arXiv:math/0001083v2 [math.RA] 2 May 2002

February 1, 2008

MAGIC SQUARES OF LIE ALGEBRAS

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Abstract. This paper is an investigation of the relation between Tit's magic square of Lie algebras and certain Lie algebras of 3×3 and 6×6 matrices with entries in alternative algebras. By reformulating Tit's definition in terms of *trialities* (a generalisation of derivations), we give a systematic explanation of the symmetry of the magic square. We show that when the columns of the magic square are labelled by the real division algebras and the rows by their split versions, then the rows can be interpreted as analogues of the matrix Lie algebras $\mathfrak{su}(3)$, $\mathfrak{sl}(3)$ and $\mathfrak{sp}(6)$ defined for each division algebra. We also define another magic square based on 2×2 and 4×4 matrices and prove that it consists of various orthogonal or (in the split case) pseudo-orthogonal Lie algebras.

1. Introduction

Semisimple Lie groups and Lie algebras are normally discussed in terms of their root systems, which makes possible a unified treatment and emphasises the common features of their underlying structures. However, some classical investigations [16] depend on particularly simple matrix descriptions of Lie groups. This creates a distinction between the classical groups (naturally enough) and the exceptional ones, which is maintained in some more recent work (e.g. [6, 9, 10]). This paper is motivated by the desire to give a similar matrix description of the exceptional groups, thus assimilating them to the classical groups, with a view to extending results like the Capelli identities to the exceptional cases.

It has long been known [9] that most exceptional Lie algebras are related to the exceptional Jordan algebra of 3×3 hermitian matrices with entries from the octonions, $\mathbb{O}$. Here we show that this relation yields descriptions of certain real forms of the complex Lie algebras F_4, E_6 and E_7 which can be interpreted as octonionic versions of, respectively, the Lie algebra of antihermitian 3×3 matrices, that of special linear 3×3 matrices and that of symplectic 6×6 matrices. To be precise, we define for each alternative algebra $\mathbb{K}$ a Lie algebra $\mathfrak{sa}(3, \mathbb{K})$ such that $\mathfrak{sa}(3, \mathbb{C}) = \mathfrak{su}(3)$ and $\mathfrak{sa}(3, \mathbb{O})$ is the compact real form of F_4; a Lie algebra $\mathfrak{sl}(3, \mathbb{K})$ equal to $\mathfrak{sl}(3, \mathbb{C})$ for $\mathbb{K} = \mathbb{C}$ and a non-compact real form of E_6 for $\mathbb{K} = \mathbb{O}$; and a Lie algebra $\mathfrak{sp}(6, \mathbb{K})$ such that $\mathfrak{sp}(6, \mathbb{C})$ is the set of 6×6 complex matrices X satisfying $X^\dagger J = -JX$, (where J is an antisymmetric real 6×6 matrix and $X^\dagger$ denotes the hermitian conjugate of X), and such that $\mathfrak{sp}(6,\mathbb{O})$ is a non-compact real form of E_7.

Our definitions can be adapted to yield Lie algebras $\mathfrak{sa}(2,\mathbb{K}), \mathfrak{sl}(2,\mathbb{K})$ and $\mathfrak{sp}(4,\mathbb{K})$ reducing to $\mathfrak{su}(2), \mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{sp}(4,\mathbb{C})$ when $\mathbb{K} = \mathbb{C}$. These Lie algebras are isomorphic to various pseudo-orthogonal algebras.

These constructions are all related to Tits's magic square of Lie algebras [15] and based on an unpublished suggestion of Ramond [12]. The magic square of Tits is a construction of a Lie algebra $L(\mathbb{J},\mathbb{K})$ for any Jordan algebra $\mathbb{J}$ and alternative algebra $\mathbb{K}$. If $\mathbb{J} = H_3(\mathbb{K}_1)$ is the Jordan algebra of 3×3 matrices with entries from an alternative algebra $\mathbb{K}_1$ and if $\mathbb{K} = \mathbb{K}_2$ is another alternative algebra, this yields a Lie algebra $L_3(\mathbb{K}_1,\mathbb{K}_2)$ for any pair of alternative algebras. Taking $\mathbb{K}_1$ and $\mathbb{K}_2$ to be real division algebras, we obtain a 4×4 square of compact Lie algebras which (magically) is symmetric and contains the compact real forms of F_4, E_6, E_7 and E_8. We will show that if the division algebra $\mathbb{K}_2$ is replaced by its split form $\tilde{\mathbb{K}}_2$, one obtains a non-symmetric square of Lie algebras whose first three rows are the sets of matrix Lie algebras described above:

$$\begin{aligned} L_3(\mathbb{K},\mathbb{R}) &= \mathfrak{sa}(3,\mathbb{K}) \\ L_3(\mathbb{K},\tilde{\mathbb{C}}) &= \mathfrak{sl}(3,\mathbb{K}) \\ L_3(\mathbb{K},\tilde{\mathbb{H}}) &= \mathfrak{sp}(6,\mathbb{K}). \end{aligned} \tag{1}$$

We will also describe magic squares of Lie algebras based on 2×2 matrices, which have similar properties.

The organisation of the paper is as follows. In Section 2 we establish notation and recall the definitions of the various kinds of algebra with which we will be concerned. In Section 3 we give Tits's definition of the Lie algebras $L_3(\mathbb{K}_1,\mathbb{K}_2)$ and state the main properties of the magic square; we also give the definition and properties of the 2×2 magic square $L_2(\mathbb{K}_1,\mathbb{K}_2)$. Section 4 is concerned with the symmetry property of the 3×3 magic square: we reformulate the definition of $L_3(\mathbb{K}_1,\mathbb{K}_2)$, using Ramond's concept of a *triality algebra*, so as to make the symmetry manifest. Section 5 contains proofs of the properties of the 2×2 magic square which were stated in Section 3, and Section 6 contains the proofs of the corresponding properties of the 3×3 magic square.

2. Notation.

We will use the notation $\dot{+}$ to denote the direct sum of vector spaces. This enables us to reserve the use of $\oplus$ to denote the direct sum of Lie algebras, i.e. $A\oplus B$ implies that $[A,B]=0$.

An algebra $\mathbb{K}$ (over $\mathbb{R}$) with a non-degenerate quadratic form, which we will denote by $x \mapsto |x|^2$, satisfying

$$|xy|^2 = |x|^2\,|y|^2 \quad x, y \in \mathbb{K}, \tag{2}$$

is known as a *composition algebra*. We consider $\mathbb{R}$ to be embedded in $\mathbb{K}$ as the set of scalar multiples of the identity element, and denote by $\mathbb{K}'$ the subspace of $\mathbb{K}$ orthogonal to $\mathbb{R}$. It can then be shown [8] that $\mathbb{K} = \mathbb{R} \dot{+} \mathbb{K}'$ and we write $x = \operatorname{Re} x + \operatorname{Im} x$ with $\operatorname{Re} x \in \mathbb{R}$ and $\operatorname{Im} x \in \mathbb{K}'$. It can also be shown that the conjugation which fixes each element of $\mathbb{R}$ and multiplies every element of $\mathbb{K}'$ by -1, denoted $x \mapsto \overline{x}$, satisfies

$$\overline{xy} = \overline{y}\,\overline{x} \tag{3}$$

as well as

$$x\overline{x} = |x|^2 . \tag{4}$$

We use the notation $[x, y, z]$ for the associator

$$[x, y, z] = (xy)z - x(yz). \tag{5}$$

Any composition algebra $\mathbb{K}$ satisfies the *alternative law*, i.e. the associator is an alternating function of x, y and z. If $|x|^2$ is positive definite then $\mathbb{K}$ is a division algebra.

A division algebra is an algebra in which we have

$$xy = 0 \Rightarrow x = 0 \text{ or } y = 0.$$

The only such positive definite composition algebras are $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ (Hurwitz's Theorem) [13] which we denote in general by $\mathbb{K}$. We denote their dimension by ν, thus $\nu = 1, 2, 4$ or 8. These algebras are obtained from the Cayley-Dickson process [13] and, using the same process with different signs, *split* forms of these algebras can also be obtained. These are so called because in $\mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ we have

$$i^2 + 1 = 0$$

but in the split algebras $\tilde{\mathbb{C}}, \tilde{\mathbb{H}}$ and $\tilde{\mathbb{O}}$ we have

$$i^2 - 1 = (i + 1)(i - 1) = 0$$

i.e. the equation can be *split* for at least one of the imaginary basis elements. Thus whilst the positive definite algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ are division algebras, the split forms $\tilde{\mathbb{C}}, \tilde{\mathbb{H}}$ and $\tilde{\mathbb{O}}$ are not.

Our notation for Lie algebras is that used in [14]. We use the notation $A^\dagger$ for the hermitian conjugate of the matrix A with entries in $\mathbb{K}$, defined in analogy to the complex case by

$$(X^\dagger)_{ij} = \overline{X}_{ji}.$$

We use $\mathfrak{su}(s, t)$ for the Lie algebra of the pseudo-unitary group,

$$\mathfrak{su}(s, t) = \{A \in \mathbb{C}^{n\times n} : A^\dagger G + GA = 0\}$$

where $G = \mathrm{diag}(-1, \ldots, -1, +1, \ldots, +1)$ with s $-$ signs and t $+$ signs; $\mathfrak{sq}(n)$ for the Lie algebra of antihermitian quaternionic matrices A,

$$\mathfrak{sq}(n) = \{A \in \mathbb{H}^{n\times n} : A^\dagger = -A\};$$

and $\mathfrak{sp}(2n, \mathbb{K})$ for the Lie algebra of the symplectic group of $2n \times 2n$ matrices with entries in $\mathbb{K}$, i.e.

$$\mathfrak{sp}(2n, \mathbb{K}) = \{A \in \mathbb{K}^{2n\times 2n} : A^\dagger J + JA = 0\}$$

where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. We also have $\mathfrak{so}(s, t)$, the Lie algebra of the pseudo-orthogonal group $\mathrm{SO}(s, t)$, given by

$$\mathfrak{so}(s, t) = \{A \in \mathbb{R}^{n\times n} : A^T G + GA = 0\}$$

where G is defined as before. We will also write $\mathrm{O}(V, q)$ for the group of linear maps of the vector space V preserving the non-degenerate quadratic form q, $\mathrm{SO}(V, q)$ for its unimodular (or special) subgroup, $\mathfrak{o}(V, q)$ and $\mathfrak{so}(V, q)$ for their Lie algebras. We omit q if it is understood from the context. Thus for any division algebra we have $\mathrm{SO}(\mathbb{K})$ and $\mathfrak{so}(\mathbb{K})$.

A Jordan algebra $\mathbb{J}$ is defined to be a commutative algebra (over a field $\mathbb{K}$) in which all products satisfy the Jordan identity

$$(xy)x^2 = x(yx^2).$$

Let $L_n(\mathbb{K})$ be the set of all $n \times n$ matrices with entries in $\mathbb{K}$, and let $H_n(\mathbb{K})$ and $A_n(\mathbb{K})$ be the sets of all hermitian and antihermitian matrices with entries in $\mathbb{K}$ respectively. We denote by $H_n'(\mathbb{K})$, $A_n'(\mathbb{K})$ and $L_n'(\mathbb{K})$ the subspaces of traceless matrices of $H_n(\mathbb{K})$, $A_n(\mathbb{K})$ and $L_n(\mathbb{K})$ respectively. We thus have $L_n(\mathbb{K}) = H_n(\mathbb{K}) \dot{+} A_n(\mathbb{K})$ and $L_n'(\mathbb{K}) = H_n'(\mathbb{K}) \dot{+} A_n'(\mathbb{K})$. We will use the fact that $H_n(\mathbb{K})$ is a Jordan algebra for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ for all n and for $\mathbb{K} = \mathbb{O}$ when $n = 2, 3$ [14], with the Jordan product as the anticommutator

$$X \cdot Y = XY + YX.$$

This is a commutative but non-associative product.

The derivation algebra, $\mathrm{Der}\, A$, of any algebra A is defined as

$$\mathrm{Der}\, A = \{D \mid D(xy) = D(x)y + xD(y)\} \tag{6}$$

for $x, y \in A$. The derivation algebras of the four positive definite composition algebras are as follows:

$$\mathrm{Der}\, \mathbb{R} = \mathrm{Der}\, \mathbb{C} = 0,$$
$$\mathrm{Der}\, \mathbb{H} = C(\mathbb{H}') = \{C_a \mid a \in \mathbb{H}'\} \text{ where } C_a(q) = aq - qa.$$

$\mathrm{Der}\, \mathbb{O}$ is an exceptional Lie algebra of type G_2.

The structure algebra $\operatorname{Str} A$ of any algebra A is defined to be the Lie algebra generated by left and right multiplication maps L_a and R_a for $a \in A$. For Jordan algebras this can be shown to be [13]

$$\operatorname{Str} \mathbb{J} = \operatorname{Der} \mathbb{J} \dot{+} L(\mathbb{J}) \tag{7}$$

where $L(\mathbb{J})$ is the set of all L_a with $a \in \mathbb{J}$. The algebra denoted by $\operatorname{Str}' \mathbb{J}$ is the structure algebra with its centre factored out. We also require another Lie algebra associated with a Jordan algebra, namely the conformal algebra as constructed by Kantor (1973) and Koecher (1967). The underlying vector space of this is

$$\operatorname{Con} \mathbb{J} = \operatorname{Str} \mathbb{J} \dot{+} \mathbb{J}^2. \tag{8}$$

We will specify the Lie brackets of these algebras at a later stage.

3. Magic Squares: Summary of Results

3.1. 3×3 **Matrices.** Let $\mathbb{K}$ be a real composition algebra and $\mathbb{J}$ a real Jordan algebra, with $\mathbb{K}'$ and $\mathbb{J}'$ the quotients of the algebras by the subspaces of scalar multiples of the identity. Define a vector space

$$M(\mathbb{J}, \mathbb{K}) = \operatorname{Der} \mathbb{J} \dot{+} (\mathbb{J}' \otimes \mathbb{K}') \dot{+} \operatorname{Der} \mathbb{K}. \tag{9}$$

Then define

$$L_3(\mathbb{K}_1, \mathbb{K}_2) = M(H_3(\mathbb{K}_1), \mathbb{K}_2).$$

Explicitly this is the vector space

$$L_3(\mathbb{K}_1, \mathbb{K}_2) = \operatorname{Der} H_3(\mathbb{K}_1) \dot{+} H_3'(\mathbb{K}_1) \otimes \mathbb{K}_2' \dot{+} \operatorname{Der} \mathbb{K}_2 \tag{10}$$

which is a Lie algebra with Lie subalgebras $\operatorname{Der} H_3(\mathbb{K}_1)$ and $\operatorname{Der} \mathbb{K}_2$ when taken with the brackets

$$\begin{aligned} &[D, A \otimes x] = D(A) \otimes x \\ &[E, A \otimes x] = A \otimes E(x) \\ &[D, E] = 0 \qquad (11) \\ &[A \otimes x, B \otimes y] = \tfrac{1}{6}\langle A, B\rangle D_{x,y} + (A * B) \otimes \tfrac{1}{2}[x, y] - \langle x, y\rangle [L_A, L_B] \end{aligned}$$

with $D \in \operatorname{Der} H_3(\mathbb{K}_1)$; $A, B \in H_3'(\mathbb{K}_1)$; $x, y \in \mathbb{K}_2'$ and $E \in \operatorname{Der} \mathbb{K}_2$. These brackets are obtained from Schafer's description of the Tits construction [13]. They require some explanation. $\langle A, B\rangle$ and (x, y) denote the symmetric bilinear forms on $H_3(\mathbb{K}_1)$ and $\mathbb{K}_2$ respectively, given by

$$\begin{aligned} \langle A, B\rangle &= \operatorname{Re}(\operatorname{tr}(A \cdot B)) = 2 \operatorname{Re}(\operatorname{tr}(AB)) \\ \langle x, y\rangle &= \tfrac{1}{2}(|x + y|^2 - |x|^2 - |y|^2) = \operatorname{Re}(x\overline{y}). \end{aligned}$$

The derivation $D_{x,y}$ is defined as

$$D_{x,y} = [L_x, L_y] + [L_x, R_y] + [R_x, R_y] \quad \in \operatorname{Der} \mathbb{K}_2. \tag{12}$$

For future reference we note that

$$D_{x,y} z = [[x, y], z] - 3[x, y, z] \tag{13}$$

which shows that $D_{x,y} = -D_{y,x}$. Finally $(A * B)$ is the traceless part of the Jordan product of A and B,

$$A * B = A \cdot B - \tfrac{1}{3}\,\mathrm{tr}(A \cdot B). \tag{14}$$

Tits [15] (see also [3, 13]) showed that this gives a unified construction leading to the so-called magic square of Lie algebras of 3×3 matrices whose complexifications are

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	A_1	A_2	C_3	F_4
$\mathbb{C}$	A_2	$A_2 \oplus A_2$	A_5	E_6
$\mathbb{H}$	C_3	A_5	B_6	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

The striking properties of this square are (a) its symmetry and (b) the fact that four of the five exceptional Lie algebras occur in its last row. The explanation of the symmetry property is the subject of section 4. The fifth exceptional Lie algebra, G_2, can be included by adding an extra row corresponding to the Jordan algebra $\mathbb{R}$.

