

“Cayley-Klein” schemes for real Lie algebras and Freudenthal Magic Squares

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Abstract

We introduce three “Cayley-Klein” families of Lie algebras through realizations in terms of either real, complex or quaternionic matrices. Each family includes simple as well as some limiting quasi-simple real Lie algebras. Their relationships naturally lead to an infinite family of 3×3 Freudenthal-like magic squares, which relate algebras in the three CK families. In the lowest dimensional cases suitable extensions involving octonions are possible, and for $N = 1, 2$, the “classical” 3×3 Freudenthal-like squares admit a 4×4 extension, which gives the original Freudenthal square and the Sudbery square.

1 The \mathfrak{sa} , \mathfrak{sl} , \mathfrak{sh} CK families

Consider the *real* matrices of order $N+1$ given by:

$$J_{ab} = \begin{pmatrix} \cdot & \dot{\cdot} & \cdot & -\dot{\omega}_{ab} & \cdot \\ \cdot & \dot{\cdot} & \cdot & \cdot & \cdot \\ \cdot & \dot{1} & \cdot & \cdot & \cdot \\ \cdot & \dot{\cdot} & \cdot & \cdot & \cdot \\ \cdot & \dot{\cdot} & \cdot & \cdot & \cdot \end{pmatrix}, \quad M_{ab} = \begin{pmatrix} \cdot & \dot{\cdot} & \cdot & \dot{\omega}_{ab} & \cdot \\ \cdot & \dot{\cdot} & \cdot & \cdot & \cdot \\ \cdot & \dot{1} & \cdot & \cdot & \cdot \\ \cdot & \dot{\cdot} & \cdot & \cdot & \cdot \\ \cdot & \dot{\cdot} & \cdot & \cdot & \cdot \end{pmatrix}, \quad H_m = \begin{pmatrix} -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \dot{1} & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad E_0 = \begin{pmatrix} 1 & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} \quad (1)$$

where $a, b = 0, 1, \dots, N$, $a < b$, $m = 1, \dots, N$, the matrix indices range from $0, 1, \dots, N$, the dotted rows and columns are those with row or column indices a, b or m , and $\omega_{ab} := \omega_{a+1}\omega_{a+2}\cdots\omega_b$ depend on N non-zero real coefficients $\omega_1, \dots, \omega_N$.

We apply two basic procedures to these matrices to build up a set of matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$ which contain sets of generators for *all* simple classical real Lie algebras:

- Let X be a real matrix in (1). In the complex case, let $X^1 := iX$. In the quaternion case, let $X^1 := iX$, $X^2 := jX$, $X^3 := kX$. Generically, these new matrices will be denoted X^α , where the range of α is none for \mathbb{R} , 1 for \mathbb{C} and 1, 2, 3 for \mathbb{H} .
- For any X in the previous list, define matrices $\mathbb{X}, \mathbb{X}_\lambda$, $\lambda = 1, 2, 3$ of order $2(N+1)$:

$$\mathbb{X} = \begin{pmatrix} X & \\ & X \end{pmatrix}, \quad \mathbb{X}_{;1} = \begin{pmatrix} & X \\ -X & \end{pmatrix}, \quad \mathbb{X}_{;2} = \begin{pmatrix} & X \\ X & \end{pmatrix}, \quad \mathbb{X}_{;3} = \begin{pmatrix} X & \\ & -X \end{pmatrix} \quad (2)$$

Let A be any matrix of order r over either $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and let G denote the symmetric or antisymmetric real matrix of an hermitian or skew-hermitian product in the space \mathbb{K}^r . The matrix A will be called G -antihermitian if $A^\dagger G + GA^\dagger = 0$, and G -hermitian if $A^\dagger G - GA^\dagger = 0$. With the choices $I_\omega = \text{diag}(1, \omega_{01}, \omega_{02}, \dots, \omega_{0N})$ and $\mathbb{I}_\omega = \mathbb{I}_{;1}$, the matrices $J_{ab}, M_{ab}^\alpha, H_m^\alpha, E_0^\alpha$ are I_ω -antihermitian, and $\mathbb{J}_{ab}, \mathbb{M}_{ab;\lambda}, \mathbb{H}_{m;\lambda}, \mathbb{E}_{0;\lambda}, \mathbb{J}_{ab;\lambda}^\alpha, \mathbb{M}_{ab}^\alpha, \mathbb{H}_m^\alpha, \mathbb{E}_0^\alpha$ are \mathbb{I}_ω -antihermitian, no matter of whether $\omega_i = 0$ or not.

