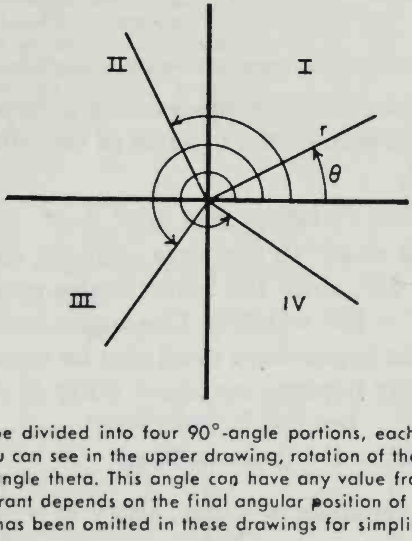
Fig. 306. *Trigonometric functions in the fourth quadrant. The angle,* [redacted] *theta, is between 270° and 360°.*

For $\theta = 90°$, x = 0, and hence, tan $\theta = \frac{y}{x} = \frac{y}{0} = \infty$ (infinity). Thus, the tangent function is positive and varies between 0 and ∞ in the first quadrant.

By referring to Fig. 304, you can easily determine the variation of the sine, cosine and tangent functions in the second quadrant for values of θ between 90° and 180°. Since r and y are both positive, the sine function is "+" and declines in value from +1 at $\theta = 90°$ to 0 at $\theta = 180°$. The cosine function is negative (–), since the x-coordinate is negative throughout the second quadrant. The function varies from 0 for $\theta = 90°$ (when x = 0) to –1 for $\theta = 180°$ (when –x = r). Since y is positive and x is negative (–) in the second quadrant, the ratio $\frac{y}{x} = \tan \theta$ is also negative (–) throughout the quadrant. The tangent function varies in magnitude from ∞ at 90° to 0 at 180°.

You can verify for yourself the variation of the three main functions in the third quadrant (Fig. 305) for angles from 180° to 270°, and in the fourth quadrant (Fig. 306) for angles from 270° to 360° (which is, of course, the same as 0°). The table (on page 68) briefly summarizes the variation in the value of the sine, cosine and tangent functions in the four quadrants (from 0° to 360°). You don't need to memorize this, since you can derive the sign and limits of each function quickly by drawing diagrams similar to Figs. 303 through 306.



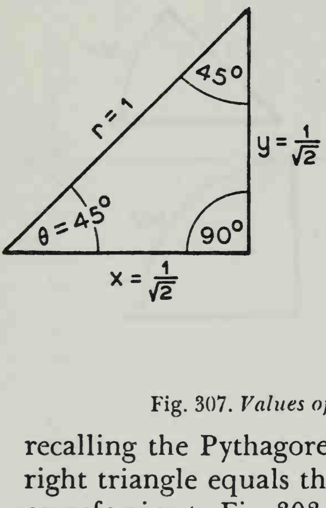


Any circle can be divided into four 90°-angle portions, each of which is known as a quadrant. As you can see in the upper drawing, rotation of the hypotenuse, r, increases the size of the angle theta. This angle can have any value from 0 to 360 degrees. The applicable quadrant depends on the final angular position of radius R. [The ordinate y has been omitted in these drawings for simplification.]

Variation in Value and Sign of Main Functions

	Sin Θ		Cos Θ		Tan Θ	
Quadrant	Sign	Value	Sign	Value	Sign	Value
I (0°–90°)	+	0 to 1	+	1 to 0	+	0 to ∞
II (90°–180°)	+	1 to 0	–	0 to 1	–	∞ to 0
III (180°–270°)	–	0 to 1	–	1 to 0	+	0 to ∞
IV (270°–360°)	–	1 to 0	+	0 to 1	–	∞ to 0

Although you can look up the value of the functions for any angle θ in the tables of natural trigonometric functions, you should commit to memory a few standard values for angles of 0°, 30°, 45°, 60° and 90°. You can verify each of the values listed in the table following by drawing the appropriate right triangle and



$$x^2 + y^2 = r^2$$

$$\frac{1}{2} + \frac{1}{2} = 1$$

$$\text{SIN } 45^\circ = \frac{y}{r} = \frac{\frac{1}{\sqrt{2}}}{1} = \frac{1}{\sqrt{2}} = 0.707$$

$$\text{COS } 45^\circ = \frac{x}{r} = \frac{\frac{1}{\sqrt{2}}}{1} = \frac{1}{\sqrt{2}} = 0.707$$

$$\text{TAN } 45^\circ = \frac{y}{x} = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = 1.000$$

Fig. 307. *Values of the functions when the phase angle is 45°.*

recalling the Pythagorean law: the square on the hypotenuse of a right triangle equals the sum of the squares of the other two sides; or, referring to Fig. 303,

$$x^2 + y^2 = r^2; \text{ hence: } r = \sqrt{x^2 + y^2}$$

For example, when $\theta = 45°$ in the right triangle, the remaining angle must also equal 45°, since the three angles must add up to 180° (that is, 45° + 45° + 90° = 180°). The angles being equal, the two sides adjacent to the hypotenuse must also be equal (Fig. 307). Letting the length of the hypotenuse equal unity ($r = 1$), we can write: $x^2 + y^2 = r^2$ but $r = 1$, and $x = y$

hence $x^2 + x^2 = 1^2$ or $2x^2 = 1$ (substituting)

and solving for x: $x^2 = \frac{1}{2}$ and $x = \frac{1}{\sqrt{2}}$; also $y = \frac{1}{\sqrt{2}}$.

We, therefore, have r = 1, $x = \frac{1}{\sqrt{2}}$ and $y = \frac{1}{\sqrt{2}}$ for the three sides of the triangle, and compute the three functions for $\theta = 45°$, as follows:

$$\sin 45° = \frac{y}{r} = \frac{1/\sqrt{2}}{1} = \frac{1}{\sqrt{2}}\ ;\ \cos 45° = \frac{x}{r} = \frac{1/\sqrt{2}}{1} = \frac{1}{\sqrt{2}};$$

and $\tan 45° = \frac{y}{x} = \frac{1/\sqrt{2}}{1/\sqrt{2}} = 1$, as shown in the table.

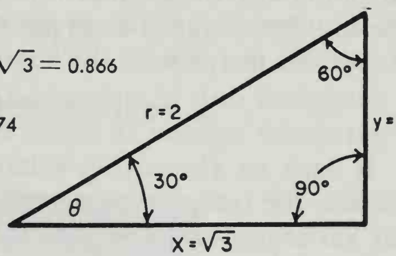
Value of Functions for Frequently Used Angles

Angle θ:	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2} = 0.5$	$1/\sqrt{2} = 0.707$	$\frac{1}{2}\sqrt{3} = 0.866$	1
$\cos\theta$	1	$\frac{1}{2}\sqrt{3} = 0.866$	$1/\sqrt{2} = 0.707$	$\frac{1}{2} = 0.5$	0
$\tan\theta$	0	$1/\sqrt{3} = 0.5774$	1	$\sqrt{3} = 1.732$	∞

$$\sin 30° = \frac{y}{r} = \frac{1}{2} = 0.5$$

$$\cos 30° = \frac{x}{r} = \frac{\sqrt{3}}{2} = 1/2\sqrt{3} = 0.866$$

$$\tan 30° = \frac{y}{x} = \frac{1}{\sqrt{3}} = 0.5774$$

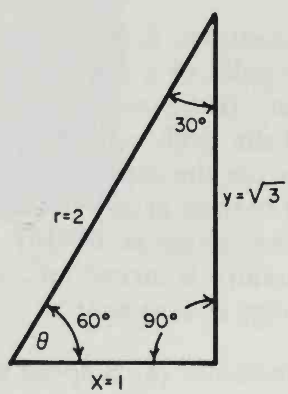


These are the values of the functions when the phase angle is 30° [see drawing above] and when the phase angle is 60° [see drawing below]. In examining the triangles you will see some apparent relationships. The sin of 30° is the same as the cos of 60°. Similarly, the cos of 30° is equivalent to the sin of 60°. However, tan 30° and tan 60° are reciprocals.

$$\sin 60° = \frac{y}{r} = \frac{\sqrt{3}}{2} = 1/2\sqrt{3} = 0.866$$

$$\cos 60° = \frac{x}{r} = \frac{1}{2} = 0.5$$

$$\tan 60° = \frac{y}{x} = \frac{\sqrt{3}}{1} = \sqrt{3} = 1.732$$



Instantaneous value of ac voltage and current

Now let us get back to the sine wave generated by a rotating armature (Fig. 302). Since we now know that the instantaneous voltage, e, generated by the armature (or rotating radius) varies as the sine of the angle of rotation, θ, and that its maximum value is E_m, we may write for the voltage generated at any instant:

$$e = E_m \sin \theta \qquad (1)$$

We further know that, for a uniformly rotating armature, the angular velocity (symbol ω) must equal the angle (θ) "swept out" per unit time (t), or $\omega = \theta/t$, where θ is the angle measured in *radians* ($360° = 2\pi$ or 6.283 radians) and t is the time in seconds. Substituting for θ in (1) above, we obtain for the instantaneous ac voltage:

$$e = E_m \sin \omega t \qquad (2)$$

Moreover, since each 360° revolution of the armature corresponds to 2π radians, the angular velocity (ω) in radians is simply 2π times the number of revolutions per second, or 2π times the frequency, f. Expressed in symbols, the angular velocity,

$$\omega = 2\pi f = 6.283f \quad (\text{since } \pi = 3.1416) \qquad (3)$$

If such an alternating voltage, e, is applied to a resistive load circuit, the instantaneous current, i, will, of course, undergo similar variations as the voltage and will be related to the maximum value of the current, I_m, in the same way. Hence, we can write the instantaneous value of the current,

$$i = I_m \sin \omega t = I_m \sin 2\pi ft \qquad (4)$$

EXAMPLE: A coil of an elementary ac generator rotates between two poles of a magnet at a rate of 3,600 revolutions per minute (rpm). If the maximum (peak) value of the induced voltage is 170 and the peak value of the current through a load is 20 amperes, (a) write the expressions for the instantaneous values of the voltage and current at any time, and (b) compute the instantaneous values of the voltage at .004167, .00833, .0125 and .0167 seconds after the generator is turned on. (Assume that the generator starts with zero voltage at zero time.)

Solution: (a) A speed of 3,600 rpm is equivalent to $\frac{3{,}600}{60} = 60$ revolutions per second (rps). The frequency, f, therefore is 60 cycles per second (cps). Hence, the angular velocity (in radians),

$$\omega = 2\pi f = 6.283 \times 60 = 377 \text{ radians/sec}$$

Hence, $e = E_m \sin \omega t = 170 \sin 377t$ volts

and $i = I_m \sin \omega t = 20 \sin 377t$ amperes

(b) NOTE: Since 2π radians $= 360°$, π rad $= 180°$, $\frac{\pi}{2}$ rad $= 90°$, etc. We can write $e = E_m \sin 2\pi ft = 170 \sin 120\pi t$.

Hence, after .004167 second,

$$e = 170 \sin 120\pi \times .004167 = 170 \sin 0.5\pi = 170 \sin \frac{\pi}{2}$$
$$= 170 \sin 90° = 170 \times 1 = 170 \text{ volts}$$

After .00833 second:

$$e = 170 \sin 120\pi \times .00833 = 170 \sin \pi = 170 \sin 180°$$
$$= 170 \times 0 = 0 \text{ volts}$$

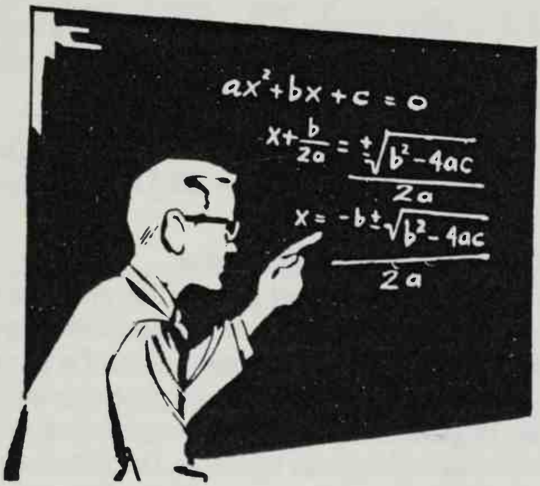
After .0125 second:

$$e = 170 \sin 120\pi \times .0125 = 170 \sin 1.5\pi = 170 \sin 270°$$
$$= 170(-1) = -170 \text{ volts}$$

And after .0167 second:

$$e = 170 \sin 120\pi \times .0167 = 170 \sin 2\pi = 170 \sin 360°$$
$$= 170 \times 0 = 0 \text{ volts}$$

It is evident from the example that the instants of time have been chosen equal to ¼, ½, ¾ and 1 cycle of the ac voltage, where the



Mathematics loses its mystery and becomes less difficult, once you know the rules and have learned how to apply them.

wave goes through its maximum and zero values. You can see that the voltage oscillates between values of 0, +170 and −170.

Effective (root-mean-square) value of ac

Since the instantaneous value of an alternating current varies continuously as a sine wave between zero, positive and negative maximum values, some way had to be found to define a value of the current and voltage that could be said to be *effective* in a circuit. The *effective value* of an alternating current is that value which produces heat at exactly the same rate as an equal quantity of direct current flowing through the same resistance. In other words, an effective alternating current of 1 ampere produces the same heat in a given resistance and given time as 1 ampere dc.

From the manner in which the effective ac value is determined, by taking the root of the mean (or average) squared current value, the term root-mean-square (rms) value arises; it means exactly the same as the effective ac value. Without going into the computations, it turns out that the effective (or rms) value of an alternating current or voltage is equal to $1/\sqrt{2}$ or 0.707 times the maximum (or peak) value of the current or voltage. Hence, the effective value of an alternating current, $I = 0.707\ I_m$, and the effective value of an ac voltage, $E = 0.707\ E_m$. Similarly, the peak or maximum ac current, $I_m = 1.414I$, and the peak or maximum ac voltage, $E_m = 1.414E$, where I and E stand for the effective values.

EXAMPLE: An ac voltage with a peak value of 162.8 causes an rms current of 20 amperes to flow. What is the effective value of the voltage and the peak value of the current?

Solution: $E = 0.707\ E_m = 0.707 \times 162.8 = 115$ volts (rms)
$I_m = 1.414I = 1.414 \times 20 = 28.3$ amperes (peak)

Phase, phase angle and phase difference

The terms phase, phase angle and phase difference are constantly and often interchangeably, used in connection with ac, with considerable confusion as to just what they mean. To get these concepts straight, you have to make a few simple distinctions.

1. When phase or phase angle is used in connection with a single alternating current or voltage, it refers to the fraction of a cycle that has elapsed since the current or voltage has passed a given reference point (usually the starting point). For example, at the start of the ac voltage in Fig. 302 (point 1), the phase is said to be zero. At point 2, the phase or phase angle is 30°, at point 3 it is 60°, at point 4 it is 90°, or a quarter cycle; at point 7 it is 180°, or a half-cycle; at point 10 it is 270°, or three-quarter cycle, and so forth.

When used in this way, the phase is significant only for a fraction of a cycle, since it repeats during each successive cycle.

2. The terms phase or phase difference are more frequently used to compare two or more alternating currents or voltages of the *same frequency* that pass through their zero and maximum values at different instants of time. For example, you know that the current in a coil or capacitor does not rise and fall together with the ac voltage applied across these components, and hence, the current is said to be out of phase with the voltage, or to have a phase angle or phase difference. When the frequency is the same, each cycle of an alternating current or voltage takes exactly the same amount of time (T = 1/f), and hence, the phase difference between two such alternating voltages, currents, or a voltage and a current, is conveniently expressed in degrees or fractions of a cycle, a measure of time being implied in either case.

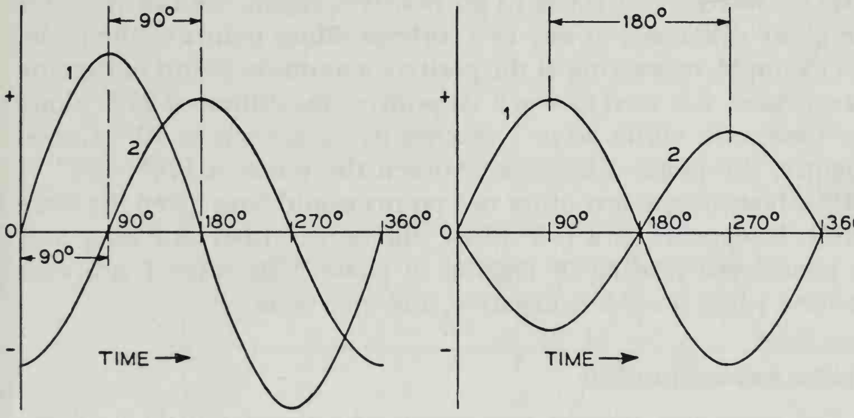


Fig. 308. *Voltages and currents can be in phase or out of phase in varying degrees.*

Sine waves out of phase

Sine waves 1 and 2 in Fig. 308 may represent two ac voltages, currents or a voltage and a current of the same frequency which we would like to compare. Evidently, waveform 1 has a larger peak value (also called amplitude) than waveform 2. Moreover, the waves do not rise and fall in unison and, hence, are out of phase with each other. We would like to know by how much.

In Fig. 308-a, sine wave 1 has a value of zero at the 0° reference point, while sine wave 2 is at its negative maximum value at the same instant. Only when sine wave 1 reaches its positive maximum value at the 90° point along the time axis does sine wave 2 finally pass through zero. Again, when sine wave 1 returns to zero at the 180° marker, sine wave 2 just reaches its positive maximum value and, when wave 1 reaches its negative maximum value, wave 2 just passes downward through zero. Clearly, sine waves 1 and 2 are out of phase with each other by one-quarter cycle or 90°. Moreover, since sine wave 1 reaches corresponding points of the cycle earlier than sine wave 2, wave 1 is said to lead wave 2 by 90° in phase. Equivalently, sine wave 2 lags sine wave 1 by 90°, or a quarter cycle. It doesn't matter at which points of the cycle you measure, the phase difference between the two sine waves will always be 90°, or a quarter-cycle.

Fig. 308-b shows two sine waves of the same frequency 180°, or one-half cycle out of phase with each other. When wave 1 rises in the positive direction from the 0° starting point, wave 2 rises in the negative direction. When wave 1 starts to go negative at the 180° marker, wave 2 just starts to go positive. Again, we can measure the phase difference at any two corresponding points of the cycle. For example, measuring at the positive maximum points of the sine waves, wave 2 is seen to reach its positive maximum at 270° along the time axis, while wave 1 reaches its maximum at 90°. Consequently, the phase difference between the waves is $270° - 90° = 180°$. Measuring at any other two points would have given the same result. Moreover, for a 180° phase difference, either sine wave may be considered leading or lagging in phase. Sine wave 1 is always positive when wave 2 is negative, and vice versa.

Vector representation

The successive vertical projections of a counterclockwise rotating radius generate a sine waveform. Conversely, it is also true that a sine wave can be represented at any instant of time by the position, at that instant, of a rotating radius. All we need to do is to make the length of the radius equal to the peak value (amplitude) of the sine wave and position the radius at such an angle that its projection upon the vertical (that is, the sine of the angle) equals the value of the sine wave at the particular instant.

In Fig. 309 we have drawn the rotating radii (usually called rotating vectors) that would be required to generate sine waves 1 and 2 of Fig. 308-a. At the top left of the figure, a counterclockwise rotating radius, R, equal in length to the amplitude of sine wave 1, is illustrated in five positions, corresponding to five successive instants of time. As in Fig. 302, the vertical projections of the radius at these successive instants yield the corresponding ordinates of the sine wave shown at the right. Thus, at zero time or 0° rotation (point 1) the vertical projection of R is zero and, hence, the ordinate of the sine wave at the right also is zero.

A moment later, after R has reached the 45° position (point 2), the vertical projection of R, transferred to the 45° position of the sine-wave plot at the right, gives the height of the ordinate at this

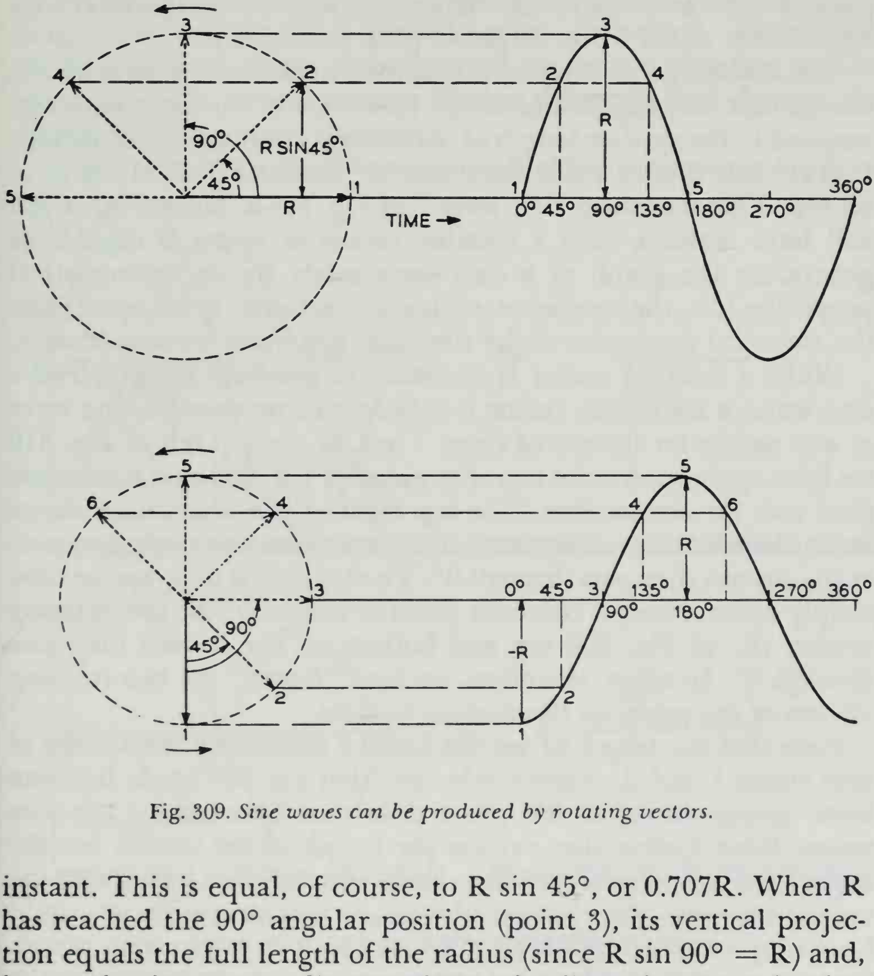


Fig. 309. *Sine waves can be produced by rotating vectors.*

instant. This is equal, of course, to R sin 45°, or 0.707R. When R has reached the 90° angular position (point 3), its vertical projection equals the full length of the radius (since R sin 90° = R) and, hence, the sine-wave ordinate at 90° (point 3) reaches its peak value equal to R.

As the radius continues to rotate, its vertical projections grow smaller again, reaching zero at 180° rotation (point 5), whereupon they rise in the negative direction. By plotting the vertical projections of the rotating radius at a sufficient number of successive instants, the smooth sine-wave trace shown at top right of Fig. 309 is obtained.

At the bottom of Fig. 309 we have again drawn a counterclockwise rotating vector, R, but of a length equal to the amplitude of sine wave 2 in Fig. 308-a. Moreover, at zero time or 0° rotation (point 1), this vector points vertically downward so that its vertical projection is a maximum in the negative direction. Accordingly, the sine wave at the right commences its cycle at 0° or (point 1) with an ordinate equal to the negative peak value (–R).

The remainder of the plot is obtained in the same way as above, except that the amplitudes of the sine wave at the right now correspond to the smaller length of the radius (or vector) R, at the left. It is evident that this sine wave lags 90° in phase behind the wave on top. This represents sine wave 2 of Fig. 308-a. Moreover, as you will have realized, such a rotating radius or vector is capable of generating the graph of a sine wave solely by its mathematical properties (i.e., the successive vertical projections), quite apart from the electrical properties of the simple ac generator we started with.

While a rotating vector is necessary to generate the graph of a sine wave, a stationary vector is sufficient to represent a sine wave at any particular instant of time. Thus, at the top left of Fig. 310 we have again drawn the two sine waves of Fig. 308-a on a common time axis for comparison. The top right of the illustration shows an equivalent vector diagram which represents the two sine waves at the instant they pass through 0°. To obtain this diagram we have simply drawn (from a common point of origin, 0) the two rotating vectors (R) of Fig. 309 top and bottom, at the instant they pass through 0°. In effect, therefore, we have "frozen" the two rotating vectors of the previous illustration in time.

Note that the length of vectors 1 and 2 equals the amplitudes of sine waves 1 and 2, respectively, and that the 90° angle between them clearly shows the 90° phase difference between the two sine waves. Note further that neither the length of the vectors nor the angle between them would have been affected if we had drawn the vectors for some other instant of time. As you can verify, we would have simply turned the entire diagram about its origin (zero point). Since in most ac problems we are interested only in comparing the amplitudes and relative phase of two or more sine waves, the instant of time chosen does not matter.

The bottom left of Fig. 310 reproduces the 180° out-of-phase sine waves of Fig. 308-b, while the bottom right shows the equivalent vector diagram at 0°. Note again that the vectors are equal in length to the sine-wave amplitudes and that they are 180° apart.

What are vectors?

We have been talking very glibly about rotating and stationary vectors without having identified them. As you may have gathered from our examples, a vector is simply a straight line of a certain length pointed in a specific direction. Such a directed line segment, or vector, is used to represent physical quantities that have both magnitude and direction (called vector quantities). The length of the line denotes the magnitude of the vector quantity, and its direction, with respect to some base or reference line, denotes the direction of the vector quantity.