In [14] it is asserted without proof that we can write this in a slightly different form and that we can include the isomorphisms listed in (1). This involves a different set of real forms obtained by taking the split composition algebras $\mathbb{R}, \tilde{\mathbb{C}}, \tilde{\mathbb{H}}$, and $\tilde{\mathbb{O}}$ rather than $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$ as the second algebra. Thus the split magic square for three by three matrices looks like

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathrm{Der}\, H_3(\mathbb{K}) \cong L_3(\mathbb{K}, \mathbb{R})$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{sq}(3)$	$F_{4,1}$
$\mathrm{Str}'\, H_3(\mathbb{K}) \cong L_3(\mathbb{K}, \tilde{\mathbb{C}})$	$\mathfrak{sl}(3, \mathbb{R})$	$\mathfrak{sl}(3, \mathbb{C})$	$\mathfrak{sl}(3, \mathbb{H})$	$E_{6,1}$
$\mathrm{Con}\, H_3(\mathbb{K}) \cong L_3(\mathbb{K}, \tilde{\mathbb{H}})$	$\mathfrak{sp}(6, \mathbb{R})$	$\mathfrak{su}(3,3)$	$\mathfrak{sp}(6, \mathbb{H})$	$E_{7,1}$
$L_3(\mathbb{K}, \tilde{\mathbb{O}})$	$F_{4,2}$	$E_{6,2}$	$E_{7,2}$	$E_{8,1}$

where the notation $_{,1}$ and $_{,2}$ (in the style of [3]) is used to distinguish between different real forms of the exceptional Lie algebras in the last row and column. These are identified by their maximal compact subalgebras as follows:

Exceptional Lie Algebra	Maximal Compact Subalgebra
$E_{6,1}$	F_4
$E_{7,1}$	$E_{6,1} \oplus \mathfrak{so}(2)$
$E_{8,1}$	$E_{7,1} \oplus \mathfrak{so}(3)$
$E_{6,2}$	$\mathfrak{sq}(3) \oplus \mathfrak{so}(3)$
$E_{7,2}$	$\mathfrak{su}(6) \oplus \mathfrak{so}(3)$
$E_{8,2}$	$\mathfrak{so}(12) \oplus \mathfrak{so}(3)$

.

3.2. 2×2 **Matrices.** The Tits construction can also be adapted for 2×2 matrix algebras. In this case we take the vector space to be

$$L_2(\mathbb{K}_1, \mathbb{K}_2) = \mathrm{Der}\, H_2(\mathbb{K}_1) \dot{+} H_2'(\mathbb{K}_1) \otimes \mathbb{K}_2' \dot{+} \mathfrak{so}(\mathbb{K}_2') \tag{15}$$

which is again a Lie algebra when taken with the brackets

$$\begin{aligned} [D, A \otimes x] &= D(A) \otimes x \\ [E, A \otimes x] &= A \otimes E(x) \\ [D, E] &= 0 \\ [A \otimes x, B \otimes y] &= \tfrac{1}{4}\langle A, B\rangle D_{x,y} - \langle x, y\rangle [R_A, R_B] \end{aligned} \tag{16}$$

where the symbols used in this set of brackets are defined in the same way as the ones used in the 3×3 case. We note that $D_{x,y} = 2s_{x,y}$, where $s_{x,y}$ is the element of $\mathfrak{so}(\mathbb{K}_2')$ that maps x to y and y to $\pm x$, depending on the metric of $\mathbb{K}_2'$ i.e.

$$S_{x,y}(z) = \langle x, z\rangle y - \langle y, z\rangle x \tag{17}$$

If $\mathbb{K}_1, \mathbb{K}_2$ are division algebras then this gives the compact magic square for 2×2 matrix algebras

$$L_2(\mathbb{K}_1, \mathbb{K}_2) = \mathfrak{so}(\nu_1 + \nu_2).$$

If $\mathbb{K}_2$ is one of the split composition algebras $\tilde{\mathbb{C}}, \tilde{\mathbb{H}}$ or $\tilde{\mathbb{O}}$ this becomes

$$L_2(\mathbb{K}_1, \mathbb{K}_2) = \mathfrak{so}(\nu_1 + \tfrac{1}{2}\nu_2, \tfrac{1}{2}\nu_2).$$

giving the magic square

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$L_2(\mathbb{K}, \mathbb{R})$	$\mathfrak{so}(2)$	$\mathfrak{so}(3)$	$\mathfrak{so}(5)$	$\mathfrak{so}(9)$
$L_2(\mathbb{K}, \tilde{\mathbb{C}})$	$\mathfrak{so}(2,1)$	$\mathfrak{so}(3,1)$	$\mathfrak{so}(5,1)$	$\mathfrak{so}(9,1)$
$L_2(\mathbb{K}, \tilde{\mathbb{H}})$	$\mathfrak{so}(3,2)$	$\mathfrak{so}(4,2)$	$\mathfrak{so}(6,2)$	$\mathfrak{so}(10,2)$
$L_2(\mathbb{K}, \tilde{\mathbb{O}})$	$\mathfrak{so}(5,4)$	$\mathfrak{so}(6,4)$	$\mathfrak{so}(8,4)$	$\mathfrak{so}(12,4)$

.

As in the 3×3 case, these Lie algebras can be identified with certain types of 2×2 matrix algebras

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathrm{Der}\, H_2(\mathbb{K}) \cong L_2(\mathbb{K}, \mathbb{R})$	$\mathfrak{so}(2)$	$\mathfrak{su}(2)$	$\mathfrak{sq}(2)$	$\mathfrak{so}(9)$
$\mathrm{Str}\, H_2(\mathbb{K}) \cong L_2(\mathbb{K}, \tilde{\mathbb{C}})$	$\mathfrak{sl}(2, \mathbb{R})$	$\mathfrak{sl}(2, \mathbb{C})$	$\mathfrak{sl}(2, \mathbb{H})$	$\mathfrak{sl}(2, \mathbb{O})$
$\mathrm{Con}\, H_2(\mathbb{K}) \cong L_2(\mathbb{K}, \tilde{\mathbb{H}})$	$\mathfrak{sp}(4, \mathbb{R})$	$\mathfrak{su}(2,2)$	$\mathfrak{sp}(4, \mathbb{H})$	$\mathfrak{sp}(4, \mathbb{O})$
$L_2(\mathbb{K}, \tilde{\mathbb{O}})$	$\mathfrak{so}(5,4)$	$\mathfrak{so}(6,4)$	$\mathfrak{so}(8,4)$	$\mathfrak{so}(12,4)$

.

Again this extends the concepts of the Lie algebras $\mathfrak{sa}(2, \mathbb{K}), \mathfrak{sl}(2, \mathbb{K})$ and $\mathfrak{sp}(2, \mathbb{K})$ to $\mathbb{K} = \mathbb{H}$ and $\mathbb{O}$. Note that $\mathfrak{su}(2,2) \cong \mathfrak{sp}(4, \mathbb{C})$.

4. Symmetry Property of the 3×3 magic square.

In this section we will rearrange the definition

$$L_3(\mathbb{K}_1, \mathbb{K}_2) = \operatorname{Der} H_3(\mathbb{K}_1) \dot{+} H_3'(\mathbb{K}_1) \otimes \mathbb{K}_2' \dot{+} \operatorname{Der} \mathbb{K}_2 \tag{18}$$

so as to make explicit the symmetry between $\mathbb{K}_1$ and $\mathbb{K}_2$. We need a new Lie algebra associated with any $\mathbb{K}$, defined as follows:

Definition 1. Let $\mathbb{K}$ be a composition algebra over $\mathbb{R}$. The *triality algebra* of $\mathbb{K}$ is

(19)

$$\operatorname{Tri} \mathbb{K} = \{(A, B, C) \in \mathfrak{so}(\mathbb{K})^3 | A(xy) = x(By) + (Cx)y, \forall x, y \in \mathbb{K}\}.$$

It is easy to verify that $\operatorname{Tri} \mathbb{K}$ is a Lie algebra with brackets defined componentwise, i.e. it is a Lie subalgebra of $\mathfrak{so}(\mathbb{K}) \oplus \mathfrak{so}(\mathbb{K}) \oplus \mathfrak{so}(\mathbb{K})$.

Lemma 1. *The triality algebras of the four positive-definite composition algebras can be identified as follows:*

$$\begin{aligned}
&\operatorname{Tri} \mathbb{R} = 0 \\
&\operatorname{Tri} \mathbb{C} \cong \mathbb{R}^2 \\
&\operatorname{Tri} \mathbb{H} \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(3) \\
&\operatorname{Tri} \mathbb{O} \cong \mathfrak{so}(8)
\end{aligned}$$

Proof. $\operatorname{Tri} \mathbb{R} = 0$ because $\mathfrak{so}(\mathbb{R}) = 0$. For $\mathbb{C}$ we can identify $\mathfrak{so}(\mathbb{C})$ with the set of multiplication maps $z \mapsto hz$ with h pure imaginary, which is isomorphic to $\mathbb{R}$ as a Lie algebra. Then $\operatorname{Tri} \mathbb{C}$ is the subspace of the abelian Lie algebra $\mathbb{R}^3$ given by

$$\operatorname{Tri} \mathbb{C} = \{(u, v, w) : u = v + w\}$$

which is two dimensional.

Antisymmetric linear maps $A : \mathbb{H} \to \mathbb{H}$ are all of the form $A = L_{a_1} + R_{a_2}$ with $a_1, a_2 \in \mathbb{H}'$. These are all independent (this is a reflection of the Lie algebra isomorphism $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$). Hence the condition for $(A, B, C) \in \operatorname{Tri} \mathbb{H}$ is of the form

$$a_1 xy + xy a_2 = c_1 xy + x(c_2 + b_1)y + xy b_2, \qquad \forall x, y \in \mathbb{H}$$

Taking $y = 1$ and using the independence of the left and right multiplication maps gives

$$a_1 = c_1 \quad \text{and} \quad a_2 = c_2 + b_1 + b_2.$$

Taking $x = 1$ gives

$$a_1 = c_1 + c_2 + b_1 \quad \text{and} \quad a_2 = b_2.$$

Hence $c_2 + b_1 = 0$ and we have

$$A = L_{a_1} + R_{a_2}, \quad B = L_{b_1} + R_{a_2}, \quad C = L_{a_1} - R_{b_1}.$$

Thus $\operatorname{Tri} \mathbb{H} \cong \mathbb{H}'^3 \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

Finally, the infinitesimal version of the principle of triality [11] asserts that for each $A \in \mathfrak{so}(8)$ there are unique $B, C \in \mathfrak{so}(8)$ such that

$$A(xy) = x(By) + (Cx)y \qquad \forall x, y, \in \mathbb{O}.$$

This establishes an isomorphism between $\mathrm{Tri}\,\mathbb{O}$ and $\mathfrak{so}(8)$. □

We will now describe the chain of inclusions

$$\mathrm{Der}\,\mathbb{K} \subset \mathrm{Tri}\,\mathbb{K} \subset \mathrm{Der}\,H_3(\mathbb{K}) \tag{20}$$

in a unified way, valid for any composition algebra $\mathbb{K}$. We will use a multiple notation to describe multiple direct sums, writing

$$nV = \underbrace{V \dot{+} V \dot{+} \ldots \dot{+} V}_{n}$$

rather than V^n (which might suggest $V \otimes \cdots \otimes V$).

Lemma 2. *For any composition algebra* $\mathbb{K}$*,*

$$\mathrm{Tri}\,\mathbb{K} = \mathrm{Der}\,\mathbb{K} \dot{+} 2\mathbb{K}'$$

in which $\mathrm{Der}\,\mathbb{K}$ *is a Lie subalgebra,*

$$\begin{aligned}
[D, (a, b)] &= (Da, Db) \in 2\mathbb{K}' \\
[(a, 0), (b, 0)] &= \tfrac{2}{3} D_{a,b} + \left(\tfrac{1}{3}[a, b], -\tfrac{2}{3}[a, b]\right), \\
[(a, 0), (0, b)] &= \tfrac{1}{3} D_{a,b} - \left(\tfrac{1}{3}[a, b], \tfrac{1}{3}[a, b]\right), \\
[(0, a), (0, b)] &= \tfrac{2}{3} D_{a,b} + \left(-\tfrac{2}{3}[a, b], \tfrac{1}{3}[a, b]\right).
\end{aligned}$$

Proof. Define $T : \mathrm{Der}\,\mathbb{K} \dot{+} 2\mathbb{K}' \to \mathrm{Tri}\,\mathbb{K}$ by

$$\begin{aligned} &T(D, a, b) = \\ &\qquad (D + L_a - R_b,\ D - L_a - L_b - R_b,\ D + L_a + R_a + R_b). \end{aligned} \tag{21}$$

This belongs to $\mathrm{Tri}\,\mathbb{K}$ as a consequence of the alternative law. The map T is injective, for $T(D, a, b) = 0$ implies

$$2L_a + L_b = 0 \quad \text{and} \quad R_a + 2R_b = 0$$

(subtracting the first component from the second and third in turn), so $2a + b = a + 2b = 0$ and hence $a + b = 0$, which implies $D = 0$. To show that T is surjective, suppose $(A, B, C) \in \mathrm{Tri}\,\mathbb{K}$ and define $a, b \in \mathbb{K}'$ by

$$\begin{aligned} B(1) &= -a - 2b, \\ C(1) &= 2a + b. \end{aligned}$$

Let

$$D = A - L_a + R_b. \tag{22}$$

Since $(A, B, C) \in \mathrm{Tri}\,\mathbb{K}$ we have

$$B(x) = 1.B(x) = A(1.x) - C(1)x.$$

Thus

$$\begin{aligned} B &= A - 2L_a - L_b \\ &= D - L_a - L_b - R_b \end{aligned} \tag{23}$$

and

$$C = D + L_a + R_a + R_b \tag{24}$$

Now

$$\begin{aligned} D(xy) &= A(xy) - a(xy) + (xy)b \\ &= x(By) + (Cx)y - a(xy) + (xy)b \\ &= (Dx)y + x(Dy) \end{aligned} \tag{25}$$

by equations (23, 24) and the alternative law. Hence D is a derivation and $(A, B, C) = T(D, a, b)$.

The first Lie bracket stated above, i.e.

$$[T(D, 0, 0), T(0, a, b)] = T(0, Da, Db),$$

follows from

$$[D, L_a] = L_{Da} \quad \text{and} \quad [D, R_a] = R_{Da}. \tag{26}$$

The other brackets follow from the commutators

$$\begin{aligned} [L_x, L_y] &= \tfrac{2}{3}D_{x,y} + \tfrac{1}{3}L_{[x,y]} + \tfrac{2}{3}R_{[x,y]} \\ [L_x, R_y] &= -\tfrac{1}{3}D_{x,y} + \tfrac{1}{3}L_{[x,y]} - \tfrac{1}{3}R_{[x,y]} \\ [R_x, R_y] &= \tfrac{2}{3}D_{x,y} - \tfrac{2}{3}L_{[x,y]} - \tfrac{1}{3}R_{[x,y]} \end{aligned}$$

which can be calculated using equation (13). □

Any two elements $x, y \in \mathbb{K}$ are associated with a derivation $D_{x,y} \in \operatorname{Der}\mathbb{K}$ and also with an antisymmetric map $S_{x,y} \in \mathfrak{so}(\mathbb{K})$, the generator of rotations in the plane of x and y given by (17). There is also an element of $\operatorname{Tri}\mathbb{K}$ associated with x and y:

Lemma 3. *For any $x, y \in \mathbb{K}$, let*

$$T_{x,y} = (4S_{x,y},\ R_yR_{\bar{x}} - R_xR_{\bar{y}},\ L_yL_{\bar{x}} - L_xL_{\bar{y}}).$$

Then $T_{x,y} \in \operatorname{Tri}\mathbb{K}$.

Proof. We can write the action of $S_{x,y}$ as

$$2S_{x,y}z = (x\bar{z} + z\bar{x})y - x(\bar{z}y + \bar{y}z) \tag{27}$$

$$= -[x, y, z] + z(\bar{x}y) - (x\bar{y})z \tag{28}$$

using the alternative law and the relation $[x, y, \bar{z}] = -[x, y, z]$. Since $\operatorname{Re}(\bar{x}y) = \operatorname{Re}(x\bar{y})$, we can write the last two terms as

$$z(\bar{x}y) - (x\bar{y})z = z\operatorname{Im}(\bar{x}y) - \operatorname{Im}(x\bar{y})z \tag{29}$$

$$= \tfrac{1}{2}z(\bar{x}y - \bar{y}x) - \tfrac{1}{2}(x\bar{y} - y\bar{x})z. \tag{30}$$

Now, by equation (13), we have

$$S_{x,y} = \tfrac{1}{6}D_{x,y} + L_a - R_b \tag{31}$$

with

$$a = -\tfrac{1}{6}[x,y] - \tfrac{1}{4}(x\bar{y} - y\bar{x}) \in \mathbb{K}'$$
$$b = -\tfrac{1}{6}[x,y] - \tfrac{1}{4}(\bar{x}y - \bar{y}x) \in \mathbb{K}'.$$

Hence, by equation (21), there is an element $(A, B, C) \in \operatorname{Tri}\mathbb{K}$ with $A = S_{x,y}$ and

$$B = \tfrac{1}{6}D_{x,y} - L_a - L_b - R_b = S_{x,y} - L_{2a+b},$$
$$C = \tfrac{1}{6}D_{x,y} + L_a + R_a + R_b = S_{x,y} + R_{a+2b},$$

Writing $[x,y] = -\frac{1}{2}([\bar{x},y] + [x,\bar{y}])$ gives

$$a + 2b = \tfrac{1}{4}(\bar{y}x - \bar{x}y)$$
$$2a + b = \tfrac{1}{4}(y\bar{x} - x\bar{y})$$

so equations (27) and (29) give

$$S_{x,y} = \tfrac{1}{2}Q_{x,y} - R_{a+2b} + L_{2a+b} \tag{32}$$

where $Q_{x,y}z = -[x,y,z]$. Hence

$$Cz = -\tfrac{1}{2}[x,y,z] + \tfrac{1}{4}(y\bar{x} - x\bar{y})z$$
$$= \tfrac{1}{4}y(\bar{x}z) - \tfrac{1}{2}x(\bar{y}z)$$

i.e.

$$C = \tfrac{1}{4}(L_y L_{\bar{x}} - L_x L_{\bar{y}})$$

and similarly

$$B = \tfrac{1}{4}(R_y R_{\bar{x}} - R_x R_{\bar{y}}).$$

Thus $T = (4S_{x,y}, 4C, 4B)$ is an element of $\operatorname{Tri}\mathbb{K}$. □

Note that if $x, y \in \mathbb{K}'$, so that $\bar{x} = -x$ and $\bar{y} = -y$, then $a = b = \frac{1}{12}[x,y]$ and so

$$T_{x,y} = (\tfrac{2}{3}D_{x,y} + \tfrac{1}{3}L_{[x,y]} - \tfrac{1}{3}R_{[x,y]}, \tfrac{2}{3}D_{x,y} + \tfrac{1}{3}L_{[x,y]} + \tfrac{2}{3}R_{[x,y]}, \tfrac{2}{3}D_{x,y} - \tfrac{2}{3}L_{[x,y]} - \tfrac{1}{3}R_{[x,y]}). \tag{33}$$

The element $T_{x,y}$ will be needed to describe $\operatorname{Tri}\mathbb{K}$ as a Lie subalgebra of $\operatorname{Der} H_3(\mathbb{K})$. We will also need an automorphism of $\operatorname{Tri}\mathbb{K}$ defined as follows. For any linear map $A : \mathbb{K} \to \mathbb{K}$, let $\overline{A} = KAK$, where $K : \mathbb{K} \to \mathbb{K}$ is the conjugation $x \mapsto \bar{x}$ in $\mathbb{K}$, i.e.

$$\overline{A}(x) = \overline{A(\bar{x})}.$$

Then $\overline{\overline{A}} = A$ and $\overline{AB} = \overline{A}\,\overline{B}$. Note also that

$$\overline{L}_x = R_{\bar{x}}$$

and $D = \overline{D}$ if $D \in \mathfrak{so}(\mathbb{K}')$, in particular if D is a derivation of $\mathbb{K}$.

