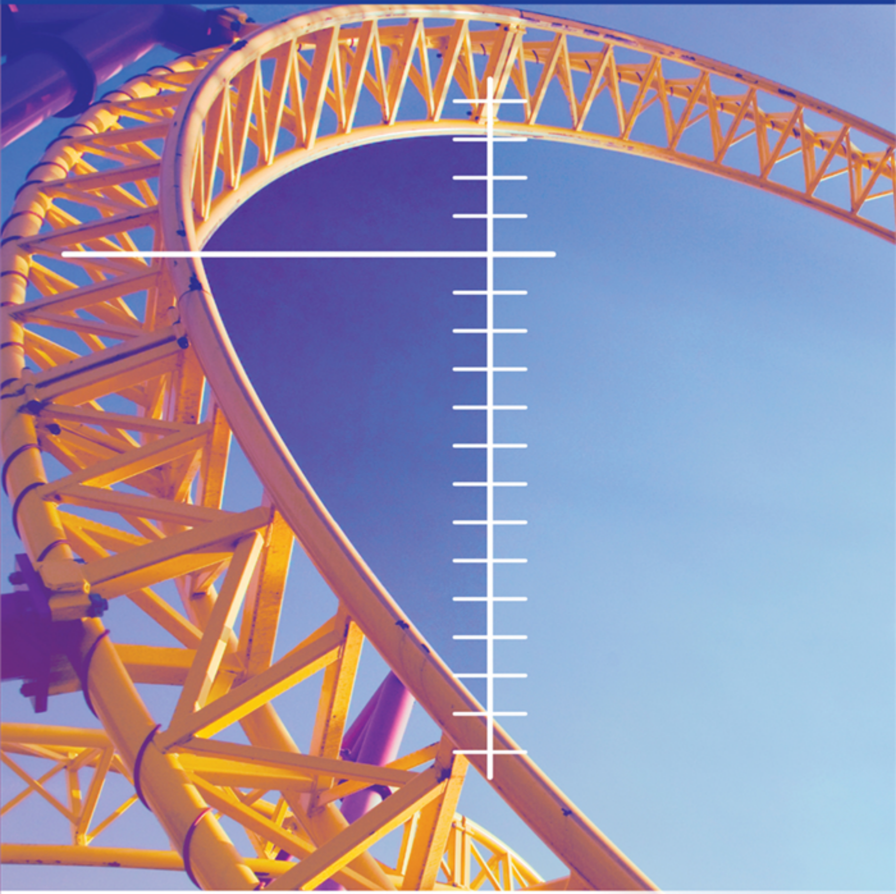
AMS / MAA | TEXTBOOKS VOL 40

Calculus in 3D

Geometry, Vectors, and Multivariate Calculus

Zbigniew Nitecki



MAA PRESS

An Imprint of the AMERICAN MATHEMATICAL SOCIETY

AMS/MAA | TEXTBOOKS

VOL **40**

Calculus in 3D

Geometry, Vectors, and Multivariate Calculus

Zbigniew Nitecki

MAA PRESS

An Imprint of the American Mathematical Society

Providence, Rhode Island

2010 *Mathematics Subject Classification*. Primary 26-01.

For additional information and updates on this book, visit
www.ams.org/bookpages/text-40

Library of Congress Cataloging-in-Publication Data

Names: Nitecki, Zbigniew, author.
Title: Calculus in 3D: Geometry, vectors, and multivariate calculus / Zbigniew Nitecki.
Description: Providence, Rhode Island: MAA Press, an imprint of the American Mathematical Society, [2018] | Series: AMS/MAA textbooks; volume 40 | Includes bibliographical references and index.
Identifiers: LCCN 2018020561 | ISBN 9781470443603 (alk. paper)
Subjects: LCSH: Calculus–Textbooks. | AMS: Real functions – Instructional exposition (textbooks, tutorial papers, etc.). msc
Classification: LCC QA303.2.N5825 2018 | DDC 515/.8–dc23
LC record available at https://lccn.loc.gov/2018020561.

10 9 8 7 6 5 4 3 2 1 23 22 21 20 19 18

Contents

Preface

The present volume is a sequel to my earlier book, *Calculus Deconstructed: A Second Course in First-Year Calculus*, published by the Mathematical Association of America in 2009. I have used versions of this pair of books for several years in the Honors Calculus course at Tufts, a two-semester "boot camp" intended for mathematically inclined freshmen who have been exposed to calculus in high school. The first semester of this course, using the earlier book, covers single-variable calculus, while the second semester, using the present text, covers multivariate calculus. However, the present book is designed to be able to stand alone as a text in multivariate calculus.

The treatment here continues the basic stance of its predecessor, combining hands-on drill in techniques of calculation with rigorous mathematical arguments. Nonetheless, there are some differences in emphasis. On one hand, the present text assumes a higher level of mathematical sophistication on the part of the reader: there is no explicit guidance in the rhetorical practices of mathematicians, and the theorem-proof format is followed a little more brusquely than before. On the other hand, the material being developed here is unfamiliar territory for the intended audience to a far greater degree than in the previous text, so more effort is expended on motivating various approaches and procedures, and a substantial number of technical arguments have been separated from the central text, as exercises or appendices.

Where possible, I have followed my own predilection for geometric arguments over formal ones, although the two perspectives are naturally intertwined. At times, this may feel like an analysis text, but I have studiously avoided the temptation to give the general, n-dimensional versions of arguments and results that would seem natural to a mature mathematician: the book is, after all, aimed at the mathematical novice, and I have taken seriously the limitation implied by the "3D" in my title. This has the advantage, however, that many ideas can be motivated by natural geometric arguments. I hope that this approach lays a good intuitive foundation for further generalization that the reader will see in later courses.

Perhaps the fundamental subtext of my treatment is the way that the theory developed earlier for functions of one variable interacts with geometry to handle higher-dimension situations. The progression here, after an initial chapter developing the tools of vector algebra in the plane and in space (including dot products and cross products), is to first view vector-valued functions of a single real variable in terms of parametrized curves—here, much of the theory translates very simply in a coordinate-wise way—then to consider real-valued functions of several variables both as functions with a vector input and in terms of surfaces in space (and level curves in the plane), and finally to vector fields as vector-valued functions of vector variables.

Idiosyncracies

There are a number of ways, some apparent, some perhaps more subtle, in which this treatment differs from the standard ones:

Conic Sections: I have included in § 2.1 a treatment of conic sections, starting with a version of Apollonius's formulation in terms of sections of a double cone (and explaining the origin of the names *parabola*, *hyperbola*, and *ellipse*), then discussing the focus-directrix formulation following Pappus, and finally sketching how this leads to the basic equations for such curves. I have taken a quasi-historical approach here, trying to give an idea of the classical Greek approach to curves which contrasts so much with our contemporary calculus-based approach. This is an example of a place where I think some historical context enriches our understanding of the subject. This can be treated as optional in class, but I personally insist on spending at least one class on it.

Parametrization: I have stressed the parametric representation of curves and surfaces far more, and beginning somewhat earlier, than many multivariate texts. This approach is essential for applying calculus to geometric objects, and it is also a beautiful and satisfying interplay between the geometric and analytic points of view. While Chapter 2 begins with a treatment of the conic sections from a classical point of view, this is followed by a catalogue of parametrizations of these curves and, in § 2.4, by a more careful study of regular curves in the plane and their relation to graphs of functions. This leads naturally to the formulation of path integrals in § 2.5. Similarly, quadric surfaces are introduced in § 3.4 as level sets of quadratic polynomials in three variables, and the (three-dimensional) Implicit Function Theorem is introduced to show that any such surface is locally the graph of a function of two variables. The notion of parametrization of a surface is then introduced and exploited in § 3.6 to obtain the tangent planes of surfaces. When we get to surface integrals in § 4.4, this gives a natural way to define and calculate surface area and surface integrals of functions. This approach comes to full fruition in Chapter 5 in the formulation of the integral theorems of vector calculus.

Linear Algebra: Linear algebra is not strictly necessary for procedural mastery of multivariate calculus, but some understanding of linearity, linear independence, and the matrix representation of linear mappings can illuminate the "hows" and "whys" of many procedures. Most (but not all) of the students in my class have already encountered vectors and matrices in their high school courses, but few of them understand these more abstract concepts. In the context of the plane and 3-space it is possible to interpret many of these algebraic notions geometrically, and I have taken full advantage of this possibility in my narrative. I have introduced these ideas piecemeal, and in close conjunction with their application in multivariate calculus.

For example, in § 3.2, the derivative, as a linear real-valued function, can be represented as a homogeneous polynomial of degree one in the coordinates of the input (as in the first Taylor polynomial), as the dot product of the (vector) input with a fixed vector (the gradient), or as multiplying the coordinate column of the input by a row (a $1 \times n$ matrix, the matrix of partials). Then in § 4.3 and § 4.5, substitution in a double or triple integral is interpreted as a coordinate transformation whose linearization is represented by the Jacobian matrix, and whose determinant reflects the effect of this transformation on area or volume. In Chapter 5, differential forms are constructed as (alternating) multilinear functionals (building on the differential of a real-valued function) and investigation of their effect on pairs or triples of vectors—especially in view of independence considerations—ultimately leads to the standard representation of these forms via wedge products.

A second example is the definition of 2×2 and 3×3 determinants. There seem to be two prevalent approaches in the literature to introducing determinants: one is formal, dogmatic and brief, simply giving a recipe for calculation and proceeding from there with little motivation for it; the other is even more formal but elaborate, usually involving the theory of permutations. I believe I have come up with an approach to introducing 2×2 and 3×3 determinants (along with cross-products) which is both motivated and rigorous, in § 1.6. Starting with the problem of calculating the area of a planar triangle from the coordinates of its vertices, we deduce a formula which is naturally written as the absolute value of a 2×2 determinant; investigation of the determinant itself leads to the notion of signed (*i.e.*, oriented) area (which has its own charm, and prophesies the introduction of 2-forms in Chapter 5). Going to the analogous problem in space, we introduce the notion of an oriented area, represented by a vector (which we ultimately take as the definition of the cross-product, an approach taken for example by David Bressoud). We note that oriented areas project nicely, and from the projections of an oriented area vector onto the coordinate planes come up with the formula for a cross-product as the expansion by minors along the first row of a 3×3 determinant. In the present treatment, various algebraic properties of determinants are developed as needed, and the relation to linear independence is argued geometrically.

Vector Fields vs. Differential Forms: A number of relatively recent treatments of vector calculus have been based exclusively on the theory of differential forms, rather than the traditional formulation using vector fields. I have tried this approach in the past, but in my experience it confuses the students at this level, so that they end up dealing with the theory on a blindly formal basis. By contrast, I find it easier to motivate the operators and results of vector calculus by treating a vector field as the velocity of a moving fluid, and so have used this as my primary approach. However, the formalism of differential forms is very slick as a calculational device, and so I have also introduced it interwoven with the vector field approach. The main strength of the differential forms approach, of course, is that it generalizes to dimensions higher than 3; while I hint at this, it is one place where my self-imposed limitation to "3D" is evident.

Appendices: My goal in this book, as in its predecessor, is to make available to my students an essentially complete development of the subject from first principles, in particular presenting (or at least sketching) proofs of all results. Of course, it is physically (and cognitively) impossible to effectively present too many technical arguments as well as new ideas in the available class time. I have therefore (adopting a practice used by among others Jerry Marsden in his various textbooks) relegated to exercises and appendices[1] a number of technical proofs which can best be approached only after the results being proven are fully understood. This has the advantage of streamlining the central narrative, and—to be realistic—bringing it closer to what the student will experience in the classroom. It is my expectation that (depending on the preference of the teacher) most of these appendices will not be directly treated in class, but they are there for reference and may be returned to later by the curious student. This format comports with the actual practice of mathematicians when confronting a new result: we all begin with a quick skim

[1]Specifically, Appendices A.1-A.2, A.4, A.6-A.7, A.9-A.10, and A.12.

focused on understanding the statement of the result, followed by several (often, very many) re-readings focused on understanding the arguments in its favor.

The other appendices present extra material which fills out the central narrative:

- Appendix A.3 presents the Principal Axis Theorem, that every symmetric matrix has an orthonormal basis of eigenvectors. Together with the (optional) last part of § 3.9, this completes the treatment of quadratic forms in three variables and so justifies the Second Derivative Test for functions of three variables. The treatment of quadratic forms in terms of matrix algebra, which is not necessary for the basic treatment of quadratic forms in the plane (where completion of the square suffices), does allow for the proof (in Exercise 4) of the fact that the locus of a quadratic equation in two variables has as its locus a conic section, a point, a line, two intersecting lines or the empty set.
  I am particularly fond of the proof of the Principal Axis Theorem itself, which is a wonderful example of synergy between linear algebra and calculus (Lagrange multipliers).
- Appendix A.5 presents the basic facts about the matrix representation, invertibility, and operator norm of a linear transformation, and a geometric argument that the determinant of a product of matrices is the product of their determinants.
- Appendix A.8 presents the example of H. Schwartz and G. Peano showing how the "natural" extension to surface area of the definition of arclength via piecewise linear approximations fails.
- Appendix A.11 clarifies the need for orientability assumptions by presenting the Möbius band.

Format

In general, I have continued the format of my previous book in this one.

As before, **exercises** come in four flavors:

Practice Problems: serve as drill in calculation.

Theory Problems: involve more ideas, either filling in gaps in the argument in the text or extending arguments to other cases. Some of these are a bit more sophisticated, giving details of results that are not sufficiently central to the exposition to deserve explicit proof in the text.

Challenge Problems: require more insight or persistence than the standard theory problems. In my class, they are entirely optional, extra-credit assignments.

Historical Notes: explore arguments from original sources. There are much fewer of these than in the previous volume, in large part because the history of multivariate calculus is not nearly as well documented and studied as is the history of single-variable calculus. Nonetheless, I strongly feel that we should strive more than we have to present mathematics in at least a certain amount of historical context: I believe that it is very helpful to students to realize that mathematics is an activity by real people in real time, and that our understanding of many mathematical phenomena has evolved over time.

Acknowledgments

As with the previous book, I want to thank Jason Richards, who as my grader in this course over several years contributed many corrections and useful comments about the text. After he graduated, several other student graders—Erin van Erp, Thomas Snarsky, Wenyu Xiong, and Kira Schuman—made further helpful comments. I also affectionately thank my students over the past few years, particularly Matt Ryan, who noted a large number of typos and minor errors in the “beta” version of this book. I have benefited greatly from much help with TEXpackages especially from the e-forum on pstricks and pst-3D solids run by Herbert Voss, as well as the “TeX on Mac OS X” elist. My colleague Loring Tu helped me better understand the role of orientation in the integration of differential forms. On the history side, Sandro Capparini helped introduce me to the early history of vectors, and Lenore Feigenbaum and especially Michael N. Fried helped me with some vexing questions concerning Apollonius’ classification of the conic sections. Scott Maclachlan helped me think through several somewhat esoteric but useful results in vector calculus. As always, what is presented here is my own interpretation of their comments, and is entirely my personal responsibility.

1 Coordinates and Vectors

1.1 Locating Points in Space

Rectangular Coordinates in the Plane. The geometry of the number line $\mathbb{R}$ is quite straightforward: the location of a real number x relative to other numbers is determined—and specified—by the inequalities between it and other numbers x': if $x < x'$, then x is to the *left* of x', and if $x > x'$, then x is to the *right* of x'. Furthermore, the **distance** between x and x' is just the difference $\triangle x = x' - x$ (*resp.* $x - x'$) in the first (*resp.* second) case, a situation summarized as the **absolute value**

$$|\triangle x| = |x - x'|\,.$$

When it comes to points in the plane, more subtle considerations are needed. The most familiar system for locating points in the plane is a **rectangular** or **Cartesian coordinate system**. We pick a distinguished point called the **origin**, denoted $\mathcal{O}$, and draw two mutually perpendicular lines through the origin, each regarded as a copy of the real line, with the origin corresponding to zero. The first line, the x**-axis**, is by convention *horizontal* with the "increasing" direction going left-to-right; the second, or y**-axis**, is *vertical*, with "up" increasing.

Given a point P in the plane, we draw a rectangle with $\mathcal{O}$ and P as opposite vertices, and the two edges emanating from $\mathcal{O}$ lying along our axes. The two edges emanating from P are parallel to the axes; each of them intersects the "other" axis at the point corresponding to a number x (*resp.* y) on the x-axis (*resp.* y-axis).[1] We say that the (rectangular or Cartesian) **coordinates** of P are the two numbers (x, y).

We adopt the notation $\mathbb{R}^2$ for the collection of all pairs of real numbers, and this with the collection of all points in the plane, referring to "the point $P(x, y)$" when we mean "the point P in the plane whose (rectangular) coordinates are (x, y)".

The idea of using a pair of numbers in this way to locate a point in the plane was pioneered in the early seventeenth cenury by Pierre de Fermat (1601-1665) and René Descartes (1596-1650). By means of such a scheme, a plane curve can be identified with the **locus** of points whose coordinates satisfy some equation; the study of curves by analysis of the corresponding equations, called **analytic geometry**, was initiated in the research of these two men.[2]

One particular advantage of a rectangular coordinate system (in which the axes are perpendicular to each other) over an oblique one (axes not mutually perpendicular) is the calculation of distances. If P and Q are points with respective rectangular coordinates (x_1, y_1) and (x_2, y_2), then we can introduce the point R which shares its last

[1]Traditionally, x (*resp.* y) is called the **abcissa** (*resp.* **ordinate**) of P.

[2]Actually, it is a bit of an anachronism to refer to rectangular coordinates as "Cartesian", since both Fermat and Descartes often used **oblique coordinates**, in which the axes make an angle other than a right one. We shall explore some of the differences between rectangular and oblique coordinates in Exercise 13. Furthermore, Descartes in particular didn't really consider the meaning of negative values for either coordinate.

coordinate with P and its first with Q—that is, R has coordinates (x_2, y_1). The "legs" PR and QR of the right triangle $\triangle PRQ$ are parallel to the coordinate axes, while the hypotenuse PQ exhibits the distance from P to Q; Pythagoras' Theorem then gives the distance formula

$$|PQ| = \sqrt{\triangle x^2 + \triangle y^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{1.1}$$

In an oblique system, the formula becomes more complicated (Exercise 13).

Rectangular Coordinates in Space. The rectangular coordinate scheme extends naturally to locating points in space. We again distinguish one point $\mathcal{O}$ as the **origin**, and construct a rectangular coordinate system on the horizontal plane through it (the xy-**plane**), and draw a third z-**axis** vertically through $\mathcal{O}$. A point P is located by the coordinates x and y of the point P_{xy} in the xy-plane that lies on the vertical line through P, together with the number z corresponding to the intersection of the z-axis with the horizontal plane through P. The "increasing" direction along the z-axis is defined by the **right-hand rule**: if our right hand is placed at the origin with the x-axis coming out of the palm and the fingers curling toward the positive y-axis, then our right thumb points in the "positive z" direction. Note the standing convention that, when we draw pictures of space, we regard the x-axis as pointing toward us (or slightly to our left) out of the page, the y-axis as pointing to the right along the page, and the z-axis as pointing up along the page (Figure 1.1).

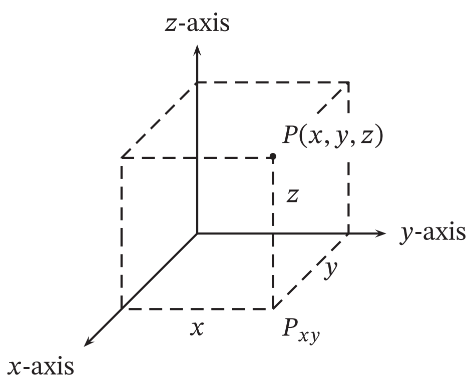


Figure 1.1. Pictures of Space

This leads to the identification of the set $\mathbb{R}^3$ of triples (x, y, z) of real numbers with the points of space, which we sometimes refer to as **three-dimensional space** (or 3-**space**).

As in the plane, the distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in $\mathbb{R}^3$ can be calculated by applying Pythagoras' Theorem to the right triangle PQR, where $R(x_2, y_2, z_1)$ shares its last coordinate with P and its other coordinates with Q. Details are left to you (Exercise 11); the resulting formula is

$$|PQ| = \sqrt{\triangle x^2 + \triangle y^2 + \triangle z^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{1.2}$$

In what follows, we will denote the distance between P and Q by $\text{dist}(P, Q)$.

Polar and Cylindrical Coordinates. Rectangular coordinates are the most familiar system for locating points, but in problems involving rotations, it is sometimes convenient to use a system based on the direction and distance to a point from the origin.

In the plane, this leads to **polar coordinates**. Given a point P in the plane, think of the line ℓ through P and $\mathcal{O}$ as a copy of the real line, obtained by rotating the x-axis θ radians counterclockwise; then P corresponds to the real number r on ℓ. The relation of the *polar* coordinates (r,θ) of P to *rectangular* coordinates (x,y) is illustrated in Figure 1.2, from which we see that

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta. \end{aligned} \tag{1.3}$$

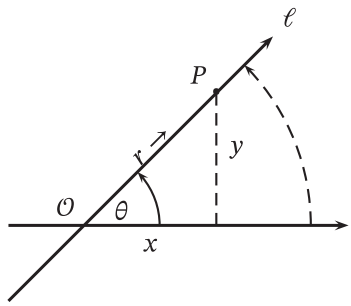


Figure 1.2. Polar Coordinates

The derivation of Equation (1.3) from Figure 1.2 requires a pinch of salt: we have drawn θ as an acute angle and x, y, and r as positive. But in our interpretation of ℓ as a rotated copy of the x-axis (and θ as the net counterclockwise rotation) all possible configurations are accounted for, and the formula remains true.

While a given geometric point P has only one pair of *rectangular* coordinates (x,y), it has many pairs of *polar* coordinates. Thus if (r,θ) is one pair of polar coordinates for P then so are $(r,\theta+2n\pi)$ and $(-r,\theta+(2n+1)\pi)$ for any integer n (positive or negative). Also, $r = 0$ precisely when P is the origin, so then the line ℓ is indeterminate: $r = 0$ together with *any* value of θ satisfies Equation (1.3), and gives the origin.

For example, to find the polar coordinates of the point P with rectangular coordinates $(-2\sqrt{3},2)$, we first note that $r^2 = (-2\sqrt{3})^2 + (2)^2 = 16$. Using the positive solution of this, $r = 4$, we have

$$\cos\theta = -\frac{2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2}, \quad \sin\theta = -\frac{2}{4} = \frac{1}{2}.$$

The first equation says that θ is, up to adding multiples of 2π, one of $\theta = 5\pi/6$ or $\theta = 7\pi/6$; the fact that $\sin\theta$ is positive picks out the first of these values. So one set of polar coordinates for P is

$$(r,\theta) = (4, \frac{5\pi}{6} + 2n\pi),$$

where n is any integer. Replacing r with its negative and adding π to the angle, we get the second set, which is most naturally written as $(-4, -\frac{\pi}{6} + 2n\pi)$.

For problems in *space* involving rotations (or rotational symmetry) about a single axis, a convenient coordinate system locates a point P relative to the origin as follows (Figure 1.3): if P is not on the z-axis, then this axis together with the line $\mathcal{O}P$ determine

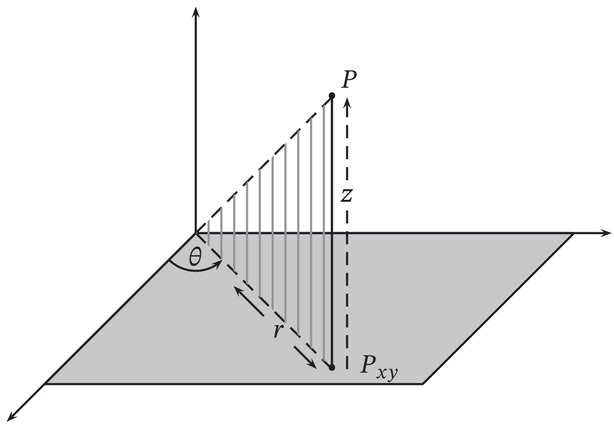


Figure 1.3. Cylindrical Coordinates

a (vertical) plane, which can be regarded as the xz-plane rotated so that the x-axis moves θ radians counterclockwise (in the horizontal plane); we take as our coordinates the angle θ together with the coordinates of P in *this* plane, which equal the distance r of the point from the z-axis and its (signed) distance z from the xy-plane. We can think of this as a hybrid: combine the *polar* coordinates (r, θ) of the projection P_{xy} with the vertical *rectangular* coordinate z of P to obtain the **cylindrical coordinates** (r, θ, z) of P. Even though in principle r could be taken as negative, in this system it is customary to confine ourselves to $r \geq 0$. The relation between the cylindrical coordinates (r, θ, z) and the rectangular coordinates (x, y, z) of a point P is essentially given by Equation (1.3):

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z. \tag{1.4}$$

We have included the last relation to stress the fact that this coordinate is the same in both systems. The inverse relations are given by

$$r^2 = x^2 + y^2, \quad \cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r} \tag{1.5}$$

and, for cylindrical coordinates, the trivial relation $z = z$.

The name “cylindrical coordinates” comes from the geometric fact that the locus of the equation $r = c$ (which in polar coordinates gives a circle of radius c about the origin) gives a vertical cylinder whose axis of symmetry is the z-axis, with radius c.

Cylindrical coordinates carry the ambiguities of polar coordinates: a point on the z-axis has $r = 0$ and θ arbitrary, while a point off the z-axis has θ determined up to adding *even* multiples of π (since r is taken to be positive).

Spherical Coordinates. Another coordinate system in space, which is particularly useful in problems involving rotations around various axes through the origin (for example, astronomical observations, where the origin is at the center of the earth) is the system of **spherical coordinates**. Here, a point P is located relative to the origin $\mathcal{O}$

by measuring the distance $\rho = |\mathcal{O}P|$ of P from the origin together with two angles: the angle θ between the xz-plane and the plane containing the z-axis and the line $\mathcal{O}P$, and the angle ϕ between the (positive) z-axis and the line $\mathcal{O}P$ (Figure 1.4). Of course, the

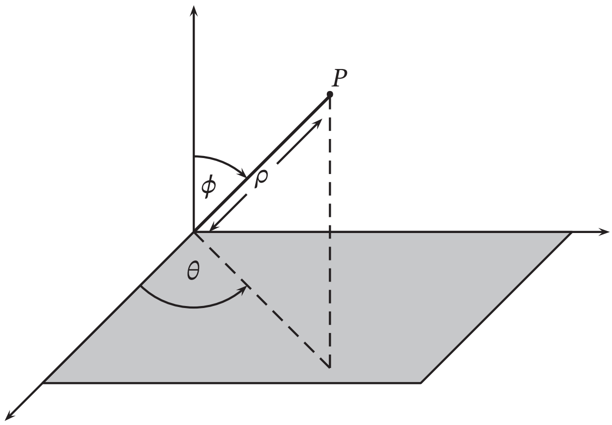


Figure 1.4. Spherical Coordinates

spherical coordinate θ of P is identical to the *cylindrical* coordinate θ, and we use the same letter to indicate this identity.[3] While θ is sometimes allowed to take on all real values, it is customary in spherical coordinates to restrict ϕ to $0 \leq \phi \leq \pi$. The relation between the cylindrical coordinates (r, θ, z) and the spherical coordinates (ρ, θ, ϕ) of a point P is illustrated in Figure 1.5 (which is drawn in the vertical plane determined by θ):

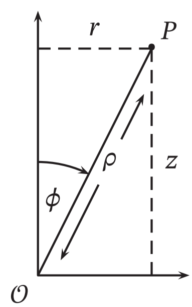


Figure 1.5. Spherical *vs.* Cylindrical Coordinates

$$r = \rho \sin \phi, \quad \theta = \theta, \quad z = \rho \cos \phi. \tag{1.6}$$

To invert these relations, we note that, since $\rho \geq 0$ and $0 \leq \phi \leq \pi$ by convention, z and r completely determine ρ and ϕ:

$$\rho = \sqrt{r^2 + z^2}, \quad \theta = \theta, \quad \phi = \arccos \frac{z}{\rho}. \tag{1.7}$$

The ambiguities in spherical coordinates are the same as those for cylindrical coordinates: the origin has $\rho = 0$ and both θ and ϕ arbitrary; any other point on the z-axis

[3]Be warned that in some of the engineering and physics literature the names of the two spherical angles are reversed, leading to potential confusion when converting between spherical and cylindrical coordinates.

($\phi = 0$ or $\phi = \pi$) has arbitrary θ, and for points off the z-axis, θ can (in principle) be augmented by arbitrary even multiples of π.

Thus, the point P with cylindrical coordinates $(r, \theta, z) = (4, \frac{5\pi}{6}, 4)$ has spherical coordinates

$$(\rho, \theta, \phi) = (4\sqrt{2}, \frac{5\pi}{6}, \frac{\pi}{4}).$$

Combining Equations (1.4) and (1.6), we can write the relation between the *spherical* coordinates (ρ, θ, ϕ) of a point P and its *rectangular* coordinates (x, y, z) as

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho\cos\phi. \tag{1.8}$$

The inverse relations are a bit more complicated, but clearly, given x, y, and z,

$$\rho = \sqrt{x^2 + y^2 + z^2} \tag{1.9}$$

and ϕ is completely determined (if $\rho \neq 0$) by the last equation in (1.8), while θ is determined by (1.5) and (1.4).

In spherical coordinates, the equation $\rho = R$ describes the sphere of radius R centered at the origin, while $\phi = \alpha$ describes a cone with vertex at the origin, making an angle α (*resp.* $\pi - \alpha$) with its axis, which is the positive (*resp.* negative) z-axis if $0 < \phi < \pi/2$ (*resp.* $\pi/2 < \phi < \pi$).

Exercises for § 1.1

Answers to Exercises 1a, 2a, 3a, and 4a are given in Appendix A.13.

Practice problems:

(1) Find the distance between each pair of points (the given coordinates are rectangular):
 (a) $(1, 1)$, $(0, 0)$ (b) $(1, -1)$, $(-1, 1)$
 (c) $(-1, 2)$, $(2, 5)$ (d) $(1, 1, 1)$, $(0, 0, 0)$
 (e) $(1, 2, 3)$, $(2, 0, -1)$ (f) $(3, 5, 7)$, $(1, 7, 5)$

(2) What conditions on the (rectangular) coordinates x, y, z signify that $P(x, y, z)$ belongs to
 (a) the x-axis? (b) the y-axis? (c) z-axis?
 (d) the xy-plane? (e) the xz-plane? (f) the yz-plane?

(3) For each point with the given rectangular coordinates, find (i) its cylindrical coordinates and (ii) its spherical coordinates:
 (a) $x = 0, y = 1, z = -1$ (b) $x = 1, y = 1, z = 1$
 (c) $x = 1, y = \sqrt{3}, z = 2$ (d) $x = 1, y = \sqrt{3}, z = -2$
 (e) $x = -\sqrt{3}, y = 1, z = 1$ (f) $x = -\sqrt{3}, y = -1, z = 1$

(4) Given the spherical coordinates of the point, find its rectangular coordinates:
 (a) $(\rho, \theta, \phi) = (2, \frac{\pi}{3}, \frac{\pi}{2})$ (b) $(\rho, \theta, \phi) = (1, \frac{\pi}{4}, \frac{2\pi}{3})$
 (c) $(\rho, \theta, \phi) = (2, \frac{2\pi}{3}, \frac{\pi}{4})$ (d) $(\rho, \theta, \phi) = (1, \frac{4\pi}{3}, \frac{\pi}{3})$

(5) What is the geometric meaning of each transformation (described in cylindrical coordinates) below?
 (a) $(r, \theta, z) \to (r, \theta, -z)$ (b) $(r, \theta, z) \to (r, \theta + \pi, z)$
 (c) $(r, \theta, z) \to (-r, \theta - \frac{\pi}{4}, z)$

(6) Describe the locus of each equation (in cylindrical coordinates) below:
 (a) $r = 1$ (b) $\theta = \frac{\pi}{3}$ (c) $z = 1$

(7) What is the geometric meaning of each transformation (described in spherical coordinates) below?
   (a) $(\rho, \theta, \phi) \to (\rho, \theta + \pi, \phi)$ (b) $(\rho, \theta, \phi) \to (\rho, \theta, \pi - \phi)$
   (c) $(\rho, \theta, \phi) \to (2\rho, \theta + \frac{\pi}{2}, \phi)$

(8) Describe the locus of each equation (in spherical coordinates) below:
   (a) $\rho = 1$ (b) $\theta = \frac{\pi}{3}$ (c) $\phi = \frac{\pi}{3}$

(9) Express the plane $z = x$ in terms of (a) cylindrical and (b) spherical coordinates.

(10) What conditions on the spherical coordinates of a point signify that it lies on:
   (a) the x-axis? (b) the y-axis? (c) z-axis?
   (d) the xy-plane? (e) the xz-plane? (f) the yz-plane?

Theory problems:

(11) Prove the distance formula for $\mathbb{R}^3$ (Equation (1.2))

$$|PQ| = \sqrt{\triangle x^2 + \triangle y^2 + \triangle z^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

as follows (see Figure 1.6). Given $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, let R be the point which shares its last coordinate with P and its first two coordinates with Q. Use the distance formula in $\mathbb{R}^2$ (Equation (1.1)) to show that

$$\text{dist}(P, R) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

and then consider the triangle $\triangle PRQ$. Show that the angle at R is a right angle, and hence by Pythagoras' Theorem again,

$$|PQ| = \sqrt{|PR|^2 + |RQ|^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

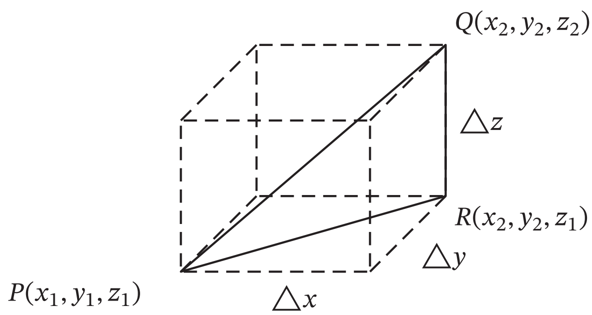


Figure 1.6. Distance in 3-Space

Challenge problems:

(12) Use Pythagoras' Theorem and the angle-summation formulas to prove the **Law of Cosines**: If ABC is any triangle with sides

$$a = |AC|, \quad b = |BC|, \quad c = |AB|$$

and the angle at C is $\angle ACB = \theta$, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta. \tag{1.10}$$

Here is one way to proceed (see Figure 1.7) Drop a perpendicular from C to AB,

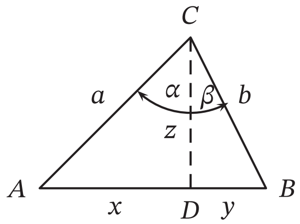


Figure 1.7. Law of Cosines

meeting AB at D. This divides the angle at C into two angles, satisfying $\alpha + \beta = \theta$ and divides AB into two intervals, with respective lengths $|AD| = x$ and $|DB| = y$, so $x + y = c$. Finally, set $|CD| = z$.

Now show the following:

$$x = a \sin \alpha, \quad y = b \sin \beta, \quad z = a \cos \alpha = b \cos \beta$$

and use this, together with Pythagoras' Theorem, to conclude that

$$a^2 + b^2 = x^2 + y^2 + 2z^2 \text{ and } c^2 = x^2 + y^2 + 2xy$$

and hence

$$c^2 = a^2 + b^2 - 2ab \cos(\alpha + \beta).$$

See Exercise 15 for the version of this which appears in Euclid.

(13) **Oblique Coordinates:** Consider an **oblique coordinate system** on $\mathbb{R}^2$, in which the vertical axis is replaced by an axis making an angle of α radians with the horizontal one; denote the corresponding coordinates by (u, v) (see Figure 1.8).

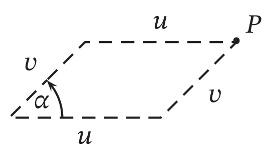


Figure 1.8. Oblique Coordinates

(a) Show that the oblique coordinates (u, v) and rectangular coordinates (x, y) of a point are related by

$$x = u + v \cos \alpha, \quad y = v \sin \alpha.$$

(b) Show that the distance of a point P with oblique coordinates (u, v) from the origin is given by

$$\operatorname{dist}(P, \mathcal{O}) = \sqrt{u^2 + v^2 + 2uv \cos \alpha}.$$

(c) Show that the distance between points P (with oblique coordinates (u_1, v_1)) and Q (with oblique coordinates (u_2, v_2)) is given by

$$\operatorname{dist}(P, Q) = \sqrt{\triangle u^2 + \triangle v^2 + 2\triangle u \triangle v \cos \alpha},$$

where $\triangle u := u_2 - u_1$ and $\triangle v := v_2 - v_1$. (*Hint:* There are two ways to do this. One is to substitute the expressions for the rectangular coordinates in terms of the oblique coordinates into the standard distance formula, the other is to use the law of cosines. Try them both.)

History note:

(14) Given a right triangle with "legs" of respective lengths a and b and hypotenuse of length c (Figure 1.9) **Pythagoras' Theorem** says that

$$c^2 = a^2 + b^2.$$

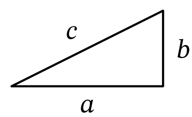


Figure 1.9. Right-angle triangle

In this problem, we outline two quite different proofs of this fact.
First Proof: Consider the pair of figures in Figure 1.10.

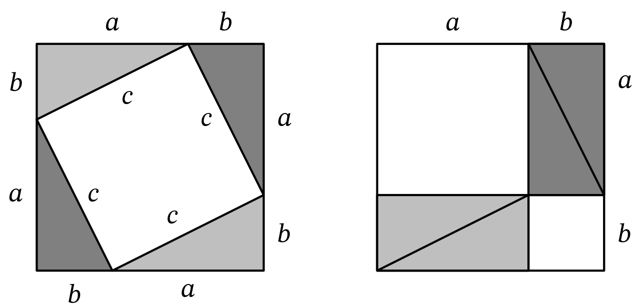


Figure 1.10. Pythagoras' Theorem by Dissection

(a) Show that the white quadrilateral on the left is a square (that is, show that the angles at the corners are right angles).
(b) Explain how the two figures prove Pythagoras' theorem.

A variant of Figure 1.10 was used by the twelfth-century Indian writer Bhāskara (b. 1114) to prove Pythagoras' Theorem. His proof consisted of a figure related to Figure 1.10 (without the shading) together with the single word "Behold!".

According to Eves [14, p. 158] and Maor [36, p. 63], reasoning based on Figure 1.10 appears in one of the oldest Chinese mathematical manuscripts, the *Caho Pei Suang Chin*, thought to date from the Han dynasty in the third century BC.

The Pythagorean Theorem appears as Proposition 47, Book I of Euclid's *Elements* with a different proof (see below). In his translation of the *Elements*, Heath has an extensive commentary on this theorem and its various proofs [28, vol. I, pp. 350-368]. In particular, he (as well as Eves) notes that the proof above has been suggested as possibly the kind of proof that Pythagoras himself might have produced. Eves concurs with this judgement, but Heath does not.

Second Proof: The proof above represents one tradition in proofs of the Pythagorean Theorem, which Maor [36] calls "dissection proofs." A second approach is via the theory of proportions. Here is an example: again, suppose $\triangle ABC$ has a right angle at C; label the sides with lower-case versions of the labels of the opposite vertices (Figure 1.11) and draw a perpendicular CD from the right angle to the hypotenuse. This cuts the hypotenuse into two pieces of respective lengths c_1 and c_2, so

$$c = c_1 + c_2. \tag{1.11}$$

Denote the length of CD by x.

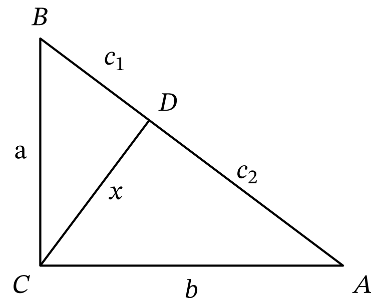


Figure 1.11. Pythagoras' Theorem by Proportions

(a) Show that the two triangles $\triangle ACD$ and $\triangle CBD$ are both similar to $\triangle ABC$.
(b) Using the similarity of $\triangle CBD$ with $\triangle ABC$, show that

$$\frac{a}{c} = \frac{c_1}{a}, \text{ or } a^2 = cc_1.$$

(c) Using the similarity of $\triangle ACD$ with $\triangle ABC$, show that

$$\frac{c}{b} = \frac{b}{c_2}, \text{ or } b^2 = cc_2.$$

(d) Now combine these equations with Equation (1.11) to prove Pythagoras' Theorem.

The basic proportions here are those that appear in Euclid's proof of Proposition 47, Book I of the *Elements*, although he arrives at these via different reasoning. However, in Book VI, Proposition 31 , Euclid presents a generalization of this theorem: draw any polygon using the hypotenuse as one side; then draw similar polygons using the legs of the triangle; Proposition 31 asserts that the sum of the areas of the two polygons on the legs equals that of the polygon on the hypotenuse. Euclid's proof of this proposition is essentially the argument given above.

(15) The Law of Cosines for an *acute* angle is essentially given by Proposition 13 in Book II of Euclid's *Elements* [28, vol. 1, p. 406]:

> *In acute-angled triangles the square on the side subtending the acute angle is less than the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle, namely that on which the perpendicular falls, and the straight line cut off within by the perpendicular towards the acute angle.*

Translated into algebraic language (see Figure 1.12, where the acute angle is $\angle ABC$) this says

$$|AC|^2 = |CB|^2 + |BA|^2 - |CB|\,|BD|\,.$$

Explain why this is the same as the Law of Cosines.

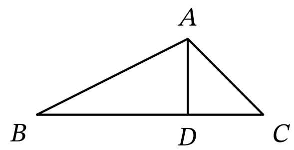


Figure 1.12. Euclid Book II, Proposition 13

1.2 Vectors and Their Arithmetic

Many quantities occurring in physics have a magnitude and a direction—for example, forces, velocities, and accelerations. As a prototype, we will consider **displacements**.

Suppose a rigid body is pushed (without being rotated) so that a distinguished spot on it is moved from position P to position Q (Figure 1.13). We represent this motion by a directed line segment, or arrow, going from P to Q and denoted $\overrightarrow{PQ}$. Note that this arrow encodes all the information about the motion of the *whole* body: that is, if we had distinguished a different spot on the body, initially located at P', then *its* motion would be described by an arrow $\overrightarrow{P'Q'}$ parallel to $\overrightarrow{PQ}$ and of the same length: in other words, the important characteristics of the displacement are its *direction* and *magnitude*, but *not* the location in space of its *initial* or *terminal points* (*i.e.*, its **tail** or **head**).

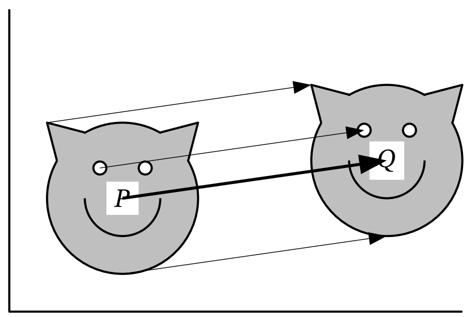


Figure 1.13. Displacement

A second important property of displacement is the way different displacements combine. If we first perform a displacement moving our distinguished spot from P to Q (represented by the arrow $\overrightarrow{PQ}$) and then perform a second displacement moving our spot from Q to R (represented by the arrow $\overrightarrow{QR}$), the net effect is the same as if we had pushed directly from P to R. The arrow $\overrightarrow{PR}$ representing this net displacement is formed by putting arrow $\overrightarrow{QR}$ with its tail at the head of $\overrightarrow{PQ}$ and drawing the arrow from the tail of $\overrightarrow{PQ}$ to the head of $\overrightarrow{QR}$ (Figure 1.14). More generally, the net effect of several successive displacements can be found by forming a broken path of arrows placed tail-to-head, and forming a new arrow from the tail of the first arrow to the head of the last.

A representation of a physical (or geometric) quantity with these characteristics is sometimes called a **vectorial representation**. With respect to velocities, the "parallelogram of velocities" appears in the *Mechanica*, a work incorrectly attributed to, but contemporary with, Aristotle (384-322 BC) [25, vol. I, p. 344], and is discussed at some length in the *Mechanics* by Heron of Alexandria (*ca.* 75 AD) [25, vol. II, p. 348]. The

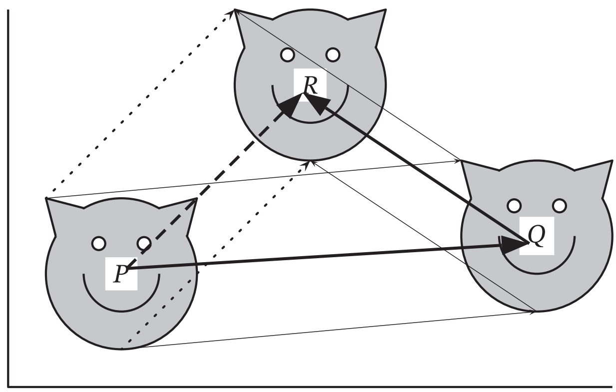


Figure 1.14. Combining Displacements

vectorial nature of some physical quantities, such as velocity, acceleration and force, was well understood and used by Isaac Newton (1642-1727) in the *Principia* [40, Corollary 1, Book 1 (p. 417)]. In the late eighteenth and early nineteenth century, Paolo Frisi (1728-1784), Leonard Euler (1707-1783), Joseph Louis Lagrange (1736-1813), and others realized that other physical quantities, associated with rotation of a rigid body (torque, angular velocity, moment of a force), could also be usefully given vectorial representations; this was developed further by Louis Poinsot (1777-1859), Siméon Denis Poisson (1781-1840), and Jacques Binet (1786-1856). At about the same time, various geometric quantities (*e.g.*, areas of surfaces in space) were given vectorial representations by Gaetano Giorgini (1795-1874), Simon Lhuilier (1750-1840), Jean Hachette (1769-1834), Lazare Carnot (1753-1823)), Michel Chasles (1793-1880) and later by Hermann Grassmann (1809-1877) and Giuseppe Peano (1858-1932). In the early nineteenth century, vectorial representations of complex numbers (and their extension, quaternions) were formulated by several researchers; the term *vector* was coined by William Rowan Hamilton (1805-1865) in 1853. Finally, extensive use of vectorial properties of electromagnetic forces was made by James Clerk Maxwell (1831-1879) and Oliver Heaviside (1850-1925) in the late nineteenth century. However, a general theory of vectors was only formulated in the very late nineteenth century; the first elementary exposition was given by Edwin Bidwell Wilson (1879-1964) in 1901 [55], based on lectures by the American mathematical physicist Josiah Willard Gibbs (1839-1903)[4] [18].

By a **geometric vector** in $\mathbb{R}^3$ (or $\mathbb{R}^2$) we will mean an "arrow" which can be moved to any position, provided its direction and length are maintained.[5] We will denote vectors with a letter surmounted by an arrow, like this: $\vec{v}$. [6] We define two operations on vectors. The **sum** of two vectors is formed by moving $\vec{w}$ so that its "tail" coincides in position with the "head" of $\vec{v}$, then forming the vector $\vec{v} + \vec{w}$ whose tail coincides with that of $\vec{v}$ and whose head coincides with that of $\vec{w}$ (Figure 1.15). If instead we place $\vec{w}$ with its tail at the position previously occupied by the tail of $\vec{v}$ and then move

[4] I learned much of this from Sandro Caparrini [6–8]. This narrative differs from the standard one, given by Michael Crowe [10]

[5] This mobility is sometimes expressed by saying it is a **free vector**.

[6] For example, all of the arrows in Figure 1.13 represent the vector $\overrightarrow{PQ}$.

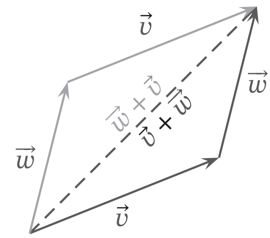


Figure 1.15. Sum of two vectors

$\vec{v}$ so that its tail coincides with the head of $\vec{w}$, we form $\vec{w}+\vec{v}$, and it is clear that these two configurations form a parallelogram with diagonal

$$\vec{v}+\vec{w}=\vec{w}+\vec{v}.$$

This is the **commutative property** of vector addition.

A second operation is **scaling** or **multiplication of a vector by a number**. We naturally define (positive integer) multiples of a vector: $1\vec{v}=\vec{v}$, $2\vec{v}=\vec{v}+\vec{v}$, $3\vec{v}=\vec{v}+\vec{v}+\vec{v}=2\vec{v}+\vec{v}$, and so on. Then we can define *rational* multiples by $\vec{v}=\frac{m}{n}\vec{w}\Leftrightarrow n\vec{v}=m\vec{w}$. Finally, to define multiplication by an arbitrary (positive) real number, suppose $\frac{m_i}{n_i}\to\ell$ is a sequence of rationals converging to the real number ℓ. For any fixed vector $\vec{v}$, if we draw arrows representing the vectors $(m_i/n_i)\vec{v}$ with all their tails at a fixed position, then the heads will form a convergent sequence of points along a line, whose limit is the position for the head of $\ell\vec{v}$. Alternatively, if we pick a unit of length, then for any vector $\vec{v}$ and any positive real number r, the vector $r\vec{v}$ has the same direction as $\vec{v}$, and its length is that of $\vec{v}$ multiplied by r. For this reason, we refer to real numbers (in a vector context) as **scalars**.

If $\vec{u}=\vec{v}+\vec{w}$ then it is natural to write $\vec{v}=\vec{u}-\vec{w}$ and from this (Figure 1.16) it is natural to define the negative $-\vec{w}$ of a vector $\vec{w}$ as the vector obtained by interchanging the head and tail of $\vec{w}$. This allows us to also define multiplication of a vector $\vec{v}$ by any *negative* real number $r=-|r|$ as

$$(-|r|)\vec{v}\coloneqq|r|\,(-\vec{v})$$

—that is, we reverse the direction of $\vec{v}$ and "scale" by $|r|$.

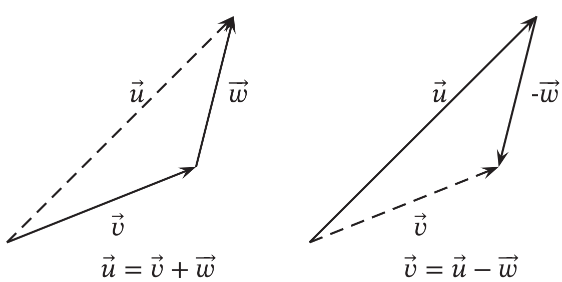


Figure 1.16. Difference of vectors

Addition of vectors (and of scalars) and multiplication of vectors by scalars have many formal similarities with addition and multiplication of numbers. We list the major ones (the first of which has already been noted above):

- Addition of vectors is
  commutative: $\vec{v}+\vec{w}=\vec{w}+\vec{v}$, and

  associative: $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$.
- Multiplication of vectors by scalars
  distributes over vector sums: $r(\vec{v}+\vec{w})=r\vec{w}+r\vec{v}$, and

  distributes over scalar sums: $(r+s)\vec{v}=r\vec{v}+s\vec{v}$.

We will explore some of these properties further in Exercise 3.

The interpretation of displacements as vectors gives us an alternative way to represent vectors. If we know that an arrow has its tail at the origin (we call this **standard position**), then the vector it represents is entirely determined by the coordinates of its head. This gives us a natural correspondence between *vectors* $\vec{v}$ in $\mathbb{R}^3$ (or $\mathbb{R}^2$) and *points* $P \in \mathbb{R}^3$ (*resp.* $\mathbb{R}^2$): the **position vector** of the point P is the vector $\overrightarrow{\mathcal{O}P}$; it represents that displacement of $\mathbb{R}^3$ which moves the origin to P. We shall make extensive use of the correspondence between vectors and points, often denoting a point by its position vector $\vec{p} \in \mathbb{R}^3$, or specifying a vector by the coordinates (x, y, z) of its head when represented in standard position. We refer to x, y and z as the **components** or **entries** of $\vec{v}$, and sometimes write $\vec{v}=(x, y, z)$. Vector arithmetic is very easy to calculate in this representation: if $\vec{w}=(\triangle x, \triangle y, \triangle z)$, then the displacement represented by $\vec{w}$ moves the origin to $(\triangle x, \triangle y, \triangle z)$; the sum of this and $\vec{v}=(x, y, z)$ is the displacement taking the origin first to (x, y, z) and then to

$$\vec{v}+\vec{w}=(x+\triangle x, y+\triangle y, z+\triangle z);$$

that is, *we add vectors componentwise.*

Similarly, if r is any scalar and $\vec{v}=(x, y, z)$, then

$$r\vec{v}=(rx, ry, rz)\ :$$

a scalar multiplies all entries of the vector.

This representation points out the presence of an exceptional vector—the **zero vector**

$$\vec{0} \coloneqq (0,0,0)$$

which is the result of either multiplying an arbitrary vector by the scalar zero ($0\vec{v}=\vec{0}$) or of subtracting an arbitrary vector from itself ($\vec{v}-\vec{v}=\vec{0}$). As a *point*, $\vec{0}$ corresponds to the origin $\mathcal{O}$ itself. As an *arrow*, its tail and head are at the same position. As a *displacement*, it corresponds to not moving at all. Note in particular that *the zero vector does not have a well-defined direction*—a feature which will be important to remember in the future. From a formal, algebraic point of view, the zero *vector* plays the role for *vector* addition that is played by the *number* zero for addition of *numbers*: it is an **additive identity element**, which means that adding it to any vector gives back that vector:

$$\vec{v}+\vec{0}=\vec{v}=\vec{0}+\vec{v}.$$

By thinking of vectors in $\mathbb{R}^3$ as triples of numbers, we can recover the entries of a vector geometrically: if $\vec{v}=(x, y, z)$ then we can write

$$\vec{v}=(x,0,0)+(0,y,0)+(0,0,z)=x(1,0,0)+y(0,1,0)+z(0,0,1).$$

This means that any vector in $\mathbb{R}^3$ can be expressed as a sum of scalar multiples (or **linear combination**) of three specific vectors, known as the **standard basis** for $\mathbb{R}^3$ (see Figure 1.17), and denoted

$$\vec{\imath} = (1,0,0), \quad \vec{\jmath} = (0,1,0), \quad \vec{k} = (0,0,1).$$

We have just seen that every vector $\vec{v} \in \mathbb{R}^3$ can be expressed as

$$\vec{v} = (x,y,z) = x\vec{\imath} + y\vec{\jmath} + z\vec{k},$$

where x, y, and z are the coordinates of $\vec{v}$.

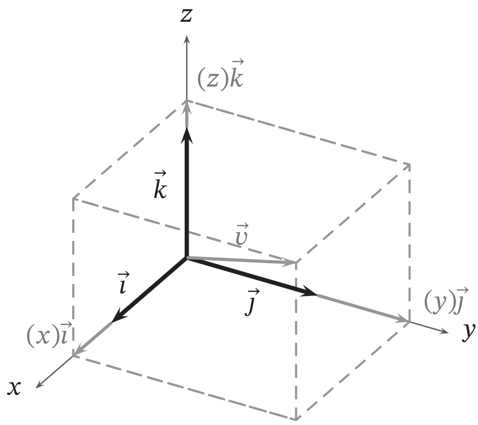


Figure 1.17. The Standard Basis for $\mathbb{R}^3$

We shall find it convenient to move freely between the coordinate notation $\vec{v} = (x,y,z)$ and the "arrow" notation $\vec{v} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$; generally, we adopt coordinate notation when $\vec{v}$ is regarded as a position vector, and "arrow" notation when we want to picture it as an arrow in space.

We began by thinking of a vector $\vec{v}$ in $\mathbb{R}^3$ as determined by its magnitude and its direction, and have ended up thinking of it as a triple of numbers. To come full circle, we recall that the vector $\vec{v} = (x,y,z)$ has as its standard representation the arrow $\overrightarrow{\mathcal{O}P}$ from the origin $\mathcal{O}$ to the point P with coordinates (x,y,z); thus its magnitude (or **length**, denoted $\|\vec{v}\|$ or $|\vec{v}|$) is given by the distance formula

$$\|\vec{v}\| = \sqrt{x^2 + y^2 + z^2}.$$

When we want to specify the **direction** of $\vec{v}$, we "*point*," using as our standard representation the **unit vector**—that is, the vector of length 1—in the direction of $\vec{v}$. From the scaling property of multiplication by real numbers, we see that the unit vector in the direction of a (nonzero[7]) vector $\vec{v}$ ($\vec{v} \neq \vec{0}$) is

$$\mathrm{u}\left(\vec{v}\right) = \frac{1}{\|\vec{v}\|}\vec{v}.$$

In particular, the standard basis vectors $\vec{\imath}, \vec{\jmath}$, and $\vec{k}$ are unit vectors in the direction(s) of the (positive) coordinate axes.

[7] A vector is **nonzero** if it is not equal to the zero vector: *some* of its entries can be zero, but *not all* of them.

Two (nonzero) vectors point in *the same* direction precisely if their respective unit vectors are the same: $\frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\|\vec{w}\|}\vec{w}$, or

$$\vec{v} = \lambda\vec{w}, \quad \vec{w} = \frac{1}{\lambda}\vec{v},$$

where the (positive) scalar λ is $\lambda = \frac{\|\vec{v}\|}{\|\vec{w}\|}$. Similarly, the two vectors point in *opposite* directions if the two unit vectors are *negatives* of each other, or $\vec{v} = \lambda\vec{w}$ (*resp.* $\vec{w} = \frac{1}{\lambda}\vec{v}$), where the *negative* scalar λ is $\lambda = -\frac{\|\vec{v}\|}{\|\vec{w}\|}$. We shall refer to two vectors as **parallel** if they point in the same or opposite directions, that is, if each is a *nonzero* (positive or negative) multiple of the other.

We can summarize this by

Remark 1.2.1. *For two nonzero vectors $\vec{v} = (x_1, y_1, z_1)$ and $\vec{w} = (x_2, y_2, z_2)$, the following are equivalent:*

- *$\vec{v}$ and $\vec{w}$ are parallel (i.e., they point in the same or opposite directions);*
- *$\vec{v} = \lambda\vec{w}$ for some nonzero scalar λ;*
- *$\vec{w} = \lambda'\vec{v}$ for some nonzero scalar λ';*
- *$\frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2} = \lambda$ for some nonzero scalar λ (where if one of the entries of $\vec{w}$ is zero, so is the corresponding entry of $\vec{v}$, and the corresponding ratio is omitted from these equalities);*
- *$\frac{x_2}{x_1} = \frac{y_2}{y_1} = \frac{z_2}{z_1} = \lambda'$ for some nonzero scalar λ' (where if one of the entries of $\vec{w}$ is zero, so is the corresponding entry of $\vec{v}$, and the corresponding ratio is omitted from these equalities).*

The values of λ (resp. λ') are the same wherever they appear above, and λ' is the reciprocal of λ.

λ (hence also λ') is positive *precisely if $\vec{v}$ and $\vec{w}$ point in the* same *direction, and* negative *precisely if they point in* opposite *directions.*

Two vectors are **linearly dependent** if, when we picture them as arrows from a common initial point, the two heads and the common tail fall on a common line. Algebraically, this means that one of them is a scalar multiple of the other. This terminology will be extended in Exercise 7—but for more than two vectors, the condition is more complicated. Vectors which are *not* linearly dependent are **linearly independent**. Remark 1.2.1 says that two *nonzero* vectors are linearly *dependent* precisely if they are *parallel*.

Exercises for § 1.2

Practice problems:

(1) In each part, you are given two vectors, $\vec{v}$ and $\vec{w}$. Find (i) $\vec{v} + \vec{w}$; (ii) $\vec{v} - \vec{w}$; (iii) $2\vec{v}$; (iv) $3\vec{v} - 2\vec{w}$; (v) the length of $\vec{v}$, $\left\|\vec{v}\right\|$; (vi) the unit vector $\vec{u}$ in the direction of $\vec{v}$:

   (a) $\vec{v} = (3, 4)$, $\vec{w} = (-1, 2)$
   (b) $\vec{v} = (1, 2, -2)$, $\vec{w} = (2, -1, 3)$
   (c) $\vec{v} = 2\vec{\imath} - 2\vec{\jmath} - \vec{k}$, $\vec{w} = 3\vec{\imath} + \vec{\jmath} - 2\vec{k}$

(2) In each case below, decide whether the given vectors are linearly dependent or linearly independent.

(a) $(1, 2), (2, 4)$
(b) $(1, 2), (2, 1)$
(c) $(-1, 2), (3, -6)$
(d) $(-1, 2), (2, 1)$
(e) $(2, -2, 6), (-3, 3, 9)$
(f) $(-1, 1, 3), (3, -3, -9)$
(g) $\vec{\imath} + \vec{\jmath} + \vec{k}, 2\vec{\imath} - 2\vec{\jmath} + 2\vec{k}$
(h) $2\vec{\imath} - 4\vec{\jmath} + 2\vec{k}, -\vec{\imath} + 2\vec{\jmath} - \vec{k}$

Theory problems:

(3) (a) We have seen that the commutative property of vector addition can be interpreted via the "parallelogram rule" (Figure A.5). Give a similar pictorial interpretation of the associative property.
(b) Give geometric arguments for the two distributive properties of vector arithmetic.
(c) Show that if $a\vec{v} = \vec{0}$ then either $a = 0$ or $\vec{v} = \vec{0}$. (*Hint:* What do you know about the relation between lengths for $\vec{v}$ and $a\vec{v}$?)
(d) Show that if a vector $\vec{v}$ satisfies $a\vec{v} = b\vec{v}$, where $a \neq b$ are two specific, distinct scalars, then $\vec{v} = \vec{0}$.
(e) Show that vector subtraction is *not* associative.

(4) **Polar notation for vectors:**
(a) Show that any planar vector $\vec{u}$ of length 1 can be written in the form

$$\vec{u} = (\cos\theta, \sin\theta),$$

where θ is the (counterclockwise) angle between $\vec{u}$ and the positive x-axis.
(b) Conclude that every nonzero planar vector $\vec{v}$ can be expressed in **polar form**

$$\vec{v} = \|\vec{v}\|(\cos\theta, \sin\theta),$$

where θ is the (counterclockwise) angle between $\vec{v}$ and the positive x-axis.

(5) (a) Show that if $\vec{v}$ and $\vec{w}$ are two linearly independent vectors in the plane, then every vector in the plane can be expressed as a linear combination of $\vec{v}$ and $\vec{w}$. (*Hint:* The independence assumption means they point along non-parallel lines. Given a point P in the plane, consider the parallelogram with the origin and P as opposite vertices, and with edges parallel to $\vec{v}$ and $\vec{w}$. Use this to construct the linear combination.)
(b) Now suppose $\vec{u}$, $\vec{v}$ and $\vec{w}$ are *three* nonzero vectors in $\mathbb{R}^3$. If $\vec{v}$ and $\vec{w}$ are linearly independent, show that every vector lying in the plane that contains the two lines through the origin parallel to $\vec{v}$ and $\vec{w}$ can be expressed as a linear combination of $\vec{v}$ and $\vec{w}$. Now show that if $\vec{u}$ does not lie in this plane, then every vector in $\mathbb{R}^3$ can be expressed as a linear combination of $\vec{u}$, $\vec{v}$ and $\vec{w}$. The two statements above are summarized by saying that $\vec{v}$ and $\vec{w}$ (*resp.* $\vec{u}$, $\vec{v}$ and $\vec{w}$) **span** $\mathbb{R}^2$ (*resp.* $\mathbb{R}^3$).

Challenge problem:

(6) Show (using vector methods) that the line segment joining the midpoints of two sides of a triangle is parallel to and has half the length of the third side.

(7) Given a collection $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_k}\}$ of vectors, consider the equation (in the unknown coefficients $c_1, \dots, c_k$)

$$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k} = \vec{0}; \tag{1.12}$$

that is, an expression for the zero vector as a linear combination of the given vectors. Of course, regardless of the vectors $\vec{v_i}$, one solution of this is

$$c_1 = c_2 = \cdots = 0;$$

the combination coming from this solution is called the **trivial combination** of the given vectors. The collection $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_k}\}$ is **linearly dependent** if there exists some **nontrivial** combination of these vectors—that is, a solution of Equation (1.12) with *at least one* nonzero coefficient. It is **linearly independent** if it is not linearly dependent—that is, if the *only* solution of Equation (1.12) is the trivial one.

(a) Show that any collection of vectors which includes the zero vector is linearly dependent.
(b) Show that a collection of *two* nonzero vectors $\{\vec{v_1}, \vec{v_2}\}$ in $\mathbb{R}^3$ is linearly independent precisely if (in standard position) they point along non-parallel lines.
(c) Show that a collection of *three* position vectors in $\mathbb{R}^3$ is linearly dependent precisely if at least one of them can be expressed as a linear combination of the other two.
(d) Show that a collection of three position vectors in $\mathbb{R}^3$ is linearly *independent* precisely if the corresponding points determine a plane in space that does *not* pass through the origin.
(e) Show that any collection of *four or more* vectors in $\mathbb{R}^3$ is linearly *dependent*. (*Hint:* Use either part (a) of this problem or part (b) of Exercise 5.)

1.3 Lines in Space

Parametrization of Lines. An equation of the form

$$Ax + By = C,$$

where A, B, and C are constants with at least one of A and B nonzero, is called a "linear" equation because if we interpret x and y as the rectangular coordinates of a point in the plane, the resulting locus is a line (at least provided A, B and C are not all zero). Via straightforward algebraic manipulation, (if $B \neq 0$)[8] we can rewrite this as the **slope-intercept formula**

$$y = mx + b, \tag{1.13}$$

where the **slope** m is the tangent of the angle the line makes with the horizontal and the y**-intercept** b is the ordinate (signed height) of its intersection with the y-axis. We can think of this formula as a two-step determination of a line: the slope determines a direction, and the intercept picks out a particular line from the family of (parallel) lines that have that slope.

The locus in space of a "linear" equation in the three rectangular coordinates x, y and z, $Ax + By + Cz = D$, is a *plane*, not a line, but we can construct a *vector* equation for a line analogous in spirit to the point-slope formula (1.13). A direction in 3-space cannot be determined by a single number, but it is naturally specified by a nonzero vector, so the three-dimensional analogue of the slope of a line is a **direction vector** $\vec{v} = a\vec{\imath} + b\vec{\jmath} + c\vec{k}$ to which it is parallel.[9] Given the direction $\vec{v}$, we can specify a

[8] $B = 0$ means we have $x = a$, a vertical line.
[9] Note that a direction vector need not be a unit vector.

particular line among all those parallel to $\vec{v}$ by giving a **basepoint** $P_0(x_0, y_0, z_0)$ through which the line is required to pass, say P_0. The line through P_0 parallel to $\vec{v}$ consists of those points whose displacement from P_0 is a scalar multiple of $\vec{v}$. This scheme is most efficiently written in terms of position vectors: if the base point has position vector $\overrightarrow{p_0} = x_0\vec{\imath} + y_0\vec{\jmath} + z_0\vec{k}$ then the point whose displacement from P_0 is $t\vec{v}$ has position vector

$$\vec{p}(t) = \overrightarrow{p_0} + t\vec{v}.$$

As the scalar t ranges over all real numbers, $\vec{p}(t)$ defines a **vector-valued function** of the real variable t. In terms of coordinates, this reads

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct. \end{aligned}$$

We refer to the vector-valued function $\vec{p}(t)$ as a **parametrization**; the coordinate equations are **parametric equations** for the line. We can think of this as the position vector at time t of a moving point whose position at time $t = 0$ is the basepoint P_0, and which travels at the constant velocity $\vec{v}$.[10] It is useful to keep in mind the distinction between the *parametrization* $\vec{p}(t)$, which represents a moving point, and the *line* ℓ being parametrized, which is the *path* of this moving point. A given line ℓ has many different parametrizations: we can take any point on ℓ as P_0, and any nonzero vector pointing parallel to ℓ as the direction vector $\vec{v}$. This ambiguity means we need to be careful when making geometric comparisons between lines given parametrically. Nonetheless, this way of presenting lines exhibits geometric information in a very accessible form.

For example, let us consider two lines in 3-space. The first, ℓ_1, is given by the parametrization $\overrightarrow{p_1}(t) = (1, -2, 3) + t(-3, -2, 1)$ or, in coordinate form,

$$\begin{array}{rcrr} x &=& 1 & -3t \\ y &=& -2 & -2t \\ z &=& 3 & +t \end{array}$$

while the second, ℓ_2, is given in coordinate form as

$$\begin{array}{rcrr} x &=& 1 & +6t \\ y &=& & 4t \\ z &=& 1 & -2t. \end{array}$$

We can easily read off from this that ℓ_2 has parametrization $\overrightarrow{p_2}(t) = (1, 0, 1) + t(6, 4, -2)$. Comparing the two direction vectors $\overrightarrow{v_1} = -3\vec{\imath} - 2\vec{\jmath} + \vec{k}$ and $\overrightarrow{v_2} = 6\vec{\imath} + 4\vec{\jmath} - 2\vec{k}$, we see that $\overrightarrow{v_2} = -2\overrightarrow{v_1}$ so the two lines have the same direction—either they are parallel, or they coincide. To decide which is the case, it suffices to decide whether the basepoint of one of the lines lies on the other line. Let us see whether the basepoint of ℓ_2, $\overrightarrow{p_2}(0) = (1, 0, 1)$ lies on ℓ_1: This means we need to see if for some value of t we have $\overrightarrow{p_2}(0) = \overrightarrow{p_1}(t)$, or

$$\begin{array}{rcrr} 1 &=& 1 & -3t \\ 0 &=& -2 & -2t \\ 1 &=& 3 & +t. \end{array}$$

[10] Of course, a line in the plane can also be represented via a parametrization, or a vector-valued function whose values are vectors in the plane. This is illustrated in the exercises.

It is easy to see that the first equation requires $t = 0$, the second requires $t = -1$, and the third requires $t = -2$; there is no way we can solve all three simultaneously. It follows that ℓ_1 and ℓ_2 are distinct, but parallel, lines.

Now, consider a third line, ℓ_3, given by

$$\begin{array}{rcrr} x & = & 1 & +3t \\ y & = & 2 & +t \\ z & = & -3 & +t. \end{array}$$

We read off its direction vector as $\vec{v_3} = 3\vec{\imath}+\vec{\jmath}+\vec{k}$ which is clearly *not* a scalar multiple of the other two. This tells us immediately that ℓ_3 is different from both ℓ_1 and ℓ_2 (it has a different direction). Now let us ask whether ℓ_2 intersects ℓ_3. It might be tempting to try to answer this by looking for a solution of the vector equation

$$\vec{p_2}(t) = \vec{p_3}(t)$$

but *this would be a mistake*. Remember that these two parametrizations describe the positions of two points—one moving along ℓ_2 and the other moving along ℓ_3—at time t. The equation above requires the two points to be at the same place *at the same time*—in other words, it describes a *collision*. But all we ask is that the two *paths* cross: it would suffice to locate a *place* occupied by both moving points, but possibly *at different times*. This means we need to distinguish the parameters appearing in the two functions $\vec{p_2}(t)$ and $\vec{p_3}(t)$, by renaming one of them (say the first) as (say) s: the vector equation we need to solve is

$$\vec{p_2}(s) = \vec{p_3}(t),$$

which amounts to the three equations in two unknowns

$$\begin{array}{rrcrr} 1 & +6s & = & 1 & +3t \\ & 4s & = & 2 & +t \\ 1 & -2s & = & -3 & +t. \end{array}$$

You can check that these equations hold for $t = 2$ and $s = 1$—that is, the lines ℓ_2 and ℓ_3 intersect at the point

$$\vec{p_2}(1) = (7, 4, -1) = \vec{p_3}(2).$$

Now let us apply the same process to see whether ℓ_1 intersects ℓ_3. The vector equation $\vec{p_1}(s) = \vec{p_3}(t)$ yields the three coordinate equations

$$\begin{array}{rrcrr} 1 & -3s & = & 1 & +3t \\ -2 & -2s & = & 2 & +t \\ 3 & +s & = & -3 & +t. \end{array}$$

You can check that these equations have no simultaneous solution, so ℓ_1 and ℓ_3 *do not intersect*, even though they are *not parallel*. Such lines are sometimes called **skew lines**.

Geometric Applications. A basic geometric fact is that *any pair of distinct points determines a line*. Given two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, how do we find a parametrization of the line ℓ they determine?

Suppose the position vectors of the two points are $\vec{p_1} = x_1\vec{\imath} + y_1\vec{\jmath} + z_1\vec{k}$ and $\vec{p_2} = x_2\vec{\imath} + y_2\vec{\jmath} + z_2\vec{k}$. The vector joining them lies along ℓ, so we can use it as a direction vector:

$$\vec{v} = \overrightarrow{P_1P_2} = \vec{p_2} - \vec{p_1} = \triangle x\vec{\imath} + \triangle y\vec{\jmath} + \triangle z\vec{k}$$

(where $\triangle x = x_2 - x_1$, $\triangle y = y_2 - y_1$, and $\triangle z = z_2 - z_1$). Using P_1 as basepoint, this leads to the parametrization

$$\vec{p}\,(t) = \overrightarrow{p_1} + t\vec{v} = \overrightarrow{p_1} + t(\overrightarrow{p_2} - \overrightarrow{p_1}) = (1-t)\overrightarrow{p_1} + t\overrightarrow{p_2}.$$

Note that we have set this up so that $\vec{p}\,(0) = \overrightarrow{p_1}$ and $\vec{p}\,(1) = \overrightarrow{p_2}$.

The full line ℓ through these points corresponds to allowing the parameter to take on all real values. However, if we restrict t to the interval $0 \le t \le 1$, the corresponding points fill out the **line segment** P_1P_2.

Remark 1.3.1 (Two-Point Formula). *Suppose $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are distinct points. The line through P_1 and P_2 is given by the parametrization*[11]

$$\vec{p}\,(t) = (1-t)\overrightarrow{p_1} + t\overrightarrow{p_2}$$

with coordinates

$$\begin{aligned} x &= (1-t)x_1 + tx_2 \\ y &= (1-t)y_1 + ty_2 \\ z &= (1-t)z_1 + tz_2. \end{aligned}$$

*The **line segment** P_1P_2 consists of the points for which $0 \le t \le 1$. The value of t gives the portion of P_1P_2 represented by the segment $P_1\vec{p}\,(t)$; in particular, the **midpoint** of P_1P_2 has position vector*

$$\frac{1}{2}(\overrightarrow{p_1} + \overrightarrow{p_2}) = \left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2), \frac{1}{2}(z_1 + z_2)\right).$$

Exercises for § 1.3

Answers to Exercises 1a, 2a, 3a, 4a, and 6a are given in Appendix A.13.

Practice problems:

(1) For each line in the plane described below, give (i) an equation in the form $Ax + By + C = 0$, (ii) parametric equations, and (iii) a parametric vector equation:
  (a) The line with slope -1 through the origin.
  (b) The line with slope -2 and y-intercept 1.
  (c) The line with slope 1 and y-intercept -2.
  (d) The line with slope 3 going through the point $(-1, 2)$.
  (e) The line with slope -2 going through the point $(-1, 2)$.

(2) Find the slope and y-intercept for each line given below:
  (a) $2x + y - 3 = 0$ (b) $x - 2y + 4 = 0$
  (c) $3x + 2y + 1 = 0$ (d) $y = 0$ (e) $x = 1$

(3) For each line in $\mathbb{R}^3$ described below, give (i) parametric equations, and (ii) a parametric vector equation:
  (a) The line through the point $(2, -1, 3)$ with direction vector $\vec{v} = -\vec{\imath} + 2\vec{\jmath} + \vec{k}$.
  (b) The line through the points $(-1, 2, -3)$ and $(3, -2, 1)$.
  (c) The line through the points $(2, 1, 1)$ and $(2, 2, 2)$.

[11]This is the parametrized analogue of the **two-point formula** for a line in the plane determined by a pair of points.

(d) The line through the point $(1, 3, -2)$ parallel to the line

$$\begin{aligned} x &= 2 - 3t \\ y &= 1 + 3t \\ z &= 2 - 2t. \end{aligned}$$

(4) In each part below, you are given a pair of lines in $\mathbb{R}^2$. Decide in each case whether these lines are parallel or if not, find their point of intersection.

(a) $x + y = 3$ and $3x - 3y = 3$

(b) $2x - 2y = 2$ and $2y - 2x = 2$

(c)
$$\begin{aligned} x &= 1 + 2t \\ y &= -1 + t \end{aligned} \quad \text{and} \quad \begin{aligned} x &= 2 - t \\ y &= -4 + 2t \end{aligned}$$

(d)
$$\begin{aligned} x &= 2 - 4t \\ y &= -1 - 2t \end{aligned} \quad \text{and} \quad \begin{aligned} x &= 1 + 2t \\ y &= -4 + t \end{aligned}$$

(5) Find the points at which the line with parametrization

$$\vec{p}(t) = (3 + 2t, 7 + 8t, -2 + t),$$

that is,

$$\begin{aligned} x &= 3 + 2t \\ y &= 7 + 8t \\ z &= -2 + t \end{aligned}$$

intersects each of the coordinate planes.

(6) Determine whether the given lines intersect:

(a)
$$\begin{array}{rcl} x &=& 3t + 2 \\ y &=& t - 1 \\ z &=& 6t + 1 \end{array} \quad \text{and} \quad \begin{array}{rcl} x &=& 3t - 1 \\ y &=& t - 2 \\ z &=& t \end{array}.$$

(b)
$$\begin{array}{rcl} x &=& t + 4 \\ y &=& 4t + 5 \\ z &=& t - 2 \end{array} \quad \text{and} \quad \begin{array}{rcl} x &=& 2t + 3 \\ y &=& t + 1 \\ z &=& 2t - 3 \end{array}.$$

(c)
$$\begin{array}{rcl} x &=& 3t + 2 \\ y &=& t - 1 \\ z &=& 6t + 1 \end{array} \quad \text{and} \quad \begin{array}{rcl} x &=& 2t + 3 \\ y &=& t + 1 \\ z &=& 2t - 3 \end{array}.$$

Theory problems:

(7) Show that if $\vec{u}$ and $\vec{v}$ are both *unit* vectors, placed in standard position, then the line through the origin parallel to $\vec{u} + \vec{v}$ bisects the angle between them.

Challenge problems:

(8) The following is implicit in the proof of Book V, Proposition 4 of Euclid's *Elements* [28, pp. 85-88]. Here, we work through a proof using vectors; we explore a proof of the same fact following Euclid in Exercise 11.

Theorem 1.3.2 (Angle Bisectors). *In any triangle, the lines which bisect the three interior angles meet in a common point.*

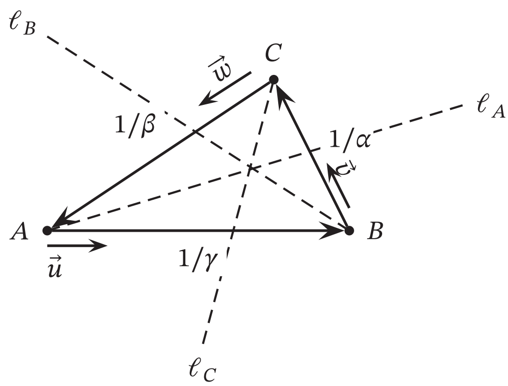


Figure 1.18. Theorem 1.3.2

Suppose the position vectors of the vertices A, B, and C are $\vec{a}$, $\vec{b}$, and $\vec{c}$, respectively.

(a) Show that the *unit* vectors pointing *counterclockwise* along the edges of the triangle (see Figure 1.18) are as follows:

$$\vec{u} = \gamma\vec{b} - \gamma\vec{a}, \quad \vec{v} = \alpha\vec{c} - \alpha\vec{b}, \quad \vec{w} = \beta\vec{a} - \beta\vec{c},$$

where

$$\alpha = \frac{1}{|BC|}, \quad \beta = \frac{1}{|AC|}, \text{ and } \gamma = \frac{1}{|AB|}$$

are the reciprocals of the lengths of the sides (each length is labelled by the Greek analogue of the name of the opposite vertex).

(b) Show that the line ℓ_A bisecting the angle $\angle A$ can be given as

$$\vec{p_A}(r) = (1 - r\beta - r\gamma)\vec{a} + r\gamma\vec{b} + r\beta\vec{c}$$

and the corresponding bisectors of $\angle B$ and $\angle C$ are

$$\vec{p_B}(s) = s\gamma\vec{a} + (1 - s\alpha - s\gamma)\vec{b} + s\alpha\vec{c}$$

$$\vec{p_C}(t) = t\beta\vec{a} + t\alpha\vec{b} + (1 - t\alpha - t\beta)\vec{c}.$$

(c) Show that the intersection of ℓ_A and ℓ_B is given by

$$r = \frac{\alpha}{\alpha\beta + \beta\gamma + \gamma\alpha}, \quad s = \frac{\beta}{\alpha\beta + \beta\gamma + \gamma\alpha}.$$

(d) Show that the intersection of ℓ_B and ℓ_C is given by

$$s = \frac{\beta}{\alpha\beta + \beta\gamma + \gamma\alpha}, \quad t = \frac{\gamma}{\alpha\beta + \beta\gamma + \gamma\alpha}.$$

(e) Conclude that all three lines meet at the point given by

$$\begin{aligned}\vec{p_A}\left(\frac{\alpha}{\alpha\beta+\beta\gamma+\gamma\alpha}\right)\\ &= \vec{p_B}\left(\frac{\beta}{\alpha\beta+\beta\gamma+\gamma\alpha}\right) = \vec{p_C}\left(\frac{\gamma}{\alpha\beta+\beta\gamma+\gamma\alpha}\right)\\ &= \frac{1}{\alpha\beta+\beta\gamma+\gamma\alpha}\left(\beta\gamma\vec{a}+\gamma\alpha\vec{b}+\alpha\beta\vec{c}\right).\end{aligned}$$

(9) **Barycentric Coordinates:** Show that if $\vec{a}$, $\vec{b}$, and $\vec{c}$ are the position vectors of the vertices of a triangle $\triangle ABC$ in $\mathbb{R}^3$, then the position vector $\vec{p}$ of every point P in that triangle (lying in the plane determined by the vertices) can be expressed as a linear combination of $\vec{a}$, $\vec{b}$ and $\vec{c}$

$$v' = \lambda_1\vec{a}+\lambda_2\vec{b}+\lambda_3\vec{c} \qquad \text{with } 0 \le \lambda_i \le 1 \text{ for } i = 1,2,3 \text{ and } \lambda_1+\lambda_2+\lambda_3 = 1.$$

(*Hint:* (see Figure 1.19) Draw a line from vertex A through P, and observe where it meets the opposite side; call this point D. Use Remark 1.3.1 to show that the position vector $\vec{d}$ of D is a linear combination of $\vec{b}$ and $\vec{c}$, with coefficients between zero and one and summing to 1. Then use Remark 1.3.1 again to show that $\vec{p}$ is a linear combination of $\vec{d}$ and $\vec{a}$.)

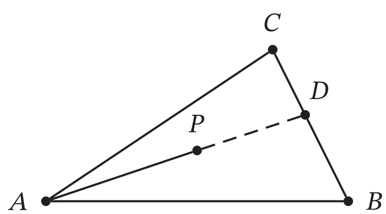


Figure 1.19. Barycentric Coordinates

The numbers λ_i are called the **barycentric coordinates** of P with respect to A, B, and C. Show that P lies on an edge of the triangle precisely if one of its barycentric coordinates is zero.

Barycentric coordinates were introduced (in a slightly different form) by August Möbius (1790-1860) in his book *Barycentrische Calcul* (1827). His name is more commonly associated with "Möbius transformations" in complex analysis and with the "Möbius band" (the one-sided surface that results from joining the ends of a band after making a half-twist) in topology.[12]

(10) Find a line that lies entirely in the set defined by the equation $x^2 + y^2 - z^2 = 1$.

History note:

(11) Heath [28, pp. 85-88] points out that the proof of Proposition 4, Book IV of the *Elements* contains the following implicit proof of Theorem 1.3.2 (see Figure 1.20). This was proved by vector methods in Exercise 8.

[12]The "Möbius band" was independently formulated by Johann Listing (1808-1882) at about the same time—in 1858, when Möbius was 68 years old. These two are often credited with beginning the study of topology. [32, p. 1165]

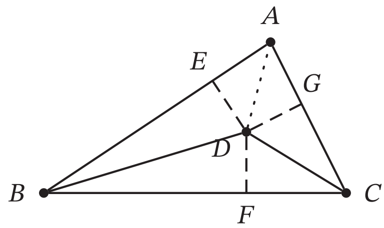


Figure 1.20. Euclid's proof of Theorem 1.3.2

(a) The lines bisecting $\angle B$ and $\angle C$ intersect at a point D above BC because of Book I, Postulate 5 (known as the **Parallel Postulate**):

> That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Why do the interior angles between BC and the two angle bisectors add up to less than a right angle? (*Hint:* What do you know about the angles of a triangle?)

(b) Drop perpendiculars from D to each edge of the triangle, meeting the edges at E, F, and G.
Show that the triangles $\triangle BFD$ and $\triangle BED$ are congruent. (*Hint:* ASA—angle, side, angle!)

(c) Similarly, show that the triangles $\triangle CFD$ and $\triangle CGD$ are congruent.

(d) Use this to show that $|DE| = |DF| = |DG|$.

(e) Now draw the line DA. Show that the triangles $\triangle AGD$ and $\triangle AED$ are congruent. (*Hint:* Both are right triangles; compare one pair of legs and the hypotenuse.)

(f) Conclude that $\angle EAD = \angle GAD$—which means that AD bisects $\angle A$. Thus D lies on all three angle bisectors.

1.4 Projection of Vectors; Dot Products

Suppose a weight is set on a ramp which is inclined θ radians from the horizontal (Figure 1.21). The gravitational force $\vec{g}$ on the weight is directed downward, and some

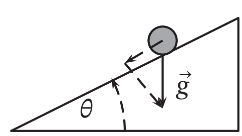


Figure 1.21. A weight on a ramp

of this is countered by the structure holding up the ramp. The effective force on the weight can be found by expressing $\vec{g}$ as a sum of two (vector) forces, $\overrightarrow{g_\perp}$ *perpendicular* to the ramp, and $\overrightarrow{g_\parallel}$ *parallel* to the ramp. Then $\overrightarrow{g_\perp}$ is cancelled by the structural forces in the ramp, and the net unopposed force is $\overrightarrow{g_\parallel}$, the *projection* of $\vec{g}$ in the direction of the ramp.

To abstract this situation, recall that a direction is specified by a unit vector. The (vector) **projection** of an arbitrary vector $\vec{v}$ in the direction specified by the unit vector $\vec{u}$ is the vector

$$\operatorname{proj}_{\vec{u}} \vec{v} \coloneqq (\|\vec{v}\| \cos\theta)\vec{u}$$

where θ is the angle between $\vec{u}$ and $\vec{v}$ (Figure 1.22). Note that replacing $\vec{u}$ with its neg-

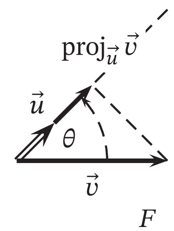


Figure 1.22. Projection of a Vector

ative replaces θ with $\pi - \theta$, and the projection is unchanged: $\operatorname{proj}_{-\vec{u}} \vec{v} = (\|\vec{v}\| \cos(\pi - \theta))(-\vec{u}) = \operatorname{proj}_{\vec{u}} \vec{v}$. This means that we can regard the projection as being onto any positive or negative scalar multiple of $\vec{u}$ (or onto any line which can be parametrized using $\vec{u}$ as its direction vector): for any nonzero vector $\vec{w}$, we define the **projection** of $\vec{v}$ onto (the direction of) $\vec{w}$ as its projection onto the unit vector $\vec{u} = \vec{w}/\|\vec{w}\|$ in the direction of $\vec{w}$:

$$\operatorname{proj}_{\vec{w}} \vec{v} = \operatorname{proj}_{\vec{u}} \vec{v} = \left(\frac{\|\vec{v}\|}{\|\vec{w}\|} \cos\theta\right) \vec{w}. \tag{1.14}$$

How do we calculate this projection from the entries of the two vectors? To this end, we perform a theoretical detour.[13]

Suppose $\vec{v} = (x_1, y_1, z_1)$ and $\vec{w} = (x_2, y_2, z_2)$; how do we determine the angle θ between them? If we put them in standard position, representing $\vec{v}$ by $\overrightarrow{\mathcal{O}P}$ and $\vec{w}$ by $\overrightarrow{\mathcal{O}Q}$ (Figure 1.23), then we have a triangle $\triangle\mathcal{O}PQ$ with angle θ at the origin, and two

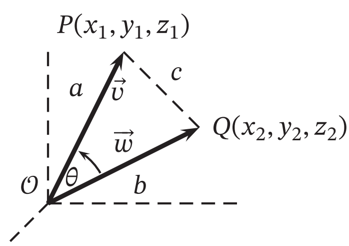


Figure 1.23. Determining the Angle θ

sides given by

$$a = \|\vec{v}\| = \sqrt{x_1^2 + y_1^2 + z_1^2}, \quad b = \|\vec{w}\| = \sqrt{x_2^2 + y_2^2 + z_2^2}.$$

The distance formula lets us determine the length of the third side:

$$c = \operatorname{dist}(P, Q) = \sqrt{\triangle x^2 + \triangle y^2 + \triangle z^2}.$$

[13]Thanks to my student Benjamin Brooks, whose questions helped me formulate the approach here.

But we also have the **Law of Cosines** (Exercise 12):

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

or

$$2ab\cos\theta = a^2 + b^2 - c^2. \tag{1.15}$$

We can compute the right-hand side of this equation by substituting the expressions for a, b and c in terms of the entries of $\vec{v}$ and $\vec{w}$:

$$a^2 + b^2 - c^2 = (x_1^2 + y_1^2 + z_1^2) + (x_2^2 + y_2^2 + z_2^2) - (\triangle x^2 + \triangle y^2 + \triangle z^2).$$

Consider the terms involving x:

$$\begin{aligned} x_1^2 + x_2^2 - \triangle x^2 &= x_1^2 + x_2^2 - (x_1 - x_2)^2 \\ &= x_1^2 + x_2^2 - (x_1^2 - 2x_1x_2 + x_2^2) \\ &= 2x_1x_2. \end{aligned}$$

Similar calculations for the y- and z-coordinates allow us to conclude that

$$a^2 + b^2 - c^2 = 2(x_1x_2 + y_1y_2 + z_1z_2)$$

and hence, substituting into Equation (1.15), factoring out 2, and recalling that $a = \|\vec{v}\|$ and $b = \|\vec{w}\|$, we have

$$\|\vec{v}\|\|\vec{w}\|\cos\theta = x_1x_2 + y_1y_2 + z_1z_2. \tag{1.16}$$

This quantity, which is easily calculated from the entries of $\vec{v}$ and $\vec{w}$ (on the right) but has a useful geometric interpretation (on the left), is called the *dot product*[14] of $\vec{v}$ and$\vec{w}$. Equation (1.16) appears already (with somewhat different notation) in Lagrange's 1788 *Méchanique Analitique* [35, N.11], and also as part of Hamilton's definition (1847) of the product of quaternions [23], although the scalar product of vectors was apparently not formally identified until Wilson's 1901 textbook [55], or more accurately Gibbs' earlier (1881) notes on the subject [18, p. 20].

Definition 1.4.1. *Given any two vectors $\vec{v} = (x_1, y_1, z_1)$ and $\vec{w} = (x_2, y_2, z_2)$ in $\mathbb{R}^3$, their **dot product** is the scalar*

$$\vec{v}\cdot\vec{w} = x_1x_2 + y_1y_2 + z_1z_2.$$

This dot product exhibits a number of algebraic properties, which we leave to you to verify (Exercise 3):

Proposition 1.4.2. *The dot product has the following algebraic properties:*

(1) *It is **commutative**:*

$$\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$$

(2) *It **distributes over vector sums**[15]:*

$$\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$$

(3) *it **respects scalar multiples**:*

$$(r\vec{v})\cdot\vec{w} = r(\vec{v}\cdot\vec{w}) = \vec{v}\cdot(r\vec{w}).$$

[14] Also the *scalar product*, *direct product*, or *inner product*

[15] In this formula, $\vec{u}$ is an arbitrary vector, not necessarily of unit length.

Also, the geometric interpretation of the dot product given by Equation (1.16) yields a number of geometric properties:

Proposition 1.4.3. *The dot product has the following geometric properties:*

(1) $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\theta$*, where* θ *is the angle between the "arrows" representing* $\vec{v}$ *and* $\vec{w}$*.*

(2) $\vec{v} \cdot \vec{w} = 0$ *precisely if the arrows representing* $\vec{v}$ *and* $\vec{w}$ *are perpendicular to each other, or if one of the vectors is the zero vector.*

(3) $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$.

(4) $\operatorname{proj}_{\vec{w}} \vec{v} = \left(\dfrac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\right) \vec{w}$ *(provided* $\vec{w} \neq \vec{0}$*).*

We note the curiosity in the second item: the dot product of the zero vector with *any* vector is zero. While the zero vector has no well-defined direction, we will find it a convenient fiction to say that *the zero vector is perpendicular to every vector, including itself.*

Proof. (1) This is just Equation (1.16).

(2) This is an (almost) immediate consequence: if $\|\vec{v}\|$ and $\|\vec{w}\|$ are both nonzero (*i.e.*, $\vec{v} \neq \vec{0} \neq \vec{w}$) then $\vec{v} \cdot \vec{w} = 0$ precisely when $\cos\theta = 0$, and this is the same as saying that $\vec{v}$ is perpendicular to $\vec{w}$ (denoted $\vec{v} \perp \vec{w}$). But if either $\vec{v}$ or $\vec{w}$ *is* $\vec{0}$, then clearly $\vec{v} \cdot \vec{w} = 0$ by either side of Equation (1.16).

(3) This is just (1) when $\vec{v} = \vec{w}$, which in particular means $\theta = 0$, or $\cos\theta = 1$.

(4) This follows from Equation (1.14) by substitution:

$$\operatorname{proj}_{\vec{w}} \vec{v} = \left(\frac{\|\vec{v}\|}{\|\vec{w}\|} \cos\theta\right) \vec{w} = \left(\frac{\|\vec{v}\|\|\vec{w}\|}{\|\vec{w}\|^2} \cos\theta\right) \vec{w}$$

$$= \left(\frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\right) \vec{w}.$$

□

These interpretations of the dot product make it a powerful tool for attacking certain kinds of geometric and mechanical problems. We consider two examples below, and others in the exercises.

Distance from a point to a line: Given a point Q with coordinate vector $\vec{q}$ and a line ℓ parametrized via

$$\vec{p}\,(t) = \overrightarrow{p_0} + t\vec{v}$$

let us calculate the distance from Q to ℓ. We will use the fact that this distance is achieved by a line segment from Q to a point R on the line such that QR is *perpendicular* to ℓ (Figure 1.24).

We have

$$\overrightarrow{P_0Q} = \vec{q} - \overrightarrow{p_0}.$$

We will denote this, for clarity, by

$$\vec{w} \coloneqq \vec{q} - \overrightarrow{p_0}; \quad \overrightarrow{P_0R} = \operatorname{proj}_{\vec{v}} \overrightarrow{P_0Q} = \operatorname{proj}_{\vec{v}} \vec{w}$$

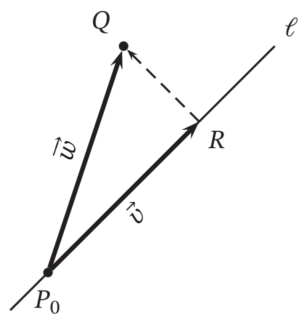


Figure 1.24. Distance from Point to Line

so $\|P_0R\| = \frac{\vec{w}\cdot\vec{v}}{\|\vec{v}\|}$ and thus by Pythagoras' Theorem

$$|QR|^2 = |P_0Q|^2 - |P_0R|^2 = \vec{w}\cdot\vec{w} - \left(\frac{\vec{w}\cdot\vec{v}}{\|\vec{v}\|}\right)^2$$

$$= \frac{(\vec{w}\cdot\vec{w})(\vec{v}\cdot\vec{v}) - (\vec{w}\cdot\vec{v})^2}{\vec{v}\cdot\vec{v}}.$$

Another approach is outlined in Exercise 7.

Angle cosines: A natural way to try to specify the direction of a line through the origin is to find the angles it makes with the three coordinate axes; these are sometimes referred to as the **Euler angles** of the line. In the plane, it is clear that the angle α between a line and the horizontal is complementary to the angle β between the line and the vertical. In space, the relation between the angles α, β and γ which a line makes with the positive x, y, and z-axes respectively is less obvious on purely geometric grounds. The relation

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{1.17}$$

was implicit in the work of the eighteenth-century mathematicians Joseph Louis Lagrange (1736-1813) and Gaspard Monge (1746-1818), and explicitly stated by Leonard Euler (1707-1783) [4, pp. 206-7]. Using vector ideas, it is almost obvious.

Proof of Equation (1.17). Let $\vec{u}$ be a unit vector in the direction of the line. Then the angles between $\vec{u}$ and the unit vectors along the three axes are

$$\vec{u}\cdot\vec{\imath} = \cos\alpha, \quad \vec{u}\cdot\vec{\jmath} = \cos\beta, \quad \vec{u}\cdot\vec{k} = \cos\gamma$$

from which it follows that

$$\vec{u} = \cos\alpha\vec{\imath} + \cos\beta\vec{\jmath} + \cos\gamma\vec{k}, \text{ in other words, } \vec{u} = (\cos\alpha, \cos\beta, \cos\gamma).$$

But then the distance formula says that

$$1 = \|\vec{u}\| = \sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma}$$

and squaring both sides yields Equation (1.17). □

Scalar Projection: The projection $\operatorname{proj}_{\vec{w}} \vec{v}$ of the vector $\vec{v}$ in the direction of the vector $\vec{w}$ is itself a vector; a related quantity is the **scalar projection** of $\vec{v}$ in the direction of $\vec{w}$, also called the **component** of $\vec{v}$ in the direction of $\vec{w}$. This is defined as

$$\operatorname{comp}_{\vec{w}} \vec{v} = \left\|\vec{v}\right\| \cos\theta,$$

where θ is the angle between $\vec{v}$ and $\vec{w}$; clearly, this can also be expressed as $\vec{v} \cdot \vec{u}$, where

$$\vec{u} := \frac{\vec{w}}{\left\|\vec{w}\right\|}$$

is the unit vector parallel to $\vec{w}$. Thus we can also write

$$\operatorname{comp}_{\vec{w}} \vec{v} = \frac{\vec{v} \cdot \vec{w}}{\left\|\vec{w}\right\|}. \tag{1.18}$$

This is a scalar, whose absolute value is the length of the vector projection, which is positive if $\operatorname{proj}_{\vec{w}} \vec{v}$ is parallel to $\vec{w}$ and negative if it points in the opposite direction.

Exercises for § 1.4

Answer to Exercise 1a is given in Appendix A.13.

Practice problems:

(1) For each pair of vectors $\vec{v}$ and $\vec{w}$ below, find (i) their dot product, (ii) their lengths, (iii) the cosine of the angle between them, and (iv) the (vector) projection of each onto the direction of the other:

(a) $\vec{v} = (2, 3)$, $\vec{w} = (3, 2)$
(b) $\vec{v} = (2, 3)$, $\vec{w} = (3, -2)$
(c) $\vec{v} = (1, 0)$, $\vec{w} = (3, 2)$
(d) $\vec{v} = (1, 0)$, $\vec{w} = (3, 4)$
(e) $\vec{v} = (1, 2, 3)$, $\vec{w} = (3, 2, 1)$
(f) $\vec{v} = (1, 2, 3)$, $\vec{w} = (3, -2, 0)$
(g) $\vec{v} = (1, 2, 3)$, $\vec{w} = (3, 0, -1)$
(h) $\vec{v} = (1, 2, 3)$, $\vec{w} = (1, 1, -1)$

(2) A point travelling at the constant velocity $\vec{v} = \vec{\imath} + \vec{\jmath} + \vec{k}$ goes through the position $(2, -1, 3)$; what is its closest distance to $(3, 1, 2)$ over the whole of its path?

Theory problems:

(3) Prove Proposition 1.4.2

(4) The following theorem (see Figure 1.25) can be proved in two ways:

Theorem 1.4.4. *In any parallelogram, the sum of the squares of the diagonals equals the sum of the squares of the (four) sides.*

(a) Prove Theorem 1.4.4 using the Law of Cosines (§ 1.2, Exercise 12).
(b) Prove Theorem 1.4.4 using vectors, as follows:
Place the parallelogram with one vertex at the origin: suppose the two adjacent vertices are P and Q and the opposite vertex is R (Figure 1.25). Represent the sides by the vectors $\vec{v} = \overrightarrow{\mathcal{O}P} = \overrightarrow{QR}$ and $\vec{w} = \overrightarrow{\mathcal{O}Q} = \overrightarrow{PR}$.
(i) Show that the diagonals are represented by $\overrightarrow{\mathcal{O}R} = \vec{v} + \vec{w}$ and $\overrightarrow{PQ} = \vec{v} - \vec{w}$.
(ii) Show that the squares of the diagonals are $|\mathcal{O}R|^2 = \|\vec{v}\|^2 + 2\vec{v} \cdot \vec{w} + \|\vec{w}\|^2$ and $|PQ|^2 = \|\vec{v}\|^2 - 2\vec{v} \cdot \vec{w} + \|\vec{w}\|^2$.

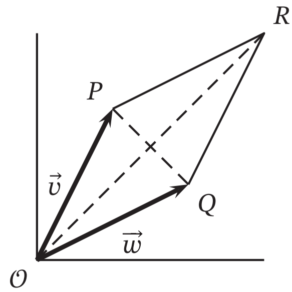


Figure 1.25. Theorem 1.4.4

(iii) Show that $|\mathcal{O}R|^2 + |PQ|^2 = 2\|\vec{v}\|^2 + 2\|\vec{w}\|^2$; but of course

$$\begin{aligned}|\mathcal{O}P|^2 + |PR|^2 + |RQ|^2 + |Q\mathcal{O}|^2 \\ &= \|\vec{v}\|^2 + \|\vec{w}\|^2 + \|\vec{v}\|^2 + \|\vec{w}\|^2 \\ &= 2\|\vec{v}\|^2 + 2\|\vec{w}\|^2.\end{aligned}$$

(5) Show that if $\vec{v} = x\vec{\imath} + y\vec{\jmath}$ is any nonzero vector in the plane, then the vector $\vec{w} = y\vec{\imath} - x\vec{\jmath}$ is perpendicular to $\vec{v}$.

(6) Consider the line ℓ in the plane defined as the locus of the linear equation $Ax + By = C$ in x and y. Define the vector $\vec{N} = A\vec{\imath} + B\vec{\jmath}$.
   (a) Show that ℓ is the set of points P whose position vector $\vec{p}$ satisfies $\vec{N} \cdot \vec{p} = C$.
   (b) Show that if $\overrightarrow{p_0}$ is the position vector of a specific point on the line, then ℓ is the set of points P whose position vector $\vec{p}$ satisfies $\vec{N} \cdot (\vec{p} - \overrightarrow{p_0}) = 0$.
   (c) Show that $\vec{N}$ is perpendicular to ℓ.

(7) Show that if ℓ is a line given by $Ax + By = C$ then the distance from a point $Q(x, y)$ to ℓ is given by the formula

$$\operatorname{dist}(Q, \ell) = \frac{|Ax + By - C|}{\sqrt{A^2 + B^2}}. \tag{1.19}$$

(*Hint:* Let $\vec{N}$ be the vector given in Exercise 6, and $\overrightarrow{p_0}$ the position vector of any point P_0 on ℓ. Show that $\operatorname{dist}(Q, \ell) = \|\operatorname{proj}_{\vec{N}} \overrightarrow{P_0Q}\| = \|\operatorname{proj}_{\vec{N}}(\vec{q} - \overrightarrow{p_0})\|$, and interpret this in terms of A, B, C, x and y.)

1.5 Planes

Equations of Planes. We noted earlier that the locus of a "linear" equation in the three rectangular coordinates x, y, and z

$$Ax + By + Cz = D \tag{1.20}$$

is a plane in space. Using the dot product, we can extract a good deal of geometric information about this plane from Equation (1.20).

Let us form a vector from the coefficients on the left of (1.20):

$$\vec{N} = A\vec{\imath} + B\vec{\jmath} + C\vec{k}.$$

Using $\vec{p} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ as the position vector of $P(x, y, z)$, we see that (1.20) can be expressed as the vector equation

$$\vec{N} \cdot \vec{p} = D. \tag{1.21}$$

In the special case that $D = 0$ this is the condition that $\vec{N}$ is perpendicular to $\vec{p}$. In general, for any two points P_0 and P_1 satisfying (1.20), the vector $\overrightarrow{P_0P_1}$ from P_0 to P_1, which is the difference of their position vectors ($\overrightarrow{P_0P_1} = \overrightarrow{\triangle p} = \vec{p_1} - \vec{p_0} = \triangle x\vec{\imath} + \triangle y\vec{\jmath} + \triangle z\vec{k}$) lies in the plane, and hence satisfies

$$\vec{N} \cdot \overrightarrow{\triangle p} = \vec{N} \cdot (\vec{p_1} - \vec{p_0}) = D - D = 0.$$

Thus, letting the second point P_1 be an arbitrary point $P(x, y, z)$ in the plane, we have

Remark 1.5.1. *If $P_0(x_0, y_0, z_0)$ is any point whose coordinates satisfy* (1.20)

$$Ax_0 + By_0 + Cz_0 = D$$

then the locus of Equation (1.20) *is the plane through P_0 perpendicular to the* ***normal vector***

$$\vec{N} := A\vec{\imath} + B\vec{\jmath} + C\vec{k}.$$

This geometric characterization of a plane from an equation is similar to the geometric characterization of a line from its parametrization: the normal vector $\vec{N}$ formed from the left side of Equation (1.20) (by analogy with the direction vector $\vec{v}$ of a parametrized line) determines the "tilt" of the plane, and then the right-hand side D picks out from among the planes perpendicular to $\vec{N}$ (which are, of course, all parallel to one another) a particular one by, in effect, determining a point that must lie in this plane.

For example, the plane $\mathcal{P}$ determined by the equation $2x - 3y + z = 5$ is perpendicular to the normal vector $\vec{N} = 2\vec{\imath} - 3\vec{\jmath} + \vec{k}$. To find an explicit point P_0 in $\mathcal{P}$, we can use one of many tricks. One such trick is to fix two of the values x y and z and then substitute to see what the third one must be. If we set $x = 0 = y$, then substitution into the equation yields $z = 5$, so we can use as our basepoint $P_0(0,0,5)$ (which is the intersection of $\mathcal{P}$ with the z-axis).

We could find the intersections of $\mathcal{P}$ with the other two axes in a similar way. Alternatively, we could notice that $x = 1$ and $y = -1$ means $2x - 3y = 5$, so then $z = 0$ and we could equally use $P_0'(1, -1, 0)$ as our base point.

Conversely, given a nonzero vector $\vec{N}$ and a basepoint $P_0(x_0, y_0, z_0)$, we can write an equation for the plane through P_0 perpendicular to $\vec{N}$ in vector form as

$$\vec{N} \cdot \vec{p} = \vec{N} \cdot \vec{p_0}$$

or equivalently

$$\vec{N} \cdot (\vec{p} - \vec{p_0}) = 0.$$

For example an equation for the plane through the point $P_0(3, -1, -5)$ perpendicular to $\vec{N} = 4\vec{\imath} - 6\vec{\jmath} + 2\vec{k}$ is

$$4(x - 3) - 6(y + 1) + 2(z + 5) = 0, \text{ or } 4x - 6y + 2z = 8.$$

Note that the point $P_0'(2, 1, 3)$ also lies in this plane. If we used this as our basepoint (and kept $\vec{N} = 4\vec{\imath} - 6\vec{\jmath} + 2\vec{k}$) the equation $\vec{N} \cdot (\vec{p} - \vec{p_0}) = 0$ would take the form

$$4(x - 2) - 6(y - 1) + 2(z - 3) = 0$$

which, you should check, is equivalent to the previous equation.

An immediate corollary of Remark 1.5.1 is

Corollary 1.5.2. *The planes given by two linear equations*

$$A_1x + B_1y + C_1z = D_1$$
$$A_2x + B_2y + C_2z = D_2$$

are parallel (or coincide) precisely if the two normal vectors

$$\vec{N}_1 = A_1\vec{\imath} + B_1\vec{\jmath} + C_1\vec{k}$$
$$\vec{N}_2 = A_2\vec{\imath} + B_2\vec{\jmath} + C_2\vec{k}$$

are (nonzero) scalar multiples of each other; when the normal vectors are equal (i.e., the two left-hand sides of the two equations are the same) then the planes coincide if $D_1 = D_2$*, and otherwise they are parallel and non-intersecting.*

For example the plane given by the equation $-6x+9y-3z = 12$ has normal vector $\vec{N} = -6\vec{\imath} + 9\vec{\jmath} - 3\vec{k} = -\frac{3}{2}(4\vec{\imath} - 6\vec{\jmath} + 2\vec{k})$ so multiplying the equation by $-2/3$, we get an equivalent equation

$$4x - 6y + 2z = -8$$

which shows that this plane is parallel to (and does not intersect) the plane specified earlier by $4x - 6y + 2z = 8$ (since $8 \neq -8$).

We can also use vector ideas to calculate the **distance from a point** $Q(x, y, z)$ **to the plane** $\mathcal{P}$ given by an equation

$$Ax + By + Cz = D.$$

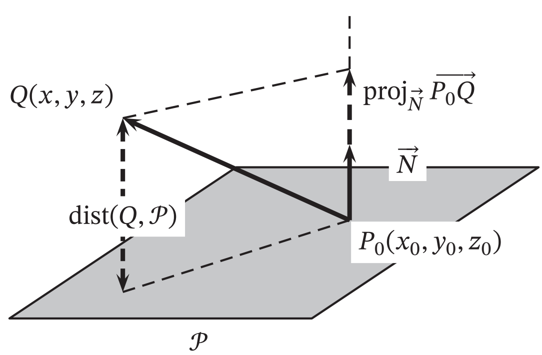


Figure 1.26. $\operatorname{dist}(Q, \mathcal{P})$

If $P_0(x_0, y_0, z_0)$ is any point on $\mathcal{P}$ (see Figure 1.26) then the (perpendicular) distance from Q to $\mathcal{P}$ is the (length of the) projection of $\overrightarrow{P_0Q} = \triangle x\vec{\imath} + \triangle y\vec{\jmath} + \triangle z\vec{k}$ in the direction of $\vec{N} = A\vec{\imath} + B\vec{\jmath} + C\vec{k}$

$$\operatorname{dist}(Q, \mathcal{P}) = \|\operatorname{proj}_{\vec{N}} \overrightarrow{P_0Q}\|$$

which we can calculate as

$$\frac{\left|\vec{N} \cdot (\vec{q} - \vec{p}_0)\right|}{\left\|\vec{N}\right\|} = \frac{\left|\vec{N} \cdot \vec{q} - \vec{N} \cdot \vec{p}_0\right|}{\sqrt{\vec{N} \cdot \vec{N}}} = \frac{|(Ax + By + Cz) - D|}{\sqrt{A^2 + B^2 + C^2}}.$$

For example, the distance from $Q(1,1,2)$ to the plane $\mathcal{P}$ given by $2x-3y+z=5$ is

$$\operatorname{dist}(Q,\mathcal{P}) = \frac{|(2)(1)-3(1)+1(2)-(5)|}{\sqrt{2^2+(-3)^2+1^2}} = \frac{4}{\sqrt{14}}.$$

The **distance between two parallel planes** is the distance from any *point* Q on *one* of the planes to the *other plane*. Thus, the distance between the parallel planes discussed earlier

$$\begin{array}{rrrcr} 4x & -6y & +2z & = & 8 \\ -6x & +9y & -3z & = & 12 \end{array}$$

is the same as the distance from $Q(3,-1,-5)$, which lies on the first plane, to the second plane, or

$$\begin{aligned} \operatorname{dist}(\mathcal{P}_1,\mathcal{P}_2) &= \operatorname{dist}(Q,\mathcal{P}_2) \\ &= \frac{|(-6)(3)+(9)(-1)+(-3)(-5)-(12)|}{\sqrt{(-6)^2+(9)^2+(-3)^2}} = \frac{24}{3\sqrt{14}}. \end{aligned}$$

Finally, the **angle θ between two planes** $\mathcal{P}_1$ and $\mathcal{P}_2$ can be defined as follows (Figure 1.27): if they are parallel, the angle is zero. Otherwise, they intersect along a line

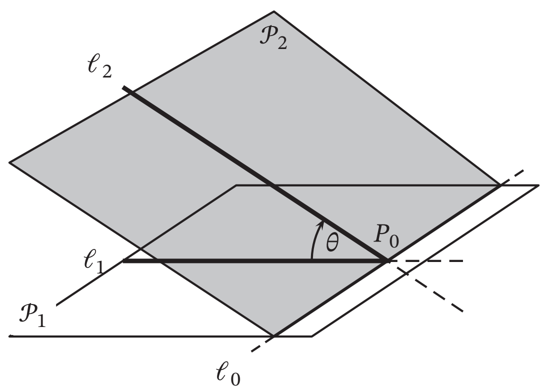


Figure 1.27. Angle between two planes

ℓ_0: pick a point P_0 on ℓ_0, and consider the line ℓ_i in $\mathcal{P}_i$ $(i=1,2)$ through P_0 and perpendicular to ℓ_0. Then θ is by definition the angle between ℓ_1 and ℓ_2.

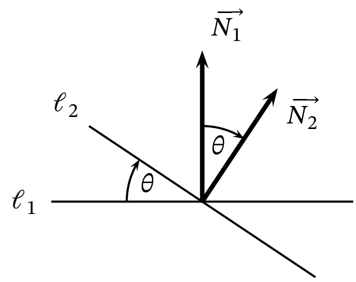


Figure 1.28. Angle between planes (cont'd)

To relate this to the equations of $\mathcal{P}_1$ and $\mathcal{P}_2$, consider the plane $\mathcal{P}_0$ (through P_0) containing the lines ℓ_1 and ℓ_2. $\mathcal{P}_0$ is perpendicular to ℓ_0 and hence contains the arrows with tails at P_0 representing the normals $\overrightarrow{N_1} = A_1\vec{\imath} + B_1\vec{\jmath} + C_1\vec{k}$ (*resp.* $\overrightarrow{N_2} = A_2\vec{\imath} + B_2\vec{\jmath} + C_2\vec{k}$) to $\mathcal{P}_1$ (*resp.* $\mathcal{P}_2$). But since $\overrightarrow{N_i}$ is perpendicular to ℓ_i for $i = 1, 2$, the angle between the vectors $\overrightarrow{N_1}$ and $\overrightarrow{N_2}$ is the same as the angle between ℓ_1 and ℓ_2 (Figure 1.28), hence

$$\cos\theta = \frac{\overrightarrow{N_1}\cdot\overrightarrow{N_2}}{\|\overrightarrow{N_1}\|\|\overrightarrow{N_2}\|}. \tag{1.22}$$

For example, the planes determined by the two equations

$$\begin{array}{rrrrr} x & +y & +z & = & 3 \\ x & +\sqrt{6}y & -z & = & 2 \end{array}$$

meet at angle θ, where

$$\cos\theta = \frac{\|(1,1,1)\cdot(1,\sqrt{6},-1)\|}{\sqrt{1^2+1^2+1^2}\sqrt{1^2+\sqrt{6}^2+(-1)^2}} = \frac{\left|1+\sqrt{6}-1\right|}{\sqrt{3}\sqrt{8}} = \frac{\sqrt{6}}{2\sqrt{6}} = \frac{1}{2}$$

so θ equals $\pi/6$ radians.

Parametrization of Planes. So far, we have dealt with planes given as loci of linear equations. This is an *implicit* specification. However, there is another way to specify a plane, which is more *explicit* and in closer analogy to the parametrizations we have used to specify lines in space.

Suppose $\vec{v} = v_1\vec{\imath} + v_2\vec{\jmath} + v_3\vec{k}$ and $\vec{w} = w_1\vec{\imath} + w_2\vec{\jmath} + w_3\vec{k}$ are two linearly independent vectors in $\mathbb{R}^3$. If we represent them via arrows in standard position, they determine a plane $\mathcal{P}_0$ through the origin. Note that any linear combination of $\vec{v}$ and $\vec{w}$, $\vec{p}\,(s,t) = s\vec{v} + t\vec{w}$ is the position vector of some point in this plane: when s and t are both positive, we draw the parallelogram with one vertex at the origin, one pair of sides parallel to $\vec{v}$, of length $s\,|\vec{v}|$, and the other pair of sides parallel to $\vec{w}$, with length $t\,|\vec{w}|$ (Figure 1.29). Then $\vec{p}\,(s,t)$ is the vertex opposite the origin in this parallelogram. Conversely, the

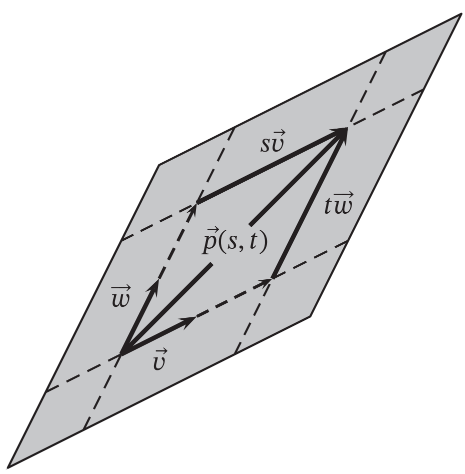


Figure 1.29. Linear Combination

position vector of any point P in $\mathscr{P}_0$ can be expressed uniquely as a linear combination of $\vec{v}$ and $\vec{w}$. We leave it to you to complete the details (see Exercise 5 in § 1.2).

Remark 1.5.3. *If $\vec{v}$ and $\vec{w}$ are linearly independent vectors in $\mathbb{R}^3$, then the set of all linear combinations of $\vec{v}$ and $\vec{w}$*

$$\mathscr{P}_0\left(\vec{v},\vec{w}\right) \coloneqq \{s\vec{v}+t\vec{w} \mid s,t \in \mathbb{R}\}$$

*is the set of position vectors for points in the plane (through the origin) determined by $\vec{v}$ and $\vec{w}$, called the **span** of $\vec{v}$ and $\vec{w}$.*

Suppose now we want to describe a plane $\mathcal{P}$ parallel to $\mathscr{P}_0\left(\vec{v},\vec{w}\right)$, but going through an arbitrarily given basepoint $P_0(x_0,y_0,z_0)$. If we let $\overrightarrow{p_0} = x_0\vec{\imath}+y_0\vec{\jmath}+z_0\vec{k}$ be the position vector of P_0, then displacement by $\overrightarrow{p_0}$ moves the origin $\mathcal{O}$ to P_0 and the plane $\mathscr{P}_0\left(\vec{v},\vec{w}\right)$ to the plane $\mathcal{P}$ through P_0 parallel to $\mathscr{P}_0\left(\vec{v},\vec{w}\right)$. It is clear from Remark 1.5.3 that the position vector $\vec{p} = x\vec{\imath}+y\vec{\jmath}+z\vec{k}$ of every point in $\mathcal{P}$ can be expressed as $\overrightarrow{p_0}$ plus some linear combination of $\vec{v}$ and $\vec{w}$,

$$\vec{p}\,(s,t) = \overrightarrow{p_0} + s\vec{v} + t\vec{w} \tag{1.23}$$

or

$$\left\{\begin{array}{ccccc} x & = & x_0 & +sv_1 & +tw_1 \\ y & = & y_0 & +sv_2 & +tw_2 \\ z & = & z_0 & +sv_3 & +tw_3 \end{array}\right. \tag{1.24}$$

for a unique pair of scalars $s,t \in \mathbb{R}$. These scalars form an oblique coordinate system for points in the plane $\mathcal{P}$. Equivalently, we can regard these equations as defining a **vector-valued function** $\vec{p}\,(s,t)$ which assigns to each point (s,t) in the "st-plane" a point $\vec{p}\,(s,t)$ of the plane $\mathcal{P}$ in $\mathbb{R}^3$. This is a **parametrization** of the plane $\mathcal{P}$. By contrast with the parametrization of a line, which uses *one* parameter t, and can be thought of as putting a copy of the real line into space, this uses *two* parameters, s and t, which live in the **parameter plane**; our parametrization puts a copy of the parameter plane into space.

We can use the vector approach sketched above to parametrize the plane determined by any three noncollinear points. Suppose $\triangle PQR$ is a nondegenerate triangle[16] in $\mathbb{R}^3$. Set $\overrightarrow{p_0} = \overrightarrow{\mathcal{O}P}$, the position vector of the vertex P, and let $\vec{v} = \overrightarrow{PQ}$ and $\vec{w} = \overrightarrow{PR}$ be two vectors representing the sides of the triangle at this vertex. Then the parametrization

$$\vec{p}\,(s,t) = \overrightarrow{p_0} + s\vec{v} + t\vec{w} = \overrightarrow{\mathcal{O}P} + s\overrightarrow{PQ} + t\overrightarrow{PR}$$

describes the plane containing our triangle; the vertices have position vectors

$$\begin{array}{llll} \overrightarrow{\mathcal{O}P} & & = \overrightarrow{p_0} & = \vec{p}\,(0,0) \\ \overrightarrow{\mathcal{O}Q} & = \overrightarrow{\mathcal{O}P} + \overrightarrow{PQ} & = \overrightarrow{p_0} + \vec{v} & = \vec{p}\,(1,0) \\ \overrightarrow{\mathcal{O}R} & = \overrightarrow{\mathcal{O}P} + \overrightarrow{PR} & = \overrightarrow{p_0} + \vec{w} & = \vec{p}\,(0,1)\,. \end{array}$$

[16]that is, the three vertices are distinct and don't all lie on a single line

For example, the three points located one unit from the origin along the three (positive) coordinate axes

$$P(1,0,0) \quad (\overrightarrow{\mathcal{O}P} = \vec{\imath})$$
$$Q(0,1,0) \quad (\overrightarrow{\mathcal{O}Q} = \vec{\jmath})$$
$$R(0,0,1) \quad (\overrightarrow{\mathcal{O}R} = \vec{k})$$

determine the plane with parametrization $\vec{p}\,(s,t) = \vec{\imath} + s(\vec{\jmath} - \vec{\imath}) + t(\vec{k} - \vec{\imath})$ or

$$\begin{array}{rcrrr} x & = & 1 & -s & -t, \\ y & = & & s & \\ z & = & & & t. \end{array}$$

To see whether the point $P(3,1,-3)$ lies in this plane, we can try to solve

$$\begin{array}{rcrrr} 3 & = & 1 & -s & -t \\ 1 & = & & s & \\ -3 & = & & & t; \end{array}$$

it is clear that the values of s and t given by the second and third equations also satisfy the first, so P does indeed lie in the plane through $\vec{\imath}, \vec{\jmath}$ and $\vec{k}$: $\overrightarrow{\mathcal{O}P} = \vec{p}\,(1,-3)$.

Given a linear equation, we can parametrize its locus by finding three noncollinear points on the locus and using the procedure above. For example, to parametrize the plane given by

$$3x - 2y + 4z = 12$$

we need to find three noncollinear points in this plane. If we set $y = z = 0$, we have $x = 4$, and so we can take our basepoint P to be $(4,0,0)$, or $\vec{p_0} = 4\vec{\imath}$. To find two other points, we could note that if $x = 4$ then $-2y + 4z = 0$, so any choice with $y = 2z$ will work, for example $Q(4,2,1)$, or $\vec{v} = \overrightarrow{PQ} = 2\vec{\jmath} + \vec{k}$ gives one such point. Unfortunately, any *third* point given by this scheme will produce $\vec{w}$ a scalar multiple of $\vec{v}$, so won't work. However, if we set $x = 0$ we have $-2y + 4z = 12$, and one solution of this is $y = -4$ and $z = 1$. Thus $R(0,-4,1)$ works, with $\vec{w} = \overrightarrow{PR} = -4\vec{\imath} - 4\vec{\jmath} + \vec{k}$. This leads to the parametrization $\vec{p}\,(s,t) = 4\vec{\imath} + s(2\vec{\jmath} + \vec{k}) + t(-4\vec{\imath} - 4\vec{\jmath} + \vec{k})$, or

$$\begin{array}{rcrrr} x & = & 4 & & -4t \\ y & = & & 2s & -4t \\ z & = & & s & +t. \end{array}$$

The converse problem—given a parametrization of a plane, to find an equation describing it—can sometimes be solved easily: for example, the plane through $\vec{\imath}, \vec{\jmath}$, and $\vec{k}$ easily leads to the relation $x + y + z = 1$. However, in general, it will be easier to handle this problem using cross products (§ 1.6).

Exercises for § 1.5

Practice problems:

(1) Write an equation for the plane through P perpendicular to $\vec{N}$:

(a) $P(2,-1,3)$, $\vec{N} = \vec{\imath} + \vec{\jmath} + \vec{k}$ (b) $P(1,1,1)$, $\vec{N} = 2\vec{\imath} - \vec{\jmath} + \vec{k}$

(c) $P(3,2,1)$, $\vec{N} = \vec{\jmath}$

(2) Find a point P on the given plane, and a vector normal to the plane:
   (a) $3x + y - 2z = 1$
   (b) $x - 2y + 3z = 5$
   (c) $5x - 4y + z = 8$
   (d) $z = 2x + 3y + 1$
   (e) $x = 5$

(3) Find a parametrization of each plane below:
   (a) $2x + 3y - z = 4$
   (b) $z = 4x + 5y + 1$
   (c) $x = 5$

(4) Find an equation for the plane through the point $(2, -1, 2)$:
   (a) parallel to the plane $3x + 2y + z = 1$
   (b) perpendicular to the line given by

$$\begin{aligned} x &= 3 - t \\ y &= 1 - 3t \\ z &= 2t. \end{aligned}$$

(5) Find the distance from the point $(3, 2, 1)$ to the plane $x - y + z = 5$.

(6) Find the angle between $\mathcal{P}_1$ and $\mathcal{P}_2$:
   (a)

$$\begin{aligned} \mathcal{P}_1 : &\quad 2x + y - z = 4 \\ \mathcal{P}_2 : &\quad 2x - y + 3z = 3 \end{aligned}$$

   (b)

$$\begin{aligned} \mathcal{P}_1 : &\quad 2x + 2y + 2\sqrt{6}z = 1 \\ \mathcal{P}_2 : &\quad \sqrt{3}x + \sqrt{3}y + \sqrt{2}z = \sqrt{5} \end{aligned}$$

Theory problems:

(7) (a) *Show:* If

$$\begin{aligned} \vec{p}\,(t) &= \overrightarrow{p_0} + t\vec{v} \\ \vec{q}\,(t) &= \overrightarrow{p_0} + t\vec{w} \end{aligned}$$

are parametrizations of two distinct lines both lying in the plane $\mathcal{P}$, and both going through a point P_0 (with position vector $\overrightarrow{p_0}$), then

$$\vec{p}\,(s, t) = \overrightarrow{p_0} + s\vec{v} + t\vec{w}$$

is a parametrization of the plane $\mathcal{P}$.

   (b) Suppose an equation for $\mathcal{P}$ is

$$Ax + By + Cz = D$$

with $C \neq 0$. *Show* that the intersections of $\mathcal{P}$ with the xz-plane and yz-plane are given by

$$\begin{aligned} z &= -\frac{A}{C}x + \frac{D}{C} \\ z &= -\frac{B}{C}y + \frac{D}{C} \end{aligned}$$

and combine this with (a) to get a parametrization of $\mathcal{P}$.

(c) Apply this to the plane $x + 2y + 3z = 9$.

(8) Find an equation for the plane $\mathcal{P}$ parametrized by

$$x = 2 + s - t$$
$$y = 1 - s + 2t$$
$$z = 3 + 2s - t.$$

1.6 Cross Products

Oriented Areas in the Plane. The standard formula for the area of a triangle

$$\mathcal{A} = \frac{1}{2}bh, \tag{1.25}$$

where b is the "base" length and h is the "height", is not always convenient to apply. Often we are presented with either the lengths of the three sides or the coordinates of the vertices (from which these lengths are easily calculated); in either case we can take a convenient side as the *base*, but calculating the *height*—the perpendicular distance from the base to the opposite vertex—can require some work.

In Exercise 5 we derive a vector formula for the area of a triangle based on the discussion (p. 28) of the distance from a point to a line, and in Exercise 16 and Exercise 17 we consider two area formulas given by Heron of Alexandria (*ca.* 75 AD) in his *Metrica*.

Here, however, we concentrate on finding a formula for the area of a triangle in $\mathbb{R}^2$ in terms of the coordinates of its vertices. Suppose the vertices are $A(a_1, a_2)$, $B(b_1, b_2)$, and $C(c_1, c_2)$. Using the side AB as the base, we have $b = \|\overrightarrow{AB}\|$ and, letting θ be the angle at vertex A, $h = \|\overrightarrow{AC}\| \sin\theta$, so

$$\mathcal{A}\left(\triangle ABC\right) = \frac{1}{2}\|\overrightarrow{AB}\|\|\overrightarrow{AC}\| \sin\theta.$$

To express this in terms of the coordinates of the vertices, note that

$$\overrightarrow{AB} = \triangle x_B \vec{\imath} + \triangle y_B \vec{\jmath},$$

where

$$\triangle x_B = b_1 - a_1, \quad \triangle y_B = b_2 - a_2$$

and similarly

$$\overrightarrow{AC} = \triangle x_C \vec{\imath} + \triangle y_C \vec{\jmath}.$$

Recall (Exercise 4 in § 1.2) that any vector $\vec{v} = x\vec{\imath} + y\vec{\jmath}$ in the plane can also be written in "polar" form as

$$\vec{v} = \|\vec{v}\|(\cos\theta_v \vec{\imath} + \sin\theta_v \vec{\jmath}),$$

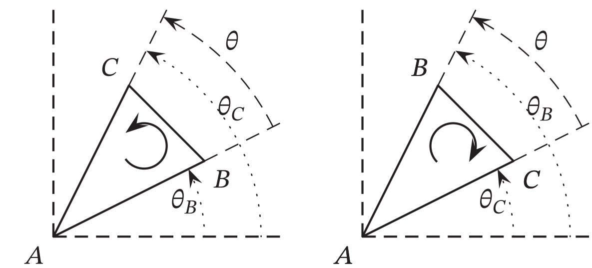
where θ_v is the counterclockwise angle between $\vec{v}$ and the horizontal vector $\vec{\imath}$. Thus, the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is $\theta = \theta_2 - \theta_1$ where θ_B and θ_C are the angles between $\vec{\imath}$ and each of the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\theta_C > \theta_B$. But the formula for the sine of a sum of angles gives us

$$\sin\theta = \cos\theta_1 \sin\theta_2 - \cos\theta_2 \sin\theta_1.$$

Thus, if $\theta_C > \theta_B$ we have

$$\begin{aligned}\mathcal{A}\left(\triangle ABC\right) &= \frac{1}{2}\|\overrightarrow{AB}\|\|\overrightarrow{AC}\|\sin\theta \\ &= \frac{1}{2}\|\overrightarrow{AB}\|\|\overrightarrow{AC}\|(\cos\theta_B\sin\theta_C - \cos\theta_C\sin\theta_B) \\ &= \frac{1}{2}[\triangle x_B \triangle y_C - \triangle x_C \triangle y_B]. \quad (1.26)\end{aligned}$$

The condition $\theta_C > \theta_B$ means that the direction of $\overrightarrow{AC}$ is a *counterclockwise* rotation (by an angle between 0 and π radians) from that of $\overrightarrow{AB}$; if the rotation from $\overrightarrow{AB}$ to $\overrightarrow{AC}$ is *clockwise*, then the two vectors trade places—or equivalently, the expression above gives us *minus* the area of $\triangle ABC$. See Figure 1.30.



(a) $\theta_C > \theta_B$: Counterclockwise (b) $\theta_C < \theta_B$: Clockwise

Figure 1.30. Orientation of $\triangle ABC$

The expression in brackets at the end of Equation (1.26) is easier to remember using a "visual" notation. An array of four numbers

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$

in two horizontal rows, with the entries vertically aligned in columns, is called a 2×2 **matrix**[17]. The **determinant** of a 2×2 matrix is the product x_1y_2 of the *downward* diagonal minus the product x_2y_1 of the *upward* diagonal. We denote the determinant by replacing the brackets surrounding the array with vertical bars:[18]

$$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = x_1y_2 - x_2y_1.$$

It is also convenient to sometimes treat the determinant as a function of its rows, which we think of as vectors:

$$\vec{v_i} = x_i\vec{\imath} + y_i\vec{\jmath}, \quad i = 1, 2;$$

treated this way, the determinant will be denoted

$$\Delta\left(\vec{v_1}, \vec{v_2}\right) = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.$$

[17]pronounced *"two by two matrix"*

[18]When a matrix is given a letter name—say A—we name its determinant $\det A$.

If we are simply given the coordinates of the vertices of a triangle in the plane, without a picture of the triangle, we can pick one of the vertices—call it A—and calculate the vectors to the other two vertices—call them B and C—and then take half the determinant. This will equal the area of the triangle *up to sign*:

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = \sigma(A,B,C)\mathcal{A}(\triangle ABC), \tag{1.27}$$

where $\sigma(A,B,C) = \pm 1$ depending on the direction of rotation from $\overrightarrow{AB}$ to $\overrightarrow{AC}$. We refer to $\sigma(A,B,C)$ as the **orientation** of the triangle (so an **oriented triangle** is one whose vertices have been assigned a specific order) and the quantity $\sigma(A,B,C)\mathcal{A}(\triangle ABC)$ as the **signed area** of the oriented triangle. You should verify that the oriented triangle $\triangle ABC$ has **positive orientation** precisely if going from A to B to C and then back to A constitutes a *counterclockwise* transversal of its periphery, and a **negative orientation** if this traversal is *clockwise*. Thus the orientation is determined by the "cyclic order" of the vertices: a **cyclic permutation** (moving everything one space over, and putting the entry that falls off the end back at the beginning) doesn't change the orientation:

$$\sigma(A,B,C) = \sigma(B,C,A) = \sigma(C,A,B).$$

For example, the triangle with vertices $A(2,-3)$, $B(4,-2)$, and $C(3,-1)$, shown in Figure 1.31, has $\overrightarrow{AB} = 2\vec{\imath} + \vec{\jmath}$ and $\overrightarrow{AC} = \vec{\imath} + 2\vec{\jmath}$, and its signed area is

$$\frac{1}{2}\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = \frac{1}{2}[(2)(2) - (1)(1)] = \frac{3}{2};$$

you can verify from Figure 1.31 that the path $A \mapsto B \mapsto C \mapsto A$ traverses the triangle *counterclockwise*.

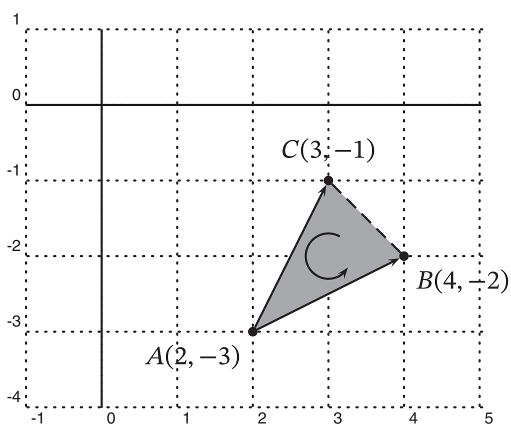


Figure 1.31. Oriented Triangle $\triangle ABC$, Positive Orientation

The triangle with vertices $A(-3,4)$, $B(-2,5)$ and $C(-1,3)$ has $\overrightarrow{AB} = \vec{\imath} + \vec{\jmath}$, $\overrightarrow{AC} = 2\vec{\imath} - \vec{\jmath}$, and signed area

$$\frac{1}{2}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = \frac{1}{2}[(1)(-1) - (2)(1)] = -\frac{3}{2};$$

you can verify that the path $A \mapsto B \mapsto C \mapsto A$ traverses the triangle *clockwise*.

These ideas can be extended to polygons in the plane: for example, a quadrilateral with vertices A, B, C, and D is positively (*resp.* negatively) oriented if the vertices in this order are consecutive in the counterclockwise (*resp.* clockwise) direction (Figure 1.32) and we can define its signed area as the area (*resp.* minus the area). By cutting the

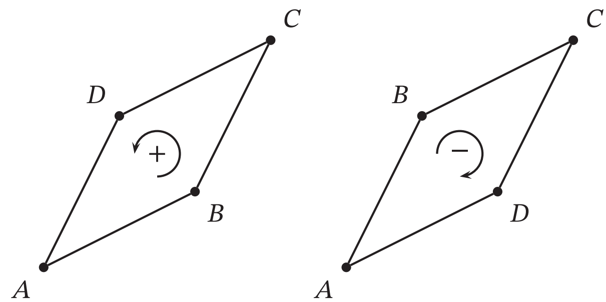


Figure 1.32. Oriented Quadrilaterals

quadrilateral into two triangles with a diagonal, and using Equation (1.27) on each, we can calculate its signed area from the coordinates of its vertices. This will be explored in Exercises 10-14.

For the moment, though, we consider a very special case. Suppose we have two nonzero vectors $\vec{v_1} = x_1\vec{\imath} + y_1\vec{\jmath}$ and $\vec{v_2} = x_2\vec{\imath} + y_2\vec{\jmath}$. Then the determinant using these rows

$$\Delta\left(\vec{v_1}, \vec{v_2}\right) = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.$$

can be interpreted geometrically as follows. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the points with position vectors $\vec{v_1}$ and $\vec{v_2}$, respectively, and let $R(x_1 + x_2, y_1 + y_2)$ be the point whose position vector is $\vec{v_1} + \vec{v_2}$ (Figure 1.33). Then the signed area of $\triangle OPQ$ equals

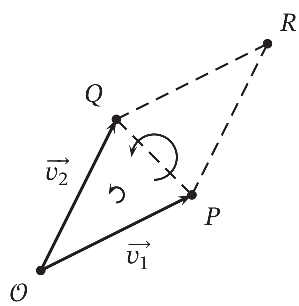


Figure 1.33. Proposition 1.6.1

$\frac{1}{2}\Delta\left(\vec{v_1}, \vec{v_2}\right)$; but note that the *parallelogram* $OPRQ$ has the same orientation as the *triangles* $\triangle OPQ$ and $\triangle PRQ$, and these two triangles $\triangle OPQ$ and $\triangle PRQ$ are congruent, hence have the same area. Thus the signed area of the parallelogram $OPRQ$ is twice the signed area of $\triangle OPQ$. In other words:

Proposition 1.6.1. *The* 2×2 *determinant*

$$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

is the signed area of the parallelogram $\mathcal{O}PRQ$*, where*

$$\overrightarrow{\mathcal{O}P} = x_1\vec{\imath} + y_1\vec{\jmath}, \quad \overrightarrow{\mathcal{O}Q} = x_2\vec{\imath} + y_2\vec{\jmath}, \text{ and } \overrightarrow{\mathcal{O}R} = \overrightarrow{\mathcal{O}P} + \overrightarrow{\mathcal{O}Q}.$$

Let us note several properties of the determinant $\Delta(\vec{v}, \vec{w})$ which make it a useful computational tool. The proof of each of these properties is a straightforward calculation (Exercise 6):

Proposition 1.6.2. *The* 2×2 *determinant* $\Delta(\vec{v}, \vec{w})$ *has the following algebraic properties:*

(1) *It is* ***additive*** *in each slot:*[19] *for any three vectors* $\vec{v_1}, \vec{v_2}, \vec{w} \in \mathbb{R}^2$

$$\Delta(\vec{v_1} + \vec{w}, \vec{v_2}) = \Delta(\vec{v_1}, \vec{v_2}) + \Delta(\vec{w}, \vec{v_2})$$
$$\Delta(\vec{v_1}, \vec{v_2} + \vec{w}) = \Delta(\vec{v_1}, \vec{v_2}) + \Delta(\vec{v_1}, \vec{w}).$$

(2) *It is* ***homogeneous*** *in each slot: for any two vectors* $\vec{v_1}, \vec{v_2} \in \mathbb{R}^2$ *and any scalar* $r \in \mathbb{R}$

$$\Delta(r\vec{v_1}, \vec{v_2}) = r\Delta(\vec{v_1}, \vec{v_2}) = \Delta(\vec{v_1}, r\vec{v_2}).$$

(3) *It is* ***skew-symmetric****: for any two vectors* $\vec{v_1}, \vec{v_2} \in \mathbb{R}^2$

$$\Delta(\vec{v_2}, \vec{v_1}) = -\Delta(\vec{v_1}, \vec{v_2}).$$

In particular, recall from § 1.2 that two vectors are **linearly dependent** if they are parallel or one of them is the zero vector. This property is detected by determinants:

Corollary 1.6.3. *A* 2×2 *determinant equals zero precisely if its rows are linearly dependent.*

Proof. If two vectors are linearly dependent, we can write both of them as scalar multiples of the same vector, and

$$\Delta(r\vec{v}, s\vec{v}) = rs\Delta(\vec{v}, \vec{v}) = \Delta(s\vec{v}, r\vec{v}) = -\Delta(r\vec{v}, s\vec{v}),$$

where the last equality comes from skew-symmetry. So $\Delta(r\vec{v}, s\vec{v})$ equals its negative, and hence must equal zero.

To prove the reverse implication, write $\vec{v_i} = x_i\vec{\imath} + y_i\vec{\jmath}$, $i = 1, 2$, and suppose $\Delta(\vec{v_1}, \vec{v_2}) = 0$. This translates to $x_1y_2 - x_2y_1 = 0$ or

$$x_1y_2 = x_2y_1.$$

Assuming that $\vec{v_1}$ and $\vec{v_2}$ are both not vertical ($x_i \neq 0$ for $i = 1, 2$), we can conclude that

$$\frac{y_2}{x_2} = \frac{y_1}{x_1}$$

which means they are dependent. We leave it to you to show that if one of them *is* vertical (and the determinant is zero), then either the other is also vertical, or else one of them is the zero vector. □

Of course, Corollary 1.6.3 can also be proved on geometric grounds, using Proposition 1.6.1 (Exercise 7).

[19]This is a kind of distributive law.

Oriented Areas in Space. Suppose now that A, B, and C are three noncollinear points in $\mathbb{R}^3$. We can think of the ordered triple of *points* (A, B, C) as defining an oriented triangle, and hence associate to it a "signed" area. But which sign should it have—positive or negative? The question is ill posed, since the words "clockwise" and "counterclockwise" have no natural meaning in space: even when A, B, and C all lie in the xy-plane, and have positive orientation in terms of the previous subsection, the motion from A to B to C will look counterclockwise only when viewed from *above* the plane; viewed from *underneath*, it will look *clockwise*. When the plane containing A, B, and C is at some cockeyed angle, it is not at all clear which viewpoint is correct.

We deal with this by turning the tables:[20] the motion, instead of being inherently "clockwise" or "counterclockwise," picks out a side of the plane—namely, the one from which the motion appears counterclockwise. We can think of this as replacing the *sign* $\sigma(A, B, C)$ with a *unit vector* $\vec{\sigma}(A, B, C)$, normal to the plane containing the three points and pointing toward the side of this plane from which the motion described by our order appears counterclockwise. One way to determine which of the two unit normals is correct is the **right-hand rule**: point the fingers of your right hand along the direction of motion; then your (right) thumb will point in the appropriate direction. In Figure 1.34 we sketch the triangle with vertices $A(2, -3, 4)$, $B(4, -2, 5)$, and $C(3, -1, 3)$;

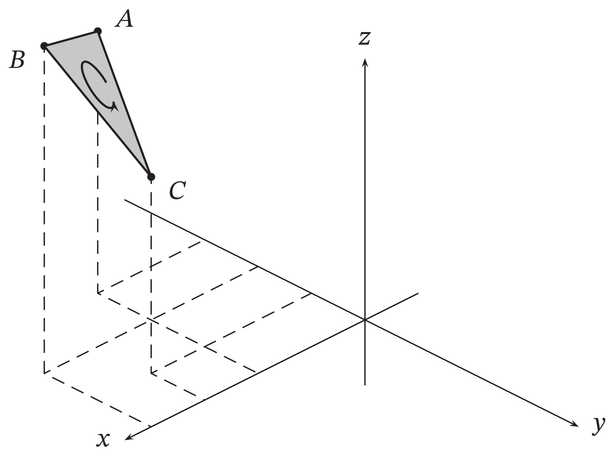


Figure 1.34. Oriented Triangle in $\mathbb{R}^3$

from our point of view (we are looking from moderately high in the first octant), the orientation appears counterclockwise.

By interpreting $\sigma(A, B, C)$ as a unit normal vector, we associate to an oriented triangle $\triangle ABC \in \mathbb{R}^3$ an **oriented area**

$$\vec{\mathcal{A}}(\triangle ABC) = \vec{\sigma}(A, B, C)\,\mathcal{A}(\triangle ABC)$$

represented by a vector normal to the triangle whose length is the ordinary area of $\triangle ABC$. Note that for a triangle in the xy-plane, this means $\vec{\sigma}(ABC) = \sigma(ABC)\,\vec{k}$: the oriented area is the vector $\vec{k}$ times the signed area in our old sense. This interpretation can be applied as well to any oriented polygon contained in a plane in space.

[20]No pun intended! :-)

In particular, by analogy with Proposition 1.6.1, we can define a function which assigns to a pair of vectors $\vec{v}, \vec{w} \in \mathbb{R}^3$ a new vector representing the oriented area of the parallelogram with two of its edges emanating from the origin along $\vec{v}$ and $\vec{w}$, and oriented in the direction of the first vector. This is called the **cross product**[21] of $\vec{v}$ and $\vec{w}$, and is denoted $\vec{v} \times \vec{w}$.

For example, the sides emanating from A in $\triangle ABC$ in Figure 1.34 are represented by $\vec{v} = \overrightarrow{AB} = 2\vec{\imath} + \vec{\jmath} + \vec{k}$ and $\vec{w} = \overrightarrow{AC} = \vec{\imath} + 2\vec{\jmath} - \vec{k}$: these vectors, along with the direction of $\vec{v} \times \vec{w}$, are shown in Figure 1.35.

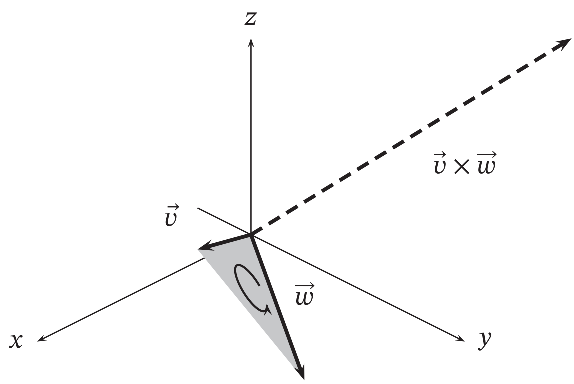


Figure 1.35. Direction of Cross Product

We stress that the cross product differs from the dot product in two essential ways: first, $\vec{v} \cdot \vec{w}$ is a *scalar*, but $\vec{v} \times \vec{w}$ is a *vector*; and second, the dot product is commutative ($\vec{w} \cdot \vec{v} = \vec{v} \cdot \vec{w}$), but the cross product is **anticommutative** ($\vec{w} \times \vec{v} = -\vec{v} \times \vec{w}$).

How do we calculate the components of the cross product $\vec{v} \times \vec{w}$ from the components of $\vec{v}$ and $\vec{w}$? To this end, we detour slightly and consider the projection of areas.

Projections in $\mathbb{R}^3$.

The (orthogonal) **projection** of points in $\mathbb{R}^3$ to a plane $\mathcal{P}'$ takes a point $P \in \mathbb{R}^3$ to the intersection with $\mathcal{P}'$ of the line through P perpendicular to $\mathcal{P}'$ (Figure 1.36). We denote this by

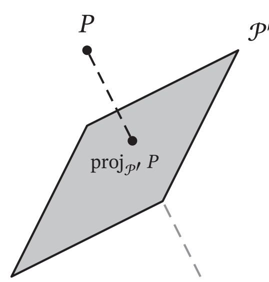


Figure 1.36. Projection of a Point P on the Plane $\mathcal{P}'$

$$P' = \mathrm{proj}_{\mathcal{P}'} P.$$

[21] Also *vector product*, or *outer product*.

Similarly, a vector $\vec{v}$ is projected onto the direction of the line where $\mathcal{P}'$ meets the plane containing both $\vec{v}$ and the normal to $\mathcal{P}'$ (Figure 1.37).

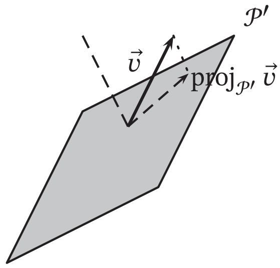


Figure 1.37. Projection of a Vector $\vec{v}$ on the Plane $\mathcal{P}'$

Suppose $\triangle ABC$ is an oriented triangle in $\mathbb{R}^3$. Its projection to $\mathcal{P}'$ is the oriented triangle $\triangle A'B'C'$, with vertices $A' = \text{proj}_{\mathcal{P}'} A$, $B' = \text{proj}_{\mathcal{P}'} B$, and $C' = \text{proj}_{\mathcal{P}'} C$. What is the relation between the oriented areas of these two triangles?

Let $\mathcal{P}$ be the plane containing $\triangle ABC$ and let $\vec{n}$ be the unit vector (normal to $\mathcal{P}$) such that $\vec{\mathcal{A}}(\triangle ABC) = \mathcal{A}\vec{n}$ where $\mathcal{A}$ is the area of $\triangle ABC$. If the two planes $\mathcal{P}$ and $\mathcal{P}'$ are parallel, then $\triangle A'B'C'$ is a parallel translate of $\triangle ABC$, and the two oriented areas are the same. Suppose the two planes are not parallel, but meet at (acute) angle θ along a line ℓ (Figure 1.38).

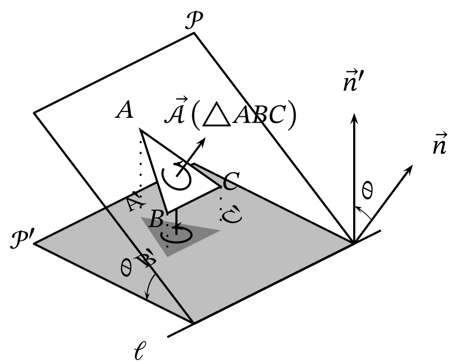


Figure 1.38. Projection of a Triangle

Then a vector $\overrightarrow{v_\ell}$ parallel to ℓ (and hence to both $\mathcal{P}$ and $\mathcal{P}'$) is unchanged by projection, while a vector $\overrightarrow{v_\perp}$ parallel to $\mathcal{P}$ but *perpendicular* to ℓ projects to a vector $\text{proj}_{\mathcal{P}'} \overrightarrow{v_\perp}$ parallel to $\mathcal{P}'$, also perpendicular to ℓ, with length

$$\| \text{proj}_{\mathcal{P}'} \overrightarrow{v_\perp} \| = \| \overrightarrow{v_\perp} \| \cos\theta.$$

The angle between these vectors is the same as between $\vec{n}$ and a unit vector $\vec{n}'$ normal to $\mathcal{P}'$; the oriented triangle $\triangle A'B'C'$ is traversed counterclockwise when viewed from the side of $\mathcal{P}'$ determined by $\vec{n}'$.

Furthermore, if $\triangle ABC$ has one side parallel to ℓ and another perpendicular to ℓ, then the same is true of $\triangle A'B'C'$; the sides parallel to ℓ have the same length, while projection scales the side perpendicular to ℓ—and hence the area—by a factor of $\cos\theta$.

Since every triangle in $\mathcal{P}$ can be subdivided (using lines through the vertices parallel and perpendicular to ℓ) into triangles of this type, the area of *any* triangle $\triangle ABC$ is multiplied by $\cos\theta$ under projection. This means

$$\vec{\mathcal{A}}\left(\triangle A'B'C'\right) = (\mathcal{A}\cos\theta)\vec{n}'$$

which is easily seen to be the projection of $\vec{\mathcal{A}}\left(\triangle ABC\right)$ onto the direction normal to the plane $\mathcal{P}'$. We have shown

Proposition 1.6.4. *For any oriented triangle $\triangle ABC$ and any plane $\mathcal{P}'$ in $\mathbb{R}^3$, the oriented area of the projection $\triangle A'B'C'$ of $\triangle ABC$ onto $\mathcal{P}'$ (as a triangle) is the projection of the oriented area $\vec{\mathcal{A}}\left(\triangle ABC\right)$ (as a vector) onto the direction normal to $\mathcal{P}'$. That is,*

$$\vec{\mathcal{A}}\left(\triangle A'B'C'\right) = \operatorname{proj}_{\vec{n}'} \vec{\mathcal{A}}\left(\triangle ABC\right).$$

Note in particular that when $\triangle ABC$ is *parallel* to $\mathcal{P}'$, its oriented area is *unchanged*, while if $\triangle ABC$ is *perpendicular* to $\mathcal{P}'$, its projection is a degenerate triangle with *zero* area.

As an example, let us consider the projections onto the coordinate planes of the triangle with vertices $A(2,-3,4)$, $B(4,-2,5)$, and $C(3,-1,3)$, which is the triangle we sketched in Figure 1.34. We reproduce this in Figure 1.39, showing the projections of $\triangle ABC$ on each of the coordinate axes.

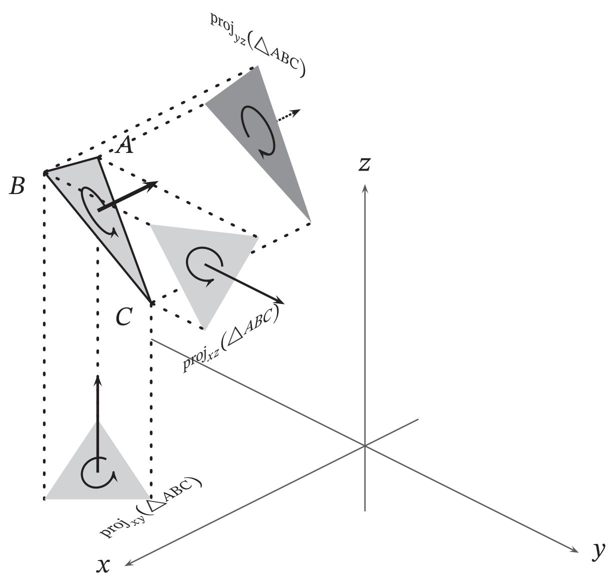


Figure 1.39. Projections of $\triangle ABC$

The projection onto the xy-plane has vertices $A(2,-3)$, $B(4,-2)$, and $C(3,-1)$, which is the triangle we sketched in Figure 1.31. This has signed area 3/2, so its oriented area is $\frac{3}{2}\vec{k}$—that is, the area is 3/2 and the orientation is counterclockwise when seen from *above* the xy-plane.

The projection onto the yz-plane has vertices $A(-3,4)$, $B(-2,5)$, and $C(-1,3)$ and we saw that its signed area is $-1/2$. If we look at the yz-plane from the direction of the positive x-axis, then we see a "clockwise" triangle, so the oriented area is $-\frac{1}{2}\vec{\imath}$—it points in the direction of the *negative* x-axis.

Finally, the projection onto the xz-plane has vertices $A(2,4)$, $B(4,5)$, and $C(3,3)$. You can verify that its signed area, calculated via Equation (1.27), is $-3/2$. Note, however, that if we look at our triangle from the direction of the positive y-axis, we see a *counterclockwise* triangle. Why the discrepancy? The reason for this becomes clear if we take into account not just the triangle, but also the *axes*: in Figure 1.39 the positive z-axis, seen from the positive y-axis, points "north," but the positive x-axis points "west," so the orientation of the x-axis and z-axis (in that order) looks *counterclockwise* only if we look from the direction of the *negative* y-axis. From *this* point of view—that is, the direction of $-\vec{\jmath}$ (which is the one we used to calculate the signed area)—the triangle looks negatively oriented, so the oriented area should be $\left(-\frac{3}{2}\right)(-\vec{\jmath}) = \frac{3}{2}\vec{\jmath}$. This agrees with the geometric observation based on Figure 1.35.

We have seen that the projections of the oriented area vector $\vec{\mathcal{A}}(\triangle ABC)$ onto the three axes are

$$\begin{aligned}\operatorname{proj}_{\vec{k}}\vec{\mathcal{A}}(\triangle ABC) &= \frac{3}{2}\vec{k}\\ \operatorname{proj}_{\vec{\imath}}\vec{\mathcal{A}}(\triangle ABC) &= -\frac{1}{2}\vec{\imath}\\ \operatorname{proj}_{\vec{\jmath}}\vec{\mathcal{A}}(\triangle ABC) &= \frac{3}{2}\vec{\jmath}.\end{aligned}$$

But these projections are simply the components of the vector, so we conclude that the oriented area $\vec{\mathcal{A}}(\triangle ABC)$ is

$$\vec{\mathcal{A}}(\triangle ABC) = -\frac{1}{2}\vec{\imath} + \frac{3}{2}\vec{\jmath} + \frac{3}{2}\vec{k}.$$

Looked at differently, the two sides of $\triangle ABC$ emanating from vertex A are represented by the vectors

$$\begin{aligned}\vec{v} &= \overrightarrow{AB} = 2\vec{\imath} + \vec{\jmath} + \vec{k}\\ \vec{w} &= \overrightarrow{AC} = \vec{\imath} + 2\vec{\jmath} - \vec{k}\end{aligned}$$

and by definition their cross product (p. 45) is the oriented area of the (oriented) parallelogram with sides parallel to $\vec{v}$ and $\vec{w}$, or twice the oriented area of the triangle $\triangle ABC$:

$$\begin{aligned}\vec{v} \times \vec{w} &= 2\vec{\mathcal{A}}(\triangle ABC)\\ &= \vec{\imath}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} - \vec{\jmath}\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}.\end{aligned}$$

Cross-products and determinants. The reasoning used in this example leads to the following general formula for the cross product of two vectors in $\mathbb{R}^3$ from their components.

Theorem 1.6.5. *The cross product of two vectors*

$$\begin{aligned}\vec{v} &= x_1\vec{\imath} + y_1\vec{\jmath} + z_1\vec{k}\\ \vec{w} &= x_2\vec{\imath} + y_2\vec{\jmath} + z_2\vec{k}\end{aligned}$$

is given by

$$\vec{v} \times \vec{w} = \vec{\imath}\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} - \vec{\jmath}\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} + \vec{k}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.$$

Proof. Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be the points in $\mathbb{R}^3$ with position vectors $\vec{v}$ and $\vec{w}$, respectively. Then

$$\vec{v} \times \vec{w} = 2\vec{\mathcal{A}}(\triangle \mathcal{O}PQ) = a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}.$$

The three components of $\vec{\mathcal{A}}(\triangle \mathcal{O}PQ)$ are its projections onto the three coordinate directions, and hence by Proposition 1.6.4 each represents the oriented area of the projection $\mathrm{proj}_{\mathcal{P}} \triangle \mathcal{O}PQ$ of $\triangle \mathcal{O}PQ$ onto the plane $\mathcal{P}$ *perpendicular* to the corresponding vector.