There are many such quantities besides alternating currents

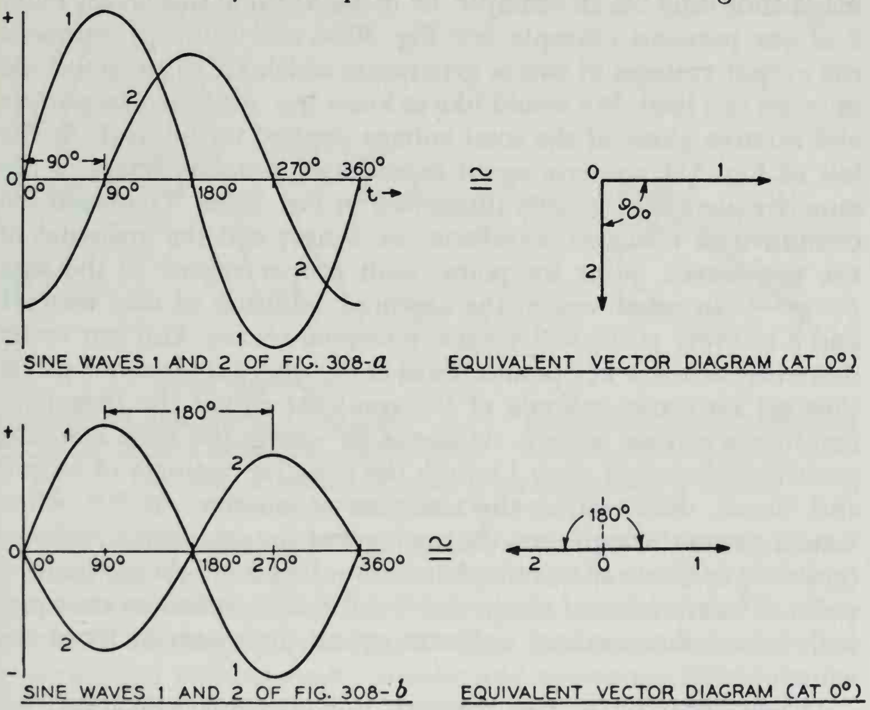


Fig. 310. *A pair of rotating vectors produces a pair of sine waves whose phase angle is the same as the phase angle of the vectors.*

and voltages. Velocity, for example, is a vector quantity, since you must specify both the speed of a vehicle (such as mph) as well as the direction in which it is going (north, east, etc.) to determine where it will be at any time. Force is another vector quantity having both magnitude and direction. If two people pull a load with the same amount of force but in opposite directions, they are not helping each other much, since the force vectors will cancel. If they pull in the same direction, however, the force vectors add, and the load is pulled easily.

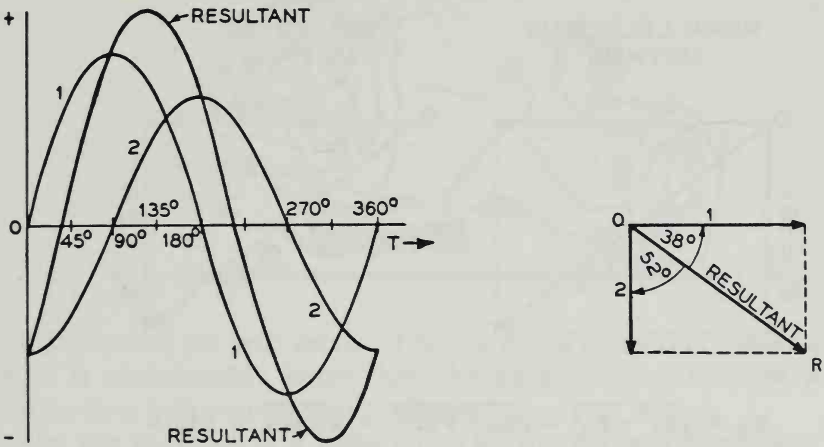
In contrast, quantities that have magnitude only are known as scalar quantities. Length, width, height, time and potential are such quantities, since a single number suffices to specify them completely. All the numerical quantities we have dealt with in arithmetic and algebra thus far have been of the scalar variety.

Vector addition

There are certain rules for combining (adding or subtracting) vectors, which differ from those for ordinary numbers that have magnitude only. As an example, let us assume that sine waves 1 and 2 of our previous example (see Fig. 308-a and 310 top) represent the output voltages of two ac generators which are to be connected in series to a load. We would like to know the combined amplitude and relative phase of the total voltage applied to the load. At the left of Fig. 311 we have again reproduced two sine waves of the same frequency, originally illustrated in Fig. 308-a. To obtain the combined or resultant waveform, we simply add the ordinates of the two waves, point for point, with proper regard to the sign (+ or –). In other words, the algebraic addition of sine waves 1 and 2 at every point will yield the resultant wave. You can verify this easily at a few key points. Thus at 0°, where sine wave 1 passes through zero, the ordinate of the resultant equals the (negative) amplitude of sine wave 2. At about 38° along the time axis, the positive ordinate of wave 1 equals the negative ordinate of wave 2 and, hence, their sum or the resultant equals zero. At 90°, when wave 2 passes through zero, the ordinate of the resultant equals the (positive) ordinate of wave 1 (that is, they intersect). At the positive point of intersection of sine waves 1 and 2 their ordinates are equal and, hence, the resultant ordinate equals their sum or twice the value of each.

If you continue in this manner, the resultant waveform will emerge. This is seen to be another sine wave, greater in amplitude than either wave 1 or 2 (but not equal to their sum), and lagging wave 1 in phase by about 38°, or equivalently leading wave 2 in phase by about 52°. (Since wave 1 leads wave 2 by 90° and the resultant lags wave 1 by 38°, it must lead wave 2 by the difference, or by $90° - 38° = 52°$.)

Now compare the cumbersome algebraic addition of the two sine waves with the extremely simple vector addition of the two waveforms shown at the right in Fig. 311. Here we have again reproduced the vector diagram of sine waves 1 and 2 from the top right of Fig. 310. To "add" the two vectors and find the resultant, we simply complete the parallelogram and draw in the diagonal 0-R. This diagonal is the resultant vector, representing the vector sum of sine waves 1 and 2. The length of the resultant 0-R represents the amplitude of the resultant sine wave, as you can verify by measuring. The direction of the resultant vector indicates the phase difference of the resultant waveform with respect to sine waves 1 and 2. By measuring the angles with a protractor, you can easily determine that the resultant vector lags behind sine wave 1 by



ADDITION OF SINE WAVES 1 AND 2 OF FIG. 308 *a* TO GIVE RESULTANT WAVEFORM

EQUIVALENT VECTOR ADDITION OF SINE WAVES 1 AND 2

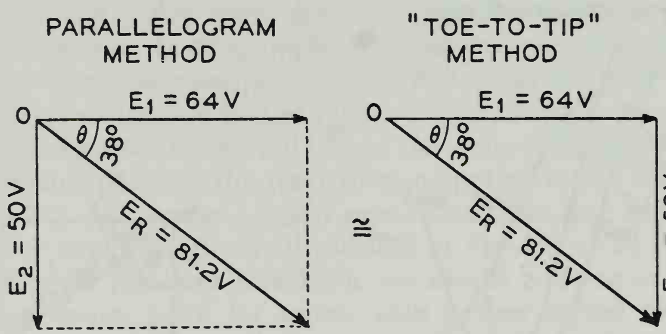
Fig. 311. *The algebraic addition of a pair of vectors produces a resultant. The resultant can be considered as a rotating vector.*

about 38° and leads sine wave 2 by 52°, as before. Moreover, since the resultant vector can be looked upon as the hypotenuse of a right triangle whose sides are made up by vectors 1 and 2, the magnitude and angle of the resultant can be determined precisely by calculation without resort to ruler and protractor. More about this presently.

The simple parallelogram method of adding two vectors can replace the laborious step-by-step method of algebraically adding two waveforms to determine the resultant. You might object that the resultant vector (at the right in Fig. 311) does not actually show the waveform of the resultant voltage (left, Fig. 311) but gives only its amplitude and relative phase. This is not really necessary, however. We know in advance, from theoretical considerations, that the resultant of sine waves with the same frequency must always be another sine wave. Hence, the vector resultant 0-R represents a sine wave, which you could easily sketch by making the amplitude equal to the length of the vector and by drawing it with the proper phase angle with respect to sine waves 1 and 2. When two or more sine waves differ in frequency, the resultant is not a sine wave and it must be obtained by the step-by-step addition of the individual waveforms. Vector addition is possible only for waves of the same frequency.

Two methods of adding vectors

Assume that the output of one of the generators in the last ex-



$$E_R = \sqrt{E_1^2 + E_2^2} = \sqrt{64^2 + 50^2} = \sqrt{6596} = 81.2\text{V}$$

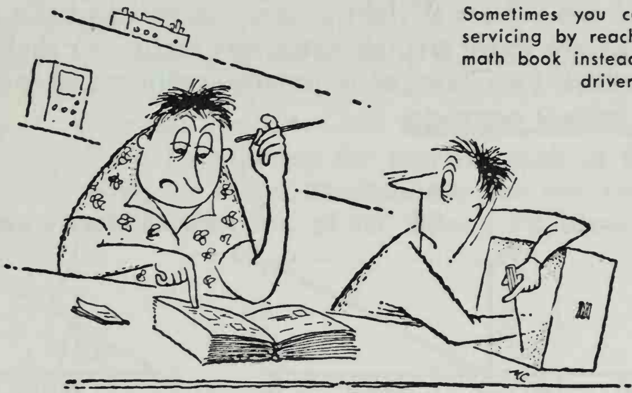
$$\text{TAN } \theta = \frac{E_2}{E_1} = \frac{50}{64} = 0.781$$

$$\therefore \theta = \text{ARC TAN } 0.781 = 38^\circ$$

Fig. 312. *Two methods of adding vectors.*

ample (Fig. 311) is sine-wave voltage $E_1 = 64$ volts, while the output sine wave of the other generator $E_2 = 50$ volts. E_2 lags E_1 by 90° in phase, as before. At the left of Fig. 312 we have again laid out the two voltage vectors (to an arbitrary scale) at right angles to each other from the common origin (0). Completing the parallelogram and drawing the diagonal, as before, you will find by measuring the length of the diagonal (to the same scale as E_1 and E_2) it represents a resultant voltage, $E_R = 81.2$ volts. With a protractor you can ascertain that E_R lags behind E_1 by 38° or, equivalently, leads E_2 by 52°.

Since opposite sides of a parallelogram are equal, it is evident that the dotted lines of the parallelogram are equal in length to vectors E_1 and E_2, respectively. We could, therefore, have obtained the resultant, E_R, quickly by laying out vector E_2 at the end of and at right angles to vector E_1, as is shown at the right of Fig. 312. The resultant, E_R, then is simply the hypotenuse of the right triangle thus formed. If the vectors are not at right angles to each other, you can still use this "toe-to-tip" method of adding vectors, and find the resultant by drawing a line from the origin (0) to the tip



Sometimes you can do faster servicing by reaching for your math book instead of a screw-driver.

of the second (or last) vector. The toe-to-tip method of vector addition is considerably faster than the parallelogram method when more than two vectors are involved.

In the example of a right triangle, you can again measure the resultant (hypotenuse) and the included angle and find that $E_R = 81.2$ volts and $\theta = 38°$, as before. However, in the case of a right triangle, it is much more precise to use the Pythagorean theorem ($c^2 = a^2 + b^2$) to find the length of the hypotenuse (or resultant), and a little elementary trigonometry to find the angle θ. Thus the length of the hypotenuse, or magnitude of the resultant,

$$E_R = \sqrt{E_1^2 + E_2^2} = \sqrt{64^2 + 50^2} = \sqrt{4{,}096 + 2{,}500}$$
$$= \sqrt{6{,}596} = 81.2 \text{ volts}$$

To find the phase angle θ, you need only realize that the ratio of the opposite side (E_2) to the adjacent side (E_1) defines the tangent of θ. Hence, $\tan \theta = \frac{E_2}{E_1} = \frac{50}{64} = 0.781$. In a table of trigonometric functions the angle (θ) corresponding to this tangent value is 38° (approximately), as before.

You could also have solved the problem by trigonometry alone, without resort to the Pythagorean theorem. First find the tangent of θ and the phase angle, θ, as above. Then, realizing that the ratio of the opposite side (E_2) to the hypotenuse (E_R) defines the sine of θ, you can write: $\sin \theta = \frac{E_2}{E_R}$, and hence, $E_R = \frac{E_2}{\sin \theta}$. Substituting,

$$E_R = \frac{50 \text{ volts}}{\sin 38^\circ} = \frac{50 \text{ volts}}{0.616 \text{ (from tables)}} = 81.2 \text{ volts.}$$

Since in most ac impedance problems you are required to find the resultant of two vectors at right angles, you will find this method of solving a right vector triangle extremely useful. We shall discuss various methods for solving ac impedance problems later on.

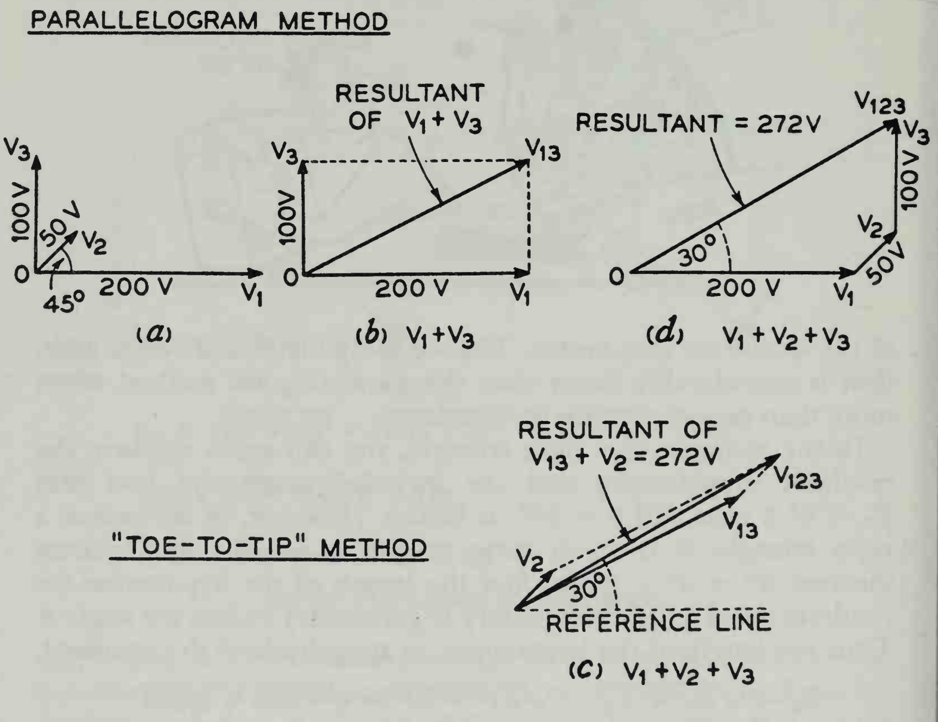


Fig. 313. *Methods of adding vectors where more than two vectors are involved.*

Adding more than two vectors

Both the parallelogram and toe-to-tip method can be used for adding more than two vectors. Using the parallelogram method, solve the problem step by step. First, obtain the resultant of any two vectors, then add this resultant vector to a third vector by completing the parallelogram, and so on. Naturally, this method can become quite cumbersome when a number of vectors are to be added. With the toe-to-tip method, in contrast, you can add a number of vectors almost in one step.

As an example, consider the vector addition of three ac voltages 45° out of phase with each other (Fig. 313). V1 = 200 volts and is represented by the horizontal vector 0–V1 in Fig. 313-a. V2, 50 volts in magnitude, is represented by vector 0–V2 and leads V1 by 45° in phase. V3 (100 volts) is represented by the vertical vector 0–V3 in Fig. 313-a. Let us find the resultant voltage acting when these three voltages are applied to an ac circuit.

First, using the parallelogram method, we find the resultant of V1 and V3, as shown in Fig. 313-b. The resultant, V13, is represented by the diagonal. Next, we add vector V2 to V13, as illustrated in Fig. 313-c. Completing the parallelogram of these two vectors and drawing the diagonal, we obtain the resultant, V123, which represents the vector sum of V1, V2 and V3. Measuring the

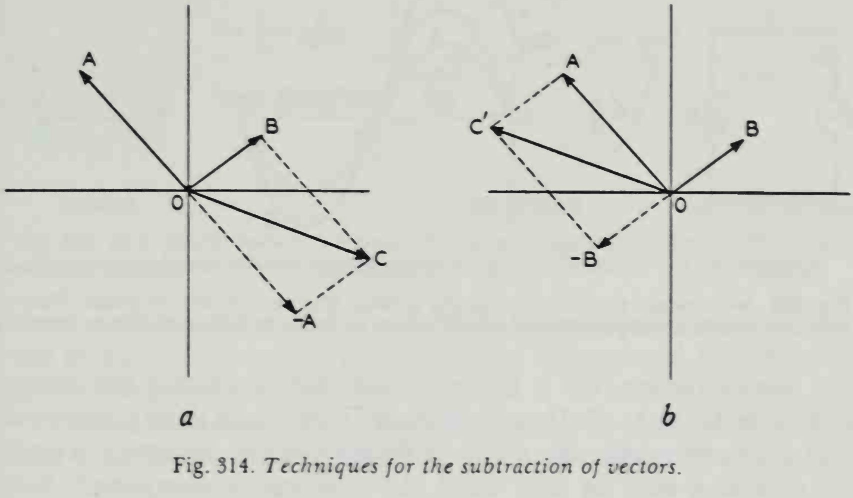


Fig. 314. *Techniques for the subtraction of vectors.*

length of this resultant to the scale of the diagram, we find the magnitude of V123 to be about 272 volts. With a protractor we determine the angle between V123 and V1 (the horizontal reference line) to be about 30°. Thus, the resultant V123 is 272 volts in magnitude and it leads V1 in phase by 30°; it also lags V2 by 45° – 30°, or 15°, and V3 by 90° – 30°, or 60°.

The toe-to-tip method of adding the vectors yields the same result much quicker, as shown in Fig. 313-d. Here we have laid off vector V2 toe-to-tip to vector V1, and vector V3 to the tip (arrow point) of V2. You have to take care, of course, that the length of the vectors and their directions are preserved when you do this. You can obtain the resultant by drawing a line from the origin (0) to the arrow point of V3. This resultant again measures 272 volts in length and leads vector V1 30° in phase.

Vector subtraction

Two vectors may be "subtracted" from each other by reversing the vector to be subtracted and then adding this reversed vector to the first. This process is very similar to algebraic subtraction, except that you have to reverse an entire line segment. For example, in Fig. 314-a, vector A is to be subtracted from vector B. To do this, simply reverse vector A and then add –A to B by the parallelogram method. The resultant vector, C, represents the vector difference, or C = B – A.

If, in contrast, vector B is to be subtracted from vector A, B is reversed and –B is added to A (see Fig. 314-b). The diagonal of the

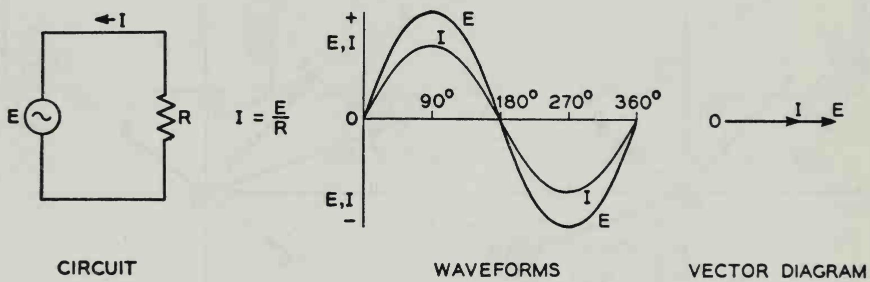


Fig. 315. *In a purely resistive circuit, the voltage and current are in phase. Note that the vectors are superimposed since there is no phase difference between them.*

new parallelogram, C′, is the resultant and represents the vector difference, C′ = A – B. If more than two vectors are to be subtracted from each other, take the vector difference of two vectors at a time and continue step by step until the problem is completed. You probably will find the toe-to-tip method of adding the reversed vectors more convenient in this case.

Vector applications: reactance and impedance calculations

We are now ready to use our knowledge of vectors in practical ac reactance and impedance calculations. But before we get into this, let us quickly review the fundamental facts of alternating-current flow in resistive, inductive and capacitive circuits.

Case 1—Pure Resistance: A resistance behaves in exactly the same way with an applied ac voltage as for dc. As shown in Fig. 315, the amount of current flow is determined by Ohm's Law and the voltage and current waveforms are in phase. The vector diagram also shows that the current, though smaller, is in line (or in phase) with the voltage. To obtain the effective (rms) values of E and I, which are the values indicated by most voltmeters and ammeters, multiply the peak values by a factor of .707.

Case 2—Pure Inductance (Fig. 316): A pure inductance doesn't exist, of course, since every coil has some winding resistance. However, the case is of interest whenever the inductance is large compared to the winding resistance, so that the latter may be neglected. The effect of a pure inductance is to make the current lag 90° behind the applied voltage and also choke down its magnitude to a value smaller than in the absence of the inductance. This property of an inductance is known as inductive reactance (X_L). The inductive reactance, $X_L = 2\pi fL = 6.283fL$ (approximately), where f is the frequency (in cycles) of the applied voltage and L is the value of the inductance in henries.

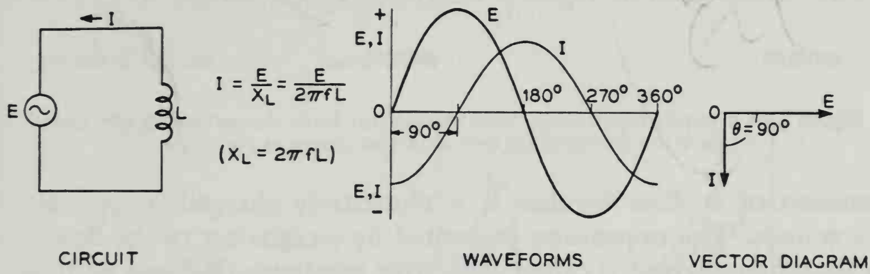


Fig. 316. *In a purely inductive circuit, the current lags the voltage by 90°. You can see the phase displacement by examining the vectors or the sine waves.*

To find the current in a pure inductance, we must modify Ohm's Law and divide the applied voltage by the (inductive) reactance rather than the resistance.

$$\text{Hence, current } I = \frac{E}{X_L} = \frac{E}{2\pi fL} = \frac{E}{6.283fL}$$

You can see in the waveforms and vector diagram of Fig. 316 that the current through an inductance coil lags the applied voltage 90° in phase

EXAMPLE: Determine the magnitude of the rms current flowing through a 5-henry choke of negligible resistance, which is connected across the 115-volt 60-cycle ac line.

Solution: $X_L = 2\pi fL = 6.283 \times 60 \times 5 = 1{,}885$ ohms

$$I = \frac{E}{X_L} = \frac{115 \text{ volts}}{1{,}885 \text{ ohms}} = .061 \text{ ampere}$$

(You could, of course, have performed the entire calculation in one step by using the formula $I = E/2\pi fL$.)

Case 3—Pure Capacitance (see Fig. 317): Though a capacitor completely blocks the flow of direct current, it permits a certain

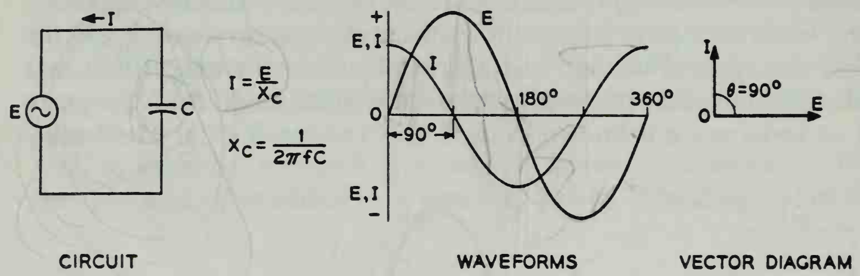


Fig. 317. *In a purely capacitive circuit, the current leads the voltage by 90°. Compare the arrangement here with that shown in Fig. 316.*

amount of ac flow because it is alternately charged in opposite directions. The opposition presented by a capacitor to the flow of alternating current is called capacitive reactance (X_c) and its magnitude is given by the formula:

$$X_c = \frac{1}{2\pi fC} = \frac{1}{6.283fC} = \frac{0.1592}{fC}$$

where C is in farads. The current in a purely capacitive circuit, therefore, is given by the ratio of the applied voltage (E) to the capacitive reactance (X_c), or $I = \frac{E}{X_c}$.

The effect of the capacitive reactance is to make the current through the capacitor lead the applied voltage by 90° in phase. This is clearly shown by the waveforms and the vector diagram of Fig. 317. To indicate the phase opposition of capacitive reactance to inductive reactance, a minus sign (–) is sometimes placed in front of the capacitive reactance value.

EXAMPLE: What is the magnitude of the current when a 220-volt 60-cycle ac voltage is applied across a 25-μf capacitor?

Solution: 25μf = 25 × 10^{-6} farads. Hence, the capacitive reactance $X_c = \frac{1}{6.283fC} = \frac{1}{6.283 \times 60 \times 25 \times 10^{-6}} = 106$ ohms

Current $I = \frac{E}{X_c} = \frac{220 \text{ volts}}{106 \text{ ohms}} = 2.075$ amperes.

