CONSTRUCTING ALL MAGIC SQUARES OF ORDER THREE

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Abstract. We find by applying MacMahon's partition analysis that all magic squares of order three, up to rotations and reflections, are of two types, each generated by three basis elements. A combinatorial proof of this fact is given.

Keywords: *magic square, linear Diophantine equations*

1. introduction

A *magic square* of *order* n is an n by n matrix with distinct nonnegative integer entries such that every row sum, column sum, and (two) diagonal sums equals to the same number m, the *magic number*. Adding 1 to every entry will give us a traditional magic square of positive integers. A magic square is *pure* if the entries are the consecutive numbers from 0 to n^2-1, and hence it has magic number $3\binom{n+1}{3}$.

Magic squares have been objects of study for centuries. As Pickover wrote in his book[5, p. 60]:

> *...the holy grail of magic squares creation would be to discover a method that would generate every possible arrangement for a square of a given size. Such a solution is probably not discoverable.*

This "holy grail" could be achieved by first finding the complete generating function (which is a rational function) for magic squares of a given size, and then writing the generating function as a sum of simple rational functions, the series expansion of which has only nonnegative coefficients.

We achieve this for magic squares of order 3, as given in Theorem 2.1.

Weak magic squares, magic squares without the restriction of distinct elements, have been studied in [1; 2; 3; 4] by using the rich theory of counting solutions of a system of linear Diophantine equations, or equivalently, counting lattice points of a convex polytope. For further references, see [6, Ch. 4.6]. These methods also apply to counting magic squares, but give no obvious reason why a simple solution as in Theorem 2.1 exists.

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We give our main result in Section 2, and give a combinatorial proof in Section 3. In Section 4, we discuss the discovery of our main result and possible future work.

2. Main Results

A magic square of order 3 is a 3 by 3 matrix of distinct nonnegative integers such that every row sum, column sum, and diagonal sum equals the magic number m.

Our main result is the following Theorem 2.1, which generates all magic square of order 3. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & 4 \\ 2 & 6 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 3 \\ 2 & 2 & 2 \\ 1 & 4 & 1 \end{bmatrix}, \tag{2.1}$$

$$T_1 = \begin{bmatrix} 7 & 0 & 5 \\ 2 & 4 & 6 \\ 3 & 8 & 1 \end{bmatrix}, \qquad T_2 = \begin{bmatrix} 8 & 0 & 7 \\ 4 & 5 & 6 \\ 3 & 10 & 2 \end{bmatrix}. \tag{2.2}$$

Then they are related as follows:

$$B = C + D, \qquad T_1 = B + C, \qquad T_2 = B + D. \tag{2.3}$$

If we let C' be obtained from C by reflecting in the vertical axis, then we have one more relation: $D = C + C'$. It is straightforward to check that A, C, and D are linearly independent.

In fact, A, C, D are the three basis elements that generate all magic squares of order 3, and T_1, T_2 are the unique magic squares with magic numbers 12 and 15, respectively, up to rotations and reflections.

Theorem 2.1. *Every magic square of order three, up to rotation and reflection, can be written uniquely as either $T_1 + iA + jB + kC$ or $T_2 + iA + jB + kD$, where i, j, k are nonnegative integers and A, B, C, D, T_1, T_2 are as in* (2.1), (2.2).

Remark 2.2. Note that traditional magic squares can be generated by either $iA + jB + kC$ or $iA + jB + kD$ for positive integers i, j, k. This description reveals a kind of symmetry.

Theorem 2.1 says that magic squares, as a set of lattice points, is a disjoint union of $16 = 8 \cdot 2$ *polyhedrons* that are isomorphic to $\mathbb{N}^3$, where the factor 8 is the order of the dihedral group of rotations and reflections. We will give a combinatorial proof of this result in the next section.

Corollary 2.3. *The number of magic squares of order* 3 *with magic number* $3s$ *and its associated generating function is given by*

$$\frac{8t^4(1+2t)}{(1-t)(1-t^2)(1-t^3)} = \sum_{s\geq 0}\left(2s^2 - \frac{20}{3}s + 1 - (-1)^s + \frac{8}{3}(s \bmod 3)\right)t^s$$
$$= 8\left(t^4 + 3t^5 + 4t^6 + 7t^7 + 10t^8 + 13t^9 + 17t^{10} + 22t^{11} + 26t^{12} + \cdots\right).$$

3. A Combinatorial Proof

In what follows, magic squares are always of order 3 unless specified otherwise.

Let M be a magic square with magic number m. We write

$$M = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}, \tag{3.1}$$

where

C1: Every row sum, column sum, and diagonal sum is equal to m.

C2: The entries of M are distinct nonnegative integers.

Rotating or reflecting M will give us different magic squares. Without loss of generality, we can assume that c_3 is smaller than a_1, a_3 and c_1, and that $c_1 < a_3$. Also by subtracting A times the minimal entry of M from M, we can assume that 0 is an entry of M. Then M satisfies the following two extra conditions:

C3: One of the entries of M is 0.

C4: $c_3 < a_1, a_3, c_1$, and $c_1 < a_3$.