Projection onto the plane perpendicular to a coordinate direction consists of taking the other two coordinates. For example, the direction of $\vec{k}$ is normal to the xy-plane, and the projection onto the xy-plane takes $P(x_1, y_1, z_1)$ onto $P(x_1, y_1)$.

Thus, the determinant

$$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

represents twice the signed area of $\triangle \mathcal{O}P_3Q_3$, the projection of $\triangle \mathcal{O}PQ$ onto the xy-plane, when viewed from above—that is, from the direction of $\vec{k}$—so the oriented area is given by

$$a_3\vec{k} = 2\vec{\mathcal{A}}(\triangle \mathcal{O}P_3Q_3) = \vec{k}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.$$

Similarly,

$$a_1\vec{\imath} = 2\vec{\mathcal{A}}(\triangle \mathcal{O}P_1Q_1) = \vec{\imath}\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}.$$

Finally, noting that the direction from which the *positive* z-axis is *counterclockwise* from the positive x-axis is $-\vec{\jmath}$, we have

$$a_2\vec{\jmath} = 2\vec{\mathcal{A}}(\triangle \mathcal{O}P_2Q_2) = -\vec{\jmath}\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}.$$

Adding these yields the desired formula. □

In each projection, we used the 2×2 determinant obtained by omitting the coordinate along whose axis we were projecting. The resulting formula can be summarized in terms of the array of coordinates of $\vec{v}$ and $\vec{w}$

$$\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix}$$

by saying: the coefficient of the standard basis vector in a given coordinate direction is the 2×2 determinant obtained by eliminating the corresponding column from the above array, and multiplying by -1 for the second column.

We can make this even more "visual" by defining 3×3 determinants.

A 3×3 **matrix** [22] is an array consisting of three rows of three entries each, vertically aligned in three columns. It is sometimes convenient to label the entries of an

[22]Pronounced "3 by 3 matrix"

abstract 3×3 matrix using a single letter with a double index: the entry in the i^{th} *row* and j^{th} *column* of a matrix A is denoted [23] a_{ij}, giving the general form for a 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

We define the **determinant** of a 3×3 matrix as follows: for each entry a_{1j} in the first row, its **minor of a matrix** is the 2×2 matrix A_{1j} obtained by deleting the row and column containing our entry. Thus

$$A_{11} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & a_{22} & a_{23} \\ \cdot & a_{32} & a_{33} \end{pmatrix}$$

$$A_{12} = \begin{pmatrix} \cdot & \cdot & \cdot \\ a_{21} & \cdot & a_{23} \\ a_{31} & \cdot & a_{33} \end{pmatrix}$$

$$A_{13} = \begin{pmatrix} \cdot & \cdot & \cdot \\ a_{21} & a_{22} & \cdot \\ a_{31} & a_{32} & \cdot \end{pmatrix}.$$

Now, the 3×3 determinant of A can be expressed as the *alternating* sum of the *entries* of the first row times the *determinants of their minors*:

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13} \\ &= \sum_{j=1}^{3} (-1)^{1+j} a_{1j} \det A_{1j}. \end{aligned}$$

For future reference, the numbers multiplying the first-row entries in the formula above are called the **cofactors** of these entries: the cofactor of a_{1j} is

$$\text{cofactor}(1j) \coloneqq (-1)^{1+j} \det A_{1j}.$$

We shall see later that this formula usefully generalizes in several ways. For now, though, we see that once we have mastered this formula, we can express the calculation of the cross product of $\vec{v} = v_1\vec{\imath} + v_2\vec{\jmath} + v_3\vec{k}$ with $\vec{w} = w_1\vec{\imath} + w_2\vec{\jmath} + w_3\vec{k}$ as

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \tag{1.28}$$

Exercises for § 1.6

Answers to Exercises 1a, 2a, and 3a are given in Appendix A.13.

Practice problems:

(1) Calculate each determinant below:

(a) $\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix}$ (b) $\begin{vmatrix} -1 & 2 \\ 3 & -4 \end{vmatrix}$ (c) $\begin{vmatrix} -1 & 2 \\ 4 & -8 \end{vmatrix}$

[23] Note that the *row* index precedes the *column* index: a_{ji} is in the j^{th} *row* and i^{th} *column*, a very different place in the matrix.

(2) Sketch the triangle $\triangle ABC$ and indicate its orientation; find $\sigma(A, B, C)\mathcal{A}(\triangle ABC)$:
  (a) $A(0,0)$, $B(2,1)$, $C(1,2)$
  (b) $A(1,2)$, $B(2,0)$, $C(3,3)$
  (c) $A(2,1)$, $B(1,3)$, $C(3,2)$

(3) Calculate $\vec{v} \times \vec{w}$:
  (a) $\vec{v} = (1,2,3)$, $\vec{w} = (3,1,2)$
  (b) $\vec{v} = (3,1,2)$, $\vec{w} = (6,5,4)$
  (c) $\vec{v} = \vec{\imath}$, $\vec{w} = \vec{\jmath}$
  (d) $\vec{v} = \vec{\imath}$, $\vec{w} = \vec{k}$
  (e) $\vec{v} = 4\vec{\imath} - 3\vec{\jmath} + 7\vec{k}$, $\vec{w} = -2\vec{\imath} - 5\vec{\jmath} + 4\vec{k}$

(4) Find the oriented area vector $\vec{\mathcal{A}}(\triangle ABC)$ and calculate the area of the triangle:
  (a) $A = (0,0,0)$, $B = (1,2,3)$, $C = (3,2,1)$
  (b) $A = (1,3,2)$, $B = (2,3,1)$, $C = (3,3,2)$
  (c) $A = (2,-1,-4)$, $B = (-1,1,0)$, $C = (3,-3,-2)$

Theory problems:

(5) Suppose that in $\triangle ABC$ the vector from B to A is $\vec{v}$ and that from B to C is $\vec{w}$. Use the vector formula for the distance from A to BC on p. 28 to prove that the area of the triangle is given by

$$\mathcal{A}(\triangle ABC) = \frac{1}{2}\sqrt{(\vec{w}\cdot\vec{w})(\vec{v}\cdot\vec{v}) - (\vec{v}\cdot\vec{w})^2}.$$

(6) Prove Proposition 1.6.2.

(7) Use Proposition 1.6.1 to prove Corollary 1.6.3. (*Hint:* If the rows are linearly dependent, what does this say about the parallelogram $\mathcal{OPRQ}$?)

(8) Show that the cross product is:
  (a) **skew-symmetric**: $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$
  (b) **additive in each slot:** $(\vec{v_1} + \vec{v_2}) \times \vec{w} = (\vec{v_1} \times \vec{w}) + (\vec{v_2} \times \vec{w})$ (use skew-symmetry to take care of the other slot: this is a kind of distributive law)
  (c) **homogeneous in each slot:** $(a\vec{v}) \times \vec{w} = a(\vec{v} \times \vec{w}) = \vec{v} \times (a\vec{w})$
  (d) Conclude that the cross product is **bilinear**: $(a_1\vec{w_1} + a_2\vec{w_2}) \times \vec{v} = a_1(\vec{w_1} \times \vec{v}) + a_2(\vec{w_2} \times \vec{v})$ and, analogously $\vec{v} \times (a_1\vec{w_1} + a_2\vec{w_2}) = a_1(\vec{v} \times \vec{w_1}) + a_2(\vec{v} \times \vec{w_2})$.

(9) (a) Prove the following cross-product formulas, using the determinant formula Equation (1.28) on p. 50:

$$\vec{\imath} \times \vec{\imath} = \vec{\jmath} \times \vec{\jmath} = \vec{k} \times \vec{k} = \vec{0}$$

$$\begin{aligned} \vec{\imath} \times \vec{\jmath} &= \vec{k} & \qquad \vec{\jmath} \times \vec{\imath} &= -\vec{k} \\ \vec{k} \times \vec{\imath} &= \vec{\jmath} & \qquad \vec{\imath} \times \vec{k} &= -\vec{\jmath} \\ \vec{\jmath} \times \vec{k} &= \vec{\imath} & \qquad \vec{k} \times \vec{\jmath} &= -\vec{\imath} \end{aligned}$$

  (b) Here is a way to remember these formulas:
  - The cross product of any vector with itself is the zero vector.
  - Label the vertices of a planar triangle with $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$, with positive orientation (see Figure 1.40). Then the cross product of two of these vertices is, up to sign, the third one; the sign is the same as the orientation of the triangle consisting of the first and second factors followed by the third vertex.

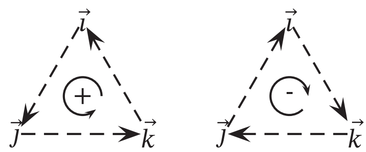


Figure 1.40. Sign of cross products.

(10) (a) Suppose A, B, and C lie on the line ℓ in $\mathbb{R}^2$, and that ℓ does not go through the origin.
Explain why, if B is between A and C, the areas satisfy

$$\mathcal{A}\left(\triangle\mathcal{O}AB\right) + \mathcal{A}\left(\triangle\mathcal{O}BC\right) - \mathcal{A}\left(\triangle\mathcal{O}AC\right) = 0.$$

(b) Show that the above is not true if B is *not* between A and C.

(c) Show that the signed areas satisfy

$$\sigma(\mathcal{O}, A, B)\mathcal{A}\left(\triangle\mathcal{O}AB\right) + \sigma(\mathcal{O}, B, C)\mathcal{A}\left(\triangle\mathcal{O}BC\right) + \sigma(\mathcal{O}, C, A)\mathcal{A}\left(\triangle\mathcal{O}CA\right) = 0$$

regardless of the order of A, B, and C along the line.

(11) Show that the oriented area of a triangle can also be calculated as half of the cross product of the vectors obtained by moving along two successive edges:

$$\vec{\mathcal{A}}\left(\triangle ABC\right) = \frac{1}{2}\overrightarrow{AB} \times \overrightarrow{BC}$$

(*Hint:* You may use Exercise 8.)

Challenge Problems:

Given a point D in the plane, and a directed line segment $\overrightarrow{AB}$, we can define the **area swept out** by the line DP as P moves from A to B along $\overrightarrow{AB}$ to be the signed area of the oriented triangle $[D, A, B]$. We can then extend this definition to the area swept out by DP as P moves along any broken-line path (*i.e.*, a path consisting of finitely many directed line segments) to be the sum of the areas swept out over each of the segments making up the path.

(12) (a) *Show* that the area swept out by DP as P travels along an oriented triangle equals the signed area of the triangle: that is, show that

$$\sigma(ABC)\mathcal{A}\left(\triangle ABC\right) = \sigma(DAB)\mathcal{A}\left(\triangle DAB\right) + \sigma(DBC)\mathcal{A}\left(\triangle DBC\right) + \sigma(DCA)\mathcal{A}\left(\triangle DCA\right).$$

(*Hint:* This can be done geometrically. Consider three cases: D lies outside, inside, or on $\triangle ABC$. See Figure 1.41.)

(b) *Show* that the area swept out by $\mathcal{O}P$ as P moves along the line segment from (x_0, y_0) to (x_1, y_1) is

$$\frac{1}{2}\begin{vmatrix} x_0 & y_0 \\ x_1 & y_1 \end{vmatrix}.$$

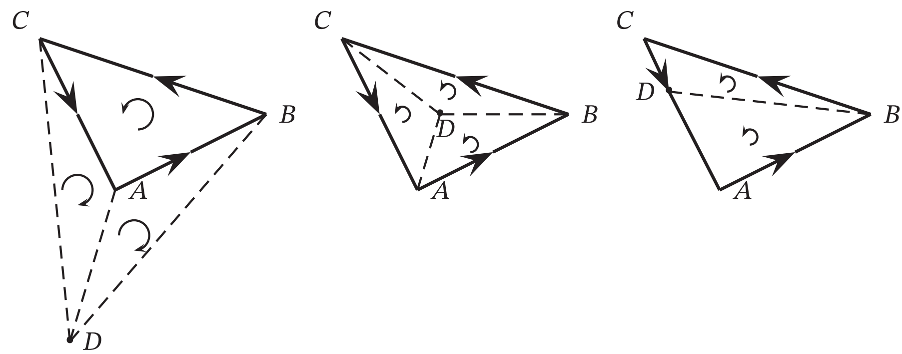


Figure 1.41. Area Swept Out by DP as P Traverses a Triangle

(c) *Show* that if $\vec{v_i} = (x_i, y_i)$, $i = 0, \ldots, 3$ with $\vec{v_0} = \vec{v_3}$ then the signed area of $[\vec{v_1}, \vec{v_2}, \vec{v_3}]$ can be calculated as

$$\sigma\left(\vec{v_1}\vec{v_2}\vec{v_3}\right)\mathcal{A}\left(\triangle\vec{v_1}\vec{v_2}\vec{v_3}\right) = \frac{1}{2}\sum_{i=1}^{3}\begin{vmatrix} x_{i-1} & y_{i-1} \\ x_i & y_i \end{vmatrix}.$$

(13) (a) Consider the three quadrilaterals in Figure 1.42. In all three cases, the orien-

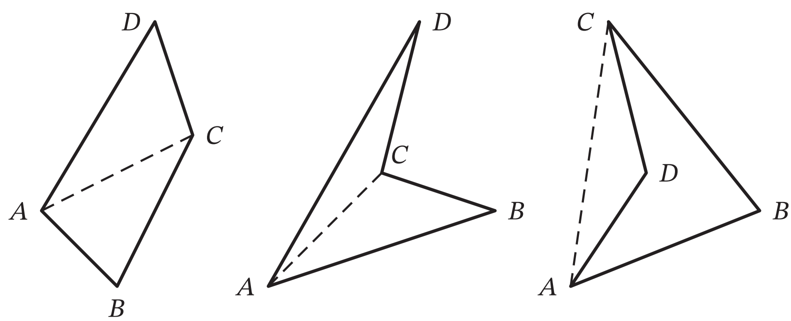


Figure 1.42. Signed Area of Quadrangles

tation of $\square[ABCD]$ and of $\triangle ABC$ is positive, but the orientation of $\triangle ACD$ is not necessarily positive. Show that in all three cases,

$$\mathcal{A}(\square ABCD) = \sigma(ABC)\mathcal{A}\left(\triangle ABC\right) + \sigma(ACD)\mathcal{A}\left(\triangle ACD\right).$$

(b) Use this to show that the signed area of a quadrilateral $\square[ABCD]$ with vertices $A(x_0, y_0)$, $B(x_1, y_1)$, $C(x_2, y_2)$, and $D(x_3, y_3)$ is given by

$$\sigma(ABCD)\mathcal{A}(\square[ABCD]) = \frac{1}{2}\{(x_2 - x_0)(y_3 - y_1) + (x_1 - x_3)(y_2 - y_0)\}$$

Note that this is the same as $\frac{1}{2}\Delta\left(\vec{v}, \vec{w}\right)$ where $\vec{v} = \overrightarrow{AC}$ and $\vec{w} = \overrightarrow{DB}$ are the diagonal vectors of the quadrilateral.

(c) What should be the (signed) area of the oriented quadrilateral $\square[ABCD]$ in Figure 1.43?

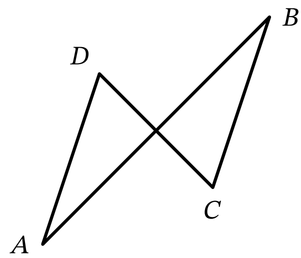


Figure 1.43. Signed Area of Quadrangles (2)

(14) Show that the area swept out by a line DP as P travels along a closed, simple[24] polygonal path equals the signed area of the polygon: that is, suppose the vertices of a polygon in the plane, traversed in counterclockwise order, are $\vec{v_i} = (x_i, y_i)$, $i = 0, ..., n$ with $\vec{v_0} = \vec{v_n}$. Show that the (signed) area of the polygon is

$$\frac{1}{2}\sum_{i=1}^{n}\begin{vmatrix} x_{i-1} & y_{i-1} \\ x_i & y_i \end{vmatrix}.$$

(15) Now extend the definition of the area swept out by a line to space, by replacing *signed area* (in the plane) with *oriented area* in space: that is, given three points $D, A, B \in \mathbb{R}^3$, the **area swept out** by the line DP as P moves from A to B along $\overrightarrow{AB}$ is defined to be the oriented area $\vec{\mathcal{A}}(\triangle DAB)$. *Show* that the oriented area of a triangle $\triangle ABC \subset \mathbb{R}^3$ in space equals the area swept out by the line DP as P traverses the triangle, for any point $D \in \mathbb{R}^3$. (*Hint:* Consider the projections on the coordinate planes, and use Exercise 12.)

History Notes:

Heron's Formulas: Heron of Alexandria (*ca.* 75 AD), in his *Metrica*, gave two formulas for the area of a triangle in terms of the lengths of its sides.

(16) **Heron's First Formula:** The first area formula given by Heron in the *Metrica* is an application of the Law of Cosines, as given in Book II, Propositions 12 and 13 in the *Elements*. Given $\triangle ABC$, we denote the (lengths of the) side opposite each vertex using the corresponding lower case letter (see Figure 1.44).

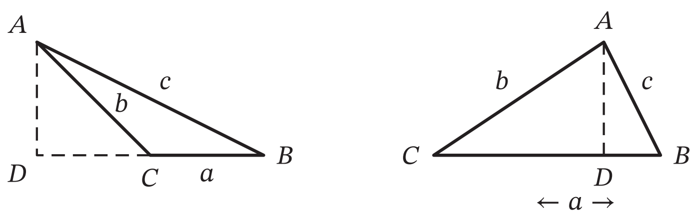


Figure 1.44. Propositions II.12-13: $c^2 = a^2 + b^2 \pm 2a \cdot CD$

(a) **Obtuse Case:** Suppose the angle at C is obtuse. Extend BC to the foot of the perpendicular from A, at D. Prove Euclid's Proposition 11.12:

$$c^2 = a^2 + b^2 + 2a \cdot CD.$$

[24] i.e., the path does not cross itself: this means the path is the boundary of a polygon.

From this, prove Heron's formula[25] in the obtuse case:

$$\mathcal{A}\left(\triangle ABC\right) = \frac{1}{4}\sqrt{2(a^2b^2 + b^2c^2 + a^2c^2) - (a^4 + b^4 + c^4)}. \tag{1.29}$$

(*Hint:* First find CD, then use the standard formula.)

(b) **Acute case:** Suppose the angle at C is acute. Let D be the foot of the perpendicular from A to BC. *Show* that

$$c^2 = a^2 + b^2 - 2a \cdot CD.$$

From this, prove that Equation (1.29) also holds in the acute case.

(17) **Heron's Second Formula:** The second (and more famous) area formula given by Heron is

$$\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)} \tag{1.30}$$

where a, b, and c are the lengths of the sides of the triangle, and s is the *semiperimeter*

$$s = \frac{1}{2}(a+b+c).$$

Equation (1.30) is known as **Heron's formula**, although it now seems clear from Arabic commentaries that it was already known to Archimedes of Syracuse (*ca.* 287-212 BC).

Prove this formula as follows: (refer to Figure 1.45; we follow the exposition in [5, p. 186] and [25, p. 322]):

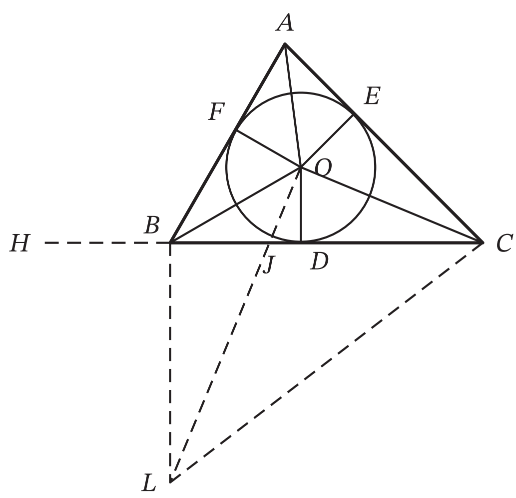


Figure 1.45. Heron's Formula

The original triangle is $\triangle ABC$.

(a) Inscribe a circle inside $\triangle ABC$, touching the sides at D, E, and F. Denote the center of the circle by O; Note that

$$OE = OF = OD.$$

[25]Of course, Heron did not give this complicated algebraic expression. Rather, he outlined the procedure we are using here.

Show that

$$AE = AF, \quad CE = CD, \text{ and } BD = BF.$$

(*Hint: e.g.*, the triangles $\triangle OAF$ and $\triangle OAE$ are similar—why?)

(b) *Show* that the area of $\triangle ABC$ equals $s \cdot OD$. (*Hint:* Consider $\triangle OBC$, $\triangle OAC$ and $\triangle OAB$.)

(c) Extend CB to H, so that $BH = AF$. *Show* that

$$s = CH.$$

(d) Let L be the intersection of the line through O perpendicular to OC with the line through B perpendicular to BC. *Show* that the points O, B, L and C all lie on a common circle. (*Hint:* Each of the triangles $\triangle CBL$ and $\triangle COL$ have right angles opposite their common edge CL, and the hypotenuse of a right triangle is a diameter of a circle containing the right angle.)

(e) It then follows by Proposition III.22 of the *Elements* (opposite angles of a quadrilateral inscribed in a circle sum to two right angles) that $\angle CLB + \angle COB$ equals two right angles.
Show that $\angle BOC + \angle AOF$ equals two right angles. (*Hint:* Each of the lines from O to a vertex of $\triangle ABC$ bisects the angle there.) It follows that

$$\angle CLB = \angle AOF.$$

(f) *Show* from this that $\triangle AOF$ and $\triangle CLB$ are similar.

(g) This leads to the proportions

$$\frac{BC}{BH} = \frac{BC}{AF} = \frac{BL}{OF} = \frac{BL}{OD} = \frac{BJ}{JD}.$$

Add one to both outside fractions to *show* that

$$\frac{CH}{BH} = \frac{BD}{JD}.$$

(h) Use this to show that

$$\frac{(CH)^2}{CH \cdot HB} = \frac{BD \cdot CD}{JD \cdot CD} = \frac{BD \cdot CD}{(OD)^2}.$$

(*Hint:* For the second equality, use the fact that $\triangle COD$ and $\triangle OJD$ are similar.) Conclude that

$$(CH)^2(OD)^2 = CH \cdot HB \cdot BD \cdot CD.$$

(i) Explain how this proves Heron's formula.

1.7 Applications of Cross Products

In this section we explore some useful applications of cross products.

Equation of a Plane. The fact that $\vec{v} \times \vec{w}$ is perpendicular to both $\vec{v}$ and $\vec{w}$ can be used to find a "linear" equation for a plane, given three noncollinear points on it.

Remark 1.7.1. *If $\vec{v}$ and $\vec{w}$ are linearly independent vectors in $\mathbb{R}^3$, then any plane containing a line ℓ_v parallel to $\vec{v}$ and a line ℓ_w parallel to $\vec{w}$ has*

$$\vec{n} = \vec{v} \times \vec{w}$$

as a normal vector.

In particular, given a nondegenerate triangle $\triangle ABC$ in $\mathbb{R}^3$, an equation for the plane $\mathcal{P}$ containing this triangle is

$$\vec{n} \cdot (\vec{p} - \vec{p_0}) = 0, \tag{1.31}$$

where $\vec{p} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$, $\vec{p_0} = \overrightarrow{OA}$, and $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$.

For example, an equation for the plane $\mathcal{P}$ containing $\triangle PQR$ with vertices $P(1, -2, 3)$, $Q(-2, 4, -1)$ and $R(5, 3, 1)$ can be found using $\vec{p_0} = \vec{\imath} - 2\vec{\jmath} + 3\vec{k}$, $\overrightarrow{PQ} = -3\vec{\imath} + 6\vec{\jmath} - 4\vec{k}$, and $\overrightarrow{PR} = 4\vec{\imath} + 5\vec{\jmath} - 2\vec{k}$: then

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -3 & 6 & -4 \\ 4 & 5 & -2 \end{vmatrix} = 8\vec{\imath} - 22\vec{\jmath} - 39\vec{k}$$

so the equation for $\mathcal{P}$ is

$$8(x-1) - 22(y+2) - 39(z-3) = 0$$

or

$$8x - 22y - 39z = -153.$$

As another example, consider the plane $\mathcal{P}'$ parametrized by

$$\begin{array}{rcrrr} x & = & 3 & -2s & +t \\ y & = & -1 & +2s & -2t \\ z & = & & 3s & -t. \end{array}$$

We can read off that $\vec{p_0} = 3\vec{\imath} - \vec{\jmath}$ is the position vector of $\vec{p}\,(0,0)$ (corresponding to $s = 0$, $t = 0$), and two vectors parallel to the plane are $\vec{v_s} = -2\vec{\imath} + 2\vec{\jmath} + 3\vec{k}$ and $\vec{v_t} = \vec{\imath} - 2\vec{\jmath} - \vec{k}$. Thus, a normal vector is

$$\vec{n} = \vec{v_s} \times \vec{v_t} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -2 & 2 & 3 \\ 1 & -2 & -1 \end{vmatrix} = 4\vec{\imath} + \vec{\jmath} + 2\vec{k}$$

and an equation for $\mathcal{P}'$ is

$$4(x-3) + 1(y+1) + 2(z-0) = 0$$

or

$$4x + y + 2z = 11.$$

Intersection of Planes. The line of intersection of two planes can be specified as the set of simultaneous solutions of two linear equations, one for each plane. How do we find a parametrization for this line?

Note that a linear equation $Ax + By + Cz = D$ for a plane immediately gives us a normal vector $\vec{n} = A\vec{\imath} + B\vec{\jmath} + C\vec{k}$. If we are given two such equations

$$\begin{aligned} A_1\vec{\imath} + B_1\vec{\jmath} + C_1\vec{k} &= D_1 \\ A_2\vec{\imath} + B_2\vec{\jmath} + C_2\vec{k} &= D_2, \end{aligned}$$

then the line of intersection ℓ (the locus of this *pair* of equations) is *perpendicular* to *both* normal vectors $\vec{n_i} = A_i\vec{\imath} + B_i\vec{\jmath} + C_i\vec{k} \quad i = 1, 2, 3$ and hence *parallel* to their cross-product

$$\vec{v} = \overrightarrow{n_1} \times \overrightarrow{n_2}.$$

Thus, given any one point $P_0(x_0, y_0, z_0)$ on ℓ (*i.e.*, one solution of the pair of equations) the line ℓ can be parametrized using P_0 as a basepoint and $\vec{v} = \overrightarrow{n_1} \times \overrightarrow{n_2}$ as a direction vector.

For example, consider the two planes

$$\begin{aligned} 3x - 2y + z &= 1 \\ 2x + y - z &= 0. \end{aligned}$$

The first has normal vector $\overrightarrow{n_1} = 3\vec{\imath} - 2\vec{\jmath} + \vec{k}$ while the second has $\overrightarrow{n_2} = 2\vec{\imath} + \vec{\jmath} - \vec{k}$. Thus, a direction vector for the intersection line is

$$\vec{v} = \overrightarrow{n_1} \times \overrightarrow{n_2} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 3 & -2 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \vec{\imath} + 5\vec{\jmath} + 7\vec{k}.$$

One point of intersection can be found by adding the equations to eliminate z

$$5x - y = 1$$

and choosing a value for x, say $x = 1$; this forces $y = 4$ and substituting back into either of the two original equations yields $z = 6$. so we can use $(1, 4, 6)$ as a basepoint; a parametrization of ℓ is

$$\vec{p}(t) = (\vec{\imath} + 4\vec{\jmath} + 6\vec{k}) + t(\vec{\imath} + 5\vec{\jmath} + 7\vec{k})$$

or

$$\begin{aligned} x &= 1 + t \\ y &= 4 + 5t \\ z &= 6 + 7t. \end{aligned}$$

If we try this when the two planes are parallel, we have linearly dependent normals, and their cross product is zero (Exercise 6 in § 1.6). In this case, the two left sides of the equations describing the planes are proportional: if the right sides have the same proportion, then we really have only one equation (the second is the first in disguise) and the two planes are the same, while if the right sides have a *different* proportion, the two equations are mutually contradictory—the planes are parallel, and have no intersection.

For example, the two equations

$$x - 2y + 3z = 1$$
$$-2x + 4y - 6z = -2$$

are equivalent (the second is the first multiplied by -2) and describe a (single) plane, while

$$x - 2y + 3z = 1$$
$$-2x + 4y - 6z = 0$$

are contradictory, and represent two parallel, nonintersecting planes.

Oriented Volumes. **Cylinders:** In common usage, a *cylinder* is the surface formed from two horizontal discs in space, one directly above the other, and of the same radius, by joining their boundaries with vertical line segments. Mathematicians generalize this, replacing the discs with horizontal copies of any plane region, and allowing the two copies to not be directly above one another (so the line segments joining their boundaries, while parallel to each other, need not be perpendicular to the two regions). Another way to say this is to define a (solid) **cylinder** on a given **base** (which is some region in a plane) to be formed by parallel line segments of equal length emanating from all points of the base (Figure 1.46). We will refer to a vector $\vec{v}$ representing these segments as a **generator** for the cylinder.

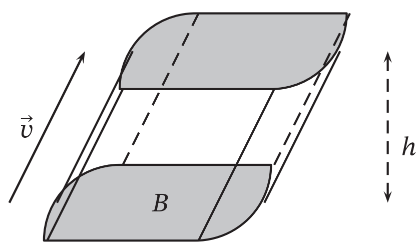


Figure 1.46. Cylinder with base B, generator $\vec{v}$, height h

Using Cavalieri's principle (see *Calculus Deconstructed*, p. 365, or another single-variable calculus text) it is fairly easy to see that the volume of a cylinder is the area of its base times its height (the perpendicular distance between the two planes containing the endpoints of the generating segments). Up to sign, this is given by orienting the base and taking the dot product of the generator with the oriented area of the base

$$V = \pm\vec{v} \cdot \vec{\mathcal{A}}(B).$$

We can think of this dot product as the "signed volume" of the oriented cylinder, where the orientation of the cylinder is given by the direction of the generator together with the orientation of the base. The signed volume is positive (*resp.* negative) if $\vec{v}$ points toward the side of the base from which its orientation appears counterclockwise (*resp.* clockwise)—in other words, the orientation of the cylinder is positive if these data obey the right-hand rule. We will denote the signed volume of a cylinder C by $\vec{V}(C)$.

A cylinder whose base is a parallelogram is called a **parallelepiped**: this has three quartets of parallel **edges**, which in pairs bound three pairs of parallel parallelograms,[26]

[26]This tongue twister was unintentional! :-)

called the **faces**. If the base parallelogram has sides represented by the vectors $\overrightarrow{w_1}$ and $\overrightarrow{w_2}$ and the generator is $\vec{v}$ (Figure 1.47) we denote the parallelepiped by $\square[\vec{v}, \overrightarrow{w_1}, \overrightarrow{w_2}]$.

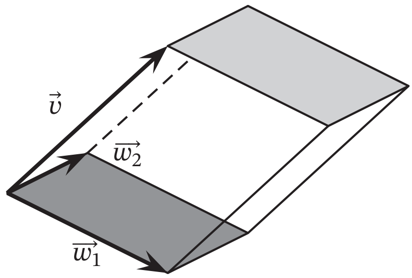


Figure 1.47. Parallelepiped

The oriented area of the base is

$$\vec{\mathcal{A}}(B) = \overrightarrow{w_1} \times \overrightarrow{w_2}$$

so the signed volume is[27]

$$\vec{\mathcal{V}}(\square[\vec{v}, \overrightarrow{w_1}, \overrightarrow{w_2}]) = \vec{v} \cdot \vec{\mathcal{A}}(B) = \vec{v} \cdot (\overrightarrow{w_1} \times \overrightarrow{w_2})$$

(where $\vec{v}$ represents the third edge, or generator).

If the components of the "edge" vectors are

$$\begin{aligned} \vec{v} &= a_{11}\vec{\imath} + a_{12}\vec{\jmath} + a_{13}\vec{k} \\ \overrightarrow{w_1} &= a_{21}\vec{\imath} + a_{22}\vec{\jmath} + a_{23}\vec{k} \\ \overrightarrow{w_2} &= a_{31}\vec{\imath} + a_{32}\vec{\jmath} + a_{33}\vec{k}, \end{aligned}$$

then

$$\begin{aligned} \overrightarrow{w_1} \times \overrightarrow{w_2} &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= \vec{\imath}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - \vec{\jmath}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + \vec{k}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \end{aligned}$$

so

$$\begin{aligned} \vec{v} \cdot (\overrightarrow{w_1} \times \overrightarrow{w_2}) &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \end{aligned}$$

This gives us a geometric interpretation of a 3×3 (numerical) determinant:

[27]The last calculation in this equation is sometimes called the **triple scalar product** of $\vec{v}$, $\overrightarrow{w_1}$ and $\overrightarrow{w_2}$.

Remark 1.7.2. *The* 3×3 *determinant*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

is the signed volume $\vec{V}(\square[\vec{v}, \vec{w_1}, \vec{w_2}])$ *of the oriented parallelepiped* $\square[\vec{v}, \vec{w_1}, \vec{w_2}]$ *whose generator is the first row*

$$\vec{v} = a_{11}\vec{\imath} + a_{12}\vec{\jmath} + a_{13}\vec{k}$$

and whose base is the oriented parallelogram with edges represented by the other two rows

$$\vec{w_1} = a_{21}\vec{\imath} + a_{22}\vec{\jmath} + a_{23}\vec{k}$$
$$\vec{w_2} = a_{31}\vec{\imath} + a_{32}\vec{\jmath} + a_{33}\vec{k}.$$

For example, the parallelepiped with base $\mathcal{O}PRQ$, with vertices the origin, $P(0,1,0)$, $Q(-1,1,0)$, and $R(-1,2,0)$ and generator $\vec{v} = \vec{\imath} - \vec{\jmath} + 2\vec{k}$ (Figure 1.48) has "top" face

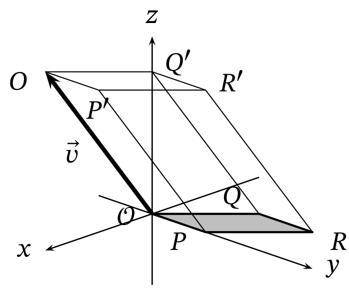


Figure 1.48. $\square\mathcal{O}PRQ$

$OP'R'Q'$, with vertices $O(1,-1,2)$, $P'(1,0,2)$, $Q'(0,0,2)$ and $R'(0,1,2)$. Its signed volume is given by the 3×3 determinant whose rows are $\vec{v}$, $\overrightarrow{\mathcal{O}P}$ and $\overrightarrow{\mathcal{O}Q}$:

$$\begin{aligned} \vec{V}(\square[\mathcal{O}PRQ]) &= \begin{vmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{vmatrix} \\ &= (1)\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} - (-1)(1)\begin{vmatrix} 0 & 0 \\ -1 & 0 \end{vmatrix} + (2)(1)\begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} \\ &= (1)(0) - (-1)(0) + (2)(0+1) \\ &= 2. \end{aligned}$$

We see from Figure 1.48 that the vectors $\overrightarrow{\mathcal{O}P}$, $\overrightarrow{\mathcal{O}Q}$, $\vec{v}$ obey the right-hand rule, so have *positive* orientation.

Simplices: Given any four points A, B, C, and D in $\mathbb{R}^3$, we can form a "pyramid" built on the triangle $\triangle ABC$, with a "peak" at D (Figure 1.49). The traditional name for such a solid is *tetrahedron*, but we will follow the terminology of combinatorial topology, calling this the **simplex** [28] with vertices A, B, C, and D, and denote it

[28] Actually, this is a **3-simplex**. In this terminology, a triangle is a **2-simplex** (it lies in a plane), and a line segment is a **1-simplex** (it lies on a line).

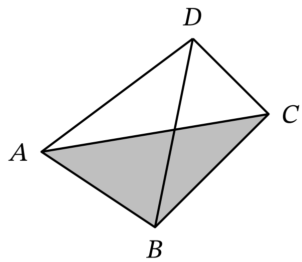


Figure 1.49. Simplex $\triangle ABCD$

$\triangle ABCD$; it is **oriented** when we pay attention to the order of the vertices. Just as for a triangle, the edges emanating from the vertex A are represented by the displacement vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$. The first two vectors determine the oriented "base" triangle $\triangle ABC$, and the simplex $\triangle ABCD$ is positively (*resp.* negatively) oriented if the orientation of $\triangle ABC$ is positive (*resp.* negative) when viewed from D, or equivalently if the dot product $\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})$ is positive (*resp.* negative).

In Exercise 8, we see that the parallelepiped $\square OPRQ$ determined by the three vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$ can be subdivided into six simplices, all congruent to $\triangle ABCD$, and its orientation agrees with that of the simplex. Thus we have

Lemma 1.7.3. *The signed volume of the oriented simplex $\triangle ABCD$ is*

$$\vec{V}(\triangle ABCD) = \frac{1}{6}\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = \frac{1}{6}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

where $\overrightarrow{AB} = a_{21}\vec{\imath} + a_{22}\vec{\jmath} + a_{23}\vec{k}$, $\overrightarrow{AC} = a_{31}\vec{\imath} + a_{32}\vec{\jmath} + a_{33}\vec{k}$, and $\overrightarrow{AD} = a_{11}\vec{\imath} + a_{12}\vec{\jmath} + a_{13}\vec{k}$.

We can use this geometric interpretation (which is analogous to Proposition 1.6.1) to establish several algebraic properties of 3×3 determinants, analogous to those in the 2×2 case which we noted in § 1.6:

Remark 1.7.4. *The* 3×3 *determinant has the following properties:*

(1) *It is **skew-symmetric**: Interchanging two rows of a* 3×3 *determinant reverses its sign (and leaves the absolute value unchanged).*
(2) *It is **homogeneous** in each row: multiplying a single row by a scalar multiplies the determinant by that scalar.*
(3) *It is **additive** in each row: Suppose two matrices (say A and B) agree in two rows (say, the second rows are the same, and the third rows are the same). Then the matrix with the same second and third rows, but with first row equal to the sum of the first rows of A and of B, has determinant* $\det A + \det B$.
(4) *A* 3×3 *determinant **equals zero** precisely if its rows are linearly dependent.*

For the first item, note that interchanging the two edges of the base reverses the sign of its oriented area and hence the sign of its oriented volume; if the first row is interchanged with one of the other two, you should check that this also reverses the orientation. Once we have the first item, we can assume in the second item that we are scaling the first row, and and in the second that A and B agree except in their first row(s). The additivity and homogeneity in this case follows from the fact that the oriented volume equals the oriented area of the base dotted with the first row. Finally, the last item follows from noting that zero determinant implies zero volume, which means the "height" measured off a plane containing the base is zero.

Rotations. So far, the physical quantities we have associated with vectors—forces, velocities—concern *displacements*. In effect, we have been talking about the motion of individual points, or the abstraction of such motion for larger bodies obtained by replacing each body with its center of mass. However, a complete description of the motion of solid bodies also involves *rotation*.

A rotation of 3-space about the z-axis is most easily described in cylindrical coordinates: a point P with cylindrical coordinates (r, θ, z), under a counterclockwise rotation (seen from above the xy-plane) by α radians does not change its r- or z-coordinates, but its θ- coordinate increases by α. Expressing this in rectangular coordinates, we see that the rotation about the z-axis by α radians counterclockwise (when seen from above) moves the point with rectangular coordinates (x, y, z), where

$$x = r\cos\theta, \quad y = r\sin\theta$$

to the point

$$\begin{aligned} x(\alpha) &= r\cos(\theta+\alpha) \\ y(\alpha) &= r\sin(\theta+\alpha) \\ z(\alpha) &= z. \end{aligned}$$

These new rectangular coordinates can be expressed in terms of the old ones, using the angle-summation formulas for sine and cosine, as

$$\begin{array}{rcll} x(\alpha) &=& x\cos\alpha & -y\sin\alpha \\ y(\alpha) &=& x\sin\alpha & +y\cos\alpha \\ z(\alpha) &=& z. & \end{array} \tag{1.32}$$

Under a steady rotation around the z-axis with angular velocity[29] $\dot{\alpha} = \omega\, radians/sec$, the velocity $\vec{v}$ of our point is given by

$$\begin{array}{lllll} \dot{x} &= \left(\left.\frac{dx(\alpha)}{d\alpha}\right|_{\alpha=0}\right)\omega & = (-x\sin 0 - y\cos 0)\omega & = -y\omega \\ \dot{y} &= \left(\left.\frac{dy(\alpha)}{d\alpha}\right|_{\alpha=0}\right)\omega & = (x\cos 0 - y\sin 0)\omega & = x\omega \\ \dot{z} & & = 0 & \end{array}$$

which can also be expressed as

$$\vec{v} = -y\omega\vec{\imath} + x\omega\vec{\jmath} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = \omega\vec{k} \times \vec{p}, \tag{1.33}$$

where $\vec{p} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ is the position vector of P.

When the rotation is about a different axis, the analogue of Equation (1.32) is rather complicated, but Equation (1.33) is relatively easy to carry over, on geometric grounds. Note first that the z coordinate does not affect the velocity in Equation (1.33): we could replace $\vec{p}$, which is the displacement $\overrightarrow{OP}$ from the origin to our point, with the displacement $\overrightarrow{P_0P}$ from *any* point on the z-axis. Second, the vector $\omega\vec{k}$ can be characterized as a vector parallel to our axis of rotation whose length equals the angular velocity, where ω is positive if the rotation is counterclockwise when viewed from above. That is, we

[29] We use a dot over a variable to indicate its time derivative.

can regard the **angular velocity** as a vector $\vec{\omega}$ analogous to an oriented area: its *magnitude* is the angular speed, and its *direction* is normal to the planes invariant under the rotation (*i.e.*, planes perpendicular to the axis of rotation) in the direction from which the rotation is counterclockwise. These considerations easily yield

Remark 1.7.5. *The (spatial) velocity $\vec{v}$ of a point P under a steady rotation (about the axis ℓ) with angular velocity $\vec{\omega}$ is*

$$\vec{v} = \vec{\omega} \times \overrightarrow{P_0P}, \tag{1.34}$$

where P_0 is an arbitrary point on ℓ, the axis of rotation.

Exercises for § 1.7

Answers to Exercises 1a, 2a, and 4a are given in Appendix A.13.

Practice problems:

(1) Find an equation for the plane $\mathcal{P}$ described in each case:
   (a) $\mathcal{P}$ goes through $(1, 2, 3)$ and contains lines parallel to each of the vectors $\vec{v} = (3, 1, 2)$ and $\vec{w} = (1, 0, 2)$.
   (b) $\mathcal{P}$ contains the three points $P(4, 5, 6)$, $Q(3, 2, 7)$, and $R(5, 1, 1)$.
   (c) $\mathcal{P}$ contains $P(3, 1, 4)$ and the line

$$\begin{array}{ccrr} x & = & 1 & +t \\ y & = & -2 & +2t \\ z & = & 3 & -t. \end{array}$$

   (d) $\mathcal{P}$ is parametrized by

$$\begin{array}{ccrrr} x & = & 1 & -2s & +3t \\ y & = & 2 & -s & +t \\ z & = & -2 & +s & +t. \end{array}$$

(2) Give a parametrization of each plane $\mathcal{P}$ described below:
   (a) $\mathcal{P}$ contains the three points $P(3, -1, 2)$, $Q(2, 1, -1)$, and $R(8, 3, 1)$.
   (b) $\mathcal{P}$ contains the lines

$$\ell_1 : \left\{ \begin{array}{lrr} x & = -2 & +t \\ y & = 1 & -2t \\ z & = 4 & +t \end{array} \right. \text{ and } \ell_2 : \left\{ \begin{array}{lrr} x & = -1 & +2t \\ y & = -1 & +t \\ z & = 5 & -3t \end{array} \right. .$$

   (c) $\mathcal{P}$ meets the plane $3x + y + z = 2$ in the line

$$\begin{array}{lrr} x & = 1 & -t \\ y & = 2 & +t \\ z & = -3 & +2t \end{array}$$

   and is perpendicular to it.
   (d) $\mathcal{P}$ is the locus of $2x - 3y + 4z = 3$.

(3) (a) Find a line in the plane $3x + 7y + z = 29$ which is perpendicular to the line

$$\begin{array}{lrr} x & = 1 & -2t \\ y & = 3 & +t \\ z & = 5 & -t. \end{array}$$

(b) Find the line in the plane $x + y + z = 0$ which meets the line ℓ given by

$$\begin{aligned} x &= -5 && +3t \\ y &= 4 && -2t \\ z &= 1 && -t \end{aligned}$$

at the point $(-2, 2, 0)$ and is perpendicular to ℓ.

(4) Parametrize the line described in each case below:

(a) ℓ is the intersection of the planes

$$\begin{aligned} \mathcal{P}_1 &: \quad 5x - 2y + 3z = 0 \\ \mathcal{P}_2 &: \quad 2x + 2y + z = 3. \end{aligned}$$

(b) ℓ is the intersection of the planes parametrized as follows:

$$\mathcal{P}_1 : \begin{cases} x = 1 + 2s + 3t \\ y = 2 - s + t \\ z = 3 + s - 2t \end{cases}$$

$$\mathcal{P}_2 : \begin{cases} x = 1 + s - t \\ y = 2 + 2s + 3t \\ z = 3 - 3s - t \end{cases}$$

(5) Find the volume of each parallelepiped described below:

(a) The origin is a vertex, and the three vertices joined to the origin by an edge are $P(1, -3, 2)$, $Q(2, 3, -1)$, and $R(3, 2, 1)$.

(b) The faces of the parallelepiped lie on the planes $z = 0$, $z = 1$, $z = 2y$, $z = 2y - 1$, $z = x$, and $z = x + 1$.

(6) Determine the orientation and volume of the simplex $\triangle ABCD$ whose vertices are $A(1, -1, 1)$, $B(2, 0, 1)$, $C(2, -2, 1)$, and $D(1, -1, 0)$.

(7) The plane $x + y + z = 3$ is continuously rotated about the line

$$\begin{aligned} x &= t \\ y &= t \\ z &= t. \end{aligned}$$

If the point $P(2, 2, -1)$ has velocity $\vec{v} = \vec{\imath} - \vec{\jmath}$, what is the angular velocity?

Challenge problem:

(8) Consider the "prism" E bounded below by the xy-plane ($z = 0$), above by the plane $z = 1$, and on the sides by the three vertical planes $x = 0$ (the yz-plane), $y = 0$ (the xz-plane), and $x + y = 1$ (see Figure 1.50).

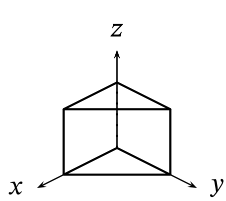


Figure 1.50. The Prism E

(a) Show that E consists of all points in $\mathbb{R}^3$ which simultaneously satisfy the inequalities $x \geq 0$, $y \geq 0$, $x + y \leq 1$, and $0 \leq z \leq 1$.

(b) Show that the six vertices of E are $P_0(0,0,0)$, $P_1(1,0,0)$, $P_2(0,1,0)$, $Q_0(0,0,1)$, $Q_1(1,0,1)$, and $Q_2(0,1,1)$. (Note that in this numbering, Q_i is directly above P_i.)

(c) Now consider the three oriented simplices $\triangle_1 = \triangle P_0P_1P_2Q_0$, $\triangle_2 = \triangle P_1P_2Q_0Q_1$, and $\triangle_3 = \triangle P_2Q_0Q_1Q_2$. Show that: (i) $\triangle_1$ consists of all points in E which also satisfy $x + y + z \leq 1$.
(ii) $\triangle_2$ consists of all points in E which also satisfy $x+y+z \geq 1$ and $y+z \leq 1$.
(iii) $\triangle_3$ consists of all points in E which also satisfy $y + z \geq 1$.

(d) Show that each of the pairs of simplices $\triangle_1$ and $\triangle_2$ (*resp.* $\triangle_2$ and $\triangle_3$) meets along a common face, while $\triangle_1$ and $\triangle_3$ meet only along the line P_2Q_0.

(e) Show that $\triangle_1$ and $\triangle_2$ have equal volume because they share a common face and the vectors $\overrightarrow{P_0Q_0}$ and $\overrightarrow{P_1Q_1}$ are equal. Analogously, $\triangle_2$ and $\triangle_3$ have the same volume, so each of these has volume $\frac{1}{6}$.

2
Curves and Vector-Valued Functions of One Variable

2.1 Conic Sections

We begin this chapter with a glimpse at the way the Greeks thought about curves beyond circles and lines. This will give us some idea of the power of the analytic methods which replaced this approach in modern times and which will occupy us for the remainder of the chapter. It will also provide us with a family of examples of curves to investigate using methods of calculus.

A major source of information about classical Greek mathematics is Pappus of Alexandria (*ca.* 300 AD), a formidable geometer of the late third century AD.[1] In his *Mathematical Collection*[2] he surveyed the work of his predecessors; many of these works have been lost. He classified mathematical problems according to the kinds of loci (curves) required for their solution: *planar* problems, which can be solved using circles and straight lines[3]; *solid* problems, which involve the intersection of a plane with a cone (**solid loci**, or **conic sections**); and *linear* problems, which[4] involve other loci, such as *spirals*, *quadratices*, and *conchoids*.

In this section, we briefly consider Pappus' second class of curves, the conic sections, first from the geometric perspective of Apollonius and Pappus and second as loci of quadratic equations in two variables.

Conics according to Apollonius. Pappus referred to two works on conic sections —one by by Euclid, the other by Aristaeus the Elder (*ca.* 320 BC)—which preceded him by six centuries. These works have been lost,[5] but in any case they were quickly eclipsed by the work of Apollonius of Perga (*ca.* 262-190 BC). His *Conics*, in eight books, was recognized by his contemporaries as the definitive work on the subject.[6] Here, we give a simplified and anachronistic sketch of the basic ideas in Book I, bowdlerizing [26, pp. 355-9].

---

[1]The work of Pappus is sometimes taken to mark the end of the classical Greek tradition in mathematics.

[2]Parts of this survive in a twelfth-century copy.

[3]These are often called **compass and straightedge constructions.**

[4]*Caution*: this is *not* the modern meaning of "linear"!

[5]Pappus refers to the "still surviving" *Solid Loci* of Aristaeus, but the *Conics* of Euclid were apparently already lost by the time of Pappus.

[6]The first four books of Apollonius' *Conics* have survived in a Greek edition with commentaries by Eutocius (*ca.* 520 AD), and the next three survived in an Arabic translation of Eutocius' edition by Thabit ibn Qurra (826-901); the eighth book is lost. A modern translation of Books I-IV is [42]. An extensive detailed and scholarly examination of the *Conics* has recently been published by Fried and Unguru [15].

Start with a horizontal circle $\mathcal{C}$; on the vertical line through the center of $\mathcal{C}$ (the **axis**[7]) pick a point A distinct from the center of $\mathcal{C}$. The union of the lines through A intersecting $\mathcal{C}$ (the **generators**) is a surface $\mathcal{K}$ consisting of two cones joined at their common vertex, A (Figure 2.1). If we put the origin at A, the axis coincides with the

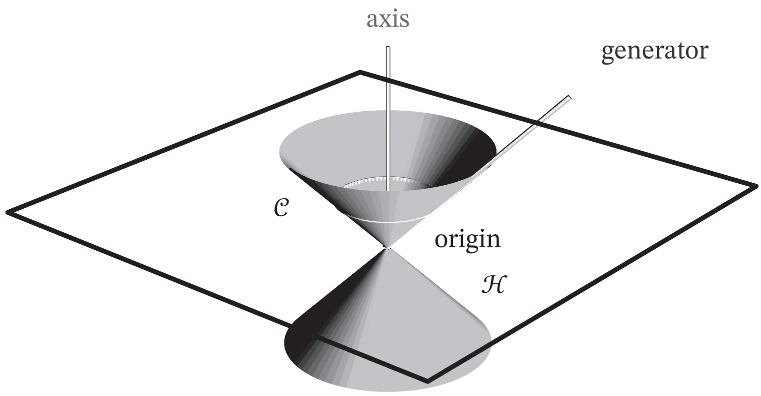


Figure 2.1. Conical Surface $\mathcal{K}$

z-axis, and $\mathcal{K}$ is the locus of the equation in rectangular coordinates

$$z^2 = m^2(x^2 + y^2), \tag{2.1}$$

where $m = \cot\alpha$ is the cotangent of the angle α between the axis and the generators.

Now consider the intersection of a plane with the conical surface $\mathcal{K}$.

If our plane contains the origin A, this intersection is rather uninteresting (Exercise 5). A *horizontal* plane $\mathcal{H}$ *not* containing A intersects $\mathcal{K}$ in a circle centered on the axis. The yz-plane intersects $\mathcal{H}$ in a line which meets this circle at two points, B and C; clearly the segment BC is a diameter of the circle. Given a point Q on this circle

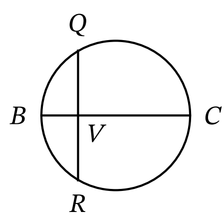


Figure 2.2. *Elements*, Book VI, Prop. 13

distinct from B and C (Figure 2.2), the line through Q parallel to the x-axis intersects the circle in a second point R, and the segment QR is bisected at V, the intersection of QR with the yz-plane. Note that QR is perpendicular to BC.

Apollonius' analysis starts from a basic property of circles, implicit in Prop. 13, Book VI of Euclid's *Elements* [28, vol. 2, p. 216] and equivalent to the equation of a circle in rectangular coordinates [8] (Exercise 6)—namely,

[7]Apollonius allows the axis to be *oblique*—not necessarily normal to the plane of $\mathcal{C}$.

[8]Since BC bisects QR, the product in Equation (2.2) equals $|QV|^2$.

The product of the segments on a chord equals the product of the segments on the diameter perpendicular to it.

In Figure 2.2, this means

$$|QV| \cdot |VR| = |BV| \cdot |VC| . \tag{2.2}$$

To understand the curve obtained by intersecting $\mathcal{K}$ with a *non*-horizontal plane $\mathcal{P}$ not containing the origin A, we can assume that horizontal lines[9] in $\mathcal{P}$ are parallel to the x-axis—that is, its equation has the form $z = My + c$ for some nonzero constants M and c. Let γ be the intersection of $\mathcal{P}$ with $\mathcal{K}$.

The yz-plane intersects $\mathcal{P}$ in a line that meets γ in one or two points; we label the first P and the second (if it exists) P'; these are the **vertices** of γ (Figure 2.3). Given

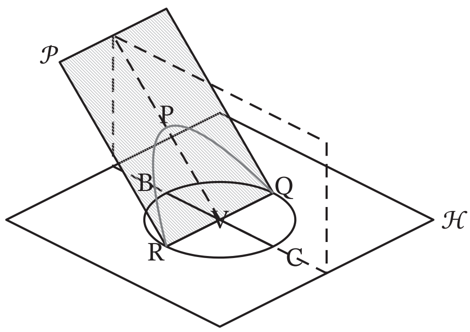


Figure 2.3. Conic Section

a point Q on γ distinct from the vertices, and $\mathcal{H}$ the *horizontal* plane through Q, define the points R, V, B, and C as in Figure 2.2. The line segments QV and PV are, respectively, the **ordinate** and **abcissa**.

There are three possible configurations, depending on the angle between the planes $\mathcal{P}$ and $\mathcal{H}$:

- **Parabolas:** If the generator AC is parallel to the plane $\mathcal{P}$, then P is the only vertex of γ. Apollonius constructs a line segment PL perpendicular to the plane $\mathcal{P}$ called the **orthia**; [10] he then formulates a relation between the square of the ordinate and the abcissa analogous to Equation (2.2) as equality of area between the rectangle LPV and a square with side $|QV|$ (recall Equation (2.2)).

$$|QV|^2 = |PL|\,|PV| . \tag{2.3}$$

  In a terminology going back to the Pythagoreans, this says that the square on the ordinate is equal to the rectangle *applied* to PL, with width equal to the abcissa. Accordingly, Apollonius calls this curve a **parabola**. (Figure 2.4) (the Greek word for "application" is παραβολή) [26, p. 359].
- **Ellipses:** If PV is not parallel to AC, then the line PV (extended) meets the line AB (extended) at a second vertex P'. If ϕ denotes the (acute) angle between $\mathcal{P}$ and

[9]Observe that every plane contains at least one horizontal direction (Exercise 4).

[10]The Latin translation of this term is **latus rectum**, although this term has come to mean a slightly different quantity, the *parameter of ordinates*.

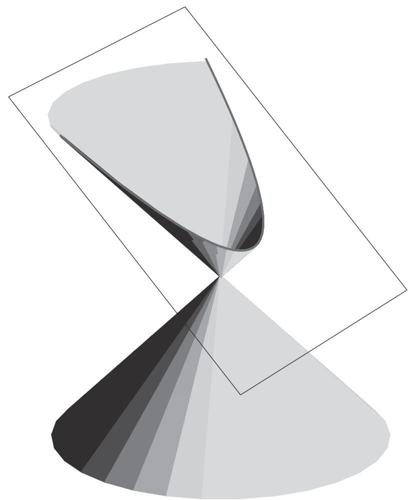


Figure 2.4. Parabola

a horizontal plane $\mathcal{H}$, then V lies between P and P' if $0 \leq \phi < \frac{\pi}{2} - \alpha$ and P lies between V and P' if $\frac{\pi}{2} - \alpha < \phi \leq \frac{\pi}{2}$.

In the first case ($0 \leq \phi \leq \frac{\pi}{2} - \alpha$) one finds a point S so that $\triangle P'VS$ is a right triangle whose hypotenuse SP' is horizontal, and derives the equation

$$|QV|^2 = |VS| \cdot |PV| . \tag{2.4}$$

This is like Equation (2.3), but $|PL|$ is replaced by the shorter length $|VS|$; in the Pythagorean terminology, the square on the ordinate is equal to the rectangle with width equal to the abcissa applied to the segment VS, *falling short* of PL. The Greek for "falling short" is ἔλλειψις, and Apollonius calls γ an **ellipse** in this case. (See Figure 2.5.)

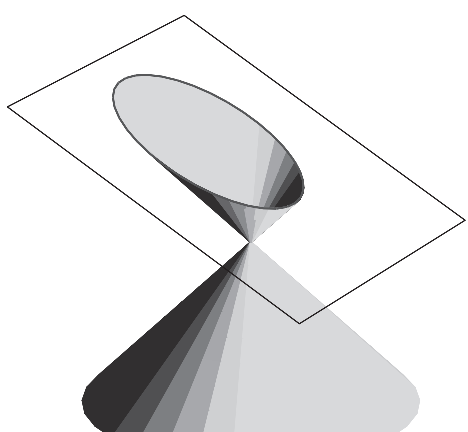


Figure 2.5. Ellipse

- **Hyperbolas:** In the final case, when When $\frac{\pi}{2} - \alpha < \phi \leq \frac{\pi}{2}$, the same arguments as in the ellipse case yield Equation (2.4), but this time the segment VS *exceeds* PL; the Greek for "excess" is ὑπερβολή, and γ is called a **hyperbola**. (See Figure 2.6.)

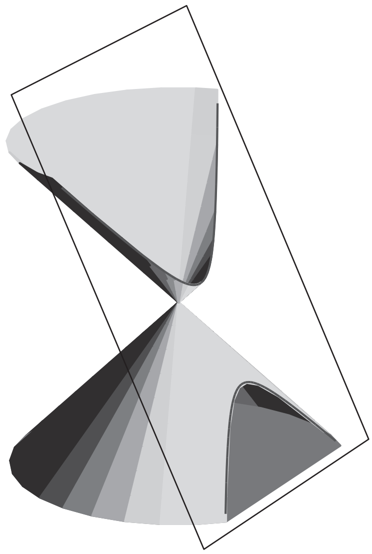


Figure 2.6. Hyperbola

The Focus-Directrix Property. Pappus, in a section of the *Collection* headed "Lemmas to the *Surface Loci*[11] of Euclid", proves the following ([26, p. 153]):

Lemma 2.1.1. *If the distance of a point from a fixed point be in a given ratio to its distance from a fixed straight line, the locus of the point is a conic section, which is an ellipse, a parabola, or a hyperbola according as the ratio is less than, equal to, or greater than, unity.*

The fixed point is called the **focus**, the line is the **directrix**, and the ratio is called the **eccentricity** of the conic section. This *focus-directrix property* of conics is not mentioned by Apollonius, but Heath deduces from the way it is treated by Pappus that this lemma must have been stated without proof, and regarded as well known, by Euclid.

The focus-directrix characterization of conic sections can be turned into an equation. This approach—treating a curve as the locus of an equation in the rectangular coordinates—was introduced in the early seventeenth century by René Descartes (1596-1650) and Pierre de Fermat (1601-1665).

Here, we explore each of the three types of conic section by considering, for each, a standard model locus with a particularly simple equation from which various geometric features are easily deduced.

Parabolas. A parabola is a curve with eccentricity $e = 1$, which means it is the locus of points equidistant from a given line (the **directrix**) and a given point (the **focus**). Taking the directrix vertical and to the left of the y-axis (say $x = -\frac{p}{4}$) and the focus on

[11]This work, like Euclid's *Conics*, is lost, and little information about its contents can be gleaned from Pappus.

the x-axis an equal distance to the right of the origin ($F(\frac{p}{4}, 0)$), we get (Exercise 7) a curve going through the origin, with equation

$$y^2 = px. \tag{2.5}$$

The origin is the point on this curve closest to the directrix, known as the **vertex** of the parabola. Since y only appears squared in the equation, replacing y with $-y$ does not change the equation, so the curve is **symmetric** about the x-axis (if (x, y) lies on the locus, so does $(x, -y)$). The curve has two branches, both going to infinity "to the right" (values of x can be arbitrarily high). See Figure 2.7.

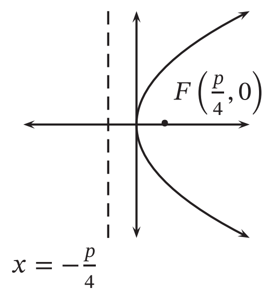


Figure 2.7. The Parabola $y^2 = px$, $p > 0$

If we interchange x and y in Equation (2.5), we obtain the graph

$$y = ax^2, \tag{2.6}$$

where $a = \frac{1}{p}$. Geometrically, this amounts to reflecting the locus across the diagonal, leading to a horizontal directrix $y = -\frac{px}{4} = -\frac{x}{4a}$ and a focus $F(0, \frac{p}{4}) = (0, \frac{1}{4a})$ on the y-axis. This curve is symmetric about the y-axis and opens up (Figure 2.8).

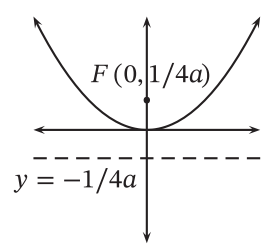


Figure 2.8. The Parabola $y = ax^2$, $a > 0$

Ellipses. The standard ellipse is given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{2.7}$$

where a and b are (by convention positive) constants.

We easily see that for any point on this curve, $|x| \leq a$ and $|y| \leq b$, so the curve is *bounded*. The two numbers a and b determine the x-intercepts $(\pm a, 0)$ and y-intercepts $(0, \pm b)$ of the curve. The intervals between the intercepts are called the **axes** of this ellipse; the larger (*resp.* smaller) is called the **major axis** (*resp.* **minor axis**); thus the larger of a and b is the **semi-major axis** while the smaller is the **semi-minor axis**.

Note that when $a = b$, Equation (2.7) is the equation of a circle, centered at the origin, whose radius equals their common value $a = b = r$. In general, the roles of a and b in the focus-directrix analysis depend on which is major and which is minor. We shall carry out this analysis assuming that a is the semi-major axis:

$$a > b > 0$$

and at the end consider how to modify this when $a < b$.

The ends of the major axis are the **vertices** of the ellipse. We are assuming these are $(\pm a, 0)$. Associated to the vertex $(a, 0)$ is a focus $F(c, 0)$ and a directrix $x = d$. We expect that $c < a < d$. If we fix the eccentricity $0 < e < 1$, the focus-directrix property at the vertex reads

$$a - c = e(d - a).$$

You should verify (Exercise 8a) that this holds if

$$c = ae \tag{2.8}$$

$$d = \frac{a}{e}. \tag{2.9}$$

For other points satisfying Equation (2.7), the condition reads

$$(x - c)^2 + y^2 = e^2(d - x)^2$$

which holds (Exercise 8b) if

$$e = \sqrt{1 - \left(\frac{b}{a}\right)^2}; \tag{2.10}$$

note that this together with Equation (2.8) gives

$$c = \sqrt{a^2 - b^2}. \tag{2.11}$$

As in the case of the parabola, the locus of Equation (2.7) is symmetric about each of the coordinate axes. In particular, reflection about the y-axis yields a second focus-directrix pair, $F(-ae, 0)$ and $x = -\frac{a}{e}$. As a consequence of Equation (2.8), we obtain the following characterization of the ellipse, sometimes called the *Gardener's Rule* (Exercise 10):

The sum of the distances of any point on the ellipse to the two foci equals the major axis.

This information is illustrated in Figure 2.9.

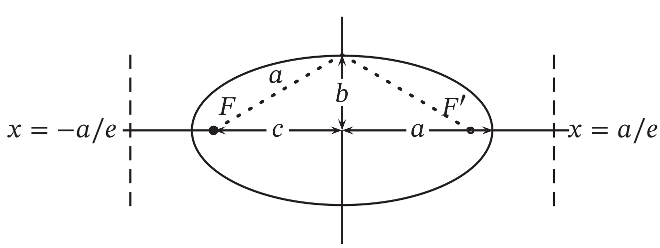


Figure 2.9. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b > 0$

Our analysis was based on the assumption that the positive constants in Equation (2.7) satisfy $a > b > 0$, that is, the major axis is horizontal. An ellipse with vertical major axis can be obtained either by interchanging x and y in Equation (2.7) (so that a remains the major semi-axis but is now associated with y) or by interchanging the roles of a and b in our analysis. The latter approach is probably more natural; it is carried out in Exercise 8c.

Hyperbolas. The standard equation for a hyperbola can be obtained from Equation (2.7) by replacing the *sum* of terms with their *difference*:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{2.12}$$

Clearly, the first term must be at least equal to 1, so $|x| \geq a$ for any point on the curve; in particular there are no y-intercepts. There are two x-intercepts, $(\pm a, 0)$, which again play the role of **vertices**. Associated to the vertex $(a, 0)$ is the focus-directrix pair $F(ae, 0)$, $x = \frac{a}{e}$. Note however that the eccentricity of a hyperbola is *greater* than 1, so the focus (*resp.* directrix) is to the *right* (*resp. left*) of the vertex. We see that the locus has two branches, one opening to the right from the vertex $(a, 0)$, the other to the left from $(-a, 0)$. You should verify that Equation (2.12) is satisfied in general if in the focus-directrix condition we take the eccentricity to be

$$e = \sqrt{1 + \left(\frac{b}{a}\right)^2}. \tag{2.13}$$

Note that by contrast with the ellipse case, a and b can have any relative (nonzero) values.

If we interchange x and y, we obtain another hyperbola, whose branches open up and down. The same effect can be obtained by leaving the left side of Equation (2.12) alone but switching the sign on the right:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1. \tag{2.14}$$

This causes a and b (as well as x and y) to switch roles; see Exercise 9b.

Finally, the equation obtained by replacing the right side with zero

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

has as its locus a pair of straight lines, crossing at the origin:

$$\frac{x}{a} = \pm\frac{y}{b}. \tag{2.15}$$

These are the **asymptotes** of the hyperbola: a point moving to infinity along one of the branches of Equation (2.12) or Equation (2.14) approaches one of the lines in Equation (2.15).

This information is illustrated in Figure 2.10.

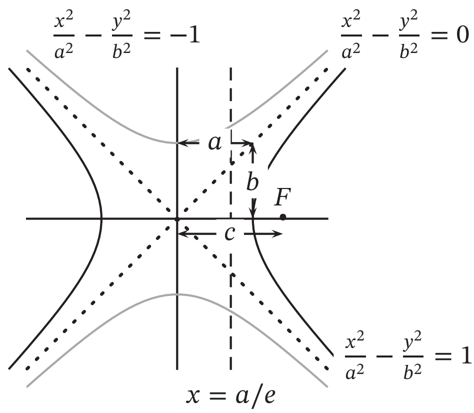


Figure 2.10. Hyperbolas and asymptotes

Moving loci.

Translation. In the model equations we have obtained for parabolas, ellipses, and hyperbolas in this section, the origin and the two coordinate axes play special roles with respect to the geometry of the locus. For the parabola given by Equation (2.6), the origin is the *vertex*, the point of closest approach to the directrix, and the y-axis is an axis of symmetry for the parabola, while the x-axis is a kind of boundary which the curve can touch but never crosses. For the ellipse given by Equation (2.7), both of the coordinate axes are axes of symmetry, containing the major and minor axes, and the origin is their intersection (the *center* of the ellipse). For the hyperbola given by Equation (2.12), the coordinate axes are again both axes of symmetry, and the origin is their intersection, as well as the intersection of the asymptotes (the *center* of the hyperbola).

Suppose we want to move one of these loci (or indeed any locus) to a new location: that is, we want to displace the locus (without rotation) so that the special point given by the origin for the model equation moves to (α, β). Any such motion is accomplished by replacing x with x plus a constant and y with y plus another constant inside the equation; we need to do this in such a way that substituting $x = \alpha$ and $y = \beta$ into the *new* equation leads to the same calculation as substituting $x = 0$ and $y = 0$ into the *old* equation. It may seem wrong that this requires replacing x with $x - \alpha$ and y with $y - \beta$ in the old equation; to convince ourselves that it is right, let us consider a few simple examples.

First, the substitution

$$x \mapsto x - 1$$
$$y \mapsto y - 2$$

into the model parabola equation

$$y = x^2$$

leads to the equation

$$y - 2 = (x - 1)^2;$$

we note that in the new equation, substitution of the point $(1, 2)$ leads to the equation $0 = 0$, and furthermore no point lies below the horizontal line through this point, $y - 2 = 0$: we have displaced the parabola so as to move its vertex from the origin to the point $(1, 2)$ (Figure 2.11).

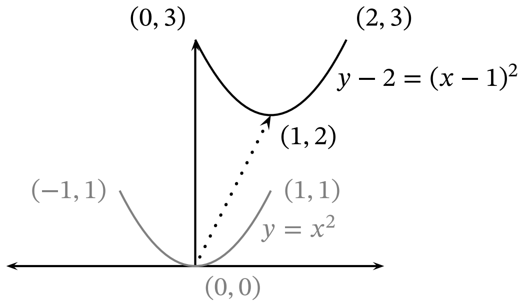


Figure 2.11. Displacing a parabola

Second, to move the ellipse

$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

so that its center moves to $(-2, 2)$, we perform the substitution

$$x \mapsto x - (-2) = x + 2$$
$$y \mapsto y - 2$$

(See Figure 2.12.)

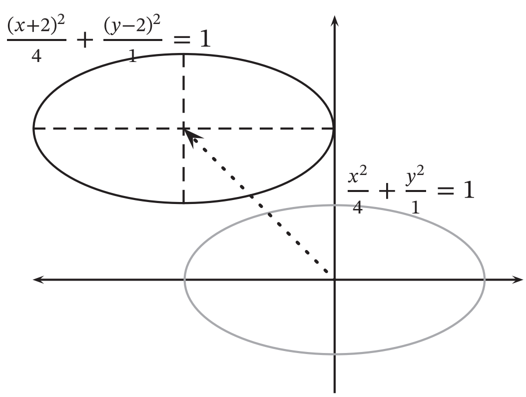


Figure 2.12. Displacing an ellipse

This process can be reversed, using completion of the square. As an example, consider the locus of the equation

$$9x^2 + 18x + 16y^2 - 64y = 71.$$

Let us complete the square in the x terms and in the y terms:

$$9(x^2+2x\quad)+16(y^2-4y\quad)=71$$
$$9(x^2+2x+1)+16(y^2-4y+4)=71+9+64$$
$$9(x+1)^2+16(y-2)^2=144;$$

now dividing both sides by 144 we obtain

$$\frac{(x+1)^2}{16}+\frac{(y-2)^2}{9}=1$$

which we recognize as obtained from the model equation

$$\frac{x^2}{4^2}+\frac{y^2}{3^2}=1$$

via the substitution

$$x \to x+1$$
$$y \to y-2.$$

Thus the locus of our original equation is an ellipse, centered at $(-1,2)$, with (horizontal) semi-major axis 4 and (vertical) semi-minor axis 3.

This procedure can be used to identify any quadratic equation with no "xy" terms (bxy, $b \in \mathbb{R}$) as a conic section or degenerate locus.

Reflection. We can also reflect a locus about a coordinate axis. Since our model ellipses and hyperbolas are symmetric about these axes, this has no effect on the curve. However, while the model parabola given by Equation (2.5) is symmetric about the y-axis, it opens *up*; we can reverse this, making it open *down*, by replacing y with $-y$, or equivalently replacing the positive coefficient p with its negative.

For example, when $p=1$ this leads to the equation

$$y=-x^2$$

whose locus opens *down*: it is the reflection of our original parabola $y=x^2$ about the x-axis (Figure 2.13).

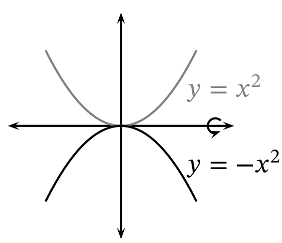


Figure 2.13. Reflecting a parabola about the x-axis

Finally, we can interchange the two variables; this effects a reflection about the diagonal line $y=x$. We have already seen the effect of this. For a parabola, the interchange $x \leftrightarrow y$ takes the parabola $y=x^2$ in Figure 2.8, which opens along the positive y-axis (*i.e., up*), to the parabola $x=y^2$ in Figure 2.7, which opens along the positive x-axis (*i.e., to the right*), and the parabola $y=-x^2$, which opens along the negative y-axis (*i.e., down*), to the parabola $x=-y^2$, which opens along the negative x-axis (*i.e.*, *to the left*). An ellipse with horizontal (*resp.* vertical) major axis is changed under interchange of variables to one with vertical (*resp.* horizontal) major axis, while a hyperbola which opens horizontally (*resp.* vertically) goes to one opening vertically (*resp.* horizontally).

Rotation. A rotation of the plane by α radians *counterclockwise* amounts to adding α to the "angular" polar coordinate of each point; thus the point $P(x, y)$ with Cartesian coordinates $(r\cos\theta, r\sin\theta)$ is taken to $P'(x', y')$, where

$$\begin{aligned} x' &= r\cos(\theta+\alpha) = r(\cos\theta\cos\alpha - \sin\theta\sin\alpha) = x\cos\alpha - y\sin\alpha \\ y' &= r\sin(\theta+\alpha) = r(\cos\theta\sin\alpha + \sin\theta\cos\alpha) = x\sin\alpha + y\cos\alpha. \end{aligned}$$

In keeping with our experience of displacements, it is reasonable that to rotate a locus by α radians in a given direction, we should substitute the values x' and y' for a point rotated in the *opposite* direction by α radians; thus to rotate a given locus α radians *counterclockwise* we should perform the substitution

$$x \mapsto x\cos\alpha + y\sin\alpha, \quad y \mapsto -x\sin\alpha + y\cos\alpha.$$

As an example, let us rotate the hyperbola given by $\frac{x^2}{2} - \frac{y^2}{2} = 1$ by $\frac{\pi}{4}$ radians (45 degrees) counterclockwise. The appropriate substitution is

$$x \mapsto x\cos\frac{\pi}{4} + y\sin\frac{\pi}{4} = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}, \quad y \mapsto -x\sin\frac{\pi}{4} + y\cos\frac{\pi}{4} = -\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}};$$

this substitution transforms the equation into

$$\frac{1}{2}\left(\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right)^2 - \frac{1}{2}\left(-\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right)^2 = 1$$

which simplifies to

$$xy = 1.$$

It turns out (Exercise 4 in § 3.9) that, with a few degenerate exceptions, every quadratic equation has as its locus one of the types of conic sections discussed here. As illustrated by the example above, "xy" terms are introduced when the locus of one of our model equations is rotated by an angle which is not a multiple of a right angle.

Exercises for § 2.1

Answers to Exercises 1a, 2a, and 3a are given in Appendix A.13.

Practice problems:

(1) In each problem below, you are given the locus of a conic section in "standard position"; give an equation for the locus resulting from the indicated motion:
   (a) The parabola $y = x^2$, moved so its vertex is at $(1, -2)$.
   (b) The ellipse $4x^2 + y^2 = 1$, moved so its center is at $(3, -2)$.
   (c) The hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$, moved so that its "right" vertex is at $(1, -2)$.
   (d) The parabola $y = x^2\sqrt{2}$, rotated counterclockwise $\frac{\pi}{4}$ radians.
   (e) The ellipse $\frac{x^2}{4} + y^2 = 1$, rotated counterclockwise $\frac{\pi}{4}$ radians.
   (f) The ellipse $\frac{x^2}{4} + y^2 = 1$, rotated counterclockwise $\frac{\pi}{3}$ radians.

(2) Identify each of the following curves as a circle, ellipse, hyperbola, parabola, or degenerate locus. For a parabola, determine the axis of symmetry and vertex. For a hyperbola, determine the vertices, asymptotes and center. For an ellipse (*resp.* circle), determine the center and semimajor and semiminor axes (*resp.* radius).
   (a) $y^2 = x + 2y$
   (b) $4x^2 + 4x + 4y^2 - 12y = 15$
   (c) $4x^2 + 4x + y^2 + 6y = 15$
   (d) $x^2 - 10x - y^2 - 6y - 2 = 0$

(3) Determine the focus, directrix and eccentricity of each conic section below:
   (a) $2x^2 - 4x - y = 0$
   (b) $4y^2 - 16y + x + 16 = 0$
   (c) $4x^2 - 8x + 9y^2 + 36y + 4 = 0$
   (d) $x^2 + 4x - 16y^2 + 32y + 4 = 0$

Theory problems:

(4) Show that every plane in space contains at least one horizontal direction. (*Hint:* Without loss of generality, you can assume your plane is not itself horizontal. Then show that it intersects every horizontal plane.)

(5) **Show** that if $\mathcal{P}$ is a plane through A, the vertex of the cone $\mathcal{K}$, then $\mathcal{P} \cap \mathcal{K}$ is (a) just the origin if $\mathcal{P}$ is horizontal or is tilted not too far off the horizontal; (b) a single generator if $\mathcal{P}$ is tangent to the cone, and (c) a pair of generators otherwise.

(6) **Show** that Equation (2.2) (the statement of Prop. 13, Book VI of the *Elements*) is equivalent to the standard equation for a circle. (*Hint:* In Figure 2.2, put the origin at the center of the circle and assume the radius of the circle is ρ. This means the coordinates of B are $(-\rho, 0)$ and those of C are $(\rho, 0)$. If the coordinates of Q are (x, y), then show that $|BV| = \rho + x$, $|CV| = \rho - x$, and $|QV| = |y|$. Substituting these values into Equation (2.2) then gives the equation of the circle.)

(7) (a) Show that a point $P(x, y)$ is equidistant from the vertical line $x = -\frac{p}{4}$ and the point $F(\frac{p}{4})$ precisely if its coordinates satisfy

$$y^2 = px.$$

   (b) Verify that the curve with equation $y = ax^2$ is a parabola with directrix $y = -\frac{1}{4a}$ and focus $F(0, \frac{1}{4a})$.

Challenge problems:

(8) (a) Verify that the point $P(a, 0)$ satisfies the eccentricity condition (that the distance to $F(c, 0)$ is e times the distance to the line $x = d$) if $c = ae$ and $d = \frac{a}{e}$; that is, if Equation (2.8) and Equation (2.9) hold then

$$a - c = e(d - a).$$

   Note that for the ellipse, $0 < e < 1$ means that $c < a < d$.

   (b) Assume the conditions of Exercise 8a hold, with $0 < e < 1$. Verify that the eccentricity condition (that the distance from $P(x, y)$ to $F(ae, 0)$ equals e times its distance to the line $x = \frac{a}{e}$) holds for every point of the locus of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{2.7}$$

provided

$$a^2(1-e^2)=b^2 \tag{2.16}$$

or equivalently,

$$e=\sqrt{1-\left(\frac{b}{a}\right)^2}. \tag{2.10}$$

Note that these last two conditions are possible only if $0<b<a$.

(c) Suppose that $0<a<b$ in Equation (2.7); then Equation (2.16) (and hence Equation (2.10)) is an impossible condition. Show, however, that in this case (2.7) is the locus of points $P(x,y)$ whose distance from the point $F(0,be)$ equals e times their distance from the line $y=\frac{b}{e}$, where

$$e=\sqrt{1-\left(\tfrac{a}{b}\right)^2}.$$

(9) (a) Note that if $e>1$ in Exercise 8a, then $d<a<c$, and that the condition in Equation (2.10) is impossible. However, show that if

$$e=\sqrt{1+\left(\frac{b}{a}\right)^2}$$

then the eccentricity condition (that the distance from $P(x,y)$ to $F(ae,0)$ equals e times its distance to the line $x=\frac{a}{e}$) does hold for every point on the locus of

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

(b) Show that replacing 1 with -1 in Equation (2.12) is equivalent to interchanging the roles of x and y–or equivalently, to creating a hyperbola which opens up and down instead of right and left.

Challenge problem:

(10) **Show** that the sum of the distances from a point on an ellipse to its two foci equals the major axis. (You may assume the equation is in standard form.) This is sometimes called the *Gardener's characterization* of an ellipse: explain how one can construct an ellipse using a piece of string.

2.2 Parametrized Curves

Parametrized Curves in the Plane. There are two distinct ways of specifying a curve in the plane. In classical geometric studies, a curve is given in a static way, either as the intersection of the plane with another surface (like the conical surface in Apollonius) or by a geometric condition (like fixing the distance from a point or the focus-directrix property in Euclid and Pappus). This approach reached its modern version in the seventeenth century with Descartes' and Fermat's formulation of a curve as the *locus of an equation* in the coordinates of a point. A second and equally important source of curves is dynamic in nature: a curve can be generated as the *path of a moving point*. This is the fundamental viewpoint in Newton's *Principia* (as well as the work of Newton's older contemporary Christian Huygens (1629-1695)), but "mechanical" constructions of curves also go back to antiquity, for example in "Archimedes' spiral" (p. 86).

We have seen in the case of lines in the plane how these two approaches interact: for example, the intersection of two lines is easier to find as the simultaneous solution of their equations, but a parametrized version more naturally encodes intrinsic geometric properties like the "direction" of a line. We have also seen that when one goes from lines in the plane to lines in space, the static formulation becomes unwieldy, requiring *two* equations, while—especially with the language of vectors—the dynamic formulation extends quite naturally. For this reason, we will adopt the dynamic approach as our primary way to specify a curve.

We can think of the position of a point moving in the plane as a **vector-valued function** assigning to each $t \in I$ the position vector $\vec{p}\,(t)$; this point of view is signified by the notation $\vec{p} : \mathbb{R} \to \mathbb{R}^2$ indicating that the function $\vec{p}$ takes real numbers as input and produces vectors in $\mathbb{R}^2$ as output. If we want to be explicit about the domain I we write $\vec{p} : I \to \mathbb{R}^2$. The component functions of a vector-valued function $\vec{p}\,(t) = (x\,(t)\,, y\,(t))$ are simply the (changing) coordinates of the moving point; thus a vector-valued function $\vec{p} : \mathbb{R} \to \mathbb{R}^2$ is the same thing as a pair of (ordinary, real-valued) functions.

We have seen how to parametrize a line in the plane. Some other standard parametrizations of curves in the plane are:

Circle: A circle in the plane with center at the origin and radius $R > 0$ is the locus of the equation $x^2 + y^2 = R^2$. A natural way to locate a point on this circle is to give the angle that the radius through the point makes with the positive x-axis; equivalently, we can think of the circle as given by the equation $r = R$ in polar coordinates, so that the point is specified by the polar coordinate θ. Translating back to rectangular coordinates we have $x = R\cos\theta$, $y = R\sin\theta$ and the parametrization of the circle is given by the vector-valued function

$$\vec{p}\,(\theta) = (R\cos\theta, R\sin\theta).$$

As θ goes through the values from 0 to 2π, $\vec{p}\,(\theta)$ traverses the circle once counterclockwise; if we allow *all* real values for θ, $\vec{p}\,(\theta)$ continues to travel counterclockwise around the circle, making a full circuit every time θ increases by 2π. Note that if we interchange the two formulas for x and y, we get another parametrization, $\vec{q}\,(\theta) = (R\sin\theta, R\cos\theta)$, which traverses the circle *clockwise*.

We can displace this circle, to put its center at any specified point $C(c_1, c_2)$, by adding the (constant) position vector of the desired center C to $\vec{p}\,(\theta)$ (or $\vec{q}\,(\theta)$):

$$\vec{r}\,(\theta) = (R\cos\theta, R\sin\theta) + (c_1, c_2) = (c_1 + R\cos\theta, c_2 + R\sin\theta).$$

Ellipse: The "model equation" for an ellipse with center at the origin (Equation (2.7) in § 2.1), $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, looks just like the equation for a circle of radius 1 centered at the origin, but with x (*resp.* y)) replaced by x/a (*resp.* y/b), so we can parametrize this locus via $\frac{x}{a}u = \cos\theta$, $\frac{y}{b} = \sin\theta$, or

$$\vec{p}\,(\theta) = (a\cos\theta, b\sin\theta). \tag{2.17}$$

Again, the ellipse is traversed once counterclockwise as θ varies by 2π.

The geometric significance of θ in this parametrization is given in Exercise 5.

As before, by adding a constant displacement vector, we can move the ellipse so that its center is at (c_1, c_2):

$$\vec{r}\,(\theta) = (a\cos\theta, b\sin\theta) + (c_1, c_2) = (c_1 + a\cos\theta, c_2 + b\sin\theta).$$

Hyperbola: The "model equation" for a hyperbola (Equation (2.12) in § 2.1)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1$$

can be parametrized using the hyperbolic functions. The functions

$$\cosh t = \frac{e^t + e^{-t}}{2}, \quad \sinh t = \frac{e^t - e^{-t}}{2} \tag{2.18}$$

are known, respectively, as the **hyperbolic cosine** and **hyperbolic sine** of t. Using Euler's formula (see *Calculus Deconstructed*, p. 475, or another single-variable calculus text), they can be interpreted in terms of the sine and cosine of an imaginary multiple of t, and satisfy variants of the usual trigonometric identities (Exercise 7):

$$\cosh t = \cos it, \quad \sinh t = -i \sin it.$$

You can verify that

$$\vec{p}\,(t) = (a \cosh t, b \sinh t) \quad -\infty < t < \infty$$

gives a curve satisfying $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. However, note that $\cosh t$ is always positive (in fact, $\cosh t \geq 1$ for all t), so this parametrizes only the "right branch" of the hyperbola; the "left branch" is parametrized by

$$\vec{p}\,(t) = (-a \cosh t, b \sinh t) \quad -\infty < t < \infty.$$

Similarly, the two branches of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ are parametrized by

$$\vec{p}\,(t) = (a \sinh t, \pm b \cosh t) \quad -\infty < t < \infty.$$

Parabolas: The model equation for a parabola with horizontal directrix (Equation (2.6) in § 2.1)

$$y = ax^2$$

is easily parametrized using x as the parameter: $x = t$, $y = at^2$ which leads to

$$\vec{p}\,(t) = \left(t, at^2\right) \quad -\infty < t < \infty.$$

This last example illustrates how to parametrize a whole class of curves. The equation for a parabola gives one of the coordinates as an explicit function of the other—that is, the curve is represented as the graph of a function.

Remark 2.2.1. *If a curve is expressed as the graph of a function*

$$y = f\,(x)$$

then using the independent variable as our parameter, we can parametrize the curve as

$$\vec{p}\,(t) = (t, f\,(t)).$$

For example,the circle $x^2 + y^2 = 1$ consists of two graphs: if we solve for y as a function of x, we obtain

$$y = \pm\sqrt{1 - x^2}, \quad -1 \leq x \leq 1.$$

The graph of the positive root is the upper semicircle, and this can be parametrized by

$$x\,(t) = t, \quad y\,(t) = \sqrt{1 - t^2}$$

or

$$\vec{p}\,(t) = (t, \sqrt{1 - t^2}), \quad t \in [-1, 1]\,.$$

Note, however, that in this parametrization, the upper semicircle is traversed *clockwise*; to get a *counterclockwise* motion, we replace t with its negative:

$$\vec{q}\,(t) = (-t, \sqrt{1-t^2}), \quad t \in [-1, 1]\,.$$

The lower semicircle, traversed counterclockwise, is the graph of the negative root:

$$\vec{p}\,(t) = (t, -\sqrt{1-t^2}), \quad t \in [-1, 1]\,.$$

Displacing and Rotating Curves. The parametrizations so far concern ellipses and hyperbolas in standard positions—in particular, they have all been centered at the origin. We saw at the end of § 2.1 how the standard equation of a conic section can be modified to give a displaced version of the standard one. Actually, displacing a curve given via a parametrization is even easier: we simply add the desired (constant) displacement vector to the standard parametrization.

For example, the standard ellipse (centered at the origin, with horizontal semi-axis a and vertical semi-axis b) is parametrized by $\vec{p}\,(\theta) = (a\cos\theta)\vec{\imath} + (b\sin\theta)\vec{\jmath}$ or $x = a\cos\theta$, $y = b\sin\theta$. Suppose we want to describe instead the ellipse with the same semi-axes (still parallel to the coordinate axes) but with center at the point (c_1, c_2). The displacement vector taking the origin to this position is simply the position vector of the new center, $\vec{c} = c_1\vec{\imath} + c_2\vec{\jmath}$, so we can obtain the new ellipse from the old simply by adding this (constant) vector to our parametrization function:

$$\vec{c} + \vec{p}\,(\theta) = (c_1 + a\cos\theta)\vec{\imath} + (c_2 + b\sin\theta)\vec{\jmath}$$

or, in terms of coordinates,

$$x = c_1 + a\cos\theta, \quad y = c_2 + b\sin\theta$$

We might also consider the possibility of a conic section obtained by rotating a standard one. This is easily accomplished for a parametrized expression: the role of $\vec{\imath}$ (*resp.* $\vec{\jmath}$) is now played by a rotated version $\overrightarrow{u_1}$ (*resp.* $\overrightarrow{u_2}$) of this vector. Two words of caution are in order here: the new vectors must still be *unit* vectors, and they must still be *perpendicular* to each other. Both of these properties are guaranteed if we make sure to rotate both $\vec{\imath}$ and $\vec{\jmath}$ by the same amount, in the same direction.

For example, suppose we want to describe the ellipse, still centered at the origin, with semi-axes a and b, but rotated counterclockwise from the coordinate axes by $\alpha = \frac{\pi}{6}$ radians. Rotating $\vec{\imath}$ leads to the unit vector making angle $\alpha = \frac{\pi}{6}$ with the positive x-axis

$$\overrightarrow{u_1} = (\cos\alpha)\vec{\imath} + (\sin\alpha)\vec{\jmath} = \frac{\sqrt{3}}{2}\vec{\imath} + \frac{1}{2}\vec{\jmath}$$

while rotating $\vec{\jmath}$ the same amount yields the vector making angle α (counterclockwise) with the positive y-axis, or equivalently making angle $\alpha + \frac{\pi}{2}$ with the positive x-axis:

$$\overrightarrow{u_2} = \cos(\alpha + \frac{\pi}{2})\vec{\imath} + \sin(\alpha + \frac{\pi}{2})\vec{\jmath} = -\frac{1}{2}\vec{\imath} + \frac{\sqrt{3}}{2}\vec{\jmath}.$$

Our parametrization of the *rotated* ellipse is obtained from the standard parametrization by replacing $\vec{\imath}$ with $\overrightarrow{u_1}$ and $\vec{\jmath}$ with $\overrightarrow{u_2}$:

$$\begin{aligned}\vec{p}\,(\theta) &= (a\cos\theta)\overrightarrow{u_1} + (b\sin\theta)\overrightarrow{u_2} \\ &= \left(\frac{a\sqrt{3}}{2}\cos\theta - \frac{b}{2}\sin\theta\right)\vec{\imath} + \left(-\frac{a}{2}\cos\theta + \frac{b\sqrt{3}}{2}\sin\theta\right)\vec{\jmath}\end{aligned}$$

or, in terms of coordinates,

$$x = \frac{a\sqrt{3}}{2}\cos\theta - \frac{b}{2}\sin\theta, \quad y = -\frac{a}{2}\cos\theta + \frac{b\sqrt{3}}{2}\sin\theta$$

Of course, we can combine these operations, but again some care is necessary: *rotate* the standard parametrization *before* adding the displacement; otherwise you will have rotated the displacement, as well. For example, a parametrization of the ellipse centered at $\vec{c} = (1,2)$ with axes rotated $\frac{\pi}{6}$ radians counterclockwise from the positive coordinate axes is given (in terms of the notation above) by

$$\begin{aligned}\vec{p}\,(\theta) &= \vec{c} + \left((a\cos\theta)\overrightarrow{u_1} + (b\sin\theta)\overrightarrow{u_2}\right)\\ &= \left(1 + \frac{a\sqrt{3}}{2}\cos\theta - \frac{b}{2}\sin\theta\right)\vec{\imath} + \left(2 - \frac{a}{2}\cos\theta + \frac{b\sqrt{3}}{2}\sin\theta\right)\vec{\jmath}\end{aligned}$$

or, in terms of coordinates,

$$x = 1 + \frac{a\sqrt{3}}{2}\cos\theta - \frac{b}{2}\sin\theta, \quad y = 2 - \frac{a}{2}\cos\theta + \frac{b\sqrt{3}}{2}\sin\theta.$$

The general relation between a plane curve, given as the locus of an equation, and its possible parametrizations will be clarified by means of the Implicit Function Theorem in Chapter 3.

Analyzing a Curve from a Parametrization. The examples in the preceding subsection all went from a static expression of a curve as the locus of an equation to a dynamic description as the image of a vector-valued function. The converse process can be difficult, but given a function $\vec{p}: \mathbb{R} \to \mathbb{R}^2$, we can try to "trace out" the path as the point moves.

As an example, consider the function $\vec{p}: \mathbb{R} \to \mathbb{R}^2$ defined by

$$x\,(t) = t^3, \quad y\,(t) = t^2$$

with domain $(-\infty, \infty)$. We note that $y\,(t) \geq 0$, with equality only for $t = 0$, so the curve lies in the upper half-plane. Note also that $x\,(t)$ takes each real value once, and that since $x\,(t)$ is an *odd* function and $y\,(t)$ is an *even* function, the curve is symmetric across the y-axis. Finally, we might note that the two functions are related by $(y\,(t))^3 = (x\,(t))^2$, or $y\,(t) = (x\,(t))^{2/3}$, so the curve is the graph of the function $x^{2/3}$—that is, it is the locus of the equation

$$y = x^{2/3}.$$

This is shown in Figure 2.14: as t goes from $-\infty$ to ∞, the point moves to the right, "bouncing" off the origin at $t = 0$.

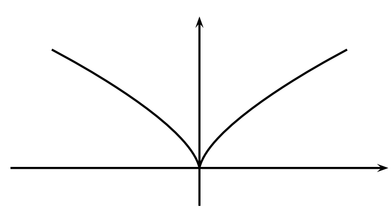


Figure 2.14. The curve $y^3 = x^2$

Curves in Polar Coordinates. A large class of curves can be given as the graph of an equation in *polar coordinates*. Usually, this takes the form $r = f(\theta)$. Using the relation between polar and rectangular coordinates, this can be parametrized as

$$\vec{p}(\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta).$$

We consider a few examples.

The polar equation $r = \sin\theta$ describes a curve which starts at the origin when $\theta = 0$; as θ increases, so does r until it reaches a maximum at $t = \frac{\pi}{2}$ (when $\vec{p}\left(\frac{\pi}{2}\right) = (0, 1)$) and then decreases, with $r = 0$ again at $\theta = \pi$ ($\vec{p}(\pi) = (-1, 0)$). For $\pi < \theta < 2\pi$, r is negative, and by examining the geometry of this, we see that the actual points $\vec{p}(\theta)$ trace out the same curve as was already traced out for $0 < \theta < \pi$. The curve is shown in Figure 2.15. In this case, we can recover an equation in rectangular coordinates for

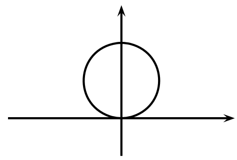


Figure 2.15. The curve $r = \sin\theta$

our curve: multiplying both sides of $r = \sin\theta$ by r, we obtain $r^2 = r\sin\theta$ and then using the identities $r^2 = x^2 + y^2$ and $y = r\sin\theta$, we can write $x^2 + y^2 = y$ which, after completing the square, can be rewritten as $x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$. We recognize this as the equation of a circle centered at $(0, \frac{1}{2})$ with radius $\frac{1}{2}$.

The polar equation $r = \sin 2\theta$ may appear to be an innocent variation of the preceding, but it turns out to be quite different. Again the curve begins at the origin when $\theta = 0$ and r increases with θ, but this time it reaches its maximum $r = 1$ when $\theta = \frac{\pi}{4}$, which is to say along the diagonal ($\vec{p}\left(\frac{\pi}{4}\right) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$), and then decreases, hitting $r = 0$ and hence the origin when $\theta = \frac{\pi}{2}$. Then r turns negative, which means that as θ goes from $\frac{\pi}{2}$ to π, the point $\vec{p}(\theta)$ lies in the fourth quadrant ($x > 0$, $y < 0$); for $\pi < \theta < \frac{3\pi}{2}$, r is again positive, and the point makes a "loop" in the third quadrant, and finally for $\frac{3\pi}{2} < \theta < 2\pi$, it traverses a loop in the second quadrant. After that, it traces out the same curve all over again. This curve is sometimes called a **four-petal rose** (Figure 2.16). Again, it is possible to express this curve as the locus of an equation in rectangular coordinates via mutliplication by r. However, it is slightly more complicated: if we multiply $r = \sin 2\theta$ by r, we obtain $r^2 = r\sin 2\theta$, whose left side is easy to interpret as $x^2 + y^2$, but whose right side is not so obvious. If we recall the identity $\sin 2\theta = 2\sin\theta\cos\theta$, we see that $r\sin 2\theta = 2r\sin\theta\cos\theta$, but to turn the right side into a recognizable expression in x and y we need to multiply through by r again; this yields $r^3 = 2(r\sin\theta)(r\cos\theta)$, or

$$\left(x^2 + y^2\right)^{3/2} = 2xy.$$

While this *is* an equation in rectangular coordinates, it is *not* particularly informative about our curve.

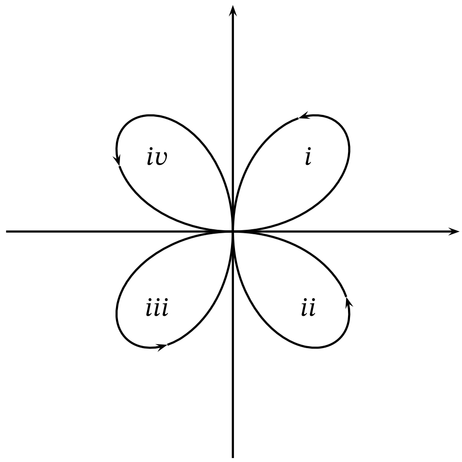


Figure 2.16. Four-petal Rose $r = \sin 2\theta$

Polar equations of the form

$$r = \sin n\theta$$

define curves known as "roses": it turns out that when n is *even* (as in the preceding example) there are $2n$ "petals", traversed as θ goes over an interval of length 2π, but when n is *odd*—as for example $n = 1$, which was the earlier example—then there are n "petals", traversed as θ goes over an interval of length π.

A different kind of example is provided by the polar equation $r = a\theta$, where $a > 0$ is a constant, which was (in different language, of course) studied by Archimedes of Syracuse (*ca.* 287-212 BC) in his work *On Spirals* [3] and is sometimes known as the **spiral of Archimedes**. Here is his own description (as translated by Heath [27, p. 154]):

> *If a straight line of which one extremity remains fixed be made to revolve at a uniform rate in a plane until it returns to the position from which it started, and if, at the same time as the straight line revolves, a point move at a uniform rate along the straight line, starting from the fixed extremity, the point will describe a spiral in the plane.*

Of course, Archimedes is describing the above curve for the variation of θ from 0 to 2π. If we continue to increase θ beyond 2π, the curve continues to spiral out, as illustrated in Figure 2.17. If we include negative values of θ, we get another spiral, going clockwise instead of counterclockwise (Figure 2.18). It is difficult to see how to write down an equation in x and y with this locus.

The Cycloid. Finally, we consider the **cycloid**, which can be described as the path of a point on the rim of a wheel rolling along a line (Figure 2.19). Let R be the radius of the wheel, and assume that at the beginning the point is located on the line—which we take to be the x-axis—at the origin, so the center of the wheel is at $(0, R)$. We take as our parameter the (clockwise) angle θ which the radius to the point makes with the downward vertical, that is, the amount by which the wheel has turned from its initial position. When the wheel turns θ radians, its center travels $R\theta$ units to the right, so the

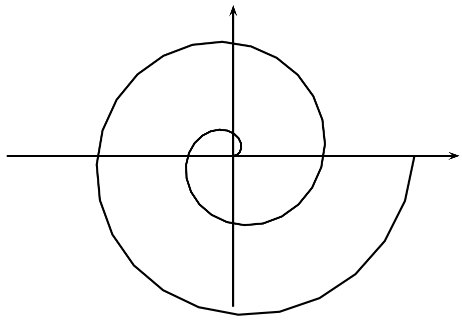


Figure 2.17. The Spiral of Archimedes, $r = \theta, \theta \geq 0$

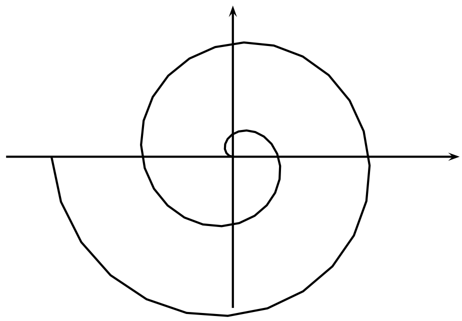


Figure 2.18. $r = \theta, \theta < 0$

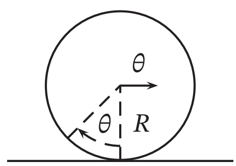


Figure 2.19. Turning Wheel

position of the *center* of the wheel corresponding to a given value of θ is

$$\vec{c}(\theta) = R\vec{j} + (R\theta)\vec{i} = (R\theta, R).$$

At that moment, the radial vector $\vec{r}(\theta)$ from the center of the wheel to the point on the rim is

$$\vec{r}(\theta) = -R(\sin\theta\vec{i} + \cos\theta\vec{j})$$

and so the position vector of the point is

$$\vec{p}(\theta) = \vec{c}(\theta) + \vec{r}(\theta) = (R\theta - R\sin\theta, R - R\cos\theta)$$

or

$$x(\theta) = R(\theta - \sin\theta), \quad y(\theta) = R(1 - \cos\theta).$$

The curve is sketched in Figure 2.20.

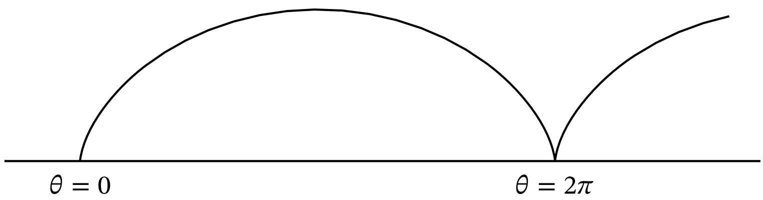


Figure 2.20. Cycloid

Curves in Space. As we have seen in the case of lines, when we go from curves in the plane to curves in space, the static formulation of a curve as the locus of an equation must be replaced by the more complicated idea of the locus of a *pair* of equations. By contrast, the dynamic view of a curve as the path of a moving point—especially when we use the language of vectors—extends very naturally to curves in space. We shall adopt this latter approach to specifying a curve in space.

The position vector of a point in space has three components, so the (changing) position of a moving point is specified by a function whose values are vectors in $\mathbb{R}^3$, which we denote by $\vec{p}: \mathbb{R} \to \mathbb{R}^3$; this can be regarded as a *triple* of functions:

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

or

$$\vec{p}(t) = (x(t), y(t), z(t)).$$

As before, it is important to distinguish the *vector-valued function* $\vec{p}(t)$, which specifies the motion of a point, from the *path* traced out by the point. Of course the same *path* can be traced out by different *motions*; the *curve* parametrized by the function $\vec{p}(t)$ is the **range** (or **image**) of the function:

$$\mathcal{C} = \{\vec{p}(t) \mid t \in \operatorname{domain}(\vec{p})\}.$$

When we are given a vector-valued function $\vec{p}: \mathbb{R} \to \mathbb{R}^3$, we can try to analyze the motion by considering its projection on the coordinate planes. As an example, consider the function defined by

$$x(t) = \cos 2\pi t, \quad y(t) = \sin 2\pi t, \quad z(t) = t$$

which describes a point whose projection on the xy-plane moves counterclockwise in a circle of radius 1 about the origin; as this projection circulates around the circle, the point itself rises in such a way that during a complete "turn" around the circle, the "rise" is one unit. The "corkscrew" curve traced out by this motion is called a **helix** (Figure 2.21).

While this can be considered as the locus of the pair of equations $x = \cos 2\pi z$, $y = \sin 2\pi z$, such a description gives us far less insight into the curve than the parametrized version.

As another example, let us parametrize the locus of the pair of equations

$$x^2 + y^2 = 1, \quad y + z = 0$$

which, geometrically, is the intersection of a vertical cylinder with a plane. The projection of the cylinder on the xy-plane is easily parametrized by $x = \cos t$, $y = \sin t$, and

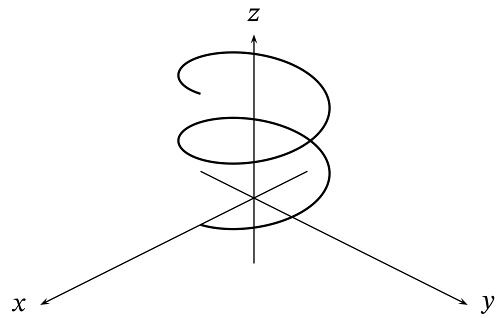


Figure 2.21. Helix

then substitution into the equation of the plane gives us $z = -\sin t$. Thus, this curve can be described by the function $\vec{p} : \mathbb{R} \to \mathbb{R}^3$

$$\vec{p}(t) = (\cos t, \sin t, -\sin t).$$

This is illustrated in Figure 2.22. Note that it is an *ellipse*, not a circle (for example, it intersects the x-axis in a line of length 2, but it intersects the yz-plane in the points $(0, \pm 1, \mp 1)$, which are distance $\sqrt{2}$ apart).

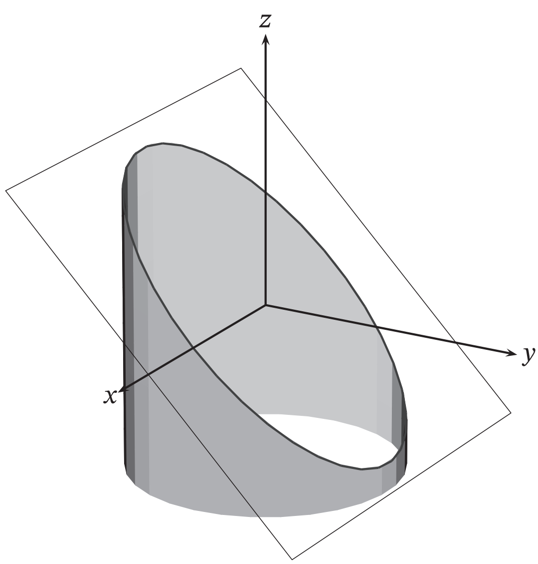


Figure 2.22. Intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 0$.

How would we parametrize a *circle* in the plane $y + z = 0$, centered at the origin? One way is to set up a rectangular coordinate system in this plane, much like we did for conic sections on p. 83, by finding two unit vectors parallel to the plane and perpendicular to each other. Start from the locus of the equation $y + z = 0$ in the yz-plane; this is a line with direction vector $\vec{\jmath} - \vec{k}$ or, normalizing (so it becomes a unit vector) $\vec{u_1} = (\vec{\jmath} - \vec{k})/\sqrt{2}$. The x-axis (defined by $y = 0$ and $z = 0$) also lies in the plane $y + z = 0$, and its direction vector $\vec{\imath}$ is a unit vector perpendicular to $\overrightarrow{u_1}$, so we can take it to be $\overrightarrow{u_2}$. Then the circle of radius 1 about the origin in the plane $y + z = 0$ consists of vectors of the form

$$\vec{p}(\theta) = (\cos\theta)\overrightarrow{u_1} + (\sin\theta)\overrightarrow{u_2} = (\cos\theta, \frac{1}{\sqrt{2}}\sin\theta, -\frac{1}{\sqrt{2}}\sin\theta).$$

This is sketched in Figure 2.23.

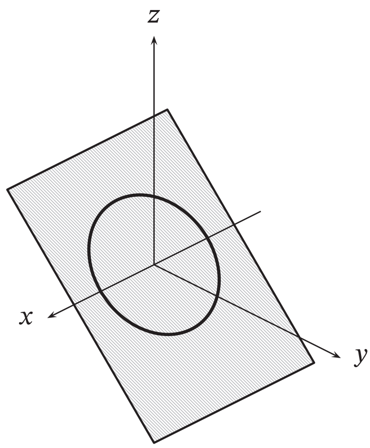


Figure 2.23. Circle of radius 1 about the Origin in the Plane $y + z = 0$.

Exercises for § 2.2

Practice problems:

(1) Parametrize each plane curve below, indicating an interval of parameter values over which the curve is traversed once:
   (a) The circle of radius 5 with center $(2, 3)$.
   (b) The ellipse centered at $(1, 2)$ with horizontal semimajor axis 3 and vertical semiminor axis 1.
   (c) The upper branch of the hyperbola $y^2 - x^2 = 4$.
   (d) The lower branch of the hyperbola $4y^2 - x^2 = 1$.

(2) Sketch the curve traced out by each function $\vec{p} : \mathbb{R} \to \mathbb{R}^2$:
   (a) $\vec{p}(t) = (t, \sin t)$
   (b) $\vec{p}(t) = (\cos t, t)$
   (c) $\vec{p}(t) = (3\cos t, \sin t)$
   (d) $\vec{p}(t) = (t\cos t, t\sin t)$
   (e) $\vec{p}(t) = (t + \sin t, t + \cos t)$

(3) Sketch the curve given by the polar equation:
   (a) $r = 3\cos\theta$
   (b) $r = \sin 3\theta$
   (c) $r = \sin 4\theta$
   (d) $r = 1 - \cos\theta$
   (e) $r = 2\cos 2\theta$

(4) Parametrize each of the curves in $\mathbb{R}^3$ described below:
   (a) The intersection of the plane $x + y + z = 1$ with the cylinder $y^2 + z^2 = 1$
   (b) The circle of radius 1, centered at $(1, 1, 1)$, and lying in the plane $x+y+z = 3$.
   (c) A curve lying on the cone $z = \sqrt{x^2 + y^2}$ which rotates about the z-axis while rising in such a way that in one rotation it rises 2 units. (*Hint:* Think cylindrical.)
   (d) The great circle[12] on the sphere of radius 1 about the origin which goes through the points $(1, 0, 0)$ and $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

Theory problems:

(5) Verify that the following construction (see Figure 2.24) yields the parametrization of the ellipse given by Equation (2.17):

Imagine a pair of circles centered at the origin, one circumscribed (with radius the semi-major axis a), the other inscribed (with radius the semi-minor axis b) in the ellipse.

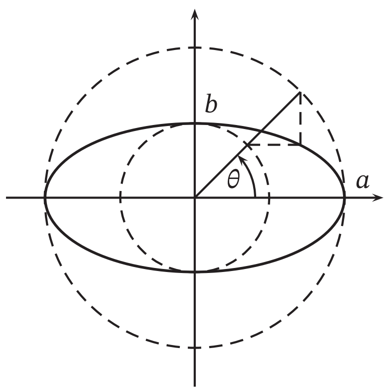


Figure 2.24. Parametrization of an Ellipse

Draw a ray at angle θ with the positive x-axis; the point $\vec{p}(\theta)$ is the intersection of two lines—one vertical, the other horizontal—through the intersections of the ray with the two circles.

(6) Using the definition of the hyperbolic cosine and sine (Equation (2.18)), prove that they satisfy the identities:
   (a) $\cosh^2 t - \sinh^2 t = 1$.
   (b) $\cosh^2 t = \frac{1}{2}(1 + \cosh 2t)$.
   (c) $\sinh^2 t = \frac{1}{2}(\cosh 2t - 1)$.

Challenge problem:

(7) Using Euler's formula

$$e^{a+bi} = e^a(\cos b + i \sin b)$$

[12]A **great circle** on a sphere is a circle whose center is the center of the sphere.

prove the identities

$$\cos t = \frac{e^{it} + e^{-it}}{2}$$
$$\sin t = \frac{e^{it} - e^{-it}}{2i}$$

and use these to justify the definitions

$$\cos it = \cosh t$$
$$\sin it = i \sinh t.$$

(8) (a) A wheel of radius 1 in the plane, rotating counterclockwise with angular velocity ω_1 rotations per second, is attached to the end of a stick of length 3 whose other end is fixed at the origin, and which itself is rotating counterclockwise with angular velocity ω_2 rotations per second. Parametrize the motion of a point on the rim of the wheel.

(b) A wheel of radius 1 in the plane rolls along the outer edge of the disc of radius 3 centered at the origin. Parametrize the motion of a point on the rim.

(9) A vertical plane $\mathcal{P}$ through the z-axis makes an angle θ radians with the xz-plane counterclockwise (seen from above). The **torus** $\mathcal{T}$ consists of all points in $\mathbb{R}^3$ at distance 1 from the circle $x^2 + y^2 = 9$, $z = 0$ in the xy-plane. Parametrize the intersection $\mathcal{P} \cap \mathcal{T}$ of these surfaces. (*Hint:* It is a circle.)

(10) Parametrize the path in space of a point on the wheel of a unicycle of radius a which is ridden along a circular path of radius b centered at the origin. (*Hint:* Note that the plane of the unicycle is vertical and contains, at any moment, the line tangent to the path at the point of contact with the wheel. Note also that as the wheel turns, it travels along the path a distance given by the amount of rotation (in radians) times the radius of the wheel.)

2.3 Calculus of Vector-Valued Functions

To apply methods of calculus to curves in $\mathbb{R}^2$ or $\mathbb{R}^3$ or equivalently to their parametrizations via vector-valued functions, we must first reformulate the basic notion of convergence, as well as differentiation and integration, in these contexts. [13]

Convergence of Sequences of Points. The convergence of sequences of points $\{\vec{p}_i\}$ in $\mathbb{R}^{2 \text{ or } 3}$ is a natural extension of the corresponding idea for numbers, or points on the line $\mathbb{R}$.

Before formulating a geometric definition of convergence, we note a few properties of the distance function on $\mathbb{R}^3$. The first property will allow us to use estimates on coordinates to obtain estimates on distances, and *vice versa*.

Lemma 2.3.1. *Suppose $P, Q \in \mathbb{R}^3$ have respective (rectangular) coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2). Let*

$$\delta := \max(|\triangle x|, |\triangle y|, |\triangle z|)$$

(where $\triangle x := x_2 - x_1$, etc.)

Then

$$\delta \le \operatorname{dist}(P, Q) \le \delta\sqrt{3}. \tag{2.19}$$

[13]Since much of what we have to say is essentially the same for the context of $\mathbb{R}^2$ or $\mathbb{R}^3$, we will sometimes adopt the notation $\mathbb{R}^{\mathbf{2\ or\ 3}}$ to talk about both contexts at once.

Proof. Since each of $(\triangle x)^2$, $(\triangle y)^2$ and $(\triangle z)^2$ is at least zero and at most δ^2 (and at least one of them equals δ^2), we have

$$\delta^2 \leq (\triangle x)^2 + (\triangle y)^2 + (\triangle z)^2 \leq \delta^2 + \delta^2 + \delta^2 = 3\delta^2$$

and taking square roots gives us Equation (2.19). □

In particular, we clearly have

$$\mathrm{dist}(P, Q) = 0 \iff P = Q. \tag{2.20}$$

The next important property is proved by a calculation which you do in Exercise 4.

Lemma 2.3.2 (Triangle Inequality). *For any three points $P, Q, R \in \mathbb{R}^3$,*

$$\mathrm{dist}(P, Q) \leq \mathrm{dist}(P, R) + \mathrm{dist}(R, Q). \tag{2.21}$$

With these properties in hand, we consider the notion of convergence for a sequence $\{\vec{p}_i\}$ of points $\vec{p}_i \in \mathbb{R}^3$. The definition is an almost verbatim translation of the corresponding notion for sequences of numbers (*i.e.*, of points in $\mathbb{R}$) (see *Calculus Deconstructed*, Dfn. 2.2.2, or another single-variable calculus text).

Definition 2.3.3. *A sequence of points $\vec{p}_i \in \mathbb{R}^3$ **converges** to a point $\vec{L} \in \mathbb{R}^3$ if for every desired accuracy $\varepsilon > 0$ there exists a place N in the sequence such that every later point of the sequence approximates $\vec{L}$ with accuracy ε:*

$$i > N \textit{ guarantees } \mathrm{dist}(\vec{p}_i, \vec{L}) < \varepsilon.$$

We will write

$$\vec{p}_i \to \vec{L}$$

in this case.

An immediate corollary of the triangle inequality is the uniqueness of limits (see Exercise 5 for a proof):

Corollary 2.3.4. *If a sequence $\{\vec{p}_i\}$ converges to $\vec{L}$ and also to $\vec{L}'$, then $\vec{L} = \vec{L}'$.*

As a result of Corollary 2.3.4, if $\vec{p}_i \to \vec{L}$ we can refer to $\vec{L}$ as *the* **limit** of the sequence, and write

$$\vec{L} = \lim \vec{p}_i.$$

A sequence is **convergent** if it has a limit, and **divergent** if it has none.

The next result lets us relate convergence of points to convergence of their coordinates. A proof is outlined in Exercise 7.

Lemma 2.3.5. *Suppose $\{\vec{p}_i\}$ is a sequence of points in $\mathbb{R}^3$ with respective coordinates (x_i, y_i, z_i) and $\vec{L} \in \mathbb{R}^3$ has coordinates (ℓ_1, ℓ_2, ℓ_3). Then the following are equivalent:*

(1) $\vec{p}_i \to \vec{L}$ *(in $\mathbb{R}^3$);*
(2) $x_i \to \ell_1$, $y_i \to \ell_2$, *and* $z_i \to \ell_3$ *(in $\mathbb{R}$).*

As in $\mathbb{R}$, we say a sequence $\{\vec{p}_i\}$ of points is **bounded** if there is a finite upper bound on the distance of all the points in the sequence from the origin—that is, there is a real number M such that

$$\mathrm{dist}(\vec{p}_i, \vec{0}) \leq M \text{ for all } i.$$

An easy analogue of a basic property of sequences of numbers is the following, whose proof we leave to you (Exercise 6):

Remark 2.3.6. *Every convergent sequence is bounded.*

A major difference between sequences of numbers and sequences of points in $\mathbb{R}^3$ is that there is no natural way to compare two points: a statement like "$P < Q$" does not make sense for points in space. As a result, there is no natural way to speak of monotone sequences, and correspondingly we cannot think about, for example, the maximum or supremum of a (bounded) sequence of points. What we *can* do, however, is think about the maximum or supremum of a sequence of *numbers* associated to a sequence of points—we have already seen an instance of this in the definition of boundedness for a sequence.

One consequence of the lack of natural inequalities between points is that we cannot translate the Completeness Axiom (see *Calculus Deconstructed*, Axiom 2.3.2, or another single-variable calculus text) directly to $\mathbb{R}^3$. However, the Bolzano-Weierstrass Theorem (see *Calculus Deconstructed*, Prop. 2.3.8, or another single-variable calculus text), which is an effective substitute for the Completeness Axiom, can easily be extended from sequences of numbers to sequences of points (see Exercise 8):

Proposition 2.3.7 (Bolzano-Weierstrass Theorem). *Every bounded sequence of points in $\mathbb{R}^3$ has a convergent subsequence.*

In the exercises, you will check a number of features of convergence (and divergence) which carry over from sequences of numbers to sequences of points.

Continuity of Vector-Valued Functions. Using the notion of convergence formulated in the previous subsection, the notion of continuity for real-valued functions extends naturally to vector-valued functions.

Definition 2.3.8. *$\vec{f}: \mathbb{R} \to \mathbb{R}^3$ is **continuous** on $D \subset \mathbb{R}$ if for every convergent sequence $t_i \to t$ in D the sequence of points $\vec{f}(t_i)$ converges to $\vec{f}(t)$.*

Every function from $\mathbb{R}$ to $\mathbb{R}^3$ can be expressed as

$$\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$$

or

$$\vec{f}(t) = f_1(t)\vec{\imath} + f_2(t)\vec{\jmath} + f_3(t)\vec{k},$$

where $f_1(t)$, $f_2(t)$, and $f_3(t)$, the **component functions** of $\vec{f}(t)$, are ordinary (real-valued) functions. Using Lemma 2.3.5, it is easy to connect continuity of $\vec{f}(t)$ with continuity of its components:

Remark 2.3.9. *A function $\vec{f}: \mathbb{R} \to \mathbb{R}^3$ is continuous on $D \subset \mathbb{R}$ precisely if each of its components $f_1(t)$, $f_2(t)$, $f_3(t)$ is continuous on D.*

A related notion, that of limits of functions, is an equally natural generalization of the single-variable idea:

Definition 2.3.10. *$\vec{f}: \mathbb{R} \to \mathbb{R}^3$ **converges** to $\vec{L} \in \mathbb{R}^3$ as $t \to t_0$ if t_0 is an accumulation point of the domain of $\vec{f}(t)$ and for every sequence $\{t_i\}$ in the domain of $\vec{f}$ which converges to, but is distinct from, t_0, the sequence of points $p_i = \vec{f}(t_i)$ converges to $\vec{L}$.*

We write

$$\vec{f}(t) \to \vec{L} \text{ as } t \to t_0$$

or

$$\vec{L} = \lim_{t \to t_0} \vec{f}(t)$$

when this holds.

Again, convergence of $\vec{f}$ relates immediately to convergence of its components:

Remark 2.3.11. *$\vec{f} : \mathbb{R} \to \mathbb{R}^3$ converges to $\vec{L}$ as $t \to t_0$ precisely when the components of $\vec{f}$ converge to the components of $\vec{L}$ as $t \to t_0$.*

If any of the component functions diverges as $t \to t_0$, then so does $\vec{f}(t)$.

The following algebraic properties of limits are easy to check (Exercise 11):

Proposition 2.3.12. *Suppose $\vec{f}, \vec{g} : \mathbb{R} \to \mathbb{R}^3$ satisfy $\vec{L}_f = \lim_{t \to t_0} \vec{f}(t)$ and $\vec{L}_g = \lim_{t \to t_0} \vec{g}(t)$, and $r : \mathbb{R} \to \mathbb{R}$ satisfies $L_r = \lim_{t \to t_0} r(t)$. Then*

(1) $\lim_{t \to t_0} \left[\vec{f}(t) \pm \vec{g}(t)\right] = \vec{L}_f \pm \vec{L}_g$

(2) $\lim_{t \to t_0} r(t)\vec{f}(t) = L_r \vec{L}_f$

(3) $\lim_{t \to t_0} \left[\vec{f}(t) \cdot \vec{g}(t)\right] = \vec{L}_f \cdot \vec{L}_g$

(4) $\lim_{t \to t_0} \left[\vec{f}(t) \times \vec{g}(t)\right] = \vec{L}_f \times \vec{L}_g$.

Derivatives of Vector-Valued Functions. When we think of a function $\vec{f} : \mathbb{R} \to \mathbb{R}^3$ as describing a moving point, it is natural to ask about its velocity, acceleration, and so on. For this, we need to extend the notion of differentiation. We shall often use the Newtonian "dot" notation for the derivative of a vector-valued function interchangeably with the "prime" notation we have used so far.

Definition 2.3.13. *The **derivative** of the function $\vec{f} : \mathbb{R} \to \mathbb{R}^3$ at an interior point t_0 of its domain is the limit*

$$\dot{\vec{f}}(t_0) = \overrightarrow{f'}(t_0) = \left.\frac{d}{dt}\right|_{t=t_0} \left[\vec{f}\right] = \lim_{h \to 0} \frac{1}{h} \left[\vec{f}(t_0 + h) - \vec{f}(t_0)\right]$$

provided it exists. (If not, the function is not differentiable at $t = t_0$.)

Again, using Lemma 2.3.5, we connect this with differentiation of the component functions:

Remark 2.3.14. *The vector-valued function*

$$\vec{f}(t) = (x(t), y(t), z(t))$$

is differentiable at $t = t_0$ precisely if all of its component functions are differentiable at $t = t_0$, and then

$$\overrightarrow{f'}(t_0) = (x'(t_0), y'(t_0), z'(t_0)).$$

In particular, every differentiable vector-valued function is continuous.

When $\vec{p}(t)$ describes a moving point, then its derivative is referred to as the **velocity** of $\vec{p}(t)$

$$\vec{v}(t_0) = \dot{\vec{p}}(t_0)$$

and the derivative of velocity is **acceleration**

$$\vec{a}\,(t_0) = \dot{\vec{v}}\,(t_0) = \ddot{\vec{p}}\,(t_0)\,.$$

The *magnitude* of the velocity is the **speed**, sometimes denoted

$$\frac{d\mathfrak{s}}{dt} = \left\|\vec{v}\,(t)\right\|.$$

Note the distinction between *velocity*, which has a direction (and hence is a vector) and *speed*, which has no direction (and is a scalar).

For example, the point moving along the helix

$$\vec{p}\,(t) = (\cos 2\pi t, \sin 2\pi t, t)$$

has velocity

$$\vec{v}\,(t) = \dot{\vec{p}}\,(t) = (-2\pi \sin 2\pi t, 2\pi \cos 2\pi t, 1)$$

speed

$$\frac{d\mathfrak{s}}{dt} = \sqrt{4\pi^2 + 1}$$

and acceleration

$$\vec{a}\,(t) = \dot{\vec{v}}\,(t) = (-4\pi^2 \cos 2\pi t, -4\pi^2 \sin 2\pi t, 0).$$

The relation of derivatives to vector algebra is analogous to the situation for real-valued functions. The proofs of these statements are outlined in Exercise 12.

Theorem 2.3.15. *Suppose the vector-valued functions $\vec{f}, \vec{g} \colon I \to \mathbb{R}^3$ are differentiable on I. Then the following are also differentiable:*

Linear Combinations: *For any real constants $\alpha, \beta \in \mathbb{R}$, the function*

$$\alpha\vec{f}\,(t) + \beta\vec{g}\,(t)$$

is differentiable on I, and

$$\frac{d}{dt}\left[\alpha\vec{f}\,(t) + \beta\vec{g}\,(t)\right] = \alpha\overrightarrow{f'}(t) + \beta\vec{g}'\,(t)\,. \tag{2.22}$$

Products: [14]

- *The product with any differentiable real-valued function $\alpha(t)$ on I is differentiable on I:*

$$\frac{d}{dt}\left[\alpha(t)\vec{f}\,(t)\right] = \alpha'(t)\vec{f}\,(t) + \alpha(t)\overrightarrow{f'}(t). \tag{2.23}$$

- *The dot product (resp. cross product) of two differentiable vector-valued functions on I is differentiable on I:*

$$\frac{d}{dt}\left[\vec{f}\,(t) \cdot \vec{g}\,(t)\right] = \overrightarrow{f'}(t) \cdot \vec{g}\,(t) + \vec{f}\,(t) \cdot \vec{g}'\,(t) \tag{2.24}$$

$$\frac{d}{dt}\left[\vec{f}\,(t) \times \vec{g}\,(t)\right] = \overrightarrow{f'}(t) \times \vec{g}\,(t) + \vec{f}\,(t) \times \vec{g}'\,(t)\,. \tag{2.25}$$

[14] These are the **product rules** or **Leibniz formulas** for vector-valued functions of one variable.

Compositions: [15] *If* $\mathfrak{t}(s)$ *is a differentiable function on* J *and takes values in* I*, then the composition* $(\vec{f}\circ\mathfrak{t})(s)$ *is differentiable on* J*:*

$$\frac{d}{ds}\left[\vec{f}(\mathfrak{t}(s))\right]=\frac{d\vec{f}}{dt}\frac{dt}{ds}=\overrightarrow{f'}(\mathfrak{t}(s))\mathfrak{t}'(s). \tag{2.26}$$

An interesting and useful corollary of this is

Corollary 2.3.16. *Suppose* $\vec{f}\colon\mathbb{R}\to\mathbb{R}^3$ *is differentiable, and let*

$$\rho(t):=\left\|\vec{f}(t)\right\|.$$

Then $\rho^2(t)$ *is differentiable, and*

(1) $\dfrac{d}{dt}\left[\rho^2(t)\right]=2\vec{f}(t)\cdot\overrightarrow{f'}(t)$.

(2) $\rho(t)$ *is constant precisely if* $\vec{f}(t)$ *is always perpendicular to its derivative.*

(3) *If* $\rho(t_0)\neq 0$*, then* $\rho(t)$ *is differentiable at* $t=t_0$*, and* $\rho'(t_0)$ *equals the component of* $\overrightarrow{f'}(t_0)$ *in the direction of* $\vec{f}(t_0)$*:*

$$\rho'(t_0)=\frac{\overrightarrow{f'}(t_0)\cdot\vec{f}(t_0)}{\left\|\vec{f}(t_0)\right\|}. \tag{2.27}$$

A proof is sketched in Exercise 13.

Linearization of Vector-Valued Functions. In single-variable calculus, an important application of the derivative of a function $f(x)$ is to define its linearization or degree-one Taylor polynomial at a point $x=a$:

$$T_a f(x):=f(a)+f'(a)(x-a).$$

This function is the affine function (*e.g.*, polynomial of degree one) which best approximates $f(x)$ when x takes values near $x=a$; one formulation of this is that the linearization has **first-order contact** with $f(x)$ at $x=a$:

$$\lim_{x\to a}\frac{|f(x)-T_a f(x)|}{|x-a|}=0;$$

or, using “little-oh” notation, $f(x)-T_a f(x)=\mathfrak{o}(x-a)$. This means that the closer x is to a, the smaller is the discrepancy between the easily calculated affine function $T_a f(x)$ and the (often more complicated) function $f(x)$, even when we measure the discrepancy as a percentage of the value. The graph of $T_a f(x)$ is the line tangent to the graph of $f(x)$ at the point corresponding to $x=a$.

Linearization has a straightforward analogue for vector-valued functions:

Definition 2.3.17. *The* ***linearization*** *of a differentiable vector-valued function* $\vec{p}(t)$ *at* $t=t_0$ *is the vector-valued function*

$$T_{t_0}\vec{p}(t)=\vec{p}(t_0)+t\vec{p}\,'(t_0)$$

whose components are the linearizations of the component functions of $\vec{p}(t)$*: if*

$$\vec{p}(t)=x(t)\vec{\imath}+y(t)\vec{\jmath},$$

[15] This is a **chain rule** for curves

then the linearization at $t = t_0$ is

$$T_{t_0}\vec{p}(t) = (T_{t_0}x(t))\vec{\imath} + (T_{t_0}y(t))\vec{\jmath}$$
$$= (x(t_0) + x'(t_0)t)\vec{\imath} + (y(t_0) + y'(t_0)t)\vec{\jmath}.$$

A component-by-component analysis (Exercise 15) easily gives

Remark 2.3.18. *The vector-valued functions $T_{t_0}\vec{p}(t)$ and $\vec{p}(t)$ have* ***first-order contact*** *at $t = t_0$:*

$$\lim_{t\to t_0} \frac{\vec{p}(t) - T_{t_0}\vec{p}(t)}{|t - t_0|} = \vec{0}.$$

When we interpret $\vec{p}(t)$ as describing the motion of a point in the plane or in space, we can interpret $T_{t_0}\vec{p}(t)$ as the constant-velocity motion which would result, according to Newton's First Law of motion, if all the forces making the point follow $\vec{p}(t)$ were instantaneously turned off at time $t = t_0$. If the velocity $\vec{p}\,'(t_0)$ is a nonzero vector, then $T_{t_0}\vec{p}(t)$ traces out a line with direction vector $\vec{p}\,'(t_0)$, which we call the **tangent line** to the motion at $t = t_0$.

Integration of Vector-Valued Functions. Integration also extends to vector-valued functions componentwise. Given $\vec{f}\colon [a,b] \to \mathbb{R}^3$ and a partition $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a,b]$, we can't form upper or lower sums, since the "sup" and "inf" of $\vec{f}(t)$ over I_j don't make sense. However we *can* form (vector-valued) Riemann sums

$$\mathcal{R}(\mathcal{P}, \vec{f}, \{t_j^*\}) = \sum_{j=1}^{n} \vec{f}\left(t_j^*\right) \triangle t_j$$

and ask what happens to these Riemann sums for a sequence of partitions whose mesh size goes to zero. If all such sequences have a common (vector) limit, we call it the **definite integral** of $\vec{f}(t)$ over $[a,b]$. It is natural (and straightforward to verify, using Lemma 2.3.5) that this happens precisely if each of the component functions $f_i(t)$, $i = 1,2,3$ is integrable over $[a,b]$, and then

$$\int_a^b \vec{f}(t)\,dt = \left(\int_a^b f_1(t)\,dt, \int_a^b f_2(t)\,dt, \int_a^b f_3(t)\,dt\right).$$

A direct consequence of this and the Fundamental Theorem of Calculus is that the integral of (vector) velocity is the net (vector) displacement:

Lemma 2.3.19. *If $\vec{v}(t) = \dot{\vec{p}}(t)$ is continuous on $[a,b]$, then*

$$\int_a^b \vec{v}(t)\,dt = \vec{p}(b) - \vec{p}(a).$$

The proof of this is outlined in Exercise 14.

Exercises for § 2.3

Answers to Exercises 1aci, 2a, and 3a are given in Appendix A.13.

Practice problems:

(1) For each sequence $\{\vec{p_n}\}$ below, find the limit, or show that none exists.

(a) $\left(\frac{1}{n}, \frac{n}{n+1}\right)$

(b) $(\cos(\frac{\pi}{n}), \sin(\frac{n\pi}{n+1}))$
(c) $(\sin(\frac{1}{n}), \cos(n))$
(d) $(e^{-n}, n^{1/n})$
(e) $\left(\frac{n}{n+1}, \frac{n}{2n+1}, \frac{2n}{n+1}\right)$
(f) $\left(\frac{n}{n+1}, \frac{n}{n^2+1}, \frac{n^2}{n+1}\right)$
(g) $(\sin \frac{n\pi}{n+1}, \cos \frac{n\pi}{n+1}, \tan \frac{n\pi}{n+1})$
(h) $(\frac{1}{n} \ln n, \frac{1}{\sqrt{n^2+1}}, \frac{1}{n} \ln \sqrt{n^2+1})$
(i) $(x_1, y_1, z_1) = (1, 0, 0), \quad (x_{n+1}, y_{n+1}, z_{n+1}) = (y_n, z_n, 1 - \frac{x_n}{n})$
(j) $(x_1, y_1, z_1) = (1, 2, 3), \quad (x_{n+1}, y_{n+1}, z_{n+1}) = (x_n + \frac{1}{2}y_n, y_n + \frac{1}{2}z_n, \frac{1}{2}z_n)$

(2) An **accumulation point** of a sequence $\{\vec{p}_i\}$ of points is any limit point of any subsequence. Find all the accumulation points of each sequence below.
(a) $\left(\frac{1}{n}, \frac{(-1)^n n}{n+1}\right)$
(b) $\left(\frac{n}{n+1} \cos n, \frac{n}{n+1} \sin n\right)$
(c) $\left(\frac{n}{n+1}, \frac{(-1)^n n}{2n+1}, (-1)^n \frac{2n}{n+1}\right)$
(d) $\left(\frac{n}{n+1} \cos \frac{n\pi}{2}, \frac{n}{n+1} \sin \frac{n\pi}{2}, \frac{2n}{n+1}\right)$

(3) For each vector-valued function $\vec{p}(t)$ and time $t = t_0$ below, find the linearization $T_{t_0}\vec{p}(t)$.
(a) $\vec{p}(t) = (t, t^2)$, $t = 1$
(b) $\vec{p}(t) = t^2\vec{\imath} - t^3\vec{\jmath}$, $t = 2$
(c) $\vec{p}(t) = (\sin t, \cos t)$, $t = \frac{4\pi}{3}$
(d) $\vec{p}(t) = (2t+1)\vec{\imath} + (3t^2-2)\vec{\jmath}$, $t = 2$
(e) $\vec{p}(t) = (\sin t, \cos t, 2t)$, $t = \frac{\pi}{6}$
(f) $\vec{p}(t) = (\sin t)\vec{\imath} + (\cos 2t)\vec{\jmath} + (\cos t)\vec{k}$, $t = \frac{\pi}{2}$

Theory problems:

(4) Prove the **triangle inequality**

$$\text{dist}(P, Q) \le \text{dist}(P, R) + \text{dist}(R, Q)$$

(a) in $\mathbb{R}^2$;
(b) in $\mathbb{R}^3$.

(*Hint:* Replace each distance with its definition. Square both sides of the inequality and expand, cancelling terms that appear on both sides, and then rearrange so that the single square root is on one side; then square again and move all terms to the same side of the equals sign (with zero on the other). Why is the given quantity non-negative?

You may find it useful to introduce some notation for differences of coordinates, for example

$$\triangle x_1 = x_2 - x_1$$
$$\triangle x_2 = x_3 - x_2;$$

note that then

$$\triangle x_1 + \triangle x_2 = x_3 - x_1.)$$

(5) Prove Corollary 2.3.4 as follows:

For any $\varepsilon > 0$, we can find integers N and N' so that $\operatorname{dist}(\vec{p}_i, \vec{L}) < \varepsilon$ for every $i > N$ and also $\operatorname{dist}(\vec{p}_i, \vec{L}') < \varepsilon$ for every $i > N'$.

Show how, given any index i beyond both N and N', we can use the triangle inequality (in $\mathbb{R}$) to write

$$\operatorname{dist}(\vec{L}, \vec{L}') < 2\varepsilon.$$

But this says that $\operatorname{dist}(\vec{L}, \vec{L}')$ is less than any positive number and hence equals zero, so $\vec{L} = \vec{L}'$ by Equation (2.20).

(6) Show that if $\vec{p}_i \to L$ in $\mathbb{R}^3$, then $\{\vec{p}_i\}$ is bounded.

(7) Prove Lemma 2.3.5 as follows:

(a) Suppose $\vec{p}_i \to \vec{L}$. Given $\varepsilon > 0$, we can find N so that $i > N$ guarantees $\operatorname{dist}(\vec{p}_i, \vec{L}) < \varepsilon$. But then by Lemma 2.3.1

$$\max(|x_i - \ell_1|, |y_i - \ell_2|, |z_i - \ell_3|) < \varepsilon,$$

showing that each of the coordinate sequences converges to the corresponding coordinate of $\vec{L}$.

(b) Conversely, suppose $x_i \to \ell_1$, $y_i \to \ell_2$, and $z_i \to \ell_3$. Given $\varepsilon > 0$, we can find

$$N_1 \text{ so that } i > N_1 \text{ guarantees } |x_i - \ell_1| < \frac{\varepsilon}{\sqrt{3}}$$
$$N_2 \text{ so that } i > N_2 \text{ guarantees } |y_i - \ell_2| < \frac{\varepsilon}{\sqrt{3}}$$
$$N_3 \text{ so that } i > N_3 \text{ guarantees } |z_i - \ell_3| < \frac{\varepsilon}{\sqrt{3}}.$$

Let $\vec{L} \in \mathbb{R}^3$ be the point with rectangular coordinates (ℓ_1, ℓ_2, ℓ_3).

$$i > N \text{ guarantees } \operatorname{dist}(\vec{p}_i, \vec{L}) < \sqrt{3}\frac{\varepsilon}{\sqrt{3}} = \varepsilon,$$

so $\vec{p}_i \to \vec{L}$.

(8) Prove Proposition 2.3.7 from the one-dimensional Bolzano-Weierstrass Theorem as follows: Suppose M is an upper bound on distances from the origin:

$$\operatorname{dist}(\vec{p}_i, \mathcal{O}) < M \text{ for all } i.$$

Show that we can pick a subsequence of $\{\vec{p}_i\}$ whose first coordinates form a convergent sequence of numbers. Then by the same reasoning, we can find a (sub-)subsequence for which the the second coordinates also converge, and finally a third (sub-sub-)subsequence for which the third coordinates *also* converge. Show that this last (sub-sub-)subsequence is in fact a convergent sequence of vectors.

(9) Suppose $\{\vec{p}_i\}$ is a sequence of points in $\mathbb{R}^3$ for which the distances between consecutive points form a convergent series:

$$\sum_0^\infty \operatorname{dist}(\vec{p}_i, \overrightarrow{p_{i+1}}) < \infty.$$

(a) Show that the sequence $\{\vec{p}_i\}$ is bounded. (*Hint:* Use the triangle inequality.)
(b) Show that the sequence is **Cauchy**—that is, for every $\varepsilon > 0$ there exists N so that $i, j > N$ guarantees $\operatorname{dist}(\vec{p}_i, \vec{p}_j) < \varepsilon$. (*Hint:* see *Calculus Deconstructed*, Exercise 2.5.9, or another single-variable calculus text.)
(c) Show that the sequence is convergent.

(10) This problem concerns some properties of accumulation points (Exercise 2).
(a) Show that a sequence with at least two distinct accumulation points diverges.
(b) Show that a *bounded* sequence has at least one accumulation point.
(c) Give an example of a sequence with *no* accumulation points.
(d) Show that a *bounded* sequence with *exactly one* accumulation point converges to that point.

(11) Prove Proposition 2.3.12.

(12) In this problem, we will prove Theorem 2.3.15.
(a) To prove Equation (2.22), apply standard differentiation formulas (in $\mathbb{R}$) to each component of

$$\vec{h}(t) = \alpha\vec{f}(t) + \beta\vec{g}(t)$$

to get

$$h_i'(t) = \alpha f_i'(t) + \beta g_i'(t), \quad i = 1, 2, 3$$

and then recombine using Remark 2.3.14.
(b) Use a similar strategy to prove Equation (2.23).
(c) Use a similar strategy to prove Equation (2.26).
(d) To prove Equation (2.24), set

$$h(t) = \vec{f}(t) \cdot \vec{g}(t);$$

then

$$\triangle h = \vec{f}(t + \triangle t) \cdot \vec{g}(t + \triangle t) - \vec{f}(t) \cdot \vec{g}(t).$$

Use the "Leibnz trick" (add and subtract a term) to get

$$\triangle h = \vec{f}(t + \triangle t) \cdot \vec{g}(t + \triangle t) - \vec{f}(t) \cdot \vec{g}(t + \triangle t) + \vec{f}(t) \cdot \vec{g}(t + \triangle t) - \vec{f}(t) \cdot \vec{g}(t).$$

Now divide by $\triangle t$ and take limits as $\triangle t \to 0$.
(e) Use a similar strategy to prove Equation (2.25).

(13) Prove Corollary 2.3.16.

(14) Prove Lemma 2.3.19. (*Hint:* Look at each component separately.)

(15) Use the fact that each component of the linearization $T_{t_0}\vec{p}(t)$ of $\vec{p}(t)$ has first-order contact with the corresponding component of $\vec{p}(t)$ to prove Remark 2.3.18.

Challenge problem:

(16) (David Bressoud) A missile travelling at constant speed is homing in on a target at the origin. Due to an error in its circuitry, it is consistently misdirected by a constant angle α. Find its path. Show that if $|\alpha| < \frac{\pi}{2}$ then it will eventually hit its target, taking $\frac{1}{\cos\alpha}$ times as long as if it were correctly aimed.

2.4 Regular Curves

Regular Plane Curves. In § 2.2 we saw how a vector-valued function $\vec{p}\,(t)$ specifies a curve by "tracing it out." This approach is particularly useful for specifying curves in space. It is also a natural setting (even in the plane) for applying calculus to curves. However, it has the intrinsic complication that a given curve can be traced out in many different ways.

Consider, for example, the upper semicircle of radius 1 centered at the origin in the plane

$$x^2 + y^2 = 1, \quad y > 0$$

which is the graph of the function $y = \sqrt{1-x^2}$ for $-1 < x < 1$. As is the case for the graph of any function, we can parametrize this curve using the input as the parameter; that is, the vector-valued function $\vec{p}\,(t) = (t, \sqrt{1-t^2})$ traces out the semicircle as t goes from -1 to 1. The velocity vector $\vec{p}\,'\,(t) = \vec{v}(t) = \vec{\imath} + \left(-\frac{t}{\sqrt{1-t^2}}\right)\vec{\jmath}$ is a direction vector for the line tangent to the graph at the point $\vec{p}\,(t)$; its slope is $f'\,(t)$. This parametrization is well-behaved from a calculus point of view; the features we are looking for are specified in the following definitions.

Definition 2.4.1. (1) *A vector-valued function $\vec{p}\,(t)$ defined on an interval I is* ***regular*** *on I if it is $\mathcal{C}^1$ and has nonvanishing velocity for every parameter value in I. (The velocity vector at an endpoint, if included in I, is the one-sided derivative, from the inside.)*

(2) *A* ***regular parametrization*** *of the curve $\mathcal{C}$ is a regular function defined on an interval I whose image equals $\mathcal{C}$ (i.e., $\vec{p}$ maps I* ***onto*** *$\mathcal{C}$).*[16]

(3) *A curve is* ***regular*** *if it can be represented by a regular parametrization.*

Note that in the case of the semicircle we cannot include the endpoints in the domain of the "graph" parametrization, since the velocity vector $\frac{d\vec{p}}{dt}$ is not defined there.

However, another natural parametrization of this semicircle uses the polar coordinate θ as the parameter: $q\,(\theta) = (\cos\theta, \sin\theta)$, with velocity vector $q\,(\theta) = (-\sin\theta, \cos\theta)$, for $0 < \theta < \pi$. How are these two parametrizations related?

Each point of the curve has a position vector $\vec{p}\,(t) = (t, \sqrt{1-t^2})$ for some $t \in (-1, 1)$, but it also has a position vector $\vec{q}\,(\theta) = (\cos\theta, \sin\theta)$ for some $\theta \in (0, \pi)$. Equating these two vectors

$$(t, \sqrt{1-t^2}) = (\cos\theta, \sin\theta)$$

we see immediately that for any angle $0 < \theta < \pi$ the point $\vec{q}\,(\theta)$ is the same as the point $\vec{p}\,(t)$ corresponding to the parameter value $t = \cos\theta$. We say that the vector-valued function $\vec{p}\,(t)$ is a **reparametrization** of $\vec{q}\,(\theta)$, and call the function $\mathbf{t}$ obtained by solving for t in terms of θ a **recalibration function**. Note in our example that

[16]The French-derived term for "onto" is **surjective**.

solving for θ in terms of t also yields a recalibration function, $\theta = \arccos t$, showing that $\vec{q}(\theta)$ is a reparametrization of $\vec{p}(t)$, as well.[17]

When we study a curve using a parametrization, we are interested in intrinsic *geometric* data presented in a way that is not changed by reparametrization.

Intuitively, reparametrizing a curve amounts to speeding up or slowing down the process of tracing it out. Since the speed with which we trace out a curve is certainly *not* an intrinsic property of the curve itself, we want to formulate data about the curve which does not change under reparametrization. One such datum is the tangent line.

Suppose $\vec{p}(t)$, $t \in I$ is a reparametrization of $\vec{q}(s)$, $s \in J$. If we differentiate and apply the Chain Rule, we see that the velocity $\vec{p}\,'(t)$ with which $\vec{p}(t)$ passes a given point and the corresponding velocity $\vec{q}\,'(s)$ for $\vec{q}(s)$ are dependent:

$$\vec{q}\,'(s) = \frac{d}{ds}\left[\vec{p}(\mathfrak{t}(s))\right] = \vec{p}\,'(\mathfrak{t}(s)) \cdot \mathfrak{t}'(s).$$

From this we can draw several conclusions:

Remark 2.4.2. *Suppose $\vec{q}(s)$ and $\vec{p}(t)$ are regular parametrizations of the same curve $\mathcal{C}$, and $\vec{p}(t)$ is a reparametrization of $\vec{q}(s)$ via the recalibration $t = \mathfrak{t}(s)$.*

(1) *Since $\vec{q}$ has nonvanishing velocity, $\mathfrak{t}'(s) \neq 0$; thus a recalibration function is strictly monotone. Also, it has a regular inverse, so $\vec{q}(s)$ is a reparametrization of $\vec{p}(t)$, as well.*
(2) *Since $\vec{q}(s)$ and $\vec{p}(\mathfrak{t}(s))$ are the same point and the velocities $\vec{q}\,'(s)$ and $\vec{p}\,'(\mathfrak{t}(s))$ are dependent, the line through this point with direction vector $\vec{q}\,'(s)$ is the same as the one with direction vector $\vec{p}\,'(\mathfrak{t}(s))$. We call this the **tangent line** to $\mathcal{C}$ at this point.*
(3) *The linearization $T_{t_0}\vec{p}(t)$ of $\vec{p}(t)$ at $t = t_0$ is a regular parametrization of the tangent line to $\mathcal{C}$ at the point $\vec{p}(t_0)$.*

We can standardize our representation of the line tangent to a curve at a point by concentrating on the **unit tangent vector** determined by a parametrization $\vec{p}(t)$,

$$\overrightarrow{T_p}(t) = \frac{\overrightarrow{v_p}(t)}{\left|\overrightarrow{v_p}(t)\right|}.$$

The unit tangent vector can be used as the direction vector for the tangent line to the curve at the point $\vec{p}(t_0)$. Remark 2.4.2 suggests that the unit tangent is unchanged if we compute it using a reparametrization $\vec{q}(s)$ of $\vec{p}(t)$. This is *almost* true, but not quite: if the derivative of the recalibration function $\mathfrak{t}(s)$ is negative, then the unit tangent vectors for $\vec{p}$ and $\vec{q}$ point in exactly opposite directions. This occurs in the example of the upper semicircle above. It reflects the fact that the "graph" parametrization $\vec{p}(t)$ traces the semicircle *left-to-right* (*i.e.*, clockwise), while the parametrization $\vec{q}(\theta)$ coming from polar coordinates traces it *right-to-left* (*i.e.*, counterclockwise): the recalibration function $\mathfrak{t}(\theta) = \cos\theta$ is strictly decreasing. Thus, a parametrization of a curve encodes not just the path of its motion, but also a *direction* along that path, which we will refer to as its **orientation**. We will say that a reparametrization **preserves orientation** if the recalibration has positive derivative (and hence is strictly increasing), and it **reverses orientation** if the recalibration has negative derivative (and hence is strictly decreasing). Since the recalibration has nonzero derivative everywhere, this derivative can't change sign, so these are the only two possibilities.

[17] It is crucial here that the substitution $\theta = \arccos t$ yields the correct functions for *both* x and y.

We shall see in § 2.5 that it is possible in principle to realize the unit tangent vector as the velocity of a suitable reparametrization of the curve.

Graphs of Functions. The parametrization of the semicircle as the graph of the function $\sqrt{1-x^2}$ is very easy to visualize: we can think of the parameter t as a point on the x-axis, and the function $\vec{p}$ simply pushes that point straight up until it hits the curve. In general, the graph of a function defined on an interval I is a very special kind of planar curve; in particular it must pass the **vertical line test**:[18] for every $x_0 \in I$, the vertical line $x = x_0$ must meet the curve in exactly one point. When $f(x)$ is a $\mathcal{C}^1$ function on the interval I, it is easy to see that the "graph" parametrization of its graph is a regular parametrization. Furthermore, it is **one-to-one**: distinct parameter values identify distinct points on the curve. Conversely, if a curve $\mathcal{C}$ has a regular parametrization whose velocity has a non vanishing horizontal component, then $\mathcal{C}$ passes the vertical line test and hence is the graph of a function (which can be shown to be $\mathcal{C}^1$). Of course, there are many curves in the plane which fail the vertical line test, beginning with the full circle.

Note that in the case of the full circle, while there are problems with using the "graph" parametrization near the two points that lie on the x-axis, we can regard the two "right" and "left" semicircles into which the y-axis divides the circle as graphs–but expressing x as a function of y instead of vice versa. This is actually a convenient general property of regular curves:

Proposition 2.4.3. *Suppose $\vec{p}(t)$ $(t \in I)$ is a regular parametrization of the curve $\mathcal{C}$ in the plane. Then $\vec{p}$ is* locally *the graph of a function: for each $t_0 \in I$, for every sufficiently small $\varepsilon > 0$, the subcurve parametrized by $\vec{p}$ restricted to $|t - t_0| \le \varepsilon$ is the graph of y as a function of x, or of x as a function of y. This function is $\mathcal{C}^1$.*

A proof of this is sketched in Exercise 5; from this we can conclude that the tangent line to a regular curve is an intrinsic, geometric item.

Theorem 3.4.2 (*resp.* Proposition A.6.2) gives a similar picture for level curves of functions $f(x, y)$.

Polar Curves. A large class of such curves are given by an equation of the form $r = f(\theta)$, which plot the "radial" polar coordinate as a function of the "angle" coordinate. It is easy to check that as long as f is a $\mathcal{C}^1$ function and $f(\theta) \neq 0$, the vector-valued function

$$\vec{p}(\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$$

is a regular parametrization of this curve (Exercise 2).

One example is the circle itself, where the "radial" coordinate is a constant function; another is the **spiral of Archimedes** (see p. 86), given by the polar equation $r = \theta$, and hence parametrized by $\vec{p}(\theta) = (\theta\cos\theta, \theta\sin\theta)$. In this case, even though the vertical and horizontal line tests fail, the parametrization (see Exercise 3) is one-to-one: each point gets "hit" once.

2.4.0.1 Arcs. We shall call a curve $\mathcal{C}$ an **arc** if it can be parametrized via a one-to-one vector-valued function on a *closed* interval $[a, b]$. One example is the graph of a (continuous) function on $[a, b]$, but there are clearly arcs which do not satisfy the vertical line test, for example the spiral of Archimedes described above, if we restrict

[18] This is the test for a curve to be the graph of y as a function of x; the analogous **horzontal line test** checks whether the curve is the graph of x as a function of y.

θ to a closed interval of length more than 2π. Arcs share many of the nice properties of graphs as curves; in particular, they have the property that any convergent sequence of points on $\mathcal{C}$ corresponds to a convergent sequence of parameter values, and as a result any two parametrizations of an arc are reparametrizations of each other. See Exercise 11 for some discussion of these properties. A corollary of Proposition 2.4.3 is that every regular curve can be represented as a sequence of arcs placed "end-to-end" (Exercise 12).

Piecewise-Regular Curves. Our definition of regularity for a parametrized curve applies to most of the curves we want to consider, but it excludes a few, notably polygons like triangles or rectangles and the cycloid (Figure 2.20). These have a few exceptional points at which the tangent vector is not well-defined. Generally, we define a parametrization (and the curve it traces out) to be **piecewise regular** on an interval if there is a finite partition such that the restriction to each closed atom[19] is regular; at the partition points we require that the parametrization be continuous and locally one-to-one. We shall not pursue this line further.

Regular Curves in Space. The theory we have outlined for planar curves applies as well to curves in space. A regular parametrization of a curve in space is a $\mathcal{C}^1$ vector-valued function of the form $\vec{p}\,(t) = (x\,(t)\,, y\,(t)\,, z\,(t))$ with non-vanishing velocity $\vec{v}\,(t) := \vec{p}\,'\,(t) = (x'\,(t)\,, y'\,(t)\,, z'\,(t)) \neq \vec{0}$ (or equivalently, non-zero speed $(x'\,(t))^2 + (y'\,(t))^2 + (z'\,(t))^2 \neq 0$). We can no longer talk about such a curve as the graph of a function, but we can get a kind of analogue of Proposition 2.4.3 which can play a similar role:

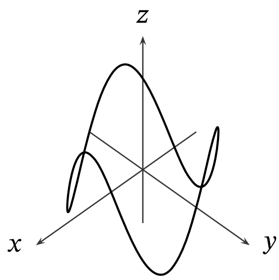
Remark 2.4.4. *If $\vec{p}\,(t)$ is a regular vector-valued function with values in $\mathbb{R}^3$, then locally its projections onto two of the three coordinate planes are graphs: more precisely, for each parameter value $t = t_0$ at least one of the component functions has nonzero derivative on an interval of the form $|t - t_0| < \varepsilon$ for $\varepsilon > 0$ sufficiently small; if the first component has this property, then the projection of the subcurve defined by this inequality onto the xy-plane (resp. xz-plane) is the graph of y (resp. of z) as a $\mathcal{C}^1$ function of x.*

From this we can conclude that, as in the planar case, the **tangent line** to the parametrization at any particular parameter value $t = t_0$ is well-defined, and is the line in space going through the point $\vec{p}\,(t_0)$ with direction vector $\vec{v}\,(t_0)$; furthermore, the linearization of $\vec{p}\,(t)$ at $t = t_0$ is a regular vector-valued function which parametrizes this line, and has first-order contact with $\vec{p}\,(t)$ at $t = t_0$.