NOTE: When the frequency is given in megacycles = (10^6 cycles) and the capacitance is in microfarads = (10^{-6} farad) then the factors 10^6 and 10^{-6} in the denominator cancel and, may be omitted.

IMPEDANCE: The alternating current in a pure resistance is in phase with the applied voltage, while the current in a pure inductance lags the impressed voltage by 90° and the current in a pure capacitance leads the applied voltage by 90°. When both inductance and capacitance are present, the current will either lag or lead the impressed voltage by 90°, depending upon whether the inductive or the capacitive reactance is larger in magnitude. The

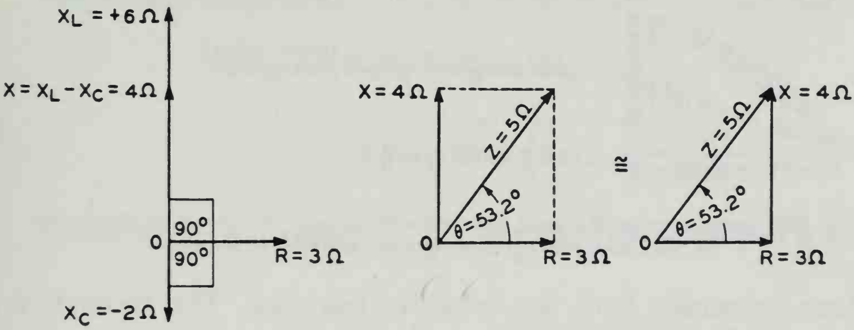


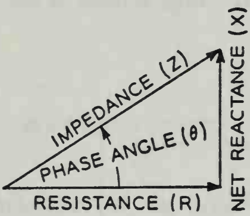
Fig. 318. *The presence of resistance, inductance and capacitance in a circuit is really a problem in vectors and can be solved as such.*

total opposition to current flow is then presented by the net reactance (X), which is equal to the vector or algebraic sum of the inductive reactance (X_L), and the capacitive reactance (X_C). Since X_L is considered positive (+) and X_C is taken as negative (–), the net reactance, X, also is simply equal to the arithmetic difference between the numerical values of X_L and X_C:

$$\text{Net Reactance } X = X_L - X_C = 2\pi fL - \frac{1}{2\pi fC}$$

When all three—resistance, inductance and capacitance—are combined in series, the total opposition to current flow, called the impedance (Z), is equal to the vector sum of the resistance and net reactance. The current, which is the same throughout the R-L-C series circuit, can either lead or lag the impressed voltage, depending upon the impedance. Moreover, the vector sum of the voltage drops across R, L and C must equal the value of the impressed voltage (emf). (The arithmetic sum of the voltage drops may well exceed the applied voltage.)

Assume, for example, that an R-L-C series circuit has a resistance of 3 ohms, an inductive reactance, X_L, of 6 ohms, and a capacitive reactance, X_C, of −2 ohms. A vector diagram of this situation is shown at the left in Fig. 318. The resistance is arbitrarily drawn as a horizontal vector with a length of 3 units (equal to 3 ohms). The inductive reactance, X_L, is a vector of 6 units in length, drawn vertically upward, since it is positive (+) and forms an angle of 90° with the resistance. The capacitive reactance, X_C, is 2 units long and drawn vertically downward, since it is negative or in



$$Z = \sqrt{R^2 + X^2} = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2}$$

$$\text{TAN } \theta = \frac{X}{R} = \frac{X_L - X_C}{R} = \frac{\left(2\pi fL - \frac{1}{2\pi fC}\right)}{R}$$

$$\text{OR } \theta = \text{ARC TAN } \frac{X}{R}$$

Fig. 319. *Impedance, or the total opposition to current flow in an ac circuit, can be calculated through the algebraic addition of vectors.*

phase opposition with the inductive reactance. The vector sum of X_L and X_C is the net reactance.

$$X = 6 + (-2) = 6 - 2 = +4 \text{ ohms},$$

which is seen to be the same as the arithmetic difference $X_L - X_C$. Since net reactance X is positive, it is drawn vertically upward and is 4 units in length (corresponding to 4 ohms).

The impedance, Z, of the circuit is the vector sum of the net reactance and resistance. It is easily obtained by completing the parallelogram of R and X and drawing in the diagonal, as shown at the right in Fig. 318.

Alternatively, you can use the toe-to-tip method, placing reactance X at the end of the resistance vector, R, and drawing the hypotenuse, Z. In either case, the magnitude of impedance vector Z turns out to be 5 units long or 5 ohms in value, while its phase angle $\theta = 53.2°$ with respect to the resistance.

The triangle (at the right in Fig. 318) is a standard configuration known as a 3–4–5 triangle. Whenever two sides of a triangle are in proportion as 3 to 4 (or 6 to 8, 15 to 20, etc.), the hypotenuse will always be in the proportion of 5 units in length (that is, 6 to 8 to 10 or 15 to 20 to 25, etc.). Another such standard triangle is in the proportion of 5 to 12 to 13.

While you could easily compute the impedance by drawing a vector diagram for any particular case, the fact that the resistance and net reactance always form a right triangle permits us to formulate a simple mathematical solution for all cases. Fig. 319 illustrates a general impedance triangle, with the resistance vector, R, forming the horizontal side; the net reactance, X, the vertical side, and the impedance, Z, the hypotenuse of the triangle. The mathematical relations are evident from the figure and, hence, we can write immediately:

$$\text{Impedance Magnitude, } Z = \sqrt{R^2 + X^2} = \sqrt{R^2 + (X_L - X_C)^2}$$

$$\text{also, since } X_L = 2\pi fL \text{ and } X_C = \frac{1}{2\pi fC}$$

$$Z = \sqrt{R^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2}$$

$$\text{Phase Angle, } \theta = \text{arc tan } \frac{X}{R} \left(\text{i.e. the angle whose tangent is } \frac{X}{R}\right)$$

$$\text{or } \tan\theta = \frac{X}{R} = \frac{X_L - X_C}{R} = \frac{\left(2\pi fL - \frac{1}{2\pi fC}\right)}{R}$$

Applying these relations to the previous example, for R = 3 ohms, X_L = 6 ohms and X_C = 2 ohms, we obtain

$$\text{Magnitude } Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{3^2 + (6-2)^2}$$
$$= \sqrt{9 + 16} = 5 \text{ ohms}$$

$$\text{and } \tan\theta = \frac{X}{R} = \frac{4}{3} = 1.333. \quad \text{Hence, from the tables, } \theta = 53°8'$$

Ohm's Law for ac

Now that we have such a convenient formula for the total opposition (Z) to the flow of alternating current, we can readily write down a modified form of Ohm's Law for ac:

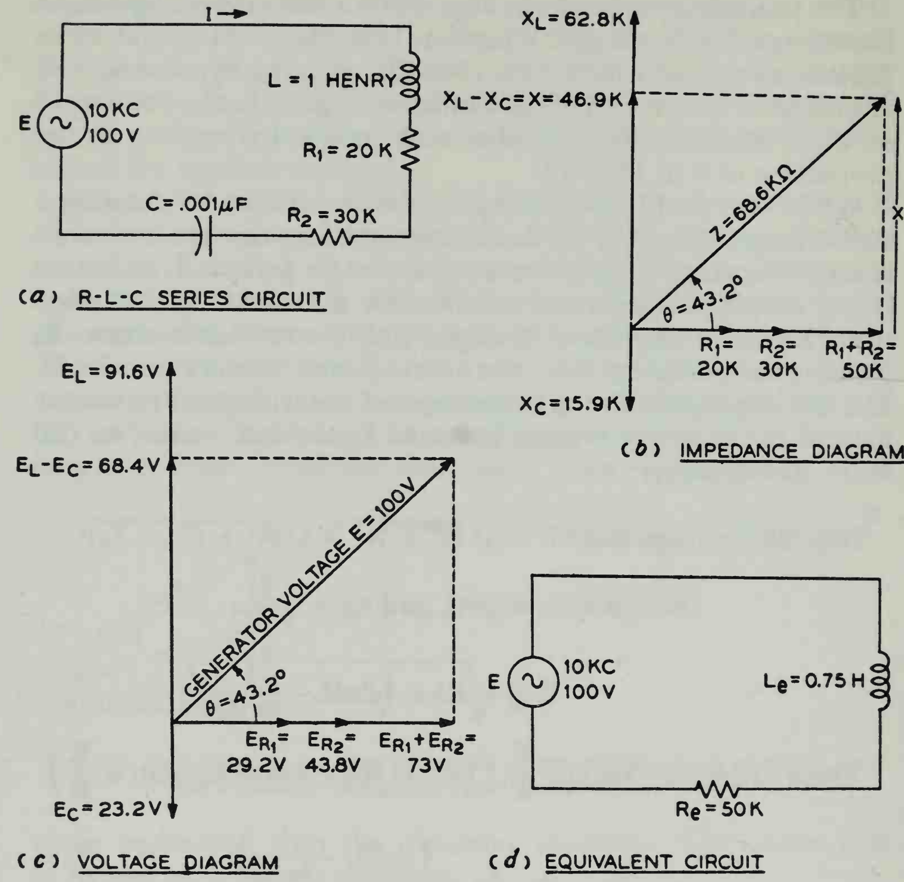


Fig. 320. *Series circuit and its equivalent impedance and voltage vector diagrams.*

$$\text{Current} = \frac{\text{Applied Voltage}}{\text{Impedance}}, \text{ or } I = \frac{E}{Z} = \frac{E}{\sqrt{R^2 + (X_L - X_c)^2}}$$

Of course, you can always find the magnitude of the impedance, Z, simply by taking the ratio of the voltage over the current, or

$$\text{magnitude } Z = \frac{E}{I}$$

Finally, the voltage drop across an impedance,

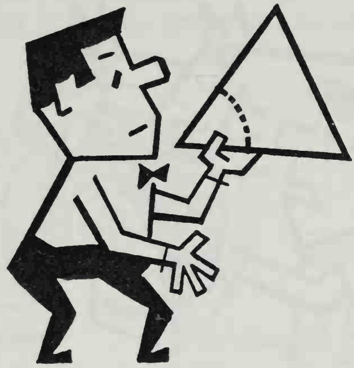
$$E = IZ = I\sqrt{R^2 + (X_L - X_c)^2}$$

The phase angle by which the current leads or lags the applied voltage is, of course, equal to the angle θ between the resistance and the impedance in the impedance triangle (Fig. 319) and, hence, is again given by:

$$\tan \theta = \frac{X}{R} = \frac{X_L - X_C}{R} \text{ or } \theta = \text{arc tan } \frac{X}{R}$$

("the angle whose tangent is" may be written either "arc tan" or "$\tan^{-1}$".)

As a check on your impedance computations, you have the fact that, in an R-L-C series circuit, the vector sum of the voltage drops



The triangle is an important figure. It shows up when we work with phase or impedance.

across the resistance (E_R), the inductance (E_L), and the capacitance (E_C) must add up to the applied voltage (E). That is,

$$E = \sqrt{E_R^2 + (E_L - E_C)^2}$$

Moreover, since the voltage drops across the circuit components are proportional to their resistance or reactance, respectively, the angle between the current and voltage is also given by

$$\text{Phase angle } \theta = \text{arc tan } \frac{E_L - E_C}{E_R}$$

By drawing a vector diagram of the separate voltage drops and taking their vector sum, you can always make this check on your calculations.

Example: In the R-L-C series circuit shown in Fig. 320-a find the impedance and phase angle, the line current and the equivalent R-C or R-L combination that will replace the circuit at a frequency of 10 kc. Check your calculations by drawing a vector diagram of the separate voltage drops across the resistors, capacitor and coil.

Solution (Fig. 320): The total series resistance,

$R = R1 + R2 = 20{,}000 \text{ ohms} + 30{,}000 \text{ ohms} = 50{,}000 \text{ ohms}.$

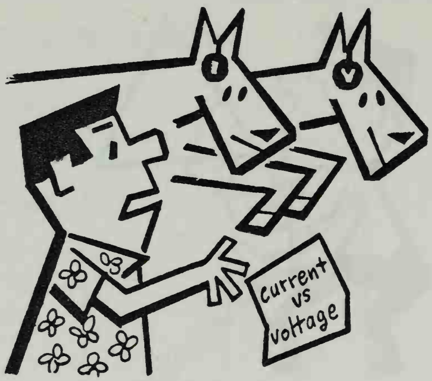
The inductive reactance of the 1-henry choke (L) at 10,000 cycles, $X_L = 2\pi fL = 6.283 \times 10{,}000 \times 1 = 62{,}800$ ohms (approximately).

The capacitive reactance of the .001-μf capacitor (C) at 10,000 cycles,

$$X_c = \frac{1}{2\pi fC} = \frac{1}{6.283 \times 10{,}000 \times .001 \times 10^{-6}}$$
$$= 15{,}900 \text{ ohms (approximately).}$$

The net reactance, X, is the difference between X_L and X_c, or

$$X = X_L - X_c = 62{,}800 \text{ ohms} - 15{,}900 \text{ ohms} = 46{,}900 \text{ ohms.}$$



Whether current or voltage leads depends on the relative amounts of inductance and capacitance in a circuit.

Since the net reactance comes out positive (+), the circuit is primarily inductive at a frequency of 10 kc. To find the impedance (Fig. 320-b), we use the formula

$$Z = \sqrt{R^2 + X^2} = \sqrt{(50{,}000)^2 + (46{,}900)^2}$$
$$= \sqrt{(25 \times 10^8) + (22 \times 10^8)}$$
$$= \sqrt{47 \times 10^8} = 6.86 \times 10^4 = 68{,}600 \text{ ohms}$$

The phase angle, $\theta = \text{arc tan } \frac{X}{R} = \text{arc tan } \frac{46{,}900}{50{,}000} = \text{arc tan } 0.938$, which turns out to be 43°10′ or 43.2° (from the tables).

Finally, the line current in the series circuit,

$$I = \frac{E}{Z} = \frac{100 \text{ volts}}{68{,}600 \text{ ohms}} = 1.46 \times 10^{-3} \text{ ampere, or } 1.46 \text{ ma.}$$

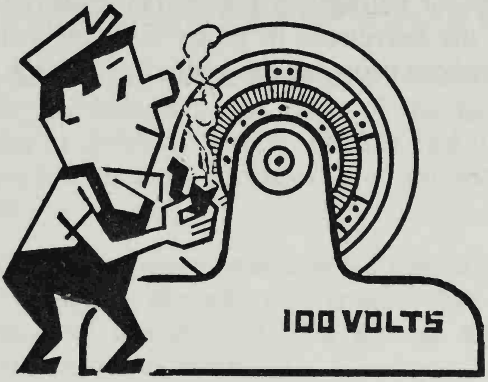
This current lags the applied voltage by 43.2° (the phase angle), but is in phase with the voltage drop across the resistors. If you measure the impedance vector, Z, and the phase angle, θ, in Fig. 320-b, you will find that the graphical results check these calculations.

Let us find the equivalent circuit at 10 kc (Fig. 320-d). We know the equivalent resistance R_e = 50,000 ohms. Since the circuit is inductive (X is +), we can use the formula $X_L = 2\pi fL$ to obtain the equivalent inductance, L_e. Hence,

$$L_e = \frac{X}{2\pi f} = \frac{46{,}900}{6.28 \times 10{,}000} = 0.75 \text{ henry (See Fig. 320-d.)}$$

Thus, a combination of a 50,000-ohm resistor and 0.75-henry choke would have the same reactance (46,900 ohms) and impedance (68,600 ohms) at a frequency of 10 kc as the actual circuit.

As a final check, let us obtain the separate voltage drops and see whether their vector sum adds up to the generator voltage (100 v).



The algebraic sum of the voltage drops in a closed network is equal to the voltage at the generator.

The voltage across R1, $E_{R1} = I\,R1 = 1.46 \times 10^{-3} \times 20{,}000 = 29.2$ volts. The drop across R2, $E_{R2} = I\,R2 = 1.46 \times 10^{-3} \times 30{,}000 = 43.8$ volts. Hence the total resistance drop $= E_{R1} + E_{R2} = 29.2 + 43.8 = 73$ volts, as is shown in the voltage vector diagram (Fig. 320-c).

The voltage drop across the coil (L),

$E_L = IX_L = 1.46 \times 10^{-3} \times 62{,}800 = 91.6$ volts. This drop leads the current by 90° and, hence, E_L is vertical (upward) in (c).

The voltage drop across the capacitor (C),

$E_C = IX_C = 1.46 \times 10^{-3} \times 15{,}900 = 23.2$ volts. This voltage drop lags behind the current by 90° and, hence, is downward. The net reactive voltage drop in the circuit is:

$E_L - E_C = 91.6 - 23.2 = 68.4$ volts. Since this voltage is positive (+), the vector is drawn vertically upward in Fig. 320-c.

Finally, the vector sum of the voltage drops,

$$\sqrt{E_R^2 + (E_L - E_C)^2} = \sqrt{(73)^2 + (68.4)^2} = \sqrt{5{,}329 + 4{,}671}$$
$$= \sqrt{10{,}000} = 100 \text{ volts}$$

which is equal to the generator voltage (E = 100 volts), as anticipated. Checking the phase angle,

$$\theta = \text{arc tan}\frac{E_L - E_C}{E_R} = \text{arc tan}\ \frac{68.4}{73} = \text{arc tan } 0.938$$

and θ turns out again to be about 43.2° (from tables), as before. Since the generator voltage (in Fig. 320-c) leads the resistive drops by 43.2°, and the current is in phase with the voltage across the resistors, the current must lag the generator voltage by 43.2°.

PRACTICE EXERCISE 5

1. Using the definitions of the trigonometric functions and Fig. 303, prove the following relations: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$; $\sin^2\theta + \cos^2\theta = 1$; $\tan\theta = \sqrt{\sec^2\theta - 1}$; $\operatorname{cosec}\theta = \sqrt{\cot^2\theta + 1}$; $\sin\theta = \cos(90 - \theta) = \sin(180 - \theta)$; $\cos\theta = \sin(90 - \theta) = -\cos(180 - \theta)$; $\tan\theta = \cot(90 - \theta) = -\tan(180 - \theta)$.

2. Using a diagram similar to Fig. 307, verify that: $\sin 30° = \cos 60° = \frac{1}{2}$; $\cos 30° = \sin 60° = \frac{1}{2}\sqrt{3}$; $\tan 30° = 1/\sqrt{3}$; $\tan 60 = \sqrt{3}$.

3. A 400-cycle sine-wave voltage has a peak value of 100. Write the expression for the instantaneous value of the voltage (e) and compute its values at .000625, .00125, .001875 and .0025 seconds after the generator is turned on. [Answer: $e = 100 \sin 800\,\pi\, t$; 100, 0, –100 and 0 volts.]

4. A 25-cycle alternating-current sine wave has an amplitude of 30 amperes. (a) Compute the effective (rms) value and (b) the instantaneous value of the current .002 second after it passes through zero in the positive direction.

[Answer: (a) 21.2 amperes; (b) 9.27 amperes.]

5. An ac voltmeter reads 385 volts (rms) drop across a load and an ammeter indicates 22 amperes load current. What are the peak values of E and I? [Answer: 545 volts, 31.1 amperes.]

6. Sine wave 1 leads sine wave 2 by 60° and lags wave 3 by 130°. Find the phase angle between waves 2 and 3, and determine which is leading. [Answer: Sine wave 2 leads wave 3 by 170°.]

7. Vector X is 200 units long and at right angles with vector Y, which is 150 units long. (a) Find the vector sum X + Y and the angle between the resultant and Y. (b) Find the vector difference X – Y and Y – X, and compare the magnitude of the resultants with that in (a). [Answer: (a) 250 units, 53.2°; (b) same length.]

8. A vector is 13 units long and forms an angle of 22.6° with the horizontal. Resolve this vector into two rectangular component vectors, whose vector resultant will equal the original vector. (Hint: the vertical component is the resultant times $\sin\theta$, and the horizontal component is the resultant times cosine θ.)

[Answer: 12 units horizontal; 5 units vertical.]

9. A 20-volt signal from a 5-mc oscillator is applied to a 100-$\mu\mu$f capacitor. Find the capacitive reactance and the current.

[Answer: X_c = 318 ohms; I = 63 ma.]

10. A 60-ohm resistor, a 33.2-μf capacitor and a 0.53-henry coil with a 30-ohm winding resistance are connected in series across a 300 volt, 60-cycle ac source. Find the inductive and capacitive reactances, the impedance, the line current, and the phase angle by which the current leads or lags the applied voltage.

[Answer: X_L = 200 ohms; X_c = 80 ohms; Z = 150 ohms; I = 2 amps; I lags E by 53.2°.]

11. A circuit contains inductive and capacitive reactances of 50 ohms each and a resistive component of 25 ohms. However, the vector diagram shows the resistive component only. Is this correct?

[Answer: The vector diagram is correct. The algebraic sum of the reactances is zero, hence need not be drawn. The circuit is in a condition of resonance.]

12. If the length of the resistance vector is 30 units, the reactive vector 40 units, what is the length of the impedance vector?

[Answer: 50 units.]

CHAPTER 4

Complex Numbers

THE man who first put down $\sqrt{-1}$ was undoubtedly shocked by his own audacity. For what was the meaning of this strange-looking quantity? What number multiplied by itself could possibly yield -1? As he kept experimenting, this adventurer discovered a whole new world of such numbers, $\sqrt{-4}$, $\sqrt{-12.59}$, $\sqrt{-x}$, $\sqrt{-5y}$, etc., which all looked equally mysterious. Since he couldn't find any numbers whose square would come out negative, he called them imaginary numbers and he gave the symbol i to $\sqrt{-1}$. Thus, by factoring $\sqrt{-1}$ from his quantities, he obtained $\sqrt{4} \times \sqrt{-1} = 2i$, $\sqrt{12.59}\sqrt{-1} = 3.55i$, $\sqrt{x}\sqrt{-1} = i\sqrt{x}$, and $\sqrt{5y} \cdot \sqrt{-1} = 2.24\sqrt{y}\,i$. The letter i looked much better than $\sqrt{-1}$ and it sort of covered up the whole unthinkable mess. Moreover, as he kept fooling with these quantities, he found out that there was nothing imaginary about them. On the contrary, they rounded out the whole number system and could be given various interesting interpretations. As the utility of the so-called "imaginaries" became obvious in engineering and electrical calculations, the symbol j came into use for the imaginary quantity ($\sqrt{-1}$), to avoid confusion with the current symbol i. Thus, in all electrical work the imaginary number $\sqrt{-1}$ is known as the j-operator and, as we shall see, it comes in very handy for the solution of alternating-current problems.

J-operator rotates a vector through 90°

One interesting interpretation of the j-operator is that it rotates any quantity multiplied by it by a counterclockwise angle of 90° (i.e., a right angle). Consider a vector, $+a$, laid out to the right of the origin (0) along the horizontal axis, in accordance with the usual rectangular notation (Fig. 401). If you multiply a by -1, the result, -a, will be a vector of length a, extending to the left of the origin, as shown in the figure.

You might say, geometrically speaking, that multiplying a vector by -1 has rotated it through 180°, or two right angles. Multiplying -a by -1 once more results again in +a, or the original vector. Hence, multiplying twice by -1 results in rotating a vector through

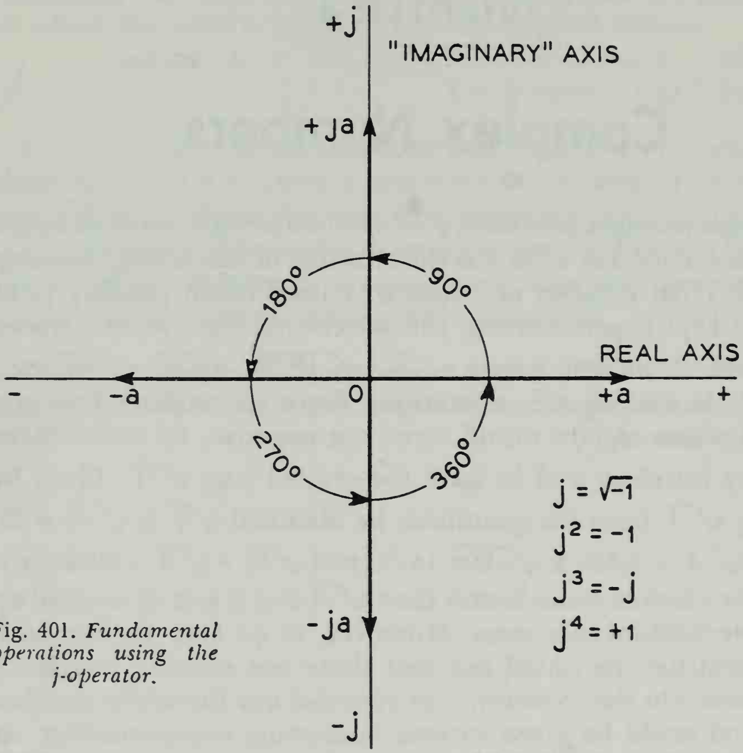


Fig. 401. *Fundamental operations using the j-operator.*

360°, or four right angles. Now, multiplying by a factor of -1 is equivalent to multiplying by j^2, since $j^2 = j \cdot j = \sqrt{-1} \cdot \sqrt{-1} = -1$, and multiplying twice by -1 is equivalent to multiplying by j^4, since $j^4 = j^2 \cdot j^2 = (-1)(-1) = +1$.

Thus, multiplying *twice* by j, or by $j^2 = -1$, results in rotation by two right angles or 180°, while multiplying four times by j, or by $j^4 = +1$, results in rotation by four right angles, or 360°. Consistent with this geometrical interpretation, it follows without further ado that multiplying a quantity once by j or $\sqrt{-1}$ is equivalent to rotating it by one right angle, or 90°, and multiplying a quantity three times by j, or $j^3 = (\sqrt{-1})(\sqrt{-1})(\sqrt{-1}) = -j$, is equivalent to rotating it by three right angles or 270°.