Lemma 4. *Given* $T = (A, B, C) \in \operatorname{Tri}\mathbb{K}$*, let*

$$\theta(T) = (\overline{B}, C, \overline{A}).$$

Then $\theta(T) \in \operatorname{Tri}\mathbb{K}$ *and* θ *is a Lie algebra automorphism.*

Proof. By Lemma 2, $T = T(D, a, b)$ for some $D \in \operatorname{Der}\mathbb{K}$ and $a, b \in \mathbb{K}'$. Then

$$\begin{aligned} A &= D + L_a - R_b \\ B &= D - L_a - L_b - R_b \\ C &= D + L_a + R_a + R_b. \end{aligned}$$

It follows that

$$\overline{B} = D + R_a + R_b + L_b = D + L_{a'} - R_{b'}$$

which is the first component of $T' = (A', B', C') \in \operatorname{Tri}\mathbb{K}$, where

$$\begin{aligned} B' &= D - L_{a'} - L_{b'} - R_{b'} \\ &= D - L_b + L_{a+b} + R_{a+b} = C \\ C' &= D + L_{a'} + R_{a'} + R_{b'} \\ &= D + L_b + R_b - R_{a+b} = \overline{A}, \end{aligned}$$

i.e. $T' = (\overline{B}, C, \overline{A}) = \theta(T)$. It is clear that θ is a Lie algebra automorphism. □

Given $T = (A, B, C) \in \operatorname{Tri}\mathbb{K}$, it is convenient to define $(T_1, T_2, T_3) = (A, \overline{B}, \overline{C})$. Then $\theta(T)_i = T_{\sigma(i)}$ where $\sigma \in \mathcal{S}_3$ is the cyclic permutation

$$\sigma(1) = 2, \quad \sigma(2) = 3, \quad \sigma(3) = 1.$$

Write $a_1 = -a - b$, $a_2 = a$, $a_3 = b$. Then the triality obtained from (a, b) can be written in the symmetric form $T(0, a, b) = (T_1, \overline{T}_2, \overline{T}_3)$ where

$$\begin{aligned} T_1 &= L_{a_2} - R_{a_3} \\ T_2 &= L_{a_3} - R_{a_1} \\ T_3 &= L_{a_1} - R_{a_2} \end{aligned}$$

i.e. $T_i = L_{a_j} - R_{a_k}$ where (i, j, k) is a cyclic permutation of $(1, 2, 3)$.

Theorem 1. *For any composition algebra* $\mathbb{K}$*,*

$$\operatorname{Der} H_3(\mathbb{K}) = \operatorname{Tri}\mathbb{K} \dot{+} 3\mathbb{K}$$

in which $\operatorname{Tri}\mathbb{K}$ *is a Lie subalgebra, and the brackets in* $[\operatorname{Tri}\mathbb{K}, 3\mathbb{K}]$ *are*

$$[T, F_i(x)] = F_i(T_i x) \quad \in 3\mathbb{K}, \tag{34}$$

if $T = (T_1, \overline{T}_2, \overline{T}_3) \in \operatorname{Tri}\mathbb{K}$ *and* $F_1(x) + F_2(y) + F_3(z) = (x, y, z) \in 3\mathbb{K}$*; and the brackets in* $[\operatorname{Tri}\mathbb{K}, \operatorname{Tri}\mathbb{K}]$ *are given by*

$$[F_i(x), F_j(y)] = F_k(\bar{y}\bar{x}) \in 3\mathbb{K}, \tag{35}$$

if $x, y \in \mathbb{K}$ and (i, j, k) is a cyclic permutation of $(1, 2, 3)$; and

$$[F_i(x), F_i(y)] = \theta^{1-i}(T_{x,y}) \in \operatorname{Tri}\mathbb{K}. \tag{36}$$

Proof. Define elements $e_i, P_i(x)$ of $H_3(\mathbb{K})$ (where $i = 1, 2, 3$; $x \in \mathbb{K}$) by the equation

$$\begin{pmatrix} \alpha & z & \bar{y} \\ \bar{z} & \beta & x \\ y & \bar{x} & \gamma \end{pmatrix} = \alpha e_1 + \beta e_2 + \gamma e_3 + P_1(x) + P_2(y) + P_3(z) \tag{37}$$

for $\alpha, \beta, \gamma \in \mathbb{R}$; $x, y, z \in \mathbb{K}$. Then the Jordan product in $H_3(\mathbb{K})$ is given by

$$e_i \cdot e_j = 2\delta_{ij} e_i \tag{38a}$$

$$e_i \cdot P_j(x) = (1 - \delta_{ij}) P_j(x) \tag{38b}$$

$$P_i(x) \cdot P_i(y) = 2(x, y)(e_j + e_k) \tag{38c}$$

$$P_i(x) \cdot P_j(y) = P_k(\bar{y}\,\bar{x}) \tag{38d}$$

where in each of the last two equations (i, j, k) is a cyclic permutation of $(1, 2, 3)$.

Now let $D : H_3(\mathbb{K}) \to H_3(\mathbb{K})$ be a derivation of this algebra. First suppose that

$$De_i = 0, \quad i = 1, 2, 3.$$

Then

$$\begin{aligned} e_i \cdot DP_i(x) &= 0 \\ e_i \cdot DP_j(x) &= DP_j(x) \quad \text{if } i \neq j \end{aligned}$$

Thus $DP_j(x)$ is an eigenvector of each of the multiplication operators L_{e_i}, with eigenvalue 0 if $i = j$ and 1 if $i \neq j$. It follows that

$$DP_j(x) = P_j(T_j x) \tag{39}$$

for some $T_j : \mathbb{K} \to \mathbb{K}$. Now

$$DP_j(x) \cdot P_j(y) + P_j(x) \cdot DP_j(y) = 0$$

gives $T_j \in \mathfrak{so}(\mathbb{K})$; and the derivation property of D applied to equation (38d) gives

$$T_k(\bar{y}\bar{x}) = \bar{y}(\overline{T_i x}) + (\overline{T_j y})\bar{x}$$

i.e. $(T_k, \overline{T_i}, \overline{T_j}) \in \operatorname{Tri}\mathbb{K}$ and therefore $(T_1, \overline{T_2}, \overline{T_3}) \in \operatorname{Tri}\mathbb{K}$.

If $De_i \neq 0$, then from equation (38a) with $i = j$,

$$2e_i \cdot De_i = 2De_i$$

so De_i is an eigenvector of the multiplication L_{e_i} with eigenvalue 1, i.e. $De_i \in P_j(\mathbb{K}) + P_k(\mathbb{K})$ where (i, j, k) are distinct. Write

$$De_i = P_j(x_{ij}) + P_k(x_{ik});$$

then equation (38a) with $i \neq j$ gives

$$e_i \cdot P_k(x_{jk}) + e_i \cdot P_i(x_{ji}) + P_j(x_{ij}) \cdot e_j + P_k(x_{ik}) \cdot e_j = 0.$$

Thus

$$P_k(x_{jk} + x_{ik}) = 0.$$

It follows that the action of any derivation on the e_i must be of the form $F_1(x) + F_2(y) + F_3(z)$ where

$$\begin{aligned} F_i(x)e_i &= 0 \\ F_i(x)e_j = -F_i(x)e_k &= P_i(x), \end{aligned} \tag{40}$$

(i, j, k) being a cyclic permutation of $(1, 2, 3)$. Hence $\operatorname{Der} H_3(\mathbb{K}) \subseteq \operatorname{Tri} \mathbb{K} \oplus \mathbb{K}^3$.

To show that such derivations $F_i(x)$ exist and therefore the inclusion just mentioned is an equality, consider the operation of commutation with the matrix

$$\begin{aligned} X &= \begin{pmatrix} 0 & -z & \bar{y} \\ \bar{z} & 0 & -x \\ -y & \bar{x} & 0 \end{pmatrix} \\ &= X_1(x) + X_2(y) + X_3(z) \end{aligned}$$

i.e. define $F_i(x) = C_{X_i(x)}$ where $C_X : H_3(\mathbb{K}) \to H_3(\mathbb{K})$ is the commutator map

$$C_X(H) = XH - HX. \tag{41}$$

This satisfies equation (40) and also

$$\begin{aligned} F_i(x)P_i(y) &= -2(x, y)(e_j - e_k) \\ F_i(x)P_j(y) &= -P_k(\bar{y}\,\bar{x}) \\ F_i(x)P_k(y) &= P_j(\bar{x}\,\bar{y}). \end{aligned} \tag{42}$$

It is a derivation of $H_3(\mathbb{K})$ by virtue of the matrix identity

$$[X, \{H, K\}] = \{[X, H], K\} + \{H, [X, K]\} \tag{43}$$

(in which square brackets denote commutators and round brackets denote anticommutators), which we will prove separately in lemma 7.

The Lie brackets of these derivations follow from another matrix identity which is also proved in lemma 7,

$$[X, [Y, H]] - [Y, [X, H]] = [[X, Y], H] - E(X, Y)H \tag{44}$$

where $E(X, Y) \in \mathfrak{so}(\mathbb{K}')$ is defined by

$$E(X, Y)z = \sum_{ij} [x_{ij}, y_{ji}, z],$$

x_{ij}, y_{ji} being the matrix elements of X and Y. If $X = X_i(x)$ and $Y = X_j(y)$ we have $D(X, Y) = 0$ and

$$[X_i(x), X_j(y)] = X_k(\bar{y}\bar{x})$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$. This yields the Lie bracket (35). If $X = X_i(x)$ and $Y = X_j(y)$, the matrix commutator $Z = [X, Y]$ is diagonal with $z_{ii} = 0$, $z_{jj} = y\bar{x} - x\bar{y}$ and $z_{kk} = \bar{y}x - \bar{x}y$ (i, j, k cyclic). Hence the action of the commutator $[F_i(x), F_i(y)] = C_z + E(X, Y)$ on $H_3(\mathbb{K})$ is

$$\begin{aligned}
[F_i(x), F_i(y)]e_m &= 0 \quad (m = i, j, k) \\
[F_i(x), F_i(y)]P_i(w) &= P_i(z_{jj}w - wz_{kk} - 2[x, y, w]) = P_i(T_1 w) \\
&= 4P_i(S_{xy}w) \quad \text{by equation (17).} \\
[F_i(x), F_i(y)]P_j(w) &= P_j(z_{kk}w - 2[x, y, w]) \\
&= P_j(\bar{y}(xw) - \bar{x}(yw)) \\
[F_i(x), F_i(y)]P_k(w) &= P_k(-wz_{jj} - 2[x, y, w]) \\
&= P_k((wx)\bar{y} - (wy)\bar{x}).
\end{aligned}$$

Comparing with lemma 3, we see that

$$\begin{aligned}
[F_i(x), F_i(y)]P_i(w) &= P_i(T_1 w) = P_i(T_i' w) \\
[F_i(x), F_i(y)]P_j(w) &= P_j(\overline{T_2} w) = P_j(\overline{T_j'} w) \\
[F_i(x), F_i(y)]P_k(w) &= P_k(\overline{T_3} w) = P_k(\overline{T_k'} w)
\end{aligned}$$

where $(T_1, T_2, T_3) = T_{xy}$, so that $T' = \theta^{1-i}(T_{xy})$. This establishes the Lie bracket (36). □

The matrix identities needed in the proof of Theorem 1 are contained in the following, in which we include a third identity for the sake of completeness:

Lemma 5. *Let $\mathbb{K}$ be a composition algebra, let H, K and L be hermitian 3×3 matrices with entries from $\mathbb{K}$, and let X, Y be traceless antihermitian matrices over $\mathbb{K}$. Then*

$$[X, \{H, K\}] = \{[X, H], K\} + \{H, [X, K]\} \tag{45a}$$

$$[X, [Y, H]] - [Y, [X, H]] = [[X, Y], H] + E(X, Y)H \tag{45b}$$

$$\{H\{K, L\}\} - \{K\{H, L\}\} = [[H, K], L] + E(H, K)L \tag{45c}$$

where $E(X, Y) \in \mathfrak{so}(\mathbb{K})$ is defined for any 3×3 matrices X, Y by

$$E(X, Y)z = \sum_{ij} [x_{ij}, y_{ji}, z]. \tag{46}$$

In these matrix identities the square brackets denote commutators and the chain brackets denote anticommutators.

Proof. We consider first part (a). The difference between the two sides can be written in terms of matrix associators, where the (i, j)th element is

$$\sum_{mn}([x_{im},h_{mn},k_{nj}]+[x_{im},k_{mn},h_{nj}] + [k_{im},h_{mn},x_{nj}]-[h_{im},x_{mn},k_{nj}]-[k_{im},x_{mn},h_{nj}]). \tag{47}$$

Suppose $i\neq j$ and let k be the third index. Since the diagonal elements of H and K are real, any associator containing them vanishes. Hence the terms containing x_{ij} or x_{ji} are

$$\sum_n([x_{ij},h_{jn},k_{nj}]+[x_{ij},k_{jn},h_{nj}])+\sum_m([h_{im},k_{mi},x_{ij}]+[k_{im},h_{mi},x_{ij}]) - [h_{ij},x_{ji},k_{ij}]+[k_{ij},x_{ji},h_{ij}]=0$$

by the alternative law, the hermiticity of H and K, and the fact that an associator changes sign when one of its elements is conjugated. The terms containing x_{ik} or x_{ki} are

$$[x_{ik},h_{ki},k_{ij}]+[x_{ik},k_{ki},h_{ij}]-[h_{ik},x_{ki},k_{ij}]-[k_{ik},x_{ki},h_{ij}]=0$$

using also $x_{ki}=-\bar{x}_{ik}$. Similarly, the terms containing x_{jk} or x_{kj} vanish. Finally, the terms containing x_{ii},x_{jj} and x_{kk} are

$$[x_{ii},h_{ik},k_{kj}]+[x_{ii},k_{ik},h_{kj}]+[h_{ik},k_{kj},x_{jj}]+[k_{ik},h_{kj},x_{jj}] - [h_{ik},x_{kk},k_{kj}]-[k_{ik},x_{kk},h_{kj}]=0$$

since $x_{ii}+x_{jj}+x_{kk}=0$.

Now consider the (i,i)th element. The last two terms of equation (47) become

$$-\sum_{mn}([h_{im},x_{mn},k_{ni}]+[k_{in},x_{nm},h_{mi}])=0.$$

Let j be one of the other two indices. The terms containing x_{ij} or x_{ji} are

$$[x_{ij},h_{jk},k_{ki}]+[x_{ij},k_{jk},h_{ki}]+[h_{ik},k_{kj},x_{ji}]+[k_{ik},h_{kj},x_{ji}]=0,$$

where k is the third index. There are no terms containing x_{jk} or x_{kj}. The terms containing x_{ii},x_{jj} or x_{kk} are

$$\sum_n([x_{ii},h_{in},k_{ni}]+[x_{ii},k_{in},h_{ni}]) + \sum_m([h_{im},k_{mi},x_{ii}]+[k_{im},h_{mi},x_{ii}])=0.$$

Thus in all cases the expression (47) vanishes, proving (a). Parts (b) and (c) are proved by similar arguments, which the reader will find more entertaining to write than to read. □

In any Jordan algebra, the commutator of two multiplication operators L_x and L_y is a derivation (this fact is used in the construction of the magic square Lie algebras $L_3(\mathbb{K}_1, \mathbb{K}_2)$; see [15]). In the case of the Jordan algebra $H_3(\mathbb{K})$, we can identify these derivations as follows:

Lemma 6. *In $H_3(\mathbb{K})$, where $\mathbb{K}$ is any composition algebra,*

$$[L_{e_i}, L_{e_j}] = 0$$
$$[L_{e_i}, L_{P_j(x)}] = \epsilon_{ij} L_{P_j(x)}$$

where $\epsilon_{ij} = 0$ if $i = j$, otherwise ϵ_{ij} is the sign of the permutation (i, j, k) of $(1, 2, 3)$ where k is the third index,

$$[L_{P_i(x)}, L_{P_i(y)}] = -T_{x,y}$$
$$[L_{P_i(x)}, L_{P_j(y)}] = F_k(\bar{y}\,\bar{x})$$

where $e_i, P_i(x) \in H_3(\mathbb{K})$ are defined by (37) and $F_k(x) \in \operatorname{Der} H_3(\mathbb{K})$ is given by (40) and (42).

Proof. Straightforward calculation from (38a- 38d). □

The proof of Theorem 1 suggests an alternative description of $\operatorname{Der} H_3(\mathbb{K})$, which leads us to identify it as $\mathfrak{sa}(3, \mathbb{K})$:

Theorem 2. *For any composition algebra $\mathbb{K}$,*

$$\operatorname{Der} H_3(\mathbb{K}) = \operatorname{Der} \mathbb{K} \dot{+} A_3'(\mathbb{K}) \tag{48}$$

in which $\operatorname{Der} \mathbb{K}$ is a Lie subalgebra, the Lie brackets between $\operatorname{Der} \mathbb{K}$ and $A_3'(\mathbb{K})$ are given by the elementwise action of $\operatorname{Der} \mathbb{K}$ on 3×3 matrices, and

$$[X, Y] = (XY - YX)' + \tfrac{1}{3} D(X, Y)$$

where $X, Y \in A_3'(\mathbb{K})$,

$$(XY - YX)' = XY - YX - \tfrac{1}{3}\operatorname{tr}(XY - YX)\mathbb{1} \quad \in A_3'(\mathbb{K})$$

and

$$D(X, Y) = \sum_{ij} D(x_{ij}, y_{ji}) \quad \in \operatorname{Der} \mathbb{K}$$

x_{ij}, y_{ji} being the matrix elements of X and Y and $D(x, y)$ being the derivation $D_{x,y}$ defined in equation (12).