Now we define the three ‘‘classical’’ CK series of algebras as follows [1]:

- $\mathfrak{sa}_{\omega_1 \dots \omega_N}(N+1, \mathbb{K})$, the *special antihermitian* CK algebra over \mathbb{K} is the quotient of the Lie algebras of $N+1 \times N+1$ I_ω -antihermitian matrices over \mathbb{K} by its center. They can be realized as the Lie algebra of all I_ω -antihermitian matrices over \mathbb{K} if $\mathbb{K} = \mathbb{R}, \mathbb{H}$ and as the subalgebra determined by the condition $\text{tr}X = 0$ if $\mathbb{K} = \mathbb{C}$.

- $\mathfrak{sl}_{\omega_1 \dots \omega_N}(N+1, \mathbb{K})$, the *special linear* CK algebra over \mathbb{K} is the quotient of the Lie algebra of all $N+1 \times N+1$ matrices over \mathbb{K} by its center. It can be realized as the Lie subalgebra of the Lie algebra of all $N+1 \times N+1$ matrices over \mathbb{K} determined by the condition $\text{tr}X = 0$ if $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and by the condition $\text{Re}(\text{tr}X) = 0$ when $\mathbb{K} = \mathbb{H}$.

- $\mathfrak{sh}_{\omega_1 \dots \omega_N}(2(N+1), \mathbb{K})$, the *special symplectic antihermitian* CK algebra over \mathbb{K} is the quotient of the Lie algebra of \mathbb{I}_ω -antihermitian matrices of order $2(N+1)$ over \mathbb{K} by its center. It is the analogous to the first family when the metric matrix is the antisymmetric \mathbb{I}_ω . It can be realized again as the Lie algebra of all \mathbb{I}_ω -antihermitian matrices if $\mathbb{K} = \mathbb{R}, \mathbb{H}$ and as the subalgebra with $\text{tr}\mathbb{X} = 0$ if $\mathbb{K} = \mathbb{C}$.

The notation \mathfrak{sa} has been used in [2] and the notation \mathfrak{sn} is new, although of course the algebras so denoted are not. The CK algebras with $\omega_1 \dots \omega_N = + \dots +$ are isomorphic to the ones usually denoted by the same symbol and without any ω subscript. The translation to the standard notation is as follows:

$$\begin{array}{llll}
& \mathbb{K} = \mathbb{R} & \mathbb{K} = \mathbb{C} & \mathbb{K} = \mathbb{H} \\
\mathfrak{sa}_{+\dots+}(N+1, \mathbb{K}) & \mathfrak{so}(N+1) & \mathfrak{su}(N+1) & \mathfrak{sp}(N+1) \\
\mathfrak{sl}_{+\dots+}(N+1, \mathbb{K}) & \mathfrak{sl}(N+1, \mathbb{R}) & \mathfrak{sl}(N+1, \mathbb{C}) & \mathfrak{su}^*(2(N+1)) \\
\mathfrak{sn}_{+\dots+}(2(N+1), \mathbb{K}) & \mathfrak{sp}(2(N+1), \mathbb{R}) & \mathfrak{su}(N+1, N+1) & \mathfrak{so}^*(4(N+1))
\end{array} \quad (3)$$

When the ω_i are not all positive, the CK algebras in the three \mathfrak{sa} series are isomorphic to the non-compact real forms $\mathfrak{so}(p, q)$, $\mathfrak{su}(p, q)$, and $\mathfrak{sp}(p, q)$. When some $\omega_i = 0$, the general CK algebras $\mathfrak{sa}_{\omega_1 \dots \omega_N}(N+1, \mathbb{K})$, etc. are defined in such a way that each $\omega_i = 0$ corresponds to a contraction; lack of space precludes giving details.