In Fig. 401, the vector +ja is vertically upward, rotated 90° counterclockwise from its original (horizontal) position, and the vector $j^3a = -ja$ is shown vertically downward and rotated 270° counterclockwise from its original position.

In other words, multiplying a vector four times in succession by the j-operator rotates it through four right angles, or 360°. Thus,

original vector, +a	(0° rotation)
multiplying by j: $j(+a) = +ja$	(vertically up, 90° rotation)
multiplying again by j: $j(ja) = j^2a = -a$	(horizontal, 180° rotation)
and again: $j(j^2a) = j^3a = -ja$	(vertically down, 270° rotation)
and for the fourth time: $j(j^3a) = j^4a = +a$	(360° rotation)

The horizontal axis in Fig. 401, along which the real quantities +a and –a are located, is called the *axis of reals,* while the vertical axis, along which the "imaginary" quantities +ja and –ja are located, is known as the *axis of imaginaries.* You can see that any real or imaginary quantity can easily be located along one of these two rectangular axes.

What are "complex numbers"?

Let us extend this graphical representation of imaginary numbers a little further. The real and imaginary axes, apparently, define a plane in which any point has a specific numerical designation, just as in the conventional rectangular coordinate plane. Let us designate the horizontal (real) axis as the R or resistance axis

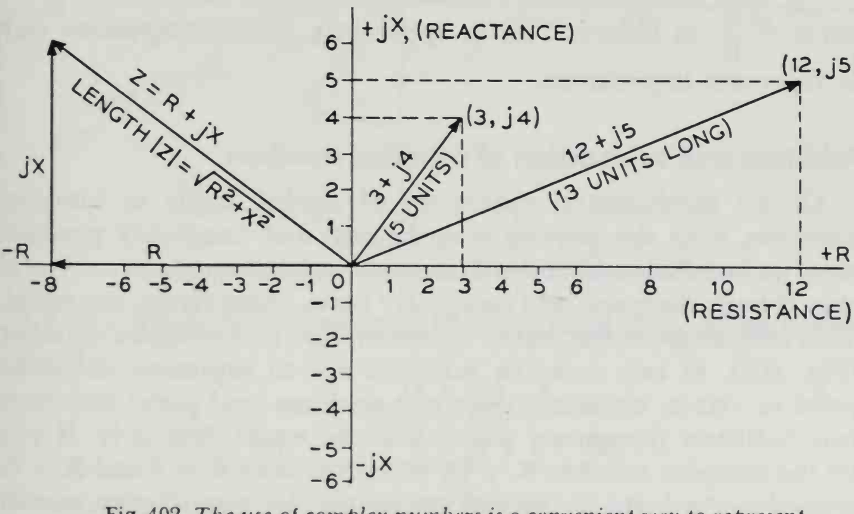


Fig. 402. *The use of complex numbers is a convenient way to represent impedances.*

(Fig. 402) and the vertical (imaginary) axis as the jX or reactance axis. (We could have equally well chosen an X-axis and jY-axis but, since we are going to deal with impedance later on, the R and jX axes will be more useful.) Thus, any point in the R-jX plane will have a horizontal or real abscissa along R and a vertical or imaginary ordinate along jX. A line drawn to such a point will represent a vector R + jX, where R and X may represent any values at all.

The quantity R + jX is called a complex number, though there is nothing complex about it. It is simply a rectangular vector presentation. Thus, the complex number 3 + j4 in Fig. 402 represents a vector that is made up of a horizontal (real) component of 3 ohms resistance and a vertical (imaginary) component of 4 ohms reactance. The line drawn to the point 3 + j4 is 5 units long, since it is the hypotenuse of a triangle with sides 3 and 4 units long ($\sqrt{3^2 + 4^2} = \sqrt{25} = 5$). Similarly, a line drawn to the point 12 + j5 represents a vector made up of a horizontal (resistance) component of 12 ohms and a vertical (reactance) component of 5 ohms. This vector is $\sqrt{12^2 + 5^2} = \sqrt{144 + 25} = \sqrt{169} = 13$ units in length.

In general, a line drawn to any complex number (or point) R + jX is a vector that is $\sqrt{R^2 + X^2}$ units in length, and hence, can represent an impedance, Z, in accordance with our previous definition. Moreover, the angle θ which the impedance vector makes with the resistance (real or horizontal) axis is given by $\tan \theta = \frac{X}{R}$, as before. This is, apparently, a very convenient way to represent impedances.

Addition and subtraction of complex numbers

All the fundamental operations of algebra apply to complex numbers, with the proviso that the real and imaginary portions must be handled separately. Thus two complex numbers are equal only if both their real and imaginary parts, respectively, are equal. This follows from the vector representation of a complex number (Fig. 402). If two complex numbers are to represent the same point or vector, obviously their two abscissas (real parts) and their two ordinates (imaginary parts) must be equal. Similarly, if you set the complex number R + jX = 0, then both R = 0 and X = 0, since the zero point (or vector) can neither have an abscissa nor an ordinate.

To add or subtract complex numbers, simply add or subtract the real and imaginary parts separately. For example, adding the two complex numbers graphed in Fig. 402,

$$\begin{array}{r} (3 + j4) \\ +(12 + j5) \\ \hline 15 + j9 \end{array}$$

Fig. 403 illustrates the vector addition of the two numbers obtained by completing the parallelogram. The result, 15 + j9, is a vector made up of a horizontal (real) component, 15 units long and a vertical (imaginary) component 9 units long. If we subtract (3 + j4) from (12 + j5), we obtain algebraically

$$\begin{array}{r} (12 + j5) \\ -(3 + j4) \\ \hline 9 + j1 \end{array}$$

The vector subtraction (in Fig. 403), obtained by adding the reversed (negative) vector –(3 + j4) to (12 + j5) yields the same result

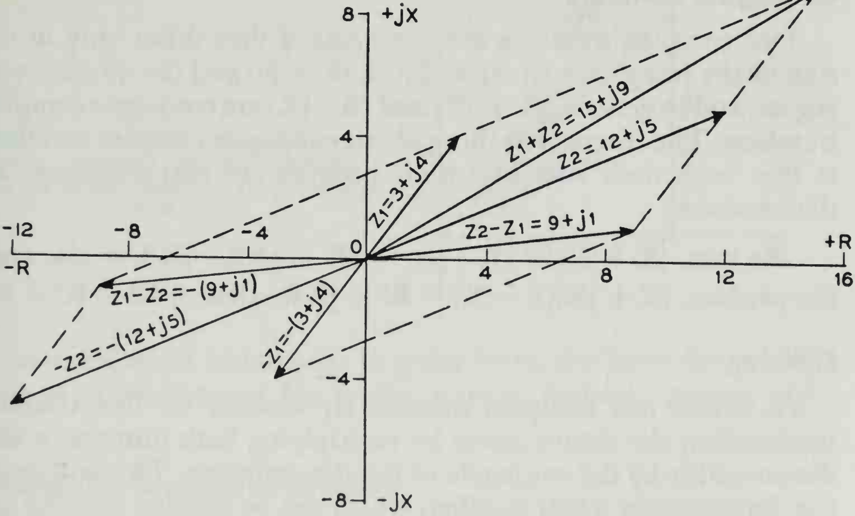


Fig. 403. *This diagram illustrates vector addition and subtraction.*

(9 + j1). Similarly, subtracting (12 + j5) from (3 + j4), we obtain –9 – j1.

$$\begin{array}{r} (3 + j4) \\ -(12 + j5) \\ \hline -9 - j1 \end{array}$$

The vector subtraction, shown in Fig. 403, again leads to exactly the same result.

Thus, we can state in general, for two complex numbers (or impedances), $Z1 = R1 + jX1$ and $Z2 = R2 + jX2$,
the sum, $Z1 + Z2 = (R1 + R2) + j\,(X1 + X2)$, and the difference, $Z1 - Z2 = (R1 - R2) + j\,(X1 - X2)$.

Multiplication and division of complex numbers

You can multiply complex numbers in exactly the same way as algebraic polynomials, except that you replace j^2 by -1, j^3 by $-j$ and j^4 by $+1$, when it occurs in the final result. Thus, to multiply $Z1 = R1 + jX1$, by $Z2 = R2 + jX2$:

$$Z1 \times Z2 = (R1 + jX1)\,(R2 + jX2) = R1R2 + jX1R2 + jR1X2 + j^2X1X2 = (R1R2 - X1X2) + j(R1X2 + X1R2)$$

Example: Multiply $(5 + j3)$ by $(2 - j2)$

$$(5 + j3)\,(2 - j2) = 10 + j6 - j10 - j^26 = 10 - j4 - (-1)(6) = 10 - j4 + 6 = 16 - j4$$

Conjugate numbers

Two complex numbers are *conjugate* if they differ only in the sign of the imaginary (j) term. Thus, $(5 + j9)$ and $(5 - j9)$ are conjugate, and in general $(R + jX)$ and $(R - jX)$ are conjugate complex numbers. The interesting thing about conjugate complex numbers is that both their sum and their product are real numbers. To demonstrate,

the sum, $(R + jX) + (R - jX) = 2R + jXR - jXR = 2R$, and
the product, $(R + jX)(R - jX) = R^2 + jXR - jXR - j^2X^2 = R^2 + X^2$

Division

To divide one complex number by another we first simplify (rationalize) the denominator by multiplying both numerator and denominator by the conjugate of the denominator. This will make the denominator a *real* number, which can be divided into the numerator. Thus, if $Z1 = R1 + jX1$ and $Z2 = R2 + jX2$, the quotient

$$\text{of } \frac{Z1}{Z2} = \frac{R1 + jX1}{R2 + jX2} = \frac{(R1 + jX1)\,(R2 - jX2)}{(R2 + jX2)\,(R2 - jX2)}$$

$$= \frac{R1R2 + jX1R2 - jR1X2 - j^2X1X2}{R2^2 - j^2X2^2}$$

$$= \frac{(R1R2 + X1X2) + j(X1R2 - R1X2)}{R2^2 + X2^2}$$

$$= \frac{R1R2 + X1X2}{R2^2 + X2^2} + j\frac{X1R2 - R1X2}{R2^2 + X2^2}$$

Example: Divide (5 + j3) by (2 – j2)

$$\frac{5 + j3}{2 - j2} = \frac{(5 + j3)(2 + j2)}{(2 - j2)(2 + j2)} = \frac{10 + j6 + j10 + j^2 6}{4 - j^2 4} = \frac{10 + j16 - 6}{4 + 4}$$

$$= \frac{4 + j16}{8} = \frac{1 + j4}{2} = 0.5 + j2$$

Polar form of complex numbers

There are two coordinate systems—the rectangular and the polar—for specifying points in a plane. In the polar coordinate system, the location of a point is specified by the length of a line (called radius vector) drawn from the origin to the point and by its inclination (angle θ) with respect to a usually horizontal reference line. (See Fig. 404.) Complex numbers and their associated vectors may

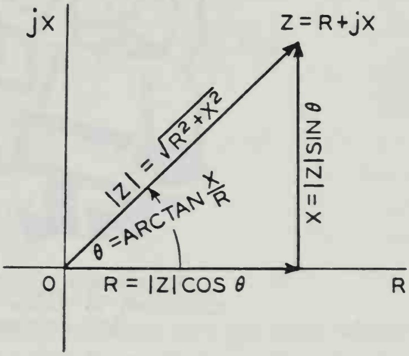


Fig. 404. Complex numbers can be stated in either polar or rectangular form.

be stated either in rectangular or polar form, the latter frequently being more convenient for multiplication, division, powers and roots.

Rectangular-to-polar conversion

The magnitude (i.e., the length of the vector) of a complex quantity, $Z = R + jX$, is given by the Pythagorean theorem. The magnitude of Z is usually symbolized $|Z|$ to distinguish it from the vector, Z, which has both magnitude and direction. Thus,

$$|Z| = \sqrt{R^2 + X^2}$$

The angle between Z and the horizontal (real) axis,

$$\theta = \arctan\frac{X}{R} \text{ or } \tan\theta = \frac{X}{R}\text{, as shown in Fig. 404.}$$

Using these relations you can express any complex number $Z = R + jX$ in polar form, $|Z|\underline{/\theta}$.

EXAMPLE: Express $12 + j5$ in polar form

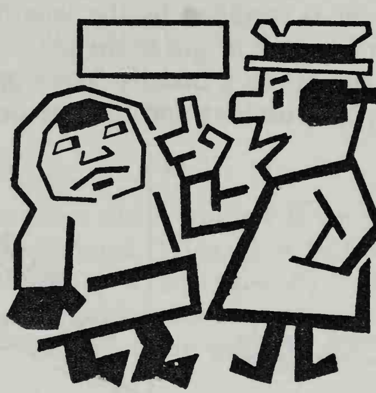
$$|Z| = \sqrt{12^2 + 5^2} = \sqrt{144 + 25} = \sqrt{169} = 13$$

$$\theta = \arctan \frac{5}{12} = \arctan .416 = 22.63° \text{ (from the tables).}$$

Hence, $12 + j5 = 13\underline{/22.63°}$

Polar-to-rectangular conversion

Fig. 404 also illustrates the relations for converting a complex number from polar form to rectangular form. The horizontal or



Sometimes it is necessary and convenient to change from polar to rectangular coordinates.

real component, R, of the vector Z is clearly the horizontal projection, or

$$R = |Z| \cos \theta$$

The vertical or imaginary component, X, is, of course, the vertical projection, or $X = |Z| \sin \theta$. Hence, the polar form

$$|Z|\underline{/\theta} = |Z| \cos \theta + j\,|Z| \sin \theta \quad \text{(in rectangular form).}$$

EXAMPLE: Convert $35\underline{/40°}$ to rectangular form.

$$35\underline{/40°} = 35 \cos 40° + j35 \sin 40°$$
$$= 35\,(.7660) + j35(.6428) = 26.81 + j22.50$$

Addition and subtraction in polar form

Though we have written down this heading nonchalantly, it just isn't possible. To add or subtract complex numbers given in polar form, you have to convert first to rectangular form, then add or subtract; you may convert the result back to polar form, if you wish. This is, obviously, inconvenient, but as an example let's add $35\underline{/40^\circ}$ and $47\underline{/55^\circ}$. We have seen above that $35\underline{/40^\circ} = 26.81 + j22.50$. Converting $47\underline{/55^\circ} = 47\cos 55^\circ + j47\sin 55^\circ = 47(.5736) + j47(.8192) = 26.96 + j38.50$. Adding $(26.81 + j22.50) + (26.96 + j38.50)$, we obtain the result $53.77 + j61.0$. You may let it go at that, but if you wish to convert back to polar form, recall that

$$\tan\theta = \frac{X}{R} = \frac{61.0}{53.77} = 1.134. \text{ Hence, } \theta = \arctan 1.134 = 48.6^\circ$$

(from the tables). You can get the magnitude by the usual relation,

$$|Z| = \sqrt{R^2 + X^2} = \sqrt{(53.77)^2 + (61)^2} = \sqrt{2890 + 3721} \doteq \sqrt{6611} = 81.4,$$

which is somewhat complicated because of the squares and square roots. It is much easier to get the magnitude from the relations

$$|Z| = \frac{R}{\cos\theta} \quad \text{or } |Z| = \frac{X}{\sin\theta} \text{ (see Fig. 404).}$$

$$\text{Hence, } |Z| = \frac{53.77}{\cos 48.6^\circ} = \frac{53.77}{.6613} = 81.4$$

$$\text{or } |Z| = \frac{61}{\sin 48.6^\circ} = \frac{61}{.75} = 81.4$$

Thus, the final result of the addition is $81.4\underline{/48.6^\circ}$.

You may realize by now the complications you get into when you try to add complex numbers in polar form. Subtraction is done in the same manner, but we won't bother to give an example of this rather inconvenient method.

Adding and subtracting complex numbers in polar form can sometimes result in mental strain.

Multiplication and division in polar form

The polar form of complex numbers really comes into its own, when you have to multiply or divide by a number of them. To multiply complex numbers (or vectors) in polar form, simply multiply their magnitudes (absolute values) and add their angles. Thus,

$$(|Z1|\angle\theta 1) \times (|Z2|\angle\theta 2) = |Z1| \times |Z2|\angle\theta 1 + \theta 2$$

You can verify this statement by converting first to the rectangular form, multiplying the quantities together, and then converting back to polar form; you will find this an interesting exercise in trigonometry.

EXAMPLE: Multiply $47\angle 55°$ by $55\angle 40°$

$$47\angle 55° \times 55\angle 40° = 47 \times 55\angle 55°+40° = 2{,}585\angle 95°$$

Division

To divide one complex number by another, divide their magnitudes and subtract the angles. (The numbers must be in polar form.) Hence, $Z1\angle\theta 1 \div Z2\angle\theta 2 = \frac{Z1}{Z2}\angle\theta - \theta 2$.

You can verify this rule in the same way, by converting to rectangular form.

EXAMPLE: Divide $47\angle 55°$ by $55\angle 40°$

$$47\angle 55° \div 55\angle 40° = \frac{47}{55}\angle 55°-40° = 0.855\angle 15°$$

Powers and roots

Powers and roots of complex numbers are also easily obtained, when they are in polar form. There is an interesting theorem, known as *De Moivre's theorem,* which we won't bother to prove. It states

$$[r(\cos\theta + j\sin\theta)]^n = r^n(\cos n\theta + j\sin n\theta),$$

where n may be an integer or a fraction, positive or negative. This theorem, therefore, applies to powers (integral exponents) as well as to roots (fractional exponents). Accordingly, we can formulate the rule: to raise a complex number (in polar form) to a power, raise its magnitude to the desired power and multiply the angle by the exponent. Thus, $(Z\angle\theta)^n = Z^n(\cos n\theta + j\sin n\theta) = Z^n\angle n\theta$

EXAMPLE: $(15\angle 20°)^3 = 15^3\angle 3\times 20° = 3{,}375\angle 60°$

Roots are extracted in similar fashion: to extract a root of complex numbers, extract the required root of the magnitude and divide the angle by the index of the root. (Square root has index 2, etc:) Hence, $(Z \underline{/\theta})^{1/n} = Z^{1/n} \underline{/\frac{\theta}{n}}$

EXAMPLE: $\sqrt{10\underline{/50^\circ}} = \sqrt{10}\ \underline{/\frac{50^\circ}{2}} = 3.16\underline{/25^\circ}$

Let's examine the result and see what it looks like graphically:

$$(Z \underline{/\theta})^n = Z^n (\cos n\theta + j \sin \theta)$$

then

$$3.16 \underline{/25^\circ} = 3.16 (\cos 25^\circ + j \sin 25^\circ)$$

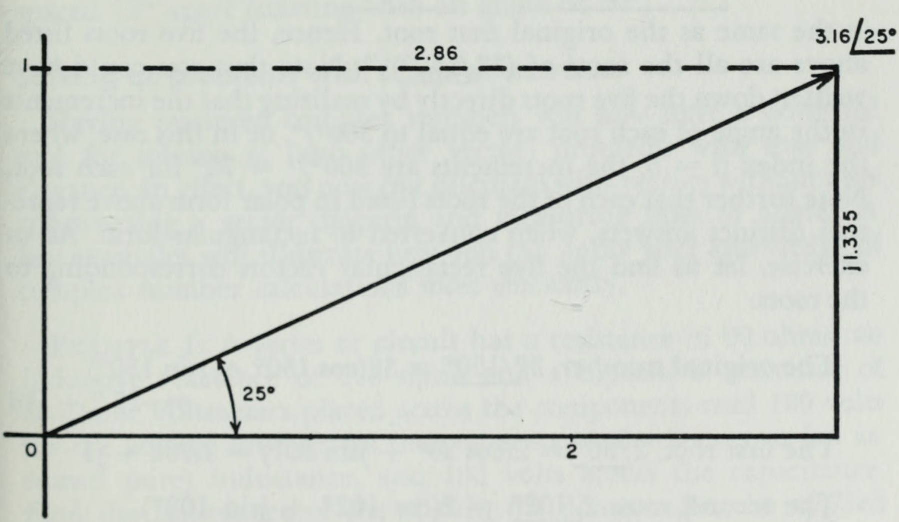
from tables of natural sines and cosines we obtain:

$$\sin 25^\circ = .4226$$
$$\cos 25^\circ = .9063$$

thus we have

$$3.16 (.9063 + j\,.4226) = 2.86 + j\,1.335$$

and we can plot this graphically as shown below:



You might well ask, what happens to the other roots, since a square root has two roots, a cube root has three roots, etc. If you want these roots, simply add 360° to the original angle (which doesn't change it) and divide again by the index (n) of the root. After you have done this (n–1) times, you will have all the roots, and the answers will start to repeat themselves. To illustrate, let's say you want to find the 5 fifth roots of $32\angle 150°$.

To find the first root: $(32\angle 150°)^{1/5} = (32)^{1/5} \angle \frac{150°}{5} = 2\angle 30°$

The second root $= (32)^{1/5} \angle \frac{150° + 360°}{5} = 2\angle \frac{510°}{5} = 2\angle 102°$

The third root $= 2\angle \frac{510° + 360°}{5} = 2\angle \frac{870°}{5} = 2\angle 174°$

The fourth root $= 2\angle \frac{870° + 360°}{5} = 2\angle \frac{1230°}{5} = 2\angle 246°$

The fifth root $= 2\angle \frac{1230° + 360°}{5} = 2\angle \frac{1590°}{5} = 2\angle 318°$

If you attempted to find a sixth root, it would be

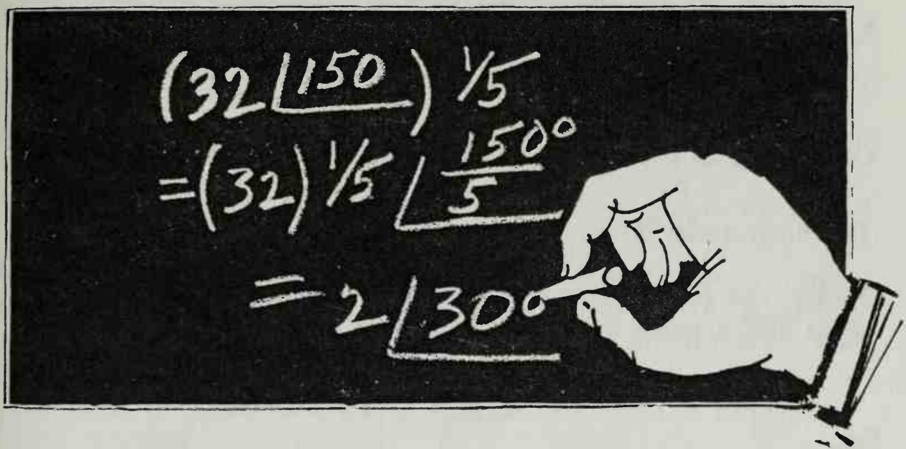
$2\angle \frac{1590° + 360°}{5} = 2\angle \frac{1950°}{5} = 2\angle 390° = 2\angle 30°$, which

is the same as the original first root. Hence, the five roots listed above are all the roots of $(32\angle 150°)^{1/5}$. Note that you could have written down the five roots directly by realizing that the increments in the angle of each root are equal to $360°/^{n}$, or in this case, where the index n = 5, the increments are $360°/^{5} = 72°$ for each root. Note further that each of the roots listed in polar form above represent distinct answers, when converted to rectangular form. As an exercise, let us find the five rectangular vectors corresponding to the roots.

The original number, $32\angle 150° = 32(\cos 150° + j\sin 150°)$
$= -27.7 + j16$

The first root, $2\angle 30° = 2(\cos 30° + j\sin 30°) = 1.732 + j1$

The second root, $2\angle 102° = 2(\cos 102° + j\sin 102°)$
$= -.4158 + j1.956$



Finding a root is the inverse of raising a number to a power.

The third root, $2\underline{/174^\circ} = 2(\cos 174^\circ + j\sin 174^\circ)$
$= -1.989 + j0.209$

The fourth root, $2\underline{/246^\circ} = 2(\cos 246^\circ + j\sin 246^\circ)$
$= -0.8134 - j1.827$

The fifth root, $2\underline{/318^\circ} = 2(\cos 318^\circ + j\sin 318^\circ)$
$= 1.486 - j1.338$

Fig. 405 is a vector representation of the five roots of $32\underline{/150^\circ}$ Note that all five roots lie in a circle of radius 2 and each are spaced 72° apart (starting with an angle of 30°).

Solving ac problems with complex numbers

Having mastered complex numbers you now have a powerful tool for solving ac (and other vector) problems with ease and elegance. In effect, you now can calculate with vectors without ever constructing a vector diagram and measuring lines or angles. A few examples will illustrate how you can apply your knowledge of complex number calculations most efficiently.

EXAMPLE 1: A series ac circuit has a resistance of 90 ohms, an inductive reactance of 200 ohms and a capacitive reactance of 80 ohms. Voltmeters placed across the components read 180 volts for the voltage drop across the resistance, 400 volts across (an assumed pure) inductance, and 160 volts across the capacitance. Find the impedance of the circuit, the phase angle, the applied voltage (E), and the line (series) current.

Solution (see Fig. 406): Since R = 90 ohms, X_L = 200 ohms and X_C = 80 ohms, the impedance,

$$Z = R + j(X_L - X_C) = 90 + j(200-80) = 90 + j120 \text{ ohms}$$

Converting to polar form, $\tan\theta = \frac{120}{90} = 1.33$; hence $\theta = 53.2°$.

The magnitude, $|Z| = \frac{R}{\cos\theta} = \frac{90}{\cos 53.2°} = \frac{90}{0.6} = 150$ ohms.