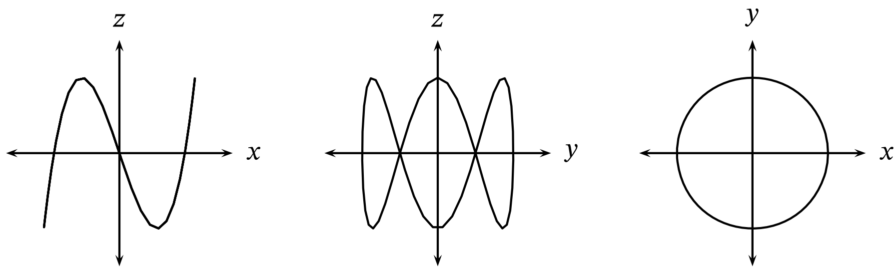
As a quick example, we consider the vector-valued function $\vec{p}\,(t) = (\cos t, \sin t, \cos 3t)$ with velocity $\vec{v}\,(t) = (-\sin t, \cos t, -3\sin 3t)$. Since $\sin t$ and $\cos t$ cannot both be zero at the same time, this is a regular parametrization of a curve in space, sketched in Figure 2.25. We note, for example, that $\frac{dx}{dt} = 0$ when t is an integer multiple of π: x is strictly decreasing as a function of t for $t \in [0, \pi]$, going from $x\,(0) = 1$ to $x\,(\pi) = -1$; as t goes from $t = 0$ to $t = \pi$, y goes from $y = 0$ to $y = 1$ and back again, and z goes from $z = 1$ (at $t = \frac{\pi}{3}$) to $z = -1$ to $z = 1$ (at $t = \frac{2\pi}{3}$) and back to $z = -1$. The projected point (x, y) traces out the upper semicircle in the projection on the xy-plane; meanwhile, (x, z) traces out the graph $z = 4x^3 - 3x$, both going right-to-left. As t goes from $t = \pi$ to $t = 2\pi$, (x, y) traces out the lower semicircle and (x, z) retraces the same graph as before, this time left-to-right.

---

[19]See footnote on p. 114



(a) 3D View



(b) Projections onto Coordinate Planes

Figure 2.25. The Parametric Curve $\vec{p}(t) = (\cos t, \sin t, \cos 3t)$

Similarly, $\frac{dy}{dt} = 0$ when t is an odd multiple of $\frac{\pi}{2}$; for $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, y is strictly increasing, and (x, y) traverses the right semicircle in the projection on the xy-plane counterclockwise, while (y, z) traverses the "W-shaped" graph $z = (1 - 4y^2)\sqrt{1 - y^2}$ in the yz-plane left-to-right.

Finally, $\frac{dt}{dx} = 0$ when t is an integer multiple of $\frac{\pi}{3}$. These correspond to six points on the curve, and correspondingly there are six subintervals on which z is strictly monotone. For example, as t goes from $t = 0$ to $t = \frac{\pi}{3}$, z goes from $z = 1$ $z = -1$, (x, z) traverses the leftmost branch of the projection on the xz-plane and (y, z) traces out the downward-sloping branch of the projection on the yz-plane, from the z-axis at $(0, 1)$ to the minimum point $\left(\frac{\sqrt{3}}{2}, -1\right)$ to its right.

Exercises for § 2.4

Practice problems:

(1) For each pair of vector-valued functions $\vec{p}\,(t)$ and $\vec{q}\,(t)$ below, find a recalibration function $\mathfrak{t}\,(s)$ so that $\vec{q}\,(s)=\vec{p}\,(\mathfrak{t}\,(s))$ and another, $\mathfrak{s}\,(t)$, so that $\vec{p}\,(t)=\vec{q}\,(\mathfrak{s}\,(t))$:
(a)

$$\vec{p}\,(t) = (t, t) \quad -1 \le t \le 1$$
$$\vec{q}\,(t) = (\cos t, \cos t) \quad 0 \le t \le \pi$$

(b)

$$\vec{p}(t) = (t, e^t) \quad -\infty < t < \infty$$
$$\vec{q}(t) = (\ln t, t) \quad 0 < t < \infty$$

(c)

$$\vec{p}(t) = (\cos t, \sin t) \quad -\infty < t < \infty$$
$$\vec{q}(t) = (\sin t, \cos t) \quad -\infty < t < \infty$$

(d)

$$\vec{p}(t) = (\cos t, \sin t, \sin 2t) \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$
$$\vec{q}(t) = (\sqrt{1-t^2}, t, t\sqrt{4-4t^2}) \quad -1 \leq t \leq 1$$

Theory problems:

(2) Let $\mathcal{C}$ be given by the polar equation $r = f(\theta)$ and set $\vec{p}(\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$. Assume that the function f is $\mathcal{C}^1$.
  (a) Show that if $f(\theta_0) \neq 0$, we have $\vec{v}(\theta_0) \neq \vec{0}$. (*Hint:* Calculate $\vec{v}(\theta)$ and the speed $|\vec{v}(\theta)|$.)
  (b) Show that if $f(\theta_0) = 0$ but $f'(\theta_0) \neq 0$, we still have $\vec{v}(\theta_0) \neq 0$.
  (c) Show that in the second case (*i.e.*, when the curve goes through the origin) the velocity makes angle θ_0 with the positive x-axis.

(3) Show directly that the vector-valued function giving the Spiral of Archimedes

$$\vec{p}(\theta) = (\theta\cos\theta, \theta\sin\theta), \qquad \theta \geq 0.$$

is regular. (*Hint:* what is its speed?) Show that it is one-to-one on each of the intervals $(-\infty, 0)$ and $(0, \infty)$.

(4) Consider the standard parametrization of the circle

$$\vec{p}(\theta) = (\cos\theta, \sin\theta).$$

  (a) Show that if I has length strictly greater than 2π, then the restriction of $\vec{p}$ to I cannot be one-to-one.
  (b) Show that if I is an interval of length strictly less than 2π, then the restriction of $\vec{p}$ to I cannot have the whole circle as its image.
  (c) Suppose I is an interval of length exactly 2π.
    (i) Show that if I is closed, then the restriction of $\vec{p}$ to I is not one-to-one.
    (ii) Show that if I is open, say $I = (a, b)$ with $b = a + 2\pi$, then $\lim_{t\to a}\vec{p}(t) = \lim_{t\to b}\vec{p}(t)$ is a point not in the image of $\vec{p}$.
    (iii) Suppose I is half-open (say $I = [a, b)$ with $b = a + 2\pi$). Show that $\vec{p}$ is one-to-one and onto. On the other hand, find a sequence of points $\vec{p}_i = \vec{p}(t_i) \to \vec{p}(a)$ such that $t_i\ 6\!\to a$.

(5) Prove Proposition 2.4.3 as follows:
   (a) Given $t_0 \in I$, at least one of $x'(t_0)$ and $y'(t_0)$ must be nonzero. Assume $x'(t_0) \neq 0$; without loss of generality, assume it is positive. Show that for some $\varepsilon > 0$ the x-coordinate $x(t)$ is strictly increasing on the interval $t_0 - \varepsilon < t < t_0 + \varepsilon$.
   (b) Show that the vertical-line test applies to $\vec{p}(t)$ on $(t_0 - \varepsilon, t_0 + \varepsilon)$. This means that this restriction lies on the graph of some function

$$y(t) = f(x(t)) \text{ for } t_0 - \varepsilon < t < t_0 + \varepsilon.$$

   (c) To show that $f(t)$ is $\mathcal{C}^1$, show that the slope of the velocity vector $\vec{v}(t_0)$ equals $f'(t_0)$: first, show that any sequence $x(t_i) \to x(t_0)$ must have $t_i \to t_0$. But then the corresponding y-values $y(t_i)$ must converge to $y(t_0)$, and the slopes of the secant lines satisfy

$$\frac{\triangle y}{\triangle x} \to \frac{y'(t_0)}{x'(t_0)},$$

   where the denominator in the last line is nonzero by assumption.
   (d) Complete the proof by showing how to modify the argument if either $x'(t_0)$ is negative, or if $x'(t_0)$ is zero but $y'(t_0)$ is not.

(6) Consider the curve $\mathcal{C}$ given by the polar equation

$$r = 2\cos\theta - 1$$

known as the **Limaçon of Pascal** (see Figure 2.26).

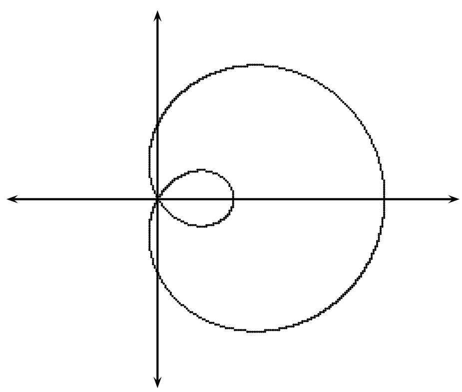


Figure 2.26. Limaçon of Pascal: $(x^2 - 2x + y^2)^2 = x^2 + y^2$

   (a) Find a regular parametrization of $\mathcal{C}$.
   (b) Verify that this curve is the locus of the equation

$$(x^2 - 2x + y^2)^2 = x^2 + y^2.$$

   (c) Find the equations of the two "tangent lines" at the crossing point at the origin.

(7) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$. *Show*:
   (a) f is one-to-one if and only if it is strictly monotone.
   (*Hint:* One direction is trivial. For the other direction, use the Intermediate Value Theorem: what does it mean to *not* be monotone?)

(b) If f is *locally* one-to-one, then it is *globally* one-to-one.
(c) Give an example of a function $f(x)$ which is one-to-one on $[-1,1]$ but is *not* strictly monotone on $[-1,1]$.

(8) (a) Suppose $\vec{p}(t)$ is a regular vector-valued function whose image satisfies the vertical line test—so that the curve it traces out is the graph of a function $f(x)$. Show that this function is $\mathcal{C}^1$, provided the velocity always has a nonzero horizontal component (that is, the velocity is never vertical). (*Hint:* Show that the "slope" of the velocity vector at any point $(x, f(x))$ equals the derivative of $f(x)$ at x.)
(b) Show that the vector-valued function $\vec{p}(t) = (t\,|t|, t^2)$ is $\mathcal{C}^1$ and one-to-one on the whole real line, but its image is the graph of the function $y = |x|$, which fails to be differentiable at $x = 0$. This shows the importance of the condition that the velocity is always nonzero in our definition of regularity.

(9) Prove Remark 2.4.4, as follows: Suppose $\mathcal{C}$ is parametrized by

$$\vec{p}(t) = (x(t), y(t), z(t))$$

with $\frac{dx}{dt} \neq 0$ at $t = t_0$, and hence nearby (*i.e.,* on J). Now consider the two plane curves parametrized by

$$\overrightarrow{p_y}(t) = (x(t), y(t))$$
$$\overrightarrow{p_z}(t) = (x(t), z(t)).$$

These are the projections of $\mathcal{C}$ onto, respectively the xy-plane and the xz-plane. Mimic the argument for Proposition 2.4.3 to show that each of these is the graph of the second coordinate as a function of the first.

Challenge problems:

(10) Suppose the planar curve $\mathcal{C}$ is the graph of a $\mathcal{C}^1$ function $f(x)$.
(a) Show that any other regular parametrization of $\mathcal{C}$ is a reparametrization of the "graph" parametrization $\vec{p}(t) = (t, f(t))$.
(b) Use this together with Proposition 2.4.3 to show that if $\vec{p}(t)$ and $\vec{q}(s)$ are one-to-one regular parametrizations of the same curve then each is a reparametrization of the other.

(11) **Arcs:** In this exercise, we study some properties of arcs.
(a) Suppose $\vec{p}: I \to \mathbb{R}^3$ is a continuous, one-to-one vector-valued function on the closed interval $I = [a, b]$ with image the arc $\mathcal{C}$, and suppose $\vec{p}(t_i) \to \vec{p}(t_0) \in \mathcal{C}$ is a convergent sequence of points in $\mathcal{C}$. *Show* that $t_i \to t_0$ in I. (*Hint:* Show that the sequence t_i must have at least one accumulation point t_* in I, and that for *every* such accumulation point t_*, we must have $\vec{p}(t_*) = \vec{p}(t_0)$. Then use the fact that $\vec{p}$ is one-to-one to conclude that the *only* accumulation point of t_i is $t_* = t_0$. But a bounded sequence with exactly one accumulation point must converge to that point.)
(b) Show that this property fails for the parametrization of the circle by $\vec{p}(\theta) = (\cos\theta, \sin\theta)$, $0 \leq \theta < 2\pi$.
(c) Give an example of an arc in space whose projections onto the three coordinate planes are not arcs. (This is a caution concerning how you answer the

next question.) *Note: this particular part of the problem is especially challenging, technically. If necessary, skip it and later read one solution, in the solution manual.*

(d) In particular, it follows from Proposition 2.4.3 that every regular curve in the plane is *locally* an arc. Show that every regular curve in space is also locally an arc.

(12) Here we will show that every regular curve can be expressed as a union of arcs placed "end-to-end". Suppose first that $\vec{p}(t) = (x(t), y(t))$, $a \leq t \leq b$ is a regular parametrization of the curve $\mathcal{C}$ in the plane.

(a) Show that if $\frac{dx}{dt} \neq 0$ for all $t \in [a, b]$ or $\frac{dx}{dt} \neq 0$ for all $t \in [a, b]$ then $\mathcal{C}$ is a regular arc.

(b) Now suppose that there are parameter values where $\frac{dx}{dt}$ vanishes and others where $\frac{dy}{dt}$ vanishes. Set $t_0 = a$. Note that by regularity there is no parameter value where both derivatives vanish. Suppose $\frac{dx}{dt} \neq 0$ at $t = a$, and let s_0 be the first parameter value for which $\frac{dx}{dt} = 0$. Then $\frac{dy}{dt} \neq 0$ there. If $\frac{dy}{dt} \neq 0$ for all $t > s_0$, stop; otherwise take t_1 to be the first parameter value above s_0 where $\frac{dy}{dt}$ vanishes. Then $\frac{dx}{dt}$ is nonzero there, and we can ask if it ever vanishes above t_1; if so, call s_1 the first such place. We can continue in this way, creating two increasing sequences of parameter values in $[a, b]$, $t_0 < s_0 < t_1 < \ldots$, such that $\frac{dx}{dt} \neq 0$ for $t_i \leq t < s_i$ and $\frac{dy}{dt} \neq 0$ for $s_i \leq t < t_i$. *Show:* that this sequence ends after finitely many steps with $b = s_k$ or $b = t_k$. (*Hint:* If not, both of the sequences t_i and s_i converge to a common limit in $[a, b]$; but this means that both derivatives vanish at the limit parameter value, a contradiction to regularity.)

(c) Show that by moving each of the points other than a and b down slightly, we obtain a partition $\mathcal{P} = a = p_0 < p_1 < \cdots < p_N = b$ such that for each atom $I_j = [p_{j-1}, p_j]$, one of the two derivatives is non vanishing in I_j.

(d) Conclude that the image of $\vec{p}$ on each atom is a regular arc.

(e) Finally, we handle the case when the domain of a regular parametrization is not a closed interval. First, suppose $\vec{p}$ is a regular parametrization of $\mathcal{C}$ but its domain of definition is a *right-open* interval $[a, b)$ (including the possibility that on of these is infinite). Then any strictly increasing sequence $\mathcal{P} = \{a = p_0 < p_1 < \cdots\}$ with $\lim p_j = b$ has the property that every point of $[a, b)$ belongs to at least one of the atoms $I_j = [p_{j-1}, p_j]$, $j = 1, s, \ldots$ (and they have disjoint interiors). Then we can apply our earlier arguments to each restriction $\vec{p}|I_j$. Finally, how do we partition an *open* interval (a, b) with closed atoms?

(13) Recall the four-petal rose illustrated in Figure 2.16 on p. 86, whose polar equation is

$$r = \sin 2\theta.$$

This is the locus of the equation

$$(x^2 + y^2)^3 = 4x^2y^2.$$

The parametrization $\vec{p}(\theta)$ associated to the polar equation is

$$\begin{cases} x = \sin 2\theta \cos\theta = 2\sin\theta\cos^2\theta \\ y = \sin 2\theta \sin\theta = 2\sin^2\theta\cos\theta \end{cases}$$

(a) Verify that as θ runs through the interval $\theta \in [0, 2\pi]$, the origin is crossed four times, at $\theta = 0$, $\theta = \frac{\pi}{2}$, $\theta = \pi$, $\theta = \frac{3\pi}{2}$, and again at $\theta = 2\pi$, with a horizontal velocity when $\theta = 0$ or π and a vertical one when $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Verify also that the four "petals" are traversed in the order i, ii, iii, iv as indicated in Figure 2.16.

(b) Now consider the vector-valued function $\vec{q}(\sigma)$ defined in pieces by

$$\begin{cases} x = \sin 2\sigma \cos\sigma = 2\sin\sigma\cos^2\sigma \\ y = \sin 2\sigma \sin\sigma = 2\sin^2\sigma\cos\sigma \end{cases} \quad 0 \le t \le \pi$$
$$\begin{cases} x = \sin 2\sigma \cos\sigma = -2\sin\sigma\cos^2\sigma \\ y = -\sin 2\sigma \sin\sigma = -2\sin^2\sigma\cos\sigma \end{cases} \quad \pi \le t \le 2\pi. \tag{2.28}$$

Verify that this function is regular (the main point is differentiability and continuity of the derivative at the crossings of the origin).

(c) Verify that the image of $\vec{q}(\sigma)$ is also the four-leaf rose. In what order are the loops traced by $\vec{q}(\sigma)$ as σ goes from 0 to 2π?

(d) Show that $\vec{p}(\theta)$ and $\vec{q}(\sigma)$ cannot be reparametrizations of each other. (*Hint:* Consider short open intervals about each of the parameter values where the origin is crossed, and show that their images under $\vec{p}(\theta)$ cannot match those under $\vec{q}(\sigma)$.)

History note:

(14) **Bolzano's curve:** A version of the following was constructed by Bernhard Bolzano (1781-1848) in the 1830s; a more complete study is given in (see *Calculus Deconstructed*, §4.11, or another single-variable calculus text).

(a) Start with the following: suppose we have an affine function f, defined over the interval $[a, b]$, with $f(a) = c$ and $f(b) = d$; thus its graph is the straight line segment from (a, c) to (b, d). Construct a new, piecewise-affine function $\bar{f}$ by keeping the endpoint values, but interchanging the values at the points one-third and two-thirds of the way across (see Figure 2.27).

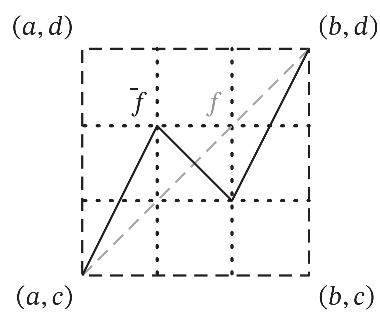


Figure 2.27. The basic construction

Thus, originally

$$f(x) = c + m(x - a),$$

where

$$m = \frac{d-c}{b-a},$$

and in particular

$$\begin{aligned} f(a) &= c \\ f(a_1) = \frac{2a+b}{3} &= \frac{2c+d}{3} \\ f(a_2) = \frac{a+2b}{3} &= \frac{c+2d}{3} \\ f(b) &= d. \end{aligned}$$

Now, $\bar{f}$ is defined by

$$\begin{aligned} \bar{f}(a) &= c \\ \bar{f}(a_1) &= c_1 = \frac{c+2d}{3} \\ \bar{f}(a_2) &= c_2 = \frac{2c+d}{3} \\ \bar{f}(b) &= d \end{aligned}$$

and $\bar{f}$ is affine on each of the intervals $I_1 = [a, a_1]$, $I_2 = [a_1, a_2]$, and $I_3 = [a_2, b]$. **Show** that the slopes m_j of the graph of $\bar{f}$ on I_j satisfy

$$\begin{aligned} m_1 = m_3 &= 2m \\ m_2 &= -m. \end{aligned}$$

(b) Now, we construct a sequence of functions f_k on $[0,1]$ via the recursive definition

$$\begin{aligned} f_0 &= id \\ f_{k+1} &= \bar{f}_k. \end{aligned}$$

Show that

$$|f_k(x) - f_{k+1}(x)| \leq \left(\frac{2}{3}\right)^{k+1} \tag{2.29}$$

for all $x \in [0,1]$. This implies that for each $x \in [0,1]$,

$$f(x) := \lim f_k(x)$$

is well-defined for each $x \in [0,1]$. We shall accept without proof the fact that Equation (2.29) (which implies a property called *uniform convergence*) also guarantees that f is continuous on $[0,1]$. Thus its graph is a continuous curve—in fact, it is an arc.

(c) **Show** that if x_0 is a **triadic rational** (that is, it has the form $x_0 = \frac{p}{3^j}$ for some j) then $f_{k+1}(x_0) = f_k(x_0)$ for k sufficiently large, and hence this is the value $f(x_0)$. In particular, **show** that f has a local extremum at each triadic rational. (*Hint:* x_0 is a local extremum for all f_k once k is sufficiently large;

furthermore, once this happens, the sign of the slope on either side does not change, and its absolute value is increasing with k.)

This shows that f has infinitely many local extrema—in fact, between any two points of $[0, 1]$ there is a local maximum (and a local minimum); in other words, the curve has infinitely many "corners". It can be shown (see *Calculus Deconstructed*, §4.11, or another single-variable calculus text) that the function f, while it is continuous on $[0, 1]$, is not differentiable at any point of the interval. In Exercise 6 in § 2.5, we will also see that this curve has infinite "length".

2.5 Integration along Curves

Arclength. How long is a curve? While it is clear that the length of a straight line segment is the distance between its endpoints, a rigorous notion of the "length" for more general curves is not so easy to formulate. We will formulate a geometric notion of length for arcs, essentially a modernization of the method used by Archimedes of Syracuse (*ca.* 287-212 BC) to measure the circumference of a circle. Archimedes realized that the length of an inscribed (*resp.* circumscribed) polygon is a natural lower (*resp.* upper) bound on the circumference, and also that by using polygons with many sides, the difference between these two bounds could be made as small as possible. Via a proof by contradiction he established in [2] that the area of a circle is the same as that of a triangle whose base equals the circumference and whose height equals the radius.[20] By using regular polygons with 96 sides, he was able to establish that π, defined as the ratio of the circumference to the diameter, is between $\frac{22}{7}$ and $\frac{221}{71}$. Archimedes didn't worry about whether the length of the circumference makes sense; he took this to be self-evident. His argument about the lengths of polygons providing bounds for the circumference was based on a set of axioms concerning *convex* curves; this was needed most for the use of the *circumscribed* polygons as an *upper* bound. The fact that *inscribed* polygons give a *lower* bound follows from the much simpler assumption, which *we* take as self-evident, that the shortest curve between two points is a straight line segment.

Suppose that $\mathcal{C}$ is an arc parametrized by $\vec{p} : \mathbb{R} \to \mathbb{R}^3$ and let $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition of the domain of $\vec{p}$. The sum

$$\ell\left(\mathcal{P}, \vec{p}\right) = \sum_{j=1}^{n} \left\| \vec{p}\left(t_j\right) - \vec{p}\left(t_{j-1}\right) \right\|$$

is the length of a path consisting of straight line segments joining successive points along $\mathcal{C}$. It is clear that, under any reasonable notion of "length", $\mathcal{C}$ is at least as long as $\ell\left(\mathcal{P}, \vec{p}\right)$. We would also think intuitively that a partition with small mesh size should give a good approximation to the "true" length of $\mathcal{C}$. We therefore say $\mathcal{C}$ is **rectifiable** if the values of $\ell\left(\mathcal{P}, \vec{p}\right)$ among all partitions are bounded, and then we define the **arclength** of $\mathcal{C}$ to be

$$\mathfrak{s}\left(\mathcal{C}\right) = \sup_{\mathcal{P}} \ell\left(\mathcal{P}, \vec{p}\right).$$

Not every curve is rectifiable. Two examples of *non*-rectifiable curves are the graph of Bolzano's nowhere-differentiable function, constructed in Exercise 14 in § 2.4 (see

---

[20] In our language, the circumference is $2\pi r$, where r is the radius, so the area of such a triangle is $\frac{1}{2}(2\pi r)(r) = \pi r^2$.

Exercise 6) and the graph of $y = x \sin \frac{1}{x}$ (see Exercise 5). In such cases, there exist partitions $\mathcal{P}$ for which $\ell(\mathcal{P}, \vec{p})$ is arbitrarily high.

We need to show that the arclength $\mathfrak{s}(\mathcal{C})$ does not depend on the parametrization we use to construct $\mathcal{C}$. Suppose $\vec{p}, \vec{q}: \mathbb{R} \to \mathbb{R}^3$ are two one-to-one continuous functions with the same image $\mathcal{C}$. As noted in Exercise 10, § 2.4, we can find a strictly monotone recalibration function $\mathfrak{t}(s)$ from the domain of $\vec{q}$ to the domain of $\vec{p}$ so that $\vec{p}(\mathfrak{t}(s)) = \vec{q}(s)$ for all parameter values of $\vec{q}$. If $\mathcal{P}$ is a partition of the domain of $\vec{p}$, then there is a unique sequence of parameter values for $\vec{q}$ defined by $\mathfrak{t}(s_j) = t_j$; this sequence is either strictly increasing (if $\mathfrak{t}(s)\uparrow$) or strictly decreasing (if $\mathfrak{t}(s)\downarrow$). Renumbering if necessary in the latter case, we see that the s_j form a partition $\mathcal{P}_s$ of the domain of $\vec{q}$, with the same succession as the t_j; in particular, $\ell(\mathcal{P}_s, \vec{q}) = \ell(\mathcal{P}, \vec{p})$, so the supremum of $\ell(\mathcal{P}', \vec{q})$ over *all* partitions $\mathcal{P}'$ of the domain of $\vec{q}$ is at least the same as that over partitions of the domain of $\vec{p}$. Reversing the roles of the two parametrizations, we see that the two suprema are actually the same.

This formulation of arclength has the advantage of being clearly based on the geometry of the curve, rather than the parametrization we use to construct it. However, as a tool for computing the arclength, it is as useful (or as useless) as the definition of the definite integral via Riemann sums is for calculating definite integrals. Fortunately, for regular curves, we can use definite integrals to calculate arclength.

Theorem 2.5.1. *Every regular arc is rectifiable, and if $\vec{p}: [a, b] \to \mathbb{R}^3$ is a regular one-to-one function with image $\mathcal{C}$, then*

$$\mathfrak{s}(\mathcal{C}) = \int_a^b \left\|\dot{\vec{p}}(t)\right\| \, dt.$$

This can be understood as saying that *the length of a regular curve is the integral of its speed*, which agrees with our understanding (for real-valued functions representing motion along an axis) that the integral of speed is the total distance traveled. If we consider the function $\mathfrak{s}(t)$ giving the arclength (or distance travelled) between the starting point and the point $\vec{p}(t)$, then our notation for the speed is naturally suggested by applying the Fundamental Theorem of Calculus to the formula above:

$$\frac{d}{dt}[\mathfrak{s}(t)] = \frac{d}{dt} \int_a^t \left\|\dot{\vec{p}}(t)\right\| \, dt = \left\|\dot{\vec{p}}(t)\right\|.$$

In other words, $\frac{d\mathfrak{s}}{dt} = \left\|\dot{\vec{p}}(t)\right\|$.

The proof of Theorem 2.5.1 relies on a technical estimate, whose proof is outlined in Exercise 4.

Lemma 2.5.2. *Suppose $\vec{p}: [a, b] \to \mathbb{R}^3$ is a regular, one-to-one function and that $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_n = b\}$ is a partition of $[a, b]$ such that the speed varies by less than $\delta > 0$ over each atom*[21] *$I_j = [t_{j-1}, t_j]$:*

$$\left|\left\|\dot{\vec{p}}(t)\right\| - \left\|\dot{\vec{p}}(t')\right\|\right| < \delta \text{ whenever } t_{j-1} \le t, t' \le t_j.$$

[21] The intervals I_j were called *component intervals* in *Calculus Deconstructed*; however, the possible confusion surrounding the use of the word "component" convinced us to instead use the term *atom*, which is standard in other contexts where partitions arise.

Then

$$\left| \int_a^b \left\| \dot{\vec{p}}(t) \right\| dt - \ell\left(\mathcal{P}, \vec{p}\right) \right| < 3\delta(b-a).$$

Proof of Theorem 2.5.1. We will use the fact that since the speed is continuous on the closed interval $[a,b]$, it is uniformly continuous, which means that given any $\delta > 0$, we can find $\mu > 0$ so that it varies by at most δ over any subinterval of $[a,b]$ of length μ or less. Put differently, this says that the hypotheses of Lemma 2.5.2 are satisfied by any partition of mesh size μ or less. We will also use the easy observation that refining the partition raises (or at least does not lower) the "length estimate" $\ell\left(\mathcal{P}, \vec{p}\right)$ associated to the partition.

Suppose now that $\mathcal{P}_k$ is a sequence of partitions of $[a,b]$ for which $\ell_k = \ell\left(\mathcal{P}_k, \vec{p}\right)$ is strictly increasing with limit $\mathfrak{s}(\mathcal{C})$ (which, *a priori* may be infinite). Without loss of generality, we can assume (refining each partition if necessary) that the mesh size of $\mathcal{P}_k$ goes to zero monotonically. Given $\varepsilon > 0$, we set

$$\delta = \frac{\varepsilon}{3(b-a)}$$

and find $\mu > 0$ such that every partition with mesh size μ or less satisfies the hypotheses of Lemma 2.5.2; eventually, $\mathcal{P}_k$ satisfies $\operatorname{mesh}(\mathcal{P}_k) < \mu$, so

$$\left| \int_a^b \left\| \dot{\vec{p}}(t) \right\| dt - \ell\left(\mathcal{P}_k, \vec{p}\right) \right| < 3\delta(b-a) = \varepsilon.$$

This shows first that the numbers ℓ_k converge to $\int_a^b \left\| \dot{\vec{p}}(t) \right\| dt$—but by assumption, $\lim \ell_k = \mathfrak{s}(\mathcal{C})$, so we are done. □

The content of Theorem 2.5.1 is encoded in a notational device: given a regular parametrization $\vec{p}\colon \mathbb{R} \to \mathbb{R}^3$ of the curve $\mathcal{C}$, we define the **differential of arclength**, denoted $d\mathfrak{s}$, to be the formal expression

$$d\mathfrak{s} := \left\| \dot{\vec{p}}(t) \right\| dt = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2}\, dt.$$

This may seem a bit mysterious at first, but we will find it very useful; using this notation, the content of Theorem 2.5.1 can be written

$$\mathfrak{s}(\mathcal{C}) = \int_a^b d\mathfrak{s}.$$

As an example, let us use this formalism to find the length of the helix parametrized by $\vec{p}(t) = (\cos 2\pi t, \sin 2\pi t, t)$ for $0 \le t \le 2$. We calculate that

$$d\mathfrak{s} = \sqrt{(-2\pi \sin 2\pi t)^2 + (2\pi \cos 2\pi t)^2 + (1)^2}\, dt = \sqrt{4\pi^2 + 1}\, dt.$$

Thus,

$$\mathfrak{s}(\mathcal{C}) = \int_0^2 d\mathfrak{s} = \int_0^2 \sqrt{4\pi^2+1}\, dt = 2\sqrt{4\pi^2+1}.$$

As a second example, the arclength of the parabola $y = x^2$ between $(0,0)$ and $(\frac{1}{2}, \frac{1}{4})$ can be calculated using the "graph" parametrization $\vec{p}(x) = (x, x^2)$, $0 \le t \le 1$. The element of arc length is $d\mathfrak{s} = \sqrt{1+4x^2}\,dx$, so we need to compute the integral

$$\mathfrak{s}(\mathcal{C}) = \int_0^{\frac{1}{2}} \sqrt{1+4x^2}\,dx.$$

The trigonometric substitution $x = \tan\theta$ leads to $\int_0^{\pi/4} \frac{1}{2}\sec^3\theta\,d\theta$ which, via integration by parts (or cheating and looking it up in a table) gives arclength $\frac{1}{4}\{\sqrt{2}+\ln(1+\sqrt{2})\}$.

As another example, let us use this formalism to compute the circumference of a circle. The circle is not an arc, but the domain of the standard parametrization $\vec{p}(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$ can be partitioned via $\mathcal{P} = \{0, \pi, 2\pi\}$ into two semicircles, $\mathcal{C}_i$, $i = 1, 2$, which meet only at the endpoints; it is natural then to say that $\mathfrak{s}(\mathcal{C}) = \mathfrak{s}(\mathcal{C}_1) + \mathfrak{s}(\mathcal{C}_2)$. We can calculate that $d\mathfrak{s} = \sqrt{(-\sin t)^2 + (\cos t)^2}\,dt = dt$ and thus

$$\mathfrak{s}(\mathcal{C}) = \int_0^{\pi} dt + \int_{\pi}^{2\pi} dt = 2\pi.$$

The example of the circle illustrates the way that we can go from the definition of arclength for an *arc* to arclength for a general *curve*. By Exercise 12 in § 2.4, any parametrized curve $\mathcal{C}$ can be partitioned into arcs $\mathcal{C}_k$, and the arclength of $\mathcal{C}$ is in a natural way the sum of the arclengths of these arcs:

$$\mathfrak{s}(\mathcal{C}) = \sum_k \mathfrak{s}(\mathcal{C}_k);$$

when the curve is parametrized over a closed interval, this is a finite sum, but it can be an infinite (positive) series when the domain is an open interval (see the last part of Exercise 12). Notice that a reparametrization of $\mathcal{C}$ is related to the original one via a strictly monotone, continuous function, and this associates to every partition of the original domain a partition of the reparametrized domain involving the same segments of the curve, and hence having the same value of $\ell(\mathcal{P}, \vec{p})$. Furthermore, when the parametrization is regular, the sum above can be rewritten as a single (possibly improper) integral. This shows

Remark 2.5.3. *The arclength of a parametrized curve $\mathcal{C}$ does not change under reparametrization. If the curve is regular, then the arclength is given by the integral of the speed (possibly improper if the domain is open)*

$$\mathfrak{s}(\mathcal{C}) = \int_a^b d\mathfrak{s} = \int_a^b \left(\frac{d\mathfrak{s}}{dt}\right) dt = \int_a^b \left\|\dot{\vec{p}}(t)\right\| dt.$$

for any regular parametrization of $\mathcal{C}$.

In retrospect, this justifies our notation for speed, and also fits our intuitive notion that the length of a curve $\mathcal{C}$ is the distance travelled by a point as it traverses $\mathcal{C}$ once.

As a final example, we calculate the arclength of one "arch" of the cycloid $\vec{p}(\theta) = (\theta - \sin\theta, 1 - \cos\theta)$, $0 \le \theta \le 2\pi$. Differentiating, we get $\vec{v}(\theta) = (1 - \cos\theta, \sin\theta)$ so $d\mathfrak{s} = \sqrt{2 - 2\cos\theta}\,d\theta$. The arclength integral $\mathfrak{s}(\mathcal{C}) = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\,d\theta$ can be rewritten, multiplying and dividing the integrand by $\sqrt{1+\cos\theta}$, as

$$\sqrt{2}\int_0^{2\pi} \frac{\sqrt{1-\cos^2\theta}}{\sqrt{1+\cos\theta}}\,d\theta$$

which suggests the substitution $u = 1 + \cos\theta,\ du = -\sin\theta\, d\theta$, since the numerator of the integrand looks like $\sin\theta$. However, there is a pitfall here: the numerator *does* equal $\sqrt{\sin^2\theta}$, but this equals $\sin\theta$ only when $\sin\theta \geq 0$, which is to say over the first half of the curve, $0 \leq \theta \leq \pi$; for the second half, it equals $-\sin\theta$. Therefore, we break the integral in two:

$$\sqrt{2}\int_0^{2\pi}\sqrt{1-\cos\theta}\,d\theta = \sqrt{2}\int_0^{\pi}\frac{\sin\theta\,d\theta}{\sqrt{1+\cos\theta}} - \sqrt{2}\int_{\pi}^{2\pi}\frac{\sin\theta\,d\theta}{\sqrt{1+\cos\theta}}$$
$$= 2\sqrt{2}\int_0^2 u^{-1/2}\,du = 4\sqrt{2}u^{1/2}\Big|_0^2 = 8.$$

We note one technical point here: strictly speaking, the parametrization of the cycloid is not regular: while it is continuously differentiable, the velocity vector is zero at the ends of the arch. To get around this problem, we can think of this as an improper integral, taking the limit of the arclength of the curve $\vec{p}\,(\theta)$, $\varepsilon \leq \theta \leq 2\pi - \varepsilon$ as $\varepsilon \to 0$. The principle here (similar, for example, to the hypotheses of the Mean Value Theorem) is that the velocity can vanish at an endpoint of an arc in Theorem 2.5.1, or more generally that it can vanish at a set of isolated points of the curve[22] and the integral formula still holds, provided we don't "backtrack" after that.

Integrating a Function along a Curve (Path Integrals). Suppose we have a wire that is shaped like an arc, but has variable thickness, and hence variable density. If we know the density at each point along the arc, how do we find the total mass? If the arc happens to be an interval along the x-axis, then we simply define a function $f(x)$ whose value at each point is the density, and integrate. We would like to carry out a similar process along an arc or, more generally, along a curve.

Our abstract setup is this: we have an arc, $\mathcal{C}$, parametrized by the (continuous, one-to-one) vector-valued function $\vec{p}\,(t)$, $a \leq t \leq b$, and we have a (real-valued) function that assigns to each point $\vec{p}$ of $\mathcal{C}$ a number $f(\vec{p})$; we want to integrate f along $\mathcal{C}$. The process is a natural combination of the Riemann integral with the arclength calculation of § 2.5. Just as for arclength, we begin by partitioning $\mathcal{C}$ via a partition of the domain $[a, b]$ of our parametrization, $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_n = b\}$. For a small mesh size, the arclength of $\mathcal{C}$ between successive points $\vec{p}\,(t_j)$ is well approximated by $\triangle\mathfrak{s}_j = \left\|\vec{p}\,(t_j) - \vec{p}\,(t_{j-1})\right\|$ and we can form lower and upper sums $\mathcal{L}(\mathcal{P}, f) = \sum_{j=1}^{n}\inf_{t\in I_j} f(\vec{p}\,(t))\triangle\mathfrak{s}_j$ and $\mathcal{U}(\mathcal{P}, f) = \sum_{j=1}^{n}\sup_{t\in I_j} f(\vec{p}\,(t))\triangle\mathfrak{s}_j$. As in the usual theory of the Riemann integral, we have for any partition $\mathcal{P}$ that $\mathcal{L}(\mathcal{P}, f) \leq \mathcal{U}(\mathcal{P}, f)$; it is less clear that refining a partition lowers $\mathcal{U}(\mathcal{P}, f)$ (although it clearly does increase $\mathcal{L}(\mathcal{P}, f)$), since the quantity $\ell(\mathcal{P}, \vec{p})$ increases under refinement. However, if the arc is rectifiable, we can modify the upper sum by using $\mathfrak{s}(\vec{p}\,(I_j))$ in place of $\triangle\mathfrak{s}_j$. Denoting this by

$$\mathcal{U}^*(\mathcal{P}, f) = \sum_{j=1}^{n}\sup_{t\in I_j} f(\vec{p}\,(t))\,\mathfrak{s}(\vec{p}\,(I_j))$$

we have, for any two partitions $\mathcal{P}_i$, $i = 1, 2$,

$$\mathcal{L}(\mathcal{P}_1, f) \leq \mathcal{U}^*(\mathcal{P}_2, f)$$

[22] With a little thought, we see that it can even vanish on a nontrivial closed interval.

We will say the function $f(\vec{p})$ is **integrable** over the arc $\mathcal{C}$ if

$$\sup_{\mathcal{P}} \mathcal{L}(\mathcal{P}, f) = \inf_{\mathcal{P}} \mathcal{U}^*(\mathcal{P}, f)$$

and in this case the common value is called the **path integral** or **integral with respect to arclength** of f along the arc $\mathcal{C}$, denoted $\int_{\mathcal{C}} f\, d\mathfrak{s}$. As in the case of the usual Riemann integral, we can show that if f is integrable over $\mathcal{C}$ then for any sequence $\mathcal{P}_k$ of partitions of $[a, b]$ with $\operatorname{mesh}(\mathcal{P}_k) \to 0$, the Riemann sums using any sample points $t_j^* \in I_j$ converge to the integral: $\mathcal{R}(\mathcal{P}_k, f, \{t_j^*\}) = \sum_{j=1}^{n} f(t_j^*) \triangle \mathfrak{s}_j \to \int_{\mathcal{C}} f\, d\mathfrak{s}$.

It is easy to see that the following analogue of Remark 2.5.3 holds for path integrals:

Remark 2.5.4. *The path integral of a function over a parametrized curve is unchanged by reparametrization; when the parametrization $\vec{p}: \mathbb{R} \to \mathbb{R}^3$ is regular, we have*

$$\int_{\mathcal{C}} f\, d\mathfrak{s} = \int_a^b f(\vec{p}(t)) \left\| \dot{\vec{p}}(t) \right\| dt.$$

As an example, let us take $\mathcal{C}$ to be the parabola $y = x^2$ between $(0, 0)$ and $(1, 1)$. First, we compute the integral $\int_{\mathcal{C}} f\, d\mathfrak{s}$ for the function $f(x, y) = x$. Using the standard parametrization in terms of x, $\vec{p}(x) = (x, x^2)$, $\quad 0 \le x \le 1$, the element of arclength is $d\mathfrak{s} = \sqrt{1 + 4x^2}\, dx$, so

$$\int_{\mathcal{C}} f\, d\mathfrak{s} = \int_{\mathcal{C}} x\, d\mathfrak{s} = \int_0^1 (x)(\sqrt{1 + 4x^2}\, dx) = \frac{5\sqrt{5} - 1}{12}.$$

Now let us calculate $\int_{\mathcal{C}} f\, d\mathfrak{s}$ for the function $f(x, y) = y$ over the same curve. If we try to use the same parametrization, we have to calculate $\int_{\mathcal{C}} f\, d\mathfrak{s} = \int_{\mathcal{C}} y\, d\mathfrak{s} = \int_{\mathcal{C}} x^2\, d\mathfrak{s} = \int_0^1 x^2 \sqrt{1 + 4x^2}\, dx$ which, while not impossible, is much harder to do. However, we can also express $\mathcal{C}$ as the graph of $x = \sqrt{y}$ and parametrize in terms of y; this yields $d\mathfrak{s} = \sqrt{\frac{1}{4y} + 1}\, dy$ and so

$$\int_{\mathcal{C}} y\, d\mathfrak{s} = \int_0^1 y \sqrt{\frac{1}{4y} + 1}\, dy = \int_0^1 \sqrt{\frac{y}{4} + y^2}\, dy$$

which we can calculate by completing the square and substituting $8y + 1 = \sec\theta$; the value turns out to be $\frac{9\sqrt{5}}{32} - \frac{1}{128} \ln(9 + 4\sqrt{5})$.

Exercises for § 2.5

Answers to Exercises 1a, 2a, 3a, and 3j are given in Appendix A.13.

Practice problems:

(1) Set up an integral expressing the arc length of each curve below. Do not attempt to integrate.
 (a) $y = x^n, \quad 0 \le x \le 1$ (b) $y = e^x, \quad 0 \le x \le 1$
 (c) $y = \ln x, \quad 1 \le x \le e$ (d) $y = \sin x, \quad 0 \le x \le \pi$
 (e) $x = a\cos\theta, y = b\sin\theta, 0 \le \theta \le 2\pi$
 (f) $x = e^t + e^{-t}, y = e^t - e^{-t}, -1 \le t \le 1$

(2) Find the length of each curve below.
 (a) $y = x^{3/2}, \quad 0 \le x \le 1$
 (b) $y = x^{2/3}, \quad 0 \le x \le 1$

(c) $y = \frac{x^3}{3} + \frac{1}{4x}, \quad 1 \le x \le 2$
(d) $y = \int_1^x \sqrt{t^4 - 1}\, dt, \quad 1 \le x \le 2$
(e) $x = \sin^3 t, y = \cos^3 t, 0 \le t \le \frac{\pi}{4}$
(f) $x = 9t^2, y = 4t^3, z = t^4, 0 \le t \le 1$
(g) $x = 8t^3, y = 15t^4, z = 15t^5, 0 \le t \le 1$
(h) $x = t^2, y = \ln t, z = 2t, 1 \le t \le 2$
(i) $x = \sin\theta, y = \theta + \cos\theta, 0 \le \theta \le \frac{\pi}{2}$
(j) $x = 3t, y = 4t \sin t, z = 4t \cos t, 0 \le t \le \frac{5}{4}$

(3) Calculate $\int_{\mathcal{C}} f \, d\mathfrak{s}$:
(a) $f(x, y) = 36x^3$, $\mathcal{C}$ is $y = x^3$ from $(0, 0)$ to $(1, 1)$.
(b) $f(x, y) = 32x^5$, $\mathcal{C}$ is $y = x^4$ from $(0, 0)$ to $(1, 1)$.
(c) $f(x, y) = x^2 + y^2$, $\mathcal{C}$ is $y = 2x$ from $(0, 0)$ to $(1, 2)$.
(d) $f(x, y) = 4(x + \sqrt{y})$, $\mathcal{C}$ is $y = x^2$ from $(0, 0)$ to $(1, 1)$.
(e) $f(x, y) = x^2$, $\mathcal{C}$ is the upper half circle $x^2 + y^2 = 1, y \ge 0$.
(f) $f(x, y) = x^2 + y^2$, $\mathcal{C}$ is given in parametric form as $x = t$, $y = \sqrt{1 - t^2}$, $0 \le t \le 1$.
(g) $f(x, y) = (1 - x^2)^{3/2}$, $\mathcal{C}$ is upper half of the circle $x^2 + y^2 = 1$.
(h) $f(x, y) = x^3 + y^3$, $\mathcal{C}$ is given in parametric form as $x = 2\cos t$, $y = 2\sin t$, $0 \le t \le \pi$.
(i) $f(x, y) = xy$, $\mathcal{C}$ is $y = x^2$ from $(0, 0)$ to $(1, 1)$.
(j) $f(x, y, z) = xy$, $\mathcal{C}$ is given in parametric form as $x = \cos t$, $y = \sin t$, $z = t$, $0 \le t \le \pi$.
(k) $f(x, y, z) = x^2 y$, $\mathcal{C}$ is given in parametric form as $x = \cos t$, $y = \sin t$, $z = t$, $0 \le t \le \pi$.
(l) $f(x, y, z) = z$, $\mathcal{C}$ is given in parametric form as $x = \cos t$, $y = \sin t$, $z = t$, $0 \le t \le \pi$.
(m) $f(x, y, z) = 4y$, $\mathcal{C}$ is given in parametric form as $x = t$, $y = 2t$, $z = t^2$, $0 \le t \le 1$.
(n) $f(x, y, z) = x^2 - y^2 + z^2$, $\mathcal{C}$ is given in parametric form as $x = \cos t$, $y = \sin t$, $z = 3t$, $0 \le t \le \pi$.
(o) $f(x, y, z) = 4x + 16z$, $\mathcal{C}$ is given in parametric form as $x = 2t$, $y = t^2$, $z = \frac{4t^3}{9}$, $0 \le t \le 3$.

Theory problems:

(4) Prove Lemma 2.5.2 as follows:
(a) Fix an atom $I_j = [t_{j-1}, t_j]$ of $\mathcal{P}$. Use the Mean Value Theorem to show that there exist parameter values s_1,s_2 and s_3 such that

$$x(t_j) - x(t_{j-1}) = \dot{x}(s_1)\triangle t_j$$
$$y(t_j) - y(t_{j-1}) = \dot{y}(s_2)\triangle t_j$$
$$z(t_j) - z(t_{j-1}) = \dot{z}(s_3)\triangle t_j.$$

(b) Show that the vector $\vec{v}_j = (\dot{x}(s_1), \dot{y}(s_2), \dot{z}(s_3))$ satisfies

$$\vec{p}(t_j) - \vec{p}(t_{j-1}) = \vec{v}_j \triangle t_j$$

and hence

$$\left\|\vec{p}\left(t_j\right)-\vec{p}\left(t_{j-1}\right)\right\|=\left\|\vec{v}_j\right\| \triangle t_j .$$

(c) Show that for any $t \in I_j$

$$\left\|\dot{\vec{p}}(t)-\vec{v}_j\right\|<3 \delta .$$

(d) Use the Triangle Inequality to show that

$$\left|\left\|\dot{\vec{p}}(t)\right\|-\left\|\vec{v}_j\right\|\right|<3 \delta \text { for all } t \in I_j .$$

(e) Show that

$$\left|\int_a^b\left\|\dot{\vec{p}}(t)\right\| d t-\ell(\mathcal{P}, \vec{p})\right|<3 \delta(b-a) .$$

(5) Consider the graph of the function defined on $[0,1]$ by $f(0)=0$ and $f(x)=x \sin \frac{1}{x}$ for $0<x \leq 1$.

(a) **Show** that $|f(x)| \leq|x|$ with equality at 0 and the points $x_k:=\frac{2}{(2 k-1) \pi}$, $k=1, \ldots$.

(b) **Show** that f is continuous. (*Hint:* the issue is $x=0$.) Thus, its graph is a curve. Note that f is differentiable except at $x=0$.

(c) Consider the piecewise linear approximation to this curve (albeit with infinitely many pieces) consisting of joining $\left(x_k, f\left(x_k\right)\right)$ to $\left(x_{k+1}, f\left(x_{k+1}\right)\right)$ with straight line segments: note that at one of these points, $f(x)=x$ while at the other $f(x)=-x$. **Show** that the line segment joining the points on the curve corresponding to $x=x_k$ and $x=x_{k+1}$ has length at least

$$\begin{aligned}
\triangle \mathfrak{s}_k & =\left|f\left(x_{k+1}\right)-f\left(x_k\right)\right|=x_{k+1}+x_k \\
& =\frac{2}{(2 k+1) \pi}+\frac{2}{(2 k-1) \pi}=\frac{2}{\pi}\left(\frac{4 k}{4 k^2-1}\right) .
\end{aligned}$$

(d) **Show** that the sum $\sum_{k=1}^{\infty} \triangle \mathfrak{s}_k$ diverges. This means that if we take (for example) the piecewise linear approximations to the curve obtained by taking the straight line segments as above to some finite value of k and then join the last point to $(0,0)$, their lengths will also diverge as the finite value increases. Thus, there exist partitions of the curve whose total lengths are arbitrarily large, and the curve is not rectifiable.

Challenge problem:

(6) **Bolzano's curve (continued):** We continue here our study of the curve described in Exercise 14 in § 2.4; we keep the notation of that exercise.

(a) **Show** that the slope of each straight piece of the graph of f_k has the form $m = \pm 2^n$ for some integer $0 \leq n \leq k$. Note that each interval over which f_k is affine has length 3^{-k}.

(b) **Show** that if two line segments start at a common endpoint and end on a vertical line, and their slopes are 2^n and 2^{n+1}, respectively, then the ratio of the second to the first length is

$$\frac{\ell_2}{\ell_1} = \sqrt{\frac{1 + 2^{n+1}}{1 + 2^n}}$$

(c) **Show** that this quantity is non-decreasing, and that therefore it is always at least equal to $\sqrt{5/3}$.

(d) Use this to **show** that the ratio of the lengths of the graphs of f_{k+1} and f_k are bounded below by $2\sqrt{5/3}\sqrt{3} + 1/3 \geq 1.19$.

(e) How does this show that the graph of f is non-rectifieable?

3 Differential Calculus for Real-Valued Functions of Several Variables

In this and the next chapter we consider functions whose input involves several variables—or equivalently, whose input is a vector—and whose output is a real number.

We shall restrict ourselves to functions of two or three variables, where the vector point of view can be interpreted geometrically.

A function of two (*resp.* three) variables can be viewed in two slightly different ways, reflected in two different notations.

We can think of the input as three separate variables; often it will be convenient to use subscript notation x_i (instead of x, y, z) for these variables, so we can write

$$f(x, y) = f(x_1, x_2)$$

in the case of two variables and

$$f(x, y, z) = f(x_1, x_2, x_3)$$

in the case of three variables.

Alternatively, we can think of the input as a single (variable) vector $\vec{x}$ formed from listing the variables in order: $\vec{x} = (x, y) = (x_1, x_2)$ or $\vec{x} = (x, y, z) = (x_1, x_2, x_3)$ and simply write our function as $f(\vec{x})$.

A third notation which is sometimes useful is that of mappings: we write

$$f : \mathbb{R}^n \to \mathbb{R}$$

(with $n = 2$ or $n = 3$) to indicate that f has inputs coming from $\mathbb{R}^n$ and produces outputs that are real numbers.[1]

In much of our expositon we will deal explicitly with the case of three variables, with the understanding that in the case of two variables one simply ignores the third variable. Conversely, we will in some cases concentrate on the case of two variables and if necessary indicate how to incorporate the third variable.

In this chapter, we consider the definition and use of derivatives in this context.

3.1 Continuity and Limits

Continuous Functions of Several Variables. Recall from § 2.3 that *a sequence of vectors converges if it converges coordinatewise.* Using this notion, we can define continuity of a real-valued function of *three (or two)* variables $f(\vec{x})$ by analogy to the definition for real-valued functions $f(x)$ of *one* variable:

[1]When the domain is a specified subset $D \subset \mathbb{R}^n$ we will write $f : D \to \mathbb{R}$.

Definition 3.1.1. *A real-valued function $f(\vec{x})$ is **continuous** on a subset $D \subset \mathbb{R}^{2\,or\,3}$ of its domain if whenever the inputs converge in D (as points in $\mathbb{R}^{2\,or\,3}$) the corresponding outputs also converge (as numbers):*

$$\overrightarrow{x_k} \to \overrightarrow{x_0} \Rightarrow f\left(\overrightarrow{x_k}\right) \to f\left(\overrightarrow{x_0}\right).$$

It is easy, using this definition and basic properties of convergence for sequences of numbers, to verify the following analogues of properties of continuous functions of one variable. First, the composition of continuous functions is continuous (Exercise 5):

Remark 3.1.2. *Suppose $f(\vec{x})$ is continuous on $D \subset \mathbb{R}^{2\,or\,3}$.*

(1) *If $g\colon \mathbb{R} \to \mathbb{R}$ is continuous on $G \subset \mathbb{R}$ and $f(\vec{x}) \in G$ for every $\vec{x} = (x, y, z) \in D$, then the composition $g \circ f\colon \mathbb{R}^{2\,or\,3} \to \mathbb{R}$, defined by $(g \circ f)(\vec{x}) = g\left(f\left(\vec{x}\right)\right)$, in other words $(g \circ f)(x, y, z) = g\left(f(x, y, z)\right)$, is continuous on D.*

(2) *If $\vec{g}\colon \mathbb{R} \to \mathbb{R}^3$ is continuous on $[a, b]$ and $\vec{g}(t) \in D$ for every $t \in [a, b]$, then $f \circ \vec{g}\colon \mathbb{R} \to \mathbb{R}$, defined by $(f \circ \vec{g})(t) = f\left(\vec{g}(t)\right)$–i.e., $(f \circ \vec{g})(t) = f\left(g_1(t), g_2(t), g_3(t)\right)$[2]– is continuous on $[a, b]$.*

Second, functions defined by reasonable formulas are continuous where they are defined:

Lemma 3.1.3. *If $f(x, y, z)$ is defined by a formula composed of arithmetic operations, powers, roots, exponentials, logarithms and trigonometric functions applied to the various components of the input, then $f(x, y, z)$ is continuous where it is defined.*

Proof. Consider the functions on $\mathbb{R}^2$

$$\begin{aligned} add(x_1, x_2) &= x_1 + x_2 & sub(x_1, x_2) &= x_1 - x_2 \\ mul(x_1, x_2) &= x_1 x_2 & div(x_1, x_2) &= \frac{x_1}{x_2}; \end{aligned}$$

each of the first three is continuous on $\mathbb{R}^2$, and the last is continuous off the x_1-axis, because of the basic laws about arithmetic of convergent sequences (see *Calculus Deconstructed*, Theorem 2.4.1, or another single-variable calculus text).

But then application of Remark 3.1.2 to these and powers, roots, exponentials, logarithms and trigonometric functions (which are all continuous where defined) yields the lemma. □

Remark 3.1.2 can also be used to get a weak analogue of the Intermediate Value Theorem (see *Calculus Deconstructed*, Theorem 3.2.1, or another single-variable calculus text). Recall that this says that, for $f\colon \mathbb{R} \to \mathbb{R}$ continuous on $[a, b]$, if $f(a) = A$ and $f(b) = B$ then for every C between A and B the equation $f(x) = C$ has at least one solution between a and b. Since the notion of a point in the plane or in space being "between" two others doesn't really make sense, there isn't really a direct analogue of the Intermediate Value Theorem, either for $\vec{f}\colon \mathbb{R} \to \mathbb{R}^{2\text{ or }3}$ or for $f\colon \mathbb{R}^{2\text{ or }3} \to \mathbb{R}$. However, we can do the following: Given two points $\vec{a}, \vec{b} \in \mathbb{R}^{2\text{ or }3}$, we define a **path** from $\vec{a}$ to $\vec{b}$ to be the image of any locally one-to-one continuous function $\vec{p}\colon \mathbb{R} \to \mathbb{R}^{2\text{ or }3}$, parametrized so that $\vec{p}(a) = \vec{a}$ and $\vec{p}(b) = \vec{b}$. Then we can talk about points "between" $\vec{a}$ and $\vec{b}$ *along this curve*.

[2](omit $g_3(t)$ if $D \subset \mathbb{R}^2$).

Proposition 3.1.4. *If $f : \mathbb{R}^{2\ or\ 3} \to \mathbb{R}$ is continuous on a set $D \subset \mathbb{R}^{2\ or\ 3}$ and $\vec{a}$ and $\vec{b}$ are points of D that can be joined by a path in D, then for every number C between $f(\vec{a})$ and $f(\vec{b})$ the equation*

$$f(\vec{x}) = C$$

has at least one solution between $\vec{a}$ and $\vec{b}$ along any path in D which joins the two points.

The proof of this is a simple application of Remark 3.1.2 to $f \circ \vec{p}$ (Exercise 6).

For example, if $f(\vec{x})$ is continuous on $\mathbb{R}^{2 \text{ or } 3}$ and $f(\vec{a})$ is positive while $f(\vec{b})$ is negative, then the function must equal zero somewhere on any path from $\vec{a}$ to $\vec{b}$.

Limits of Functions. To study discontinuities for a real-valued function of one variable, we defined the limit of a function at a point. In this context, we always ignored the value of the function *at* the point in question, looking only at the values at points *nearby*. The old definition carries over verbatim:

Definition 3.1.5. *Suppose the function $f(\vec{x})$ is defined on a set $D \subset \mathbb{R}^{2\ or\ 3}$ and $\vec{x_0}$ is an accumulation point[3] of D; we say that the function* **converges** *to $L \in \mathbb{R}$ as $\vec{x}$ goes to $\vec{x_0}$ if whenever $\{\vec{x_k}\}$ is a sequence of points in D, all distinct from $\vec{x_0}$, which converges to $\vec{x_0}$, the corresponding sequence of values of $f(x_0)$ converges to L: $\vec{x_0} \neq \vec{x_k} \to \vec{x_0} \Rightarrow f(\vec{x_k}) \to L$.*

The same arguments that worked before show that a function converges to at most one number at any given point, so we can speak of "the" **limit** of the function at $\vec{x} = \vec{x_0}$, denoted $L = \lim_{\vec{x} \to \vec{x_0}} f(\vec{x})$.

For functions of one variable, we could sometimes understand (ordinary, two-sided) limits in terms of "one-sided limits". Of course, this idea does not really work for functions of more than one variable, since the "right" and "left" sides of a point in the plane or space don't make much sense. One way we might try to adapt this idea is to think in terms of limits from different "directions", that is, we might test what happens as we approach the point along different lines through the point. For example, the function defined for $\vec{x} \neq \vec{0} \in \mathbb{R}^2$ by

$$f(x, y) = \frac{xy}{x^2 + y^2}, \quad (x, y) \neq (0, 0)$$

is constant (and hence approaches a limit) along each line through the origin, but these limits depend on the slope of the line (Exercise 3), and so the limit $\lim_{\vec{x} \to \vec{0}} f(\vec{x})$ does not exist. However, this kind of test may not be enough. For example, the function defined on the plane except the origin by

$$f(x, y) = \frac{x^2 y}{x^4 + y^2}, \quad (x, y) \neq (0, 0)$$

approaches 0 along *every* line through the origin, but along the *parabola* $y = mx^2$ we see a different behavior: the function has a constant value which unfortunately depends on the parameter m (Exercise 3). Thus the limit along a *parabola* depends on which parabola we use to approach the origin. In fact, we *really* need to require that the limit of the function along *every curve* through the origin is the same. This is even harder to think about than looking at every *sequence* converging to $\vec{0}$.

[3]A point $\vec{x_0}$ is an **accumulation point** of the set $D \subset \mathbb{R}^{2 \text{ or } 3}$ if there exists a sequence of points in D, all distinct from $\vec{x_0}$, which converge to $\vec{x_0}$.

The definition of limits in terms of δ's and ε's, which we downplayed in the context of single variable calculus, is a much more useful tool in the context of functions of several variables.

Remark 3.1.6. *(ε-δ Definition of limit:)*
For a function $f(\vec{x})$ defined on a set $D \subset \mathbb{R}^{2 \text{ or } 3}$ with $\vec{x_0}$ an accumulation point of D, the following conditions are equivalent:

(1) *For every sequence $\{\vec{x_k}\}$ of points in D distinct from $\vec{x_0}$, $f(\vec{x_k}) \to L$;*
(2) *For every $\varepsilon > 0$ there exists $\delta > 0$ such that for points $\vec{x} \in D$, $0 < \operatorname{dist}(\vec{x}, \vec{x_0}) < \delta$ guarantees $|f(\vec{x}) - L| < \varepsilon$.*

The Polar Trick. The ε-δ formulation can sometimes be awkward to apply, but for finding limits of functions of two variables at the origin in $\mathbb{R}^2$, we can sometimes use a related trick, based on polar coordinates. To see how it works, consider the example

$$f(x, y) = \frac{x^3}{x^2 + y^2}, \quad (x, y) \neq (0, 0).$$

If we express this in the polar coordinates of (x, y) (that is, use the substitution $x = r\cos\theta$ and $y = r\sin\theta$), we have

$$f(r\cos\theta, r\sin\theta) = \frac{r^3\cos^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = r\cos^3\theta.$$

Now, the distance of (x, y) from the origin is r, so convergence to a limit at the origin would mean that by making $r < \delta$ we can ensure that $|f(x, y) - L| < \varepsilon$; in other words, we want to know whether $r\cos^3\theta$ approaches a limit as $r \to 0$, regardless of the behavior of θ. But this is clear: since $|\cos^3\theta| \leq 1$, for any sequence of points $\vec{p_i}$ converging to the origin the polar coordinates (r_i, θ_i) satisfy $r_i\cos^3\theta_i \to 0$ and so

$$\lim_{(x,y)\to\vec{0}} \frac{x^3}{x^2 + y^2} = 0.$$

We explore some features of this method in Exercise 4.

Discontinuities. Recall that a function is **continuous at a point** x_0 in its domain if

$$\lim_{x\to x_0} f(x) = f(x_0).$$

This carries over verbatim to functions of several variables: a function $f : \mathbb{R}^{2 \text{ or } 3} \to \mathbb{R}$ is continuous at a point $\vec{x_0}$ in its domain if

$$\lim_{\vec{x}\to\vec{x_0}} f(\vec{x}) = f(\vec{x_0}).$$

If a function has a limit at $\vec{x_0}$ but fails to be continuous at $\vec{x_0}$ either because the limit as $\vec{x} \to \vec{x_0}$ differs from the value at $\vec{x} = \vec{x_0}$, or because $f(\vec{x_0})$ is undefined, then we can restore continuity at $\vec{x} = \vec{x_0}$ simply by redefining the function at $\vec{x} = \vec{x_0}$ to equal its limit there; we call this a **removable discontinuity**. If on the other hand the limit as $\vec{x} \to \vec{x_0}$ fails to exist, there is no way (short of major revisionism) of getting the function to be continuous at $\vec{x} = \vec{x_0}$, and we have an **essential discontinuity**.

Our divergent examples above show that the behavior of a rational function (a ratio of polynomials) in several variables near a zero of its denominator can be much more complicated than for one variable, if the discontinuity is essential.

Exercises for § 3.1

Answers to Exercises 1a, 1c, and 2a are given in Appendix A.13.

Practice problems:

(1) For each function below, find its limit as $(x, y) \to (0, 0)$:

(a) $\frac{\sin(x^2 + y^2)}{x^2 + y^2}$ (b) $\frac{x^2}{\sqrt{x^2 + y^2}}$ (c) $\frac{x^2}{x^2 + y^2}$

(d) $\frac{2x^2y}{x^2 + y^2}$ (e) $e^x y$ (f) $\frac{(x + y)^2 - (x - y)^2}{xy}$

(g) $\frac{x^3 - y^3}{x^2 + y^2}$ (h) $\frac{\sin(xy)}{y}$ (i) $\frac{e^{xy} - 1}{y}$

(j) $\frac{\cos(xy) - 1}{x^2y^2}$ (k) $\frac{xy}{x^2 + y^2 + 2}$ (l) $\frac{(x - y)^2}{x^2 + y^2}$

(2) Find the limit of each function as $(x, y, z) \to (0, 0, 0)$:

(a) $\frac{2x^2y \cos z}{x^2 + y^2}$ (b) $\frac{xyz}{x^2 + y^2 + z^2}$

Theory problems:

(3) (a) Show that the function $f(x, y) = \frac{xy}{x^2+y^2}$ for $(x, y) \neq (0, 0)$ is constant along each line $y = mx$ through the origin, but that the constant value along each such line is different.

(b) Show that the function $f(x, y) = \frac{x^2y}{x^4+y^2}$ for $(x, y) \neq (0, 0)$ approaches zero along any line through the origin.

(c) Show that the function $f(x, y) = \frac{x^2y}{x^4+y^2}$ for $(x, y) \neq (0, 0)$ is constant along each of the parabolas $y = mx^2$ going through the origin, but that this constant value varies with the parameter m.

(4) (a) Use polar coordinates to show that the function $f(x, y) = \frac{xy}{x^2+y^2}$, $(x, y) \neq (0, 0)$ diverges at the origin.

(b) Explore what happens when you try the "polar trick" on the function $f(x, y) = \frac{x^2y}{x^4+y^2}$.

(5) Prove Remark 3.1.2.

(6) Prove Proposition 3.1.4.

3.2 Linear and Affine Functions

So far we have seen the derivative in two settings. For a real-valued function $f(x)$ of one variable, the derivative $f'(x_0)$ at a point x_0 first comes up as a number, which turns out to be the slope of the tangent line. This in turn is the line which best approximates the graph $y = f(x)$ near the point, in the sense that it is the graph of the polynomial of degree one, $T_{x_0}f = f(x_0) + f'(x_0)(x - x_0)$, which has **first-order contact** with the curve at the point $(x_0, f(x_0))$:

$$\left|f(x) - T_{x_0}f(x)\right| = \mathfrak{o}(|x - x_0|)$$

or

$$\frac{|f(x) - T_{x_0}f(x)|}{|x - x_0|} \to 0 \text{ as } x \to x_0.$$

If we look back at the construction of the derivative $\vec{p}\,'(t)$ of a vector-valued function $\vec{p}\colon \mathbb{R} \to \mathbb{R}^{2 \text{ or } 3}$ in § 2.3, we see a similar phenomenon: $\vec{p}\,'(t_0) = \vec{v}(t_0)$ is the direction vector for a parametrization of the tangent line, and the resulting vector-valued function, $T_{t_0}\vec{p}(t) = \vec{p}(t_0) + \vec{v}(t_0)(t - t_0)$, expresses how the point would move if the constraints keeping it on the curve traced out by $\vec{p}(t)$ were removed after $t = t_0$. In complete analogy to the real-valued case, $T_{t_0}\vec{p}(t)$ has first-order contact with $\vec{p}(t)$ at $t = t_0$:

$$\frac{\left\|\vec{p}(t) - T_{t_0}\vec{p}(t)\right\|}{|t - t_0|} \to 0 \text{ as } t \to t_0$$

or, in "little oh" notation,

$$\left\|\vec{p}(t) - T_{t_0}\vec{p}(t)\right\| = \mathfrak{o}(|t - t_0|).$$

It is really this last approximation property of the derivative in both cases that is at the heart of the way we use derivatives. So it would be useful to find an analogous formulation for derivatives in the case of a real-valued function $f(\vec{x})$ of a *vector* variable. This section is devoted to formulating what kind of approximation we are looking for (the analogue of having a parametrization of a line in the vector-valued case); then in the next section we will see how this gives us the right kind of approximation to $f(\vec{x})$.

Linearity. In both of the cases reviewed above, the tangent approximation to a function (real- or vector-valued) is given by polynomials of degree one in the variable. Analogously, in trying to approximate a function $f(x_1, x_2, x_3)$ of 3 variables, we would expect to look for a polynomial of degree one in these variables: $p(x_1, x_2, x_3) = a_1x_1 + a_2x_2 + a_3x_3 + c$, where the coefficients a_i, $i = 1, 2, 3$ and c are real constants. To formulate this in vector terms, we begin by ignoring the constant term (which in the case of our earlier approximations is just the value of the function being approximated, at the time of approximation). A degree one polynomial with zero constant term,[4] $h(x_1, x_2, x_3) = a_1x_1 + a_2x_2 + a_3x_3$, has two important properties:

Scaling: If we multiply each variable by some common real number α, the value of the function is multiplied by α:

$$h(\alpha x_1, \alpha x_2, \alpha x_3) = \alpha \cdot h(x_1, x_2, x_3);$$

in vector terms, this can be written $h(\alpha\vec{x}) = \alpha h(\vec{x})$. This property is often referred to as **homogeneity of degree one**.

Additivity: If the value of each variable is a sum of two values, the value of the function is the same as its value over the first summands plus its value over the second ones:

$$h(x_1 + y_1, x_2 + y_2, x_3 + y_3) = h(x_1, x_2, x_3) + h(y_1, y_2, y_3)$$

or in vector terms, $h(\vec{x} + \vec{y}) = h(\vec{x}) + h(\vec{y})$.

[4] also called a **homogeneous polynomial** of degree one

These two properties together can be summarized by saying that $h(\vec{x})$ **respects linear combinations**: for any two vectors $\vec{x}$ and $\vec{y}$ and any two numbers α and β,

$$h(\alpha\vec{x}+\beta\vec{y}) = \alpha h(\vec{x}) + \beta h(\vec{y}).$$

A function which respects linear combinations is called a **linear function**.

The preceding discussion shows that every homogeneous polynomial of degree one is a linear function.

Recall that the **standard basis** for $\mathbb{R}^3$ is the collection $\vec{\imath}, \vec{\jmath}, \vec{k}$ of unit vectors along the three positive coordinate axes; we will find it useful to replace the "alphabetical" notation for the standard basis with an indexed one: $\vec{e_1} = \vec{\imath}$, $\vec{e_2} = \vec{\jmath}$, and $\vec{e_3} = \vec{k}$. The basic property[5] of the standard basis is that every vector $\vec{x} \in \mathbb{R}^3$ is, in a standard way, a linear combination of these specific vectors: $\vec{x} = (x, y, z) = x\vec{\imath}+y\vec{\jmath}+z\vec{k}$, or $(x_1, x_2, x_3) = x_1\vec{e_1} + x_2\vec{e_2} + x_3\vec{e_3}$.

Then combining this with the fact that linear functions respect linear combinations, we easily see (Exercise 8) that all linear functions are homogeneous polynomials in the coordinates of their input:

Remark 3.2.1. *Every linear function $\ell : \mathbb{R}^3 \to \mathbb{R}$ is determined by its effect on the elements of the standard basis for $\mathbb{R}^3$: if $\ell(\vec{e_i}) = a_i$ for $i = 1, 2, 3$, then $\ell(x_1, x_2, x_3)$ is the degree one homogeneous polynomial*

$$\ell(x_1, x_2, x_3) = a_1x_1 + a_2x_2 + a_3x_3.$$

Matrix Representation of Linear Functions. We are now going to set up what will at first look like an unnecessary complication of the picture above, but in time it will open the door to appropriate generalizations. The essential data concerning a linear function (*a.k.a.* a homogeneous polynomial of degree one) is the set of values taken by ℓ on the standard basis of $\mathbb{R}^3$: $a_i = \ell(\vec{e_i})$, $i = 1, 2, 3$. We shall form these numbers into a 1×3 matrix (a **row matrix**), called the **matrix representative** of ℓ:

$$[\ell] = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}.$$

We shall also create a 3×1 matrix (a **column matrix**) whose entries are the components of the vector $\vec{x}$, called the **coordinate matrix** of $\vec{x}$:

$$[\vec{x}] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

We then define the **product** of a row with a column as the result of substituting the entries of the column into the homogeneous polynomial whose coefficients are the entries of the row; equivalently, we match the i^{th} entry of the row with the i^{th} entry of the column, multiply each matched pair, and add:

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = a_1x_1 + a_2x_2 + a_3x_3.$$

Of course, in this language, we are representing the linear function $\ell : \mathbb{R}^3 \to \mathbb{R}$ as the product of its matrix representative with the coordinate matrix of the input

$$\ell(\vec{x}) = [\ell][\vec{x}].$$

[5]No pun intended.

Another way to think of this representation is to associate, to any **row**, a **vector** $\vec{a}$ (just put commas between the entries of the row matrix), and then to notice that the product of the *row* with the *coordinate matrix* of $\vec{x}$ is the same as the *dot product* of the *vector* $\vec{a}$ with $\vec{x}$:

$$\begin{aligned}\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= (a_1, a_2, a_3) \cdot (x_1, x_2, x_3) \\ &= \vec{a} \cdot \vec{x}.\end{aligned}$$

Thus we see that there are three ways to think of the action of the linear function $\ell : \mathbb{R}^3 \to \mathbb{R}$ on a vector $\vec{x} \in \mathbb{R}^3$:

- Substitute the components of $\vec{x}$ into a homogeneous polynomial of degree one, whose coefficients are the values of ℓ on the standard basis;
- Multiply the coordinate matrix of $\vec{x}$ by the matrix representative of ℓ;
- Take the dot product of the vector $\vec{a}$ (obtained from the row matrix $[\ell]$ by introducing commas) with the vector $\vec{x}$.

Affine Functions. Finally, we introduce one more piece of terminology: an **affine function** is the sum of a constant and a linear function:

$$\phi(\vec{x}) = c + \ell(\vec{x}).$$

In other words, an affine function is the same thing as a polynomial of degree one (with no homogeneity conditions—that is, without any restriction on the constant term).

Note that if $\phi(\vec{x}) = c + \ell(\vec{x})$ is an affine function, then for any two vectors $\vec{x}$ and $\vec{y}$,

$$\phi(\vec{y}) - \phi(\vec{x}) = \ell(\vec{y}) - \ell(\vec{x}) = \ell(\vec{y} - \vec{x}).$$

Let $\triangle\vec{x} = \vec{y} - \vec{x}$ be the displacement of $\vec{y}$ from $\vec{x}$: $\vec{y} = \vec{x} + \triangle\vec{x}$. Then we can write

$$\phi(\vec{x} + \triangle\vec{x}) = \phi(\vec{x}) + \ell(\triangle\vec{x}). \tag{3.1}$$

Remark 3.2.2. *Given any "basepoint" $\overrightarrow{x_0} \in \mathbb{R}^3$, the affine function $\phi : \mathbb{R}^3 \to \mathbb{R}$ can be written in the form of Equation* (3.1)*, as its value at $\overrightarrow{x_0}$ plus a linear function of the displacement from $\overrightarrow{x_0}$:*

$$\phi(\overrightarrow{x_0} + \triangle\vec{x}) = \phi(\overrightarrow{x_0}) + \ell(\triangle\vec{x}) \tag{3.2}$$

or, stated differently, the displacement of $\phi(\vec{x})$ from $\phi(\overrightarrow{x_0})$ is a linear function of the displacement of $\vec{x}$ from $\overrightarrow{x_0}$:

$$\phi(\overrightarrow{x_0} + \triangle\vec{x}) - \phi(\overrightarrow{x_0}) = \ell(\triangle\vec{x}).$$

In light of this observation, we can use Remark 3.2.1 to determine an affine function from its value at a point $\overrightarrow{x_0}$ together with its values at the points $\overrightarrow{x_0} + \vec{e_j}$ obtained by displacing the original point in a direction parallel to one of the coordinate axes. A brief calculation shows that

$$\phi(\overrightarrow{x_0} + \triangle\vec{x}) = a_0 + \sum_{j=1}^{3} a_j \triangle x_j, \tag{3.3}$$

where $\triangle\vec{x} = (\triangle x_1, \triangle x_2, \triangle x_3)$ and for $j = 1, 2, 3$ $a_j = \phi(\overrightarrow{x_0} + \vec{e_j}) - \phi(\overrightarrow{x_0})$.

Of course, everything we have said about linear and affine functions on $\mathbb{R}^3$ applies just as well to functions on $\mathbb{R}^2$. A geometric way to think about linear versus affine functions on $\mathbb{R}^2$ is the following:

Remark 3.2.3. *The graph of a linear function $\ell : \mathbb{R}^2 \to \mathbb{R}$ is a (non-vertical) plane through the origin; the graph of an affine function $\phi(\vec{x}) = c + \ell(\vec{x})$ is a plane, crossing the z-axis at $z = c$, parallel to the graph of ℓ.*

Finally, we note that, in addition to the one-dimensional examples of derivatives and of tangent lines to graphs of functions, our standard approach to parametrizing a plane in $\mathbb{R}^3$, as given in Equation (1.24) expresses each of the three coordinates x, y and z as affine functions of the two parameters s and t. In fact, it would be natural to think of Equation (1.23) as defining an affine vector-valued function of the vector $(s,t) \in \mathbb{R}^2$—a viewpoint we will adopt in Chapter 5.

Exercises for § 3.2

Answers to Exercises 1a and 4a are given in Appendix A.13.

Practice problems:

(1) For each linear function $\ell(\vec{x})$ below, you are given the values on the standard basis. Find $\ell(1,-2,3)$ and $\ell(2,3,-1)$.
  (a) $\ell(\vec{\imath}) = 2, \ell(\vec{\jmath}) = -1, \ell\left(\vec{k}\right) = 1$.
  (b) $\ell(\vec{\imath}) = 1, \ell(\vec{\jmath}) = 1, \ell\left(\vec{k}\right) = 1$.
  (c) $\ell(\vec{\imath}) = 3, \ell(\vec{\jmath}) = 4, \ell\left(\vec{k}\right) = -5$.

(2) Is there a *linear* function $\ell : \mathbb{R}^3 \to \mathbb{R}$ for which $\ell(1,1,1) = 0$, $\ell(1,-1,2) = 1$, and $\ell(2,0,3) = 2$? Why or why not? Is there an *affine* function with these values? If so, give one. Are there others?

(3) If $\ell : \mathbb{R}^3 \to \mathbb{R}$ is linear with $\ell(1,1,1) = 3$, $\ell(1,2,0) = 5$, and $\ell(0,1,2) = 2$, then
  (a) Find $\ell(\vec{\imath})$, $\ell(\vec{\jmath})$, and $\ell\left(\vec{k}\right)$.
  (b) Express $\ell(x,y,z)$ as a homogeneous polynomial.
  (c) Express $\ell(\vec{x})$ as a matrix multiplication.
  (d) Express $\ell(\vec{x})$ as a dot product.

(4) Consider the affine function $\phi : \mathbb{R}^3 \to \mathbb{R}$ given by the polynomial $3x - 2y + z + 5$. Express $\phi(\vec{x})$ in the form given by Remark 3.2.2, when $\vec{x_0}$ is each of the vectors given below:
  (a) $\vec{x_0} = (1,2,1)$ (b) $\vec{x_0} = (-1,2,1)$ (c) $\vec{x_0} = (2,1,1)$

Theory problems:

(5) **Show** that an affine function $f : \mathbb{R}^2 \to \mathbb{R}$ is determined by its values on the vertices of any nondegenerate triangle.

(6) Suppose $\vec{p}(s,t)$ is a parametrization of a plane in $\mathbb{R}^3$ of the form given by Equation (1.23) and Equation (1.24) in § 1.5, and $f : \mathbb{R}^3 \to \mathbb{R}$ is linear. Show that $f \circ \vec{p} : \mathbb{R}^2 \to \mathbb{R}$ is an affine function.

(7) A *level set* of a function is the set of points where the function takes a particular value. Show that any level set of an affine function on $\mathbb{R}^2$ is a line, and a level set of an affine function on $\mathbb{R}^3$ is a plane. When does the line/plane go through the origin?

(8) Prove Remark 3.2.1.

(9) Prove Remark 3.2.2.
(10) Carry out the calculation that establishes Equation (3.3).

3.3 Derivatives

In this section we carry out the program outlined at the beginning of § 3.2, trying to formulate the derivative of a real-valued function of several variables $f(\vec{x})$ in terms of an affine function making first-order contact with $f(\vec{x})$.

Definition 3.3.1. *A real-valued function of three variables $f : \mathbb{R}^{2\,or\,3} \to \mathbb{R}$ is **differentiable** at $\overrightarrow{x_0} \in \mathbb{R}^{2\,or\,3}$ if f is defined for all $\vec{x}$ sufficiently near $\overrightarrow{x_0}$ and there exists an affine function $T_{\overrightarrow{x_0}} f(\vec{x}) : \mathbb{R}^{2\,or\,3} \to \mathbb{R}$ which has first-order contact with $f(\vec{x})$ at $\vec{x} = \overrightarrow{x_0}$:*

$$\left|f(\vec{x}) - T_{\overrightarrow{x_0}} f(\vec{x})\right| = \mathfrak{o}(\left\|\vec{x} - \overrightarrow{x_0}\right\|) \tag{3.4}$$

which is to say

$$\lim_{\vec{x} \to \overrightarrow{x_0}} \frac{\left|f(\vec{x}) - T_{\overrightarrow{x_0}} f(\vec{x})\right|}{\left\|\vec{x} - \overrightarrow{x_0}\right\|} = 0. \tag{3.5}$$

*When such an affine function exists, we call it the **linearization** of $f(\vec{x})$ or the **linear approximation** to $f(\vec{x})$, at $\vec{x} = \overrightarrow{x_0}$.*[6]

Since functions with first-order contact must agree at the point of contact, we know that

$$T_{\overrightarrow{x_0}} f(\overrightarrow{x_0}) = f(\overrightarrow{x_0});$$

then Remark 3.2.2 tells us that

$$T_{\overrightarrow{x_0}} f(\overrightarrow{x_0} + \triangle\vec{x}) = f(x_0) + \ell(\triangle\vec{x}), \tag{3.6}$$

where ℓ is a linear function.

Furthermore, since $\ell(\triangle\vec{x})$ is a polynomial, it is continuous, so that

$$\lim_{\triangle\vec{x} \to \vec{0}} \ell(\triangle\vec{x}) = 0$$

and $\lim_{\vec{x} \to \overrightarrow{x_0}} \left(f(\vec{x}) - T_{\overrightarrow{x_0}} f(\vec{x})\right) = \lim_{\vec{x} \to \overrightarrow{x_0}} \left[f(\vec{x}) - f(\overrightarrow{x_0})\right]$. But since the denominator in Equation (3.5) goes to zero, so must the numerator, which says that the last limit above is zero. This shows:

Remark 3.3.2. *If $f(\vec{x})$ is differentiable at $\vec{x} = \overrightarrow{x_0}$ then it is continuous there.*

To calculate the "linear part" $\ell(\triangle\vec{x})$ of $T_{\overrightarrow{x_0}} f(\vec{x})$ (if it exists), we consider the action of $f(\vec{x})$ along the line through $\overrightarrow{x_0}$ with a given direction vector $\vec{v}$: this is parametrized by $\vec{p}(t) = \overrightarrow{x_0} + t\vec{v}$ and the restriction of $f(\vec{x})$ to this line is given by the composition $f(\vec{p}(t)) = f(\overrightarrow{x_0} + t\vec{v})$. Then setting $\triangle\vec{x} = t\vec{v}$ in Equation (3.6) we have

$$T_{\overrightarrow{x_0}} f(\overrightarrow{x_0} + t\vec{v}) = f(\overrightarrow{x_0}) + \ell(t\vec{v}) = f(\overrightarrow{x_0}) + t\ell(\vec{v}).$$

Equation (3.5) then says that, if we let $t \to 0$,

$$\frac{\left|f(\overrightarrow{x_0} + t\vec{v}) - f(\overrightarrow{x_0}) - t\ell(\vec{v})\right|}{\left\|t\vec{v}\right\|} \to 0,$$

[6] Properly speaking, it *should* be called the **affine approximation**.

from which it follows that

$$\ell(\vec{v}) = \lim_{t\to 0} \frac{1}{t}\left(f(\overrightarrow{x_0} + t\vec{v}) - f(\overrightarrow{x_0})\right).$$

This formula shows that, *if it exists*, the affine approximation to $f(\vec{x})$ at $\vec{x} = \overrightarrow{x_0}$ is unique; we call the "linear part" $\ell(\triangle\vec{x})$ of $T_{\overrightarrow{x_0}} f(\vec{x})$ the **derivative** or **differential** of $f(\vec{x})$ at $\vec{x} = \overrightarrow{x_0}$, and denote it $d_{\overrightarrow{x_0}} f$. Note that this equation can also be interpreted in terms of the derivative at $t = 0$ of the composite function $f(\vec{p}(t))$:

$$d_{\overrightarrow{x_0}} f(\vec{v}) = \lim_{t\to 0} \frac{1}{t}\left(f(\overrightarrow{x_0} + t\vec{v}) - f(\overrightarrow{x_0})\right) = \left.\frac{d}{dt}\right|_{t=0} \left[f(\overrightarrow{x_0} + t\vec{v})\right]. \tag{3.7}$$

For example, if $f(x, y) = x^2 - xy$, $\overrightarrow{x_0} = (3, 1)$, and $\vec{v} = (v_1, v_2)$, then, using the limit formula,

$$\begin{aligned} d_{(3,1)} f((v_1, v_2)) &= \lim_{t\to 0} \frac{1}{t}\left[f(3 + v_1 t, 1 + v_2 t) - f(3, 1)\right] \\ &= \lim_{t\to 0} \left[(5v_1 - 3v_2) + t(v_1^2 - v_1 v_2)\right] \\ &= 5v_1 - 3v_2 \end{aligned}$$

or we could use the differentiation formula:

$$d_{(3,1)} f((v_1, v_2)) = \left.\frac{d}{dt}\right|_{t=0} \left[f(3 + v_1 t, 1 + v_2 t)\right] = 5v_1 - 3v_2.$$

Partial Derivatives. Equation (3.7), combined with Remark 3.2.1, gives us a way of expressing the differential $d_{\overrightarrow{x_0}} f(\vec{v})$ as a homogeneous polynomial in the components of $\vec{v}$. The quantity given by Equation (3.7) in the special case that $\vec{v} = \vec{e_j}$ is an element of the standard basis for $\mathbb{R}^3$, is called a **partial derivative**. It corresponds to moving through $\overrightarrow{x_0}$ parallel to one of the coordinate axes with unit speed—that is, the motion parametrized by $\vec{p_j}(t) = \overrightarrow{x_0} + t\vec{e_j}$:

Definition 3.3.3. *The j^{th} **partial derivative** (or **partial with respect to** x_j) of a function $f(x_1, x_2, x_3)$ of three variables at $\vec{x} = \overrightarrow{x_0}$ is the derivative (if it exists) of the function $(f \circ \vec{p_j})(t)$ obtained by fixing all variables except the j^{th} at their values at $\overrightarrow{x_0}$, and letting x_j vary:*[7]

$$f_{x_j}(\overrightarrow{x_0}) = \frac{\partial f}{\partial x_j}(\overrightarrow{x_0}) \coloneqq \left.\frac{d}{dt}\right|_{t=0} \left[f(\vec{p_j}(t))\right] = \left.\frac{d}{dt}\right|_{t=0} \left[f(\overrightarrow{x_0} + t\vec{e_j})\right]$$

in other words,

$$\begin{aligned} f_x(x, y, z) &= \frac{\partial f}{\partial x}(x, y, z) \coloneqq \lim_{t\to 0} \frac{1}{t}\left[f(x + t, y, z) - f(x, y, z)\right] \\ f_y(x, y, z) &= \frac{\partial f}{\partial y}(x, y, z) \coloneqq \lim_{t\to 0} \frac{1}{t}\left[f(x, y + t, z) - f(x, y, z)\right] \\ f_z(x, y, z) &= \frac{\partial f}{\partial z}(x, y, z) \coloneqq \lim_{t\to 0} \frac{1}{t}\left[f(x, y, z + t) - f(x, y, z)\right]. \end{aligned}$$

In practice, partial derivatives are easy to calculate: we just differentiate, treating all but one of the variables as a constant. For example, if

$$f(x, y) = x^2 y + 3x + 4y$$

[7]The symbol $\frac{\partial f}{\partial x_j}$ is pronounced as if the ∂'s were d's.

then $\frac{\partial f}{\partial x}$, the partial with respect to x, is obtained by treating y as the name of some constant:

$$f_x(x,y) := \frac{\partial f}{\partial x}(x,y) = 2xy + 3$$

while the partial with respect to y is found by treating x as a constant:

$$f_y(x,y) := \frac{\partial f}{\partial y}(x,y) = x^2 + 4;$$

similarly, if

$$g(x,y,z) = \sin 2x \cos y + xyz^2,$$

then

$$\begin{aligned}
g_x(x,y,z) &:= \frac{\partial g}{\partial x}(x,y,z) = 2\cos 2x \cos y + yz^2 \\
g_y(x,y,z) &:= \frac{\partial g}{\partial y}(x,y,z) = -\sin 2x \sin y + xz^2 \\
g_z(x,y,z) &:= \frac{\partial g}{\partial z}(x,y,z) = 2xyz.
\end{aligned}$$

Remark 3.2.1 tells us that the differential of f, being linear, is determined by the partials of f:

$$d_{\vec{x_0}} f(\vec{v}) = \left(\frac{\partial f}{\partial x_1}(\vec{x_0})\right) v_1 + \left(\frac{\partial f}{\partial x_2}(\vec{x_0})\right) v_2 + \left(\frac{\partial f}{\partial x_3}(\vec{x_0})\right) v_3 = \sum_{j=1}^{3} \frac{\partial f}{\partial x_j} v_j.$$

So far, we have avoided the issue of existence: all our formulas above assume that $f(\vec{x})$ is differentiable at $\vec{x} = \vec{x_0}$. Since the partial derivatives of a function are essentially derivatives as we know them from single-variable calculus, it is usually pretty easy to determine whether they exist and if so to calculate them formally. However, the existence of the *partials* is not by itself a guarantee that the function is *differentiable*. For example, the function we considered in § 3.1

$$f(x) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{at } (0,0) \end{cases}$$

has the constant value zero along both axes, so certainly its two partials at the origin exist and equal zero

$$\frac{\partial f}{\partial x}(0,0) = 0 = \frac{\partial f}{\partial y}(0,0)$$

but if we try to calculate $d_{(0,0)}f(\vec{v})$ for the vector $\vec{v} = (1,m)$ using Equation (3.7),

$$d_{(0,0)}f(1,m) = \lim_{t\to 0} \frac{1}{t}(f(t,mt) - f(0,0))$$

then along any line $y = mx$ the function has a constant value $f(t,mt) = m/(1+m^2)$ for $x \neq 0$, while $f(0,0) = 0$. If $m \neq 0$ (that is, the line is not one of the axes) we see that the limit above does not exist:

$$\lim_{t\to 0} \frac{1}{t}(f(t,mt) - f(0,0)) = \lim_{t\to 0} \frac{1}{t}\left(\frac{m}{1+m^2}\right)$$

diverges, and the differential cannot be evaluated along the vector $\vec{v} = \vec{\imath} + m\vec{\jmath}$ if $m \neq 0$. In fact, we saw before that this function is not continuous at the origin, which already contradicts differentiability, by Remark 3.3.2.

Another example, this time one which *is* continuous at the origin, is

$$f(x, y) = \begin{cases} \frac{2xy}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{at } (0, 0) \end{cases}.$$

This function is better understood when expressed in polar coordinates, where it takes the form

$$f(r\cos\theta, r\sin\theta) = \frac{2r^2\cos\theta\sin\theta}{r} = r\sin 2\theta.$$

From this we see that along the line making angle θ with the x-axis, $f(x, y)$ is a constant $(\sin 2\theta)$ times the distance from the origin: geometrically, the graph of $f(x, y)$ over this line is itself a line through the origin of slope $m = \sin 2\theta$. Along the two coordinate axes, this slope is zero, but for example along the line $y = x$ $(\theta = \pi/4)$, the slope is $\sin\pi/2 = 1$. So this time, the function defined by Equation (3.7) (without asking about differentiability) exists at the origin, but *it is not linear* (since again it is zero on each of the standard basis elements $\vec{\imath}$ and $\vec{\jmath}$).

A third example is defined by a straightforward formula (no "cases"):

$$f(x, y) = x^{1/3}y^{1/3}.$$

Again, the function is constant along the coordinate axes, so both partials are zero. However, if we try to evaluate the limit in Equation (3.7) using any vector not pointing along the axes, we get

$$d_{(0,0)}f\left(\alpha\vec{\imath} + \beta\vec{\jmath}\right) = \left.\frac{d}{dt}\right|_{t=0}\left[\alpha^{1/3}\beta^{1/3}t^{2/3}\right];$$

since $t^{2/3}$ is definitely not differentiable at $t = 0$, the required linear map $d_{(0,0)}f\left(\alpha\vec{\imath} + \beta\vec{\jmath}\right)$ cannot exist.

From all of this, we see that having the partials at a point $\overrightarrow{x_0}$ is not enough to guarantee differentiability of $f\left(\vec{x}\right)$ at $\vec{x} = \overrightarrow{x_0}$. It is not even enough to also have partials at every point near $\overrightarrow{x_0}$—all our examples above have this property. However, a slight tweaking of this last condition *does* guarantee differentiability. We call $f\left(\vec{x}\right)$ **continuously differentiable** at $\overrightarrow{x_0}$ (or $\mathcal{C}^1$) if all the partial derivatives exist for every point near $\overrightarrow{x_0}$ (including $\overrightarrow{x_0}$ itself), *and are continuous at* $\overrightarrow{x_0}$. Then we can assert:

Theorem 3.3.4. *If $f\left(\vec{x}\right)$ is continuously differentiable at $\overrightarrow{x_0}$, then it is differentiable there.*

Proof. For notational convenience, we concentrate on the case of a function of two variables; the modification of this proof to the case of three variables is straightforward (Exercise 11).

We know that, if it exists, the linearization of $f\left(\vec{x}\right)$ at $\vec{x} = \overrightarrow{x_0} = (x, y)$ is determined by the partials to be

$$T_{\overrightarrow{x_0}}f\left(x + \triangle x, y + \triangle y\right) = f(x, y) + \frac{\partial f}{\partial x}(x, y)\triangle x + \frac{\partial f}{\partial y}(x, y)\triangle y; \tag{3.8}$$

so we need to show that

$$\frac{1}{\left\|(\triangle x, \triangle y)\right\|}\left|f\left(x+\triangle x, y+\triangle y\right)-\left(f\left(x, y\right)+\frac{\partial f}{\partial x}\left(x, y\right)\triangle x+\frac{\partial f}{\partial y}\left(x, y\right)\triangle y\right)\right| \to 0 \tag{3.9}$$

as $(\triangle x, \triangle y) \to 0$. When we remove the parentheses inside the absolute value, we have an expression whose first two terms are $f\left(x+\triangle x, y+\triangle y\right)-f\left(x, y\right)$; we rewrite this as follows. By adding and subtracting the value of f at a point that shares one coordinate with each of these two points—say $f\left(x, y+\triangle y\right)$—we can write

$$\begin{aligned} f\left(x+\triangle x, y+\triangle y\right)-f\left(x, y\right) &= \left\{f\left(x+\triangle x, y+\triangle y\right)-f\left(x, y+\triangle y\right)\right\} \\ &\quad +\left\{f\left(x, y+\triangle y\right)-f\left(x, y\right)\right\} \end{aligned}$$

and then proceed to analyze each of the two quantities in braces. Note that the first quantity is the difference between the values of f along a horizontal line segment, which can be parametrized by $\vec{p}\left(t\right)=\left(x+t\triangle x, y+\triangle y\right), \quad 0 \le t \le 1$; the composite function

$$g\left(t\right)=f\left(\vec{p}\left(t\right)\right)=f\left(x+t\triangle x, y+\triangle y\right)$$

is an ordinary function of one variable, whose derivative is related to a partial derivative of f (Exercise 6):

$$g'\left(t\right)=\frac{\partial f}{\partial x}\left(x+t\triangle x, y+\triangle y\right)\triangle x.$$

Thus, we can apply the Mean Value Theorem to conclude that there is a value $t=t_1$ between 0 and 1 for which $g\left(1\right)-g\left(0\right)=g'\left(t_1\right)$. Letting $t_1\triangle x=\delta_1$, we can write

$$\begin{aligned} f\left(x+\triangle x, y+\triangle y\right)-f\left(x, y+\triangle y\right) &= g\left(1\right)-g\left(0\right) \\ &= g'\left(t_1\right)=\frac{\partial f}{\partial x}\left(x+\delta_1, y+\triangle y\right)\triangle x, \end{aligned}$$

where $|\delta_1| \le |\triangle x|$. A similar argument applied to the second term in parentheses (Exercise 6) yields

$$f\left(x, y+\triangle y\right)-f\left(x, y\right)=\frac{\partial f}{\partial y}\left(x, y+\delta_2\right)\triangle y,$$

where $|\delta_2| \le |\triangle y|$. This allows us to rewrite the quantity inside the absolute value in Equation (3.9) as

$$\begin{aligned} &f\left(x+\triangle x, y+\triangle y\right)-\left(f\left(x, y\right)+\frac{\partial f}{\partial x}\left(x, y\right)\triangle x+\frac{\partial f}{\partial y}\left(x, y\right)\triangle y\right) \\ &\quad =\left(f\left(x+\triangle x, y+\triangle y\right)-f\left(x, y\right)\right)-\left(\frac{\partial f}{\partial x}\left(x, y\right)\triangle x+\frac{\partial f}{\partial y}\left(x, y\right)\triangle y\right) \\ &=\left(\frac{\partial f}{\partial x}\left(x+\delta_1, y+\triangle y\right)\triangle x+\frac{\partial f}{\partial y}\left(x, y+\delta_2\right)\triangle y\right)-\left(\frac{\partial f}{\partial x}\left(x, y\right)\triangle x+\frac{\partial f}{\partial y}\left(x, y\right)\triangle y\right) \\ &\quad =\left(\frac{\partial f}{\partial x}\left(x+\delta_1, y+\triangle y\right)-\frac{\partial f}{\partial x}\left(x, y\right)\right)\triangle x+\left(\frac{\partial f}{\partial y}\left(x, y+\delta_2\right)-\frac{\partial f}{\partial y}\left(x, y\right)\right)\triangle y. \end{aligned}$$

Now, we want to show that this quantity, divided by $\left\|(\triangle x, \triangle y)\right\|=\sqrt{\triangle x^2+\triangle y^2}$, goes to zero as $(\triangle x, \triangle y) \to (0,0)$. Clearly,

$$\frac{|\triangle x|}{\sqrt{\triangle x^2+\triangle y^2}} \le 1 \text{ and } \frac{|\triangle y|}{\sqrt{\triangle x^2+\triangle y^2}} \le 1,$$

so it suffices to show that each of the quantities in parentheses goes to zero. But as $(\triangle x, \triangle y) \to 0$, all of the quantities $\triangle x$, $\triangle y$, δ_1 and δ_2 go to zero, which means that all of the points at which we are evaluating the partials are tending to $\overrightarrow{x_0} = (x, y)$; in particular, the difference inside each pair of (large) parentheses is going to zero. Since each such quantity is being multiplied by a bounded quantity ($\triangle x/\sqrt{\triangle x^2 + \triangle y^2}$, or $\triangle y/\sqrt{\triangle x^2 + \triangle y^2}$), the whole mess goes to zero.

This proves our assertion, that the affine function $T_{\overrightarrow{x_0}} f(\vec{x})$ as defined by Equation (3.8) has first-order contact with $f(\vec{x})$ at $\vec{x} = \overrightarrow{x_0}$. □

This result ensures that functions defined by algebraic or analytic expressions such as polynomials (in two or three variables) or combinations of trigonometric, exponential, logarithmic functions and roots are generally differentiable, since by the formal rules of differentiation the partials are again of this type, and hence are continuous wherever they are defined; the only difficulties arise in cases where differentiation introduces a denominator which becomes zero at the point in question.

The Gradient and Directional Derivatives. Recall from § 3.2 that a linear function can be viewed in three different ways: as a homogeneous *polynomial* of degree one, as multiplication of the coordinate matrix by its *matrix representative*, and as the *dot product* of the input with a fixed vector. We have seen that when $f(\vec{x})$ is differentiable at $\vec{x} = \overrightarrow{x_0}$, then the coefficients of the differential $d_{\overrightarrow{x_0}} f(\vec{v})$, as a polynomial in the entries of $\vec{v}$, are the partial derivatives of f at $\vec{x} = \overrightarrow{x_0}$; this tells us that the matrix representative of $d_{\overrightarrow{x_0}} f$ is

$$\left[d_{\overrightarrow{x_0}} f\right] = \left[\frac{\partial f}{\partial x}\left(\overrightarrow{x_0}\right) \ \frac{\partial f}{\partial y}\left(\overrightarrow{x_0}\right) \ \frac{\partial f}{\partial z}\left(\overrightarrow{x_0}\right)\right].$$

This matrix is sometimes referred to as the **Jacobian** of f at $\vec{x} = \overrightarrow{x_0}$, and denoted Jf. Equivalently, when we regard this row as a vector, we get the **gradient**[8] of f:

$$\vec{\nabla} f\left(\overrightarrow{x_0}\right) = \left(\frac{\partial f}{\partial x}\left(\overrightarrow{x_0}\right), \frac{\partial f}{\partial y}\left(\overrightarrow{x_0}\right), \frac{\partial f}{\partial z}\left(\overrightarrow{x_0}\right)\right).$$

That is, *the gradient is the vector whose entries are the partials.* It is worth reiterating how these two objects represent the differential:

$$\begin{aligned} d_{\overrightarrow{x_0}} f(\vec{v}) &= \left(Jf\left(\overrightarrow{x_0}\right)\right)[\vec{v}] \quad \text{(Matrix product)} && (3.10)\\ &= \vec{\nabla} f\left(\overrightarrow{x_0}\right) \cdot \vec{v} \quad \text{(Dot Product).} && (3.11)\end{aligned}$$

These ways of representing the differential carry essentially the same information. However, the gradient in particular has a nice geometric interpretation.

Recall that we represent a direction in the plane or space by means of a unit vector $\vec{u}$. When the differential is applied to such a vector, the resulting number is called the **directional derivative** of the function at the point. From Equation (3.7), we see that *the directional derivative gives the rate at which $f(\vec{x})$ changes as we move in the direction $\vec{u}$ at speed one.* In the plane, a unit vector has the form $\overrightarrow{u_\alpha} = (\cos\alpha)\vec{\imath} + (\sin\alpha)\vec{\jmath}$ where α is the angle our direction makes with the x-axis. In this case, the directional derivative in the direction given by α is

$$d_{\overrightarrow{x_0}} f\left(\overrightarrow{u_\alpha}\right) = \frac{\partial f}{\partial x}\left(\overrightarrow{x_0}\right)\cos\alpha + \frac{\partial f}{\partial y}\left(\overrightarrow{x_0}\right)\sin\alpha.$$

[8]The symbol $\vec{\nabla} f$ is pronounced "grad f"; another notation for the gradient is $\mathbf{grad}\, f$.

Equation (3.11) tells us that the directional derivative in the direction of the unit vector $\vec{u}$ is the dot product $d_{\overrightarrow{x_0}} f\left(\vec{u}\right) = \vec{\nabla} f \cdot \vec{u}$ which is related to the angle θ between the two vectors, so also

$$\vec{\nabla} f \cdot \vec{u} = \left\|\vec{\nabla} f\left(\overrightarrow{x_0}\right)\right\| \left\|\vec{u}\right\| \cos\theta = \left\|\vec{\nabla} f\left(\overrightarrow{x_0}\right)\right\| \cos\theta$$

since $\vec{u}$ is a unit vector. Now, $\cos\theta$ reaches its *maximum* value, which is 1, when $\theta = 0$, which is to say when $\vec{u}$ points in the direction of $\vec{\nabla} f\left(\overrightarrow{x_0}\right)$, and its *minimum* value of -1 when $\vec{u}$ points in the *opposite* direction. This gives us a geometric interpretation of the gradient, which will prove very useful.

Remark 3.3.5. *The gradient vector $\vec{\nabla} f\left(\overrightarrow{x_0}\right)$ points in the* direction *in which the directional derivative has its highest value, known as the* ***direction of steepest ascent****, and its* length *is the* value *of the directional derivative in that direction.*

As an example, consider the function $f(x,y) = 49 - x^2 - 3y^2$ at the point $\overrightarrow{x_0} = (4,1)$. The graph of this function can be viewed as a hill whose peak is above the origin, at height $f(0,0) = 49$. The gradient of this function is $\vec{\nabla} f(x,y) = (-2x)\vec{\imath} + (-6y)\vec{\jmath}$. At the point $(4,1)$,$\vec{\nabla} f(4,1) = -8\vec{\imath} - 6\vec{\jmath}$ has length $\left\|\vec{\nabla} f(4,1)\right\| = \sqrt{8^2+6^2} = 10$ and the unit vector parallel to $\vec{\nabla} f(4,1)$ is $\vec{u} = -\frac{4}{5}\vec{\imath} - \frac{3}{5}\vec{\jmath}$. This means that at the point 4 units east and one unit north of the peak, a climber who wishes to gain height as fast as possible should move in the direction given on the map by $\vec{u}$; by moving in this direction, the climber will be ascending at 10 units of height per unit of horizontal motion from an initial height of $f(4,1) = 30$. Alternatively, if a stream flowing down the mountain passes the point 4 units east and one unit north of the peak, its direction of flow on the map will be in the opposite direction, the *direction of steepest descent*.

The analogue of Remark 3.3.5 for a function of three variables holds for the same reasons. Note that in either case, the gradient "lives" in the domain of the function; thus, although the *graph* of a function of two variables is a surface in *space*, its gradient vector at any point is a vector in the *plane*.

Chain Rules. For two differentiable real-valued functions of a (single) real variable, the Chain Rule tells us that the derivative of the composition is the product of the derivatives of the two functions:

$$\left.\frac{d}{dt}\right|_{t=t_0} [f \circ g] = f'\left(g\left(t_0\right)\right) \cdot g'\left(t_0\right).$$

Similarly, if g is a differentiable real-valued function of a real variable and $\vec{f}$ is a differentiable vector-valued function of a real variable, the composition $\vec{f} \circ g$ is another vector-valued function, whose derivative is the product of the derivative of $\vec{f}$ (*i.e.*, its velocity) and the derivative of g:

$$\left.\frac{d}{dt}\right|_{t=t_0} \left[\vec{f} \circ g\right] = \overrightarrow{f'}\left(g\left(t_0\right)\right) \cdot g'\left(t_0\right).$$

We would now like to turn to the case when $\vec{g}$ is a vector-valued function of a real variable, and f is a real-valued function of a vector variable, so that their composition $f \circ \vec{g}$ is a real-valued function of a real variable. We have already seen that if $\vec{g}$ is steady motion along a straight line, then the derivative of the composition is the same as the action of the differential of f on the derivative (*i.e.*, the velocity) of $\vec{g}$. We would like to say that this is true in general. For ease of formulating our result, we shall use the notation $\vec{p}(t)$ in place of $\vec{g}(t)$, $\vec{v}$ for the velocity of $\vec{p}(t)$ at $t = t_0$, and the representation of $d_{\overrightarrow{x_0}} f(\vec{v})$ as $\vec{\nabla} f(\overrightarrow{x_0}) \cdot \vec{v}$.

Proposition 3.3.6 (Chain Rule for $\mathbb{R} \to \mathbb{R}^{2\,or\,3} \to \mathbb{R}$). *Suppose $f : \mathbb{R}^{2\,or\,3} \to \mathbb{R}$ is differentiable at $\vec{x} = \overrightarrow{x_0}$ and $\vec{p} : \mathbb{R} \to \mathbb{R}^{2\,or\,3}$ is a vector-valued function which is differentiable at $t = t_0$, where $\vec{p}(t_0) = \overrightarrow{x_0}$.*

Then the composite function $(f \circ \vec{p}) : \mathbb{R} \to \mathbb{R}$, $(f \circ \vec{p})(t) = f(\vec{p}(t))$ is differentiable at $t = t_0$, and

$$\left.\frac{d}{dt}\right|_{t=t_0} [f \circ \vec{p}] = \vec{\nabla} f(\overrightarrow{x_0}) \cdot \vec{v}, \tag{3.12}$$

where $\vec{v} = \dot{\vec{p}}(t_0)$ is the velocity with which the curve passes $\overrightarrow{x_0}$ at $t = t_0$.

Proof. For the purpose of this proof, it will be convenient to write the condition that $f(\vec{x})$ and $T_{\overrightarrow{x_0}} f(\vec{x})$ have first-order contact at $\vec{x} = \overrightarrow{x_0}$ in a somewhat different form. If we set

$$\varepsilon = \frac{f(\overrightarrow{x_0} + \triangle\vec{x}) - T_{\overrightarrow{x_0}} f(\overrightarrow{x_0} + \triangle\vec{x})}{\left\|\triangle\vec{x}\right\|},$$

where $\triangle\vec{x} = \vec{x} - \overrightarrow{x_0}$, then Equation (3.5) can be rewritten in the form

$$f(\overrightarrow{x_0} + \triangle\vec{x}) = T_{\overrightarrow{x_0}} f(\overrightarrow{x_0} + \triangle\vec{x}) + \left\|\triangle\vec{x}\right\| \varepsilon,$$
$$\text{where } |\varepsilon| \to 0 \text{ as } \triangle\vec{x} \to \vec{0}.$$

If we substitute into this the expression for the affine approximation

$$T_{\overrightarrow{x_0}} f(\overrightarrow{x_0} + \triangle\vec{x}) = f(\overrightarrow{x_0}) + d_{\overrightarrow{x_0}} f(\triangle\vec{x})$$

we obtain the following version of Equation (3.5):

$$f(\overrightarrow{x_0} + \triangle\vec{x}) - f(\overrightarrow{x_0}) = d_{\overrightarrow{x_0}} f(\triangle\vec{x}) + \left\|\triangle\vec{x}\right\| \varepsilon,$$
$$\text{where } \varepsilon \to 0 \text{ as } \triangle\vec{x} \to \vec{0}.$$

Using the representation of $d_{\overrightarrow{x_0}} f(\triangle\vec{x})$ as a dot product, we can rewrite this in the form

$$f(\overrightarrow{x_0} + \triangle\vec{x}) - f(\overrightarrow{x_0}) = \vec{\nabla} f(\overrightarrow{x_0}) \cdot \triangle\vec{x} + \left\|\triangle\vec{x}\right\| \varepsilon,$$
$$\text{where } \varepsilon \to 0 \text{ as } \triangle\vec{x} \to \vec{0}.$$

In a similar way, we can write the analogous statement for $\vec{p}(t)$, using $\vec{v} = \dot{\vec{p}}(t_0)$:

$$\vec{p}(t_0 + \triangle t) - \vec{p}(t_0) = \vec{v}\triangle t + |\triangle t|\,\vec{\delta},$$
$$\text{where } \vec{\delta} \to \vec{0} \text{ as } \triangle t \to 0.$$

Now, we consider the variation of the composition $f(\vec{p}(t))$ as t goes from $t = t_0$ to $t = t_0 + \triangle t$:

$$\begin{aligned} f(\vec{p}(t_0 + \triangle t)) - f(\vec{p}(t_0)) &= \vec{\nabla} f(\overrightarrow{x_0}) \cdot \left(\vec{v}\triangle t + |\triangle t|\,\vec{\delta}\right) + \left\|\triangle\vec{x}\right\| \varepsilon \\ &= \vec{\nabla} f(\overrightarrow{x_0}) \cdot (\vec{v}\triangle t) + |\triangle t|\,\vec{\nabla} f(\overrightarrow{x_0}) \cdot \vec{\delta} + \left\|\triangle\vec{x}\right\| \varepsilon. \end{aligned}$$

Subtracting the first term on the right from both sides, we can write

$$\begin{aligned} f\left(\vec{p}\left(t_0+\triangle t\right)\right)-f\left(\vec{p}\left(t_0\right)\right)-\vec{\nabla} f\left(\overrightarrow{x_0}\right) \cdot(\vec{v} \triangle t) \\ =(\triangle t) \vec{\nabla} f\left(\overrightarrow{x_0}\right) \cdot \vec{\delta}+|\triangle t|\left\|\frac{\triangle \vec{x}}{\triangle t}\right\| \varepsilon . \end{aligned}$$

Taking the absolute value of both sides and dividing by $|\triangle t|$, we get

$$\begin{aligned} & \frac{1}{|\triangle t|}\left|f\left(\vec{p}\left(t_0+\triangle t\right)\right)-f\left(\vec{p}\left(t_0\right)\right)-\vec{\nabla} f\left(\overrightarrow{x_0}\right) \cdot(\triangle t \vec{v})\right| \\ & =\left|\vec{\nabla} f\left(\overrightarrow{x_0}\right) \cdot \vec{\delta} \pm\left\|\frac{\triangle \vec{x}}{\triangle t}\right\| \varepsilon\right| \leq\left\|\vec{\nabla} f\left(\overrightarrow{x_0}\right)\right\|\|\vec{\delta}\|+\left\|\frac{\triangle \vec{x}}{\triangle t}\right\||\varepsilon| . \end{aligned}$$

In the first term above, the first factor is fixed and the second goes to zero as $\triangle t \rightarrow 0$, while in the second term, the first factor is bounded (since $\triangle \vec{x} / \triangle t$ converges to $\vec{v}$) and the second goes to zero. Thus, the whole mess goes to zero, proving that the affine function inside the absolute value in the numerator on the left above represents the linearization of the composition, as required. □

An important aspect of Proposition 3.3.6 (perhaps *the* important aspect) is that the rate of change of a function applied to a moving point depends *only* on the gradient of the function and the velocity of the moving point at the given moment, *not* on how the motion might be accelerating, etc.

For example, consider the distance from a moving point $\vec{p}(t)$ to the point $(1,2)$: the distance from (x, y) to $(1,2)$ is given by $f(x, y)=\sqrt{(x-1)^2+(y-2)^2}$ with gradient $\vec{\nabla} f(x, y)=\frac{(x-1)}{\sqrt{(x-1)^2+(y-1)^2}} \vec{\imath}+\frac{(y-2)}{\sqrt{(x-1)^2+(y-1)^2}} \vec{\jmath}$. If at a given moment our point has position $\vec{p}\left(t_0\right)=(5,-3)$ and velocity $\vec{v}\left(t_0\right)=-2 \vec{\imath}-3 \vec{\jmath}$ then regardless of acceleration and so on, the rate at which its distance from $(1,2)$ is changing is given by

$$\begin{aligned} \left.\frac{d}{d t}\right|_{t=t_0}[f(\vec{p}(t))] & =\vec{\nabla} f(5,-3) \cdot \vec{v}\left(t_0\right) \\ & =\left(\frac{4}{5} \vec{\imath}-\frac{3}{5} \vec{\jmath}\right) \cdot(-2 \vec{\imath}-3 \vec{\jmath}) \\ & =-\frac{8}{5}+\frac{9}{5} \\ & =\frac{1}{5} . \end{aligned}$$

The other kind of chain rule that can arise is when we compose a real-valued function f of a vector variable with a real-valued function g of a real variable:

Proposition 3.3.7 (Chain Rule $\mathbb{R}^{2 \text{ or } 3} \rightarrow \mathbb{R} \rightarrow \mathbb{R}$). *Suppose $f(\vec{x})$ is a real-valued function of two or three variables, differentiable at $v_{=} \overrightarrow{x_0}$, and $g(y)$ is a real-valued function of a real variable, differentiable at $y=y_0=f\left(\overrightarrow{x_0}\right)$.*

Then the composition $(g \circ f)(\vec{x})$ is differentiable at $\vec{x}=\overrightarrow{x_0}$, and

$$\vec{\nabla}(g \circ f)\left(\overrightarrow{x_0}\right)=g^{\prime}\left(y_0\right) \vec{\nabla} f\left(\overrightarrow{x_0}\right) . \tag{3.13}$$

Proof. This is formally very similar to the preceding proof. Let $\triangle y = f(\overrightarrow{x_0} + \triangle\vec{x}) - f(\overrightarrow{x_0})$; then $\triangle y = \vec{\nabla} f(\overrightarrow{x_0}) \cdot \triangle\vec{x} + \delta \left\|\triangle\vec{x}\right\|$, where $\delta \to 0$ as $\triangle\vec{x} \to \vec{0}$. Note for future reference that $|\triangle y| \leq \left(\left\|\vec{\nabla} f(\overrightarrow{x_0})\right\| + \delta\right)\left\|\triangle\vec{x}\right\|$. Now,

$$g(f(\overrightarrow{x_0} + \triangle\vec{x})) = g(y_0 + \triangle y) = g(y_0) + g'(y_0)\triangle y + \varepsilon|\triangle y|,$$

where $\varepsilon \to 0$ as $\triangle y \to 0$. From this we can conclude

$$g(f(\overrightarrow{x_0} + \triangle\vec{x})) - g(f(\overrightarrow{x_0})) - \left[g'(y_0)\vec{\nabla} f(\overrightarrow{x_0})\right] = g'(y_0)\delta\left\|\triangle\vec{x}\right\| + \varepsilon|\triangle y|.$$

Taking absolute values and dividing by $\left\|\triangle\vec{x}\right\|$, we have

$$\begin{aligned}\frac{1}{\triangle\vec{x}}&\left|g(f(\overrightarrow{x_0} + \triangle\vec{x})) - g(f(\overrightarrow{x_0})) - \left[g'(y_0)\vec{\nabla} f(\overrightarrow{x_0})\right]\right| \\ &\leq |g'(y_0)|\,|\delta| + \varepsilon\frac{|\triangle y|}{\left\|\triangle\vec{x}\right\|} = |g'(y_0)|\,|\delta| + \varepsilon\left(\left\|\vec{\nabla} f(\overrightarrow{x_0})\right\| + \delta\right).\end{aligned}$$

Both terms consist of a bounded quantity times a quantity that goes to zero as $\triangle\vec{x} \to \vec{0}$, and we are done. □

Finally, we note that, as a corollary of Proposition 3.3.7, we get a formula for the partial derivatives of the composite function $g \circ f$:

$$\frac{\partial g \circ f}{\partial x_i}(\overrightarrow{x_0}) = g'(y_0)\frac{\partial f}{\partial x_i}(\overrightarrow{x_0}). \tag{3.14}$$

For example, suppose we consider the function that expresses the rectangular coordinate y in terms of spherical coordinates, $f(\rho, \phi, \theta) = \rho\sin\phi\sin\theta$. Its gradient is $\vec{\nabla} f(\rho, \phi, \theta) = (\sin\phi\sin\theta, \rho\cos\phi\sin\theta, \rho\sin\phi\cos\theta)$. Suppose further that we are interested in $z = g(y) = \ln y$. To calculate the partial derivatives of z with respect to the spherical coordinates when $\rho = 2$, $\phi = \frac{\pi}{4}$, and $\theta = \frac{\pi}{3}$, we calculate the value and gradient of f at this point:

$$f\left(2, \frac{\pi}{4}, \frac{\pi}{3}\right) = (2)\left(\frac{1}{\sqrt{2}}\right)(\tfrac{1}{2}) = \frac{1}{\sqrt{2}}$$

$$\vec{\nabla} f\left(2, \frac{\pi}{4}, \frac{\pi}{3}\right) = \left(\frac{\sqrt{3}}{2\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

or

$$\frac{\partial f}{\partial \rho} = \frac{\sqrt{3}}{2\sqrt{2}}, \quad \frac{\partial f}{\partial \phi} = \frac{\sqrt{3}}{\sqrt{2}}, \quad \frac{\partial f}{\partial \theta} = \frac{1}{\sqrt{2}}.$$

The value and derivative of $g(y)$ at $y = f\left(2, \frac{\pi}{4}, \frac{\pi}{3}\right) = \frac{1}{\sqrt{2}}$ are

$$g\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\ln 2, \quad g'\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}$$

and from this we get

$$\frac{\partial z}{\partial \rho} = g'\left(\frac{1}{\sqrt{2}}\right)\frac{\partial f}{\partial \rho} = \sqrt{2}\left(\frac{\sqrt{3}}{2\sqrt{2}}\right) = \frac{\sqrt{3}}{2}$$
$$\frac{\partial z}{\partial \phi} = g'\left(\frac{1}{\sqrt{2}}\right)\frac{\partial f}{\partial \phi} = \sqrt{2}\left(\frac{\sqrt{3}}{\sqrt{2}}\right) = \sqrt{3}$$
$$\frac{\partial z}{\partial \theta} = g'\left(\frac{1}{\sqrt{2}}\right)\frac{\partial f}{\partial \theta} = \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 1.$$

Note that this formula could have been found directly, using Definition 3.3.3 (Exercise 7): the substantive part of the proof above was to show that the composite function is differentiable.

Approximation and Estimation. Just as for functions of one variable, the linearization of a function can be used to get "quick and dirty" estimates of the value of a function when the input is close to a place where the exact value is known.

For example, consider the function $f(x, y) = \sqrt{x^2 + 5xy + y^2}$; you can check that $f(3, 1) = 5$; what is $f(2.9, 1.2)$? We calculate the partial derivatives at $(3, 1)$:

$$\frac{\partial f}{\partial x}(x, y) = \frac{2x + 5y}{2\sqrt{x^2 + 5xy + y^2}}, \quad \frac{\partial f}{\partial y}(x, y) = \frac{5x + 2y}{2\sqrt{x^2 + 5xy + y^2}},$$

so

$$\frac{\partial f}{\partial x}(3, 1) = \frac{11}{10} = 1.1, \quad \frac{\partial f}{\partial y}(3, 1) = \frac{17}{10} = 1.7.$$

since $(2.9, 1.2) = (3, 1) + (-0.1, 0.2)$, we use $\triangle x = -0.1$, $\triangle y = 0.2$ to calculate the linearization

$$\begin{aligned} T_{(3,1)}f(2.9, 1.2) &= f(3, 1) + \frac{\partial f}{\partial x}(3, 1)\triangle x + \frac{\partial f}{\partial y}(3, 1)\triangle y \\ &= 5 + (1.1)(-0.1) + (1.7)(0.2) = 5.23. \end{aligned}$$

This is an easy calculation, but the answer is only an estimate; by comparison, a calculator "calculation" of $f(2.9, 1.2)$ gives $\sqrt{27.25} \approx 5.220$.

As a second example, we consider the accuracy of the result of the calculation of a quantity whose inputs are only known approximately. Suppose, for example, that we have measured the height of a rectangular box as 2 feet, with an accuracy of $\pm 0.1 ft$, and its a base as 5×10 feet, with an accuracy in each dimension of $\pm 0.2 ft$. We calculate the volume as $100 ft^3$; how accurate is this? Here we are interested in how far the actual value of $f(x, y, z) = xyz$ can vary from $f(5, 10, 2) = 100$ when x and y can vary by at most $\triangle x = \triangle y = \pm 0.2$ and z can vary by at most $\triangle z = \pm 0.1$. The best estimate of this is the differential: $f(x, y, z) = xyz$ has $\frac{\partial f}{\partial x}(x, y, z) = yz$, $\frac{\partial f}{\partial y}(x, y, z) = xz$, $\frac{\partial f}{\partial z}(x, y, z) = xy$. In particular, at our point, $\frac{\partial f}{\partial x}(5, 10, 2) = 20$, $\frac{\partial f}{\partial y}(5, 10, 2) = 10$, and $\frac{\partial f}{\partial z}(5, 10, 2) = 50$, so the differential is $d_{(5,10,2)}f\left(\triangle x, \triangle y, \triangle z\right) = 20\triangle x + 10\triangle y + 50\triangle z$ which is at most $(20)(0.2) + (10)(0.2) + (50)(0.1) = 4 + 2 + 5 = 11$. We conclude that the figure of 100 cubic feet is correct to within ± 11 cubic feet.

Exercises for § 3.3

Answers to Exercises 1a, 2a, 3a, and 5 are given in Appendix A.13.

Practice problems:

(1) Find all the partial derivatives of each function below:

(a) $f(x, y) = x^2y - 2xy^2$ (b) $f(x, y) = x\cos y + y\sin x$

(c) $f(x, y) = e^x\cos y + y\tan x$ (d) $f(x, y) = (x+1)^2y^2 - x^2(y-1)^2$

(e) $f(x, y, z) = x^2y^3z$ (f) $f(x, y, z) = \dfrac{xy + xz + yz}{xyz}$

(2) For each function below, find its derivative $d_{\vec{a}}f(\triangle\vec{x})$, the linearization $T_{\vec{a}}f(\vec{x})$, and the gradient $\operatorname{grad} f(\vec{a}) = \vec{\nabla} f(\vec{a})$ at the given point $\vec{a}$.

(a) $f(x, y) = x^2 + 4xy + 4y^2$, $\vec{a} = (1, -2)$

(b) $f(x, y) = \cos(x^2 + y)$, $\vec{a} = (\sqrt{\pi}, \frac{\pi}{3})$

(c) $f(x, y) = \sqrt{x^2 + y^2}$, $\vec{a} = (1, -1)$

(d) $f(x, y) = x\cos y - y\cos x$, $\vec{a} = (\frac{\pi}{2}, -\frac{\pi}{2})$

(e) $f(x, y, z) = xy + xz + yz$, $\vec{a} = (1, -2, 3)$

(f) $f(x, y, z) = (x+y)^2 - (x-y)^2 + 2xyz$, $\vec{a} = (1, 2, 1)$

(3) (a) Use the linearization of $f(x, y) = \sqrt{xy}$ at $\vec{a} = (9, 4)$ to find an approximation to $\sqrt{(8.9)(4.2)}$. (Give your approximation to four decimals.)

(b) A cylindrical tin can is $h = 3$ inches tall and its base has radius $r = 2$ inches. If the can is made of tin that is 0.01 inches thick, use the differential of $V(r, h) = \pi r^2 h$ to estimate the total volume of tin in the can.

(4) If two resistors with respective resistance R_1 and R_2 are hooked up in parallel, the net resistance R is related to R_1 and R_2 by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

(a) Show that the differential of $R = R(R_1, R_2)$, as a function of the two resistances, is given by

$$dR = \left(\frac{R}{R_1}\right)^2 \triangle R_1 + \left(\frac{R}{R_2}\right)^2 \triangle R_2.$$

(b) If we know $R_1 = 150$ ohms and $R_2 = 400$ ohms, both with a possible error of 10%, what is the net resistance, and what is the possible error?

(5) A moving point starts at location $(1, 2)$ and moves with a fixed speed; in which of the following directions is the sum of its distances from $(-1, 0)$ and $(1, 0)$ increasing the fastest?

$$\vec{v_1} \text{ is parallel to } \vec{\imath}$$
$$\vec{v_2} \text{ is parallel to } \vec{\jmath}$$
$$\vec{v_3} \text{ is parallel to } \vec{\imath} + \vec{\jmath}$$
$$\vec{v_4} \text{ is parallel to } \vec{\jmath} - \vec{\imath}.$$

In what direction (among *all* possible directions) will this sum increase the fastest?

Theory problems:

(6) Fill in the following details in the proof of Theorem 3.3.4:

(a) Show that if $f(x,y)$ is differentiable at (x,y) and $g(t)$ is defined by $g(t) = f(x + t\triangle x, y + \triangle y)$ then g is differentiable at $t = 0$ and

$$g'(t) = \frac{\partial f}{\partial x}(x + t\triangle x, y + \triangle y)\triangle x.$$

(b) Show that we can write

$$f(x, y + \triangle y) - f(x,y) = \frac{\partial f}{\partial y}(x, y + \delta_2)\triangle y,$$

where $|\delta_2| \leq |\triangle y|$.

(7) (a) Use Proposition 3.3.7 to prove Equation (3.14).

(b) Use Definition 3.3.3 to prove Equation (3.14) directly.

(8) Show that if $f(x,y)$ and $g(x,y)$ are both differentiable real-valued functions of two variables, then so is their product $h(x,y) = f(x,y)\,g(x,y)$ and the following Leibniz formula holds:

$$\vec{\nabla} h = f\vec{\nabla} g + g\vec{\nabla} f.$$

Challenge problem:

(9) Show that if $f(x,y) = g(ax + by)$, where $g(t)$ is a differentiable function of one variable, then for every point (x,y) in the plane, $\vec{\nabla} f(x,y)$ is perpendicular to the line $ax + by = c$ through (x,y).

(10) (a) Show that if $f(x,y)$ is a function whose value depends only on the product xy then

$$x\frac{\partial f}{\partial x} = y\frac{\partial f}{\partial y}.$$

(b) Is the converse true? That is, suppose $f(x,y)$ is a function satisfying the condition above on its partials. Can it be expressed as a function of the product

$$f(x,y) = g(xy)$$

for some real-valued function $g(t)$ of a real variable? (*Hint:* First, consider two points in the same quadrant, and join them with a path on which the product xy is constant. Note that this cannot be done if the points are in different quadrants.)

(11) Adapt the proof of Theorem 3.3.4 given in this section for functions of two variables to get a proof for functions of three variables.

3.4 Level Curves

A **level set** of a function f is any subset of its domain of the form

$$\mathcal{L}(f,c) := \{\vec{x} \mid f(\vec{x}) = c\},$$

where $c \in \mathbb{R}$ is some constant. This is nothing other than the solution set of the equation in two (*resp.* three) variables $f(x,y) = c$ (*resp.* $f(x,y,z) = c$).

For a function of two variables, we expect this set to be a curve in the plane and for three variables we expect a surface in space.

Level Curves and Implicit Differentiation. For a function of two variables, there is another way to think about the level set, which in this case is called a **level curve**: the *graph* of $f(x, y)$ is the locus of the equation $z = f(x, y)$, which is a surface in space, and $\mathcal{L}(f, c)$ is found by intersecting this surface with the horizontal plane $z = c$, and then projecting the resulting curve onto the xy-plane. Of course, this is a "generic" picture: if for example the function itself happens to be constant, then its level set is the xy-plane for one value, and the empty set for all others. We can cook up other examples for which the level set is quite exotic. However, for many functions, the level sets really are curves.

For example (see Figure 3.1):

- The level curves of a non-constant affine function are parallel straight lines.
- The level curves of the function $f(x, y) = x^2 + y^2$ are concentric circles centered at the origin for $c > 0$, just the origin for $c = 0$, and the empty set for $c < 0$.
- For the function $f(x, y) = \frac{x^2}{4} + y^2$, the level sets $\mathcal{L}(f, c)$ for $c > 0$ are the ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a = 2\sqrt{c}$ and $b = \sqrt{c}$, which are all centered at the origin with the same eccentricity. For $c = 0$, we again get just the origin, and for $c < 0$ the empty set.
- The level curves of the function $f(x, y) = x^2 - y^2$ are hyperbolas:
  - for $c = a^2 > 0$, $\mathcal{L}(f, c)$ is the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$ which "opens" left and right,
  - and for $c = -a^2 < 0$ we have $\frac{x^2}{a^2} - \frac{y^2}{a^2} = -1$, which "opens" up and down.

  For $c = 0$ we have the common asymptotes of all these hyperbolas.

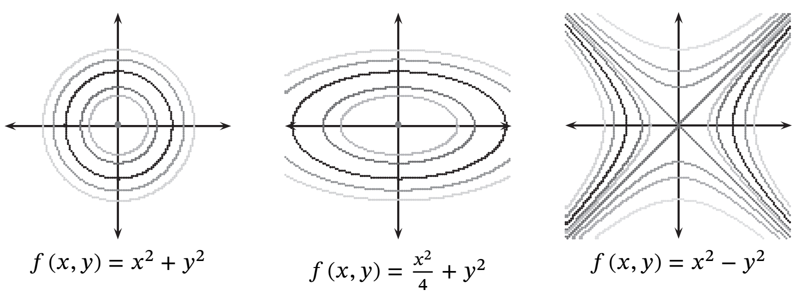


Figure 3.1. Level Curves

We would like to establish criteria for when a level set of a function $f(x, y)$ will be a regular curve. This requires in particular that the curve have a well-defined tangent line. We have often found the slope of the tangent to the locus of an equation via implicit differentiation: for example to find the slope of the tangent to the ellipse (Figure 3.2)

$$x^2 + 4y^2 = 8 \tag{3.15}$$

at the point $(2, -1)$, we think of y as a function of x and differentiate both sides to obtain

$$2x + 8y\frac{dy}{dx} = 0; \tag{3.16}$$

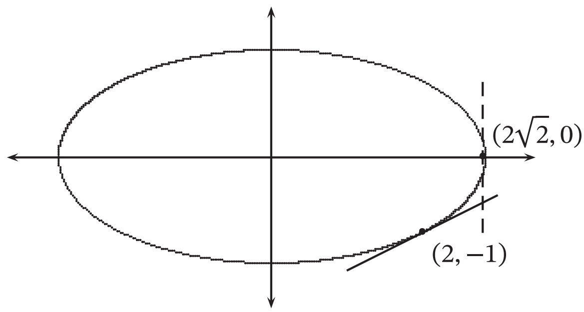


Figure 3.2. The Curve $x^2 + 4y^2 = 8$

then substituting $x = 2$ and $y = -1$ yields $4 - 8\frac{dy}{dx} = 0$ which we can solve for dy/dx: $\frac{dy}{dx} = \frac{4}{8} = \frac{1}{2}$. However, the process can break down: at the point $(2\sqrt{2}, 0)$, substitution into (3.16) yields $4\sqrt{2} + 0\frac{dy}{dx} = 0$, which has no solutions. Of course, here we can instead differentiate (3.15) treating x as a function of y, to get $2x\frac{dx}{dy} + 8y = 0$. Upon substituting $x = 2\sqrt{2}$, $y = 0$, this yields $4\sqrt{2}\frac{dx}{dy} + 0 = 0$ which *does* have a solution, $\frac{dx}{dy} = 0$. In this case, we can see the reason for our difficulty by explicitly solving the original equation (3.15) for y in terms of x: near $(2, -1)$, y can be expressed as the function of x

$$y = -\sqrt{\frac{8 - x^2}{4}} = -\sqrt{2 - \frac{x^2}{4}}.$$

(We need the minus sign to get $y = -1$ when $x = 2$.) Note that this solution is *local*: near $(2, 1)$ we would need to use the positive root. Near $(2\sqrt{2}, 0)$, we cannot solve for y in terms of x, because the "vertical line test" fails: for any x-value slightly below $x = 2\sqrt{2}$, there are two distinct points with this abcissa (corresponding to the two signs for the square root). However, near this point, the "*horizontal* line test" works: to each y-value near $y = 0$, there corresponds a unique x-value near $x = 2\sqrt{2}$ yielding a point on the ellipse, given by

$$x = \sqrt{8 - 4y^2}$$

While we are able in this particular case to determine what works and what doesn't, in other situations an explicit solution for one variable in terms of the other is not so easy. For example, the curve

$$x^3 + xy + y^3 = 13 \tag{3.17}$$

contains the point $(3, -2)$. We cannot easily solve this for y in terms of x, but implicit differentiation yields

$$3x^2 + y + x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0 \tag{3.18}$$

and substituting $x = 3$, $y = -2$ we get the equation $27 - 2 + 3\frac{dy}{dx} + 12\frac{dy}{dx} = 0$, which is easily solved for dy/dx: $15\frac{dy}{dx} = -25$, or $\frac{dy}{dx} = -\frac{5}{3}$. It seems that we have found the slope of the line tangent to the locus of Equation (3.17) at the point $(3, -2)$; but how do we know that this line even exists? Figure 3.3 illustrates what we think we have found.

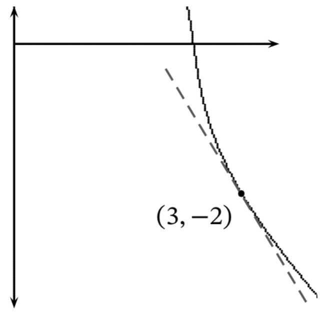


Figure 3.3. The Curve $x^3 + y^3 + xy = 13$

A clue to what is going on can be found by recasting the process of implicit differentiation in terms of level curves. Suppose that near the point (x_0, y_0) on the level set $\mathcal{L}(f, c)$ given by

$$f(x, y) = c \tag{3.19}$$

we can (in principle) solve for y in terms of x: $y = \phi(x)$. Then the graph of this function can be parametrized as $\vec{p}(x) = (x, \phi(x))$. Since this function is a solution of Equation (3.19) for y in terms of x, its graph lies on $\mathcal{L}(f, c)$–that is, $f(\vec{p}(x)) = f(x, \phi(x)) = c$. Applying the Chain Rule to the composition, we can differentiate both sides of this to get $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$ and, *provided the derivative $\partial f/\partial y$ is not zero*, we can solve this for $\phi'(x) = dy/dx$: $\phi'(x) = \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$. This process breaks down if $\partial f/\partial y = 0$: either there are no solutions, if $\partial f/\partial x \neq 0$, or, if $\partial f/\partial x = 0$, the equation tells us nothing about the slope.

Of course, as we have seen, even when $\partial f/\partial y$ is zero, all is not lost, for if $\partial f/\partial x$ is nonzero, then we can interchange the roles of y and x, solving for the derivative of x as a function of y. So the issue seems to be: is at least *one* of the partials nonzero? If so, we seem to have a perfectly reasonable way to calculate the direction of a line tangent to the level curve at that point. All that remains is to establish our original assumption—that one of the variables can be expressed as a function of the other—as valid. This is the purpose of the Implicit Function Theorem.

The Implicit Function Theorem in the Plane. We want to single out points for which at least one partial is nonzero, or what is the same, at which the gradient is a nonzero vector. Note that to even talk about the gradient or partials, we need to assume that $f(x, y)$ is defined not just at the point in question, but at all points nearby: such a point is called an **interior point** of the domain.

Definition 3.4.1. *Suppose $f(x, y)$ is a differentiable function of two variables. An interior point $\vec{x}$ of the domain of f is a **regular point** if*

$$\vec{\nabla} f(\vec{x}) \neq \vec{0},$$

that is, at least one *partial derivative at $\vec{x}$ is nonzero. $\vec{x}$ is a **critical point** of $f(x, y)$ if*

$$\frac{\partial f}{\partial x}(\vec{x}) = 0 = \frac{\partial f}{\partial y}(\vec{x}).$$

Our result will be a *local* one, describing the set of solutions to the equation $f(x, y) = c$ *near* a given solution. The examples listed at the beginning of this section exhibit level sets that are completely reasonable curves except at the origin:[9] for the function x^2+y^2, the level "curve" corresponding to $c=0$ is a single point, while for the function x^2-y^2 it crosses itself at the origin. Both of these are cases in which the origin is a critical point of $f(x,y)$, where we already know that the formal process of implicit differentiation fails; we can only expect to get a reasonable result near *regular points* of $f(x,y)$.

The following result is a fundamental fact about regular points of functions.[10]

Theorem 3.4.2 (Implicit Function Theorem for $\mathbb{R}^2 \to \mathbb{R}$)**.** *The level set of a continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ can be expressed near each of its regular points as the graph of a function.*

Specifically, suppose

$$f(x_0,y_0)=c$$

and

$$\frac{\partial f}{\partial y}(x_0,y_0)\neq 0.$$

Then there exists a rectangle

$$R=[x_0-\delta_1,x_0+\delta_1]\times[y_0-\delta_2,y_0+\delta_2]$$

centered at $\overrightarrow{x_0}=(x_0,y_0)$ (where $\delta_1,\delta_2>0$), such that the intersection of $\mathcal{L}(f,c)$ with R is the graph of a C^1 function $\phi(x)$, defined on $[x_0-\delta_1,x_0+\delta_1]$ and taking values in $[y_0-\delta_2,y_0+\delta_2]$.

In other words, if $(x,y)\in R$, (i.e., $|x-x_0|\leq\delta_1$ and $|y-y_0|\leq\delta_2$), then

$$f(x,y)=c \iff \phi(x)=y. \tag{3.20}$$

Furthermore, at any point $x\in(x_0-\delta_1,x_0+\delta_1)$, the derivative of $\phi(x)$ is

$$\frac{d\phi}{dx}=-\left[\frac{\partial f}{\partial x}(x,\phi(x))\right]\Big/\left[\frac{\partial f}{\partial y}(x,\phi(x))\right]. \tag{3.21}$$

The proof of this theorem is in two steps. Here, we give a geometric proof that the level set is locally the graph of a function $\phi(x)$ (that is, the level set passes the vertical line test). The second step is to show that this function is differentiable and that its derivative satisfies Equation (3.21). This is a rather technical argument, which we defer to Appendix A.1.

Proof of existence of $\phi(x)$. For notational convenience, we assume without loss of generality that $f(x_0,y_0)=0$ (that is, $c=0$), and $\frac{\partial f}{\partial y}(x_0,y_0)>0$. Since $f(x,y)$ is continuous, we know that $\frac{\partial f}{\partial y}(\vec{x})>0$ at all points $\vec{x}=(x,y)$ sufficiently near $\overrightarrow{x_0}$, say for $|x-x_0|\leq\delta$ and $|y-y_0|\leq\delta_2$. For any $a\in[x-\delta,x+\delta]$, consider the function of y obtained by fixing the value of x at $x=a$, say $g_a(y)=f(a,y)$. Then $g_a'(y)=\frac{\partial f}{\partial y}(a,y)>0$ so $g_a(y)$ is strictly increasing on $[y-\delta_2,y+\delta_2]$. In particular, when $a=x_0$, it changes sign: $g_{x_0}(y_0-\delta_2)<0<g_{x_0}(y_0+\delta_2)$. Then by continuity $f(x,y)$ remains positive (*resp.* negative) along a short horizontal segment through

[9]Of course, the first, affine example was equally reasonable at the origin as well.

[10]For a detailed study of the Implicit Function Theorem in its many incarnations, including some history, and the proof on which the one we give is modeled, see [33].

$(x_0, y_0 + \delta_2)$ (*resp.* through $(x_0, y_0 - \delta_2)$), so we have $g_a(y_0 - \delta_2) < 0 < g_a(y_0 + \delta_2)$ for each $a \in [x_0 - \delta_1, x_0 + \delta_1]$. The Intermediate Value Theorem ensures that for each such a there is *at least one* $y \in [y_0 - \delta_2, y_0 + \delta_2]$ for which $g_a(y) = f(a, y) = 0$, and the fact that $g_a(y)$ is strictly increasing ensures that there is *precisely one*. Writing x in place of a, we see that the definition

$$\phi(x) = y \iff f(a, y) = 0, \quad |x - x_0| \leq \delta_1, \text{and } |y - y_0| \leq \delta_2$$

gives a well-defined function $\phi(x)$ on $[x_0 - \delta_1, x_0 + \delta_1]$ satisfying Equation (3.21). □

We note some features of this theorem:

- Equation (3.21) is simply implicit differentiation: using $y = \phi(x)$ and setting $z = f(x, y)$, we can[11] differentiate the relation $z = f(x, \phi(x))$ using the Chain Rule and the fact that z is constant to get

$$0 = \frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\phi'(x)$$

  which we can solve for $\phi'(x)$ to get $\phi'(x) = -\frac{\partial f/\partial x}{\partial f/\partial y}$ as required. The proof in Appendix A.1 is what justifies this formal manipulation.

  A mnemonic device to remember which partial goes on top of this fraction and which goes on the bottom is to write Equation (3.21) formally as

$$\frac{dy}{dx} = -\frac{dy}{dz}\frac{dz}{dx}$$

  —that is, we formally (and unjustifiably) "cancel" the dz terms of the two "fractions". (Of course, we have to remember separately that we need the minus sign up front.)
- Equation (3.21) can also be interpreted as saying that a vector tangent to the level curve has slope

$$\phi'(x) = -\left[\frac{\partial f}{\partial x}(x, \phi(x))\right] \Big/ \left[\frac{\partial f}{\partial y}(x, \phi(x))\right],$$

  which means that it is perpendicular to $\vec{\nabla} f(x, \phi(x))$. Of course, this could also be established directly using the Chain Rule (Exercise 3); the point of the proof in Appendix A.1 is that one can *take* a vector tangent to $\mathcal{L}(f, c)$, or equivalently that $\phi(x)$ is differentiable.
- In the statement of the theorem, the roles of x and y can be interchanged: if $\frac{\partial f}{\partial x}(x_0, y_0) \neq 0$, then the level set can be expressed as the graph of a function $x = \psi(y)$. At a regular point, at least one partial is nonzero, and Theorem 3.4.2 says that if the partial of f at $\overrightarrow{x_0}$ with respect to *one* of the variables is nonzero, then near $\overrightarrow{x_0}$ we can solve the equation

$$f(x, y) = f(x_0, y_0)$$

  for *that* variable in terms of the *other*.

As an illustration of this last point, we again consider the function $f(x, y) = x^2 + y^2$. The level set $\mathcal{L}(f, 1)$ is the circle of radius 1 about the origin, $x^2 + y^2 = 1$. We can solve this equation for y in terms of x on any open arc which does not include either

[11] This assumes, of course, that $\phi(x)$ is differentiable–which is the content of Appendix A.1.

of the points $(\pm 1, 0)$: if the point (x_0, y_0) with $|x_0| < 1$ has $y_0 > 0$, the solution near (x_0, y_0) is

$$\phi(x) = \sqrt{1 - x^2}$$

whereas if $y_0 < 0$ it is

$$\phi(x) = -\sqrt{1 - x^2}.$$

Since $\frac{\partial f}{\partial y} = 2y$, at the two points $(\pm 1, 0)$, the partial with respect to y is zero, and the theorem does not guarantee the possibility of solving for y in terms of x; in fact, for x near ± 1 there are two values of y for a point on the curve, given by the two formulas above. However, since at these points

$$\frac{\partial f}{\partial x}(\pm 1, 0) = \pm 2 \neq 0,$$

the theorem *does* guarantee a solution for x in terms of y; in fact, near any point other than $(0, \pm 1)$ (the "north pole" and "south pole") we can write $x = \psi(y)$, where

$$\psi(y) = \sqrt{1 - y^2}$$

for points on the right semicircle and

$$\psi(y) = -\sqrt{1 - y^2}$$

on the left semicircle.

Reconstructing Surfaces from Slices. The level curves of a function $f(x, y)$ can be thought of as a "topographical map" of the graph of $f(x, y)$: a sketch of several level curves $\mathcal{L}(f, c)$, labeled with their corresponding c-values, allows us to formulate a rough idea of the shape of the graph: these are "slices" of the graph by horizontal planes at different heights. By studying the intersection of the graph with suitably chosen *vertical* planes, we can see how these horizontal pieces fit together to form the surface.

Consider for example the function $f(x, y) = x^2 + y^2$. We know that the horizontal slice at height $c = a^2 > 0$ is the circle $x^2 + y^2 = a^2$ of radius $a = \sqrt{c}$ about the origin; in particular, $\mathcal{L}(f, a^2)$ crosses the y-axis at $x = 0$; the intersection is found by substituting the second equation in the first to get $z = y^2$ and we see that the "profile" of our surface is a parabola, with vertex at the origin, opening up. (See Figure 3.4.)

If instead we consider the function $f(x, y) = 4x^2 + y^2$, the horizontal slice at height $c = a^2 > 0$ is the ellipse $\frac{x^2}{(a/2)^2} + \frac{y^2}{a^2} = 1$ centered at the origin, with major axis along the y-axis and minor axis along the x-axis. $\mathcal{L}(f, a^2)$ again crosses the y-axis at the pair of points $(0, \pm a)$, and it crosses the x-axis at the pair of points $(\pm a/2, 0)$. To see how these ellipses fit together to form the graph of $f(x, y)$, we consider the intersection of the graph $z = 4x^2 + y^2$ with the yz-plane ($x = 0$); the intersection is found by substituting the second equation in the first to get the parabola $z = y^2$. Similarly, the intersection of the graph with the xz-plane ($y = 0$) is a different parabola, $z = 4x^2$. One might say that the "shadow" of the graph on the xz-plane is a narrower parabola than the shadow on the yz-plane. (See Figure 3.5.) This surface is called an **elliptic paraboloid**.

A more interesting example is given by the function $f(x, y) = x^2 - y^2$. The horizontal slice at height $c \neq 0$ is a hyperbola which opens along the x-axis if $c > 0$ and along the y-axis if $c < 0$; the level set $\mathcal{L}(f, 0)$ is the pair of diagonal lines (the common

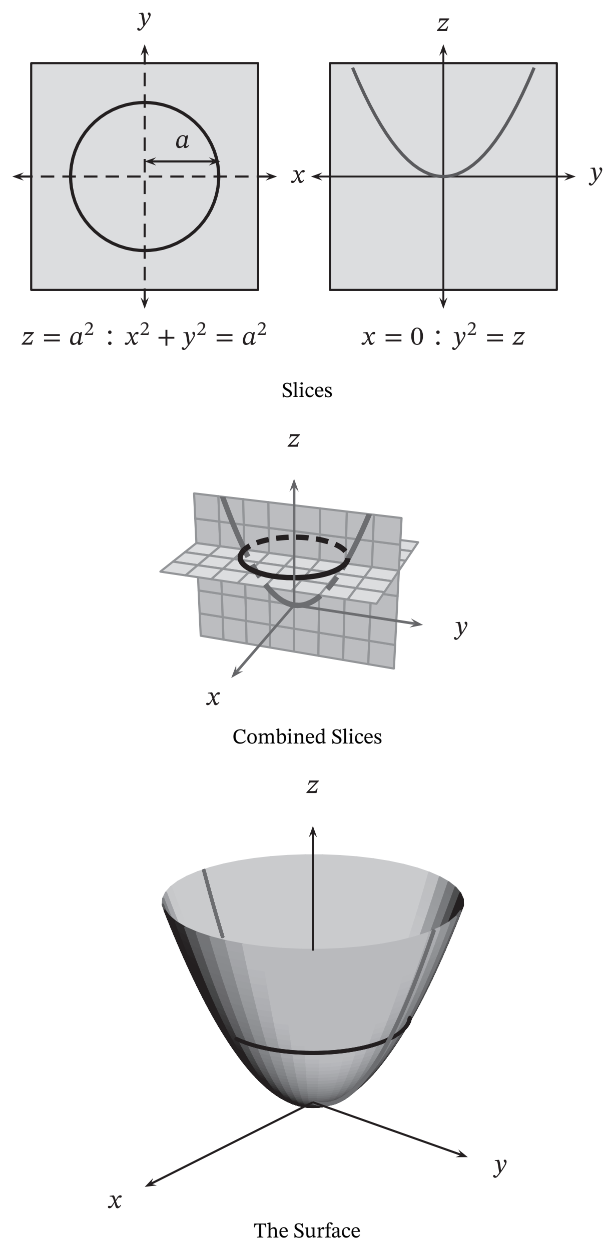


Figure 3.4. The Surface $x^2 + y^2 = z$

asymptotes of these hyperbolas) $y = \pm x$. which are the common asymptotes of each of these hyperbolas.

To see how these fit together to form the graph, we again slice along the coordinate planes. The intersection of the graph $z = x^2 - y^2$ with the xz-plane ($y = 0$) is a parabola opening *up*: these points are the "vertices" of the hyperbolas $\mathcal{L}(f, c)$ for *positive* c. The intersection with the yz-plane ($x = 0$) is a parabola opening *down*, going through the

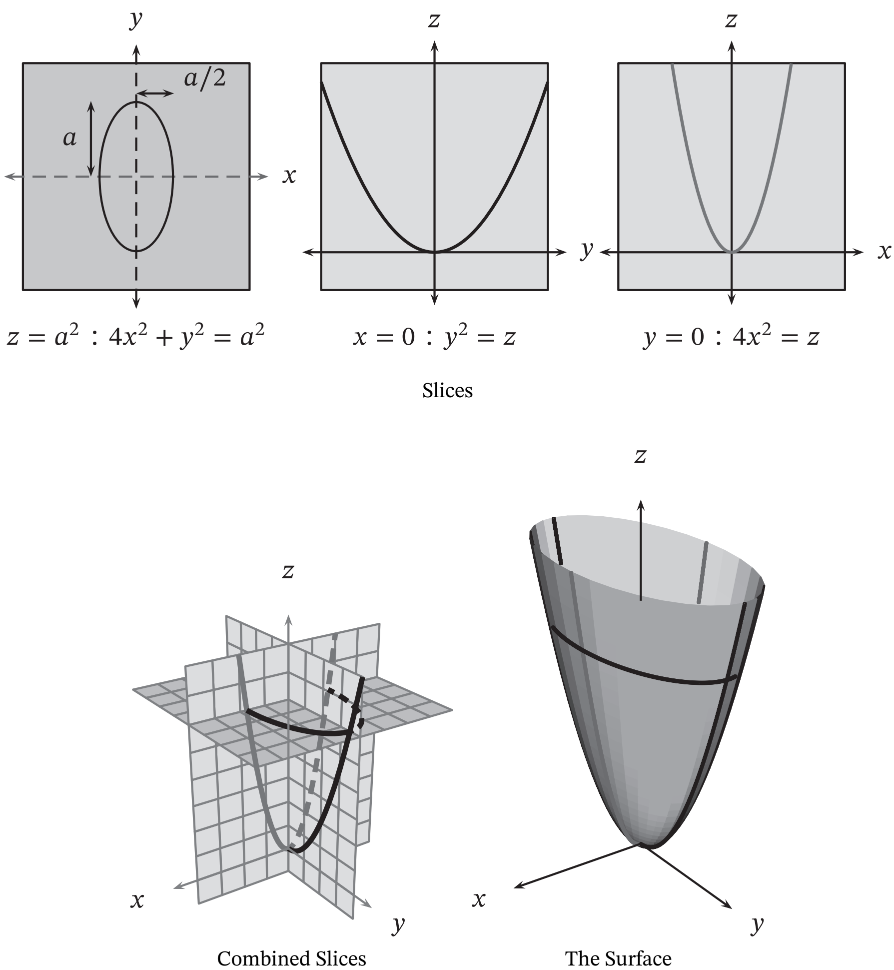
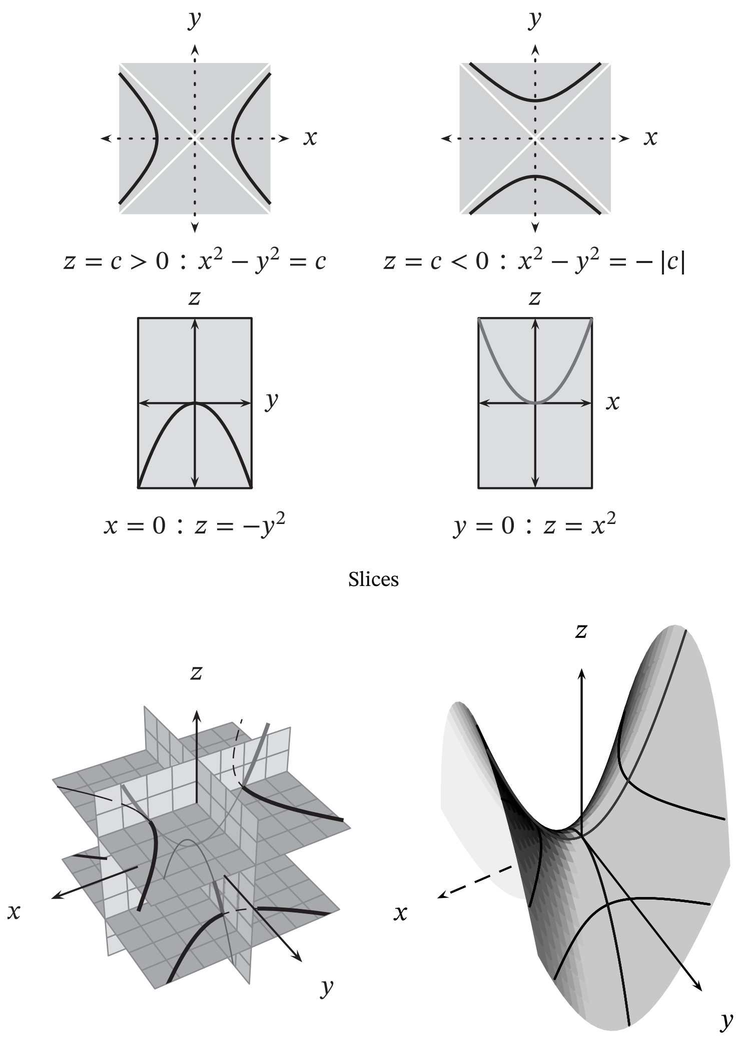


Figure 3.5. Elliptic Paraboloid $4x^2 + y^2 - z = 0$

vertices of the hyperbolas $\mathcal{L}(f, c)$ for *negative* c. Fitting these pictures together, we obtain a surface shaped like a saddle (imagine the horse's head facing parallel to the x-axis, and the rider's legs parallel to the yz-plane). It is often called the **saddle surface**, but its official name is the **hyperbolic paraboloid**. (See Figure 3.6.)

These slicing techniques can also be used to study surfaces given by equations in x, y, and z which are not explicitly graphs of functions. We consider three examples.

The first is given by the equation $\frac{x^2}{4} + y^2 + z^2 = 1$. The intersection of this with the xy-plane $z = 0$ is the ellipse $\frac{x^2}{4} + y^2 = 1$, centered at the origin and with the ends of the axes at $(\pm 2, 0, 0)$ and $(0, \pm 1, 0)$; the intersection with any other horizontal plane $z = c$ for which $|c| < 1$ is an ellipse similar to this and with the same center, but scaled down: $\frac{x^2}{4} + y^2 = 1 - c^2$ or $\frac{x^2}{4(1-c^2)} + \frac{y^2}{1-c^2} = 1$. There are no points on this surface with $|z| > 1$.



Combined Slices, and Surface

Figure 3.6. Hyperbolic Paraboloid $x^2 - y^2 - z = 0$

Similarly, the intersection with a vertical plane parallel to the xz-plane, $y = c$ (again with $|c| < 1$) is a scaled version of the same ellipse, but in the xz-plane: $\frac{x^2}{4} + z^2 = 1 - c^2$ and again no points with $|y| > 1$.

Finally, the intersection with a plane parallel to the yz-plane, $x = c$, is nonempty provided $\left|\frac{x}{2}\right| < 1$ or $|x| < 2$, and in that case is a circle centered at the origin in the yz-plane of radius $r = \sqrt{1 - \frac{c^2}{4}}$, $y^2 + z^2 = 1 - \frac{c^2}{4}$.

For sketching purposes, it is enough to sketch the intersections with the three coordinate planes. This surface is like a sphere, but "elongated" in the direction of the x-axis by a factor of 2 (see Figure 3.7); it is called an **ellipsoid**.