Proof. By Lemma 2 and Theorem 1

$$\operatorname{Der} H_3(\mathbb{K}) = \operatorname{Der} \mathbb{K} \dot{+} 2\mathbb{K}' \dot{+} 3\mathbb{K}. \tag{49}$$

Identify $(a, b) + (x, y, z) \in 2\mathbb{K}' \dot{+} 3\mathbb{K}$ with the traceless antihermitian matrix

$$X = \begin{pmatrix} -a-b & -z & \bar{y} \\ \bar{z} & a & -x \\ -y & \bar{x} & b \end{pmatrix} \quad \in A_3'(\mathbb{K});$$

then the actions of $2\mathbb{K}'$ and $3\mathbb{K}$ on $H_3(\mathbb{K})$ defined in Theorem 1 are together equivalent to the commutator action C_x defined by equation (41). By Lemma 5(b),

$$[C_X, C_Y] = C_{(XY-YX)'} + C_{t\mathbb{1}} + E(X,Y)$$

where

$$\begin{aligned} t &= \tfrac{1}{3}\operatorname{tr}(XY - YX) \\ &= \tfrac{1}{3}\sum_{ij}(x_{ij}y_{ji} - y_{ji}x_{ij}). \end{aligned}$$

Now $C_{t\mathbb{1}} + E(X,Y)$ acts elementwise on matrices in $H_3(\mathbb{K})$ according to the map $D : \mathbb{K} \to \mathbb{K}$ given by

$$\begin{aligned} Dz &= [t, z] + E(X,Y)z \\ &= \sum_{ij}\left(\tfrac{1}{3}[[x_{ij}, y_{ji}], z] - [x_{ij}, y_{ji}, z]\right) \\ &= \tfrac{1}{3}D(X,Y)Z. \end{aligned}$$

Hence the bracket $[X,Y]$ is as stated. □

Finally we use Theorem 1 to give a description of $L_3(\mathbb{K}_1, \mathbb{K}_2)$ which makes manifest the symmetry between $\mathbb{K}_1$ and $\mathbb{K}_2$.

Theorem 3. *For any two composition algebras* $\mathbb{K}_1, \mathbb{K}_2$,

$$L_3(\mathbb{K}_1, \mathbb{K}_2) = \operatorname{Tri}\mathbb{K}_1 \oplus \operatorname{Tri}\mathbb{K}_2 \dot{+} 3\mathbb{K}_1 \otimes \mathbb{K}_2 \tag{50}$$

in which $\operatorname{Tri}\mathbb{K}_1 \oplus \operatorname{Tri}\mathbb{K}_2$ *is a Lie subalgebra;*

$$[T_1, F_i(x \otimes y)] = F_i(T_{1i}x_1 \otimes x_2) \quad \in 3\mathbb{K}_1 \otimes \mathbb{K}_2 \tag{51}$$

$$[T_2, F_i(x \otimes y)] = F_i(x_1 \otimes T_{2i}x_2) \quad \in 3\mathbb{K}_1 \otimes \mathbb{K}_2 \tag{52}$$

if $T_\alpha = (T_{\alpha 1}, \overline{T}_{\alpha 2}, \overline{T}_{\alpha 3}) \in \operatorname{Tri}\mathbb{K}\ (\alpha = 1, 2)$, *and*

$$F_1(x_1 \otimes x_2) + F_2(y_1 \otimes y_2) + F_3(z_1 \otimes z_2) = (x_1 \otimes x_2, y_1 \otimes y_2, z_1 \otimes z_2) \in 3\mathbb{K}_1 \otimes \mathbb{K}_2;$$

$$[F_i(x_1 \otimes x_2), F_j(y_1 \otimes y_2)] = F_k(\bar{y}_1\bar{x}_1 \otimes \bar{y}_2\bar{x}_2) \in 3\mathbb{K}_1 \otimes \mathbb{K}_2 \tag{53}$$

if $x_\alpha, y_\alpha \in \mathbb{K}_2$ *and* (i,j,k) *is a cyclic permutation of* $(1,2,3)$; *and*

$$[F_i(x_1 \otimes x_2), F_i(y_1 \otimes y_2)] = \langle x_2, y_2\rangle \theta^{1-i} T_{x_1 y_1} + \langle x_1, y_1\rangle \theta^{1-i} T_{x_2 y_2} \in \operatorname{Tri}\mathbb{K}_1 \oplus \operatorname{Tri}\mathbb{K}_2 \tag{54}$$

Proof. We can write

$$H_3'(\mathbb{K}_1) = 2\mathbb{R} \oplus 3\mathbb{K}$$

by identifying $(\alpha, \beta) + (x, y, z) \in 2\mathbb{R} \oplus 3\mathbb{K}$ with the matrix

$$\begin{pmatrix} -\alpha-\beta & z & \bar{y} \\ \bar{z} & \alpha & x \\ y & \bar{x} & \beta \end{pmatrix} \in H_3'(\mathbb{K}),$$
$$= \alpha(e_2 - e_1) + \beta(e_3 - e_1) + P_1(x) + P_2(y) + P_3(z)$$

in the notation of theorem 1. Then the vector space structure (10) of $L_3(\mathbb{K}_1, \mathbb{K}_2)$ can be written using Theorem 1, as

$$\begin{aligned} L_3(\mathbb{K}_1, \mathbb{K}_2) &= \operatorname{Der} H_3(\mathbb{K}_1) \dot{+} H_3'(\mathbb{K}_1) \otimes \mathbb{K}_2' \dot{+} \operatorname{Der} \mathbb{K}_2 \\ &= (\operatorname{Tri} \mathbb{K} \dot{+} 3\mathbb{K}_1) \dot{+} (2\mathbb{K}_2' \dot{+} 3\mathbb{K}_1 \otimes 2\mathbb{K}_2') \dot{+} \operatorname{Der} \mathbb{K}_2 \\ &= \operatorname{Tri} \mathbb{K}_1 \dot{+} (\operatorname{Der} \mathbb{K}_2 \dot{+} 2\mathbb{K}_2') \dot{+} (3\mathbb{K}_1 \otimes \mathbb{K}_2' \dot{+} 3\mathbb{K}_1) \\ &\cong \operatorname{Tri} \mathbb{K}_1 \dot{+} \operatorname{Tri} \mathbb{K}_2 \dot{+} 3\mathbb{K}_1 \otimes \mathbb{K}_2. \end{aligned}$$

We use the following notation for the elements of the five subspaces of $L_3(\mathbb{K}_1, \mathbb{K}_2)$:

1. $\operatorname{Tri} \mathbb{K} \subset \operatorname{Der} H_3(\mathbb{K}_1)$ contains elements $T = (T_1, \overline{T}_2, \overline{T}_3)$ acting on $H_3'(\mathbb{K}_1)$ as in Theorem 1:
$$Te_i = 0, \quad TP_i(x) = P_i(T_i x) \quad (x \in \mathbb{K}; i = 1, 2, 3)$$
2. $3\mathbb{K}_1$ is the subspace of $\operatorname{Der} H_3(\mathbb{K}_1)$ containing the elements $F_i(x)$ defined in Theorem 1; these will be identified with the elements $F_i(x \otimes 1) \in 3\mathbb{K}_1 \otimes \mathbb{K}_2'$.
3. $2\mathbb{K}_2'$ is the subspace $\Delta \otimes \mathbb{K}_2'$ of $H_3(\mathbb{K}_1) \otimes \mathbb{K}_2'$, where $\Delta \subset H_3'(\mathbb{K}_1)$ is the subspace of real, diagonal, traceless matrices and is identified with the subspace of $\operatorname{Tri} \mathbb{K}$ as described in Lemma 2. We will regard $2\mathbb{K}_2'$ as a subspace of $3\mathbb{K}_2'$, namely
$$2\mathbb{K}_2' = \{(a_1, a_2, a_3) \in 3\mathbb{K}_2' : a_1 + a_2 + a_3 = 0\}$$
and identify $\mathbf{a} = (a_1, a_2, a_3)$ with the 3×3 matrix
$$\Delta(\mathbf{a}) = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \in H_3'(\mathbb{K}_1) \otimes \mathbb{K}_2'$$
and with the triality $T(\mathbf{a}) = (T_1, \overline{T}_2, \overline{T}_3)$ where $T_i = L_{a_j} - R_{a_k}$ (see the remark after the proof of Lemma 4).
4. $3\mathbb{K}_1 \otimes \mathbb{K}_2'$ is the subspaces of $H_3(\mathbb{K}_1 \otimes \mathbb{K}_2'$ spanned by elements $P_i(x) \otimes a$ $(i = 1, 2, 3 : x \in \mathbb{K}_1, a \in \mathbb{K}_2')$; it is also a subspace of $3\mathbb{K}_1 \otimes \mathbb{K}_2$ in the obvious way.
5. $\operatorname{Der} \mathbb{K}_2$ is a subspace of $\operatorname{Tri} \mathbb{K}_2$, a derivation D being identified with $(D, D, D) \in \operatorname{Tri} \mathbb{K}_2$.

To complete the proof we must verify that the Lie brackets defined by Tits (see Section 3.1) coincide with those in the statement of the theorem. The above decomposition of $L_3(\mathbb{K}_1, \mathbb{K}_2)$ into five parts gives us fifteen types of bracket to examine. We will write $[\,,\,]_{\text{Tits}}$ for the bracket defined in section 3.1 and $[\,,\,]_{\text{here}}$ for that defined above.

1. $[\operatorname{Tri}\mathbb{K}_1, \operatorname{Tri}\mathbb{K}_2]$: For $T_1, T_2 \in \operatorname{Tri}\mathbb{K}_1$, $[T_1, T_2]_{\text{Tits}}$ is the bracket in $\operatorname{Der} H_3(\mathbb{K}_1)$, which by theorem 1 is the same as $[T_1, T_2]_{\text{here}}$.
2. $[\operatorname{Tri}\mathbb{K}_1, 3\mathbb{K}_1]$: For $T \in \operatorname{Tri}\mathbb{K}_1$, $F_i(x \otimes 1) \in 3\mathbb{K}_1$,
$$\begin{aligned}[T, F_i(x)]_{\text{Tits}} &= F_1(T_i x) \text{ see Theorem 1} \\ &= F_i(T_i x \otimes 1) = [T, F_i(x \otimes 1)]_{\text{here}}.\end{aligned}$$
3. $[\operatorname{Tri}\mathbb{K}_1, 2\mathbb{K}_2]$: For $T_1 \in \operatorname{Tri}\mathbb{K}_1$, $(a, b, c) \in 2\mathbb{K}_2'$,
$$[T_1, (a, b, c)]_{\text{Tits}} = [T, e_1 \otimes a + e_2 \otimes b + e_3 \otimes c] = 0$$
since in Theorem 1 $\operatorname{Tri}\mathbb{K}_1$ was obtained as the subspace of derivations which annihilate the diagonal matrices e_i. On the other hand,
$$[T_1, (a, b, c)]_{\text{here}} = [T_1, T_2(a, b, c)] = 0.$$
4. $[\operatorname{Tri}\mathbb{K}_1, 3\mathbb{K}_1 \otimes \mathbb{K}_2']$: For $T_1 \in \operatorname{Tri}\mathbb{K}_1$, $P_i(x \otimes a) \in 3\mathbb{K}_1 \otimes \mathbb{K}_2'$,
$$[T_1, P_i(x \otimes a) = P_i(T_{1i} x \otimes a) = [T_1, P_i(x \otimes a)]_{\text{here}}. \tag{55}$$
5. $[\operatorname{Tri}\mathbb{K}_1, \operatorname{Der}\mathbb{K}_2]_{\text{Tits}} \subset [\operatorname{Der} H_3(\mathbb{K}_1), \operatorname{Der}\mathbb{K}_2] = 0$, while $[\operatorname{Tri}\mathbb{K}_1, \operatorname{Der}\mathbb{K}_2]_{\text{here}} \subset [\operatorname{Tri}\mathbb{K}_1, \operatorname{Tri}\mathbb{K}_2] = 0$.
6. $[3\mathbb{K}_1, 3\mathbb{K}_1]$: $3\mathbb{K}_1 = 3\mathbb{K}_1 \otimes \mathbb{R}$ is spanned by $F_i(x) = F_i(x \otimes 1)$ $(i = 1, 2, 3; x \in \mathbb{K}_1)$, and $[F_i(x), F_j(y)]_{\text{Tits}}$ is given by Theorem 1, while $[F_i(x \otimes 1), F_j(y \otimes 1)]_{\text{here}}$ is the same since $T_{x_2, y_2} = 0$ if $x_2, y_2 \in \mathbb{R}$.
7. $[3\mathbb{K}_1, 2\mathbb{K}_2']$: For $F_i(x) \in 3\mathbb{K}_1$, $\mathbf{a} = (a_1, a_2, a_3) \in 2\mathbb{K}_2'$ with $(a_1 + a_2 + a_3 = 0)$,
$$\begin{aligned}[F_i(x), \mathbf{a}]_{\text{Tits}} &= [F_i(x), \sum e_i \otimes a_i] \in [\operatorname{Der} H_3(\mathbb{K}_1), H_3'(\mathbb{K}_1) \otimes \mathbb{K}_2'] \\ &= P_i(x) \otimes (a_j - a_k) \qquad \text{by (40)}\end{aligned}$$
while
$$\begin{aligned}[F_i(x), \mathbf{a}]_{\text{here}} &= [F_i(x \otimes 1), T(\mathbf{a})] \in [3\mathbb{K}_1 \otimes \mathbb{K}_2, \operatorname{Tri}\mathbb{K}_2] \\ &= F_i(x \otimes (a_j - a_k)).\end{aligned}$$
8. $[3\mathbb{K}_1, 3\mathbb{K}_1 \otimes \mathbb{K}_2']$: For $F_i(x) \in 3\mathbb{K}_1, F_j(y \otimes a) \in 3\mathbb{K}_1 \otimes \mathbb{K}_2'$,
$$\begin{aligned}[F_i(x), F_i(y \otimes a)]_{\text{Tits}} &= [F_i(x), P_i(y) \otimes a] \in [\operatorname{Der} H_3(\mathbb{K}_1), H_3'(\mathbb{K}_1) \otimes \mathbb{K}_2'] \\ &= -2\langle x, y\rangle (e_j - e_k) \otimes a \in H_3'(\mathbb{K}_1 \otimes \mathbb{K}_2' \\ &= -2\langle x, y\rangle (a_1, a_2, a_3) \in 2\mathbb{K}_2'\end{aligned}$$
where $a_i = 0, a_j = a, a_k = -a$ $(i, j, k$ cyclic). On the other hand
$$\begin{aligned}[F_i(x), F_i(y \otimes a)]_{\text{here}} &= [F_i(x \otimes 1), F_i(y \otimes a)] \\ &= -\langle x, y\rangle \theta^{1-i} T_{1,a} \\ &= [F_i(x), F_i(y \otimes a)]_{\text{Tits}}\end{aligned}$$

since $T_{1,a} = (2L_a + R_a, 2R_a, 2L_a)$ which is identified with $(0, 2a, -2a) \in \mathbb{K}_2'$ in paragraph 3 above. If $i \neq j$ and (i, j, k) is a cyclic permutation of $1, 2, 3)$,

$$\begin{aligned}
[F_i(x), F_j(y \otimes a)]_{\text{Tits}} &= [F_i(x), P_j(y \otimes a)] \ \in [\operatorname{Der} H_3(\mathbb{K}_1), H_3'(\mathbb{K}_1) \otimes \mathbb{K}_2] \\
&= -P_k(\bar{y}\bar{x}) \otimes a \in H_3'(\mathbb{K}_1) \otimes \mathbb{K}_2' \\
&= -F_k(\bar{y}\bar{x}) \otimes a \in 3\mathbb{K}_1 \otimes \mathbb{K}_2'
\end{aligned}$$

while $[F_i(x), F_j(y \otimes a)]_{\text{here}} = -F_k(\bar{y}\bar{x}) \otimes a$ since $\bar{a} = -a$. Similarly,

$$[F_i(x), F_k(y \otimes a)]_{\text{Tits}} = F_k(\bar{x}\bar{y}) \otimes a = [F_i(x), F_k(y \otimes a)]_{\text{here}}.$$

9. $[3\mathbb{K}_1, \operatorname{Der} \mathbb{K}_2]_{\text{Tits}} \in [\operatorname{Der} H_3(\mathbb{K}_1), \operatorname{Der} \mathbb{K}_2] = 0$
and

$$[3\mathbb{K}_1, \operatorname{Der} \mathbb{K}_2]_{\text{here}} \in [3\mathbb{K}_1 \otimes \mathbb{R}, \operatorname{Der} \mathbb{K}_2] = 0.$$

10. $[2\mathbb{K}_2', 2\mathbb{K}_2']$: For $\mathbf{a}, \mathbf{b} \in 2\mathbb{K}_2'$, with $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ where $a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = 0$,

$$\begin{aligned}
[\mathbf{a}, \mathbf{b}]_{\text{Tits}} &= [\sum e_i \otimes a_i, \sum e_j \otimes b_j] \\
&= \sum_{i,j} \big(\langle e_i, e_j\rangle D_{a_i,b_j} + (e_i * e_j) \otimes \operatorname{Im}(a_i b_j) + \langle a_i, b_j\rangle [L_{e_i}, L_{e_j}]\big) \\
&= \sum_{ij} \big(2\delta_{ij} D_{a_i,b_j} + \delta_{ij}(2e_i - \tfrac{2}{3}\mathbb{1}) \otimes \tfrac{1}{2}[a_i, b_j]\big) \\
&= \sum_i \big(2D_{a_i,b_i} + \tfrac{1}{3} e_i \otimes (2[a_i, b_i] - [a_j, b_j] - [a_k, b_k])\big) \\
&= [\mathbf{a}, \mathbf{b}]_{\text{here}}
\end{aligned}$$

11. $[2\mathbb{K}_2', 3\mathbb{K}_1 \otimes \mathbb{K}_2']$: For $\mathbf{a} = (a_1, a_2, a_3) \in 2\mathbb{K}_2'$ and $F_i(x \otimes b) \in 3\mathbb{K}_1 \otimes \mathbb{K}_2'$,

$$\begin{aligned}
[\mathbf{a}, F_i(x \otimes b)]_{\text{Tits}} &= [e_i \otimes a_i + e_j \otimes a_j + e_k \otimes a_k, P_i(x) \otimes b] \\
&= P_i(x) \otimes \tfrac{1}{2}[a_j, b] + P_i(x) \otimes \tfrac{1}{2}[a_k, b] - \langle a_j - a_k, b\rangle F_i(x)
\end{aligned}$$

using Lemma 6. The first two terms belong to the subspace $3\mathbb{K}_1 \otimes \mathbb{K}_2'$ of $H_3' \otimes \mathbb{K}_2'$ and the third to the subspace $3\mathbb{K}_1$ of $\operatorname{Der} H_3(\mathbb{K}_1)$, so together they constitute an element of $3\mathbb{K}_1 \otimes \mathbb{K}_2$:

$$\begin{aligned}
[\mathbf{a}, F_i(x \otimes b)]_{\text{Tits}} &= F_i(x \otimes (\tfrac{1}{2}[a_j, b] = \tfrac{1}{2}[a_k, b] - \langle a_j - a_k, b\rangle)) \\
&= F_i(x \otimes (a_j b - b a_k)) \\
&= F_i(x \otimes T(\mathbf{a})_i b) \\
&= [\mathbf{a}, F_i(x \otimes b)]_{\text{here}}
\end{aligned}$$

12. $[2\mathbb{K}_2', \operatorname{Der} \mathbb{K}_2]$: Tits's bracket (11) coincides with the bracket in $\operatorname{Tri} \mathbb{K}_2$ as given by Lemma 2.