Hence, the impedance, $Z = 150\underline{/53.2°}$, has a magnitude of 150 ohms and a phase angle of 53.2°. (See Fig. 406-a.) Since the net

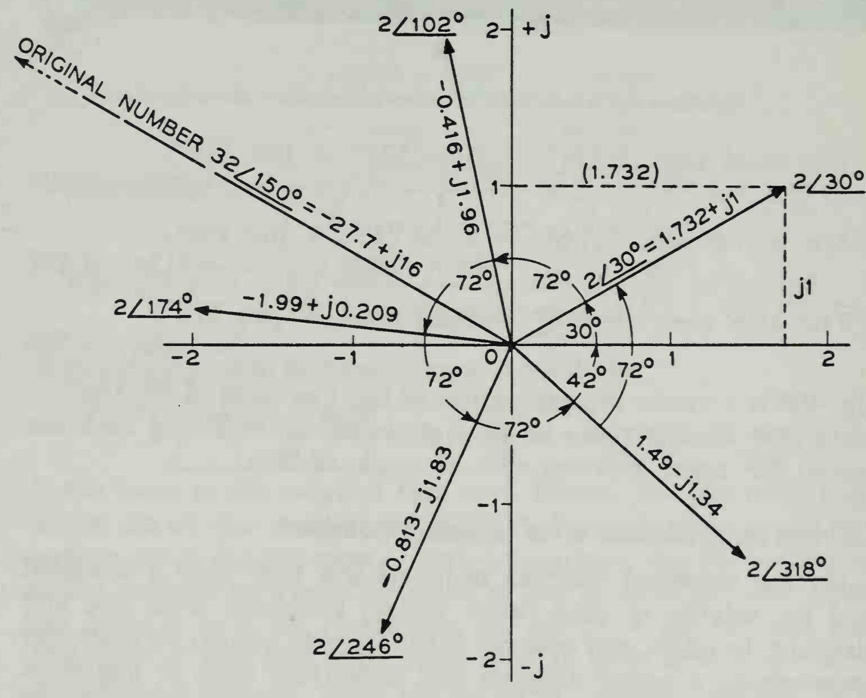


Fig. 405. *Vector representation of the five roots of* $32\underline{/150°}$.

reactance (X) is positive, the circuit is inductive, and the current lags the voltage by 53.2°.

The applied voltage in a series circuit must equal the vector sum of the voltage drops. Here, V_R = 180 V, V_L = 400 V and V_C = 160 V. Hence, the applied voltage (or emf),

$$E = V_R + j(V_L - V_C) = 180 + j(400-160) = 180 + j240 \text{ volts.}$$

Again, converting to polar form, $\tan\theta = \frac{240}{180} = 1.33$, and $\theta =$ 53.2°, as before. The magnitude of $|E| = \sqrt{(180)^2 + (240)^2} = \sqrt{90{,}000} = 300$ volts, or equivalently, $|E| = \frac{V_R}{\cos\theta} = \frac{180}{\cos 53.2°}$

$$= \frac{180}{0.6} = 300 \text{ volts.}$$

Hence, the applied voltage, $E = 300\underline{/53.2°}$ volts (Answer), where the positive angle indicates that the applied voltage (E) leads the voltage drop across R (V_R) by 53.2°. (See Fig. 406-b.) Since the current is in phase with the voltage across the resistance, this is equivalent to the statement that E leads I by 53.2° or I lags E by 53.2°, as before.

Finally, the line current, $I = \frac{E}{Z} = \frac{300\underline{/53.2°}}{150\underline{/53.2°}} = 2\underline{/0°}$ amperes.

Thus, the line current is 2 amperes and its angle is 0°. Since the voltage has a positive angle of +53.2°, this again means that E

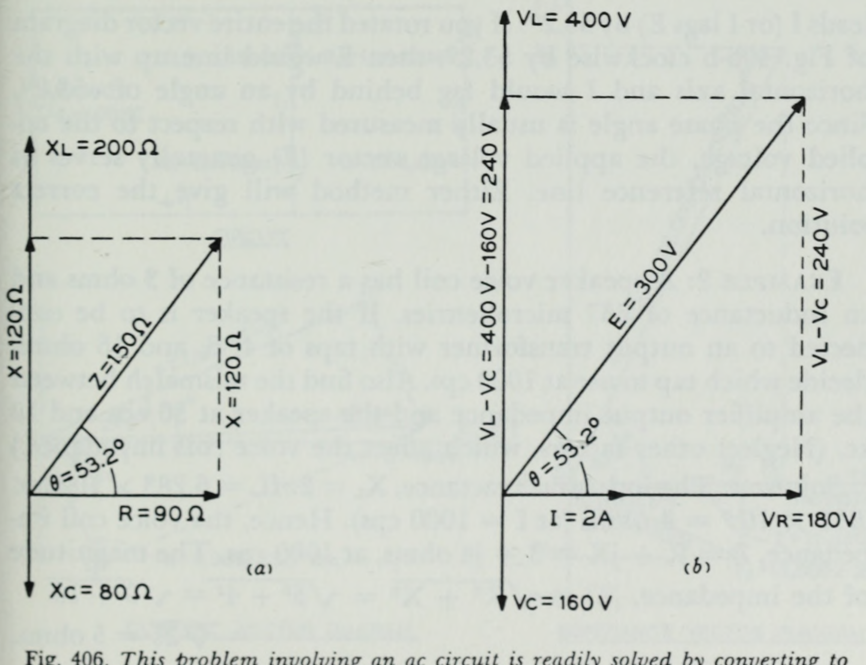
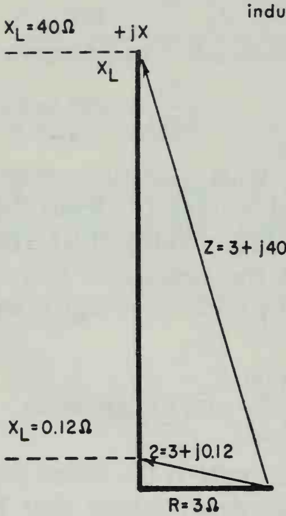
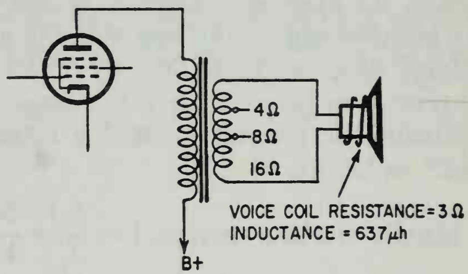


Fig. 406. *This problem involving an ac circuit is readily solved by converting to polar form.*



Graphic representation of Example 2, below, for extreme values of 30 cps and 10 kc. Note the effect of increasing frequency on the impedance. For lower frequencies, the impedance approaches the resistance, hence inductive reactance plays a role of decreasing importance.



leads I (or I lags E) by 53.2°. If you rotated the entire vector diagram of Fig. 406-b clockwise by 53.2°, then E would line up with the horizontal axis and I would lag behind by an angle of –53.2°. Since the phase angle is usually measured with respect to the applied voltage, the applied voltage vector (E) generally serves as horizontal reference line. Either method will give the correct solution.

EXAMPLE 2: A speaker voice coil has a resistance of 3 ohms and an inductance of 637 microhenries. If the speaker is to be connected to an output transformer with taps of 4, 8, and 16 ohms, decide which tap to use at 1000 cps. Also find the mismatch between the amplifier output impedance and the speaker at 30 cps and 10 kc. (Neglect other factors, which affect the voice coil impedance.)

Solution: The inductive reactance, $X_L = 2\pi fL = 6.283 \times 1000 \times .637 \times 10^{-3} = 4$ ohms (at f = 1000 cps). Hence, the voice coil impedance, $Z = R + jX = 3 + j4$ ohms, at 1000 cps. The magnitude of the impedance, $|Z| = \sqrt{R^2 + X^2} = \sqrt{3^2 + 4^2} = \sqrt{9 + 16}$

$$= \sqrt{25} = 5 \text{ ohms.}$$

Thus, the 4-ohm tap of the output transformer provides the best match at the mid-frequency range (1000 cps).

At 30 cps, $X_L = 6.283 \times 30 \times 0.637 \times 10^{-3} = 0.12$ ohms. Hence, $Z = R + jX = 3 + j0.12 = 3$ ohms, approximately. (Since the inductive reactance is less than 1/20 of the resistance, we can neglect it.) At 30 cps, therefore, the voice coil presents an almost pure resistance of 3 ohms. The mismatch of 1 ohm (at the 4-ohm tap) is the same as at the mid-frequency of 1000 cps.

At 10,000 cps (10 kc), $X_L = 6.283 \times 10^4 \times 6.37 \times 10^{-4} = 40$ ohms. Hence, the voice coil impedance, $Z = 3 + j40$ ohms. The absolute value (magnitude), $|Z| = \sqrt{3^2 + (40)^2} = \sqrt{9 + 1600} = \sqrt{1609}$
$= 40.1$ ohms.

(We could have neglected R in comparison with X, in this case.) Thus, at 10 kc the voice coil impedance is practically purely inductive, and it presents a mismatch of 10:1 (i.e., 40:4). Other factors are usually present, however, which reduce the magnitude of mismatch.

EXAMPLE 3: In the series-parallel ac circuit, illustrated in Fig. 407, find the impedance of each the parallel branches, the total

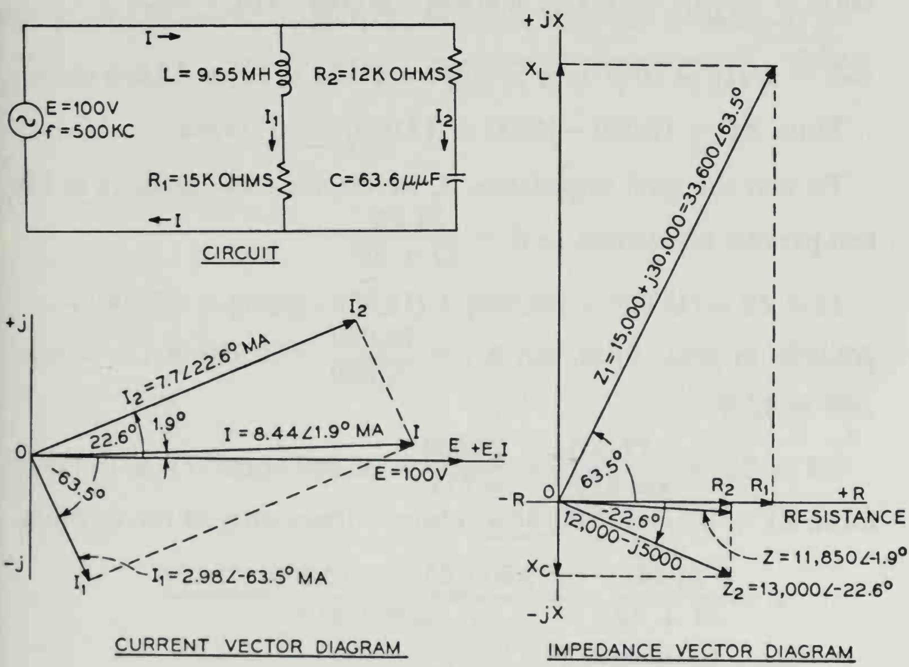


Fig. 407. *Series-parallel circuit and its current and impedance vector diagrams.*

impedance, the branch currents, the total (line) current, and the voltage drops across the resistors, the capacitor and the coil.

Solution: The given circuit constants are: E = 100 V; f = 500 kc; L = 9.55 millihenries; C = 63.6 $\mu\mu$f; R1 = 15,000 ohms; R2 = 12,000 ohms. The inductive reactance, $X_L = 2\pi fL = 6.283 \times 5 \times 10^5 \times 9.55 \times 10^{-3} = 30{,}000$ ohms. The capacitive reactance,

$$X_c = \frac{1}{2\pi fC} = \frac{1}{6.283 \times 5 \times 10^5 \times 63.6 \times 10^{-12}} = 5{,}000 \text{ ohms.}$$

The impedance of branch 1, $Z1 = R1 + jX_L = 15{,}000 + j30{,}000$ ohms; in polar form, $\tan\theta_1 = \frac{30{,}000}{15{,}000} = 2$, hence $\theta_1 = \arctan 2 = 63.5°$; the magnitude, $|Z| = \sqrt{(1.5 \times 10^4)^2 + (3 \times 10^4)^2} = \sqrt{11.25 \times 10^8} = 33{,}600$ ohms. (We could also have used the relation $|Z1| = R1/\cos\theta_1$.)

Hence, $Z1 = 15{,}000 + j30{,}000 = 33{,}600\underline{/63.5}$ ohms.

The impedance of branch 2, $Z2 = R2 - jX_c = 12{,}000 - j5000$ ohms (since X_c is negative); again converting to polar form, $\tan\theta_2 = \frac{-5000}{12{,}000} = -0.416$; hence $\theta_2 = \arctan -.416 = -22.6°$;

$$|Z2| = \sqrt{(12 \times 10^3)^2 + (5 \times 10^3)^2} = \sqrt{169 \times 10^6} = 13{,}000 \text{ ohms.}$$

Thus, $Z2 = 12{,}000 - j5000 = 13{,}000\underline{/-22.6°}$ ohms.

To find the total impedance, Z, we use the same formula as for two parallel resistances, or $Z = \frac{Z1\,Z2}{Z1 + Z2}$.

$Z1 + Z2 = (15{,}000 + j30{,}000) + (12{,}000 - j5000) = 27{,}000 + j25{,}000$; in polar form, $\tan\theta_{1\text{-}2} = \frac{25{,}000}{27{,}000} = 0.926$; $\theta_{1\text{-}2} = \arctan .926 = 42.8°$.

$|Z1 + Z2| = \frac{27{,}000}{\cos 42.8°} = \frac{27{,}000}{0.734} = 36{,}800$ ohms. Thus, in polar form, $Z1 + Z2 = 36{,}800\underline{/42.8°}$ ohms. Substituting in the formula,

$$Z = \frac{Z1\,Z2}{Z1 + Z2} = \frac{(33{,}600\underline{/63.5°})\,(13{,}000\underline{/-22.6°})}{36{,}800\underline{/42.8°}}$$

$$= \frac{436{,}000\underline{/40.9°}}{36.8\underline{/42.8°}} = 11{,}850\,\underline{/-1.9°} \text{ ohms.}$$

The total line current, $I = \frac{E}{Z} = \frac{100\underline{/0^\circ}}{11{,}850\underline{/-1.9^\circ}}$

$= 0.00844\,\underline{/1.9^\circ}$ amperes.

the branch current, $I_1 = \frac{E}{Z1} = \frac{100\underline{/0^\circ}}{33{,}600\underline{/63.5^\circ}}$

$= 0.00298\underline{/-63.5^\circ}$ amperes.

and branch current $I_2 = \frac{E}{Z2} = \frac{100\underline{/0^\circ}}{13{,}000\underline{/-22.6^\circ}}$

$= 0.0077\underline{/22.6^\circ}$ amperes.

Finally, let us compute the voltage drops across R1, R2, L and C.

$E_{R1} = I_1R1 = (2.98 \times 10^{-3})\underline{/-63.5^\circ} \times 15 \times 10^3 = 44.6\underline{/-63.5^\circ}$ volts.

$E_{R2} = I_2R2 = (7.7 \times 10^{-3})\underline{/22.6^\circ} \times 12 \times 10^3 = 92.5\underline{/22.6^\circ}$ volts.

$E_L = I_1X_L = (2.98 \times 10^{-3})\underline{/-63.5^\circ} \times (30 \times 10^3)\underline{/90^\circ} = 89.5\underline{/26.5^\circ}$ volts.

and $E_C = I_2X_C = (7.7 \times 10^{-3})\underline{/22.6^\circ} \times 5 \times 10^3\underline{/-90^\circ} = 38.5\underline{/-67.4^\circ}$ volts.

PRACTICE EXERCISE 6

1. First add the following pairs of complex numbers; then subtract the second from the first. Also, plot each pair of numbers on graph paper and take the vector sum and difference graphically:

$5 + j3$ and $2 - j2$; $6 - j4$ and $-5 - j3$; $3 + j2$ and $2 + j3$.

(Answers: $7 + j1, 3 + j5$; $1 - j7, 11 - j1$; $5 + j5, 1 - j1$.)

2. Multiply the following numbers: $(2 + j5)(3 - j2)$; $(3 - j1)(3 + j1)$; $(a + j\sqrt{b})(a - j\sqrt{b})$; $(2 + j3 - j2\sqrt{2})(1 + j3 + j4\sqrt{2})$.

(Answers: $16 + j11$; 10; $a^2 + b$; $(9 - 6\sqrt{2}) + j(9 + 6\sqrt{2})$.

3. Divide $(6 - j4)$ by $(-5 - j3)$; $(2 + j5)$ by $(2 - j5)$; $(x + jy)$ by $(x + j2y)$; $(a - jb)$ by $(a + jb)$.

(Answers: $\frac{-9 + j19}{17}$; $\frac{-21 + j20}{29}$; $\frac{x^2 + 2y^2 - jxy}{x^2 + 4y^2}$; $\frac{a^2 - b^2 - j2ab}{a^2 + b^2}$.)

4. Draw a diagram to derive the relations between the rectangular and polar form of a complex number and convert the following numbers into rectangular or polar form, as required:

$42.4 + j16.7$; $51.4\underline{/-10.5°}$; $6 + j5$; $3\underline{/45°}$; $-1 - j4$; $2\underline{/300°}$.

(Answers: $45.6\underline{/21.5°}$; $50.5 - j9.36$; $7.8\underline{/39.8°}$; $2.12 + j2.12$; $4.13\underline{/256°}$; $1 - j1.73$).

5. First add $45.6\underline{/21.5°}$ to $51.4\underline{/-10.5°}$, then subtract the former from the latter number. (Answers: $92.9 + j7.34$; $8.1 - j26.06$.)

6. By using the relation $Z\underline{/\theta} = Z(\cos\theta + j\sin\theta)$ prove that

$$Z1\underline{/\theta1} \times Z2\underline{/\theta2} = (Z1\,Z2)\underline{/\theta1 + \theta2} \text{ and } \frac{Z1\underline{/\theta1}}{Z2\underline{/\theta2}} = \frac{Z1}{Z2}\underline{/\theta1 - \theta2}.$$

7. Reduce to polar form and find the product of

$(1 + j)(3 + j\sqrt{3})$ (Answer: $2\sqrt{6}\underline{/75°}$.)

8. Divide in polar form $(3 + j\sqrt{3})$ by $(1 + j1)$.
(Answer: $2.45\angle{-15^\circ}$.)

9. Find $(2 - j1)^4$; $(1 + j2)^5$; the three cube roots of 27; the five roots of $(2 + j2)^{1/5}$.

(Answers: $25\angle{253^\circ 44'}$; $56\angle{316^\circ 25'}$; $3, -1.5 + j2.6, -1.5 - j2.6$; $1.232\angle{9^\circ}$, $1.232\angle{81^\circ}$, $1.232\angle{153^\circ}$, $1.232\angle{225^\circ}$, and $1.232\angle{297^\circ}$.)

10. Rework problem 10 in Exercise 5, using complex number notation.

11. A 48-volt, 100 cps ac source is connected across a parallel circuit, consisting of three branches. Branch 1 is a 6-ohm resistor; branch 2 a 66 microfarad capacitor, and branch 3 a 19 millihenry choke coil. Compute the branch impedances, the total imped-

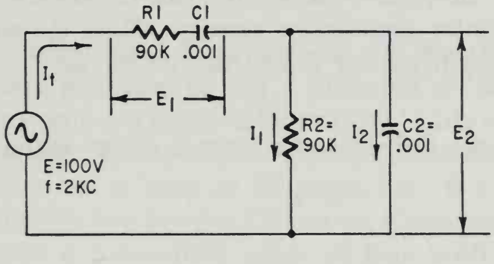


Fig. 408. *This is the series-parallel circuit for problem 12.*

ance (Z), the total (line) current (I_t), and the three branch currents. Draw vector diagrams.

(Answers:

$Z1 = R = 6\angle{0^\circ}$ ohms; $Z_2 = X_c = 24\angle{-90^\circ}$; $Z3 = X_L = 12\angle{90^\circ}$;

$Z = 5.65 + j1.414 = 5.82\angle{14^\circ}$; $I_t = 8.25\angle{-14^\circ}$; $I_R = 8$ amperes;

$I_c = 2\angle{-90^\circ}$ amperes; $I_L = 4\angle{90^\circ}$ amperes.)

12. Part of a Wien-bridge oscillator circuit consists of a series R1-C1 combination and a parallel R2-C2 combination, both connected in series with an applied voltage, E = 100 volts at 2,000 cps.

If R1 = R2 = 90,000 ohms, and C1 = C2 = 1,000 $\mu\mu$f, compute the total impedance, Z, at 2,000 cps; the total line current, I_t; the branch currents, I_1 and I_2, through R2 and C2; the voltage drop, E1, across R1-C1, and the drop, E2, across R2-C2.

(Answers: Z = 129,400 − j124,300 = 179,000 $\underline{/-44°}$ ohms; I_t = 0.559 $\underline{/44°}$ ma; I_1 through R2 = 0.37 $\underline{/-4.6°}$ ma; I_2 through C2 = 0.418 $\underline{/85.4°}$ ma; E1 (across R1–C1) = 67 $\underline{/2.6°}$ volts; E2 (across R2-C2) = 33.3 $\underline{/-4.6°}$ volts.)

13. What is meant by conjugate complex numbers? Demonstrate by setting up a problem that the sum and product of conjugate complex numbers are real numbers.

14. In the division of one complex number by another, how is the denominator formed into a real number?

15. Explain, in your own words, the difference between rectangular and polar coordinate systems. How do we distinguish between the magnitude of Z and the vector Z?

16. Explain the difference between a power and a root.

CHAPTER 5

Logarithms

IF somebody told you to add instead of multiplying, to subtract instead of dividing, and to multiply and divide in place of raising to a power or extracting a root, you would probably think of the suggestion as fantastic. But this is exactly what John Napier (1550–1617) told the world in 1614 when he published his work on logarithms.

Logarithms (abbreviated logs) will do all the things we mentioned above, but you have to pay a price for all this simplicity: you have to learn to use a table of logarithms. This may be as brief as a single sheet for "four-place" logs, which are accurate to four significant figures, or as long as 20 pages for "five-place" tables, which are accurate to five figures. Of course, if you carry a slide rule around, you have a convenient table of logs built right into it, though you may not know it. When you multiply with a slide rule, you are adding logs expressed as distances along the C and D scales, and when you divide, you are subtracting logs or distances along the scales. A slide rule can give you an "engineering accuracy" of about 1%, depending upon how well you read it, while five-place log tables are accurate to about .01%, again depending upon how well you can use (interpolate) them. Thus, a table of logs is a good investment for accurate calculations.

What are logarithms?

The definition of logarithms sounds like double talk: the logarithm of a number is the exponent, or the power, to which the base (another number) must be raised to produce the given number. To illustrate, in the expression

$$2^4 = 16 \quad (\text{i.e., } 2 \times 2 \times 2 \times 2 = 16)$$

The number 2 is the base, 4 is the exponent (power) or logarithm, and 16 is the resulting number. In accordance with the definition, therefore, the logarithm of 16 to the base 2 is 4. This is written

$$\log_2 16 = 4$$

Thus, we can write the following short-hand definition of logs:

$$\text{If } b^x = N, \text{ then } \log_b N = x$$

In practice, only two base numbers (b) are used for logs. In 90% of your work you will use the number 10 as base, since it is also

Multiplying and dividing large numbers? If you use ordinary arithmetic, you'll need reams of paper.

the base of the number system. Logs to the base 10 are called common logarithms, and when no other base is stated, 10 is always implied. The other base is the number $\epsilon = 2.7182818285$ (approximately), which is called the natural number. Logs to the base ϵ are known as natural or Napierian logarithms, after their inventor, John Napier. Natural logs are useful in electrical theory and particularly for "exponential" capacitor charge and discharge calculations. The fundamental operations of calculating with logs are exactly the same for both systems, but there is a difference in finding the logs in a table. Let us first deal with common logs.

Finding common logs in the tables

You do not need a table to find the logs of exact power of 10, since these must have integers as exponents (logs). Thus,

$$\log_{10} 10 = 1, \text{ since } 10^1 = 10 \quad \text{(by definition of logs)}$$
$$\log 100 = 2, \text{ since } 10^2 = 100$$
$$\log 1{,}000 = 3, \text{ since } 10^3 = 1{,}000$$
$$\log 10{,}000 = 4, \text{ since } 10^4 = 10{,}000$$
$$\log 100{,}000 = 5, \text{ since } 10^5 = 100{,}000$$
$$\log 1{,}000{,}000 = 6, \text{ since } 10^6 = 1{,}000{,}000, \text{ and so forth.}$$

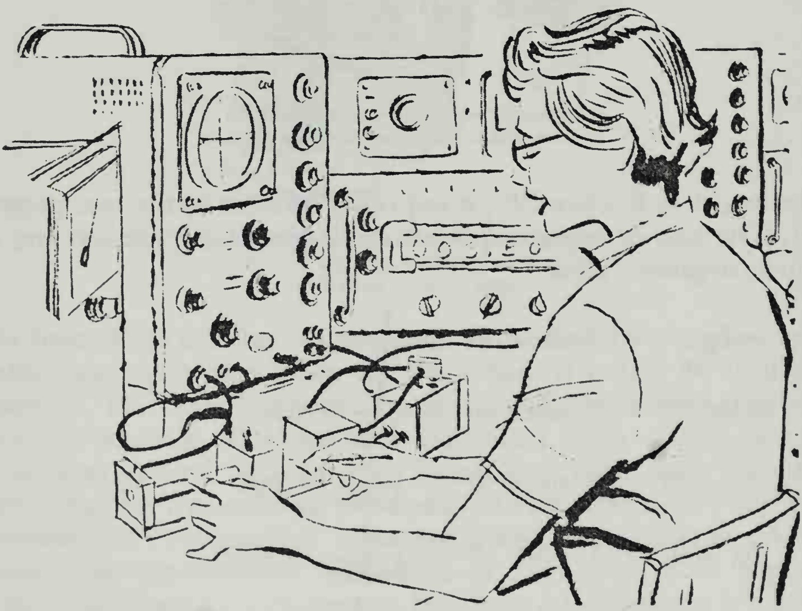
The easier way is to turn your back on those stacks of paper and use logs.