Our second example is the surface given by the equation $x^2 + y^2 - z^2 = 1$. The intersection with any horizontal plane $z = c$ is a circle, $x^2 + y^2 = c^2 + 1$, of radius $r = \sqrt{c^2 + 1}$ about the origin (actually, about the intersection of the plane $z = c$ with the z-axis). Note that always $r \geq 1$; the smallest circle is the intersection with the xy-plane. If we slice along the xz-plane ($y = 0$) we get the hyperbola $x^2 - z^2 = 1$ whose vertices lie on the small circle in the xy-plane. Slicing along the yz-plane we get a similar picture, since x and y play exactly the same role in the equation. The shape we get, like a cylinder that has been squeezed in the middle, is called a **hyperboloid of one sheet** (Figure 3.8).

Now, let us simply change the sign of the constant in the previous equation: $x^2 + y^2 - z^2 = -1$. The intersection with the horizontal plane $z = c$ is a circle, $x^2 + y^2 = c^2 - 1$ of radius $r = \sqrt{c^2 + 1}$ about the "origin", *provided* $c^2 > 1$; for $c = \pm 1$ we get a single point, and for $|c| < 1$ we get the empty set. In particular, our surface consists of two pieces, one for $z \geq 1$ and another for $z \leq -1$. If we slice along the xz-plane ($y = 0$) we get the hyperbola $x^2 - z^2 = -1$ or $z^2 - x^2 = 1$ which opens up and down; again, it is clear that the same thing happens along the yz-plane. Our surface consists of two "bowl"-like surfaces whose shadow on a vertical plane is a hyperbola. This is called a **hyperboloid of two sheets** (see Figure 3.9).

The reader may have noticed that the equations we have considered are the three-variable analogues of the model equations for parabolas, ellipses and hyperbolas, the quadratic curves; in fact, these are the basic models for equations given by quadratic polynomials in three coordinates, and are known collectively as the **quadric surfaces**.

Exercises for § 3.4

Answers to Exercises 1a and 2a are given in Appendix A.13.

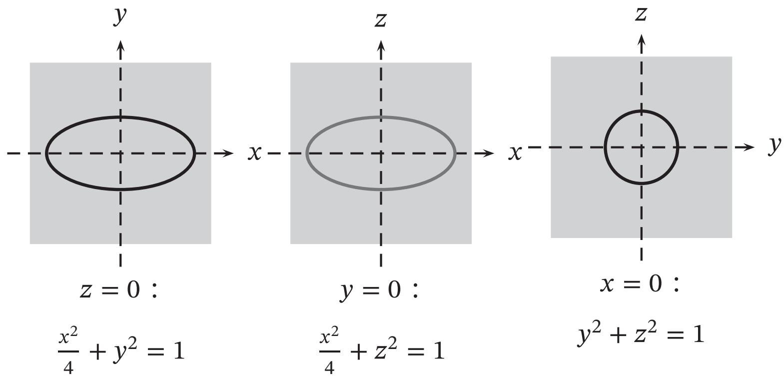
Practice problems:

(1) For each curve defined implicitly by the given equation, decide at each given point whether one can solve locally for (a) $y = \phi(x)$, (b) $x = \psi(y)$, and find the derivative of the function if it exists:
   (a) $x^3 + 2xy + y^3 = -2$, at $(1, -1)$ and at $(2, -6)$.
   (b) $(x - y)e^{xy} = 1$, at $(1, 0)$ and at $(0, -1)$.
   (c) $x^2y + x^3y^2 = 0$, at $(1, -1)$ and at $(0, 1)$

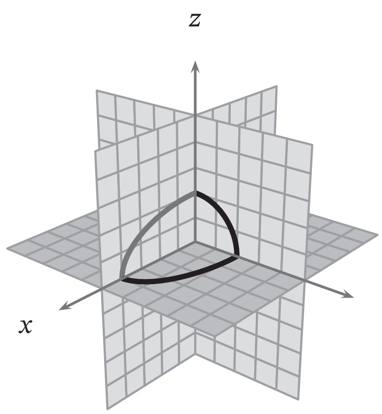
(2) For each equation below, investigate several slices and use them to sketch the locus of the equation. For quadric surfaces, decide which kind it is (*e.g.*, hyperbolic paraboloid, ellipsoid, hyperboloid of one sheet, etc.)
   (a) $z = 9x^2 + 4y^2$ (b) $z = 1 - x^2 - y^2$
   (c) $z = x^2 - 2x + y^2$ (d) $x^2 + y^2 - z = 1$
   (e) $9x^2 = y^2 + z$ (f) $x^2 - y^2 - z^2 = 1$
   (g) $x^2 - y^2 + z^2 = 1$ (h) $z^2 = x^2 + y^2$
   (i) $x^2 + 4y^2 + 9z^2 = 36$

Theory problems:

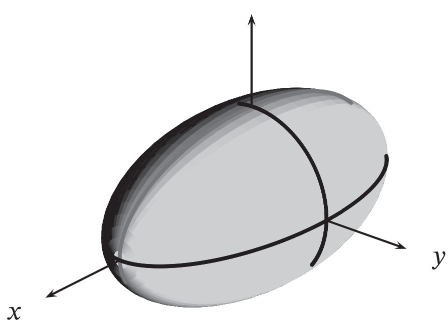
(3) Show that the gradient vector $\vec{\nabla} f$ is perpendicular to the level curves of the function $f(x, y)$, using the Chain Rule instead of implicit differentiation.



Slices



Combined Slices



The Surface

Figure 3.7. The Surface $\frac{x^2}{4} + y^2 + z^2 = 1$

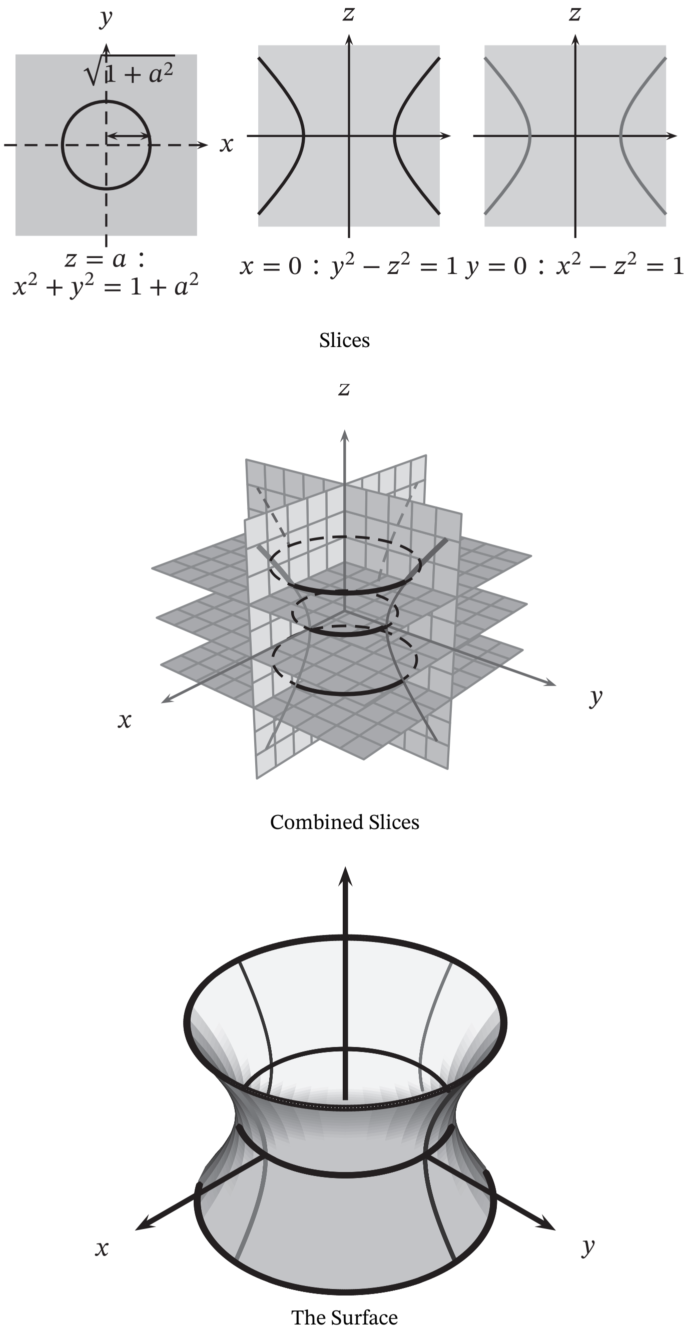
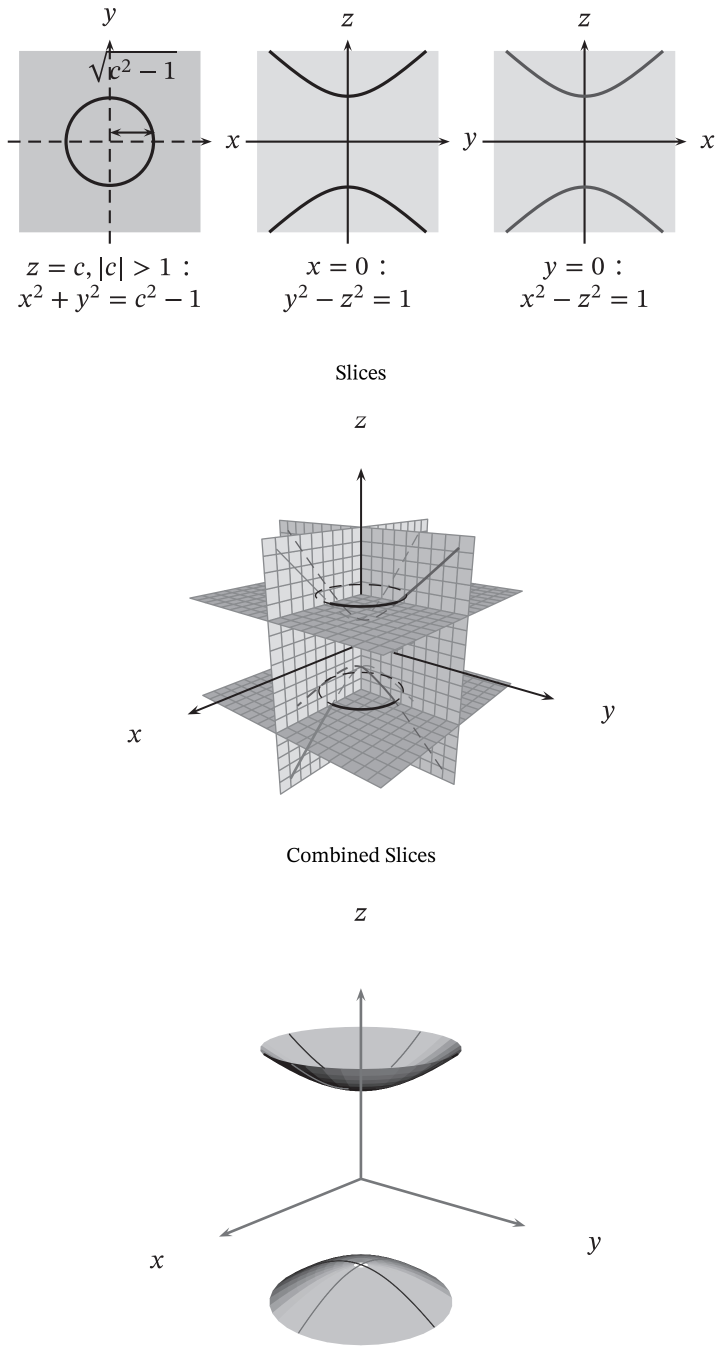


Figure 3.8. The Surface $x^2 + y^2 - z^2 = 1$



Slices

Combined Slices

The Surface

Figure 3.9. The Surface $x^2 + y^2 - z^2 = -1$

3.5 Surfaces and Tangent Planes I: Graphs and Level Surfaces

In this and the next section, we study various ways of specifying a surface, and finding its tangent plane (when it exists) at a point. We deal first with surfaces defined as graphs of functions of two variables.

Graph of a Function. The graph of a real-valued function $f(x)$ of one real variable is the subset of the plane defined by the equation $y = f(x)$, which is of course a curve—in fact an arc (at least if $f(x)$ is continuous, and defined on an interval). Similarly, the graph of a function $f(x, y)$ of two real variables is the locus of the equation $z = f(x, y)$, which is a surface in $\mathbb{R}^3$, at least if $f(x, y)$ is continuous and defined on a reasonable region in the plane.

For a curve in the plane given as the graph of a differentiable function $f(x)$, the tangent to the graph at the point corresponding to $x = x_0$ is the line through that point, $P(x_0, f(x_0))$, with slope equal to the derivative $f'(x_0)$. Another way to look at this, though, is that *the tangent at $x = x_0$ to the graph of $f(x)$ is the graph of the linearization $T_{x_0}f(x)$ of $f(x)$ at $x = x_0$*. We can take this as the definition in the case of a general graph:

Definition 3.5.1. *The **tangent plane** at $\vec{x} = \overrightarrow{x_0}$ to the graph $z = f(\vec{x})$ of a differentiable function $f : \mathbb{R}^3 \to \mathbb{R}$ is the graph of the linearization of $f(\vec{x})$ at $\vec{x} = \overrightarrow{x_0}$; that is, it is the locus of the equation*

$$z = T_{\overrightarrow{x_0}}f(\vec{x}) = f(\overrightarrow{x_0}) + d_{\overrightarrow{x_0}}f(\triangle\vec{x}),$$

where $\triangle\vec{x} = \vec{x} - \overrightarrow{x_0}$.

Note that in the definition above we are specifying where the tangent plane is being found by the value of the *input* $\vec{x}$; when we regard the graph as simply a surface in space, we should really think of the plane at $(x, y) = (x_0, y_0)$ as the tangent plane at the *point* $P(x_0, y_0, z_0)$ in space, where $z_0 = f(x_0, y_0)$.

For example, consider the function

$$f(x, y) = \frac{x^2 - 3y^2}{2} :$$

the partials are $\frac{\partial f}{\partial x} = x$ and $\frac{\partial f}{\partial y} = -3y$, so at the point $\overrightarrow{x_0} = \left(1, \frac{1}{2}\right)$, we find $f\left(1, \frac{1}{2}\right) = \frac{1}{8}$, $\frac{\partial f}{\partial x}\left(1, \frac{1}{2}\right) = 1$, $\frac{\partial f}{\partial y}\left(1, \frac{1}{2}\right) = -\frac{3}{2}$, and the linearization of $f(x, y)$ at $\vec{x} = (1, \frac{1}{2})$ is

$$T_{(1,\frac{1}{2})}f(x, y) = \frac{1}{8} + (x - 1) - \frac{3}{2}\left(y - \frac{1}{2}\right).$$

If we use the parameters $s = \triangle x = x - 1$, $t = \triangle y = y - \frac{1}{2}$, then the tangent plane is parametrized by

$$\begin{array}{rclccc} x & = & 1 & +s & \\ y & = & \frac{1}{2} & & +t \\ z & = & \frac{1}{8} & +s & -\frac{3}{2}t; \end{array} \tag{3.22}$$

the basepoint of this parametrization is $P(1, \frac{1}{2}, \frac{1}{8})$.

If we want to specify this tangent plane by an equation, we need to find a normal vector. To this end, note that the parametrization above has the natural direction vectors $\vec{v_1} = \vec{\imath} + \vec{k}$ and $\vec{v_2} = \vec{\jmath} - \frac{3}{2}\vec{k}$. Thus, we can find a normal vector by taking their cross product

$$\vec{N} = \vec{v_1} \times \vec{v_2} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 0 & 1 \\ 0 & 1 & -\frac{3}{2} \end{vmatrix} = -\vec{\imath} + \frac{3}{2}\vec{\jmath} + \vec{k}.$$

It follows that the tangent plane has the equation

$$0 = -(x-1) + \frac{3}{2}\left(y - \frac{1}{2}\right) + \left(z - \frac{1}{8}\right)$$

which we recognize as a restatement of Equation (3.22) identifying this plane as a graph:

$$z = \frac{1}{8} + (x-1) - \frac{3}{2}\left(y - \frac{1}{2}\right) = T_{(1,\frac{1}{2})} f(x,y).$$

These formulas have a geometric interpretation. The parameter $s = x - 1$ represents a displacement of the input from the base input $(1, \frac{1}{2})$ parallel to the x-axis—that is, holding y constant (at the base value $y = \frac{1}{2}$). The intersection of the graph $z = f(x,y)$ with this plane $y = \frac{1}{2}$ is the curve $z = f\left(x, \frac{1}{2}\right)$, which is the graph of the function $z = \frac{x^2}{2} - \frac{3}{8}$; at $x = 1$, the derivative of this function is $\left.\frac{dz}{dx}\right|_1 = \frac{\partial f}{\partial x}\left(1, \frac{1}{2}\right) = 1$ and the line through the point $x = 1$, $z = \frac{9}{4}$ in this plane with slope 1 lies in the plane tangent to the graph of $f(x,y)$ at $(1, \frac{1}{2})$; the vector $\vec{v_1} = \vec{\imath} + \vec{k}$ is a direction vector for this line: the line itself is parametrized by

$$\begin{aligned} x &= 1 & &+s \\ y &= \tfrac{1}{2} & & \\ z &= \tfrac{1}{8} & &+s \end{aligned}$$

which can be obtained from the parametrization of the full tangent plane by fixing $t = 0$. (See Figure 3.10.)

Similarly, the intersection of the graph $z = f(x,y)$ with the plane $x = 1$ is the curve $z = f(1,y)$, which is the graph of the function $z = \frac{1}{2} - \frac{3y^2}{2}$; at $y = \frac{1}{2}$, the derivative of this function is $\left.\frac{dz}{dy}\right|_{\frac{1}{2}} = \frac{\partial f}{\partial y}\left(1, \frac{1}{2}\right) = -\frac{3}{2}$ and $\vec{v_2} = \vec{\jmath} - \frac{3}{2}\vec{k}$ is the direction vector for the line of slope $-\frac{3}{2}$ through $y = \frac{1}{2}$, $z = \frac{1}{8}$ in this plane—a line which also lies in the tangent plane. This line is parametrized by

$$\begin{aligned} x &= 1 & & \\ y &= \tfrac{1}{2} & &+t \\ z &= \tfrac{1}{8} & &-\tfrac{3}{2}t \end{aligned}$$

which can be obtained from the parametrization of the full tangent plane by fixing $s = 0$. (See Figure 3.11.)

The combined picture, together with the normal vector and tangent plane, is given in Figure 3.12.

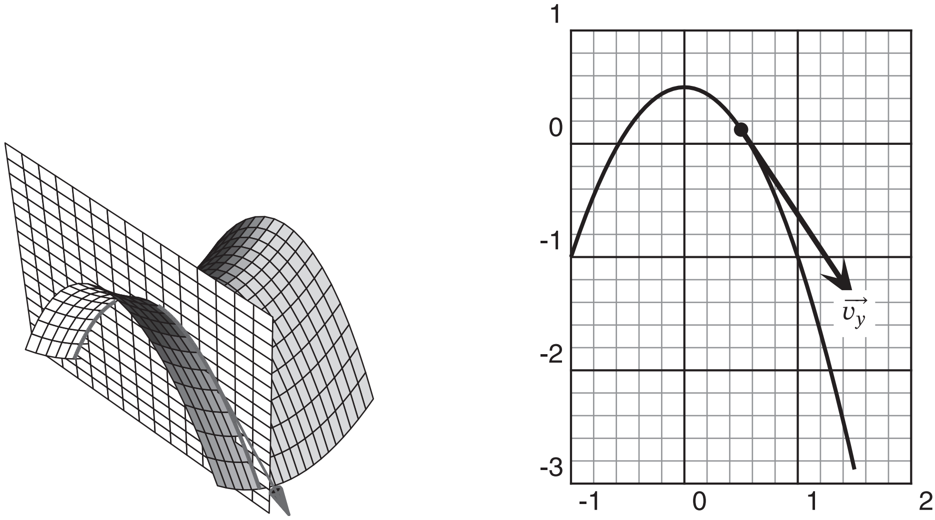


Figure 3.10. Slicing the graph of $\frac{x^2-3y^2}{2}$ at $x = 1$

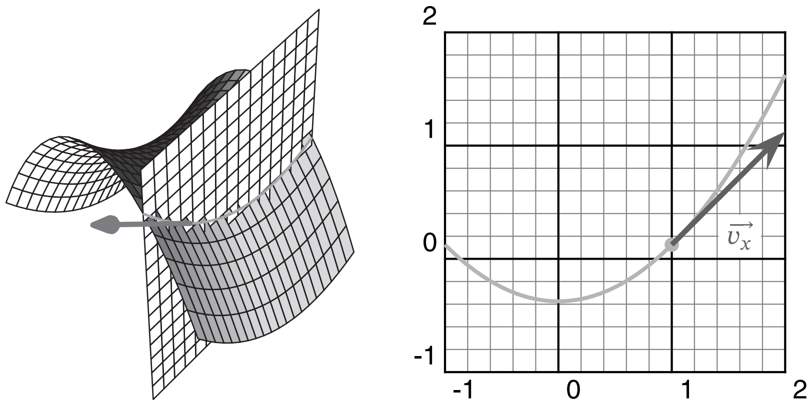


Figure 3.11. Slicing graph of $\frac{x^2-3y^2}{2}$ at $y = \frac{1}{2}$

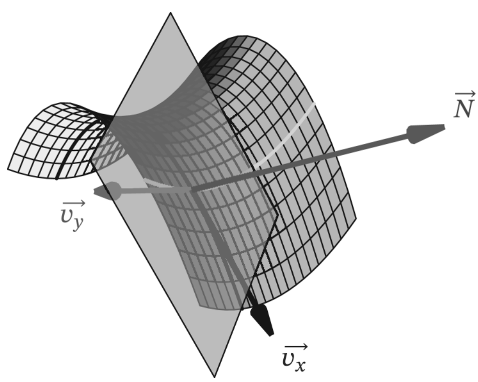


Figure 3.12. Tangent plane and normal vector to graph of $\frac{x^2-3y^2}{2}$

The alert reader (this means *you*!) will have noticed that the whole discussion above could have been applied to the graph of *any* differentiable function of two variables. We summarize it below.

Remark 3.5.2. *If the function $f(x, y)$ is differentiable at $\vec{x_0} = (x_0, y_0)$, then the plane tangent to the graph $z = f(x, y)$ at $x = x_0$, $y = y_0$, which is the graph of the linearization of $f(x, y)$,*

$$z = T_{(x_0,y_0)} f(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

is the plane through the point $P(x_0, y_0, z_0)$, where $z_0 = f(x_0, y_0)$, with direction vectors

$$\vec{v_1} = \vec{\imath} + \frac{\partial f}{\partial x}(x_0, y_0)\vec{k}, \quad \vec{v_2} = \vec{\jmath} + \frac{\partial f}{\partial y}(x_0, y_0)\vec{k}.$$

These represent the direction vectors of the lines tangent at $P(x_0, y_0, z_0)$ to the intersection of the planes $y = y_0$ and $x = x_0$ respectively, with our graph. A parametrization of the tangent plane is

$$\begin{aligned} x &= x_0 + s \\ y &= y_0 + t \\ z &= z_0 + \frac{\partial f}{\partial x}(x_0, y_0)\, s + \frac{\partial f}{\partial y}(x_0, y_0)\, t \end{aligned}$$

and the two lines are parametrized by setting t (resp. s) equal to zero. A vector normal to the tangent plane is given by the cross product

$$\vec{n} = \vec{v_1} \times \vec{v_2} = -\frac{\partial f}{\partial x}(x_0, y_0)\vec{\imath} - \frac{\partial f}{\partial y}(x_0, y_0)\vec{\jmath} + \vec{k}.$$

The adventurous reader is invited to think about how this extends to graphs of functions of more than two variables.

Level Surfaces: The Implicit Function Theorem in $\mathbb{R}^3$. For a real-valued function $f(x, y, z)$ of *three* variables, the level set $\mathcal{L}(f, c)$ is defined by an equation in *three* variables, and we expect it to be a *surface*.

For example, the level sets $\mathcal{L}(f, c)$ of the function $f(x, y, z) = x^2 + y^2 + z^2$ are spheres (of radius $\sqrt{c}$) centered at the origin if $c > 0$; again for $c = 0$ we get a single point and for $c < 0$ the empty set: the origin is the one place where $\vec{\nabla} f(x, y, z) = 2x\vec{\imath} + 2y\vec{\jmath} + 2z\vec{k}$ vanishes.

Similarly, the function $f(x, y, z) = x^2 + y^2 - z^2$ can be seen, following the analysis in § 3.4, to have as its level sets $\mathcal{L}(f, c)$ a family of hyperboloids[12]—of one sheet for $c > 0$ and two sheets for $c < 0$. (See Figure 3.13.)

For $c = 0$, the level set is given by the equation $x^2 + y^2 = z^2$ which can be rewritten in polar coordinates as $r^2 = z^2$; we recognize this as the conical surface we used to study the conics in § 2.1. This is a reasonable surface, except at the origin, which again is the only place where the gradient grad f vanishes.

This might lead us to expect an analogue of Theorem 3.4.2 for functions of *three* variables. Before stating it, we introduce a useful bit of notation. By the **ε-ball** or **ball of radius** ε about $\vec{x_0}$, we mean the set of all points at distance at most $\varepsilon > 0$ from $\vec{x_0}$:

$$B_\varepsilon(\vec{x_0}) := \{\vec{x} \mid \operatorname{dist}(\vec{x}, \vec{x_0}) \leq \varepsilon\}.$$

[12]Our analysis in § 3.4 clearly carries through if 1 is replaced by any positive number $|c|$.

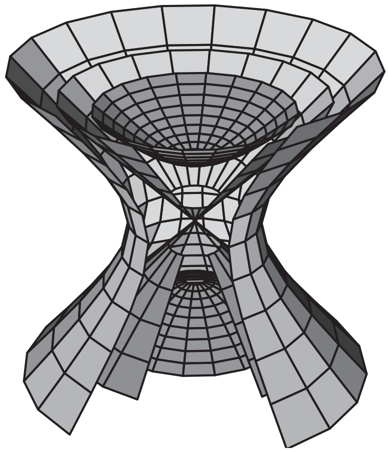


Figure 3.13. Level Sets of $f(x, y, z) = x^2 + y^2 - z^2$

For points on the line, this is the interval $[x_0 - \varepsilon, x_0 + \varepsilon]$; in the plane, it is the disc $\{(x, y) \mid (x - x_0)^2 + (y - y_0)^2 \le \varepsilon^2\}$, and in space it is the actual ball $\{(x, y, z) | (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \le \varepsilon^2\}$.

Theorem 3.5.3 (Implicit Function Theorem for $\mathbb{R}^3 \to \mathbb{R}$). *The level set of a continuously differentiable function $f : \mathbb{R}^3 \to \mathbb{R}$ can be expressed near each of its regular points as the graph of a function.*

Specifically, suppose that at $\overrightarrow{x_0} = (x_0, y_0, z_0)$ we have $f(\overrightarrow{x_0}) = c$ and $\frac{\partial f}{\partial z}(\overrightarrow{x_0}) \neq 0$. Then there exists a set of the form

$$R = B_\varepsilon((x_0, y_0)) \times [z_0 - \delta, z_0 + \delta]$$

(where $\varepsilon > 0$ and $\delta > 0$), such that the intersection of $\mathcal{L}(f, c)$ with R is the graph of a C^1 function $\phi(x, y)$, defined on $B_\varepsilon((x_0, y_0))$ and taking values in $[z_0 - \delta, z_0 + \delta]$. In other words, if $\vec{x} = (x, y, z) \in R$, then

$$f(x, y, z) = c \iff z = \phi(x, y). \tag{3.23}$$

Furthermore, at any point $(x, y) \in B_\varepsilon(\overrightarrow{x_0})$, the partial derivatives of ϕ are

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= -\frac{\partial f / \partial x}{\partial f / \partial z} \\ \frac{\partial \phi}{\partial y} &= -\frac{\partial f / \partial y}{\partial f / \partial z}, \end{aligned} \tag{3.24}$$

where the partial on the left is taken at $(x, y) \in B_\varepsilon \subset \mathbb{R}^2$ and the partials on the right are taken at $(x, y, \phi(x, y)) \in R \subset \mathbb{R}^3$.

Note that the statement of the general theorem says when we can solve for z in terms of x and y, but an easy argument (Exercise 7) shows that we can replace this with *any* variable whose partial is nonzero at $\vec{x} = \overrightarrow{x_0}$.

Proof sketch: This is a straightforward adaptation of the proof of Theorem 3.4.2 for functions of two variables.

Recall that the original proof had two parts. The first was to show simply that $\mathcal{L}(f, c) \cap R$ is the graph of a function on $B_\varepsilon(\overrightarrow{x_0})$. The argument for this in the three-variable case is almost verbatim the argument in the original proof: assuming that

$\frac{\partial f}{\partial z} > 0$ for all $\vec{x}$ near $\overrightarrow{x_0}$, we see that F is strictly increasing along a short vertical line segment through any point (x', y', z') near $\overrightarrow{x_0}$:

$$I_{(x',y')} = \{(x', y', z) \mid z' - \delta \leq z \leq z' + \delta\}.$$

In particular, assuming $c = 0$ for convenience, we have at (x_0, y_0) $f(x_0, y_0, z_0 - \delta) < 0 < f(x_0, y_0, z_0 + \delta)$ and so for $x = x_0 + \triangle x, y = y_0 + \triangle y$, $\triangle x$ and $\triangle y$ small $(\left\|(\triangle x, \triangle y)\right\| < \varepsilon)$, we also have f positive at the top and negative at the bottom of the segment $I_{(x_0+\triangle x, y_0+\triangle y)}$:

$$f\left(x_0 + \triangle x, y_0 + \triangle y, z_0 - \delta\right) < 0 < f\left(x_0 + \triangle x, y_0 + \triangle y, z_0 + \delta\right).$$

The Intermediate Value Theorem then guarantees that $f = 0$ for at least one point on each vertical segment in R, and the strict monotonicity of f along each segment also guarantees that there is *precisely* one such point along each segment. This analogue of the "vertical line test" proves that the function $\phi(x, y)$ is well-defined in $B_\varepsilon(x_0, y_0)$.

The second part of the original proof, showing that this function ϕ is continuously differentiable, could be reformulated in the three variable case, although it is perhaps less clear how the various ratios could be handled. But there is an easier way. The original proof that $\phi'(x)$ is the negative ratio of $\partial f/\partial x$ and $\partial f/\partial y$ in the two variable case is easily adapted to prove that the restriction of our new function $\phi(x, y)$ to a line parallel to either the x-axis or y-axis is differentiable, and that the derivative of the *restriction* (which is nothing other than a partial of $\phi(x, y)$) is the appropriate ratio of partials of f, as given in Equation (3.24). But then, rather than trying to prove *directly* that ϕ is differentiable as a function of two variables, we can appeal to Theorem 3.3.4 to conclude that, since its partials are continuous, the function is differentiable. This concludes the proof of the Implicit Function Theorem for real-valued functions of three variables. □

As an example, consider the level surface (Figure 3.14) $\mathcal{L}(f, 1)$, where $f(x, y, z) = 4x^2 + y^2 - z^2$. The partial derivatives of $f(x, y, z)$ are $\frac{\partial f}{\partial x}(x, y, z) = 8x$, $\frac{\partial f}{\partial y}(x, y, z) =$

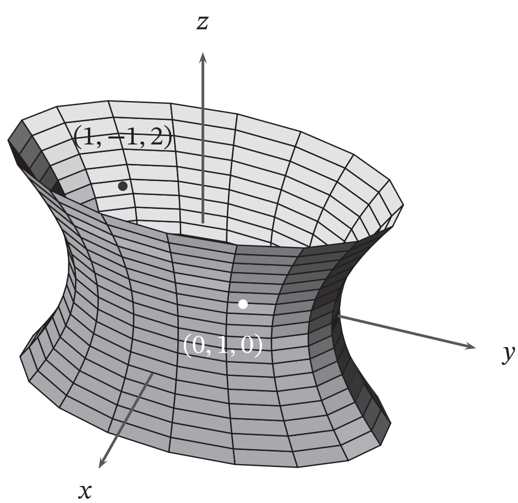


Figure 3.14. The Surface $\mathcal{L}\left(4x^2 + y^2 - z^2, 1\right)$

$2y$, and $\frac{\partial f}{\partial z}(x,y,z) = -2z$. At the point $(1,-1,2)$, these values are $\frac{\partial f}{\partial x}(1,-1,2) = 8$, $\frac{\partial f}{\partial y}(1,-1,2) = -2$, and $\frac{\partial f}{\partial z}(1,-1,2) = -4$, so we see from the Implicit Function Theorem that we can solve for any one of the variables in terms of the other two. For example, near this point we can write $z = \phi(x,y)$, where $4x^2 + y^2 - \phi(x,y)^2 = 1$ and $\phi(1,-1) = 2$; the theorem tells us that $\phi(x,y)$ is differentiable at $x = 1$, $y = -1$, with $\frac{\partial \phi}{\partial x}(1,-1) = -\frac{\partial f/\partial x}{\partial f/\partial z} = -\frac{8}{-4} = 2$ and $\frac{\partial \phi}{\partial y}(1,-1) = -\frac{\partial f/\partial y}{\partial f/\partial z} = -\frac{-2}{-4} = \frac{1}{2}$. Of course, in this case, we can verify the conclusion by solving explicitly: $\phi(x,y) = \sqrt{4x^2+y^2-1}$. You should check that the properties of this function are as advertised. However, at $(0,1,0)$, the situation is different: since $\frac{\partial f}{\partial x}(0,1,0) = 0$, $\frac{\partial f}{\partial y}(0,1,0) = -2$, and $\frac{\partial f}{\partial z}(0,1,0) = 0$. We can only hope to solve for y in terms of x and z; the theorem tells us that in this case $\frac{\partial y}{\partial x}(0,0) = 0$ and $\frac{\partial y}{\partial z}(0,0) = 0$.

We note in passing that Theorem 3.5.3 can be formulated for a function of any number of variables, and the passage from three variables to more is very much like the passage from two to three. However, some of the geometric setup to make this rigorous would take us too far afield. There is also a very slick proof of the most general version of this theorem based on the "contraction mapping theorem"; this is the version that you will probably encounter in higher math courses.

Tangent Planes of Level Surfaces. When a surface is defined by an equation in x, y, and z, it is being presented as a level surface of a function $f(x,y,z)$. Theorem 3.5.3 tells us that in theory, we can express the locus of such an equation near a regular point of f as the graph of a function expressing one of the variables in terms of the other two. From this, we can in principle find the tangent plane to the level surface at this point. However, this can be done directly from the defining equation, using the gradient or linearization of f.

Suppose $P(x_0, y_0, z_0)$ is a regular point of f, and suppose $\vec{p}(t)$ is a differentiable curve in the level surface $\mathcal{L}(f,c)$ through P (so $c = f(P)$), with $\vec{p}(0) = P$. Then the velocity vector $\vec{p}\,'(0)$ lies in the plane tangent to the surface $\mathcal{L}(f,c)$ at $\vec{p}(0)$.

Now on one hand, by the Chain Rule (3.3.6) we know that $\left.\frac{d}{dt}\right|_{t=0}[f(\vec{p}(t))] = \vec{\nabla} f(P) \cdot \vec{p}\,'(0)$; on the other hand, since $\vec{p}(t)$ lies in the level set $\mathcal{L}(f,c)$, $f(\vec{p}(t)) = c$ for all t, and in particular, $\left.\frac{d}{dt}\right|_{t=0}[f(\vec{p}(t))] = 0$. It follows that $\vec{\nabla} f(P) \cdot \vec{p}\,'(0) = 0$ for every vector tangent to $\mathcal{L}(f,c)$ at P; in other words,[13]

Remark 3.5.4. *If P is a regular point of $f(x,y,z)$, then the tangent plane to the level set $\mathcal{L}(f,c)$ through P is the plane through P perpendicular to the gradient vector $\vec{\nabla} f(P)$ of f at P.*

If we write this out in terms of coordinates, we find that a point $(x,y,z) = (x_0 + \triangle x, y_0 + \triangle y, z_0 + \triangle z)$ lies on the plane tangent at (x_0,y_0,z_0) to the surface $f(x,y,z) = c = f(x_0,y_0,z_0)$ if and only if

$$\left(\frac{\partial f}{\partial x}(x_0,y_0,z_0)\right)\triangle x + \left(\frac{\partial f}{\partial y}(x_0,y_0,z_0)\right)\triangle y + \left(\frac{\partial f}{\partial z}(x_0,y_0,z_0)\right)\triangle z = 0,$$

[13]Strictly speaking, we have only shown that every tangent vector is perpendicular to $\vec{\nabla} f$; we need to also show that every vector which is perpendicular to $\vec{\nabla} f$ is the velocity vector of some curve in $\mathcal{L}(f,c)$ as it goes through P. See Exercise 6.

in other words, if

$$d_{(x_0,y_0,z_0)}f\,(x - x_0, y - y_0, z - z_0) = 0.$$

Yet a third way to express this is to add $c = f\,(x_0, y_0, z_0)$ to both sides, noting that the left side then becomes the linearization of f at P:

$$T_{(x_0,y_0,z_0)}f\,(x, y, z) = f\,(x_0, y_0, z_0)\,.$$

We summarize all of this in

Proposition 3.5.5. *Suppose $P(x_0, y_0, z_0)$ is a regular point of the real-valued function $f\,(x, y, z)$ and $f\,(x_0, y_0, z_0) = c$. Then the level set of f through P,*

$$\mathcal{L}\,(f, c) \coloneqq \{(x, y, z) \mid f\,(x, y, z) = c\}\,,$$

has a tangent plane $\mathcal{P}$ at P, which can be characterized in any of the following ways:

- *$\mathcal{P}$ is the plane through P with normal vector $\vec{\nabla} f\,(P)$;*
- *$\mathcal{P}$ is the set of all points $P + \vec{v}$ where $d_P f\,(\vec{v}) = 0$;*
- *$\mathcal{P}$ is the level set $\mathcal{L}\,(T_P f, f\,(P))$ through P of the linearization of f at P:*

$$\mathcal{P} = \mathcal{L}\,(T_P f, f\,(P))\,.$$

Let us see how this works out in practice for a few examples.

First, let us find the plane tangent to the ellipsoid

$$x^2 + 3y^2 + 4z^2 = 20$$

at the point $P(2, -2, -1)$ (Figure 3.15).

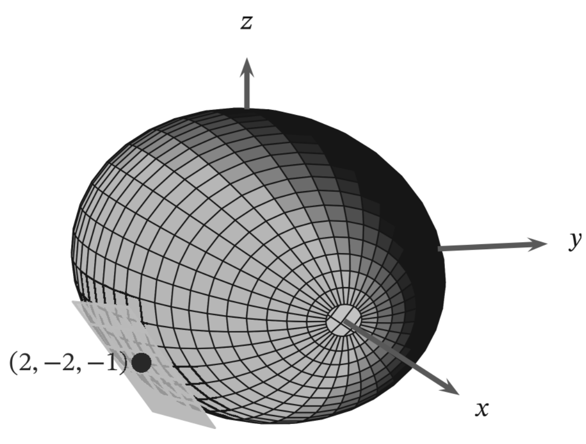


Figure 3.15. The surface $x^2 + 3y^2 + 4z^2 = 20$ with tangent plane at $(2, -2, -1)$

This can be regarded as the level set $\mathcal{L}\,(f, 20)$ of the function $f\,(x, y, z) = x^2 + 3y^2 + 4z^2$. We calculate the partials: $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 6y$, and $\frac{\partial f}{\partial z} = 8sz$, which gives the gradient $\vec{\nabla} f\,(2, -2, -1) = 4\vec{\imath} - 12\vec{\jmath} - 8\vec{k}$. Thus the tangent plane is the plane through $P(2, -2, -1)$ perpendicular to $4\vec{\imath} - 12\vec{\jmath} - 8\vec{k}$, which has equation $4(x - 2) - 12(y + 2) - 8(z + 1) = 0$ or $4x - 12y - 8z = 8 + 24 + 8 = 40$. We note that this is the same as

$$d_{(2,-2,-1)}f\,(\triangle x, \triangle y, \triangle z) = 0$$

with $\triangle x = x - 2$, $\triangle y = y - (-2)$, and $\triangle z = z - (-1)$ or, calculating the linearization $T_{(2,-2,-1)}f(x,y,z) = 20 + 4(x-2) - 12(y+2) - 8(z+1) = 4x - 12y - 8z - 20$ the tangent plane is the level set of the linearization

$$\mathcal{L}\left(T_{(2,-2,-1)}f, 20\right) = \left\{(x,y,z) \,|\, T_{(2,-2,-1)}f(x,y,z) = 20\right\}.$$

We note in passing that in this case we could also have solved for z in terms of x and y: $4z^2 = 20 - x^2 - 3y^2$, or $z^2 = 5 - \frac{x^2}{4} - \frac{3y^2}{4}$ yields $z = \pm\sqrt{5 - \frac{x^2}{4} - \frac{3y^2}{4}}$ and since at our point z is negative, the nearby solutions are $z = -\sqrt{5 - \frac{x^2}{4} - \frac{3y^2}{4}}$ This would have given us an expression for the ellipsoid near $(2,-2,-1)$ as the graph $z = \phi(x,y)$ of the function of x and y: $\phi(x,y) = -\sqrt{5 - \frac{x^2}{4} - \frac{3y^2}{4}}$. The partials of this function are $\frac{\partial\phi}{\partial x} = -\frac{-x/4}{\sqrt{5 - \frac{x^2}{4} - \frac{3y^2}{4}}}$ and $\frac{\partial\phi}{\partial y} = -\frac{-3y/4}{\sqrt{5 - \frac{x^2}{4} - \frac{3y^2}{4}}}$. At our point, these have values $\frac{\partial\phi}{\partial x}(2,-2) = \frac{1}{2}$ and $\frac{\partial\phi}{\partial y}(2,-2) = -\frac{3}{2}$, so the parametric form of the tangent plane is

$$\begin{cases} x &= & 2 & +s & \\ y &= & -2 & & +t \\ z &= & -1 & +\frac{s}{2} & -\frac{3t}{2} \end{cases}$$

while the equation of the tangent plane can be formulated in terms of the normal vector

$$\vec{n} = \vec{v_x} \times \vec{v_y} = (\vec{\imath} + \frac{1}{2}\vec{k}) \times (\vec{\jmath} - \frac{3}{2}\vec{k}) = -\left(\frac{1}{2}\right)\vec{\imath} - \left(-\frac{3}{2}\right)\vec{\jmath} + \vec{k}$$

as $-\frac{1}{2}(x-2) + \frac{3}{2}(y+2) + (z+1) = 0$, or $-\frac{1}{2}x + \frac{3}{2}y + z = -1 - 3 - 1 = -5$, which we recognize as our earlier equation, divided by -8.

As a second example, we consider the surface

$$x^3y^2z + x^2y^3z + xyz^3 = 30$$

near the point $P(-2,3,1)$. This time, it is not feasible to solve for any one of the variables in terms of the others; our only choice is to work directly with this as a level surface of the function $f(x,y,z) = x^3y^2z + x^2y^3z + xyz^3$. The partials of this function are $\frac{\partial f}{\partial x} = 3x^2y^2z + 2zy^3z + yz^3$, $\frac{\partial f}{\partial y} = 2x^3yz + 3x^2y^2z + xz^3$, and $\frac{\partial f}{\partial z} = x^3y^2 + x^2y^3 + 3xyz^2$. The values of these at our point are $\frac{\partial f}{\partial x}(-2,3,1) = 3$, $\frac{\partial f}{\partial y}(-2,3,1) = 58$, and $\frac{\partial f}{\partial z}(-2,3,1) = 18$, giving as the equation of the tangent plane $3(x+2) + 58(y-3) + 18(z-1) = 0$ or $3x + 58y + 18z = 186$. You should check that this is equivalent to any one of the forms of the equation given in Proposition 3.5.5.

Exercises for § 3.5

Practice Problems:

For each given surface, express the tangent plane at each given point (a) as the locus of an equation in x, y and z (b) in parametrized form:

(1) $z = x^2 - y^2$, $(1,-2,-3)$, $(2,-1,3)$
(2) $z^2 = x^2 + y^2$, $(1,1,\sqrt{2})$, $(2,-1,\sqrt{5})$
(3) $x^2 + y^2 - z^2 = 1$, $(1,-1,1)$, $(\sqrt{3},0,\sqrt{2})$

(4) $x^2 + y^2 + z^2 = 4$, $(1, 1, \sqrt{2})$, $(\sqrt{3}, 1, 0)$
(5) $x^3 + 3xy + z^2 = 2$, $(1, \frac{1}{3}, 0)$, $(0, 0, \sqrt{2})$

Theory problems:

(6) For each surface defined implicitly, decide at each given point whether one can solve locally for (i) z in terms of x and y; (ii) x in terms of y and z; (iii) y in terms of x and z. Find the partials of the function if it exists.
  (a) $x^3z^2 - z^3xy = 0$ at $(1, 1, 1)$ and at $(0, 0, 0)$.
  (b) $xy + z + 3xz^5 = 4$ at $(1, 0, 1)$
  (c) $x^3 + y^3 + z^3 = 10$ at $(1, 2, 1)$ and at $(\sqrt[3]{5}, 0, \sqrt[3]{5})$.
  (d) $\sin x \cos y - \cos x \sin z = 1$ at $(\pi, 0, \frac{\pi}{2})$.

(7) Mimic the argument for Theorem 3.5.3 to show that we can solve for *any* variable whose partial does not vanish at our point.

3.6 Surfaces and Tangent Planes II: Parametrized Surfaces

Regular Parametrizations. In § 2.2 we saw how to go beyond *graphs* of *real-valued* functions of a real variable to express more general curves as images of *vector-valued* functions of a real variable. In this subsection, we will explore the analogous representation of a surface in space as the image of a *vector-valued* function of *two* variables. Of course, we have already seen such a representation for planes.

Just as continuity and limits for functions of several variables present new subtleties compared to their single-variable cousins, an attempt to formulate the idea of a "surface" in $\mathbb{R}^3$ using only continuity notions will encounter a number of difficulties. We shall avoid these by starting out immediately with differentiable parametrizations.

Definition 3.6.1. *A vector-valued function*

$$\vec{p}\,(s,t) = (x_1\,(s,t)\,, x_2\,(s,t)\,, x_3\,(s,t))$$

*of two real variables is **differentiable** (resp. **continuously differentiable**, or $\mathcal{C}^1$) if each of the coordinate functions $x_j : \mathbb{R}^2 \to \mathbb{R}$ is differentiable (resp. continuously differentiable). We know from Theorem* 3.3.4 *that a C^1 function is automatically differentiable.*

*We define the **partial derivatives** of a differentiable function $\vec{p}\,(s,t)$ to be the vectors*

$$\frac{\partial \vec{p}}{\partial s} = \left(\frac{\partial x_1}{\partial s}, \frac{\partial x_2}{\partial s}, \frac{\partial x_3}{\partial s} \right)$$
$$\frac{\partial \vec{p}}{\partial t} = \left(\frac{\partial x_1}{\partial t}, \frac{\partial x_2}{\partial t}, \frac{\partial x_3}{\partial t} \right).$$

*We will call $\vec{p}\,(s,t)$ **regular** if it is C^1 and at every pair of parameter values (s,t) in the domain of $\vec{p}$ the partials are linearly independent—that is, neither is a scalar multiple of the other. The image of a regular parametrization*

$$\mathfrak{S} := \left\{ \vec{p}\,(s,t) \mid (s,t) \in \operatorname{dom}(\vec{p}) \right\}$$

*is a surface in $\mathbb{R}^3$, and we will refer to $\vec{p}\,(s,t)$ as a **regular parametrization** of $\mathfrak{S}$.*

As an example, you should verify (Exercise 4a) that the graph of a (continuously differentiable) function $f\,(x,y)$ is a surface parametrized by $\vec{p}\,(s,t) = (s, t, f\,(s,t))$.

As another example, consider the function $\vec{p}\,(\theta, t) = (\cos\theta, \sin\theta, t)$; this can also be written

$$x = \cos\theta, \quad y = \sin\theta, \quad z = t.$$

The first two equations give a parametrization of the circle of radius one about the origin in the xy-plane, while the third moves such a circle vertically by t units: we see that this parametrizes a cylinder with axis the z-axis, of radius 1 (Figure 3.16).

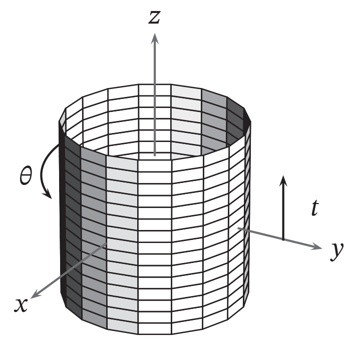


Figure 3.16. Parametrized Cylinder

The partials are

$$\frac{\partial\vec{p}}{\partial\theta}\,(\theta, t) = -(\sin\theta)\vec{\imath} + (\cos\theta)\vec{\jmath}$$
$$\frac{\partial\vec{p}}{\partial t}\,(\theta, t) = \vec{k}$$

Another function is $\vec{p}\,(r, \theta) = (r\cos\theta, r\sin\theta, 0)$, or

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = 0$$

which describes the xy-plane in polar coordinates; the partials are

$$\frac{\partial\vec{p}}{\partial r}\,(r, \theta) = (\cos\theta)\vec{\imath} + (\sin\theta)\vec{\jmath}$$
$$\frac{\partial\vec{p}}{\partial\theta}\,(r, \theta) = -(r\sin\theta)\vec{\imath} + (r\cos\theta)\vec{\jmath};$$

these are independent unless $r = 0$, so we get a regular parametrization of the xy-plane provided we stay away from the origin.

We can similarly parametrize the sphere of radius R by using spherical coordinates:

$$\vec{p}\,(\theta, \phi) = (R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi) \tag{3.25}$$

or

$$x = R\sin\phi\cos\theta, \quad y = R\sin\phi\sin\theta, \quad z = R\cos\phi;$$

the partials are

$$\frac{\partial\vec{p}}{\partial\phi}\,(\phi, \theta) = (R\cos\phi\cos\theta)\vec{\imath} + (R\cos\phi\sin\theta)\vec{\jmath} - (R\sin\phi)\vec{k}$$
$$\frac{\partial\vec{p}}{\partial\theta}\,(\phi, \theta) = -(R\sin\phi\sin\theta)\vec{\imath} + (R\sin\phi\cos\theta)\vec{\jmath}$$

which are independent provided $R \neq 0$ and ϕ is not a multiple of π; the latter is required because $\frac{\partial \vec{p}}{\partial \theta}(n\pi, \theta) = \vec{0}$.

Regular parametrizations of surfaces share a pleasant property with regular parametrizations of curves:

Proposition 3.6.2. *A regular function $\vec{p} : \mathbb{R}^2 \to \mathbb{R}^3$ is **locally one-to-one**—that is, for every point (s_0, t_0) in the domain there exists $\delta > 0$ such that the restriction of $\vec{p}(s, t)$ to parameter values with*

$$|s - s_0| < \delta$$
$$|t - t_0| < \delta$$

is one-to-one: $(s_1, t_1) \neq (s_2, t_2)$ guarantees that $\vec{p}(s_1, t_1) \neq \vec{p}(s_2, t_2)$.

Note as before that the condition $(s_1, t_1) \neq (s_2, t_2)$ allows *one* pair of coordinates to be equal, provided the *other* pair is not; similarly, $\vec{p}(s_1, t_1) \neq \vec{p}(s_2, t_2)$ requires only that they differ in *at least one* coordinate.

A proof of Proposition 3.6.2 is sketched in Exercise 5.

The parametrization of the sphere (Equation (3.25)) shows that the conclusion of Proposition 3.6.2 breaks down if the parametrization is not regular: when $\phi = 0$ we have

$$\vec{p}(\phi, \theta) = (0, 0, 1)$$

independent of θ; in fact, the curves corresponding to fixing ϕ at a value slightly above zero are circles of constant latitude around the North Pole, while the curves corresponding to fixing θ are great circles, all going through this pole. This is reminiscent of the breakdown of polar coordinates at the origin.

A point at which a C^1 function $\vec{p} : \mathbb{R}^2 \to \mathbb{R}^3$ has dependent partials (including the possibility that at least one partial is the zero vector) is called a **singular point**; points at which the partials are independent are **regular points**. Proposition 3.6.2 can be rephrased as saying that $\vec{p} : \mathbb{R}^2 \to \mathbb{R}^3$ is locally one-to-one at each of its regular points. Of course, continuity says that every point sufficiently near a given regular point (that is, corresponding to nearby parameter values) is also regular; a region in the domain of $\vec{p} : \mathbb{R}^2 \to \mathbb{R}^3$ consisting of regular points, and on which $\vec{p}$ is one-to-one is sometimes called a **coordinate patch** for the surface it is parametrizing.

We consider one more example. Let us start with a circle in the xy-plane of radius $a > 0$, centered at the origin: this can be expressed in cylindrical coordinates as $r = a$, and the point on this circle which also lies in the vertical plane corresponding to a fixed value of θ has rectangular coordinates $(a \cos \theta, a \sin \theta, 0)$. We are interested, however, not in this circle, but in the surface consisting of points in $\mathbb{R}^3$ at distance b from this circle, where $0 < b < a$; this is called a **torus**. It is reasonable to assume (and this will be verified later) that for any point P not on the circle, the nearest point to P *on* the circle lies in the vertical plane given by fixing θ at its value for P, say $\theta = \alpha$. This means that if P has cylindrical coordinates (r, α, z) then the nearest point to P on the circle is the point $Q(a \cos \alpha, a \sin \alpha, 0)$ as given above. The vector $\overrightarrow{QP}$ lies in the plane $\theta = \alpha$; its length is, by assumption, b, and if we denote the angle it makes with the radial line $\mathcal{O}Q$ by β (Figure 3.17), then we have

$$\overrightarrow{QP} = (b \cos \beta)\overrightarrow{v_\alpha} + (b \sin \beta)\vec{k},$$

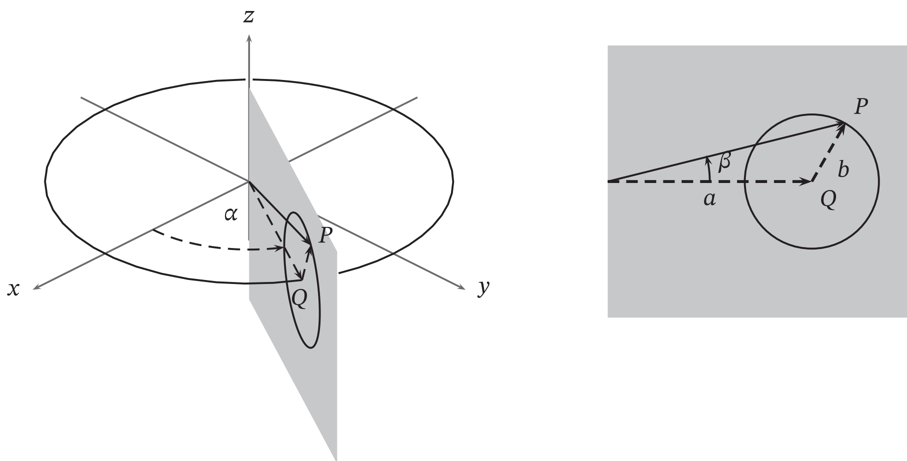


Figure 3.17. Parametrization of Torus

where $\overrightarrow{v_\alpha} = (\cos\alpha)\vec{\imath} + (\sin\alpha)\vec{\jmath}$ is the horizontal unit vector making angle α with the x-axis. Since $\overrightarrow{\mathcal{O}Q} = a\overrightarrow{v_\alpha} = (a\cos\alpha)\vec{\imath} + (a\sin\alpha)\vec{\jmath}$, we see that the position vector of P is

$$\begin{aligned}\overrightarrow{\mathcal{O}P} &= \overrightarrow{\mathcal{O}Q} + \overrightarrow{QP} \\ &= [(a\cos\alpha)\vec{\imath} + (a\sin\alpha)\vec{\jmath}] + [(b\cos\beta)[(\cos\alpha)\vec{\imath} + (\sin\alpha)\vec{\jmath}] + (b\sin\beta)\vec{k}\end{aligned}$$

so the torus (sketched in Figure 3.18) is parametrized by the vector-valued function

$$\vec{p}\,(\alpha,\beta) = (a + b\cos\beta)[(\cos\alpha)\vec{\imath} + (\sin\alpha)\vec{\jmath}] + (b\sin\beta)\vec{k} \tag{3.26}$$

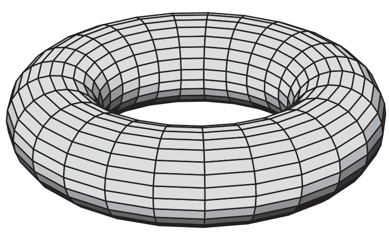


Figure 3.18. Torus

The partial derivatives of this function are

$$\begin{aligned}\frac{\partial\vec{p}}{\partial\alpha} &= (a + b\cos\beta)[(-\sin\alpha)\vec{\imath} + (\cos\alpha)\vec{\jmath}] \\ \frac{\partial\vec{p}}{\partial\beta} &= (-b\sin\beta)[(\cos\alpha)\vec{\imath} + (\sin\alpha)\vec{\jmath}] + (b\cos\beta)\vec{k}.\end{aligned}$$

To see that these are independent, we note first that if $\cos\beta \neq 0$ this is obvious, since $\frac{\partial\vec{p}}{\partial\beta}$ has a nonzero vertical component while $\frac{\partial\vec{p}}{\partial\alpha}$ does not. If $\cos\beta = 0$, we simply note that the two partial derivative vectors are perpendicular to each other (in fact, in retrospect, this is true whatever value β has). Thus, every point is a regular point. Of course, increasing either α or β by 2π will put us at the same position, so to get a coordinate patch we need to restrict each of our parameters to intervals of length $< 2\pi$.

Tangent Planes. To define the plane tangent to a regularly parametrized surface, we can think, as we did for the graph of a function, in terms of slicing the surface and finding lines tangent to the resulting curves. A more fruitful view, however, is to think in terms of arbitrary curves in the surface. Suppose $\vec{p}\,(r,s)$ is a C^1 function parametrizing the surface $\mathfrak{S}$ in $\mathbb{R}^3$ and $P = \vec{p}\,(r_0, s_0)$ is a regular point; by restricting the domain of $\vec{p}$ we can assume that we have a coordinate patch for $\mathfrak{S}$. Any curve in $\mathfrak{S}$ can be represented as $\vec{\gamma}\,(t) = \vec{p}\,(r\,(t)\,, s\,(t))$, or

$$x = x\,(r\,(t)\,, s\,(t))\,, \quad y = y\,(r\,(t)\,, s\,(t))\,, \quad z = z\,(r\,(t)\,, s\,(t))$$

—that is, we can "pull back" the curve on $\mathfrak{S}$ to a curve in the parameter space. If we want the curve to pass through P when $t = 0$, we need to require $r\,(0) = r_0$ and $s\,(0) = s_0$. If $r\,(t)$ and $s\,(t)$ are differentiable, then by the Chain Rule $\gamma\,(t)$ is also differentiable, and its velocity vector can be found via $\vec{v}\,(t) = \dot{\vec{\gamma}}\,(t) = (dx/dt, dy/dy, dz/dt)$, where

$$\begin{aligned}
\frac{dx}{dt} &= \frac{\partial x}{\partial r}\frac{dr}{dt} + \frac{\partial x}{\partial s}\frac{ds}{dt} \\
\frac{dy}{dt} &= \frac{\partial y}{\partial r}\frac{dr}{dt} + \frac{\partial y}{\partial s}\frac{ds}{dt} \\
\frac{dz}{dt} &= \frac{\partial z}{\partial r}\frac{dr}{dt} + \frac{\partial z}{\partial s}\frac{ds}{dt}.
\end{aligned}$$

We expect that for *any* such curve, $\vec{v}\,(0)$ will be parallel to the tangent plane to $\mathfrak{S}$ at P. In particular, the two curves obtained by holding one of the parameters constant will give a vector in this plane: holding s constant at $s = s_0$, we can take $r = r_0 + t$ to get $\vec{\gamma}\,(t) = \vec{p}\,(r_0 + t, s_0)$, whose velocity at $t = t_0$ is

$$\vec{v_r}\,(0) = \frac{\partial \vec{p}}{\partial r}$$

and similarly, the velocity obtained by holding $r = r_0$ and letting $s = s_0 + t$ will be

$$\vec{v_s}\,(0) = \frac{\partial \vec{p}}{\partial s}.$$

Because P is a regular point, these are linearly independent and so form direction vectors for a parametrization of a plane

$$T_{(r_0,s_0)}\vec{p}\,(r_0 + \triangle r, s_0 + \triangle s) = \vec{p}\,(r_0, s_0) + \triangle r\frac{\partial \vec{p}}{\partial r} + \triangle s\frac{\partial \vec{p}}{\partial s}.$$

By looking at the components of this vector equation, we easily see that each component of $T_{(r_0,s_0)}\vec{p}\,(r_0 + \triangle r, s_0 + \triangle s)$ is the linearization of the corresponding component of $\vec{p}\,(r, s)$, and so has first order contact with it at $t = 0$. It follows, from arguments that are by now familiar, that for *any* curve in $\mathfrak{S}$ $\vec{\gamma}\,(t) = \vec{p}\,(r\,(t)\,, s\,(t)) = (x\,(r\,(t)\,, s\,(t))\,, y\,(r\,(t)\,, s\,(t))\,, z\,(r\,(t)\,, s\,(t)))$ the velocity vector $\vec{v}\,(0) = \frac{\partial \vec{p}}{\partial r}\frac{dr}{dt} + \frac{\partial \vec{p}}{\partial s}\frac{ds}{dt}$ lies in the plane parametrized by $T\vec{p}$. It is also a straightforward argument to show that this parametrization of the tangent plane has **first order contact** with $\vec{p}\,(r, s)$ at $(r, s) = (r_0, s_0)$, in the sense that

$$\left\|\vec{p}\,(r_0 + \triangle r, s_0 + \triangle s) - T_{(r_0,s_0)}\vec{p}\,(r_0 + \triangle r, s_0 + \triangle s)\right\| = \mathfrak{o}(\left\|(\triangle r, \triangle s)\right\|)$$

as $(\triangle r, \triangle s) \to \vec{0}$. The parametrization $T_{(r_0,s_0)}\vec{p}$ assigns to each vector $\vec{v} \in \mathbb{R}^2$ a vector $T_{(r_0,s_0)}\vec{p}\,(\vec{v})$ in the tangent plane at (r_0, s_0): namely if $\gamma\,(\tau)$ is a curve in the (s,t)-plane going through (r_0, s_0) with velocity $\vec{v}$, then the corresponding curve $\vec{p}\,(\gamma\,(\tau))$ in $\mathfrak{S}$ goes through $\vec{p}\,(r_0, s_0)$ with velocity $T_{(r_0,s_0)}\vec{p}\,(\vec{v})$. This is sometimes called the **tangent map** at (r_0, s_0) of the parametrization $\vec{p}$.

We can also use the two partial derivative vectors $\frac{\partial\vec{p}}{\partial r}$ and $\frac{\partial\vec{p}}{\partial s}$ to find an equation for the tangent plane to $\mathfrak{S}$ at P. Since they are direction vectors for the plane, their cross product gives a normal to the plane:

$$\vec{N} = \frac{\partial\vec{p}}{\partial r} \times \frac{\partial\vec{p}}{\partial s}$$

and then the equation of the tangent plane is given by

$$\vec{N} \cdot [(x, y, z) - \vec{p}\,(r_0, s_0)] = 0.$$

You should check that in the special case when $\mathfrak{S}$ is the graph of a function $f\,(x, y)$, and $\vec{p}$ is the parametrization of $\mathfrak{S}$ as $\vec{p}\,(x, y) = (x, y, f\,(x, y))$ then $\vec{N} = -\frac{\partial f}{\partial x}\vec{\imath} - \frac{\partial f}{\partial y}\vec{\jmath} + \vec{k}$, yielding the usual equation for the tangent plane.

We summarize these observations in the following

Remark 3.6.3. *If $\vec{p} : \mathbb{R}^2 \to \mathbb{R}^3$ is regular at (r_0, s_0), then*

(1) *The linearization of $\vec{p}\,(r, s)$ at $r = r_0$, $s = s_0$*

$$T_{(r_0,s_0)}\vec{p}\,(r_0 + \triangle r, s_0 + \triangle s) = \vec{p}\,(r_0, s_0) + \triangle r\frac{\partial\vec{p}}{\partial r} + \triangle s\frac{\partial\vec{p}}{\partial s}$$

has first-order contact with $\vec{p}\,(r, s)$ at $r = r_0$, $s = s_0$.

(2) *It parametrizes a plane through $P = \vec{p}\,(r_0, s_0) = (x_0, y_0, z_0)$ which contains the velocity vector of any curve passing through P in the surface $\mathfrak{S}$ parametrized by $\vec{p}$.*

(3) *The equation of this plane is*

$$\vec{N} \cdot (x - x_0, y - y_0, z - z_0) = 0,$$

where $\vec{N} = \frac{\partial\vec{p}}{\partial r} \times \frac{\partial\vec{p}}{\partial s}$.

*This plane is the **tangent plane** to $\mathfrak{S}$ at P.*

Let us consider two quick examples.

First, we consider the sphere parametrized using spherical coordinates in Equation (3.25); using $R = 1$ we have $\vec{p}\,(\theta, \phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$ (see Figure 3.19).

Let us find the tangent plane at $P\left(\frac{\sqrt{3}}{2\sqrt{2}}, -\frac{\sqrt{3}}{2\sqrt{2}}, \frac{1}{2}\right)$, which corresponds to $\phi = \frac{\pi}{3}$ and $\theta = -\frac{\pi}{4}$. The partials are

$$\frac{\partial\vec{p}}{\partial\phi}\left(\frac{\pi}{3}, -\frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}\vec{\imath} - \frac{1}{2\sqrt{2}}\vec{\jmath} - \frac{\sqrt{3}}{2}\vec{k}$$

$$\frac{\partial\vec{p}}{\partial\theta}\left(\frac{\pi}{3}, -\frac{\pi}{4}\right) = -\frac{\sqrt{3}}{2\sqrt{2}}\vec{\imath} + \frac{\sqrt{3}}{2\sqrt{2}}\vec{\jmath}$$

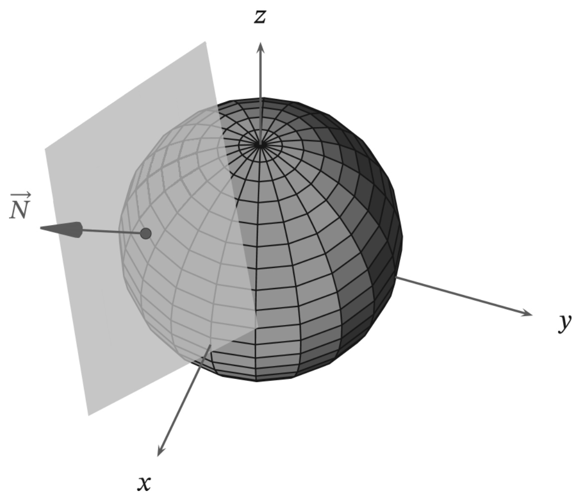


Figure 3.19. Tangent Plane to Sphere at $\left(\frac{\sqrt{3}}{2\sqrt{2}}, -\frac{\sqrt{3}}{2\sqrt{2}}, \frac{1}{2}\right)$

so a parametrization of the tangent plane is given by

$$x = \frac{\sqrt{3}}{2\sqrt{2}} + \left(\frac{1}{2\sqrt{2}}\right)\triangle r + \left(\frac{\sqrt{3}}{2\sqrt{2}}\right)\triangle s$$

$$y = -\frac{\sqrt{3}}{2\sqrt{2}} - \left(\frac{1}{2\sqrt{2}}\right)\triangle r + \left(\frac{\sqrt{3}}{2\sqrt{2}}\right)\triangle s$$

$$z = \frac{1}{2} + \left(\frac{\sqrt{3}}{2}\right)\triangle r.$$

The normal vector is $\vec{N} = \frac{3}{4\sqrt{2}}\vec{\imath} - \frac{3}{4\sqrt{2}}\vec{\jmath} + \frac{\sqrt{3}}{4}\vec{k}$ so the equation of the tangent plane is

$$\frac{3}{4\sqrt{2}}\left(x - \frac{\sqrt{3}}{2\sqrt{2}}\right) - \frac{3}{4\sqrt{2}}\left(y + \frac{\sqrt{3}}{2\sqrt{2}}\right) + \frac{\sqrt{3}}{4}\left(z - \frac{1}{2}\right) = 0.$$

Next, we consider the torus with outer radius $a = 2$ and inner radius $b = 1$ parametrized by

$$\vec{p}\,(\alpha, \beta) = (2 + \cos\beta)[(\cos\alpha)\vec{\imath} + (\sin\alpha)\vec{\jmath}] + (\sin\beta)\vec{k}$$

at $P\left(\frac{5\sqrt{3}}{4}, \frac{5}{4}, \frac{\sqrt{3}}{2}\right)$, which corresponds to $\alpha = \frac{\pi}{6}$ and $\beta = \frac{\pi}{3}$ (seeFigure 3.20).

The partials are

$$\frac{\partial\vec{p}}{\partial\alpha} = -\left(\frac{5}{4}\right)\vec{\imath} + \left(\frac{5\sqrt{3}}{4}\right)\vec{\jmath}$$

$$\frac{\partial\vec{p}}{\partial\beta} = \left(\sqrt{3} - \frac{3}{4}\right)\vec{\imath} + \left(1 - \frac{\sqrt{3}}{4}\right)\vec{\jmath} + \frac{1}{2}\vec{k}$$

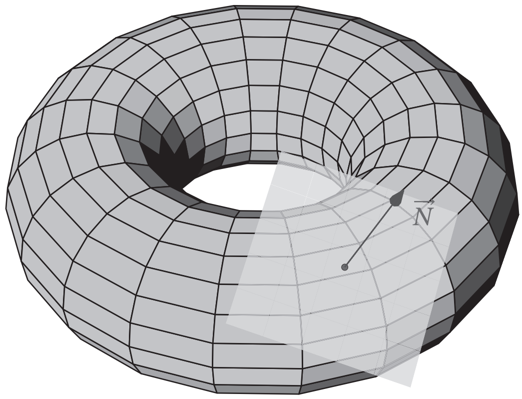


Figure 3.20. Tangent Plane to Torus at $\left(\frac{5\sqrt{3}}{4}, \frac{5}{4}, \frac{\sqrt{3}}{2}\right)$

so a parametrization of the tangent plane is

$$\begin{aligned}
x &= \frac{5\sqrt{3}}{4} - \left(\frac{5}{4}\right)\triangle\alpha + \left(\sqrt{3} - \frac{3}{4}\right)\triangle\beta \\
y &= \frac{5}{4} + \left(\frac{5\sqrt{3}}{4}\right)\triangle\alpha + \left(1 - \frac{\sqrt{3}}{4}\right)\triangle\beta \\
z &= \frac{\sqrt{3}}{2}\triangle\alpha + \frac{1}{2}\triangle\beta.
\end{aligned}$$

The normal to the tangent plane is

$$\vec{N} = \left(\frac{5\sqrt{3}}{8}\right)\vec{\imath} + \left(\frac{5}{8}\right)\vec{\jmath} + \left(\frac{65\sqrt{3}}{16} - 5\right)\vec{k}$$

so an equation for the plane is

$$\left(\frac{5\sqrt{3}}{8}\right)\left(x - \frac{5\sqrt{3}}{4}\right) + \left(\frac{5}{8}\right)\left(y - \frac{5}{4}\right) + \left(\frac{65\sqrt{3}}{16} - 5\right)\left(z - \frac{\sqrt{3}}{2}\right) = 0.$$

Exercises for § 3.6

Practice Problems:

For each given surface, express the tangent plane (a) as the locus of an equation in x, y and z (b) in parametrized form:

(1)

$$x = s, \quad y = s^2 + t, \quad z = t^2 + 1 \quad \text{at } (-1, 0, 2)$$

(2)

$$x = u^2 - v^2, \quad y = u + v, \quad z = u^2 + 4v \quad \text{at } (-\frac{1}{4}, \frac{1}{2}, 2).$$

(3)

$$x = (2 - \cos v)\cos u, \quad y = (2 - \cos v)\sin u, \quad z = \sin v$$

at any point (give in terms of u and v).

Theory problems:

(4) (a) Verify that $\vec{p}(s,t) = (s,t,f(s,t))$ is a regular parametrization of the graph $z = f(x,y)$ of any $\mathcal{C}^1$ function $f(x,y)$ of two variables.
(b) What is the appropriate generalization for $n > 2$ variables?

Challenge problem:

(5) Prove Proposition 3.6.2 as follows:
(a) Suppose $\vec{v}$ and $\vec{w}$ are linearly independent vectors, and consider the function
$$f(\theta) = \left\|(\cos\theta)\vec{v} + (\sin\theta)\vec{w}\right\|.$$
Since $f(\theta)$ is periodic ($f(\theta + 2\pi) = f(\theta)$ for all θ), it achieves its minimum for some value $\theta_0 \in [0, 2\pi]$, and since $\vec{v}$ and $\vec{w}$ are linearly independent, $f(\theta_0) > 0$. Of course, this value depends on the choice of $\vec{v}$ and $\vec{w}$, so we write it as $K(\vec{v},\vec{w}) > 0$. We would like to show that $K(\vec{v},\vec{w})$ depends continuously on the vectors $\vec{v}$ and $\vec{w}$:
(i) Show that
$$f(\theta)^2 = \frac{1}{2}\left\|\vec{v}\right\|^2(1 + \cos 2\theta) + \vec{v}\cdot\vec{w}\sin 2\theta + \frac{1}{2}\left\|\vec{w}\right\|^2(1 - \cos 2\theta).$$
(ii) Show that the extreme values of $f(\theta)$ occur when
$$\tan 2\theta = \frac{2\vec{v}\cdot\vec{w}}{\left\|\vec{v}\right\|^2 - \left\|\vec{w}\right\|^2}.$$
From this we can see that there is a way to solve for θ_0 as a (continuous) function of $\vec{v}$ and $\vec{w}$, and hence to obtain $K(\vec{v},\vec{w})$ as a continuous function of the two vectors. (The exact formula is not particularly useful.)
(b) Apply this to the vectors
$$\vec{v} = \frac{\partial\vec{p}}{\partial s}, \quad \vec{w} = \frac{\partial\vec{p}}{\partial t}$$
to find a positive, continuous function $K(s,t)$ defined on the domain of $\vec{p}$ such that for every θ the vector
$$\vec{v}(s,t,\theta) = (\cos\theta)\frac{\partial\vec{p}}{\partial s}(s,t) + (\sin\theta)\frac{\partial\vec{p}}{\partial t}(s,t)$$
has $\left\|\vec{v}(s,t,\theta)\right\| \geq K(s,t)$.
In particular, show that, given s, t, and an angle θ, some *component* of the vector $\vec{v}(s,t,\theta)$ must have absolute value exceeding $K(s,t)/2$:
$$\left|v_j(s,t,\theta)\right| > \frac{K(s,t)}{2}.$$
(c) Identify three (overlapping) sets of θ-values, say Θ_j ($j = 1,2,3$) such that every θ belongs to at least one of them, and for every $\theta \in \Theta_j$ the estimate above works at (s_0,t_0) using the j^{th} coordinate:
$$\left|v_j(s_0,t_0,\theta)\right| > \frac{K}{2},$$
and by continuity this continues to hold if (s,t) is sufficiently close to (s_0,t_0).

(d) Suppose (s_i, t_i), $i = 1, 2$ are distinct pairs of parameter values near (s_0, t_0), and consider the straight line segment joining them in parameter space; parametrize this line segment by τ. Assume without loss of generality that $\theta \in \Theta_1$, and show that

$$\begin{aligned} x'(\tau) &= \triangle s \frac{\partial x}{\partial s}(s(\tau), t(\tau)) + \triangle t \frac{\partial x}{\partial t}(s(\tau), t(\tau)) \\ &= \left(\sqrt{\triangle s^2 + \triangle t^2}\right) v_j(s, t, \theta) \end{aligned}$$

which has absolute value at least $(K/2)\sqrt{\triangle s^2 + \triangle t^2}$, and in particular is nonzero.

(e) Explain why this shows the points $\vec{p}(s_1, t_1)$ and $\vec{p}(s_2, t_2)$ are distinct.

(6) Suppose $P(x_0, y_0, z_0)$ is a regular point of the $\mathcal{C}^1$ function $f(x, y, z)$; for definiteness, assume $\frac{\partial f}{\partial z}(P) \neq 0$. Let $\vec{v}$ be a nonzero vector perpendicular to $\vec{\nabla} f(P)$.

(a) Show that the projection $\vec{w} = (v_1, v_2)$ of $\vec{v}$ onto the xy-plane is a nonzero vector.

(b) By the Implicit Function Theorem, the level set $\mathcal{L}(f, c)$ of f through P near P can be expressed as the graph $z = \phi(x, y)$ of some $\mathcal{C}^1$ function $\phi(x, y)$. Show that (at least for $|t| < \varepsilon$ for some $\varepsilon > 0$) the curve $\vec{p}(t) = (x_0 + v_1 t, y_0 + v_2 t, \phi(x_0 + v_1 t, y_0 + v_2 t))$ lies on $\mathcal{L}(f, c)$, and that $\vec{p}\,'(0) = \vec{v}$.

(c) This shows that every vector in the plane perpendicular to the gradient is the velocity vector of some curve in $\mathcal{L}(f, c)$ as it goes through P, at least if $\vec{\nabla} f(P)$ has a nonzero z-component. What do you need to show this assuming only that $\vec{\nabla} f(P)$ is a nonzero vector?

3.7 Extrema

Bounded Functions. The notions associated to boundedness of a function of one variable on an interval or other set of real numbers can be applied to a real-valued function on a set of points in $\mathbb{R}^{2 \text{ or } 3}$.

Definition 3.7.1. *Suppose $f : \mathbb{R}^{2 \text{ or } 3} \to \mathbb{R}$ is a real-valued function with domain* dom(f) $\subset \mathbb{R}^{2 \text{ or } 3}$*, and let $S \subset$* dom(f) *be any subset of the domain of f.*

(1) *A* ***lower bound*** *for f on S is any number $\alpha \in \mathbb{R}$ such that $\alpha \leq f(x)$ for every $x \in S$.*

f is ***bounded below*** *on S if there exists a lower bound for f on S.*

An ***upper bound*** *for f on S is any number $\beta \in \mathbb{R}$ such that $f(x) \leq \beta$ for every $x \in S$.*

f is ***bounded above*** *on S if there exists an upper bound for f on S.*

f is ***bounded*** *on S if it is bounded below on S* ***and*** *bounded above on S.*

(2) *If f is bounded below (resp. bounded above) on S then there exists a unique lower (resp. upper) bound $A \in \mathbb{R}$ (resp. $B \in \mathbb{R}$) such that every other lower bound α (resp. upper bound β) satisfies $\alpha \leq A$ (resp. $B \leq \beta$).*

A (resp. B) is called the ***infimum*** *(resp.* ***supremum****) of f on S, and denoted* $\inf_{x \in S} f(x) = A$ *(resp.* $\sup_{x \in S} f(x) = B$*).*

(3) *We say that f* ***achieves its minimum*** *(resp.* ***achieves its maximum****) on S at $x_0 \in S$ if $f(x_0)$ is a lower (resp. upper) bound for f on S.*

We say that x_0 is an ***extreme point*** *of f on S if f achieves its minimum or maximum on S at x_0, and then the value $f(x_0)$ will be referred to as an* ***extreme value*** *of f on S.*

In all the statements above, when the set S is not mentioned explicitly, it is understood to be the whole domain of f.

An alternative way to formulate these definitions is to reduce them to corresponding definitions for sets of real numbers. The **image** of S under f is the set of values taken on by f among the points of S: $f(S) \coloneqq \{f(s) \mid s \in S\}$. This is a set of real numbers, and it is easy to see (Exercise 11) that, for example, a lower bound for the function f on the set S is the same as a lower bound for the image of S under f.

A useful observation for determining the infimum or supremum of a set or function is:

Remark 3.7.2. *If α (resp. β) is a lower (resp. upper) bound for f on S, then it equals $\inf_{x\in S} f(x)$ (resp. $\sup_{x\in S} f(x)$) precisely if there exists a sequence of points $s_k \in S$ such that $f(s_k) \to \alpha$ (resp. $f(s_k) \to \beta$).*

The Extreme Value Theorem. A basic result in single-variable calculus is the Extreme Value Theorem, which says that a continuous function achieves its maximum and minimum on any closed, bounded interval $[a, b]$. We wish to extend this result to real-valued functions defined on subsets of $\mathbb{R}^{2 \text{ or } 3}$. First, we need to set up some terminology.

Definition 3.7.3. *A set $S \subset \mathbb{R}^{2 \text{ or } 3}$ of points in $\mathbb{R}^{2 \text{ or } 3}$ is **closed** if for any convergent sequence s_i of points in S, the limit also belongs to S:*

$$s_i \to L \text{ and } s_i \in S \text{ for all } i \Rightarrow L \in S.$$

It is an easy exercise (Exercise 9) to show that each of the following are examples of closed sets:

(1) closed intervals $[a, b]$ in $\mathbb{R}$, as well as half-closed intervals of the form $[a, \infty)$ or $(-\infty, b]$;
(2) level sets $\mathcal{L}(g, c)$ of a continuous function g, as well as sets defined by weak inequalities like $\{x \in \mathbb{R}^{2 \text{ or } 3} \mid g(x) \le c\}$ or $\{x \in \mathbb{R}^{2 \text{ or } 3} \mid g(x) \ge c\}$;
(3) any set consisting of a convergent sequence s_i together with its limit, or any set consisting of a sequence together with all of its accumulation points.

We also want to formulate the idea of a bounded set in $\mathbb{R}^{2 \text{ or } 3}$. We cannot talk about such a set being "bounded above" or "bounded below"; the appropriate definition is:

Definition 3.7.4. *A set $S \subset \mathbb{R}^{2 \text{ or } 3}$ is **bounded** if the set of distances from the origin to elements of S $\{\|s\| \mid s \in S\}$ is bounded—that is, if there exists $M \in \mathbb{R}$ such that*

$$\|s\| \le M \text{ for all } s \in S.$$

(This is the same as saying that there exists some ball $B_\varepsilon(\mathcal{O})$—where $\varepsilon = M$ is in general not assumed small—that contains S.)

A basic and important property of $\mathbb{R}^{2 \text{ or } 3}$ is stated in the following.

Proposition 3.7.5. *For a subset $S \subset \mathbb{R}^{2 \text{ or } 3}$, the following are equivalent:*

(1) *S is closed and bounded;*
(2) *S is **sequentially compact**: every sequence s_i of points in S has a subsequence that converges to a point of S.*

We shall abuse terminology and refer to such sets as **compact** sets.[14]

[14]The property of being *compact* has a specific definition in very general settings; however, in the context of $\mathbb{R}^{2 \text{ or } 3}$, this is equivalent to either sequential compactness or being closed and bounded.

Proof. If S is bounded, then by the Bolzano-Weierstrass Theorem (Proposition 2.3.7) every sequence in S has a convergent subsequence, and if S is also closed, then the limit of this subsequence must also be a point of S.

Conversely, if S is *not bounded*, it cannot be sequentially compact since there must exist a sequence s_k of points in S with $\|s_k\| > k$ for $k = 1, 2, \ldots$; such a sequence has no convergent subsequence. Similarly, if S is *not closed*, there must exist a convergent sequence s_k of points in S whose limit L lies outside S; since every subsequence also converges to L, S cannot be sequentially compact. □

With these definitions, we can formulate and prove the following.

Theorem 3.7.6 (Extreme Value Theorem). *If $S \subset \mathbb{R}^{2\,or\,3}$ is compact, then every real-valued function f that is continuous on S achieves its minimum and maximum on S.*

Note that this result includes the Extreme Value Theorem for functions of one variable, since closed intervals are compact, but even in the single variable setting, it applies to functions continuous on sets more general than intervals.

Proof. The strategy of this proof is: first, we show that f must be bounded on S, and second, we prove that there exists a point $s \in S$ where $f(s) = \sup_{x \in S} f(x)$ (*resp.* $f(s) = \inf_{x \in S} f(x)$).[15]

Step 1: *$f(x)$ is bounded on S:* Suppose $f(x)$ is *not* bounded on S: this means that there exist points in S at which $|f(x)|$ is arbitrarily high: thus we can pick a sequence $s_k \in S$ with $|f(s_k)| > k$. Since S is (sequentially) compact, we can find a subsequence—which without loss of generality can be assumed to be the whole sequence—that converges to a point of S: $s_k \to s_0 \in S$. Since $f(x)$ is continuous on S, we must have $f(s_k) \to f(s_0)$; but this contradicts the assumption that $|f(s_k)| > k$.

Step 2: *$f(x)$ achieves its maximum and minimum on S:* We will show that $f(x)$ achieves its maximum on S; the case of the minimum is entirely analogous. Since $f(x)$ is bounded on S, the set of values on S has a supremum, say $\sup_{x \in S} f(x) = A$; by the remarks in Definition 3.7.1, there exists a sequence $f(s_i)$ converging to A, where s_i all belong to S; pick a subsequence of s_i which converges to $s_0 \in S$; by continuity $f(s_0) = A$ and we are done. □

Local Extrema. How do we find the extreme values of a function on a set? For a function of one variable on an interval, we looked for local extrema interior to the interval and compared them to the values at the ends. Here we need to formulate the analogous notions. The following is the natural higher-dimension analogue of local extrema for single-variable functions.

Definition 3.7.7. *The function $f(x)$ has a **local maximum** (resp. **local minimum**) at $\overrightarrow{x_0} \in \mathbb{R}^{2\,or\,3}$ if there exists a ball $B_\varepsilon\left(\overrightarrow{x_0}\right)$, $\varepsilon > 0$, such that*

(1) $f(x)$ is defined on all of $B_\varepsilon\left(\overrightarrow{x_0}\right)$; and

(2) $f(x)$ achieves its maximum (resp. minimum) on $B_\varepsilon\left(\overrightarrow{x_0}\right)$ at $\vec{x} = \overrightarrow{x_0}$.

*A **local extremum** of $f(x)$ is a local maximum or local minimum.*

To handle sets more complicated than intervals, we need to formulate the analogues of interior points and endponts.

Definition 3.7.8. *Let $S \subset \mathbb{R}^{2\,or\,3}$ be any set in $\mathbb{R}^{2\,or\,3}$.*

[15] A somewhat different proof, based on an idea of Daniel Reem, is worked out in Exercise 14.

(1) *A point $\vec{x} \in \mathbb{R}^{2\,or\,3}$ is an **interior point** of S if S contains some ball about $\vec{x}$: $B_\varepsilon(\vec{x}) \subset S$ or in other words all points within distance ε of $\vec{x}$ belong to S. The set of all interior points of S is called the **interior** of S, denoted* int S.

*A set S is **open** if every point is an interior point: $S = \text{int}\, S$.*

(2) *A point $\vec{x} \in \mathbb{R}^{2\,or\,3}$ is a **boundary point** of S if every ball $B_\varepsilon(\vec{x})$, $\varepsilon > 0$ contains points* in *S as well as points* not in *S: both $B_\varepsilon(\vec{x}) \cap S$ and $B_\varepsilon(\vec{x}) \setminus S$ are nonempty. The set of boundary points of S is called the **boundary** and denoted ∂S.*

The following are relatively easy observations (Exercise 10):

Remark 3.7.9. (1) *For any set $S \subset \mathbb{R}^{2\,or\,3}$, $S \subseteq (\text{int}\, S) \cup (\partial S)$.*

(2) *The boundary ∂S of any set is closed.*

(3) *S is closed precisely if it contains its boundary points: S closed $\Leftrightarrow \partial S \subset S$.*

(4) *$S \subset \mathbb{R}^{2\,or\,3}$ is* closed *precisely if its complement, $\mathbb{R}^{2\,or\,3} \setminus S := \{x \in \mathbb{R}^{2\,or\,3} \mid x \notin S\}$, is* open.

The lynchpin of our strategy for finding extrema in the case of single-variable functions was that every local extremum is a critical point, and in most cases there are only finitely many of these. The analogue for our present situation is the following.

Theorem 3.7.10 (Critical Point Theorem). *If $f : \mathbb{R}^3 \to \mathbb{R}$ has a local extremum at $\vec{x} = \overrightarrow{x_0}$ and is differentiable there, then $\overrightarrow{x_0}$ is a critical point of $f(\vec{x})$:*

$$\vec{\nabla} f(\overrightarrow{x_0}) = \vec{0}.$$

Proof. If $\vec{\nabla} f(\overrightarrow{x_0})$ is not the zero vector, then some partial derivative, say $\frac{\partial f}{\partial x_j}$, is nonzero. But this means that along the line through $\overrightarrow{x_0}$ parallel to the x_j-axis, the function is locally monotone:

$$\frac{d}{dt}\left[f(\overrightarrow{x_0} + t\vec{e_j})\right] = \frac{\partial f}{\partial x_j}(\overrightarrow{x_0}) \neq 0$$

means that there are nearby points where the function exceeds, and others where it is less than, the value at $\overrightarrow{x_0}$; therefore $\overrightarrow{x_0}$ is *not* a local extreme point of $f(\vec{x})$. □

Finding Extrema. Putting all this together, we can formulate a strategy for finding the extreme values of a function on a subset of $\mathbb{R}^{2\text{ or }3}$, analogous to the strategy used in single-variable calculus:

Given a function $f(\vec{x})$ defined on the set $S \subset \mathbb{R}^{2\text{ or }3}$, search for extreme values as follows:

(1) **Critical Points:** Locate all the critical points of $f(\vec{x})$ interior to S, and evaluate $f(\vec{x})$ at each.

(2) **Boundary Behavior:** Find the maximum and minimum values of $f(\vec{x})$ on the boundary ∂S; if the set is unbounded, study the limiting values as $\left\|\vec{x}\right\| \to \infty$ in S.

(3) **Comparison:** Compare these values: the lowest (*resp.* highest) of all the values is the infimum (*resp.* supremum), and if the point at which it is achieved lies in S, it is the minimum (*resp.* maximum) value of f on S.

In practice, this strategy is usually applied to sets of the form $S = \{\vec{x} \in \mathbb{R}^{2\text{ or }3} | g(\vec{x}) \leq c\}$. We consider a few examples.

First, let us find the maximum and minimum of the function $f(x, y) = x^2 - 2x + y^2$ inside the disc of radius 2; that is, among points satisfying $x^2 + y^2 \leq 4$. (See Figure 3.21.)

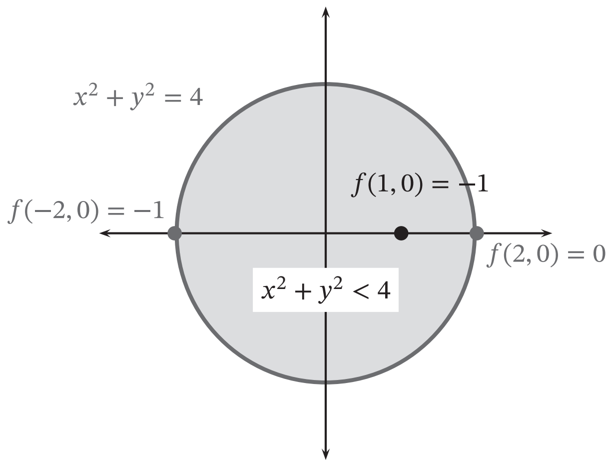


Figure 3.21. Critical Points and Boundary Behavior of $f(x,y) = x^2-2x+y^2$ on $\{(x,y) \mid x^2+y^2 \le 4\}$

Critical Points: $\vec{\nabla} f(x,y) = (2x-2)\vec{\imath} + 2y\vec{\jmath}$ vanishes only for $x = 1$ and $y = 0$, and the value of $f(x,y)$ at the critical point $(1,0)$ (which lies inside the disc), is $f(1,0) = 1-2+0 = -1$.

Boundary Behavior: The boundary is the circle of radius 2, given by $x^2+y^2 = 4$, which we can parametrize as

$$x = 2\cos\theta, \quad y = 2\sin\theta$$

so the function restricted to the boundary can be written $g(\theta) = f(2\cos\theta, 2\sin\theta) = 4-4\cos\theta$. To find the extrema of this, we can either use common sense (how?) or take the derivative: $\frac{dg}{d\theta} = 4\sin\theta$. This vanishes when $\theta = 0$ or π. The values at these places are $g(0) = 0$ and $g(\pi) = 8$.

Comparison: Since $-1 < 0 < 8$, we can conclude that $\max_{x^2+y^2\le 4} x^2-2x+y^2 = 8 = g(\pi) = f(-2,0)$, while $\min_{x^2+y^2\le 4} x^2-2x+y^2 = -1 = f(1,0)$.

Next, let's find the extreme values of the same function on the *unbounded* set (see Figure 3.22) defined by $x \le y$:

Critical Points: The lone critical point $(1,0)$ lies *outside* the set, so all the extreme behavior is "at the boundary".

Boundary Behavior: There are two parts to this: first, we look at the behavior on the boundary points of S, which is the line $x = y$. Along this line we can write $g(x) = f(x,x) = 2x^2-2x$; the derivative $g'(x) = 4x-2$ vanishes at $x = \frac{1}{2}$ and the value there is $g\left(\frac{1}{2}\right) = f\left(\frac{1}{2},\frac{1}{2}\right) = -\frac{1}{2}$.

Behavior "at Infinity": But we also need to consider what happens when $\|(x,y)\| \to \infty$ in our set. It is easy to see that for *any* point (x,y), $f(x,y) \ge x^2-2x \ge -1$, and also that $x^2-2x \to \infty$ if $|x| \to \infty$. For any sequence (x_j,y_j) with $\left\|(x_j,y_j)\right\| \to \infty$, either $|x| \to \infty$ (so $f(x,y) \ge x^2-2x \to \infty$) or $|y| \to \infty$ (so $f(x,y) \ge y^2-1 \to \infty$);

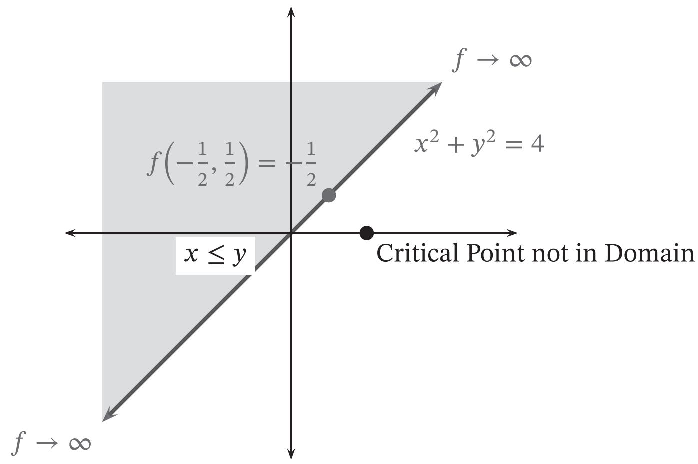


Figure 3.22. Critical Points and Boundary Behavior of $f(x, y) = x^2 - 2x + y^2$ on $\{(x, y) \mid x \leq y\}$

in either case, $f(x_j, y_j) \to \infty$. Since there exist such sequences with $x_j \leq y_j$, the function is not bounded above.

Comparison: Now, if $\vec{s_i} = (x_i, y_i)$ is a sequence with $x_i \leq y_i$ and $f(\vec{s_i}) \to \inf_{x \leq y} f(x, y)$, either $\vec{s_i}$ have no convergent subsequence, and hence $\left\|\vec{s_i}\right\| \to \infty$, or some accumulation point of $\vec{s_i}$ is a local minimum for f. The first case is impossible, since we already know that then $f(\vec{s_i}) \to \infty$, while in the second case this accumulation point must be $\left(\frac{1}{2}, \frac{1}{2}\right)$, and then $f(\vec{s_i}) \to -\frac{1}{2}$. From this it follows that

$$\min_{x \leq y}(x^2 - 2x + y^2) = -\frac{1}{2} = f\left(\frac{1}{2}, \frac{1}{2}\right).$$

Lagrange Multipliers. For problems in two variables, the boundary is a curve, which can often be parametrized, so that the problem of optimizing the function on the boundary is reduced to a one-variable problem. However, when three or more variables are involved, the boundary can be much harder to parametrize. Fortunately, there is an alternative approach, pioneered by Joseph Louis Lagrange (1736-1813) in connection with isoperimetric problems (for example, find the triangle of greatest area with a fixed perimeter). [16]

The method is applicable to problems of the form: find the extreme values of the function $f(\vec{x})$ on a level set $\mathcal{L}(g, c)$ of the differentiable function $g(\vec{x})$ containing no critical points of g (we call c a **regular value** of $g(\vec{x})$ if $\vec{\nabla} g(\vec{x}) \neq \vec{0}$ whenever $g(\vec{x}) = c$). These are sometimes called **constrained extremum** problems.

The idea is this: suppose the function $f(\vec{x})$ *when restricted to the level set* $\mathcal{L}(g, c)$ has a local maximum at $\vec{x_0}$: this means that, while it might be possible to find nearby points where the function takes values higher than $f(\vec{x_0})$, they cannot lie on the level set. Thus, we are interested in finding those points for which the function has a local

[16] According to [49, pp. 169-170], when Lagrange communicated his method to Euler in 1755 (at the age of 18!), the older master was so impressed that he delayed publication of some of his own work on inequalities to give the younger mathematician the credit he was due for this elegant method.

maximum along any curve through the point *which lies in the level set*. Suppose that $\vec{p}(t)$ is such a curve; that is, we are assuming that $g(\vec{p}(t)) = c$ for all t, and that $\vec{p}(0) = \overrightarrow{x_0}$. In order for $f(\vec{p}(t))$ to have a local maximum at $t = 0$, the derivative must vanish—that is, $0 = \frac{d}{dt}\Big|_{t=0}\left[f(\vec{p}(t))\right] = \vec{\nabla} f(\overrightarrow{x_0}) \cdot \vec{v}$, where $\vec{v} = \dot{\vec{p}}(0)$ is the velocity vector of the curve as it passes $\overrightarrow{x_0}$: the velocity must be perpendicular to the gradient of f. This must be true for *any* curve in the level set as it passes through $\overrightarrow{x_0}$, which is the same as saying that it must be true for any vector in the plane tangent to the level set $\mathcal{L}(g, c)$ at $\overrightarrow{x_0}$: in other words, $\vec{\nabla} f(\overrightarrow{x_0})$ must be normal to this tangent plane. But we already know that the gradient of g is normal to this tangent plane; thus the two gradient vectors must point along the same line—they must be linearly dependent! This proves (see Figure 3.23).

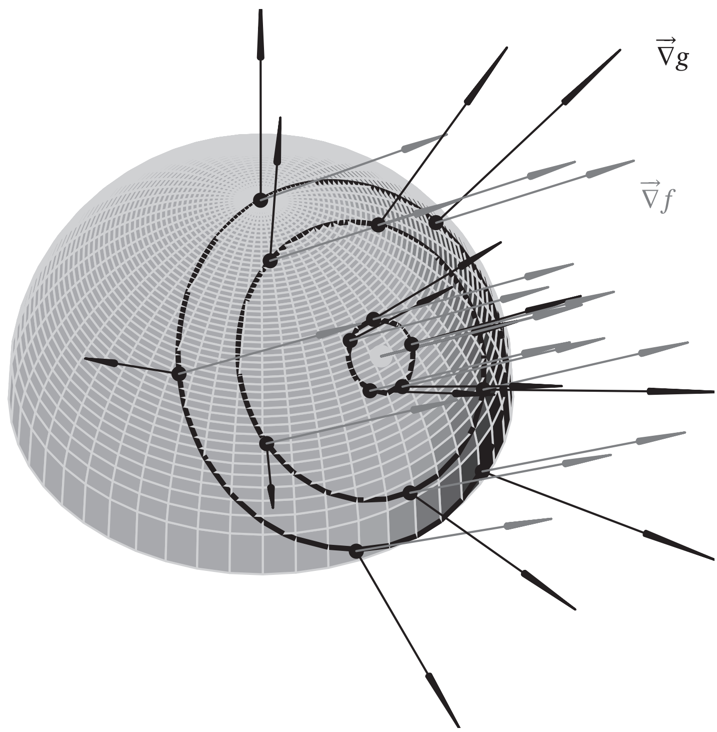


Figure 3.23. The Geometry of Lagrange Multipliers

Proposition 3.7.11 (Lagrange Multipliers). *If $\overrightarrow{x_0}$ is a local extreme point of the restriction of the function $f(\vec{x})$ to the level set $\mathcal{L}(g, c)$ of the function $g(\vec{x})$, and c is a regular value of g, then $\vec{\nabla} f(\overrightarrow{x_0})$ and $\vec{\nabla} g(\overrightarrow{x_0})$ must be linearly dependent:*

$$\vec{\nabla} f(\overrightarrow{x_0}) = \lambda \vec{\nabla} g(\overrightarrow{x_0}) \tag{3.27}$$

for some real number λ.

The number λ is called a **Lagrange multiplier**. We have formulated the linear dependence of the gradients as $\vec{\nabla} f$ being a multiple of $\vec{\nabla} g$, rather than the other way around, because we assume that $\vec{\nabla} g$ is nonvanishing, while this formulation allows $\vec{\nabla} f$ to vanish—that is, this equation holds automatically if $\overrightarrow{x_0}$ is a genuine critical point of f. We will refer to this weaker situation by saying $\overrightarrow{x_0}$ is a **relative critical point** of $f(\vec{x})$—that is, it is critical relative to the constraint $g(\vec{x}) = c$.

To see this method in practice, we consider a few examples.

First, let us find the extreme values of $f(x, y, z) = x - y + z$ on the sphere defined by $x^2 + y^2 + z^2 = 4$ (see Figure 3.24). We have $\vec{\nabla} f(x, y, z) = \vec{\imath} - \vec{\jmath} + \vec{k}$ and $g(x, y, z) =$

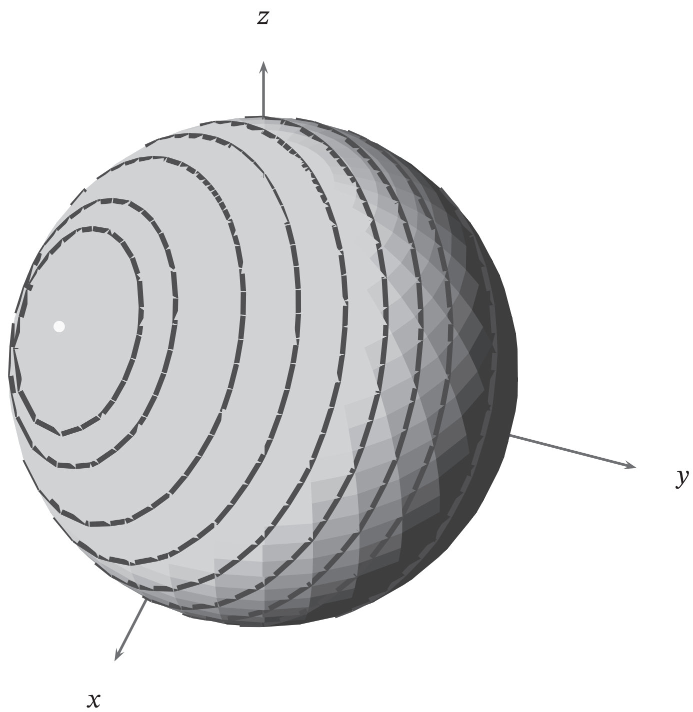


Figure 3.24. Level Curves of $f(x, y, z) = x - y + z$ on the Sphere $x^2 + y^2 + z^2 = 4$

$x^2 + y^2 + z^2$, so $\vec{\nabla} g(x, y, z) = 2x\vec{\imath} + 2y\vec{\jmath} + 2z\vec{k}$. The Lagrange Multiplier equation $\vec{\nabla} f(\overrightarrow{x_0}) = \lambda \vec{\nabla} g(\overrightarrow{x_0})$ amounts to the three scalar equations

$$
\begin{aligned}
1 &= 2\lambda x \\
-1 &= 2\lambda y \\
1 &= 2\lambda z
\end{aligned}
$$

which constitute 3 equations in 4 unknowns; a fourth equation is the specification that we are on $\mathcal{L}(g, 4)$,

$$x^2 + y^2 + z^2 = 4.$$

Note that none of the four variables can equal zero (why?), so we can rewrite the three Lagrange equations in the form

$$x = \frac{1}{2\lambda}$$
$$y = -\frac{1}{2\lambda}$$
$$z = \frac{1}{2\lambda}.$$

Substituting this into the fourth equation, we obtain $\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = 4$ or $3 = 16\lambda^2$, so

$$\lambda = \pm\frac{\sqrt{3}}{4}.$$

This yields two relative critical points: $\lambda = \frac{\sqrt{3}}{4}$ gives the point $\left(\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$ where $f = 2\sqrt{3}$, while $\lambda = -\frac{\sqrt{3}}{4}$ gives the point $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$ where $f = -2\sqrt{3}$. Thus,

$$\max_{x^2+y^2+z^2} f(x, y, z) = f\left(\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) = 2\sqrt{3}$$
$$\min_{x^2+y^2+z^2} f(x, y, z) = f\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right) = -2\sqrt{3}.$$

As a second example, let us find the point on the surface $xyz = 1$ closest to the origin. We characterize the surface as $\mathcal{L}(g, 1)$, where $g(x, y, z) = xyz$, and $\vec{\nabla} g(x, y, z) = (yz, xz, xy)$. As is usual in distance-optimizing problems, it is easier to work with the square of the distance; this is minimized at the same place(s) as the distance, so we take $f(x, y, z) = \text{dist}((x, y, z), (0, 0, 0))^2 = x^2+y^2+z^2$, with $\vec{\nabla} f(x, y, z) = (2x, 2y, 2z)$. (See Figure 3.26.)

The Lagrange Multiplier Equation $\vec{\nabla} f = \lambda\vec{\nabla} g$ reads

$$2x = \lambda yz$$
$$2y = \lambda xz$$
$$2z = \lambda xy.$$

Note first that if $xyz = 1$, all three coordinates must be nonzero. Thus, we can solve each of these equations for λ:

$$\lambda = \frac{2x}{yz}$$
$$\lambda = \frac{2y}{xz}$$
$$\lambda = \frac{2z}{xy}.$$

We can eliminate λ—whose value is of no direct importance to us—by setting the three right-hand sides equal: $\frac{2x}{yz} = \frac{2y}{xz} = \frac{2z}{xy}$. Cross-multiplying the first equation yields $2x^2z = 2y^2z$ and since $z \neq 0$ (why?) $x^2 = y^2$. Similarly, we cross-multiply the second equation to get $y^2 = z^2$. In particular, all three have the same absolute value, so $|x|^3 = 1$ implies $|x| = |y| = |z| = 1$ and an even number of the variables can be negative. This yields four relative critical points, at all of which $f(x, y, z) = 3$: $(1, 1, 1)$, $(1, -1, -1)$,

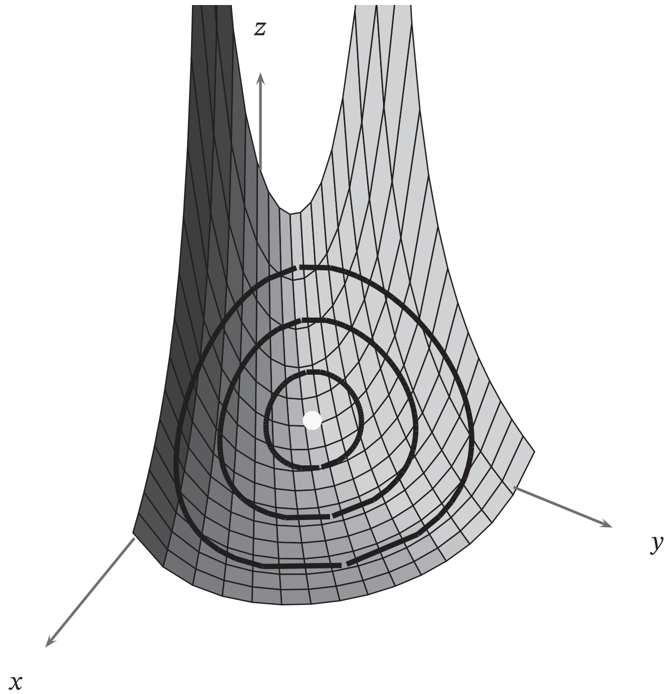


Figure 3.25. Level Curves of $f(x, y, z) = x^2 + y^2 + z^2$ on the surface $xyz = 1$

$(-1, -1, 1)$, and $(-1, 1, -1)$. To see that they are the closest (not the furthest) from the origin, simply note that there are points on this surface arbitrarily far from the origin, so the distance to the origin is not bounded above.

Finally, let us consider a "full" optimization problem: to find the extreme values of $f(x, y, z) = 2x^2 + y^2 - z^2$ inside the unit ball $x^2 + y^2 + z^2 \leq 1$ (see Figure 3.26).

Critical Points: The partials of f, $\frac{\partial f}{\partial x} = 4x$, $\frac{\partial f}{\partial y} = 2y$, and $\frac{\partial f}{\partial z} = 2z$, all vanish only at the origin, and $f(0, 0, 0) = 0$.

Boundary Behavior: To find the extrema on the boundary, we use Lagrange multipliers. The two gradients are $\vec{\nabla} f(x, y, z) = 4x\vec{\imath} + 2y\vec{\jmath} - 2z\vec{k}$ and $\vec{\nabla} g(x, y, z) = 2x\vec{\imath} + 2y\vec{\jmath} + 2z\vec{k}$; the Lagrange Multiplier Equations read

$$\begin{aligned} 4x &= 2\lambda x \\ 2y &= 2\lambda y \\ -2z &= 2\lambda z. \end{aligned}$$

The first equation tells us that *either* $\lambda = 2$ *or* $x = 0$; the second says that *either* $\lambda = 1$ *or* $y = 0$, while the third says that *either* $\lambda = -1$ *or* $z = 0$. Since only one of the three named λ-values can hold, two of the coordinates must be zero, which means in terms of the constraint that the third is ± 1. Thus we have six relative

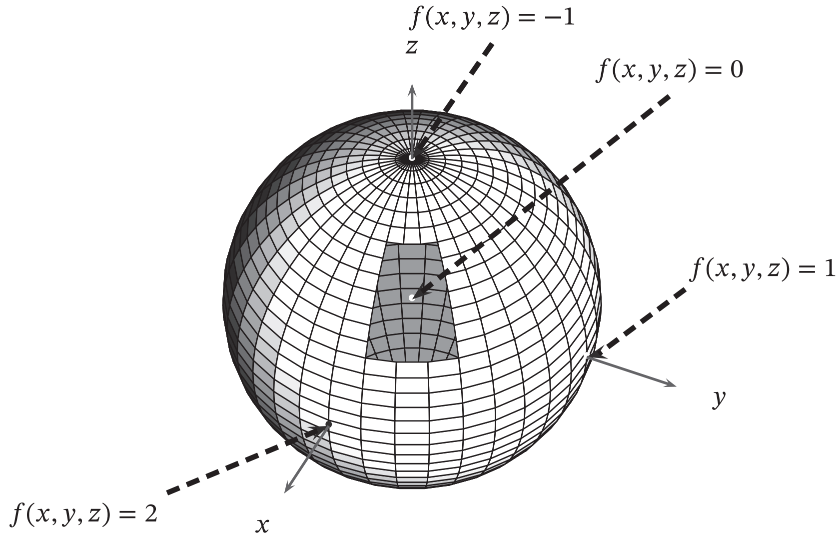


Figure 3.26. Critical Points of $f(x, y) = 2x^2 + y^2 - z^2$ inside the Ball $x^2 + y^2 + z^2 \le 1$

critical points, with respective f-values

$$f(\pm 1, 0, 0) = 2$$
$$f(0, \pm 1, 0) = 1$$
$$f(0, 0, \pm 1) = -1.$$

Comparison: Combining this with the critical value 0 at the origin, we have

$$\min_{x^2+y^2+z^2 \le 1} (2x^2 + y^2 - z^2) = f(0, 0, \pm 1) = -1$$
$$\max_{x^2+y^2+z^2 \le 1} (2x^2 + y^2 - z^2) = f(\pm 1, 0, 0) = 2.$$

Multiple Constraints (Optional). The method of Lagrange Multipliers can be extended to problems in which there is more than one constraint present. We illustrate this with a single example, involving two constraints.

The intersection of the cylinder $x^2 + y^2 = 4$ with the plane $2x + 2y + 2z = 2$ is an ellipse; we wish to find the points on this ellipse nearest and farthest from the origin. Again, we will work with the square of the distance from the origin, $f(x, y, z) = x^2 + y^2 + z^2$, with gradient $\vec{\nabla} f(x, y, z) = (2x, 2y, 2z)$. We are looking for the extreme values of this function on the curve of intersection of two level surfaces. In principle, we could parametrize the ellipse, but instead we will work directly with the constraints and their gradients: $g_1(x, y, z) = x^2 + y^2$, with gradient $\vec{\nabla} g_1 = (2x, 2y, 0)$, and $g_2(x, y, z) = 2x + 2y + 2z$, with gradient $\vec{\nabla} g_2 = (2, 2, 2)$. Since our curve lies in the intersection of the two level surfaces $\mathcal{L}(g_1, 4)$ and $\mathcal{L}(g_2, 1)$, its velocity vector must be perpendicular to both gradients: $\vec{v} \cdot \vec{\nabla} g_1 = 0$ and $\vec{v} \cdot \vec{\nabla} g_2 = 0$. At a place where the restriction of f to this curve achieves a local (relative) extremum, the velocity must also be perpendicular to the gradient of f: $\vec{v} \cdot \vec{\nabla} f = 0$. But the two gradient vectors $\vec{\nabla} g_1$ and $\vec{\nabla} g_2$ are linearly independent, and hence span the plane perpendicular to $\vec{v}$. It follows that $\vec{\nabla} f$ must lie in this plane, or stated differently, it must be a linear combination of the $\vec{\nabla} g$'s: $\vec{\nabla} f = \lambda_1 \vec{\nabla} g_1 + \lambda_2 \vec{\nabla} g_2$. (See Figure 3.27.)

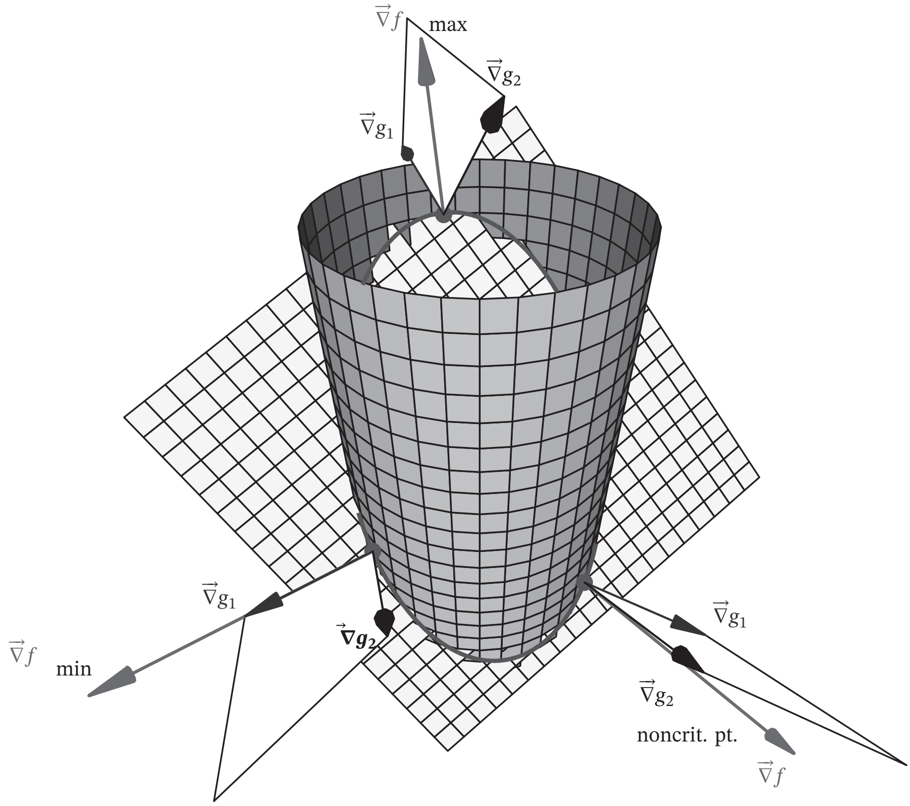


Figure 3.27. Lagrange Multipliers with Two Constraints

Written out, this gives us three equations in the five unknowns x, y, z, λ_1, and λ_2:

$$\begin{aligned} 2x &= 2\lambda_1 x + \lambda_2 \\ 2y &= 2\lambda_1 y + \lambda_2 \\ 2z &= \lambda_2. \end{aligned}$$

The other two equations are the constraints:

$$\begin{aligned} x^2 + y^2 &= 4 \\ x + y + z &= 1. \end{aligned}$$

We can solve the first three equations for λ_2 and eliminate it:

$$(1 - \lambda_1)x = (1 - \lambda_1)y = 2z.$$

The first of these equalities says that either $\lambda_1 = 1$ or $x = y$. If $\lambda_1 = 1$, then the second equality says that $z = 0$, so $y = 1 - x$. In this case the first constraint gives us $x^2 + (1 - x)^2 = 4$, that is $2x^2 - 2x - 3 = 0$, so $x = \frac{1}{2}(1 \pm \sqrt{7}) = y$, yielding two relative critical points, at which the function f has value $f\left(\frac{1}{2}(1 \pm \sqrt{7}), \frac{1}{2}(1 \mp \sqrt{7}), 0\right) = \frac{9}{4}$. If $x = y$, then the first constraint tells us that $x^2 + x^2 = 4$, so $x = y = \pm\sqrt{2}$, and then the second constraint says $z = 1 - 2x = 1 \mp 2\sqrt{2}$, yielding another pair of relative critical points, with respective values for f $f\left(\sqrt{2}, \sqrt{2}, 1 - 2\sqrt{2}\right) = 13 - 4\sqrt{2}$ and $f\left(-\sqrt{2}, -\sqrt{2}, 1 + 2\sqrt{2}\right) = 13 + 4\sqrt{2}$. Comparing these various values, we see that the point farthest from the origin is $(-\sqrt{2}, -\sqrt{2}, 1 + 2\sqrt{2})$ and the closest are the two points $\left(\frac{1}{2}(1 \pm \sqrt{7}), \frac{1}{2}(1 \mp \sqrt{7}), 0\right)$.

Exercises for § 3.7

Answer to Exercise 1 is given in Appendix A.13.

Practice problems:

(1) Find the minimum and maximum values of $f(x, y) = x^2 + xy + 2y^2$ inside the unit disc $x^2 + y^2 \leq 1$.
(2) Find the minimum and maximum values of $f(x, y) = x^2 - xy + y^2$ inside the disc $x^2 + y^2 \leq 4$.
(3) Find the minimum and maximum values of $f(x, y) = x^2 - xy + y^2$ inside the elliptic disc $x^2 + 4y^2 \leq 4$.
(4) Find the minimum and maximum values of $f(x, y) = \sin x \sin y \sin(x + y)$ inside the square $0 \leq x \leq \pi, 0 \leq y \leq \pi$.
(5) Find the minimum and maximum values of $f(x, y) = (x^2 + 2y^2)e^{-(x^2+y^2)}$ in the plane.
(6) Find the minimum and maximum values of $f(x, y, z) = xyz$ on the sphere $x^2 + y^2 + z^2 = 1$.
(7) Find the point on the sphere $x^2 + y^2 + z^2 = 1$ which is farthest from the point $(1, 2, 3)$.
(8) Find the rectangle of greatest perimeter inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Theory problems:

(9) Show that each of the following is a closed set, according to Definition 3.7.3:
   (a) Any closed interval $[a, b]$ in $\mathbb{R}$;
   (b) any half-closed interval of the form $[a, \infty)$ or $(-\infty, b]$;
   (c) any level set $\mathcal{L}(g, c)$ of a continuous function g;
   (d) any set defined by weak inequalities like $\{x \in \mathbb{R}^3 \mid g(x) \leq c\}$ or $\{x \in \mathbb{R}^3 | g(x) \geq c\}$;
(10) Prove Remark 3.7.9:
   (a) For any set $S \subset \mathbb{R}^3$, $S \subseteq \operatorname{int} S \cup \partial S$.
   (b) The boundary ∂S of any set is closed.
   (c) S is closed precisely if it contains its boundary points: S closed $\Leftrightarrow \partial S \subset S$.
   (d) $S \subset \mathbb{R}^3$ is *closed* precisely if its complement $\mathbb{R}^3 \setminus S := \{x \in \mathbb{R}^3 \mid x \notin S\}$ is *open*.
(11) Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a function and S is a subset of its domain. Recall that the image of S under f is the set $f(S) := \{y \in \mathbb{R} \mid y = f(x) \text{ for some } x \in S\}$. Explain why each statement below (relating inequalities for the function f on S to inequalities for the set $f(S)$) is true. (These are a matter of chasing definitions, but there are a few subtleties.)

(a) $\alpha \in \mathbb{R}$ is a lower bound for f on S if and only if α is a lower bound for $f(S)$.
(b) f is bounded below on S if and only if $f(S)$ has a (finite) lower bound.
(c) $\inf_S f = \inf(f(S))$.
(d) f achieves its minimum on S if and only if $f(S)$ has a minimum.

Challenge problems:

(12) (a) Show that any set consisting of a *convergent* sequence s_i together with its limit is a closed set;
(b) Show that any set consisting of a (not necessarily convergent) sequence together with all of its accumulation points is a closed set.

(13) Prove that if $\alpha, \beta > 0$ satisfy $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then for all $x, y \geq 0$

$$xy \leq \frac{1}{\alpha}x^{\alpha} + \frac{1}{\beta}y^{\beta}$$

as follows:
(a) The inequality is clear for $xy = 0$, so we can assume $xy \neq 0$.
(b) If it is true (given α and β) for a given pair (x, y), then it is also true for the pair $(t^{1/\alpha}x, t^{1/\beta}y)$ (verify this!), and so we can assume without loss of generality that $xy = 1$
(c) Prove the inequality in this case by minimizing

$$f(x, y) = \frac{1}{\alpha}x^{\alpha} + \frac{1}{\beta}y^{\beta}$$

over the hyperbola $xy = 1$.

(14) Here is a somewhat different proof of Theorem 3.7.6, based on an idea of Daniel Reem [45]. Suppose $S \subset \mathbb{R}^3$ is compact.
(a) Show that for every integer $k = 1, 2, \ldots$ there is a *finite* subset $S_k \subset S$ such that for every point $x \in S$ there is at least one point in S_k whose coordinates differ from those of x by at most 10^{-k}. In particular, for every $x \in S$ there is a sequence of points $\{x_k\}_{k=1}^{\infty}$ such that $x_k \in S_k$ for $k = 1, \ldots$ and $x = \lim x_k$.
(b) Show that these sets can be picked to be nested: $S_k \subset S_{k+1}$ for all k.
(c) Now, each of the sets S_k is finite, so f has a minimum $\min_{s \in S_k} f(s) = f(m_k)$ and a maximum $\max_{s \in S_k} f(s) = f(M_k)$. Show that $f(m_k) \geq f(m_{k+1})$ and $f(M_k) \leq f(M_{k+1})$.
(d) Also, by the Bolzano-Weierstrass Theorem, each of the sequences $\{m_k\}_{k=1}^{\infty}$ and $\{M_k\}_{k=1}^{\infty}$ has a convergent subsequence. Let m (*resp.* M) be the limit of such a subsequence. Show that $m, M \in S$ and

$$f(m) = \inf f(m_k) = \lim f(m_k)$$
$$f(M) = \sup f(M_k) = \lim f(M_k).$$

(e) Finally, show that $f(m) \leq f(x) \leq f(M)$ for every $x \in S$, as follows: given $x \in S$, by part (a), there is a sequence $x_k \to x$ with $x_k \in S_k$. Thus,

$$f(m_k) \leq f(x_k) \leq f(M_k)$$

and so by properties of limits (which?) the desired conclusion follows.

(15) Suppose $\vec{a}$ satisfies $f(\vec{a}) = b$ and $g(\vec{a}) = c$ and is not a critical point of either function; suppose furthermore that $\vec{\nabla} g \neq \vec{0}$ everywhere on the level set $\mathcal{L}(g, c)$ (that is, c is a regular value of g), and $\max_{\mathcal{L}(g,c)} f(x) = b$.

(a) Show that $\mathcal{L}(f,b)$ and $\mathcal{L}(g,c)$ are tangent at $\vec{a}$.
(b) As a corollary, show that the restriction of $g(\vec{x})$ to $\mathcal{L}(f,b)$ has a local extremum at $\vec{x}=\vec{a}$.

3.8 Higher Derivatives

For a function of one variable, the higher-order derivatives give more subtle information about the function near a point: while the first derivative specifies the "tilt" of the graph, the second derivative tells us about the way the graph curves, and so on. Specifically, the second derivative can help us decide whether a given critical point is a local maximum, local minimum, or neither.

In this section we develop the basic theory of higher-order derivatives for functions of several variables, which can be a bit more complicated than the single-variable version. Most of our energy will be devoted to second-order derivatives.

Higher-order Partial Derivatives. The partial derivatives of a function of several variables are themselves functions of several variables, and we can try to find *their* partial derivatives. Thus, if $f(x,y)$ is differentiable, it has two first-order partials

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}$$

and, if they are also differentiable, each has two partial derivatives, which are the **second-order partials** of f:

$$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right]=\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]=\frac{\partial^2 f}{\partial y\partial x},$$
$$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]=\frac{\partial^2 f}{\partial x\partial y}, \quad \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right]=\frac{\partial^2 f}{\partial y^2}.$$

In subscript notation, the above would be written

$$(f_x)_x = f_{xx}, \quad (f_x)_y = f_{xy},$$
$$(f_y)_x = f_{yx}, \quad (f_y)_y\, f_{yy}.$$

Notice that in the "partial" notation, the order of differentiation is right-to-left, while in the subscript version it is left-to-right. (We shall see shortly that for $\mathcal{C}^2$ functions, this is not an issue.)

For example, the function $f(x,y)=x^2+2xy+y-1+xy^3$ has first-order partials

$$f_x=\frac{\partial f}{\partial x}=2x+2y+y^3, \quad f_y=\frac{\partial f}{\partial y}=2x+1+3xy^2$$

and second-order partials

$$f_{xx}=\frac{\partial^2 f}{\partial x^2}=2, \quad f_{xy}=\frac{\partial^2 f}{\partial y\partial x}=2+3y^2,$$
$$f_{yx}=\frac{\partial^2 f}{\partial x\partial y}=2+3y^2, \quad f_{yy}=\frac{\partial^2 f}{\partial y^2}=6xy.$$

It is clear that the game of successive differentiation can be taken further; in general a sufficiently smooth function of two (*resp.* three) variables will have 2^r (*resp.* 3^r) partial derivatives of order r. Recall that a function is called **continuously differentiable**, or $\mathcal{C}^1$, if its (first-order) partials exist and are continuous; Theorem 3.3.4 tells us that such functions are automatically differentiable (in the sense of Definition 3.3.1). We shall extend this terminology to higher derivatives: a function is r **times continuously differentiable** or $\mathcal{C}^r$ if all of its partial derivatives of order $1, 2, ..., r$ exist and are continuous. In practice, we shall seldom venture beyond the second-order partials.

The alert reader will have noticed that the two *mixed partials* of the function above are equal. This is no accident; the phenomenon was first noted around 1718 by Nicolaus I Bernoulli (1687-1759);[17] in 1734 Leonard Euler (1707-1783) and Alexis-Claude Clairaut (1713-1765) published proofs of the following result.

Theorem 3.8.1 (Equality of Mixed Partials)**.** *If a real-valued function f of two or three variables is twice continuously differentiable (C^2), then for any pair of indices i, j*

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

While it is formulated for second-order partials, this theorem automatically extends to partials of higher order (Exercise 6): if f is C^r, then the order of differentiation in any mixed partial derivative of order up to r does not affect its value. This reduces the number of *different* partial derivatives of a given order tremendously.

We outline the proof of this (for functions of two variables) in Appendix A.2. At first glance, it might seem that a proof for functions of more than two variables might need some extra work. However, when we are looking at the equality of two specific mixed partials, say $\frac{\partial^2 f}{\partial x_i \partial x_j}$ and $\frac{\partial^2 f}{\partial x_j \partial x_i}$, we are holding *all other variables* constant, so the case of functions of more than two variables is a corollary of the proof for two variables in Appendix A.2 (Exercise 5).

Taylor Polynomials. The higher derivatives of a function of one variable can be used to construct a polynomial that has high-order contact with the function at a point, and hence is a better local approximation to the function than the linear approximation, whose graph is the line tangent at this point to the graph of the function. An analogous construction is possible for functions of several variables, but more work is needed to combine the various partial derivatives of a given order into the appropriate polynomial.

A polynomial in several variables consists of monomial terms, each involving powers of the different variables. The degree of each term is its **exponent sum**: the sum of the exponents of all the variables appearing in that term.[18] Thus, each of the monomial terms $3x^2yz^3$, $2xyz^4$, and $5x^6$ has exponent sum 6. We group the terms of a polynomial according to their exponent sums: the group with exponent sum k on its own is a **homogeneous** function of degree k. This means that inputs scale via the k^{th} power of the scalar. We already saw that homogeneity of degree one is exhibited by linear functions: $\ell\left(c\vec{x}\right) = c\ell\left(\vec{x}\right)$. The degree k analogue is $\varphi\left(c\vec{x}\right) = c^k\varphi\left(\vec{x}\right)$; for example, $\varphi\left(x, y, z\right) = 3x^2yz^3 + 2xyz^4 + 5x^6$ satisfies $\varphi\left(cx, cy, cz\right) = 3(cx)^2(cy)(cz)^3 + 2(cx)(cy)(cz)^4 + 5(cx)^6 = c^6(3x^2yz^3 + 2xyz^4 + 5x^6)$ so this function is homogeneous

[17]There were at least six Bernoullis active in mathematics in the late seventeenth and early eighteenth century: the brothers Jacob Bernoulli (1654-1705) and Johann Bernoulli (1667-1748)—who was the tutor to L'Hôpital—their nephew, son of the painter Nicolaus and also named Nicolaus—who is denoted Nicolaus I—and Johann's three sons, Nicolaus II Bernoulli (1695-1726), Daniel Bernoulli (1700-1782) and Johann II Bernoulli (1710-1790). I am following the numeration given by [13, pp. 92-94], which has a brief biographical account of Nicolaus I in addition to a detailed study of his contributions to partial differentiation.

[18]The variables that don't appear have exponent zero.

of degree 6. In general, it is easy to see that a polynomial (in any number of variables) is homogeneous precisely if the exponent sum of each term appearing in it is the same, and this sum equals the degree of homogeneity.

For functions of one variable, the k^{th} derivative determines the term of degree k in the Taylor polynomial, and similarly for a function of several variables the partial derivatives of order k determine the part of the Taylor polynomial which is homogeneous of degree k. Here, we will concentrate on degree two.

For a $\mathcal{C}^2$ function $f(x)$ of one variable, the Taylor polynomial of degree two, $T^2_{\vec{a}} f(\vec{x}) := f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$, has contact of order two with $f(x)$ at $x = a$, and hence is a closer approximation to $f(x)$ (for x near a) than the linearization (or degree one Taylor polynomial). To obtain the analogous polynomial for a function f of two or three variables, given $\vec{a}$ and a nearby point $\vec{x}$, we consider the restriction of f to the line segment from $\vec{a}$ to $\vec{x}$, parametrized as

$$g(t) = f(\vec{a} + t\triangle\vec{x}), \quad 0 \leq t \leq 1,$$

where $\triangle\vec{x} = \vec{x} - \vec{a}$. Taylor's Theorem with Lagrange Remainder for functions of one variable (see *Calculus Deconstructed*, Theorem 6.1.7, or another single-variable calculus text) tells us that

$$g(t) = g(0) + tg'(0) + \frac{t^2}{2} g''(s) \tag{3.28}$$

for some $0 \leq s \leq t$. By the Chain Rule (Proposition 3.3.6)

$$g'(t) = \vec{\nabla} f(\vec{a} + t\triangle\vec{x}) \cdot \triangle\vec{x} = \sum_j \frac{\partial f}{\partial x_j}(\vec{a} + t\triangle\vec{x}) \triangle_j \vec{x}$$

and so

$$g''(s) = \sum_i \sum_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a} + s\triangle\vec{x}) \triangle_i \vec{x} \triangle_j \vec{x}.$$

This is a homogeneous polynomial of degree two, or **quadratic form**, in the components of $\triangle\vec{x}$. By analogy with our notation for the total differential, we denote it by

$$d^2_{\vec{a}} f(\triangle\vec{x}) := \sum_i \sum_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}) \triangle_i \vec{x} \triangle_j \vec{x}.$$

We shall refer to this particular quadratic form—the analogue of the second derivative—as the **Hessian form** of f, after Ludwig Otto Hesse (1811-1874), who introduced it in 1857 [29].

Again by analogy with the single-variable setting, we define the degree two **Taylor polynomial** of f at $\vec{a}$ as the sum of the function with its (total) differential and half the quadratic form at $\vec{a}$, both applied to $\triangle\vec{x} = \vec{x} - \vec{a}$. Note that in the quadratic part, equality of cross-partials allows us to combine any pair of terms involving the same indices (in different order) into one term, whose coefficient is precisely the relevant partial derivative; we use this in writing the last expression below. (We write the version for a function of three variables; for a function of two variables, we simply omit any terms that are supposed to involve x_3.)

$$T^2_{\vec{a}} f(\vec{x}) = f(a) + d_{\vec{a}} f(\triangle\vec{x}) + \frac{1}{2} d^2_{\vec{a}} f(\triangle\vec{x})$$

$$= f(a) + \sum_{j=1}^{3} \frac{\partial f}{\partial x_j}(\vec{a}) \triangle x_j + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}) \triangle_i \vec{x} \triangle_j \vec{x}$$

$$= f(a) + \sum_{j=1}^{3} \frac{\partial f}{\partial x_j}(\vec{a}) \triangle_j \vec{x} + \frac{1}{2}\sum_{i=1}^{3} \frac{\partial^2 f}{\partial x_i^2}(\vec{a}) \triangle x_i^2 + \sum_{1 \le i < j \le 3} \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}) \triangle x_i \triangle x_j.$$

We consider two examples.

The function $f(x, y) = e^{2x} \cos y$ has

$$\frac{\partial f}{\partial x}(x_0, y_0) = 2e^{2x} \cos y, \quad \frac{\partial f}{\partial y}(x_0, y_0) = -e^{2x} \sin y$$

$$\frac{\partial^2 f}{\partial x^2}(x_0, y_0) = 4e^{2x} \cos y, \quad \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = -2e^{2x} \sin y, \quad \frac{\partial^2 f}{\partial y^2}(x_0, y_0) = -e^{2x} \cos y.$$

At $\vec{a} = \left(0, \frac{\pi}{3}\right)$, these values are

$$f\left(0, \frac{\pi}{3}\right) = \frac{1}{2},$$

$$\frac{\partial f}{\partial x}\left(0, \frac{\pi}{3}\right) = 1, \quad \frac{\partial f}{\partial y}\left(0, \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2},$$

$$\frac{\partial^2 f}{\partial x^2}\left(0, \frac{\pi}{3}\right) = 2 \quad \frac{\partial^2 f}{\partial x \partial y}\left(0, \frac{\pi}{3}\right) = -\sqrt{3}, \quad \frac{\partial^2 f}{\partial y^2}\left(0, \frac{\pi}{3}\right) = -\frac{1}{2}$$

so the degree two Taylor polynomial at $\vec{a} = \left(0, \frac{\pi}{3}\right)$ is

$$T_2 f\left(\left(0, \frac{\pi}{3}\right)\right) \triangle x, \triangle y = \frac{1}{2} + \triangle x - \left(\frac{\sqrt{3}}{2}\right) \triangle y + \triangle x^2 - \frac{1}{4} \triangle y^2 - \sqrt{3} \triangle x \triangle y.$$

Let us compare the value $f\left(0.1, \frac{\pi}{2}\right)$ with $f\left(0, \frac{\pi}{3}\right) = 0.5$:

- The exact value is $f\left(0.1, \frac{\pi}{2}\right) = e^{0.2} \cos \frac{\pi}{2} = 0$.
- The linearization (degree one Taylor polynomial) gives an estimate of

$$T_{\left(0, \frac{\pi}{3}\right)} f\left(0.1, \frac{\pi}{6}\right) = \frac{1}{2} + 0.1 - \left(\frac{\sqrt{3}}{2}\right) \frac{\pi}{6} \approx 0.14655.$$

- The quadratic approximation (degree two Taylor polynomial) gives

$$\begin{aligned} T_2 f\left(\left(0, \frac{\pi}{3}\right)\right) 0.1, \frac{\pi}{6} &= \frac{1}{2} + 0.1 - \left(\frac{\sqrt{3}}{2}\right) \frac{\pi}{6} \\ &\quad + (0.1)^2 - \frac{1}{4}\left(\frac{\pi}{6}\right)^2 - \sqrt{3}(0.1)\left(\frac{\pi}{6}\right) \\ &\approx -.00268 \end{aligned}$$

a much better approximation.

As a second example, consider the function $f(x, y, z) = x^2y^3z$, which has

$$f_x = 2xy^3z, \quad f_y = 3x^2y^2z, \quad f_z = x^2y^3$$
$$f_{xx} = 2y^3z, \quad f_{xy} = 6xy^2z, \quad f_{xz} = 2xy^3$$
$$f_{yy} = 6x^2yz, \quad f_{yz} = 3x^2y^2 \quad f_{zz} = 0.$$

Evaluating these at $\vec{a} = \left(1, \frac{1}{2}, 2\right)$ yields

$$f\left(1, \frac{1}{2}, 2\right) = \frac{1}{4},$$
$$f_x\left(1, \frac{1}{2}, 2\right) = \frac{1}{2}, \quad f_y\left(1, \frac{1}{2}, 2\right) = \frac{3}{2}, \quad f_z\left(1, \frac{1}{2}, 2\right) = \frac{1}{8},$$
$$f_{xx}\left(1, \frac{1}{2}, 2\right) = \frac{1}{2}, \quad f_{xy}\left(1, \frac{1}{2}, 2\right) = 3, \quad f_{xz}\left(1, \frac{1}{2}, 2\right) = \frac{1}{4}$$
$$f_{yy}\left(1, \frac{1}{2}, 2\right) = 6, \quad f_{yz}\left(1, \frac{1}{2}, 2\right) = \frac{3}{4}, \quad f_{zz}\left(1, \frac{1}{2}, 2\right) = 0.$$

The degree two Taylor polynomial is

$$\begin{aligned}
&T^2_{\vec{a}}f\,(x_0 + \triangle x, y_0 + \triangle y, z_0 + \triangle z) \\
&\quad = \frac{1}{4} + \left(\frac{1}{2}\triangle x + \frac{3}{2}\triangle y + \frac{1}{8}\triangle z\right) \\
&\qquad + \frac{1}{2}\left(\frac{1}{2}\triangle x^2 + 6\triangle y^2 + 0\triangle z^2 + 2(3\triangle x\triangle y + \frac{1}{4}\triangle x\triangle z + \frac{3}{4}\triangle y\triangle z)\right) \\
&\quad = 0.25 + 0.5\triangle x + 1.5\triangle y + 0.125\triangle z \\
&\qquad + 0.25\triangle x^2 + 3\triangle y^2 + 3\triangle x\triangle y + 0.25\triangle x\triangle z + 0.75\triangle y\triangle z.
\end{aligned}$$

Let us compare the values of $T^n_{\vec{a}}f\,(1.1, 0.4, 1.8)$, $n = 1, 2$, with $f\,(1.1, 0.4, 1.8)$:

- The exact value is $f\,(1.1, 0.4, 1.8) = (1.1)^2(0.4)^3(1.8) = 0.139392$.
- The linear approximation, with $\vec{a} = (1.0, 0.5, 2.0)$, $\triangle x = 0.1$, $\triangle y = -0.1$ and $\triangle z = -0.2$ is $T_{\vec{a}}f\,(0.1, -0.1, -0.2) = 0.125$.
- The quadratic approximation is $T^2_{\vec{a}}f\,(0.1, -0.1, -0.2) = 0.1375$, again a better approximation.

These examples illustrate that the quadratic approximation, or degree two Taylor polynomial $T^2_{\vec{a}}f\,(\vec{x})$, provides a better approximation than the linearization $T_{\vec{a}}f\,(\vec{x})$. This was the expected effect, as we designed $T^2_{\vec{a}}f\,(\vec{x})$ to have contact of order two with $f\,(x)$ at $\vec{x} = \vec{a}$. Let us confirm that this is the case.

Proposition 3.8.2 (Taylor's Theorem for $f : \mathbb{R}^3 \to \mathbb{R}$ (degree 2))**.** *If $f : \mathbb{R}^3 \to \mathbb{R}$ is $\mathcal{C}^2$ (f has continuous second-order partials), then $T^2_{\vec{a}}f\,(\vec{x})$ and $f\,(\vec{x})$ have contact of order two at $\vec{x} = \vec{a}$:*

$$\lim_{\vec{x}\to\vec{a}} \frac{\left|f\,(\vec{x}) - T^2_{\vec{a}}f\,(\vec{x})\right|}{\left\|\vec{x} - \vec{a}\right\|^2} = 0.$$

Proof. Equation (3.28), evaluated at $t = 1$ and interpreted in terms of f, says that, fixing $\vec{a} \in \mathbb{R}^3$, for any $\vec{x}$ in the domain of f,

$$f(\vec{x}) = f(\vec{a}) + d_{\vec{a}} f(\triangle\vec{x}) + \frac{1}{2} d^2_{\vec{s}} f(\triangle\vec{x}),$$

where $\vec{s}$ lies on the line segment from $\vec{a}$ to $\vec{x}$. Thus,

$$\begin{aligned} f(\vec{x}) - T^2_{\vec{a}} f(\vec{x}) &= \frac{1}{2}\left(d^2_{\vec{a}} f(\triangle\vec{x}) - d^2_{\vec{s}} f(\triangle\vec{x})\right) \\ &= \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}) - \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{s})\right)\triangle x_i \triangle x_j \end{aligned}$$

and so

$$\begin{aligned} &\frac{\left|f(\vec{x}) - T^2_{\vec{a}} f(\vec{x})\right|}{\left\|\vec{x} - \vec{a}\right\|^2} \\ &\qquad \leq \frac{1}{2}\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\left|\frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}) - \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{s})\right|\right)\frac{|\triangle x_i \triangle x_j|}{\triangle\vec{x}^2} \\ &\qquad \leq \frac{n^2}{2}\max_{i,j}\left|\frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}) - \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{s})\right|\max_{i,j}\frac{|\triangle x_i \triangle x_j|}{\triangle\vec{x}^2}. \end{aligned} \tag{3.29}$$

By an argument analogous to that giving Equation (2.19) on p. 92 (Exercise 7), we can say that

$$\max_{i,j}\frac{|\triangle x_i \triangle x_j|}{\left\|\triangle\vec{x}\right\|^2} \leq 1$$

and by continuity of the second-order partials, for each i and j

$$\lim_{\vec{x}\to\vec{a}}\frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x}) = \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}).$$

Together, these arguments show that the right-hand side of Equation (3.29) goes to zero as $\vec{x} \to \vec{a}$ (since also $\vec{s} \to \vec{a}$), proving our claim. □

We note in passing that higher-order "total" derivatives, and the corresponding higher-degree Taylor polynomials, can also be defined and shown to satisfy higher-order contact conditions. However, the formulation of these quantities involves more complicated multi-index formulas, and since we shall not use derivatives beyond order two in our theory, we leave these constructions and proofs to your imagination.

Exercises for § 3.8

Answers to Exercises 1a, 2a, and 3a are given in Appendix A.13.

Practice problems:

(1) Find $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, and $\frac{\partial^2 f}{\partial y \partial x}$ for each function below:

(a) $f(x,y) = x^2y$ (b) $f(x,y) = \sin x + \cos y$
(c) $f(x,y) = x^3y + 3xy^2$ (d) $f(x,y) = \sin(x^2y)$
(e) $f(x,y) = \sin(x^2 + 2y)$ (f) $f(x,y) = \ln(x^2y)$
(g) $f(x,y) = \ln(x^2y + xy^2)$ (h) $f(x,y) = \dfrac{x+y}{x^2+y^2}$
(i) $f(x,y) = \dfrac{xy}{x^2+y^2}$

(2) Find all second-order derivatives of each function below:

(a) $f(x,y,z) = x^2 + y^2 + z^2$ (b) $f(x,y,z) = xyz$
(c) $f(x,y,z) = \sqrt{xyz}$ (d) $f(x,y,z) = \dfrac{1}{x^2+y^2+z^2}$
(e) $f(x,y,z) = e^{x^2+y^2+z^2}$ (f) $f(x,y,z) = \dfrac{xyz}{x^2+y^2+z^2}$

(3) Find the degree two Taylor polynomial of the given function at the given point:

(a) $f(x,y) = x^3y^2$ at $(-1,-2)$.
(b) $f(x,y) = \frac{x}{y}$ at $(2,3)$.
(c) $f(x,y,z) = \frac{xy}{z}$ at $(2,-3,5)$.

(4) Let

$$f(x,y) = \frac{xy}{x+y}.$$

(a) Calculate an approximation to $f(0.8, 1.9)$ using the degree one Taylor polynomial at $(1,2)$, $T_{(1,2)}f\left(\triangle x, \triangle y\right)$.
(b) Calculate an approximation to $f(0.8, 1.9)$ using the degree two Taylor polynomial at $(1,2)$, $T_2 f\left((1,2)\right)\triangle x, \triangle y$.
(c) Compare the two to the calculator value of $f(0.8, 1.9)$; does the second approximation improve the accuracy, and by how much?

Theory problems:

(5) (a) Show that for any $\mathcal{C}^2$ function $f(x,y,z)$, $\frac{\partial^2 f}{\partial x \partial z}(x_0,y_0,z_0) = \frac{\partial^2 f}{\partial z \partial x}(x_0,y_0,z_0)$.
(b) How many *distinct* partial derivatives of order 2 can a $\mathcal{C}^2$ function of n variables have?

(6) (a) Show that for any $\mathcal{C}^3$ function $f(x,y)$, $\frac{\partial^3 f}{\partial x \partial y \partial x}(x_0,y_0) = \frac{\partial^3 f}{\partial y \partial^2 x}(x_0,y_0)$.
(b) How many *distinct* partial derivatives of order n can a $\mathcal{C}^n$ function of two variables have?
(c) How many *distinct* partial derivatives of order n can a $\mathcal{C}^n$ function of three variables have?

(7) Prove the comment in the proof of Proposition 3.8.2, that

$$\max_{(i,j)} \frac{\left|\triangle x_i \triangle x_j\right|}{\left\|\triangle \vec{x}\right\|^2} \leq 1.$$

Challenge problem:

(8) Consider the function of two variables

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } x^2+y^2 \neq 0, \\ 0 & \text{at } (x,y) = (0,0). \end{cases}$$

(a) Calculate $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ for $(x,y) \neq (0,0)$.
(b) Calculate $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$.
(c) Calculate the second-order partial derivatives of $f(x,y)$ at $(x,y) \neq (0,0)$.
(d) Calculate the second-order partial derivatives of $f(x,y)$ at the origin. Note that $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$. Explain.

3.9 Local Extrema

The Critical Point Theorem (Theorem 3.7.10) tells us that a local extremum must be a critical point: if a differentiable function $f : \mathbb{R}^{2 \text{ or } 3} \to \mathbb{R}$ has a local maximum (or local minimum) at $\vec{a}$, then $d_{\vec{a}} f(\vec{v}) = 0$ for all $\vec{v} \in \mathbb{R}^{2 \text{ or } 3}$. The converse is not true: for example, the restriction of $f(xy) = x^2 - y^2$ to the x-axis has a *minimum* at the origin, while its restriction to the y-axis has a *maximum* there. So to determine whether a critical point $\vec{a}$ is a local extremum, we need to study the (local) behavior in all directions—in particular, we need to study the Hessian $d^2_{\vec{a}} f$, which is a quadratic form (*i.e.*, a homogeneous polynomial of degree two).

Definite Quadratic Forms. Since it is homogeneous, every quadratic form is zero at the origin. We call the quadratic form Q **definite** if it is nonzero everywhere else: $Q(\vec{x}) \neq 0$ for $\vec{x} \neq \vec{0}$. For example, the forms $Q(x,y) = x^2+y^2$ and $Q(x,y) = -x^2-2y^2$ are definite, while $Q(x,y) = x^2 - y^2$ and $Q(x,y) = xy$ are not. We shall see that the form $Q(x,y) = 2(x+y)^2 + y(y-6x) = 2x^2 - 2xy + 3y^2$ is definite, but *a priori* this is not entirely obvious.

Suppose Q is a definite quadratic form. Given two points $\vec{x} \neq \vec{0}$ and $\vec{y} \neq \vec{0}$, there is a path joining them which misses the origin, so $Q \neq 0$ along this path. By Proposition 3.1.4, this means that $Q(\vec{x})$ and $Q(\vec{y})$ have the same sign. Therefore,

Remark 3.9.1. *If $Q(\vec{x})$ is a definite quadratic form, then one of the following inequalities holds:*

- $Q(\vec{x}) > 0$ *for all* $\vec{x} \neq \vec{0}$ *(Q is* ***positive definite****), or*
- $Q(\vec{x}) < 0$ *for all* $\vec{x} \neq \vec{0}$ *(Q is* ***negative definite****)*

Actually, in this case we can say more:

Lemma 3.9.2. *If $Q(\vec{x})$ is a positive definite (resp. negative definite) quadratic form, then there exists $K > 0$ such that*

$$Q(\vec{x}) \geq K \left\|\vec{x}\right\|^2 \text{ (resp. } Q(\vec{x}) \leq -K \left\|\vec{x}\right\|^2\text{) for all } x.$$

Proof. The inequality is trivial for $\vec{x} = \vec{0}$. If $\vec{x} \neq \vec{0}$, let $\vec{u} = \vec{x}/\left\|\vec{x}\right\|$ be the unit vector parallel to $\vec{x}$; then

$$Q(\vec{x}) = Q(\vec{u}) \left\|\vec{x}\right\|^2$$

and so we need only show that $|Q(\vec{x})|$ is bounded away from zero on the **unit sphere**, $\mathcal{S} = \left\{\vec{u} \mid \left\|\vec{u}\right\| = 1\right\}$. In the plane, $\mathcal{S}$ is the unit circle $x^2 + y^2 = 1$, while in space it is the unit sphere $x^2 + y^2 + z^2 = 1$. Since $\mathcal{S}$ is closed and bounded (Exercise 3), it is sequentially compact, so $|Q(\vec{x})|$ achieves its minimum on $\mathcal{S}$, which is not zero, since Q is definite. It is easy to see that

$$K = \min_{\left\|\vec{u}\right\|=1} |Q(\vec{u})|$$

has the required property. □

Using Lemma 3.9.2 and Taylor's theorem (Proposition 3.8.2), we can show that a critical point with definite Hessian is a local extremum.

Proposition 3.9.3. *Suppose f is a $\mathcal{C}^2$ function and $\vec{a}$ is a critical point for f where the Hessian form $d^2_{\vec{a}}f$ is definite.*

Then f has a local extremum at $\vec{a}$:

- *If $d^2_{\vec{a}}f$ is positive definite, then f has a local minimum at $\vec{a}$;*
- *If $d^2_{\vec{a}}f$ is negative definite, then f has a local maximum at $\vec{a}$.*

Proof. The fact that the quadratic approximation $T^2_{\vec{a}}f(\vec{x})$ has second order contact with $f(\vec{x})$ at $\vec{x} = \vec{a}$ can be written in the form

$$f(\vec{x}) = T^2_{\vec{a}}f(\vec{x}) + \varepsilon(\vec{x})\left\|\vec{x} - \vec{a}\right\|^2, \quad \text{where } \lim_{\vec{x}\to\vec{a}} \varepsilon(\vec{x}) = 0.$$

Since $\vec{a}$ is a critical point, $d_{\vec{a}}f(\triangle\vec{x}) = 0$, so $T^2_{\vec{a}}f(\vec{x}) = f(\vec{a}) + \frac{1}{2}d^2_{\vec{a}}f(\triangle\vec{x})$, or

$$f(\vec{x}) - f(\vec{a}) = \frac{1}{2}d^2_{\vec{a}}f(\triangle\vec{x}) + \varepsilon(\vec{x})\left\|\triangle\vec{x}\right\|^2.$$

Suppose $d^2_{\vec{a}}f$ is positive definite, and let $K > 0$ be the constant given in Lemma 3.9.2, such that

$$d^2_{\vec{a}}f(\triangle\vec{x}) \geq K\left\|\triangle\vec{x}\right\|^2.$$

Since $\varepsilon(\vec{x}) \to 0$ as $\vec{x} \to \vec{a}$, for $\left\|\triangle\vec{x}\right\|$ sufficiently small we have $|\varepsilon(\vec{x})| < \frac{K}{4}$ and hence

$$f(\vec{x}) - f(\vec{a}) \geq \{\frac{K}{2} - \varepsilon(\vec{x})\}\left\|\triangle\vec{x}\right\|^2 > \frac{K}{4}\left\|\triangle\vec{x}\right\|^2 > 0$$

or

$$f(\vec{x}) > f(\vec{a}) \text{ for } \vec{x} \neq \vec{a} \quad (\left\|\triangle\vec{x}\right\| \text{ sufficiently small}).$$

The argument when $d^2_{\vec{a}}f$ is negative definite is analogous (Exercise 4a). □

An analogous argument (Exercise 4b) gives:

Lemma 3.9.4. *If $d^2_{\vec{a}}f$ takes both positive and negative values at the critical point $\vec{x} = \vec{a}$ of f, then f does* not *have a local extremum at $\vec{x} = \vec{a}$.*

Quadratic Forms in $\mathbb{R}^2$. To take advantage of Proposition 3.9.3 we need a way to decide whether or not a given quadratic form Q is positive definite, negative definite, or neither.

In the planar case, there is an easy and direct way to decide this.

A monomial of degree two in x and y has one of the following forms: αx^2, βxy, $\beta' yx$, or γy^2. We can arrange the four coefficients in a 2×2 matrix; conversely, every 2×2 matrix defines a quadratic form:

$$\begin{bmatrix} \alpha & \beta \\ \beta' & \gamma \end{bmatrix} A = \begin{bmatrix} \alpha & \beta \\ \beta' & \gamma \end{bmatrix} \leftrightarrow Q(x, y) = \alpha x^2 + \beta xy + \beta' yx + \gamma y^2.$$

Of course, the xy terms can be combined into one, or more generally we can replace the pair of coefficients of xy and of yx with any pair of numbers summing to $\beta + \beta'$ without changing the quadratic form $Q(x, y)$ represented by the matrix. We adopt the presentation in which the two cross terms have equal coefficients, $b = (\beta + \beta')/2$. This means the matrix is **symmetric**—reflection across the diagonal does not change the matrix. Written this way, a quadratic form

$$Q(x, y) = ax^2 + 2bxy + cy^2 \tag{3.30}$$

corresponds to the matrix

$$[Q] = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \tag{3.31}$$

we call $[Q]$ the **matrix representative** of Q.[19] Using the presentation Equation (3.30) we can factor out "a" from the first two terms and complete the square:

$$\begin{aligned} Q(x, y) &= a\left(\left(x + \frac{b}{a}y\right)^2 - \frac{b^2}{a^2}y^2\right) + cy^2 \\ &= a\left(x + \frac{b}{a}y\right)^2 + \left(c - \frac{b^2}{a}\right)\left(y\right)^2. \end{aligned}$$

Thus, Q is definite provided the two coefficients in the last line have the same sign, or equivalently, if their product is positive:[20]

$$(a)\left(c - \frac{b^2}{a}\right) = ac - b^2 > 0.$$

The quantity in this inequality will be denoted Δ_2; it can be written as the determinant of the matrix representative of Q.

$$\Delta_2 = \det[Q].$$

If $\Delta_2 > 0$, then Q is *definite*, which is to say the two coefficients in the expression for $Q(x, y)$ have the same sign; to tell whether it is *positive* definite or *negative* definite, we need to decide if this sign is positive or negative, and this is most easily seen by looking at the sign of a, which we will denote Δ_1. The significance of this notation will become clear later.

---

[19]Remember that the off-diagonal entries in $[Q]$ are each *half* of the coefficient of xy in our presentation.

[20]Note that if either coefficient is zero, then there is a whole line along which $Q = 0$, so it is not definite.

With this notation, we have

Proposition 3.9.5. *A quadratic form*

$$Q(x,y) = ax^2 + 2bxy + cy^2$$

is definite only if $\Delta_2 \coloneqq ac - b^2 > 0$*; it is* positive *definite if in addition* $\Delta_1 \coloneqq a > 0$ *and* negative *definite if* $\Delta_1 < 0$.

If $\Delta_2 < 0$*, then* $Q(\vec{x})$ *takes both (strictly) positive and (strictly) negative values.*

Let us see what this tells us about the forms we introduced at the beginning of this section:

(1) $Q(x,y) = x^2 + y^2$ has
$$A = [Q] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
so $\Delta_1 = 1 > 0$, $\Delta_2 = 1 > 0$, and Q is positive definite.

(2) $Q(x,y) = -x^2 - 2y^2$ has
$$A = [Q] = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$
so $\Delta_1 = -1 < 0$, $\Delta_2 = 2 > 0$ and Q is negative definite.

(3) $Q(x,y) = x^2 - y^2$ has
$$A = [Q] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
so $\Delta_2 = -1 < 0$ and Q is not definite.

(4) $Q(x,y) = xy$ has
$$A = [Q] = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$$
so $\Delta_2 = -\frac{1}{4} < 0$ and Q is not definite.

(5) Finally, for the one we couldn't decide in an obvious way: $Q(x,y) = 2x^2 - 2xy + 3y^2$ has
$$A = [Q] = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$$
so $\Delta_1 = 2 > 0$, $\Delta_2 = 5 > 0$ and Q is positive definite.

The matrix representative of the Hessian form of a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is the matrix of partials of f, sometimes called the **Hessian matrix** of f:

$$Hf(\vec{a}) = \begin{bmatrix} f_{xx}(\vec{a}) & f_{xy}(\vec{a}) \\ f_{xy}(\vec{a}) & f_{yy}(\vec{a}) \end{bmatrix}.$$

Applying Proposition 3.9.5 to this form gives us [21]

Theorem 3.9.6 (Second Derivative Test, Two Variables)**.** *If* $f\colon \mathbb{R}^2 \to \mathbb{R}$ *is* $\mathcal{C}^2$ *and has a critical point at* $\vec{x} = \vec{a}$*, consider the determinant of the Hessian matrix* [22]

$$\Delta = \Delta_2(\vec{a}) = f_{xx}(\vec{a})\, f_{yy}(\vec{a}) - f_{xy}(\vec{a})^2,$$

and its upper left entry

$$\Delta_1(\vec{a}) = f_{xx}.$$

[21] The Second Derivative Test was published by Joseph Louis Lagrange (1736-1813) in his very first mathematical paper [34] ([20, p. 323]).