13. $[3\mathbb{K}_1 \otimes \mathbb{K}_2', 3\mathbb{K}_1 \otimes \mathbb{K}_2']$: For $P_i(x), P_i(y) \in 3\mathbb{K}_1$ and $a, b \in \mathbb{K}_2'$, if $i \neq j$ and (i, j, k) is a cyclic permutation of $(1, 2, 3)$ then

$$[P_i(x) \otimes a, P_j(y) \otimes b]_{\text{Tits}} = P_k(\bar{y}\bar{x}) \otimes \tfrac{1}{2}[a, b] - \langle a, b\rangle F_k(\bar{y}\bar{x})$$

by equation (38d) and Lemma 6. This is an element of $3\mathbb{K}_1 \otimes \mathbb{K}_2' \dot{+} 3\mathbb{K}_1 \otimes \mathbb{R}$ which is identified with the following element of $3\mathbb{K}_1 \otimes \mathbb{K}_2$:

$$\begin{aligned} F_k(\bar{y}\bar{x} \otimes \tfrac{1}{2}\left([a, b] - (a\bar{b} + b\bar{a})\right) &= F_k(\bar{y}\bar{x} \otimes \bar{b}\bar{a}) \\ &= [F_i(x \otimes a), F_j(y \otimes b)]_{\text{here}}. \end{aligned}$$

If $i = j$, then

$$\begin{aligned} [P_i(x) \otimes a, P_i(y) \otimes b]_{\text{Tits}} = 4\langle x, y\rangle D_{a,b} + \\ \tfrac{1}{2}\langle x, y\rangle(-2e_i + e_j + e_k) \otimes [a, b] - \langle a, b\rangle \theta^{1-i} T_{x,y} \end{aligned}$$

by Lemma 6. The second term belongs to the subspace $2\mathbb{K}_2'$ and it is to be identified with the triality $\frac{1}{3}\langle x, y\rangle\theta^{1-i}T$ where $T_1 = \frac{1}{3}(L_{[a,b]} - R_{[a,b]})$, $\overline{T}_2 = -\frac{1}{3}(R_{[a,b]} + 2L_{[a,b]})$ and $\overline{T}_3 = \frac{1}{3}(2R_{[a,b]} + L_{[a,b]})$. By (33), $T = T_{a,b}$. Hence

$$[P_i(x) \otimes a, P_i(y) \otimes b]_{\text{Tits}} = \langle x, y\rangle\theta^{1-i}T_{a,b} - \langle a, b\rangle\theta^{1-i}T_{x,y}.$$

14. $[3\mathbb{K}_1 \otimes \mathbb{K}_2', \operatorname{Der}\mathbb{K}_2]$ is given by the action of $\operatorname{Der}\mathbb{K}_2$ on the second factor of the tensor product in both cases.
15. $[\operatorname{Der}\mathbb{K}_2, \operatorname{Der}\mathbb{K}_2]$ is given by the Lie bracket of $\operatorname{Der}\mathbb{K}_2$ in both cases.

□

5. Magic Squares of 2 × 2 Matrix Algebras: Proofs

In this section we prove the following theorems.

Theorem 4. *For $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$,*

$$\begin{aligned} L_2(\mathbb{K}_1, \mathbb{K}_2) &\cong \mathfrak{so}(\mathbb{K}_1 \dot{+} \mathbb{K}_2) \\ L_2(\mathbb{K}_1, \tilde{\mathbb{K}}_2) &\cong \mathfrak{so}(\tfrac{1}{2}(\nu_1 + \nu_2), \tfrac{1}{2}\nu_2). \end{aligned}$$

Theorem 5. *The following isomorphisms are true for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$.*

$$L_2(\mathbb{K}, \mathbb{R}) \cong \operatorname{Der} H_2(\mathbb{K}) \tag{56a}$$

$$L_2(\mathbb{K}, \tilde{\mathbb{C}}) \cong \operatorname{Str} H_2(\mathbb{K}) \tag{56b}$$

$$L_2(\mathbb{K}, \tilde{\mathbb{H}}) \cong \operatorname{Con} H_2(\mathbb{K}) \tag{56c}$$

We prove Theorem 4 by first showing that $L_2(\mathbb{K}_1, \mathbb{K}_2)$ is isomorphic to the Lie algebra of the pseudo-orthogonal group $\mathrm{O}(\mathbb{K}_1 \dot{+} \mathbb{K}_2)$ of linear transformations of $\mathbb{K}_1 \dot{+} \mathbb{K}_2$ preserving the quadratic form

$$|x_1 + x_2|^2 = |x_1|^2 + |x_2|^2. \quad (x_1 \in \mathbb{K}_1, x_2 \in \mathbb{K}_2)$$

We then prove equation (56c) and notice that the proof of this contains the isomorphisms for equations (56a) and (56b). The proofs of these equations will require the use of the following Theorem, and associated Lemmas.

Theorem 6. *The derivation algebra of $H_2(\mathbb{K})$ can be expressed in the form*

$$\operatorname{Der} H_2(\mathbb{K}) = A_2'(\mathbb{K}) \dot{+} \mathfrak{so}(\mathbb{K}'). \tag{57}$$

Our proof requires the use of the lemma

Lemma 7. *Let $A \in A_n(\mathbb{K})$ and $X, Y \in H_n(\mathbb{K})$ where $\mathbb{K}$ is any alternative algebra. The identity*

$$[A, \{X, Y\}] = \{[A, X], Y\} + \{X, [A, Y]\}, \tag{58}$$

(where the brackets denote commutators of matrices) holds if $n = 2$ or if $n = 3$ and $\operatorname{tr} A = 0$.

Proof. The 3×3 case was proved in Lemma 5. The 2×2 case can be deduced from it by considering the 3×3 matrices

$$\widetilde{A} = \begin{pmatrix} A & 0 \\ 0 & -\operatorname{tr} A \end{pmatrix}, \quad \widetilde{X} = \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}, \quad \widetilde{Y} = \begin{pmatrix} Y & 0 \\ 0 & 1 \end{pmatrix}.$$

□

Proof of theorem 6. From lemma 7 we see that for each $A \in A_2(\mathbb{K})$ there is a derivation $D(A)$ of $H_2(\mathbb{K})$ given by

$$D(A)(X) = AX - XA.$$

We consider $H_2(\mathbb{K})$ as a Jordan algebra with product

$$\begin{pmatrix} \alpha & x \\ \bar{x} & \beta \end{pmatrix} \cdot \begin{pmatrix} \gamma & y \\ \bar{y} & \delta \end{pmatrix} = \begin{pmatrix} 2\alpha\gamma + 2\operatorname{Re}(x\bar{y}) & (\gamma+\delta)x + (\alpha+\beta)y \\ (\gamma+\delta)\bar{x} + (\alpha+\beta)\bar{y} & 2\beta\delta + 2\operatorname{Re}(\bar{x}y) \end{pmatrix}.$$

We can write a matrix $A \in H_2(\mathbb{K})$ as follows

$$\begin{pmatrix} \alpha & x \\ \bar{x} & \beta \end{pmatrix} = \lambda I + \mu E + P(x)$$

where $\lambda = \frac{1}{2}(\alpha+\beta)$, $\mu = \frac{1}{2}(\alpha-\beta)$, $E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $P(x) = \begin{pmatrix} 0 & x \\ \bar{x} & 0 \end{pmatrix}$. Then the Jordan multiplication can be rewritten as

$$E \cdot E = I$$
$$P(x) \cdot P(y) = 2\langle x, y\rangle I$$
$$E \cdot P(x) = 0.$$

Thus $H_2(\mathbb{K})$ can be identified with $\mathbb{J}(V)$, the Jordan algebra associated with the inner product space $V = \mathbb{K} \oplus \mathbb{R}$. $\mathbb{J}(V)$ is a subalgebra of the anticommutator algebra of $\mathrm{Cl}(V)$, where $\mathbf{v}\cdot\mathbf{w} = \langle \mathbf{v}, \mathbf{w}\rangle 1$. Derivations of this algebra must satisfy

$$D(1) = 0$$
$$\langle 1, D(\mathbf{v})\rangle = 0.$$

Thus

$$\langle D(\mathbf{v}), \mathbf{w}\rangle + \langle D(\mathbf{w}), \mathbf{v}\rangle = 0$$

i.e. D is an antisymmetric map of $\mathbf{v}$. Hence $\mathrm{Der}\, H_2(\mathbb{K}) = \mathfrak{o}(\mathbb{K}\dot{+}\mathbb{R})$.

Considering the matrix structure of $\mathfrak{o}(\mathbb{K}\dot{+}\mathbb{R})$ we can write this as $\mathfrak{o}(\mathbb{K})\dot{+}\mathbb{K}$. Consider the action of $\mathbb{K}$ on the $(\nu+1)\times 1$ column vectors $\begin{pmatrix}0\\1\end{pmatrix}$ and $\begin{pmatrix}x\\0\end{pmatrix}$. We express $k \in \mathbb{K}$ as the final row and column in a $(\nu+1)\times(\nu+1)$ block matrix. Then

$$\begin{pmatrix}0 & k\\ -k^t & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}k\\0\end{pmatrix}$$
$$\begin{pmatrix}0 & k\\ -k^t & 0\end{pmatrix}\begin{pmatrix}x\\0\end{pmatrix} = \begin{pmatrix}0\\-k^t x\end{pmatrix}$$

Thus k maps E to $P(k)$ and $P(x)$ to $-\langle k, x\rangle E$. Now

$$\left[\begin{pmatrix}0 & k\\ -\bar{k} & 0\end{pmatrix}, \begin{pmatrix}0 & x\\ \bar{x} & 0\end{pmatrix}\right] = 2\begin{pmatrix}\langle k, x\rangle & -k\\ \bar{k} & -\langle k, x\rangle\end{pmatrix}$$

i.e. multiplication by $\begin{pmatrix}0 & k\\ -k^t & 0\end{pmatrix}$ in $\mathbb{J}(V)$ is equivalent to commutation with $\begin{pmatrix}0 & -\frac{k}{2}\\ \frac{\bar{k}}{2} & 0\end{pmatrix}$ in $H_2(\mathbb{K})$.

We can split $\mathfrak{o}(\mathbb{K}) = \mathfrak{o}(\mathbb{K}'\oplus\mathbb{R})$ into $\mathfrak{o}(\mathbb{K}')\dot{+}\mathbb{K}$. Consider the action of the $\nu\times\nu$ matrix $\begin{pmatrix}0 & l\\ -l^t & 0\end{pmatrix}$ with $l \in \mathbb{K}'$ on the vectors $\begin{pmatrix}y\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ with $y \in \mathbb{K}'$:

$$\begin{pmatrix}0 & l\\ -l^t & 0\end{pmatrix}\begin{pmatrix}y\\0\end{pmatrix} = \begin{pmatrix}-l^t x\\0\end{pmatrix}$$
$$\begin{pmatrix}0 & l\\ -l^t & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\l\end{pmatrix}$$

and we obtain (by a similar method) that multiplication by $\begin{pmatrix}0 & l\\ -l^t & 0\end{pmatrix}$ in $\mathbb{J}(V)$ is equivalent to commutation with $\begin{pmatrix}\frac{l}{2} & 0\\ 0 & -\frac{l}{2}\end{pmatrix}$ in $H_2(\mathbb{K})$. Further $\mathfrak{so}(\mathbb{K}')$ acts in $\mathbb{J}(V)$ precisely as it does in $H_2(\mathbb{K})$. Thus we have

$$\mathrm{Der}\, H_2(\mathbb{K}) = A_2'(\mathbb{K})\dot{+}\mathfrak{so}(\mathbb{K}')$$

as required. □

The brackets in $A'_2(\mathbb{K}_1)\dot{+}\mathfrak{so}(\mathbb{K}_1)$ are given by

$$\begin{aligned}[A, A'] &= AA' - A'A\\ [S, A] &= S(A)\\ [S, S'] &= SS' - S'S\end{aligned}$$

with $A, A' \in A'_2(\mathbb{K}_1)$ and $S, S' \in \mathfrak{so}(\mathbb{K}_1)$ and $S(A)$ describes S acting elementwise on A. When they arise in calculations we consider multiples of the 2×2 identity matrix I_2 to be elements of $\mathfrak{so}(\mathbb{K}_1)$. Finally, the following lemma holds.

Lemma 8. *The Jacobi identity*

$$[A, [B, H]] + [B, [H, A]] + [H, [A, B]] = 0$$

holds for $A, B \in A'_2(\mathbb{K})$ and $X \in H_2(\mathbb{K})$.

Proof of Theorem 4. Using Theorem 6 we can write $L_2(\mathbb{K}_1, \mathbb{K}_2)$ as

$$L_2(\mathbb{K}_1, \mathbb{K}_2) = A'_2(\mathbb{K}_1)\dot{+}\mathfrak{so}(\mathbb{K}_1)\dot{+}H'_2(\mathbb{K}_1) \otimes \mathbb{K}'_2\dot{+}\mathfrak{so}(\mathbb{K}'_2).$$

This can be considered to contain the following elements

$$\begin{aligned}
J &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in A'_2(\mathbb{R})\\
A_1 &= \begin{pmatrix} a_1 & 0 \\ 0 & -a_1 \end{pmatrix} \in A'_2(\mathbb{K}'_1)\\
A_2 &= \begin{pmatrix} 0 & a_2 \\ a_2 & 0 \end{pmatrix} \in A'_2(\mathbb{K}'_1)\\
F &\in \mathfrak{so}(\mathbb{K}'_1)\\
S &\in \mathfrak{so}(\mathbb{K}'_2)\\
B_1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes b_1 \ \in H'_2(\mathbb{R}) \otimes \mathbb{K}'_2\\
B_2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes b_1 \ \in H'_2(\mathbb{R}) \otimes \mathbb{K}'_2\\
C &= \begin{pmatrix} 0 & c_1 \\ -c_1 & 0 \end{pmatrix} \otimes c_2 \ \in H'_2(\mathbb{K}'_1) \otimes \mathbb{K}'_2
\end{aligned} \tag{59}$$

with $a_1, b_1, c_1 \in \mathbb{K}'_1$ and $a_2, b_2, c_2 \in \mathbb{K}'_2$.

We define $\varphi : L_2(\mathbb{K}_1, \mathbb{K}_2) \to \mathfrak{so}(\nu_1 + \nu_2)$ by

$$\varphi(J + A_1 + A_2 + F + S + B_1 + B_2 + C) = \begin{pmatrix} S & 2Gb_1 & 2Gb_2 & 2Gc_2c_1^t \\ -2b_1^tG & 0 & 2 & 2a_2^t \\ -2b_2^tG & -2 & 0 & -2a_1^t \\ -2c_1c_2^tG & -2a_2 & 2a_1 & F \end{pmatrix} \tag{60}$$

where now elements of $\mathbb{K}'_i$ are identified with column vectors in $\mathbb{R}^{\nu_i - 1}$ and G is the metric matrix for $\mathbb{K}'_2$. In the Euclidean case it is merely the identity matrix whereas in the non-Euclidean case it is the diagonal matrix consisting of $(\frac{\nu_2}{2})$ positive 1's and $(\frac{\nu_2}{2} - 1)$ negative 1's. The order of the positive and negative elements is determined by the choice of ± 1 in the Cayley-Dickson calculation for $\mathbb{K}_2$. We show that ψ is a Lie algebra isomorphism by calculating the multiplication tables for the Lie brackets between the elements listed in $L_2(\mathbb{K}_1, \mathbb{K}_2)$. We then calculate the equivalent brackets in $\mathfrak{so}(\mathbb{K}_1 \oplus \mathbb{K}_2)$ and show that they are equivalent. The relevant tables are found on the following two pages and are obtained simply by applying the stated Lie brackets for each algebra to these basis elements.