The $\log_{10} 1 = 0$, since 10^0 (or any other number to the zero power) is 1. The logs of negative powers of 10 (decimal fractions) are, of course, negative. Thus,

$$\log_{10} .1 = -1, \text{ since } 10^{-1} = \frac{1}{10} = .1$$
$$\log .01 = -2, \text{ since } 10^{-2} = \frac{1}{100} = .01$$
$$\log .001 = -3, \text{ since } 10^{-3} = \frac{1}{1{,}000} = .001$$
$$\log .0001 = -4, \text{ since } 10^{-4} = \frac{1}{10{,}000} = .0001$$
$$\log .00001 = -5, \text{ since } 10^{-5} = \frac{1}{100{,}000} = .00001, \text{ and so forth.}$$

Thus, we can find the log of all exact powers of 10 directly, as shown above. You would hardly bother with logs, however, to calculate with powers of 10. What about inexact powers of 10, numbers like 5, 46, 365, 3,897, 12.57, .153, .0024, etc.? Let's examine them in turn. Since the number 5 is between 1 and 10, it must have a log between 0 and 1, or 0.???; 46 is between 10 and 100 and, thus, the log is between 1 and 2, or 1.???; 365 is between 100 and 1,000, and hence, the log is between 2 and 3, or 2.???; 3,897 is between 1,000 and 10,000, so that the log is between 3 and 4, or 3.????; 12.57 is again between 10 and 100, and thus the log is 1.???. Note that we can always determine the integral portion of the log to the left of the decimal point by an inspection of the number, but we cannot ascertain the decimal portion to the right of the point (i.e., the question marks). The integer preceding the decimal point is called the characteristic of the log, while the decimal portion is called the mantissa. You can determine the characteristic directly, but you'll have to look up the mantissa in the log tables.

A test setup is the result of planning and mathematics is always part of that planning. [Spectran Electronics Corp.]



Determining the characteristic

You could always determine the characteristic by writing down the exponent of the nearest power of 10 of the number, as we have done in the preceding table. But you may also note from the characteristics of the numbers we have just written that the characteristic is always one less than the number of digits to the left of the decimal point. Hence, you can use the following rule for the characteristic of numbers greater than 1:

Rule 1: For numbers greater than 1 the characteristic is positive and is numerically one less than the number of digits to the left of the decimal point.

This is the sad but inevitable result of failing to follow our advice. Final warning — save time, trouble and paper — learn to use logs.

We already know that for numbers less than 1 the characteristic is negative and we can formulate the following rule to determine it:

Rule 2: For numbers less than 1 (decimals) the characteristic is negative and is numerically one greater than the number of zeros immediately following the decimal point. For example, using the same numbers as before, the characteristic of .153 is –1, since there are no zeros after the decimal point; the characteristic of .0024 is –3, since there are two zeros. Thus, the characteristics of 72,350, 7,235, 72.35, 7.235, .7235, and .0007235 are 4, 3, 1, 0, –1, and –4, respectively, while the mantissas are exactly the same.

There is one more point you have to observe concerning characteristics. Since the mantissa (in the tables) is always positive, while the characteristic may be either positive or negative, you cannot place a minus (–) sign in front of the entire logarithm. You may place the "–" sign above the characteristic, or better, add 10 to the negative characteristic and then subtract 10 again after the logarithm. For example, if the characteristic is –4 and the mantissa is 0.6571 (the log of .000454), you may write either $\bar{4}.6571$, or 6.6571 –10 (adding and subtracting 10), but not –4.6571.

Finding the mantissa

You can find the mantissa of all numbers in the log tables, either the brief four-place tables, which list four decimal places, or five- or higher-place tables of greater accuracy. The mantissa depends only on the significant figures in the number whose log is desired and does not depend on the decimal point. The characteristic takes

Log tables will look attractive if you're involved in any work requiring computations.

care of that, as we have seen. Thus, the mantissas of 72,350, 7,235, 7.235 and .0007235 are all the same and equal to 0.85944 (from tables). The logs of these numbers are, therefore, 4.85944, 3.85944, 0.85944, and 6.85944 –10, respectively.

In finding the mantissa from four-place log tables, the left-hand column of the table, headed N, lists the first two significant figures of the given number, while the top row shows the third figure. Hence, in finding a log, you first look for two significant digits in the left-hand column under N, then follow the row across the page until you reach the intersection with the column listing the third digit. This is the desired mantissa. The process is the same for five-place tables, except that you will find three digits under N. Looking up the log of 595 in four-place tables, for example, first find 59 in the left-hand (N) column and 5 in the top row; then follow the 59 row across to the right until it intersects with the 5 column (or go down the 5 column to the intersection with the 59 row). At the intersection point you will read 7,745. The mantissa, therefore, is 0.7745 (the decimal point is implied) and the log of 595 is 2.7745. Similarly, looking up the log of 7,235 in five-place tables, find 723 in the column under N, then go across to the intersection with column marked "5" on top, and find the number 85,944. (You may find only 944, the first two digits 85 being listed the first time they appear.) The mantissa, therefore, is 0.85944 and the log of 7,235 is 3.85944. (See pages 152–153 for four-place log tables.)

As an exercise, let us look up the logs of the numbers we listed with question marks, a little while ago:

Number	Characteristic	Mantissa	Logarithm	Remarks
5	0	0.6990	0.6990	4-place tables
46	1	0.6628	1.6628	4-place tables
365	2	0.5623	2.5623	4-place tables
3,897	3	0.59073	3.59073	5-place tables
12.57	1	0.09934	1.09934	5-place tables
0.153	–1	0.1847	9.1847 –10	4-place tables
0.0024	–3	0.3802	7.3802 –10	4-place tables

Interpolation

We looked up the numbers 3,897 and 12.57, which have four digits each, in five-place log tables, since numbers with more than three digits are not listed in the four-place log tables. What, if we had only four-place tables available and wanted to find the value of the extra (fourth) digit? Well, we would use an arithmetical process known as interpolation, which assumes that logs increase proportionately (linearly) with the number. (This isn't quite correct.) By arranging the mantissas in a tabular manner, you can interpolate easily. Thus to find the log of 3897, write:

$$\begin{array}{r} \text{mantissa of } 3900 = 0.5911 \\ \text{mantissa of } 3890 = 0.5899 \\ \hline \text{the difference} = .0012 \end{array}$$

Since 3,897 is 7/10 of the way between 3,890 and 3,900, the mantissa must be 7/10 of the way between 0.5899 and 0.5911. But $7/10 \times .0012 = .00084$; hence, we must add the increment, .00084, to 0.5899, obtaining 0.59074. The log of 3,897, therefore, is 3.59074 when interpolated on four-place tables. Compare this with the value 3.59073 from five-place tables, which differs by no more than 1 part in 100,000.

Similarly, to find the log of 12.57, we write

$$\begin{array}{r} \text{mantissa of } 1{,}260 = 0.1004 \\ \text{mantissa of } 1{,}250 = 0.0969 \\ \hline \text{difference} = .0035 \end{array}$$

Again 1,257 is 7/10 between 1,250 and 1,260. Hence, 7/10 × .0035 = .00245 is the increment. Adding .00245 to .0969 we obtain a mantissa of .09935. Hence the log of 12.57 from four-place tables is 1.09935, compared to 1.09934 obtained directly from five-place tables. You can, of course, apply interpolation to five-place tables also, thus obtaining the log for numbers of five significant figures. You will find that commercial tables usually list the proportional parts for various differences between logs on the margin of each page. This makes possible interpolation between logs by inspection.

Looking up the antilog

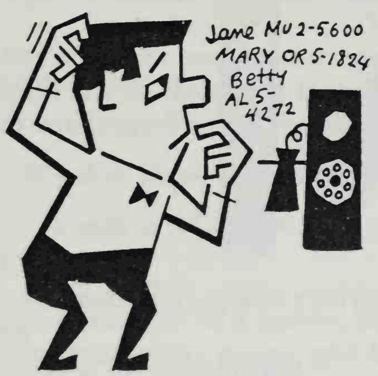
After you have completed your calculations with logs, the result will be another logarithm. The number that corresponds to this logarithm, called the antilogarithm or antilog, is the numerical result we're interested in, of course. To find the antilog, or the number corresponding to a log, you must use the tables in reverse, so to speak. For example, to find the antilog of 1.8987, first look in the four-place log tables for the mantissa 0.8987. You will find it in the "79" row (of N) and under the "2" column to the right. Thus, the significant digits of the number are 792. Since the characteristic is 1, there must be two digits to the left of the decimal point, by our previous rule. Thus, the antilog of 1.8987 is 79.2.

As another example, let's find the antilog of 2.4325. The mantissa 0.4325 is not listed in the four-place tables. However, mantissas 0.4314, corresponding to 270, and 0.4330, corresponding to 271, are listed. Hence, we must interpolate. The difference between the two mantissas is 0.4330 – 0.4314 = .0016, and the difference between the given mantissa (0.4325) and the one listed for 270 (0.4314) is 0.4325 – .4314 = .0011. Hence the desired number is $\frac{.0011}{.0016} = \frac{11}{16}$ of the way between 270 and 271. Adding $\frac{11}{16} = 0.69$ (approximately) to 270, we obtain the significant figures 27,069. Since the characteristic is 2, in this case, there must be three figures preceding the decimal point, and hence, the desired number is 270.69. In practice, it is sufficient to carry the interpolation to the nearest tenth (0.1), so that the antilog of 2.4325 is 270.7, to the nearest tenth.

Finally, as an example of a negative characteristic, let's find the antilog of $\bar{4}.74846$ or 6.74846 –10. Since the mantissa has five places, let us try to find it in five-place log tables. The exact mantissa is not listed, however. The nearest two mantissas we can find, and the numbers corresponding to them are:

	0.74850	corresponding to	5604
and	0.74842	corresponding to	5603
difference	0.00008	(corresponding to	1)

The difference between the desired mantissa and the one listed for 5603 is 0.74846 – 0.74842 = .00004. Hence, the desired number is $\frac{.00004}{.00008} = \frac{4}{8}$ or one-half of the way between 5603 and 5604; the significant figures, thus, are 56035. Since the characteristic is –4.



Having trouble finding the desired number? Interpolation will help.

three zeros must follow the decimal point (that is, one less than the characteristic), by the rule for negative characteristics. Hence, the desired antilog of –4.74846 is .00056035. Though our step-by-step presentation of the required interpolation makes the calculations appear lengthy, they are actually extremely simple, and often can be performed mentally.

Computations with logarithms

After having spent all this time looking up logs and antilogs, let us now see what we can do with them. Remember, we promised at the outset of the chapter that we would be able to add instead of multiplying, subtract instead of dividing, and so forth. The following brief rules finally fulfill this promise. Though we have dealt

Once you learn the rules, working with logs is easy. But it does take practice — enough practice to get you out of the head-scratching department.

only with common logs (to the base 10) thus far, the following rules are valid for operations with any system of logs, regardless of the base.

Rule 1: To multiply two or more numbers add their logarithms and look up the antilog of the result. If the numbers to be multiplied are M and N, this rule may be formulated mathematically

$$\log MN = \log M + \log N$$

To prove this rule, let $x = \log_b M$, so that $b^x = M$ (by definition of logs)

and let $y = \log_b N$, so that $b^y = N$, where b is any base.

Multiplying, $M \times N = b^x \times b^y = b^{x+y}$

Taking the log of both sides, $\log_b MN = \log (b^{x+y}) = x + y$ (by definition).

Hence, $\log_b MN = x + y = \log_b M + \log_b N$ (by substitution).

EXAMPLE: Multiply 69.52 by 4.37

Solution: $\log (69.52 \times 4.37) = \log 69.52 + \log 4.37$

$$\begin{aligned} \log 69.52 &= 1.8421 \\ \log\ \ 4.37 &= \underline{0.6405} \\ \text{sum} &= 2.4826 \end{aligned}$$

Thus the product is the antilog of 2.4826 = 303.8. The actual product of 69.52 × 4.37 turns out to be 303.8024. The error in using logs in this case is about 8 parts in 1,000,000 or .0008%.

Rule 2: To divide one number by another, subtract the logarithm of the divisor from the log of the dividend and find the antilog of the result.

$$\text{Mathematically,} \quad \log \frac{M}{N} = \log M - \log N$$

You can prove this rule in the same way as the multiplication rule.

EXAMPLE: Divide 69.52 by 4.37

Solution: $\log (69.52 \div 4.37) = \log 69.52 - \log 4.37$

$$\begin{aligned} \log 69.52 &= 1.8421 \\ \log \ \ 4.37 &= \underline{0.6405} \\ \text{difference} &= 1.2016 \end{aligned}$$

The quotient is the antilog of 1.2016 = 15.907

A slight case of log m minus log n.

Rule 3: To raise a number to a power, multiply the log of the number by the exponent of the power and find the antilog of the result. Thus, $\log M^n = n \log M$

EXAMPLE 1: Compute the value of $(5.2)^6$

Solution: $\log (5.2)^6 = 6 \log 5.2$

$\log 5.2 = .7160$; $6 \times .7160 = 4.2960$; the antilog of $4.2960 = 19{,}765$; hence, $(5.2)^6 = 19{,}765$.

EXAMPLE 2: Find the value of $(45.6)^{-3}$

Solution: $\log (45.6)^{-3} = -3 \log 45.6$

$\log 45.6 = 1.65896$

$-3 \log 45.6 = -3 \times 1.65896 = -4.97688$

Since the log tables list only positive mantissas, we add and subtract 10, obtaining

$$\begin{array}{r} 10.00000 - 10 \\ -\ 4.97688 \quad\;\; \\ \hline 5.02312 - 10 \end{array}$$

The quantity 5.02312 – 10, which is the negative of the log above, is called the *cologarithm* of the number. *Cologs* are occasionally useful, since, during division, you can add the colog of the divisor instead of subtracting its logarithm. If a large number of quantities

A big problem doesn't mean a big work sheet — if you use logs.

are to be variously multiplied and divided, it is easier to add the cologs in one column, rather than to subtract in separate calculations. In this particular case, the antilog of 5.02312 – 10 = .000010547. (Since the characteristic is –5, the number of zeros after the decimal point is 4.)

$$\text{Hence, } (45.6)^{-3} = .000010547 = 1.0547 \times 10^{-5}.$$

Rule 4: To extract the root of a number, divide the logarithm of the number by the index of the root and find the antilog of the result. (Since roots are powers with fractional exponents, this is, of course, the same as multiplying the log of the number by the fractional exponent; hence, rule 3 above covers roots also, if properly applied.)

$$\text{Mathematically, } \log \sqrt[n]{M} = \frac{1}{n} \log M$$

EXAMPLE: Find the third root of 1.572 ($\sqrt[3]{1.572}$).

Solution: $\log \sqrt[3]{1.572} = \frac{1}{3} \log 1.572$

$$\log 1.572 = .19645$$
$$\frac{1}{3} \log 1.572 = \frac{.19645}{3} = .065483$$
$$\text{antilog } .065483 = 1.1628$$

Hence, $\sqrt[3]{1.572} = 1.1628$

Using logs

As an example of the combined use of these four rules, let us carry out the calculations in the following example.

EXAMPLE: Using logs, compute the value of $\frac{\sqrt[3]{167.2} \times (8.16)^3}{3.921\ (.027)^2}$

Solution: $\log \left[\frac{\sqrt[3]{167.2} \times (8.16)^3}{3.921\ (.027)^2}\right]$

$$= \frac{1}{3} \log 167.2 + 3 \log 8.16 - (\log 3.921 + 2 \log .027)$$

Tabulating: $\log 167.2 = 2.22324$; hence $\frac{1}{3} \log 167.2 \ldots = 0.74108$

$\log 8.16 = 0.91169$; hence $3 \log 8.16 \ldots = 2.73507$

$\frac{1}{3} \log 167.2 + 3 \log 8.16 = \text{sum} \ldots = 3.47615$

$\log 3.921 = 0.59340 \ldots = 0.59340$

$\log 0.027 = 8.43136 - 10$; $2 \log .027 \ldots = 16.86272 - 20$

$\log 3.921 + 2 \log .027 = \text{sum} \ldots = 17.45612 - 20$

$= 7.45612 - 10$

difference $(3.47615) - (7.45612 - 10)$

$= 3.47615 - 7.45612 + 10 = 13.47615$

$- \ 7.45612$

difference $= 6.02003$

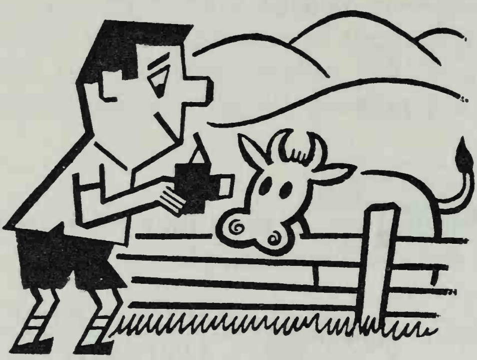
Thus, the resulting log answer is 6.02003, and the antilog of 6.02003 = 1,047,200 is the numerical answer to the problem.

Natural logarithms and exponentials

John Napier, the inventor of logarithms, used as base the natural number ϵ, which is an irrational number like π. Thus, he defined his system of logs:

$$\text{if } \epsilon^x = N, \text{ then } \log_\epsilon N = x$$

The Napierian system of natural logs is still used extensively in engineering theory and higher mathematics, primarily because the



Limits aren't just a mathematical invention. A fence is a good, practical example — especially if the cow turns out to be a bull.

slope of an exponential curve ($y = A\epsilon^{bx}$) at any point equals the ordinate at that point. The basis of differential calculus is the slope of the tangent of a function of y and, if this slope is equal to y, differentiation becomes a very simple matter. The natural number, ϵ, is usually defined in this way:

$$\epsilon = \frac{\text{Lim}}{n \to \infty}\left(1 + \frac{1}{n}\right)^n$$

which is read "epsilon equals the limit of one plus one over n to the nth power, as n approaches infinity." This limit can be obtained by taking the sum of the infinite series

$$\epsilon = 1 + \frac{1}{1} + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2 \cdot 1} + \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} + \frac{1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} + \cdots\cdots$$

By adding extra terms you can get the value of ϵ to any desired accuracy. Computing sum of the six terms listed above, we obtain

$$\epsilon = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = 2.71667$$

A more accurate value for ϵ is 2.7182818285; this is usually rounded off in engineering problems to 2.7183.

Finding natural logs

Standard, commercially available mathematical tables usually list natural logarithms ($\log_\epsilon N$), as well as the exponential function, ϵ^x. You will find the complete logarithm, characteristic and mantissa in natural log tables rather than the mantissa alone. This makes looking up natural logs extremely simple. What can you

Lost in the mathematical woods? Look to natural log tables for help.

do, however, if you don't have natural log tables available? You can easily derive for yourself the following simple relations between common and natural logarithms:

$$\text{let } \log_\epsilon N = x, \text{ so that } \epsilon^x = N$$
$$\text{and } \log_{10} M = y, \text{ so that } 10^y = M$$

To find the log of the same number in both systems, $N = M$, or

$$\epsilon^x = 10^y$$

Taking the log of both sides of this equation, first to the base ϵ,

$$\log_\epsilon \epsilon^x = x = \log_\epsilon 10^y = y \log_\epsilon 10, \text{ or } x = y \log_\epsilon 10.$$

Substituting, $x = \log_\epsilon N$, $y = \log_{10} M$ and $M = N$, we obtain

$$\log_\epsilon N = \log_{10} N \cdot \log_\epsilon 10, \qquad (1)$$

which gives the conversion from common to natural logs. Similarly, taking the log of $\epsilon^x = 10^y$ to the base 10,

$$\log_{10}\epsilon^x = \log_{10}(10^y) = y$$

$$\text{or } x \log_{10}\epsilon = y$$

Again, substituting for $x = \log_\epsilon N$ and for $y = \log_{10} M = \log_{10} N$, we obtain

$$\log_\epsilon N \times \log_{10} \epsilon = \log_{10} N,$$

or

$$\log_{10} N = \log_\epsilon N \times \log_{10} \epsilon, \qquad (2)$$

which gives the conversion from natural to common logs. Combining these relations (1) and (2), we finally derive

$$\log_\epsilon N = \log_{10} N \times \log_\epsilon 10 = \frac{\log_{10} N}{\log_{10}\epsilon} \qquad (3)$$

and $$\log_{10} N = \log_\epsilon N \times \log_{10} \epsilon = \frac{\log_\epsilon N}{\log_\epsilon 10} \qquad (4)$$

Looking up the two constants, $\log_\epsilon 10 = 2.3026$ and $\log_{10} \epsilon = 0.4343$, and substituting in (3) and (4) above, we can summarize the conversion formulas (recalling that $\log_\epsilon N$ = natural logs, and $\log_{10} N$ = common logs):

$$\text{natural log} = \text{common log} \times 2.3026 = \frac{\text{common log}}{0.4343} \qquad (5)$$

and $$\text{common log} = \text{natural log} \times 0.4343 = \frac{\text{natural log}}{2.3026} \qquad (6)$$

Thus, you can always obtain the natural log by multiplying the common log of the number by 2.3026 or dividing it by 0.4343. Conversely, you can obtain the common log by multiplying the natural log of a number by 0.4343 or by dividing it by 2.3026. There is one other point you must observe when looking up natural logs. These are usually given only for numbers from 1 to 10. If the number whose log is desired is 10 times or 1/10 as great as the number listed, you'll have to add or subtract, respectively, the $\log_\epsilon 10$ or 2.3026. Thus, each time you shift the decimal point of the listed number one place to the right, you must add 2.3026 to the natural log of the number in the tables, and each time you shift the decimal point one place to the left, you must subtract 2.3026 from the listed natural log. All other operations are the same as for common logs.

EXAMPLE: Find the natural logs of 5.46 and 85, both from natural log tables and by conversion from common log tables. Then find the product of the two numbers by means of natural logs and check the result by use of common logs.

Solution: Looking up first in natural log tables, log 5.46 = 1.69745 and log 8.5 = 2.14007. To find log 85 we must add 2.3026 to log 8.5, or 2.14007 + 2.3026 = 4.44267

Hence, $\log_\epsilon 5.46 = 1.69745$
and $\log_\epsilon 85 = 4.44267$

In 4-place common log tables we find

$$\log_{10} 5.46 = 0.7372$$

and

$$\log_{10} 85 = 1.9294$$

No magic is needed to convert common logs to natural logs.

To convert common logs to natural logs, we multiply by 2.3026. Hence, $\log_{\epsilon} 5.46 = 0.7372 \times 2.3026 = 1.69747$ (approximately), and

$$\log_{\epsilon} 85 = 1.9294 \times 2.3026 = 4.44264 \text{ (approximately)}$$

Hence, the natural logs found by either method check closely. To find the product of 5.46×85 by natural logs (using the values from the tables), we add the natural logs. Thus, the sum of the two logs is 6.14012, as shown.

$$\begin{array}{rr} & 1.69745 \\ & +\ 4.44267 \\ \hline \text{sum} & 6.14012 \end{array}$$

But the natural log table from 1 to 10 lists only values to 2.3026. Hence, we look up 1/100 of the number by subtracting twice 2.3026, or 4.60520, from the log. The difference is 1.53492.

$$\begin{array}{rr} & 6.14012 \\ & -4.60520 \\ \hline \text{difference} & 1.53492 \end{array}$$

The antilog of 1.53492 is 4.641 by interpolation from the table Since this is only 1/100 of the value, however, the result is 100×4.641 or 464.1.

Checking this result with common logs, we obtain a sum of $0.7372 + 1.9294 = 2.6666$. The antilog of this number, by interpolation from common log tables, again is 464.1. The exact result of 5.46×85, by arithmetic, is 464.10.

Using natural logs and exponentials in electrical problems

The natural number ϵ, natural logs and exponentials (ϵ^x) pop up continuously in electronic pulse work, computations of time constants, transmission lines, etc. Let us study one such problem—capacitor charge and discharge—to become familiar with the handling of exponentials and natural logs. The charge and discharge of a capacitor is one of the classic problems in electricity, which is solved with elegance by the methods of elementary calculus. Though we won't go into calculus here, we can take a look at the characteristic equations and curves of this problem.