The CK Lie algebras \mathfrak{sa} , \mathfrak{sl} , \mathfrak{sn} over the three associative division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ can be generated by means of adequate choices of matrices, as given in the Table:

Lie algebra	has a linear basis given by the matrices		
	$\mathbb{K}=\mathbb{R}$	$\mathbb{K}=\mathbb{C}$	$\mathbb{K}=\mathbb{H}$
$\mathfrak{sa}_{\omega_1 \dots \omega_N}(N+1, \mathbb{K})$	J_{ab}	J_{ab}, M_{ab}^1, H_m^1	$J_{ab}, M_{ab}^\alpha, H_m^\alpha, E_0^\alpha$
$\mathfrak{sl}_{\omega_1 \dots \omega_N}(N+1, \mathbb{K})$	J_{ab}, M_{ab}, H_m	$J_{ab}, M_{ab}, H_m,$ $J_{ab}^1, M_{ab}^1, H_m^1$	$J_{ab}, M_{ab}, H_m,$ $J_{ab}^\alpha, M_{ab}^\alpha, H_m^\alpha, E_0^\alpha$
$\mathfrak{sn}_{\omega_1 \dots \omega_N}(2(N+1), \mathbb{K})$	$\mathbb{J}_{ab}, \mathbb{M}_{ab;\lambda}, \mathbb{H}_{m;\lambda},$ $\mathbb{E}_{0;\lambda}$	$\mathbb{J}_{ab}, \mathbb{M}_{ab;\lambda}, \mathbb{H}_{m;\lambda}, \mathbb{E}_{0;\lambda},$ $\mathbb{J}_{ab}^1, \mathbb{M}_{ab}^1, \mathbb{H}_m^1$	$\mathbb{J}_{ab}, \mathbb{M}_{ab;\lambda}, \mathbb{H}_{m;\lambda}, \mathbb{E}_{0;\lambda},$ $\mathbb{J}_{ab}^\alpha, \mathbb{M}_{ab}^\alpha, \mathbb{H}_{ma}^\alpha, \mathbb{E}_0^\alpha$

2 The “classical” $(N+1)$ -d Freudenthal-like square

The former table looks simpler if each Lie algebra is given as the Lie span (instead of the linear span) of as few elements as possible. A minimal choice is:

Lie algebra	is the Lie span of the generators		
	$\mathbb{K}=\mathbb{R}$	$\mathbb{K}=\mathbb{C}$	$\mathbb{K}=\mathbb{H}$
$\mathfrak{sa}_{\omega_1 \dots \omega_N}(N+1, \mathbb{K})$	J_{ab}	J_{ab}, M_{ab}^1	$J_{ab}, M_{ab}^1, M_{ab}^2$
$\mathfrak{sl}_{\omega_1 \dots \omega_N}(N+1, \mathbb{K})$	J_{ab}, M_{ab}	J_{ab}, M_{ab}, M_{ab}^1	$J_{ab}, M_{ab}, M_{ab}^1, M_{ab}^2$
$\mathfrak{sn}_{\omega_1 \dots \omega_N}(2(N+1), \mathbb{K})$	$\mathbb{J}_{ab}, \mathbb{M}_{ab;2}, \mathbb{M}_{ab;1}$	$\mathbb{J}_{ab}, \mathbb{M}_{ab;2}, \mathbb{M}_{ab;1}, \mathbb{M}_{ab}^1$	$\mathbb{J}_{ab}, \mathbb{M}_{ab;2}, \mathbb{M}_{ab;1}, \mathbb{M}_{ab}^1, \mathbb{M}_{ab}^2$

This Table shows a rather unexpected and remarkable symmetry between rows and columns. Since $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$, each algebra is in the obvious way a subalgebra of those at its left. And each algebra is also a subalgebra of those below it, provided we have made the isomorphic identifications $J_{ab} \rightarrow \mathbb{J}_{ab}, M_{ab} \rightarrow \mathbb{M}_{ab;2}, M_{ab}^1 \rightarrow \mathbb{M}_{ab}^1, M_{ab}^2 \rightarrow \mathbb{M}_{ab}^2$. This is required since \mathfrak{sn} is a group of matrices of dimension twice that of \mathfrak{sl} .