[22] sometimes called the **discriminant** of f

Then:

(1) *if* $\Delta > 0$*, then* $\vec{a}$ *is a local extremum of* f*:*
   (a) *it is a strong*[23] *local minimum if* $\Delta_1(\vec{a}) = f_{xx} > 0$
   (b) *it is a strong local maximum if* $\Delta_1(\vec{a}) = f_{xx} < 0$*;*
(2) *if* $\Delta < 0$*, then* $\vec{a}$ *is not a local extremum of* f*;*
(3) $\Delta = 0$ *does not give enough information to distinguish the possibilities.*

Proof. (1) We know that $d^2_{\vec{a}}f$ is positive (*resp.* negative) definite by Proposition 3.9.5, and then apply Proposition 3.9.3.
(2) Apply Proposition 3.9.5 and then Lemma 3.9.4 in the same way.
(3) Consider the following three functions:

$$f(x,y) = (x+y)^2 = x^2 + 2xy + y^2$$
$$g(x,y) = f(x,y) + y^4 = x^2 + 2xy + y^2 + y^4$$
$$h(x,y) = f(x,y) - y^4 = x^2 + 2xy + y^2 - y^4.$$

They all have second order contact at the origin, which is a critical point, and all have Hessian matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

so all have $\Delta = 0$. However:
- f has a weak local minimum at the origin: the function is non-negative everywhere, but equals zero along the whole line $y = -x$;
- g has a strict minimum at the origin: $g(\vec{x}) > 0$ for all $\vec{x} \neq \vec{0}$, and
- h has saddle behavior: its restriction to the x-axis has a minimum at the origin, while its restriction to the line $y = -x$ has a maximum at the origin.

□

As an application of the Second Derivative Test, consider the function $f(x,y) = 5x^2 + 6xy + 5y^2 - 8x - 8y$. We calculate the first partials, $f_x(x,y) = 10x + 6y - 8$ and $f_y(x,y) = 6x + 10y - 8$, and set both equal to zero to find the critical points: the pair of equations

$$10x + 6y = 8$$
$$6x + 10y = 8$$

has the unique solution $(x,y) = \left(\frac{1}{2}, \frac{1}{2}\right)$. Now we calculate the second partials, $f_{xx}(x,y) = 10$, $f_{xy}(x,y) = 6$, and $f_{yy}(x,y) = 10$. Thus, the discriminant is

$$\Delta_2(x,y) := f_{xx}f_{yy} - (f_{xy})^2 = (10)\cdot(10) - (6)^2 > 0,$$

and since also $\Delta_1(x,y) = f_{xx}(x,y) = 10 > 0$, the function has a (strong) local minimum at $\left(\frac{1}{2}, \frac{1}{2}\right)$. As another example, $f(x,y) = 5x^2 + 26xy + 5y^2 - 36x - 36y + 12$ has $f_x(x,y) = 10x + 26y - 36$ and $f_y(x,y) = 26x + 10y - 36$, so the sole critical point is $(1,1)$; the second partials are $f_{xx}(x,y) = 10$, $f_{xy}(x,y) = 26$, and $f_{yy}(x,y) = 10$, so the discriminant is

$$\Delta_2(1,1) = (10)\cdot(10) - (13)^2 < 0$$

and the function has a saddle point at $(1,1)$.

---

[23] In general, at a local minimum we only require that the weak inequality $f(x_0) \leq f(x)$ holds for all x near x_0; we call it a **strong** local minimum if equality holds only for $x = x_0$.

Finally, consider $f(x,y) = x^3 - y^3 + 3x^2 + 3y$. Its partial derivatives, $f_x(x,y) = 3x^2+6x = 3x(x+2)$ and $f_y(x,y) = -3y^2+3$ both vanish when $x = 0$ or -2 and $y = \pm 1$, yielding four critical points. The second partials are $f_{xx}(x,y) = 6x+6$, $f_{xy}(x,y) = 0$, and $f_{yy}(x,y) = -6y$, so the discriminant is

$$\Delta_2(x,y) = (6x+6)(-6y) - 0 = -36(x+1)y.$$

The respective values at the four critical points are: $\Delta_2(0,-1) = 36 > 0$, $\Delta_2(0,1) = -36 < 0$, $\Delta_2(-2,-1) = -36 < 0$, and $\Delta_2(-2,1) = 36 > 0$. So $(0,1)$ and $(-2,-1)$ are saddle points, while $(0,-1)$ and $(-2,1)$ are local extrema; for further information about the extrema, we consider the first partials there: $\Delta_1(0,-1) = f_{xx}(0,-1) = 6 > 0$, so $f(x,y)$ has a local minimum there, while $\Delta_1(-2,1) = f_{xx}(-2,1) = -12 < 0$ so $f(x,y)$ has a local maximum there.

Quadratic Forms in $\mathbb{R}^3$ (Optional). The analysis of quadratic forms in three or more variables is an extension of what we have just described, but involves some new ideas and subtleties. Here, we give what amounts to a sketch of the situation for three variables, without trying to justify every step or explain all the practical details of calculation in the general case. Some of these details are explained more fully in Appendix A.3.

In what follows, it will be useful to adopt subscript notation for the three coordinates in $\mathbb{R}^3$: $x_1 = x$, $x_2 = y$, and $x_3 = z$. A homogeneous polynomial of degree two in three variables, x_1, x_2 and x_3 consists of monomial terms, of which there are *a priori* nine possibilities: $a_{i,j}x_ix_j$ for all possible choices of the (ordered) pair of indices chosen from $1, 2, 3$. Thus a quadratic form in $\mathbb{R}^3$ has the form

$$Q(\vec{x}) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{i,j}x_ix_j.$$

As in the case of $\mathbb{R}^2$, we can rewrite this expression by replacing each of the two coefficients involving the products of $x_ix_{,j}$ and $x_jx_{,i}$ $(i \neq j)$ with their average, $b_{i,j} = b_{j,i} = (a_{i,j} + a_{j,i})/2$. We then have a "balanced" presentation of Q, corresponding to a symmetric 3×3 matrix:

$$\begin{aligned} Q(x_1,x_2,x_3) &= a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + 2b_{1,2}x_1x_2 + 2b_{1,3}x_1x_3 + 2b_{2,3}x_2x_3 \\ \leftrightarrow [Q] &= \begin{bmatrix} a_1 & b_{1,2} & b_{1,3} \\ b_{1,2} & a_2 & b_{2,3} \\ b_{1,3} & b_{2,3} & a_3 \end{bmatrix}. \end{aligned}$$

The simplest example of a quadratic form is the norm squared, $Q(\vec{x}) = \|\vec{x}\|^2 = \vec{x}\cdot\vec{x} = x_1^2 + x_2^2 + x_3^2$. Its matrix representative is the identity matrix. More generally, when $[Q]$ is **diagonal** (that is, all the entries off the diagonal are zero),

$$[Q] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

the quadratic form is a simple weighted sum of the coordinates of $\vec{x}$, squared:

$$Q(\vec{x}) = \lambda_1x_1^2 + \lambda_2x_2^2 + \lambda_3x_3^2.$$

It is clear that this form is positive definite (*resp.* negative definite) if and only if all three diagonal entries are positive (*resp.* negative). This can be formulated in terms of determinants, in a way parallel to the 2×2 case. Given the symmetric matrix $[Q]$,

$$[Q] = \begin{bmatrix} a_1 & b_{1,2} & b_{1,3} \\ b_{1,2} & a_2 & b_{2,3} \\ b_{1,3} & b_{2,3} & a_3 \end{bmatrix}$$

we define three numbers:

$$\Delta_3 = \det \begin{bmatrix} a_1 & b_{1,2} & b_{1,3} \\ b_{1,2} & a_2 & b_{2,3} \\ b_{1,3} & b_{2,3} & a_3 \end{bmatrix}, \quad \Delta_2 = \det \begin{bmatrix} a_1 & b_{1,2} \\ b_{1,2} & a_2 \end{bmatrix} \quad \Delta_1 = \begin{bmatrix} a_1 \end{bmatrix}.$$

In the diagonal case ($b_{1,2} = b_{1.3} = b_{2,3} = 0$ and $a_i = \lambda_i$, $i = 1, 2, 3$) we have

$$\Delta_3 = \lambda_1\lambda_2\lambda_3; \quad \Delta_2 = \lambda_1\lambda_2; \quad \Delta_1 = \lambda_1$$

and so Q is positive definite if and only if Δ_3, Δ_2, and Δ_1 are all positive, and negative definite if and only if Δ_3 and Δ_1 are negative, while Δ_2 is positive.

It turns out that an analogous situation holds for *any* quadratic form. The **Principal Axis Theorem** (Theorem A.3.1) tells us that, given a quadratic form Q, we can, by an appropriate rotation of the axes, get a new rectangular coordinate system, with coordinates ξ_i ($i = 1, 2, 3$) such that

$$Q\left(\vec{x}\right) = \lambda_1\xi_1^2 + \lambda_2\xi_2^2 + \lambda_3\xi_1^2.$$

Again, it is clear that Q is positive (*resp.* negative) definite precisely if all three of the numbers λ_j are positive (*resp.* negative). Via arguments beyond the scope of this book, the determinant-based criteria we have noted for diagonal matrices carry over to general symmetric matrices:

We state (without proof) the following fact:

Proposition 3.9.7. *Suppose Q is a quadratic form on $\mathbb{R}^3$. Define Δ_i for $i = 1, 2, 3$ as follows:*

- $\Delta_3 = \det\ [Q]$, *the determinant of the matrix representative of Q;*
- Δ_2 *is the determinant obtained by discarding the last row and last column of* $[Q]$*;*
- Δ_1 *is the upper left entry (row 1, column 1) of* $[Q]$.

Then Q is positive definite if and only if Δ_1, Δ_2, and Δ_3 are all positive. It is negative definite if and only if Δ_2 is positive while Δ_1 and Δ_3 are both negative.

From this, we can formulate the three-dimensional version of the Second Derivative Test:

Theorem 3.9.8 (Second Derivative Test, Three Variables)**.** *Suppose $\overrightarrow{x_0}$ is a critical point of the $\mathcal{C}^2$ function $f(x, y, z)$; define the three numbers*

$$\Delta_1 = f_{xx}\left(\overrightarrow{x_0}\right);$$

$$\Delta_2 = \det \begin{bmatrix} f_{xx}\left(\overrightarrow{x_0}\right) & f_{xy}\left(\overrightarrow{x_0}\right) \\ f_{yx}\left(\overrightarrow{x_0}\right) & f_{yy}\left(\overrightarrow{x_0}\right) \end{bmatrix};$$

$$\Delta_3 = \det \begin{bmatrix} f_{xx}\left(\overrightarrow{x_0}\right) & f_{xy}\left(\overrightarrow{x_0}\right) & f_{xz}\left(\overrightarrow{x_0}\right) \\ f_{yx}\left(\overrightarrow{x_0}\right) & f_{yy}\left(\overrightarrow{x_0}\right) & f_{yz}\left(\overrightarrow{x_0}\right) \\ f_{zx}\left(\overrightarrow{x_0}\right) & f_{zy}\left(\overrightarrow{x_0}\right) & f_{zz}\left(\overrightarrow{x_0}\right) \end{bmatrix}.$$

Then f has a strict local minimum at $\overrightarrow{x_0}$ if all three are positive; it has a strict local maximum if Δ_2 is positive and Δ_1 and Δ_3 are both negative.

We illustrate with one example. Let $f(x,y,z) = x^2 + 6xy + y^2 - (x+y)^4 - z^2$. The first partials are $f_x = 2x + 6y - 4(x+y)^3$, $f_y = 6x + 2y - 4(x+y)^3$, $f_z = -2z$. There are three critical points, $(0,0,0)$, $(2,2,0)$ and $-2,-2,0)$. The Hessian matrix at (x,y,z) is

$$Hf(x,y,z) = \begin{bmatrix} 2-12(x+y)^2 & 6-12(x+y)^2 & 0 \\ 6-12(x+y)^2 & 2-12(x+y)^2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

At $(0,0,0)$, we have $\Delta_1 = 2$, $\Delta_2 = -32$, and $\Delta_3 = 64$. Since $\Delta_2 < 0$, this point is a saddle point. At the other two critical points, $\pm(2,2,0)$, $\Delta_1 = -190$, $\Delta_2 = 190^2 - 186^2$, and $\Delta_3 = -2\Delta_2$. Since Δ_2 is positive and the other two are both negative, these points are local maxima.

Exercises for § 3.9

Answers to Exercises 1a and 2a are given in Appendix A.13.

Practice problems:

(1) For each quadratic form below, find its matrix representative, and use Proposition 3.9.5 to decide whether it is *positive definite*, *negative definite*, or not definite.

(a) $Q(x,y) = x^2 - 2xy + y^2$ (b) $Q(x,y) = x^2 + 4xy + y^2$
(c) $Q(x,y) = 2x^2 + 2xy + y^2$ (d) $Q(x,y) = x^2 - 2xy + 2y^2$
(e) $Q(x,y) = 2xy$ (f) $Q(x,y) = 4x^2 + 4xy$
(g) $Q(x,y) = 4x^2 - 2xy$ (h) $Q(x,y) = -2x^2 + 2xy - 2y^2$

(2) For each function below, locate all critical points and classify each as a local maximum, local minimum, or saddle point.

(a) $f(x,y) = 5x^2 - 2xy + 10y^2 + 1$
(b) $f(x,y) = 3x^2 + 10xy - 8y^2 + 2$
(c) $f(x,y) = x^2 - xy + y^2 + 3x - 2y + 1$
(d) $f(x,y) = x^2 + 3xy + y^2 + x - y + 5$
(e) $f(x,y) = 5x^2 - 2xy + y^2 - 2x - 2y + 25$
(f) $f(x,y) = 5y^2 + 2xy - 2x - 4y + 1$
(g) $f(x,y) = (x^3 - 3x)(y^2 - 1)$
(h) $f(x,y) = x + y\sin x$

Theory problems:

(3) Show that the unit sphere $\mathcal{S}$ is a closed and bounded set.

(4) (a) Mimic the proof given in the *positive* definite case of Proposition 3.9.3 to prove the *negative* definite case.

(b) Prove Lemma 3.9.4.

4 Integral Calculus for Real-Valued Functions of Several Variables

In this chapter, we consider integrals of functions of several variables.

4.1 Integration over Rectangles

In this section, we will generalize the process of integration from functions of one variable to functions of two variables. As we shall see, the passage from one to several variables presents new difficulties, although the basic underlying ideas from single-variable integration remain.

Integrals in One Variable: A Review. Let us recall the theory behind the Riemann integral for a function of one variable, which is motivated by the idea of finding the area underneath the graph.

Given a function $f(x)$ defined and bounded on the closed interval $[a, b]$, we consider the partition of $[a, b]$ into n subintervals via the partition points

$$\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\};$$

the j^{th} **atom**[1] is then $I_j = \left[x_{j-1}, x_j\right]$; its length is denoted $\triangle x_j = \left\|I_j\right\| = x_j - x_{j-1}$ and the **mesh size** of $\mathcal{P}$ is $\mathrm{mesh}(\mathcal{P}) = \max_{j=1,\dots,n} \triangle x_j$. From this data we form two sums: the **lower sum** $\mathcal{L}(\mathcal{P}, f) = \sum_{j=1}^{n} \left(\inf_{I_j} f\right) \triangle x_j$ and the **upper sum** $\mathcal{U}(\mathcal{P}, f) = \sum_{j=1}^{n} \left(\sup_{I_j} f\right) \triangle x_j$. It is clear that the lower sum is less than or equal to the upper sum for $\mathcal{P}$; however, we can also compare the sums obtained for different partitions: we show that *every* lower sum is lower than *every* upper sum: that is, for any pair of partitions $\mathcal{P}$ and $\mathcal{P}'$, $\mathcal{L}(\mathcal{P}, f) \leq \mathcal{U}(\mathcal{P}', f)$ by comparing each of these to the lower (*resp.* upper) sum for their mutual refinement $\mathcal{P} \vee \mathcal{P}'$ (the partition points of $\mathcal{P} \vee \mathcal{P}'$ are the union of the partition points of $\mathcal{P}$ with those of $\mathcal{P}'$). In particular this means that we can define the **lower integral** $\underline{\int} f(x)\, dx = \sup_{\mathcal{P}} \mathcal{L}(\mathcal{P}, f)$ and the **upper integral** $\overline{\int} f(x)\, dx = \inf_{\mathcal{P}} \mathcal{U}(\mathcal{P}, f)$; clearly, $\underline{\int} f(x)\, dx \leq \overline{\int} f(x)\, dx$. If the two are equal, we say that $f(x)$ is **Riemann integrable** (or just *integrable*) over $[a, b]$, and define the **definite integral** (or *Riemann integral*) of $f(x)$ over $[a, b]$ to be the common value of the lower and upper integrals: $\int_{[a,b]} f(x)\, dx = \underline{\int} f(x)\, dx = \overline{\int} f(x)\, dx$. A few important observations about the Riemann integral are:

- If $f(x)$ is integrable over $[a, b]$ then it is integrable over any subinterval of $[a, b]$.

[1] In *Calculus Deconstructed* I used the term "component interval", but since "component" has a different meaning here, I am borrowing terminology from ergodic theory.

- If $f(x)$ is continuous on $[a,b]$ (with the possible exception of a finite number of points), then for any sequence of partitions $\mathcal{P}_k$ with $\operatorname{mesh}(\mathcal{P}_k) \to 0$, the corresponding lower (or upper) sums converge to the integral.
- In fact, for any partition we can replace the infimum (*resp.* supremum) of $f(x)$ in the lower (*resp.* upper) sum with its value at an arbitrary sample point $s_j \in I_j$ to form a **Riemann sum**

$$\mathcal{R}(\mathcal{P}, f) = \mathcal{R}(\mathcal{P}, f, \{s_j\}) := \sum_{j=1}^{n} f(s_j) \triangle x_j.$$

Then, if $f(x)$ is continuous at all but a finite number of points in $[a,b]$, and $\mathcal{P}_k$ is a sequence of partitions with $\operatorname{mesh}(\mathcal{P}_k) \to 0$, the sequence of Riemann sums corresponding to any choice(s) of sample points for each $\mathcal{P}_k$ converges to the integral:

$$\mathcal{R}(\mathcal{P}_k, f) \to \int_{[a,b]} f(x)\, dx.$$

Integrals over Rectangles. Let us now see how this line of reasoning can be mimicked to define the integral of a function $f(x,y)$ of *two* variables. A major complication arises at the outset: we integrate a function of one variable over an *interval*: what is the analogue for functions of two variables? In different terms, a "piece" of the real line is, in a natural way, a subinterval[2], but a "piece" of the plane is a *region* whose shape can be quite complicated. We shall start with the simplest regions and then explore the generalization to other regions later. By a **rectangle** in the plane we will mean something more specific: a rectangle whose sides are parallel to the coordinate axes. This is defined by its projections, $[a,b]$ (onto the x-axis) and $[c,d]$ (onto the y-axis); it is referred to as their *product*:[3]

$$[a,b] \times [c,d] := \{(x,y) \mid x \in [a,b] \text{ and } y \in [c,d]\}.$$

A natural way to partition the "product" rectangle $[a,b] \times [c,d]$ is to partition each of the "factors" separately (see Figure 4.1): that is, a **partition** $\mathcal{P}$ of the product rectangle $[a,b] \times [c,d]$ is defined by a partition of $[a,b]$, $\mathcal{P}_1 = \{a = x_0 < x_1 < \cdots < x_m = b\}$ and a partition[4] of $[c,d]$, $\mathcal{P}_2 = \{c = y_0 < y_1 < \cdots < y_n = d\}$. This defines a subdivision of $[a,b] \times [c,d]$ into mn subrectangles R_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$: $R_{ij} = I_i \times J_j = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ whose respective areas are $\triangle A_{ij} = \triangle x_i \triangle y_j = (x_i - x_{i-1})(y_j - y_{j-1})$.

Now we can again form lower and upper sums[5]

$$\mathcal{L}(\mathcal{P}, f) = \sum_{i,j=1}^{i=m,\, j=n} \left(\inf_{R_{ij}} f\right) \triangle A_{ij}$$

$$\mathcal{U}(\mathcal{P}, f) = \sum_{i,j=1}^{i=m,\, j=n} \left(\sup_{R_{ij}} f\right) \triangle A_{ij}.$$

---

[2] or maybe a union of subintervals

[3] In general, the **product** of two sets A and B is the set of pairs (a,b) consisting of an element a of A and an element b of B.

[4] Note that the number of elements in the two partitions is not assumed to be the same.

[5] Of course, to form these sums we must assume that $f(x,y)$ is bounded on $[a,b] \times [c,d]$.

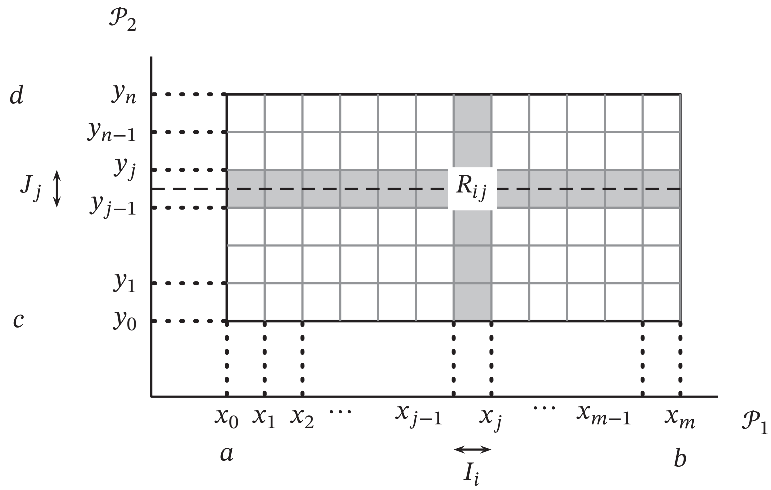


Figure 4.1. Partitioning the Rectangle $[a,b] \times [c,d]$

If $f(x,y) > 0$ over $[a,b] \times [c,d]$, we can picture the lower (*resp.* upper) sum as the total volume of the rectilinear solid formed out of rectangles with base R_{ij} and height $h_{ij}^{-} = \inf_{R_{ij}} f(x,y)$ (*resp.* $h_{ij}^{+} = \sup_{R_{ij}} f(x,y)$) (see Figure 4.2 (*resp.* Figure 4.3)).

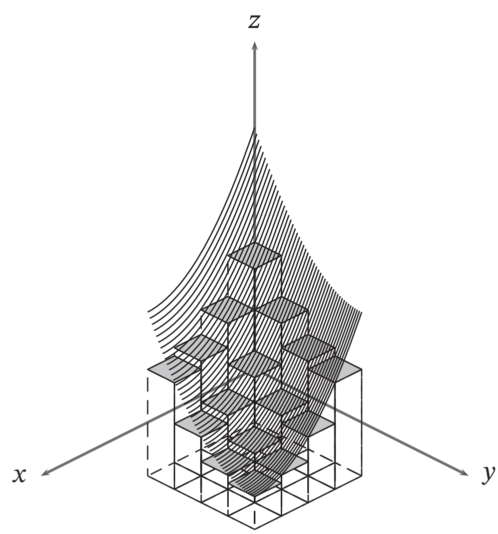


Figure 4.2. Lower sum

As before, we can show (Exercise 2) that for any two partitions $\mathcal{P}$ and $\mathcal{P}'$ of $[a,b] \times [c,d]$, $\mathcal{L}(\mathcal{P}, f) \leq \mathcal{U}(\mathcal{P}', f)$ and so can define $f(x,y)$ to be **integrable** if

$$\underline{\iint}_{[a,b]\times[c,d]} f(x,y)\, dA := \sup_{\mathcal{P}} \mathcal{L}(\mathcal{P}, f) \quad \text{and} \quad \overline{\iint}_{[a,b]\times[c,d]} f(x,y)\, dA := \inf_{\mathcal{P}} \mathcal{U}(\mathcal{P}, f)$$

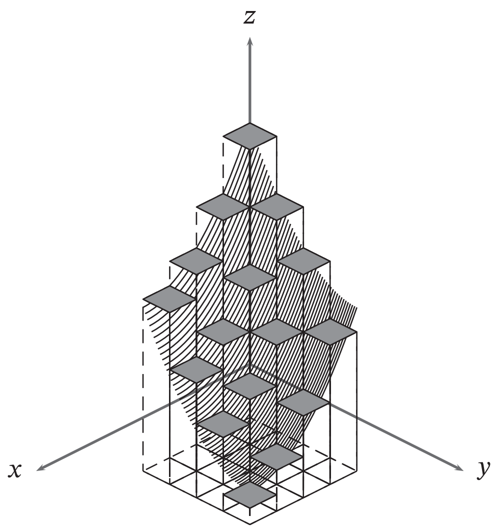


Figure 4.3. Upper sum

are equal, in which case their common value is the **integral**[6] of $f(x, y)$ over the rectangle $[a, b] \times [c, d]$ $\iint_{[a,b]\times[c,d]} f\, dA = \underline{\iint}_{[a,b]\times[c,d]} f(x, y)\, dA = \overline{\iint}_{[a,b]\times[c,d]} f(x, y)\, dA$. Again, given a collection of sample points $\overrightarrow{s_{ij}} = (x^*_{ij}, y^*_{ij}) \in R_{ij}$, $i = 1, \dots, m$, $j = 1, \dots, n$, we can form a Riemann sum

$$\mathcal{R}(\mathcal{P}, f) = \mathcal{R}(\mathcal{P}, f, \{\overrightarrow{s_{ij}}\}) = \sum_{i=1,j=1}^{m,n} f\left(\overrightarrow{s_{ij}}\right)\, dA_{ij}.$$

Since (Exercise 3) $\mathcal{L}(\mathcal{P}, f) \leq \mathcal{R}(\mathcal{P}, f) \leq \mathcal{U}(\mathcal{P}, f)$ for every partition $\mathcal{P}$ (and every choice of sample points), it follows (Exercise 4) that if $f(x, y)$ is integrable over $[a, b] \times [c, d]$ and $\mathcal{P}_k$ is a sequence of partitions with $\mathcal{L}(\mathcal{P}_k, f) \to \iint_{[a,b]\times[c,d]} f\, dA$ and $\mathcal{U}(\mathcal{P}_k, f) \to \iint_{[a,b]\times[c,d]} f\, dA$, then the Riemann sums converge to the definite integral:

$$\mathcal{R}(\mathcal{P}_k, f) \to \iint_{[a,b]\times[c,d]} f\, dA.$$

Which functions $f(x, y)$ are Riemann integrable? For functions of one variable, there are several characterizations of precisely which functions are Riemann integrable; in particular, we know that every monotone function and every continuous function (in fact, every bounded function with a finite number of points of discontinuity) is Riemann integrable (see *Calculus Deconstructed*, §5.2 and §5.9, or another single-variable calculus text). We shall not attempt such a general characterization in the case of several variables; however, we wish to establish that continuous functions, as well as functions with certain kinds of discontinuity, are integrable. To this end, we need to refine our notion of continuity. There are two ways to define continuity of a function at a point $\vec{x} \in \mathbb{R}^2$: in terms of sequences converging to $\vec{x}$, or the "$\varepsilon - \delta$" definition. It is

[6]The double integral sign in this notation indicates that we are integrating over a two-dimensional region; the significance of this notation will be clarified below, when we consider how to calculate the definite integral using iterated integrals.

the latter that we wish to refine. In light of Remark 3.1.6, we can re-cast our usual definition of continuity at a point in terms of ε and δ:

Definition 4.1.1 (ε-δ Defintion of Continuity)**.** *$f(x,y)$ is **continuous** at $\overrightarrow{x_0} = (x_0, y_0)$ if we can guarantee any required accuracy in the output by requiring some specific, related accuracy in the input: that is, given $\varepsilon > 0$, there exists $\delta > 0$ such that for every point $\vec{x} = (x,y)$ in the domain of $f(x,y)$ with $\operatorname{dist}(\vec{x}, \overrightarrow{x_0}) < \delta$, the value of f at $\vec{x}$ is within ε of the value at $\overrightarrow{x_0}$:*

$$\left\|\vec{x} - \overrightarrow{x_0}\right\| < \delta \Rightarrow |f(\vec{x}) - f(\overrightarrow{x_0})| < \varepsilon. \tag{4.1}$$

Suppose we know that f is continuous at every point of some set $S \subset \operatorname{dom}(f)$ in the sense of the definition above. What this says, precisely formulated, is

Given a point $\overrightarrow{x_0}$ in S and $\varepsilon > 0$, there exists $\delta > 0$ such that (4.1) *holds.*

The thing to note is that the accuracy required of the input—that is, δ—depends on *where* we are trying to apply the definition: that is, continuity at another point $\overrightarrow{x_1}$ may require a different value of δ to guarantee the estimate $|f(\vec{x}) - f(\overrightarrow{x_1})| < \varepsilon$, even for the same $\varepsilon > 0$. (An extensive discussion of this issue can be found in *Calculus Deconstructed*, §3.7, or another single-variable calculus text. We say that f is *uniformly* continuous on a set S if δ can be chosen in a way that is independent of the "basepoint" $\overrightarrow{x_0}$; that is,[7]

Definition 4.1.2. *f is **uniformly continuous** on a set $S \subset \operatorname{dom}(f)$ if, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(\vec{x}) - f(\vec{x}')| < \varepsilon$ whenever $\vec{x}$ and $\vec{x}'$ are points of S satisfying $\left\|\vec{x} - \vec{x}'\right\| < \delta$:*

$$\vec{x},\ \vec{x}' \in S \text{ and } \left\|\vec{x} - \vec{x}'\right\| < \delta \Rightarrow |f(\vec{x}) - f(\vec{x}')| < \varepsilon. \tag{4.2}$$

The basic fact that allows us to prove integrability of continuous functions is the following.

Lemma 4.1.3. *If S is a compact set and f is continuous on S, then it is uniformly continuous on S.*

Proof. The proof is by contradiction. Suppose that f is continuous, but not uniformly continuous, on the compact set S. This means that for some required accuracy $\varepsilon > 0$, there is *no* $\delta > 0$ which guarantees (4.2). In other words, no matter how small we pick $\delta > 0$, there is at least one pair of points in S with $\left\|\vec{x} - \vec{x}'\right\| < \delta$ but $|f(\vec{x}) - f(\vec{x}')| \geq \varepsilon$. More specifically, for each positive integer k, we can find a pair of points $\overrightarrow{x_k}$ and $\vec{x}'_k$ in S with $\left\|\overrightarrow{x_k} - \vec{x}'_k\right\| < \frac{1}{k}$, but $|f(\vec{x}) - f(\vec{x}')| \geq \varepsilon$. Now, since S is (sequentially) compact, there exists a subsequence of the $\overrightarrow{x_k}$ (which we can assume is the full sequence) that converges to some point v_0 in S; furthermore, since $\left\|\overrightarrow{x_k} - \vec{x}'_k\right\| \to 0$, the $\vec{x}'_k$ also converge to the same limit $\overrightarrow{x_0}$. Since f is continuous, this implies that $f(\overrightarrow{x_k}) \to f(\overrightarrow{x_0})$ and $f(\vec{x}'_k) \to f(\overrightarrow{x_0})$. But this is impossible, since $|f(\overrightarrow{x_k}) - f(\vec{x}'_k)| \geq \varepsilon > 0$, and provides us with the contradiction that proves the lemma. □

[7]Technically, there is a leap here: when $\overrightarrow{x_0} \in S$, the definition of continuity at $\overrightarrow{x_0}$ given above allows the other point $\vec{x}$ to be *any* point of the domain, not just a point of S. However, as we use it, this distinction will not matter.

Using this, we can prove that continuous functions are Riemann integrable. However, we need first to define one more notion generalizing the one-variable situation: the mesh size of a partition. For a partition of an interval, the length of a atom I_j also controls the distance between points in that interval; however, the *area* of a rectangle R_{ij} can be small and still allow some pairs of points in it to be far apart (if, for example, it is tall but extremely thin). Thus, we need to separate out a measure of distances from area. The **diameter** of a rectangle (or of any other set) is the supremum of the pairwise distances of points in it; for a rectangle, this is the same as the length of the diagonal (Exercise 5). A more convenient measure in the case of a rectangle, though, is the maximum of the lengths of its sides, that is, we define $\left\|R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]\right\| := \max\{\triangle x_i, \triangle y_j\}$. This is always less than the diagonal; it also controls the possible distance between pairs of points in $R_{i,j}$, but has the virtue of being easy to calculate. Then we define the **mesh size** (or just *mesh*) of a partition $\mathcal{P}$ to be the maximum of these "diameters": $\operatorname{mesh}(\mathcal{P}) := \max_{i,j} \left\|R_{ij}\right\| = \max_{i \le m, j \le n}\{\triangle x_i, \triangle y_j\}$.

With this, we can formulate the following.

Theorem 4.1.4. *Every function f which is continuous on the rectangle $[a,b] \times [c,d]$ is Riemann integrable on it. More precisely, if $\mathcal{P}_k$ is any sequence of partitions of $[a,b]\times[c,d]$ for which $\operatorname{mesh}(\mathcal{P}_k) \to 0$, then the sequence of corresponding lower sums (and upper sums—in fact any Riemann sums) converges to the definite integral:*

$$\lim \mathcal{L}(\mathcal{P}_k, f) = \lim \mathcal{U}(\mathcal{P}_k, f) = \lim \mathcal{R}(\mathcal{P}_k, f) = \iint_{[a,b]\times[c,d]} f\, dA.$$

Proof. Note first that it suffices to show that

$$\mathcal{U}(\mathcal{P}_k, f) - \mathcal{L}(\mathcal{P}_k, f) \to 0 \tag{4.3}$$

since for every k, $\mathcal{L}(\mathcal{P}_k, f) \le \sup_{\mathcal{P}} \mathcal{L}(\mathcal{P}, f) \le \inf_{\mathcal{P}} \mathcal{U}(\mathcal{P}, f) \le \mathcal{U}(\mathcal{P}_k, f)$ and for every sample choice $\mathcal{L}(\mathcal{P}_k, f) \le \mathcal{R}(\mathcal{P}_k, f) \le \mathcal{U}(\mathcal{P}_k, f)$ (see Exercise 6 for details).

Now let A be the area of $[a,b] \times [c,d]$ (that is, $A = |b-a| \cdot |d-c|$). Given $\varepsilon > 0$, use the uniform continuity of f on the compact set $[a,b]\times[c,d]$ to find $\delta > 0$ such that

$$\vec{x}, \vec{x}\,' \in S \text{ and } \left\|\vec{x} - \vec{x}\,'\right\| < \delta \Rightarrow |f(\vec{x}) - f(\vec{x}\,')| < \frac{\varepsilon}{A}.$$

Then if a partition $\mathcal{P}$ has $\operatorname{mesh}(\mathcal{P}) < \delta$, we can guarantee that any two points in the same subrectangle R_{ij} are at distance at most δ apart, which guarantees that the values of f at the two points are at most ε/A apart, so

$$\begin{aligned}\mathcal{U}(\mathcal{P}, f) - \mathcal{L}(\mathcal{P}, f) &= \sum_{i,j} \left(\sup_{R_{ij}} f - \inf_{R_{ij}} f\right) \triangle A_{ij} \\ &\le \left(\frac{\varepsilon}{A}\right) \sum_{i,j} \triangle A_{ij} = \frac{\varepsilon}{A} \cdot A = \varepsilon.\end{aligned}$$

Thus if $\mathcal{P}_k$ are partitions satisfying $\operatorname{mesh}(\mathcal{P}_k) \to 0$, then for every $\varepsilon > 0$ we eventually have $\operatorname{mesh}(\mathcal{P}_k) < \delta$ and hence $\mathcal{U}(\mathcal{P}_k, f) - \mathcal{L}(\mathcal{P}_k, f) < \varepsilon$; that is, Equation (4.3) holds, and f is Riemann integrable. □

Iterated Integrals. After all of this nice theory, we need to come back to Earth. How, in practice, can we compute the integral $\iint_{[a,b]\times[c,d]} f\,dA$ of a given function f over the rectangle $[a,b]\times[c,d]$?

The intuitive idea is to consider how we might calculate a Riemann sum for this integral. If we are given a partition $\mathcal{P}$ of $[a,b]\times[c,d]$ defined by $\mathcal{P}_1 = \{a = x_0 < x_1 < \cdots < x_m = b\}$ for $[a,b]$ and $\mathcal{P}_2 = \{c = y_0 < y_1 < \cdots < y_n = d\}$ for $[c,d]$, the simplest way to pick a sample set $\{\overrightarrow{s_{ij}}\}$ for the Riemann sum $\mathcal{R}(\mathcal{P},f)$ is to pick a sample x-coordinate $x'_i \in I_i$ in each atom of $\mathcal{P}_1$ and a sample y-coordinate $y'_j \in J_j$ in each atom of $\mathcal{P}_2$, and then to declare the sample point in the subrectangle $R_{ij} = I_i \times J_j$ to be $\overrightarrow{s_{ij}} = (x'_i, y'_j) \in R_{ij}$ (Figure 4.4).

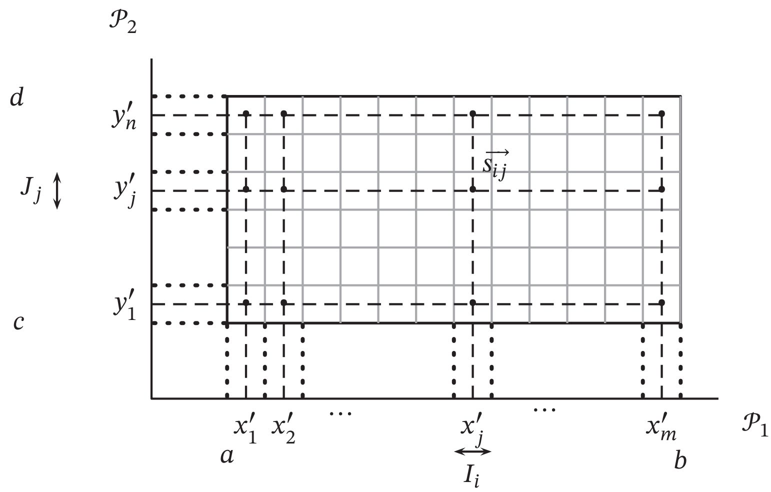


Figure 4.4. Picking a Sample Set

Then we can sum $\mathcal{R}(\mathcal{P},f)$ by first adding up along the i^{th} "column" of our partition

$$S_i = \sum_{j=1}^{n} f\left(x'_i, y'_j\right) \triangle A_{ij} = \sum_{j=1}^{n} f\left(x'_i, y'_j\right) \triangle x_i \triangle y_j = \triangle x_i \sum_{j=1}^{n} f\left(x'_i, y'_j\right) \triangle y_j$$

and then adding up these column sums:

$$\begin{aligned}
\mathcal{R}(\mathcal{P},f) &= \sum_{i=1}^{m} S_i = \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x'_i, y'_j\right)\, dA_{ij} \\
&= \sum_{i=1}^{m}\left(\triangle x_i \sum_{j=1}^{n} f\left(x'_i, y'_j\right) \triangle y_j\right) = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} f\left(x'_i, y'_j\right) \triangle y_j\right)\triangle x_i
\end{aligned}$$

Notice that in the sum S_i, the x-value is fixed at $x = x'_i$, and S_i can be viewed as $\triangle x_i$ times a Riemann sum for the integral $\int_c^d g(y)\,dy$, where $g(y) = f\left(x'_i, y\right)$ is the function of y alone obtained from $f(x,y)$ by fixing the value of x at x'_i: we denote this integral by

$$\int_c^d f\left(x'_i, y\right)\,dy := \int_c^d g(y)\,dy.$$

This gives us a number that depends on x'_i; call it $G\left(x'_i\right)$.

For example, if $f(x,y) = x^2+2xy$ and $[a,b]\times[c,d] = [0,1]\times[1,2]$ then fixing $x = x_i'$ for some $x_i' \in [0,1]$, $g(y) = (x_i')^2 + 2x_i'y$ and $\int_c^d g(y)\,dy = \int_1^2 \left((x_i')^2 + 2x_i'y\right)\,dy$ – which, since x_i' is a constant, equals $\left[(x_i')^2 y + 2x_i'\frac{y^2}{2}\right]_1^2 = \left[2(x_i')^2 + 4x_i'\right] - \left[(x_i')^2 + x_i'\right] = (x_i')^2 + 3x_i'$. That is, $G\left(x_i'\right) = (x_i')^2 + 3x_i'$.

But now, when we sum over the i^{th} "column", we add up S_i over all values of i; since S_i is an approximation of $G\left(x_i'\right)\triangle x_i$, we can regard $\mathcal{R}(\mathcal{P}, f)$ as a Riemann sum for the integral of the function $G(x)$ over the interval $[a,b]$. In our example, this means

$$\begin{aligned}
\mathcal{R}(\mathcal{P}, x^2+2xy) &= \sum_{i=1}^{m}\sum_{j=1}^{n}\left[(x_i')^2 + 2x_i'y_j'\right]\triangle x_i \triangle y_j \\
&\approx \sum_{i=1}^{m}\left[\int_1^2 \left((x_i')^2 + 2x_i'y\right)\,dy\right]\triangle x_i \\
&= \sum_{i=1}^{m}\left[(x_i')^2 + 3x_i'\right]\triangle x_i \\
&\approx \int_0^1 \left[x^2 + 3x\right]\,dx \\
&= \left[\frac{x^3}{3} + 3\frac{x^2}{2}\right]_0^1 \\
&= \left(\frac{1}{3} + \frac{3}{2}\right) - (0) \\
&= \frac{11}{6}.
\end{aligned}$$

Ignoring for the moment the fact that we have made two approximations here, the process can be described as: first, we integrate our function treating x as a constant, so that $f(x,y)$ looks like a function of y alone: this is denoted $\int_c^d f(x,y)\,dy$ and for each value of x, yields a number—in other words, this **partial integral** is a function of x. Then we integrate *this* function (with respect to x) to get the presumed value $\iint_{[a,b]\times[c,d]} f\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy\right)\,dx$. We can drop the parentheses, and simply write the result of our computation as the **iterated integral** or **double integral**

$$\int_a^b \int_c^d f(x,y)\,dy\,dx.$$

Of course, our whole process could have started by summing first over the j^{th} "*row*", and then adding up the row sums; the analogous notation would be another iterated integral,

$$\int_c^d \int_a^b f(x,y)\,dx\,dy.$$

In our example, this calculation would go as follows:

$$\int_1^2 \int_0^1 (x^2 + 2xy)\, dx\, dy = \int_1^2 \left[\int_0^1 (x^2 + 2xy)\, dx \right] dy$$
$$= \int_1^2 \left[\left(\frac{x^3}{3} + x^2 y \right)_{x=0}^{1} \right] dy = \int_1^2 \left[\left(\frac{1}{3} + y \right) - (0) \right] dy = \int_1^2 \left[\frac{1}{3} + y \right] dy$$
$$= \left[\frac{y}{3} + \frac{y^2}{2} \right]_1^2 = \left[\frac{2}{3} + \frac{4}{2} \right] - \left[\frac{1}{3} + \frac{1}{2} \right] = \frac{16}{6} - \frac{5}{6} = \frac{11}{6}.$$

Let us justify our procedure.

Theorem 4.1.5 (Fubini's Theorem[8]). *If f is continuous on $[a,b]\times[c,d]$, then its integral can be computed via double integrals:*

$$\iint_{[a,b]\times[c,d]} f\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy. \tag{4.4}$$

Proof. We will show the first equality above; the proof that the second iterated integral equals the definite integral is analogous.

Define a function $F(x)$ on $[a,b]$ via $F(x) = \int_c^d f(x,y)\, dy$. Given a partition $\mathcal{P}_2 = \{c = y_0 < y_1 < \cdots < y_n = d\}$ of $[c,d]$, we can break the integral defining $F(x)$ into integrals over the atoms J_j of $\mathcal{P}_2$: $F(x) = \sum_{j=1}^{n} \int_{y_{j-1}}^{y_j} f(x,y)\, dy$ and since $f(x,y)$ is continuous, we can apply the Integral Mean Value Theorem (see *Calculus Deconstructed*, Proposition 5.2.10, or another single-variable calculus text) on J_j to find a point $Y_j(x) \in J_j$ where the value of $f(x,y)$ equals its average (with x fixed) over J_j; it follows that the sum above equals $\sum_{j=1}^{n} f(x, Y_j(x)) \triangle y_j$.Now if $\mathcal{P}_1 = \{a = x_0 < x_1 < \cdots < x_m = b\}$ is a partition of $[a,b]$ then a Riemann sum for the integral $\int_a^b F(x)\, dx$, using the sample coordinates $x_i \in I_i$, is

$$\sum_{i=1}^{m} F(x_i) \triangle x_i = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} f(x_i, Y_j(x_i)) \triangle y_j \right) \triangle x_i;$$

but this is also a Riemann sum for the integral $\iint_{[a,b]\times[c,d]} f\, dA$ using the "product" partition $\mathcal{P}$ generated by $\mathcal{P}_1$ and $\mathcal{P}_2$, and the sample coordinates $s_{ij} = (x_i, Y_i(x_i))$. Thus, if we pick a sequence of partitions of $[a,b]\times[c,d]$ with mesh going to zero, the left sum above converges to $\int_a^b F(x)\, dx$ while the right sum converges to $\iint_{[a,b]\times[c,d]} f\, dA$. □

If the function $f(x,y)$ is positive over the rectangle $[a,b]\times[c,d]$, then the definite integral $\iint_{[a,b]\times[c,d]} f\, dA$ is interpreted as the volume between the graph of $f(x,y)$ and the xy-plane. In this case, the calculation via iterated integrals can be interpreted as finding this volume by "slicing" parallel to one of the vertical coordinate planes (see Figure 4.5) and "integrating" the areas of the resulting slices; this is effectively an application of **Cavalieri's Principle**:

[8]It is something of an anachronism to call this Fubini's Theorem. The result actually proved by Guido Fubini (1879-1943) [16] is far more general, and far more complicated than this. However, "Fubini's Theorem" is used generically to refer to all such results about expressing integrals over multi-dimensional regions via iterated integrals.

If two solid bodies intersect each of a family of parallel planes in regions with equal areas, then the volumes of the two bodies are equal.

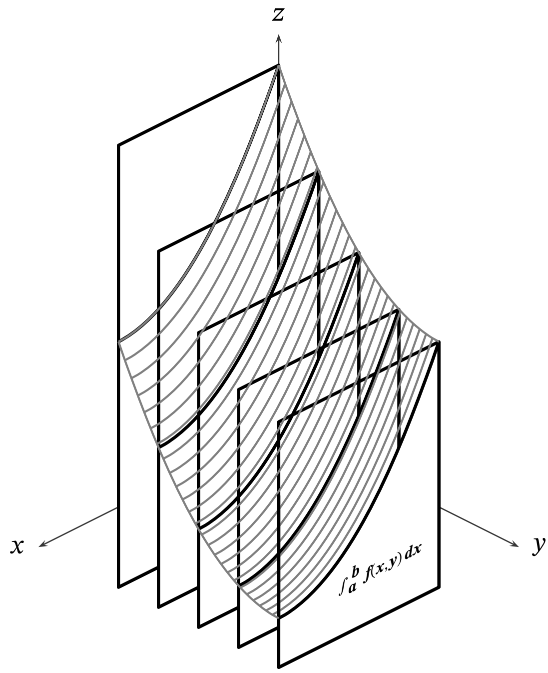


Figure 4.5. Fubini's Theorem: Volume via Slices

Let us consider a few more examples of this process.

The integral

$$\iint_{[-1,1]\times[0,1]} (x^2+y^2)\,dA$$

can be calculated via two different double integrals:

$$\int_0^1 \int_{-1}^1 (x^2+y^2)\,dA = \int_0^1 \left[\frac{x^3}{3}+xy^2\right]_{x=-1}^1 dy = \int_0^1 \left[\left(\frac{1}{3}+y^2\right)-\left(-\frac{1}{3}-y^2\right)\right] dy$$
$$= \int_0^1 \left[\frac{2}{3}+2y^2\right] dy = \left[\frac{2}{3}y+\frac{2y^3}{3}\right]_{y=0}^1 = \frac{4}{3}$$

or

$$\int_{-1} 1 \int_0^1 \left(x^2+y^2\right)\,dy\,dx = \int_{-1}^1 \left[x^2y+\frac{y^3}{3}\right]_{y=0}^1 dy = \int_{-1}^1 \left[\left(x^2+\frac{1}{3}\right)-(0)\right] dy$$
$$= \left[\frac{x^3}{3}+\frac{x}{3}\right]_{x=-1}^1 = \left[\frac{1}{3}+\frac{1}{3}\right]-\left[-\frac{1}{3}-\frac{1}{3}\right] = \frac{4}{3}.$$

A somewhat more involved example shows that the order in which we do the double integration can affect the difficulty of the process. The integral $\iint_{[1,4]\times[0,1]} y\sqrt{x+y^2}\,dA$ can be calculated two ways. To calculate the double integral $\int_0^1\int_1^4 y\sqrt{x+y^2}\,dx\,dy$ we start with the "inner" partial integral, in which y is treated as a constant: using the substitution $u = x + y^2$, $du = dx$, we calculate the indefinite integral as

$$\int y\sqrt{x+y^2}\,dx = \int y\,u^{1/2}\,du = \frac{2}{3}y\,u^{3/2} + C = \frac{2}{3}y(x + y^2)^{3/2} + C$$

so the (inner) definite integral is

$$\int_1^4 y\sqrt{x+y^2}\,dx = \frac{2}{3}y\left(y^2+x\right)_{x=1}^4 = \frac{2}{3}\left[y(y^2+4)^{3/2} - y(y^2+1)^{3/2}\right].$$

Thus the "outer" integral becomes

$$\frac{2}{3}\int_0^1 \left[y(y^2+4)^{3/2} - y(y^2+1)^{3/2}\right] dy.$$

Using the substitution $u = y^2 + 4$, $du = 2y\,dy$, we calculate the indefinite integral of the first term as

$$\int y(y^2+4)\,dy = \frac{1}{2}\int u^{3/2}\,du = \frac{1}{5}u^{5/2} + C = \frac{1}{5}(y^2+4)^{5/2} + C;$$

similarly, the indefinite integral of the second term is

$$\int y(y^2+1)\,dy = \frac{1}{5}(y^2+1)^{5/2} + C.$$

It follows that the whole "outer" integral is

$$\frac{2}{3}\int_0^1 \left[y(y^2+4)^{3/2} - y(y^2+1)^{3/2}\right] dy = \frac{2}{15}\left[(y^2+4)^{5/2} - (y^2+1)^{5/2}\right]_{y=0}^1$$
$$= \frac{2}{15}\left[\left(5^{5/2} - 2^{5/2}\right) - \left(4^{5/2} - 1^{5/2}\right)\right] = \frac{2}{15}\left[25\sqrt{5} - 4\sqrt{2} - 31\right].$$

If instead we perform the double integration in the opposite order

$$\int_1^4 \int_0^1 y\sqrt{x+y^2}\,dy\,dx$$

the "inner" integral treats x as constant; we use the substitution $u = x + y^2$, $du = 2y\,dy$ to find the "inner" indefinite integral

$$\int y\sqrt{x+y^2}\,dy = \int \frac{1}{2}u^{1/2}\,du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(x+y^2)^{3/2} + C$$

so the definite "inner" integral is

$$\int_0^1 y\sqrt{x+y^2}\,dy = \frac{1}{3}(x+y^2)^{3/2}\Big|_{y=0}^1 = \frac{1}{3}\left[(x+1)^{3/2} - (x)^{3/2}\right].$$

Now the "outer" integral is

$$\frac{1}{3}\int_1^4 \left[(x+1)^{3/2} - (x)^{3/2}\right] dx = \frac{1}{3}\left[\frac{2}{5}(x+1)^{5/2} - \frac{2}{5}x^{5/2}\right]_{x=1}^4$$
$$= \frac{2}{15}\left[(5^{5/2} - 4^{5/2}) - (2^{5/2} - 1^{5/2})\right] = \frac{2}{15}\left[25\sqrt{5} - 4\sqrt{2} - 31\right].$$

As a final example, let us find the volume of the solid with vertical sides whose base is the rectangle $[0,1] \times [0,1]$ in the xy-plane and whose top is the planar quadrilateral with vertices $(0,0,4)$, $(1,0,2)$, $(0,1,3)$, and $(1,1,1)$ (Figure 4.6).

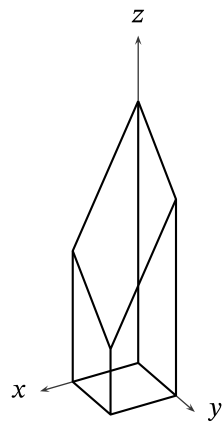


Figure 4.6. A Volume

First, we should find the equation of the top of the figure. Since it is planar, it has the form $z = ax + by + c$; substituting each of the four vertices into this yields four equations in the three unknowns a, b and c:

$$\begin{aligned} 4 &= c \\ 2 &= a + c \\ 3 &= b + c \\ 1 &= a + b + c. \end{aligned}$$

The first three equations have the solution

$$\begin{aligned} a &= -2 \\ b &= -1 \\ c &= 4 \end{aligned}$$

and you can check that this also satisfies the fourth equation; so the top is the graph of $z = 4 - 2x - 3y$. Thus, our volume is given by the integral

$$\iint_{[0,1]\times[0,1]} (4-2x-3y)\,dA = \int_0^1 \int_0^1 (4-2x-3y)\,dx\,dy = \int_0^1 \left[4x - x^2 - 3xy\right]_{x=0}^1 dy$$
$$= \int_0^1 [(3-3y)-(0)]\,dy = \left[3y - \frac{3y^2}{2}\right]_{y=0}^1 = \frac{3}{2}.$$

Exercises for § 4.1

Answer to Exercise 1a is given in Appendix A.13.

Practice problems:

(1) Calculate each integral below:

(a) $\iint_{[0,1]\times[0,2]} 4x\,dA$

(b) $\iint_{[0,1]\times[0,2]} 4xy\,dA$

(c) $\iint_{[0,1]\times[0,\pi]} x\sin y\,dA$

(d) $\iint_{[0,1]\times[-\frac{\pi}{2},\frac{\pi}{2}]} e^x\cos y\,dA$

(e) $\iint_{[0,1]\times[1,2]} (2x+4y)\,dA$

(f) $\iint_{[1,2]\times[0,1]} (2x+4y)\,dA$

Theory problems:

(2) Let $\mathcal{P}$ and $\mathcal{P}'$ be partitions of the rectangle $[a,b]\times[c,d]$.

(a) Show that if every partition point of $\mathcal{P}$ is also a partition point of $\mathcal{P}'$ (that is, $\mathcal{P}'$ is a refinement of $\mathcal{P}$) then for any function f

$$\mathcal{L}(\mathcal{P},f)\le\mathcal{L}(\mathcal{P}',f)\le\mathcal{U}(\mathcal{P}',f)\le\mathcal{U}(\mathcal{P},f).$$

(b) Use this to show that for *any* two partitions $\mathcal{P}$ and $\mathcal{P}'$,

$$\mathcal{L}(\mathcal{P},f)\le\mathcal{U}(\mathcal{P}',f).$$

(*Hint:* Use the above on the mutual refinement $\mathcal{P}\vee\mathcal{P}'$, whose partition points consist of all partition points of $\mathcal{P}$ together with those of $\mathcal{P}'$.)

(3) Let $\mathcal{P}$ be a partition of $[a,b]\times[c,d]$ and f a function on $[a,b]\times[c,d]$. Show that the Riemann sum $\mathcal{R}(\mathcal{P},f)$ corresponding to any choice of sample points is between the lower sum $\mathcal{L}(\mathcal{P},f)$ and the upper sum $\mathcal{U}(\mathcal{P},f)$.

(4) Show that if $\mathcal{P}_k$ is a sequence of partitions of $[a,b]\times[c,d]$ for which $\mathcal{L}(\mathcal{P}_k,f)$ and $\mathcal{U}(\mathcal{P}_k,f)$ both converge to $\iint_{[a,b]\times[c,d]} f\,dA$ then for any choice of sample points in each partition, the corresponding Riemann sums also converge there.

(5) Show that the diameter of a rectangle equals the length of its diagonal, and that this always lies between the maximum of the sides and $\sqrt{2}$ times the maximum of the sides.

(6) Let f be any function on $[a,b]\times[c,d]$.

(a) Show that for any partition $\mathcal{P}$ of $[a,b]\times[c,d]$, $\mathcal{L}(\mathcal{P},f)\le\sup_{\mathcal{P}}\mathcal{L}(\mathcal{P},f)$ and $\inf_{\mathcal{P}}\mathcal{U}(\mathcal{P},f)\le\mathcal{U}(\mathcal{P},f)$.

(b) Use this, together with Exercise 3, to show that if

$$\sup_{\mathcal{P}}\mathcal{L}(\mathcal{P},f)=\inf_{\mathcal{P}}\mathcal{U}(\mathcal{P},f)$$

then there exists a sequence $\mathcal{P}_k$ of partitions such that

$$\mathcal{U}(\mathcal{P}_k,f)-\mathcal{L}(\mathcal{P}_k,f)\to 0$$

and conversely that the existence of such a sequence implies that $\sup_{\mathcal{P}}\mathcal{L}(\mathcal{P},f) = \inf_{\mathcal{P}}\mathcal{U}(\mathcal{P},f)$.

(c) Use this, together with Exercise 4, to show that if f is integrable, then for any such sequence, the Riemann sums corresponding to any choices of sample points converge to the integral of f over $[a,b]\times[c,d]$.

4.2 Integration over General Planar Regions

In this section we extend our theory of integration to more general regions in the plane. By a "region" we mean a bounded set defined by a finite set of inequalities of the form $g_i(x,y)\le c_i$, $i=1,\dots,k$, where the functions g_i are continuous or continuously differentiable. When the Implicit Function Theorem (Theorem 3.4.2) applies, the region is bounded by a finite set of curves, each of which is part of a level curve ($g_i(x, y) \leq c_i$), and can be viewed as a graph of the form $y = \phi_i(x)$ or $x = \psi_i(y)$. In fact, the most general kind of "region" over which such an integration can be performed was the subject of considerable study in the 1880s and early 1890s [24, pp. 86-96]. The issue was finally resolved by Camille Jordan (1838-1922) in 1892; his solution is well beyond the scope of this book.

Integration over Elementary Regions. Suppose we have a function $f(x, y)$ defined and positive on a rectangle $[a, b] \times [c, d]$, and we wish to find a volume under its graph—not the volume over the whole rectangle, but only the part above a subregion $D \subset [a, b] \times [c, d]$. One way to do this is to "crush" the part of the graph outside $\mathcal{D}$ down to the xy-plane and integrate the resulting function defined in pieces

$$f \restriction_{\mathcal{D}} (\vec{x}) := \begin{cases} f(\vec{x}) & \text{if } \vec{x} \in \mathcal{D}, \\ 0 & \text{otherwise.} \end{cases}$$

Of course, this definition makes sense even if f is not positive on $[a, b] \times [c, d]$. And this process can be turned around: the definition above extends any function which is defined at least on $\mathcal{D}$ to a function defined on the whole plane.

Definition 4.2.1. *If $f(x, y)$ is defined at every point of the bounded set $\mathcal{D}$, then the integral of f over $\mathcal{D}$ is defined as*

$$\iint_{\mathcal{D}} f\, dA := \iint_{[a,b]\times[c,d]} f \restriction_{\mathcal{D}} dA,$$

where $[a, b] \times [c, d]$ is any rectangle containing $\mathcal{D}$ (provided this integral exists, i.e., provided $f \restriction_{\mathcal{D}}$ is Riemann integrable on $[a, b] \times [c, d]$).

We know from Theorem 4.1.4 that a function which is continuous on a rectangle is Riemann integrable on this rectangle. However, if we "crush" part of the function as in Definition 4.2.1, the new function will have discontinuities at (almost) every point on the boundary of $\mathcal{D}$. We need to prove that the kind of discontinuity created by this process does not destroy the integrability of this function. This is the content of

Theorem 4.2.2 (Integrability with Jump Discontinuities)**.** *If a function f is bounded on $[a, b] \times [c, d]$ and continuous except possibly for jump discontinuities along a finite union of graphs (curves of the form $y = \phi(x)$ or $x = \psi(y)$), then f is Riemann integrable over $[a, b] \times [c, d]$.*

The proof is given in Appendix A.4.

An immediate consequence of Theorem 4.2.2 is the following.

Remark 4.2.3. *If f is continuous on a region $\mathcal{D}$ bounded by finitely many graphs of continuous functions $y = \phi(x)$ or $x = \psi(y)$, then Theorem 4.2.2 guarantees that $\iint_{\mathcal{D}} f\, dA$ is well-defined.*

Any such region can be broken down into regions of a particularly simple type:

Definition 4.2.4. *A region $D \subset \mathbb{R}^2$ is y-**regular** if it can be specified by inequalities of the form*

$$\mathcal{D} = \{(x, y) \mid c(x) \leq y \leq d(x), \quad a \leq x \leq b\},$$

where $c(x)$ and $d(x)$ are continuous and satisfy $c(x) \leq d(x)$ on $[a, b]$. (See Figure 4.7.)

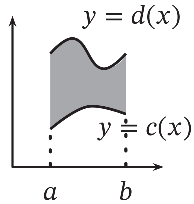


Figure 4.7. y-regular region

*It is x-**regular** if it can be specified by inequalities on x of the form*

$$\mathcal{D} = \{(x, y) \mid a(y) \le x \le b(y), \quad c \le y \le d\},$$

where $a(y)$ and $b(y)$ are continuous and satisfy $a(y) \le b(y)$ on $[c, d]$. (See Figure 4.8.)

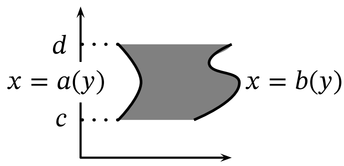


Figure 4.8. x-regular region

A region which is both *x- and y-regular is (simply)* ***regular*** *(see Figure 4.9), and regions of* either *type are called* ***elementary regions****.*

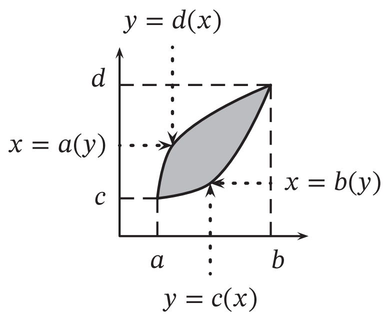


Figure 4.9. Regular region

Basically, a region is y-regular if first, every line parallel to the y-axis intersects the region, if at all, in a closed interval, and second, if each of the endpoints of this interval vary continuously as functions of the x-intercept of the line.

Integration over elementary regions can be done via iterated integrals. We illustrate with an example.

Let $\mathcal{D}$ be the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$. $\mathcal{D}$ is is a regular region, bounded by the line $y = x$, the x-axis ($y = 0$) and the line $x = 1$ (Figure 4.10). Slicing vertically, it can be specified by the pair of inequalities

$$0 \le y \le x$$
$$0 \le x \le 1$$

or, slicing horizontally, by the inequalities

$$y \le x \le 1$$
$$0 \le y \le 1.$$

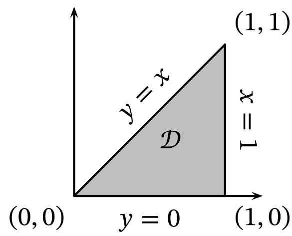


Figure 4.10. The triangle $\mathcal{D}$

To integrate the function $f(x) = 12x^2 + 6y$ over $\mathcal{D}$, we can enclose $\mathcal{D}$ in the rectangle $[0,1]\times[0,1]$ and then integrate $f \restriction_{x,y}$ over this rectangle, slicing vertically. This leads to the double integral

$$\int_0^1 \int_0^1 f \restriction_{\mathcal{D}} \, dy\, dx.$$

Now, since the function $f \restriction_{\mathcal{D}}$ is defined in pieces,

$$f \restriction_{\mathcal{D}} (x,y) = \begin{cases} 12x^2 + 6y & \text{if } 0 \le y \le x \text{ and } x \in [0,1], \\ 0 & \text{otherwise,} \end{cases}$$

the "inner" integral $\int_0^1 f \restriction_{\mathcal{D}} \, dy$ (with $x \in [0,1]$ fixed) can be broken into two parts

$$\int_0^1 f \restriction_{\mathcal{D}} \, dy = \int_0^x (12x^2 + 6y)\, dy + \int_x^1 (12x^2 + 6y)\, dy$$

and since the integrand is zero in the second integral, we can write

$$\int_0^1 f \restriction_{\mathcal{D}} \, dy = \int_0^x (12x^2 + 6y)\, dy = \left(12x^2 y + 3y^2\right)_{y=0}^{y=x} = (12x^3 + 3x^2) - (0).$$

Now, we can write the "outer" integral as

$$\int_0^1 \left(\int_0^1 f \restriction_{\mathcal{D}} \, dy \right) dx = \int_0^1 \int_0^x (12x^2 + 6y)\, dy\, dx = \int_0^1 (12x^3 + 3x^2)\, dx$$
$$= (3x^4 + x^3)_{x=0}^{x=1} = (3 + 1) - (0) = 4.$$

Alternatively, if we slice horizontally, we get the "inner" integral (with $y \in [0,1]$ fixed)

$$\int_0^1 f \restriction_{\mathcal{D}} \, dx = \int_0^y (12x^2 + 6y)\, dy + \int_y^1 (12x^2 + 6y)\, dx = \int_y^1 (12x^2 + 6y)\, dx$$

(since $f \restriction_{\mathcal{D}} (x)$ is zero to the left of $x = y$)

$$= (4x^3 + 6xy)_{x=y}^{x=1} = (4 + 6y) - (4y^3 + 6y^2).$$

Then the outer integral is

$$\int_0^1 \left(\int_0^1 f \restriction_{\mathcal{D}} dx \right) dy = \int_0^1 \int_y^1 (12x^2) = \int_0^1 (4 + 6y - 4y^3 - 6y^2)\, dy$$
$$= (4y + 3y^2 - y^4 - 2y^3)_{y=0}^{y=1} = (4 + 3 - 1 - 2) - (0) = 4.$$

The procedure illustrated by this example is only a slight modification of what we do to integrate over a rectangle. In the case of a rectangle, the "inner" integral has fixed limits, and we integrate regarding all but one variable in the integrand as constant. The result is a function of the other variable. When integrating over a y-regular region, the limits of the inner integration may also depend on x, but we regard x as fixed in both the limits and the integrand; this still yields an integral that depends on the value of x—that is, it is a function of x alone—and in the "outer" integral we simply integrate this function with respect to x, with fixed (numerical) limits. The analogous procedure, with the roles of x and y reversed, results from slicing horizontally, when $\mathcal{D}$ is x-regular.

We illustrate with some further examples.

Let $\mathcal{D}$ be the region bounded by the curves $y = x + 1$ and $y = x^2 - 1$; to find their intersection, we solve $x + 1 = x^2 - 1$ or $x^2 - x - 2 = 0$ whose solutions are $x = -1, 2$. (see Figure 4.11).

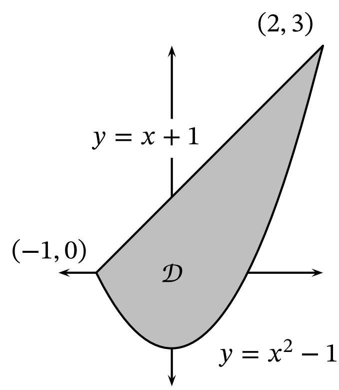


Figure 4.11. The region $\mathcal{D}$

To calculate the integral

$$\iint_{\mathcal{D}} (x + 2y)\, dA$$

over this region, which is presented to us in y-regular form, we slice vertically: a vertical slice is determined by an x-value, and runs from $y = x^2 - 1$ up to $y = x + 1$; the possible x-values run from $x = -1$ to $x = 2$ (Figure 4.12).

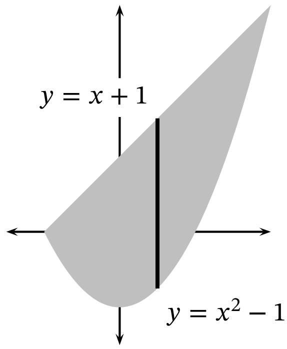


Figure 4.12. Vertical Slice

This leads to the double integral

$$\int_1^2 \int_{x^2-1}^{x+1} (x + 2y)\,dy\,dx = \int_{-1}^2 (xy + y^2)_{y=x^2-1}^{y=x+1}\,dx$$
$$= \int_{-1}^2 \left[\{x(x+1) + (x+1)^2\} - \{x(x^2-1) + (x^2-1)^2\}\right] dx$$
$$= \int_{-1}^2 \left[\{x^2 + x + x^2 + 2x + 1\} - \{x^3 - x + x^4 - 2x^2 + 1\}\right] dx$$
$$= \int_{-1}^2 \left[-x^4 - x^3 + 4x^2 + 4x\right] dx = \left[-\frac{x^5}{5} - \frac{x^4}{4} + \frac{4x^3}{3} + 2x^2\right]_{-1}^2$$
$$= \left[-\frac{32}{5} - 4 + \frac{32}{3} + 8\right] - \left[\frac{1}{5} - \frac{1}{4} - \frac{4}{3} + 2\right] = \frac{153}{20} = 7\frac{13}{20}.$$

Now technically, the region $\mathcal{D}$ is also x-regular, but horizontal slices are much more cumbersome: horizontal slices *below* the x-axis run between the two solutions of $y = x^2 - 1$ for x in terms of y, which means the horizontal slice at height $-1 \le y \le 0$ runs from $x = -\sqrt{y+1}$ to $x = \sqrt{y+1}$, while horizontal slices *above* the x-axis at height $0 \le y \le 3$ run from $x = y - 1$ to $x = \sqrt{y+1}$ (Figure 4.13).

This leads to the pair of double integrals

$$\int_{-1}^0 \int_{-\sqrt{y+1}}^{\sqrt{y+1}} (x + 2y)\,dx\,dy + \int_0^3 \int_{y-1}^{\sqrt{y+1}} (x + 2y)\,dx\,dy$$

which is a lot messier than the previous calculation.

As another example, let us find the volume of the simplex (or "pyramid") cut off from the first octant by the triangle with vertices one unit out along each coordinate axis (Figure 4.14). The triangle is the graph of a linear function $z = ax + by + c$

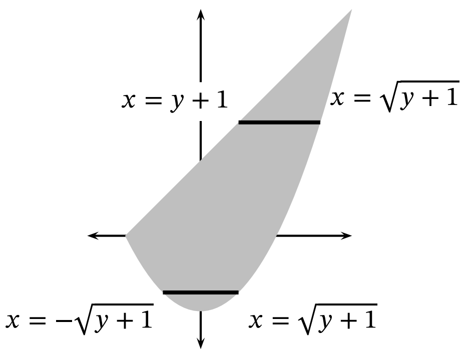


Figure 4.13. Horizontal Slices

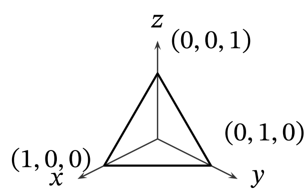


Figure 4.14. Simplex

satisfying

$$\begin{aligned} 0 &= a + c \\ 0 &= b + c \\ 1 &= c \end{aligned}$$

so the graph in question is $z = -x - y + 1$. We are interested in the integral of this function over the triangle T in the xy-plane with vertices at the origin, $(1, 0)$ and $(0, 1)$ (Figure 4.15). It is fairly easy to see that the upper edge of this triangle has equation

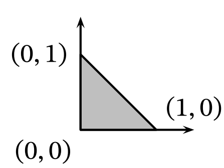


Figure 4.15. The base of the simplex, the triangle T

$x + y = 1$, so T is described by the (y-regular) inequalities $0 \le y \le 1 - x$, $0 \le x \le 1$; that is, a vertical slice at $0 \le x \le 1$ runs from $y = 0$ to $y = 1 - x$. Hence the volume in question is given by the integral

$$\iint_T (1-x-y)\,dA = \int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx$$
$$= \int_0^1 \left(y - xy - \frac{y^2}{2}\right)_{y=0}^{y=1-x} dx = \int_0^1 \left((1-x) - x(1-x) - \frac{(1-x)^2}{2}\right) dx$$
$$= \int_0^1 \left(\frac{(1-x)^2}{2}\right) dx = -\frac{(1-x)^3}{6}\Big|_0^1 = \frac{1}{6}.$$

Sometimes the integral dictates which way we slice. For example, consider the integral $\iint_{\mathcal{D}} \sqrt{a^2-y^2}\,dA$ where $\mathcal{D}$ is the part of the circle $x^2+y^2 = a^2$ in the first quadrant (Figure 4.16). The y-regular description of this region is $0 \le y \le \sqrt{a^2-x^2}$, $0 \le x \le a$

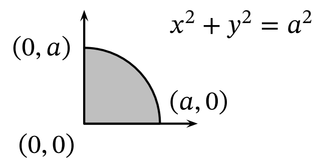


Figure 4.16. The quarter-circle $\mathcal{D}$

which leads to the double integral $\iint_{\mathcal{D}} \sqrt{a^2-y^2}\,dA = \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-y^2}\,dy\,dx$; the inner integral can be done, but requires a trigonometric substitution (and the subsequent evaluation at the limits is a real mess). However, if we consider the region as x-regular, with description $0 \le x \le \sqrt{a^2-y^2}$, $0 \le y \le a$ we come up with the double integral $\iint_{\mathcal{D}} \sqrt{a^2-y^2}\,dA = \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-y^2}\,dx\,dy$; since the integrand is constant as far as the inner integral is concerned, we can easily integrate this:

$$\int_0^a \left(\int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-y^2}\,dx\right) dy = \int_0^a \left(x\sqrt{a^2-y^2}\right)_{x=0}^{x=\sqrt{a^2-y^2}} dy$$
$$= \int_0^a \left(\sqrt{a^2-y^2}\right)^2 dy = \int_0^a (a^2-y^2)\,dy$$
$$= \left(a^2y - \frac{y^3}{3}\right)_{y=0}^{a} = \left(a^3 - \frac{a^3}{3}\right) - (0) = \frac{2a^3}{3}.$$

This illustrates the usefulness of reinterpreting a double integral geometrically and then switching the order of iterated integration. As another example, consider the double integral

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\,dx\,dy.$$

Here, the inner integral is impossible.[9] However, the double integral is the x-regular version of

$$\iint_{\mathcal{D}} \frac{\sin x}{x}\,dA,$$

[9] Of course, it is also an improper integral.

where $\mathcal{D}$ is the triangle in Figure 4.10, and $\mathcal{D}$ can also be described in y-regular form

$$\begin{aligned} 0 \leq &y \leq x \\ 0 \leq &x \leq 1 \end{aligned}$$

leading to the double integral

$$\iint_{\mathcal{D}} \frac{\sin x}{x}\, dA = \int_0^1 \int_0^x \frac{\sin x}{x}\, dy\, dx.$$

Since the integrand in the *inner* integral is regarded as constant, this can be integrated easily:

$$\begin{aligned} \int_0^1 \left(\int_0^x \frac{\sin x}{x}\, dy \right) dx = \int_0^1 \left(\frac{\sin x}{x} \cdot y \right)_{y=0}^{y=x} dx \\ = \int_0^1 \sin x\, dx = -\cos x \Big|_0^1 = 1 - \cos 1. \end{aligned}$$

Symmetry Considerations. You may recall from single-variable calculus that some integrals can be simplified with the help of symmetry considerations.

The clearest instance is that of an **odd** function—that is, a function satisfying $f(-x) = -f(x)$ for all x integrated over an interval that is symmetric about the origin, $[-a, a]$: the integral is necessarily zero. To see this, we note that given a partition $\mathcal{P}$ of $[-a, a]$, we can refine it by throwing in the negative of each point of $\mathcal{P}$ together with zero; for this refinement, every atom $I_j = [p_{j-1}, p_j]$ to the right of zero ($p_j > 0$) is matched by another atom $I_{j^*} = [-p_j, -p_{j-1}]$ to the left of zero. If we use sample points also chosen symmetrically (the point in I_{j^*} is the negative of the one in I_j), then in the resulting Riemann sum, the contributions of matching atoms will cancel. Thus for example, even though we can't find an antiderivative for $f(x) = \sin x^3$, we know that it is odd, so automatically $\int_{-1}^{1} \sin x^3\, dx = 0$.

A related argument says that the integral of an **even** function over a symmetric interval $[-a, a]$ equals twice its integral over either of its halves, say $[0, a]$.

These kinds of arguments can be extended to multiple integrals. A planar region $\mathcal{D}$ is x-**symmetric** if it is invariant under reflection across the y-axis—that is, if $(-x, y) \in \mathcal{D}$ whenever $(x, y) \in \mathcal{D}$. A function $f(x, y)$ is **odd in** x if $f(-x, y) = -f(x, y)$ for all (x, y); it is **even in** x if $f(-x, y) = f(x, y)$ for all (x, y). In particular, a polynomial in x and y is odd (*resp.* even) in x if every power of x which appears is odd (*resp.* even).

The one-variable arguments can be applied to an iterated integral (Exercise 7) to give

Remark 4.2.5. *If $\mathcal{D}$ is an x-regular region which is x-symmetric, then*

(1) *For any function $f(x, y)$ which is odd in x,*

$$\iint_{\mathcal{D}} f\, dA = 0.$$

(2) *If $f(x, y)$ is even in x, then*

$$\iint_{D} f(x, y)\, dA = 2 \iint_{D^+} f(x, y)\, dA,$$

where

$$D^+ = \{\vec{x} = (x, y) \in D \mid x \geq 0\}$$

is the part of D to the right of the y-axis.

Of course, x can be replaced by y in the above definitions, and then also in Remark 4.2.5. One can also consider symmetry involving both x and y; see Exercise 10.

Exercises for § 4.2

Answers to Exercises 1a, 2a, 3a, 4a, and 9a are given in Appendix A.13.

Practice problems:

(1) In the following, specify any region which is elementary by inequalities of the type given in Definition 4.2.4; subdivide any non-elementary region into non-overlapping elementary regions and describe each by such inequalities.

(a) The region bounded above by $y = 9 - x^2$, below by $y = x^2 - 1$ and on the sides by the y-axis and the line $x = 2$.

(b) The unit disc $\{(x, y) \mid x^2 + y^2 \leq 1\}$.

(c) The part of the unit disc above the x-axis.

(d) The part of the unit disc in the first quadrant.

(e) The part of the unit disc in the second quadrant (to the left of the first).

(f) The triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 3)$.

(g) The triangle with vertices $(-1, 0)$, $(1, 0)$, and $(0, 1)$.

(h) The region bounded above by $x + y = 5$ and below by $y = x^2 - 1$.

(i) The region bounded by $y = x^2$ and $x = y^2$.

(j) The region bounded by the curve $y = x^3 - 4x$ and the x-axis.

(2) Each region described below is regular. If it is described as a y-regular (*resp.* x-regular) region, give its description as an x-regular (*resp.* y-regular) region.

(a) $\begin{cases} 0 \leq y \leq 2x \\ 0 \leq x \leq 1 \end{cases}$ (b) $\begin{cases} 0 \leq y \leq 2 - x \\ 0 \leq x \leq 2 \end{cases}$

(c) $\begin{cases} x^2 \leq y \leq x \\ 0 \leq x \leq 1 \end{cases}$ (d) $\begin{cases} -\sqrt{4 - y^2} \leq x \leq \sqrt{4 - y^2} \\ -2 \leq y \leq 2 \end{cases}$

(e) $\begin{cases} 0 \leq x \leq \sqrt{4 - y^2} \\ -2 \leq y \leq 2 \end{cases}$

(3) Calculate each iterated integral below.

(a) $\int_0^1 \int_x^1 (x^2 y + x y^2)\, dy\, dx$ (b) $\int_1^e \int_0^{\ln x} x\, dy\, dx$

(c) $\int_1^2 \int_x^{x^2} (x - 5y)\, dy\, dx$ (d) $\int_0^2 \int_0^y (2xy - 1)\, dx\, dy$

(4) Calculate $\iint_D f\, dA$ as indicated.

(a) $f(x, y) = 4x^2 - 6y$, D described by $\begin{cases} 0 \leq y \leq x \\ 0 \leq x \leq 2 \end{cases}$

(b) $f(x, y) = y\sqrt{x^2 + 1}$, D described by $\begin{cases} 0 \leq y \leq \sqrt{x} \\ 0 \leq x \leq 1 \end{cases}$

(c) $f(x, y) = 4y + 15$, D described by $\begin{cases} y^2 + 2 \leq x \leq 3y \\ 1 \leq y \leq 2 \end{cases}$

(d) $f(x, y) = x$, D is the region bounded by $y = \sin x$, the x-axis and $x = \pi/2$.

(e) $f(x, y) = xy$, D is the region bounded by $x + y = 5$ and $y = x^2 - 1$.

(f) $f(x,y) = 1$, D is the intersection of the discs given by $x^2 + y^2 \leq 1$ and $x^2 + (y-1)^2 \leq 1$.

(5) Rewrite each iterated integral with the order of integration reversed, and calculate it both ways (note that you should get the same answer both ways!)

(a) $\int_0^2 \int_x^2 xy\,dy\,dx$ (b) $\int_0^1 \int_{x^2}^{\sqrt{x}} x\,dy\,dx$

(c) $\int_0^1 \int_{1-y}^{\sqrt{1-y^2}} y\,dx\,dy$ (d) $\int_{-1}^2 \int_1^{\sqrt{3-y}} x\,dx\,dy$

(6) For each region below, decide whether it is x-symmetric, y-symmetric, or neither:

(a) $\{(x,y)\,|\,x^2 + y^2 \leq 1\}$

(b) $\left\{(x,y)\,|\,\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\right\}$ $(a^2 \neq b^2)$

(c) $\{(x,y)\,|\,-1 \leq xy \leq 1\}$

(d) $\{(x,y)\,|\,0 \leq xy \leq 1\}$

(e) $\{(x,y)\,|\,|y| \leq |x|,\quad |x| \leq 1\}$

(f) The region bounded by the lines $x + y = 1$, $x + y = -1$, and the coordinate axes.

(g) The region bounded by the lines $x + y = 1$, $x + y = -1$, $x - y = -1$ and $x - y = 1$.

(h) The region bounded by the lines $y = x + 1$, $y = 1 - x$, and $y = 0$.

(i) The region bounded by the lines $x = 1$, $y = 2$, $x = -1$, and $y = -2$.

(j) The triangle with vertices $(-1,0)$, $(0,-1)$, and $(1,1)$.

(k) The triangle with vertices $(-1,-1)$, $(-1,1)$, and $(1,0)$.

(l) The inside of the rose $r = \cos 2\theta$

(m) The inside of the rose $r = \sin 2\theta$

(n) The inside of the rose $r = \cos 3\theta$

(o) The inside of the rose $r = \sin 3\theta$

Theory problems:

(7) (a) Show that a polynomial in x and y is odd (*resp.* even) in x if and only if each power of x which appears is odd (*resp.* even).

(b) Prove Remark 4.2.5.

(c) Formulate the analogous concepts and results for regions symmetric in y, etc.

(8) (a) Show that if $f(x,y)$ is even in x, then the region $\{(x,y)\,|\,f(x,y) \leq c\}$ for any c is x-symmetric.

(b) Show that if $f(x,y)$ is even, then the region $\{(x,y)\,|\,f(x,y) \leq c\}$ for any c is symmetric with respect to the origin.

(9) Use symmetry considerations either to conclude that the given iterated integral is zero, or to rewrite it as twice a different iterated integral.

(a) $\int_{-1}^1 \int_{x^2-1}^{1-x^2} xy\,dy\,dx$ (b) $\int_{-1}^1 \int_{-\cos x}^{\cos x} (x+y)\,dy\,dx$

(c) $\int_{-2}^1 \int_{-x^3-3x^2-1}^{x^3+3x^2+1} x^2y\,dy\,dx$ (d) $\int_{-2}^2 \int_{x^2-6}^{2-x^2} (xy^2 + x^3y)\,dy\,dx$

(e) $\int_{-1}^1 \int_{1-x^2}^{4-4x^2} x^2y\,dy\,dx$ (f) $\int_{-1}^1 \int_{x^2-1}^{|1-x|} \sin x^3\,dy\,dx$

Challenge problem:

(10) (a) A planar region $\mathcal{D}$ is **symmetric with respect to the origin** if $(-x,-y) \in \mathcal{D}$ whenever $(x,y) \in \mathcal{D}$.

(i) Show that a region which is both x-symmetric and y-symmetric is also symmetric with respect to the origin.
(ii) Give an example of a region which is symmetric with respect to the origin but neither x-symmetric nor y-symmetric.

(b) For each of the regions in Exercise 6, decide whether or not it is symmetric about the origin.

(c) A function $f(x,y)$ of two variables is **odd** (*resp.* **even**) if

$$f(-x,-y) = -f(x,y) \quad (\textit{resp. } f(-x,-y) = f(x,y))$$

for all (x,y).
(i) Show that a function which is both even in x and even in y is even.
(ii) What about a function which is both odd in x and odd in y?
(iii) Show that a polynomial is odd (*resp.* even) precisely if each term has even (*resp.* odd) degree (the degree of a term is the sum of the powers appearing in it).

(d) Show that the integral of an odd function over an elementary region which is symmetric with respect to the origin equals zero.

4.3 Changing Coordinates

Substitution in a Double Integral. Recall that when we perform a substitution $x = \varphi(t)$ inside an integral $\int_a^b f(x)\,dx$, it is not enough to just rewrite $f(x)$ in terms of t (as $f(\varphi(t))$); we also need to express the limits of integration and the "differential" term dx in terms of t. For double (and triple) integrals, this process is a little more complicated; this section is devoted to understanding what is needed.

A substitution in a double integral $\iint_D f(x,y)\,dA$ consists of a pair of substitutions,

$$x = \varphi_1(s,t), \quad y = \varphi_2(s,t). \tag{4.5}$$

This can be viewed as a vector-valued function $\Phi(s,t) = (\varphi_1(s,t), \varphi_2(s,t))$ on the (s,t)-plane, taking values in the (x,y)-plane (also referred to as a **mapping** or **transformation** of the plane). We will use the notation $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ to indicate that the vector-valued function Φ has inputs and outputs in $\mathbb{R}^2$; if we wish to specify its domain (and possibly its image) we will write $\Phi\colon \mathcal{D} \to \tilde{\mathcal{D}}$, where $\tilde{\mathcal{D}} = \Phi(\mathcal{D})$.

The transformation $\Phi(s,t) = (\varphi_1(s,t), \varphi_2(s,t))$ is **linear** (*resp.* **affine**) if each of the coordinate functions $\varphi_i(s,t)$, $i = 1,2$ is linear (*resp.* affine). This means that the coordinate functions of a linear transformation are homogeneous polynomials of degree one in s and t. In vector terms, this means that for any two vectors $\vec{v_i}$ and scalars a_i, $i = 1,2$,

$$\Phi\left(a_1\vec{v_1} + a_2\vec{v_2}\right) = a_1\Phi\left(\vec{v_1}\right) + a_2\Phi\left(\vec{v_2}\right).$$

An affine transformation is simply a linear one plus a (vector) constant.

If $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation of the plane, with coordinate functions $\phi_i(s,t) = a_{i1}s + a_{i2}t$ for $i = 1,2$, a convenient way to display this data is the 2×2 matrix whose i^{th} row is the matrix representative of ϕ_i; this is the **matrix representative** of Φ

$$[\Phi] := \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

We summarize a few useful observations about the use of $[\Phi]$ which are discussed in more detail in Appendix A.5:

- The coordinate column of $\Phi(\vec{v})$, where $\vec{v} = v_1\vec{\imath}+v_2\vec{\jmath}$, is given by the matrix product
$$[\Phi(\vec{v})] = [\Phi]\,[\vec{v}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} a_{11}v_1 & a_{12}v_2 \\ a_{21}v_1 & a_{22}v_2 \end{bmatrix}.$$
- The columns of $[\Phi]$ are the coordinate columns of the standard basis:
$$[\Phi] = \begin{bmatrix} [\Phi(\vec{\imath})] & [\Phi(\vec{\jmath})] \end{bmatrix}.$$

We can think of a transformation $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ as a parametrization of (part of) the (x, y)-plane, regarded as a surface, and the analysis in § 3.6 carries over naturally to our setting: the vector-valued function Φ is **differentiable** (*resp.* $\mathcal{C}^1$) if each of its coordinate functions $\varphi_i(s,t)$, $i = 1, 2$ is differentiable (*resp.* $\mathcal{C}^1$); the linear transformation $D\Phi_{\vec{a}}(\triangle s, \triangle t) = (d\varphi_1(\triangle s, \triangle t), d\varphi_2(\triangle s, \triangle t))$ is the **derivative** of Φ at $\vec{a}$, and is characterized by the fact that the affine transformation $T_{\vec{a}}\Phi(s,t) = \Phi(\vec{a}) + D\Phi_{\vec{a}}(\triangle s, \triangle t)$ has first-order contact with Φ at $\vec{a}$.

The **partials** of Φ are the two vectors
$$\frac{\partial\Phi}{\partial s} = \left(\frac{\partial\varphi_1}{\partial s}, \frac{\partial\varphi_2}{\partial s}\right), \quad \frac{\partial\Phi}{\partial t} = \left(\frac{\partial\varphi_1}{\partial t}, \frac{\partial\varphi_2}{\partial t}\right);$$
it is easy to see that
$$D\Phi_{\vec{a}}(\triangle s, \triangle t) = \triangle s\frac{\partial\Phi}{\partial s} + \triangle t\frac{\partial\Phi}{\partial t}. \tag{4.6}$$

A point in the domain of a $\mathcal{C}^1$ transformation is a **regular point** if its partials at that point are linearly independent; it is a **critical point** or **singular point** if they are dependent (including one of them being the zero vector). The $\mathcal{C}^1$ condition implies that all points sufficiently close to a regular point are also regular.

In our situation, we need to be able to solve the pair of substitution equations Equation (4.5) for s and t in terms of x and y, which means our function Φ must be **one-to-one**:[10] different vector inputs (pairs (s,t) of values for the input) must lead to different vector outputs. The regularity condition at $\vec{a}$ is (by Equation (4.6)) the requirement that every nonzero vector is taken by $D\Phi_{\vec{a}}$ to a nonzero vector. A little work (see Proposition A.5.1) shows that this is the same as requiring either one of: (i) different vectors go to different vectors under $D\Phi_{\vec{a}}$ (*i.e.*, $D\Phi_{\vec{a}}$ is one-to-one); or equivalently, (ii) every vector in the plane is the image of some vector under $D\Phi_{\vec{a}}$ (*i.e.*, $D\Phi_{\vec{a}}$ maps the plane onto the plane). Φ is **regular** if every point in its domain is a regular point. Since $D\Phi_{\vec{a}}$ is a linear approximation of Φ near $\vec{a}$, an easy adaptation of the proof of Proposition 3.6.2 shows that a regular transformation Φ is *locally* one-to-one.

However, we need to require a stronger, *global* one-to-one condition: that *any* pair of distinct inputs to Φ results in distinct outputs.

Definition 4.3.1. *A **coordinate transformation** on a region $\mathcal{D} \subset \mathbb{R}^2$ is a regular $\mathcal{C}^1$ vector-valued function Φ with planar inputs and outputs, which is one-to-one on its domain $\mathcal{D}$. It maps $\mathcal{D}$ onto the planar region $\Phi(\mathcal{D})$*

The global one-to-one condition allows us to define an inverse function on $\Phi(\mathcal{D})$ by
$$\Phi^{-1}(x,y) = (s,t) \Leftrightarrow (x,y) = \Phi(s,t)$$

---

[10]The French-derived term for one-to-one is **injective**.

The **Inverse Mapping Theorem** (Theorem A.6.1), which we discuss (without proof) in Appendix A.6 shows that if the original map Φ is at least $\mathcal{C}^1$ (all partials are continuous), then the regularity condition ensures that the inverse will also be differentiable, in analogy with the recalibration function for parametrizations of curves in § 2.2.

This lets us move back and forth between expressions in s and t and expressions in x and y without worrying about technical issues of differentiability.

We expect the integrand $f(x, y)$ in our integral to be replaced by a function of s and t; in fact it is pretty clear that the natural choice is $(f \circ \Phi)(s, t) = f(\Phi(s, t))$. It is also pretty clear that we need to take as our new domain of integration the domain of our parametrization, and hence we need $\Phi(\mathcal{D}_{s,t})$ to equal our original region of inegration, $\mathcal{D}_{x,y}$.

An example of such a substitution is the switch from rectangular to polar coordinates,

$$x = r\cos\theta, \quad y = r\sin\theta,$$

(so s is r and t is θ) provided we stay within a region $\mathcal{D}_{r,\theta}$ where θ does not increase by as much as 2π radians.

So far, we have seen how to express the integrand $f(x, y)$ of a double integral, as well as the domain of integration, in terms of s and t. It remains to see what to do with the element of area dA. Recalling that this corresponds to the areas $\triangle A_{ij}$ of the atoms of a partition in the construction of the double integral, we need to see how the change of coordinates affects area. That is, we need to know the relation between the area of a set $\mathcal{D}$ and that of its image $\Phi(\mathcal{D})$ under a coordinate transformation.

Coordinate Transformations and Area. Following the philosophy that $T_\Phi\vec{a}$ is a good approximation to Φ near $\vec{a}$, we first determine the effect of an *affine* transformation on area. Note that adding a constant to a linear transformation means simply displacing the image of the linear transformation, a move which does not affect area. So it suffices to look at the effect on area of the *linear* transformation $D\Phi_{\vec{a}}$, the derivative of Φ at $\vec{a}$.

If we look at what happens to the unit square $[0,1] \times [0,1]$, whose sides are given by the standard basis vectors $\vec{\imath}$ and $\vec{\jmath}$, it is easy to see that

$$D\Phi_{\vec{a}}(\vec{\imath}) = (\partial\Phi/\partial x), \quad D\Phi_{\vec{a}}(\vec{\jmath})(\partial\Phi/\partial y)$$

and the image of the unit square is a parallelogram whose sides are these two image vectors. But the (signed) area of a parallelogram is given by a determinant, which in this case is the determinant of the partial derivatives of the two component functions, called the **Jacobian determinant** of Φ:

$$J\Phi = \det\begin{pmatrix} \partial\varphi_1/\partial x & \partial\varphi_1/\partial y \\ \partial\varphi_2/\partial x & \partial\varphi_2/\partial y \end{pmatrix}.$$

To get the unsigned area, we take absolute values:

$$\mathcal{A}\left(D\Phi_{\vec{a}}([0,1] \times [0,1])\right) = \left|J\Phi(\vec{a})\right|.$$

It is fairly easy to see from this calculation that in general if R is a rectangle with sides parallel to the coordinate axes (and hence given by scalar multiples of $\vec{\imath}$ and $\vec{\jmath}$), its image under $D\Phi_{\vec{a}}$ scales by the same amount, and so we see that the area of $D\Phi_{\vec{a}}(R)$ is the area of R multiplied by $\left|J\Phi(\vec{a})\right|$. By tiling arguments, the same is true of *any* region:

Proposition 4.3.2. *If R is any planar region, then the effect of the linear transformation $D\Phi_{\vec{a}}$ on area is to multiply it by $|J\Phi(\vec{a})|$:*

$$\mathcal{A}\left(D\Phi_{\vec{a}}(R)\right) = |J\Phi(\vec{a})| \cdot \mathcal{A}(R).$$

Our next goal is to decide what happens to areas under *differentiable* transformations. The description can be complicated for transformations which either have critical points or overlap images of different regions. Thus, we will consider only **coordinate transformations**, as defined in Definition 4.3.1. Since the effect of a (nonlinear) coordinate transformation on area can be different in different parts of a region, we expect to find the overall effect by getting a good handle on the *local* effect and then integrating this over the region. We also expect the local effect to be related to the linearization of the transformation. In § A.7 we establish the following estimate:

Proposition 4.3.3. *Suppose R is a rectangle and Φ is a $\mathcal{C}^1$ coordinate transformation defined on R; then the area of the image of R under Φ satisfies*

$$\min_{\vec{x}\in R} |J\Phi(\vec{x})|\, \mathcal{A}(R) \le \mathcal{A}(\Phi(R)) \le \max_{\vec{x}\in R} |J\Phi(\vec{x})|\, \mathcal{A}(R). \tag{4.7}$$

Change of Coordinates in Double Integrals. A consequence of Proposition 4.3.3 is the following important result.

Theorem 4.3.4 (Change of Coordinates). *Suppose $\mathcal{D}$ is an elementary region, $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is a coordinate transformation defined on $\mathcal{D}$, and $f : \mathbb{R}^2 \to \mathbb{R}$ is a real-valued function which is integrable on $\Phi(\mathcal{D})$.*

Then

$$\iint_{\Phi(\mathcal{D})} f(\vec{x})\, dA = \iint_{\mathcal{D}} f(\Phi(\vec{x}))\, |J\Phi(\vec{x})|\, dA. \tag{4.8}$$

A proof is given in § A.7 in Appendix A.7.

The most frequent example of the situation in the plane handled by Theorem 4.3.4 is calculating an integral in polar instead of rectangular coordinates. You may already know how to integrate in polar coordinates, but here we will see this as part of a larger picture.

Consider the mapping Φ taking points in the (r,θ)-plane to points in the (x,y)-plane (Figure 4.17)

$$\Phi\left(\begin{bmatrix} r \\ \theta \end{bmatrix}\right) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix};$$

this takes horizontal lines (θ constant) in the (r,θ)-plane to rays in the (x,y)-plane and vertical lines (r constant) to circles centered at the origin. Its Jacobian determinant is

$$J\Phi(r,\theta) = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r,$$

so every point except the origin is a regular point. It is one-to-one on any region $\mathcal{D}$ in the (r,θ) plane for which r is always positive and θ does not vary by 2π or more, so on such a region it is a coordinate transformation.

Thus, to switch from a double integral expressing $\iint_{\mathcal{D}} f\, dA$ in rectangular coordinates to one in polar coordinates, we need to find a region $\mathcal{D}'$ in the r,θ-plane on which Φ is one-to-one, and then calculate the alternative integral $\iint_{\mathcal{D}'} (f \circ \Phi)\, r\, dA$. this amounts to expressing the quantity $f(x,y)$ in polar coordinates and then using $r\, dr\, d\theta$ in place of $dx\, dy$.

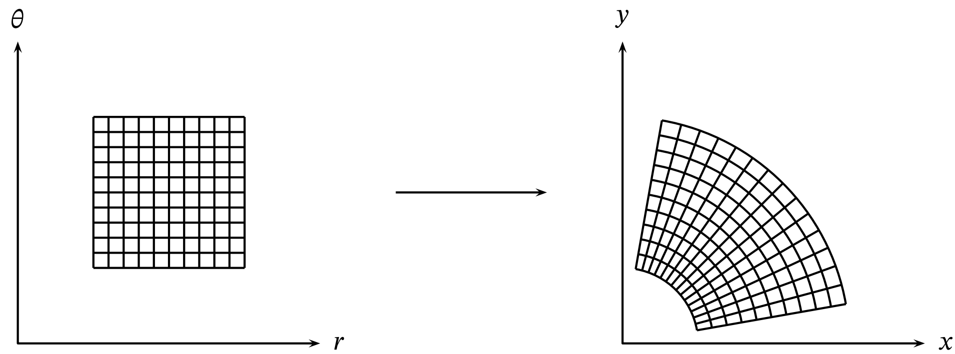


Figure 4.17. The Coordinate Transformation from Polar to Rectangular Coordinates

For example, suppose we want to integrate the function $f(x, y) = 3x + 16y^2$ over the region in the first quadrant between the circles of radius 1 and 2, respectively (Figure 4.18). In rectangular coordinates, this is fairly difficult to describe. Technically, it

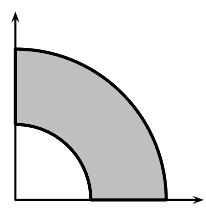


Figure 4.18. Region Between Concentric Circles in the First Quadrant

is x-simple (every vertical line crosses it in an interval), and the top is easily viewed as the graph of $y = \sqrt{4 - x^2}$; however, the bottom is a function defined in pieces:

$$y = \begin{cases} \sqrt{1 - x^2} & \text{for } 0 \leq x \leq 1, \\ 0 & \text{for } 1 \leq x \leq 2. \end{cases}$$

The resulting specification of $\mathcal{D}$ in effect views this as a union of two regions: $\sqrt{1 - x^2} \leq y \leq \sqrt{4 - x^2}$, $0 \leq x \leq 1$ and $0 \leq y \leq \sqrt{4 - x^2}$, $1 \leq x \leq 2$. this leads to the pair of double integrals

$$\iint_{\mathcal{D}} 3x + 16y^2 \, dA = \int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} (3x + 16y^2) \, dy \, dx + \int_1^2 \int_0^{\sqrt{4-x^2}} (3x + 16y^2) \, dy \, dx.$$

By contrast, the description of our region in polar coordinates is easy: $1 \leq r \leq 2$, $0 \leq \theta \leq \frac{p}{2}$, and (using the formal equivalence $dx\,dy = r\,dr\,d\theta$) the integral is

$$\iint_{\mathcal{D}} 3x + 16y^2\,dA = \int_0^{\pi/2} \int_1^2 (3r\cos\theta + 16r^2\sin^2\theta)\,r\,dr\,d\theta$$
$$= \int_0^{\pi/2} \int_1^2 (3r^2\cos\theta + 16r^3\sin^2\theta)\,dr\,d\theta = \int_0^{\pi/2} (r^3\cos\theta + 4r^4\sin^2\theta)_1^2\,d\theta$$
$$= \int_0^{\pi/2} (7\cos\theta + 60\sin^2\theta)\,d\theta = 7\sin\theta\Big|_0^{\pi/2} + 30\int_0^{\pi/2} (1-\cos 2\theta)\,d\theta$$
$$= 7 + 30\left(\theta - \frac{1}{2}\sin 2\theta\right)_0^{\pi/2} = 7 + 15\pi.$$

We note in passing that the requirement that the coordinate transformation be regular and one-to-one on the whole domain can be relaxed slightly: we can allow critical points on the boundary, and also we can allow the boundary to have multiple points for the map. In other words, we need only require that every *interior* point of $\mathcal{D}$ is a regular point of Φ, and that the *interior* of $\mathcal{D}$ maps in a one-to-one way to its image.

Remark 4.3.5. *Suppose $\mathcal{D}$ is an elementary region (or is tiled by a finite union of elementary regions) and $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is a $\mathcal{C}^1$ mapping defined on $\mathcal{D}$ such that*

(1) *Every point interior to $\mathcal{D}$ is a regular point of Φ*

(2) *Φ is one-to-one on the interior of $\mathcal{D}$.*

Then for any function f which is integrable on $\Phi(\mathcal{D})$, Equation (4.8) *still holds.*

To see this, Let $P_k \subset \mathcal{D}$ be polygonal regions formed as nonoverlapping unions of squares inside $\mathcal{D}$ whose areas converge to that of $\mathcal{D}$. Then Theorem 4.3.4 applies to each, and the integral on either side of Equation (4.8) over P_k converges to the same integral over $\mathcal{D}$ (because the function is bounded, and the difference in areas goes to zero).

For example, suppose we want to calculate the volume of the upper hemisphere of radius R. One natural way to do this is to integrate the function $f(x, y) = \sqrt{x^2 + y^2}$ over the disc $\mathcal{D}$ of radius R, which in rectangular coordinates is described by $-\sqrt{R^2 - x^2} \leq y \leq \sqrt{R^2 - x^2}$, $-R \leq x \leq R$, leading to the double integral

$$\iint_{\mathcal{D}} \sqrt{x^2 + y^2}\,dA = \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \sqrt{x^2 + y^2}\,dy\,dx.$$

This is a fairly messy integral. However, if we describe $\mathcal{D}$ in polar coordinates (r, θ), we have the much simpler description $0 \leq r \leq R$, $0 \leq \theta \leq 2\pi$. Now, the coordinate transformation Φ has a critical point at the origin, and identifies the two rays $\theta = 0$ and $\theta = 2\pi$; however, this affects only the boundary of the region, so we can apply our remark and rewrite the integral in polar coordinates. The quantity $\sqrt{x^2 + y^2}$ expressed in polar coordinates is $f(r, \theta) = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} = r$. Then, replacing $dx\,dy$

with $r\,dr\,d\theta$, we have the integral

$$\iint_{\mathcal{D}} (r)(r\,dr\,d\theta)\,dA = \int_0^{2\pi}\int_0^R r^2\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{r^3}{3}\right)_0^R d\theta = \int_0^{2\pi} \left(\frac{R^3}{3}\right) d\theta = \frac{2\pi R^3}{3}.$$

Exercises for § 4.3

Answers to Exercises 1a and 3 are given in Appendix A.13.

Practice problems:

(1) Use polar coordinates to calculate each integral below:
 (a) $\iint_D (x^2+y^2)\,dA$ where D is the annulus specified by $1 \le x^2+y^2 \le 4$.
 (b) The area of one "petal" of the "rose" given in polar coordinates as $r = \sin n\theta$, where n is a positive integer.
 (c) The area of the lemniscate given in rectangular coordinates by $(x^2+y^2)^2 = 2a^2(x^2-y^2)$ where a is a constant. (*Hint:* Change to polar coordinates, and note that there are two equal "lobes"; find the area of one and double.)

(2) Calculate the area of an ellipse in terms of its semiaxes. (*Hint:* There is a simple linear mapping taking a circle centered at the origin to an ellipse with center at the origin and horizontal and vertical axes.)

(3) Calculate the integral $\iint_{[0,1]\times[0,1]} \frac{1}{\sqrt{1+2x+3y}}\,dA$ using the transformation $\varphi(x,y) = (2x, 3y)$, that is, using the substitution $u = 2x$, $v = 3y$.

(4) Calculate $\iint_D (x^2+y^2)\,dA$, where D is the parallelogram with vertices $(0,0)$, $(2,1)$, $(3,3)$, and $(1,2)$, by noting that D is the image of the unit square by the linear transformation $\varphi(s,t) = (2s+t, s+2t)$.

(5) Calculate $\iint_D \frac{1}{(x+y)^2}\,dA$, where D is the region in the first quadrant cut off by the lines $x+y=1$ and $x+y=2$, using the substitution $x = s - st$, $y = st$.

Theory problems:

(6) **Normal Distribution:** In probability theory, when the outcome of a process is measured by a real variable, the statistics of the outcomes is expressed in terms of a **density function** $f(x)$: the probability of an outcome occurring in a given interval $[a,b]$ is given by the integral $\int_a^b f(x)\,dx$. Note that since the probability of *some* outcome is 100% (or, expressed as a fraction, 1), a density function must satisfy

$$\int_{-\infty}^{\infty} f(x)\,dx = 1. \tag{4.9}$$

In particular, when a process consists of many independent trials of an experiment whose outcome can be thought of as "success" or "failure" (for example, a coin toss, where "success" is "heads") then a standard model has a density function of the form

$$f(x) = Ce^{-x^2/2a^2}. \tag{4.10}$$

The constants C and a determine the vertical and horizontal scaling of the graph of $f(x)$, which however is always a "bell curve": the function is positive and even (*i.e.*, its graph is symmetric about $x = 0$—which is the **mean** or expected value—where it has a maximum), and $f(x) \to 0$ as $x \to \pm\infty$.

(a) Show that the function $f(x)$ given by Equation (4.10) has inflection points at $x = \pm a$: this is called the **standard deviation** of the distribution (a^2 is the **variance**).

(b) Given the variance a, we need to *normalize* the distribution: that is, we need to adjust C so that condition (4.9) holds. The Fundamental Theorem of Calculus ensures that the function $f(x)$ does have an antiderivative (*i.e.*, indefinite integral), but it is not **elementary**: it cannot be expressed by a formula using only rational functions, exponentials and logarithms, trigonometric functions and roots. Thus, the Fundamental Theorem of Calculus can't help us calculate C. However, the *definite* integral can be computed directly without going through the antiderivative, by means of a trick:

(i) First, we can regard our integral (which is an improper integral) as coming from a double integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x) f(y)\ dx\,dy = \left(\int_{-\infty}^{\infty} f(x)\ dx\right)\left(\int_{-\infty}^{\infty} f(y)\ dy\right).$$

This improper double integral can be interpreted as the limit, as $R \to \infty$, of the integral of $g(x, y) = f(x) f(y)$ over the square with vertices at $(\pm R, \pm R)$ (that is, the square of side $2R$ centered at the origin:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x) f(y)\ dx\,dy = \lim_{R\to\infty}\int_{-R}^{R}\int_{-R}^{R} f(x) f(y)\ dx\,dy = \lim_{R\to\infty}\iint_{[-R,R]\times[-R,R]} f(x) f(y)\ dA$$

(ii) Justify the claim that this limit is the same as the limit for the double integrals over the circle of radius R centered at the origin:

$$\lim_{R\to\infty}\iint_{[-R,R]\times[-R,R]} f(x) f(y)\ dA = \lim_{R\to\infty}\iint_{\{(x,y)\,|\,x^2+y^2\le R^2\}} f(x) f(y)\ dA.$$

(*Hint:* For a given value of R, find R_- and R_+ so that the circle of radius R lies between the two squares with sides $2R_-$ and $2R_+$, respectively.)

(iii) Calculate the second double integral using polar coordinates, and find the limit as $R \to \infty$.

(iv) This limit is the square of the original integral of $f(x)$. Use this to determine the value of C for which (4.9) holds.

(7) Suppose $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear mapping and the determinant of its matrix representative $[L]$ is positive. Suppose $\triangle ABC$ is a positively oriented triangle in the plane.

(a) Show that the image $L(\triangle ABC)$ is a triangle with vertices $L(A)$, $L(B)$, and $L(C)$.

(b) Show that $\sigma(L(A)L(B)L(C))$ is positive. (*Hint:* Consider the effect of L on the two vectors $\vec{v} = \overrightarrow{AB}$ and $\vec{w} = \overrightarrow{AC}$.)

(c) Show that if the determinant of $[L]$ were *negative*, then $\sigma(L(A)L(B)L(C))$ would be *negative*.

(d) Use this to show that the signed area of $[L(A), L(B), L(C)]$ equals det $[L]$ times the signed area of $[A, B, C]$.

Challenge problem:

(8) Calculate $\iint_D xy^3(1-x^2)\,dA$, where D is the region in the first quadrant between the circle $x^2+y^2=1$ and the curve $x^4+y^4=1$. (*Hint:* Start with the substitution $u=x^2$, $v=y^2$. Note that this is possible only because we are restricted to the first quadrant, so the map is one-to-one.)

4.4 Integration Over Surfaces

Surface Area. In trying to define the area of a surface in $\mathbb{R}^3$, it is natural to try to mimic the procedure we used in § 2.5 to define the length of a curve: recall that we define the length of a curve $\mathcal{C}$ by partitioning it, then joining successive partition points with straight line segments, and considering the total length of the resulting polygonal approximation to $\mathcal{C}$ as an underestimate of its length (since a straight line gives the shortest distance between two points). The length of $\mathcal{C}$ is defined as the supremum of these underestimates, and $\mathcal{C}$ is *rectifiable* if the length is finite. Unfortunately, an example found (simultaneously) in 1892 by Herman Amandus Schwarz (1843-1921) and Giuseppe Peano (1858-1932) says that if we try to define the area of a surface analogously, by taking the supremum of areas of polygonal approximations to the surface, we get the nonsense result that an ordinary cylinder has infinite area. The details are given in Appendix A.8.

As a result, we need to take a somewhat different approach to defining surface area. A number of different theories of area were developed in the period 1890-1956 by, among others, Peano, Lebesgue, Gœ̈czes, Radó, and Cesari. We shall not pursue these general theories of area, but will instead mimic the arclength formula for *regular* curves. All of the theories of area agree on the formula we obtain this way in the case of *regular* surfaces.

Recall that in finding the circumference of a circle, Archimedes used two kinds of approximation: *inscribed* polygons and *circumscribed* polygons. The naive approach above is the analogue of the inscribed approximation: in approximating a (differentiable) planar curve, the Mean Value Theorem ensures that a line segment joining two points on the curve is parallel to the tangent at some point in between, and this ensures that the projection of the arc onto this line segment joining them does not distort distances too badly (provided the arc is not too long). However, as the Schwarz-Peano example shows, this is no longer true for polygons inscribed in surfaces: inscribed triangles, even small ones, can make a large angle (near perpendicularity) with the surface, so projection distorts areas badly, and our intuition that the "area" of a piece of the surface projects nicely onto an inscribed polygon fails. But by definition, *circumscribed* polygons will be tangent to the surface at some point; this means that the projection of every curve in the surface that stays close to the point of tangency onto the tangent plane will make a relatively small angle with the surface, so that projection will not distort lengths or angles on the surface too badly. This is of course just an intuitive justification, but it suggests that we regard the projection of a (small) piece of surface onto the tangent plane at one of its points as a good approximation to the "actual" area.

To be more specific, let us suppose for the moment that our surface $\mathfrak{S}$ is the graph of a differentiable function $z = f(x, y)$ over the planar region $\mathcal{D}$, which for simplicity we take to be a rectangle $[a, b] \times [c, d]$. A partition of $[a, b] \times [c, d]$ divides $\mathcal{D}$ into subrectangles R_{ij}, and we denote the part of the graph above each such subrectangle as a subsurface $\mathfrak{S}_{ij}$ (Figure 4.19). Now we pick a sample point $(x_i, y_j) \in R_{ij}$ in each

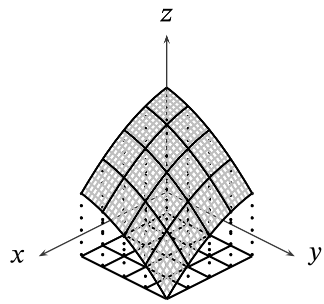


Figure 4.19. Subdividing a Graph

subrectangle of $\mathcal{D}$, and consider the plane tangent to $\mathfrak{S}$ at the corresponding point (x_i, y_j, z_{ij}) $(z_{ij} = f(x_i, x_j))$ of $\mathfrak{S}_{ij}$. The part of this plane lying above R_{ij} is a parallelogram whose area we take as an approximation to the area of $\mathfrak{S}_{ij}$, and we take these polygons as an approximation to the area of $\mathfrak{S}$ (Figure 4.20).

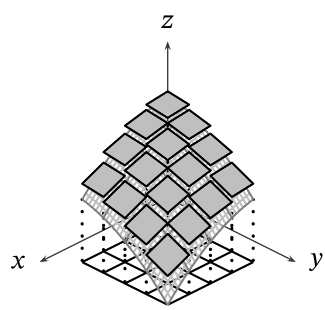


Figure 4.20. Approximating the Area of a Graph

To find the area $\triangle S_{ij}$ of the parallelogram over R_{ij}, we can take as our sample point in R_{ij} its lower left corner; the sides of R_{ij} are parallel to the coordinate axes, so can be denoted by the vectors $\triangle x_i \vec{\imath}$ and $\triangle y_j \vec{\jmath}$. The edges of the parallelogram over R_{ij} are then given by vectors $\overrightarrow{v_x}$ and $\overrightarrow{v_y}$ which project down to these two, but lie in the tangent plane, which means their slopes are the two partial derivatives of f at the sample point (Figure 4.21). Thus,

$$\overrightarrow{v_x} = \left(\vec{\imath} + \frac{\partial f}{\partial x}\vec{k}\right)\triangle x_i = (1, 0, f_x)\triangle x_i$$

$$\overrightarrow{v_y} = \left(\vec{\jmath} + \frac{\partial f}{\partial y}\vec{k}\right)\triangle y_j = (0, 1, f_x)\triangle y_j$$

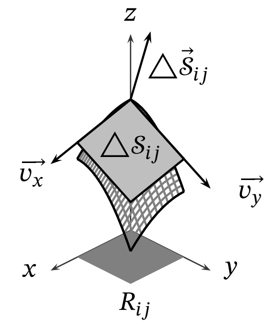


Figure 4.21. Element of Surface Area for a Graph

and the signed area of the parallelogram is

$$\triangle\vec{\mathcal{S}}_{ij} = \vec{v_x} \times \vec{v_y} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} \triangle x_i \triangle y_j = \left(-f_x\vec{\imath} - f_y\vec{\jmath} + \vec{k}\right) \triangle x_i \triangle y_j$$

while the unsigned area is the length of this vector

$$\triangle\mathcal{S}_{ij} = \|\triangle\vec{\mathcal{S}}_{ij}\| = \sqrt{f_x^2 + f_y^2 + 1}\, \triangle x_i \triangle y_j.$$

An alternative interpretation of this is to note that when we push a piece of $\mathcal{D}$ "straight up" onto the tangent plane at (x_i, y_j), its area gets multiplied by the factor $\sqrt{f_x^2 + f_y^2 + 1}$.

Adding up the areas of our parallelograms, we get as an approximation to the area of $\mathfrak{S}$

$$\sum_{i,j} \triangle\mathcal{S}_{ij} \triangle x_i \triangle y_j = \sum_{i,j} \sqrt{f_x^2 + f_y^2 + 1}\, \triangle x_i \triangle y_j.$$

But this is clearly a Riemann sum for an integral, which we take to be the definition of the area

$$\mathcal{A}(\mathfrak{S}) := \iint_{\mathcal{D}} d\mathcal{S}, \quad \text{where } d\mathcal{S} := \sqrt{f_x^2 + f_y^2 + 1}\; ds\, dy. \tag{4.11}$$

$d\mathcal{S}$ is called the **element of surface area** for the graph.

For example, to find the area of the surface $z = \frac{2}{3}\left(x^{3/2} + y^{3/2}\right)$ over the rectangle $\mathcal{D} = [0,1] \times [0,1]$ (Figure 4.22), we calculate the partials of $f(x,y) = \frac{2}{3}\left(x^{3/2} + y^{3/2}\right)$ as $f_x = x^{1/2}$, $f_y = y^{1/2}$, so $d\mathcal{S} = \sqrt{x + y + 1}\, dx\, dy$.

$$\begin{aligned}
\mathcal{A}(\mathfrak{S}) &= \iint_{\mathcal{D}} dS = \int_0^1 \int_0^1 \sqrt{x + y + 1}\, dx\, dy \\
&= \int_0^1 \frac{2}{3}\left((x + y + 1)^{3/2}\right)_{x=0}^{x=1} dy = \int_0^1 \frac{2}{3}\left((y + 2)^{3/2} - (y + 1)^{3/2}\right) dy \\
&= \frac{2}{3} \cdot \frac{2}{5}\left((y + 2)^{5/2} - (y + 1)^{5/2}\right)_0^1 = \frac{2}{15}\left((3^{5/2} - 2^{5/2}) - (2^{5/2} - 1^{5/2})\right) \\
&= \frac{2}{15}(9\sqrt{3} - 8\sqrt{2} + 1).
\end{aligned}$$

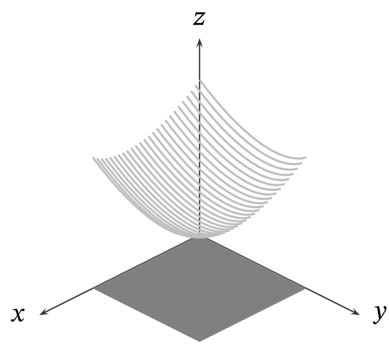


Figure 4.22. $z = \frac{2}{3}(x^{3/2} + y^{3/2})$

We wish to extend our analysis to a general parametrized surface. The starting point of this analysis is the fact that if $\vec{p}\,(s,t)$ is a regular parametrization of the surface $\mathfrak{S}$, $x = x\,(s,t)$, $y = y\,(s,t)$, $z = z\,(s,t)$, then a parametrization of the tangent plane to $\mathfrak{S}$ at $P = \vec{p}\,(s_0, t_0)$ is $T_P\vec{p}\,(s,t) = P + \frac{\partial \vec{p}}{\partial s}\triangle s + \frac{\partial \vec{p}}{\partial t}\triangle t$, that is,

$$\begin{aligned}
x &= x\,(s_0,t_0) + \frac{\partial x}{\partial s}\,(P)\,(s-s_0) + \frac{\partial x}{\partial t}\,(P)\,(t-t_0)\\
y &= y\,(s_0,t_0) + \frac{\partial y}{\partial s}\,(P)\,(s-s_0) + \frac{\partial y}{\partial t}\,(P)\,(t-t_0)\\
z &= z\,(s_0,t_0) + \frac{\partial z}{\partial s}\,(P)\,(s-s_0) + \frac{\partial z}{\partial t}\,(P)\,(t-t_0).
\end{aligned}$$

This defines the **tangent map** $T_P\vec{p}$ of the parametrization, which by analogy with the case of the graph analyzed above corresponds to "pushing" pieces of $\mathcal{D}$, the domain of the parametrization, to the tangent plane. To understand its effect on areas, we note that the edges of a rectangle in the domain of $\vec{p}$ with sides parallel to the s-axis and t-axis, and lengths $\triangle s$ and $\triangle t$, respectively, are taken by the tangent map to the vectors $\frac{\partial \vec{p}}{\partial s}\triangle s$ and $\frac{\partial \vec{p}}{\partial t}\triangle t$, which play the roles of $\overrightarrow{v_x}$ and $\overrightarrow{v_y}$ from the graph case. Thus, the signed area of the corresponding parallelogram in the tangent plane is given by the cross product (Figure 4.23)

$$\triangle\vec{\mathcal{S}} = \left(\frac{\partial \vec{p}}{\partial s}\triangle s\right) \times \left(\frac{\partial \vec{p}}{\partial t}\triangle t\right) = \left(\frac{\partial \vec{p}}{\partial s} \times \frac{\partial \vec{p}}{\partial t}\right)\triangle s\triangle t.$$

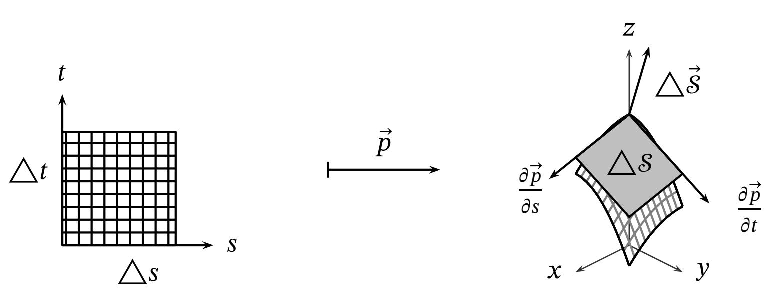


Figure 4.23. Element of Surface Area for a Parametrization

The (unsigned) area is the length of this vector

$$\triangle\mathcal{S} = \left\|\triangle\vec{\mathcal{S}}\right\| = \left\|\frac{\partial\vec{p}}{\partial s}\times\frac{\partial\vec{p}}{\partial t}\right\|\triangle s\triangle t.$$

Again, if we partition the domain of $\vec{p}$ into such rectangles and add up their areas, we are forming a Riemann sum, and as the mesh size of the partition goes to zero, these Riemann sums converge to the integral, over the domain $\mathcal{D}$ of $\vec{p}$, of the function $\left\|\frac{\partial\vec{p}}{\partial s}\times\frac{\partial\vec{p}}{\partial t}\right\|$:

$$\mathcal{A}(\mathfrak{S}) = \iint_{\mathcal{D}}\left\|\frac{\partial\vec{p}}{\partial s}\times\frac{\partial\vec{p}}{\partial t}\right\| dA. \tag{4.12}$$

By analogy with the element of arclength $d\mathfrak{s}$, we denote the integrand above $d\mathcal{S}$; this is the **element of surface area**:

$$d\mathcal{S} = \left\|\frac{\partial\vec{p}}{\partial s}\times\frac{\partial\vec{p}}{\partial t}\right\| dA = \left\|\frac{\partial\vec{p}}{\partial s}\times\frac{\partial\vec{p}}{\partial t}\right\| ds\,dt.$$

We shall see below that the integral in Equation (4.12) is independent of the (regular) parametrization $\vec{p}$ of the surface $\mathfrak{S}$, and we write

$$\mathcal{A}(\mathfrak{S}) = \iint_{\mathfrak{S}} d\mathcal{S}.$$

For future reference, we also set up a vector-valued version of $d\mathcal{S}$, which could be called the **element of oriented surface area**

$$d\vec{\mathcal{S}} = \left(\frac{\partial\vec{p}}{\partial s}\times\frac{\partial\vec{p}}{\partial t}\right) ds\,dt.$$

To see that the definition of surface area given by Equation (4.12) is independent of the parametrization, it suffices to consider two parametrizations of the same coordinate patch, say $\vec{p}(u,v)$ and $\vec{q}(s,t)$. By Corollary A.6.3, we can write $\vec{q} = \vec{p}\circ T$, where $T(s,t) = (u(s,t), v(s,t))$.

By the Chain Rule,

$$\frac{\partial\vec{q}}{\partial s} = \frac{\partial\vec{p}}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial\vec{p}}{\partial v}\frac{\partial v}{\partial s}, \quad \frac{\partial\vec{q}}{\partial t} = \frac{\partial\vec{p}}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial\vec{p}}{\partial v}\frac{\partial v}{\partial t}$$

so the cross product is

$$\begin{aligned}\frac{\partial\vec{q}}{\partial s}\times\frac{\partial\vec{q}}{\partial t} &= \left(\frac{\partial\vec{p}}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial\vec{p}}{\partial v}\frac{\partial v}{\partial s}\right)\times\left(\frac{\partial\vec{p}}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial\vec{p}}{\partial v}\frac{\partial v}{\partial t}\right)\\ &= \left(\frac{\partial u}{\partial s}\frac{\partial u}{\partial t}\right)\left(\frac{\partial\vec{p}}{\partial u}\times\frac{\partial\vec{p}}{\partial u}\right) + \left(\frac{\partial u}{\partial s}\frac{\partial v}{\partial t}\right)\left(\frac{\partial\vec{p}}{\partial u}\times\frac{\partial\vec{p}}{\partial v}\right)\\ &\quad + \left(\frac{\partial v}{\partial s}\frac{\partial u}{\partial t}\right)\left(\frac{\partial\vec{p}}{\partial v}\times\frac{\partial\vec{p}}{\partial u}\right) + \left(\frac{\partial v}{\partial s}\frac{\partial v}{\partial t}\right)\left(\frac{\partial\vec{p}}{\partial v}\times\frac{\partial\vec{p}}{\partial v}\right)\\ &= \left(\frac{\partial u}{\partial s}\frac{\partial v}{\partial t} - \frac{\partial v}{\partial s}\frac{\partial u}{\partial t}\right)\left(\frac{\partial\vec{p}}{\partial u}\times\frac{\partial\vec{p}}{\partial v}\right)\\ &= (\det JT)\left(\frac{\partial\vec{p}}{\partial u}\times\frac{\partial\vec{p}}{\partial v}\right).\end{aligned}$$

Now, by Theorem 4.3.4 (or, if necessary, Remark 4.3.5) we see that the integral over the domain of $\vec{p}$ of the first cross product equals the integral over the domain of $\vec{q}$ of the last cross product, which is to say the two surface area integrals are equal.

As an example, let us find the surface area of the cylinder

$$x^2 + y^2 = 1, \quad 0 \le z \le 1.$$

We use the natural parametrization (writing θ instead of s)

$$x = \cos\theta, \quad y = \sin\theta, \quad z = t$$

with domain $\mathcal{D} = [0, 2\pi]\times[0, 1]$. The partial derivatives of the parametrization $\vec{p}(\theta, t) = (\cos\theta, \sin\theta, t)$ are $\frac{\partial\vec{p}}{\partial\theta} = (-\sin\theta, \cos\theta, 0)$, $\frac{\partial\vec{p}}{\partial t} = (0, 0, 1)$; their cross-product is

$$\frac{\partial\vec{p}}{\partial\theta} \times \frac{\partial\vec{p}}{\partial t} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta)\vec{\imath} + (\sin\theta)\vec{\jmath}$$

so the element of area is

$$d\mathcal{S} = \left\|(\cos\theta)\vec{\imath} + (\sin\theta)\vec{\jmath}\right\| \, d\theta\, dt = d\theta\, dt$$

and its integral, giving the surface area, is

$$\mathcal{A}(\mathfrak{S}) = \iint_{\mathfrak{S}} d\mathcal{S} = \iint_{[0,2\pi]\times[0,1]} d\theta\, dt = \int_0^1 \int_0^{2\pi} d\theta\, dt = \int_0^1 2\pi\, dt = 2\pi$$

which is what we would expect (you can form the cylinder by rolling the rectangle $[0, 2\pi] \times [0, 1]$ into a "tube").

As a second example, we calculate the surface area of a sphere $\mathcal{S}$ of radius R; we parametrize via spherical coordinates:

$$\begin{aligned}
\vec{p}(\phi, \theta) &= (R\sin\phi\cos\theta, R\cos\phi\sin\theta, R\cos\phi); \\
\frac{\partial\vec{p}}{\partial\phi} &= (R\cos\phi\cos\theta, R\cos\phi\sin\theta, -R\sin\phi) \\
\frac{\partial\vec{p}}{\partial\theta} &= (-R\sin\phi\sin\theta, R\sin\phi\cos\theta, 0) \\
\frac{\partial\vec{p}}{\partial\phi} \times \frac{\partial\vec{p}}{\partial\theta} &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ R\cos\phi\cos\theta & R\cos\phi\sin\theta & -R\sin\phi \\ -R\sin\phi\sin\theta & R\sin\phi\cos\theta & 0 \end{vmatrix} \\
&= R^2(\sin^2\phi\cos\theta)\vec{\imath} + R^2(\sin^2\phi\sin\theta)\vec{\jmath} \\
&\quad + R^2(\sin\phi\cos\phi\cos^2\theta + \sin\phi\cos\phi\sin^2\theta)\vec{k}
\end{aligned}$$

so the element of oriented area is

$$d\vec{\mathcal{S}} = R^2(\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi)\, d\phi\, d\theta$$

and the element of area is

$$\begin{aligned}
d\mathcal{S} &= R^2\sqrt{\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \sin^2\phi\cos^2\phi}\, d\phi\, d\theta \\
&= R^2\sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi}\, d\phi\, d\theta = R^2\sqrt{\sin^2\phi}\, d\phi\, d\theta \\
&= R^2\sin\phi\, d\phi\, d\theta
\end{aligned}$$

(where the last equality is justified by the fact that $0 \leq \phi \leq \pi$, so $\sin\phi$ is always non-negative). From this, we have the area integral

$$\mathcal{A}(\mathcal{S}) = \iint_{\mathcal{S}} d\mathcal{S} = \int_0^{2\pi} \int_0^{\pi} R^2 \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} (-R^2 \cos\theta)_0^{\pi} \, d\theta = \int_0^{2\pi} 2R^2 \, d\theta = 4\pi R^2.$$

The alert reader (you!) has undoubtedly noticed a problem with this last calculation. We have used a "spherical coordinates" parametrization of the sphere, but the mapping fails to be one-to-one on parts of the boundary of our domain of integration: $\theta = 0$ and $\theta = 2\pi$, for any value of ϕ, represent the same point, and even worse, at either of the two extreme values of ϕ, $\phi = 0$ and $\phi = \pi$ *all* values of θ represent the same point (one of the "poles" of the sphere). The resolution of this problem is to think in terms of *improper* integrals. If we restrict the domain of integration to a closed rectangle $\alpha \leq \theta \leq \beta$, $\gamma \leq \phi \leq \delta$ with $0 < \alpha < \beta < 2\pi$, $0 < \gamma < \delta < \pi$ then we are using an honest parametrization of a piece of the sphere, and the integral is perfectly OK. Now, we can define *our* integral to be the limit, as $\alpha \to 0$, $\beta \to 2\pi$, $\gamma \to 0$, and $\delta \to \pi$. The important thing to note is that the set of points on the sphere corresponding to the boundary of our domain (the **singular points** of the parametrization) is contained in a curve which can be thought of as the graph of a function, and hence by arguments similar to those in § 4.2 the values on this set will have no effect on the integral. While this is not an entirely rigorous argument, it can be made more precise; we will slide over these kinds of difficulties in other integrals of this kind.

Finally, let us calculate the area of the helicoid (Figure 4.24) $x = r\cos\theta$, $y = r\sin\theta$, $z = \theta$, with domain $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$.

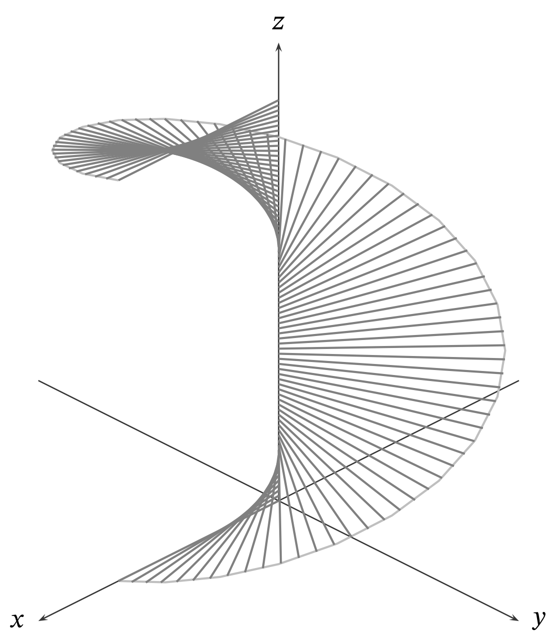


Figure 4.24. Helicoid

The partials of the parametrization $\vec{p}(r,\theta) = (r\cos\theta, r\sin\theta, \theta)$ are $\frac{\partial\vec{p}}{\partial r} = (\cos\theta, \sin\theta, 0)$, $\frac{\partial\vec{p}}{\partial\theta} = (-r\sin\theta, r\cos\theta, 1)$, so

$$\begin{aligned} d\vec{\mathcal{S}} &= (r\cos\theta, r\sin\theta, \theta) \times (-r\sin\theta, r\cos\theta, 1)\, dr\, d\theta \\ &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 1 \end{vmatrix} dr\, d\theta = (\sin\theta)\vec{\imath} - (\cos\theta)\vec{\jmath} + r\vec{k} \end{aligned}$$

and

$$d\mathcal{S} = \left\| (\sin\theta)\vec{\imath} - (\cos\theta)\vec{\jmath} + r\vec{k} \right\| dr\, d\theta = \sqrt{1+r^2}\, dr\, d\theta.$$

The surface area is given by the integral

$$\iint_{\mathfrak{S}} d\mathcal{S} = \int_0^{2\pi} \int_0^1 \sqrt{1+r^2}\, dr\, d\theta;$$

using the substitution $r = \tan\alpha$, $dr = \sec^2\alpha\, d\alpha$, so $\sqrt{1+r^2} = \sec\alpha$, $r = 0 \leftrightarrow \alpha = 0$ and $r = 1 \leftrightarrow \alpha = \frac{\pi}{4}$, the inner integral becomes

$$\begin{aligned} \int_0^1 \sqrt{1+r^2}\, dr &= \int_0^{\pi/4} \sec^3\alpha\, d\alpha \\ &= \frac{1}{2}(\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha|) = \frac{1}{2}\left(\sqrt{2} + \ln(\sqrt{2}+1)\right) \end{aligned}$$

turning the outer integral into

$$\begin{aligned} \int_0^{2\pi} \int_0^1 \sqrt{1+r^2}\, dr\, d\theta &= \int_0^{2\pi} \int_0^{\pi/4} \sec^3\alpha\, d\alpha\, d\theta \\ &= \int_0^{2\pi} \frac{1}{2}\left(\sqrt{2} + \ln(\sqrt{2}+1)\right) d\theta = \pi\left(\sqrt{2} + \ln(\sqrt{2}+1)\right). \end{aligned}$$

We note that for a surface given as the graph $z = f(x,y)$ of a function over a domain $\mathcal{D}$, the natural parametrization is $\vec{p}(s,t) = (s, t, f(s,t))$, with partials $\frac{\partial\vec{p}}{\partial s} = (1, 0, f_x)$ and $\frac{\partial\vec{p}}{\partial t} = (0, 1, f_y)$, so the element of oriented surface area is

$$\begin{aligned} d\vec{\mathcal{S}} &= (1, 0, f_x) \times (0, 1, f_y)\, ds\, dt \\ &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} ds\, dt = -f_x\vec{\imath} - f_y\vec{\jmath} + \vec{k} \end{aligned}$$

and in particular the element of (unoriented) surface area is

$$d\mathcal{S} = \left\| -f_x\vec{\imath} - f_y\vec{\jmath} + \vec{k} \right\| ds\, dt = \sqrt{(f_x)^2 + (f_y)^2 + 1}\, ds\, dt.$$

That is, we recover the formula (4.11) we obtained earlier for this special case.

Another special situation in which the element of surface area takes a simpler form is that of a **revolute** or **surface of revolution**—that is, the surface formed from a plane curve $\mathcal{C}$ when the plane is rotated about an axis that does not cross $\mathcal{C}$ (Figure 4.25). Let us assume that the axis of rotation is the x-axis, and that the curve $\mathcal{C}$ is parametrized by $x = x(t)$, $y = y(t)$, for $a \le t \le b$. Then our assumption is that the axis does not

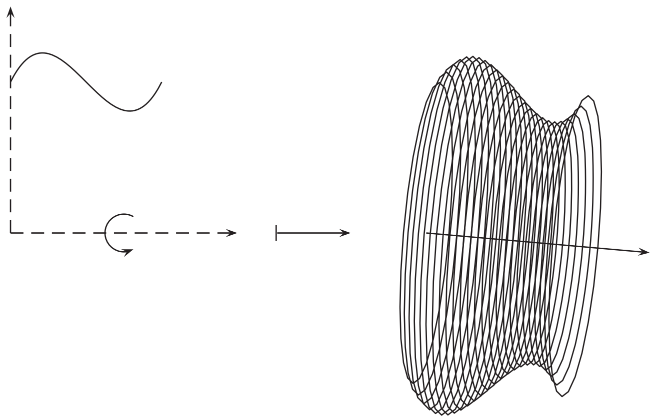


Figure 4.25. Revolute

cross $\mathcal{C}$ is $y(t) \geq 0$. A natural parametrization of the surface of revolution is obtained by replacing the point $(x(t), y(t))$ with a circle, centered at $(x(t), 0, 0)$ and parallel to the yz-plane, of radius $y(t)$; this yields the parametrization $\vec{p}(t, \theta)$ of the revolute given by $x = x(t)$, $y = y(t)\cos\theta$, and $z = y(t)\sin\theta$ with $a \leq t \leq b$, $0 \leq \theta \leq 2\pi$. The partials are $\frac{\partial\vec{p}}{\partial t} = (x'(t), y'(t)\cos\theta, y'(t)\sin\theta)$ and $\frac{\partial\vec{p}}{\partial\theta} = (0, -y(t)\sin\theta, y(t)\cos\theta)$, with cross product

$$\frac{\partial\vec{p}}{\partial t} \times \frac{\partial\vec{p}}{\partial\theta} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ x'(t) & y'(t)\cos\theta & y'(t)\sin\theta) \\ 0 & -y(t)\sin\theta & y(t)\cos\theta) \end{vmatrix} = (yy')\vec{\imath} + (yx')\left[-(\cos\theta)\vec{\jmath} + (\sin\theta)\vec{k}\right]$$

with length $\left\|\frac{\partial\vec{p}}{\partial t} \times \frac{\partial\vec{p}}{\partial\theta}\right\| = y\sqrt{(y')^2 + (x')^2}$. Thus the element of surface area for a surface of revolution is

$$d\mathcal{S} = \left[y\sqrt{(y')^2 + (x')^2}\right] dt\, d\theta \tag{4.13}$$

and the surface area is

$$\begin{aligned} \mathcal{A}(\mathfrak{S}) &= \int_0^{2\pi} \int_a^b \left[y\sqrt{(y')^2 + (x')^2}\right] dt\, d\theta \\ &= 2\pi \int_a^b \left[y\sqrt{(y')^2 + (x')^2}\right] dt. \end{aligned} \tag{4.14}$$

More generally, we should replace $x(t)$ with the projection of a point $\vec{p}(t)$ on the curve $\mathcal{C}$ onto the axis of rotation, and $y(t)$ with its distance from that axis.

For example, the area of the surface obtained by rotating the curve $y = x^2$, $0 \leq x \leq 1$ about the x-axis, using the natural parametrization $x = t$, $y = t^2$, for $0 \leq t \leq 1$ is

$$\begin{aligned} 2\pi \int_0^1 t^2\sqrt{t^2+1}\, dt &= 2\pi \left[\frac{t}{8}(1 + 2t^2\sqrt{t^2+1}) - \frac{t^4}{8}\ln(t + \sqrt{t^2+1})\right]_0^1 \\ &= \frac{\pi}{4}(3\sqrt{2} - \ln(1+\sqrt{2}), \end{aligned}$$

while for the surface obtained by rotating the same surface about the y-axis (using the same parametrization) is

$$2\pi \int_0^1 t\sqrt{1+t^2}\,dt = 2\pi \left[\frac{1}{3}(1+t^2)^{3/2}\right]_0^1$$
$$= \frac{2\pi}{3}\left(2^{3/2} - 1^{3/2}\right) = \frac{2\pi}{3}\left(2\sqrt{2} - 1\right).$$

Surface Integrals. Just as we could use the element of arclength to integrate a function f on $\mathbb{R}^3$ over a curve, so can we integrate this function over a (regular) surface. This can be thought of in terms of starting from a (possibly negative as well as positive) density function to calculate the total mass. Going through the same approximation process that we used to define the surface area itself, this time we sum up the area of small rectangles in the tangent plane at partition points *multiplied by the values of the function* there; this gives a Riemann sum for the **surface integral** of f over the surface, denoted $\iint_{\mathfrak{S}} f\ d\mathcal{S}$. Given a parametrization $\vec{p}(s,t)$ ($(s,t) \in \mathcal{D}$) of the surface $\mathfrak{S}$, the process of calculating the surface integral above is exactly the same as before, except that we also throw in the value $f\left(\vec{p}(s,t)\right)$ of the function.

For example, to calculate the integral of $f(x,y,z) = \sqrt{x^2+y^2+1}$ over the helicoid parametrized by $x = r\cos\theta$, $y = r\sin\theta$, and $z = \theta$, with $0 \le r \le 1$ and $0 \le \theta \le 2\pi$ (which we studied earlier), we recall that $d\mathcal{S} = \sqrt{1+r^2}\,dr\,d\theta$, and clearly

$$f(r\cos\theta, r\sin\theta, \theta) = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + 1} = \sqrt{r^2+1},$$

so our integral becomes

$$\iint_{\mathfrak{S}} \sqrt{x^2+y^2+1}\ d\mathcal{S} = \int_0^{2\pi}\int_0^1 \left(\sqrt{r^2+1}\right)\left(\sqrt{r^2+1}\,dr\,d\theta\right)$$
$$= \int_0^{2\pi}\int_0^1 \left(r^2+1\right)\,dr\,d\theta = \int_0^{2\pi}\left(\frac{r^3}{3}+r\right)_0^1 d\theta = \int_0^{2\pi}\left(\frac{4}{3}\right)d\theta = \frac{8\pi}{3}.$$

As another example, let us calculate the surface integral $\iint_{\mathcal{S}^2} z^2\ d\mathcal{S}$ where $\mathcal{S}^2$ is the unit sphere in $\mathbb{R}^3$.

We can parametrize the sphere via spherical coordinates $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$, $z = \cos\phi$, where $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$; the partials are $\frac{\partial\vec{p}}{\partial\phi} = (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi)$ and $\frac{\partial\vec{p}}{\partial\theta} = (-\sin\phi\sin\theta, \sin\phi\cos\theta, 0)$, so

$$d\vec{\mathcal{S}} = \frac{\partial\vec{p}}{\partial\phi}\times\frac{\partial\vec{p}}{\partial\theta}\,d\phi\,d\theta = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & ,-\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix} d\phi\,d\theta$$
$$= \left(\sin^2\phi\cos\theta\right)\vec{\imath} + \left(\sin^2\phi\sin\theta\right)\vec{\jmath} + (\sin\phi\cos\phi)\,\vec{k}\,d\phi\,d\theta$$

and

$$d\mathcal{S} = \left\|d\vec{\mathcal{S}}\right\| = \sqrt{\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \sin^2\phi\cos^2\phi}\,d\phi\,d\theta$$
$$= \sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi}\,d\phi\,d\theta = |\sin\phi|\ d\phi\,d\theta$$

which, in the range $0 \le \phi \le \pi$, equals $\sin\phi\, d\phi\, d\theta$. Now, our function $f(x,y,z) = z^2$ is $z^2 = \cos^2\phi$, so the integral becomes

$$\iint_{\mathcal{S}^2} z^2\, d\mathcal{S} = \int_0^{2\pi}\int_0^{\pi} (\cos^2\phi)(\sin\phi\, d\phi\, d\theta) = \int_0^{2\pi} -\frac{\cos^3\phi}{3}\Big|_0^{\pi} d\theta = \frac{4\pi}{3}.$$

Exercises for § 4.4

Answers to Exercises 1a, 1f, and 2a are given in Appendix A.13.

Practice problems:

(1) Find the area of each surface below.
 (a) The graph of $f(x,y) = 1 - \frac{x^2}{2}$ over the rectangle $[-1,1]\times[-1,1]$.
 (b) The graph of $f(x,y) = xy$ over the unit disc $x^2 + y^2 \le 1$.
 (c) The part of the paraboloid $z = a^2 - x^2 - y^2$ above the xy-plane.
 (d) The part of the saddle surface $z = x^2 - y^2$ inside the cylinder $x^2 + y^2 = 1$.
 (e) The cone given in cylindrical coordinates by $z = mr$, $r \le R$.
 (f) The part of the sphere $x^2 + y^2 + z^2 = 8$ cut out by the cone $z = \sqrt{x^2 + y^2}$.
 (g) The part of the sphere $x^2 + y^2 + z^2 = 9$ outside the cylinder $4x^2 + 4y^2 = 9$.
 (h) The surface parametrized by

$$\begin{cases} x = s^2 + t^2 \\ y = s - t \\ z = s + t \end{cases}, \quad s^2 + t^2 \le 1.$$

(2) Evaluate each surface integral $\iint_{\mathfrak{S}} f\, d\mathcal{S}$ below.
 (a) $f(x,y,z) = x^2 + y^2$, $\mathfrak{S}$ is the part of the plane $z = x + 2y$ lying over the square $[0,1]\times[0,1]$.
 (b) $f(x,y,z) = xy + z$, $\mathfrak{S}$ is the part of the hyperboloid $z = xy$ over the square $[0,1]\times[0,1]$.
 (c) $f(x,y,z) = xyz$, $\mathfrak{S}$ is the triangle with vertices $(1,0,0)$, $(0,2,0)$, and $(0,0,1)$.
 (d) $f(x,y,z) = z$, $\mathfrak{S}$ is the upper hemisphere of radius R centered at the origin.
 (e) $f(x,y,z) = x^2 + y^2$, $\mathfrak{S}$ is the surface of the cube $[0,1]\times[0,1]\times[0,1]$. (*Hint:* Calculate the integral over each face separately, and add.)
 (f) $f(x,y,z) = \sqrt{x^2 + y^2 + 1}$, $\mathfrak{S}$ is the part of the surface $z = xy$ inside the cylinder $x^2 + y^2 = 1$.
 (g) $f(x,y,z) = z$, $\mathfrak{S}$ is the cone given in cylindrical coordinates by $z = 2r$, $0 \le z \le 2$.

Theory problems:

(3) (a) Give a parametrization of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

 (b) Set up, but do not attempt to evaluate, an integral giving the surface area of the ellipsoid.

(4) (a) Let $\mathfrak{S}$ be the surface obtained by rotating a curve $\mathcal{C}$ which lies in the upper half-plane about the x-axis. Show that the surface area as given by Equation (4.14) is just the path integral

$$\mathcal{A}(\mathfrak{S}) = 2\pi \int_{\mathcal{C}} y \, d\mathfrak{s}.$$

(b) The **centroid** of a curve $\mathcal{C}$ can be defined as the "average" position of points on the curve with respect to arc length; that is, the x-coordinate of the centroid is given by

$$\bar{x} := \frac{\int_{\mathcal{C}} x \, d\mathfrak{s}}{\mathfrak{s}(\mathcal{C})}$$

with analogous definitions for the other two coordinates. This is the "center of gravity" of the curve if it has constant density.

Pappus' First Theorem, given in the *Mathematical Collection* of Pappus of Alexandria (*ca.* 300 AD) and rediscovered in the sixteenth century by Paul (Habakkuk) Guldin (1577-1643), says that the area of a surface of revolution equals the length of the curve being rotated times the distance travelled by its centroid. Prove this result from the preceding.

(5) Suppose $f(x, y, z)$ is a $\mathcal{C}^1$ function for which the partial derivative $\partial f/\partial z$ is nonzero in the region $\mathfrak{D} \subset \mathbb{R}^3$, so that the part of any level surface in $\mathfrak{D}$ can be expressed as the graph of a function $z = \phi(x, y)$ over a region $\mathcal{D}$ in the x, y-plane. Show that the area of such a level surface is given by

$$\mathcal{A}(\mathcal{L}(f, c) \cap \mathfrak{D}) = \iint_{\mathcal{D}} \frac{\left\|\vec{\nabla} f\right\|}{|\partial f/\partial z|} \, dx \, dy.$$

Challenge problems:

(6) (a) Use the parametrization of the torus given in Equation (3.26) to find its surface area.

(b) Do the same calculation using Pappus' First Theorem.

(7) Given a surface $\mathfrak{S}$ parametrized by $\vec{p}(u, v)$, $(u, v) \in \mathcal{D}$, define the functions

$$E = \left\|\frac{\partial \vec{p}}{\partial u}\right\|^2, \quad F = \frac{\partial \vec{p}}{\partial u} \cdot \frac{\partial \vec{p}}{\partial v}, \quad G = \left\|\frac{\partial \vec{p}}{\partial v}\right\|^2.$$

(a) Suppose $\mathcal{C}$ is a curve in $\mathfrak{S}$ given in (u, v) coordinates as $\vec{g}(t) = (u(t), v(t))$, $t_0 \le t \le t_1$—that is, it is parametrized by

$$\gamma(t) = \vec{p}(\vec{g}(t)) = \vec{p}(u(t), v(t)).$$

Show that the speed of $\gamma(t)$ is given by

$$\frac{d\mathfrak{s}}{dt} = \|\gamma'(t)\| = Q(g'(t)),$$

where Q is the quadratic form with matrix representative

$$[Q] = \begin{bmatrix} E & F \\ F & G \end{bmatrix}.$$

This means the length of $\mathcal{C}$ is

$$\mathfrak{s}(\mathcal{C}) = \int_{\mathcal{C}} d\mathfrak{s} = \int_{t_0}^{t_1} \sqrt{E(u')^2 + 2F(u')(v') + G(v')^2} \, dt.$$

The quadratic form Q is called the **first fundamental form** of $\mathfrak{S}$.

(b) Show that for any two vectors $\vec{v}, \vec{w} \in \mathbb{R}^3$,

$$|\vec{v} \times \vec{w}| = (\vec{v} \cdot \vec{v})(\vec{w} \cdot \vec{w}) - (\vec{v} \cdot \vec{w})^2.$$

(c) Show that the surface area of $\mathfrak{S}$ is given by

$$\mathcal{A}(\mathfrak{S}) = \iint_{\mathcal{D}} \sqrt{EG - F^2}\, du\, dv$$

or

$$d\mathcal{S} = \det\,[Q]\; du\, dv.$$

4.5 Integration in Three Variables

Triple Integrals. In theory, the extension of integration from two to three variables is a simple matter: the role of rectangles $[a,b] \times [c,d]$ is now played by rectangular solids with faces parallel to the coordinate planes

$$[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \coloneqq \{(x_1, x_2, x_3) \mid x_i \in [a_i, b_i] \text{ for } i = 1, 2, 3\};$$

a partition $\mathcal{P}$ of $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ is determined by three coordinate partitions

$$\mathcal{P}_1 \coloneqq \{a_1 = x_0 < x_1 < \cdots < x_m = b_1\}$$
$$\mathcal{P}_2 \coloneqq \{a_2 = y_0 < y_1 < \cdots < y_n = b_2\}$$
$$\mathcal{P}_3 \coloneqq \left\{a_3 = z_0 < z_1 < \cdots < z_p = b_3\right\}$$

which subdivide $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ into $m \cdot n \cdot p$ subsolids R_{ijk}, $i = 1, \dots, m$, $j = 1, \dots, n$, $k = 1, \dots, p$

$$R_{ijk} = [x_{i-1}, x_i] \times \left[y_{j-1}, y_j\right] \times [z_{k-1}, z_k]$$

with respective volumes

$$\triangle V_{ijk} = \triangle x_i \triangle y_j \triangle z_k = (x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1}).$$

Now given a function f bounded on $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ we can form the lower and upper sums

$$\mathcal{L}(\mathcal{P}, f) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} \left(\inf_{R_{ijk}} f \right) \triangle V_{ijk}$$
$$\mathcal{U}(\mathcal{P}, f) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} \left(\sup_{R_{ijk}} f \right) \triangle V_{ijk}.$$

If the lower integral

$$\underline{\iiint}_{[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]} f(x, y, z)\; dV \coloneqq \sup_{\mathcal{P}} \mathcal{L}(\mathcal{P}, f)$$

equals the upper integral

$$\overline{\iiint}_{[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]} f(x, y, z)\; dV \coloneqq \inf_{\mathcal{P}} \mathcal{U}(\mathcal{P}, f)$$

then the function is **integrable** over $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$, with **integral**

$$\iiint_{[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]} f\ dV = \underline{\iiint}_{[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]} f(x,y,z)\ dV$$
$$= \overline{\iiint}_{[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]} f(x,y,z)\ dV.$$

We shall not retrace all the details of the theory beyond this. As before, in practice such integrals are calculated via iterated integrals, but (not surprizingly) they are *triple* integrals. We shall call a region $\mathfrak{D} \subset \mathbb{R}^3$ in space z-**regular** if (see Figure 4.26):

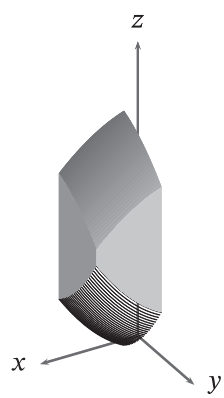


Figure 4.26. A z-regular region $\mathfrak{D}$

- a line parallel to the z-axis (*i.e.*, the set $\ell(x,y)$ defined by fixing the x- and y-coordinates and allowing the z-coordinate to vary) intersects $\mathfrak{D}$ in an interval $[\alpha(x,y), \beta(x,y)]$:
$$\mathfrak{D} \cap \ell(x,y) = \{(x,y,z) \mid z \in [\alpha(x,y), \beta(x,y)],\ x, y \text{ fixed}\}$$
- the set of pairs (x,y) for which $\ell(x,y)$ intersects $\mathfrak{D}$ forms an elementary region $\mathcal{D}$ in the x,y-plane.
- the endpoints $\alpha(x,y)$ and $\beta(x,y)$ are continuous functions of $(x,y) \in \mathcal{D}$.

If in turn the region $\mathcal{D}$ is y-regular, then we can specify $\mathfrak{D}$ via three inequalities of the form

$$\begin{cases} \alpha(x,y) & \leq z \leq & \beta(x,y) \\ c(x) & \leq y \leq & d(x) \\ a & \leq x \leq & b, \end{cases} \tag{4.15}$$

while if $\mathcal{D}$ is x-regular, we can specify it via

$$\begin{cases} \alpha(x,y) & \leq z \leq & \beta(x,y) \\ a(y) & \leq x \leq & b(y) \\ c & \leq y \leq & d. \end{cases} \tag{4.16}$$

Note the pattern here: the limits in the first inequality (for z) are functions of x and y, the limits in the second inequality (for y, respectively x) are functions of x (*resp.* y), and the limits in the third inequality (for x, respectively y) are just (constant) numbers. Analogous definitions can be formulated for x-regular or y-regular regions in $\mathbb{R}^3$ (Exercise 8).

When the region $\mathfrak{D}$ is z-regular in the sense of the definition above, and f is integrable over $\mathfrak{D}$, then the integral can be calculated in terms of the partial integral

$$\int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)\, dz$$

in which x and y (so also the limits of integration $\alpha(x,y)$ and $\beta(x,y)$) are treated as constant, as far as the integration is concerned; this results in a function of x and y (defined over $\mathcal{D}$) and the full triple integral is the (double) integral of this function over $\mathcal{D}$. Thus, from the specification (4.15) we obtain the triple integral

$$\begin{aligned}\iiint_{[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]} f\, dV &= \iint_{\mathcal{D}} \left(\int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)\, dz \right) dA \\ &= \int_a^b \int_{c(x)}^{d(x)} \int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)\, dz\, dy\, dx\end{aligned}$$

while from (4.16) we obtain

$$\begin{aligned}\iiint_{[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]} f\, dV &= \iint_{\mathcal{D}} \left(\int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)\, dz \right) dA \\ &= \int_c^d \int_{a(y)}^{b(y)} \int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)\, dz\, dx\, dy.\end{aligned}$$

As a first example, let us find the integral of $f(x,y,z) = 3x^2 - 3y^2 + 2z$ over the rectangular solid $\mathfrak{D} = [1,3] \times [1,2] \times [0,1]$ shown in Figure 4.27.

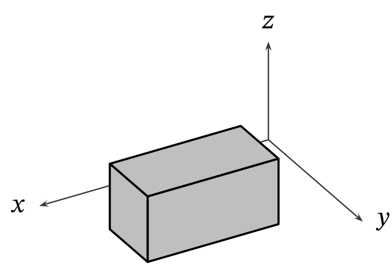


Figure 4.27. The rectangular region $\mathfrak{D} = [1,3] \times [1,2] \times [0,1]$

The region is specified by the inequalities $0 \le z \le 1$, $1 \le y \le 2$, and $1 \le x \le 3$, yielding the triple integral

$$\begin{aligned}
&\iiint_{[1,3]\times[1,2]\times[0,1]} 3(x^2 - 3y^2 + 2z)\, dV \\
&= \int_1^3 \int_1^2 \int_0^1 (3x^2 - 2y^2 + 2z)\, dz\, dy\, dx = \int_1^3 \int_1^2 (3x^2 z + 3y^2 z + z^2)_{z=0}^{z=1}\, dy\, dx \\
&\qquad = \int_1^3 \int_1^2 (3x^2 + 3y^2 + 1)\, dy\, dx = \int_1^3 (3x^2 y + y^3 + y)_{y=1}^{2}\, dx \\
&\qquad = \int_1^3 (\{6x^2 + 8 + 2\} - \{3x^2 + 1 + 1\}]\, dx = \int_1^3 (3x^2 + 8)\, dx \\
&\qquad\qquad = (x^3 + 8x)_1^3 = (27 + 24) - (1 + 8) = 42.
\end{aligned}$$

As a second example, let us integrate the function $f(x, y, z) = x + y + 2z$ over the region $\mathfrak{D}$ bounded by the xz-plane, the yz-plane, the plane $z = x + y$, and the plane $z = 2$ (Figure 4.28).