	J	A_1	A_2	B_1	B_2	C	F	S
J	0	$-2\begin{pmatrix}0 & a_1\\ a_1 & 0\end{pmatrix}$	$2\begin{pmatrix}a_2 & 0\\ 0 & -a_2\end{pmatrix}$	$-2\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \otimes b_1$	$2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes b_2$	0	0	0
A_1'	$2\begin{pmatrix}0 & a_1'\\ a_1' & 0\end{pmatrix}$	$2\,\mathrm{Im}(a_1 a_1')I$	$-2\,\mathrm{Re}(a_1' a_1)J$	0	$2\begin{pmatrix}0 & a_1'\\ a_1' & 0\end{pmatrix} \otimes b_2$	$2\,\mathrm{Re}(c_1 a_1') \times \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \otimes c_2$	$F(A_1')$	0
A_2'	$-2\begin{pmatrix}a_2' & 0\\ 0 & -a_2'\end{pmatrix}$	$-2\,\mathrm{Re}(a_1 a_2')J$	$2\,\mathrm{Im}(a_2 a_2')I$	$2\begin{pmatrix}0 & -a_2'\\ a_2' & 0\end{pmatrix} \otimes b_1$	0	$2\,\mathrm{Re}(a_2' c_1) \times \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes c_2$	$F(A_2'$	0
B_1'	$2\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \otimes b_1'$	0	$2\begin{pmatrix}0 & a_2\\ a_2 & 0\end{pmatrix} \otimes b_1'$	D_{b_1', b_1}	$-2(b_1', b_2)J$	$-2(b_1', c_2) \times \begin{pmatrix}0 & c_1\\ c_1 & 0\end{pmatrix}$	0	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes S(b_1')$
B_2'	$-2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes b_2'$	$2\begin{pmatrix}0 - a_1 & \\ a_1 & 0\end{pmatrix} \otimes b_2'$	0	$2(b_1, b_2')J$	D_{b_2', b_2}	$-2(b_2', c_2) \times \begin{pmatrix}-c_1 & 0\\ 0 & c_1\end{pmatrix}$	0	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \otimes S(b_2')$
C'	0	$-2\,\mathrm{Re}(c_1' a_1) \times \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \otimes c_2'$	$2\,\mathrm{Re}(a_2 c_1') \times \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes c_2'$	$2(b_1, c_2') \times \begin{pmatrix}0 & c_1'\\ c_1' & 0\end{pmatrix}$	$-2(b_2, c_2') \times \begin{pmatrix}c_1' & 0\\ 0 & -c_1'\end{pmatrix}$	$(c_1, c_1')D_{c_2', c_2} - 2(c_2, c_2') \times \mathrm{Im}(c_1' c_1)I$	$F\begin{pmatrix}0 & c_1'\\ c_1' & 0\end{pmatrix} \otimes c_2'$	$\begin{pmatrix}0 & c_1'\\ -c_1' & 0\end{pmatrix} \otimes S(c_2')$
F'	0	$-F'(A_1)$	$-F'(A_2)$	0	0	$-F'\begin{pmatrix}0 & c_1\\ c_1 & 0\end{pmatrix} \otimes c_2$	$F'F - FF'$	0
S'	0	0	0	$-\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes S'(b_1)$	$-\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \otimes S'(b_2)$	$-\begin{pmatrix}0 & c_1\\ c_1 & 0\end{pmatrix} \otimes S'(c_2)$	0	$S'S - SS'$

	$\psi(J)$	$\psi(A_1)$	$\psi(A_2)$	$\psi(B_1)$	$\psi(B_2)$	$\psi(C)$	$\psi(F)$	$\psi(S)$
$\psi(J')$	0	$-4a_1^tG_{24}+4a_1G_{42}$	$-4a_2^tG_{34}+4a_2G_{43}$	$-4b_1GG_{13}+4Gb_1^tG_{31}$	$4b_2GG_{12}-4Gb_2^tG_{21}$	0	0	0
$\psi(A_1')$	$4a_1'^tG_{24}-4a_1'G_{42}$	$(-4a_1'a_1^t+4a_1a_1'^t)G_{44}$	$-4a_2^ta_1'G_{23}+4a_1'^ta_2G_{32}$	0	$4Gb_2a_1'^tG_{14}-4a_1'b_2^tGG_{41}$	$-4Gc_2c_1^ta_1'G_{13}+4a_1'^tc_1c_2^tGG_{31}$	$-2a_1'^tFG_{34}+2F^ta_1'G_{43}$	0
$\psi(A_2')$	$4a_2'^tG_{34}-4a_2'G_{43}$	$4a_2'^ta_1G_{23}-4a_1^ta_2'G_{32}$	$(-4a_2'a_2^t+4a_2a_2'^t)G_{44}$	$4a_2'b_1^tGG_{41}-4Gb_1a_2'^tG_{14}$	0	$4Gc_2c_1^ta_2'G_{12}-4a_2'^tc_1c_2^tGG_{21}$	$2a_2'^tFG_{24}-2Fa_2G_{42}$	0
$\psi(B_1')$	$4b_1'GG_{13}-4b_1'^tGG_{31}$	0	$4Gb_1'a_2^tG_{14}-4a_2b_1'^tGG_{41}$	$(-4b_1'b_1^t+4b_1b_1'^t)G_{11}$	$4b_2^tGGb_1'G_{32}-4b_1'^tGGb_2G_{23}$	$4Gc_2c_1^tGb_1'G_{42}-4b_1'^tGGc_2c_1^tG_{24}$	0	$2b_1'^tGS^tG_{21}-2SGb_1'G_{12}$
$\psi(B_2')$	$4b_2'^tGG_{21}-4Gb_2'G_{12}$	$4a_1b_2'^tGG_{41}-4b_2'Ga_1^tG_{14}$	0	$4b_1^tGGb_2'G_{23}-4b_2'^tGGb_1G_{32}$	$4(Gb_2b_2'^tG-Gb_2'b_2^tG)G_{11}$	$4c_1c_2^tGGb_2'G_{43}-4b_2'^tGGc_2c_1^tG_{34}$	0	$2b_2'^tGS^tG_{31}-2Sb_2'GG_{13}$
$\psi(C')$	0	$4Gc_2'c_1'^ta_1G_{13}-4a_1^tc_1'c_2'^tGG_{31}$	$4a_2^tc_1'c_2'^tGG_{21}-4Gc_2'c_1'^ta_2G_{12}$	$4b_1^tGGc_2'c_1'^tG_{24}-4Gc_2'c_1'^tGb_1G_{42}$	$4b_2^tGGc_2'c_1'^tG_{34}-4c_1'c_2'^tGGb_2G_{43}$	$4(Gc_2c_1^tc_1'c_2'^tG-Gc_2'c_1'^tc_1c_2^tG)G_{11}+4(c_1c_2^tGGc_2'c_1'^t-c_1'c_2'^tGGc_2c_1^t)G_{44}$	$2Gc_2'c_1'^tFG_{14}-2Fc_1'c_2'^tGG_{41}$	$2c_1'c_2'^tGS^tG_{41}-2SGc_2'c_1'^tG_{14}$
$\psi(F')$	0	$2a_1^tF'G_{34}-2F'^ta_1G_{43}$	$2F'^ta_2G_{42}-2a_2^tF'G_{24}$	0	0	$2F'c_1c_2^tGG_{41}-2Gc_2c_1^tF'G_{14}$	$(F'F-FF')G_{44}$	0
$\psi(S')$	0	0	0	$2S'Gb_1G_{12}-2b_1^tGS'^tG_{21}$	$2S'Gb_2G_{13}-2b_2^tGS'^tG_{31}$	$2S'Gc_2c_1^tG_{14}-2c_1c_2^tGS'^tG_{41}$	0	$(S'S-SS')G_{11}$

Since Lie brackets are antisymmetric the tables show that there are 25 non-zero brackets to compare. We will use the following lemma.

Lemma 9. *If* $\mathbf{a}_1,\ \mathbf{a}_2 \in \mathbb{R}^{\nu-1}$ *are the vector representations of the hypercomplex numbers* $a_1, a_2 \in \mathbb{K}'$ *then*

1. $4(\mathbf{a}_1\mathbf{a}_2^t + \mathbf{a}_2\mathbf{a}_1^t) \in \mathfrak{so}(\mathbb{K}_1 \dot{+} \mathbb{K}_2)$ *is equivalent to* $2\,\mathrm{Im}(a_2a_1)I \in L_2(\mathbb{K}_1, \mathbb{K}_2)$,
2. $\mathbf{a}_2^t\mathbf{a}_1 \in \mathfrak{so}(\mathbb{K}_1 \dot{+} \mathbb{K}_2)$ *is equivalent to* $(a_1, a_2) = -\,\mathrm{Re}(a_1a_2) \in L_2(\mathbb{K}_1, \mathbb{K}_2)$.

Proof. We shall use the notation that if $a = a_ie_i$ is a hypercomplex number in $\mathbb{K}'$ then $\mathbf{a} = (a_i)$ is its vector representation as a column vector in $\mathbb{R}^{\nu-1}$ where ν is the dimension of $\mathbb{K}$.

1. The (i,j)th component of the matrix $(\mathbf{a}\mathbf{b}^t + \mathbf{b}\mathbf{a}^t)$ is the element $a_ib_j + b_ia_j$. If this multiplies a third hypercomplex number, say $\mathbf{c} = (c_k)$ then the nth element of the resulting vector will be
$$\sum_{k=1}^{\nu-1}(a_nb_k + b_na_k)c_k.$$
Now we can write $2\,\mathrm{Im}(ba) = 2\sum_i e_i(b_ja_k - b_ka_j)$ where (i,j,k) is a cyclic permutation of a quaternionic triple. Commuting this with c (which is equivalent to the action of $2\,\mathrm{Im}(ba)I$ on an element in a matrix $H_2(\mathbb{K}_1)$) gives
$$2\,\mathrm{Im}(ba) = 2(b_ja_k - a_jb_k)c_m(e_ie_m - e_me_i)$$
Now $e_ie_m = -e_me_i$, thus
$$[2\,\mathrm{Im}(ba), c] = \sum_j 4((b_ja_k - b_ka_j)c_me_j$$
where (e_i, e_m, e_j) form a quaternionic triple. Therefore $4(\mathbf{a}_1\mathbf{a}_2^t + \mathbf{a}_2\mathbf{a}_1^t)$ is equivalent to $2\,\mathrm{Im}(a_2a_1)I$.
2. Now $\mathbf{b}^t\mathbf{a} = \sum_{i=1}^{\nu-1} b_ia_i$. But
$$\begin{aligned}(a,b) &= \mathrm{Re}(a\bar{b})\\ &= -\sum_{i=1}^{\nu-1} a_ib_i(e_i)^2\\ &= \sum_{i=1}^{\nu-1} a_ib_i\end{aligned}$$
as required. Clearly $\mathrm{Re}(ab) = -(a,b)$.

□

Using these results and the fact that $G^2 = I_{\nu-1}$ the two tables are clearly equivalent and ψ is a Lie algebra isomorphism. □

5.1. **Proof of equation (56c).** We start by defining the structure and conformal algebras as they apply to the 2×2 hermitian matrix case. We will see that the structure algebra is a subalgebra of the conformal algebra and the derivation algebra a subalgebra of the structure algebra. This means that our isomorphism for the conformal algebra includes the isomorphisms for the structure algebra, and trivially for the derivation algebra. From equations (7) and (57) we can deduce that

$$\mathrm{Str}'\, H_2(\mathbb{K}) = H_2'(\mathbb{K}) \dot{+} A_2'(\mathbb{K}) \dot{+} \mathfrak{so}(\mathbb{K}') \tag{61}$$

which is a Lie algebra with brackets defined by the statement that $\mathrm{Der}(H_2(\mathbb{K})) = A_2'(\mathbb{K}) \dot{+} \mathfrak{so}(\mathbb{K}')$ is a subalgebra. Denoting the elements of the algebra by

$$D \in \mathrm{Der}(H_2(\mathbb{K}))$$
$$H \in H_2'(\mathbb{K})$$

we then have the brackets

$$[D, H] = D(H)$$
$$[H, H'] = [L_H, L_{H'}]$$

where we consider the derivation $[L_H, L_{H'}]$ to be an element of $\mathrm{Der}\,\mathbb{K}$. For the conformal algebra given by (8) we define the Lie brackets by taking $\mathrm{Str}\,\mathbb{J}$ as a subalgebra and the brackets as below. We take the elements of $\mathrm{Con}\, H_2(\mathbb{K})$ to be

$$T \in \mathrm{Str}\,\mathbb{J}$$
$$(X, Y) \in [H_2(\mathbb{K})]^2.$$

Then if $R \to R^*$ is an involutive automorphism which is the identity on $\mathrm{Der}\,\mathbb{J}$ and multiplies elements of $\mathbb{J}^2$ by -1 then the Lie brackets are

$$[T, (X, Y)] = (TX, T^*Y)$$
$$[(X, 0), (Y, 0)] = [(0, X), (0, Y)] = 0$$
$$[(X, 0), (0, Y)] = 2L_{XY} + 2[L_X, L_Y].$$

In the case of the conformal algebra for 2×2 hermitian matrices we substitute $H_2(\mathbb{K})$ for $\mathbb{J}$ in equation (8) to obtain

$$\mathrm{Con}\, H_2(\mathbb{K}) = \mathrm{Str}\, H_2(\mathbb{K}) \dot{+} [H_2(\mathbb{K})]^2$$

which we can expand to

$$\mathrm{Con}\, H_2(\mathbb{K}) = H_2'(\mathbb{K}) \dot{+} A_2'(\mathbb{K}) \dot{+} \mathfrak{so}(\mathbb{K}') \dot{+} \mathbb{R} \dot{+} [H_2(\mathbb{K})]^2. \tag{62}$$

We can now define the Lie brackets explicitly for $\mathrm{Con}\, H_2(\mathbb{K})$ by considering the following elements

$$\begin{aligned} D &\in \mathrm{Der}\, H_2(\mathbb{K}) \\ H &\in H_2'(\mathbb{K}) \\ r, r' &\in \mathbb{R} \\ (X, Y) &\in [H_2(\mathbb{K})]^2. \end{aligned}$$

Then the brackets are

$$\begin{aligned} [D, r] = [r, H] = [r, r'] &= 0 \\ [D, H] &= D(H) \\ [D, (X, Y)] &= (D(X), D(Y)) \\ [r, (X, Y)] &= (rX, -rY) \\ [H, (X, Y)] &= (H \cdot X, -H \cdot Y) \end{aligned}$$

with the brackets for (X, Y) defined as above. We can also think of $\mathrm{Str}\, H_2(\mathbb{K})$ and $\mathrm{Con}\, H_2(\mathbb{K})$ in terms of 2×2 matrices over $H_2(\mathbb{K})$. Writing $\mathrm{Str}\, H_2(\mathbb{K})$ and $\mathrm{Con}\, H_2(\mathbb{K})$ in this way gives

$$\begin{aligned} \mathrm{Str}\, H_2(\mathbb{K}) &= \mathrm{Der}\, H_2(\mathbb{K}) \dot{+} \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \quad A \in H_2(\mathbb{K}) \right\} \\ \mathrm{Con}\, H_2(\mathbb{K}) &= \mathrm{Der}\, H_2(\mathbb{K}) \dot{+} \left\{ \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \quad A, B, C \in H_2(\mathbb{K}) \right\}. \end{aligned}$$

In this section we denote the elements of $L_2(\mathbb{K}, \tilde{\mathbb{H}})$ by

$$\begin{aligned} A &\in A_2'(\mathbb{K}) \\ S &\in \mathfrak{so}(\mathbb{K}') \\ B \otimes \tilde{i}, C \otimes \tilde{j}, D \otimes \tilde{k} &\in H_2'(\mathbb{K}) \otimes \tilde{\mathbb{H}}' \end{aligned}$$

along with the basis elements of $\mathfrak{so}(\tilde{\mathbb{H}}') \cong \mathfrak{so}(2,1)$ which we will call s_{12}, s_{13} and s_{23}, where

$$\begin{aligned} s_{12} &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ s_{13} &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\ s_{23} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \end{aligned}$$

Now we define an isomorphism $\psi : L_2(\mathbb{K}, \tilde{\mathbb{H}}) \to \operatorname{Con} H_2(\mathbb{K})$ by

$$\begin{aligned} \psi(A) &= A & \psi(S) &= S \\ \psi(B \otimes \tilde{i}) &= B & & \\ \psi(C \otimes \tilde{j}) &= \tfrac{1}{2}(C, C), & \psi(D \otimes \tilde{k}) &= \tfrac{1}{2}(D, -D) \\ \psi(s_{12}) &= \tfrac{1}{2}(I, -I), & \psi(s_{13}) &= \tfrac{1}{2}(I, I) \\ \psi(s_{23}) &= 1. & & \end{aligned}$$

We can also define this in terms of our 4×4 matrices by

$$\begin{aligned} \psi(A) &= A & \psi(S) &= S \\ \psi(B \otimes \tilde{i}) &= \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix} & & \\ \psi(C \otimes \tilde{j}) &= \begin{pmatrix} 0 & \frac{1}{2}C \\ 0 & 0 \end{pmatrix}, & \psi(D \otimes \tilde{k}) &= \begin{pmatrix} 0 & 0 \\ \frac{1}{2}D & 0 \end{pmatrix} \\ \psi(s_{12}) &= \begin{pmatrix} 0 & 0 \\ \frac{1}{2}I & 0 \end{pmatrix}, & \psi(s_{13}) &= \begin{pmatrix} 0 & \frac{1}{2}I \\ 0 & 0 \end{pmatrix} \\ \psi(s_{23}) &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. & & \end{aligned}$$

The proof is a series of routine calculations of each product in both algebras showing that the isomorphism holds in all cases,which can be found on the next page.

Now $L_2(\mathbb{K}, \tilde{\mathbb{C}})$ is embedded in $L_2(\mathbb{K}, \tilde{\mathbb{H}})$ because $\tilde{\mathbb{C}}$ is embedded in $\tilde{\mathbb{H}}$ and ψ maps $L_2(\mathbb{K}, \tilde{\mathbb{C}})$ to $\operatorname{Str}' H_2(\mathbb{K})$. Thus if we define $\psi_1 = \psi \mid L_2(\mathbb{K}, \tilde{\mathbb{C}})$ then ψ_1 is an isomorphism between $L_2(\mathbb{K}, \tilde{\mathbb{C}})$ and $\operatorname{Str}' H_2(\mathbb{K})$. Similarly we can define $\psi_2 = \psi \mid L_2(\mathbb{K}, \mathbb{R})$ as an isomorphism between $L_2(\mathbb{K}, \mathbb{R})$ and $\operatorname{Der} H_2(\mathbb{K})$ by restricting ψ to $L_2(\mathbb{K}, \mathbb{R})$. Thus we have obtained proofs for all the 2×2 isomorphisms.