In fact, **C4** can be replaced with

C4′: $c_3 < c_1 < a_3 < a_1$,

which follows from the sum of the two diagonals.

If M satisfies the above four conditions, then we say that M is a *reduced* magic square. It is well-known that the magic number m is $m = 3b_2$. Let $m = 3s$ or equivalently $s = b_2$.

Lemma 3.1. *If M is a reduced magic square, then $a_2 = 0$, and $c_2 = 2s$.*

Proof. Since $b_2 = s = m/3$, where m is the magic number, the last statement $c_2 = 2s$ follows from $a_2 = 0$, which is what we are going to show now.

In a reduced magic square M, a_1 and a_3 are the largest two entries among the four corners a_1, a_3, c_1, c_3. It follows from the first and third row sums and column sums that $a_2 < b_1, c_2, b_3$.

We see that $b_2 = s \geq 4$, since all entries are distinct nonnegative integers. It remains to show that c_3 cannot be 0. Assuming that c_3 equals 0, then $a_1 = 2s$ by the diagonal sum $a_1 + b_2 + c_3 = 3s$. By investigating the first row sum and the first column sum, we get $a_3 < s - 1$ and $c_1 < s - 1$, contradicting the condition for the diagonal (a_3, b_2, c_1). □

Lemma 3.2. *A reduced magic square M can be uniquely written as $T_1 + \alpha C + \beta D$, where $\alpha \geq -1$ and $\beta \geq 0$ are integers.*

Proof. To see the existence, we use Lemma 3.1. Assuming that $c_3 = r$ and $b_2 = s$, we obtain all the entries of M by the condition **C1** for row sums, column sums, and diagonal sums:

$$M = \begin{bmatrix} 2s - r & 0 & s + r \\ 2r & s & 2s - 2r \\ s - r & 2s & r \end{bmatrix}.$$

Comparing the above matrix with

$$T1 + \alpha C + \beta D = \begin{bmatrix} 7 + 2\alpha + 3\beta & 0 & 5 + \alpha + 3\beta \\ 2 + 2\beta & 4 + \alpha + 2\beta & 6 + 2\alpha + 2\beta \\ 3 + \alpha + \beta & 8 + 2\alpha + 4\beta & 1 + \beta \end{bmatrix},$$

we solve uniquely for α and β:

$$\alpha = s - 2r - 2, \text{ and } \beta = r - 1.$$

Consequently,

$$s = \alpha + 2\beta + 4 \text{ and } r = \beta + 1.$$

We see that $c_3 \geq 1$ and $c_1 > c_3$ implies that $\alpha \geq -1$ and $\beta \geq 0$, completing the proof of the existence.

The uniqueness follows from the above proof, and also from the fact that C and D are linearly independent. □

We are now ready to give the proof of our main theorem.

Proof of Theorem 2.1. It is straightforward to check that

$$T_1 + iA + jB + kC = \begin{bmatrix} 7+i+5j+2k & i & 5+i+4j+k \\ 2+i+2j & 4+i+3j+k & 6+i+4j+2k \\ 3+i+2j+k & 8+i+6j+2k & 1+i+j \end{bmatrix}, \tag{3.2}$$

$$T_2 + iA + jB + kD = \begin{bmatrix} 8+i+5j+3k & i & 7+i+4j+3k \\ 4+i+2j+2k & 5+i+3j+2k & 6+i+4j+2k \\ 3+i+2j+k & 10+i+6j+4k & 2+i+j+k \end{bmatrix} \tag{3.3}$$

give different magic squares for all nonnegative integers i, j, k.

Given a magic square M, we need to show that M equals either (3.2) or (3.3).

Let i be the minimum of the entries of M. Then up to rotations and reflections, we can assume $M' = M - iA$ is a reduced magic square. By Lemma 3.2, M' can be uniquely written as $T_1 + \alpha C + \beta D$, with $\alpha \geq -1$ and $\beta \geq 0$.

If $\alpha \geq \beta \geq 0$, M' can be rewritten (recall that $B = C + D$) as $T_1 + \beta B + (\alpha - \beta)C$. Hence we let $j = \beta \geq 0$ and $k = \alpha - \beta \geq 0$.

If $\alpha < \beta$, M' can be rewritten (recall that $T_1 + D = T_2 + C$) as

$$T_1 + \alpha B + (\beta - \alpha)D = T_2 + C + \alpha B + (\beta - \alpha - 1)D = T_2 + (\alpha + 1)B + (\beta - \alpha - 2)D.$$

Thus we let $j = \alpha + 1 \geq 0$ and $k = \beta - \alpha - 2 \geq -1$.

The only remaining case is $k = -1$, which is equivalent to $\beta = \alpha + 1$. But in this case

$$M' = T_1 + (\beta - 1)C + \beta D = \begin{bmatrix} 5+5\beta & 0 & 4+4\beta \\ 2+2\beta & 3+3\beta & 4+4\beta \\ 2+2\beta & 6+6\beta & 1+\beta \end{bmatrix},$$

which is not a magic square because it has equal entries. □

4. Further Discussion

The combinatorial proof in the previous section seems unlikely to be applicable to magic squares of higher order. We describe how we discovered Theorem 2.1 by using MacMahon's partition analysis, which has been restudied by Andrews and his coauthors in a series of papers (see e.g., [3]).