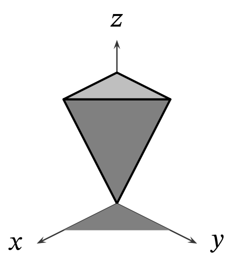


Figure 4.28. The region $\mathfrak{D}$ (and its shadow, $\mathcal{D}$)

The "shadow" of $\mathfrak{D}$, that is, its projection onto the xy-plane, is the triangular region $\mathcal{D}$ determined by the inequalities $0 \le y \le 2$, $0 \le x \le 2$, and $0 \le x + y \le 2$, which is a y-regular region. The corresponding specification is $0 \le y \le 2 - x$, $0 \le x \le 2$. A vertical line intersects the three-dimensional region $\mathfrak{D}$ if and only if it goes through this shadow, and then it runs from $z = x + y$ to $z = 2$. Thus, $\mathfrak{D}$ is z-regular, with corresponding inequalities $x + y \le z \le 2$, $0 \le y \le 2 - x$, and $0 \le x \le 2$. This leads to the triple integral

$$\iiint_{\mathfrak{D}} (x+y+z)\, dV$$

$$= \int_0^2 \int_0^{2-x} \int_{x+y}^2 (x+y+2z)\, dz\, dy\, dx$$

$$= \int_0^2 \int_0^{2-x} \{(x+y)z + z^2\}_{z=x+y}^{z=2}\, dy\, dx$$

$$= \int_0^2 \int_0^{2-x} \{[2(x+y)+4] - [(x+y)^2 + (x+y)^2]\}\, dy\, dx$$

$$= \int_0^2 \int_0^{2-x} \{2(x+y)^2 - 2(x+y)^2 + 4\}\, dy\, dx.$$

The inner integral is best done using the substitution $u = x+y$, $du = dy$ (So $y = 0 \leftrightarrow u = x$ and $y = 2-x \leftrightarrow u = 2$.) This leads to the inner integral

$$\int_0^{2-x} \{2(x+y) - 2(x+y)^2 + 4\}\, dy = \int_x^2 \{2u - 2u^2 + 4\}\, du$$

$$= \left\{u^2 - \frac{2u^3}{3} + 4u\right\}_{u=x}^2 = \left\{4 - \frac{16}{3} + 8\right\} - \left\{x^2 - \frac{2x^3}{3} + 4x\right\}$$

$$= \frac{20}{3} - x^2 + \frac{2x^3}{3} - 4x.$$

Substituting this into the outer integral yields

$$\int_0^2 \int_0^{2-x} \{2(x+y) - 2(x+y)^2 + 4\}\, dy\, dx = \int_0^2 \left(\frac{20}{3} - x^2 + \frac{2x^3}{3} - 4x\right) dx$$

$$= \left(\frac{20}{3}x - \frac{x^3}{3} + \frac{x^4}{6} - 2x^2\right)_0^2 = \frac{24}{3} = 8.$$

As a final example, let us integrate $f(x,y,z) = x+y+1$ over the region $\mathfrak{D}$ bounded below by the surface $z = x^2+3y^2$ and above by the surface $z = 8-x^2-y^2$ (Figure 4.29).

The two surfaces intersect where $8 - x^2 - 5y^2 = x^2 + 3y^2$, or $x^2 + 4y^2 = 4$. This defines the shadow $\mathcal{D}$, which can be specified in the y-regular form $-\frac{1}{4}\sqrt{4-x^2} \leq y \leq \frac{1}{4}\sqrt{4-x^2}$, $-2 \leq x \leq 2$ or, in the x-regular form, $-\sqrt{4-4y^2} \leq x \leq \sqrt{4-4y^2}$, $-1 \leq y \leq 1$.

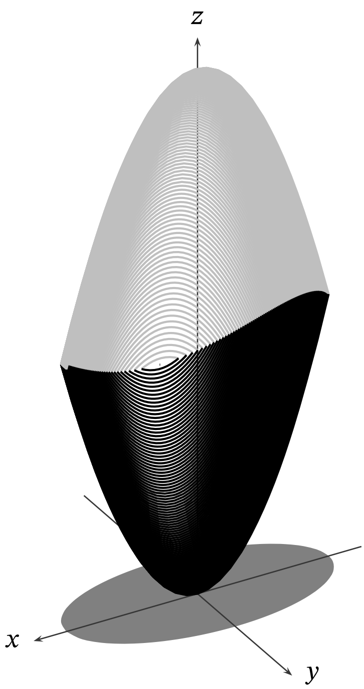


Figure 4.29. The region $\mathfrak{D}$

We choose the latter, so our integral becomes

$$\iiint_{\mathfrak{D}} f\ dV = \int_{-1}^{1}\int_{-\sqrt{4-4y^2}}^{\sqrt{4-4y^2}}\int_{x^2+3y^2}^{8-x^2-5y^2(x+y+1)} dz\,dx\,dy$$

$$= \int_{-1}^{1}\int_{-\sqrt{4-4y^2}}^{\sqrt{4-4y^2}} (x+y++1)z\Big|_{z=x^2+3y^2}^{z=8-x^2-5y^2} dx\,dy$$

$$= \int_{-1}^{1}\int_{-\sqrt{4-4y^2}}^{\sqrt{4-4y^2}} (x+y+1)(8-2x^2-8y^2)\,dx\,dy$$

$$= \int_{-1}^{1}\int_{-\sqrt{4-4y^2}}^{\sqrt{4-4y^2}} [-2x^3-(2y+1)x^2+8(1-y^2)x+8(1+y-y^2-y^3)]\,dx\,dy$$

$$= \int_{-1}^{1}\int_{-\sqrt{4-4y^2}}^{\sqrt{4-4y^2}} \left[-\frac{x^4}{2}-\frac{(2y+1)x^3}{3}+4(1-y^2)x^2\right]_{x=-\sqrt{4-4y^2}}^{x=\sqrt{4-4y^2}} dy$$

$$= \int_{-1}^{1}\left[-\frac{2(2y+1)}{3}(4-4y^2)^{3/2}+16(1+y-y^2-y^3)\sqrt{4-4y^2}\right] dy$$

$$= \int_{-1}^{1}\left[-\frac{4}{3}(2y+1)(4-4y^2)+16(1+y-y^2-y^3)\right]\sqrt{4-4y^2}\,dy$$

$$= \int_{-1}^{1}\left[\frac{32}{3}(1-y^2)+\frac{16}{3}y(1-y^2)\right]\sqrt{4-4y^2}\,dy = \frac{16\sqrt{2}}{3}\int_{-1}^{1}(2+y)(1-y^2)^{3/2}\,dy.$$

Using the substitution $x = \sin\theta \quad (\theta = \arcsin x)$, $dx = \cos\theta\, d\theta$, (so $x = -1 \leftrightarrow \theta = -\frac{\pi}{2}$ and $x = 1 \leftrightarrow \theta = \frac{\pi}{2}$) we get the integral

$$\frac{16\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2}(2+\sin\theta)(\cos^4\theta)\,d\theta = \frac{32\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2}2\cos^4\theta\,d\theta + \frac{16\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2}\cos^4\theta\sin\theta\,d\theta.$$

The first of these two integrals is done via the half-angle identities

$$\begin{aligned}\frac{32\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2}2\cos^4\theta\,d\theta &= \frac{8}{3}\sqrt{2}\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)^2\,d\theta \\ &= \frac{8}{3}\sqrt{2}\int_{-\pi/2}^{\pi/2}(1+2\cos 2\theta+\frac{1}{2}(1+\cos 4\theta))\,d\theta = 4\sqrt{2}\theta\Big|_{-\pi/2}^{\pi/2} = 4\pi\sqrt{2}.\end{aligned}$$

The second integral is an easy substitution of the form $u = \cos\theta$, yielding

$$\frac{16\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2}\cos^4\theta\sin\theta\,d\theta = \frac{16}{15}\sqrt{2}\cos^5\theta\Big|_{-\pi/2}^{\pi/2} = 0.$$

Combining these, we have the full integral

$$\begin{aligned}&\int_{-1}^{1}\int_{-\sqrt{4-4y^2}}^{\sqrt{4-4y^2}}\int_{x^2+3y^2}^{8-x^2-5y^2(x+y+1)}dz\,dx\,dy \\ &\qquad = \frac{32\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2}2\cos^4\theta\,d\theta + \frac{16\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2}\cos^4\theta\sin\theta\,d\theta \\ &\qquad = 4\pi\sqrt{2}+0 = 4\pi\sqrt{2}.\end{aligned}$$

Change of Coordinates in Triple Integrals. The theory behind changing coordinates in triple integrals again follows the lines of the two-dimensional theory which we set out in detail in § 4.3. We shall simply outline the basic features of this theory.

The first observation is that for any linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$, the absolute value of the determinant of its matrix representative

$$\Delta(L) \coloneqq |\det[L]|$$

gives the volume of the parallelepiped whose edges are the images of the three unit vectors along the axes (Exercise 10); this in turn is the image under L of the "unit box" $[0,1]\times[0,1]\times[0,1]$. From this we can argue as in § 4.3 to obtain the following analogue of Proposition 4.3.2: [11]

Proposition 4.5.1. *If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is an affine map ($T(\vec{x}) = \vec{y_0} + L(\vec{x})$, where L is linear) and $\mathfrak{D}$ is an elementary region in $\mathbb{R}^3$, then*

$$\Delta(T) \coloneqq \Delta(L)$$

gives the ratio between the volume of the image of $\mathfrak{D}$ and the volume of $\mathfrak{D}$ itself:

$$\mathcal{V}(T(\mathfrak{D})) = \Delta(T)\cdot\mathcal{V}(\mathfrak{D}).$$

Using this, we can establish the three-dimensional analogue of Proposition 4.3.3:

[11] By analogy with the two-dimensional case, a region $\mathfrak{D} \subset \mathbb{R}^3$ is **elementary** if it can be specified by inequalities on the three coordinates in which the first is bounded by functions of the other two, the second by functions of the third, and the third by constants. This is the definition of z regular when the three are in the order z, y, x.

Proposition 4.5.2. *Suppose R is a rectangular solid in $\mathbb{R}^3$ and Φ is a $\mathcal{C}^1$ coordinate transformation defined on R; then the volume of the image of R under Φ satisfies*

$$\min_{\vec{x}\in R} \left|J\Phi\left(\vec{x}\right)\right| \mathcal{V}(R) \leq \mathcal{V}(\Phi(R)) \leq \max_{\vec{x}\in R} \left|J\Phi\left(\vec{x}\right)\right| \mathcal{V}(R). \tag{4.17}$$

The notion of a coordinate transformation carries over practically verbatim from Definition 4.3.1:

Definition 4.5.3. *A **coordinate transformation** on a region $\mathfrak{D} \subset \mathbb{R}^3$ is a $\mathcal{C}^1$ mapping $\Phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ satisfying:*

(1) *Φ has no critical points in $\mathfrak{D}$ (i.e., its Jacobian determinant*

$$\Delta\left(\Phi\right)\left(\vec{x}\right) := \left|J\Phi\left(\vec{x}\right)\right|$$

is nonzero at every point $\vec{x}$ of $\mathfrak{D}$).

(2) *Φ maps $\mathfrak{D}$ onto $\Phi\left(\mathfrak{D}\right)$ in a one-to-one manner.*

A modification of the argument in Appendix A.7 for Theorem 4.3.4 gives the three-dimensional analogue:

Theorem 4.5.4 (Change of Coordinates in Triple Integrals)**.** *Suppose $\mathfrak{D} \subset \mathbb{R}^3$ is an elementary region, $\Phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a coordinate transformation defined on $\mathfrak{D}$, and $f\colon \mathbb{R}^3 \to \mathbb{R}$ is a real-valued function which is integrable on $\Phi\left(\mathfrak{D}\right)$.*

Then

$$\iiint_{\Phi(\mathfrak{D})} f\left(\vec{x}\right)\, dV = \iiint_{\mathfrak{D}} f\left(\Phi\left(\vec{x}\right)\right)\Delta\left(\Phi\right)\left(\vec{x}\right)\, dV \tag{4.18}$$

where

$$\Delta\left(\Phi\right)\left(\vec{x}\right) := \left|J\Phi\left(\vec{x}\right)\right|.$$

Triple Integrals in Cylindrical and Spherical Coordinates. An important application of Theorem 4.5.4 is the calculation of triple integrals in cylindrical and spherical coordinates.

The case of cylindrical coordinates has essentially been covered in § 4.3, since these involve replacing x and y with polar coordinates in the plane and then keeping z as the third coordinate. However, let us work this out directly from Theorem 4.5.4. If the region $\mathfrak{D}$ is specified by a set of inequalities in cylindrical coordinates, like $z_1 \leq z \leq z_2$, $r_1 \leq r \leq r_2$, $\theta_1 \leq \theta \leq \theta_2$ (where it is understood that the inequalities can be listed in a different order, and that some of the limits can be functions of variables appearing further down the list), then we can regard these inequalities as specifying a new region $\mathfrak{D}_{cyl} \subset \mathbb{R}^3$, which we think of as living in a different copy of $\mathbb{R}^3$, "(r,θ,z) space", and think of $\mathfrak{D}$ (in "(x,y,z) space") as the image of $\mathfrak{D}_{cyl}$ under the mapping $\Phi_{Cyl}\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\Phi_{Cyl}\left(r,\theta,z\right) = \left(r\cos\theta, r\sin\theta, z\right).$$

The Jacobian determinant of Φ_{Cyl} is easily calculated to be

$$\det\left(J(\Phi_{Cyl})\right)(r,\theta,z) = (\cos\theta)(r\cos\theta - 0) - (-r\sin\theta)(\sin\theta) = r\cos^2\theta + r\sin^2\theta = r.$$

To ensure that Φ_{Cyl} is a coordinate transformation on $\mathfrak{D}_{cyl}$, we need r to be nonzero (and usually positive) in the interior of this region. Then Theorem 4.5.4 tells us that an integral of the form $\iiint_{\mathfrak{D}} f\, dV$ can be rewritten as

$$\iiint_{\mathfrak{D}} f\, dV = \iiint_{\mathfrak{D}_{cyl}} (f \circ \Phi_{Cyl}) \cdot \Delta\left(\Phi_{Cyl}\right)\, dV.$$

The factor $f \circ \Phi_{Cyl}$ is simply the function f, thought of as assigning a real value to every point of $\mathbb{R}^3$, expressed as a formula in terms of the cylindrical coordinates of that point. Strictly speaking, this means we need to substitute the expressions for x and y in terms of polar coordinates in the appropriate places: if $f(x, y, z)$ denotes the formula for f in terms of rectangular coordinates, then

$$(f \circ \Phi_{Cyl})(r, \theta, z) = f\left(\Phi_{Cyl}(r, \theta, z)\right) = f(r\cos\theta, r\sin\theta, z)$$

but in writing out abstract statements, we allow abuse of notation and write simply $f(r, \theta, z)$. Using this naughty abbreviation, we can state the following special case of Theorem 4.5.4:

Corollary 4.5.5 (Triple Integrals in Cylindrical Coordinates). *If the region $\mathfrak{D} \subset \mathbb{R}^3$ is described by inequalities in cylindrical coordinates, say*[12] $z_1 \leq z \leq z_2$, $r_1 \leq r \leq r_2$, $\theta_1 \leq \theta \leq \theta_2$ *corresponding to the region $\mathfrak{D}_{cyl}$ in (r, θ, z) space, then*

$$\iiint_{\mathfrak{D}} f\, dV = \iiint_{\mathfrak{D}_{cyl}} (f \circ \Phi_{Cyl}) \cdot \Delta\left(\Phi_{Cyl}\right)\, dV = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} \int_{z_1}^{z_2} f(r, \theta, z)\, r\, dz\, dr\, d\theta,$$

where $f(r, \theta, z) = (f \circ \Phi_{Cyl})(r, \theta, z)$ is simply f expressed in terms of cylindrical coordinates.

In other words, to switch from rectangular to cylindrical coordinates in a triple integral, we replace the limits in x, y, z with corresponding limits in r, θ, z, rewrite the integrand in terms of cylindrical coordinates, and substitute $dV = r\, dz\, dr\, d\theta$.

For example, let us calculate the integral $\iiint_{\mathfrak{D}} x\, dV$ where $\mathfrak{D}$ is the part of the "shell" between the cylinders of radius 1 and 2, respectively, about the z-axis, above the xy-plane, in front of the yz-plane, and below the plane $y + z = 3$ (Figure 4.30). In

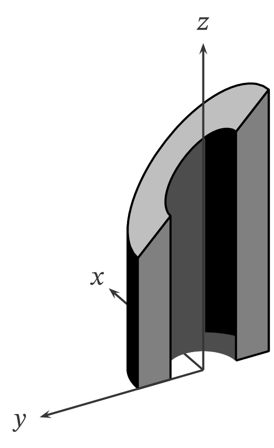


Figure 4.30. Half-Cylindrical Shell Cut by the Plane $y + z = 3$

rectangular coordinates, the region can be described by $0 \leq z \leq 3 - y$, $1 \leq x^2 + y^2 \leq 4$, $x \geq 0$. However, the region is more naturally specified by the inequalities in cylindrical coordinates $0 \leq z \leq 3 - r\sin\theta$, $1 \leq r \leq 2$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

[12]In general, the order of the inequalities can be different, and some limits can be functions of variables appearing below them, rather than constants.

Then Corollary 4.5.5 tells us that

$$\iiint_{\mathfrak{D}} x\ dV = \iiint_{\mathfrak{D}_{cyl}} (r\cos\theta)\cdot r\ dV$$
$$= \int_{-\pi/2}^{\pi/2}\int_1^2\int_0^{3-r\sin\theta}(r\cos\theta)(r\,dz\,dr\,d\theta) = \int_{-\pi/2}^{\pi/2}\int_1^2\int_0^{3-r\sin\theta} r^2\cos\theta\,dz\,dr\,d\theta$$
$$= \int_{-\pi/2}^{\pi/2}\int_1^2 (3-r\sin\theta)(r^2\cos\theta)\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_1^2 (3r^2\cos\theta - r^3\sin\theta\cos\theta)\,dr\,d\theta$$
$$= \int_{-\pi/2}^{\pi/2}\left(r^3\cos\theta - \frac{r^4}{4}\sin\theta\cos\theta\right)_{r=1}^2 d\theta = \int_{-\pi/2}^{\pi/2}(7\cos\theta - \frac{15}{4}\sin\theta\cos\theta)\,d\theta$$
$$= \left(7\sin\theta - \frac{15}{8}\sin^2\theta\right)_{\theta=-\pi/2}^{\pi/2} = 14.$$

A region $\mathfrak{D} \subset \mathbb{R}^3$ specified by inequalities in spherical coordinates, say $\rho_1 \le \rho \le \rho_2, \phi_1 \le \phi \le \phi_2, \theta_1 \le \theta \le \theta_2$ can, in a similar way, be regarded as the image of a region $\mathfrak{D}_{spher}$ in "(ρ,ϕ,θ) space" under the mapping $\Phi_{Sph} : \mathbb{R}^3 \to \mathbb{R}^3$, which can be written

$$\Phi_{Sph}(\rho,\phi,\theta) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$$

and whose Jacobian determinant can be calculated by expansion along the last row:

$$\Delta(\Phi_{Sph})(\rho,\phi,\theta) = |J\Phi_{Sph}(\rho,\phi,\theta)|$$
$$= (\cos\phi)[(\rho\cos\phi\cos\theta)(\rho\sin\phi\cos\theta) - (-\rho\sin\phi\sin\theta)(\rho\cos\phi\sin\theta)]$$
$$- (-\rho\sin\phi)[(\sin\phi\cos\theta)(\rho\sin\phi\cos\theta) - (-\rho\sin\phi\sin\theta)(\sin\phi\sin\theta)] + 0$$
$$= (\cos\phi)[\rho^2\sin\phi\cos\phi\cos^2\theta + \rho^2\sin\phi\cos\phi\sin^2\theta]$$
$$+ (\rho\sin\phi)[\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta]$$
$$= (\cos\phi)[\rho^2\sin\phi\cos\phi] + (\rho\sin\phi)[\rho\sin^2\phi]$$
$$= \rho^2\sin\phi\cos^2\phi + \rho^2\sin\phi\sin^2\phi$$
$$= \rho^2\sin\phi.$$

So in a way exactly analogous to Corollary 4.5.5 we have

Corollary 4.5.6 (Triple Integrals in Spherical Coordinates). *If a region $\mathfrak{D} \subset \mathbb{R}^3$ is specified by inequalities in spherical coordinates, say*[13] $\rho_1 \le \rho \le \rho_2, \phi_1 \le \phi \le \phi_2, \theta_1 \le \theta \le \theta_2$, *then for any function f defined on $\mathfrak{D}$ we have* [14]

$$\iiint_{\mathfrak{D}} f\ dV = \iiint_{\mathfrak{D}_{spher}} (f\circ\Phi_{Sph})\cdot\Delta(\Phi_{Sph})\ dV$$
$$= \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} f(\rho,\phi,\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

As an example, let us calculate $\iiint_{\mathfrak{D}} f\ dV$ where $f(x,y,z)$=z and $\mathfrak{D}$ is the region in the first octant bounded by the spheres of radius 1.5 and 2 (centered at the origin), the xy-plane, the xz-plane, and the plane $x = y$ (Figure 4.31).

[13]As before, the order of the inequalities can be different, and some limits can be functions of variables appearing below them, rather than constants.

[14]where again we write $(f\circ\Phi_{Sph})(\rho,\phi,\theta)$ as $f(\rho,\phi,\theta)$

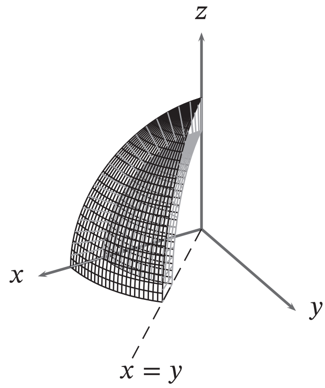


Figure 4.31. Spherical shell

The region $\mathfrak{D}$ corresponds to the region $\mathfrak{D}_{spher}$ specified in spherical coordinates by $1.5 \le \rho \le 2$, $0 \le \phi \le \frac{\pi}{2}$, $0 \le \theta \le \frac{\pi}{4}$. Thus we can set up the integral as a triple integral

$$\iiint_{\mathfrak{D}} f\ dV = \iiint_{\mathfrak{D}_{spher}} (f \circ \Phi_{Sph}) \cdot \Delta\left(\Phi_{Sph}\right)\ dV$$

$$= \int_0^{\pi/4} \int_0^{\pi/2} \int_{1.5}^2 (\rho\cos\phi)(\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta) = \int_0^{\pi/4} \int_0^{\pi/2} \int_{1.5}^2 \rho^3 \cos\phi \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{\pi/4} \int_0^{\pi/2} \left(\frac{\rho^4}{4}\right)_{1.5}^2 \cos\phi \sin\phi\, d\phi\, d\theta = \left(\frac{175}{16}\right) \int_0^{\pi/4} \int_0^{\pi/2} \cos\phi \sin\phi\, d\phi\, d\theta$$

$$= \left(\frac{175}{16}\right) \int_0^{\pi/4} \frac{1}{2} \sin^2\phi\Big|_0^{\pi/2} d\theta = \left(\frac{175}{32}\right) \int_0^{\pi/4} d\theta = \frac{175\pi}{128}.$$

Exercises for § 4.5

Answers to Exercises 1a and 2a are given in Appendix A.13.

Practice problems:

(1) Calculate each triple integral $\iiint_{\mathfrak{D}} f\ dV$ below:
  (a) $f(x, y, z) = x^3$, $\mathfrak{D}$ is $[0, 1] \times [0, 1] \times [0, 1]$.
  (b) $f(x, y, z) = 3x^3y^2z$, $\mathfrak{D}$ is $[0, 2] \times [2, 3] \times [1, 2]$.
  (c) $f(x, y, z) = e^{x-2y+3z}$, $\mathfrak{D}$ is $[0, 1] \times [0, 1] \times [0, 1]$.
  (d) $f(x, y, z) = 1$, $\mathfrak{D}$ is the region bounded by the coordinate planes and the plane $x + y + 2z = 2$.
  (e) $f(x, y, z) = x + y + z$, $\mathfrak{D}$ is the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 1$, and $x + z = 2 - y$.
  (f) $f(x, y, z) = 1$, $\mathfrak{D}$ is the region bounded by the two surfaces $z = 24 - 5x^2 - 2y^2$ and $z = x^2 + y^2$.
  (g) $f(x, y, z) = 1$, $\mathfrak{D}$ is the region inside the cylinder $2x^2 + y^2 = 4$, bounded below by the xy-plane and above by the plane $x + y + 2z = 6$.

(h) $f(x,y,z) = x + yz$, $\mathfrak{D}$ is specified by

$$\begin{aligned} 0 \le &z \le y \\ 0 \le &y \le x \\ 0 \le &x \le 1. \end{aligned}$$

(i) $f(x,y,z) = z + 2y$, $\mathfrak{D}$ is the pyramid with top vertex $(0,0,1)$ and base vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(1,1,0)$.

(j) $f(x,y,z) = 1 - y$, $\mathfrak{D}$ is the part of the inside of the cylinder $x^2 + y^2 = 1$ above the xy-plane and below the plane $y + z = 1$.

(k) $f(x,y,z) = 1$, $\mathfrak{D}$ is the part of $\mathfrak{D}$ from problem (1j) in the first octant.

(l) $f(x,y,z) = x^2$, $\mathfrak{D}$ is the part of the inside of the cylinder $x^2 + y^2 = 1$ above the xy-plane and below the paraboloic sheet $z = y^2$.

(m) $f(x,y,z) = z$, $\mathfrak{D}$ is the "cap" cut off from the top of the sphere of radius 2 about the origin by the plane $z = 1$.

(n) $f(x,y,z) = x^2+y^2+z^2$, $\mathfrak{D}$ is the sector ("lemon wedge") cut out of the sphere $x^2 + y^2 + z^2 = 1$ by the two half-planes $y = x\sqrt{3}$ and $x = y\sqrt{3}$, $x, y \ge 0$.

(o) $f(x,y,z) = z$, $\mathfrak{D}$ is the part of the "front" ($x \ge 0$) hemisphere of radius 1 centered at the origin which lies above the downward cone with vertex at the origin whose edge makes an angle α ($0 < \alpha < \frac{\pi}{2}$) with the z-axis.

(2) Express each region below by inequalities of the form

$$\begin{aligned} a_1(x,y) \le z \le a_2(x,y) \\ b_1(x) \le y \le b_2(x) \\ c_1 \le x \le c_2. \end{aligned}$$

(a) $\mathfrak{D} = \left\{(x,y,z) \mid x^2 + y^2 + z^2 \le 4, \quad z \ge \sqrt{2}\right\}$

(b) $\mathfrak{D} = \left\{(x,y,z) \mid x^2 + y^2 + z^2 \le 1, \quad |x| \le y\right\}$

(c) $\mathfrak{D} = \left\{(x,y,z) \mid x^2 + y^2 \le z \le \sqrt{x^2 + y^2}\right\}$

(d) $\mathfrak{D}$ is the region bounded by the surfaces $z = 6x^2 - 6y^2$ and $10x^2 + 10y^2 + z = 4$.

(3) Show that the region in the first octant in which $x + y \le 1$ and $x \le z \le y$ is the simplex with vertices $(0,0,0)$, $(0,1,0)$, $(0,1,1)$, and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Find its volume.

(4) Consider the region specified by

$$\begin{aligned} 0 \le z \le y \\ 0 \le y \le x \\ 0 \le x \le 1. \end{aligned}$$

Give inequalities expressing the same region in the form

$$\begin{aligned} a_1(y,z) \le x \le a_2(y,z) \\ b_1(z) \le y \le b_2(z) \\ c_1 \le z \le c_2. \end{aligned}$$

(5) Express the volume of the pyramid with base $[-1,1] \times [-1,1]$ and vertex $(0,0,1)$ in two ways:

(a) As an iterated integral of the form $\iiint dy\,dx\,dz$

(b) As a sum of four iterated integrals of the form $\iiint dz\,dy\,dx$.

Then evaluate one of these expressions.

(6) (a) Let $\mathfrak{D}$ be the intersection of the two regions $x^2+y^2 \leq 1$ and $x^2+z^2 \leq 1$. Sketch the part of $\mathfrak{D}$ lying in the first octant, and set up a triple integral expressing the volume of $\mathfrak{D}$.

(b) Do the same for the intersection of the *three* regions $x^2 + y^2 \leq 1$, $x^2 + z^2 \leq 1$, and $y^2 + z^2 \leq 1$. (*Hint:* First consider the part of $\mathfrak{D}$ in the first octant, and in particular the two parts into which it is divided by the vertical plane $x = y$.)

(7) Find the volume of each region below:

(a) The region between the paraboloids $z = 1 - x^2 - y^2$ and $z = x^2 + y^2 - 1$.

(b) The region bounded below by the upper hemisphere of radius 2 centered at the origin and above by the paraboloid $z = 4 - x^2 - y^2$.

(c) The "ice cream cone" cut out of the sphere of radius 1 by a cone whose edge makes an angle α ($0 < \alpha < \frac{\pi}{2}$) with its axis.

Theory problems:

(8) (a) Formulate a definition of x-regular and y-regular regions in $\mathbb{R}^3$ parallel to that given on p. 249 for z-regular regions.

(b) For each of these give the possible ways such a region can be specified by inequalities.

(9) **Symmetry in Three Dimensions:**

(Refer to Exercise 7 in § 4.2.)

(a) Formulate a definition of x-symmetric (*resp.* y-symmetric or z-symmetric) regions in $\mathbb{R}^3$.

(b) Define what it means for a function $f(x, y, z)$ of three variables to be odd (*resp.* even) in x (*resp.* y or z).

(c) Show that if $f(x, y)$ is odd in x on an x-symmetric, x-regular region in $\mathbb{R}^3$, its integral is zero.

(d) Show that if $f(x, y)$ is even in x on an x-symmetric, x-regular region in $\mathbb{R}^3$, its integral is twice its integral in the part of the region on the positive side of the yz-plane.

(e) Suppose $f(x, y, z)$ is even in all three variables, and D is regular and symmetric in all three variables. Then the integral of f over D is a multiple of its integral over the intersection of D with the first octant: what multiple?

(10) Prove Proposition 4.5.1 as follows:
Suppose $T: \mathbb{R}^3 \to \mathbb{R}^3$ is affine, say

$$T(\vec{x}) = \overrightarrow{y_0} + L(\vec{x}),$$

where $L: \mathbb{R}^3 \to \mathbb{R}^3$ is linear.

(a) Use Remark 1.7.2 to prove that the volume of $T([0,1] \times [0,1] \times [0,1])$ is $\Delta(L) = \Delta(T)$.

(b) Use linearity to show that for any rectangular box $B = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$,

$$\mathcal{V}(T(B)) = \Delta(T)\, \mathcal{V}(B).$$

(c) Suppose now that $\mathfrak{D}$ is an elementary region in $\mathbb{R}^3$. As in § 4.3, let $P_\pm$ be two regions, each formed from rectangular boxes (with disjoint boundaries), with $P_- \subset \mathfrak{D} \subset P_+$; given $\varepsilon > 0$, assume we can construct these regions so that [15]

$$\mathcal{V}(P_+) \leq (1+\varepsilon)\mathcal{V}(P_-).$$

Then

$$T(P_-) \subset T(\mathfrak{D}) \subset T(P_+)$$

and hence

$$\begin{aligned}
\Delta(T) \cdot \mathcal{V}(P_-) &= \mathcal{V}(T(P_-)) \\
&\leq \mathcal{V}(T(\mathfrak{D})) \\
&\leq \mathcal{V}(T(P_+)) = \Delta(T) \cdot \mathcal{V}(P_+) \\
&\leq (1+\varepsilon)\mathcal{V}(T(P_-)) = (1+\varepsilon)\Delta(T) \cdot \mathcal{V}(P_-).
\end{aligned}$$

Then use the squeeze theorem to show that

$$\mathcal{V}(T(\mathfrak{D})) = \Delta(T) \cdot \mathcal{V}(\mathfrak{D}).$$

Challenge problem:

(11) Suppose f is continuous on $\mathbb{R}^3$, and let B_δ be the ball of radius $\delta > 0$ centered at (x_0, y_0, z_0), and let $\mathcal{V}(B_\delta)$ denote the volume of the ball. Show that

$$\lim_{\delta \to 0} \frac{1}{\mathcal{V}(B_\delta)} \iiint_{B_\delta} f(x,y,z)\, dV = f(x_0, y_0, z_0).$$

[15]This is a bit fussier to prove than in the two-dimensional case, but we will slide over this technicality.

5 Integral Calculus for Vector Fields and Differential Forms

In this chapter, we will consider a family of results known collectively as **Generalized Stokes' Theorem**, which can be regarded as a far-reaching generalization of the Fundamental Theorem of Calculus. These results can be formulated in several languages; we shall follow two of these: the language of *vector fields* and the language of *differential forms*; along the way, we shall develop a dictionary for passing from either one of these languages to the other.

5.1 Line Integrals of Vector Fields and 1-Forms

A **vector field** on $D \subset \mathbb{R}^n$ is simply a mapping $\vec{F}: D \to \mathbb{R}^n$ assigning to each point $p \in D$ a vector $\vec{F}(p)$. However, our point of view is somewhat different from that in § 4.3 and Appendix A.7. We think of the domain and range of a *mapping* as essentially separate collections of vectors or points (even when they are the same space), whereas in the *vector field* setting we think of the *input* as a *point*, and the *output* as a *vector*; we picture this vector as an arrow "attached" to the point. The distinction is emphasized by our use of an arrow over the name of the vector field, and dropping the arrow over the input point.

One way to formalize this point of view is to take a leaf from our study of surfaces in space (particularly Lagrange multipliers in § 3.7). If a curve $\vec{p}(t)$ lies on the surface $\mathfrak{S}$, then its velocity is everywhere tangent to the surface; turning this around, we can think of the tangent plane to $\mathfrak{S}$ at $p \in \mathfrak{S}$ as consisting of all the possible velocity vectors for points moving in $\mathfrak{S}$ through p. Analogously, we can formulate the **tangent space to** $\mathbb{R}^n$ at $p \in \mathbb{R}^n$ as the set $T_p\mathbb{R}^n$ of all velocity vectors for points moving in $\mathbb{R}^n$ through p. This is of course a copy of $\mathbb{R}^n$, but we think of these vectors as all "attached" to p. Examples of physical quantities for which this interpretation is appropriate include forces which vary from point to point (such as interplanetary gravitation), velocity of fluids (such as wind velocity on weather maps), and forces acting on rigid bodies.

We can visualize vector fields in the plane and in 3-space as, literally, "fields of arrows". For example, the vector field in the plane given by $\vec{F}(x, y) = y\vec{\imath} + x\vec{\jmath}$ assigns to every point $(x, 0)$ on the x-axis a vertical arrow of length $|x|$ (pointing up for $x > 0$ and down for $x < 0$) and similarly a horizontal arrow of length $|y|$ to every point $(0, y)$ on the y-axis; at a generic point (x, y), the arrow is the sum of these. The resulting field is pictured in Figure 5.1. Note that when $y = \pm x$, the vector points along the diagonal (or antidiagonal).

By contrast, the vector field $\vec{F}(x, y) = y\vec{\imath} - x\vec{\jmath}$ is everywhere perpendicular to the position vector (x, y), so $\vec{F}(x, y)$ is tangent to the circle through (x, y) centered at the origin (Figure 5.2).

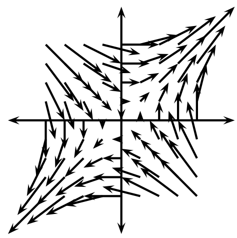


Figure 5.1. The vector field $\vec{F}(x,y) = y\vec{\imath} + x\vec{\jmath}$

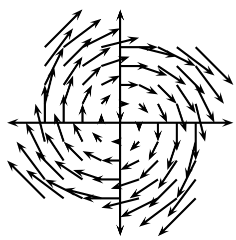


Figure 5.2. The vector field $\vec{F}(x,y) = y\vec{\imath} - x\vec{\jmath}$

Work and Line Integrals. If you have to push your stalled car a certain distance, the work you do is intuitively proportional to how hard you need to push, and also to how far you have to push it. This intuition is formalized in the physics concept of **work**: if a (constant) force of magnitude F is applied to move an object over a straight line distance $\triangle\mathfrak{s}$, then the work W is given by

$$W = F\triangle\mathfrak{s};$$

more generally, if the force is not directed parallel to the direction of motion, we write the force and the displacement as vectors $\vec{F}$ and $\triangle\vec{\mathfrak{s}}$, respectively, and consider only the component of the force in the direction of the displacement:

$$W = \left(\text{comp}_{\triangle\vec{\mathfrak{s}}}\vec{F}\right)\triangle\mathfrak{s} = \vec{F}\cdot\triangle\vec{\mathfrak{s}}.$$

When the displacement occurs over a curved path $\mathcal{C}$, and the force varies along the path, then to calculate the work we need to go through a process of integration. We pick partition points p_j, $j = 0, \ldots, n$, along the curve and make two approximations. First, since the force should vary very little along a short piece of curve, we replace the varying force by its value $\vec{F}\left(x_j\right)$ at some representative point x_j between p_{j-1} and p_j along $\mathcal{C}$. Second, we use the vector $\triangle\vec{\mathfrak{s}}_j = \vec{p_j} - \overrightarrow{p_{j-1}}$ as the displacement. Thus, the work done along one piece is approximated by the quantity $\triangle_j W = \vec{F}\left(x_j\right)\cdot\triangle\vec{\mathfrak{s}}_j$ and the total work over $\mathcal{C}$ is approximated by the sum

$$W \approx \sum_{j=1}^{n}\triangle_j W = \sum_{j=1}^{n}\vec{F}\left(x_j\right)\cdot\triangle\vec{\mathfrak{s}}_j.$$

As usual, we consider progressively finer partitions of $\mathcal{C}$, and expect the approximations to converge to an integral

$$W = \int_{\mathcal{C}}\vec{F}\cdot d\vec{\mathfrak{s}}.$$

This might look like a new kind of integral, but we can see it as a path integral of a function over $\mathcal{C}$, as in § 2.5. For this, it is best to think in terms of a parametrization of $\mathcal{C}$, say $\vec{p}\,(t)$, $a \leq t \leq b$. We can write

$$p_j = \vec{p}\,(t_j)\,.$$

Then the vector $\triangle\vec{\mathfrak{s}}_j$ is approximated by the vector $\vec{v}\,(t_j)\,\triangle t_j$ where $\triangle t_j = t_j - t_{j-1}$ and $\vec{v}\,(t) = \frac{d\vec{p}}{dt}$ is the velocity of the parametrization. As in § 2.5, we can write $\vec{v}\,(t) = \left\|\vec{v}\,(t)\right\| \vec{T}\,(t)$ where $\vec{T}$ is a unit vector tangent to $\mathcal{C}$ at $\vec{p}\,(t)$. Thus, we can write $\triangle\vec{\mathfrak{s}}_j \approx \left\|\vec{v}\,(t)\right\| \vec{T}\,(t_j)\,\triangle t_j$ and the integral for work can be rewritten

$$W = \int_{\mathcal{C}} \vec{F} \cdot d\vec{\mathfrak{s}} = \int_a^b \vec{F} \cdot \vec{T} \left\|\vec{v}\,(t)\right\| \, dt$$

which we can recognize as a line integral $W = \int_{\mathcal{C}} \vec{F} \cdot \vec{T}\, d\mathfrak{s}$ of the function given by the tangential component of $\vec{F}$, that is $W = \int_{\mathcal{C}} f\, d\mathfrak{s}$ where

$$f\left(\vec{p}\,(t)\right) = \vec{F}\left(\vec{p}\,(t)\right) \cdot \vec{T}\left(\vec{p}\,(t)\right) = \operatorname{comp}_{\vec{v}(t)} \vec{F}\left(\vec{p}\,(t)\right).$$

Let us work this out for an example. Suppose our force is given by the planar vector field $\vec{F}\,(x, y) = \vec{\imath} + y\vec{\jmath}$ and $\mathcal{C}$ is the semicircle $y = \sqrt{1 - x^2}$, $-1 \leq x \leq 1$. We can write $\vec{p}\,(t) = t\vec{\imath} + \sqrt{1 - t^2}\vec{\jmath}$, or equivalently, $x = t$ and $y = \sqrt{1 - t^2}$ for $-1 \leq t \leq 1$. Then $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = -\frac{t}{\sqrt{1-t^2}}$, or equivalently $\vec{v}\,(t) = \vec{\imath} - \frac{t}{\sqrt{1-t^2}}\vec{\jmath}$, and $\left\|\vec{v}\,(t)\right\| = \sqrt{1 + \frac{t^2}{1-t^2}} = \frac{1}{\sqrt{1-t^2}}$, so the unit tangent vector is

$$\vec{T} = \frac{\vec{v}}{\left\|\vec{v}\right\|} = (\sqrt{1 - t^2})\left(\vec{\imath} - \frac{t}{\sqrt{1 - t^2}}\vec{\jmath}\right) = \sqrt{1 - t^2}\vec{\imath} - t\vec{\jmath}.$$

The value of the vector field along the curve is

$$\vec{F}\,(t) = \vec{F}\left(t, \sqrt{1 - t^2}\right) = \vec{\imath} + \sqrt{1 - t^2}\vec{\jmath}$$

so the function we are integrating is

$$f\,(t) = \vec{F} \cdot \vec{T} = \sqrt{1 - t^2} - t\sqrt{1 - t^2} = (1 - t)\sqrt{1 - t^2}.$$

Meanwhile, $d\mathfrak{s} = \left\|\vec{v}\right\| \, dt = \frac{1}{\sqrt{1-t^2}}\, dt$ and our integral becomes

$$\int_{\mathcal{C}} \vec{F} \cdot \vec{T}\, d\mathfrak{s} = \int_{-1}^{1} [(1 - t)\sqrt{1 - t^2}][\frac{1}{\sqrt{1 - t^2}}\, dt] = \int_{-1}^{1} (1 - t)\, dt$$
$$= -\frac{(1 - t)^2}{2}\Big|_{-1}^{1} = -\frac{(0)^2}{2} + \frac{(2)^2}{2} = 2.$$

In the calculation above, you undoubtedly noticed that the factor $\left\|\vec{v}\right\| = \sqrt{1 - t^2}$, which appeared in the numerator when calculating the unit tangent, also appeared in the denominator when calculating the differential of arclength, so they cancelled. A

moment's thought should convince you that this is always the case: formally, $\vec{T} = \frac{\vec{v}}{\left\|\vec{v}\right\|}$.
and $d\mathfrak{s} = \left\|\vec{v}\right\| dt$ means that

$$\vec{T}\,d\mathfrak{s} = \left(\frac{\vec{v}}{\left\|\vec{v}\right\|}\right)\left(\left\|\vec{v}\right\|\,dt\right) = \vec{v}\,dt$$

so

$$\vec{F}\cdot d\vec{\mathfrak{s}} = \vec{F}\cdot\vec{T}\,d\mathfrak{s} = \vec{F}\cdot\left(\vec{v}\,dt\right);$$

in other words, we can write, formally,

$$d\vec{\mathfrak{s}} = \vec{v}\,dt = \left(\frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath}\right)\,dt.$$

If we allow ourselves the indulgence of formal differentials, we can use the relations $dx = \frac{dx}{dt}\,dt$ and $dy = \frac{dy}{dt}\,dt$ to write $d\vec{\mathfrak{s}} = dx\vec{\imath} + dy\vec{\jmath}$. Now, if the vector field $\vec{F}$ is given by $\vec{F}(x,y) = P(x,y)\vec{\imath} + Q(x,y)\vec{\jmath}$ then (again formally) $\vec{F}\cdot d\vec{\mathfrak{s}} = P(x,y)\,dx + Q(x,y)\,dy$ leading us to the formal integral

$$\int_{\mathcal{C}}\vec{F}\cdot d\vec{\mathfrak{s}} = \int_{\mathcal{C}} P(x,y)\,dx + Q(x,y)\,dy.$$

While the geometric interpretation of this is quite murky at the moment, this way of writing things leads, via the rules of formal integrals, to a streamlined way of calculating our integral. Let us apply it to the example considered earlier.

The vector field $\vec{F}(x,y) = \vec{\imath} + y\vec{\jmath}$ has components $P(x,y) = 1$ and $Q(x,y) = y$, so our integral can be written formally as

$$\int_{\mathcal{C}}\vec{F}\cdot d\vec{\mathfrak{s}} = \int_{\mathcal{C}}(\,dx + y\,dy).$$

Using the parametrization from before, $x = t$, $y = \sqrt{1-t^2}$ for $-1 \le t \le 1$, we use the rules of formal differentials to write $dx = \frac{dx}{dt}\,dt = dt$ and $dy = \frac{dy}{dt}\,dt = -\frac{t}{\sqrt{1-t^2}}\,dt$, so

$$P\,dx + Q\,dy = \,dx + y\,dy = (1)(\,dt) + (\sqrt{1-t^2})\left(-\frac{t}{\sqrt{1-t^2}}\,dt\right) = (1-t)\,dt$$

and the integral becomes

$$\int_{\mathcal{C}} P\,dx + Q\,dy = \int_{\mathcal{C}}\,dx + y\,dy = \int_{-1}^{1}(1-t)\,dt = 2$$

as before.

But there is another natural parametrization of the upper half-circle: $x = \cos\theta$ and $y = \sin\theta$ for $0 \le \theta \le \pi$. This leads to the differentials $dx = -\sin\theta\,d\theta$ and $dy = \cos\theta\,d\theta$. The components of the vector field, expressed in terms of our parametrization, are $P = 1$ and $Q = \sin\theta$ so

$$P\,dx + Q\,dy = (-\sin\theta)(\,d\theta) + (\sin\theta)(\cos\theta\,d\theta) = (-\sin\theta + \sin\theta\cos\theta)\,d\theta$$

and our integral becomes

$$\int_{\mathcal{C}} P\,dx + Q\,dy = \int_0^{\pi} (-\sin\theta + \sin\theta\cos\theta)\,d\theta$$
$$= \left(\cos\theta + \frac{\sin^2\theta}{2}\right)_0^{\pi} = (-1+0) - (1+0) = -2.$$

Note that this has the opposite sign from our previous calculation. Why?

The answer becomes clear if we think in terms of the expression for the work integral

$$W = \int_{\mathcal{C}} \vec{F}\cdot d\vec{\mathfrak{s}} = \int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,d\mathfrak{s}.$$

Clearly, the vector field $\vec{F}$ does not change when we switch parametrizations for $\mathcal{C}$. However, our first parametrization (treating $\mathcal{C}$ as the graph of the function $y = \sqrt{1-t^2}$) traverses the semicircle *clockwise*, while the second one traverses it *counterclockwise*. This means that the unit tangent vector $\vec{T}$ determined by the first parametrization is the negative of the one coming from the second, as a result of which the two parametrizations yield path integrals of functions that differ in sign. Thus, even though the path integral of a *scalar-valued function* $\int_{\mathcal{C}} f\,d\mathfrak{s}$ depends only on the geometric curve $\mathcal{C}$ and not on how we parametrize it, the work integral $\int_{\mathcal{C}} \vec{F}\cdot d\vec{\mathfrak{s}}$ depends also on the direction in which we move along the curve: in other words, it depends on the *oriented* curve given by $\mathcal{C}$ together with the direction along it—which determines a choice between the two unit tangents at each point of $\mathcal{C}$. To underline this distinction, we shall refer to *path* integrals of (scalar-valued) *functions*, but *line* integrals of *vector fields*.

Definition 5.1.1. (1) *An **orientation** of a curve $\mathcal{C}$ is a continuous unit vector field $\vec{T}$ defined on $\mathcal{C}$ and tangent to $\mathcal{C}$ at every point. Each regular curve has two distinct orientations.*

(2) *An **oriented curve**[1] is a curve $\mathcal{C}$ together with a choice of orientation $\vec{T}$ of $\mathcal{C}$.*

(3) *The **line integral** of a vector field $\vec{F}$ defined on $\mathcal{C}$ over the oriented curve determined by the unit tangent field $\vec{T}$ is the work integral*

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{\mathfrak{s}} = \int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,d\mathfrak{s}.$$

Since the function $\vec{F}\cdot\vec{T}$ determined by a vector field along an oriented curve is the same for all parametrizations yielding the orientation $\vec{T}$, we have the following invariance principle.

Remark 5.1.2. *The line integral of a vector field $\vec{F}$ over an oriented curve is the same for any parametrization whose velocity points in the same direction as the unit tangent field $\vec{T}$ determined by the orientation. Switching orientation switches the sign of the line integral.*

Differential Forms. So far, we have treated expressions like dx as purely formal expressions, sometimes mysteriously related to each other by relations like $dy = y'\,dx$. An exception has been the notation df for the derivative of a real-valued function

[1] This is also sometimes called a **directed curve**.

$f : \mathbb{R}^n \to \mathbb{R}$ on $\mathbb{R}^n$. This exception will be the starting point of a set of ideas which makes sense of other expressions of this sort.

Recall that the derivative $d_p f$ of $f : \mathbb{R}^n \to \mathbb{R}$ at a point p in its domain is itself a linear function—that is, it respects linear combinations:

$$d_p f\left(a_1\vec{v_1} + a_2\vec{v_2}\right) = a_1 d_p f\left(\vec{v_1}\right) + a_2 d_p f\left(\vec{v_2}\right).$$

Furthermore, if we consider the way it is used, this linear function is applied only to velocity vectors of curves as they pass through the point p. In other words, we should think of the derivative as a linear function $d_p f : T_p\mathbb{R}^n \to \mathbb{R}$ acting on the tangent space to $\mathbb{R}^n$ at p. To keep straight the distinction between the underlying function f, which acts on $\mathbb{R}^n$, and its derivative at p, which acts on the tangent space $T_p\mathbb{R}^n$, we refer to the latter as a **linear functional** on $T_p\mathbb{R}^n$. Now, as we vary the basepoint p, the derivative gives us different linear functionals, acting on different tangent spaces. We abstract this notion in:

Definition 5.1.3. *A **differential form** on $\mathbb{R}^n$ is a rule ω assigning to each point $p \in \mathbb{R}^n$ a linear functional $\omega_p : T_p\mathbb{R}^n \to \mathbb{R}$ on the tangent space to $\mathbb{R}^n$ at p.*

We will in the future often deal with differential forms defined only at points in a subregion $D \subset \mathbb{R}^n$, in which case we will refer to a differential form on D.

Derivatives of functions aside, what do other differential forms look like?

Let us consider the case $n = 2$. We know that a linear functional on $\mathbb{R}^2$ is just a homogeneous polynomial of degree 1; since the functional can vary from basepoint to basepoint, the coefficients of this polynomial are actually functions of the basepoint. To keep the distinction between $\mathbb{R}^2$ and $T_p\mathbb{R}^2$, we will denote points in $\mathbb{R}^2$ by $p = (x, y)$ and vectors in $T_p\mathbb{R}^2$ by $\vec{v} = (v_1, v_2)$; then a typical form acts on a tangent vector $\vec{v}$ at p via

$$\omega_p\left(\vec{v}\right) = P(x, y)\, v_1 + Q(x, y)\, v_2.$$

To complete the connection between formal differentials and differential forms, we notice that the first term on the right above is a multiple (by the scalar P, which depends on the basepoint) of the component of $\vec{v}$ parallel to the x-axis. This component is a linear functional on $T_p\mathbb{R}^n$, which we can think of as the derivative of the function on $\mathbb{R}^2$ that assigns to a point p its x-coordinate; we denote it[2] by dx. Similarly, the linear functional on $T_p\mathbb{R}^2$ assigning to each tangent vector its y-component is denoted dy. We call these the **coordinate forms**:

$$dx(\vec{v}) = v_1$$
$$dy(\vec{v}) = v_2.$$

Then, using this notation, we can write any form ω on $\mathbb{R}^2$ as

$$\omega = P\,dx + Q\,dy.$$

(Of course, it is understood that ω, P and Q all depend on the basepoint p at which they are applied.)

Using this language, we can systematize our procedure for finding work integrals using forms. Given a curve $\mathcal{C}$ parametrized by $\vec{p}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath}, \quad t_0 \le t \le t_1$ and a form defined along $\mathcal{C}$, $\omega = P\,dx + Q\,dy$, we apply the form to the velocity vector

[2] Strictly speaking, we should include a subscript indicating the basepoint p, but since the action on any tangent space is effectively the same, we suppress it.

$\vec{p}'(t) = (x'(t), y'(t))$ of the parametrization. The result can be expressed as a function of the parameter alone:

$$w(t) = \omega_{\vec{p}(t)}\left(\vec{p}'(t)\right) = P(x(t), y(t))\, x'(t) + Q(x(t), y(t))\, y'(t);$$

we then integrate this over the domain of the parametrization:

$$\begin{aligned}\int_{\mathcal{C}} \omega &= \int_{t_0}^{t_1} \left(\omega_{\vec{p}(t)}\left(\vec{p}'(t)\right)\right) dt \\ &= \int_{t_0}^{t_1} \left[P(x(t), y(t))\, x'(t) + Q(x(t), y(t))\, y'(t)\right] dt.\end{aligned} \tag{5.1}$$

The expression appearing inside either of the two integrals itself looks like a form, but now it "lives" on the real line. In fact, we can also regard it as a coordinate form on $\mathbb{R}^1$ in the sense of Definition 5.1.3, using the convention that dt acts on a velocity along the line (which is now simply a real number) by returning the number itself. At this stage—when we have a form on $\mathbb{R}$ rather than on a curve in $\mathbb{R}^2$—we simply interpret our integral in the normal way, as the integral of a function over an interval.

However, the interpretation of this expression as a form can still play a role, when we compare different parametrizations of the same curve. We will refer to the form on parameter space obtained from a parametrization of a curve by the process above as the **pullback** of ω by $\vec{p}$:

$$\left[\vec{p}^*(\omega)\right]_t = \omega_{\vec{p}(t)}\left(\vec{p}'(t)\right) dt. \tag{5.2}$$

Then we can summarize our process of integrating a form along a curve by saying *the integral of a form ω along a parametrized curve is the integral, over the domain in parameter space, of the pullback $\vec{p}^*(\omega)$ of the form by the parametrization.*

Suppose now that $\vec{q}(s)$, $s_0 \le s \le s_1$ is a reparametrization of the same curve. By definition, this means that there is a continuous, strictly monotone function $\mathfrak{t}(s)$ such that $\vec{q}(s) = \vec{p}(\mathfrak{t}(s))$. In dealing with regular curves, we assume that $\mathfrak{t}(s)$ is differentiable, with non-vanishing derivative. We shall call this an **orientation-preserving reparametrization** if $\frac{dt}{ds}$ is positive at every point, and **orientation-reversing** if $\frac{dt}{ds}$ is negative.[3]

Suppose first that our reparametrization is orientation-preserving. To integrate ω over our curve using $\vec{q}(s)$ instead of $\vec{p}(t)$, we take the pullback of ω by $\vec{q}$,

$$\left[\vec{q}^*(\omega)\right]_s = \omega_{\vec{q}(s)}\left(\vec{q}'(s)\right) ds.$$

By the Chain Rule, setting $t = \mathfrak{t}(s)$,

$$\vec{q}'(s) = \frac{d}{ds}\left[\vec{q}(s)\right] = \frac{d}{ds}\left[\vec{p}(\mathfrak{t}(s))\right] = \frac{d}{dt}\left[\vec{p}(\mathfrak{t}(s))\right]\frac{dt}{ds} = \vec{p}'(\mathfrak{t}(s))\, \mathfrak{t}'(s)\, ds.$$

Now if we think of the change-of-variables map $\mathfrak{t}: \mathbb{R} \to \mathbb{R}$ as describing a point moving along the t-line, parametrized by $t = \mathfrak{t}(s)$, we see that the pullback of any form $\alpha_t = P(t)\, dt$ by $\mathfrak{t}$ is given by

$$\left[\mathfrak{t}^*(\alpha_t)\right]_s = \alpha_{\mathfrak{t}(s)}\left(\mathfrak{t}'(s)\right) ds = P(\mathfrak{t}(s))\, \mathfrak{t}'(s)\, ds.$$

---

[3]You may note that this is the same as a "direction-preserving" (*resp.* "direction-reversing") reparametrization of the curve. We adopt the "orientation" terminology here in anticipation of a parallel terminology for surfaces.

Applying this to $\alpha_t = [\vec{p}^*(\omega)]_t$ we see that

$$[\mathfrak{t}^*(\vec{p}^*(\omega))]_s = [\vec{p}^*(\omega)]_{\mathfrak{t}(s)}\mathfrak{t}'(s)\,ds = \omega_{\vec{p}(\mathfrak{t}(s))}\left(\vec{p}\,'(\mathfrak{t}(s))\right)\mathfrak{t}'(s)\,ds$$
$$= \omega_{\vec{q}(s)}\left(\vec{p}\,'(\mathfrak{t}(s))\right)\mathfrak{t}'(s)\,ds = \omega_{\vec{q}(s)}\left(\vec{p}\,'(\mathfrak{t}(s))\mathfrak{t}'(s)\right)ds$$
$$= \omega_{\vec{q}(s)}\left(\vec{q}\,'(s)\right)ds = [\vec{q}^*(\omega)]_s;$$

in other words,

$$\vec{q}^*(\omega) = \mathfrak{t}^*\left(\vec{p}^*(\omega)\right). \tag{5.3}$$

Clearly, the two integrals coming from pulling ω back by $\vec{p}$ and $\vec{q}$, respectively, are the same:

$$\int_{s_0}^{s_1}[\vec{q}^*(\omega)]_s = \int_{t_0}^{t_1}[\vec{p}^*(\omega)]_t.$$

In other words, the definition of $\int_{\mathcal{C}}\omega$ via Equation (5.1) yields the same quantity for a given parametrization as for any orientation-preserving reparametrization.

What changes in the above argument when $\mathfrak{t}$ has *negative* derivative? The integrand in the calculation using $\vec{q}$ is the same: we still have Equation (5.3). However, since the reparametrization is order-*reversing*, $\mathfrak{t}$ is strictly *decreasing*, which means that it interchanges the endpoints of the domain: $\mathfrak{t}(s_0) = t_1$ and $\mathfrak{t}(s_1) = t_0$. Thus,

$$\int_{t_0}^{t_1}[\vec{p}^*(\omega)]_t = \int_{s_1}^{s_0}[\vec{q}^*(\omega)]_s = -\int_{s_0}^{s_1}[\vec{q}^*(\omega)]_s\;:$$

the integral given by applying Equation (5.1) to $\vec{q}$ has the same *integrand*, but the limits of integration are reversed: the resulting integral is the negative of what we would have gotten had we used $\vec{p}$.

Now let us relate this back to our original formulation of work integrals in terms of vector fields. Recall from § 3.2 that a linear functional on $\mathbb{R}^n$ can be represented as taking the dot product with a fixed vector. In particular, the form $\omega = P\,dx + Q\,dy$ corresponds to the vector field $\vec{F} = P\vec{\imath} + Q\vec{\jmath}$ in the sense that $\omega_p(\vec{v}) = P(p)v_1 + Q(p)v_2 = \vec{F}(p)\cdot\vec{v}$. In fact, using the formal vector $d\vec{\mathfrak{s}} = dx\vec{\imath} + dy\vec{\jmath}$ which can itself be thought of as a "vector-valued" form, we can write $\omega = \vec{F}\cdot d\vec{\mathfrak{s}}$.

Our whole discussion carries over practically *verbatim* to $\mathbb{R}^3$. A vector field $\vec{F}$ on $\mathbb{R}^3$ can be written as $\vec{F}(x,y,z) = P(x,y,z)\vec{\imath} + Q(x,y,z)\vec{\jmath} + R(x,y,z)\vec{k}$, and the corresponding form on $\mathbb{R}^3$ is $\omega = \vec{F}\cdot d\vec{\mathfrak{s}} = P\,dx + Q\,dy + R\,dz$. Let us see an example of how the line integral works out in this case.

The vector field $\vec{F}(x,y,z) = z\vec{\imath} - y\vec{\jmath} + x\vec{k}$ corresponds to the form $\omega = \vec{F}\cdot d\vec{\mathfrak{s}} = z\,dx - y\,dy + x\,dz$. Let us integrate this over the curve given parametrically by $\vec{p}(t) = \cos t\vec{\imath} + \sin t\vec{\jmath} + \sin 2t\vec{k}$ for $0 \le t \le \frac{\pi}{2}$. The velocity of this parametrization is given by $\vec{p}\,'(t) = -\sin t\vec{\imath} + \cos t\vec{\jmath} + 2\cos 2t\vec{k}$ and its pullback by the form ω is

$$[\vec{p}^*(\omega)]_t = \omega_{\vec{p}(t)}\left(\vec{p}\,'(t)\right)dt$$
$$= [(\sin 2t)(-\sin t) - (\sin t)(\cos t) + (\cos t)(2\cos 2t)]\,dt$$
$$= [-2\sin^2 t\cos t - \sin t\cos t + 2(1-2\sin^2 t)\cos t]\,dt$$
$$= [-6\sin^2 t - \sin t + 2]\cos t\,dt.$$

Thus,

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{\mathfrak{s}} = \int_{\mathcal{C}} z\,dx - y\,dy + x\,dz = \int_{\mathcal{C}} \omega = \int_0^{\pi/2} \omega^*\left(\vec{p}\,'\right)$$
$$= \int_0^{\pi/2} [-6\sin^2 t - \sin t + 2]\cos t\,dt$$
$$= [-2\sin^3 t - \frac{1}{2}\sin^2 t + 2\sin t]_0^{\pi/2} = [-2 - \frac{1}{2} + 2] = -\frac{1}{2}.$$

Exercises for § 5.1

Answers to Exercises 1a, 1b, 2a, 3a, and 4a are given in Appendix A.13.

Practice problems:

(1) Sketch each vector field below, in the style of Figures 5.1 and 5.2.
  (a) $x\vec{\imath}$
  (b) $x\vec{\jmath}$
  (c) $y\vec{\imath} - y\vec{\jmath}$
  (d) $x\vec{\imath} + y\vec{\jmath}$
  (e) $x\vec{\imath} - y\vec{\jmath}$
  (f) $-y\vec{\imath} + x\vec{\jmath}$

(2) Evaluate $\int_{\mathcal{C}} \vec{F} \cdot d\vec{\mathfrak{s}}$:
  (a) $\vec{F}(x,y) = x\vec{\imath} + y\vec{\jmath}$, $\mathcal{C}$ is the graph $y = x^2$ from $(-2,4)$ to $(1,1)$.
  (b) $\vec{F}(x,y) = y\vec{\imath} + x\vec{\jmath}$, $\mathcal{C}$ is the graph $y = x^2$ from $(1,1)$ to $(-2,4)$.
  (c) $\vec{F}(x,y) = (x+y)\vec{\imath} + (x-y)\vec{\jmath}$, $\mathcal{C}$ is given by $x = t^2$, $y = t^3$, $-1 \le t \le 1$.
  (d) $\vec{F}(x,y) = x^2\vec{\imath} + y^2\vec{\jmath}$, $\mathcal{C}$ is the circle $x^2 + y^2 = 1$ traversed counterclockwise.
  (e) $\vec{F}(x,y,z) = x^2\vec{\imath} + xz\vec{\jmath} - y^2\vec{k}$, $\mathcal{C}$ is given by $x = t$, $y = t^2$, $z = t^3$, $-1 \le t \le 1$.
  (f) $\vec{F}(x,y,z) = y\vec{\imath} - x\vec{\jmath} + ze^x\vec{k}$, $\mathcal{C}$ is the line segment from $(0,0,0)$ to $(1,1,1)$.
  (g) $\vec{F}(x,y,z) = yz\vec{\imath} + xz\vec{\jmath} + xy\vec{k}$, $\mathcal{C}$ is given by $\vec{p}(t) = (t^2, t, -t^2)$, $-1 \le t \le 1$.
  (h) $\vec{F}(x,y,z) = yz\vec{\imath} + xz\vec{\jmath} + xy\vec{k}$, $\mathcal{C}$ is the polygonal path from $(1,-1,1)$ to $(2,1,3)$ to $(-1,0,0)$.

(3) Evaluate $\int_{\mathcal{C}} P\,dx + Q\,dy$ (or $\int_{\mathcal{C}} P\,dx + Q\,dy + R\,dz$):
  (a) $P(x,y) = x^2 + y^2$, $Q(x,y) = y - x$, $\mathcal{C}$ is the y-axis from the origin to $(0,1)$.
  (b) $P(x,y) = x^2 + y^2$, $Q(x,y) = y - x$, $\mathcal{C}$ is the x-axis from $(-1,0)$ to $(1,0)$.
  (c) $P(x,y) = y$, $Q(x,y) = -x$, $\mathcal{C}$ is given by $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$.
  (d) $P(x,y) = xy$, $Q(x,y) = y^2$, $\mathcal{C}$ is $y = \sqrt{1-x^2}$ from $(-1,0)$ to $(1,0)$.
  (e) $P(x,y) = xy$, $Q(x,y) = y^2$, $\mathcal{C}$ is given by $x = t^2$, $y = t$, $-1 \le t \le 1$
  (f) $P(x,y) = -x$, $Q(x,y) = y$, $\mathcal{C}$ is given by $\vec{p}(t) = (\cos^3 t, \sin^3 t))$, $0 \le t \le 2\pi$.
  (g) $P(x,y,z) = xy$, $Q(x,y,z) = xz$, $R(x,y,z) = yz$, $\mathcal{C}$ is given by $x = \cos t$, $y = \sin t$, $z = -\cos t$, $0 \le t \le \frac{\pi}{2}$.
  (h) $P(x,y,z) = z$, $Q(x,y,z) = x^2 + y^2$, $R(x,y,z) = x + z$, $\mathcal{C}$ is given by $x = t^{1/2}$, $y = t$, $z = t^{3/2}$, $1 \le t \le 2$.
  (i) $P(x,y,z) = y + z$, $Q(x,y,z) = -x$, $R(x,y,z) = -x$, $\mathcal{C}$ is given by $x = \cos t$, $y = \sin t$, $z = \sin t + \cos t$, $0 \le t \le 2\pi$.

(4) Let $\mathcal{C}$ be the upper semicircle $x^2 + y^2 = 1$ from $(1,0)$ to $(-1,0)$, followed by the x-axis back to $(1,0)$. For each 1-form below, set up the integral two ways:
  - using the parametrization $(x,y) = (\cos\theta, \sin\theta)$, $0 \le \theta \le \pi$ for the upper semicircle;

- using the fact that the upper semicircle is the graph $y = \sqrt{1-x^2}$, $-1 \leq x \leq 1$ (*Caution:* make sure your curve goes in the right direction!);

Then evaluate one of these versions.

(a) $\omega = x\,dy + y\,dx$

(b) $\omega = (x^2 + y)\,dx + (x + y^2)\,dy$

5.2 The Fundamental Theorem for Line Integrals

The Fundamental Theorem for Line Integrals in the Plane. Recall the *Fundamental Theorem of Calculus*, which says in part that if a function f is continuously differentiable on the interior of an interval (a, b) (and continuous at the endpoints), then the integral over $[a, b]$ of its derivative is the difference between the values of the function at the endpoints:

$$\int_a^b \frac{df}{dt}\,dt = f\Big|_a^b := f(b) - f(a).$$

The analogue of this for functions of several variables is called the *Fundamental Theorem for Line Integrals*. The derivative of a real-valued function on $\mathbb{R}^2$ is our first example of a form;

$$d_{(x,y)}f(v_1, v_2) = \left(\frac{\partial f}{\partial x}(x, y)\right)v_1 + \left(\frac{\partial f}{\partial y}(x, y)\right)v_2.$$

We shall call a form ω **exact** if it equals the differential of some function f: $\omega = df$. Let us integrate such a form over a curve $\mathcal{C}$, parametrized by $\vec{p}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath}$, $a \leq t \leq b$. We have

$$\begin{aligned}[\vec{p}^*(\omega)]_t &= \omega_{\vec{p}(t)}\left(\frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath}\right)dt \\ &= \left[\left(\frac{\partial f}{\partial x}(x(t), y(t))\right)\frac{dx}{dt} + \left(\frac{\partial f}{\partial y}(x(t), y(t))\right)\frac{dy}{dt}\right]dt.\end{aligned}$$

By the Chain Rule, this is $\frac{d}{dt}[f(x(t), y(t))]\,dt = g'(t)\,dt$, where $g(t) = f(\vec{p}(t)) = f(x(t), y(t))$. Thus,

$$\int_{\mathcal{C}} df = \int_a^b \frac{d}{dt}[f(x(t), y(t))]\,dt = \int_a^b g'(t)\,dt.$$

Provided this integrand is continuous (that is, the partials of f are continuous), the Fundamental Theorem of Calculus tells us that this equals $g(t)\Big|_a^b = g(b) - g(a)$, or, writing this in terms of our original function,

$$f(\vec{p}(t))\Big|_a^b = f(\vec{p}(b)) - f(\vec{p}(a)).$$

Let us see how this translates to the language of vector fields. The vector field corresponding to the differential of a function is its gradient

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = \vec{\nabla} f \cdot d\vec{\mathfrak{s}}.$$

A vector field $\vec{F}$ is called **conservative** if it equals the gradient of some function f; the function f is then a **potential** for $\vec{F}$.

The bilingual statement (that is, in terms of both vector fields and forms) of this fundamental result is:

Theorem 5.2.1 (Fundamental Theorem for Line Integrals). *Suppose $\mathcal{C}$ is an oriented curve starting at p_{start} and ending at p_{end}, and f is a continuously differential function defined along $\mathcal{C}$. Then the integral of its differential df (resp. the line integral of its gradient vector field $\vec{\nabla} f$) over $\mathcal{C}$ equals the difference between the values of f at the endpoints of $\mathcal{C}$:*

$$\int_{\mathcal{C}} \vec{\nabla} f \cdot d\vec{\mathfrak{s}} = \int_{\mathcal{C}} df = f(x)\Big|_{p_{start}}^{p_{end}} = f(p_{end}) - f(p_{start}). \tag{5.4}$$

This result leads to a rather remarkable observation. We saw that the line integral of a vector field over an oriented curve $\mathcal{C}$ depends only on the curve (as a set of points) and the direction of motion along $\mathcal{C}$—it does not change if we reparametrize the curve before calculating it. But the Fundamental Theorem for Line Integrals tells us that if the vector field is conservative, then the line integral depends only on where the curve starts and where it ends, *not on how we get from one to the other.* Saying this a little more carefully,

Corollary 5.2.2. *Suppose f is a $\mathcal{C}^1$ function defined on the region D.*

Then the line integral $\int_{\mathcal{C}} \vec{\nabla} f \cdot d\vec{\mathfrak{s}} = \int_{\mathcal{C}} df$ is independent of the curve $\mathcal{C}$—that is, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are two curves in D with a common starting point and a common ending point, then

$$\int_{\mathcal{C}_1} \vec{\nabla} f \cdot d\vec{\mathfrak{s}} = \int_{\mathcal{C}_2} \vec{\nabla} f \cdot d\vec{\mathfrak{s}}.$$

A second consequence of Equation (5.4) concerns a **closed** curve—that is, one that starts and ends at the same point ($p_{start} = p_{end}$). In this case,

$$\int_{\mathcal{C}} \vec{\nabla} f \cdot d\vec{\mathfrak{s}} = \int_{\mathcal{C}} df = f(x)\Big|_{p_{start}}^{p_{end}} = f(p_{end}) - f(p_{start}) = 0$$

Corollary 5.2.3. *Suppose f is a $\mathcal{C}^1$ function defined on the region D. Then the line integral of df around any closed curve $\mathcal{C}$ is zero:*

$$\int_{\mathcal{C}} \vec{\nabla} f \cdot d\vec{\mathfrak{s}} = \int_{\mathcal{C}} df = 0.$$

Sometimes, the integral of a vector field $\vec{F}$ over a *closed* curve is denoted $\oint_{\mathcal{C}} \vec{F} \cdot d\vec{\mathfrak{s}}$, to emphasize the fact that the curve is closed.

Actually, Corollary 5.2.2 and Corollary 5.2.3 are easily shown to be equivalent, using the fact that reversing orientation switches the sign of the integral (Exercise 4).

How do we decide whether or not a given vectorfield $\vec{F}$ is conservative?

The most direct way is to try to find a potential function f for $\vec{F}$. Let us investigate a few examples.

An easy one is $\vec{F}(x,y) = y\vec{\imath} + x\vec{\jmath}$. The condition that $\vec{F} = \vec{\nabla} f = \frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath}$ consists of the two equations $\frac{\partial f}{\partial x}(x,y) = y$ and $\frac{\partial f}{\partial y}(x,y) = x$. The first is satisfied by $f(x,y) = xy$ and we see that it also satisfies the second. Thus, we know that one potential for $\vec{F}$ is

$$f(x,y) = xy.$$

However, things are a bit more complicated if we consider

$$\vec{F}(x,y) = (x+y)\vec{\imath} + (x+y)\vec{\jmath}.$$

It is easy enough to guess that a function satisfying the first condition, $\frac{\partial f}{\partial x}(x,y) = x+y$, is $f(x,y) = \frac{x^2}{2} + xy$, but when we try to fit the second condition, which requires $\frac{\partial}{\partial y}\left[\frac{x^2}{2} + xy\right] = x + y$, we come up with the impossible condition $x = x + y$.

Does this mean our vector field is not conservative? Well, no. We need to think more systematically.

Note that our guess for $f(x,y)$ is not the *only* function satisfying the condition $\frac{\partial f}{\partial x} = x + y$; we need a function which is an antiderivative of $x + y$ when y is treated as a constant. This means that a complete list of antiderivatives consists of our specific antiderivative *plus an arbitray "constant"*—which in our context means any expression that does not depend on x. So we should write the "constant" as a function of y:

$$f(x,y) = \frac{x^2}{2} + xy + C(y).$$

Now, when we try to match the second condition, we come up with

$$x + y = \frac{\partial f}{\partial y} = x + C'(y),$$

or $C'(y) = y$, which leads to

$$C(y) = \frac{y^2}{2} + C$$

(where this time, C is an honest constant—it depends on neither x nor y). Thus the list of all functions satisfying *both* conditions is

$$f(x,y) = \frac{x^2}{2} + xy + \frac{y^2}{2} + C,$$

showing that indeed $\vec{F}$ is conservative.

This example illustrates the general procedure. If we seek a potential $f(x,y)$ for the vector field

$$\vec{F}(x,y) = P(x,y)\vec{\imath} + Q(x,y)\vec{\jmath},$$

we first look for a complete list of functions satisfying the first condition

$$\frac{\partial f}{\partial x} = P(x,y);$$

this is a process much like taking the "inner" integral in an iterated integral, but without specified "inner" limits of integration: we treat y as a constant, and (provided we can do the integration) end up with an expression that looks like

$$f(x,y) = f_1(x,y) + C(y)$$

as a list of all functions satisfying the first condition. To decide which of these *also* satisfy the second condition, we take the partial with respect to y of our expression above, and match it to the second component of $\vec{F}$:

$$\frac{\partial}{\partial y}\left[f_1(x,y)\right] + C'(y) = Q(x,y).$$

If this match is possible (we shall see below how this might fail), then we end up with a list of all potentials for $\vec{F}$ that looks like

$$f(x,y) = f_1(x,y) + f_2(y) + C,$$

where $f_2(y)$ does not involve x, and C is an arbitrary constant.

Let's try this on a slightly more involved vector field,

$$\vec{F}(x,y) = (2xy + y^3 + 2)\vec{\imath} + (x^2 + 3xy^2 - 3)\vec{\jmath}.$$

The list of functions satisfying

$$\frac{\partial f}{\partial x} = 2xy + y^3 + 2$$

is obtained by integrating, treating y as a constant:

$$f(x,y) = x^2y + xy^3 + 2x + C(y);$$

differentiating with respect to y (and of course now treating x as constant) we obtain $x^2 + 3xy^2 + C'(y)$. Matching this with the second component of $\vec{F}$ gives

$$x^2 + 3xy^2 - 3 = \frac{\partial f}{\partial y} = x^2 + 3xy^2 + C'(y)$$

or $-3 = C'(y)$, so $C(y) = -3y + C$, and our list of potentials for $\vec{F}$ is

$$f(x,y) = x^2y + xy^3 + 2x - 3y + C.$$

Now let us see how such a procedure can fail. If we look for potentials of

$$\vec{F}(x,y) = (x + 2xy)\vec{\imath} + (x^2 + xy)\vec{\jmath}\ :$$

the first condition, $\frac{\partial f}{\partial x} = x + 2xy$, means

$$f(x,y) = \frac{x^2}{2} + x^2y + C(y);$$

the partial with respect of y of such a function is

$$\frac{\partial f}{\partial y} = x^2 + C'(y).$$

But when we try to match this to the second component of $\vec{F}$, we require $x^2 + xy = x^2 + C'(y)$, or, cancelling the first term on both sides,

$$xy = C'(y).$$

This requires $C(y)$, which is explicitly a function not involving x, to equal something that *does* involve x, an impossibility. This means no function can satisfy *both* of the conditions required to be a potential for $\vec{F}$; thus $\vec{F}$ is *not* conservative.

It is hardly obvious at first glance why our last example failed when the others succeeded. So we might ask if there is another way to decide whether a given vector field $\vec{F}(x,y) = P(x,y)\vec{\imath} + Q(x,y)\vec{\jmath}$ is conservative.

A necessary condition follows from the equality of cross-partials (Theorem 3.8.1). If $\vec{F}(x,y) = P(x,y)\vec{\imath} + Q(x,y)\vec{\jmath}$ is the gradient of the function $f(x,y)$, that is, $P(x,y) = \frac{\partial f}{\partial x}(x,y)$ and $Q(x,y) = \frac{\partial f}{\partial y}(x,y)$, then $\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$– and equality of cross-partials then says that these are equal: $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Technically, Theorem 3.8.1 requires that the two second-order partials be continuous, which means that the components of $\vec{F}$ (or of the form $\omega = P\,dx + Q\,dy$) have $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ continuous. In particular, it applies to any continuously differentiable, or $\mathcal{C}^1$, vector field.

Remark 5.2.4. *For any conservative $\mathcal{C}^1$ vector field $\vec{F}(x,y) = P(x,y)\vec{\imath} + Q(x,y)\vec{\jmath}$ (resp. $\mathcal{C}^1$ exact form $\omega_{(x,y)} = P(x,y)\,dx + Q(x,y)\,dy$),*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{5.5}$$

A vector field $\vec{F} = P\vec{\imath} + Q\vec{\jmath}$ (*resp.* differential form $\omega = P\,dx + Q\,dy$) is called **irrotational**[4] (*resp.* **closed**) if it satisfies Equation (5.5); Remark 5.2.4 then says that every conservative vector field (*resp.* exact form) is irrotational (*resp.* closed).

How about the converse—if this condition holds, is the vector field (*resp.* form) necessarily conservative (*resp.* exact)? Well...almost.

We discuss the details in Appendix A.9, but the upshot is that if we are looking at plane region without "holes" (called a **simply connected** region—see Definition A.9.4) then the condition is necessary and sufficient. This is summarized in

Proposition 5.2.5. *If $D \subset \mathbb{R}^2$ is a simply connected region, then any differential form $\omega = P\,dx + Q\,dy$ (resp. vector field $\vec{F}$) on D is exact precisely if it is closed (resp. irrotational).*

Line Integrals in Space. The situation for forms and vector fields in $\mathbb{R}^3$ is completely analogous to that in the plane.

A vector field on $\mathbb{R}^3$ assigns to a point $(x,y,z) \in \mathbb{R}^3$ a vector $\vec{F}(x,y,z) = P(x,y,z)\vec{\imath} + Q(x,y,z)\vec{\jmath} + R(x,y,z)\vec{k}$ while a form on $\mathbb{R}^3$ assigns to $(x,y,z) \in \mathbb{R}^3$ the functional $\omega_{(x,y,z)} = P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$.

The statement of Theorem 5.2.1 that we gave holds in $\mathbb{R}^3$: *the line integral of the gradient of a function (resp. of the differential of a function) over any curve equals the difference between the values of the function at the endpoints of the curve.*

It is instructive to see how the process of finding a potential function for a vector field or form works in $\mathbb{R}^3$. Let us consider the vector field

$$\vec{F}(x,y,z) = (y^2 + 2xz + 2)\vec{\imath} + (2xy + z^3)\vec{\jmath} + (x^2 + 3yz^2 + 6z)\vec{k}$$

or equivalently, the form

$$\omega_{(x,y,z)} = (y^2 + 2xz + 2)\,dx + (2xy + z^3)\,dy + (x^2 + 3yz^2 + 6z)\,dz.$$

A potential for either one is a function $f(x,y,z)$ satisfying the three conditions

$$\frac{\partial f}{\partial x}(x,y,z) = P(x,y,z) = y^2 + 2xz + 2$$
$$\frac{\partial f}{\partial y}(x,y,z) = Q(x,y,z) = 2xy + z^3$$
$$\frac{\partial f}{\partial z}(x,y,z) = R(x,y,z) = x^2 + 3yz^2 + 6z.$$

[4]The reason for this terminology will become clear later.

The first condition leads to $f(x, y, z) = xy^2 + x^2z + 2x$; more accurately, the list of *all* functions satisfying the first condition consists of this function plus any function depending only on y and z:

$$f(x, y, z) = xy^2 + x^2z + 2x + C(y, z).$$

Differentiating this with respect to y, $\frac{\partial f}{\partial y}(x, y, z) = 2xy + \frac{\partial C}{\partial y}$, turns the second condition into $2xy + z^3 = 2xy + \frac{\partial C}{\partial y}$ so the function $C(y, z)$ must satisfy $z^3 = \frac{\partial C}{\partial y}$. This tells us that

$$C(y, z) = yz^3 + C(z)$$

(since a term depending only on z will not show up in the partial with respect to y). Substituting back, we see that the list of all functions satisfying the *first two* conditions is

$$f(x, y, z) = xy^2 + x^2z + 2x + yz^3 + C(z).$$

Now, taking the partial with respect to z and substituting into the third condition yields

$$x^2 + 3yz^2 + \frac{dC}{dz} = \frac{\partial f}{\partial z}(x, y, z) = x^2 + 3yz^2 + 6z,$$

or $\frac{dC}{dz} = 6z$; hence

$$C(z) = 3z^2 + C,$$

where this time C is an honest constant. Thus, the list of all functions satisfying all three conditions—that is, all the potential functions for $\vec{F}$ or ω—is

$$f(x, y, z) = xy^2 + x^2z + 2x + yz^3 + 3z^2 + C,$$

where C is an arbitrary constant.

If we recall that Equation (5.5)—that every conservative vectorfield (*resp.* exact form) must be irrotational (*resp.* closed)—came from the equality of cross-partials (Theorem 3.8.1), it is natural that the corresponding condition in $\mathbb{R}^3$ consists of three equations (Exercise 5):

$$\begin{aligned}\frac{\partial P}{\partial y} &= \frac{\partial Q}{\partial x}\\ \frac{\partial P}{\partial z} &= \frac{\partial R}{\partial x}\\ \frac{\partial Q}{\partial z} &= \frac{\partial R}{\partial y}.\end{aligned} \tag{5.6}$$

Again, this condition is necessary *and* sufficient for a vector field (*resp.* form) in $\mathbb{R}^3$ to be conservative (*resp.* exact) in a simply-connected region of 3-space. The meaning of this is discussed in Appendix A.9.

Exercises for § 5.2

Answers to Exercises 1a, 2a, and 3a are given in Appendix A.13.

Practice problems:

(1) For each vectorfield below, determine whether it is conservative, and if it is, find a potential function; in either case, evaluate $\int_{\mathcal{C}} \vec{F} \cdot \vec{T}\, d\mathfrak{s}$ over the given curve:

   (a) $\vec{F}(x, y) = (2xy + y^2)\vec{\imath} + (2xy + x^2)\vec{\jmath}$, $\mathcal{C}$ is the straight line segment from $(0, 0)$ to $(1, 1)$.

(b) $\vec{F}(x,y) = (x^2y + x)\vec{\imath} + (x^2y + y)\vec{\jmath}$, $\mathcal{C}$ is the straight line segment from $(0,0)$ to $(1,1)$.

(c) $\vec{F}(x,y) = (x^2 + x + y)\vec{\imath} + (x + \pi \sin \pi y)\vec{\jmath}$, $\mathcal{C}$ is the straight line segment from $(0,0)$ to $(1,1)$.

(d) $\vec{F}(x,y) = (x^2 - y^2)\vec{\imath} + (x^2 - y^2)\vec{\jmath}$, $\mathcal{C}$ is the circle $x^2 + y^2 = 1$, traversed counterclockwise.

(2) Each vector field below is conservative. Find a potential function, and evaluate $\int_{\mathcal{C}} \vec{F} \cdot d\vec{s}$.

(a) $\vec{F}(x,y,z) = (2xy + z)\vec{\imath} + (x^2 + z)\vec{\jmath} + (x + y)\vec{k}$, $\mathcal{C}$ is the straight line segment from $(1,0,1)$ to $(1,2,2)$.

(b) $\vec{F}(x,y,z) = y \cos xy\vec{\imath} + (x \cos xy - z \sin yz)\vec{\jmath} - y \sin yz\vec{k}$, $\mathcal{C}$ is the straight line segment from $(0,\pi,-1)$ to $(1,\frac{\pi}{2},4)$.

(c) $\vec{F}(x,y,z) = y^2z^3\vec{\imath} + (2xyz^3 + 2z)\vec{\jmath} + (3xy^2z^2 + 2(y + z))\vec{k}$, $\mathcal{C}$ is given by $\vec{p}(t) = (\sin \frac{\pi t}{2}, te^t, te^t \sin \frac{\pi t}{2})$, $0 \le t \le 1$.

(d) $\vec{F}(x,y,z) = (2xy - y^2z)\vec{\imath} + (x^2 - 2xyz)\vec{\jmath} + (1 - xy^2)\vec{k}$, $\mathcal{C}$ is given by $x = \cos \pi t$, $y = t$, $z = t^2$, $0 \le t \le 2$.

(e) $\vec{F}(x,y,z) = ze^x cosy\vec{\imath} - ze^x \sin y\vec{\jmath} + e^x \cos y\vec{k}$, $\mathcal{C}$ is the broken line curve from $(0,0,0)$ to $(2,\pi,1)$ to $(1,\pi,1)$.

(3) For each 1-form ω, determine whether it is exact, and if so, find a potential function. In either case, evaluate $\int_{\mathcal{C}} \omega$, where $\mathcal{C}$ is the straight-line segment from $(-1,1,-1)$ to $(1,2,2)$.

(a) $\omega = 2xyz^3\,dx + x^2z^3\,dy + 3x^2yz^2\,dz$

(b) $\omega = (2xy + yz)\,dx + (x^2 + xz + 2y)\,dy + (xy + 2z)\,dz$

(c) $\omega = (y - z)\,dx + (x - z)\,dy + (x - y)\,dz$

Theory problems:

(4) **Show** that for a continuous vector field $\vec{F}$ defined in the region $D \subset \mathbb{R}^2$, the following are equivalent:

- The line integral of $\vec{F}$ over any closed curve in D is zero;
- For any two paths in D with a common starting point and a common endpoint, the line integrals of $\vec{F}$ over the two paths are equal.

(5) Prove that the equations

$$\begin{aligned} \frac{\partial P}{\partial y} &= \frac{\partial Q}{\partial x} \\ \frac{\partial P}{\partial z} &= \frac{\partial R}{\partial x} \\ \frac{\partial Q}{\partial z} &= \frac{\partial R}{\partial y}. \end{aligned} \tag{5.7}$$

are satisfied by any $\mathcal{C}^3$ conservative vector field in $\mathbb{R}^3$.

5.3 Green's Theorem

We saw in § 5.2 that the line integral of a conservative vector field (or of an exact form) around a closed curve is zero. Green's Theorem tells us what happens when a planar vector field is not conservative. This is related to the proof in Appendix A.9 of the sufficiency of the "cross-partials" condition for the existence of a potential function in a simply-connected region, where the difference between integrating the form $Q\,dx$ (*resp.* $P\,dy$) along the sides of a right triangle and integrating it along the hypotenuse was related to the integral of the partial $\frac{\partial Q}{\partial x}$ (*resp.* $\frac{\partial P}{\partial y}$) over the inside of the triangle. Here, we need to reformulate this more carefully, and do so in terms of a closed curve.

Recall that in § 1.6 we defined the orientation of a triangle in the plane, and its associated signed area. A triangle or other polygon has **positive orientation** if its vertices are traversed in counterclockwise order. We now extend this notion to a closed, simple curve.[5] An intuitively plausible observation, but one which is very difficult to prove rigorously, is known as the **Jordan Curve Theorem**: it says that a simple, closed curve $\mathcal{C}$ in the plane divides the plane into two regions (the "inside" and the "outside"): any two points in the same region can be joined by a curve disjoint from $\mathcal{C}$, but it is impossible to join a point *inside* the curve to one *outside* the curve without crossing $\mathcal{C}$. The "inside" is a bounded set, referred to as the region **bounded** by $\mathcal{C}$; the "outside" is unbounded. This result was formulated by Camille Jordan (1838-1922) in 1887 [30, 1st ed., Vol. 3, p. 593], but first proved rigorously by the American mathematician Oswald Veblen (1880-1960) in 1905 [51].[6]

We shall formulate the notion of positive orientation first for a *regular* simple closed curve. Recall from Definition 5.1.1 that an *orientation* of a regular curve $\mathcal{C}$ is a continuous choice of unit tangent vector $\vec{T}$ at each point of $\mathcal{C}$; there are exactly two such choices.

Definition 5.3.1. (1) *If* $\vec{T} = (\cos\theta, \sin\theta)$ *is a unit vector in* $\mathbb{R}^2$, *then the* ***leftward normal*** *to* $\vec{T}$ *is the vector*

$$\overrightarrow{N_+} = \left(\cos(\theta + \frac{\pi}{2}), \sin(\theta + \frac{\pi}{2})\right) = (-\sin\theta, \cos\theta).$$

(2) *Suppose $\mathcal{C}$ is a regular, simple, closed curve in the plane. The* ***postitive orientation*** *of $\mathcal{C}$ is the choice $\vec{T}$ for which the leftward normal points into the region bounded by $\mathcal{C}$— in other words, if $\vec{p}$ is the position vector for the basepoint of $\vec{T}$, then for small $\varepsilon > 0$, the point $\vec{p} + \varepsilon\overrightarrow{N_+}$ belongs to the* inside *region. (Figure* 5.3*). The other orientation (for which $\overrightarrow{N_+}$ points into the* unbounded *region) is the* ***negative orientation***.

Recall also, from § 4.2, that a region D in the plane is *regular* if it is both x-regular and y-regular—meaning that any horizontal or vertical line intersects D in either a single point or a single interval. The theory of multiple integrals we developed in Chapter 4 was limited to regions which are either regular or can be subdivided into regular subregions.

Unfortunately, a regular region need not be bounded by a regular curve. For example, a polygon such as a triangle or rectangle is not a regular curve, since it has "corners" where there is no well-defined tangent line. As another example, if D is defined by the inequalities $x^2 \leq y \leq \sqrt{x}$ and $0 \leq x \leq 1$, then its boundary consists of two pieces: the lower edge is part of the graph of $y = x^2$, while the upper edge is

[5] Recall that a curve is **closed** if it starts and ends at the same point. A curve is **simple** if it does not intersect itself: that is, if it can be parametrized over a closed interval, say by $\vec{p}(t)$, $t_0 \leq t \leq t_1$ so that the only instance of $\vec{p}(s) = \vec{p}(t)$ with $s \neq t$ is $s = a$, $t = b$. A simple, closed curve can also be thought of as parametrized over a circle, in such a way that distinct points correspond to distinct parameter values on the circle.

[6] There is some controversy concerning the rigor of Jordan's original proof [21].

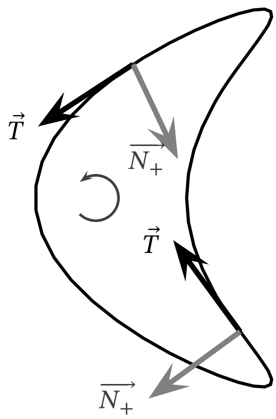


Figure 5.3. Positive Orientation of a Simple Closed Curve

part of the graph of $y = \sqrt{x}$. Each piece is naturally parametrized as a regular curve. The natural parametrization of the lower edge, $x = t$, $y = t^2$, $0 \le t \le 1$, is clearly regular. If we try to parametrize the upper edge analogously as $x = t$, $y = \sqrt{t}$, we have a problem at $t = 0$, since $\sqrt{t}$ is not differentiable there. We can, however, treat it as the graph of $x = y^2$, leading to the regular parametrization $x = t^2$, $y = t$, $0 \le t \le 1$. Unfortunately, these two regular parametrizations do not fit together in a "smooth" way: their velocity vectors at the two points where they meet—$(0, 0)$ and $(1, 1)$—point in different directions, and there is no way of "patching up" this difference to get a regular parametrization of the full boundary curve.

But for our purposes, this is not a serious problem: we can allow this kind of discrepancy at finitely many points, and extend our definition to this situation:

Definition 5.3.2. *A locally one-to-one curve $\mathcal{C}$ in $\mathbb{R}^2$ is **piecewise regular** if it can be partitioned into finitely many arcs $\mathcal{C}_i$, $i = 1, \ldots, k$ such that*

(1) *Each $\mathcal{C}_i$ is the image of a regular parametrization $\vec{p}_i$ defined on a closed interval $[a_i, b_i]$ (in particular, the tangent vectors $\vec{p}_i\,'(a_i)$ and $\vec{p}_i\,'(b_i)$ at the endpoints are nonzero, and each is the limit of the tangent vectors at nearby points of $\mathcal{C}_i$), and*

(2) *the arcs abut at endpoints: for $i = 1, \ldots, k-1$, $\vec{p}_i\,(1) = \overrightarrow{p_{i+1}}\,(0)$.*

*Thus, we allow, at each of the finitely many common endpoints of these arcs, that there are two "tangent" directions, each defined in terms of one of the two arcs that abut there. We will refer to points where such a discrepancy occurs as **corners** of the curve.*

The discrepancy between tangent vectors at a corner can amount to as much as π radians; see Figure 5.4. This means that Definition 5.3.1 cannot be applied at such points; however, we can still apply it at all other points, and have a coherent definition.

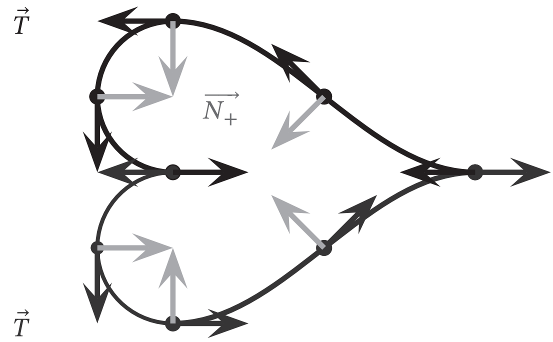


Figure 5.4. Positive Orientation for a Piecewise-Regular Curve with Corners

Definition 5.3.3. *Suppose $\mathcal{C}$ is a piecewise regular, simple, closed curve in $\mathbb{R}^2$. Then the* ***positive orientation*** *is the choice of unit tangent vector $\vec{T}$ at all non-corners such that the leftward normal $\overrightarrow{N_+}$ points into the region bounded by $\mathcal{C}$.*

With these definitions, we can formulate Green's Theorem. This was originally formulated and proved by George Green (1793-1841), a self-taught mathematician whose exposition was contained in a self-published pamphlet on the use of mathematics in the study of electricity and magnetism [19] in 1828.[7]

Theorem 5.3.4 (Green's Theorem)**.** *Suppose $\mathcal{C}$ is a piecewise regular, simple, closed curve with positive orientation in the plane, bounding the regular region D.*

Then for any pair of $\mathcal{C}^1$ functions P and Q defined on D,

$$\oint_{\mathcal{C}} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA. \tag{5.8}$$

The proof of Theorem 5.3.4 is given in Appendix A.10. Here, we consider two examples that illustrate how it is used.

First, let us consider the line integral

$$\oint_{\mathcal{C}} (x^2 - y)\,dx + (y^2 + x)\,dy,$$

where $\mathcal{C}$ is the ellipse $\frac{x^2}{4} + y^2 = 1$, traversed counterclockwise.

[7]The son of a successful miller in Nottingham, he entered his father's business instead of going to university, but studied privately. He finally went to Cambridge at the age of 40, obtaining his degree in 1837, and subsequently published six papers. Interest in his 1828 Essay on the part of William Thomson (later Lord Kelvin) got him a Fellowship at Caius College in 1839. He remained for only two terms, then returned home, dying the following year. [1, p. 202]

The ellipse can be parametrized as $x = 2\cos\theta$ and $y = \sin\theta$ for $0 \le \theta \le 2\pi$. Then $dx = -2\sin\theta\,d\theta$ and $dy = \cos\theta\,d\theta$, so

$$\oint_{\mathcal{C}} (x^2 - y)\,dx + (y^2 + x)\,dy$$

$$= \int_0^{2\pi} \{\left[(2\cos\theta)^2 - \sin\theta\right](-2\sin\theta\,d\theta) + \left[\sin^2\theta + 2\cos\theta\right](\cos\theta\,d\theta)\}$$

$$= \int_0^{2\pi} \{2 - 4\cos^2\sin\theta + \sin^2\theta\cos\theta\}\,d\theta$$

$$= \left(2\theta + \frac{4\cos^3\theta}{3} + \frac{\sin^3\theta}{3}\right)_0^{2\pi} = 4\pi.$$

If instead we use Green's Theorem, we need to integrate

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}\left[y^2 + x\right] - \frac{\partial}{\partial y}\left[x^2 - y\right] = 1 + 1 = 2$$

so we can write

$$\oint_{\mathcal{C}} (x^2 - y)\,dx + (y^2 + x)\,dy = \iint_{\frac{x^2}{4} + y^2 \le 1} 2\,dA$$

which is just twice the area of the ellipse; we know that this area is 2π, so our integral equals 4π.

As a second example, let us calculate

$$\oint_{\mathcal{C}} x(y^2 + 1)\,dx + (x^2 - y^2)\,dy,$$

where $\mathcal{C}$ is the square with corners at the origin, $(1, 0)$, $(1, 1)$, and $(0, 1)$.

To calculate this directly, we need to split $\mathcal{C}$ into four pieces:

$\mathcal{C}_1$: **$(0, 0)$ to $(1, 0)$:** This can be parametrized as $x = t$, $y = 0$, so $dx = dt$ and $dy = 0$. Then $x(y^2 + 1)\,dx + (x^2 - y^2)\,dy = t\,dt + (t^2)(0)$, so

$$\oint_{\mathcal{C}_1} x(y^2 + 1)\,dx + (x^2 - y^2)\,dy = \int_0^1 t\,dt = \frac{1}{2}.$$

$\mathcal{C}_2$: **$(1, 0)$ to $(1, 1)$:** This can be parametrized as $x = 1$, $y = t$, so $dx = 0t$ and $dy = dt$. Then $x(y^2 + 1)\,dx + (x^2 - y^2)\,dy = (1)(0) + (1 - t^2)(\,dt)$, so

$$\oint_{\mathcal{C}_2} x(y^2 + 1)\,dx + (x^2 - y^2)\,dy = \int_0^1 (1 - t^2)\,dt = \frac{2}{3}.$$

$\mathcal{C}_3$: **$(1, 1)$ to $(0, 1)$:** This can be parametrized as $x = 1 - t$, $y = 1$, so $dx = -\,dt$ and $dy = 0$. Then $x(y^2 + 1)\,dx + (x^2 - y^2)\,dy = [(1 - t)(2)](-\,dt) + [(1 - t^2) - 1](0)$ so

$$\oint_{\mathcal{C}_3} x(y^2 + 1)\,dx + (x^2 - y^2)\,dy = \int_0^1 2(1 - t)\,dt = -1.$$

$\mathcal{C}_4$: **$(0, 1)$ to $(0, 0)$:** This can be parametrized as $x = 0$, $y = 1 - t$, so $dx = 0$ and $dy = -\,dt$. Then $x(y^2 + 1)\,dx + (x^2 - y^2)\,dy = 0 + [-(1 - t)^2](-\,dt)$ so

$$\oint_{\mathcal{C}_4} x(y^2 + 1)\,dx + (x^2 - y^2)\,dy = \int_0^1 (1 - t)^2\,dt = \frac{1}{3}.$$

Summing these four integrals, we have

$$\oint_{\mathcal{C}} x(y^2+1)\,dx + (x^2-y^2)\,dy$$
$$= \oint_{\mathcal{C}_1} x(y^2+1)\,dx + (x^2-y^2)\,dy + \oint_{\mathcal{C}_1} x(y^2+1)\,dx + (x^2-y^2)\,dy$$
$$+ \oint_{\mathcal{C}_1} x(y^2+1)\,dx + (x^2-y^2)\,dy + \oint_{\mathcal{C}_1} x(y^2+1)\,dx + (x^2-y^2)\,dy$$
$$= \frac{1}{2} + \frac{2}{3} - 1 + \frac{1}{3} = \frac{1}{2}.$$

If a region is not regular, it can often be subdivided into regular regions. One approach is to draw a grid (Figure 5.5): most of the interior is cut into rectangles (which are certainly regular) and what is left are regions with some straight sides and others given by pieces of the bounding curve. With a careful choice of grid lines, these regions will also be regular.[8]

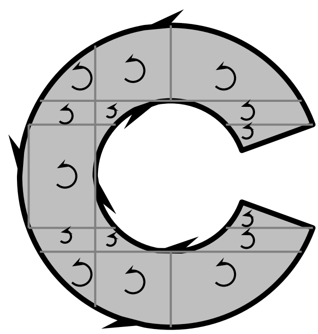


Figure 5.5. Subdivision of a Region into Regular Ones

Clearly, the double integral of $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ over all of D equals the sum of its integrals over each of the regular subregions, and Equation (5.8) applies to each of these individually, so that we can replace each such double integral in this sum with the corresponding line integral of $P\,dx + Q\,dy$ over the edge of that piece, oriented positively. Note that positive orientation of two adjacent pieces induces *opposite* directions along their common boundary segment, so when we sum up all these line integrals, the ones corresponding to pieces of the grid cancel, and we are left with only the sum of line integrals along pieces of our original bounding curve, $\mathcal{C}$. This shows that Equation (5.8) holds for the region bounded by a single closed curve—even if it is not regular—as long as it can be subdivided into regular regions.

We can take this one step further. Consider for example the region between two concentric circles[9] (Figure 5.6). This is bounded by not one, but *two* closed curves.

[8] If the curve has vertical and horizontal tangents at only finitely many points, and only finitely many "corners", then it suffices to make sure the grid lines go through all of these points. The only difficulty is when there are infinitely many horizontal or vertical tangents; in this case we can try to use a slightly rotated grid system. This is always possible if the curve is $\mathcal{C}^2$; the proof of this involves a sophisticated result in differential topology, the Morse-Sard Theorem.

[9] This is called an **annulus**.

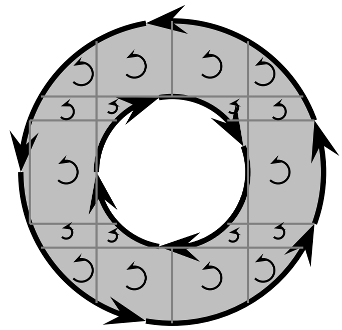


Figure 5.6. Subdivision of an Annulus into Regular Regions

If we subdivide this region into regular subregions via a grid, and orient the edge of each subregion positively, we can apply the same reasoning as above to conclude that the sum of the line integrals of $P\,dx + Q\,dy$ over the edges of the pieces (each oriented positively) equals the integral of $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$ over the whole region, and that furthermore each piece of edge coming from the grid appears twice in this sum, but with opposite directions, and hence is cancelled. Thus, the only line integrals contributing to the sum are those coming from the two boundary curves. We know that the positive orientation of the outer circle is *counterclockwise*—but we see from Figure 5.6 that the *inner* circle is directed *clockwise*. However, this is exactly the orientation we get if we adopt the phrasing in Definition 5.3.1: that the leftward normal must point into the region. Thus we see that the appropriate orientation for a boundary curve is determined by where the region lies relative to that curve. To avoid confusion with our earlier definition, we formulate the following:

Definition 5.3.5. *Suppose $D \subset \mathbb{R}^2$ is a region whose boundary ∂D consists of finitely many piecewise regular closed curves. Then for each such curve, the* ***boundary orientation*** *is the one for which the leftward normal at each non-corner points into the region D.*

With this definition, we see that Green's Theorem can be extended as follows:

Theorem 5.3.6 (Green's Theorem, Extended Version)**.** *Suppose $D \subset \mathbb{R}^2$ is a region whose boundary ∂D consists of a finite number of piecewise regular closed curves, and which can be decomposed into a finite number of regular regions.*

Then for any pair of $\mathcal{C}^1$ functions P and Q defined on D,

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA, \tag{5.9}$$

where the line integral over the boundary ∂D is interpreted as the sum of line integrals over its constituent curves, each with boundary orientation.

As an example, consider the case $P(x,y) = x + y$, $Q(x,y) = -x$, and take as our region the annulus $D = \{(x,y) \mid 1 \leq x^2 + y^2 \leq 4\}$. This has two boundary components, the *outer* circle $\mathcal{C}_1 = \{(x,y) \mid x^2 + y^2 = 4\}$, for which the boundary orientation is *counterclockwise*, and the *inner* circle, $\mathcal{C}_2 = \{(x,y) \mid x^2 + y^2 = 1\}$, for which the boundary orientation is *clockwise*.

We parametrize the *outer* circle $\mathcal{C}_1$ via $x = 2\cos\theta$, $y = 2\sin\theta$ for $0 \le \theta \le 2\pi$, with $dx = -2\sin\theta\, d\theta$ and $dy = 2\cos\theta\, d\theta$. Also, along $\mathcal{C}_1$, $P(2\cos\theta, 2\sin\theta) = 2(\cos\theta + \sin\theta)$ and $Q(2\cos\theta, 2\sin\theta) = -2\cos\theta$, so along $\mathcal{C}_1$, the form is

$$\begin{aligned} P\,dx + Q\,dy &= 2(\cos\theta + \sin\theta)(-2\sin\theta\, d\theta) + (-2\cos\theta)(2\cos\theta\, d\theta) \\ &= (-4\cos\theta\sin\theta - 4)\, d\theta \end{aligned}$$

leading to the integral

$$\int_{\mathcal{C}_1} P\,dx + Q\,dy = \int_0^{2\pi} -4(\sin\theta\cos\theta + 1)\, d\theta = \pi.$$

Now, the *inner* circle $\mathcal{C}_2$ needs to be parametrized *clockwise*; one way to do this is to reverse the two functions: $x = \sin\theta$, $y = \cos\theta$, for $0 \le \theta \le 2\pi$, so $dx = \cos\theta\, d\theta$ and $dy = -\sin\theta\, d\theta$. Then $P(\sin\theta, \cos\theta) = (\sin\theta + \cos\theta)$ and $Q(\sin\theta, \cos\theta) = -\sin\theta$, so along $\mathcal{C}_2$, the form is

$$\begin{aligned} P\,dx + Q\,dy &= (\sin\theta + \cos\theta)(\cos\theta\, d\theta) + (-\sin\theta)(-\sin\theta\, d\theta) \\ &= (\sin\theta\cos\theta + 1)\, d\theta \end{aligned}$$

with integral

$$\int_{\mathcal{C}_2} P\,dx + Q\,dy = \int_0^{2\pi} (\sin\theta\cos\theta + 1)\, d\theta = 2\pi.$$

Combining these, we have

$$\oint_{\partial D} P\,dx + Q\,dy = \int_{\mathcal{C}_1} P\,dx + Q\,dy + \int_{\mathcal{C}_2} P\,dx + Q\,dy = -8\pi + 2\pi = -6\pi.$$

Let us compare this to the double integral:

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}[-x] = -1; \quad -\frac{\partial P}{\partial y} = -\frac{\partial}{\partial y}[x + y] = -1$$

so

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D (-2) dA = -2\mathcal{A}(D) = -2(4\pi - \pi) = -6\pi.$$

Green's Theorem in the Language of Vector Fields. If we think of the planar vector field $\vec{F}(x, y) = P(x, y)\vec{\imath} + Q(x, y)\vec{\jmath}$ as the velocity of a fluid, then the line integral $\oint_{\mathcal{C}} P\,dx + Q\,dy = \oint_{\mathcal{C}} \vec{F} \cdot d\vec{\mathfrak{s}}$ around a closed curve $\mathcal{C}$ is the integral of the tangent component of $\vec{F}$: thus it can be thought of as measuring the tendency of the fluid to flow around the curve; it is sometimes referred to as the **circulation**, and the integral $\oint_{\mathcal{C}} \vec{F} \cdot d\vec{\mathfrak{s}}$ is referred to as the **circulation integral**, of $\vec{F}$ around $\mathcal{C}$.

The double integral in Green's Theorem is a bit more subtle. One way is to consider a "paddle wheel" immersed in the fluid, in the form of two line segments through a given point (a, b)—one horizontal, the other vertical (Figure 5.7).

When will the wheel tend to turn? Let us first concentrate on the horizontal segment. Intuitively, the horizontal component of velocity will have no turning effect (rather it will tend to simply displace the paddle horizontally). Similarly, a vertical velocity field which is *constant* along the length of the paddle will result in a vertical (parallel) displacement. A *turning* of the paddle will result from a monotone *change* in the vertical component of the velocity as one moves left-to-right along the paddle. In particular, counterclockwise turning requires that the vertical component Q of the

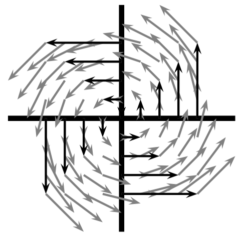


Figure 5.7. Rotation of a Vector Field: the "Paddle Wheel"

velocity *increases* as we move left-to-right: that is, the horizontal paddle will tend to turn counterclockwise around (a, b) if $\frac{\partial Q}{\partial x}(a, b) > 0$. This is sometimes referred to as a **shear** effect of the vector field.

Now consider the vertical "paddle". Again, the velocity component tangent to the paddle, as well as a *constant* horizontal velocity will effect a parallel displacement: to obtain a shear effect, we need the *horizontal* component of velocity to be changing monotonically as we move vertically. Note that in this case, a *counterclockwise* rotation results from a vertical velocity component that is *decreasing* as we move up along the paddle: $\frac{\partial P}{\partial y}(a, b) < 0$.

Since the paddle wheel is rigid, the effect of these two shears will be cumulative, and the net counterclockwise rotation effect of the two shears will be given by $\frac{\partial Q}{\partial x}(a, b) - \frac{\partial P}{\partial y}(a, b)$.

This discussion comes with an immediate disclaimer: it is purely intuitive; a more rigorous derivation of this expression as representing the tendency to turn is given in Exercise 6 using Green's Theorem. However, it helps motivate our designation of this as the (planar) **curl**[10] of the vector field $\vec{F}$:

$$\operatorname{curl} \vec{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}. \tag{5.10}$$

With this terminology, we can formulate Green's Theorem in the language of vector fields as follows:

Theorem 5.3.7 (Green's Theorem: Vector Version)**.** *If $\vec{F} = P\vec{\imath} + Q\vec{\jmath}$ is a $\mathcal{C}^1$ vector field on the planar region $D \subset \mathbb{R}^2$, and D has a piecewise regular boundary and can be subdivided into regular regions, then the circulation of $\vec{F}$ around the boundary of D (each constituent simple closed curve of ∂D given the boundary orientation) equals the integral over the region D of the (planar) curl of $\vec{F}$:*

$$\oint_{\partial D} \vec{F} \cdot d\vec{\mathfrak{s}} = \oint_{\partial D} \vec{F} \cdot \vec{T}\, d\mathfrak{s} = \iint_D \operatorname{curl} \vec{F}\, dA.$$

We note that there is a second version of Green's Theorem, in which the unit tangent is replaced by the unit outward normal, and the curl is replaced by the "divergence". This is sketched in Exercise 7; the full implication of this switch is discussed in § 5.8.

[10]We shall see in § 5.6 that the "true" curl of a vector field is a vector in $\mathbb{R}^3$; the present quantity is just one of its components.

Exercises for § 5.3

Answers to Exercises 1a, 2a, and 3a are given in Appendix A.13.

Practice problems:

(1) Evaluate $\oint_{\mathcal{C}} \omega$ for each form below, where $\mathcal{C}$ is the circle $x^2 + y^2 = R^2$ traversed counterclockwise, two ways: directly, and using Green's Theorem:

(a) $\omega = y\,dx + x\,dy$ (b) $\omega = x\,dx + y\,dy$

(c) $\omega = xy^2\,dx + x^2y\,dy$ (d) $\omega = (x-y)\,dx + (x+y)\,dy$

(e) $\omega = xy\,dx + xy\,dy$

(2) Evaluate $\oint_{\mathcal{C}} \vec{F} \cdot d\vec{s}$ for each vector field below, where $\mathcal{C}$ is the circle $x^2 + y^2 = 1$ traversed counterclockwise, two ways: directly, and using Green's Theorem:

(a) $\vec{F} = x\vec{\imath} - (x+y)\vec{\jmath}$ (b) $\vec{F} = 3y\vec{\imath} - x\vec{\jmath}$

(c) $\vec{F} = 3x\vec{\imath} - y\vec{\jmath}$ (d) $\vec{F} = -x^2y\vec{\imath} + xy^2\vec{\jmath}$

(e) $\vec{F} = y^3\vec{\imath} - x^3\vec{\jmath}$

(f) $\vec{F} = A\vec{x}$, where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

(3) Calculate the line integral $\oint_{\mathcal{C}} (e^x - y)\,dx + (e^y + x)\,dy$, where

(a) $\mathcal{C}$ is the polygonal path from $(0,0)$ to $(1,0)$ to $(1,1)$ to $(0,1)$ to $(0,0)$.

(b) $\mathcal{C}$ is the circle $x^2 + y^2 = R^2$, traversed counterclockwise.

Theory problems:

(4) (a) Show that the area of the region bounded by a simple closed curve $\mathcal{C}$ is given by any one of the following three integrals:

$$A = \int_{\mathcal{C}} x\,dy = -\int_{\mathcal{C}} y\,dx = \frac{1}{2}\int_{\mathcal{C}} x\,dy - y\,dx.$$

(b) Use this to find the area bounded by the x-axis and one arch of the cycloid

$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta).$$

(*Hint:* Pay attention to the direction of integration!)

(5) (a) Show that the area of the region bounded by a curve $\mathcal{C}$ expressed in polar coordinates as $r = f(\theta)$ is given by

$$A = \frac{1}{2}\int_{\mathcal{C}} (f(\theta))^2\,d\theta.$$

(b) Use this to find the area of the rose $r = \sin n\theta$. (Caution: the answer is different for n even and n odd; in particular, when n is even, the curve traverses the $2n$ leaves once as θ goes from 0 to 2π, while for n odd, it traverses the n leaves twice in that time interval.)

Challenge problems:

(6) (a) Show that a rotation of the plane (about the origin) with angular velocity ω gives a (spatial) velocity vector field which at each point away from the origin is given by $r\vec{\omega}(x,y) = r\omega\vec{T}$, where

$$\vec{T}(x,y) = -\frac{y}{r}\vec{\imath} + \frac{x}{r}\vec{\jmath}$$

is the unit vector, perpendicular to the ray through (x, y), pointing counterclockwise, and r is the distance from the origin.

(b) Show that the circulation of $r\vec{\omega}(x, y)$ around the circle of radius r centered at the origin is $2\pi r^2\omega$.

(c) Now suppose the vector field $\vec{F} = P\vec{\imath}+Q\vec{\jmath}$ is the velocity vector field of a planar fluid. Given a point $\overrightarrow{p_0}$ and $\vec{p} \neq \overrightarrow{p_0}$, let $\overrightarrow{T_0}(\vec{p})$ be the unit vector perpendicular to the ray from $\overrightarrow{p_0}$ to $\vec{p}$, pointing counterclockwise, and define $\omega_0(\vec{p})$ by

$$r\omega_0(\vec{p}) = \vec{F}(\vec{p}) \cdot \overrightarrow{T_0}(\vec{p}),$$

where $r = \left\|\vec{p} - \overrightarrow{p_0}\right\|$ is the distance of $\vec{p}$ from $\overrightarrow{p_0}$; in other words,

$$r\overrightarrow{\omega_0}(\vec{p}) := r\omega_0(\vec{p})\overrightarrow{T_0}(\vec{p})$$

is the component of $\vec{F}$ perpendicular to the ray from $\overrightarrow{p_0}$ to $\vec{p}$.

Let $\mathcal{C}_r$ be the circle of radius r about $\overrightarrow{p_0}$. Show that the circulation of $\vec{F}$ around $\mathcal{C}_r$ equals the circulation of $r\overrightarrow{\omega_0}(\vec{p})$ around $\mathcal{C}_r$, and hence the average value of $\omega_0(\vec{p})$ around $\mathcal{C}_r$ is

$$\omega(r) = \frac{1}{2\pi r^2}\oint_{\mathcal{C}_r} P\,dx + Q\,dy.$$

(d) Use Green's Theorem to show that this equals half the average value of the scalar curl of $\vec{F}$ over the disc of radius r centered at $\overrightarrow{p_0}$.

(e) Use the Integral Mean Value Theorem to show that

$$\lim_{r\to 0}\omega(r) = \frac{1}{2}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).$$

(7) **Green's Theorem, Flux Version:** Given the region $D \subset \mathbb{R}^2$ bounded by a simple closed curve $\mathcal{C}$ (with positive orientation) and a vector field $\vec{F} = P\vec{\imath}+Q\vec{\jmath}$ on $\mathcal{C}$, show that

$$\oint_C \vec{F} \cdot \vec{N}\,d\mathfrak{s} = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA,$$

where $\vec{N}$ is the outward pointing unit normal to $\mathcal{C}$.

(*Hint:* Rotate $\vec{F}$ and $\vec{N}$ in such a way that $\vec{N}$ is rotated into the tangent vector $\vec{T}$. What happens to $\vec{F}$? Now apply Green's Theorem.)

(8) **Green's identities**: Given a $\mathcal{C}^2$ function, define the **Laplacian** of f as

$$\nabla^2 f := \operatorname{div}\vec{\nabla} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

Furthermore, if $D \subset \mathbb{R}^2$ is a regular region, define $\frac{\partial f}{\partial n} = \vec{\nabla} f \cdot \vec{N}$ on ∂D, where $\vec{N}$ is the outward unit normal to ∂D.

Suppose f and g are $\mathcal{C}^2$ functions on D.

(a) Prove

$$\iint_D (f\nabla^2 g + \vec{\nabla} f \cdot \vec{\nabla} g)\,dA = \oint_{\partial D} f\frac{dg}{dn}\,d\mathfrak{s}$$

(*Hint:* Use Exercise 7 with $P = f\frac{\partial g}{\partial x}$, $Q = f\frac{\partial g}{\partial y}$.)

(b) Use this to prove

$$\iint_D (f\nabla^2 g - g\nabla^2 f)\,dA = \oint_{\partial D} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) d\mathfrak{s}.$$

5.4 Green's Theorem and 2-forms in $\mathbb{R}^2$

Bilinear Functions and 2-Forms on $\mathbb{R}^2$. In § 5.1 we defined a differential form on $\mathbb{R}^2$ as assigning to each point $p \in \mathbb{R}^2$ a linear functional on the tangent space $T_p\mathbb{R}^2$ at p; we integrate these objects over curves. Green's Theorem (Theorem 5.3.4) relates the line integral of such a form over the *boundary* of a region to an integral over the region itself. In the language of forms, the objects we integrate over two-dimensional regions are called 2-*forms*. These are related to *bilinear functions*.

Definition 5.4.1. *A* ***bilinear function*** *on $\mathbb{R}^2$ is a function of two vector variables $B(\vec{v},\vec{w})$ such that fixing one of the inputs results in a linear function of the other input:*

$$\begin{aligned} B(a_1\vec{v_1} + a_2\vec{v_2}, \vec{w}) &= a_1 B(\vec{v_1}, \vec{w}) + a_2 B(\vec{v_2}, \vec{w}) \\ B(\vec{v}, b_1\vec{w_1} + b_2\vec{w_2}) &= b_1 B(\vec{v}, \vec{w_1}) + b_2 B(\vec{v}, \vec{w_2}) \end{aligned} \tag{5.11}$$

for arbitrary vectors in $\mathbb{R}^2$ and real scalars.

One example of a bilinear function, by Proposition 1.4.2, is the dot product: $B(\vec{v},\vec{w}) = \vec{v}\cdot\vec{w}$. More generally, a bilinear function on $\mathbb{R}^2$ is a special kind of homogeneous degree two polynomial in the coordinates of its entries: using Equation (5.11), we see that if $\vec{v} = (x_1, y_1) = x_1\vec{\imath} + y_1\vec{\jmath}$ and $\vec{w} = (x_2, y_2) = x_2\vec{\imath} + y_2\vec{\jmath}$, then

$$\begin{aligned} B(\vec{v},\vec{w}) &= B(x_1\vec{\imath} + y_1\vec{\jmath}, \vec{w}) = x_1 B(\vec{\imath}, \vec{w}) + y_1 B(\vec{\jmath}, \vec{w}) \\ &= x_1 B(\vec{\imath}, x_2\vec{\imath} + y_2\vec{\jmath}) + y_1 B(\vec{\jmath}, x_2\vec{\imath} + y_2\vec{\jmath}) \\ &= x_1x_2 B(\vec{\imath},\vec{\imath}) + x_1y_2 B(\vec{\imath},\vec{\jmath}) + y_1x_2 B(\vec{\jmath},\vec{\imath}) + y_1y_2 B(\vec{\jmath},\vec{\jmath}). \end{aligned}$$

So if we write the values of B on the four pairs of basis vectors as

$$\begin{aligned} b_{11} &= B(\vec{\imath},\vec{\imath}), \quad b_{12} = B(\vec{\imath},\vec{\jmath}), \\ b_{21} &= B(\vec{\jmath},\vec{\imath}), \quad b_{22} = B(\vec{\jmath},\vec{\jmath}), \end{aligned}$$

then we can write B as the homogeneous degree two polynomial

$$B(\vec{v},\vec{w}) = b_{11}x_1x_2 + b_{12}x_1y_2 + b_{21}y_1x_2 + b_{22}y_1y_2. \tag{5.12}$$

As an example, the dot product satisfies Equation (5.12) with $b_{ij} = 1$ when $i = j$ and $b_{ij} = 0$ when $i \neq j$. The fact that in this case the coefficient for v_iw_j is the same as that for v_jw_i ($b_{ij} = b_{ji}$) reflects the additional property of the dot product, that it is **commutative** ($\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$).

By contrast, the bilinear functions which come up in the context of 2-forms are **anti-commutative**: for every pair of vectors $\vec{v}$ and $\vec{w}$, we require

$$B(\vec{w},\vec{v}) = -B(\vec{v},\vec{w}).$$

An anti-commutative, bilinear function on $\mathbb{R}^2$ will be referred to as a 2-**form** on $\mathbb{R}^2$.

Note that an immediate consequence of anti-commutativity is that $B(\vec{v},\vec{w}) = 0$ if $\vec{v} = \vec{w}$ (Exercise 4). Applied to the basis vectors, these conditions tell us that $b_{11} = 0 =$

b_{22} and $b_{21} = -b_{12}$ Thus, a 2-form on $\mathbb{R}^2$ is determined by the value $B(\vec{\imath},\vec{\jmath}) = b_{12} = -b_{21}$:

$$B(\vec{v},\vec{w}) = B(\vec{\imath},\vec{\jmath})(x_1y_2 - x_2y_1).$$

You might recognize the quantity in parentheses as the determinant

$$\Delta(\vec{v},\vec{w}) = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

from § 1.6, which gives the signed area of the parallelogram with sides $\vec{v}$ and $\vec{w}$: this is in fact a 2-form on $\mathbb{R}^2$, and every other 2-form is a constant multiple of it:

$$B(\vec{v},\vec{w}) = B(\vec{\imath},\vec{\jmath})\,\Delta(\vec{v},\vec{w})\,. \tag{5.13}$$

As an example, let us fix a linear transformation $L\colon \mathbb{R}^2 \to \mathbb{R}^2$, and set $B(\vec{v},\vec{w})$ to be the signed area of the *image under* L of the parallelogram with sides $\vec{v}$ and $\vec{w}$:

$$B(\vec{v},\vec{w}) = \Delta(L(\vec{v}), L(\vec{w}))\,.$$

The linearity of L easily gives us the bilinearity of B. To express it as a multiple of Δ, we use the matrix representative of L:

$$[L] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The calculation above tells us that

$$\begin{aligned} B(\vec{v},\vec{w}) &= B(\vec{\imath},\vec{\jmath})\,\Delta(\vec{v},\vec{w}) = \Delta(L(\vec{\imath}), L(\vec{\jmath}))\,\Delta(\vec{v},\vec{w}) \\ &= \Delta(a\vec{\imath} + b\vec{\jmath}, c\vec{\imath} + d\vec{\jmath})\,\Delta(\vec{v},\vec{w}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}\Delta(\vec{v},\vec{w}) \\ &= (ad - bc)\Delta(\vec{v},\vec{w})\,. \end{aligned}$$

This can also be worked out directly (Exercise 5).

To bring this in line with our notation for 1-forms as $P\,dx + Q\,dy$, we reinterpret the entries in the determinant above as the values of the 1-forms dx and dy on $\vec{v}$ and $\vec{w}$; in general, we define the **wedge product** of two 1-forms α and β to be the determinant formed from applying them to a pair of vectors:

$$(\alpha \wedge \beta)(\vec{v},\vec{w}) \coloneqq \begin{vmatrix} \alpha(\vec{v}) & \beta(\vec{v}) \\ \alpha(\vec{w}) & \beta(\vec{w}) \end{vmatrix}. \tag{5.14}$$

You should check that this is bilinear and anti-commutative in $\vec{v}$ and $\vec{w}$—that is, it is a 2-form (Exercise 6)—and that as a product, $\wedge$ is anti-commutative: for any two 1-forms α and β,

$$\beta \wedge \alpha = -\alpha \wedge \beta. \tag{5.15}$$

Using this language, we can say:

Remark 5.4.2. *The wedge product of two* 1*-forms is a* 2*-form, and every* 2*-form on* $\mathbb{R}^2$ *is a scalar multiple of* $dx \wedge dy$.

Now, we define a **differential** 2**-form** on a region $D \subset \mathbb{R}^2$ to be a mapping Ω which assigns to each point $p \in D$ a 2-form Ω_p on the tangent space $T_p\mathbb{R}^2$. From Remark 5.4.2, a differential 2-form on $D \subset \mathbb{R}^2$ can be written $\Omega_p = b(p)\,dx \wedge dy$ for some function b on D.

Finally, we define the integral of a differential 2-form Ω over a region $D \subset \mathbb{R}^2$ to be

$$\iint_D \Omega := \iint_D \Omega_p\left(\vec{\imath},\vec{\jmath}\right) dA;\ \text{that is,}\ \iint_D b\,(p)\ dx \wedge dy = \iint_D b\,dA.$$

For example, if $\Omega = (x+y)\,dx \wedge (2x-y)\,dy = (2x^2+xy-y^2)\,dx \wedge dy$, then

$$\begin{aligned}\iint_{[0,1]\times[0,1]} \Omega &= \iint_{[0,1]\times[0,1]} (2x^2+xy-y^2)\,dA \\ &= \int_0^1 \int_0^1 (2x^2+xy-y^2)\,dy\,dx \\ &= \int_0^1 \left(2x^2y + \frac{xy^2}{2} - \frac{y^3}{3}\right)_{y=0}^1 dx = \int_0^1 \left(2x^2 + \frac{x}{2} - \frac{1}{3}\right) dx \\ &= \left[\frac{2x^3}{3} + \frac{x^2}{4} - \frac{x}{3}\right]_0^1 = \left(\frac{2}{3} + \frac{1}{4} - \frac{1}{3}\right) = \frac{7}{12}.\end{aligned}$$

Green's Theorem in the Language of Forms. To formulate Theorem 5.3.4 in terms of forms, we need two more definitions.

First, we define the **exterior product** of two differential 1-forms α and β on $D \subset \mathbb{R}^2$ to be the mapping $\alpha \wedge \beta$ assigning to $p \in D$ the wedge product of α_p and β_p:

$$(\alpha \wedge \beta)_p = \alpha_p \wedge \beta_p.$$

Second, we define the **exterior derivative** $d\omega$ of a 1-form ω. There are two basic kinds of 1-form on $\mathbb{R}^2$: $P\,dx$ and $Q\,dy$, where P (*resp.* Q) is a function of x and y. The differential of a function is a 1-form, and we take the exterior derivative of one of our basic 1-forms by finding the differential of the function and taking its exterior product with the coordinate 1-form it multiplied. This yields a 2-form:

$$\begin{aligned}d(P\,dx) &= (dP) \wedge dx \\ &= \left(\frac{\partial P}{\partial x}\,dx + \frac{\partial P}{\partial y}\,dy\right) \wedge dx = \frac{\partial P}{\partial x}\,dx \wedge dx + \frac{\partial P}{\partial y}\,dy \wedge dx \\ &= -\frac{\partial P}{\partial y}\,dx \wedge dy\end{aligned}$$

$$\begin{aligned}d(Q\,dy) &= (dQ) \wedge dy \\ &= \left(\frac{\partial Q}{\partial x}\,dx + \frac{\partial Q}{\partial y}\,dy\right) \wedge dy = \frac{\partial Q}{\partial x}\,dx \wedge dy + \frac{\partial Q}{\partial y}\,dy \wedge dy \\ &= \frac{\partial Q}{\partial x}\,dx \wedge dy.\end{aligned}$$

We extend this definition to arbitrary 1-forms by making the derivative respect sums: if $\omega = P\,(x,y)\ dx + Q\,(x,y)\ dy$, then

$$d\omega = d\,(P\,dx + Q\,dy) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy.$$

Thus, for example, if $\omega = x^2y^2\,dx + 3xy\,dy$, then its exterior derivative is

$$\begin{aligned} d\omega = d\left(x^2y^2\,dx + 3xy\,dy\right) &= d(x^2y^2) \wedge dx + d(3xy) \wedge dy \\ &= (2xy^2\,dx + 2x^2y\,dy) \wedge dx + (3y\,dx + 3x\,dy) \wedge dy \\ &= 2xy^2\,dx \wedge dx + 2x^2y\,dy \wedge dx + 3y\,dx \wedge dy + 3x\,dy \wedge dy \\ &= 0 + 2x^2y\,dy \wedge dx + 3y\,dx \wedge dy + 0 \\ &= (3y - 2x^2y)\,dx \wedge dy. \end{aligned}$$

To complete the statement of Theorem 5.3.4 in terms of forms, we recall the notation ∂D for the boundary of a region $D \subset \mathbb{R}^2$. Then we can restate Green's Theorem as

Theorem 5.4.3 (Green's Theorem, Differential Form)**.** *Suppose $D \subset \mathbb{R}^2$ is a region bounded by the curve*

$$\mathcal{C} = \partial D$$

and D and ∂D are both positively oriented. Then for any differential 1-form ω on D,

$$\oint_{\partial D} \omega = \iint_D d\omega. \tag{5.16}$$

Exercises for § 5.4

Answers to Exercises 1a, 2a, 3a, and 3b are given in Appendix A.13.

Practice problems:

(1) Evaluate $\omega_p\left(\vec{\imath}, \vec{\jmath}\right)$ and $\omega_p\left(\vec{v}, \vec{w}\right)$, where ω, $\vec{p}$, $\vec{v}$, and $\vec{w}$ are as given.
  (a) $\omega = dx \wedge dy$, $p = (2,1)$, $\vec{v} = 2\vec{\imath} - 3\vec{\jmath}$, $\vec{w} = 3\vec{\imath} - 2\vec{\jmath}$.
  (b) $\omega = x^2\,dx \wedge dy$, $p = (2,1)$, $\vec{v} = \vec{\imath} + \vec{\jmath}$, $\vec{w} = 2\vec{\imath} - \vec{\jmath}$.
  (c) $\omega = x^2\,dx \wedge dy$, $p = (-2,1)$, $\vec{v} = 2\vec{\imath} + \vec{\jmath}$, $\vec{w} = 4\vec{\imath} + 2\vec{\jmath}$.
  (d) $\omega = (x^2 + y^2)\,dx \wedge dy$, $p = (1,-1)$, $\vec{v} = 3\vec{\imath} - \vec{\jmath}$, $\vec{w} = \vec{\jmath} - \vec{\imath}$.
  (e) $\omega = (x\,dx) \wedge (y\,dy)$, $p = (1,1)$, $\vec{v} = 2\vec{\imath} - \vec{\jmath}$, $\vec{w} = 2\vec{\imath} + \vec{\jmath}$.
  (f) $\omega = (y\,dy) \wedge (y\,dx)$, $p = (1,1)$, $\vec{v} = 2\vec{\imath} - \vec{\jmath}$, $\vec{w} = 2\vec{\imath} + \vec{\jmath}$.
  (g) $\omega = (y\,dy) \wedge (x\,dx)$, $p = (1,1)$, $\vec{v} = 2\vec{\imath} - \vec{\jmath}$, $\vec{w} = 2\vec{\imath} + \vec{\jmath}$.
  (h) $\omega = (x\,dx + y\,dy) \wedge (x\,dx - y\,dy)$, $p = (1,1)$, $\vec{v} = 2\vec{\imath} - \vec{\jmath}$, $\vec{w} = 2\vec{\imath} + \vec{\jmath}$.

(2) Evaluate $\iint_{[0,1]\times[0,1]} \omega$:
  (a) $\omega = x\,dx \wedge y\,dy$
  (b) $\omega = x\,dy \wedge y\,dx$
  (c) $\omega = y\,dx \wedge x\,dy$
  (d) $\omega = y\,dy \wedge y\,dx$
  (e) $\omega = (x\,dx + y\,dy) \wedge (x\,dy - y\,dx)$

(3) Find the exterior derivative of each differential 1-form below (that is, write it as a multiple of $dx \wedge dy$).
  (a) $\omega = xy\,dx + xy\,dy$
  (b) $\omega = x\,dx + y\,dy$
  (c) $\omega = y\,dx + x\,dy$
  (d) $\omega = (x^2 + y^2)\,dx + 2xy\,dy$
  (e) $\omega = \cos xy\,dx + \sin x\,dy$
  (f) $\omega = y \sin x\,dx + \cos x\,dy$
  (g) $\omega = y\,dx - x\,dy$
  (h) $\omega = \frac{y\,dx - x\,dy}{\sqrt{x^2+y^2}}$
  (i) $\omega = \frac{y\,dx + x\,dy}{\sqrt{x^2+y^2}}$

Theory problems:

(4) Show that if B is an anti-commutative 2-form, then for any vector $\vec{v}$, $B\left(\vec{v},\vec{v}\right) = 0$.

(5) Given a linear transformation $L: \mathbb{R}^2 \to \mathbb{R}^2$ verify the formula

$$\Delta\left(L\left(\vec{v}\right), L\left(\vec{w}\right)\right) = (ad - bc)\Delta\left(\vec{v},\vec{w}\right)$$

via direct substitution of the values

$$L\left(\vec{v}\right) = (av_1 + bv_2, cv_1 + dv_2)$$
$$L\left(\vec{w}\right) = (aw_1 + bw_2, cw_1 + dw_2)$$

into the determinant Δ.

(6) (a) Show that Equation (5.14) defines a 2-form: that is, the wedge product of two 1-forms is a bilinear and anti-commutative functional.
  (b) Show that, as a product, the wedge product is anti-commutative (*i.e.*, Equation (5.15)).

(7) Show that if $f(x, y)$ is a $\mathcal{C}^2$ function, and $\omega = df$ is its differential, then $d\omega = 0$.

5.5 Oriented Surfaces and Flux Integrals

Oriented Surfaces. We saw in § 5.1 that a vector field can be usefully integrated over a curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ by taking the path integral of its component tangent to the curve; the resulting line integral (Definition 5.1.1) depends on the orientation of the curve, but otherwise depends only on the curve as a point-set.

There is an analogous way to integrate a vector field in $\mathbb{R}^3$ over a *surface*, by taking the surface integral of the component *normal* to the surface. There are two choices of normal vector at any point of a surface; if one makes a choice continuously at all points of a surface, one has an orientation of the surface.

Definition 5.5.1. *Suppose $\mathfrak{S}$ is a regular surface in $\mathbb{R}^3$.*

*An **orientation** of $\mathfrak{S}$ in $\mathbb{R}^3$ is a vector field $\vec{n}$ defined at all points of $\mathfrak{S}$ such that*

(1) *$\vec{n}(p) \in T_p\mathbb{R}^3$ is normal to $\mathfrak{S}$ (that is, it is perpendicular to the plane tangent to $\mathfrak{S}$ at p);*

(2) *$\vec{n}(p)$ is a unit vector ($\left\|\vec{n}(p)\right\| = 1$ for all $p \in \mathfrak{S}$);*

(3) *$\vec{n}(p)$ varies continuously with $p \in \mathfrak{S}$.*

*An **oriented surface** is a regular surface $\mathfrak{S} \subset \mathbb{R}^3$, together with an orientation $\vec{n}$ of $\mathfrak{S}$.*

Recall (from § 3.6) that a **coordinate patch** is a regular, one-to-one mapping $\vec{p}: \mathbb{R}^2 \to \mathbb{R}^3$ of a plane region D into $\mathbb{R}^3$; by abuse of terminology, we also refer to the image $\mathfrak{S} \subset \mathbb{R}^3$ of such a mapping as a coordinate patch. If we denote the parameters in the domain of $\vec{p}$ by $(s, t) \in D$, then since by regularity $\frac{\partial \vec{p}}{\partial s}$ and $\frac{\partial \vec{p}}{\partial t}$ are linearly independent at each point of D, their cross product gives a vector normal to $\mathfrak{S}$ at $\vec{p}(s, t)$. Dividing this vector by its length gives an orientation of $\mathfrak{S}$, determined by the order of the parameters: the cross product in reverse order gives the "opposite" orientation of $\mathfrak{S}$.

At any point of $\mathfrak{S}$, there are only two directions normal to $\mathfrak{S}$, and once we have picked this direction at *one* point, there is only one way to extend this to a continuous vector field normal to $\mathfrak{S}$ at *nearby* points of $\mathfrak{S}$. Thus:

Remark 5.5.2. *A coordinate patch $\vec{p}: \mathbb{R}^2 \to \mathbb{R}^3$ with domain in (s,t)-space and image $\mathfrak{S} \in \mathbb{R}^3$ has two orientations. The orientation*

$$\vec{n} = \frac{\frac{\partial \vec{p}}{\partial s} \times \frac{\partial \vec{p}}{\partial t}}{\left\| \frac{\partial \vec{p}}{\partial s} \times \frac{\partial \vec{p}}{\partial t} \right\|} \tag{5.17}$$

*is the **local orientation** of $\mathfrak{S}$ **induced** by the mapping $\vec{p}$, while the opposite orientation is*

$$-\vec{n} = \frac{\frac{\partial \vec{p}}{\partial t} \times \frac{\partial \vec{p}}{\partial s}}{\left\| \frac{\partial \vec{p}}{\partial t} \times \frac{\partial \vec{p}}{\partial s} \right\|}$$

In general, a regular surface in $\mathbb{R}^3$ is a union of (overlapping) coordinate patches, and each can be given a local orientation; if two patches overlap, we say the two corresponding local orientations are **coherent** if at each overlap point the normal vectors given by the two local orientations are the same. In that case we have an orientation on the *union* of these patches. If we have a family of coordinate patches such that on any overlap the orientations are coherent, then we can fit these together to give a **global orientation** of the surface. Conversely, if we have an orientation of a regular surface, then we can cover it with overlapping coordinate patches for which the induced local orientations are coherent (Exercise 3).

However, not every regular surface in $\mathbb{R}^3$ can be given a global orientation. The famous example of the **Möbius band** is given in Appendix A.11. We shall henceforth consider only **orientable** surfaces in our theory.

Flux Integrals. With this definition, we can proceed to define the flux integral of a vector field over an oriented surface. Recall that in § 4.4, to define the surface integral $\iint_{\mathfrak{S}} f \, d\mathcal{S}$ of a function f over a regular surface $\mathfrak{S}$, we subdivided the domain of a parametrization into rectangles, approximating the area of each rectangle by the area $\triangle\mathcal{S}$ of a corresponding parallelogram in the tangent space, then multiplied each such area by the value of the function at a representative point of the rectangle, and finally added these to form a Riemann sum; as the mesh size of our subdivision went to zero, these Riemann sums converged to an integral independent of the parametrization from which we started.

To define the flux integral of a vector field $\vec{F}$ on an oriented regular surface $\mathfrak{S}$, we replace the element of surface area

$$d\mathcal{S} = \left\| \frac{\partial \vec{p}}{\partial s} \times \frac{\partial \vec{p}}{\partial t} \right\| ds\,dt$$

with the **element of oriented surface area**

$$d\vec{\mathcal{S}} = \left(\frac{\partial \vec{p}}{\partial s} \times \frac{\partial \vec{p}}{\partial t} \right) ds\,dt.$$

We know that the vector $\frac{\partial \vec{p}}{\partial s} \times \frac{\partial \vec{p}}{\partial t}$ is perpendicular to $\mathfrak{S}$, so either it points in the same direction as the unit normal $\vec{n}$ defining the orientation of $\mathfrak{S}$ or it points in the opposite direction. In the latter case, we modify our definition of $d\vec{\mathcal{S}}$ by taking the cross product in the opposite order. With this modification (if necessary) we can write

$$d\vec{\mathcal{S}} = \vec{n}\, d\mathcal{S} \tag{5.18}$$

and instead of multiplying the (scalar) element of surface area by the (scalar) function f, we take the *dot* product of the vector field $\vec{F}$ with the (vector) element of oriented surface area $d\vec{\mathcal{S}}$; the corresponding limit process amounts to taking the surface integral of the function obtained by dotting the vector field with the unit normal giving the orientation:

Definition 5.5.3. *Suppose $\vec{F}$ is a $\mathcal{C}^1$ vector field on $\mathbb{R}^3$, defined on a region $D \subset \mathbb{R}^3$, and $\mathfrak{S}$ is an oriented surface contained in D.*

*The **flux integral** of $\vec{F}$ over $\mathfrak{S}$ is defined as*

$$\iint_{\mathfrak{S}} \vec{F} \cdot d\vec{\mathcal{S}} = \iint_{\mathfrak{S}} \vec{F} \cdot \vec{n}\; d\mathcal{S},$$

where $\vec{n}$ is the unit normal defining the orientation of $\mathfrak{S}$, and $d\vec{\mathcal{S}}$ is the element of oriented surface area defined by Equation (5.18)*.*

If we think of the vector field $\vec{F}$ as the velocity field of a fluid (say with constant density), then the flux integral is easily seen to express the amount of fluid crossing the surface $\mathfrak{S}$ per unit time. This also makes clear the fact that reversing the orientation of $\mathfrak{S}$ reverses the sign of the flux integral. On a more formal level, replacing $\vec{n}$ with its negative in the flux integral means we are taking the surface integral of the negative of our original function, so the integral also switches sign.

We saw in Corollary A.6.3 that two different regular parametrizations $\vec{p}$ and $\vec{q}$ of the same surface $\mathfrak{S}$ differ by a change-of-coordinates transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ whose Jacobian determinant is nowhere zero, and from this we argued that the surface integral of a function does not depend on the parametrization. Thus, provided we pick the correct unit normal $\vec{n}$, the flux integral is independent of parametrization.

Note that calculating the unit normal vector $\vec{n}$ in the surface-integral version of the flux integral involves finding the cross product $\frac{\partial \vec{p}}{\partial s} \times \frac{\partial \vec{p}}{\partial t}$ and then dividing by its length; but then the element of surface area $d\mathcal{S}$ equals that same length times $ds\,dt$, so these lengths cancel and at least the calculation of the length is redundant. If we just use the formal definition of the element of oriented area

$$d\vec{\mathcal{S}} = \left(\frac{\partial \vec{p}}{\partial s} \times \frac{\partial \vec{p}}{\partial t}\right) ds\,dt$$

and take its formal dot product with the vector field $\vec{F}$ (expressed in terms of the parametrization), we get the correct integrand without performing the redundant step.

However, we do need to pay attention to the direction of the unit normal $\vec{n}$, which is the same as the direction of the vector $\frac{\partial \vec{p}}{\partial s} \times \frac{\partial \vec{p}}{\partial t}$. It is usually a fairly simple matter to decide whether this cross product points in the correct direction; if it does not, we simply use its negative, which is the same as the cross product in the opposite order.

To see how this works, consider the vector field $\vec{F}(x,y,z) = x^2y\vec{\imath} + yz^2\vec{\jmath} + xyz\vec{k}$ over the surface $z = xy$ over $0 \le x \le 1$, $0 \le y \le 1$ with *upward* orientation—that is, we want $\vec{n}$ to have a positive z-component. (See Figure 5.8.)

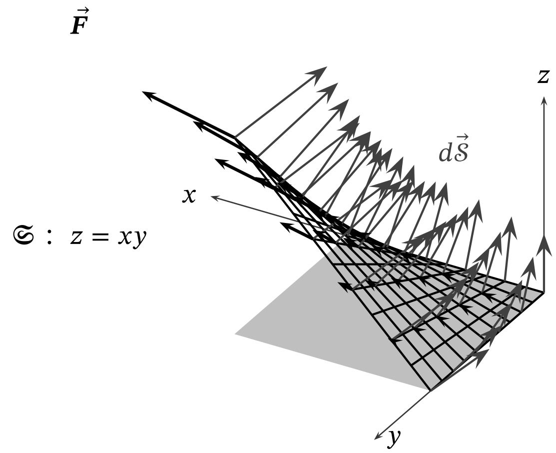


Figure 5.8. $\vec{F}(x, y, z) = x^2y\vec{\imath} + yz^2\vec{\jmath} + xyz\vec{k}$ on $z = xy$

Since this surface is the graph of a function, it is a coordinate patch, with the natural parametrization $x = s$, $y = t$, and $z = st$, for $0 \le s \le 1$ and $0 \le t \le 1$. In vector terms, this is $\vec{p}(s, t) = (s, t, st)$, so $\frac{\partial \vec{p}}{\partial s} = (1, 0, t)$ and $\frac{\partial \vec{p}}{\partial t} = (0, 1, s)$. Then

$$\frac{\partial \vec{p}}{\partial s} \times \frac{\partial \vec{p}}{\partial t} = (1, 0, t) \times (0, 1, s) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 0 & t \\ 0 & 1 & s \end{vmatrix} = -t\vec{\imath} - s\vec{\jmath} + \vec{k}.$$

Note that this has an upward vertical component, so corresponds to the correct (upward) orientation. Thus we can write

$$d\vec{\mathcal{S}} = (-t\vec{\imath} - s\vec{\jmath} + \vec{k})\, ds\, dt.$$

In terms of the parametrization, the vector field along $\mathfrak{S}$ becomes

$$\vec{F}\left(\vec{p}(s, t)\right) = (s)^2(t)\vec{\imath} + (t)(st)^2\vec{\jmath} + (s)(t)(st)\vec{k} = s^2t\vec{\imath} + s^2t^3\vec{\jmath} + s^2t^2\vec{k}$$

giving

$$\begin{aligned} \vec{F} \cdot d\vec{\mathcal{S}} &= \left[\left(s^2t\right)(-t) + \left(s^2t^3\right)(-s) + \left(s^2t^2\right)(1)\right] ds\, dt \\ &= \left[-s^2t^2 - s^3t^3 + s^2t^2\right] ds\, dt = -s^3t^3\, ds\, dt \end{aligned}$$

and the flux integral becomes

$$\iint_{\mathfrak{S}} \vec{F} \cdot d\vec{\mathcal{S}} = \int_0^1 \int_0^1 \left(-s^3t^3\right)\, ds\, dt = -\frac{1}{4}\int_0^1 t^3\, dt = -\frac{t^4}{16}\Big|_0^1 = -\frac{1}{16}.$$

We note in passing that for a surface given as the graph of a function, $z = f(x, y)$, the natural parametrization using the input to the function as parameters $x = s$, $y = t$, and $z = f(s, t)$ leads to a particularly simple form for the element of oriented surface area $d\vec{\mathcal{S}}$. The proof is a straightforward calculation, which we leave to you (Exercise 4):

Remark 5.5.4. *If $\mathfrak{S}$ is the graph of a function $z = f(x, y)$, then the natural parametrization $\vec{p}(s, t) = s\vec{\imath} + t\vec{\jmath} + f(s, t)\vec{k}$ with orientation* upward *has element of surface area*

$$d\vec{\mathcal{S}} = \left(\frac{\partial \vec{p}}{\partial x} \times \frac{\partial \vec{p}}{\partial y}\right) dx\, dy = \left(-f_x\vec{\imath} - f_y\vec{\jmath} + \vec{k}\right) dx\, dy.$$

As a second example, let $\vec{F}(x,y,z) = 2x\vec{\imath} + 2y\vec{\jmath} + 8z\vec{k}$, and take as $\mathfrak{S}$ the portion of the sphere of radius 1 about the origin lying between the xy-plane and the plane $z = 0.5$, with orientation *into* the sphere (Figure 5.9).

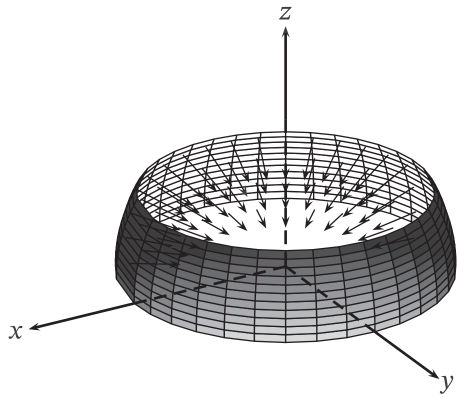


Figure 5.9. Inward Orientation for a Piece of the Sphere

The surface is most naturally parametrized using spherical coordinates: $x = \sin\phi$, $\cos\theta$, $y = \sin\phi\sin\theta$, $z = \cos\phi$, with $\frac{\pi}{3} \leq \phi \leq \frac{\pi}{2}$ and $0 \leq \theta \leq 2\pi$; the partial derivatives of this parametrization are

$$\frac{\partial\vec{p}}{\partial\phi} = \cos\phi\cos\theta\vec{\imath} + \cos\phi\sin\theta\vec{\jmath} - \sin\phi\vec{k}$$
$$\frac{\partial\vec{p}}{\partial\theta} = -\sin\phi\sin\theta\vec{\imath} + \sin\phi\cos\theta\vec{\jmath}$$

leading to

$$\begin{aligned}\frac{\partial\vec{p}}{\partial\phi} \times \frac{\partial\vec{p}}{\partial\theta} &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix} \\ &= \sin^2\phi\cos\theta\vec{\imath} + \sin^2\phi\sin\theta\vec{\jmath} + \sin\phi\cos\phi(\cos^2\theta + \sin^2\theta)\vec{k} \\ &= \sin^2\phi(\cos\theta\vec{\imath} + \sin\theta\vec{\jmath}) + \sin\phi\cos\phi\vec{k}.\end{aligned}$$

Does this give the appropriate orientation? Since the sphere is orientable, it suffices to check this at one point: say at $\vec{p}\left(\frac{\pi}{2},0\right) = (1,0,0)$: here $\frac{\partial\vec{p}}{\partial\phi} \times \frac{\partial\vec{p}}{\partial\theta} = \vec{\imath}$, which points *outward* instead of inward. Thus we need to use the cross product in the other order (which means the negative of the vector above) to set

$$d\vec{S} = -\{\sin^2\phi(\cos\theta\vec{\imath} + \sin\theta\vec{\jmath}) + \sin\phi\cos\phi\vec{k}\}\,d\phi\,d\theta.$$

In terms of this parametrization,

$$\vec{F} = 2\sin\phi\cos\theta\vec{\imath} + 2\sin\phi\sin\theta\vec{\jmath} + 8\cos\phi\vec{k}$$

so

$$\begin{aligned}\vec{F}\cdot d\vec{S} &= (-2\sin^3\phi\cos^2\theta - 2\sin^3\phi\sin^2\theta - 8\sin\phi\cos^2\phi)\,d\phi\,d\theta \\ &= (-2\sin^3\phi - 8\sin\phi\cos^2\phi)\,d\phi\,d\theta\end{aligned}$$

and the integral becomes

$$\iint_{\mathfrak{S}} \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} (-2\sin^3\phi - 8\sin\phi\cos^2\phi)\, d\phi\, d\theta$$
$$= \int_0^{2\pi} \int_{\pi/3}^{\pi/2} -2\sin\phi(1 - \cos^2\phi + 4\cos^2\phi)\, d\phi\, d\theta$$
$$= 2\int_0^{2\pi} \int_{\pi/3}^{\pi/2} (1 + 3\cos^2\phi)(d\,(\cos\phi))\, d\theta = 2\int_0^{2\pi} \left(\cos\phi + \cos^3\phi\right)_{\pi/3}^{\pi/2} d\theta$$
$$= 2\int_0^{2\pi} -\left(\frac{1}{2} + \frac{1}{8}\right) d\theta = -\frac{5}{4}\int_0^{2\pi} d\theta = -\frac{5\pi}{2}.$$

Exercises for § 5.5

Answers to Exercises 1a and 2a are given in Appendix A.13.

Practice problems:

(1) Evaluate each flux integral $\iint_{\mathfrak{S}} \vec{F} \cdot d\vec{S}$ below:
   (a) $\vec{F}(x,y,z) = x\vec{\imath} + y\vec{\jmath} + 2z\vec{k}$, $\mathfrak{S}$ is the graph of $z = 3x + 2y$ over $[0,1] \times [0,1]$, oriented up.
   (b) $\vec{F}(x,y,z) = yz\vec{\imath} + x\vec{\jmath} + xy\vec{k}$, $\mathfrak{S}$ is the graph of $z = x^2 + y^2$ over $[0,1] \times [0,1]$, oriented up.
   (c) $\vec{F}(x,y,z) = x\vec{\imath} - y\vec{\jmath} + z\vec{k}$, $\mathfrak{S}$ is the part of the plane $x + y + z = 1$ in the first octant, oriented up.
   (d) $\vec{F}(x,y,z) = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$, $\mathfrak{S}$ is the upper hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, oriented up.
   (e) $\vec{F}(x,y,z) = z\vec{k}$, $\mathfrak{S}$ is the part of the sphere $x^2 + y^2 + z^2 = 1$ between the xy-plane and the plane $z = \frac{1}{2}$, oriented outward.
   (f) $\vec{F}(x,y,z) = x\vec{\imath} - y\vec{\jmath} + z\vec{k}$, $\mathfrak{S}$ is the unit sphere $x^2 + y^2 + z^2 = 1$, oriented outward.
   (g) $\vec{F}(x,y,z) = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$, $\mathfrak{S}$ is the surface parametrized by

$$\begin{cases} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= 1 - r^2 \end{cases}, \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{cases},$$

   oriented up.
   (h) $\vec{F}(x,y,z) = (y+z)\vec{\imath} + (x+y)\vec{\jmath} + (x+z)\vec{k}$, $\mathfrak{S}$ is the surface parametrized by

$$\begin{cases} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= \theta \end{cases}, \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 4\pi, \end{cases},$$

   oriented up.

Theory problems:

(2) (a) Evaluate the flux integral $\iint_{\mathfrak{S}} \vec{F} \cdot d\vec{S}$, where $\vec{F}(x,y,z) = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ and $\mathfrak{S}$ is the plane $z = ax + by$ over $[0,1] \times [0,1]$, oriented up.
   (b) Give a geometric explanation for your result.

(c) What happens if we replace this plane by the parallel plane $z = ax + by + c$?

(3) Suppose $\mathfrak{S}$ is a regular surface and $\vec{n}$ is a continuous choice of unit normal vectors (*i.e.*, an orientation of $\mathfrak{S}$). Explain how we can cover $\mathfrak{S}$ with overlapping coordinate patches for which the induced local orientations are coherent. (Note that by definition, we are given a family of overlapping coordinate patches covering $\mathfrak{S}$. The issue is how to modify them so that their induced local orientations are coherent.)

(4) Prove Remark 5.5.4.

5.6 Stokes' Theorem

The Curl of a Vector Field. Let us revisit the discussion of (planar) curl for a vector field in the plane, from the end of § 5.3. There, we looked at the effect of a local shear in a vector field, which tends to rotate a line segment around a given point. The main observation was that for a segment parallel to the x-axis, the component of the vector field in the direction of the x-axis, as well as the actual value of the component in the direction of the y-axis, are irrelevant: the important quantity is the *rate of change* of the y-component in the x-direction. A similar analysis applies to a segment parallel to the y-axis: the important quantity is then the rate of change in the y-direction of the x-component of the vector field—more precisely, of the component of the vector field in the direction of the unit vector which is a right angle *counterclockwise* from the unit vector $\vec{\jmath}$. That is, we are looking at the directional derivative, in the direction of $\vec{\jmath}$, of the component of our vector field in the direction of $-\vec{\imath}$.