	A'	D'	$B' \otimes \tilde{i}$	$C' \otimes \tilde{j}$	$E' \otimes \tilde{k}$	s_{12}	s_{13}	s_{23}
A	$AA' - A'A$	$-D'(A)$	$(AB' - B'A) \otimes\tilde{i}$	$(AC'-C'A) \otimes\tilde{j}$	$(AE' - E'A) \otimes\tilde{k}$	0	0	0
D	$D(A')$	$DD' - D'D$	$D(B') \otimes \tilde{i}$	$D(C') \otimes \tilde{j}$	$D(E') \otimes \tilde{k}$	0	0	0
$B \otimes \tilde{i}$	$(BA'-A'B) \otimes\tilde{i}$	$-D'(B) \otimes \tilde{i}$	$[L_B, L_{B'}]$	$\frac{1}{2}\langle B, C'\rangle s_{12}+ (B * C') \otimes \tilde{k}$	$\frac{1}{2}\langle B, E'\rangle s_{13}+ (B * E') \otimes \tilde{j}$	$B \otimes \tilde{j}$	$B \otimes \tilde{k}$	0
$C \otimes \tilde{j}$	$(CA'-A'C) \otimes\tilde{j}$	$-D'(C) \otimes \tilde{j}$	$-\frac{1}{2}\langle C, B'\rangle s_{12}- (C * B') \otimes \tilde{k}$	$[L_C, L_{C'}]$	$-\frac{1}{2}\langle C, E'\rangle s_{23}- (C * E') \otimes \tilde{i}$	$-C \otimes \tilde{i}$	0	$C \otimes \tilde{k}$
$E \otimes \tilde{k}$	$(EA'-A'E) \otimes\tilde{k}$	$-D'(E) \otimes \tilde{k}$	$-\frac{1}{2}\langle E, B'\rangle s_{13}- (E * B') \otimes \tilde{j}$	$\frac{1}{2}\langle E, C'\rangle s_{23}+ (E * C') \otimes \tilde{i}$	$[L_E, L_{E'}]$	0	$E \otimes \tilde{i}$	$E \otimes \tilde{j}$
s_{12}	0	0	$-B' \otimes \tilde{j}$	$C' \otimes \tilde{i}$	0	0	s_{23}	$-s_{13}$
s_{13}	0	0	$-B' \otimes \tilde{k}$	0	$-E' \otimes \tilde{i}$	$-s_{23}$	0	$-s_{12}$
s_{23}	0	0	0	$-C' \otimes \tilde{k}$	$-E' \otimes \tilde{j}$	s_{13}	s_{12}	0

The multiplication table for $L_2(\mathbb{K}, \tilde{\mathbb{H}})$.

	A'	D'	B'	$\frac{1}{2}(C',C')$	$\frac{1}{2}(E',-E')$	$\frac{1}{2}(I,-I)$	$\frac{1}{2}(I,I)$	1
A	$AA'-A'A$	$-D'(A)$	$AB'-B'A$	$\frac{1}{2}(AC'-C'A,$ $AC'-C'A)$	$\frac{1}{2}(AE'-E'A,$ $E'A-AE')$	0	0	0
D	$D(A')$	$DD'-D'D$	$D(B')$	$\frac{1}{2}(D(C'),D(C'))$	$\frac{1}{2}(D(E'),-D(E'))$	0	0	0
B	$BA'-A'B$	$-D'(B)$	$[L_B,L_{B'}]$	$\frac{1}{4}\langle B,C'\rangle(I,-I)+$ $(B*C',B*C')$	$\frac{1}{4}\langle B,E'\rangle(I,I)+$ $(B*E',B*E')$	$\frac{1}{2}(B,B)$	$\frac{1}{2}(B,-B)$	0
$\frac{1}{2}(C,C)$	$\frac{1}{2}(CA'-A'C,$ $CA'-A'C)$	$-\frac{1}{2}(D'(C),D'(C))$	$-\frac{1}{4}\langle C,B'\rangle(I,-I)-$ $(C*B',C*B')$	$[L_C,L_{C'}]$	$-\frac{1}{2}\langle C,E'\rangle 1-C*E'$	$-C$	0	$\frac{1}{2}(C,-C)$
$\frac{1}{2}(E,-E)$	$\frac{1}{2}(EA'-A'E,$ $A'E-EA')$	$-\frac{1}{2}(D'(E),D'(E))$	$-\frac{1}{4}\langle E,B'\rangle(I,I)-$ $(E*B',E*B')$	$\frac{1}{2}\langle E,C'\rangle 1+E*C'$	$[L_E,L_{E'}]$	0	E	$\frac{1}{2}(E,E)$
$\frac{1}{2}(I,-I)$	0	0	$-\frac{1}{2}(B',B')$	C'	0	0	1	$-\frac{1}{2}(I,I)$
$\frac{1}{2}(I,I)$	0	0	$-\frac{1}{2}(B',-B')$	0	$-E'$	-1	0	$-\frac{1}{2}(I,-I)$
1	0	0	0	$-\frac{1}{2}(C',-C')$	$-\frac{1}{2}(E',E')$	$\frac{1}{2}(I,I)$	$\frac{1}{2}(I,-I)$	0

The multiplication table for $\mathrm{Con}\, H_2(\mathbb{K})$.

6. Magic squares of 3×3 Matrix Algebras: Proofs

In this section we extend the results of the last section to the 3×3 matrix case. We then develop these ideas by showing the maximal compact subalgebras for each of the exceptional Lie algebras that appear in the magic square.

We begin by showing that

$$L_3(\mathbb{K}, \tilde{\mathbb{H}}) \cong \operatorname{Con} H_3(\mathbb{K}) \tag{63}$$

and then show the maximal compact subalgebras are as stated previously.

6.1. **Proof of equation (63).**

In the case of 3×3 matrices we know from (7) and Theorem 2 that

$$\operatorname{Str}' H_3(\mathbb{K}) = L_3'(\mathbb{K}) \dot{+} \operatorname{Der} \mathbb{K}$$

and also from (8),

$$\operatorname{Con} H_3(\mathbb{K}) = \operatorname{Str} H_3(\mathbb{K}) \dot{+} [H_3(\mathbb{K})]^2.$$

Thus

$$\operatorname{Con} H_3(\mathbb{K}) = \operatorname{Der} \mathbb{K} \dot{+} A_3'(\mathbb{K}) \dot{+} H_3'(\mathbb{K}) \dot{+} \mathbb{R} \dot{+} [H_3(\mathbb{K})]^2.$$

We have

$$L_3(\mathbb{K}, \tilde{\mathbb{H}}) = \operatorname{Der} H_3(\mathbb{K}) \dot{+} H_3'(\mathbb{K}) \otimes \tilde{\mathbb{H}}' \dot{+} \operatorname{Der} \tilde{\mathbb{H}}.$$

It is known from [14] that $\operatorname{Der} \mathbb{H} = C(\mathbb{H}')$ and can be shown similarly that $\operatorname{Der} \tilde{\mathbb{H}} = C(\tilde{\mathbb{H}}')$. Thus, using Theorem 2

$$L_3(\mathbb{K}, \tilde{\mathbb{H}}) = \operatorname{Der} \mathbb{K} \dot{+} A_3'(\mathbb{K}) \dot{+} H_3'(\mathbb{K}) \otimes \tilde{\mathbb{H}}' \dot{+} C(\tilde{\mathbb{H}}').$$

If we consider the elements of $L_3(\mathbb{K}, \tilde{\mathbb{H}})$

$$\begin{gathered} A \in A_3'(\mathbb{K}) \\ D \in \operatorname{Der}(\mathbb{K}) \\ B \otimes \tilde{i}, C \otimes \tilde{j}, E \otimes \tilde{k} \in H_3'(\mathbb{K}) \otimes \tilde{\mathbb{H}}' \\ C_{\tilde{i}}, C_{\tilde{j}}, C_{\tilde{k}} \in C(\tilde{\mathbb{H}}') \end{gathered}$$

and the elements of $\operatorname{Con} H_3(\mathbb{K})$

$$\begin{gathered} A \in A_3'(\mathbb{K}) \\ B \in H_3'(\mathbb{K}) \\ D \in \operatorname{Der}(\mathbb{K}) \\ r \in \mathbb{R} \\ (X, Y) \in [H_3(\mathbb{K})]^2 \end{gathered}$$

then we can define an isomorphism $\phi : L_3(\mathbb{K}, \tilde{\mathbb{H}}) \to \mathrm{Con}\, H_3(\mathbb{K})$ by :

$$\begin{aligned} \phi(A) &= A & \phi(D) &= D \\ \phi(B \otimes \tilde{i}) &= B & & \\ \phi(C \otimes \tilde{j}) &= \tfrac{1}{2}(C, C) & \phi(E \otimes \tilde{k}) &= \tfrac{1}{2}(E, -E) \\ \phi(C_{\tilde{k}}) &= (I, -I) & \phi(C_{\tilde{j}}) &= (I, I) \\ \phi(C_{\tilde{i}}) &= 2. & & \end{aligned}$$

Again we can express $\mathrm{Con}\, H_3(\mathbb{K})$ and $\mathrm{Str}'\, H_3(\mathbb{K})$ in terms of 2×2 matrices over $H_3(\mathbb{K})$ given by

$$\mathrm{Str}'\, H_3(\mathbb{K}) = \mathrm{Der}\, H_3(\mathbb{K}) \dot{+} \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \quad A \in H_3'(\mathbb{K}) \right\}$$
$$\mathrm{Con}\, H_3(\mathbb{K}) = \mathrm{Der}\, H_3(\mathbb{K}) \dot{+} \left\{ \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \quad A, B, C \in H_3(\mathbb{K}) \right\}.$$

Further we can also define our isomorphism in terms of these matrices

$$\begin{aligned} \phi(A) &= A & \phi(D) &= D \\ \phi(B \otimes \tilde{i}) &= \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix} & & \\ \phi(C \otimes \tilde{j}) &= \begin{pmatrix} 0 & \frac{1}{2}C \\ 0 & 0 \end{pmatrix} & \phi(E \otimes \tilde{k}) &= \begin{pmatrix} 0 & 0 \\ \frac{1}{2}E & 0 \end{pmatrix} \\ \phi(C_{\tilde{k}}) &= \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} & \phi(C_{\tilde{j}}) &= \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \\ \phi(C_{\tilde{i}}) &= \begin{pmatrix} 2I & 0 \\ 0 & -2I \end{pmatrix}. & & \end{aligned}$$

We can again show this is a Lie algebra isomorphism by comparisons of the multiplication tables on the following page for $L_3(\mathbb{K}, \tilde{\mathbb{H}})$ and $\mathrm{Con}\, H_3(\mathbb{K})$.

	A'	D'	$B'\otimes\tilde{i}$	$C'\otimes\tilde{j}$	$E'\otimes\tilde{k}$	$C_{\tilde{k}}$	$C_{\tilde{j}}$	$C_{\tilde{i}}$
A	$AA'-A'A$	$-D'(A)$	$(AB'-B'A)\otimes\tilde{i}$	$(AC'-C'A)\otimes\tilde{j}$	$(AE'-E'A)\otimes\tilde{k}$	0	0	0
D	$D(A')$	$DD'-D'D$	$D(B')\otimes\tilde{i}$	$D(C')\otimes\tilde{j}$	$D(E')\otimes\tilde{k}$	0	0	0
$B\otimes\tilde{i}$	$(BA'-A'B)\otimes\tilde{i}$	$-D'(B)\otimes\tilde{i}$	$[L_B, L_{B'}]$	$\frac{1}{6}\langle B,C'\rangle C_{\tilde{k}}$ $+$ $(B*C')\otimes\tilde{k}$	$\frac{1}{6}\langle B,E'\rangle C_{\tilde{j}}$ $+$ $(B*E')\otimes\tilde{j}$	$2B\otimes\tilde{j}$	$2B\otimes\tilde{k}$	0
$C\otimes\tilde{j}$	$(CA'-A'C)\otimes\tilde{j}$	$-D'(C)\otimes\tilde{j}$	$-\frac{1}{6}\langle C,B'\rangle C_{\tilde{k}}-$ $(C*B')\otimes\tilde{k}$	$[L_C, L_{C'}]$	$-\frac{1}{6}\langle C,E'\rangle C_{\tilde{i}}-$ $(C*E')\otimes\tilde{i}$	$-2C\otimes\tilde{i}$	0	$2C\otimes\tilde{k}$
$E\otimes\tilde{k}$	$(EA'-A'E)\otimes\tilde{k}$	$-D'(E)\otimes\tilde{k}$	$-\frac{1}{6}\langle E,B'\rangle C_{\tilde{j}}-$ $(E*B')\otimes\tilde{j}$	$\frac{1}{6}\langle E,C'\rangle C_{\tilde{i}}$ $+$ $(E*C')\otimes\tilde{i}$	$[L_E, L_{E'}]$	0	$2E\otimes\tilde{i}$	$2E\otimes\tilde{j}$
$C_{\tilde{k}}$	0	0	$-2B'\otimes\tilde{j}$	$2C'\otimes\tilde{i}$	0	0	$2C_{\tilde{i}}$	$-2C_{\tilde{j}}$
$C_{\tilde{j}}$	0	0	$-2B'\otimes\tilde{k}$	0	$-2E'\otimes\tilde{i}$	$-2C_{\tilde{i}}$	0	$-2C_{\tilde{k}}$
$C_{\tilde{i}}$	0	0	0	$-2C'\otimes\tilde{k}$	$-2E'\otimes\tilde{j}$	$2C_{\tilde{j}}$	$2C_{\tilde{k}}$	0

The multiplication table for $L_3(\mathbb{K},\tilde{\mathbb{H}})$.

	A'	D'	B'	$\frac{1}{2}(C',C')$	$\frac{1}{2}(E',-E')$	$(I,-I)$	(I,I)	2
A	$AA'-A'A$	$-D'(A)$	$AB'-B'A$	$\frac{1}{2}(AC'-C'A,$ $AC'-C'A)$	$(AE'-E'A,$ $-AE'+E'A)$	0	0	0
D	$D(A')$	$DD'-D'D$	$D(B')$	$\frac{1}{2}(D(C'),D(C'))$	$\frac{1}{2}(D(E'),-D(E'))$	0	0	0
B	$BA'-A'B$	$-D'(B)$	$[L_B,L_{B'}]B$	$\frac{1}{6}\langle B,C'\rangle(I,-I)+$ $(B*C',B*C')$	$\frac{1}{6}\langle B,E'\rangle(I,I)+$ $(B*E',B*E')$	(B,B)	$(B,-B)$	0
$\frac{1}{2}(C,C)$	$\frac{1}{2}(CA'-A'C,$ $CA'-A'C)$	$-\frac{1}{2}(D'(C),D'(C))$	$-\frac{1}{6}\langle C,B'\rangle(I,-I)-$ $(C*B',C*B')$	$[L_C,L_{C'}]$	$-\frac{1}{3}\langle C,E'\rangle-C*E'$	$-2C$	0	$(C,-C)$
$\frac{1}{2}(E,-E)$	$\frac{1}{2}(EA'-A'E,$ $-EA'+A'E)$	$-\frac{1}{2}(D'(E),D'(E))$	$-\frac{1}{6}\langle E,B'\rangle(I,I)-$ $(E*B',E*B')$	$\frac{1}{3}\langle E,C'\rangle+E*C'$	$[L_E,L_{E'}]$	0	$2E$	(E,E)
$(I,-I)$	0	0	$-(B',B')$	$2C'$	0	0	4	$-2(I,I)$
(I,I)	0	0	$-(B',B')$	0	$-2E'$	-4	0	$-2(I,-I)$
2	0	0	0	$-(C',-C')$	$-(E',E')$	$2(I,I)$	$2(I,-I)$	0

The multiplication table for $\mathrm{Con}\, H_3(\mathbb{K})$.

Invoking the same method as used in the similar proof for 2×2 matrices we note that we can define $\phi_1 : L_3(\mathbb{K}, \tilde{\mathbb{C}}) \to \mathrm{Str}' H_3(\mathbb{K})$ by:

$$\phi_1(A) = A, \quad \phi_1(D) = D$$
$$\phi_1(B \otimes \tilde{i}) = B$$

and also, trivially, $\phi_2 : L_3(\mathbb{K}, \mathbb{R}) \to \mathrm{Der}\, H_3(\mathbb{K})$ by :

$$\phi_2(A) = A, \quad \phi_2(D) = D$$

Thus we have proved the following.

Theorem 7.

$$L_3(\mathbb{K}, \mathbb{R}) \cong \mathrm{Der}\, H_3(\mathbb{K})$$
$$L_3(\mathbb{K}, \tilde{\mathbb{C}}) \cong \mathrm{Str}'\, H_3(\mathbb{K})$$
$$L_3(\mathbb{K}, \tilde{\mathbb{H}}) \cong \mathrm{Con}\, H_3(\mathbb{K})$$

hold for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ *and* $\mathbb{O}$.

The relation to the matrix algebras described in the Introduction in equation (1) is:

Theorem 8.

$$L_3(\mathbb{K}, \mathbb{R}) \cong \mathfrak{su}(3, \mathbb{K})$$
$$L_3(\mathbb{K}, \tilde{\mathbb{C}}) \cong \mathfrak{sl}(3, \mathbb{K})$$
$$L_3(\mathbb{K}, \tilde{\mathbb{H}}) \cong \mathfrak{sp}(6, \mathbb{K}).$$

Thus we have the table

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathrm{Der}\, H_3(\mathbb{K}) \cong L_3(\mathbb{K}, \mathbb{R})$	$\mathfrak{su}(3, \mathbb{R})$	$\mathfrak{su}(3, \mathbb{C})$	$\mathfrak{su}(3, \mathbb{H})$	$\mathfrak{su}(3, \mathbb{O})$
$\mathrm{Str}'\, H_3(\mathbb{K}) \cong L_3(\mathbb{K}, \tilde{\mathbb{C}})$	$\mathfrak{sl}(3, \mathbb{R})$	$\mathfrak{sl}(3, \mathbb{C})$	$\mathfrak{sl}(3, \mathbb{H})$	$\mathfrak{sl}(3, \mathbb{O})$
$\mathrm{Con}\, H_3(\mathbb{K}) \cong L_3(\mathbb{K}, \tilde{\mathbb{H}})$	$\mathfrak{sp}(6, \mathbb{R})$	$\mathfrak{sp}(6, \mathbb{C})$	$\mathfrak{sp}(6, \mathbb{H})$	$\mathfrak{sp}(6, \mathbb{O})$

Proof.