Capacitor charge

The circuit for charging or discharging a capacitor through a resistance is shown in Fig. 501. An uncharged capacitor, C, in

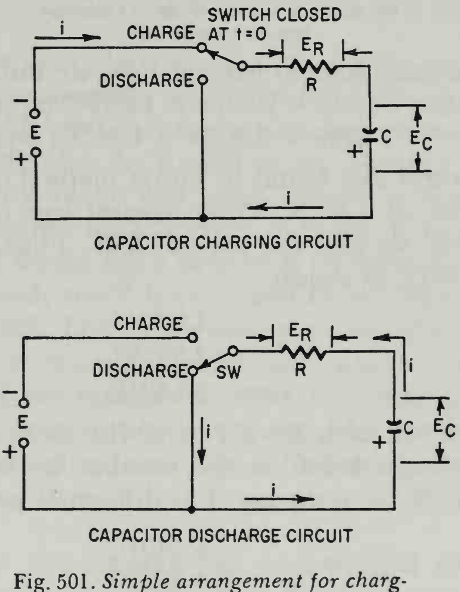


Fig. 501. *Simple arrangement for charging and discharging a capacitor.*

series with a limiting resistor, R, is connected through a switch to a source of constant potential, E. (This may be a battery.) At a certain time, t = 0, the switch is placed in "charge" position, connecting the battery voltage E across the R-C combination. At the moment of closing the switch, the current rushes into the capacitor and is limited only by the resistor, R. By Ohm's Law, therefore, the current $i = \frac{E}{R}$ at time $t = 0$ (when the switch is closed). After a certain (theoretically infinite) time interval, the charging current

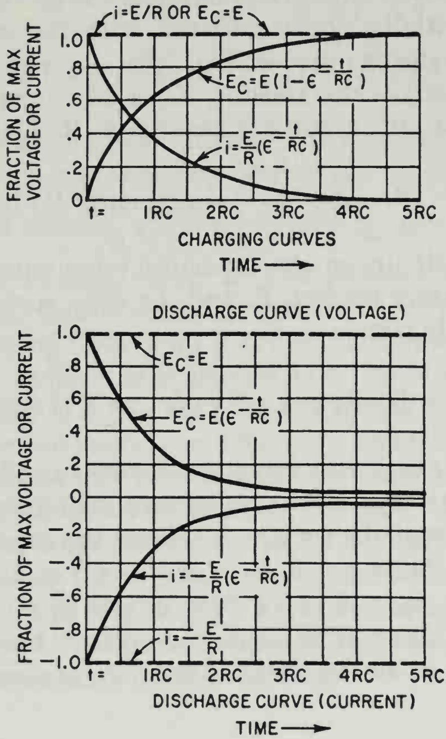


Fig. 502. *Capacitor charge and discharge curves.*

drops to zero, when the capacitor has been fully charged to the battery voltage, E. Hence, at $t = \infty$ (infinity) $i = 0$. The charging current at any time (t) is given by

$$i = \frac{E}{R}\,\epsilon^{-\frac{t}{RC}} \qquad (1)$$

You can verify this equation for the two conditions we have stated: at $t = 0$, $i = \frac{E}{R}\,\epsilon^{-\frac{0}{RC}} = \frac{E}{R}\,\epsilon^{0} = \frac{E}{R}$ (since a number raised to the zero power is 1)

and at $t = \infty$, $i = \frac{E}{R}\,\epsilon^{-\frac{\infty}{RC}} = \frac{E}{R}\epsilon^{-\infty} = \frac{E}{R} \times \frac{1}{\epsilon^{\infty}} = \frac{E}{R} \times \frac{1}{\infty}$

$$= \frac{E}{R} \times 0 = 0$$

Thus, the charging current is $\frac{E}{R}$ when the switch is first closed, and it is zero after an infinite time, in accordance with the previous conditions. Between these two extreme values, the current i declines "exponentially" in accordance with the equation, and as shown in the charging curve of Fig. 502.

The voltage across the resistor, E_R, is of course the product of the charging current, i, and the resistance, R. Hence, at any time

$$E_R = iR = E\epsilon^{-\frac{t}{RC}} \quad \text{[multiplying (1) by R]} \qquad (2)$$

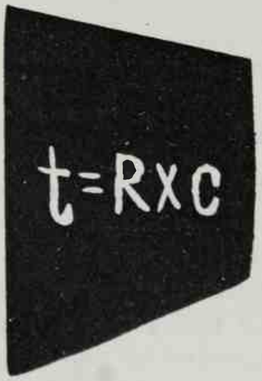
The voltage built up on the capacitor must equal the difference between the battery voltage, E, and the drop across the resistance, E_R. Hence, at any time,

$$E_C = E - E_R = E - E\epsilon^{-\frac{t}{RC}} = E(1 - \epsilon^{-\frac{t}{RC}}) \qquad (3)$$

Again, you can check this equation for two conditions, since the voltage across the capacitor must be zero initially (at $t = 0$) and it must equal E eventually (at $t = \infty$), when the capacitor is charged. Substituting in (3), at $t = 0$, $E_C = E(1 - \epsilon^{-0}) = E(1 - 1) = 0$ and at a time $t = \infty$, $E_C = E(1 - \epsilon^{-\infty}) = E(1 - 0) = E$, which checks the stated conditions. The capacitor voltage builds up between these two values in an exponential manner, as shown in the curve.

Time constant

Let us find the charging current and capacitor voltage at a certain time, T, when $t = RC$. This time interval (after the switch



A time constant [in seconds] is the product of the resistance [in ohms] and the capacitance [in farads].

has been closed) is known as the time constant of the circuit. For the condition, $t = T = RC$, we obtain

$$i = \frac{E}{R} \epsilon^{-\frac{RC}{RC}} = \frac{E}{R}\epsilon^{-1} = \frac{E}{R} \times \frac{1}{\epsilon} = \frac{E}{R \times 2.718} = 0.368 \frac{E}{R}$$

and $E_c = E(1 - \epsilon^{-\frac{RC}{RC}}) = E(1 - \epsilon^{-1}) = E(1 - 0.368) = 0.632\,E.$

Thus, we see that after a time interval equal to one time constant ($t = RC$) has elapsed, the charging current has declined to 0.368 or 36.8% of its initial value, $\frac{E}{R}$, and the capacitor voltage has reached 0.632 or 63.2% of its final value, E. The product of R and C, or the time constant, therefore, is a convenient way to gauge the approximate charging time of an R-C circuit. The charging curves (top right, Fig. 514) have been plotted in terms of time constants, rather than seconds. You can see from the curves that after a time equal to two time constants ($t = 2RC$), the capacitor voltage (E_c) reaches approximately 0.865 or 86.5% of its final value, after three time constants (3RC) about 95%, after four time constants, 98.2%, and after five time constants it reaches 99.3% (approximately) of its final value. During the same time intervals the current declines to 13.5% at 2RC, 5% at 3RC, 1.8% at 4RC, and to 0.7% of its initial value at 5RC. For all practical purposes, therefore, the charging process can be considered completed after five time constants, or 5RC.

Example: As a practical illustration, assume that the battery voltage (E) in Fig. 503 is 200 volts, $R = 100{,}000$ ohms, and $C =$

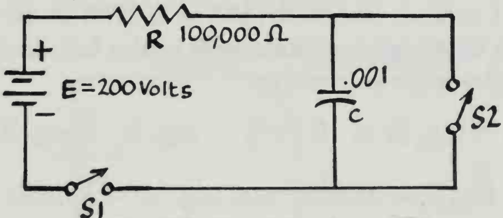
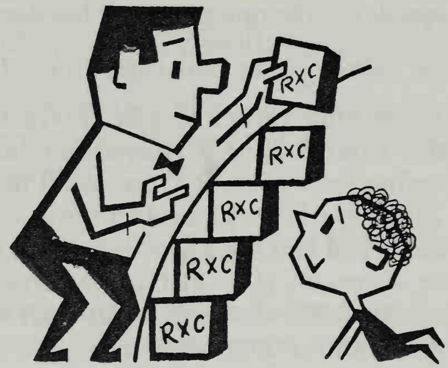


Fig. 503. *Circuit involving problem in time constant.*

0.001 μf. Determine the time constant; find the charging current, capacitor voltage and resistor voltage drop 200 microseconds after the switch (S1) is closed; determine the time when the voltage across the resistor will be 50 volts, and the time it takes for the capacitor to be fully charged (practically).

Solution: The time constant = RC = $10^5 \times .001 \times 10^{-6}$ = .0001 second (10^{-4} second) or 100 microseconds. The charging process will be completed in about five time constants, or 5RC = 500 microseconds.

You could, of course, determine the current and voltages after 200 microseconds, or 2RC, from the charging curves, but substitut-



Charging a capacitor produces a smooth curve.

ing in the equations will give more accurate results. Hence, when $t = 200 \times 10^{-6}$,

$$i = \frac{E}{R} \epsilon^{-\frac{t}{RC}} = \frac{200}{10^5} \epsilon^{-\frac{200 \times 10^{-6}}{100 \times 10^{-6}}} = 2 \times 10^{-3} \epsilon^{-2}$$

From the tables of the exponential function, $\epsilon^{-2} = 0.1353$ (approximately). Hence, $i = 2 \times 10^{-3} \times 0.1353 = 0.2706 \times 10^{-3}$ ampere = 0.2706 ma. If you don't have exponential tables but do have natural log tables available, you could write:

$$\log i = \log_\epsilon (2 \times 10^{-3} \epsilon^{-2}) = \log_\epsilon 2 - 3 \log_\epsilon 10 - 2$$

From log tables, $\log_\epsilon 2 = 0.69315$ and $\log_\epsilon 10 = 2.3026$;

$$\text{hence, } \log i = 0.69315 - 3 \times 2.3026 - 2 = 0.69315 - 8.9078$$
$$= -8.21465$$

Since negative logarithms are not listed in the natural log table, we add $4 \times \log_\epsilon 10 = 4 \times 2.3026 = 9.2104$, thus obtaining the log of 10,000 times the desired number. Adding $(-8.21465 + 9.2104) = 0.99575$ and looking up the antilog of 0.99575, we obtain 2.707 by interpolation. The desired current, therefore, is $\frac{2.707}{10,000} = 0.2707 \times 10^{-3}$ ampere.

The voltage across R after 200 microseconds is

$$E_R = iR = 0.2706 \times 10^{-3} \times 10^5 = 27.06 \text{ volts}$$

and the voltage across the capacitor,

$$E_c = E - E_R = 200 - 27.06 = 172.94 \text{ volts.}$$

Alternatively, substituting in equation (3)

$$E_C = E\,(1 - \epsilon^{-\frac{t}{RC}}) = 200\,(1 - \epsilon^{-2}) = 200\,(1 - 0.1353)$$
$$= 200 \times 0.8647 = 172.94 \text{ volts.}$$

To find the time when $E_R = 50$ volts, we substitute in equation 2

$$E_R = E\,\epsilon^{-\frac{t}{RC}} = 200\,\epsilon^{-\frac{t}{10^{-4}}} = 50 \text{ volts}$$

$$\text{or } \frac{50}{200} = 0.25 = \epsilon^{-\frac{t}{10^{-4}}}$$

$$\text{hence, } \log_\epsilon 0.25 = -\frac{t}{10^{-4}}. \text{ But, } \log_\epsilon 0.25 = 8.614 - 10$$

$$\text{thus, } 8.614 - 10 = -1.386 = -\frac{t}{10^{-4}}$$

and, hence, $t = 1.386 \times 10^{-4} = 138.6$ microseconds.

Capacitor discharge

Assume that the capacitor in the R-C circuit of Fig. 504 has been fully charged to the battery voltage, E. If the switch is now placed

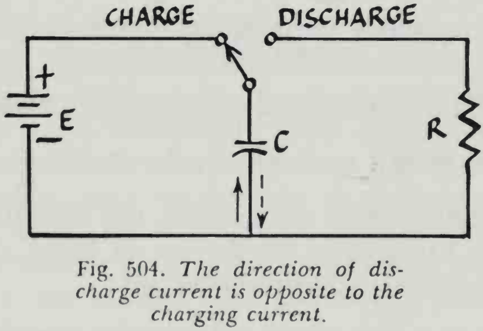


Fig. 504. *The direction of discharge current is opposite to the charging current.*

in the discharge position, we have provided a direct path and the capacitor discharges through the resistor (R). The discharge current is, of course, opposite in direction to the charging current, though it must be the same in magnitude, declining again from E to 0. Hence, we can write directly the equation for the discharge current

$$i = -\frac{E}{R}\,\epsilon^{-\frac{t}{RC}} \qquad (4)$$

where the minus (–) sign stands for the reversed current direction. Fig. 502 shows the current discharge curve, which is seen to have mirror symmetry with the current charging curve.

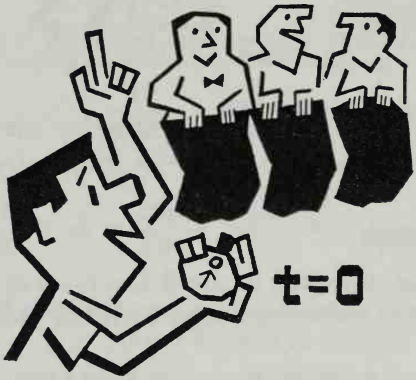
The voltage across the resistor, E_R, again is equal to iR, or

$$E_R = iR = -E\epsilon^{-\frac{t}{RC}} \qquad (5)$$

Since the battery is short-circuited, the sum of the capacitor voltage and the resistance drop must equal zero, or

$$E_c + E_R = 0, \text{ and hence, } E_c = -E_R = E\epsilon^{-\frac{t}{RC}} \qquad (6)$$

Thus, the capacitor voltage E_c declines from its initial value, E, at $t = 0$, to zero at $t = \infty$. The relative values of the discharge current



Starting time in an RC circuit or a sack race is when $T = 0$.

and voltages after a time interval of one time constant ($t = T = RC$) are all the same in this case and equal to $0.368 \frac{E}{R}$ or 0.368 E, respectively. In other words, the current, the capacitor voltage and the drop across the resistor all decline to 36.8% of their initial values after an interval of one time constant.

Example: If the capacitor in the previous example has been charged to 200 volts and is then discharged, find the current and the capacitor voltage 300 microseconds after the switch has been closed.

Solution: Substituting in equation 4,

$$i = -\frac{E}{R}\epsilon^{-\frac{t}{RC}} = -\frac{200}{10^5}\epsilon^{-\frac{300 \times 10^{-6}}{100 \times 10^{-6}}} = -2 \times 10^{-3} \times \epsilon^{-3}$$

from tables of the exponential function, $\epsilon^{-3} = .0498$ (approximately). Hence, $i = -2 \times 10^{-3} \times .0498 = -.0996 \times 10^{-3}$ ampere. (The minus sign indicates the negative discharge current.)

The capacitor voltage, $E_c = -E_R = -iR$

hence, $E_c = -(-.0996 \times 10^{-3}) \times 10^5 = 9.96$ volts.

Decibels for comparing power and voltage levels

In communication and acoustical work where enormous ranges of power are involved, power ratios are usually compared by a logarithmic unit, known as the decibel (db). The logarithm of a ratio increases by only 1 for each tenfold increase in the ratio, thus permitting very large ratios to be expressed by conveniently small numbers. (The log of 10,000,000 : 1, for example is only 7.) The use of a logarithmic unit, like the decibel, has an additional significance in acoustics, since the response of the human ear increases approximately as the logarithm of the acoustic power. Thus, the increase in the number of decibels of a sound corresponds approximately to the actual increase in loudness experienced.

Relative levels of sound power are measured in decibels.

The mathematical definition of the decibel is very simple. If the two powers being compared are P1 and P2, respectively, then the

$$\text{number of decibels} = 10 \log_{10} \frac{P1}{P2},$$

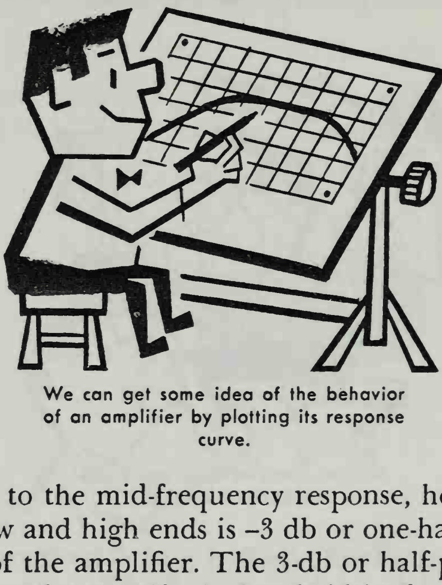
that is, decibels are 10 times the log of the power ratio. If P1 is greater than P2, then the ratio P1/P2 is larger than 1, and the number of decibels will be positive (+); if P1 is less than P2, however, the number of db will be negative (–). Thus, if P1 is twice P2, then

$$\text{db} = 10 \log \frac{P1}{P2} = 10 \log 2 = 10 \times 0.301 = 3 \text{ db (approximately)},$$

but if P1 is one-half of P2 (i.e., P1 = ½ P2), then

$$\text{db} = 10 \log \frac{P1}{P2} = 10 \log \tfrac{1}{2} = 10 \log 1 - 10 \log 2$$
$$= 0 - 10 \times .3 = -3 \text{ db}$$

(since the log of 1 is zero). Hence, if you hear someone say that the frequency response of an amplifier is 3 db down at the low and high



We can get some idea of the behavior of an amplifier by plotting its response curve.

ends, compared to the mid-frequency response, he means that the power at the low and high ends is –3 db or one-half of the midfrequency power of the amplifier. The 3-db or half-power points are also frequently used to specify the bandwidth of a tuned circuit. It may also be useful to keep in mind that 1 db represents a power ratio of about 5 to 4, while 60 db corresponds to a ratio of 1,000,000 to 1. Electrical handbooks give tables of db versus the power ratio, so you won't even have to bother to look up the logarithm.

Voltage and current ratios

Regrettably, the decibel is frequently used in electrical work to express voltage and current ratios, and it is here that the confusion begins. Since the decibel is defined as expressing a power ratio, it can be used for voltage and current ratios only when the power is an exact function of the voltage or current and the resistances are equal. Thus, in electrical circuits where the power is proportional to the square of the current or the voltage, we may write

$$P1 = I_1^2R_1 = \frac{E_1^2}{R_1} \text{ and } P2 = I_2^2R_2 = \frac{E_2^2}{R_2}$$

Substituting for the current ratio in the defining equation,

$$db = 10 \log \frac{P1}{P2} = 10 \log \frac{I_1^2R_1}{I_2^2R_2} = 20 \log \frac{I_1}{I_2} + 10 \log \frac{R_1}{R_2}$$

and for the voltage ratio

$$db = 10 \log \frac{P1}{P2} = 10 \log \frac{\frac{E_1^2}{R_1}}{\frac{E_2^2}{R_2}} = 10 \log \frac{E_1^2R_2}{E_2^2R_1}$$

$$\text{Hence, } db = 20 \log \frac{E_1}{E_2} + 10 \log \frac{R_2}{R_1}$$

If the resistances are equal, the decibel formulas for the comparison of voltages, currents and powers becomes simplified.

If the resistances or impedances associated with the two currents or voltages are equal ($R_1 = R_2$), *and only then,*

$$\log \frac{R_1}{R_2} = \log \frac{R_2}{R_1} = \log 1 = 0$$

$$\text{and hence, } db = 10 \log \frac{P1}{P2} = 20 \log \frac{I_1}{I_2} = 20 \log \frac{E_1}{E_2}$$

You will frequently find in practice that resistance or impedance values are ignored and the voltage or current levels are compared on a db basis by taking 20 times the log of their ratio. As an example, consider an audio amplifier which produces 20 watts output across an 8-ohm impedance if a 5-millivolt signal is applied across a 1-megohm input resistor (Fig. 505). To determine the gain of the amplifier in decibels, let $R_1 = 8$ ohms, $P1 = 20$ watts, $E_2 = 5$ millivolts $= .005$ volt, $R_2 = 1$ megohm $= 10^6$ ohms. The power level at the input, $P2 = \frac{E_2^2}{R_2} = \frac{(5 \times 10^{-3})^2}{10^6} = \frac{25 \times 10^{-6}}{10^6} = 25 \times 10^{-12}$ watt;

also, the output voltage, $E_1 = \sqrt{P1\ R_1} = \sqrt{20 \times 8} = \sqrt{160}$
$= 12.65$ volts.

Hence, the gain in db $= 20 \log \frac{E_1}{E_2} + 10 \log \frac{R_2}{R_1}$

$$= 20 \log \frac{12.65}{.005} + 10 \log \frac{10^6}{8}$$

$$= 20 \log 2530 + 10 \log (1.25 \times 10^5)$$

$$= 20 \times 3.4031 + 10 \times 5.0969 = 68.06 + 50.97 = 119.03 \text{ db}$$

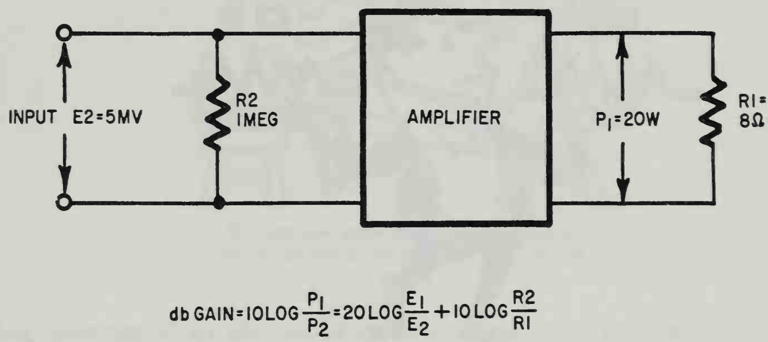


Fig. 505. *Problem involving the calculation of voltage gain of an amplifier.*

If you ignored the ratio of the resistances in this case, you would get an erroneous gain figure of 68 db (approximately) instead of 119 db. If you don't want to run into this kind of trouble, use only the power ratio to obtain the db gain. In this case,

$$\text{db} = 10 \log \frac{P1}{P2} = 10 \log \frac{20}{25 \times 10^{-12}} = 10 \log 8 \times 10^{11}$$

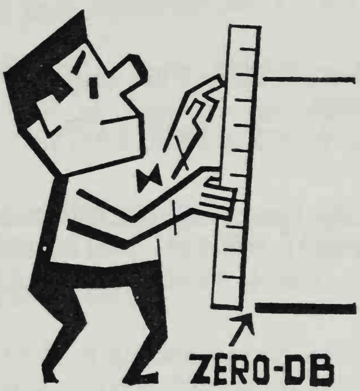
$$= 10 \times 11.903 = 119.03 \text{ decibels.}$$

The answer is, of course, the same as in the previous computation.

Decibel gain figures are especially useful for finding the stage-by-stage gain of a number of cascaded amplifier stages, since you can *add db* instead of multiplying the numerical voltage gain. Thus, if each stage of an amplifier has 20 db gain, corresponding to a voltage ratio of 1 to 10 across equal impedances, the gain of five such cascaded stages is 5×20, or 100 db, corresponding to an overall voltage ratio of 1 to 100,000.

DB reference levels

Decibels are also occasionally used in acoustics and electronics to express a power (sound or electrical) level with reference to some



The use of decibels often requires reference to a standard level.

standard zero-db reference level. Thus, in acoustics the zero-db reference level is set at 10^{-16} watt per square centimeter, which is approximately the faintest sound one can hear. All other sound powers may then be expressed with reference to this zero level. For example, the rustle of leaves in a gentle breeze is about 15 db above the zero-db level of 10^{-16} watts. The noise in an average residence is about 40 db above the zero level, a subway train has a level of about 100 db and a propeller-driven airplane at 18 feet from the ground produces about 125 db of noise. Similarly, in electrical work a zero-db level of 1 milliwatt across an impedance of 600 ohms is frequently used. However, other logarithmic units, such as VU (volume units), have slightly different definitions.

PRACTICE EXERCISE 7

1. Write the following in logarithmic form: $2^4 = 16$; $3^2 = 9$; $4^{3/2} = 8$; $5^0 = 1$; $3^{-1} = \frac{1}{3}$; $16^{-3/4} = \frac{1}{8}$; $25^{-1/2} = \frac{1}{5}$.

   (Answers: $\log_2 16 = 4$; $\log_3 9 = 2$; $\log_4 8 = \frac{3}{2}$; $\log_5 1 = 0$;

   $$\log_3 \frac{1}{3} = -1;\ \log_{16} \frac{1}{8} = -\frac{3}{4};\ \log_{25} \frac{1}{5} = -\frac{1}{2}.)$$

2. What are the characteristics of $\log_{10}$ of 82.16, 3,285, 1.845, 50,436, 0.62, .0865, .00006 and 3,890.05.
   (Answers: 1; 3; 0; 4; –1; –2; –5; 3.)

3. Given log 386.5 = 2.5872, what is log 38.65, log 3.865, log 38,650, log .0003865.
   (Answers: 1.5872; 0.5872; 4.5872; $\bar{4}.5872$.)

4. Find the log to the base 10 of the following numbers: 28.69; 4.8692; 1238.6; 0.8691; 0.004981; and 86,432,000.
   (Answers: 1.4578; 0.6874; 3.0929; $\bar{1}.9391$; $\bar{3}.6973$; 7.9367.)

5. Look up the antilog corresponding to the following logs (base 10): 1.8162; 7.5670 –10; 0.8197; 3.0067; 8.6283 –10; 9.1287 –10.
   (Answers: 65.5; .00369; 6.603; 1016; .04249; 0.1345.)

6. Compute by common logs the value of each of the following:

   (a) $(36.42)(111.42)(3.46)^2(.0176)$; (b) $\frac{1.58}{4{,}326}$; (c) $\left(\frac{8.643}{1.21}\right)^2$;

   (d) $(3.087)^4 \times (1.234)^2 \div (-3.5421)^3$; (e) $\frac{(861.42)\sqrt[6]{21.477}}{\sqrt[5]{-121.4}}$.

   (Answers: (a) 854.8; (b) .0003652; (c) 51.03; (d) –3.112; (e) –550.)

7. Solve the following: $3.7^x = 111.42$; $2.14^x = 62.875$; and $(3.86)^{x+1} = 0.4863$.
   (Answers: $x = 3.602$; $x = 5.443$; $x = -1.533$.)