How do we extend this analysis to a segment and vector field in 3-space? Fix a point $\vec{p} \in \mathbb{R}^3$, and consider a vector field

$$\vec{F}(x, y, z) = P(x, y, z)\vec{\imath} + Q(x, y, z)\vec{\jmath} + R(x, y, z)\vec{k}$$

acting on points near $\vec{p}$. If a given segment through $\vec{p}$ rotates under the influence of $\vec{F}$, its angular velocity, following the ideas at the end of § 1.7, will be represented by the vector $\vec{\omega}$ whose direction gives the axis of rotation, and whose magnitude is the angular velocity. Now, we can try to decompose this vector into components. The vertical component of $\vec{\omega}$ represents precisely the rotation about a vertical axis through $\vec{p}$, with an *upward* direction corresponding to *counterclockwise* rotation. We can also think of this in terms of the projection of the line segment onto the horizontal plane through $\vec{p}$ and its rotation about $\vec{p}$. We expect the vertical component, $R(x, y, z)$, of $\vec{F}$ to have no effect on this rotation, as it is "pushing" along the length of the vertical axis. So we expect the planar curl $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ to be the correct measure of the tendency of the vector field to produce rotation about a vertical axis. As with directed area in § 1.7, we make this into a *vector* pointing along the axis of rotation (more precisely, pointing along that axis toward the side of the horizontal plane from which the rotation being induced appears counterclockwise). This leads us to multiply the (scalar) planar curl by the vertical unit vector $\vec{k}$:

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}.$$

Now we extend this analysis to the other two components of rotation. To analyze the tendency for rotation about the x-axis, we stare at the yz-plane from the positive x-axis: the former role of the x-axis (*resp.* y-axis) is now played by the y-axis (*resp.* z-axis), and in a manner completely analogous to the argument in § 5.3 and its reinterpretation in the preceding paragraph, we represent the tendency toward rotation about the x-axis by the vector

$$\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\vec{\imath}.$$

Finally, the tendency for rotation about the y-axis requires us to look from the direction of the *negative* y-axis, and we represent this tendency by the vector

$$\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)(-\vec{\jmath}).$$

This way of thinking may remind you of our construction of the cross product from oriented areas in § 1.6; however, in this case, instead of multiplying certain components of two vectors, we seem to be taking different partial derivatives. We can formally recover the analogy by creating an abstract "vector" whose components are differentiations

$$\vec{\nabla} := \vec{\imath}\frac{\partial}{\partial x}+\vec{\jmath}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z} \tag{5.19}$$

and interpreting "multiplication" by one of these components as performing the differentiation it represents: it is a **differential operator**—a "function of functions", whose input is a function, and whose output depends on derivatives of the input. This formal idea was presented by William Rowan Hamilton (1805-1865) in his *Lectures on Quaternions* (1853) [22, Lecture VII, pp. 610-11].[11] We pronounce the symbol $\vec{\nabla}$ as "del".[12]

At the most elementary formal level, when we "multiply" a function of three variables by this, we get the gradient vector:

$$\vec{\nabla}f=\left(\vec{\imath}\frac{\partial}{\partial x}+\vec{\jmath}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)f=\frac{\partial f}{\partial x}\vec{\imath}+\frac{\partial f}{\partial y}\vec{\jmath}+\frac{\partial f}{\partial z}\vec{k}.$$

However, we can also apply this operator to a vector field in several ways. For present purposes, we can take the formal cross product of this vector with a vector field, to get a different operator: if

$$\vec{F}=P\vec{\imath}+Q\vec{\jmath}+R\vec{k}$$

[11] Quaternions were invented by Hamilton, and their use in physics was championed by Peter Guthrie Tait (1831–1901) [50] and James Clark Maxwell (1831–1879) [39], [39]. Algebraically, quaternions are like complex numbers on steroids: while complex numbers have the form $a+bi$ ($a,b\in\mathbb{R}$) and multiply formally with the interpretation $i^2=-1$, quaternions have the form $q=a+bi+cj+dk$ ($a,b,c,d\in\mathbb{R}$) and products of i's, j's and k's are interpreted according to the (non-commutative) relations we associate with cross products (Equation (1.28) in Exercise 9, § 1.6). Geometrically, quaternions were interpreted as operations acting on vectors in $\mathbb{R}^3$. From this point of view, $Tq=a$ is distinguished from $Uq=bi+cj+dk$; Tq (called the "tensor" by Hamilton and Tait, the "scalar" by Maxwell) "stretches" a vector while Uq (called the "versor" by Hamilton and the "vector" part by Maxwell) involves rotation [50, §48, p. 33] and [39, §11, p.10]. Hamilton introduced a differentiation operator (acting on functions of a quaternion variable) which he denoted $\lhd$ [22, p. 610]. Maxwell called the scalar part of this the "convergence" and its vector part the "rotation" [39, §17, p.16 & §25, p. 30]. Later William Kingdon Clifford (1845-1879) referred to the negative of convergence as "divergence". I am not sure who introduced the term "curl" for the rotation part of $\lhd$.

[12] This symbol appears in Maxwell's *Treatise on Electricity and Magnetism* [39, vol. 1, p. 16]—as well as an earlier paper [38]—but it is not given a name until Wilson's version of Gibbs' Lectures in 1901 [55, p. 138]: here he gives the "del" pronunciation, and mentions that "Some use the term *Nabla* owing to its fancied resemblance to an Assyrian harp..." (*nabla* is the Hebrew word for harp).

then the **curl** of $\vec{F}$ is

$$\begin{aligned}\overrightarrow{curl}\,\vec{F} &= \vec{\nabla}\times\vec{F}\\ &= \begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ P & Q & R\end{vmatrix}\\ &= \vec{\imath}\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)-\vec{\jmath}\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+\vec{k}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right).\end{aligned}$$

The expression on the right in the first line above is pronounced "del cross F". Note that if $R = 0$ and P and Q depend only on x and y—that is, $\vec{F} = P(x,y)\vec{\imath} + Q(x,y)\vec{\jmath}$ is essentially a planar vector field—then the only nonzero component of $\vec{\nabla}\times\vec{F}$ is the vertical one, and it equals what we called the planar curl of the associated planar vector field in § 5.3. When necessary, we distinguish between the vector $\vec{\nabla}\times\vec{F}$ and the planar curl (a scalar) by calling this the *vector* curl.

Boundary Orientation. Suppose $\mathfrak{S}$ is an oriented surface in space, with orientation defined by the unit normal vector $\vec{n}$, and bounded by one or more curves. We would like to formulate an orientation for these curves which corresponds to the boundary orientation for ∂D when D is a region in the plane. Recall that in that context, we took the unit vector $\vec{T}$ tangent to a boundary curve and rotated it by $\frac{\pi}{2}$ radians counterclockwise to get the "leftward normal" $\overrightarrow{N_+}$; we then insisted that $\overrightarrow{N_+}$ point into the region D. It is fairly easy to see that such a rotation of a vector in the plane is accomplished by setting $\overrightarrow{N_+} = \vec{k}\times\vec{T}$, and we can easily mimic this by replacing $\vec{k}$ with the unit normal $\vec{n}$ defining our orientation (that is, we rotate $\vec{T}$ counterclockwise when viewed from the direction of $\vec{n}$). However, when we are dealing with a surface in space, the surface might "curl away" from the plane in which this vector sits, so that it is harder to define what it means for it to "point into" $\mathfrak{S}$.

One way to do this is to invoke Proposition 3.6.2, which tells us that we can always parametrize a surface as the graph of a function, locally. If a surface is the graph of a function, then its boundary is the graph of the restriction of this function to the boundary of its domain. Thus we can look at the projection of $\overrightarrow{N_+}$ onto the plane containing the domain of the function, and ask that *it* point into the domain. This is a particularly satisfying formulation when we use the second statement in Proposition 3.6.2, in which we regard the surface as the graph of a function whose domain is in the tangent plane of the surface—which is to say the plane perpendicular to the normal vector $\vec{n}$—since it automatically contains $\overrightarrow{N_+}$.

We will adopt this as a definition.

Definition 5.6.1. *Given an oriented surface $\mathfrak{S}$ with orientation given by the unit normal vector field $\vec{n}$, and $\gamma(t)$ a boundary curve of $\mathfrak{S}$, with unit tangent vector $\vec{T}$ (parallel to the velocity), we say that $\gamma(t)$ has the* ***boundary orientation*** *if for every boundary point $\gamma(t)$ the leftward normal $\overrightarrow{N_+} = \vec{n}\times\vec{T}$ points into the projection of $\mathfrak{S}$ on its tangent plane at $\gamma(t)$.*

Stokes' Theorem in the Language of Vector Fields. Using the terminology worked out above, we can state Stokes' Theorem as an almost verbatim restatement, in the context of 3-space, of Theorem 5.3.7:

Theorem 5.6.2 (Stokes' Theorem). [13] *If $\vec{F} = P\vec{\imath} + Q\vec{\jmath} + R\vec{k}$ is a $\mathcal{C}^1$ vector field defined in a region of 3-space containing the oriented surface with boundary $\mathfrak{S}$, then the circulation of $\vec{F}$ around the boundary of $\mathfrak{S}$ (each constituent piecewise regular, simple, closed curve of $\partial\mathfrak{S}$ given the boundary orientation) equals the flux integral over $\mathfrak{S}$ of the (vector) curl of $\vec{F}$:*

$$\oint_{\partial\mathfrak{S}} \vec{F} \cdot d\vec{\mathfrak{s}} = \iint_{\mathfrak{S}} (\vec{\nabla} \times \vec{F}) \cdot d\vec{\mathcal{S}}.$$

The proof, involving a calculation which reduces to Green's Theorem (Theorem 5.3.4), is sketched in Exercise 3.

Stokes' Theorem, like Green's Theorem, allows us to choose between integrating a vector field along a curve and integrating its curl over a surface bounded by that curve. Let us compare the two approaches in a few examples.

First, we consider the vector field $\vec{F}(x, y, z) = (x - y)\vec{\imath} + (x + y)\vec{\jmath} + z\vec{k}$ and the surface $\mathfrak{S}$ given by the part of the graph $z = x^2 - y^2$ inside the cylinder $x^2 + y^2 \leq 1$. We take the orientation of $\mathfrak{S}$ to be *upward*. To integrate the vector field over the boundary $x^2 + y^2 = 1$, $z = x^2 - y^2$, we parametrize the boundary curve $\partial\mathfrak{S}$ as $x = \cos\theta$, $y = \sin\theta$, $z = \cos^2\theta - \sin^2\theta$, with differentials $dx = -\sin\theta\, d\theta$, $dy = \cos\theta\, d\theta$, and $dz = (-2\cos\theta\sin\theta - 2\sin\theta\cos\theta)\, d\theta = -4\sin\theta\cos\theta\, d\theta$, so the element of arclength is

$$d\vec{\mathfrak{s}} = \left\{(-\sin\theta)\vec{\imath} + (\cos\theta)\vec{\jmath} - (4\sin\theta\cos\theta)\vec{k}\right\} d\theta.$$

Along this curve, the vector field is

$$\begin{aligned} \vec{F}(\theta) &= \vec{F}\left(\cos\theta, \sin\theta, \cos^2\theta - \sin^2\theta\right) \\ &= (\cos\theta - \sin\theta)\vec{\imath} + (\cos\theta + \sin\theta)\vec{\jmath} + (\cos^2\theta - \sin^2\theta)\vec{k}. \end{aligned}$$

Their dot product is

$$\begin{aligned} \vec{F} \cdot d\vec{\mathfrak{s}} &= \{(\cos\theta - \sin\theta)(-\sin\theta) + (\cos\theta + \sin\theta)(\cos\theta) \\ &\qquad + (\cos^2\theta - \sin^2\theta)(-4\sin\theta\cos\theta)\big\}\, d\theta \\ &= \big\{-\cos\theta\sin\theta + \sin^2\theta + \cos^2\theta + \sin\theta\cos\theta \\ &\qquad -4\sin\theta\cos^3\theta + 4\sin^3\theta\cos\theta\big\}\, d\theta \\ &= (1 - 4\cos^3\theta\sin\theta + 4\sin^3\theta\cos\theta)\, d\theta \end{aligned}$$

and the line integral of $\vec{F}$ over $\partial\mathfrak{S}$ is

$$\begin{aligned} \oint_{\partial\mathfrak{S}} \vec{F} \cdot d\vec{\mathfrak{s}} &= \int_0^{2\pi} (1 - 4\cos^3\theta\sin\theta + 4\sin^3\theta\cos\theta)\, d\theta \\ &= (\theta + \cos^4\theta + \sin^4\theta)_0^{2\pi} = 2\pi. \end{aligned}$$

Now let us consider the alternative calculation, as a flux integral. From Remark 5.5.4 we know that the natural parametrization of the surface $x = s$, $y = t$, $z = s^2 - t^2$ has element of surface area (with upward orientation)

$$d\vec{\mathcal{S}} = [-(2s)\vec{\imath} - (-2t)\vec{\jmath} + \vec{k}]\, ds\, dt.$$

[13]This result was published by George Gabriel Stokes (1819-1903) as a problem on the examination for the Smith Prize at Cambridge in 1854; it was originally communicated to him in a letter from William Thomson (Lord Kelvin) (1824-1907) in July 1850 ([1, p. 208], [32, p. 790]).

The curl of our vector field is

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x-y & x+y & z \end{vmatrix} = 0\vec{\imath} - 0\vec{\jmath} + 2\vec{k} = 2\vec{k}.$$

Thus, the flux integral of the curl is

$$\iint_{\mathfrak{S}} (\vec{\nabla}\times\vec{F})\cdot d\vec{\mathcal{S}} = \iint_{\mathfrak{S}} 2\vec{k}\cdot d\vec{\mathcal{S}} = \iint_{s^2+t^2\leq 1} 2\,ds\,dt$$

which we recognize as the area of the unit disc, or 2π.

As a second example, we consider the line integral

$$\oint_{\mathcal{C}} -y^3\,dx + x^3\,dy - z^3\,dz,$$

where the curve $\mathcal{C}$ is given by the intersection of the cylinder $x^2+y^2=1$ with the plane $x+y+z=1$, circumvented counterclockwise when seen from above.

If we attack this directly, we parametrize $\mathcal{C}$ by $x=\cos\theta$, $y=\sin\theta$, $z=1-\cos\theta-\sin\theta$ with differentials $dx=-\sin\theta\,d\theta$, $dy=\cos\theta\,d\theta$, $dz=(\sin\theta-\cos\theta)\,d\theta$, and the form becomes

$$\begin{aligned} &-y^3\,dx + x^3\,dy - z^3\,dz \\ &= (-\sin^3\theta)[-\sin\theta\,d\theta] + (\cos^3\theta)[\cos\theta\,d\theta] + (1-\cos\theta-\sin\theta)^3[(\sin\theta-\cos\theta)\,d\theta] \end{aligned}$$

leading to the integral

$$\int_0^{2\pi} \left(\sin^4\theta + \cos^4\theta - (1-\cos\theta-\sin\theta)^3(\sin\theta-\cos\theta)\right)\,d\theta$$

which is not impossible to do, but clearly a mess to try.

Note that this line integral corresponds to the circulation integral $\oint_{\mathcal{C}} \vec{F}\cdot d\vec{\mathfrak{s}}$ where $\vec{F}(x,y,z) = -y^3\vec{\imath} + x^3\vec{\jmath} - z^3\vec{k}$.

If instead we formulate this as a flux integral, we take the curl of the vector field

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^3 & x^3 & -z^3 \end{vmatrix} = 0\vec{\imath} + 0\vec{\jmath} + (3x^2+3y^2)\vec{k}.$$

Note that $\mathcal{C}$ is the boundary of the part of the plane $x+y+z=1$ over the disc $x^2+y^2\leq 1$; to make the given orientation on $\mathcal{C}$ the boundary orientation, we need to make sure that the disc is oriented *up*. It can be parametrized using polar coordinates as $\vec{p}(r,\theta) = (r\cos\theta)\vec{\imath} + (r\sin\theta)\vec{\jmath} + (1-r\cos\theta-r\sin\theta)\vec{k}$, with partials

$$\begin{aligned} \frac{\partial\vec{p}}{\partial r} &= (\cos\theta)\vec{\imath} + (\sin\theta)\vec{\jmath} - (\cos\theta+\sin\theta)\vec{k} \\ \frac{\partial\vec{p}}{\partial\theta} &= (-r\sin\theta)\vec{\imath} + (r\cos\theta)\vec{\jmath} + (r\sin\theta - r\cos\theta)\vec{k}. \end{aligned}$$

We calculate the element of oriented surface area[14] in terms of the cross product

$$\begin{aligned}\frac{\partial \vec{p}}{\partial r} \times \frac{\partial \vec{p}}{\partial \theta} &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \cos\theta & \sin\theta & -\cos\theta - \sin\theta \\ -r\sin\theta & r\cos\theta & r\sin\theta - r\cos\theta \end{vmatrix} \\ &= \vec{\imath}(r\sin^2\theta - r\sin\theta\cos\theta + r\cos^2\theta + r\sin\theta\cos\theta) \\ &\qquad -\vec{\jmath}(r\cos\theta - r\cos^2\theta - r\sin\theta\cos\theta - r\sin^2\theta) \\ &\qquad +\vec{k}(r\cos^2\theta + r\sin^2\theta) \\ &= r\vec{\imath} + r\vec{\jmath} + r\vec{k};\end{aligned}$$

in particular, the element of oriented surface area is

$$d\vec{\mathcal{S}} = r(\vec{\imath} + \vec{\jmath} + \vec{k})\,dr\,d\theta$$

which, we note, has an *upward* vertical component, as desired. Since $\vec{\nabla} \times \vec{F}$ has only a $\vec{k}$ component,

$$(\vec{\nabla} \times \vec{F}) \cdot d\vec{\mathcal{S}} = (3x^2 + 3y^2)(r)\,dr\,d\theta = 3r^3\,dr\,d\theta$$

so the flux integral is given by

$$\iint_{\mathfrak{S}} (\vec{\nabla} \times \vec{F}) \cdot d\vec{\mathcal{S}} = \int_0^{2\pi}\int_0^1 3r^3\,dr\,d\theta = \int_0^{2\pi} \frac{3}{4}\,d\theta = \frac{3\pi}{2}.$$

We note that a consequence of Stokes' Theorem, like the Fundamental Theorem for Line Integrals, is that the flux integral is the same for any two surfaces that have the same boundary. However, in practice, this is only useful if we can recognize the integrand as a curl, an issue we will delay until we have the Divergence Theorem and Proposition 5.8.4 in § 5.8.

Exercises for § 5.6

Practice problems:

(1) Find the curl of each vector field below:
  (a) $\vec{F}(x,y,z) = (xy)\vec{\imath} + (yz)\vec{\jmath} + (xz)\vec{k}$
  (b) $\vec{F}(x,y,z) = (y^2 + z^2)\vec{\imath} + (x^2 + z^2)\vec{\jmath} + (x^2 + y^2)\vec{k}$
  (c) $\vec{F}(x,y,z) = (e^y\cos z)\vec{\imath} + (x^2 z)\vec{\jmath} + (x^2 y^2)\vec{k}$
  (d) $\vec{F}(x,y,z) = (y)\vec{\imath} + (-x)\vec{\jmath} + (z)\vec{k}$
  (e) $\vec{F}(x,y,z) = (z)\vec{\imath} + (y)\vec{\jmath} + (x)\vec{k}$
  (f) $\vec{F}(x,y,z) = (e^y\cos x)\vec{\imath} + (e^y\sin z)\vec{\jmath} + (e^y\cos z)\vec{k}$

(2) Evaluate each circulation integral $\oint_{\mathcal{C}} \vec{F} \cdot \vec{T}\,d\mathfrak{s}$ two different ways: (i) directly, and (ii) using Stokes' Theorem and the fact that $\mathcal{C}$ is the boundary of $\mathfrak{S}$:
  (a) $\vec{F}(x,y,z) = (-y, x, z)$, $\mathcal{C}$ is given by $\vec{p}(\theta) = (\cos\theta, \sin\theta, 1 - \cos\theta - \sin\theta)$, $0 \le \theta \le 2\pi$, and $\mathfrak{S}$ is given by $\vec{p}(s,t) = (s, t, 1 - s - t)$, $s^2 + t^2 \le 1$.
  (b) $\vec{F}(x,y,z) = y^2\vec{\imath} + z^2\vec{\jmath} + x^2\vec{k}$, $\mathcal{C}$ is given by $\vec{p}(\theta) = (\cos\theta, \sin\theta, \cos 2\theta)$, $0 \le \theta \le 2\pi$ and $\mathfrak{S}$ is given by $\vec{p}(s,t) = (s, t, s^2 - t^2)$, $s^2 + t^2 \le 1$.

[14] Note that we can't apply Remark 5.5.4 directly here, because our input is not given in rectangular coordinates

(c) $\vec{F}(x,y,z) = (z, xz, y)$, $\mathcal{C}$ is the boundary of $\mathfrak{S}$, which in turn is the part of the plane $x + y + z = 1$ over the rectangle $[0,1] \times [0,1]$, oriented up. ($\mathcal{C}$ has the boundary orientation.)

Theory problems:

(3) Proof of Theorem 5.6.2:

Note first that, by an argument similar to the proof used there, it suffices to prove the result for a coordinate patch with one boundary component: that is, we will assume that $\mathfrak{S}$ is parametrized by a regular, $\mathcal{C}^2$ function $\vec{p}: \mathbb{R}^2 \to \mathbb{R}^3$ which is one-to-one on its boundary. Instead of using s and t for the names of the parameters, we will use u and v (so as not to conflict with the parameter t in the parametrization of $\partial\mathfrak{S}$):

$$\vec{p}(u,v) = (x(u,v), y(u,v), z(u,v)), \quad (u,v) \in D \subset \mathbb{R}^2$$

and assume that the boundary ∂D of the domain D is given by a curve

$$\gamma(t) = \vec{p}(u(t), v(t)), \quad t \in [t_0, t_1].$$

It will be useful in the first two parts of this proof to adopt an old-fashioned notation for certain 2×2 determinants that occur in this context. Given two functions of two variables (in our case, two of the three coordinates x, y, and z, written as functions of u and v), say $f_1(u_1, u_2)$ and $f_2(u_1, u_2)$, we write

$$\begin{aligned}\left|\frac{\partial(f_1, f_2)}{\partial(u_1, u_2)}\right| &:= \det \begin{bmatrix} \partial f_1/\partial u_1 & \partial f_1/\partial u_2 \\ \partial f_2/\partial u_1 & \partial f_2/\partial u_2 \end{bmatrix} \\ &= \frac{\partial f_1}{\partial u_1}\frac{\partial f_2}{\partial u_2} - \frac{\partial f_1}{\partial u_2}\frac{\partial f_2}{\partial u_1}.\end{aligned}$$

(a) Show that

$$(\vec{\nabla} \times \vec{F}) \cdot d\vec{\mathcal{S}} = \left\{\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\left|\frac{\partial(y,z)}{\partial(u,v)}\right| + \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\left|\frac{\partial(x,z)}{\partial(u,v)}\right| + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\right\} du\,dv.$$

(b) This mess is best handled by separating out the terms involving each of the components of $\vec{F}$ and initially ignoring the "$du\,dv$" at the end. Consider the terms involving the first component, P: they are

$$-\frac{\partial P}{\partial z}\left|\frac{\partial(x,z)}{\partial(u,v)}\right| - \frac{\partial P}{\partial y}\left|\frac{\partial(x,y)}{\partial(u,v)}\right|;$$

add to this the term $-\frac{\partial P}{\partial x}\left|\frac{\partial(x,x)}{\partial(u,v)}\right|$ which equals zero (right?) and expand to get

$$\begin{aligned}-\frac{\partial P}{\partial z}\left(\frac{\partial x}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial x}{\partial v}\right) - \frac{\partial P}{\partial y}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right) - \frac{\partial P}{\partial x}\left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial x}{\partial v}\right) \\ = \frac{\partial x}{\partial v}\left(\frac{\partial P}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial P}{\partial x}\frac{\partial x}{\partial u}\right) \\ - \frac{\partial x}{\partial u}\left(\frac{\partial P}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial P}{\partial x}\frac{\partial x}{\partial v}\right)\end{aligned}$$

and apply the Chain Rule to rewrite this as

$$\frac{\partial x}{\partial v}\frac{\partial P}{\partial u} - \frac{\partial x}{\partial u}\frac{\partial P}{\partial v}.$$

(c) Use the equality of cross-partials to interpret the above as a planar curl (in terms of the (u, v)-plane)

$$\frac{\partial x}{\partial v}\frac{\partial P}{\partial u} - \frac{\partial x}{\partial u}\frac{\partial P}{\partial v} = \frac{\partial}{\partial u}\left[\frac{\partial x}{\partial v}P\right] - \frac{\partial}{\partial v}\left[\frac{\partial x}{\partial u}P\right].$$

(d) Use Theorem 5.3.4 (Green's Theorem) to calculate the integral

$$\iint_D \left\{\frac{\partial}{\partial u}\left[\frac{\partial x}{\partial v}P\right] - \frac{\partial}{\partial v}\left[\frac{\partial x}{\partial u}P\right]\right\} du\, dv = \int_{\partial D} \frac{\partial x}{\partial u}P\, du + \frac{\partial x}{\partial v}P\, dv = \oint_{\partial D} P\, dx.$$

(e) In a similar way, adding $\frac{\partial Q}{\partial y}\left|\frac{\partial(y,y)}{\partial(u,v)}\right|$ (*resp.* $\frac{\partial R}{\partial z}\left|\frac{\partial(z,z)}{\partial(u,v)}\right|$) to the sum of the terms involving Q (*resp.* R) we can calculate integrals of those terms using Green's Theorem:

$$\begin{aligned}\iint_D \left\{-\frac{\partial Q}{\partial z}\left|\frac{\partial (y,z)}{\partial (u,v)}\right| + \frac{\partial Q}{\partial x}\left|\frac{\partial (x,y)}{\partial (u,v)}\right|\right\} du\, dv \\ = \iint_D \left\{\frac{\partial}{\partial u}\left[\frac{\partial y}{\partial v}Q\right] - \frac{\partial}{\partial v}\left[\frac{\partial y}{\partial u}Q\right]\right\} du\, dv \\ = \oint_{\partial D} Q\, dy\end{aligned}$$

and

$$\begin{aligned}\iint_D \left\{\frac{\partial R}{\partial y}\left|\frac{\partial (y,z)}{\partial (u,v)}\right| + \frac{\partial R}{\partial x}\left|\frac{\partial (x,z)}{\partial (u,v)}\right|\right\} du\, dv \\ = \iint_D \left\{\frac{\partial}{\partial u}\left[\frac{\partial z}{\partial v}R\right] - \frac{\partial}{\partial v}\left[\frac{\partial z}{\partial u}R\right]\right\} du\, dv \\ = \oint_{\partial D} R\, dz.\end{aligned}$$

Adding these three equations yields the desired equality.

5.7 2-forms in $\mathbb{R}^3$

The formalism introduced in § 5.4 can be extended to $\mathbb{R}^3$, giving a new language for formulating Stokes' Theorem as well as many other results.

Bilinear Functions and 2-forms on $\mathbb{R}^3$. The notion of a bilinear function given in Definition 5.4.1 extends naturally to $\mathbb{R}^3$:

Definition 5.7.1. *A **bilinear function** on $\mathbb{R}^3$ is a function of two vector variables $B(\vec{v}, \vec{w})$ such that fixing one of the inputs results in a linear function of the other input:*

$$\begin{aligned} B(a_1\vec{v_1} + a_2\vec{v_2}, \vec{w}) &= a_1 B(\vec{v_1}, \vec{w}) + a_2 B(\vec{v_2}, \vec{w}) \\ B(\vec{v}, b_1\vec{w_1} + b_2\vec{w_2}) &= b_1 B(\vec{v}, \vec{w_1}) + b_2 B(\vec{v}, \vec{w_2}) \end{aligned} \tag{5.20}$$

for arbitrary vectors in $\mathbb{R}^3$ and real scalars.

As in $\mathbb{R}^2$, the dot product is one example of a bilinear function on $\mathbb{R}^3$. Using Equation (5.20) we can see that just as in the case of the plane, a general bilinear function $B(\vec{v}, \vec{w})$ on $\mathbb{R}^3$ can be expressed as a polynomial in the coordinates of its entries, with coefficients coming from the values of the bilinear function on the standard basis elements: if $\vec{v} = (x, y, z) = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ and $\vec{w} = (x', y', z') = x'\vec{\imath} + y'\vec{\jmath} + z'\vec{k}$ then

$$\begin{aligned}
B(\vec{v}, \vec{w}) &= B\left(x\vec{\imath} + y\vec{\jmath} + z\vec{k}, x'\vec{\imath} + y'\vec{\jmath} + z'\vec{k}\right) \\
&= B(\vec{\imath}, \vec{w})\, x + B(\vec{\jmath}, \vec{w})\, y + B\left(\vec{k}, \vec{w}\right) z \\
&= B(\vec{\imath}, \vec{\imath})\, xx' + B(\vec{\imath}, \vec{\jmath})\, xy' + B\left(\vec{\imath}, \vec{k}\right) xz' \\
&\quad + B(\vec{\jmath}, \vec{\imath})\, yx' + B(\vec{\jmath}, \vec{\jmath})\, yy' + B\left(\vec{\jmath}, \vec{k}\right) zz' \\
&\quad + B\left(\vec{k}, \vec{\imath}\right) zx' + B\left(\vec{k}, \vec{\jmath}\right) zy' + B\left(\vec{k}, \vec{k}\right) zz'.
\end{aligned}$$

This is rather hard on the eyes; to make patterns clearer, we will adopt a different notation, using indices and subscripts instead of different letters to denote components, etc. Let us first change our notation for the standard basis, writing

$$\vec{\imath} = \vec{e}_1, \quad \vec{\jmath} = \vec{e}_2, \quad \vec{k} = \vec{e}_3$$

and also use subscripts for the components of a vector: instead of writing $\vec{v} = (x, y, z)$, we will write

$$\vec{v} = (v_1, v_2, v_3).$$

Finally, if we use a double-indexed notation for the coefficients above

$$B(\vec{e}_i, \vec{e}_j) = b_{ij}$$

we can write the formula above in summation form

$$B(\vec{v}, \vec{w}) = \sum_{i=1}^{3} \sum_{j=1}^{3} b_{ij} v_i w_j.$$

There is another useful way to represent a bilinear function, with matrix notation. If we write

$$[B] = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

then much in the same way as we wrote a quadratic form in Appendix A.3, we can write the formula above as[15]

$$B(\vec{v}, \vec{w}) = [\vec{v}]^T [B] [\vec{w}],$$

where $[\vec{v}]$ is the column of coordinates of $\vec{v}$ and $[\vec{v}]^T$ is its transpose

$$[\vec{v}] = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$[\vec{v}]^T = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}.$$

It is natural to call the matrix $[B]$ the **matrix representative** of B. In particular, the dot product has as its matrix representative the **identity matrix** , which has 1 on the diagonal ($b_{ii} = 1$) and 0 off it ($b_{ij} = 0$ for $i \neq j$). As in the two-dimensional case,

[15]The interpretation of $[B]$ in terms of a triple matrix product is not needed in the rest of this book, you can simply regard the matrix $[B]$ a a convenient array of numbers representing the coefficients in the preceding formula.

the fact that the matrix representative of this bilinear function is symmetric reflects the fact that the function is **commutative**: $B(\vec{v}, \vec{w}) = B(\vec{w}, \vec{v})$ for any pair of vectors in $\mathbb{R}^3$.

Again as in the two-dimensional case, we require *anti*-commutativity for a 2-form (in this context, this property is often called skew-symmetry):

Definition 5.7.2. *A* 2-***form*** *on* $\mathbb{R}^3$ *is an* ***anti-commutative*** *bilinear function: a function* $\Omega(\vec{v}, \vec{w})$ *of two vector variables satisfying*

(1) ***bilinearity:***
$$\Omega(\alpha\vec{v} + \beta\vec{v}', \vec{w}) = \alpha\Omega(\vec{v}, \vec{w}) + \beta\Omega(\vec{v}', \vec{w})$$

(2) ***anti-commutativity=skew-symmetry:***
$$\Omega(\vec{v}, \vec{w}) = -\Omega(\vec{w}, \vec{v}).$$

The skew-symmetry of a 2-form is reflected in its matrix representative: it is easy to see that this property requires (and is equivalent to) the fact that $b_{ij} = -b_{ji}$ for every pair of indices, and in particular $b_{ii} = 0$ for every index i.

However, 2-forms in $\mathbb{R}^3$ differ from those on $\mathbb{R}^2$ in one very important respect: we saw in § 5.4 that every 2-form on $\mathbb{R}^2$ is a constant multiple of the 2×2 determinant, which we denoted using the wedge product. This **wedge product** can be easily extended to 1-forms on $\mathbb{R}^3$: if α and β are two 1-forms on $\mathbb{R}^3$, their wedge product is the 2-form defined by
$$(\alpha \wedge \beta)(\vec{v}, \vec{w}) \coloneqq \det \begin{pmatrix} \alpha(\vec{v}) & \beta(\vec{v}) \\ \alpha(\vec{w}) & \beta(\vec{w}) \end{pmatrix}.$$
Now, all 1-forms in the plane are linear combinations of the two coordinate forms dx and dy; thus since the wedge product of any form with itself is zero and the wedge product is anti-commutative, every 2-form in the plane is a multiple of $dx \wedge dy$. However, there are *three* coordinate forms in $\mathbb{R}^3$: dx, dy, and dz, and these can be paired in *three* different ways (up to order); $dx \wedge dy$, $dx \wedge dz$, and $dy \wedge dz$. This means that instead of all being multiples of a single one, 2-forms on $\mathbb{R}^3$ are in general linear combinations of these three **basic** 2-**forms**:
$$\Omega(\vec{v}, \vec{w}) = a(dx \wedge dy)(\vec{v}, \vec{w}) + b(dx \wedge dz)(\vec{v}, \vec{w}) + c(dy \wedge dz)(\vec{v}, \vec{w}). \tag{5.21}$$

There is another way to think of this. If we investigate the action of a basic 2-form on a typical pair of vectors, we see that each of the forms $dx \wedge dy$, $dx \wedge dz$, and $dy \wedge dz$ acts as a 2×2 determinant on certain coordinates of the two vectors:
$$(dx \wedge dy)(\vec{v}, \vec{w}) = \det \begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}$$
$$(dx \wedge dz)(\vec{v}, \vec{w}) = \det \begin{pmatrix} v_1 & v_3 \\ w_1 & w_3 \end{pmatrix}$$
$$(dy \wedge dz)(\vec{v}, \vec{w}) = \det \begin{pmatrix} v_2 & v_3 \\ w_2 & w_3 \end{pmatrix}$$
which we might recognize as the minors in the definition of the cross product $\vec{v} \times \vec{w}$. Note that the "middle" minor, corresponding to $dx \wedge dz$, gets multiplied by -1 when we calculate the cross-product determinant; we can incorporate this into the form by replacing alphabetical order $dx \wedge dz$ with "circular" order $dz \wedge dx$. If we recall the motivation for the cross-product in the first place (§ 1.6), we see that these three basic

forms represent the projections onto the coordinate planes of the oriented area of the parallelepiped spanned by the input vectors. In any case, we can write

$$\vec{v}\times\vec{w} = \vec{\imath}((dy\wedge dz)(\vec{v},\vec{w})) + \vec{\jmath}((dz\wedge dx)(\vec{v},\vec{w})) + \vec{k}((dx\wedge dy)(\vec{v},\vec{w})).$$

But then the 2-form given by Equation (5.21) can be expressed as the dot product of $\vec{v}\times\vec{w}$ with a vector determined by the coefficients in that equation: you should check that for the expression as given in Equation (5.21), this vector is $c\vec{\imath} - b\vec{\jmath} + a\vec{k}$. Again, it is probably better to use a notation via subscripts: we rewrite the basic 1-forms as

$$dx = dx_1, \quad dy = dx_2, \quad dz = dx_3;$$

then, incorporating the modifications noted above, we rewrite Equation (5.21) as

$$\Omega = a_1\,dx_2\wedge dx_3 + a_2\,dx_3\wedge dx_1 + a_3\,dx_1\wedge dx_2.$$

With this notation, we can state the following equivalent representations of an arbitrary 2-form on $\mathbb{R}^3$:

Lemma 5.7.3. *Associated to every 2-form Ω on $\mathbb{R}^3$ is a vector $\vec{a}$, defined by*

$$\Omega(\vec{v},\vec{w}) = \vec{a}\cdot\vec{v}\times\vec{w}, \tag{5.22}$$

where

$$\begin{aligned}\Omega &= a_1\,dx_2\wedge dx_3 + a_2\,dx_3\wedge dx_1 + a_3\,dx_1\wedge dx_2\\ \vec{a} &= a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3.\end{aligned}$$

The action of this 2-form on an arbitrary pair of vectors is given by the determinant formula

$$\Omega(\vec{v},\vec{w}) = \det\begin{pmatrix} a_1 & a_2 & a_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{pmatrix}. \tag{5.23}$$

Pay attention to the numbering here: the coefficient a_i with index i is paired with the basic form $dx_j\wedge dx_k$ corresponding to the *other two* indices, and these appear in an order such that i, j, k constitutes a *cyclic* permutation of 1, 2, 3. In practice, we shall often revert to the non-subscripted notation, but this version is the best one to help us remember which vectors correspond to which 1-forms. The representation given by Equation (5.22) can be viewed as a kind of analogue of the gradient vector as a representation of the 1-form given by the derivative $d_{\vec{p}}f$ of a function $f\colon \mathbb{R}^3\to\mathbb{R}$ at the point $\vec{p}$.

We saw in § 3.2 that the action of every linear function on $\mathbb{R}^3$ can be represented as the dot product with a fixed vector, and in § 5.1 we saw that this gives a natural correspondence between differential 1-forms and differentiable vector fields on $\mathbb{R}^3$

$$\omega = P\,dx + Q\,dy + R\,dz \leftrightarrow \vec{F} = P\vec{\imath} + Q\vec{\jmath} + R\vec{k}. \tag{5.24}$$

Now we have a correspondence between 2-forms Ω and vectors $\vec{F}$ on $\mathbb{R}^3$, defined by viewing the action of Ω on a pair of vectors as the dot product of a fixed vector with their cross product, leading to the correspondence between differential 2-forms and differential vector fields on $\mathbb{R}^3$

$$\Omega = A_1\,dx_2\wedge dx_3 + A_2\,dx_3\wedge dx_1 + A_3\,dx_1\wedge dx_2 \leftrightarrow \vec{F} = a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}. \tag{5.25}$$

The wedge product now assigns a 2-form to each ordered pair of 1-forms, and it is natural to ask how this can be represented as an operation on the corresponding vectors. The answer is perhaps only a little bit surprizing:

Remark 5.7.4. *Suppose α and β are two 1-forms,*

$$\alpha = a_1\,dx + a_2\,dy + a_3\,dz$$
$$\beta = b_1\,dx + b_2\,dy + b_3\,dz,$$

corresponding to the vectors

$$\vec{a} = (a_1, a_2, a_3), \vec{b} = (b_1, b_2, b_3).$$

Then their wedge product corresponds to the cross product $\vec{a} \times \vec{b}$:

$(\alpha \wedge \beta)(\vec{v}, \vec{w}) = (\vec{a} \times \vec{b}) \cdot (\vec{v} \times \vec{w})$:

$$\alpha \wedge \beta = (a_2b_3 - a_3b_2)\,dy \wedge dz + (a_3b_1 - a_1b_3)\,dz \wedge dx + (a_1b_2 - a_2b_1)\,dx \wedge dy.$$

The proof of this is a straightforward calculation (Exercise 5).

Orientation and Integration of Differential 2-forms on $\mathbb{R}^3$. Again by analogy with the case of 2-forms on the plane, we define a **differential 2-form** on a region $D \subset \mathbb{R}^3$ to be a mapping Ω which assigns to each point $p \in D$ a 2-form Ω_p on the tangent space $T_p\mathbb{R}^3$. From Lemma 5.7.3 we can write Ω_p as a linear combination of the basic 2-forms

$$\Omega_p = a_1(p)\ dx_2 \wedge dx_3 + a_2(p)\ dx_3 \wedge dx_1 + a_3(p)\ dx_1 \wedge dx_2$$

or represent it via the associated vectorfield

$$\vec{F}(p) = a_1(p)\vec{\imath} + a_2(p)\vec{\jmath} + a_3(p)\vec{k}.$$

We shall call the form $\mathcal{C}^r$ if each of the three functions $a_i(p)$ is $\mathcal{C}^r$ on D.

The integration of 2-forms in $\mathbb{R}^3$ is carried out over surfaces in a manner analogous to the integration of 1-forms over curves described in § 5.1. There, we saw that the integral of a 1-form over a curve $\mathcal{C}$ depends on a choice of orientation for $\mathcal{C}$; reversing the orientation also reverses the sign of the integral. The same issue arises here, but in a more subtle way.

Suppose the orientation of $\mathfrak{S}$ is given by the unit normal vector field $\vec{n}$, and $\vec{p}(s,t)$ is a regular parametrization of $\mathfrak{S}$. We can define the **pullback** of a form Ω by $\vec{p}$ as the 2-form on the domain $D \subset \mathbb{R}^2$ of $\vec{p}$ defined for $(s,t) \in D$ and $\vec{v}, \vec{w} \in T_{(s,t)}\mathbb{R}^2$ by

$$[\vec{p}^*(\Omega)]_{(s,t)}(\vec{v}, \vec{w}) = \Omega_{\vec{p}(s,t)}\left(T_{(s,t)}\vec{p}(\vec{v}), T_{(s,t)}\vec{p}(\vec{w})\right). \tag{5.26}$$

This pullback will at each point be a multiple of the basic form $ds \wedge dt$, say $[\vec{p}^*(\Omega)]_{(s,t)} = f(s,t)\ ds \wedge dt$, and we define the integral of Ω over $\mathfrak{S}$ as the (usual double) integral of f over D:

$$\int_{\mathfrak{S}} \Omega := \iint_D f(s,t)\ ds\,dt. \tag{5.27}$$

So where does the orientation come in? This is a subtle and rather confusing point, going back to the distinction between area and *signed* area in the plane.

When we initially talked about "positive" orientation of an oriented triangle in the plane, we had a "natural" point of view on the standard xy-plane: a positive rotation was a counterclockwise one, which meant the direction from the positive x-axis toward the positive y-axis. Thus, we implicitly thought of the xy-plane as being the plane $z = 0$

in $\mathbb{R}^3$, and viewed it from the direction of the positive z-axis: in other words, we gave the xy-plane the orientation determined by the unit normal $\vec{k}$. Another way to say this is that our orientation amounted to choosing x as the *first* parameter and y as the *second*. With this orientation, the *signed* area of a positively oriented triangle $[A, B, C]$, coming from a determinant, agrees with the ordinary area of $\triangle ABC$, coming from the double integral $\iint_{\triangle ABC} dx\, dy$ (which is always non-negative). If we had followed Alice through the looking-glass and seen the xy-plane from *below* (that is, with the orientation reversed), then the same oriented triangle would have had *negative* signed area. Recall that this actually happens in a different plane—the xz-plane—where the orientation coming from alphabetical order (x before z) corresponds to viewing the plane from the *negative* y-axis, which is why, when we calculated the cross-product, we preceded the minor involving x and z with a minus sign.

But what is the orientation of the domain of a parametrization $\vec{p}\,(s,t)$ of $\mathfrak{S}$? You might say that counterclockwise, or positive, rotation is from the positive s-axis toward the positive t-axis, but this means we are automatically adopting alphabetical order, which is an artifact of our purely arbitrary choice of names for the parameters. We need to have a more "natural"—which is to say geometric—choice of orientation for our parameters. It stands to reason that this choice should be related to the orientation we have chosen for $\mathfrak{S}$. So here's the deal: we start with the orientation on $\mathfrak{S}$ given by the unit normal vector field $\vec{n}$ on $\mathfrak{S}$. This vector field can be viewed as the vector representative of a 2-form acting on pairs $\vec{v}, \vec{w}$ of vectors tangent to $\mathfrak{S}$ (at a common point: $\vec{v}, \vec{w} \in T_p\mathfrak{S}$) defined, following Equation (5.22), by

$$\Omega_p\left(\vec{v}, \vec{w}\right) = \vec{n} \cdot (\vec{v} \times \vec{w}).$$

When we pull this back by $\vec{p}$, we have a form $\vec{p}^*\,(\Omega)$ on the parameter space, so it is a nonzero multiple of $ds \wedge dt$, and of course the opposite multiple of $dt \wedge ds$. The orientation of parameter space corresponding to the order of the parameters *for which this multiple is positive* is the orientation **induced** by the parametrization $\vec{p}$. In other words, the "basic" 2-form on parameter space is the wedge product of ds and dt *in the order specified by the induced orientation*: when we chose the function $f\,(s,t)$ in Definition 5.27 which we integrate over the domain D of our parametrization (in the ordinary double-integral sense) to calculate $\iint_{\mathfrak{S}} \Omega$, we should have defined it as $[\vec{p}^*\,(\Omega)]_{(s,t)} = f\,(s,t)\; dt \wedge ds$ if the order given by the induced parametrization corresponded to t before s.

How does this work in practice? Given the parametrization $\vec{p}\,(s,t)$ of $\mathfrak{S}$, let us denote the unit vector along the positive s-axis (*resp.* positive t-axis) in parameter space by $\vec{e}_s$ (*resp.* $\vec{e}_t$). On one hand, $ds \wedge dt$ can be characterized as the unique 2-form on parameter space such that $(ds \wedge dt)\left(\vec{e}_s, \vec{e}_t\right) = 1$ (while $dt \wedge ds$ is characterized by $(dt \wedge ds)\left(\vec{e}_t, \vec{e}_s\right) = 1$). On the other hand, the pullback $\vec{p}^*\,(\Omega)$ acts on these vectors via

$$\vec{p}^*\Omega\left(\vec{e}_s, \vec{e}_t\right) = \Omega\left(T\vec{p}\left(\vec{e}_s\right), T\vec{p}\left(\vec{e}_t\right)\right).$$

Note that the two vectors in the last expression are by definition the partials of the parametrization:

$$T\vec{p}\left(\vec{e}_s\right) = \frac{\partial \vec{p}}{\partial s}, \quad T\vec{p}\left(\vec{e}_t\right) = \frac{\partial \vec{p}}{\partial t},$$

and substituting this into the calculation above yields

$$\vec{p}^*\Omega\left(\vec{e_s},\vec{e_t}\right)=\Omega\left(\frac{\partial\vec{p}}{\partial s},\frac{\partial\vec{p}}{\partial t}\right)=\vec{n}\cdot\left(\frac{\partial\vec{p}}{\partial s}\times\frac{\partial\vec{p}}{\partial t}\right).$$

If this is positive, then our orientation puts s before t, while if it is negative, we should put t before s.

Let's formalize this in a definition.

Definition 5.7.5. *Suppose $\vec{p}\,(s,t)$ is a regular parametrization of the surface $\mathfrak{S}$ oriented by the unit normal vector field $\vec{n}$.*

(1) *The **basic form** on parameter space induced by $\vec{p}$ is the choice $dA = ds\wedge dt$ or $dA = dt\wedge ds$, where the order of ds and dt is chosen so that the cross product of the partials of $\vec{p}$ in the same order has a positive dot product with $\vec{n}$.*

(2) *Suppose Ω is a 2-form defined on $\mathfrak{S}$. Then its pullback via $\vec{p}$ is a function times the basic form induced by $\vec{p}$:*

$$\vec{p}^*\left(\Omega\right)=f\left(s,t\right)\,dA.$$

(3) *We define the integral of Ω over the surface $\mathfrak{S}$ with orientation given by $\vec{n}$ as the (ordinary) integral of f over the domain D of $\vec{p}$:*

$$\iint_{\mathfrak{S}}\Omega:=\iint_D f\,dA=\iint_D f\left(s,t\right)\,ds\,dt.$$

Let's see how this works in a couple of examples.

First, let $\mathfrak{S}$ be the part of the plane $x+y+z=1$ in the first quadrant, oriented *up*, and take $\Omega = dx\wedge dz$. The natural parametrization of $\mathfrak{S}$ comes from regarding it as the graph of $z=1-x-y$: $x=s$, $y=t$, and $z=1-s-t$. The part of this in the first quadrant, $x\geq 0$, $y\geq 0$, $z\geq 0$, is the image of the domain D in parameter space specified by the inequalities $0\leq t\leq 1-s$ and $0\leq s\leq 1$. The standard normal to the plane $x+y+z=1$ is $\vec{N}=\vec{\imath}+\vec{\jmath}+\vec{k}$, which clearly has a positive vertical component. This is not a unit vector (we would have to divide by $\sqrt{3}$) but this is immaterial; it is only the direction that matters. The partials of the parametrization are $\frac{\partial\vec{p}}{\partial s}=\vec{\imath}-\vec{k}$ and $\frac{\partial\vec{p}}{\partial t}=\vec{\jmath}-\vec{k}$, with cross product

$$\frac{\partial\vec{p}}{\partial s}\times\frac{\partial\vec{p}}{\partial t}=\begin{vmatrix}\vec{\imath}&\vec{\jmath}&\vec{k}\\1&0&-1\\0&1&-1\end{vmatrix}=\vec{\imath}+\vec{\jmath}+\vec{k}$$

so of course its dot product with $\vec{N}$ is positive; thus our basic 2-form on parameter space is

$$dA = ds\wedge dt.$$

Now, the pullback of Ω is simply a matter of substitution: the differentials of the components of the parametrization are $dx = ds$, $dy = dt$, and $dz = -ds - dt$, so the pullback of Ω, which is simply the expression for Ω in terms of our parameters and their differentials, is

$$\Omega = dx\wedge dz=\left(ds\right)\wedge\left(-ds-dt\right)=-ds\wedge ds-ds\wedge dt=-dA$$

so $f(s,t) = -1$ and

$$\iint_{\mathfrak{S}} \Omega = \iint_D -1\, dA = -\iint_D ds\, dt = -\int_0^1 \int_0^{(1-s)} dt\, ds = -\int_0^1 (1-s)\, ds = -\frac{1}{2}.$$

As a second example, we take $\mathfrak{S}$ to be the part of the sphere $x^2 + y^2 + z^2 = 1$ cut out by the horizontal planes $z = -\frac{1}{\sqrt{2}}$ and $z = \frac{1}{2}$, the xz-plane, and the vertical half-plane containing the z-axis together with the vector $\vec{\imath} + \vec{\jmath}$. We orient $\mathfrak{S}$ *inward* (that is, toward the origin) and let $\Omega = z\, dx \wedge dy$. The natural way to parametrize this is using spherical coordinates (with $\rho = 1$): $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$, $z = \cos\phi$. The domain of this parametrization is specified by $\frac{\pi}{3} \le \phi \le \frac{3\pi}{4}$ and $0 \le \theta \le \frac{\pi}{4}$. The partials of the parametrization are $\frac{\partial\vec{p}}{\partial\phi} = (\cos\phi\cos\theta)\vec{\imath} + (\cos\phi\sin\theta)\vec{\jmath} - (\sin\phi)\vec{k}$ and $\frac{\partial\vec{p}}{\partial\theta} = (-\sin\phi\sin\theta)\vec{\imath} + (\sin\phi\cos\theta)\vec{\jmath}$; their cross product is

$$\begin{aligned}\frac{\partial\vec{p}}{\partial\phi} \times \frac{\partial\vec{p}}{\partial\theta} &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix} \\ &= (\sin^2\phi\cos\theta)\vec{\imath} + (\sin^2\phi\sin\theta)\vec{\jmath} + (\sin\phi\cos\phi)\vec{k}.\end{aligned}$$

It is hard to see how this relates to the inward normal from this formula; however, we need only check the sign of the dot product at one point. At $(1,0,0)$, where $\phi = \frac{\pi}{2}$ and $\theta = 0$, the cross product is $\vec{\imath}$, while the *inward* pointing normal is $-\vec{\imath}$. Therefore, the basic form is

$$dA = d\theta \wedge d\phi.$$

To calculate the pullback of Ω, we first find the differentials of the components of $\vec{p}$: $dx = \cos\phi\cos\theta\, d\phi - \sin\phi\sin\theta\, d\theta$, $dy = \cos\phi\sin\theta\, d\phi + \sin\phi\cos\theta\, d\theta$, $dz = -\sin\phi\, d\phi$. Then

$$\begin{aligned}\Omega &= z\, dx \wedge dy \\ &= (\cos\phi)\{(\cos\phi\cos\theta\, d\phi - \sin\phi\sin\theta\, d\theta) \wedge (\cos\phi\sin\theta\, d\phi + \sin\phi\cos\theta\, d\theta)\} \\ &= (\cos\phi)\{(\cos\phi\cos\theta\sin\phi\cos\theta)\, d\phi \wedge d\theta - (\sin\phi\sin\theta\cos\phi\cos\theta)\, d\theta \wedge d\phi\} \\ &= (\cos\phi)\{(\cos\phi\sin\phi\cos^2\theta + \sin\phi\cos\phi\sin^2\theta)\, d\phi \wedge d\theta \\ &= (\cos^2\phi\sin\phi)\, d\phi \wedge d\theta = -\cos^2\phi\sin\phi\, dA.\end{aligned}$$

Thus,

$$\begin{aligned}\iint_{\mathfrak{S}} \Omega &= \iint_D -\cos^2\phi\sin\phi\, dA = \int_0^{\pi/4} \int_{\pi/3}^{3\pi/4} -\cos^2\phi\sin\phi\, d\phi\, d\theta \\ &= \int_0^{\pi/4} \left(\frac{1}{3}\cos^3\phi\right)_{\pi/3}^{3\pi/4} d\theta = \frac{1}{3}\int_0^{\pi/4} \left(-\frac{1}{2\sqrt{2} - \frac{1}{8}}\right) d\theta \\ &= -\frac{1}{3}\left(\frac{1}{2\sqrt{2}} + \frac{1}{8}\right)\frac{\pi}{4} = -\frac{\pi}{12}\left(\frac{1}{2\sqrt{2}} + \frac{1}{8}\right) = -\frac{\pi(4+\sqrt{2})}{96\sqrt{2}}.\end{aligned}$$

Stokes' Theorem in the Language of Forms. To translate between flux integrals of vector fields and integrals of forms over oriented surfaces, we first look more closely at the "basic form" dA induced by a parametrization $\vec{p}(s,t)$ of the oriented surface $\mathfrak{S}$. This was defined in terms of the pullback of the form Ω which acted on a pair of vectors tangent to $\mathfrak{S}$ at the same point by dotting their cross product with the unit normal $\vec{n}$ defining the orientation of $\mathfrak{S}$. To calculate this pullback, let us take two vectors in parameter space and express them in terms of the unit vectors $\vec{e_s}$ and $\vec{e_t}$ in the direction of the s-axis and t-axis, respectively:

$$\vec{v} = v_s\vec{e_s} + v_t\vec{e_t}, \quad \vec{w} = w_s\vec{e_s} + w_t\vec{e_t}.$$

We note for future reference that these coordinates can be regarded as the values of the coordinate forms ds and dt on the respective vectors: $v_s = ds(\vec{v})$, $v_t = dt(\vec{v})$. Now, the pullback of Ω acts on $\vec{v}$ and $\vec{w}$ as follows:

$$\vec{p}^*\Omega\left(\vec{v},\vec{w}\right) = \vec{p}^*\Omega\left(v_s\vec{e_s} + v_t\vec{e_t}, w_s\vec{e_s} + w_t\vec{e_t}\right)$$

and using the linearity and antisymmetry of the form (or just Equation (5.13)) we can write this as

$$= \vec{p}^*\Omega\left(\vec{e_s},\vec{e_t}\right)\det\begin{pmatrix} v_s & v_t \\ w_s & w_t \end{pmatrix}.$$

By definition of the pullback, the first factor is given by the action of Ω on the images of $\vec{e_s}$ and$\vec{e_t}$ under the linearization of the parametrization, which are just the partials of the parametrization. Also, using our earlier observation concerning the coordinate forms together with the definition of the wedge product, we see that the second factor is simply the action of $ds \wedge dt$ on $\vec{v}$ and $\vec{w}$:

$$\begin{aligned} &= \Omega\left(\frac{\partial\vec{p}}{\partial s}, \frac{\partial\vec{p}}{\partial t}\right)(ds\wedge dt)\left(\vec{v},\vec{w}\right) \\ &= \left\{\vec{n}\cdot\left(\frac{\partial\vec{p}}{\partial s}\times\frac{\partial\vec{p}}{\partial t}\right)\right\}\left\{(ds\wedge dt)\left(\vec{v},\vec{w}\right)\right\}. \end{aligned}$$

Note that if we reverse the roles of s and t in both factors, we introduce two changes of sign, so we can summarize the calculation above as

$$\vec{p}^*(\Omega) = \left\{\vec{n}\cdot\left(\frac{\partial\vec{p}}{\partial s}\times\frac{\partial\vec{p}}{\partial t}\right)\right\} ds\wedge dt = \left\{\vec{n}\cdot\left(\frac{\partial\vec{p}}{\partial t}\times\frac{\partial\vec{p}}{\partial s}\right)\right\} dt\wedge ds.$$

This says that

$$\vec{p}^*(\Omega) = \left\|\frac{\partial\vec{p}}{\partial s}\times\frac{\partial\vec{p}}{\partial t}\right\| dA,$$

where dA is the "basic form" on parameter space determined by the orientation of $\mathfrak{S}$. This looks suspiciously like the element of surface area which we use to calculate surface integrals:

$$d\mathcal{S} = \left\|\frac{\partial\vec{p}}{\partial s}\times\frac{\partial\vec{p}}{\partial t}\right\| ds\,dt;$$

in fact, the latter is precisely the expression we would put inside a double integral to calculate $\iint_{\mathfrak{S}}\Omega$:

$$\iint_{\mathfrak{S}}\Omega = \int_D\left\|\frac{\partial\vec{p}}{\partial s}\times\frac{\partial\vec{p}}{\partial t}\right\| dA = \iint_D\left\|\frac{\partial\vec{p}}{\partial s}\times\frac{\partial\vec{p}}{\partial t}\right\| ds\,dt = \iint_{\mathfrak{S}} 1\ d\mathcal{S}.$$

So Ω is the 2-form version of the element of surface area; we will refer to it as the **area form** of the oriented surface $\mathfrak{S}$.

The following is a simple matter of chasing definitions (Exercise 7):

Remark 5.7.6. *If the 2-form Ω corresponds, according to Equation* (5.25)*, to the vector field $\vec{F}$, then the integral of Ω over an oriented surface equals the flux integral of $\vec{F}$ over the same surface:*

$$\Omega \leftrightarrow \vec{F} \Rightarrow \int_{\mathfrak{S}} \Omega = \iint_{\mathfrak{S}} \vec{F} \cdot d\vec{\mathcal{S}}. \tag{5.28}$$

We also need to extend the notion of exterior derivatives to differential 1-forms in $\mathbb{R}^3$. Formally, we do just what we did in § 5.4 for differential 1-forms in $\mathbb{R}^2$: a differential 1-form on $\mathbb{R}^3$ can be written

$$\omega = P(x,y,z)\ dx + Q(x,y,z)\ dy + R(x,y,z)\ dz$$

and we define its exterior derivative by wedging the differential of each coefficient function with the coordinate form it is associated to:

$$\begin{aligned} d\omega &= (dP) \wedge dx + (dQ) \wedge dy + (dR) \wedge dz \\ &= \left(\frac{\partial P}{\partial x}\,dx + \frac{\partial P}{\partial y}\,dy + \frac{\partial P}{\partial z}\,dz\right) \wedge dx + \left(\frac{\partial Q}{\partial x}\,dx + \frac{\partial Q}{\partial y}\,dy + \frac{\partial Q}{\partial z}\,dz\right) \wedge dy \\ &\qquad + \left(\frac{\partial R}{\partial x}\,dx + \frac{\partial R}{\partial y}\,dy + \frac{\partial R}{\partial z}\,dz\right) \wedge dz \\ &= \frac{\partial P}{\partial y}\,dy \wedge dx + \frac{\partial P}{\partial z}\,dz \wedge dx + \frac{\partial Q}{\partial x}\,dx \wedge dy + \frac{\partial Q}{\partial z}\,dz \wedge dy \\ &\qquad + \frac{\partial R}{\partial x}\,dx \wedge dz + \frac{\partial R}{\partial y}\,dy \wedge dz \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy. \end{aligned}$$

As with the wedge product, it is straightforward to show that this corresponds in our dictionary for translating between vector fields and forms to the curl (Exercise 6):

Remark 5.7.7. *If the 1-form ω corresponds, according to Equation* (5.24)*, to the vector field $\vec{F}$, then its exterior derivative $d\omega$ corresponds, according to Equation* (5.25)*, to the curl of $\vec{F}$:*

$$\omega \leftrightarrow \vec{F} \quad \Leftrightarrow \quad d\omega \leftrightarrow \vec{\nabla} \times \vec{F}.$$

Using this dictionary, we can now state Stokes' Theorem in terms of forms:

Theorem 5.7.8 (Stokes' Theorem, Differential Form)**.** *Suppose ω is a differential 1-form defined on an open set in $\mathbb{R}^3$ containing the surface $\mathfrak{S}$ with boundary $\partial\mathfrak{S}$.*

Then

$$\oint_{\partial\mathfrak{S}} \omega = \iint_{\mathfrak{S}} d\omega.$$

Exercises for § 5.7

Answers to Exercises 1a, 1b, 2a, 2g, 3a, and 4a are given in Appendix A.13.

Practice problems:

(1) Which of the following polynomials give bilinear functions on $\mathbb{R}^3$? (Here, we regard the polynomial as a function of the two vectors $\vec{v} = x_1\vec{\imath} + y_1\vec{\jmath} + z_1\vec{k}$ and $\vec{w} = x_2\vec{\imath} + y_2\vec{\jmath} + z_2\vec{k}$.) For each one that does, give the matrix representative and decide whether it is commutative, anti-commutative, or neither.
 (a) $x_1x_2 + y_1y_2 - z_1z_2$
 (b) $x_1y_1 + x_2y_2 - y_1z_1 + y_2z_2$
 (c) $x_1y_2 - y_1z_2 + x_2z_1 + z_1y_2 + y_1x_2 - z_2x_1$
 (d) $(x_1 + 2y_1 + 3z_1)(x_2 - y_2 + 2z_2)$
 (e) $(x_1 + y_1 + z_1)(2x_2 + y_2 + z_2) - (2x_1 + y_1 + z_1)(x_2 + y_2 + z_2)$
 (f) $(x_1 - 2y_1 + 3z_1)(x_2 - y_2 - z_2) - (x_1 - y_1 - z_1)(2y_2 - x_2 - 3z_2)$

(2) For each vector field $\vec{F}$ below, write the 2-form Ω associated to it via Equation (5.22) as $\Omega = A\,dx \wedge dy + B\,dy \wedge dz + C\,dz \wedge dx$.
 (a) $\vec{F} = \vec{\imath}$ (b) $\vec{F} = \vec{\jmath}$ (c) $\vec{F} = \vec{k}$
 (d) $\vec{F} = \vec{\imath} + \vec{\jmath} + \vec{k}$
 (e) $\vec{F} = 2\vec{\imath} - 3\vec{\jmath} + 4\vec{k}$
 (f) $\vec{F} = (x + y)\vec{\imath} + (x - y)\vec{\jmath} + (y + z)\vec{k}$
 (g) $\vec{F} = y\vec{\imath} + z\vec{\jmath} + x\vec{k}$
 (h) $\vec{F} = x^2\vec{\imath} + z^2\vec{\jmath} + (x + y)\vec{k}$
 (i) $\vec{F} = \vec{\nabla} f$, where $f(x, y, z)$ is a $\mathcal{C}^2$ function.

(3) For each differential 2-form Ω below, find the vector field $\vec{F}$ corresponding to it via Equation (5.22).
 (a) $\Omega = dx \wedge dy$ (b) $\Omega = dx \wedge dz$ (c) $\Omega = dy \wedge dz$
 (d) $\Omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$
 (e) $\Omega = df \wedge (dx + dy + dz)$, where df is the differential of the $\mathcal{C}^2$ function $f(x, y, z)$. (Write the answer in terms of the partial derivatives of f.)

(4) Evaluate $\iint_{\mathfrak{S}} \Omega$.
 (a) $\Omega = x\,dy \wedge dz$, $\mathfrak{S}$ is the plane $x + y + z = 1$ in the first octant, oriented up.
 (b) $\Omega = z\,dx \wedge dy$, $\mathfrak{S}$ is the graph $z = x^2 + y^2$ over $[0, 1] \times [0, 1]$, oriented up.
 (c) $\Omega = x\,dy \wedge dz$, $\mathfrak{S}$ is the graph $z = x^2 + y^2$ over $[0, 1] \times [0, 1]$, oriented up.
 (d) $\Omega = x^2\,dx \wedge dz$, $\mathfrak{S}$ is the graph $z = x^2 + y^2$ over $[0, 1] \times [0, 1]$, oriented down.
 (e) $\Omega = dx \wedge dy$, $\mathfrak{S}$ is the part of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant, oriented outward.
 (f) $\Omega = dx \wedge dz$, $\mathfrak{S}$ is the part of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant, oriented outward.
 (g) $\Omega = x\,dy \wedge dz$, $\mathfrak{S}$ is the part of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant, oriented inward.
 (h) $\Omega = x\,dy \wedge dz - y\,dx \wedge dz + z\,dx \wedge dy$, $\mathfrak{S}$ is given by the parametrization $x = r\cos\theta$, $y = r\sin\theta$, $z = \theta$, for $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$ with the orientation induced by the parametrization.
 (i) $\Omega = z\,dx \wedge dy$, $\mathfrak{S}$ is parametrized by $x = \cos^3 t$, $y = \sin^3 t$, $z = s$, for $0 \leq t \leq 1$ and $0 \leq s \leq 2\pi$ with the orientation induced by the parametrization.
 (j) $\Omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$, $\mathfrak{S}$ is the surface of the cube $[0, 1] \times [0, 1] \times [0, 1]$, oriented outward.

Theory problems:

(5) Prove Remark 5.7.4. (*Hint:* Carry out the formal wedge product, paying careful attention to order, and compare with the correspondence on 2-forms.)
(6) Prove Remark 5.7.7.
(7) Prove Remark 5.7.6

Challenge problem:

(8) Show that every 2-form on $\mathbb{R}^3$ can be expressed as the wedge product of two 1-forms. This shows that the notion of a "basic" 2-form on p. 308 depends on the coordinate system we use.

5.8 The Divergence Theorem

So far, we have seen how the Fundamental Theorem for Line Integrals (Theorem 5.2.1) relates the line integral of a gradient vector field $\vec{F} = \vec{\nabla} f$ over a directed curve $\mathcal{C}$ to the values of the potential function f at the ends of $\mathcal{C}$, and how Green's Theorem (Theorem 5.3.4) and its generalization, Stokes' Theorem (Theorem 5.6.2) relate the flux integral of the curl $\vec{\nabla} \times \vec{F}$ of a vector field $\vec{F}$ on a domain D in $\mathbb{R}^2$ or a surface $\mathfrak{S}$ in $\mathbb{R}^3$ to its circulation integral around the boundary ∂D (*resp.* $\partial\mathfrak{S}$) of D (*resp.* $\mathfrak{S}$). In both cases, we have a relation between the integral in a *domain* of something obtained via an operation involving derivatives (or **differential operator**) applied to a function (in the case of the Fundamental Theorem for Line Integrals) or vector field (in the case of Green's and Stokes' Theorems) and the "integral" of that object on the *boundary* of the domain. In this section, we consider the third great theorem of integral calculus for vector fields, relating the flux integral of a vector field $\vec{F}$ over the boundary of a three-dimensional region to the integral of a related object, obtained via another differential operator from $\vec{F}$, over the region. This is known variously as the *Divergence Theorem*, *Gauss's Theorem*, or the *Ostrogradsky Theorem*; the differential operator in this case is the *divergence* of the vector field.

Green's Theorem Revisited:

Divergence of a Planar Vector Field. A two-dimensional version of the Divergence Theorem is outlined in Exercise 7 in § 5.3. We recall it here:

Theorem 5.8.1 (Green's Theorem, Divergence Form). *Suppose $D \subset \mathbb{R}^2$ is a regular planar region bounded by a simple, closed regular curve $\mathcal{C} = \partial D$ with positive orientation, and $\vec{F}(x,y) = P(x,y)\vec{\imath} + Q(x,y)\vec{\jmath}$ is a $\mathcal{C}^1$ vector field on D. Let $\overrightarrow{N_-}$ denote the* outward *pointing unit normal vector field to $\mathcal{C}$.*

Then

$$\oint_{\mathcal{C}} \vec{F} \cdot \overrightarrow{N_-}\, d\mathfrak{s} = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA. \tag{5.29}$$

We note that the left side of Equation (5.29), the line integral around $\mathcal{C}$ of the *normal* component of $\vec{F}$ (by contrast with the *tangential* component which appears in Theorem 5.3.4), is the analogue in one lower dimension of the flux integral of $\vec{F}$; if we imagine a simplified two-dimensional model of fluid flow, with $\vec{F}$ the velocity (or momentum) field, then this measures the amount of "stuff" leaving D per unit time. The right side of Equation (5.29) differs from Theorem 5.3.4 in that instead of the *difference* of *cross*-derivatives of the components of $\vec{F}$ we have the *sum* of the "pure" derivatives—the x-partial of the x-component of $\vec{F}$ plus the y-partial of the y-component of $\vec{F}$. This is called the **divergence** of $\vec{F}$:

$$\operatorname{div}(P\vec{\imath} + Q\vec{\jmath}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

To gain some intuition about the divergence, we again think of $\vec{F}$ as the velocity field of a fluid flow, and consider the effect of this flow on the area of a small square with sides parallel to the coordinate axes (Figure 5.10). As in our intuitive discussion of curl on p. 286, a constant vector field will not affect the area; it will be the *change* in $\vec{F}$ which affects the area. In the previous discussion, we saw that the *vertical* change $\frac{\partial P}{\partial y}$ in the *horizontal* component of $\vec{F}$ (*resp.* the *horizontal* change $\frac{\partial Q}{\partial x}$ in the *vertical* component of $\vec{F}$) tends to a *shear* effect, and these effects will not change the area (by Cavalieri's principle). However, the the *horizontal* change $\frac{\partial P}{\partial x}$ in the *horizontal* component of $\vec{F}$ will tend to "stretch" the projection of the base of the square onto the x-axis, and similarly the the *vertical* change $\frac{\partial Q}{\partial y}$ in the *vertical* component of $\vec{F}$ will "stretch" the height, which is to say the vertical dimension of the square. A stretch in either of these directions increases the area of the square. Thus, we see, at least on a purely heuristic level, that $\operatorname{div}\vec{F}$ measures *the tendency of the velocity field to increase areas*. As before, this argument comes with a disclaimer: rigorously speaking, this interpretation of divergence is a *consequence* of Theorem 5.8.1 (Exercise 15 gives a proof based on the Change-of-Variables formula, Theorem 4.3.4).

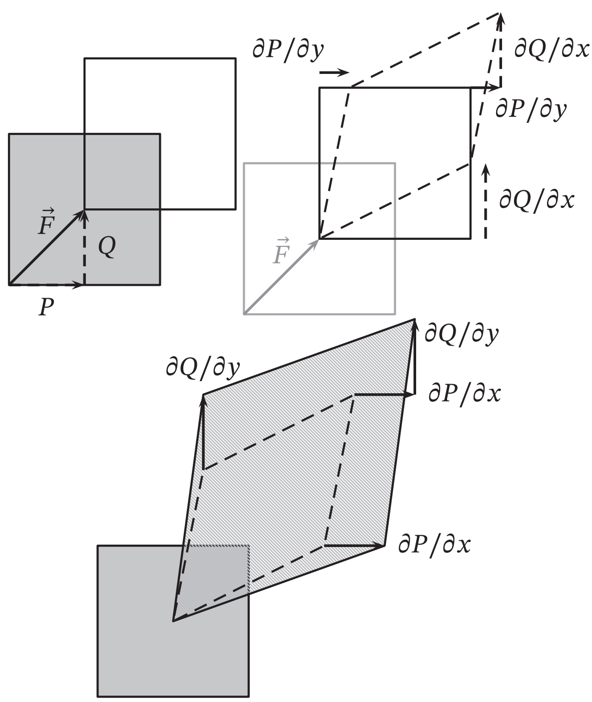


Figure 5.10. Interpretation of planar divergence

Divergence of a Vector Field in $\mathbb{R}^3$. For a vector field in space, there are three components, and it is formally reasonable that the appropriate extension of divergence to this case involves adding the partial of the *third* component.

Definition 5.8.2. *The* ***divergence*** *of a vector field*

$$\vec{F}(x,y,z) = P(x,y,z)\vec{\imath} + Q(x,y,z)\vec{\jmath} + R(x,y,z)\vec{k}$$

is

$$\operatorname{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

The heuristic argument we gave in the planar case can be extended, with a little more work, to an interpretation of this version of divergence as measuring the tendency of a fluid flow in $\mathbb{R}^3$ to increase *volumes* (Exercise 9). Note that the divergence of a vector field is a *scalar*, by contrast with its curl, which is a *vector*. If one accepts the heuristic argument that this reflects change in volume, then this seems natural: rotation has a direction (given by the axis of rotation), but volume is itself a scalar, and so its rate of change should also be a scalar. Another, deeper reason for this difference will become clearer when we consider the version of this theory using differential forms.

We can use the "del" operator $\vec{\nabla} = \vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$ to fit divergence into the formal scheme we used to denote the calculation of the differential of a function and the curl of a vector field: the divergence of $\vec{F}$ is the *dot* product of $\vec{\nabla}$ with $\vec{F}$:

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F}.$$

Just to solidify our sense of this new operator, let us compute a few examples: if

$$\vec{F}(x,y,z) = ax\vec{\imath} + by\vec{\jmath} + cz\vec{k}$$

then

$$\operatorname{div}\vec{F} = a + b + c.$$

This makes sense in terms of our heuristic: a fluid flow with this velocity increases the scale of each of the coordinates by a constant increment per unit time, and so we expect volume to be increased by $a + b + c$.

By contrast, the vector field

$$\vec{F}(x,y,z) = -y\vec{\imath} + x\vec{\jmath}$$

has divergence

$$\begin{aligned}\operatorname{div}\vec{F} &= 0 + 0\\ &= 0.\end{aligned}$$

A heuristic explanation for this comes from the geometry of the flow: this vectorfield represents a "pure" rotation about the z-axis, and rotating a body does not change its volume. In fact, the same is true of the "screwlike" (technically, *helical*) motion associated to the vector field obtained by adding a constant field to the vector field above. In fact, the vector fields which we use to represent the "infinitesimal" rotation induced by a flow—in other words, the vector fields which are themselves the curl of some other vector field—all have zero divergence. This is an easy if somewhat cumbersome calculation which we leave to you (Exercise 7):

Remark 5.8.3. *Every curl is **divergence-free**: if*

$$\vec{F} = \vec{\nabla} \times \vec{G}$$

for some $\mathcal{C}^2$ vector field $\vec{G}$, then

$$\operatorname{div} \vec{F} = 0.$$

Using our heuristic above, if the velocity vector field of a fluid is divergence-free, this means that the fluid has a kind of rigidity: the volume of a moving "blob" of the fluid neither increases nor decreases with the flow: such a fluid flow is called **incompressible**. A physical example is water, by contrast with a gas, which is highly compressible. [16]

This result has a converse:

Proposition 5.8.4. *A $\mathcal{C}^1$ vector field whose divergence vanishes on a simply-connected region $D \subset \mathbb{R}^3$ is the curl of some other vector field in D. That is, if*

$$\vec{F}(x,y,z) = P(x,y,z)\vec{\imath} + Q(x,y,z)\vec{\jmath} + R(x,y,z)\vec{k}$$

satisfies

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0$$

then there exists a $\mathcal{C}^2$ vector field

$$\vec{G}(x,y,z) = g_1(x,y,z)\vec{\imath} + g_2(x,y,z)\vec{\jmath} + g_3(x,y,z)\vec{k}$$

such that $\vec{F} = \vec{\nabla} \times \vec{G}$—that is,

$$\frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial z} = P \tag{5.30}$$

$$\frac{\partial g_1}{\partial z} - \frac{\partial g_3}{\partial x} = Q \tag{5.31}$$

$$\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y} = R. \tag{5.32}$$

A proof of this converse statement is outlined in Exercises 11-13. We call the vector field $\vec{G}$ a **vector potential** for $\vec{F}$ if $\vec{F} = \vec{\nabla} \times \vec{G}$. There is also a theorem, attributed to Hermann von Helmholtz (1821-1894), which states that every vector field can be written as the sum of a conservative vector field and a divergence-free one: this is called the **Helmholtz decomposition**. A proof of this is beyond the scope of this book.

The Divergence Theorem. Recall that a region $\mathfrak{D} \subset \mathbb{R}^3$ is z-regular if we can express it as the region between two graphs of z as a continuous function of x and y, in other words if we can specify $\mathfrak{D}$ by an inequality of the form

$$\varphi(x,y) \leq z \leq \psi(x,y), \quad (x,y) \in \mathcal{D},$$

where $\mathcal{D}$ is some elementary region in $\mathbb{R}^2$; the analogous notions of y-regular and x-regular regions $\mathfrak{D}$ are fairly clear. We shall call $\mathfrak{D} \subset \mathbb{R}^3$ **fully regular** if it is simultaneously regular in all three directions, with the further proviso that the graphs $z = \varphi(x,y)$ and $z = \psi(x,y)$ (and their analogues for the conditions of x- and y-regularity) are both regular surfaces. This ensures that we can take flux integrals across the faces of

[16] A divergence-free vectorfield is also sometimes referred to as a **solenoidal** vector field.

the region. We shall always assume that our region is regular, so that the boundary is piecewise regular; for this theorem we orient the boundary *outward*.

Theorem 5.8.5 (Divergence Theorem)**.** [17] *Suppose*

$$\vec{F}(x,y,z) = P(x,y,z)\vec{\imath} + Q(x,y,z)\vec{\jmath} + R(x,y,z)\vec{k}$$

is a $\mathcal{C}^1$ vector field on the regular region $\mathfrak{D} \subset \mathbb{R}^3$.

Then the flux integral of $\vec{F}$ over the boundary $\partial\mathfrak{D}$, oriented outward, equals the (triple) integral of its divergence over the interior of $\mathfrak{D}$:

$$\iint_{\partial\mathfrak{D}} \vec{F} \cdot d\vec{S} = \iiint_{\mathfrak{D}} \operatorname{div}\vec{F}\, dV. \tag{5.33}$$

The proof of this theorem is given in Appendix A.12

We note that, as in the case of Green's Theorem, we can extend the Divergence Theorem to any region which can be partitioned into regular regions.

It can also be extended to regular regions with "holes" that are themselves regular regions. For example, suppose $\mathfrak{D}$ is a regular region and $B_\varepsilon\left(\overrightarrow{x_0}\right)$ is a ball of radius $\varepsilon > 0$ centered at $\overrightarrow{x_0}$ and contained in the interior of $\mathfrak{D}$ (that is, it is inside $\mathfrak{D}$ and disjoint from its boundary). Then the region $\mathfrak{D} \setminus B_\varepsilon\left(\overrightarrow{x_0}\right)$ consisting of points in $\mathfrak{D}$ but at distance at least ε from $\overrightarrow{x_0}$ is "$\mathfrak{D}$ with a hole at $\overrightarrow{x_0}$"; it has two boundary components: one is $\partial\mathfrak{D}$, oriented outward, and the other is the sphere of radius ε centered at $\overrightarrow{x_0}$, and oriented *into* the ball. Suppose for a moment that F is defined inside the ball, as well. Then the flux integral over the boundary of $\mathfrak{D} \setminus B_\varepsilon\left(\overrightarrow{x_0}\right)$ is the flux integral over the boundary of $\mathfrak{D}$, oriented outward, *minus* the flux integral over the boundary of the ball (also oriented outward). The latter is the integral of $\operatorname{div}\vec{F}$ over the ball, so it follows that the flux integral over the boundary of $\mathfrak{D} \setminus B_\varepsilon\left(\overrightarrow{x_0}\right)$ is the integral of $\operatorname{div}\vec{F}$ over its interior. Now, this last integral is independent of what F does inside the hole, provided it is $\mathcal{C}^1$ and agrees with the given value along the boundary. Any $\mathcal{C}^1$ vector field F defined on and outside the sphere can be extended to its interior (Exercise 14), so we have

Corollary 5.8.6. *If the ball $B_\varepsilon\left(\overrightarrow{x_0}\right)$ is interior to the regular region $\mathfrak{D}$, then the flux integral of a $\mathcal{C}^1$ vector field $\vec{F}$ over the* boundary *of $\mathfrak{D}$ with a hole $\iint_{\partial(\mathfrak{D}\setminus B_\varepsilon(\overrightarrow{x_0}))} \vec{F} \cdot d\vec{S}$ equals the integral of* $\operatorname{div}\vec{F}$ *over the interior of $\mathfrak{D} \setminus B_\varepsilon\left(\overrightarrow{x_0}\right)$:*

$$\iint_{\partial(\mathfrak{D}\setminus B_\varepsilon(\overrightarrow{x_0}))} \vec{F} \cdot d\vec{S} = \iiint_{\mathfrak{D}\setminus B_\varepsilon(\overrightarrow{x_0})} \operatorname{div}\vec{F}\, dV. \tag{5.34}$$

In particular, if $\vec{F}$ s divergence-free in $\mathfrak{D} \setminus B_\varepsilon\left(\overrightarrow{x_0}\right)$ then the outward flux of $\vec{F}$ over $\partial(\mathfrak{D} \setminus B_\varepsilon\left(\overrightarrow{x_0}\right))$ equals the outward flux of $\vec{F}$ over the sphere of radius ε centered at $\overrightarrow{x_0}$.

Like Stokes' Theorem, the Divergence Theorem allows us to compute the same integral two different ways. We consider a few examples.

First, let us calculate directly the flux of the vector field

$$\vec{F}(x,y,z) = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$$

out of the sphere $\mathfrak{S}$ of radius R about the origin.

---

[17]This theorem was published by Carl Friedrich Gauss (1777-1855) in 1838 [17] and independently by Mikhail Vasilevich Ostrogradski (1801-1862) in 1831. (see [1]). It is often called *Gauss's Theorem* or the *Gauss-Ostrogradsky Theorem.*

The natural parametrization of this sphere uses spherical coordinates: $x = R\sin\phi\cos\theta$, $y = R\sin\phi\sin\theta$, $z = R\cos\phi$, for $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$. The partials are $\frac{\partial\vec{p}}{\partial\phi} = (R\cos\phi\cos\theta)\vec{\imath} + (R\cos\phi\sin\theta)\vec{\jmath} - (R\sin\phi)\vec{k}$ and $\frac{\partial\vec{p}}{\partial\theta} = (-R\sin\phi\sin\theta)\vec{\imath} + (R\sin\phi\cos\theta)\vec{\jmath}$, with cross product

$$\frac{\partial\vec{p}}{\partial\phi} \times \frac{\partial\vec{p}}{\partial\theta} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ R\cos\phi\cos\theta & R\cos\phi\sin\theta & -R\sin\phi \\ -R\sin\phi\sin\theta & R\sin\phi\cos\theta & 0 \end{vmatrix}$$
$$= (R^2\sin^2\phi\cos\theta)\vec{\imath} + (R^2\sin^2\phi\sin\theta)\vec{\jmath} + (R^2\sin\phi\cos\phi)\vec{k}.$$

To check whether this gives the outward orientation, we compute the direction of this vector at a point where it is easy to find, for example at $(1,0,0) = \vec{p}\left(\frac{\pi}{2},0\right)$:

$$\left(\frac{\partial\vec{p}}{\partial\phi} \times \frac{\partial\vec{p}}{\partial\theta}\right)\left(\frac{\pi}{2},0\right) = R^2\vec{\imath}$$

which points out of the sphere at $(1,0,0)$. Thus, the element of outward oriented surface area is

$$d\vec{\mathcal{S}} = \left((R^2\sin^2\phi\cos\theta)\vec{\imath} + (R^2\sin^2\phi\sin\theta)\vec{\jmath} + (R^2\sin\phi\cos\phi)\vec{k}\right)d\phi\,d\theta.$$

On the surface, the vector field is $\vec{F}(\phi,\theta) = (R\sin\phi\cos\theta)\vec{\imath} + (R\sin\phi\sin\theta)\vec{\jmath} + (R\cos\phi)\vec{k}$, so

$$\begin{aligned}\vec{F}\cdot d\vec{\mathcal{S}} &= \Big((R\sin\phi\cos\theta)(R^2\sin^2\phi\cos\theta) + (R\sin\phi\sin\theta)(R^2\sin^2\phi\sin\theta) \\ &\qquad + (R\cos\phi)(R^2\sin\phi\cos\phi)\Big)\,d\phi\,d\theta \\ &= R^3(\sin^3\phi\cos^2\theta + \sin^3\phi\sin^2\theta + \sin\phi\cos^2\phi)\,d\phi\,d\theta \\ &= R^3\sin\phi(\sin^2\phi + \cos^2\phi)\,d\phi\,d\theta = R^3\sin\phi\,d\phi\,d\theta.\end{aligned}$$

The flux integral is therefore

$$\begin{aligned}\iint_{\mathfrak{S}} \vec{F}\cdot d\vec{\mathcal{S}} &= \int_0^{2\pi}\int_0^{\pi} R^3\sin\phi\,d\phi\,d\theta = \int_0^{2\pi} \left(-R^3\cos\phi\right)_{\phi=0}^{\pi}\,d\theta \\ &= 2R^3\int_0^{2\pi} d\theta = 2R^3(2\pi) = 4\pi R^3.\end{aligned}$$

Now let us see how the same calculation looks using the Divergence Theorem. The divergence of our vector field is $\operatorname{div}\vec{F} = 1+1+1 = 3$, so the Divergence Theorem tells us that

$$\iint_{\mathfrak{S}} \vec{F}\cdot d\vec{\mathcal{S}} = \iiint_{\mathfrak{D}} \operatorname{div}\vec{F}\,dV = \iiint_{\mathfrak{D}} 3\,dV = 3\mathcal{V}(\mathfrak{D}),$$

where $\mathfrak{D}$ is the sphere of radius R, with volume $\frac{4\pi R^3}{3}$, and our integral is

$$\iint_{\mathfrak{S}} \vec{F}\cdot d\vec{\mathcal{S}} = 4\pi R^3.$$

As another example, let us calculate the flux of the vector field $\vec{F}(x,y,z) = x^2\vec{\imath} + y^2\vec{\jmath} + z^2\vec{k}$ over the same surface. We have already calculated that the element of outward oriented surface area is

$$d\vec{\mathcal{S}} = \left((R^2 \sin^2\phi \cos\theta)\vec{\imath} + (R^2 \sin^2 \sin\theta)\vec{\jmath} + (R^2 \sin\phi \cos\phi)\vec{k}\right) d\phi\, d\theta.$$

This time, our vector field on the surface is

$$\vec{F}(\phi,\theta) = (R^2 \sin^2\phi \cos^2\theta)\vec{\imath} + (R^2 \sin^2\phi \sin^2\theta)\vec{\jmath} + (R^2 \cos^2\phi)\vec{k}$$

and its dot product with $d\vec{\mathcal{S}}$ is

$$\begin{aligned}
\vec{F}\cdot d\vec{\mathcal{S}} = \Big((R^2 \sin^2\phi \cos^2\theta)(R^2 \sin^2\phi \cos\theta) &+ (R^2 \sin^2\phi \sin^2\theta)(R^2 \sin^2 \sin\theta) \\
&+ (R^2 \cos^2\phi)(R^2 \sin\phi\cos\phi)\Big)\, d\phi\, d\theta \\
&= R^4\left(\sin^4\phi \cos^3\theta + \sin^4\phi \sin^3\theta + \sin\phi\cos^3\phi\right) d\phi\, d\theta \\
&= R^4\Big(\frac{1}{4}\left(1 - 2\cos 2\phi + \cos^2\phi\right)(\cos^3\theta + \sin^3\theta) + \sin\phi\cos^3\phi\Big)\, d\phi\, d\theta \\
&= R^4\Big(\Big(\frac{3}{8} - \frac{1}{2}\cos 2\phi + \frac{1}{8}\cos 4\phi\Big)\Big)(\cos^3\theta + \sin^3\theta) + \sin\phi\cos^3\phi\, d\phi\, d\theta
\end{aligned}$$

and our flux integral is

$$\begin{aligned}
&\iint_{\mathfrak{S}} \vec{F}\cdot d\vec{\mathcal{S}} \\
&= \int_0^{2\pi}\int_0^{\pi} R^4\Big(\Big(\frac{3}{8} - \frac{1}{2}\cos 2\phi + \frac{1}{8}\cos 4\phi\Big)\Big)(\cos^3\theta + \sin^3\theta) + \sin\phi\cos^{3\phi}\ d\phi\, d\theta \\
&= R^4\int_0^{2\pi}\Big(\Big(\frac{3}{8}\theta - \frac{1}{4}\sin 2\phi + \frac{1}{32}\sin 4\phi\Big)(\cos^3\theta + \sin^3\theta) - \frac{1}{4}\cos^4\phi\Big)_{\phi=0}^{\pi} d\phi\, d\theta \\
&\qquad = R^4\int_0^{2\pi}\left(\frac{3\pi}{8}\right)(\cos^3\theta + \sin^3\theta)\cos^3 + \sin^3)\, d\theta \\
&\qquad = \frac{3\pi}{8}R^4\int_0^{2\pi}\left((1-\sin^2\theta)\cos\theta + (1-\sin^2\theta)\cos\theta\right) d\theta \\
&\qquad\qquad = \frac{3\pi}{8}R^4\Big(\sin\theta - \frac{1}{3}\sin^3\theta - \cos\theta + \frac{1}{3}\cos^3\theta\Big)_0^{2\pi} = 0.
\end{aligned}$$

Now if we use the Divergence Theorem instead, we see that the divergence of our vector field is $\operatorname{div}\vec{F} = 2x+2y+2z$ so by the Divergence Theorem $\iint_{\mathfrak{S}} \vec{F}\cdot d\vec{\mathcal{S}} = \iiint_{\mathfrak{D}} \operatorname{div}\vec{F}\ dV = \iiint_{\mathfrak{D}} 2(x+y+z)\ dV$, which is easier to do in spherical coordinates:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{R} 2(\rho\sin\phi\cos\theta+\rho\sin\phi\sin\theta+\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$
$$=\int_0^{2\pi}\int_0^{\pi}\int_0^{R} 2\rho^3\sin\phi(\sin\phi\cos\theta+\sin\phi\sin\theta+\cos\phi)\,d\rho\,d\phi\,d\theta$$
$$=\int_0^{2\pi}\int_0^{\pi}\left(\frac{\rho^4}{4}\right)_0^R\sin\phi(\sin\phi\cos\theta+\sin\phi\sin\theta+\cos\phi)\,d\rho\,d\phi\,d\theta$$
$$=\frac{R^4}{4}\int_0^{2\pi}\int_0^{\pi}\left(\sin^2\phi(\cos\theta+\sin\theta)+\sin\phi\cos\phi\right)d\phi\,d\theta$$
$$=\frac{R^4}{4}\int_0^{2\pi}\int_0^{\pi}\left(\frac{1}{2}(1-\cos2\phi)(\cos\theta+\sin\theta)+\sin\phi\cos\phi\right)d\theta$$
$$=\frac{R^4}{4}\int_0^{2\pi}\left(\left(\frac{\phi}{2}-\frac{1}{2}\sin2\phi\right)(\cos\theta+\sin\theta)+\frac{1}{2}\sin^2\phi\right)_{\phi=0}^{\pi}d\theta$$
$$=\frac{R^4}{4}\int_0^{2\pi}\left(\frac{\pi}{2}(\cos\theta+\sin\theta)+0\right)d\theta$$
$$=\frac{\pi R^4}{8}\left(\sin\theta-\cos\theta\right)_0^{2\pi}=0.$$

We note in passing that this triple integral could have been predicted to equal zero on the basis of symmetry considerations. Recall that the integral of an odd function of one real variable $f(t)$ (*i.e.*, if $f(-t)=-f(t)$) over a symmetric interval $[-a,a]$ is zero. We call a region $\mathfrak{D}\subset\mathbb{R}^3$ **symmetric in** z if it is unchanged by reflection across the xy-plane, that is, if whenever the point (x,y,z) belongs to $\mathfrak{D}$, so does $(x,y,-z)$. (The adaptation of this definition to symmetry in x or in y is left to you in Exercise 10.) We say that a function $f(x,y,z)$ is **odd in** z (*resp.* **even in** z) if reversing the sign of z but leaving x and y unchanged reverses the sign of f (*resp.* does not change f): for odd, this means $f(x,y,-z)=-f(x,y,z)$ while for even it means $f(x,y,-z)=f(x,y,z)$.

Remark 5.8.7. *If $f(x,y,z)$ is odd in z and $\mathfrak{D}$ is z-regular and symmetric in z, then*

$$\iiint_{\mathfrak{D}} f(x,y,z)\;dV=0.$$

(To see this, just set up the triple integral and look at the innermost integral.)

Recall that one of the useful consequences of the Fundamental Theorem for Line Integrals was that the line integral of a conservative vector field depends only on the endpoints of the curve, not on the curve itself; more generally, if the curl of a vector field is zero in a region, then the line integral of the field over a curve is not changed if we deform it within that region, holding the endpoints fixed. A similar use can be made of the Divergence Test. We illustrate with an example.

Let us find the flux integral over $\mathfrak{S}$ the upper hemisphere $z=\sqrt{1-x^2-y^2}$, oriented up, of the vector field $\vec{F}(x,y,z)=(1+z)e^{y^2}\vec{\imath}-(z+1)e^{x^2}\vec{\jmath}+(x^2+y^2)\vec{k}$. Whether we think of the hemisphere as the graph of a function or parametrize it using spherical coordinates, the terms involving exponentials of squares are serious trouble. However, note that this vector field is divergence-free:

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}\left[(1+z)e^{y^2}\right] + \frac{\partial}{\partial y}\left[(z+1)e^{x^2}\right] + \frac{\partial}{\partial z}\left[(x^2+y^2)\right] = 0.$$

Thus, if we consider the half-ball $\mathfrak{D}$ bounded above by the hemisphere and below by the unit disc in the xy-plane, the Divergence Theorem tells us that

$$\iint_{\partial\mathfrak{D}} \vec{F}\cdot d\vec{\mathcal{S}} = \iiint_{\mathfrak{D}} \operatorname{div}\vec{F}\, dV = 0.$$

Now, the boundary of $\mathfrak{D}$ consists of two parts: the hemisphere, $\mathfrak{S}$, and the disc, $\mathcal{D}$. The outward orientation on $\partial\mathfrak{D}$ means an upward orientation on the hemisphere $\mathfrak{S}$, but a *downward* orientation on the disc $\mathcal{D}$. Thus, the flux integral over the whole boundary equals the flux integral over the upward-oriented hemisphere, plus the flux integral over the downward-oriented disc—which is to say, *minus* the flux over the *upward*-oriented disc. Since the difference of the two upward-oriented discs equals zero, they are equal. Thus

$$\iint_{\mathfrak{S}} \vec{F}\cdot d\vec{\mathcal{S}} = \iint_{\mathcal{D}} \vec{F}\cdot d\vec{\mathcal{S}}.$$

But on $\mathcal{D}$, $d\vec{\mathcal{S}} = \vec{k}\,dA$, so substituting $z=0$ we see that the vector field on the disc is $\vec{F}(x,y) = e^{y^2}\vec{\imath} - e^{x^2}\vec{\jmath} + (x^2+y^2)\vec{k}$ and $\vec{F}\cdot d\vec{\mathcal{S}} = (x^2+y^2)\,dA$. This is easy to integrate, especially when we use polar coordinates:

$$\iint_{\mathcal{D}} \vec{F}\cdot d\vec{\mathcal{S}} = \int_0^{2\pi}\int_0^1 (r^2)(r\,dr\,d\theta) = \int_0^{2\pi} (\frac{r^4}{4})\,d\theta = \int_0^{2\pi} \frac{1}{4}\,d\theta = \frac{\pi}{2}.$$

Exercises for § 5.8

Answers to Exercises 1a, 2a, 3a, and 5a are given in Appendix A.13.

Practice problems:

(1) Use Green's Theorem to calculate the integral $\int_{\mathcal{C}} \vec{F}\cdot\vec{N}\,d\mathfrak{s}$, where $\vec{N}$ is the outward unit normal and $\mathcal{C}$ is the ellipse $x^2+4y^2=4$, traversed counterclockwise:
   (a) $\vec{F}(x,y) = x\vec{\imath} + y\vec{\jmath}$
   (b) $\vec{F}(x,y) = y\vec{\imath} + x\vec{\jmath}$
   (c) $\vec{F}(x,y) = x^2\vec{\imath} + y^2\vec{\jmath}$
   (d) $\vec{F}(x,y) = x^3\vec{\imath} + y^3\vec{\jmath}$

(2) Find the flux integral $\iint_{\mathfrak{S}} \vec{F}\cdot d\vec{\mathcal{S}}$, where $\mathfrak{S}$ is the unit sphere oriented outward, for each vector field below:
   (a) $\vec{F}(x,y,z) = (x+y^2)\vec{\imath} + (y-z^2)\vec{\jmath} + (x+z)\vec{k}$
   (b) $\vec{F}(x,y,z) = (x^3+y^3)\vec{\imath} + (y^3+z^3)\vec{\jmath} + (z^3-x^3)\vec{k}$
   (c) $\vec{F}(x,y,z) = 2xz\vec{\imath} + y^2\vec{\jmath} + xz\vec{k}$
   (d) $\vec{F}(x,y,z) = 3xz^2\vec{\imath} + y^3\vec{\jmath} + 3x^2z\vec{k}$

(3) For each vector field $\vec{F}$ below, find the flux integral $\iint_{\mathfrak{S}} \vec{F}\cdot d\vec{\mathcal{S}}$, where $\mathfrak{S}$ is the boundary of the unit cube $[0,1]\times[0,1]\times[0,1]$, oriented outward, in two different ways: (i) directly (you will need to integrate over each face separately and then add up the results) and (ii) using the Divergence Theorem.
   (a) $\vec{F}(x,y,z) = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$
   (b) $\vec{F}(x,y,z) = \vec{\imath} + \vec{\jmath} + \vec{k}$
   (c) $\vec{F}(x,y,z) = x^2\vec{\imath} + y^2\vec{\jmath} + z^2\vec{k}$

(4) Find the flux of the vector field $\vec{F}(x, y, z) = 10x^3y^2\vec{\imath} + 3y^5\vec{\jmath} + 15x^4z\vec{k}$ over the outward-oriented boundary of the solid cylinder $x^2 + y^2 \leq 1, 0 \leq z \leq 1$.

(5) Find the flux integral $\iint_{\mathfrak{S}} \vec{F} \cdot d\vec{S}$ for the vector field $\vec{F}(x, y, z) = yz\vec{\imath} + x\vec{\jmath} + xz\vec{k}$ over the boundary of each region below:
   (a) $x^2 + y^2 \leq z \leq 1$
   (b) $x^2 + y^2 \leq z \leq 1$ and $x \geq 0$
   (c) $x^2 + y^2 \leq z \leq 1$ and $x \leq 0$.

(6) Calculate the flux of the vector field $\vec{F}(x, y, z) = 5yz\vec{\imath} + 12xz\vec{\jmath} + 16x^2y^2\vec{k}$ over the surface of the cone $z^2 = x^2 + y^2$ above the xy-plane and below the plane $z = 1$.

Theory problems:

(7) Prove Remark 5.8.3. (*Hint:* Start with an expression for $\vec{G}$, calculate its curl, then take the divergence of that.)

(8) Fill in the details of the argument for P and Q needed to complete the proof of Theorem 5.8.5.

(9) Extend the heuristic argument given on p. 318 to argue that the divergence of a vector field in $\mathbb{R}^3$ reflects the tendency of a fluid flow to increase volumes.

(10) (a) Formulate a definition of what it means for a region $\mathfrak{D} \subset \mathbb{R}^3$ to be symmetric in x (*resp.* in y).
   (b) Formulate a definition of what it means for a function $f(x, y, z)$ to be even, or odd, in x (*resp.* in y).
   (c) Prove that if a function $f(x, y, z)$ is odd in x then its integral over a region which is x-regular and symmetric in x is zero.
   (d) What can you say about $\iiint_{\mathfrak{D}} f(x, y, z)\ dV$ if f is *even* in x and $\mathfrak{D}$ is x-regular and symmetric in x?

Challenge Problems:

In Exercises 11-13, you will prove Proposition 5.8.4, that every divergence-free vector field $\vec{F}$ is the curl of some vector field $\vec{G}$, by a direct construction based on [56] and [37, p. 560]. Each step will be illustrated by the example $\vec{F}(x, y, z) = yz\vec{\imath} + xz\vec{\jmath} + xy\vec{k}$.

(11) (a) Given a continuous function $\phi(x, y, z)$, show how to construct a vector field whose divergence is ϕ. (*Hint:* This can even be done with a vector field parallel to a predetermined coordinate axis.)
   (b) Given a continuous function $\varphi(x, y)$, show how to construct a *planar* vector field $\vec{G}(x, y) = g_1(x, y)\vec{\imath} + g_2(x, y)\vec{\jmath}$ whose planar curl equals φ. (*Hint:* Consider the divergence of the related vector field $\vec{G}'^{\perp}(x, y) = g_2(x, y)\vec{\imath} - g_1(x, y)\vec{\jmath}$.)
   (c) Construct a planar vector field $\vec{G}(x, y) = g_1(x, y)\vec{\imath} + g_2(x, y)\vec{\jmath}$ with planar curl
$$\frac{\partial g_2}{\partial x}(x, y) - \frac{\partial g_1}{\partial y}(x, y) = xy.$$

(12) Note that in this problem, we deal with horizontal vector fields in $\mathbb{R}^3$.
   (a) Show that the curl of a horizontal vector field $\vec{G}(x, y, z) = g_1(x, y, z)\vec{\imath} + g_2(x, y, z)\vec{\jmath}$ is determined by the planar curl of its restriction to each horizontal plane together with the derivatives of its components with respect

to z:
$$\vec{\nabla}\times\vec{G} = -\frac{\partial g_2}{\partial z}\vec{\imath} + \frac{\partial g_2}{\partial z}\vec{\jmath} + \left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial x}\right)\vec{k}.$$

(b) Construct a horizontal vector field whose restriction to the xy-plane agrees with your solution to Exercise 11c—that is, such that $\frac{\partial g_2}{\partial x}(x,y,0) - \frac{\partial g_1}{\partial y}(x,y,0) = xy$ which also satisfies $\frac{\partial g_1}{\partial z}(x,y,z) = xz$ and $\frac{\partial g_2}{\partial z}(x,y,z) = -yz$ at all points (x,y,z). Verify that the resulting vector field $\vec{G}(x,y,z)$ satisfies
$$\vec{\nabla}\times\vec{G} = yz\vec{\imath} + xz\vec{\jmath} + xy\vec{k}.$$

(13) Now suppose that $\vec{F}(x,y,z) = P(x,y,z)\vec{\imath} + Q(x,y,z)\vec{\jmath} + R(x,y,z)\vec{k}$ is any $\mathcal{C}^1$ vector field satisfying $\operatorname{div}\vec{F} = 0$.

Show that if $\vec{G}(x,y,z) = g_1(x,y,z)\vec{\imath} + g_2(x,y,z)\vec{\jmath}$ is a $\mathcal{C}^2$ horizontal vector field satisfying $\frac{\partial g_1}{\partial z}(x,y,z) = Q(x,y,z)$, $\frac{\partial g_2}{\partial z}(x,y,z) = -P(x,y,z)$, and $\frac{\partial g_2}{\partial x}(x,y,0) - \frac{\partial g_1}{\partial y}(x,y,0) = R(x,y,0)$, then
$$\vec{\nabla}\times\vec{G} = \vec{F}$$
by showing that the extension of the third condition off the xy-plane
$$\frac{\partial g_2}{\partial x}(x,y,z) - \frac{\partial g_1}{\partial y}(x,y,z) = R(x,y,z)$$
holds for all z.

(14) **Filling holes:** In this problem, you will show that given a vector field $\vec{F}$ defined and $\mathcal{C}^1$ on a neighborhood of a sphere, there exists a new vector field $\vec{G}$, defined and $\mathcal{C}^1$ on the neighborhood "filled in" to include the ball bounded by the sphere, such that $\vec{F} = \vec{G}$ on the sphere and its exterior. Thus, we can replace $\vec{F}$ with $\vec{G}$ on both sides of Equation (5.34), justifying our argument extending the Divergence Theorem to regions with holes (Corollary 5.8.6).

(a) Suppose $\phi(t)$ is a $\mathcal{C}^1$ function defined on an open interval containing $[a,b]$ satisfying $\phi(a) = 0$, $\phi'(a) = 0$, $\phi(b) = 1$, and $\phi'(b) = 0$.
Show that the function defined for all t by
$$\psi(t) = \begin{cases} 0 & \text{for } t \le a, \\ \phi(t) & \text{for } a \le t \le b, \\ 1 & \text{for } t \ge b \end{cases}$$
is $\mathcal{C}^1$ on the whole real line.

(b) Given $a < b$, find values of α and β such that
$$\phi(t) = \frac{1}{2}(1 - \cos(\alpha t + \beta))$$
satisfies the conditions above.

(c) Given $a < b$ and $\phi(t)$ as above, as well as $f(t)$ defined and $\mathcal{C}^1$ on a neighborhood $(b-\varepsilon, b+\varepsilon)$ of b, show that
$$g(t) = \begin{cases} 0 & \text{for } t < a, \\ \psi(t)\,f(t) & \text{for } a \le t < b+\varepsilon \end{cases}$$
is $\mathcal{C}^1$ on $(-\infty, b+\varepsilon)$.

(d) Given a $\mathcal{C}^1$ vector field $\vec{F}$ on a neighborhood $\mathcal{N}$ of the sphere $\mathcal{S}$ of radius R centered at $\vec{c}$

$$\mathcal{S} = \{\vec{x} \mid (\vec{x}-\vec{c})^2 = R^2\}$$
$$\mathcal{N} = \{\vec{x} \mid R^2 - \varepsilon \le (\vec{x}-\vec{c})^2 \le R^2 + \varepsilon\}$$

(where $(\vec{x}-\vec{c})^2 := (\vec{x}-\vec{c}) \cdot (\vec{x}-\vec{c})$) show that the vector field $\vec{G}$ defined by

$$\vec{G}(\vec{x}) = \begin{cases} \vec{0} & \text{for } (\vec{x}-\vec{c})^2 \le R^2 - \varepsilon, \\ \psi\left((\vec{x}-\vec{c})^2\right)\vec{F}(\vec{x}) & \text{for } \vec{x} \in \mathcal{N} \end{cases}$$

is $\mathcal{C}^1$ on

$$B_R(\vec{c}) \cup \mathcal{N} = \{\vec{x} \mid (\vec{x}-\vec{c})^2 \le R^2 + \varepsilon\}.$$

(e) Sketch how to use this to show that a $\mathcal{C}^1$ vector field defined on a region $\mathfrak{D}$ with holes can be extended to a $\mathcal{C}^1$ vector field on the region with the holes filled in. (You may assume that the vector field is actually defined on a neighborhood of each internal boundary sphere.)

(15) In this problem (based on [31, pp. 362-3]), you will use the Change-of-Variables Formula (Theorem 4.3.4) to show that the divergence of a planar vector field gives the rate of change of area under the associated flow. The analogous three-dimensional proof is slightly more involved; it is given in the work cited above.

We imagine a fluid flow in the plane: the position (u, v) at time t of a point whose position at time $t = 0$ was (x, y) is given by $u = u(x, y, t)$, $v = v(x, y, t)$, or, combining these into a mapping $F : \mathbb{R}^3 \to \mathbb{R}^2$, $(u, v) = F(x, y, t) = F_t(x, y)$, where $F_t(x, y)$is the transformation $F_t : \mathbb{R}^2 \to \mathbb{R}^2$ taking a point located at (x, y) when $t = 0$ to its position at time t; that is, it is the mapping F with t fixed. The velocity of this flow is the vector field $V(u, v) = \left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}\right) = (u', v')$. V may also vary with time, but we will suppress this in our notation.

Let $\mathcal{D} \subset \mathbb{R}^2$ be a regular planar region; we denote the area of its image under F_t as

$$\mathcal{A}(t) = \mathcal{A}(F_t(D));$$

by Theorem 4.3.4, this is

$$\mathcal{A}(t) = \iint_{\mathcal{D}} |J_t| \, dx\, dy,$$

where

$$J_t = JF_t = \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{bmatrix}$$

is the Jacobian matrix of F_t, and

$$|J_t| = \det JF_t = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

is its determinant. (Strictly speaking, we should take the absolute value, but it can be shown that for a continuous flow, this determinant is always positive.)

(a) Show that

$$\frac{d}{dt}[\det J_t] = \frac{\partial u'}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u'}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial v'}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial v'}{\partial x}\frac{\partial u}{\partial y}.$$

(b) Show that

$$\begin{aligned}\operatorname{div} V &:= \frac{\partial u'}{\partial u} + \frac{\partial v'}{\partial v} \\ &= \frac{\partial u'}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial v'}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial v}.\end{aligned}$$

(c) Show that the inverse of JF_t is

$$\begin{aligned}JF_t^{-1} &:= \begin{bmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{bmatrix} \\ &= \frac{1}{|J_t|}\begin{bmatrix} \partial v/\partial y & -\partial u/\partial y \\ -\partial v/\partial x & \partial u/\partial x \end{bmatrix}.\end{aligned}$$

(*Hint:* Use the Chain Rule, and show that the product of this with JF_t is the identity matrix.)

(d) Regarding this matrix equation as four equations (between corresponding entries of the two matrices), substitute into the previous formulas to show that

$$\operatorname{div} V = \frac{1}{|J_t|}\frac{d}{dt}\left[|J_t|\right].$$

(e) Use this to show that

$$\frac{d}{dt}\left[\mathcal{A}(t)\right] = \iint_{F_t(\mathcal{D})} \operatorname{div} V \, du\, dv.$$

5.9 3-forms and the Generalized Stokes Theorem (Optional)

Multilinear Algebra. In §§5.4 and 5.7 we encountered the notion of a *bilinear function*: a function of two vector variables which is "linear in each slot": it is linear as a function of one of the variable when the other is held fixed. This has a natural extension to more vector variables:

Definition 5.9.1. *A* ***trilinear function*** *on $\mathbb{R}^3$ is a function $T(\vec{x}, \vec{y}, \vec{z})$ of three vector variables $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^3$ such that fixing the values of two of the variables results in a linear function of the third: given $\vec{a}, \vec{b}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and $\alpha, \beta \in \mathbb{R}$,*

$$\begin{aligned}T\left(\alpha\vec{v} + \beta\vec{w}, \vec{a}, \vec{b}\right) &= \alpha T\left(\vec{v}, \vec{a}, \vec{b}\right) + \beta T\left(\vec{w}, \vec{a}, \vec{b}\right) \\ T\left(\vec{a}, \alpha\vec{v} + \beta\vec{w}, \vec{b}\right) &= \alpha T\left(\vec{a}, \vec{v}, \vec{b}\right) + \beta T\left(\vec{a}, \vec{w}, \vec{b}\right) \\ T\left(\vec{a}, \vec{b}, \alpha\vec{v} + \beta\vec{w}\right) &= \alpha T\left(\vec{a}, \vec{b}, \vec{v}\right) + \beta T\left(\vec{a}, \vec{b}, \vec{w}\right).\end{aligned}$$

As is the case for linear and bilinear functions, knowing what a trilinear function does when all the inputs are basis vectors lets us determine what it does to any inputs. This is most easily expressed using indexed notation: Let us write

$$\vec{\imath} = \vec{e}_1, \quad \vec{\jmath} = \vec{e}_2, \quad \vec{k} = \vec{e}_3$$

and for each triple of indices $i_1, i_2, i_3 \in \mathbb{R}^3$

$$c_{i_1,i_2,i_3} := T\left(e_{i_1}, e_{i_2}, e_{i_3}\right).$$

Then the function T can be expressed as a homogeneous degree three polynomial in the components of its inputs as follows: for $\vec{x} = x_1\vec{\imath} + x_2\vec{\jmath} + x_3\vec{k} = \sum_{i=1}^{3} x_i\vec{e_i}$, $\vec{y} = y_1\vec{\imath} + y_2\vec{\jmath} + y_3\vec{k} = \sum_{j=1}^{3} y_j\vec{e_j}$, and $\vec{z} = z_1\vec{\imath} + z_2\vec{\jmath} + z_3\vec{k} = \sum_{k=1}^{3} z_k\vec{e_k}$, we have

$$T\left(\vec{x},\vec{y},\vec{z}\right) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} c_{ijk}x_iy_jz_k. \tag{5.35}$$

This can be proved by a tedious but straightforward calculation (Exercise 6).

Unfortunately, there is no nice trilinear analogue to the matrix representation of a bilinear function. However, we are not interested in *arbitrary* trilinear functions, only the ones satisfying the following additional condition, the appropriate extension of anti-commutativity:

Definition 5.9.2. *A trilinear function $T\left(\vec{x},\vec{y},\vec{z}\right)$ is* ***alternating*** *if interchanging any pair of inputs reverses the sign of the function:*

$$\begin{aligned}
T\left(\vec{y},\vec{x},\vec{z}\right) &= -T\left(\vec{x},\vec{y},\vec{z}\right)\\
T\left(\vec{x},\vec{z},\vec{y}\right) &= -T\left(\vec{x},\vec{y},\vec{z}\right)\\
T\left(\vec{z},\vec{y},\vec{x}\right) &= -T\left(\vec{x},\vec{y},\vec{z}\right).
\end{aligned}$$

A 3-***form*** *on $\mathbb{R}^3$ is an alternating trilinear function on $\mathbb{R}^3$.*

Several properties follow immediately from these definitions (Exercise 7):

Remark 5.9.3. *If the trilinear function $T\left(\vec{x},\vec{y},\vec{z}\right)$ is alternating, the coefficients c_{ijk} in Equation* (5.35) *satisfy:*

(1) *If any pair of indices is equal, then $c_{ijk} = 0$;*
(2) *The six coefficients with distinct indices are equal up to sign; more precisely,*

$$\begin{aligned}
c_{123} &= c_{231} = c_{312}\\
c_{132} &= c_{321} = c_{213}
\end{aligned}$$

and the coefficients in each list are the negatives of those in the other.

In particular, every 3*-form on $\mathbb{R}^3$ is a constant multiple of the determinant*

$$T\left(\vec{x},\vec{y},\vec{z}\right) = c\Delta\left(\vec{x},\vec{y},\vec{z}\right) = c\det\begin{vmatrix} x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\\ z_1 & z_2 & z_3\end{vmatrix},$$

where c is the common value of the coefficients in the first list above.

Regarded as a 3-form, the determinant in this remark assigns to three vectors in $\mathbb{R}^3$ the oriented volume of the parallelepiped they determine; we will refer to this as the **volume form** on $\mathbb{R}^3$, and denote it by

$$dx \wedge dy \wedge dz \coloneqq \Delta.$$

We then consider any such formal “triple wedge product” of dx, dy, and dz in another order to be plus or minus the volume form, according to the alternating rule: that is, we posit that swapping neighboring entries in this product reverses its sign, giving us the following list of the six possible wedge products of all three coordinate forms:

$$\begin{aligned}
dx \wedge dy \wedge dz &= -\,dy \wedge dx \wedge dz &&= dy \wedge dz \wedge dx = -\,dz \wedge dy \wedge dx\\
&= dz \wedge dx \wedge dy &&= -\,dz \wedge dx \wedge dy.
\end{aligned}$$

Now, we can define the wedge product of a basic 1-form and a basic 2-form by removing the parentheses and comparing with the list above: for example,

$$dx \wedge (dy \wedge dz) = (dx \wedge dy) \wedge dz = dx \wedge dy \wedge dz$$

and, in keeping with the alternating rule, a product in which the same coordinate form appears twice is automatically zero. Finally, we extend this product to an arbitrary 1-form and an arbitrary 2-form on $\mathbb{R}^3$ by making the product distribute over linear combinations. As an example, if $\alpha = 3\,dx + dy + dz$ and $\beta = dx \wedge dy + 2\,dx \wedge dz + dy \wedge dz$, then

$$\begin{aligned}
\alpha \wedge \beta &= (3\,dx + dy + dz) \wedge (dx \wedge dy + 2\,dx \wedge dz + dy \wedge dz) \\
&= 3\,dx \wedge (dx \wedge dy) + 6\,dx \wedge (dx \wedge dz) + 3\,dx \wedge (dy \wedge dz) \\
&\quad + dy \wedge (dx \wedge dy) + 2\,dy \wedge (dx \wedge dz) + dy \wedge (dy \wedge dz) \\
&\quad + dz \wedge (dx \wedge dy) + 2\,dz \wedge (dx \wedge dz) + dz \wedge (dy \wedge dz) \\
&= 0 + 0 + dx \wedge dy \wedge dz + 0 + 2\,dy \wedge dx \wedge dz + 0 + dz \wedge dx \wedge dy + 0 + 0 \\
&= 3\,dx \wedge dy \wedge dz - 2\,dx \wedge dy \wedge dz + dx \wedge dy \wedge dz \\
&= 2\,dx \wedge dy \wedge dz.
\end{aligned}$$

Calculus of Differential Forms. Now, in the spirit of § 5.7, we can define a **differential 3-form** on a region $\mathfrak{D} \subset \mathbb{R}^3$ to be a mapping Λ which assigns to each point $p \in \mathcal{D}$ a 3-form Λ_p on the tangent space $T_p\mathbb{R}^3$ to $\mathbb{R}^3$ at p. By the discussion above, any such mapping can be expressed as

$$\Lambda_p = c\,(p)\; dx \wedge dy \wedge dz.$$

We can also extend the idea of an exterior derivative to 2-forms: if $\Omega = f\,(x, y, z)$ $dx_1 \wedge dx_2$ (where each of x_i, $i = 1, 2$ is x, y or z), then its **exterior derivative** is the 3-form

$$d\Omega = d\,(f\,(x, y, z)\; dx_1 \wedge dx_2) = df \wedge dx_1 \wedge dx_2.$$

The differential df of f involves three terms, corresponding to the three partial derivatives of f, but two of these lead to triple wedge products in which some coordinate form is repeated, so only one nonzero term emerges. We then extend the definition to general 2-forms using a distributive rule. For example, if $\Omega = (x^2 + xyz)\,dy \wedge dz + (y^2 + 2xyz)\,dz \wedge dx + (z^2 + xyz)\,dx \wedge dy$ then

$$\begin{aligned}
d\Omega &= \big((2x + yz)\,dx + xz\,dy + xy\,dz\big) \wedge dy \wedge dz \\
&\quad + \big(2yz\,dx + (2y + 2xz)\,dy + 2xy\,dz\big) \wedge dz \wedge dx \\
&\quad + \big(yz\,dx + xz\,dy + (2z + xy)\,dz\big) \wedge dx \wedge dy \\
&= (2x + yz)\,dx \wedge dy \wedge dz + 0 + 0 \\
&\quad + 0 + (2y + 2xz)\,dy \wedge dzx + 0 \\
&\quad + 0 + 0 + (2z + xy)\,dz \wedge dxy \\
&= (2x + yz)\,dx \wedge dy \wedge dz + (2y + 2xz)\,dx \wedge dy \wedge dz \\
&\quad + (2z + xy)\,dx \wedge dy \wedge dz \\
&= (2x + 2y + 2z + yz + 2xz + xy)\,dx \wedge dy \wedge dz.
\end{aligned}$$

It is a straightforward calculation to check the following.

Remark 5.9.4. *If the* 2*-form*

$$\Omega_{(x,y,z)} = a\,(x,y,z)\; dy \wedge dz + b\,(x,y,z)\; dz \wedge dx + c\,(x,y,z)\; dx \wedge dy$$

corresponds to the vector field

$$\vec{F}\,(x,y,z) = a\,(x,y,z)\vec{\imath} + b\,(x,y,z)\vec{\jmath} + c\,(x,y,z)\,\vec{k}$$

then its exterior derivative corresponds to the divergence of $\vec{F}$*:*

$$d\Omega = (\operatorname{div}\vec{F})\, dx \wedge dy \wedge dz.$$

Finally, we define the integral of a 3-form Ω over a region $\mathcal{D} \subset \mathbb{R}^3$ by formally identifying the basic volume form with dV: if $\Omega_p = f\,(p)\; dx \wedge dy \wedge dz$ then

$$\int_{\mathcal{D}} \Omega = \iiint_{\mathfrak{D}} f\; dV.$$

Pay attention to the distinction between the 3-*form* $dx \wedge dy \wedge dz$ and the element of volume $dV = dx\,dy\,dz$: changing the order of dx, dy and dz in the 3-form affects the sign of the integral, while changing the order of integration in a triple integral does not. The form is associated to the standard **right-handed orientation** of $\mathbb{R}^3$; the 3-forms obtained by transposing an odd number of the coordinate forms, like $dy \wedge dx \wedge dz$, are associated to the *opposite*, **left-handed orientation** of $\mathbb{R}^3$.

As an example, consider the 3-form $\Omega_{(x,y,z)} = xyz\, dx \wedge dy \wedge dz$; its integral over the "rectangle" $[0,1] \times [0,2] \times [1,2]$ is

$$\int_{[0,1]\times[0,2]\times[1,2]} \Omega = \iiint_{[0,1]\times[0,2]\times[1,2]} xyz\; dV$$

which is given by the triple integral

$$\int_0^1 \int_0^2 \int_1^2 xyz\, dz\, dy\, dx = \int_0^1 \int_0^2 \left(\frac{xyz^2}{2}\right)_{z=1}^{2} dy\, dx$$
$$= \int_0^1 \int_0^2 \left(\frac{3xy}{2}\right) dy\, dx = \int_0^1 \left(\frac{3xy^2}{4}\right)_{y=0}^{2} dx = \int_0^1 3x\, dx = \frac{3x^2}{2}\Big|_0^1 = \frac{3}{2}.$$

Finally, with all these definitions, we can reformulate the Divergence Theorem in the language of forms:

Theorem 5.9.5 (Divergence Theorem, Differential Form)**.** *If* Ω *is a* $\mathcal{C}^2$ 2*-form defined on an open set containing the regular region* $\mathcal{D} \subset \mathbb{R}^3$ *with boundary surface(s)* $\partial\mathcal{D}$*, then the integral of* Ω *over the boundary* $\partial\mathcal{D}$ *of* $\mathcal{D}$ *(with boundary orientation) equals the integral of its exterior derivative over* $\mathcal{D}$*:*

$$\int_{\partial\mathcal{D}} \Omega = \int_{\mathcal{D}} d\Omega.$$

Generalized Stokes Theorem. Looking back at Theorem 5.4.3, Theorem 5.7.8 and Theorem 5.9.5, we see that Green's Theorem, Stokes' Theorem and the Divergence Theorem, which look so different from each other in the language of vector fields (Theorem 5.3.4, Theorem 5.6.2, and Theorem 5.8.5), can all be stated as one unified result in the language of differential forms. To smooth the statement, we will abuse terminology and refer to a region $\mathcal{D} \subset \mathbb{R}^n$ ($n = 2$ or 3) as an "n-dimensional surface in $\mathbb{R}^n$":

Theorem 5.9.6 (Generalized Stokes Theorem). *If $\mathfrak{S}$ is an oriented k-dimensional surface in $\mathbb{R}^n$ ($k \leq n$) with boundary $\partial\mathfrak{S}$ (given the boundary orientation) and Ω is a $\mathcal{C}^2$ $(k-1)$-form on $\mathbb{R}^n$ defined on $\mathfrak{S}$, then*

$$\int_{\partial\mathfrak{S}} \Omega = \int_{\mathfrak{S}} d\Omega.$$

So far we have understood k to be 2 or 3 in the above, but we can also include $k = 1$ by regarding a directed curve as an oriented "1-dimensional surface", and defining a "0-form" to be a function $f : \mathbb{R}^n \to \mathbb{R}$; a "0-dimensional surface" in $\mathbb{R}^n$ to be a point or finite set of points, and an orientation of a point to be simply a sign $\pm$: the "integral" of the 0-form associated to the function f is simply the value of the function at that point, preceded with the sign given by its orientation. Then the boundary of a directed curve in $\mathbb{R}^n$ ($n = 2$ or 3) is its pair of endpoints, oriented as $p_{end} - p_{start}$, and the statement above becomes the Fundamental Theorem for Line Integrals; furthermore, the same formalism gives us the Fundamental Theorem of Calculus when $n = 1$, given that we regard an interval as a "1-dimensional surface" in $\mathbb{R}^1$.

In fact, this statement has a natural interpretation in abstract n-space $\mathbb{R}^n$ (where cross products, and hence the language of vector calculus, do not have a natural extension), and gives a powerful tool for the study of functions and differential equations, as well as the topology of manifolds.

Exercises for § 5.9

Answers to Exercises 1a, 2a, 3a, and 4a are given in Appendix A.13.

Practice problems:

(1) Calculate the exterior product $\alpha \wedge \beta$:
   (a) $\alpha = 3\,dx + 2x\,dy$, $\beta = 2\,dx \wedge dy - dy \wedge dz + x\,dx \wedge dz$
   (b) $\alpha = 3\,dx \wedge dy + 2x\,dy \wedge dz$, $\beta = 2x\,dx - dy + z\,dz$
   (c) $\alpha = x\,dx + y\,dy + z\,dz$, $\beta = dx \wedge dy - 2x\,dy \wedge dz$
   (d) $\alpha = x\,dx \wedge dy + xy\,dy \wedge dz + xyz\,dx \wedge dz$, $\beta = x\,dx - yz\,dy + xy\,dz$
(2) Express the given form as $c(x, y, z)\;dx \wedge dy \wedge dz$:
   (a) $(dx + dy + dz) \wedge (2\,dx - dy + dz) \wedge (dx + dy)$
   (b) $(dx - dy) \wedge (2\,dx + dz) \wedge (dx + dy + dz)$
   (c) $(x\,dy + y\,dz) \wedge d(x^2y\,dy - xz\,dx)$
   (d) $d((x\,dy + y\,dz) \wedge dg)$, where $g(x, y, z) = xyz$.

(3) Calculate the exterior derivative $d\Omega$:
   (a) $\Omega = dx \wedge dy + x\,dy \wedge dz$
   (b) $\Omega = xy\,dx \wedge dy + xz\,dy \wedge dz$
   (c) $\Omega = xyz(dx \wedge dy + dx \wedge dz + dy \wedge dz)$
   (d) $\Omega = (xz - 2y)\,dx \wedge dy + (xy - z^2)\,dx \wedge dz$
(4) Calculate the integral $\int_{\mathfrak{D}} \Lambda$:
   (a) $\Lambda = (xy + yz)\,dx \wedge dy \wedge dz$, $\mathfrak{D} = [0,1] \times [0,1] \times [0,1]$
   (b) $\Lambda = (x - y)\,dx \wedge dy \wedge dz$, $\mathfrak{D}$ is the region cut out of the first octant by the plane $x + y + z = 1$.
   (c) $\Lambda = (x^2 + y^2 + z^2)\,dx \wedge dy \wedge dz$, $\mathfrak{D}$ is the unit ball $x^2 + y^2 + z^2 \leq 1$.
(5) Calculate $\int_{\mathfrak{S}} \Omega$ two ways: (i) directly, and (ii) using the Generalized Stokes Theorem.
   (a) $\Omega = z\,dx \wedge dy$, $\mathfrak{S}$ is the cube with vertices $(0,0,0)$, $(1,0,0)$, $(1,1,0)$, $(0,1,0)$, $(0,0,1)$, $(1,0,1)$, $(1,1,1)$, and$(0,1,1)$, oriented outward.
   (b) $\Omega = x\,dy \wedge dz - y\,dx \wedge dz + z\,dx \wedge dy$, $\mathfrak{S}$ is the sphere $x^2 + y^2 + z^2 = 1$, oriented outward.

Theory problems:

(6) Verify Equation (5.35).
(7) Prove Remark 5.9.3.
(8) Show that the only alternating trilinear function on $\mathbb{R}^2$ is the constant zero function.
(9) Show that if

$$\alpha = P\,dx + Q\,dy + R\,dz$$

is the 1-form corresponding to the vector $\vec{v} = P\vec{\imath} + Q\vec{\jmath} + R\vec{\jmath}$ and

$$\beta = a\,dy \wedge dz + b\,dz \wedge dx + c\,dx \wedge dy$$

is the 2-form corresponding to the vector $\vec{w} = a\vec{\imath} + b\vec{\jmath} + c\vec{k}$, then

$$\alpha \wedge \beta = (\vec{v} \cdot \vec{w})\,dx \wedge dy \wedge dz = \beta \wedge \alpha.$$

Note that, unlike the product of two 1-forms, the wedge product of a 1-form and a 2-form is commutative.
(10) Prove Remark 5.9.4.
(11) Show that if $\Omega = d\omega$ is the exterior derivative of a 1-form ω, then

$$d\Omega = 0.$$

A

Appendix

A.1 Differentiability in the Implicit Function Theorem

In this appendix we complete the proof of Theorem 3.4.2 by showing that the function $y = \phi(x)$ whose graph agrees with the level set of $f(x, y)$ near (x_0, y_0) is differentiable and satisfies Equation (3.20).

Proof of differentiability of $\phi(x)$. We fix $(x, y) = (x, \phi(x))$ in our rectangle and consider another point $(x + \triangle x, y + \triangle y) = (x + \triangle x, \phi(x + \triangle x)$ on the graph of $\phi(x)$.

Since f is differentiable,

$$f(x + \triangle x, y + \triangle y) - f(x, y) = \triangle x \frac{\partial f}{\partial x}(x, y) + \triangle y \frac{\partial f}{\partial y}(x, y) + \left\|(\triangle x, \triangle y)\right\| \varepsilon,$$

where $\varepsilon \to 0$ as $(\triangle x, \triangle y) \to (0, 0)$.

Since both points lie on the graph of $\phi(x)$, and hence on the same level set of f, the left side of this equation is zero:

$$0 = \triangle x \frac{\partial f}{\partial x}(x, y) + \triangle y \frac{\partial f}{\partial y}(x, y) + \left\|(\triangle x, \triangle y)\right\| \varepsilon. \tag{A.1}$$

We will exploit this equation in two ways. For notational convenience, we will drop reference to where a partial is being taken: *for the rest of this proof, $\frac{\partial f}{\partial x}$ is understood to mean $\frac{\partial f}{\partial x}(x, y)$ and $\frac{\partial f}{\partial y}$ means $\frac{\partial f}{\partial y}(x, y)$*, where $\vec{x} = (x, y)$ is the point at which we are trying to prove differentiability of ϕ.

Moving the first two terms to the left side, dividing by $(\triangle x)(\frac{\partial f}{\partial y})$, and taking absolute values, we have

$$\left|\frac{\triangle y}{\triangle x} + \frac{\partial f/\partial x}{\partial f/\partial y}\right| = \frac{|\varepsilon|}{|\partial f/\partial y|} \frac{\left\|(\triangle x, \triangle y)\right\|}{|\triangle x|} \leq \frac{|\varepsilon|}{|\partial f/\partial y|}\left[1 + \left|\frac{\triangle y}{\triangle x}\right|\right] \tag{A.2}$$

(since $\left\|(\triangle x, \triangle y)\right\| \leq |\triangle x| + |\triangle y|$). To complete the proof, we need to find an upper bound for $\left|1 + \frac{\triangle y}{\triangle x}\right|$ on the right side.

To this end, we come back to Equation (A.1), this time moving just the second term to the left, and then take absolute values, using the triangle inequality (and $\left\|(\triangle x, \triangle y)\right\| \leq |\triangle x| + |\triangle y|$):

$$|\triangle y| \left|\frac{\partial f}{\partial y}\right| \leq |\triangle x| \left|\frac{\partial f}{\partial x}\right| + |\varepsilon| \, |\triangle x| + |\varepsilon| \, |\triangle y| \, .$$

Gathering the terms involving $\triangle x$ on the left and those involving $\triangle y$ on the right, we can write

$$|\triangle y|\left(\left|\frac{\partial f}{\partial y}\right| - |\varepsilon|\right) \le |\triangle x|\left(\left|\frac{\partial f}{\partial x}\right| + |\varepsilon|\right)$$

or, dividing by the term on the left,

$$|\triangle y| \le |\triangle x|\left(\frac{|\partial f/\partial x| + |\varepsilon|}{|\partial f/\partial y| - |\varepsilon|}\right). \tag{A.3}$$

Now, since $\varepsilon \to 0$, the ratio on the right converges to the ratio of the partials, and so is bounded–by, say, that ratio plus one–for $\triangle x$ sufficiently near zero:

$$\left|\frac{\triangle y}{\triangle x}\right| \le \left(\left|\frac{\partial f/\partial x}{\partial f/\partial y}\right| + 1\right).$$

This in turn says that the term multiplying $|\varepsilon|$ in Equation (A.2) is bounded, so $\varepsilon \to 0$ implies the desired equation

$$\phi'(x) = \lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x} = -\frac{\partial f/\partial x}{\partial f/\partial y}.$$

This shows that ϕ is differentiable, with partials given by Equation (3.21), and since the right hand side is a continuous function of x, ϕ is *continuously* differentiable. □

A.2 Equality of Mixed Partials

In this appendix we prove Theorem 3.8.1 on the equality of cross partials.

Theorem A.2.1 (Equality of Mixed Partials)**.** *If a real-valued function f of two or three variables is twice continuously differentiable (C^2), then for any pair of indices i, j*

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

Proof. We shall give the proof for a function of two variables; after finishing the proof, we shall note how this actually gives the same conclusion for three variables.

The proof is based on looking at **second-order differences**: given two points (x_0, y_0) and $(x_1, y_1) = (x_0 + \triangle x, y_0 + \triangle y)$, we can go from the first to the second in two steps: increase one of the variables, holding the other fixed, then increase the other variable. This can be done in two ways, depending on which variable we change first; the two paths form the sides of a rectangle with (x_i, y_i), $i = 1, 2$ at opposite corners (Figure A.1). Let us now consider the difference between the values of $f(x, y)$ at the

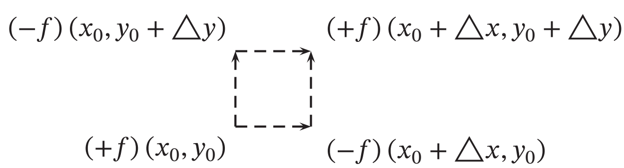


Figure A.1. Second order differences

ends of one of the horizontal edges of the rectangle: the difference along the bottom edge

$$\triangle_x f(y_0) = f(x_0 + \triangle x, y_0) - f(x_0, y_0)$$

represents the change in $f(x,y)$ when y is held at $y = y_0$ and x increases by $\triangle x$ from $x = x_0$, while the difference along the *top* edge

$$\triangle_x f(y_0 + \triangle y) = f(x_0 + \triangle x, y_0 + \triangle y) - f(x_0, y_0 + \triangle y)$$

represents the change in $f(x,y)$ when y is held at $y = y_0 + \triangle y$ and x increases by $\triangle x$ from $x = x_0$. We wish to compare these two changes, by subtracting the first from the second:

$$\begin{aligned}\triangle_y \triangle_x f &= \triangle_x f(y_0 + \triangle y) - \triangle_x f(y_0) \\ &= [f(x_0 + \triangle x, y_0 + \triangle y) - f(x_0, y_0 + \triangle y)] \\ &\quad - [f(x_0 + \triangle x, y_0) - f(x_0, y_0)] \\ &= f(x_0 + \triangle x, y_0 + \triangle y) - f(x_0, y_0 + \triangle y) \\ &\quad - f(x_0 + \triangle x, y_0) + f(x_0, y_0).\end{aligned}$$

(Note that the signs attached to the four values of $f(x,y)$ correspond to the signs in Figure A.1.) Each of the first-order differences $\triangle_x f(y_0)$ (*resp.* $\triangle_x f(y_0 + \triangle y)$) is an approximation to $\frac{\partial f}{\partial x}$ at (x_0, y_0) (*resp.* $(x_0, y_0 + \triangle y)$), multiplied by $\triangle x$; their difference is then an approximation to $\frac{\partial^2 f}{\partial y \partial x}$ at (x_0, y_0), multiplied by $\triangle y \triangle x$; we shall use the Mean Value Theorem to make this claim precisely.

But first consider the *other* way of going: the differences along the two *vertical* edges

$$\begin{aligned}\triangle_y f(x_0) &= f(x_0, y_0 + \triangle y) - f(x_0, y_0) \\ \triangle_y f(x_0 + \triangle x) &= f(x_0 + \triangle x, y_0 + \triangle y) - f(x_0 + \triangle x, y_0)\end{aligned}$$

represent the change in $f(x,y)$ as x is held constant at one of the two values $x = x_0$ (*resp.* $x = x_0 + \triangle x$) and y increases by $\triangle y$ from $y = y_0$; this roughly approximates $\frac{\partial f}{\partial y}$ at (x_0, y_0) (*resp.* $(x_0 + \triangle x, y_0)$), multiplied by $\triangle y$, and so the difference of *these* two differences

$$\begin{aligned}\triangle_x \triangle_y f &= \triangle_y f(x_0 + \triangle x) - \triangle_y f(x_0) \\ &= [f(x_0 + \triangle x, y_0 + \triangle y) - f(x_0 + \triangle x, y_0)] \\ &\quad - [f(x_0, y_0 + \triangle y) - f(x_0, y_0)] \\ &= f(x_0 + \triangle x, y_0 + \triangle y) - f(x_0 + \triangle x, y_0) \\ &\quad - f(x_0, y_0 + \triangle y) + f(x_0, y_0)\end{aligned}$$

approximates $\frac{\partial^2 f}{\partial x \partial y}$ at (x_0, y_0), multiplied by $\triangle x \triangle y$. But a close perusal shows that these two second-order differences are the same—and this will be the punch line of our proof.

Actually, for technical reasons, we don't follow the strategy suggested above precisely. Let's concentrate on the first (second-order) difference: counterintuitively, our goal is to show that

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \lim_{(\triangle x, \triangle y) \to (0,0)} \frac{\triangle_y \triangle_x f}{\triangle y \triangle x}.$$

To this end, momentarily fix $\triangle x$ and $\triangle y$ and define

$$g(t) = \triangle_x f\left(y_0 + t\triangle y\right) = f\left(x_0 + \triangle x, y_0 + t\triangle y\right) - f\left(x_0, y_0 + t\triangle y\right);$$

then

$$g'(t) = \left[\frac{\partial f}{\partial y}\left(x_0 + \triangle x, y_0 + t\triangle y\right) - \frac{\partial f}{\partial y}\left(x_0, y_0 + t\triangle y\right)\right]\triangle y.$$

Now,

$$\triangle_y \triangle_x f = g(1) - g(0)$$

and the Mean Value Theorem applied to $g(t)$ tells us that for some $\tilde{t} \in (0,1)$, this difference is

$$g'(\tilde{t}) = \left[\frac{\partial f}{\partial y}\left(x_0 + \triangle x, y_0 + \tilde{t}\triangle y\right) - \frac{\partial f}{\partial y}\left(x_0, y_0 + \tilde{t}\triangle y\right)\right]\triangle y$$

or, writing $\tilde{y} = y_0 + \tilde{t}\triangle y$, and noting that $\tilde{y}$ lies between y_0 and $y_0 + \triangle y$, we can say that

$$\triangle_y \triangle_x f = \left[\frac{\partial f}{\partial y}\left(x_0 + \triangle x, \tilde{y}\right) - \frac{\partial f}{\partial y}\left(x_0, \tilde{y}\right)\right]\triangle y,$$

where $\tilde{y}$ is some value between y_0 and $y_0 + \triangle y$. $h(t) = \frac{\partial f}{\partial y}\left(x_0 + t\triangle x, \tilde{y}\right)$ with derivative $h'(t) = \frac{\partial^2 f}{\partial x \partial y}(x_0 + t\triangle x, \tilde{y})\triangle x$ so for some $t' \in (0,1)$

$$\left[\frac{\partial f}{\partial y}\left(x_0 + \triangle x, \tilde{y}\right) - \frac{\partial f}{\partial y}\left(x_0, \tilde{y}\right)\right] = h(1) - h(0) = h'(t') = \frac{\partial^2 f}{\partial x \partial y}(x_0 + t'\triangle x, \tilde{y})\triangle x$$

and we can say that

$$\begin{aligned}\triangle_y \triangle_x f &= \left[\frac{\partial f}{\partial y}\left(x_0 + \triangle x, \tilde{y}\right) - \frac{\partial f}{\partial y}\left(x_0, \tilde{y}\right)\right]\triangle y \\ &= \frac{\partial^2 f}{\partial x \partial y}(\tilde{x}, \tilde{y})\triangle x \triangle y,\end{aligned}$$

where $\tilde{x} = x_0 + t'\triangle x$ is between x_0 and $x_0 + \triangle x$, and $\tilde{y} = y_0 + \tilde{t}\triangle y$ lies between y_0 and $y_0 + \triangle y$. Now, if we divide both sides of the equation above by $\triangle x \triangle y$, and take limits, we get the desired result:

$$\lim_{(\triangle x, \triangle y) \to (0,0)} \frac{\triangle_y \triangle_x f}{\triangle x \triangle y} = \lim_{(\triangle x, \triangle y) \to (0,0)} \frac{\partial^2 f}{\partial x \partial y}(\tilde{x}, \tilde{y}) = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$$

because $(\tilde{x}, \tilde{y}) \to (x_0, y_0)$ as $(\triangle x, \triangle y) \to (0,0)$ and the partial is assumed to be continuous at (x_0, y_0). But now it is clear that by reversing the roles of x and y we get, in the same way,

$$\lim_{(\triangle x, \triangle y) \to (0,0)} \frac{\triangle_x \triangle_y f}{\triangle y \triangle x} = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$$

which, together with our earlier observation that

$$\triangle_y \triangle_x f = \triangle_x \triangle_y f$$

completes the proof. $\square$

At first glance, it might seem that a proof for functions of more than two variables might need some work over the one given above. However, when we are looking at the equality of two specific mixed partials, say $\frac{\partial^2 f}{\partial x_i \partial x_j}$ and $\frac{\partial^2 f}{\partial x_j \partial x_i}$, we are holding *all other variables* constant, so the proof above goes over verbatim, once we replace x with x_i and y with x_j (Exercise 5 in § 3.8).

A.3 The Principal Axis Theorem

We begin by revisiting the matrix representative of a quadratic form. Just as every linear function $\ell(\vec{x})$ can be calculated as the product $[\ell][\vec{x}]$ of the matrix representative of ℓ (a row) with the coordinate column of $\vec{x}$, a quadratic form $Q(\vec{x})$ can be calculated as a double product[1] $[\vec{x}]^T [Q] [\vec{x}]$

$$[\vec{x}]^T [Q] [\vec{x}] = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + cy^2 = Q(\vec{x}).$$

Note that the matrix $[Q]$ is **symmetric**–reflection across the diagonal does not change the matrix.

In precisely the same way, a symmetric 3×3 matrix is the matrix representative of a quadratic form in three variables.[2] For any 3×3 matrix A, the double product

$$[\vec{x}]^T A [\vec{x}] = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

is the quadratic form

$$Q(\vec{x}) = \alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2 + \beta_{12} x_1 x_2 + \beta_{13} x_1 x_3 + \beta_{23} x_2 x_3,$$

where $\alpha_i = a_{ii}$ for $i = 1, 2, 3$ and

$$\beta_{12} = a_{12} + a_{21}, \quad \beta_{13} = a_{13} + a_{31}, \quad \beta_{23} = a_{23} + a_{32}.$$

The same form results from the symmetric matrix

$$[Q] = \begin{bmatrix} \alpha_1 & \beta_{12}/2 & \beta_{13}/2 \\ \beta_{12}/2 & \alpha_2 & \beta_{23}/2 \\ \beta_{13}/2 & \beta_{23}/2 & \alpha_3 \end{bmatrix}.$$

which is the matrix representative of Q.

In general, the double product $[\vec{x}]^T [Q] [\vec{x}]$ can also be interpreted as the dot product $\vec{x} \cdot ([Q]\vec{x})$ of the vector $\vec{x}$ with the vector whose coordinate column is $[Q][\vec{x}]$; we can think of the latter as a vector-valued function of $\vec{x}$. When $[Q]$ is **diagonal**

$$[Q] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

---

[1]The superscript in $[\vec{x}]^T$ indicates the **transpose** of $[\vec{x}]$; that is, we interchange rows with columns and vice versa.

[2]We shall find it more convenient to write $\vec{x} = (x_1, x_2, x_3)$ in place of (x, y, z); this will allow us to make efficient use of indices.

the quadratic form is a simple weighted sum of the coordinates of $\vec{x}$, squared:

$$Q(\vec{x}) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2.$$

It is clear that this form is positive definite (*resp*. negative definite) if and only if all three diagonal entries are positive (*resp*. negative).

An analogous situation holds for *any* quadratic form. To explain this, we need a slight diversion.

When $[Q]$ is diagonal, the function $[Q]\vec{x}$ preserves each of the three coordinate axes–or, in different language, each of the standard basis vectors $\vec{e_1} = \vec{\imath}$, $\vec{e_2} = \vec{\jmath}$ and $\vec{e_3} = \vec{k}$ is taken to a scalar multiple of itself ($[Q]\vec{e_i} = \alpha_i\vec{e_i}$). In general given a 3×3 matrix A, a vector $\vec{v} \neq \vec{0}$ is an **eigenvector** of A (with associated **eigenvalue** $\lambda \in \mathbb{R}$) if the product $A\vec{v}$ is parallel to $\vec{v}$:

$$A\vec{v} = \lambda\vec{v} \text{ for some } \lambda \in \mathbb{R}. \tag{A.4}$$

Furthermore, the standard basis is **orthonormal**: they are mutually orthogonal (perpendicular) and have length one. This can be summarized in the equation

$$\vec{e_i} \cdot \vec{e_j} = \delta_{ij}, \tag{A.5}$$

where δ_{ij} is the **Dirac delta**, which equals 1 if the indices agree and is zero otherwise.

The **Principal Axis Theorem** can be stated as follows:

Theorem A.3.1 (Principal Axis Theorem for $\mathbb{R}^3$). *Every symmetric* 3×3 *matrix has three orthonormal eigenvectors; that is, there are three vectors* $\vec{u_i}$, $i = 1, 2, 3$ *satisfying*

(1) *They are eigenvectors–they satisfy an analogue of Equation* (A.4)*: for some* $\lambda_i \in \mathbb{R}$,

$$A\vec{u_i} = \lambda_i\vec{u_i}. \tag{A.6}$$

(2) *They are orthonormal:*

$$\vec{u_i} \cdot \vec{u_j} = \delta_{ij}. \tag{A.7}$$

(Analogue of Equation (A.5)*.)*

A proof of this theorem, using Lagrange multipliers, is sketched in Exercise 3.

To understand the significance of this result, we make two observations. First, it is easy to see (Exercise 1) that because the vectors $\vec{u_i}$ are orthonormal, every vector $\vec{x} \in \mathbb{R}^3$ can be easily expressed as a linear combination of the $\vec{u_i}$:

$$\vec{x} = \xi_1\overrightarrow{u_1} + \xi_2\overrightarrow{u_2} + \xi_3\overrightarrow{u_3}, \text{ where } \xi_i = \vec{x} \cdot \vec{u_i}. \tag{A.8}$$

Second, using this together with Equation (A.6) and Equation (A.7), we can calculate (Exercise 2) that if $A = [Q]$ for the quadratic form Q,

$$Q(\vec{x}) = \vec{x} \cdot A\vec{x} = \lambda_1\xi_1^2 + \lambda_2\xi_2^2 + \lambda_3\xi_3^2. \tag{A.9}$$

From this it is easy to see that Q is positive (*resp*. negative) definite if and only if the eigenvalues λ_i are all positive (*resp*. negative).

Theorem A.3.1 is stated rather abstractly: it is not clear how to go about finding the orthonormal eigenvectors in any specific situation. We give here a procedure, without justifying why it works.

Curiously, we start by finding the eigen*values* λ_i before looking for the eigen*vectors*. To do this, we define a (formal) 3×3 matrix $A - \lambda I$ by subtracting the variable λ from each diagonal entry of A. The (formal) determinant of this matrix is a polynomial of degree 3 in λ, called the **characteristic polynomial** of the matrix A. The eigenvalues λ_i are the three zeroes of this polynomial. Once we have found one of these zeroes, λ_i, we find the corresponding eigenvector $\vec{v_i}$ by solving the vector equation $A\vec{v_i} = \lambda_i \vec{v_i}$, which can also be written as

$$(A - \lambda_i I)\vec{v_i} = 0. \tag{A.10}$$

This is a system of three equations in three unknowns; one solution is always the zero vector, but the fact that $\det(A - \lambda_i I) = 0$ ensures that there will be other solutions. Any one of these (nonzero) solutions can be normalized: dividing $\vec{v}$ by its length yields a parallel vector $\vec{u}$ of length one.

For example, the matrix representative of the quadratic form $Q(\vec{x}) = 4x_1^2 - 4x_1x_2 - x_2^2 + 4x_1x_3 - x_3^2 - 6x_2x_3$ has matrix representative

$$A = [Q] = \begin{bmatrix} 4 & -2 & 2 \\ -2 & -1 & -3 \\ 2 & -3 & -1 \end{bmatrix}.$$

Its characteristic polynomial is

$$\det \begin{bmatrix} 4-\lambda & -2 & 2 \\ -2 & -1-\lambda & -3 \\ 2 & -3 & -1-\lambda \end{bmatrix} = \lambda(\lambda+4)(6-\lambda),$$

so the eigenvalues are $\lambda = 0, -4, 6$. We can calculate that the vector $\vec{v_1} = (1, 1, -1)$ is an eigenvector for eigenvalue $\lambda_1 = 0$, $\vec{v_2} = (0, 1, 1)$ is an eigenvector for eigenvalue $\lambda_2 = -4$, and $\vec{v_3} = (2, -1, 1)$ is an eigenvector for $\lambda_3 = 6$. These three vectors are orthogonal, but not of length one; to normalize, we divide each by its length, to get the orthonormal set of eigenvectors

$$\vec{u_1} = \frac{(1,1,-1)}{\sqrt{3}}, \quad \vec{u_2} = \frac{(0,1,1)}{\sqrt{2}}, \quad \vec{u_3} = \frac{(2,-1,1)}{\sqrt{6}}$$

leading to the representation of our form as

$$\begin{aligned} Q(x_1, x_2, x_3) = \\ 0\left(\frac{x_1 + x_2 - x_3}{\sqrt{3}}\right)^2 - 4\left(\frac{x_2 + x_3}{\sqrt{2}}\right)^2 + 6\left(\frac{2x_1 - x_2 + x_3}{\sqrt{3}}\right)^2 \\ = -2(x_2 + x_3)^2 + (2x_1 - x_2 + x_3)^2. \end{aligned}$$

We see immediately (and could have seen just from the eigenvalues) that this form takes both positive and negative values.

The observation that the character of a quadratic form as positive definite, negative definite, or neither can be determined purely from the eigenvalues of its matrix representative (without looking for the eigenvectors) leads to an extension of the Second Derivative Test to functions of three variables. In fact, an examination of the characteristic polynomial of a matrix A shows that its constant term is just the determinant $\det A$—but the constant term of any polynomial is the product of its roots, so it follows that $\det A$ *is the product of the eigenvalues of* A. This immediately tells us, for example, that in order for a quadratic form Q to be positive definite, the determinant of $[Q]$ must be positive. This is, of course, not enough to ensure that all eigenvalues are positive: even for a 2×2 matrix, we get a positive determinant if and only if both eigenvalues have the same sign (and then the form is definite—possibly positive definite, but also possibly negative definite). In the 3×3 case, we also have the possibility that the determinant is positive because two eigenvalues are negative and one is positive; thus even definiteness is not guaranteed if the determinant is positive. In the 2×2 case, we saw that the trick was to also look at the sign of the determinant of a sub matrix. A similar scheme works here, although the analysis justifying it is beyond the scope of our exposition. The scheme is stated in Proposition 3.9.7.

The analysis of quadratic forms in terms of eigenvectors and eigenvalues can equally well be applied to forms $Q(x, y)$ in the plane. Exercise 4 shows how such an analysis can be used to classify the loci of quadratic equations in x and y, and in particular to show that every such locus is one of the following:

- The empty set.
- A line.
- A pair of lines that cross.
- A conic section (ellipse or circle, parabola, hyperboloid) of one or two sheets.

The proof of this is given in Exercise 4.

Exercises for Appendix A.3

(1) Prove Equation (A.8): that is, show that if $\overrightarrow{u_1}$, $\overrightarrow{u_2}$, and $\overrightarrow{u_3}$ are an orthonormal set of vectors in $\mathbb{R}^3$, then for every $\vec{x} \in \mathbb{R}^3$,

$$\vec{x} = (\vec{x} \cdot \overrightarrow{u_1})\overrightarrow{u_1} + (\vec{x} \cdot \overrightarrow{u_2})\overrightarrow{u_2} + (\vec{x} \cdot \overrightarrow{u_3})\overrightarrow{u_3}.$$

(*Hint:* First, show that they are linearly independent, so that every vector in $\mathbb{R}^3$ is a linear combination of them. Then show that if $\vec{x} = \sum_{i=1}^{3} a_i \vec{u}_i$, then $\vec{x} \cdot \vec{u}_i = a_i$.)

(2) (a) Prove Equation (A.9): if $\vec{u}_i$ are orthonormal eigenvectors for $[Q]$ with associated eigenvalues λ_i ($i = 1, 2, 3$), then

$$Q\left(\sum_{i=1}^{3} \xi_i \vec{u}_i\right) = \sum_{i=1}^{3} \lambda_i \xi_i^2.$$

(b) Justify the conclusion that Q is positive (*resp.* negative) definite if and only if the eigenvalues λ_i of $[Q]$ are all positive (*resp.* negative).

Challenge problem:

(3) Prove Theorem A.3.1, as follows[3]

(a) Let $f(\vec{x}) = Q(\vec{x})$; show that $\vec{\nabla} f(\vec{x}) = 2[Q]\vec{x}$.

(b) Let $g_1(\vec{x}) = \vec{x} \cdot \vec{x}$; show that $\vec{\nabla} g_1(\vec{x}) = 2\vec{x}$.

(c) Consider the problem of finding the minimum value of $f(\vec{x})$ on the unit sphere defined by $g_1(\vec{x}) = 1$; show that a solution of this problem occurs at a unit eigenvector $\overrightarrow{u_1}$ of $[Q]$, with associated eigenvalue $\lambda_1 = Q(\overrightarrow{u_1})$.

(d) Now consider the problem of finding the minimum and maximum values of $f(\vec{x})$[4] on the intersection of the unit sphere with the plane through the origin perpendicular to $\overrightarrow{u_1}$; that is, the set of vectors satisfying $g_1(\vec{x}) = 1$ and $g_2(\vec{x}) = 0$, where $g_2(\vec{x}) = \vec{x} \cdot \overrightarrow{u_1}$. Show that $\vec{\nabla} g_2(\vec{x}) = \overrightarrow{u_1}$.

[3]Note that this uses the Lagrange Multiplier equation for two constraints, which is the optional part of § 3.7.

[4]This is simply a circle in $\mathbb{R}^3$.

(e) Show that a solution $\vec{u}$ of the Lagrange Multiplier equation for these two constraints satisfies the equation

$$[Q]\,\vec{u} = \lambda\vec{u} + \mu\overrightarrow{u_1}.$$

Use the fact that $\vec{u}$ is perpendicular to $\overrightarrow{u_1}$ to show that $\lambda = Q\left(\vec{u}\right)$. Let $\overrightarrow{u_2}$ and $\overrightarrow{u_3}$ be the places where the restriction of f to this circle achieves its minimum and maximum; if f is constant on this circle, pick these vectors to be perpendicular to each other.

(f) You may assume without proof that for any two vectors $\vec{v}$ and $\vec{w}$ and any symmetric matrix A, $\vec{v}\cdot A\vec{w} = \vec{w}\cdot A\vec{v}$. Using this, show that the vectors $\overrightarrow{u_1}$, $\overrightarrow{u_2}$ and $\overrightarrow{u_3}$ are an orthonormal set of eigenvectors for Q.

(4) **Quadratic Curves:** In this exercise, you will show that every equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey = F \tag{A.11}$$

defines an empty set, a single point, a line, two lines, or a conic section in the plane.

(a) Write the left side of Equation (A.11) as the sum of the quadratic form $Q\left(x,y\right) = Ax^2 + Bxy + Cy^2$ and the linear function $\ell\left(x,y\right) = Dx + Ey$, so that it reads

$$Q\left(x,y\right) + \ell\left(x,y\right) = F.$$

(b) Set $\vec{x} = (x,y)$. As a special case of Theorem A.3.1 (which can be proved directly, or regarding Q as a quadratic form in three variables for which every term involving z has coefficient zero), there exist two orthonormal vectors in the plane, $\vec{u_i}$, $i = 1,2$, such that

$$Q\left(\vec{x}\right) = \lambda_1(\overrightarrow{u_1}\cdot\vec{x})^2 + \lambda_2(\overrightarrow{u_2}\cdot\vec{x})^2 \tag{A.12}$$

for some choice of $\lambda_i \in \mathbb{R}$.

(c) We saw in § 3.2 that the linear function $\ell\left(\vec{x}\right)$ can be represented as the dot product $\ell\left(\vec{x}\right) = \vec{a}\cdot\vec{x}$, where $\vec{a} = (D,E)$. Thus we can rewrite Equation (A.11) as

$$\lambda_1(\overrightarrow{u_1}\cdot\vec{x})^2 + \lambda_2(\overrightarrow{u_2}\cdot\vec{x})^2 + \vec{a}\cdot\vec{x} = F.$$

(d) Since $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$ are an orthonormal set in the plane, we can write $\vec{a}$ as a linear combination of $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$:

$$\vec{a} = \alpha_1\overrightarrow{u_1} + \alpha_2\overrightarrow{u_2}.$$

This allows us to rewrite Equation (A.12) as

$$\lambda_1(\vec{x}\cdot\overrightarrow{u_1})^2 + \alpha_1(\vec{x}\cdot\overrightarrow{u_1}) + \lambda_2(\vec{x}\cdot\overrightarrow{u_2})^2 + \alpha_2(\vec{x}\cdot\overrightarrow{u_2}) = F. \tag{A.13}$$

(e) The orthonormal pair of planar vectors $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$ can be obtained from the standard basis $\vec{\imath},\vec{\jmath}$ for $\mathbb{R}^2$ by rotating the plane. Assume without loss of generality that this rotation (say, rotation counterclockwise by θ radians) takes $\vec{\imath}$ to $\overrightarrow{u_1}$ and $\vec{\jmath}$ to $\overrightarrow{u_2}$. Then the vector $\vec{x}$ is taken by this rotation to the vector $\vec{X} = X_1\overrightarrow{u_1} + X_2\overrightarrow{u_2}$, where $X_i = \vec{x}\cdot\vec{u_i}$. As an equation in the "rotated coordinates" X_i, Equation (A.13) reads

$$\lambda_1X_1^2 + \alpha_1X_1 + \lambda_2X_2^2 + \alpha_2X_2 = F. \tag{A.14}$$

(f) If $\lambda_i \neq 0$, let $\beta_i = \alpha_i/2\lambda_i$. Then we can complete the square for the pair of terms involving X_i:

$$\lambda_iX_i^2 + \alpha_iX_i + \lambda_i\beta_i^2 = \lambda_i\left(X_i + \beta_i\right)^2$$

If $\lambda_i = 0$, set $\beta_i = 0$. Adding $\lambda_1\beta_1^2 + \lambda_2\beta_2^2$ to both sides of Equation (A.14), we have on the left a sum of two expressions, each involving a single X_i, which is either $\lambda_i (X_i + \beta_i)^2$ or $\alpha_i X_i$.