1. We recall that $\mathfrak{su}(3, \mathbb{K})$ is the set of matrices with entries in $\mathbb{K}$ satisfying $A^\dagger G + GA = 0$, i.e. the set of anti-hermitian matrices. In this case we have $L_3(\mathbb{K}, \mathbb{R}) = A_3'(\mathbb{K}) \dot{+} \mathrm{Der}\, \mathbb{K}$. Take $A \in A_3'(\mathbb{K})$ and $D \in \mathrm{Der}\, \mathbb{K}$. Then $\psi : L_3(\mathbb{K}, \mathbb{R}) \to \mathfrak{su}(3, \mathbb{K})$ by

$$\psi(A + D) = A + DI$$

where I is the 3×3 matrix identity.

2. We define $\mathfrak{sl}(3, \mathbb{K})$ to be the family of 3×3 matrices with entries in $\mathbb{K}$ with the real part of the trace equal to zero. Also

$$L_3(\mathbb{K}, \tilde{\mathbb{C}}) = A_3'(\mathbb{K}) \dot{+} \mathrm{Der}\, \mathbb{K} \dot{+} H_3'(\mathbb{K}) \otimes \tilde{\mathbb{C}}.$$

If we take $A \in A'_3(\mathbb{K})$, $D \in \operatorname{Der}\mathbb{K}$ and $H \in H'_3(\mathbb{K})$ (since we can regard the tensor product $H'_3(\mathbb{K}) \otimes \tilde{\mathbb{C}}$ as being one copy of $H'_3(\mathbb{K})$) then the isomorphism $\phi : L_3(\mathbb{K}, \tilde{\mathbb{C}}) \to \mathfrak{sl}(3, \mathbb{K})$ can be written

$$\phi(A + D + H) = A + DI + H.$$

3. The Lie algebra $\mathfrak{sp}(6, \mathbb{K})$ is defined to be the Lie algebra of 6×6 matrices satisfying the equation $A^\dagger J + JA = 0$ with the added condition that the trace is also zero. This can be written as the matrix $\begin{pmatrix} A & B \\ C & -A^\dagger \end{pmatrix}$ where A, B and C are 3×3 block matrices and B and C are hermitian. We have

$$L_3(\mathbb{K}, \tilde{\mathbb{H}}) = A'_3(\mathbb{H}) \dotplus \operatorname{Der}\mathbb{H} \dotplus H'_3(\mathbb{H}) \otimes \tilde{\mathbb{H}}' \dotplus \operatorname{Der}\tilde{\mathbb{H}}.$$

Taking $A \in A'_3(\mathbb{K})$, $D \in \operatorname{Der}\mathbb{K}$, $H_1 \otimes \tilde{i}$, $H_2 \otimes \tilde{j}$ and $H_3 \otimes \tilde{k} \in H'_3(\mathbb{K}) \otimes \tilde{\mathbb{H}}$ and $r_1 C_{\tilde{i}} + r_2 C_{\tilde{j}} + r_3 C_{\tilde{k}} \in \operatorname{Der}\tilde{\mathbb{H}}$ then the isomorphism $\chi : L_3(\mathbb{K}, \tilde{\mathbb{H}}) \to \mathfrak{sp}(6, \mathbb{K})$ can be written explicitly as

$$\chi(A + DI + H_1 \otimes \tilde{i} + H_2 \otimes \tilde{j} + H_3 \otimes \tilde{k} + r_1 C_{\tilde{i}} + r_2 C_{\tilde{j}} + r_3 C_{\tilde{k}}) = \begin{pmatrix} A + DI + H_1 + \frac{1}{3} r_1 I & (H_2 + r_2 I - H_3 - \frac{1}{3} r_3 I) \\ (H_2 + r_2 I + H_3 + \frac{1}{3} r_3 I) & A + DI - H_1 - \frac{1}{3} r_1 I \end{pmatrix}.$$

□

6.2. **Maximal Compact Subalgebras.** A semi-simple Lie algebra is called *compact* if it has a negative-definite killing form. It is called *non-compact* if its killing form is not negative-definite.

A non-compact real form, $\mathfrak{g}$, of a semi-simple complex Lie algebra, L, has a *maximal compact subalgebra* $\mathfrak{k}$ with an *orthogonal complementary subspace* $\mathfrak{p}$ such that $\mathfrak{g} = \mathfrak{k} \dotplus \mathfrak{p}$ and the brackets

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$$
$$[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$$
$$[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$$
$$(\mathfrak{k}, \mathfrak{p}) = 0$$

are satisfied. We denote by $(,)$ the killing form of L. There exists an involutive automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ such that $\mathfrak{k}$ and $\mathfrak{p}$ are eigenspaces of σ with eigenvalues $+1$ and -1 respectively. A compact real form, $\mathfrak{g}'$, of L will also contain $\mathfrak{k}$ as a compact subalgebra of $\mathfrak{g}'$ but clearly in this case the maximal compact subalgebra will be $\mathfrak{g}'$ itself. We can obtain $\mathfrak{g}'$ from $\mathfrak{g}$ by keeping the same brackets in $[\mathfrak{k}, \mathfrak{k}]$ and $[\mathfrak{k}, \mathfrak{p}]$ but multiplying the brackets in $[\mathfrak{p}, \mathfrak{p}]$ by -1, i.e. by performing the *Weyl unitary trick* (putting $\mathfrak{g}' = \mathfrak{k} \dotplus i\mathfrak{p}$).

We will now give an overview of the method used to show that the algebras given in the table on page 6 are maximal compact, which is essentially the same in each case. We know that $L_3(\mathbb{K}_1, \mathbb{K}_2)$ gives a compact real form of each Lie algebra (from [7]). Thus if $L_3(\mathbb{K}_1,\tilde{\mathbb{K}}_2)$ shares a common subalgebra with $L_3(\mathbb{K}_1,\mathbb{K}_2)$, say $\mathfrak{f}$, where

$$L_3(\mathbb{K}_1,\mathbb{K}_2) = \mathfrak{f}\dot{+}\mathfrak{p}_1$$
$$L_3(\mathbb{K}_1,\tilde{\mathbb{K}}_2) = \mathfrak{f}\dot{+}\mathfrak{p}_2,$$

and the brackets in $[\mathfrak{f},\mathfrak{p}_1]$ are the same as those in $[\mathfrak{f},\mathfrak{p}_2]$ but the brackets in $[\mathfrak{p}_1,\mathfrak{p}_1]$ are -1 times the equivalent brackets in $[\mathfrak{p}_2,\mathfrak{p}_2]$, then $\mathfrak{f}$ will be the maximal compact subalgebra of $L_3(\mathbb{K}_1,\tilde{\mathbb{K}}_2)$ and $\mathfrak{p}_2$ will be its orthogonal complementary subspace. Moreover, because of the nature of the split composition algebras, this sign change in the brackets will reflect precisely the change in sign in the Cayley-Dickson process when moving from the non-split to the split form of the composition algebra.

We will briefly consider the nature of $\operatorname{Der}\mathbb{O}$ and $\operatorname{Der}\tilde{\mathbb{O}}$ since the structure of these algebras form a fundamental part of this proof. It is well known that $\operatorname{Der}\mathbb{O}\cong G_2$ (see [14]). The derivation algebra of $\mathbb{O}$ is the Lie algebra of the automorphism group of $\mathbb{O}$. The derivations can be split into two types, those that are the infinitesimal versions of the automorphisms of $\mathbb{O}$ fixing the complex subspace $\mathbb{C}=\mathbb{R}\dot{+}i\mathbb{R}$ and those which are the infinitesimal versions of automorphisms fixing $\mathbb{R}$. The derivations of the first type (leaving i invariant) form a subalgebra isomorphic to $\mathfrak{su}(3)$.

Further we can express the elements of $\operatorname{Der}\mathbb{O}$ in terms of pairs of basis elements s_{ij} of $\mathfrak{so}(7)$, as defined previously, since $\operatorname{Der}\mathbb{O}\subset\mathfrak{so}(7)$. Similarly $\operatorname{Der}\tilde{\mathbb{O}}\subset\mathfrak{so}(4,3)$, giving a representation of $\operatorname{Der}\tilde{\mathbb{O}}$ in terms of pairs of basis elements of $\mathfrak{so}(4,3)$. It can be shown that the 14 elements of $\operatorname{Der}\mathbb{O}$ (by a method similar to that found in [1]) are

$$\begin{aligned}
g_1 &= s_{23}-s_{45} & g_2 &= s_{45}+s_{67}\\
g_3 &= s_{25}-s_{34} & g_4 &= -s_{27}-s_{36}\\
g_5 &= -s_{47}-s_{56} & g_6 &= -s_{24}-s_{35}\\
g_7 &= -s_{26}+s_{37} & g_8 &= -s_{46}+s_{57}\\
g_9 &= -s_{12}+s_{47} & g_{10} &= s_{13}+s_{57}\\
g_{11} &= s_{14}+s_{27} & g_{12} &= -s_{15}+s_{37}\\
g_{13} &= s_{16}+s_{25} & g_{14} &= -s_{17}+s_{24}.
\end{aligned}$$

The derivations in $\operatorname{Der}\tilde{\mathbb{O}}$ are obtained directly from these by writing, for example

$$\tilde{g_1} = \tilde{s}_{23}-\tilde{s}_{45},$$

where, if S_{ij} is the matrix representation of s_{ij} then G_2S_{ij} is the matrix representation of $\tilde{s_{ij}}$ (recall that G_2 is the metric for $\mathbb{K}_2$). The other elements of $\operatorname{Der}\tilde{\mathbb{O}}$ are obtained by a similar method. Then $\operatorname{Der}\mathbb{O}$ and $\operatorname{Der}\tilde{\mathbb{O}}$ have a common subalgebra $\mathfrak{so}(3)\dot{+}\mathfrak{so}(3)$ (see, for example, [4]) which is the subalgebra with basis elements $\{g_1,g_9,g_{10},g_2,g_5,g_8\}$, these being invariant under multiplication by the metric G_2. We will denote by $\tilde{G}_2$ the form of the exceptional Lie algebra G_2 isomorphic to $\operatorname{Der}\tilde{\mathbb{O}}$.

We now state explicity the result we are about to prove.

Theorem 9. *The maximal compact subalgebras, stated explicity, are of the forms given in the table below.*

$\mathfrak{g}$	$\mathfrak{k}$	
$E_{6,1}$	F_4	$\operatorname{Der} H_3(\mathbb{O})$
$E_{7,1}$	$E_6 \oplus \mathfrak{so}(2)$	$\operatorname{Der} H_3(\mathbb{O}) \dot{+} H_3'(\mathbb{O}) \otimes \{i\} \dot{+} \{C_i\}$
$E_{8,1}$	$E_7 \oplus \mathfrak{so}(3)$	$\operatorname{Der} H_3(\mathbb{O}) \dot{+} H_3'(\mathbb{O}) \otimes \mathbb{H}' \dot{+} M_{G_2}$
$E_{6,2}$	$\mathfrak{sq}(3) \oplus \mathfrak{so}(3)$	$\operatorname{Der} H_3(\mathbb{C}) \dot{+} H_3'(\mathbb{C}) \otimes \mathbb{H}' \dot{+} M_{G_2}$
$E_{7,2}$	$\mathfrak{su}(6) \oplus \mathfrak{so}(3)$	$\operatorname{Der} H_3(\mathbb{H}) \dot{+} H_3'(\mathbb{H}) \otimes \mathbb{H}' \dot{+} M_{G_2}$

.

where $M_{G_2} (= \{g_1, g_9, g_{10}, g_2, g_5, g_8\})$*, is the maximal compact subalgebra of* $\tilde{G}_2$*.*

Proof. We consider first each of the $_{,1}$ type algebras and then move on to the $_{,2}$ type. Denote by A^N the non- compact form of the algebra A and by A^C the compact form of A.

1. For $E_{6,1}^N$
$$\mathfrak{k} = \operatorname{Der} H_3(\mathbb{O})$$
$$\mathfrak{p} = H_3'(\mathbb{O}) \otimes \tilde{\mathbb{C}}'.$$
E_6^C also has $\mathfrak{k}$ as a subalgebra but in this case the remaining subspace is $H_3'(\mathbb{O}) \otimes \tilde{\mathbb{C}}'$. Thus there is only one set of brackets to check. If we consider $H_1 \otimes \tilde{i}, H_2 \otimes \tilde{i} \in H_3(\mathbb{O}) \otimes \tilde{\mathbb{C}}$ and $H_1 \otimes i, H_2 \otimes i \in H_3(\mathbb{O}) \otimes \mathbb{C}$ then clearly, using the definitions for the brackets found on page 5
$$[H_1 \otimes \tilde{i}, H_2 \otimes \tilde{i}] = [L_{H_1}, L_{H_2}]$$
$$[H_1 \otimes i, H_2 \otimes i] = -[L_{H_1}, L_{H_2}],$$
as required.
2. In the case of $E_{7,1}^N$ the orthogonal complementary subspace is
$$\tilde{\mathfrak{p}} = H_3'(\mathbb{O}) \otimes \{\tilde{j}, \tilde{k}\} \dot{+} \{C_{\tilde{j}}, C_{\tilde{k}}\},$$
where $\mathfrak{k}$ is the maximal compact subalgebra given in the table above. Then $\mathfrak{k}$ is also a subalgebra in E_7^C and the remaining subspace is $\mathfrak{p} = H_3'(\mathbb{O}) \otimes \{j, k\} \dot{+} \{C_j, C_k\}$. Now
$$[C_{\tilde{j}}, C_{\tilde{k}}] = -2C_i \qquad [C_j, C_k] = 2C_i$$
$$[C_{\tilde{j}}, H_1 \otimes \tilde{k}] = -H_1 \otimes 2i \qquad [C_j, H_1 \otimes k] = H_1 \otimes 2i$$
$$[C_{\tilde{k}}, H_1 \otimes \tilde{j}] = H_1 \otimes 2i \qquad [C_k, H_1 \otimes j] = -H_1 \otimes 2i.$$

Recall that

$$[H_1 \otimes x, H_2 \otimes y] = 2\operatorname{tr}(H_1 H_2) D_{x,y} + (H * G) \otimes \operatorname{Im}(xy) - Re(x\bar{y})[L_{H_1}, L_{H_2}].$$

We have to consider the two cases (1) when $x = y$ and (2) when $x \perp y$. Case (1) is considered in E_6. Case (2) gives

$$[H_1 \otimes \tilde{j}, H_2 \otimes \tilde{k}] = -2\operatorname{tr}(H_1 H_2) C_i - (H_1 * H_2) \otimes i$$
$$[H_1 \otimes j, H_2 \otimes k] = 2\operatorname{tr}(H_1 H_2) C_i + (H_1 * H_2) \otimes i.$$

Clearly, $\mathfrak{p}$ and $\tilde{\mathfrak{p}}$ are orthogonal complementary subspaces with their brackets with themselves giving opposite signs and thus we can deduce that the choice of maximal compact subalgebra is correct.

3. In E_8^N we have the orthogonal complementary subspace

$$\tilde{\mathfrak{p}} = H_3'(\mathbb{O}) \otimes \{\tilde{l}, \tilde{il}, \tilde{jl}, \tilde{kl}\} \dot{+} \{\tilde{g_a} \mid a = 3, 4, 6, 7, 11, 12, 13, 14\}$$

to the maximal compact subalgebra $\mathfrak{f}$ as shown in the previous table. Then $\mathfrak{f}$ is also a subalgebra of E_8^C and the remaining subspace in E_8^C will be $\mathfrak{p} = H_3'(\mathbb{O}) \otimes \{l, il, jl, kl\} \dot{+} \{g_a \mid a = 3, 4, 6, 7, 11, 12, 13, 14\}$. For convenience we will label the orthogonal complementary subspaces of G_2 $\mathfrak{p}_{G_2}$ and $\tilde{\mathfrak{p}}_{G_2}$ for the compact and non-compact cases respectively. The calculations for E_8 are much the same as those for E_7. For brackets between $H_3'(\mathbb{O}) \otimes \{l, il, jl, kl\}$ and itself we again have two cases with $x = y$ and $x \perp y$. These are resolved in the same way as before. To calculate the brackets between $H_3'(\mathbb{O}) \otimes \{l, il, jl, kl\}$ and $\mathfrak{p}_{G_2}$ and between $\mathfrak{p}_{G_2}$ and itself involves a set of long but relatively simple calculations involving s_{ij} and $\tilde{s_{ij}}$. These produce the signs as expected, however, in the interest of the rainforests, we have not reproduced them here.

Now notice that these proofs do not in fact involve the matrices in $H_3(\mathbb{K})$ since the change between compactness and non-compactness does not involve $\mathbb{K}_1$ but only $\mathbb{K}_2$. Thus since the orthogonal compact subspaces of $E_{6,2}$ and $E_{7,2}$ are

$$\tilde{\mathfrak{p}_1} = H_3'(\mathbb{C}) \otimes \{\tilde{l}, \tilde{il}, \tilde{jl}, \tilde{kl}\} \dot{+} \tilde{\mathfrak{p}}_{G_2}$$
$$\tilde{\mathfrak{p}_1} = H_3'(\mathbb{H}) \otimes \{\tilde{l}, \tilde{il}, \tilde{jl}, \tilde{kl}\} \dot{+} \tilde{\mathfrak{p}}_{G_2},$$

the proofs for these maximal compact subalgebras are contained within the that of E_8

Thus we have covered all of the subalgebras and our proof is complete. □

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