8. Compute the value of ϵ correct to three decimal places by means of the infinite series given in the text. (Answer 2.718)

9. If $\log_a N = y$ and $\log_c N = x$, find two expressions for $\log_a N$ as a function of $\log_c N$.
(Answers: $\log_a N = \log_a c \cdot \log_c N = \dfrac{\log_c N}{\log_c a}$)

10. Find the natural logarithm ($\log_\epsilon$) of the following numbers: .01; 0.86; 3.74; 8.2; 44; 85; 143; 804; and 1,107.
(Answers: 5.395 –10; 9.849 –10; 1.31909; 2.10413; 3.78419; 4.44265; 4.6284; 6.8960; 7.0941.)

11. By means of natural logs compute: $\epsilon^{.04}$; $\epsilon^{.75}$; $\epsilon^{-0.61}$; $\epsilon^{1.25}$; $\epsilon^{-2.76}$; ϵ^{10} and ϵ^{-10}.
(Answers: 1.0408; 2.1170; 0.54335; 3.4903; .063292; 22,026 and .000045.)

12. If the battery voltage E in Fig. 506 is 250 volts, R = 50,000 ohms and C= .001 μf, find (a) the time constant of the cir-

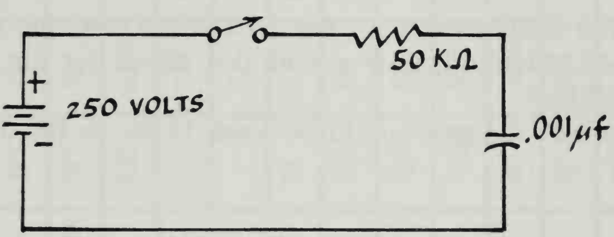


Fig. 506. *Circuit arrangement for problem 12.*

cuit; (b) E_R and E_C at 100 microseconds after the switch is closed; (c) the time (in microseconds) when E_R = 50 volts, and (d) the value of the charging current the instant the switch is closed, assuming that the capacitor was initially already charged to 100 volts (i.e., at t = 0, E_C = 100 volts).
(Answers: (a) 50 microseconds; (b) E_R = 33.7 volts, E_C = 216.3 volts; (c) 80.5 microseconds; (d) 3 milliamperes.)

13. In a gas-tube sweep oscillator, a capacitor (C) is charged through a resistor (R) from a dc plate-supply voltage, E_b. When the voltage across C reaches the ionization potential, E_i, of the tube, the capacitor is practically instantly discharged (R = 0) through the tube to a lower, deionization potential,

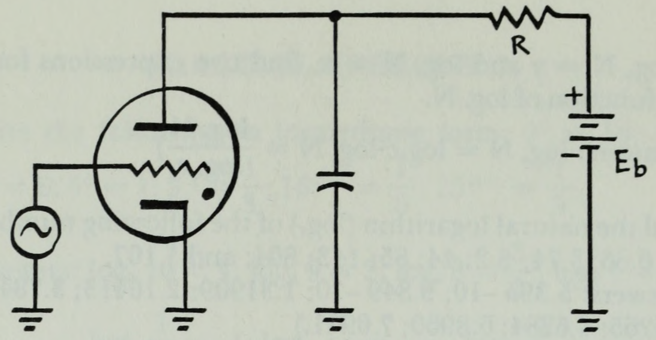


Fig. 507. *Circuit arrangement for problem 13.*

E_d. The cycle then repeats, the capacitor again being charged to E_i and then discharged to E_d. If the charge curve is exponential, as stated in the text, and the discharge takes zero time, derive an expression for the period, T, of the oscillation. (See Fig. 507.)

(Answer: $T = RC \log_\epsilon \left[\frac{(E_b - E_i)}{(E_b - E_d)}\right]$)

14. If the input resistor (in Fig. 505) of the amplifier was reduced to 50,000 ohms, how would this affect the db gain of the amplifier?
(Answer: the gain would be down 13 db, to 106 db.)

METRIC PREFIXES

TO CONVERT THESE TO —THESE. MULTIPLY BY THE FIGURES BELOW

	Pico—	Nano—	Micro—	Milli—	Centi—	Deci—	Units	Deka—	Hekto—	Kilo—	Myria—	Mega—	Giga—	Tera—
Pico—		0.001	10^{-6}	10^{-9}	10^{-10}	10^{-11}	10^{-12}	10^{-13}	10^{-14}	10^{-15}	10^{-16}	10^{-18}	10^{-21}	10^{-24}
Nano—	1000		0.001	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}	10^{-11}	10^{-12}	10^{-13}	10^{-15}	10^{-18}	10^{-21}
Micro—	10^{6}	1000		0.001	0.0001	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}	10^{-12}	10^{-15}	10^{-18}
Milli—	10^{9}	10^{6}	1000		0.1	0.01	0.001	0.0001	10^{-5}	10^{-6}	10^{-7}	10^{-9}	10^{-12}	10^{-15}
Centi—	10^{10}	10^{7}	10,000	10		0.1	0.01	0.001	0.0001	10^{-5}	10^{-6}	10^{-8}	10^{-11}	10^{-14}
Deci—	10^{11}	10^{8}	10^{5}	100	10		0.1	0.01	0.001	0.0001	10^{-5}	10^{-7}	10^{-10}	10^{-13}
Units	10^{12}	10^{9}	10^{6}	1000	100	10		0.1	0.01	0.001	0.0001	10^{-6}	10^{-9}	10^{-12}
Deka—	10^{13}	10^{10}	10^{7}	10,000	1000	100	10		0.1	0.01	0.001	10^{-5}	10^{-8}	10^{-11}
Hekto—	10^{14}	10^{11}	10^{8}	10^{5}	10,000	1000	100	10		0.1	0.01	0.0001	10^{-7}	10^{-10}
Kilo—	10^{15}	10^{12}	10^{9}	10^{6}	10^{5}	10,000	1000	100	10		0.1	0.001	10^{-6}	10^{-9}
Myria—	10^{16}	10^{13}	10^{10}	10^{7}	10^{6}	10^{5}	10,000	1000	100	10		0.01	10^{-5}	10^{-8}
Mega—	10^{18}	10^{15}	10^{12}	10^{9}	10^{8}	10^{7}	10^{6}	10^{5}	10,000	1000	100		0.001	10^{-6}
Giga—	10^{21}	10^{18}	10^{15}	10^{12}	10^{11}	10^{10}	10^{9}	10^{8}	10^{7}	10^{6}	10^{5}	1000		0.001
Tera—	10^{24}	10^{21}	10^{18}	10^{15}	10^{14}	10^{13}	10^{12}	10^{11}	10^{10}	10^{9}	10^{8}	10^{6}	1000	

COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

NATURAL LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
1.0	0.0000	0100	0198	0296	0392	0488	0583	0677	0770	0862
1.1	0953	1044	1133	1222	1310	1398	1484	1570	1655	1740
1.2	1823	1906	1989	2070	2151	2231	2311	2390	2469	2546
1.3	2624	2700	2776	2852	2927	3001	3075	3148	3221	3293
1.4	3365	3436	3507	3577	3646	3716	3784	3853	3920	3988
1.5	0.4055	4121	4187	4253	4318	4383	4447	4511	4574	4637
1.6	4700	4762	4824	4886	4947	5008	5068	5128	5188	5247
1.7	5306	5365	5423	5481	5539	5596	5653	5710	5766	5822
1.8	5878	5933	5988	6043	6098	6152	6206	6259	6313	6366
1.9	6419	6471	6523	6575	6627	6678	6729	6780	6831	6881
2.0	0.6931	6981	7031	7080	7129	7178	7227	7275	7324	7372
2.1	7419	7467	7514	7561	7608	7655	7701	7747	7793	7839
2.2	7885	7930	7975	8020	8065	8109	8154	8198	8242	8286
2.3	8329	8372	8416	8459	8502	8544	8587	8629	8671	8713
2.4	8755	8796	8838	8879	8920	8961	9002	9042	9083	9123
2.5	0.9163	9203	9243	9282	9322	9361	9400	9439	9478	9517
2.6	9555	9594	9632	9670	9708	9746	9783	9821	9858	9895
2.7	0.9933	9969	0006	0043	0080	0116	0152	0188	0225	0260
2.8	1.0296	0332	0367	0403	0438	0473	0508	0543	0578	0613
2.9	0647	0682	0716	0750	0784	0818	0852	0886	0919	0953
3.0	1.0986	1019	1053	1086	1119	1151	1184	1217	1249	1282
3.1	1314	1346	1378	1410	1442	1474	1506	1537	1569	1600
3.2	1632	1663	1694	1725	1756	1787	1817	1848	1878	1909
3.3	1939	1969	2000	2030	2060	2090	2119	2149	2179	2208
3.4	2238	2267	2296	2326	2355	2384	2413	2442	2470	2499
3.5	1.2528	2556	2585	2613	2641	2669	2698	2726	2754	2782
3.6	2809	2837	2865	2892	2920	2947	2975	3002	3029	3056
3.7	3083	3110	3137	3164	3191	3218	3244	3271	3297	3324
3.8	3350	3376	3403	3429	3455	3481	3507	3533	3558	3584
3.9	3610	3635	3661	3686	3712	3737	3762	3788	3813	3838
4.0	1.3863	3888	3913	3938	3962	3987	4012	4036	4061	4085
4.1	4110	4134	4159	4183	4207	4231	4255	4279	4303	4327
4.2	4351	4375	4398	4422	4446	4469	4493	4516	4540	4563
4.3	4586	4609	4633	4656	4679	4702	4725	4748	4770	4793
4.4	4816	4839	4861	4884	4907	4929	4951	4974	4996	5019
4.5	1.5041	5063	5085	5107	5129	5151	5173	5195	5217	5239
4.6	5261	5282	5304	5326	5347	5369	5390	5412	5433	5454
4.7	5476	5497	5518	5539	5560	5581	5602	5623	5644	5665
4.8	5686	5707	5728	5748	5769	5790	5810	5831	5851	5872
4.9	5892	5913	5933	5953	5974	5994	6014	6034	6054	6074
5.0	1.6094	6114	6134	6154	6174	6194	6214	6233	6253	6273
5.1	6292	6312	6332	6351	6371	6390	6409	6429	6448	6467
5.2	6487	6506	6525	6544	6563	6582	6601	6620	6639	6658
5.3	6677	6696	6715	6734	6752	6771	6790	6808	6827	6845
5.4	6864	6882	6901	6919	6938	6956	6974	6993	7011	7029

NATURAL LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
5.5	1.7047	7066	7084	7102	7120	7138	7156	7174	7192	7210
5.6	7228	7246	7263	7281	7299	7317	7334	7352	7370	7387
5.7	7405	7422	7440	7457	7475	7492	7509	7527	7544	7561
5.8	7579	7596	7613	7630	7647	7664	7681	7699	7716	7733
5.9	7750	7766	7783	7800	7817	7834	7851	7867	7884	7901
6.0	1.7918	7934	7951	7967	7984	8001	8017	8034	8050	8066
6.1	8083	8099	8116	8132	8148	8165	8181	8197	8213	8229
6.2	8245	8262	8278	8294	8310	8326	8342	8358	8374	8390
6.3	8405	8421	8437	8453	8469	8485	8500	8516	8532	8547
6.4	8563	8579	8594	8610	8625	8641	8656	8672	8687	8703
6.5	1.8718	8733	8749	8764	8779	8795	8810	8825	8840	8856
6.6	8871	8886	8901	8916	8931	8946	8961	8976	8991	9006
6.7	9021	9036	9051	9066	9081	9095	9110	9125	9140	9155
6.8	9169	9184	9199	9213	9228	9242	9257	9272	9286	9301
6.9	9315	9330	9344	9359	9373	9387	9402	9416	9430	9445
7.0	1.9459	9473	9488	9502	9516	9530	9544	9559	9573	9587
7.1	9601	9615	9629	9643	9657	9671	9685	9699	9713	9727
7.2	9741	9755	9769	9782	9796	9810	9824	9838	9851	9865
7.3	1.9879	9892	9906	9920	9933	9947	9961	9974	9988	0001
7.4	2.0015	0028	0042	0055	0069	0082	0096	0109	0122	0136
7.5	2.0149	0162	0176	0189	0202	0215	0229	0242	0255	0268
7.6	0281	0295	0308	0321	0334	0347	0360	0373	0386	0399
7.7	0412	0425	0438	0451	0464	0477	0490	0503	0516	0528
7.8	0541	0554	0567	0580	0592	0605	0618	0631	0643	0656
7.9	0669	0681	0694	0707	0719	0732	0744	0757	0769	0782
8.0	2.0794	0807	0819	0832	0844	0857	0869	0882	0894	0906
8.1	0919	0931	0943	0956	0968	0980	0992	1005	1017	1029
8.2	1041	1054	1066	1078	1090	1102	1114	1126	1138	1150
8.3	1163	1175	1187	1199	1211	1223	1235	1247	1258	1270
8.4	1282	1294	1306	1318	1330	1342	1353	1365	1377	1389
8.5	2.1401	1412	1424	1436	1448	1459	1471	1483	1494	1506
8.6	1518	1529	1541	1552	1564	1576	1587	1599	1610	1622
8.7	1633	1645	1656	1668	1679	1691	1702	1713	1725	1736
8.8	1748	1759	1770	1782	1793	1804	1815	1827	1838	1849
8.9	1861	1872	1883	1894	1905	1917	1928	1939	1950	1961
9.0	2.1972	1983	1994	2006	2017	2028	2039	2050	2061	2072
9.1	2083	2094	2105	2116	2127	2138	2148	2159	2170	2181
9.2	2192	2203	2214	2225	2235	2246	2257	2268	2279	2289
9.3	2300	2311	2322	2332	2343	2354	2364	2375	2386	2396
9.4	2407	2418	2428	2439	2450	2460	2471	2481	2492	2502
9.5	2.2513	2523	2534	2544	2555	2565	2576	2586	2597	2607
9.6	2618	2628	2638	2649	2659	2670	2680	2690	2701	2711
9.7	2721	2732	2742	2752	2762	2773	2783	2793	2803	2814
9.8	2824	2834	2844	2854	2865	2875	2885	2895	2905	2915
9.9	2925	2935	2946	2956	2966	2976	2986	2996	3006	3016
10.0	2.3026									

MATHEMATICAL SYMBOLS

Symbol	Name	Description
1, 2, 3, 4, 5, 6, 7, 8, 9, 0	Numerals	The numbers used in mathematics
∞	Infinity	A quantity larger than any number
.	Decimal point	A period used to point off numbers
$+$	Plus sign	Addition, positive
$-$	Minus sign	Subtraction, negative
$\pm$	Plus or minus	Addition or subtraction
$\mp$	Minus or plus	Subtraction or addition
$\times$	Multiply by	Multiplication (arithmetic)
$a \cdot b$	a is multipled by b	Multiplication (algebra)
$\div$	Divided by	Division
a/b	a is divided by b or "a is to b"	Division (fraction or ratio form)
$\frac{a}{b}$	a is divided by b or "a is to b"	Division (fraction or ratio form)
$a:b$	a is to b	Ratio form of division
$=$	Equal to	Equality
$\equiv$	Identical with	Identity
$\backsimeq$	Approximately equal to	Approximate equality in equations
$\neq$	Not equal to	Inequality
$<$	Less than	Inequality
$>$	Greater than	Inequality
$\leqq$	Less than or equal	
$\geqq$	Greater than or equal	
$\rightarrow$	Approaches (as a limit)	
$\perp$	Perpendicular	Geometry
$\parallel$	Parallel to	Geometry
$\angle$	Positive angle sign	Geometry
$\ulcorner$	Negative angle sign	Geometry
$\triangle$	Triangle	Geometry
$\odot$	Circle	Geometry
$\therefore$	Therefore	Geometry, Logic
# or No̤	Number	When placed before numerals
%	Percent	Placed after number expressing percent
. . .	Continued	To indicate the continuation of a sequence numbers or series

MATHEMATICAL SYMBOLS

Symbol	Name	*Description*
ϵ	Epsilon	$\epsilon = 2.71828183\ldots$, the base of natural logarithms (Napierian base)
$\frac{1}{\epsilon}$		0.3679
$4!$ or $\underline{4\rvert}$	Factorial	A number multiplied by all smaller numbers, thus $4! = \underline{4\rvert} = (4)\ (3)\ (2)\ (1) = 24$
Σ	Sigma, Summation	To add a series of terms
$\sum_{o}^{n}$	Summation from *o* to *n*	To add a series of terms ranging from *o* to *n*
( )	Parentheses	A sign at both ends of a grouping
[ ]	Brackets	A sign at both ends of a grouping
{ }	Braces	A sign at both ends of a grouping
—	Vinculum	A line placed over a grouping
$\sqrt{\ }$	Radical	Extract the square root of a number
$\sqrt{\overline{\quad}}$	Radical and vinculum	Extract the square root of a group of quantities or numbers
$\sqrt[4]{A}$ *or* $A^{1/4}$	Fourth root of the quantity *A*	Extract the fourth root of *A* or raise *A* to the fourth power
A^2	The second power of the quantity *A*	The power to which a quantity is to be raised. The number 2 is called the exponent of *A*
$\lvert A \rvert$	The absolute value of vector **A**	The magnitude of a vector quantity
$\dot{A}$ or $\bar{A}$	*A* is a vector	A quantity that has both magnitude and direction
A	*A* is a vector	When the magnitude is expressed $\lvert A \rvert$, **A** can be used to express a vector quantity
A_x	*x* is a subscript	To identify the quantity *A*, as, for example, lying along the *x*-axis
j	$(\sqrt{-1})$	The j operator in electrical engineering
i	$(\sqrt{-1})$	Imaginary number
°	Degree sign	Angles in degrees
′	Minute sign	Angles in minutes
″	Second sign	Angles in seconds
$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$	Determinant sign	The determinant of two simultaneous equations

TABLE OF DECIMAL EQUIVALENTS

.0156– 1/64	.2656–17/64	.5156–33/64	.7656–49/64
1/32–.0312	9/32–.2812	17/32–.5312	25/32–.7812
.0469– 3/64	.2969–19/64	.5469–35/64	.7969–51/64
1/16–.0625	5/16–.3125	9/16–.5625	13/16–.8125
.0781– 5/64	.3281–21/64	.5781–37/64	.8281–53/64
3/32–.0937	11/32–.3437	19/32–.5937	27/32–.8437
.1094– 7/64	.3594–23/64	.6094–39/64	.8594–55/64
1/8 –.1250	3/8–.3750	5/8–.6250	7/8–.8750
.1406– 9/64	.3906–25/64	.6406–41/64	.8906–57/64
5/32–.1562	13/32–.4062	21/32–.6562	29/32–.9062
.1719–11/64	.4219–27/64	.6719–43/64	.9219–59/64
3/16–.1875	7/16–.4375	11/16–.6875	15/16–.9375
.2031–13/64	.4531–29/64	.7031–45/64	.9531–61/64
7/32–.2187	15/32–.4687	23/32–.7187	31/32–.9687
.2344–15/64	.4844–31/64	.7344–47/64	.9844–63/64
1/4–.2500	1/2–.5000	3/4–.7500	1–1.000

POWER AND VOLTAGE RATIOS in db

Voltage Ratio	Power Ratio	Decibels	Inverse Voltage Ratio	Inverse Power Ratio
1.000	1.000	0	1.0000	1.0000
1.122	1.259	1	.8913	.7943
1.259	1.585	2	.7943	.6310
1.413	1.995	3	.7079	.5012
1.585	2.512	4	.6310	.3981
1.778	3.162	5	.5623	.3162
1.995	3.981	6	.5012	.2512
2.239	5.012	7	.4467	.1995
2.512	6.310	8	.3981	.1585
2.818	7.943	9	.3548	.1259
3.162	10.000	10	.3162	.1000
3.548	12.59	11	.2818	.07943
3.981	15.85	12	.2512	.06310
4.467	19.95	13	.2239	.05012
5.012	25.12	14	.1995	.03981
5.623	31.62	15	.1778	.03162
6.310	39.81	16	.1585	.02512
7.079	50.12	17	.1413	.01995
7.943	63.10	18	.1259	.01585
8.913	79.43	19	.1122	.01259
10.000	100.00	20	.1000	.01
17.78	316.2	25	.056	.00316
31.62	1,000	30	.03162	.001
56.23	3,162	35	.01778	.000316
100.0	10,000	40	.010	.0001
177.8	31,620	45	.0056	.0000316
316.2	100,000	50	.003162	.00001
1,000	1,000,000	60	.001	.000001
3,162	10,000,000	70	.0003162	.0000001
10,000	100,000,000	80	.0001	.00000001
31,620	1,000,000,000	90	.00003162	.000000001
100,000	10,000,000,000	100	.00001	.0000000001

ALGEBRAIC FORMULAS

(1) $a + b = b + a$

(2) $(a + b) + c = a + (b + c)$

(3) $(a + b + c) = a + b + c$

(4) $-(a - b + c) = -a + b - c$

(5) $(a + b - c) = a + b - c$

(6) $a + c = b + d$, if $a = b$ and $c = d$

(7) $ab = ba$

(8) $(ab)c = a(bc) = abc = (a)(b)(c)$

(9) $a(b + c) = ab + ac$

(10) $ac = bd$, if $a = b$ and $c = d$

(11) $a + (-b) = a - b$

(12) $-a - (-b) = -a + b$

(13) $\frac{0}{a} = 0$; $(0)(a) = 0$

14) $(\infty)(a) = \infty$

(15) $\frac{\infty}{a} = \infty$

(16) $a(-b) = -ab$

(17) $(-a)(-b) = ab$

(18) $-a(-b) = ab$

(19) $\frac{a}{b} = \frac{ac}{bc}$

(20) $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$

(21) $\frac{a}{-b} = \frac{-a}{b} = -\frac{a}{b}$

(22) $\frac{a}{c} + \frac{b}{c} = \frac{ac + bc}{c^2} = \frac{a + b}{c}$

(23) $\frac{a}{c} - \frac{b}{c} = \frac{ac - bc}{c^2} = \frac{a - b}{c}$

(24) $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

(25) $\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd}$

(26) $\frac{\frac{a}{b}}{\frac{c}{d}} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc}$

(27) $(a^m)(a^n) = a^{m+n}$

(28) $(a^m)^n = a^{mn}$

(29) $(abc)^m = a^m b^m c^m$

(30) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

(31) $\frac{a^m}{a^n} = a^{m-n}$

(32) $a^0 = 1$

(33) $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

(34) $a^{-m} = \frac{1}{a^m}$

(35) $(a + b)(a + b) = a^2 + 2ab + b^2 = (a + b)^2$

(36) $(a + b)(a - b) = a^2 - b^2$

(37) $(a - b)(a - b) = a^2 - 2ab + b^2 = (a - b)^2$

(38) $(a + b)(c - d) = ac + bc - ad - bd$

MISCELLANEOUS

(1) $\epsilon = 2.71828183$

(2) $\frac{1}{\epsilon} = 0.36787944$

(3) $\log \epsilon = 0.43429448$

(4) $\pi = 3.14159265$

(5) $2\pi = 6.28318530$

(6) $\frac{1}{\pi} = 0.31830989$

(7) $\pi^2 = 9.8690440$

(8) $\sqrt{\pi} = 1.77245385$

(9) $\log \pi = 0.49714987$

(10) $\log 1 = 0.000000$

(11) $\log 2 = 0.301030$

(12) $\log 3 = 0.477121$

(13) $\log 4 = 0.602060$

(14) $\log 5 = 0.698970$

(15) $\log 6 = 0.778151$

(16) $\log 7 = 0.845098$

(17) $\log 8 = 0.903090$

(18) $\log 9 = 0.954243$

(19) $\log 10 = 1.000000$

(20) $1 \text{ radian} = 180°/\pi = 57°17'44.8''$

(21) $1° = \pi/180° = 0.01745329 \text{ radian}$

(22) $\sqrt{1} = 1.0000$

(23) $\sqrt{2} = 1.4142$

(24) $\sqrt{3} = 1.7321$

(25) $\sqrt{4} = 2.0000$

(26) $\sqrt{5} = 2.2361$

(27) $\sqrt{6} = 2.4495$

(28) $\sqrt{7} = 2.6458$

(29) $\sqrt{8} = 2.8284$

(30) $\sqrt{9} = 3.0000$

(31) $\sqrt{10} = 3.1623$

(32) $j = \sqrt{-1}$

[logs are to base 10]

MISCELLANEOUS

(1)	$e = 2.7182818$	(17)	$\log 8 = 0.903090$
(2)	$\frac{1}{e} = 0.3678794$	(18)	$\log 9 = 0.954243$
(3)	$\log e = 0.4342945$	(19)	$\log 10 = 1.000000$
(4)	$\pi = 3.1415927$	(20)	$1 \text{ radian} = 180^\circ/\pi = 57^\circ 17' 44.8''$
(5)	$2\pi = 6.2831853$	(21)	$1^\circ = \pi/180^\circ = 0.01745329$ radian
(6)	$\frac{1}{\pi} = 0.3183099$	(22)	$\sqrt{1} = 1.0000$
(7)	$\pi^2 = 9.8696044$	(23)	$\sqrt{2} = 1.4142$
(8)	$\sqrt{\pi} = 1.7724539$	(24)	$\sqrt{3} = 1.7321$
(9)	$\log \pi = 0.4971499$	(25)	$\sqrt{4} = 2.0000$
(10)	$\log 1 = 0.000000$	(26)	$\sqrt{5} = 2.2361$
(11)	$\log 2 = 0.301030$	(27)	$\sqrt{6} = 2.4495$
(12)	$\log 3 = 0.477121$	(28)	$\sqrt{7} = 2.6458$
(13)	$\log 4 = 0.602060$	(29)	$\sqrt{8} = 2.8284$
(14)	$\log 5 = 0.698970$	(30)	$\sqrt{9} = 3.0000$
(15)	$\log 6 = 0.778151$	(31)	$\sqrt{10} = 3.1623$
(16)	$\log 7 = 0.845098$	(32)	$i = \sqrt{-1}$

(Logs are to base 10)