Since the original equation was assumed to be of degree two, at least one of the two λ_i is nonzero; assume it is λ_1. Multiplying the whole equation by -1 if necessary, we can assume that λ_1 is positive.

Then we have the following possibilities for Equation (A.13):

- If $\lambda_2 = 0 = \alpha_2$, it reads
$$\lambda_1 (X_1 + \beta_1)^2 = F + \lambda_1\beta_1^2$$
which is equivalent to the equation of a line if $F + \lambda_1\beta_1^2 = 0$, of two parallel lines if $F + \lambda_1\beta_1^2$ and λ_1 have the same sign, or the empty set if they have opposite sign.
- If $\lambda_2 = 0$ but $\alpha_2 \neq 0$, it reads
$$\lambda_1 (X_1 + \beta_1)^2 + \alpha_2 X_2 = F + \lambda_1\beta_1^2$$
which is the equation of a parabola.
- If $\lambda_2 \neq 0$, it reads
$$\lambda_1 (X_1 + \beta_1)^2 + \lambda_2 (X_2 + \beta_2)^2 = F + \lambda_1\beta_1^2 + \lambda_2\beta_2^2.$$
If λ_2 is also positive, this is empty if the right side is negative, a single point if it is zero, and either a circle (if $\lambda_1 = \lambda_2$) or an ellipse if the right side is positive.
If λ_2 is negative, then this is the equation of two lines crossing at the origin (if the right side is zero) and of a hyperbola otherwise.

As a final note, remember that the loci of our model equations for the conics had major and minor axes, and directrices, either horizontal or vertical. However, the equations which we are classifying at the end are in terms of coordinates with respect to the two orthonormal vectors $\vec{u}_i$, which come from rotating the standard basis θ radians counterclockwise. To properly apply the analysis in § 2.1 to our equations (written in x and y), we need to rotate back, *clockwise* θ radians.

A.4 Discontinuities and Integration

The basic idea for integrating a function $f(x, y)$ over a general region takes its inspiration from our idea of the area of such a region: we try to "subdivide" the region into rectangles (in the sense of § 4.1) and add up the integrals over them. Of course, this is essentially impossible for most regions, and instead we need to think about two kinds of *approximate* calculations: "inner" ones using rectangles entirely contained inside the region, and "outer" ones over unions of rectangles which contain our region (rather like the inscribed and circumscribed polygons Archimedes used to find the area of a circle). For the theory to make sense, we need to make sure that these two calculations give rise to the same value for the integral. This is done via the following technical lemma.

Lemma A.4.1. *Suppose a curve $\mathcal{C}$ is the graph of a continuous function, $y = \phi(x)$, $a \leq x \leq b$. Then given any $\varepsilon > 0$ we can find a finite family of rectangles $B_i = [a_i, b_i] \times [c_i, d_i]$,*

$i = 1, \ldots, k$, covering the curve (Figure A.2*)*

$$\mathcal{C} \subset \bigcup_{i=1}^{k} B_i$$

such that

(1) *Their total area is at most ε*

$$\sum_{i=1}^{k} \mathcal{A}(B_i) \leq \varepsilon.$$

(2) *The horizontal edges of each B_i are disjoint from $\mathcal{C}$*

$$c_i < \phi(x) < d_i \text{ for } a_i \leq x \leq b_i.$$

A proof of Lemma A.4.1 is sketched in Exercise 1.

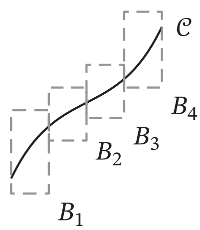


Figure A.2. Lemma A.4.1

Using this result, we can extend the class of functions which are Riemann integrable beyond those continuous on the whole rectangle (as given in Theorem 4.1.4), allowing certain kinds of discontinuity. This will in turn allow us to define the integral of a function over a more general region in the plane.

The main issue we need to face is the two-dimensional analogue of jump discontinuities. Recall that a function $f(x)$ has a "jump discontinuity" at a point if its one-sided limits at the point exist, but are not equal. While there is no natural extension to functions of several variables of the idea of a limit from the right or left, there is a situation when an analogue can be usefully defined. Suppose $f(x, y)$ is a function defined on $[a, b] \times [c, d]$ and $\mathcal{C}$ is a curve which divides $[a, b] \times [c, d]$ into two regions. If p_0 is a point on $\mathcal{C}$, we can define the notion of the limit of $f(x, y)$ at p_0 from one side of $\mathcal{C}$. For example, if $\mathcal{C}$ is the graph of a continuous function $y = \phi(x)$ for $x \in [a, b]$ (and $\phi([a, b]) \subset (c, d)$), We can then say that a number L_- is the **limit of** $f(x, y)$ **at** p_0 **from below** $\mathcal{C}$ if $L_- = \lim f(p_i)$ for every sequence $p_i = (x_i, y_i) \to p_0$ with $y_i < \phi(x_i)$ for every index i. In an analogous way, we could define the limit of $f(x, y)$ at p_0 from *above* $\mathcal{C}$.

Suppose now that $f(x, y)$ is (uniformly) continuous on the complement of $\mathcal{C}$ and at every point of $\mathcal{C}$the limits of $f(x, y)$ from both sides of $\mathcal{C}$ exist. Then we can extend $f(x, y)$ continuously from either side of $\mathcal{C}$ by defining the values on $\mathcal{C}$ to be the respective one-sided limits there. At every point of $\mathcal{C}$ where the two one-sided limits agree, we can say that $f(x, y)$ is continuous, while at points where the two disagree, we can say that $f(x, y)$ has a **jump discontinuity**.

Using this language, we can assert:

Theorem A.4.2 (Theorem 4.2.2: Integrability with Jump Discontinuities)**.** *If a function f is bounded on $[a,b]\times[c,d]$ and continuous except possibly for some points lying on a finite union of graphs (curves of the form $y=\phi(x)$ or $x=\psi(y)$), then f is Riemann integrable over $[a,b]\times[c,d]$.*

Proof. For ease of notation, we shall assume that f is bounded on $[a,b]\times[c,d]$ and that any points of discontinuity lie on a single graph $\mathcal{C}: \quad y=\phi(x), a\le x\le b$.

Given $\varepsilon>0$, we need to find a partition $\mathcal{P}$ of $[a,b]\times[c,d]$ for which

$$\mathcal{U}(\mathcal{P},f)-\mathcal{L}(\mathcal{P},f)<\varepsilon.$$

First, since f is bounded on $[a,b]\times[c,d]$, pick an upper bound for $|f|$ on $[a,b]\times[c,d]$, say

$$M>\max\{1, sup_{[a,b]\times[c,d]}|f|\}.$$

Next, use Lemma A.4.1 to find a finite family B_i, $i=1,\dots,k$, of rectangles covering the graph $y=\phi(x)$ such that

$$\sum_{i=1}^{k}\mathcal{A}(B_i)<\frac{\varepsilon}{2M}.$$

Now extend each edge of each B_i to go completely across the rectangle $[a,b]\times[c,d]$ (horizontally or verticaly)—there are finitely many such lines, and they define a partition $\mathcal{P}$ of $[a,b]\times[c,d]$ such that each B_i (and hence the union of all the B_i) is itself a union of subrectangles R_{ij} for $\mathcal{P}$. Note that if we refine this partition further by adding more (horizontal or vertical) lines, it will still be true that $\mathcal{B}=\bigcup_{i=1}^{k}B_i$ is a union of subrectangles, and

$$\left(\sum_{R_{ij}\subset\mathcal{B}}\sup_{R_{ij}}f\triangle A_{ij}\right)-\left(\sum_{R_{ij}\subset\mathcal{B}}\inf_{R_{ij}}f\triangle A_{ij}\right)=\sum_{R_{ij}\subset\mathcal{B}}\left(\sup_{R_{ij}}f-\inf_{R_{ij}}f\right)\triangle A_{ij}$$
$$<M\cdot\mathcal{A}(\mathcal{B})<M\left(\frac{\varepsilon}{2M}\right)=\frac{\varepsilon}{2}.$$

Finally, consider the union $\mathcal{D}$ of the rectangles of $\mathcal{P}$ which are disjoint from $\mathcal{C}$. This is a compact set on which f is continuous, so f is *uniformly* continuous on $\mathcal{D}$; hence as in the proof of Theorem 4.1.4 we can find $\delta>0$ such that for any of the subrectangles R_{ij} contained in $\mathcal{D}$ we have

$$\sup_{R_{ij}}f-\inf_{R_{ij}}f<\frac{\varepsilon}{2\mathcal{A}([a,b]\times[c,d])}$$

so that

$$\left(\sum_{R_{ij}\subset\mathcal{D}}\sup_{R_{ij}}f\triangle A_{ij}\right)-\left(\sum_{R_{ij}\subset\mathcal{D}}\inf_{R_{ij}}f\triangle A_{ij}\right)<\frac{\varepsilon}{2\mathcal{A}([a,b]\times[c,d])}\mathcal{A}(\mathcal{D}).$$

From this it follows that for our final partition,

$$\mathcal{U}(\mathcal{P},f)-\mathcal{L}(\mathcal{P},f)=\sum_{R_{ij}\subset\mathcal{B}}\left(\sup_{R_{ij}}f-\inf_{R_{ij}}f\right)\triangle A_{ij}+\sum_{R_{ij}\subset\mathcal{D}}\left(\sup_{R_{ij}}f-\inf_{R_{ij}}f\right)\triangle A_{ij}$$
$$<\frac{\varepsilon}{2}+\frac{\varepsilon}{2\mathcal{A}([a,b]\times[c,d])}\mathcal{A}(\mathcal{D})<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

as required.

$\square$

Exercises for Appendix A.4

(1) Prove Lemma A.4.1 as follows: Given $\varepsilon > 0$, use the uniform continuity of the function ϕ to pick $\delta > 0$ such that

$$\|x - x'\| < \delta \Rightarrow |\phi(x) - \phi(x')| < \frac{\varepsilon}{3\,|b-a|},$$

and let $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_k\}$ be a partition of $[a, b]$ with $\operatorname{mesh}(\mathcal{P}) < \delta$. Denote the atoms of $\mathcal{P}$ by I_i, $i = 1, \ldots, k$. Then, for each $i = 1, \ldots, k$, let

$$J_i = \left[\min_{I_i} \phi - \frac{\varepsilon}{3\,|b-a|}, \max_{I_i} \phi + \frac{\varepsilon}{3\,|b-a|}\right]$$

and set

$$B_i \coloneqq I_i \times J_i.$$

(a) Show that $\|J_i\| \leq \frac{\varepsilon}{|b-a|}$ for all $i = 1, \ldots, k$.

(b) Use this to show that

$$\sum_{i=1}^{k} \mathcal{A}(B_i) \leq \varepsilon.$$

(c) Show that for each $i = 1, \ldots, k$,

$$c_i < \min_{I_i} \phi \leq \max_{I_i} \phi < d_i.$$

A.5 Linear Transformations, Matrices, and Determinants

As noted in § 4.3, a transformation of the plane is linear if its two coordinate functions are homogeneous polynomials of degree one, or equivalently if as a vector-valued function of a vector variable it respects linear combinations. The first characterization says that a linear transformation Φ has the form

$$\Phi(x_1, x_2) = (a_{11}x_1 + a_{12}x_2, a_{21}x_1 + a_{22}x_2). \tag{A.15}$$

This information can be packaged as a 2×2 **matrix**

$$[\Phi] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{A.16}$$

called the **matrix representative** of Φ. This array can be viewed in several ways: the two *rows* can be identified with the matrix representatives of the two coordinate functions of Φ, meaning that to calculate each coordinate of the output we can multiply the corresponding row by the coordinate column of the input. If we regard $[\Phi]$ as a “column of rows”, this yields a natural notion of multiplying a column of height two (the coordinate column of the input) by a 2×2 matrix to obtain a new column, which is the coordinate column of the output: if $\vec{x} = (x_1, x_2)$,

$$[\Phi(\vec{x})] = [\Phi] \cdot [\vec{x}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}. \tag{A.17}$$

From this we also see that the *columns* of $[\Phi]$ are exactly the (coordinate columns of) the images of the standard basis:

$$[\Phi(\vec{\imath})] = [\Phi] \cdot [\vec{\imath}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix};$$

$$[\Phi(\vec{\jmath})] = [\Phi] \cdot [\vec{\jmath}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}.$$

By playing these interpretations of $[\Phi]$ against each other, we can deduce some useful facts about linear transformations of the plane:

- The determinant of $[\Phi]$ is $a_{11}a_{22} - a_{21}a_{12}$, which is unchanged if we replace $[\Phi]$ with its **transpose**, obtained by switching rows with columns:

$$[\Phi]^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}. \tag{A.18}$$

  That is, $\det[\Phi] = \det[\Phi]^T$.

  But $\det[\Phi]^T$ can be interpreted as the signed area of the parallelogram whose sides are the vectors $\Phi(\vec{\imath})$ and $\Phi(\vec{\jmath})$. This parallelogram is the image under Φ of the unit square, with sides $\vec{\imath}$ and $\vec{\jmath}$.

  Since displacement and rotation don't change area, and linearity of Φ says that scaling the sides of the square to form a rectangle results in a similar scaling of the image parallelogram,

  The effect of applying Φ to any rectangle is to produce a parallelogram whose area is $\Delta(\Phi) = |\det[\Phi]|$ times that of the rectangle.

- We know that a pair of vectors is linearly independent precisely if the parallelogram they span has nonzero area. From this we can deduce

  The images $\Phi(\vec{\imath})$ and $\Phi(\vec{\jmath})$ are linearly independent precisely if $\Delta(\Phi) \neq 0$.

- The basic linearity property of Φ can be formulated as

$$\Phi(x_1, x_2) = \Phi(x_1\vec{\imath} + x_2\vec{\jmath}) = x_1\Phi(\vec{\imath}) + x_2\Phi(\vec{\jmath}); \tag{A.19}$$

  the linear independence of the two images $\Phi(\vec{\imath})$ and $\Phi(\vec{\jmath})$ says that the only values of x_1 and x_2 for which the vector $\Phi(\vec{x})$ is the zero vector is $x_1 = 0 = x_2$, so

  $\Phi(\vec{\imath})$ and $\Phi(\vec{\jmath})$ are linearly independent precisely if the only solution of $\Phi(\vec{x}) = \vec{0}$ is $\vec{x} = \vec{0}$.

  Another way of interpreting this is to say (using linearity) that for any two vectors, $\vec{v}$ and $\vec{w}$, $\Phi(\vec{v}) = \Phi(\vec{w})$ only if $\Phi(\vec{v} - \vec{w}) = \Phi(\vec{v}) - \Phi(\vec{w}) = \vec{0}$, and if $\Delta(\Phi) \neq 0$, this can happen only if $\vec{v} - \vec{w} = \vec{0}$, which is to say only if $\vec{v} = \vec{w}$. Stated differently,

  $\Delta(\Phi) \neq 0$ if and only if Φ is one-to-one.

- We also know that any pair of linearly independent vectors spans a plane, so every vector in the plane can be expressed as a linear combination of $\Phi(\vec{\imath})$ and $\Phi(\vec{\jmath})$ whenever these are linearly independent. This tells us that

  $\Delta(\Phi) \neq 0$ if and only if Φ maps the plane onto itself.

Pulling these observations together we have the following result:

Proposition A.5.1. *If $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation of the plane, then the following properties are equivalent:*

(1) *$\Delta(\Phi) \neq 0$;*
(2) *Φ is (globally) one-to-one;*
(3) *Φ maps the plane onto itself.*

We call a linear transformation **nonsingular** if the first property holds; the other two properties amount to saying that we can define an **inverse transformation** Φ^{-1} by $(x_1, x_2) = \Phi^{-1}(y_1, y_2)$ if $\Phi(x_1, x_2) = (y_1, y_2)$. You can check that the inverse of a nonsingular linear transformation is itself a linear transformation. A transformation is **invertible** on a set $\mathcal{D} \subset \mathbb{R}^2$ if it has an inverse there; so Proposition A.5.1 says that a nonsingular *linear* transformation is invertible on $\mathbb{R}^2$.

Since Φ^{-1} is linear, it has a matrix representative, the **inverse** of the matrix representative of Φ: $[\Phi]^{-1} = [\Phi^{-1}]$. There is an easy formula for the inverse of a 2×2 matrix: if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The inverse of $[\Phi]$ can be understood as follows. Recall that the composition of two transformations Φ and Ψ is the transformation obtained by applying both successively; in general there are two possible compositions depending on the order of application:

$$(\Phi \circ \Psi)(\vec{x}) = \Phi(\Psi(\vec{x})), \quad (\Psi \circ \Phi)(\vec{x}) = \Psi(\Phi(\vec{x})).$$

The matrix representative of the composition $[\Phi \circ \Psi]$ is the **product** of $[\Phi]$ and $[\Psi]$, defined in general by:

The entry in row i and column j of AB is the product of the i^{th} row of A with the j^{th} column of B.

It is easy to see that the composition of a transform and its inverse in either order is the **identity matrix**, the matrix representative [id] of the identity transformation id, defined by $\mathrm{id}\,\vec{x} = \vec{x}$, has 1 in every diagonal spot, and zeroes everywhere else; it is usually denoted I. Thus, the inverse of a matrix A is defined by

$$A \cdot A^{-1} = I.$$

A linear transformation Φ of $\mathbb{R}^2$ is clearly continuous everywhere. It follows that the norm of its value on the unit circle in $\mathbb{R}^2$ achieves its maximum, which we denote

$$\|\Phi\| := \max\{\|\Phi(\vec{x})\| \mid \|\vec{x}\| = 1\};$$

this is called the **norm** (or **operator norm**operator norm of a linear transformation) of Φ. From the homogeneity of Φ, we can assert

Remark A.5.2. $\|\Phi\| = \max\left\{\frac{\|\Phi(\vec{x})\|}{\|\vec{x}\|} \mid \vec{x} \neq \vec{0}\right\}$.

Note that if Φ is invertible, the norm of its inverse, $\|\Phi^{-1}\|$, is different from the reciprocal of its norm, $\frac{1}{\|\Phi\|}$; in fact the latter can be characterized as the *minimum* of the ratio $\|(L)^{-1}(\vec{x})\|/\|\vec{x}\|$ for $\vec{x} \neq \vec{0}$.

We can also extend our observation about the effect of linear transformations on area from rectangles to general areas. If $\mathcal{D}$ is a y-simple region over an interval $[a, b]$ on the x-axis, we can geometrically represent the lower and upper sums corresponding to any partition of $[a, b]$ by polygons consisting of rectangles built on the atoms of the partition: the image of either of these polygons will on one hand have area equal to $\Delta(\Phi)$

times the area of the polygon, and on the other hand the image $\Phi(\mathcal{D})$ of our region will contain the "inner" polygon and be contained in the "outer" one (see Figure A.8 in Appendix A.7). From this it follows that

Suppose Φ is a linear transformation. For any y-simple region $\mathcal{D}$, the area of the image $\Phi(\mathcal{D})$ equals $\Delta(\Phi)$ times the area of $\mathcal{D}$.

This is the content of Proposition 4.3.2.

A corollary of this observation is

Remark A.5.3. *For any 2×2 matrices A and B, the determinant of their product is the product of their determinants:*

$$\det AB = (\det A)(\det B). \tag{A.20}$$

The 3D Case. The preceding discussion of linear transformations of the plane can be easily extended to linear transformations $\Phi\colon \mathbb{R}^3 \to \mathbb{R}^3$. This is particularly useful when thinking about change of coordinates in a triple integral. Here we will simply point out the basic features of this extension.

A **transformation** of $\mathbb{R}^3$ assigns to each point (x, y, z) in space another point in space

$$\Phi(x, y, z) = (\phi_1(x, y, z), \phi_2(x, y, z), \phi_3(x, y, z));$$

it is **linear** if the three coordinate functions $\phi_i(x, y, z)$ are all linear–that is, they are all homogeneous polynomials of degree one:

$$\phi_i(x, y, z) = a_{i,1}x + a_{i,2}y + a_{i,3}z, \quad i = 1, 2, 3.. \tag{A.21}$$

Linearity can also be formulated in vector terms–a transformation is linear if it respects linear combinations:

$$\Phi\left(c_1\vec{v_1} + c_2\vec{v_2}\right) = c_1\Phi\left(\vec{v_1}\right) + c_2\Phi\left(\vec{v_2}\right).$$

The **matrix representative** of the transformation given by Equation (A.21) is the 3×3 matrix

$$[\Phi] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

The determinant of $[\Phi]$ represents the effect of the transformation Φ on *volume*; in particular, the analogues of Proposition A.5.1, Remark A.5.2 and Remark A.5.3 all carry over to the 3×3 setting: $\Phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ maps all of $\mathbb{R}^3$ onto itself, and equivalently is (globally) one-to-one. precisely if the determinant is nonzero; the operator norm of a linear transformation equals the maximum factor by which Φ multiplies lengths of vectors; and the determinant of a composition of linear transformations is the product of the individual determinants, and is therefore independent of the order of composition.

Exercises for Appendix A.5

Practice problems:

(1) Which of the following transformations $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ are linear? Give the matrix representative for those which are linear.
  (a) $\Phi(x, y) = (y, x)$
  (b) $\Phi(x, y) = (x, x)$
  (c) $\Phi(x, y) = (e^x \cos y, e^x \sin y)$
  (d) $\Phi(x, y) = (x^2 + y^2, 2xy)$

(e) $\Phi(x, y) = (x + y, x - y)$
(f) $\Phi(x, y) = (x, y, x^2 - y^2)$

(2) Which of the following transformations $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ are linear? Give the matrix representative for those which are linear.
(a) $\Phi(x, y, z) = (2x + 3y + 4z, x + z, y + z)$
(b) $\Phi(x, y, z) = (y + z, x + z, x + y)$
(c) $\Phi(x, y, z) = (x - 2y + 1, y - z + 2, x - y - z)$
(d) $\Phi(x, y, z) = (x + 2y, z - y + 1, x)$
(e) Rotation of $\mathbb{R}^3$ around the z-axis by θ radians counterclockwise, seen from above.

(3) Find the operator norm $\|L\|$ for each linear map L below:
(a) $L(x, y) = (y, x)$.
(b) $L(x, y) = (x + y, x - y)$.
(c) $L(x, y) = (x + y\sqrt{2}, x)$.
(d) $L : \mathbb{R}^2 \to \mathbb{R}^2$ is reflection across the diagonal $x = y$.
(e) $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $L(x, y, z) = (x, x - y, x + y)$.

Theory problems:

(4) Find the matrix representative for each kind of linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$ described below:
(a) HORIZONTAL SCALING: horizontal component gets scaled (multiplied) by $\lambda > 0$, vertical component is unchanged.
(b) VERTICAL SCALING: vertical component gets scaled (multiplied) by $\lambda > 0$, horizontal component is unchanged.
(c) HORIZONTAL SHEAR: Each horizontal line $y = c$ is translated (horizontally) by an amount proportional to c.
(d) VERTICAL SHEAR: Each vertical line $x = c$ is translated (vertically) by an amount proportional to c.
(e) REFLECTION ABOUT THE DIAGONAL: x and y are interchanged.
(f) ROTATION: Each vector is rotated θ radians counterclockwise.

(5) Prove Remark A.5.3 as follows: suppose $A = [L]$ and $B = [L']$. Then $AB = [L \circ L']$. Consider the unit square S with vertices $(0, 0), (1, 0), (1, 0)$, and $(1, 0)$. (In that order, it is positively oriented.) Its signed area is

$$\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$

Now consider the parallelogram $S' = L'(S)$. The two directed edges $\vec{\imath}$ and $\vec{\jmath}$ of S map to the directed edges of S', which are $\vec{v} = L'(\vec{\imath})$ and $\vec{w} = L'(\vec{\jmath})$. **Show** that the first column of B is $[\vec{v}]$ and its second column is $[\vec{w}]$, so the signed area of $L'(S)$ is $\det B$. Now, consider $L(S')$: its directed edges are $L(\vec{v})$ and $L(\vec{w})$. **Show** that the coordinate columns of these two vectors are the columns of AB, so the signed area of $L(S')$ is $\det AB$. But it is also (by Proposition 4.3.2) $\det A$ times the area of S', which in turn is $\det B$. Combine these operations to **show** that $\det AB = \det A \det B$.

(6) An **affine transformation** of $\mathbb{R}^2$ is a transformation

$$T(x, y) = (\tau_1(x, y), \tau_2(x, y))$$

each of whose coordinate functions $\tau_i(x,y)$ is affine, as in § 3.2. This amounts to saying that $T(\vec{x}) = \vec{C} + L(\vec{x})$, where $\vec{C}$ is a constant (vector) and L is linear. By applying Remark 3.2.2 to the coordinate functions $\tau_i(x,y)$ we obtain the vector analogue of Equation (3.2): for any "base point" $\overrightarrow{x_0}$ the transformation T can be written in the form

$$T\left(\overrightarrow{x_0} + \triangle\vec{x}\right) = T\left(\overrightarrow{x_0}\right) + L\left(\triangle\vec{x}\right), \tag{A.22}$$

where $\triangle\vec{x} = \vec{x} - \overrightarrow{x_0}$.

Express each affine transformation T below as $T\left(\overrightarrow{x_0} + \triangle\vec{x}\right) = T\left(\overrightarrow{x_0}\right) + L\left(\triangle\vec{x}\right)$ with the given $\overrightarrow{x_0}$ and linear map L.

(a) $T(x,y) = (x+y-1, x-y+2)$, $\overrightarrow{x_0} = (1,2)$
(b) $T(x,y) = (3x-2y+2, x-y)$, $\overrightarrow{x_0} = (-2,-1)$
(c) $T(x,y,z) = (x+2y, z-y+1, x)$, $\overrightarrow{x_0} = (2,-1,1)$
(d) $T(x,y,z) = (x-2y+1, y-z+2, x-y-z)$, $\overrightarrow{x_0} = (1,-1,2)$
(e) $T(x,y,z) = (x+2y-z-2, 2x-y+1, z-2)$, $\overrightarrow{x_0} = (1,1,2)$

(7) Show that the composition of two affine maps is again affine.

Challenge problems:

(8) Suppose $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ is linear.
(a) **Show** that the determinant of $[L]$ is nonzero iff the image vectors $L(\vec{\imath})$ and $L(\vec{\jmath})$ are independent.
(b) **Show** that if $L(\vec{\imath})$ and $L(\vec{\jmath})$ are linearly independent, then L is an onto map.
(c) **Show** that if $L(\vec{\imath})$ and $L(\vec{\jmath})$ are linearly *dependent*, then L maps $\mathbb{R}^2$ into a line, and so is *not* onto.
(d) **Show** that if L is *not* one-to-one, then there is a nonzero vector $\vec{x}$ with $L(\vec{x}) = \vec{0}$.
(e) **Show** that if L is not one-to-one, then $L(\vec{\imath})$ and $L(\vec{\jmath})$ are linearly dependent.
(f) **Show** that if $L(\vec{\imath})$ and $L(\vec{\jmath})$ are dependent, then there is some nonzero vector sent to $\vec{0}$ by L.
(g) Use this to prove that the following are equivalent:
(i) the determinant of $[L]$ is nonzero;
(ii) $L(\vec{\imath})$ and $L(\vec{\jmath})$ are linearly independent;
(iii) L is onto;
(iv) L is one-to-one.
(h) L is **invertible** if there exists another map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $L(F(x,y)) = (x,y) = F(L(x,y))$. Show that if F exists it must be linear.

(9) In this problem you will show that every invertible linear map $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ can be expressed as a composition of the kinds of mappings described in Exercise 4. The idea is this: all we need to do is to get the two basis vectors $\vec{\imath}$ and $\vec{\jmath}$ to the right place, since these determine the columns of $[L]$. So let the desired values of L on the basis vectors be

$$L(\vec{\imath}) = \vec{a} = (a_1, a_2)$$
$$L(\vec{\jmath}) = \vec{b} = (b_1, b_2).$$

(equivalently, the columns of $[L]$ are $[\vec{a}]$ and $[\vec{b}]$). We will need to move $\vec{\imath}$ and $\vec{\jmath}$ so that the angle between them corresponds to the angle between $\vec{a}$ and $\vec{b}$, then adjust the lengths of the two vectors to correspond to the lengths of $\vec{a}$ and $\vec{b}$, and finally we will rotate the configuration into the desired pair of images.

(a) Show that the horizontal shear $(x, y) \mapsto (x + cy, y)$ takes the y-axis to the line through the origin of slope $\frac{1}{c}$. Thus if the angle between $\vec{a}$ and $\vec{b}$ is θ, then the horizontal shear with $c = \cot\theta$ will make the angle between the two image vectors correct. (What happens if $\vec{a}$ and $\vec{b}$ are orthogonal?)

(b) Before we do this, however, we want to adjust the lengths of the two image vectors correctly. It is easy to use horizontal and vertical scaling to independently change the lengths of $\vec{\imath}$ and $\vec{\jmath}$, but note that the shear will change the length of the non-horizontal vector coming from $\vec{\jmath}$, multiplying its length by $\sqrt{1+c^2}$. So our first step should be to horizontally rescale by $\left\|\vec{a}\right\|$ and vertically by $\left\|\vec{b}\right\|/\sqrt{1+c^2}$. After that, we apply the horizontal shear as in (a).

(c) At this point, we have moved the two basis vectors to vectors with the correct lengths and the correct angle between them. We have to now consider the orientation of the triangle $\triangle\vec{0}\vec{a}\vec{b}$. If it is positive (that is, $\vec{b}$ is θ radians counterclockwise from $\vec{a}$) the we can rotate the whole configuration into place. If it is negative, we need to go back to the very beginning, reflect across the diagonal, and interchange the roles of $\vec{\imath}$ and $\vec{\jmath}$ (*resp.* $\vec{a}$ and $\vec{b}$).

(d) Carry out this process for the two transformations given by the matrices

(i) $\begin{bmatrix} \sqrt{3} & 1/2 \\ 1 & \sqrt{3}/2 \end{bmatrix}$ ii $\begin{bmatrix} 0 & \sqrt{3} \\ 1 & 1 \end{bmatrix}$

A.6 The Inverse Mapping Theorem

In this appendix we discuss the extension to transformations $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ of the *Inverse Function Theorem*, which says that if a $\mathcal{C}^1$ real-valued function $f(t)$ of a real variable has a nonzero derivative at a point t_0, then there is an open interval $I = (t_0 - \delta, t_0 + \delta)$ about t_0 which is mapped injectively onto an open interval J about the image $f(t_0)$, so f has an inverse function $f^{-1}\colon J \to I$, and this inverse is itself $\mathcal{C}^1$ (see *Calculus Deconstructed*, Prop. 3.2.5, Thm. 4.4.1, or another single-variable calculus text)—in other words, that *a $\mathcal{C}^1$ function is locally invertible, with $\mathcal{C}^1$ inverse, at any of its regular points*.

A system of two equations in two unknowns

$$\begin{cases} \varphi_1(x, y) &= a \\ \varphi_2(x, y) &= b \end{cases}$$

can be interpreted as the vector equation $\Phi(\vec{x}) = \vec{y}$, where

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} a \\ b \end{bmatrix},$$

and $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\Phi(\vec{x}) = \begin{bmatrix} \varphi_1(\vec{x}) \\ \varphi_2(\vec{x}) \end{bmatrix}.$$

The analogous situation for one equation in one unknown is that if the real-valued function f of one real variable (*i.e.*, $f\colon \mathbb{R}^1 \to \mathbb{R}^1$) has nonvanishing derivative $f'(x_0)$

at x_0 then it has an inverse $g = f^{-1}$ defined (at least) on a neighborhood $(x_0 - \varepsilon, x_0 + \varepsilon)$ of x_0, and the derivative of the inverse is the reciprocal of the derivative: writing $f(x_0) = y_0$,

$$g'(y_0) = 1/f'(x_0).$$

In other words, if x_0 is a regular point of f then f is locally invertible there, and the derivative of the inverse is the inverse of the derivative.

For a mapping $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$

$$\Phi(\vec{x}) = \begin{bmatrix} \varphi_1(\vec{x}) \\ \varphi_2(\vec{x}) \end{bmatrix}$$

a point is regular if the two gradients $\vec{\nabla}\varphi_1$ and $\vec{\nabla}\varphi_2$ are linearly independent. Thus a point can be critical in two ways: if it is a critical point of one of the component functions, or if it is regular for both, but their gradients at the point are parallel (this is equivalent to saying that the two component functions have first-order contact at the point). Again, if the mapping is *continuously* differentiable (*i.e.*, both component functions are $\mathcal{C}^1$), then every regular point has a neighborhood consisting of regular points.

To generalize the differentiation formula from one to two variables, we should reinterpret the derivative $f'(x_0)$ of a function of one variable—which is a number—as the (1×1) matrix representative of the derivative "mapping" $Df_{x_0} : \mathbb{R}^1 \to \mathbb{R}^1$, which multiplies every input by $f'(x_0)$. The inverse of *multiplying* by a number is *dividing* by it, which is *multiplying* by its *reciprocal*. The analogue of the "reciprocal" for a larger matrix is its inverse, as defined in Appendix A.5.

This point of view leads naturally to the following formulation for plane mappings analogous to the situation for mappings of the real line.

Theorem A.6.1 (Inverse Mapping Theorem for $\mathbb{R}^2$). *Suppose*

$$\Phi(\vec{x}) = \begin{bmatrix} \varphi_1(\vec{x}) \\ \varphi_2(\vec{x}) \end{bmatrix}$$

is a $\mathcal{C}^1$ mapping of the plane to itself, and $\vec{x_0}$ is a regular point for Φ—that is, its Jacobian determinant at $\vec{x_0}$ is nonzero:

$$\begin{aligned} \left|\frac{\partial(\varphi_1, \varphi_2)}{\partial(x, y)}\right|(\vec{x_0}) &:= \det \begin{bmatrix} \partial\varphi_1/\partial x(\vec{x_0}) & \partial\varphi_1/\partial y(\vec{x_0}) \\ \partial\varphi_2/\partial x(\vec{x_0}) & \partial\varphi_2/\partial y(\vec{x_0}) \end{bmatrix} \\ &= \frac{\partial\varphi_1}{\partial x}(\vec{x_0})\frac{\partial\varphi_2}{\partial y}(\vec{x_0}) - \frac{\partial\varphi_1}{\partial y}(\vec{x_0})\frac{\partial\varphi_2}{\partial x}(\vec{x_0}) \\ &\neq 0. \end{aligned}$$

Then Φ is locally invertible at $\vec{x_0}$: there exist neighborhoods V of $\vec{x_0}$ and W of $\vec{y_0} = \Phi(\vec{x_0}) = (c, d)$ such that $\Phi(V) = W$, together with a $\mathcal{C}^1$ mapping $\Psi = \Phi^{-1} : W \to V$ which is the inverse of Φ (restricted to V):

$$\Psi(\vec{y}) = \vec{x} \Leftrightarrow \vec{y} = \Phi(\vec{x_0}).$$

Furthermore, the derivative of Ψ at y_0 is the inverse of the derivative of Φ at $\vec{x_0}$:

$$D\Phi^{-1}_{\vec{y_0}} = \left(D\Phi_{\vec{x_0}}\right)^{-1} \tag{A.23}$$

or in matrix terms

$$J\Phi^{-1}(\vec{y_0}) = \left(J\Phi(\vec{x_0})\right)^{-1}$$

(equivalently, the linearization of the inverse is the inverse of the linearization).

Our prime example in § 3.4 of a regular (parametrized) surface was the graph of a function of two variables. As an application of Theorem A.6.1, we see that *every* regular surface can be viewed locally as the graph of a function.

Proposition A.6.2. *Suppose $\mathfrak{S}$ is a regular surface in $\mathbb{R}^3$, and $\overrightarrow{x_0} \in \mathfrak{S}$ is a point on $\mathfrak{S}$. Let P be the plane tangent to $\mathfrak{S}$ at $\overrightarrow{x_0}$.*

Then there is a neighborhood $V \subset \mathbb{R}^3$ of $\overrightarrow{x_0}$ such that the following hold:

(1) *If P is not vertical (i.e., P is not perpendicular to the xy-plane), then $\mathfrak{S} \cap V$ can be expressed as the graph $z = \varphi(x, y)$ of a $\mathcal{C}^1$ function defined on a neighborhood of (x_0, y_0), the projection of $\overrightarrow{x_0}$ on the xy-plane. Analogously, if P is not perpendicular to the xz-plane (resp. yz-plane), then locally $\mathfrak{S}$ is the graph of y (resp. x) as a function of the other two variables. (Figure* A.3*)*

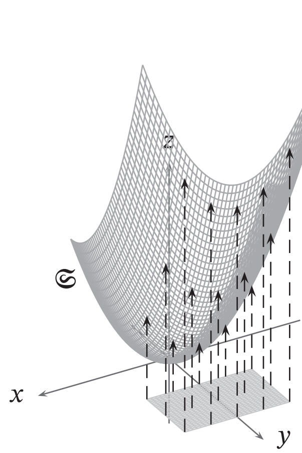


Figure A.3. $\mathfrak{S}$ parametrized by projection on the xy-plane

(2) *$\mathfrak{S} \cap V$ can be parametrized via its projection on P: there is a real-valued function f defined on $P \cap V$ such that*

$$\mathfrak{S} \cap V = \{\vec{x} + f(\vec{x})\,\vec{n} \mid \vec{x} \in V \cap P\},$$

where $\vec{n}$ is a vector normal to P (Figure A.4*).*

As a corollary of Proposition A.6.2, we can establish an analogue for parametrized surfaces of Exercise 10 in § 2.4. Recall that a **coordinate patch** for a parametrization $\vec{p} : \mathbb{R}^2 \to \mathbb{R}^3$ of a surface is a region in the domain of $\vec{p}$ consisting of regular points, on which the mapping is one-to-one. By abuse of terminology, we will also use this term to refer to the image of such a region: that is, a (sub)surface such that every point is a regular value, and such that no point corresponds to two different pairs of coordinates. This is, of course, the two-dimensional analogue of an arc (but with further conditions on the derivative).

Corollary A.6.3. *Suppose $\mathfrak{S}$ is simultaneously a coordinate patch for two regular parametrizations, $\vec{p}$ and $\vec{q}$. Then there exists a differentiable mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ which has no critical points, is one-to-one, and such that*

$$\vec{q} = \vec{p} \circ T. \tag{A.24}$$

We will refer to T as a **reparametrization** of $\mathfrak{S}$.

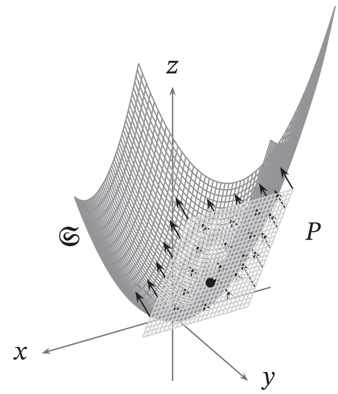


Figure A.4. $\mathfrak{S}$ parametrized by projection on the tangent plane P

A.7 Change of Coordinates: Technical Details

Coordinate Transformations: Local Estimates. In this section we extend the results of Appendix A.5 to (nonlinear) coordinate transformations. In keeping with our general philosophy, we expect the behavior of a coordinate transformation Φ to reflect the behavior of its linearization with respect to area, at least locally.

To sharpen this expectation, we establish some technical estimates. We know what the linearization map $T_{\overrightarrow{x_0}}\Phi$ at a point does to a square: it maps it to a parallelogram whose area is the original area times $\Delta\left(T_{\overrightarrow{x_0}}\Phi\right)$, which is the same as the (absolute value of) the determinant of partials, or **Jacobian determinant**, $\left|J\Phi\left(\overrightarrow{x_0}\right)\right|$. We would like to see how far the image of the same square under the nonlinear transformation Φ deviates from this parallelogram. Of course, we only expect to say something when the square is small.

Suppose P is a parallelogram whose sides are generated by the vectors $\vec{v}$ and $\vec{w}$. We will say the **center** of P is the intersection of the line joining the midpoints of the two edges parallel to $\vec{v}$ (this line is parallel to $\vec{w}$) with the line (parallel to $\vec{v}$) joining the midpoints of the other two sides (Figure A.5).[5] If the center of P is $\overrightarrow{x_0}$, then it is easy to see that

$$P = \left\{\overrightarrow{x_0} + \alpha\vec{v} + \beta\vec{w} \mid |\alpha|, |\beta| \leq 0.5\right\}.$$

Now we can **scale** P by a factor $\lambda > 0$ simply by multiplying all distances by λ. The scaled version will be denoted

$$\lambda P \coloneqq \left\{\overrightarrow{x_0} + \alpha\vec{v} + \beta\vec{w} \mid |\alpha|, |\beta| \leq 0.5\lambda\right\}.$$

When we scale a parallelogram by a factor λ, its area scales by λ^2; in particular, if λ is close to 1, then the area of the scaled parallelogram is close to that of the original. Our immediate goal is to establish that if a square $\mathcal{D}$ is small enough, then its image under Φ is contained between two scalings of its image under the linearization $T\Phi = T_{\overrightarrow{x_0}}\Phi$ of Φ at some point in the square—that is, for some $\varepsilon > 0$, $\Phi(\mathcal{D})$ *contains* $(1-\varepsilon)T\Phi(\mathcal{D})$ and *is contained in* $(1+\varepsilon)T\Phi(\mathcal{D})$.[6] (See Figure A.6.) Note that affine maps respect scaling: for any parallelogram P, $T\Phi(\lambda P) = \lambda T\Phi(P)$.

[5] Equivalently, it is the intersection of the lines joining opposite vertices of P.

[6] Our argument here is motivated by [12, pp. 178-9, 248-51].

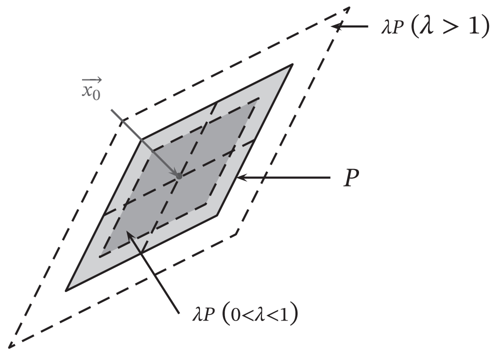


Figure A.5. Center of a Parallelogram, Scaling.

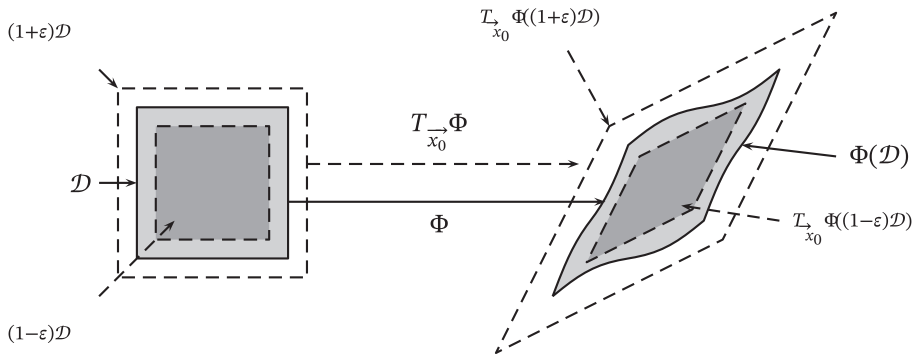


Figure A.6. Nonlinear Image Between two scaled affine images

The general argument is easiest to see when the linear part of $T_{\vec{x_0}}\Phi$ is the identity map and the region is a square; after working this through, we will return to the general case. Given a point $\vec{x_0} = (x_0, y_0)$, we will refer to the square $[x_0 - r, x_0 + r] \times [y_0 - r, y_0 + r]$ as the **square of radius r centered at $\vec{x_0}$.**

Remark A.7.1. *If $\mathcal{D}$ is a square of radius r centered at $\vec{x_0}$, then any point $\vec{x}$ whose distance from the boundary of $\mathcal{D}$ is less than $r\varepsilon$ is* inside $(1+\varepsilon)\mathcal{D}$ *and* outside $(1-\varepsilon)\mathcal{D}$.

(See Figure A.7.)

Lemma A.7.2. *Suppose $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at $\vec{x_0}$ and its derivative at $\vec{x_0}$ is the identity map. Suppose furthermore that $\mathcal{D}$ is a square of radius r, centered at $\vec{x_0}$, such that for all $\vec{x} \in \mathcal{D}$ the first-order contact condition*

$$\left|\Phi(\vec{x}) - T_{\vec{x_0}}\Phi(\vec{x})\right| < \delta\,|\vec{x} - \vec{x_0}| \tag{A.25}$$

holds, where

$$0 < \delta < \frac{\varepsilon}{\sqrt{2}}. \tag{A.26}$$

Then (provided $0 < \varepsilon < 1$*)*

$$T_{\vec{x_0}}\Phi((1-\varepsilon)\mathcal{D}) \subset \Phi(\mathcal{D}) \subset T_{\vec{x_0}}\Phi((1+\varepsilon)\mathcal{D}).$$

Proof. The main observation here is that the distance from the center to any point on the boundary of a square of radius r is between r and $r\sqrt{2}$; the latter occurs at the

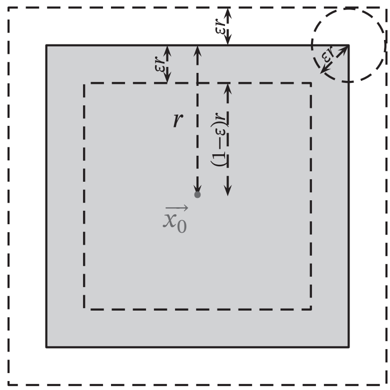


Figure A.7. Remark A.7.1

corners. Thus, for any point $\vec{x}$ on the boundary of $\mathcal{D}$, Equation (A.25) tells us that

$$\left|\Phi\left(\vec{x}\right) - T_{\overrightarrow{x_0}}\Phi\left(\vec{x}\right)\right| \delta(r\sqrt{2}) < \left(\frac{\varepsilon}{\sqrt{2}}\right)(r\sqrt{2}) = r\varepsilon$$

and since $T_{\overrightarrow{x_0}}\Phi\left(\vec{x}\right) = \vec{x}$ by assumption, it follows from Remark A.7.1 that the boundary of $\Phi\left(\mathcal{D}\right)$ (which is the image of the boundary of $\mathcal{D}$) lies entirely inside $T_{\overrightarrow{x_0}}\Phi\left((1+\varepsilon)\mathcal{D}\right)$ and entirely outside $T_{\overrightarrow{x_0}}\Phi\left((1-\varepsilon)\mathcal{D}\right)$, from which the desired conclusion follows. □

To remove the assumption that $D\Phi_{\overrightarrow{x_0}}$ is the identity map in Lemma A.7.2, suppose we are given Φ with arbitrary invertible derivative mapping $L = D\Phi_{\overrightarrow{x_0}}$. Consider the mapping

$$G = L^{-1} \circ \Phi.$$

By the Chain Rule, $DG_{\overrightarrow{x_0}}$ is the identity map, so Lemma A.7.2 says that if the first-order contact condition $\left|G\left(\vec{x}\right) - T_{\overrightarrow{x_0}}G\left(\vec{x}\right)\right| < \delta\left|\vec{x} - \overrightarrow{x_0}\right|$ applies on $\mathcal{D}$ with $0 < \delta < \frac{\varepsilon}{\sqrt{2}}$, then

$$T_{\overrightarrow{x_0}}G\left((1-\varepsilon)\mathcal{D}\right) \subset G\left(\mathcal{D}\right) \subset T_{\overrightarrow{x_0}}G\left((1+\varepsilon)\mathcal{D}\right).$$

Since $\Phi = L \circ G$, we can simply apply L to all three sets above to see that this conclusion implies the corresponding conclusion for Φ:

$$\begin{aligned} T_{\overrightarrow{x_0}}\Phi\left((1-\varepsilon)\mathcal{D}\right) = L\left(T_{\overrightarrow{x_0}}G\left((1-\varepsilon)\mathcal{D}\right)\right) \\ \subset \Phi\left(\mathcal{D}\right) = L\left(G\left(\mathcal{D}\right)\right) \\ \subset T_{\overrightarrow{x_0}}\Phi\left((1+\varepsilon)\mathcal{D}\right) = L\left(T_{\overrightarrow{x_0}}G\left((1+\varepsilon)\mathcal{D}\right)\right). \end{aligned}$$

To formulate the hypotheses in terms of Φ, we note that what is required is

$$\begin{aligned} \left|G\left(\vec{x}\right) - T_{\overrightarrow{x_0}}G\left(\vec{x}\right)\right| = \left|L^{-1}\left(\Phi\left(\vec{x}\right) - T_{\overrightarrow{x_0}}\Phi\left(\vec{x}\right)\right)\right| \\ \leq \|L^{-1}\|\left|\Phi\left(\vec{x}\right) - T_{\overrightarrow{x_0}}\Phi\left(\vec{x}\right)\right| < \delta\left|\vec{x} - \overrightarrow{x_0}\right|, \end{aligned}$$

where $\|L^{-1}\|$ is the maximum value of the ratio $\|\vec{x}\|/\|L(\vec{x})\|$, over all nonzero vectors. (See Remark A.5.2 in Appendix A.5) So dividing both sides of the last inequality by

$||L^{-1}||$, we see that our hypothesis should be

$$\left|\Phi\left(\vec{x}\right) - T_{\overrightarrow{x_0}}\Phi\left(\vec{x}\right)\right| < \frac{\delta}{||L^{-1}||}\left|\vec{x} - \overrightarrow{x_0}\right|, \tag{A.27}$$

where δ satisfies (A.26). So we can say, without any assumptions on $D\Phi_{\overrightarrow{x_0}}$, the following:

Lemma A.7.3. *Suppose $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ has invertible derivative at $\overrightarrow{x_0}$ and R is a square of radius r, centered at $\overrightarrow{x_0}$, such that for all $\vec{x} \in \mathcal{D}$ the first-order contact condition*

$$\left|\Phi\left(\vec{x}\right) - T_{\overrightarrow{x_0}}\Phi\left(\vec{x}\right)\right| < \delta\left|\vec{x} - \overrightarrow{x_0}\right| \tag{(A.30)}$$

holds, where

$$0 < \delta < \frac{\varepsilon}{\sqrt{2}||\left(D\Phi_{\overrightarrow{x_0}}\right)^{-1}||}. \tag{A.28}$$

Then (provided $0 < \varepsilon < 1$)

$$T_{\overrightarrow{x_0}}\Phi\left((1-\varepsilon)\mathcal{D}\right) \subset \Phi\left(\mathcal{D}\right) \subset T_{\overrightarrow{x_0}}\Phi\left((1+\varepsilon)\mathcal{D}\right).$$

In particular, under these conditions, and recalling that scaling a planar region by λ scales its area by λ^2, we have an estimate of area

$$(1-\varepsilon)^2\left|J\Phi\left(\overrightarrow{x_0}\right)\right| \cdot \mathcal{A}\left(R\right) \leq \mathcal{A}\left(\Phi\left(R\right)\right) \leq (1+\varepsilon)^2\left|J\Phi\left(\overrightarrow{x_0}\right)\right| \cdot \mathcal{A}\left(R\right). \tag{A.29}$$

So far, what we have is a *local* result: it only applies to a square that is small enough to guarantee the first-order contact condition (A.30). To globalize this, we need to approximate $\mathcal{D}$ with a non-overlapping union of squares all small enough to guarantee condition (A.30), relative to its center, on each of them individually. So far, though, we only know that if Φ is differentiable at a given point $\overrightarrow{x_0}$, the first-order contact condition holds on *some* sufficiently small square about $\overrightarrow{x_0}$; *a priori* the required size may vary with the point. We would like to get a *uniform* condition: to guarantee (A.30) for *any* square whose radius is less than some fixed number that depends only on the desired ε. When Φ is $\mathcal{C}^1$, this can be established by an argument similar to that used to show that continuity on a compact region guarantees *uniform* continuity there (see *Calculus Deconstructed*, Theorem 3.7.6, or another single-variable calculus text).

Lemma A.7.4. *Suppose $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is $\mathcal{C}^1$ on a compact region $\mathcal{D}$. Given $\delta > 0$, there exists $\delta' > 0$ such that the first-order contact condition*

$$\left|\Phi\left(\vec{x}\right) - T_{\vec{x}'}\Phi\left(\vec{x}\right)\right| < \delta\left|\vec{x} - \vec{x}'\right| \tag{A.30}$$

holds for any pair of points $\vec{x}, \vec{x}' \in \mathcal{D}$ whose distance apart is less than δ'.

Proof. The proof is by contradiction. Suppose no such δ' exists; then for each choice of δ', there exists a pair of points $\vec{x}, \vec{x}' \in \mathcal{D}$ with

$$\left|\vec{x} - \vec{x}'\right| < \delta'$$

but

$$\left|\Phi\left(\vec{x}\right) - T_{\vec{x}'}\Phi\left(\vec{x}\right)\right| \geq \delta\left|\vec{x} - \vec{x}'\right|.$$

We pick a sequence of such pairs, $\overrightarrow{x_k}, \vec{x}'_k \in \mathcal{D}$ corresponding to $\delta' = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$.

By the Bolzano-Weierstrass Theorem, the sequence $\vec{x}'_k$ has a convergent subsequence, which we can assume to be the full sequence: say $\vec{x}'_k \to \overrightarrow{x_0}$. Since Φ is $\mathcal{C}^1$, we can also say that the Jacobian matrices of Φ at $\vec{x}'_k$ converge to the matrix at $\overrightarrow{x_0}$, which means that for k sufficiently high,

$$\left|D\Phi_{\vec{x}'_k}\left(\vec{x}-\vec{x}'_k\right)-D\Phi_{\overrightarrow{x_0}}\left(\vec{x}-\overrightarrow{x_0}\right)\right| \leq \frac{\delta}{2}\left|\vec{x}-\overrightarrow{x_0}\right|$$

for all $\vec{x}$. In particular, the points $\overrightarrow{x_k}$ converge to $\overrightarrow{x_0}$, but

$$\left|\Phi\left(\overrightarrow{x_k}\right)-T_{\overrightarrow{x_0}}\Phi\left(\overrightarrow{x_k}\right)\right| \geq \delta\left|\overrightarrow{x_k}-\overrightarrow{x_0}\right|$$

contradicting the definition of differentiability at $\overrightarrow{x_0}$. □

Combining this with Lemma A.7.3 (or more accurately its rectangular variant), we can prove:

Proposition A.7.5. *Suppose $\Phi: \mathbb{R}^2 \to \mathbb{R}^2$ is a coordinate transformation on the (compact) elementary region $\mathcal{D}$. Then given $\varepsilon > 0$ there exists $\delta > 0$ such that if $R \subset \mathcal{D}$ is any square of radius $r < \delta$,*

$$(1-\varepsilon)^2\left|J\Phi\left(\overrightarrow{x_0}\right)\right|\mathcal{A}(R) < \mathcal{A}(\Phi(R)) < (1+\varepsilon)^2\left|J\Phi\left(\overrightarrow{x_0}\right)\right|\mathcal{A}(R), \tag{A.31}$$

where $\overrightarrow{x_0}$ is the center of R.

Proof. Note that since Φ is $\mathcal{C}^1$ on $\mathcal{D}$, there is a uniform upper bound on the norm $\|\left(D\Phi_{\overrightarrow{x_0}}\right)\|$ for all $\overrightarrow{x_0} \in \mathcal{D}$. Then we can use Lemma A.7.4 to find a bound on the radius which ensures that the first-order contact condition (A.30) needed to guarantee (A.28) holds on any square whose radius satisfies the bound. But then Lemma A.7.3 gives us Equation (A.29), which is precisely what we need. □

As a corollary of Proposition A.7.5 together with Proposition 4.3.2, we can prove the following:

Proposition A.7.6 (Proposition 4.3.3). *Suppose Φ is a $\mathcal{C}^1$ coordinate transformation defined on the rectangle R. Then*

$$\inf_{\vec{x}\in R}\left|J\Phi\left(\vec{x}\right)\right|\cdot\mathcal{A}(R) \leq \mathcal{A}(\Phi(R)) \leq \sup_{\vec{x}\in R}\left|J\Phi\left(\vec{x}\right)\right|\cdot\mathcal{A}(R). \tag{4.7}$$

Proof of Proposition 4.3.3*:*

Given $\varepsilon > 0$, let $\mathcal{P}$ be a partition of R with mesh size $\frac{\delta}{\sqrt{2}}$, where δ is as in Proposition A.7.5. Then each atom R_{ij} of $\mathcal{P}$ has radius at most δ, so Equation (A.31) holds (for R_{ij}, with $\overrightarrow{x_0}$ replaced by $\overrightarrow{x_{ij}}$, the center of R_{ij}). By definition, we can replace $\left|J\Phi\left(\overrightarrow{x_0}\right)\right|$ with $\inf_{\vec{x}\in R}\left|J\Phi\left(\vec{x}\right)\right|$ (*resp.* $\sup_{\vec{x}\in R}\left|J\Phi\left(\vec{x}\right)\right|$) in the first (*resp.* third) term of Equation (A.31) to obtain the inequality

$$(1-\varepsilon)^2\inf_{\vec{x}\in R}\left|J\Phi\left(\vec{x}\right)\right|\mathcal{A}\left(R_{ij}\right) < \mathcal{A}\left(\Phi\left(R_{ij}\right)\right) < (1+\varepsilon)^2\sup_{\vec{x}\in R}\left|J\Phi\left(\vec{x}\right)\right|\mathcal{A}\left(R_{ij}\right).$$

Adding over the atoms of $\mathcal{P}$, we can replace R_{ij} with R. Finally, taking the limit as $\varepsilon \to 0$, we obtain the desired (weak) inequality. □

Change of Coordinates. *Proof of Theorem* 4.3.4*:*

For notational convenience, let us write

$$g = f \circ \Phi.$$

Since both g and $|J\Phi(\vec{x})|$ are continuous, they are bounded and *uniformly* continuous on $\mathcal{D}$.

We first prove Equation (4.8) under the assumption that $\mathcal{D}$ is a rectangle (with sides parallel to the coordinate axes).

Our proof will be based on the following arithmetic observation, which is proved in Exercise 1:

Remark A.7.7. *Suppose $\varepsilon > 0$ and $\delta > 0$ satisfy $\delta < a\varepsilon$, where $a > 0$. Then any pair of numbers $x, x' \geq a$ with $|x' - x| < \delta$ has ratio between $1 - \varepsilon$ and $1 + \varepsilon$.*

This is proved in Exercise 1.

Let $a = \min_{\vec{x}\in\mathcal{D}} |J\Phi(\vec{x})| > 0$ and, given $\varepsilon > 0$, let $\mathcal{P}$ be a partition of $\mathcal{D}$ with mesh size sufficiently small that on each atom R_{ij} of $\mathcal{P}$, $|J\Phi(\vec{x})|$ varies by less that $a\varepsilon$. By Remark A.7.7, this says that for *any* point $\vec{x_{ij}} \in R_{ij}$, the two numbers $(1\pm\varepsilon)^2 \cdot |J\Phi(\vec{x_{ij}})|$ are upper and lower bounds for the values of $|J\Phi|$ at *all* points of R_{ij} (and so for their maximum and minimum). In view of Proposition 4.3.3, this says that for any particular point $\vec{x_{ij}} \in R_{ij}$, $(1\pm\varepsilon)^2 \cdot |J\Phi(\vec{x_{ij}})| \cdot \mathcal{A}(R_{ij})$ are bounds for the area of the image $\Phi(R_{ij})$ of R_{ij}. Letting $\vec{y_{ij}} = \Phi(\vec{x_{ij}})$ be the image of $\vec{x_{ij}}$, we can multiply these inequalities by $g(\vec{x_{ij}}) = f(\vec{y_{ij}})$ and write

$$\begin{aligned}(1-\varepsilon)^2 g(\vec{x_{ij}})\, |J\Phi(\vec{x_{ij}})|\, \mathcal{A}(R_{ij}) \\ < f(\vec{y_{ij}})\, \mathcal{A}(\Phi(R_{ij})) \\ < (1+\varepsilon)^2 g(\vec{x_{ij}})\, |J\Phi(\vec{x_{ij}})|\, \mathcal{A}(R_{ij}).\end{aligned}$$

In particular, by the Integral Mean Value Theorem (Exercise 2), we can pick $\vec{x_{ij}}$ so that

$$\iint_{\Phi(R_{ij})} f(\vec{x})\, dA = f(\vec{y_{ij}}) \cdot \mathcal{A}(\Phi(R_{ij})),$$

and conclude that

$$\begin{aligned}(1-\varepsilon)^2 g(\vec{x_{ij}})\, |J\Phi(\vec{x_{ij}})|\, \mathcal{A}(R_{ij}) \\ < \iint_{\Phi(R_{ij})} f(\vec{x})\, dA \\ < (1+\varepsilon)^2 g(\vec{x_{ij}})\, |J\Phi(\vec{x_{ij}})|\, \mathcal{A}(R_{ij}). \quad \text{(A.32)}\end{aligned}$$

A Riemann sum for the integral $\iint_{\mathcal{D}} g(\vec{x})\, |J\Phi(\vec{x})|\, dA$ is a sum over atoms R_{ij} of $\mathcal{P}$

$$\mathcal{R}(g\,|J\Phi|, \mathcal{P}, \{\vec{x_{ij}}\}) = \sum_{i,j} g(\vec{x_{ij}})\, |J\Phi(\vec{x_{ij}})|\, \mathcal{A}(R_{ij})$$

while the sum of the integrals of f over images of these atoms is the integral over $\mathcal{D}$

$$\sum_{i,j} \iint_{\Phi(R_{ij})} f(\vec{x})\, dA = \iint_{\Phi(\mathcal{D})} f(\vec{x})\, dA.$$

Thus by adding Equation (4.7) over all atoms R_{ij} of $\mathcal{P}$ we have, for each sufficiently fine partition $\mathcal{P}$ of the rectangle $\mathcal{D}$, an inequality of the form

$$(1-\varepsilon)^2\mathcal{R}(g\,|J\Phi|\,,\mathcal{P},\{\overrightarrow{x_{ij}}\}) < \iint_{\Phi(\mathcal{D})} f\left(\vec{x}\right)\,dA < (1+\varepsilon)^2\mathcal{R}(g\,|J\Phi|\,,\mathcal{P},\{\overrightarrow{x_{ij}}\}).$$

Taking a limit over partitions with mesh size going to zero, we have the inequality (for any $\varepsilon > 0$)

$$(1-\varepsilon)^2\iint_{\mathcal{D}} g\left(\vec{x}\right)\left|J\Phi\left(\vec{x}\right)\right|\,dA < \iint_{\Phi(\mathcal{D})} f\left(\vec{x}\right)\,dA < (1+\varepsilon)^2\iint_{\mathcal{D}} g\left(\vec{x}\right)\left|J\Phi\left(\vec{x}\right)\right|\,dA$$

and taking $\varepsilon \to 0$ we obtain the desired equality between the two integrals, when $\mathcal{D}$ is a rectangle.

For a general y-simple region $\mathcal{D}$, we know that we can "sandwich" $\mathcal{D}$ between polygons formed as unions of rectangles (Figure A.8) whose areas differ by an arbitrarily small amount; this establishes the change-of-variables formula for integrals over any y-simple region $\mathcal{D}$.

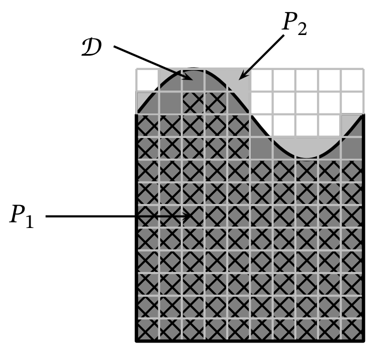


Figure A.8. "Sandwiching" $\mathcal{D}$ between two unions of rectangles

□

Exercises for Appendix A.7

(1) Prove Remark A.7.7, as follows:
   (a) Show that for any two positive numbers x, x',
$$1-\frac{|x'-x|}{x} \leq \frac{x'}{x} \leq 1+\frac{|x'-x|}{x}.$$
   (b) Show that if $a > 0$ is a lower bound for both x and x', and $|x'-x| < \delta$, then
$$\frac{|x'-x|}{x} < \frac{\delta}{a}.$$
   (c) Deduce the remark from this.

(2) Prove the **Integral Mean Value Theorem** in $\mathbb{R}^2$: if $f(\vec{x})$ is a continuous function on a compact region $\mathcal{D} \subset \mathbb{R}^2$, then there is a point $\overrightarrow{x_0}$ where f achieves its average over $\mathcal{D}$: $\iint_{\mathcal{D}} f(\vec{x})\, dA = f(\overrightarrow{x_0}) \cdot \mathcal{A}(\mathcal{D})$. (*Hint:* Use the Intermediate Value Theorem on $f(\vec{x}) \cdot \mathcal{A}(\mathcal{D})$.)

A.8 Surface Area: The Counterexample of Schwarz and Peano

We present here an example, due to Herman Amandus Schwarz (1843-1921) [47] and Giuseppe Peano (1858-1932) [43][7] which shows that the analogue for surfaces of the usual definition of the length of a curve cannot work.

Recall that if a curve $\mathcal{C}$ was given as the path of a moving point $\vec{p}(t)$, $a \le t \le b$, we partitioned $[a, b]$ via $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_n\}$ and approximated $\mathcal{C}$ by a piecewise-linear path consisting of the line segments $[\vec{p}(t_{i-1}), \vec{p}(t_i)]$, $i = 1, \ldots, n$. Since a straight line is the shortest distance between two points, the distance travelled by $\vec{p}(t)$ between $t = t_{i-1}$ and $t = t_i$ is at least the length of this segment, which is $\left\|\vec{p}\left(t_i - \vec{p}(t_{i-1})\right)\right\|$. Thus, the total length of the piecewise-linear approximation is a lower bound for the length of the actual path: we say $\mathcal{C}$ is *rectifiable* if the supremum of the lengths of all the piecewise-linear paths arising from different partitions of $[a, b]$ is finite, and in that case define the (arc)length of the curve to be this supremum. We saw in § 2.5 that every regular arc (that is, a curve given by a one-to-one differentiable parametrization with non-vanishing velocity) the length can be calculated from any regular parametrization as

$$\mathfrak{s}(\mathcal{C}) = \int_a^b \left\|\dot{\vec{p}}(t)\right\| dt.$$

The analogue of this procedure could be formulated for surface area as follows. [8] Let us suppose for simplicity that a surface $\mathfrak{S}$ in $\mathbb{R}^3$ is given by the parametrization $\vec{p}(s, t)$, with domain a rectangle $[a, b] \times [c, d]$. If we partition $[a, b] \times [c, d]$ as we did in § 4.1, we would like to approximate $\mathfrak{S}$ by rectangles in space whose vertices are the images of "vertices" $p_{i,j} = \vec{p}(x_i, y_j)$ of the subrectangles R_{ij}. This presents a difficulty, since *four* points in $\mathbb{R}^3$ need not be coplanar. However, we can easily finesse this problem if we note that *three* points in $\mathbb{R}^3$ are *always* contained in some plane. Using diagonals (see Figure A.9)[9] we can divide each subrectangle R_{ij} into two triangles, say

$$T_{ij1} = \triangle p_{i-1,j-1} p_{i,j-1} p_{i,j}$$
$$T_{ij2} = \triangle p_{i-1,j-1} p_{i-1,j} p_{i,j}.$$

[7]Schwarz tells the story of this example in a note [48, pp. 369-370]. Schwarz initially wrote down his example in a letter to one Angelo Gnocchi in December 1880, with a further clarification in January 1881. In May, 1882 Gnocchi wrote to Schwarz that in a conversation with Peano, the latter had explained that Serret's definition of surface area (to which Schwarz's example is a counterexample) could not be correct, giving detailed reasons why it was wrong; Gnocchi had then told Peano of Schwarz's letters. Gnocchi reported the example in the Notices of the Turin Academy, at which point it came to the attention of Charles Hermite (1822-1901), who published the correspondence in his *Cours d'analyse*. Meanwhile, Peano published his example. After seeing Peano's article, Schwarz contacted him and learned that Peano had independently come up with the same example in 1882.

[8]This approach was in fact followed by J. M. Serret.

[9]There are two ways to do divide each rectangle, but as we shall see, this will not prove to be an issue.

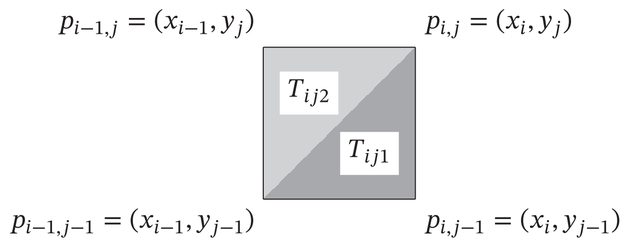


Figure A.9. Dividing R_{ij} into triangles

This tiling $\{T_{ijk} \mid i = 1, \dots, m, \quad j = 1, \dots, n, \quad k = 1, 2\}$ of the rectangle $[a, b] \times [c, d]$ is called a **triangulation**. Now, it would be natural to try to look at the total of the areas of the triangles whose vertices are the points $\overrightarrow{p^*{}_{i,j}} = \vec{p}\,(p_{i,j})$ on the surface

$$T^*_{ij1} = \triangle \overrightarrow{p^*{}_{i-1,j-1}}\,\overrightarrow{p^*{}_{i,j-1}}\,\overrightarrow{p^*{}_{i,j}}$$
$$T^*_{ij2} = \triangle \overrightarrow{p^*{}_{i-1,j-1}}\,\overrightarrow{p^*{}_{i-1,j}}\,\overrightarrow{p^*{}_{i,j}}.$$

and define the area of $\mathfrak{S}$ to be the supremum of these over all triangulations of $[a, b] \times [c, d]$.

Unfortunately, this approach doesn't work; an example found (simultaneously) in 1892 by Herman Amandus Schwarz (1843-1921) and Giuseppe Peano (1858-1932) shows

Proposition A.8.1. *There exist triangulations for the standard parametrization of the cylinder such that the total area of the triangles is arbitrarily high.*

Proof. Consider the finite cylinder surface

$$x^2 + y^2 = 1$$
$$0 \leq z \leq 1.$$

We partition the interval $[0, 1]$ of z-values into m equal parts using the $m+1$ horizontal circles $z = \frac{k}{m}$, $k = 0, \dots, m$. Then we divide each circle into n equal arcs, but in such a way that the endpoints of arcs on any particular circle are directly above or below the midpoints of the arcs on the neighboring circles. One way to do this is to define the angles

$$\theta_{jk} = \begin{cases} \frac{2\pi j}{n} & \text{for } k \ \textit{even}, \\ \frac{2\pi j}{n} - \frac{\pi}{n} & \text{for } k \ \textit{odd} \end{cases}$$

and then the points

$$p_{jk} = (\cos\theta_{jk}, \sin\theta_{jk}, \frac{k}{m}).$$

That is, the points $\{p_{jk}\}$ for k fixed and $j = 1, \dots, n$ divide the k^{th} circle into n equal arcs. Now consider the triangles whose vertices are the endpoints of an arc and the point on a neighboring circle directly above or below the midpoint of that arc (Figure A.10).

The resulting triangulation of the cylinder is illustrated in Figure A.11.

To calculate the area of a typical triangle, we note first (Figure A.12) that its base

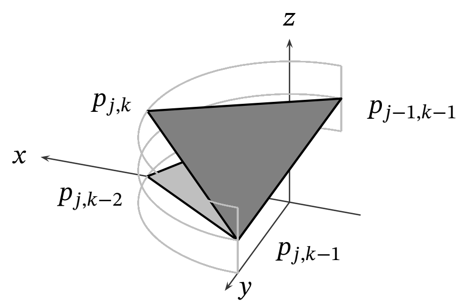


Figure A.10. Typical Triangles

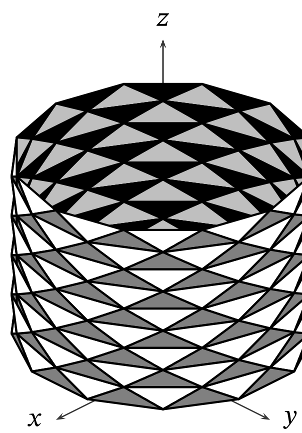


Figure A.11. Triangulation of the Cylinder

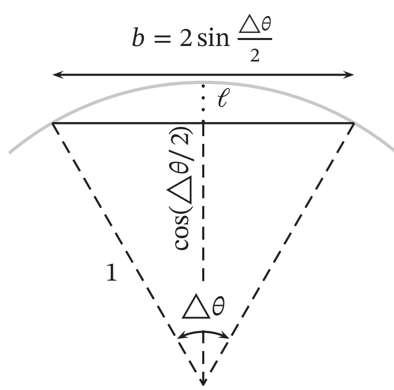


Figure A.12. The Base of a Typical Triangle

is a chord of the unit circle subtending an arc of $\triangle\theta = \frac{2\pi}{n}$ radians; it follows from general principles that the length of the chord is

$$b_n = 2\sin\frac{\triangle\theta}{2} = 2\sin\frac{\pi}{n}.$$

We also note for future reference that the part of the perpendicular bisector of the chord from the chord to the circle has length

$$\ell = 1 - \cos\frac{\triangle\theta}{2} = 1 - \cos\frac{\pi}{n}.$$

To calculate the height of a typical triangle, we note that the vertical distance between the plane containing the base of the triangle and the other vertex is $\frac{1}{m}$, while the distance (in the plane containing the base) from the base to the point below the vertex (the dotted line in Figure A.13) is $\ell = 1-\cos\frac{\pi}{n}$; it follows that the height of the triangle (the dashed line in Figure A.13) is itself the hypotenuse of a right triangle with sides ℓ and $\frac{1}{m}$, so

$$h_{m,n} = \sqrt{\left(\frac{1}{m}\right)^2 + \ell^2} = \sqrt{\left(\frac{1}{m}\right)^2 + \left(1 - \cos\frac{\pi}{n}\right)^2}.$$

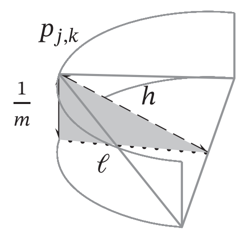


Figure A.13. Calculating the Height of a Typical Triangle

Thus the area of a single triangle of our triangulation (for a given choice of m and n) is

$$\begin{aligned}\triangle A_{m,n} &= \frac{1}{2} b_n h_{m,n} \\ &= \frac{1}{2}\left[2\sin\frac{\pi}{n}\right]\sqrt{\left(\frac{1}{m}\right)^2 + \left(1 - \cos\frac{\pi}{n}\right)^2} \\ &= \left[\sin\frac{\pi}{n}\right]\sqrt{\left(\frac{1}{m}\right)^2 + \left(1 - \cos\frac{\pi}{n}\right)^2}.\end{aligned}$$

Now let us count the number of triangles. There are $m+1$ horizontal circles, each cut into n arcs, and the chord subtending each arc is the base of exactly two triangles of our triangulation, except for the two "end" circles, for which each chord is the base of *one* triangle. This means there are $2mn$ triangles, giving a total area of

$$\begin{aligned}A_{m,n} &= 2mn\triangle Am, n \\ &= 2mn\left[\sin\frac{\pi}{n}\right]\sqrt{\left(\frac{1}{m}\right)^2 + \left(1 - \cos\frac{\pi}{n}\right)^2} \\ &= 2\left[n\sin\frac{\pi}{n}\right]\sqrt{1 + m^2\left(1 - \cos\frac{\pi}{n}\right)^2}.\end{aligned}$$

Now, the quantity in brackets converges to π as $n \to \infty$ and, for n fixed, the square root goes to ∞ as $m \to \infty$; it follows that we can fix a sequence of pairs of values

$\{(m_k, n_k)\}$ (for example, as Schwarz suggests, $m = n^3$) such that the quantity A_{m_k,n_k} diverges to infinity, establishing that the supremum of the total area of piecewise-linear approximations of a cylinder is infinite, and hence gives a bad definition for the area of the cylinder itself. □

To see what is going on here, we might note that at the vertex opposite the chord of each triangle, the plane tangent to the cylinder is vertical, while the plane of the triangle makes an angle with it of size $\theta(m, n)$, where

$$\tan\theta(m,n) = \frac{\ell}{1/m} = \frac{1 - \cos\frac{\pi}{n}}{1/m} = m\left(1 - \cos\frac{\pi}{n}\right).$$

If for example $m = n^3$, one can check (using, *e.g.*, L'Hôpital's rule) that the tangent goes to infinity with n, so in the limit the triangles approach being *perpendicular* to the cylinder.

It turns out that the approach of Serret (using these triangulations) can be made to work, provided we replace the *supremum* of all such total areas with the *limit*, as $\varepsilon \to 0$, of the *infimum* of all such total areas for triangulations with mesh size less than ε. In effect, this means we are looking at triangulations in which the triangles are as close as possible to being tangent to the cylinder.

A.9 The Poincare Lemma

In this appendix, we discuss the issues surrounding sufficiency of the "cross-partials" condition

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

which is necessary for a vector field (*resp.* form) to be irrotational (*resp.* exact). We shall explore this in a sequence of technical lemmas.

Lemma A.9.1. *Suppose D is a right triangle whose legs are parallel to the coordinate axes, and $P(x, y)$ and $Q(x, y)$ are $\mathcal{C}^1$ functions which satisfy Equation* (5.5) *on D:*

$$\frac{\partial Q}{\partial x}(x,y) = \frac{\partial P}{\partial y}(x,y) \text{ for all } (x,y) \in D.$$

Let $\mathcal{C}_1$ be the curve formed by the legs of the triangle, and $\mathcal{C}_2$ its hypotenuse, both oriented so that they start at at a common vertex of the triangle (and end at a common vertex: Figure A.14*). Then*

$$\int_{\mathcal{C}_1} P\,dx + Q\,dy = \int_{\mathcal{C}_2} P\,dx + Q\,dy.$$

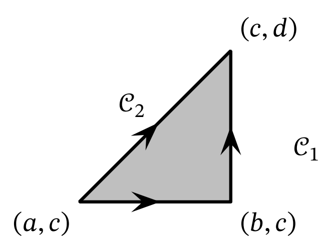


Figure A.14. Integrating along the sides of a triangle

Note that the statement of the theorem allows either the situation in Figure A.14 or the complementary one in which $\mathcal{C}_1$ goes up to (a,d) and then across to (c,d). We give the proof in the situation of Figure A.14 below, and leave to you the modifications necessary to prove the complementary case. (Exercise 1a).

Proof. The integral along $\mathcal{C}_1$ is relatively straightforward: on the horizontal part, y is constant (so, formally, $dy = 0$), while on the vertical part, x is constant ($dx = 0$); it follows that

$$\int_{\mathcal{C}_1} P\,dx + Q\,dy = \int_a^b P(x,c)\,dx + \int_c^d Q(b,y)\,dy.$$

To integrate $P\,dx$ over $\mathcal{C}_2$, we write the curve as the graph of an affine function $y = \varphi(x)$, then use this to write

$$\int_{\mathcal{C}_2} P\,dx = \int_a^b P(x,\varphi(x))\,dx.$$

Similarly, to integrate $Q\,dy$ over $\mathcal{C}_2$ we write it as $x = \psi(y)$, to obtain

$$\int_{\mathcal{C}_2} Q\,dy = \int_c^d Q(\psi(y),y)\,dy.$$

Combining these three expressions, we can express the difference between the two integrals as

$$\begin{aligned}\int_{\mathcal{C}_1} P\,dx + Q\,dy - \int_{\mathcal{C}_2} P\,dx + Q\,dy \\ = \int_a^b [P(x,c) - P(x,\varphi(x))]\,dx + \int_c^d [Q(b,y) - Q(\psi(y),y)]\,dy.\end{aligned}$$

Applying Fundamental Theorem of Calculus to the integrand in the second integral, we write the difference of integrals as an iterated integral and then interpret it as a double integral:

$$\int_{\mathcal{C}_1} Q\,dy - \int_{\mathcal{C}_2} Q\,dy = \int_c^d \int_{\psi(y)}^b \frac{\partial Q}{\partial x}(x,y)\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA \tag{A.33}$$

Similarly, we can apply the Fundamental Theorem of Calculus to the integrand in the first integral to write the difference as an iterated integral; note however that the orientation of the inner limits of integration is backward, so this gives the negative of the appropriate double integral:

$$\int_{\mathcal{C}_1} P\,dx - \int_{\mathcal{C}_2} P\,dx = \int_a^b \int_{\varphi(x)}^c \frac{\partial P}{\partial y}(x,y)\,dy = -\iint_D \frac{\partial P}{\partial y}\,dA. \tag{A.34}$$

But our hypothesis says that these two integrands are equal, so we have

$$\int_{\mathcal{C}_1} P\,dx + Q\,dy - \int_{\mathcal{C}_2} P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA - \iint_D \frac{\partial P}{\partial y}\,dA = 0.$$

□

An immediate corollary of Lemma A.9.1 is the following:

Corollary A.9.2. *Suppose Equation* (5.5) *holds on the rectangle* $D = [a,b]\times[c,d]$*; then*

$$\int_{\mathcal{C}_1} P\,dx + Q\,dy = \int_{\mathcal{C}_2} P\,dx + Q\,dy$$

for any two polygonal curves in D *going from* (a,c) *to* (b,d) *(Figure* A.15*).*

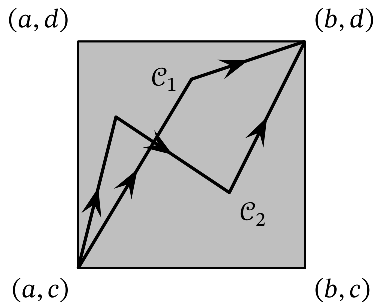


Figure A.15. Polygonal curves with common endpoints in D.

Proof. First, by Lemma A.9.1, we can replace each straight segment of $\mathcal{C}_1$ with a broken line curve consisting of a horizontal and a vertical line segment (Figure A.16) yielding $\mathcal{C}_3$.

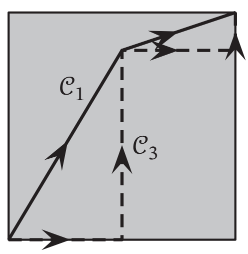


Figure A.16. $\int_{\mathcal{C}_1} P\,dx + Q\,dy = \int_{\mathcal{C}_3} P\,dx + Q\,dy$

Then, we can replace $\mathcal{C}_3$ with $\mathcal{C}_4$, the diagonal of the rectangle (Figure A.17).

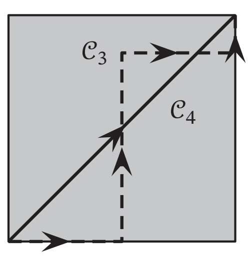


Figure A.17. $\int_{\mathcal{C}_3} P\,dx + Q\,dy = \int_{\mathcal{C}_4} P\,dx + Q\,dy$

Applying the same argument to $\mathcal{C}_2$, we end up with

$$\int_{\mathcal{C}_1} P\,dx + Q\,dy = \int_{\mathcal{C}_4} P\,dx + Q\,dy = \int_{\mathcal{C}_2} P\,dx + Q\,dy.$$

□

We note that the statement of Corollary A.9.2 can be loosened to allow the rectangle $[a, b] \times [c, d]$ to be replaced by any polygonal region containing both points, and then allow any polygonal curves $\mathcal{C}_i$ in this polygonal region which join these points (Exercise 1b).

Using this, we can prove our main result.

Proposition A.9.3 (Poincaré Lemma). [10] *Suppose $P(x, y)$ and $Q(x, y)$ are $\mathcal{C}^1$ functions on the disk centered at (a, b), $D \coloneqq \{(x, y) \mid \operatorname{dist}((x, y), (a, b)) < r\}$, satisfying Equation* (5.5)*: $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$. Then there exists a function f defined on D such that*

$$\frac{\partial f}{\partial x}(x, y) = P(x, y) \tag{A.35}$$

and

$$\frac{\partial f}{\partial y}(x, y) = Q(x, y) \tag{A.36}$$

at every point $(x, y) \in D$.

Proof. Define a function on the disc by

$$f(x, y) = \int_{\mathcal{C}} P\,dx + Q\,dy, \tag{A.37}$$

where $\mathcal{C}$ is any polygonal curve in D from (a, b) to (x, y); by Corollary A.9.2, this is well-defined.

We need to show that equations (A.35) and (A.36) both hold.

To this end, fix a point $(x_0, y_0) \in D$; we shall interpret the definition of f at points on a short horizontal line segment centered at (x_0, y_0) $\{(x_0 + t, y_0) \mid -\varepsilon \le t \le \varepsilon\}$ as given by the curve $\mathcal{C}$ consisting of a fixed curve from (a, b) to (x_0, y_0), followed by the horizontal segment $H(t)$ to $(x_0 + t, y_0)$. Then we can write

$$f(x_0 + t, y_0) - f(x_0, y_0) = \int_{H(t)} P\,dx + Q\,dy = \int_0^t P(x_0 + x, y_0)\,dx;$$

then we can apply the Fundamental Theorem of Calculus to this last integral to see that

$$\frac{\partial f}{\partial x}(x_0, y_0) = \left.\frac{\partial}{\partial t}\right|_{t=0}\left[\int_0^t P(x_0 + x, y_0)\,dx\right] = P(x_0, y_0),$$

proving Equation (A.35). The proof of Equation (A.36) is analogous (Exercise 2). □

This shows that if Equation (5.5) holds everywhere inside some disc, then there is a function f defined on this disc satisfying $df = P\,dx + Q\,dy$ or equivalently, $\vec{\nabla} f = P\vec{\imath} + Q\vec{\jmath}$ at every point of this disc. So if ω (*resp.* $\vec{F}$) is an exact form (*resp.* irrotational vector field) in some planar region D, then given any point in D, there is a function defined *locally* (that is, on some disc around that point) which acts as a potential.

There is, however, a subtle problem with extending this conclusion *globally*—that is, to the whole region—illustrated by the following example.

[10] This result in the two-dimensional case was stated casually, without indication of proof, by Jules Henri Poincaré (1854-1912) in [44], but it turns out to have been stated and proved earlier in the general case by Vito Volterra (1860-1940) in [52], [53]. A version in the language of forms was given by Élie Joseph Cartan (1869-1951) [9] and Édouard Jean-Baptiste Goursat (1858-1936)[11] in 1922. See [46] for a fuller discussion of this history.

Recall that the polar coordinates of a point in the plane are not unique—distinct values of (r, θ) can determine the same geometric point. In particular, the angular variable θ is determined only up to adding an integer multiple of π (an *odd* multiple corresponds to changing the sign of the other polar coordinate, r). Thus, θ is not really a function on the complement of the origin, since its value at any point is ambiguous. However, once we pick out one value $\theta(x, y)$ at a particular point $(x, y) \neq (0, 0)$, then there is only one way to define a *continuous* function that gives a legitimate value for θ at nearby points. Any such function will have the form $\theta(x, y) = \arctan \frac{y}{x} + n\pi$ for some (constant) integer n (why?). When we take the differential of this, the constant term disappears, and we get

$$d\theta = \frac{\frac{dy}{x} - \frac{y\,dx}{x^2}}{1 + \left(\frac{y}{x}\right)^2} = \frac{x\,dy - y\,dx}{x^2 + y^2}.$$

So even though the "function" $\theta(x, y)$ is not uniquely defined, its "differential" *is*. Furthermore, from the preceding discussion, Equation (5.5) holds (you should check this directly, at least once in your life).

Now let us try integrating $d\theta$ around the unit circle $\mathcal{C}$, oriented counterclockwise. The parametrization $x = \cos t$, $y = \sin t$ for $0 \le t \le 2\pi$ leads to $dx = -\sin t\,dt$ and $dy = \cos t\,dt$, so

$$d\theta = \frac{(\cos t)(\cos t\,dt) - (\sin t)(-\sin t\,dt)}{\cos^2 t + \sin^2 t} = \frac{\cos^2 t + \sin^2 t}{\cos^2 t + \sin^2 t}\,dt = dt$$

and thus

$$\int_{\mathcal{C}} d\theta = \int_0^{2\pi} dt = 2\pi$$

which of course would contradict Corollary 5.2.3, if $d\theta$ were exact. In fact, we can see that integrating $d\theta$ along the curve $\mathcal{C}$ amounts to continuing θ along the circle: that is, starting from the value we assign to θ at the starting point $(1, 0)$, we use the fact that there is only one way to continue θ along a short arc through this point; when we get to the end of that arc, we *still* have only one way of continuing θ along a further arc through *that* point, and so on. But when we have come all the way around the circle, the angle has steadily increased, and is now at 2π more than it was when we started!

Another way to look at this phenomenon is to cut the circle into its upper and lower semicircles, and consider the continuation of θ along each from $(1, 0)$ to $(-1, 0)$. Supposing we start with $\theta = 0$ at $(1, 0)$, the continuation along the upper semicircle lands at $\theta = \pi$ at $(-1, 0)$. However, when we continue it along the lower semicircle, our angle goes negative, and we end up with $\theta = -\pi$ at $(-1, 0)$. Thus, the two continuations do not agree.

Now, the continuation of θ is determined not just along an arc through a point, but on a whole *neighborhood* of that point. In particular, we can deform our original semicircle continuously—so long as we keep the two endpoints $(1, 0)$ and $(-1, 0)$, and as long as our deformation never goes through the origin—without changing the effect of the continuation along the curve: continuing θ along any of these deformed curves will still lead to the value π for θ at the end (Figure A.18; see Exercise 3).

We see, then, that our problem with continuing θ (or equivalently, integrating $d\theta$) around the upper and lower semicircles is related to the fact that we cannot deform the

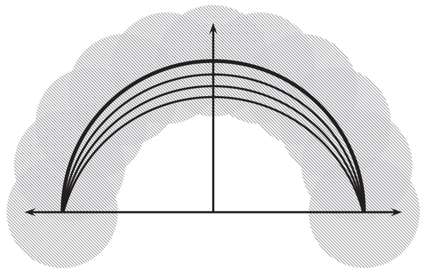


Figure A.18. Continuation along deformed curves

upper semicircle into the lower semicircle without going through the origin—where our form is undefined. A region in which this problem does not occur is called *simply connected*:

Definition A.9.4. *A region $D \subset \mathbb{R}^n$ is* ***simply connected*** *if any pair of curves in D with a common start point and a common end point can be deformed into each other through a family of curves in D (without moving the start point and end point).*

An equivalent definition is: D is simply connected if any closed *curve in D can be deformed (through a family of closed curves in D) to a single point.*[11]

From the discussion above, we can construct a proof of Proposition 5.2.5 in § 5.2:

If $D \subset \mathbb{R}^2$ is a simply connected region, then any differential form $\omega = P\,dx + Q\,dy$ (resp. vector field $\vec{F}$) on D is exact precisely if it is closed (resp. irrotational).

Conservativity and Exactness in Space. The version of Proposition A.9.3 remains true in 3-space—condition (5.6) implies the existence of a potential function, provided the region in question is simply-connected. However, simple-connectedness in $\mathbb{R}^3$ is a bit more subtle than in the plane. In the plane, a closed simple curve encloses a simply-connected region, and a region fails to be simply connected precisely if it has a "hole". In $\mathbb{R}^3$, a hole need not destroy simple connectedness: for example, any curve in a ball with the center excised can be shrunk to the point without going through the origin (Figure A.19); the kind of hole that *does* destroy this property is more like a tunnel through the ball (Figure A.20).

[11]That is, to a curve defined by a constant vector-valued function.

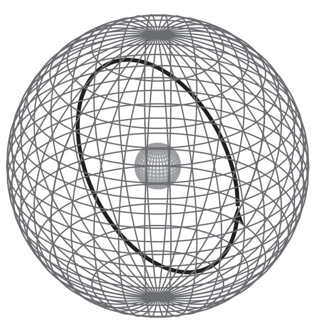


Figure A.19. Simply Connected

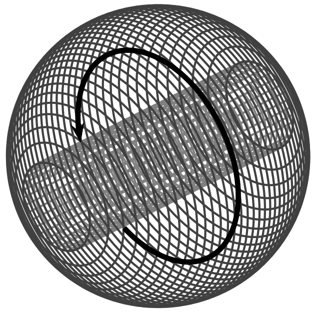


Figure A.20. Not Simply Connected

We shall not prove the version of Proposition A.9.3 for $\mathbb{R}^3$ here, but it will follow from Stokes' Theorem in § 5.6.

Exercises for Appendix A.9

(1) (a) Mimic the proof given for Lemma A.9.1 to prove the complementary case when the curve goes up to (a, d) and then across to (c, d).

(b) Extend the proof given for Corollary A.9.2 when the rectangle is replaced by an arbitrary polygonal region. (*Note*: this is a lot harder than it looks. It is really a Challenge Problem.)

(2) Mimic the proof of Equation (A.35) in the Poincaré Lemma (Proposition A.9.3) to prove Equation (A.36).

Challenge problem:

(3) **Show** that the line integral of the form $d\theta$ over the upper semicircle is unchanged if we replace the semicircle with a curve obtained by deforming the semicircle, keeping the endpoints fixed, as long as the curve doesn't go through the origin during the deformation, as follows:

(a) For any given angle θ_0, let $\mathcal{D}_{\theta_0}$ be the complement of the ray making angle θ_0 with the positive x-axis; that is,

$$\mathcal{D}_{\theta_0} := \{(x, y) \mid (x, y) \neq |(x, y)| (\cos \theta_0, \sin \theta_0)\}.$$

(Note that the origin is excluded from $\mathcal{D}_{\theta_0}$.) Let α be any *other* angle (that is, $\alpha - \theta_0$ is not an even multiple of π); **show** that there is a unique continuous function $\theta(x, y)$ defined on $\mathcal{D}_{\theta_0}$ which equals α along the ray making angle α with the positive x-axis and gives the polar coordinate at every point of $\mathcal{D}_{\theta_0}$: $(x, y) = |(x, y)| (\cos \alpha(x, y), \sin \alpha(x, y))$.

(b) Use the Fundamental Theorem for Line Integrals to conclude that

$$\int_{\mathcal{C}} d\theta = 0$$

for any closed curve contained in $\mathcal{D}_{\theta_0}$, or equivalently, that $\int_{\mathcal{C}} d\theta$ depends only on the endpoints of $\mathcal{C}$, provided $\mathcal{C}$ is contained in $\mathcal{D}_{\theta_0}$. In particular, for any curve in $\mathcal{D}_{\frac{3\pi}{2}}$ from $(1, 0)$ to $(-1, 0)$, this integral equals π.

(c) Suppose now that $\mathcal{C}$ (starting at $(1, 0)$ and ending at $(-1, 0)$) crosses the negative y-axis exactly twice, once clockwise and once counterclockwise. **Show** that $\int_{\mathcal{C}} d\theta = \pi$ as follows: Suppose $\vec{p}(t)$, $0 \leq t \leq 1$, is a parametrization of $\mathcal{C}$, and that these crossings occur at $t_1 < t_2$. Let $\mathcal{C}_1$ (*resp.* $\mathcal{C}_2$ and $\mathcal{C}_3$) be the parts of $\mathcal{C}$ parametrized by $\vec{p}(t)$ with $t \in [0, t_1]$ (*resp.* $[t_1, t_2]$, $[t_2, 1]$), and let $\mathcal{C}'$ be the segment of the negative y-axis from $\vec{p}(t_1)$ to $\vec{p}(t_2)$. Then

$$\begin{aligned}
\int_{\mathcal{C}} d\theta &= \int_{\mathcal{C}_1} d\theta + \int_{\mathcal{C}_2} d\theta + \int_{\mathcal{C}_3} d\theta \\
&= \int_{\mathcal{C}_1} d\theta + \int_{\mathcal{C}_2} d\theta + \int_{\mathcal{C}_3} d\theta + \int_{\mathcal{C}'} d\theta - \int_{\mathcal{C}'} d\theta \\
&= \left(\int_{\mathcal{C}_1} d\theta + \int_{\mathcal{C}'} d\theta + \int_{\mathcal{C}_2} d\theta \right) + \left(\int_{\mathcal{C}_3} d\theta - \int_{\mathcal{C}'} d\theta \right).
\end{aligned}$$

Then the sum of integrals in the first set of parentheses is the integral of $d\theta$ over a curve which consists of going along $\mathcal{C}$ until the first intersection with the negative y-axis, then along this axis, and then back along $\mathcal{C}$; this is not contained in $\mathcal{D}_{\frac{3\pi}{2}}$, but it is contained in $\mathcal{D}_{\theta_0}$ for θ_0 slightly *above* $\frac{3\pi}{2}$. Thus this sum of integrals is still π. The other pair of integrals represents a closed curve consisting of $\mathcal{C}_3$ followed by going *back* along $\mathcal{C}'$; this curve is contained in $\mathcal{D}_{\theta_0}$ for θ_0 slightly *below* $\frac{3\pi}{2}$, and hence equals zero.
Conclude that $\int_{\mathcal{C}} d\theta = \pi$.

(d) Now suppose in general that we have a continuous family of curves $\mathcal{C}_s$, all going from $(1, 0)$ to $(0, 1)$, none of them going through the origin, and starting from the semicircle. More precisely, assume we have a mapping $\vec{p}(s, \theta)$, $(s, t) \in [0, 1] \times [0, \pi]$, so that holding s fixed gives a regular parametrization of $\mathcal{C}_s$, such that

$$\begin{aligned}
\vec{p}(0, \theta) &= (\cos\theta, sin\theta) \\
\vec{p}(s, 0) &= (1, 0) \\
\vec{p}(s, 1) &= (-1, 0)
\end{aligned}$$

and

$$\vec{p}(s, \theta) \neq (0, 0)$$

for all s and θ. We can assume without loss of generality[12] that there are only finitely many points (s, θ) where the curve $\mathcal{C}_s$ is tangent to the negative y-axis, and that such a point of tangency $\vec{p}(s, \theta)$ is crossing the axis as s changes. From this it follows that the number of points at which $\mathcal{C}_s$ crosses the negative axis changes by an even number, if at all, as s increases, and that the extra crossings occur in adjacent pairs, with one crossing in each pair going left-to-right and the other going right-to-left. Explain how the argument of the previous section then shows that $\int_{\mathcal{C}_s} d\theta = \pi$ for all s.

A.10 Proof of Green's Theorem

In this appendix, we prove Theorem 5.3.4:

[12] A rigorous justification of this intuitively reasonable assertion is beyond the scope of this book; it involves the notion of *transversality*.

Suppose $\mathcal{C}$ is a piecewise regular, simple, closed curve with positive orientation in the plane, bounding the regular region D.

Then for any pair of $\mathcal{C}^1$ functions P and Q defined on D,

$$\oint_{\mathcal{C}} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA. \qquad \text{(Equation (5.8))}$$

Proof. First, let us describe D as a y-regular region (Figure A.21): $\varphi(x) \le y \le \psi(x)$ for $a \le x \le b$, and use it to calculate $\oint_{\mathcal{C}} P\,dx$.

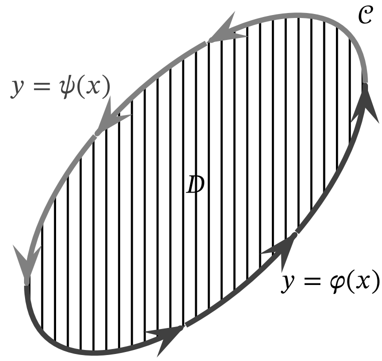


Figure A.21. y-regular version of D

Note that while the *bottom* edge ($y = \varphi(x)$) is traversed with x *increasing*, the *top* edge ($y = \psi(x)$) has x *decreasing*, so the line integral of $P\,dx$ along the bottom edge has the form

$$\int_{y=\varphi(x)} P(x,y)\,dx = \int_a^b P(x,\varphi(x))\,dx,$$

the integral along the *top* edge is reversed, so it has the form

$$\int_{y=\psi(x)} P(x,y)\,dx = \int_a^b -P(x,\psi(x))\,dx.$$

Also, if $\varphi(a) < \psi(a)$ (*resp.* $\varphi(b) < \psi(b)$)—so that $\mathcal{C}$ has a vertical segment corresponding to $x = a$ (*resp.* $x = b$)—then since x is constant, $dx = 0$ along these pieces, and they contribute nothing to $\oint_{\mathcal{C}} P\,dx$. Thus we can write

$$\oint_{\mathcal{C}} P\,dx = \int_a^b P(x,\varphi(x))\,dx + \int_a^b -P(x,\psi(x))\,dx$$
$$= \int_a^b \big(-P(x,\psi(x)) + P(x,\varphi(x))\big)\,dx.$$

But for each fixed value of x, the quantity in parentheses above is the difference between the values of P at the ends of the vertical slice of D corresponding to that x-value. Thus we can write

$$-P(x,\psi(x)) + P(x,\varphi(x)) = \int_{\varphi(x)}^{\psi(x)} -\frac{\partial P}{\partial y}\,dy$$

and hence we have the analogue of Equation (A.34) in § 5.1:

$$\oint_{\mathcal{C}} P\,dx = \int_a^b \int_{\varphi(x)}^{\psi(x)} \left(-\frac{\partial P}{\partial y}\right) dy\,dx = \iint_D \left(-\frac{\partial P}{\partial y}\right) dA. \tag{A.38}$$

Now, to handle $\oint_{\mathcal{C}} Q\,dy$, we revert to the description of D as an x-regular region (Figure A.22): $\alpha(y) \leq x \leq \beta(y)$, for $c \leq y \leq d$.

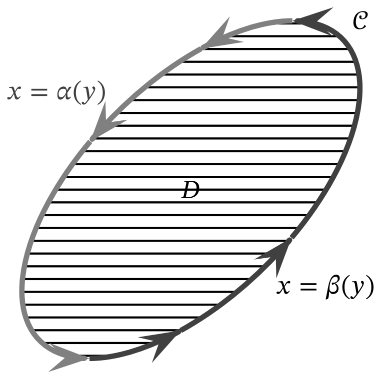


Figure A.22. x-regular version of D

The argument is analogous to that involving $P\,dx$: this time, y is *increasing* on the *right* edge ($x = \beta(y)$) of D and *decreasing* on the *left* ($x = \alpha(y)$). There is no contribution to $\oint_{\mathcal{C}} Q\,dy$ from horizontal segments in $\mathcal{C}$. This leads to the calculation

$$\oint_{\mathcal{C}} Q\,dy = \int_{x=\beta(y)} Q\,dy + \int_{x=\alpha(y)} -Q\,dy$$
$$= \int_c^d \left(Q(\beta(y), y) - Q(\alpha(y), y)\right) dy = \int_c^d \left(\int_{\alpha(y)}^{\beta(y)} \frac{\partial Q}{\partial x}\,dx\right) dy$$

from which we have the analogue of Equation (A.33) in § 5.1:

$$\oint_{\mathcal{C}} Q\,dy = \int_a^b \int_{\alpha(y)}^{\beta(y)} \frac{\partial Q}{\partial x}\,dx\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA. \tag{A.39}$$

Combining these, we get Green's Theorem

$$\oint_{\mathcal{C}} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

when D is a regular region. □

A.11 Non-Orientable Surfaces: The Möbius Band

While every coordinate patch can be given a *local* orientation as discussed in § 5.5, not every surface can be given a *global* orientation. The first (and most famous) example of this phenomenon is the **Möbius band**, discovered by Augustus Ferdinand Möbius (1790-1860). [13]

[13] See footnote on p. 24.

The Möbius band is obtained by taking a rectangle and joining a pair of parallel sides but with a twist (Figure A.23).

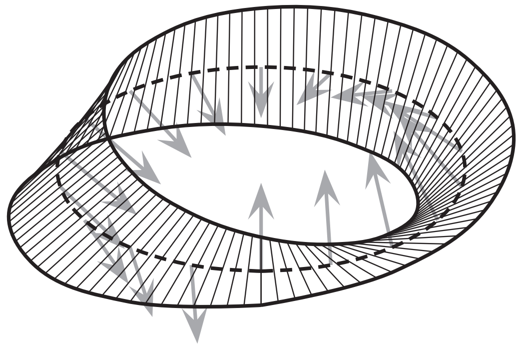


Figure A.23. A Möbius Band

One version of the Möbius band is the image of the mapping defined by

$$x(s,t) = \left(3 + t\cos\frac{s}{2}\right)\cos s$$
$$y(s,t) = \left(3 + t\cos\frac{s}{2}\right)\sin s$$
$$z(s,t) = t\sin\frac{s}{2},$$

where t is restricted to $|t| \le 1$. Geometrically, the central circle corresponding to $t = 0$ is a horizontal circle of radius 3 centered at the origin. For a fixed value of s, the interval $-1 \le t \le 1$ is mapped to a line segment, centered on this circle: as s increases over an interval of length 2π, this segment rotates in the plane perpendicular to the circle by an angle of π. This means that the two intervals corresponding to s and to $s + 2\pi$ are mapped to the same line segment, but in opposite directions. In different terms, the vector $\frac{\partial\vec{p}}{\partial s}(s,0)$ always points along the central circle, in a counterclockwise direction (viewed from above); the vector $\frac{\partial\vec{p}}{\partial t}(s,0)$ is always perpendicular to it: the two points $\vec{p}(s,0)$ and $\vec{p}(s+2\pi,0)$ are the same, but the two vectors $\frac{\partial\vec{p}}{\partial t}(s,0)$ and $\frac{\partial\vec{p}}{\partial t}(s+2\pi,0)$ point in *opposite* directions. Now, if we start at $\vec{p}(s,0)$ with a normal parallel to the cross product $\frac{\partial\vec{p}}{\partial s} \times \frac{\partial\vec{p}}{\partial t}$, then a continuation of this normal field along the central circle continues to point in the direction of $\frac{\partial\vec{p}}{\partial s} \times \frac{\partial\vec{p}}{\partial t}$; however, when we return to the same position (but corresponding to an s-value 2π higher), this direction is opposite to the one we have already chosen there. This surface is **non-orientable**: it is impossible to give it a global orientation.

A.12 Proof of Divergence Theorem

In this appendix we prove Theorem 5.8.5 from § 5.8:

Suppose

$$\vec{F}(x,y,z) = P(x,y,z)\vec{\imath} + Q(x,y,z)\vec{\jmath} + R(x,y,z)\vec{k}$$

is a $\mathcal{C}^1$ *vector field on the regular region* $\mathfrak{D} \subset \mathbb{R}^3$.

Then the flux integral of $\vec{F}$ over the boundary $\partial\mathfrak{D}$, oriented outward, equals the (triple) integral of its divergence over the interior of $\mathfrak{D}$:

$$\iint_{\partial\mathfrak{D}} \vec{F} \cdot d\vec{S} = \iiint_{\mathfrak{D}} \operatorname{div} \vec{F}\, dV.$$

Proof. Equation (5.33) can be written in terms of coordinates:

$$\iint_{\partial\mathfrak{D}} (P\vec{\imath} + Q\vec{\jmath} + R\vec{k}) \cdot d\vec{S} = \iiint_{\mathfrak{D}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV$$

and this in turn can be broken into three separate statements:

$$\iint_{\partial\mathfrak{D}} P\vec{\imath} \cdot d\vec{S} = \iiint_{\mathfrak{D}} \frac{\partial P}{\partial x}\, dV$$

$$\iint_{\partial\mathfrak{D}} Q\vec{\jmath} \cdot d\vec{S} = \iiint_{\mathfrak{D}} \frac{\partial Q}{\partial y}\, dV$$

$$\iint_{\partial\mathfrak{D}} R\vec{k} \cdot d\vec{S} = \iiint_{\mathfrak{D}} \frac{\partial R}{\partial z}\, dV.$$

We shall prove the third of these; the other two are proved in essentially the same way (Exercise 8). For this statement, we view $\mathfrak{D}$ as a z-regular region, which means that we can specify it by a set of inequalities of the form

$$\begin{aligned} \varphi(x,y) &\le z \le \psi(x,y) \\ c(x) &\le y \le d(x) \\ a &\le x \le b. \end{aligned}$$

Of course the last two inequalites might also be written as limits on x in terms of functions of y, but the assumption that $\mathfrak{D}$ is simultaneously y-regular means that an expression as above is possible; we shall not dwell on this point further. In addition, the regularity assumption on $\mathfrak{D}$ means that we can assume the functions $\varphi(x,y)$ and $\psi(x,y)$ as well as the functions $c(x)$ and $d(x)$ are all $\mathcal{C}^2$.

With this in mind, let us calculate the flux integral of $R(x,y,z)\vec{k}$ across $\partial\mathfrak{D}$. The boundary of a z-regular region consists of the graphs $z = \psi(x,y)$ and $z = \varphi(x,y)$ forming the top and bottom boundary of the region and the vertical cylinder built on the boundary of the region $\mathcal{D}$ in the xy-plane determined by the second and third inequalities above. Note that the normal vector at points on this cylinder is horizontal, since the cylinder is made up of vertical line segments. This means that the dot product $R\vec{k} \cdot \vec{n}$ is zero at every point of the cylinder, so that this part of the boundary contributes nothing to the flux integral $\iint_{\partial\mathfrak{D}} R\vec{k} \cdot d\vec{S}$. On the top graph $z = \psi(x,y)$ the outward normal has a positive vertical component, while on the bottom graph $z = \varphi(x,y)$ the outward normal has a *negative* vertical component. In particular, the element of oriented surface area on the top has the form

$$d\vec{S} = \left(-\psi_x\vec{\imath} - \psi_y\vec{\jmath} + \vec{k} \right) dA$$

while on the bottom it has the form

$$d\vec{S} = \left(\varphi_x\vec{\imath} + \varphi_y\vec{\jmath} - \vec{k} \right) dA.$$

Pulling this together with our earlier observation, we see that

$$\begin{aligned}
\iint_{\partial\mathfrak{D}} R\vec{k}\cdot d\vec{S} &= \iint_{z=\psi(x,y)} R\vec{k}\cdot d\vec{S} + \iint_{z=\varphi(x,y)} R\vec{k}\cdot d\vec{S} \\
&= \iint_{\mathcal{D}} \left(R\left(x,y,\psi\left(x,y\right)\right)\vec{k}\right)\cdot\left(-\psi_x\vec{\imath}-\psi_y\vec{\jmath}+\vec{k}\right)\,dA \\
&\quad + \iint_{\mathcal{D}} \left(R\left(x,y,\varphi\left(x,y\right)\right)\vec{k}\right)\cdot\left(\varphi_x\vec{\imath}+\varphi_y\vec{\jmath}-\vec{k}\right)\,dA \\
&= \iint_{\mathcal{D}} \left(R\left(x,y,\varphi\left(x,y\right)\right)-R\left(x,y,\psi\left(x,y\right)\right)\right)\,dA.
\end{aligned}$$

The quantity in parentheses can be interpreted as follows: given a vertical "stick" through $(x,y)\in\mathcal{D}$, we take the difference between the values of R at the ends of its intersection with $\mathfrak{D}$. Fixing (x,y), we can apply the Fundamental Theorem of Calculus to the function $f(z)=R(x,y,z)$ and conclude that for each $(x,y)\in\mathcal{D}$,

$$R\left(x,y,\varphi\left(x,y\right)\right)-R\left(x,y,\psi\left(x,y\right)\right)=\int_{\varphi(x,y)}^{\psi(x,y)}\frac{\partial R}{\partial z}\left(x,y,z\right)\,dz$$

so that

$$\begin{aligned}
\iint_{\partial\mathfrak{D}} R\vec{k}\cdot d\vec{S} &= \iint_{\mathcal{D}} \left(R\left(x,y,\varphi\left(x,y\right)\right)-R\left(x,y,\psi\left(x,y\right)\right)\right)\,dA \\
&= \iint_{\mathcal{D}}\left(\int_{\varphi(x,y)}^{\psi(x,y)}\frac{\partial R}{\partial z}\left(x,y,z\right)\,dz\right)dA \\
&= \iiint_{\mathfrak{D}}\frac{\partial R}{\partial z}\left(x,y,z\right)\,dV
\end{aligned}$$

as required. The other two statements are proved by a similar argument, which we leave to you (Exercise 8). □

A.13 Answers to Selected Exercises

Chapter 1.

§1.1: 1a: $\operatorname{dist}((1,1),(0,0))=\sqrt{(1-0)^2+(1-0)^2}=\sqrt{2}$

2: $y=z=0$

3a: (i) $\theta=\frac{\pi}{2}$; $z=-1$.

(ii) $\rho=\sqrt{1^2+(-1)^2}=\sqrt{2}$; $\theta=\frac{\pi}{4}$; $\phi=\frac{3\pi}{4}$.

4a: $x=1$; $y=\sqrt{3}$; $z=0$.

§1.2: 1a: $\vec{v}+\vec{w}=(2,6)$; $\vec{v}-\vec{w}=(4,2)$; $2\vec{v}=(6,8)$; $3\vec{v}-2\vec{w}=(11,8)$; $\left\|\vec{v}\right\|=5$; $\vec{u}=\frac{1}{5}(3,4)=\left(\frac{3}{5},\frac{4}{5}\right)$

2a,b: dependent; independent.

§1.3: 1a: (i) $y=-x$, or $x+y=0$; (ii) $\begin{array}{rcl} x &=& t \\ y &=& -t \end{array}$ (iii) $\vec{p}(t)=(0,0)+t(1,-1)$

2a: $m=-2$, $b=3$;

3a: (i) $\begin{array}{rcl} x &=& 2-t \\ y &=& -1+2t \\ z &=& 3+t \end{array}$; (ii) $\vec{p}\,(t) = (2,-1,3) + t(-1,2,1)$

4a: Not parallel; intersect at $(2,1)$

6a: Intersect at $(2,-1,1)$

§1.4: 1a $\vec{v}\cdot\vec{w} = 12$; $\left\|\vec{v}\right\| = \sqrt{13}$; $\left\|\vec{w}\right\| = \sqrt{13}$; $\cos\theta = \frac{12}{13}$; $\operatorname{proj}_{\vec{w}}\vec{v} = \left(\frac{36}{13},\frac{24}{13}\right)$; $\operatorname{proj}_{\vec{v}}\vec{w} = \left(\frac{24}{13},\frac{36}{13}\right)$

§1.5: 1a: $x+y+z=4$

3a: $x=s; y=t, z=4-2s-3t$, or $\vec{p}\,(s,t) = (0,0,4)+s(1,0,-2)+t(0,1,-3)$.

§1.6: 1a: 10

2a: Positive orientation: $\sigma(A,B,C)\mathcal{A}\left(\triangle ABC\right) = \frac{3}{2}$

3a: $(1)\vec{\imath} - (-7)\vec{\jmath} + (-5)\vec{k} = \vec{\imath} + 7\vec{\jmath} - 5\vec{k}$

§1.7: 1a: $2(x-1)-4(y-2)-(z-3)=0$ or $2x-4y-z=-9$.

2a: $x=3-s+5t, y=-1+2s+4t; z=2-3s-t$.

4a: $x=1-8t, y=1+t, z=-1+14t$.

Chapter 2.

§2.1: 1a: $y+2=(x-1)^2$ or $x^2-2x-y=1$

2a: Parabola with vertex $(-1,1)$ and axis $y=1, z=\sin\theta$ over $0\le\theta\le 2\pi$.

6a:

$$\begin{aligned}
\cosh^2 t - \sinh^2 t &= \left(\frac{e^t+e^{-t}}{2}\right)^2 - \left(\frac{e^t-e^{-t}}{2}\right)^2 \\
&= [\left(\frac{e^t}{2}\right)^2 + 2\left(\frac{e^t}{2}\right)\left(\frac{e^{-t}}{2}\right) + \left(\frac{e^t}{2}\right)^2] \\
&\quad - [\left(\frac{e^t}{2}\right)^2 - 2\left(\frac{e^t}{2}\right)\left(\frac{e^{-t}}{2}\right) + \left(\frac{e^t}{2}\right)^2] \\
&= [\frac{e^{2t}}{4} + \frac{1}{2} + \frac{e^{-2t}}{4}] \\
&\quad - [\frac{e^{2t}}{4} + \frac{1}{2} + \frac{e^{-2t}}{4}] \\
&= \frac{1}{2} + \frac{1}{2} \\
&= 1.
\end{aligned}$$

§2.3: 1a: $(0,1)$

1c: The second coordinate diverges; there is no limit.

1i: $L=(1,1,1)$.

2a: $(0,1)$ and $(0,-1)$

3a: $T_1\vec{p}\,(t) = (1,1)+t(1,2) = (1+t, 1+2t)$.

§2.4: 1a: Clearly, $\vec{q}(s) = \vec{p}(\cos s)$; in the other direction, $\vec{p}(t) = \vec{q}(\arccos t)$ (note that the domains and images match).

§2.5: 1a: $\int_0^1 \sqrt{1 + (nx^{n-1})^2}\, dx$

2a: $\frac{13\sqrt{13}-8}{27}$

3a: $\int_1^{10} u^{1/2}\, du = \frac{2}{3}\left(10\sqrt{10} - 1\right)$

3j: $\int_0^{\pi} (\cos t \sin t)(\sqrt{2}\, dt) = 0.$

Chapter 3.

§3.1: 1a: 1

1c: The limit does not exist.

2a: 0

§3.2: 1a: 0

4a: $\phi\,(() - 1 + \triangle x, 2 + \triangle y, 1 + \triangle z) = 5 + 3\triangle x - 2\triangle y + \triangle z.$

§3.3: 1a: $\frac{\partial f}{\partial x} = 2xy - 2y^2, \quad \frac{\partial f}{\partial y} = x^2 - 4xy$

2a: $\vec{\nabla} f\,(1,2) = -6\vec{\imath} - 12\vec{\jmath}$; $d_{(1,-2)}f\,(\triangle x, \triangle y) = -6(\triangle x + 2\triangle y)$;$T_{(1,-2)}f\,(x,y) = -9 - 6x - 12y$.

3a: $T_{(9,4)}f\,(8.9, 4.2) = 6 + \frac{1}{3}(-0.1) + \frac{3}{4}(0.2) \approx 6.1167.$

5: (i) $\vec{\nabla} f \cdot \vec{\imath} = \frac{1}{\sqrt{2}} \approx 0.71$ (ii) $\vec{\nabla} f \cdot \vec{\jmath} = \frac{1}{\sqrt{2}} + 1 \approx 1.7$ (iii) $\vec{\nabla} f \cdot \left(\frac{1}{\sqrt{2}}(\vec{\imath} + \vec{\jmath})\right) = 1 + \frac{1}{\sqrt{2}} \approx 1.71$ (iv) $\vec{\nabla} f \cdot \left(\frac{1}{\sqrt{2}}(\vec{\jmath} - \vec{\imath})\right) = \frac{1}{\sqrt{2}} \approx 0.71$ (v) $\frac{1}{\left\|\vec{\nabla} f\right\|}\vec{\nabla} f = \frac{1}{\sqrt{2+\sqrt{2}}} \approx (0.38)\vec{\imath} + 0.78\vec{\jmath}$; $\left\|\vec{\nabla} f\right\| = \sqrt{2 + \sqrt{2}} \approx 1.85$

§3.4: 1a: $(1,-1)$: $\frac{\partial f}{\partial x}(1,-1) = 1$, $\frac{\partial f}{\partial y}(1,-1) = 5$; $\frac{d\phi}{dx} = -\frac{1}{5}$, $\frac{d\psi}{dy} = -5$.

$(2,-6)$: $\frac{\partial f}{\partial x}(2,-6) = 0$, $\frac{\partial f}{\partial y}(2,-6) = 112$; No $\psi\,(y)$; $\frac{d\phi}{dx} = 0$.

2a: The level set for

$$z = c$$

has equation

$$9x^2 + 4y^2 = c, \text{ or } \frac{x^2}{c/9} + \frac{y^2}{c/4} = 1.$$

We cannot have $c < 0$ (since the left side is never negative) and for $c = 0$, this gives just the origin in the plane. For $c > 0$, it can be rewritten in the standard form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } a = \frac{\sqrt{c}}{3}, \quad b = \frac{\sqrt{c}}{2}$$

which determines an ellipse with major axis along the y-axis and minor axis along the x-axis, with the ratio of the two $3 : 2$. To get the profile, if we set $y = 0$ we get $9x^2 = z$ which is a parabola in the xz-plane opening up from the origin. Similarly, setting $x = 0$ gives $4y^2 = z$, a parabola in the yz-plane opening up from the origin. This is an elliptic paraboloid. (See Figure A.24.)

§3.5: 1a: At $(1,-2,-3)$: (a) $z = -3+2(x-1)+4(y+2)$; (b) $x = 2+s$, $y = -1+t$, $z = -3+2s+4t$ at $(2,-1,3)$: (a) $z = 3+4(x-2)+2(y+1)$; (b) $x = -2+s$, $y = -1+t$, $z = 3+4s+2t$

§3.6: 1: (a) $z - 2y = -1$; (b) $x = s$, $y = -1+t$, $z = 1+2t$

§3.7: 1: $\max_D f(x,y) = \frac{3}{2} = f\left(\cos(\frac{\pi}{8} + n\frac{\pi}{2}), \sin(\frac{\pi}{8} + n\frac{\pi}{2})\right)$, $n = 0,1,2,3$, $\min_D f(x,y) = 0 = f(0,0)$.

§3.8: 1a: $\frac{\partial f}{\partial x}(x,y) = \cos x$, $\frac{\partial f}{\partial y}(x,y) = -\sin y$; $\frac{\partial^2 f}{\partial x^2}(x,y) = -\sin x$, $\frac{\partial^2 f}{\partial y \partial x}(x,y) = 0$, $\frac{\partial^2 f}{\partial y^2}(x,y) = -\cos y$.

2a: $f_{xx}(x,y,z) = 2$, $f_{xy}(x,y,z) = 0$, $f_{yy}(x,y,z) = 2$, $f_{xz}(x,y,z) = 0$, $f_{yz}(x,y,z) = 0$, $f_{zz}(x,y,z) = 2$.

3a: $T^2_{(-1,-2)}f(\triangle x, \triangle y) = -4+12\triangle x+4\triangle y-12\triangle x^2-12\triangle x\triangle y-\triangle y^2$.

§3.9: 1a: $[Q] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$; $\Delta_2 = \det\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 0$. Not definite (in fact, $Q(x,y) = 0$ along the whole line $x = y$).

2a: Critical point at $(0,0)$: $\Delta_2(0,0) = (10)(20) - (-2)^2 = 194 > 0$ and $\Delta_1(0,0) = 10 > 0$: Hessian is positive definite, so critical point is local minimum.

Chapter 4.

§4.1: 1a: $\iint_{[0,1]\times[0,2]} 4x\, dA = 4$

§4.2: 1a: This is y-regular:

$$\begin{cases} x^2 - 1 & \leq y \leq & 9 - x^2 \\ 0 & \leq x \leq & 2 \end{cases}$$

2a:

$$\begin{cases} \frac{y}{2} & \leq x \leq & 1 \\ 0 & \leq y \leq & 1 \end{cases}$$

3a: $\frac{1}{6}$

4a: $\int_0^2 \int_0^x \left(4x^2 - 6y\right) dy\, dx = 8$

9a: Region is y-symmetric, xy is odd in y, so integral equals zero.

§4.3: 1a: $\frac{15\pi}{2}$

3:

$$\begin{aligned} \iint_{[0,1]\times[0,1]} \frac{1}{\sqrt{1+2x+3y}}\, dA &= \iint_{[0,2]\times[0,3]} \left(\frac{1}{\sqrt{1+u+v}}\right)\left(\frac{1}{6}\, dA_{u,v}\right) \\ &= \frac{4}{3}\sqrt{6} - \frac{2}{3}\sqrt{3} - \frac{14}{9}. \end{aligned}$$

§4.4: 1a:

$$\iint_{[-1,1]\times[-1,1]} d\mathcal{S} = \int_{-1}^{1}\int_{-1}^{1}\sqrt{1+x^2}\,dx\,dy$$
$$= \frac{1}{4}\left[\ln\frac{1+\sqrt{2}}{1-\sqrt{2}} + 2\sqrt{2}\right].$$

1f:

$$\iint_{x^2+y^2\leq 4} d\mathcal{S} = \iint_{x^2+y^2\leq 4} \frac{2\sqrt{2}}{\sqrt{8-x^2-y^2}}\,dx\,dy$$
$$= 8\pi\sqrt{2}(\sqrt{2}-1).$$

2a:

$$\int_0^1\int_0^1 (x^2+y^2)(\sqrt{6}\,dx\,dy) = \frac{2\sqrt{6}}{3}.$$

§4.5: 1a:

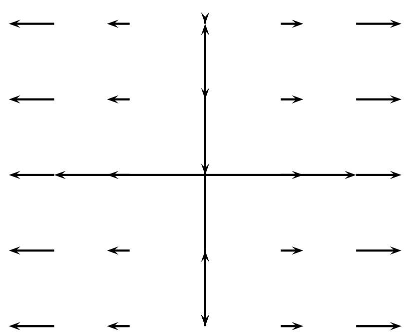
$$\iiint_{\mathcal{D}} x^3\ dV = \int_0^1\int_0^1\int_0^1 x^3\,dx\,dy\,dz$$
$$= \frac{138}{2}.$$

2a:

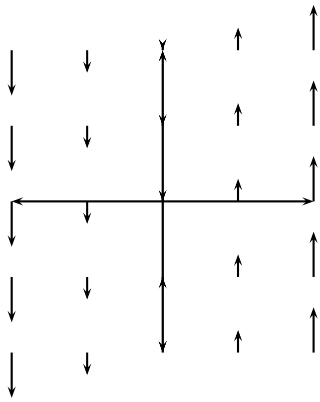
$$\begin{cases} \sqrt{2} & \leq z \leq & \sqrt{4-x^2-y^2} \\ -\sqrt{2-x^2} & \leq y \leq & \sqrt{2-x^2} \\ -\sqrt{2} & \leq x \leq & \sqrt{2} \end{cases}$$

Chapter 5.

§5.1: 1a: $x\vec{\imath}$:



1b: $x\vec{\jmath}$:



2a:

$$\begin{aligned}\int_{\mathcal{C}} \vec{F} \cdot d\vec{s} &= \int_{-2}^{1} (t + 2t^3)\, dt \\ &= \left(\frac{t^2}{2} + \frac{t^4}{2}\right)_{-2}^{1} \\ &= \left(\frac{1}{2}\frac{1}{2}\right) - \left(\frac{1}{8} + \frac{1}{32}\right) \\ &= \frac{27}{32}\end{aligned}$$

3a:

$$\begin{aligned}P\, dx + Q\, dy &= (x^2 + y^2)\, dx + (y - x)\, dy \\ &= (0^2 + t^2) \cdot 0t + (t - 0)\, dt\end{aligned}$$

and

$$\begin{aligned}\int_{\mathcal{C}} P\, dx + Q\, dy &= \int_0^1 t\, dt \\ &= \frac{t^2}{2}\Big|_0^1 \\ &= \frac{1}{2}.\end{aligned}$$

4a:

$$\int_{\mathcal{C}_1} \omega = \int_0^{\pi} dt = \pi$$

$$\begin{aligned}\int_{\mathcal{C}_2} \omega &= \int_{-1}^{1} t\, dt \\ &= \frac{t^2}{2}\Big|_{-1}^{1} = 0;\end{aligned}$$

thus,

$$\int_{\mathcal{C}} \omega = \int_{\mathcal{C}_1} \omega + \int_{\mathcal{C}_2} \omega$$
$$= \pi + 0 = \pi.$$

§5.2: 1a: The list of all potential functions for $\vec{F}$ is $f(x,y) = x^2y + xy^2 + C$, and $\int_{\mathcal{C}} \vec{F} \cdot \vec{T}\, d\mathfrak{s} = f(1,1) - f(0,0) = 2.$

2a: The list of all potential functions for $\vec{F}$ is $f(x,y,z) = x^2y + xz + yz + C$, and $\int_{\mathcal{C}} \vec{F} \cdot d\vec{\mathfrak{s}} = f(1,2,2]) - f(1,0.1) = 8 - 1 = 7.$

3a: The list of all potential functions for $\vec{F}$ is $f(x,y,z) = x^2yz^3 + C$; $\int_{\mathcal{C}} \omega = f(1,2,2) - f(-1,1,-1) = 24 - 3 = 21.$

§5.3: 1a: $\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}[x] = 1$, $\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}[y] = 1$, so $\iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{\mathcal{D}} (1-1)\, dA = 0.$

2a:

$$\oint_{\mathcal{C}} \vec{F} \cdot d\vec{\mathfrak{s}} = \int_0^{2\pi} \left(-\sin 2\theta - \frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$$
$$= \left(\frac{1}{2}\cos 2\theta - \frac{\theta}{2} - \frac{1}{2}\sin 2\theta\right)_0^{2\pi}$$
$$= -\pi$$

Now, to use Green's Theorem, we need to calculate $\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}[-(x+y)] = -1$ and $\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}[x] = 0$, so $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -1$ and $\oint_{\mathcal{C}} \vec{F} \cdot d\vec{\mathfrak{s}} = \iint_{\mathcal{D}} -dA = -\pi$.

§5.4: 1a:

$$(dx \wedge dy)_{(2,1)}(2\vec{\imath} - 3\vec{\jmath}, 3\vec{\imath} - 2\vec{\jmath}) = \begin{vmatrix} 2 & -3 \\ 3 & -2 \end{vmatrix} = -4 + 9 = 5.$$

2a:

$$\iint_{[0,1]\times[0,1]} x\, dx \wedge y\, dy = \iint_{[0,1]\times[0,1]} xy\, dA = \int_0^1 \int_0^1 xy\, dx\, dy$$
$$= \int_0^1 \frac{x^2}{2} y\Big|_{x=0}^1 dy = \int_0^1 \frac{1}{2} y\, dy$$
$$= \frac{y^2}{4}\Big|_0^1 = \frac{1}{4}.$$

3a:

$$d[xy\, dx + xy\, dy] = (y\, dx + x\, dy) \wedge dx + (y\, dx + x\, dy) \wedge dy$$
$$= y\, dx \wedge dx + x\, dy \wedge dx + y\, dx \wedge dy + x\, dy \wedge dy$$
$$= 0 - x\, dx \wedge dy + y\, dx \wedge dy + 0$$
$$= (y - x)\, dx \wedge dy.$$

3b:

$$\begin{aligned} d[x\,dx + y\,dy] &= (\,dx) \wedge \,dx + (\,dy) \wedge \,dy \\ &= 0. \end{aligned}$$

§5.5: 1a: $d\vec{S} = (-g_x\vec{\imath} - g_y\vec{\jmath} + \vec{k})\,dA = (-3\vec{\imath} - 2\vec{\jmath} + \vec{k})\,dA$; $\vec{F} \cdot \,d\vec{S} = (3x + 2y)\,dA$, so $\iint_{\mathfrak{S}} \vec{F} \cdot \,d\vec{S} = \iint_{[0,1]\times[0,1]}(3x + 2y)\,dA = \frac{5}{2}$.

2a: $d\vec{S} = (-g_x\vec{\imath} - g_y\vec{\jmath} + \vec{k})\,dA = (-a\vec{\imath} - b\vec{\jmath} + \vec{k})\,dA$ and $\vec{F}(x, y, ax + by) = x\vec{\imath} + y\vec{\jmath} + (ax + by)\vec{k}$ so $\vec{F} \cdot \,d\vec{S} = 0\,dA$ and the flux integral is zero.

§5.6: 1a:

$$\begin{aligned} \vec{\nabla} \times \vec{F} &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & yz & xz \end{vmatrix} \\ &= \vec{\imath}[\frac{\partial}{\partial y}\,[xz] - \frac{\partial}{\partial z}\,[y]\,z] - \vec{\jmath}[\frac{\partial}{\partial x}\,[xz] - \frac{\partial}{\partial z}\,[xy]] + \vec{k}[\frac{\partial}{\partial x}\,[yz] - \frac{\partial}{\partial y}\,[xy]] \\ &= (-y)\vec{\imath} - (z)\vec{\jmath} + (-x)\vec{k}. \end{aligned}$$

2a: (1) The boundary parametrization

$$\begin{aligned} x &= \cos\theta \\ y &= \sin\theta \\ z &= 1 - \cos\theta - \sin\theta \\ 0 \leq &\theta \leq 2\pi \end{aligned}$$

leads to the differentials

$$\begin{aligned} dx &= -\sin\theta\,d\theta \\ dy &= \cos\theta\,d\theta \\ dz &= (\sin\theta - \cos\theta)\,d\theta. \end{aligned}$$

Thus

$$\vec{T}\,d\mathfrak{s} = \,d\vec{\mathfrak{s}} = (-\sin\theta, \cos\theta, \sin\theta - \cos\theta)\,d\theta$$

$$\vec{F}(\cos\theta, \sin\theta, 1 - \cos\theta - \sin\theta) = (-\sin\theta, \cos\theta, 1 - \cos\theta - \sin\theta)$$

so

$$\begin{aligned} \vec{F} \cdot \vec{T}\,d\mathfrak{s} &= \vec{F} \cdot \,d\vec{\mathfrak{s}} = [1 + \cos^2\theta - \sin^2\theta + \sin\theta - \cos\theta]\,d\theta \\ \oint_{\mathcal{C}} \vec{F} \cdot \vec{T}\,d\mathfrak{s} &= \int_0^{2\pi} [1 + \cos^2\theta - \sin^2\theta + \sin\theta - \cos\theta]\,d\theta \\ &= 2\pi. \end{aligned}$$

(2)

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & z \end{vmatrix}$$
$$= \vec{\imath}\left(\frac{\partial z}{\partial y} - \frac{\partial x}{\partial z}\right) - \vec{\jmath}\left(\frac{\partial z}{\partial x} - \frac{\partial(-1)}{\partial z}\right) + \vec{k}\left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right)$$
$$= 2\vec{k}.$$

Meanwhile, the parametrization of $\mathfrak{S}$ yields

$$\frac{\partial\vec{p}}{\partial s}\times\frac{\partial\vec{p}}{\partial t} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \vec{\imath}+\vec{\jmath}+\vec{k}$$

Thus

$$\iint_{\mathfrak{S}} \vec{\nabla}\times\vec{F}\cdot d\vec{\mathcal{S}} = \iint_{s^2+t^2\leq 1} 2\,dA = 2\pi.$$

§5.7: 1a: Bilinear, with

$$[B] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix};$$

commutative.

1b: *Not* bilinear (notice that there are terms which are quadratic in the entries of the first vector: if we replace $\vec{v} = x_1\vec{\imath} + y_1\vec{\jmath} + z_1\vec{k}$ with say $2\vec{v}$, but leave $\vec{w} = x_2\vec{\imath} + y_2\vec{\jmath} + z_2\vec{k}$ alone, then the first and third terms get multiplied by 4 while the other two are unchanged).

2a: $\vec{F} = \vec{\imath}$: $\omega = dy \wedge dz$

2g: $\vec{F} = \vec{\nabla}f$, where $f(x,y,z)$ is a $\mathcal{C}^2$ function: $\omega = \frac{\partial f}{\partial z}\,dx\wedge dy + \frac{\partial f}{\partial x}\,dy\wedge dz + \frac{\partial f}{\partial y}\,dz\wedge dx$

3a: $\omega = dx\wedge dy$: $\vec{F} = \vec{k}$

4a: We can parametrize $\mathfrak{S}$ as

$$\begin{cases} x = s \\ y = t \\ z = 1-s-t \end{cases}, \quad \begin{cases} 0 \leq t \leq 1-s \\ 0 \leq s \leq 1 \end{cases}$$

$$\begin{aligned} \omega &= x\,dy\wedge dz \\ &= (s)(dt)\wedge(-ds - dt) \\ &= -s\,dt\wedge ds \\ &= s\,ds\wedge dt \end{aligned}$$

and hence

$$\begin{aligned}\iint_{\mathfrak{S}} \omega &= \int_0^1 \int_0^{1-s} s\,dt\,ds \\ &= \int_0^1 s(1-s)\,ds \\ &= \left(\frac{s^2}{2} - \frac{s^3}{3}\right)_0^1 \\ &= \frac{1}{6}.\end{aligned}$$

§5.8: 1a:

$$\begin{aligned}\frac{\partial P}{\partial x} &= 1 \\ \frac{\partial Q}{\partial y} &= 1\end{aligned}$$

so

$$\begin{aligned}\int_{\mathcal{C}} \vec{F} \cdot \vec{N}\,d\mathfrak{s} &= \iint_{\mathcal{D}} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA \\ &= \iint_{\mathcal{D}} 2\,dA = 4\pi.\end{aligned}$$

2a: We will use the Divergence Theorem; note that the volume of the ball $\mathcal{D}$ bounded by the sphere is

$$\iint_{\mathfrak{S}} 1 \cdot d\vec{\mathcal{S}} = \mathcal{V}(\mathcal{D}) = \frac{4}{3}\pi.$$

$$\begin{aligned}\operatorname{div} \vec{F} &= \frac{\partial}{\partial x}(x + y^2) + \frac{\partial}{\partial y}(y - z^2) + \frac{\partial}{\partial z}(x + z) \\ &= 3\end{aligned}$$

so

$$\begin{aligned}\iint_{\mathfrak{S}} \vec{F} \cdot d\vec{\mathcal{S}} &= \iint_{\mathcal{D}} 3\,dV \\ &= 3\mathcal{V}(\mathcal{D}) \\ &= 4\pi.\end{aligned}$$

3a: (1) Direct integration:

$\mathfrak{S}_1 = [0,1] \times [0,1] \times \{0\}$**:** directed down:

$$\begin{aligned}d\vec{\mathcal{S}} &= -\vec{k}\,dx\,dy \\ \vec{F} &= x\vec{\imath} + y\vec{\jmath}\end{aligned}$$

so

$$\iint_{\mathfrak{S}_1} \vec{F} \cdot d\vec{S} = \iint_{[0,1]\times[0,1]} (x\vec{\imath} + y\vec{\jmath}) \cdot (-\vec{k}\, dx\, dy)$$
$$= \iint_{[0,1]\times[0,1]} 0\, dx\, dy$$
$$= 0$$

$\mathfrak{S}_2 = [0,1] \times [0,1] \times \{1\}$: directed up:

$$d\vec{S} = \vec{k}\, dx\, dy$$
$$\vec{F} = x\vec{\imath} + y\vec{\jmath} + \vec{k}$$

so

$$\iint_{\mathfrak{S}_2} \vec{F} \cdot d\vec{S} = \iint_{[0,1]\times[0,1]} (x\vec{\imath} + y\vec{\jmath} + \vec{k}) \cdot (\vec{k}\, dx\, dy)$$
$$= \iint_{[0,1]\times[0,1]} dx\, dy$$
$$= 1$$

$\mathfrak{S}_3 = [0,1] \times \{0\} \times [0,1]$: direction of -$\vec{\jmath}$:

$$d\vec{S} = -\vec{\jmath}\, dx\, dz$$
$$\vec{F} = x\vec{\imath} + z\vec{k}$$

so

$$\iint_{\mathfrak{S}_3} \vec{F} \cdot d\vec{S} = \iint_{[0,1]\times\{0\}\times[0,1]} (x\vec{\imath} + z\vec{k}) \cdot (-\vec{\jmath}\, dx\, dz)$$
$$= \iint_{[0,1]\times[0,1]} 0\, dx\, dz$$
$$= 0$$

$\mathfrak{S}_4 = [0,1] \times \{1\} \times [0,1]$: direction of $\vec{\jmath}$:

$$d\vec{S} = \vec{\jmath}\, dx\, dz$$
$$\vec{F} = x\vec{\imath} + \vec{\jmath} + z\vec{k}$$

so

$$\iint_{\mathfrak{S}_4} \vec{F} \cdot d\vec{S} = \iint_{[0,1]\times\{1\}\times[0,1]} (x\vec{\imath} + \vec{\jmath} + z\vec{k}) \cdot (\vec{\jmath}\, dx\, dz)$$
$$= \iint_{[0,1]\times[0,1]} dx\, dz$$
$$= 1$$

$\mathfrak{S}_5 = \{0\} \times [0,1] \times [0,1]$: direction of $-\vec{\imath}$:

$$d\vec{\mathcal{S}} = -\vec{\imath}\, dy\, dz$$
$$\vec{F} = y\vec{\jmath} + z\vec{k}$$

so

$$\iint_{\mathfrak{S}_5} \vec{F} \cdot d\vec{\mathcal{S}} = \iint_{\{0\}\times[0,1]\times[0,1]} (y\vec{\jmath} + z\vec{k}) \cdot (-\vec{\imath}\, dy\, dz)$$
$$= \iint_{[0,1]\times[0,1]} 0\, dy\, dz$$
$$= 0$$

$\mathfrak{S}_6 = \{1\} \times [0,1] \times [0,1]$: direction of $\vec{\imath}$:

$$d\vec{\mathcal{S}} = \vec{\imath}\, dy\, dz$$
$$\vec{F} = \vec{\imath} + y\vec{\jmath} + z\vec{k}$$

so

$$\iint_{\mathfrak{S}_5} \vec{F} \cdot d\vec{\mathcal{S}} = \iint_{\{1\}\times[0,1]\times[0,1]} (\vec{\imath} + y\vec{\jmath} + z\vec{k}) \cdot (\vec{\imath}\, dy\, dz)$$
$$= \iint_{[0,1]\times[0,1]} dy\, dz = 1;$$

Thus

$$\iint_{\mathfrak{S}} \vec{F} \cdot d\vec{\mathcal{S}} =$$
$$\iint_{\mathfrak{S}_1} \vec{F} \cdot d\vec{\mathcal{S}} + \iint_{\mathfrak{S}_2} \vec{F} \cdot d\vec{\mathcal{S}} + \iint_{\mathfrak{S}_3} \vec{F} \cdot d\vec{\mathcal{S}}$$
$$+ \iint_{\mathfrak{S}_4} \vec{F} \cdot d\vec{\mathcal{S}} + \iint_{\mathfrak{S}_5} \vec{F} \cdot d\vec{\mathcal{S}} + \iint_{\mathfrak{S}_6} \vec{F} \cdot d\vec{\mathcal{S}}$$
$$= 0 + 1 + 0 + 1 + 0 + 1 = 3.$$

Using Divergence Theorem:

$$\begin{aligned}\operatorname{div}\vec{F} &= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \\ &= 1 + 1 + 1 = 3\end{aligned}$$

so, denoting the cube $[0,1] \times [0,1] \times [0,1]$ by $\mathfrak{D}$,

$$\iint_{\mathfrak{S}} \vec{F} \cdot d\vec{\mathcal{S}} = \iint_{\mathfrak{D}} 3\,dV = 3.$$

5c:

$$\begin{aligned}\iint_{\mathfrak{S}} \vec{F} \cdot d\vec{\mathcal{S}} &= \int_{\pi/2}^{3\pi/2} \int_0^1 \int_0^{\sqrt{z}} (r\cos\theta)(r\,dr\,dz\,d\theta) \\ &= \frac{2}{15} \int_{\pi/2}^{3\pi/2} \cos\theta\,d\theta \\ &= \frac{2}{15} \sin\theta \Big|_{\pi/2}^{3\pi/2} \\ &= -\frac{4}{15}.\end{aligned}$$

§5.9: 1a: $(3\,dx + 2x\,dy) \wedge (2\,dx \wedge dy - dy \wedge dz + x\,dx \wedge dz) = -(3 + 2x^2)\,dx \wedge dy \wedge dz.$

2a: $(dx + dy + dz) \wedge (2\,dx - dy + dz) \wedge (dx + dy) = 3\,dx \wedge dy \wedge dz.$

3a: $d\,(dxy + xdyz) = 0 + dx \wedge dy \wedge dz = dx \wedge dy \wedge dz.$

4a: $\int_{[0,1]\times[0,1]\times[0,1]} (xy + yz)\,dx \wedge dy \wedge dz = \frac{1}{2}.$

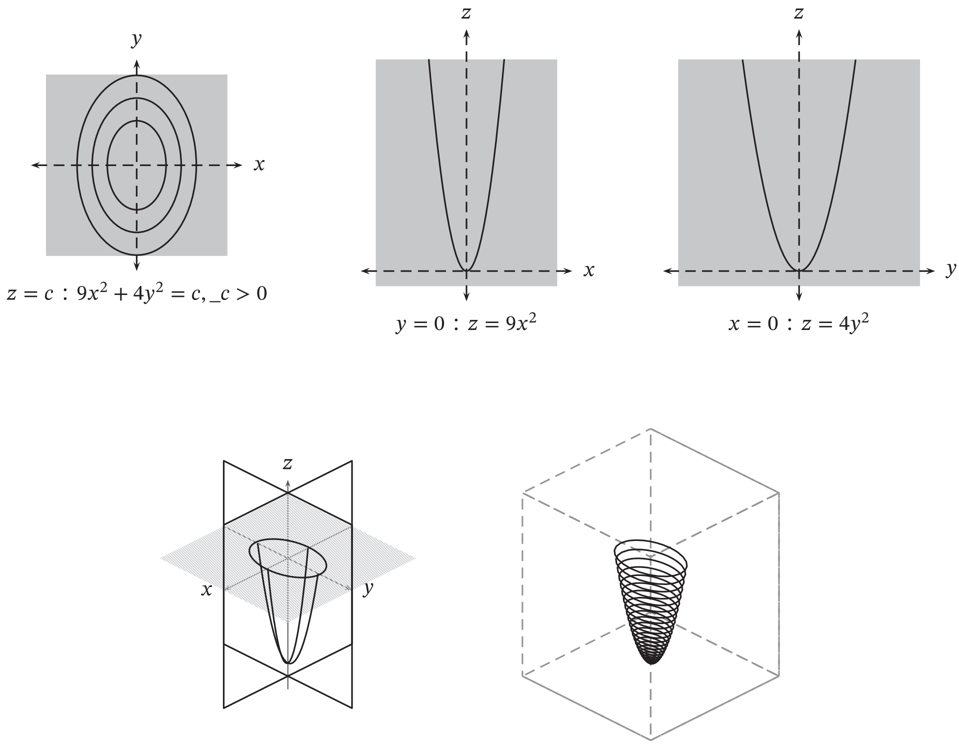


Figure A.24. Elliptic Paraboloid $9x^2 + 4y^2 - z = 0$

Bibliography

[1] Tom Archibald, *Analysis and physics in the nineteenth century: the case of boundary-value problems*, A history of a nalysis, Hist. Math., vol. 24, Amer. Math. Soc., Providence, RI, 2003, pp. 197–211. MR1998249

[2] Archimedes, *The works of Archimedes*, Dover Publications, Inc., Mineola, NY, 2002. Reprint of the 1897 edition and the 1912 supplement; Edited by T. L. Heath. MR2000800

[3] Archimedes, *The works of Archimedes*, Dover Publications, Inc., Mineola, NY, 2002. Reprint of the 1897 edition and the 1912 supplement; Edited by T. L. Heath. MR2000800

[4] Carl B. Boyer, *History of analytic geometry*, Dover Publications, Inc., Mineola, NY, 2004. Reprint of the 1956 original. MR2108489

[5] David M. Burton, *The history of mathematics*, 2nd ed., W. C. Brown Publishers, Dubuque, IA, 1991. An introduction. MR1223776

[6] Sandro Caparrini, *The discovery of the vector representation of moments and angular velocity*, Arch. Hist. Exact Sci. **56** (2002), no. 2, 151–181, DOI 10.1007/s004070200001. MR1882468

[7] Sandro Caparrini, *Early theories of vectors*, Essays on the history of mechanics, Between Mech. Archit., Birkhäuser, Basel, 2003, pp. 179–198. MR2052767

[8] Sandro Caparrini, *The theory of vectors at the beginning of the 19th century*, Variar para encontrar/Varier pour mieux trouver/The lore of variation: finding pathways to scientific knowledge, Univ. Nac. Autónoma México, México, 2004, pp. 235–257. MR2100964

[9] Élie Cartan, *Leçons sur les invariants intégraux* (French), Hermann, Paris, 1971. Troisième tirage. MR0355764

[10] Michael J. Crowe, *A history of vector analysis. The evolution of the idea of a vectorial system*, University of Notre Dame Press, Notre Dame, Ind.-London, 1967. MR0229496

[11] Édotard Goursat. *Leçons sur le Problème de Pfaff.* Hermann, 1922.

[12] C. H. Edwards Jr., *Advanced calculus of several variables*, Academic Press (a subsidiary of Harcourt Brace Jovanovich, Publishers), New York-London, 1973. MR0352341

[13] Steven B. Engelsman, *Families of curves and the origins of partial differentiation*, North-Holland Mathematics Studies, vol. 93, North-Holland Publishing Co., Amsterdam, 1984. MR756235

[14] Howard Eves, *An introduction to the history of mathematics*, 5th ed., Saunders Series, Saunders College Publishing, Philadelphia, Pa., 1983. MR684360

[15] Michael N. Fried and Sabetai Unguru, *Apollonius of Perga's Conica*, Mnemosyne. Bibliotheca Classica Batava. Supplementum, vol. 222, Brill, Leiden, 2001. Text, context, subtext. MR1929435

[16] Guido Fubini. Sugli integrali multipli. *R. Acc. Lincei Rend., Roma*, 16:608–14, 1907.

[17] Carl Friedrich Gauss. Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse des Quadrats der Entfernung wirkender Anziehungs- aund Abstossungskräfte. In *Werke*, volume V, pages 197–242. Königliche Gesellschaft der Wissenschaften zu Göttingen, 1863-1933.

[18] Edwin Bidwell Wilson, *Vector analysis*, A textbook for the use of students of mathematics and physics, founded upon the lectures of J. Willard Gibbs, Dover Publications, Inc., New York, 1960. MR0120233

[19] George Green. *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*. privately published, 1828.

[20] E. Hairer and G. Wanner, *Analysis by its history*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 2008. Corrected reprint of the 1996 original [MR1410751]; Undergraduate Texts in Mathematics. Readings in Mathematics. MR2490639

[21] Thomas C. Hales, *The Jordan curve theorem, formally and informally*, Amer. Math. Monthly **114** (2007), no. 10, 882–894, DOI 10.1080/00029890.2007.11920481. MR2363054

[22] William Rowan Hamilton. *Lectures on Quaternions*. Hodges and Smith, 1853.

[23] William Rowan Hamilton, *Elements of quaternions. Vols. I, II*, Edited by Charles Jasper Joly, Chelsea Publishing Co., New York, 1969. MR0237284

[24] Thomas Hawkins, *Lebesgue's theory of integration*, 2nd ed., AMS Chelsea Publishing, Providence, RI, 2001. Its origins and development. MR1857634

[25] Thomas Heath, *A history of Greek mathematics. Vol. I*, Dover Publications, Inc., New York, 1981. From Thales to Euclid; Corrected reprint of the 1921 original. MR654679

[26] Thomas L. Heath, *A manual of Greek mathematics*, Dover Publications, Inc., New York, 1963. MR0156760

[27] Archimedes, *The works of Archimedes*, Dover Publications, Inc., Mineola, NY, 2002. Reprint of the 1897 edition and the 1912 supplement; Edited by T. L. Heath. MR2000800

[28] Euclid, *Euclid's Elements*, Green Lion Press, Santa Fe, NM, 2002. All thirteen books complete in one volume; The Thomas L. Heath translation; Edited by Dana Densmore. MR1932864

[29] Otto Hesse, *Über die Criterien des Maximums und Minimums der einfachen Integrale* (German), J. Reine Angew. Math. **54** (1857), 227–273, DOI 10.1515/crll.1857.54.227. MR1579042

[30] Camille Jordan, *Cours d'analyse de l'École polytechnique. Tome I* (French), Les Grands Classiques Gauthier-Villars. [Gauthier-Villars Great Classics], Éditions Jacques Gabay, Sceaux, 1991. Calcul différentiel. [Differential calculus]; Reprint of the third (1909) edition. MR1188186

[31] Wilfred Kaplan, *Advanced calculus*, Addison-Wesley Press, Inc., Cambridge, Mass., 1952. MR0054676

[32] Morris Kline, *Mathematical thought from ancient to modern times*, Oxford University Press, New York, 1972. MR0472307

[33] Steven G. Krantz and Harold R. Parks, *The implicit function theorem*, Birkhäuser Boston, Inc., Boston, MA, 2002. History, theory, and applications. MR1894435

[34] Joseph Louis Lagrange, *Oeuvres. Tome 1* (French), Georg Olms Verlag, Hildesheim-New York, 1973. Publiées par les soins de J.-A. Serret; Avec une notice sur la vie et les ouvrages de J.-L. Lagrange par J.-B. J. Delambre; Nachdruck der Ausgabe Paris 1867. MR0439546

[35] Joseph-Louis Lagrange. *Méchanique Analitique*. Desaint, 1788.

[36] Eli Maor, *The Pythagorean theorem*, Paperback edition, Princeton Science Library, Princeton University Press, Princeton, NJ, 2007. A 4,000-year history. MR2683004

[37] Richard Murray, Tony Bloch, P. S. Krishnaprasad, and Naomi Leonard, *Jerrold Eldon Marsden*, IEEE Control Syst. Mag. **31** (2011), no. 2, 105–108, DOI 10.1109/MCS.2010.939943. MR2809157

[38] J. Clerk-Maxwell, *Remarks on the Mathematical Classification of Physical Quantities,* Proc. Lond. Math. Soc. **3** (1869/71), 224–233, DOI 10.1112/plms/s1-3.1.224. MR1577193

[39] James Clerk Maxwell, *A treatise on electricity and magnetism*, Dover Publications, Inc., New York, 1954. 3d ed; Two volumes bound as one. MR0063293

[40] Isaac Newton. Mathematical principles of natural philosophy. In *Isaac Newton: The Principia*, pages 401–944. Univ. of California Press, 1999. Translation from the Latin by I. Bernard Cohen and Anne Whitman of *Philosophiae Naturalis Principia Mathematica* (1687, 1713, 1726).

[41] James Clerk Maxwell, *The scientific letters and papers of James Clerk Maxwell. Vol. II*, Cambridge University Press, Cambridge, 1995. 1862–1873; Edited and with a preface, note and introduction by P. M. Harman. MR1337275

[42] Apollonius of Perga, *Conics. Books I–III*, Revised edition, Green Lion Press, Santa Fe, NM, 1998. Translated and with a note and an appendix by R. Catesby Taliaferro; With a preface by Dana Densmore and William H. Donahue, an introduction by Harvey Flaumenhaft, and diagrams by Donahue; Edited by Densmore. MR1660991

[43] Giuseppe Peano. Sulle definizione dell'area di una superficie. *Rendiconti Acc. Lincei*, 6:54–57, 1890.

[44] H. Poincaré, *Sur les résidus des intégrales doubles* (French), Acta Math. **9** (1887), no. 1, 321–380, DOI 10.1007/BF02406742. MR1554721

[45] Daniel Reem. New proofs of basic theorems in calculus. *Math Arxiv*, 0709.4492v1, 2007.

[46] Hans Samelson, *Differential forms, the early days; or the stories of Deahna's theorem and of Volterra's theorem*, Amer. Math. Monthly **108** (2001), no. 6, 522–530, DOI 10.2307/2695706. MR1840658

[47] H. A. Schwarz, *Gesammelte mathematische Abhandlungen. Band I, II* (German), Chelsea Publishing Co., Bronx, N.Y., 1972. Nachdruck in einem Band der Auflage von 1890. MR0392470

[48] H. A. Schwarz, *Gesammelte mathematische Abhandlungen. Band I, II* (German), Chelsea Publishing Co., Bronx, N.Y., 1972. Nachdruck in einem Band der Auflage von 1890. MR0392470

[49] George F. Simmons, *Calculus gems*, McGraw-Hill, Inc., New York, 1992. Brief lives and memorable mathematics; With portraits by Maceo Mitchell. MR1254212

[50] Patrick Du Val, *Homographies, quaternions and rotations*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964. MR0169108

[51] Oswald Veblen, *Theory on plane curves in non-metrical analysis situs*, Trans. Amer. Math. Soc. **6** (1905), no. 1, 83–98, DOI 10.2307/1986378. MR1500697

[52] Francesco M. Scudo, *Vito Volterra, "ecology" and the quantification of "Darwinism"*, International Conference in Memory of Vito Volterra (Italian) (Rome, 1990), Atti Convegni Lincei, vol. 92, Accad. Naz. Lincei, Rome, 1992, pp. 313–344. MR1783041

[53] Francesco M. Scudo, *Vito Volterra, "ecology" and the quantification of "Darwinism"*, International Conference in Memory of Vito Volterra (Italian) (Rome, 1990), Atti Convegni Lincei, vol. 92, Accad. Naz. Lincei, Rome, 1992, pp. 313–344. MR1783041

[54] Vito Volterra, *Opere matematiche: Memorie e note. Vol. V: 1926–1940* (Italian), Pubblicate a cura

dell'Accademia Nazionale dei Lincei col concorso del Consiglio Nazionale delle Ricerche, Accademia Nazionale dei Lincei, Rome, 1962. MR0131353

[55] Edwin Bidwell Wilson, *Vector analysis*, A textbook for the use of students of mathematics and physics, founded upon the lectures of J. Willard Gibbs, Dover Publications, Inc., New York, 1960. MR0120233

[56] Shirley Llamado Yap, *The Poincaré lemma and an elementary construction of vector potentials*, Amer. Math. Monthly **116** (2009), no. 3, 261–267, DOI 10.4169/193009709X470100. MR2491982

Index

AMS / MAA | TEXTBOOKS

Calculus in 3D is an accessible, well-written textbook for an honors course in multivariable calculus for mathematically strong first- or second-year university students. The treatment given here carefully balances theoretical rigor, the development of student facility in the procedures and algorithms, and inculcating intuition into underlying geometric principles. The focus throughout is on two or three dimensions. All of the standard multivariable material is thoroughly covered, including vector calculus treated through both vector fields and differential forms. There are rich collections of problems ranging from the routine through the theoretical to deep, challenging problems suitable for in-depth projects. Linear algebra is developed as needed. Unusual features include a rigorous formulation of cross products and determinants as oriented area, an in-depth treatment of conics harking back to the classical Greek ideas, and a more extensive than usual exploration and use of parametrized curves and surfaces.

Zbigniew Nitecki is Professor of Mathematics at Tufts University and a leading authority on smooth dynamical systems. He is the author of *Differentiable Dynamics*, MIT Press; *Differential Equations, A First Course* (with M. Guterman), Saunders; *Differential Equations with Linear Algebra* (with M. Guterman), Saunders; and *Calculus Deconstructed*, MAA Press.

ISBN 978-1-4704-4360-3

9 781470 443603

TEXT/40

For additional information and updates on this book, visit
www.ams.org/bookpages/text-40

AMS
www.ams.org