We shall better observe this symmetry if we move from the left to the right and from the top to the bottom. We realize that in each step new generators appears. Let us illustrate this idea as follows. As we move from the top \mathfrak{sa} to the bottom \mathfrak{sh} , M_{ab} appears in the first step ($\mathfrak{sa} \rightarrow \mathfrak{sl}$) and $\mathbb{M}_{ab;1}$ in the second ($\mathfrak{sl} \rightarrow \mathfrak{sh}$). This behaviour is the *same* for the three columns $\mathbb{R}, \mathbb{C}, \mathbb{H}$. On the other hand, moving from left to right, in the transition $\mathbb{R} \rightarrow \mathbb{C}$ we always add M_{ab}^1 , and in the transition $\mathbb{C} \rightarrow \mathbb{H}$ we add M_{ab}^2 . This behaviour appears in the *three* rows.

This symmetry suggests to consider a 3×3 square of CK algebras, with rows labeled by $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and columns by $\mathfrak{sa}, \mathfrak{sl}, \mathfrak{sh}$. Each site in the Table below contains the generic algebra in the CK family, the complete list of their basis generators, and the Cartan class of the corresponding simple Lie algebras.

$B_{N/2}$ or $D_{(N+1)/2}$	A_N	C_{N+1}
$\mathfrak{sa}_{\omega_1 \dots \omega_N}(N+1, \mathbb{R})$	$\mathfrak{sa}_{\omega_1 \dots \omega_N}(N+1, \mathbb{C})$	$\mathfrak{sa}_{\omega_1 \dots \omega_N}(N+1, \mathbb{H})$
J_{ab}	J_{ab}, M_{ab}^1, H_m^1	$J_{ab}, M_{ab}^\alpha, H_m^\alpha, E_0^\alpha$
A_N	$A_N \oplus A_N$	$A_{2(N+1)-1}$
$\mathfrak{sl}_{\omega_1 \dots \omega_N}(N+1, \mathbb{R})$	$\mathfrak{sl}_{\omega_1 \dots \omega_N}(N+1, \mathbb{C})$	$\mathfrak{sl}_{\omega_1 \dots \omega_N}(N+1, \mathbb{H})$
J_{ab}, M_{ab}, H_m	$J_{ab}, M_{ab}, H_m,$ $J_{ab}^1, M_{ab}^1, H_m^1$	$J_{ab}, M_{ab}, H_m,$ $J_{ab}^\alpha, M_{ab}^\alpha, H_m^\alpha, E_0^\alpha$
C_{N+1}	$A_{2(N+1)-1}$	$D_{2(N+1)}$
$\mathfrak{sh}_{\omega_1 \dots \omega_N}(2(N+1), \mathbb{R})$	$\mathfrak{sh}_{\omega_1 \dots \omega_N}(2(N+1), \mathbb{C})$	$\mathfrak{sh}_{\omega_1 \dots \omega_N}(2(N+1), \mathbb{H})$
$\mathbb{J}_{ab}, \mathbb{M}_{ab;\lambda}, \mathbb{H}_{m;\lambda}, \mathbb{E}_{0;\lambda}$	$\mathbb{J}_{ab}, \mathbb{M}_{ab;\lambda}, \mathbb{H}_{m;\lambda}, \mathbb{E}_{0;\lambda},$ $\mathbb{J}_{ab;\lambda}^1, \mathbb{M}_{ab;\lambda}^1, \mathbb{H}_{m;\lambda}^1$	$\mathbb{J}_{ab}, \mathbb{M}_{ab;\lambda}, \mathbb{H}_{m;\lambda}, \mathbb{E}_{0;\lambda},$ $\mathbb{J}_{ab;\lambda}^\alpha, \mathbb{M}_{ab;\lambda}^\alpha, \mathbb{H}_{m;\lambda}^\alpha, \mathbb{E}_{0;\lambda}^\alpha$

