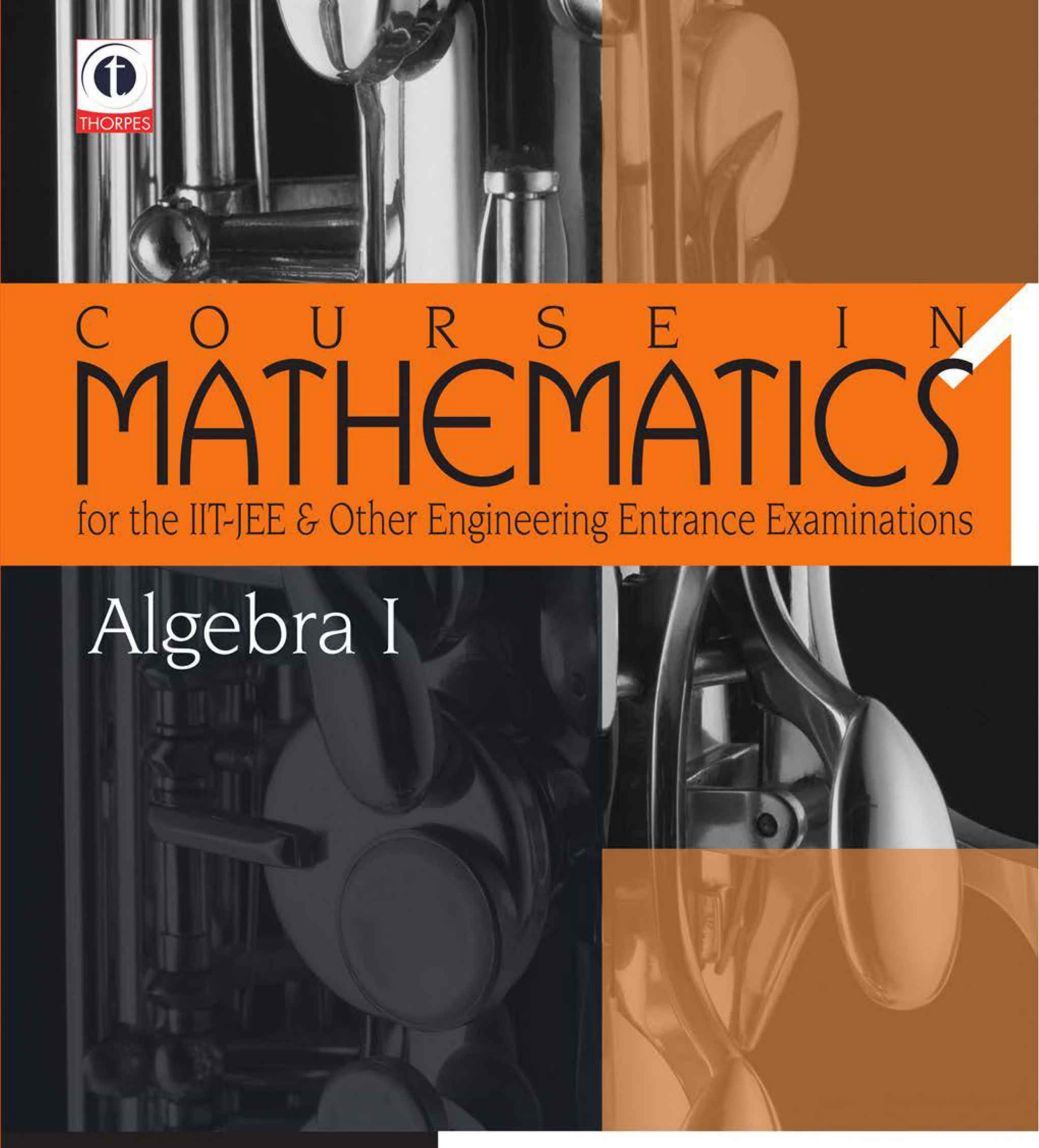


THORPES

COURSE IN MATHEMATICS 1

for the IIT-JEE & Other Engineering Entrance Examinations

Algebra I

K. R. Choubey
Ravikant Choubey
Chandrakant Choubey

ALWAYS LEARNING

PEARSON

COURSE IN MATHEMATICS

(FOR IIT JEE AND OTHER ENGINEERING ENTRANCE EXAMINATIONS)

ALGEBRA-I

K.R. CHOUBEY

RAVIKANT CHOUBEY

CHANDRAKANT CHOUBEY

PEARSON

Chandigarh • Delhi • Chennai

ISBN 9788131758670
eISBN 9789332510241

Head Office: A-8(A), Sector 62, Knowledge Boulevard, 7th Floor, NOIDA 201 309, India
Registered Office: 11 Local Shopping Centre, Panchsheel Park, New Delhi 110 017, India

CONTENTS

This page is intentionally left blank.

PREFACE

When a new book is written on a well known subject like *Algebra* for class XI/XII Academics/AIEEE/IIT/State engineering entrance exams and NDA, several questions arise like—why, what, how and for whom? What is new in it? How is it different from other books? For whom is it meant? The answers to these questions are often not mutually exclusive. Neither are they entirely satisfactory except perhaps to the authors. We are certainly not under the illusion that there are no good books. There are many good books available in the market.