MacMahon's idea is to use new variables to replace linear constraints. For example, if we want to count nonnegative integral solutions of the linear equation $a_1 + a_2 - a_3 = 0$, we can simply write the generating function as

$$\sum_{\substack{a_1,a_2,a_3\geq 0\\ a_1+a_2-a_3=0}} x_1^{a_1}x_2^{a_2}x_3^{a_3} = \sum_{a_1,a_2,a_3\geq 0} \operatorname*{CT}_{\lambda} \lambda^{a_1+a_2-a_3}x_1^{a_1}x_2^{a_2}x_3^{a_3} = \operatorname*{CT}_{\lambda} \frac{1}{(1-\lambda x_1)(1-\lambda x_2)(1-x_3/\lambda)},$$

where CT_λ means to take the constant term in λ. Then the counting problem is converted to evaluating the constant term of a special rational function, which can be done be computer package as in [7]. For a rigorous description about how the above works in general situation, i.e., in a field of iterated Laurent series, the reader is referred to [7].

Using a computer we can easily obtain the generating function of weak magic squares of order 3:

$$G = \frac{(1-tx_4x_7x_9x_6x_2x_3x_5x_8x_1)\,(1+tx_4x_7x_9x_6x_2x_3x_5x_8x_1)^2}{(1-tx_1x_5x_9{x_4}^2{x_8}^2{x_3}^2)\,(1-tx_7x_5x_3{x_4}^2{x_2}^2{x_9}^2)} \times \frac{1}{(1-tx_7x_5x_3{x_1}^2{x_8}^2{x_6}^2)\,(1-tx_1x_5x_9{x_7}^2{x_2}^2{x_6}^2)},$$

where t records $m/3$ since the m is always divisible by 3, and the exponents in $x_1, \ldots x_9$ represents $a_1, a_2, a_3, b_1, \ldots$.

To obtain the generating function for magic squares, we shall take only terms in G that have different exponents in the x's. To eliminate those terms with same exponents in x_1 and x_2, we subtract by the diagonal $\mathrm{diag}_{x_1,x_2} G$ with respect to x_1 and x_2, where

$$\mathrm{diag}_{x,y} \sum_{r\in\mathbb{N}}\sum_{s\in\mathbb{N}} b_{r,s}x^ry^s = \sum_{r\in\mathbb{N}} b_{r,r}x^ry^r,$$

and we use the formula for a rational power series $F(x,y)$:

$$\mathrm{diag}_{x,y}F(x,y) = \operatorname*{CT}_{\lambda_1,\lambda_2} \frac{1}{1-xy/(\lambda_1\lambda_2)} F(\lambda_1,\lambda_2).$$

Similarly, we can eliminate those terms with same exponents in x_i and x_j for all i and j.

The generating function of all magic squares of order 3 is still complicated. We can add the extra constraints that $c_3 < c_1 < a_3 < a_1$ to eliminate rotations and reflections. It suffices to find a way to add the constraint that the exponent of x_9 is smaller than that of x_7. The other constraints can be added iteratively. We omit the details here.

Finally we obtain the generating function of desired magic squares:

$$\frac{t^4{x_7}^3{x_5}^4{x_3}^5{x_1}^7{x_8}^8{x_6}^6x_9{x_4}^2\,(1+tx_1x_5x_9{x_4}^2{x_8}^2{x_3}^2-2t^2{x_5}^2x_9{x_4}^2{x_8}^4{x_1}^3{x_3}^3x_7{x_6}^2)}{(1-tx_7x_5x_3{x_1}^2{x_8}^2{x_6}^2)\,(1-tx_4x_7x_9x_6x_2x_3x_5x_8x_1)} \times \frac{1}{(1-t^2{x_5}^2x_9{x_4}^2{x_8}^4{x_1}^3{x_3}^3x_7{x_6}^2)\,(1-t^3{x_7}^2{x_5}^3{x_1}^5{x_8}^6{x_3}^4{x_6}^4x_9{x_4}^2)}. \quad (4.1)$$

We observe that part of the numerator can be rewritten as

$$1 + tx_1x_5x_9{x_4}^2{x_8}^2{x_3}^2 - 2t^2{x_5}^2x_9{x_4}^2{x_8}^4{x_1}^3{x_3}^3x_7{x_6}^2$$
$$= (1 - t^2{x_5}^2x_9{x_4}^2{x_8}^4{x_1}^3{x_3}^3x_7{x_6}^2) + tx_1x_5x_9{x_4}^2{x_8}^2{x_3}^2(1 - tx_7x_5x_3{x_1}^2{x_8}^2{x_6}^2),$$

where both terms on the right-hand side will cancel with the denominator of (4.1). Theorem 2.1 then follows.

The order 4 case would be really hard. The difficulty lies in the fact that there are 880 pure magic square of order 4 (up to rotations and reflections), which suggests that there will be at least 880 simple rational functions. Our current package as provided in [7] is not powerful enough to find an explicit generating function for magic squares of order 4 analogous to (4.1).

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