Dimension checking is easy. Each J_{ab} or M_{ab} counts as $N(N+1)/2$, each H_m as N , and each E_0 as one. If the three columns are labeled by $p = 1, 2, 4$ and the three rows by $q = 1, 2, 4$, the dimension of the algebra at site p, q is

$$\dim(p, q) = pq \frac{N(N+1)}{2} + (p+q-2)N + (0, 0, 3)_p + (0, 0, 3)_q \quad (4)$$

where the symbols $(0, 0, 3)_p$ or $(0, 0, 3)_q$ refers to the 3 extra generators (E_0^α) or ($\mathbb{E}_{0;\lambda}$) appearing respectively when $\mathbb{K} = \mathbb{H}$ or in the \mathfrak{sh} case. There are similar expressions for the characters of the CK algebras involved. We emphasize that for each choice of values $\omega_1 \dots \omega_N$, the algebras included in the square are different. Finally, when some $\omega_i = 0$ then the algebras are not simple; nevertheless the properties of the square are maintained in all cases, so this $(\omega_1 \dots \omega_N)$ -square includes a ‘‘compact’’, several ‘‘non-compact’’ and additional ‘‘non-simple’’ versions.

3 The lowest-dimensional ‘‘classical’’ Freudenthal squares and their extensions to include exceptional algebras.

The first two squares, $N = 1, 2$ allow an extension by introducing an additional $p = 8$ column, associated to octonions \mathbb{O} , and an additional $q = 8$ row with the so-

called metasymplectic algebras, denoted here $\mathfrak{m}\eta$. Lie algebras worth of the names $\mathfrak{sa}(N+1, \mathbb{O})$, $\mathfrak{sl}(N+1, \mathbb{O})$, $\mathfrak{sh}(2(N+1), \mathbb{O})$, $\mathfrak{m}\eta(2(N+1), \mathbb{O})$ are only possible when $N = 1, 2$ and due to the nonassociativity of \mathbb{O} its proper definition requires some approach alternative to the one sketched in Sec. 1 (for \mathfrak{sa} and \mathfrak{sl} this is done in [2]).

In the $N = 2$ case, we get the original Freudenthal square [3], which has several versions most of which are obtained by particular choices of the constants ω_1, ω_2 in:

$B_1 \equiv A_1 \equiv C_1 \equiv E_1$	A_2	$C_3 \equiv B_2$	F_4
$\mathfrak{sa}_{\omega_1\omega_2}(3, \mathbb{R})$	$\mathfrak{sa}_{\omega_1\omega_2}(3, \mathbb{C})$	$\mathfrak{sa}_{\omega_1\omega_2}(3, \mathbb{H})$	$\mathfrak{sa}_{\omega_1\omega_2}(3, \mathbb{O})$
A_2	$A_2 \oplus A_2$	A_5	E_6
$\mathfrak{sl}_{\omega_1\omega_2}(3, \mathbb{R})$	$\mathfrak{sl}_{\omega_1\omega_2}(3, \mathbb{C})$	$\mathfrak{sl}_{\omega_1\omega_2}(3, \mathbb{H})$	$\mathfrak{sl}_{\omega_1\omega_2}(3, \mathbb{O})$
$C_3 \equiv B_2$	A_5	D_6	E_7
$\mathfrak{sh}_{\omega_1\omega_2}(6, \mathbb{R})$	$\mathfrak{sh}_{\omega_1\omega_2}(6, \mathbb{C})$	$\mathfrak{sh}_{\omega_1\omega_2}(6, \mathbb{H})$	$\mathfrak{sh}_{\omega_1\omega_2}(6, \mathbb{O})$
F_4	E_6	E_7	E_8
$\mathfrak{m}\eta_{\omega_1\omega_2}(6, \mathbb{R})$	$\mathfrak{m}\eta_{\omega_1\omega_2}(6, \mathbb{C})$	$\mathfrak{m}\eta_{\omega_1\omega_2}(6, \mathbb{H})$	$\mathfrak{m}\eta_{\omega_1\omega_2}(6, \mathbb{O})$

Notice that reflection in the main diagonal corresponds to a change of real form. When $N = 1$, the extended “classical” Freudenthal-like square is:

D_1	$A_1 \equiv B_1 \equiv C_1 \equiv E_1$	$C_2 \equiv B_2$	B_4
$\mathfrak{sa}_{\omega_1}(2, \mathbb{R})$	$\mathfrak{sa}_{\omega_1}(2, \mathbb{C})$	$\mathfrak{sa}_{\omega_1}(2, \mathbb{H})$	$\mathfrak{sa}_{\omega_1}(2, \mathbb{O})$
$A_1 \equiv B_1 \equiv C_1 \equiv E_1$	$A_1 \oplus A_1 \equiv E_2$	$A_3 \equiv D_3$	$D_5 \equiv E_5$
$\mathfrak{sl}_{\omega_1}(2, \mathbb{R})$	$\mathfrak{sl}_{\omega_1}(2, \mathbb{C})$	$\mathfrak{sl}_{\omega_1}(2, \mathbb{H})$	$\mathfrak{sl}_{\omega_1}(2, \mathbb{O})$
$C_2 \equiv B_2$	$A_3 \equiv D_3$	D_4	D_6
$\mathfrak{sh}_{\omega_1}(4, \mathbb{R})$	$\mathfrak{sh}_{\omega_1}(4, \mathbb{C})$	$\mathfrak{sh}_{\omega_1}(4, \mathbb{H})$	$\mathfrak{sh}_{\omega_1}(4, \mathbb{O})$
B_4	$D_5 \equiv E_5$	D_6	D_8
$\mathfrak{m}\eta_{\omega_1}(4, \mathbb{R})$	$\mathfrak{m}\eta_{\omega_1}(4, \mathbb{C})$	$\mathfrak{m}\eta_{\omega_1}(4, \mathbb{H})$	$\mathfrak{m}\eta_{\omega_1}(4, \mathbb{O})$

In the case $\omega_1 = 1$, and through some of the low-dimension isomorphisms of Lie algebras, this square coincides with the very nice form $\mathfrak{so}(m, n)$, $m = 2, 3, 5, 9$, $n = 0, 1, 2, 4$ proposed by Sudbery [2]. Only the first row is different when $\omega_1 = -1$, and a degenerate form correspond to $\omega_1 = 0$.

4 Conclusions

Consideration of the three \mathfrak{sa} , \mathfrak{sl} , \mathfrak{sh} CK series leads in a rather natural way to a “tower” of “classical” Freudenthal-like squares relating different algebras in these CK families. In the lowest dimensional cases $N = 1, 2$, octonions are also allowed, and in these cases these squares can be extended to 4×4 . When the constants ω_i are different from zero, we get simple Lie algebras in these three series and the “classical” $N = 2$ Freudenthal square reduces to the different versions (compact and non-compact) of the original Freudenthal square, while $N = 1$ gives the one proposed by Sudbery.

The introduction of an adequate notation is the key to open the view to this

square: in the standard conventional notation and in the special case where all constants ω_i are different from zero, this square would read in a rather uninspiring form whose direct relation to reals, complex numbers and quaternions is rather remote:

$$\begin{array}{lll}
\mathfrak{so}(p, q) & \mathfrak{su}(p, q) & \mathfrak{sp}(p, q) \\
\mathfrak{sl}(N+1, \mathbb{R}) & \mathfrak{sl}(N+1, \mathbb{C}) & \mathfrak{su}^*(2(N+1)) \\
\mathfrak{sp}(2(N+1), \mathbb{R}) & \mathfrak{su}(N+1, N+1) & \mathfrak{so}^*(4(N+1))
\end{array} \tag{5}$$

This approach also displays the basic role played by the orthogonal CK family, $\mathfrak{so}_{\omega_1 \dots \omega_N}(N+1) \equiv \mathfrak{sa}_{\omega_1 \dots \omega_N}(N+1, \mathbb{R})$. This is the antihermitian CK algebra over the reals, and appears as a subfamily in all other CK series. To gain familiarity with all CK families, the orthogonal one should be first studied at depth.

We finally mention that similar ideas might also be pursued for the remaining (orthogonal and symplectic) CK families, which are associate to Lie algebras of “antisymmetric” or “symplectic-antisymmetric” (instead of antihermitian) matrices. Details for the case $\mathbb{K} = \mathbb{H}$ are involved. Three more towers (linear, orthogonal and symplectic) appear. Each tower makes sense for CK algebras with a fixed set of constant values ω_i , and there are as many such towers as the 3^N essentially different sets of constants ω_i . Different real forms, either compact or not, of *all* simple real Lie algebras are related among themselves by several of these Freudenthal-like squares.

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