However, none of them caters specifically to the needs of students. Students find it difficult to solve most of the problems of any of the books in the absence of proper planning. This inspired us to write this book *Algebra–I*, to address the requirements of students of class XI/XII CBSE and State Board Academics. In this book, we have tried to give a connected and simple account of the subject. It gives a detailed, lecture wise description of basic concepts with many numerical problems and innovative tricks and tips. Theory and problems have been designed in such a way that the students can themselves pursue the subject. We have also tried to keep this book self contained. In each lecture all relevant concepts, prerequisites and definitions have been discussed in a lucid manner and also explained with suitable illustrated examples including tests.

Due care has been taken regarding the Board (CBSE/ State) examination need of students and nearly 100 per cent articles and problems set in various examinations including the IIT-JEE have been included.

The presentation of the subject matter is lecturewise, intelligent and systematic, the style is lucid and rational, and the approach is comprehensible with emphasis on improving speed and accuracy. The basic motive is to attract students towards the study of mathematics by making it simple, easy and interesting and on a day-to-day basis. The instructions and method for grasping the lectures are clearly outlined topic wise. The presentation of each lecture is planned for better experiential learning of mathematics which is as follows:

1. Basic Concepts: Lecture Wise
2. Solved Subjective Problems (XII Board (C.B.S.E./State): For Better Understanding and Concept Building of the Topic.
3. Unsolved Subjective Problems (XII Board (C.B.S.E./State): To Grasp the Lecture Solve These Problems.
4. Solved Objective Problems: Helping Hand.
5. Objective Problem: Important Questions with Solutions.
6. Unsolved Objective Problems (Identical Problems for Practice) For Improving Speed with Accuracy.

7. Worksheet: To Check Preparation Level
8. Assertion-Reason Problems : Topic Wise Important Questions and Solutions with Reasoning
9. Mental Preparation Test: 01
10. Mental Preparation Test: 02
11. Topic Wise Warm Up Test: 01: Objective Test
12. Topic Wise Warm Up Test: 02: Objective Test
13. Objective Question Bank Topic Wise: Solve These to Master.

This book will serve the need of the students of class XI/XII board, NDA, AIEEE and SLEEE (state level engineering entrance exam) and IIT-JEE. We suggest each student to attempt as many exercises as possible without looking up the solutions. However, one should not feel discouraged if one needs frequent help of the solutions as there are many questions that are either tough or lengthy. Students should not get frustrated if they fail to understand some of the solutions in the first attempt. Instead they should go back to the beginning of the solution and try to figure out what is being done At the end of every topic, some harder problems with 100 per cent solutions and Question Bank are also given for better understanding of the subject.

There is no end and limit to the improvement of the book. So, suggestions for improving the book are always welcome.

We thank our publisher, Pearson Education for their support and guidance in completing the project in record time.

K.R. CHOUBEY
RAVIKANT CHOUBEY
CHANDRAKANT CHOUBEY

Part A

Binomial Theorem

This page is intentionally left blank.

LECTURE

1 Binomial Expansion

BASIC CONCEPTS

1. **Binomial Expression** An algebraic expression consisting of only two terms is called a binomial expression. For example, expressions such as $x + a$, $4x + 3y$, $\left(2x - \frac{4}{y}\right)$ are allbinomial expressions.

2. **Binomial Theorem** This theorem gives a formula by which any power of a binomial expression can be expanded was first given by **Isaac Newton.**

2.1 Binomial Theorem for Positive Integral Index If x and a are real numbers, then for all $n \in N$,

$$(x + a)^n = {}^nC_0\, x^n a^0 + {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 + \ldots.. \ldots. + {}^nC_{n-1}\, x^1 a^{n-1} + {}^nC_n\, x_0\, a^n \quad (1)$$

OR

$$(x+a)^n = ({}^nC_0x^{n-0}a^0 + {}^nC_2x^{n-2}a^2 + {}^nC_4x^{n-4}a^4 + \ldots..) + ({}^nC_1x^{n-1}a_1 + {}^nC_3x^{n-3}a^3 + nC_5x^{n-5}a^5 + \ldots\ldots.)$$

$= A + B$ (say)

= Sum of odd terms + Sum of even terms

Here ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ..., ${}^nC_n$ are called **binomial coefficients.** These are generally denoted by C_0, C_1, C_2, ...,C_n.

Note 1: The positive integer n is called the index of the binomial.

Note 2: The number of terms in the expansion of $(x + a)^n$ is $n + 1$, i.e., one more than the index n.

Note 3: In the expansion of $(x + a)^n$, the power of x goes on decreasing by 1 and that of a goes on increasing by 1 so that the sum of powers of x and a in any term is n.

Note 4: The binomial coefficients of the terms are equidistant from the beginning i.e., ${}^nC_r = {}^nC_{n-r}$.

Note 5: Binomial coefficient of $(r + 1)$th term is $= {}^nC_r$ i.e. the number of terms is one more than the value of ***r.***

2.2 General Term in the Expansion of $(x + a)^n$ In the binomial expansion of $(x + a)^n$ the $(r + 1)$th term from the beginning is usually called the general term and it is denoted by T_{r+1}, i.e., $T_{r+1} = {}^nC_r x^{n-r} a^r = {}^nC_r$ (first term)$^{n-r}$. (second term)r

It is obvious to note that the binomial coefficient of the general term i.e., $(r + 1)$th term $= {}^nC_r$.

2.3 General term in the expansion of $(a + x)^n$ is

$$T_{r+1} = {}^nC_r\, a^{n-r} x^r.$$

3. **Special Cases**

(i) Replacing a by $-a$, in (1), we get

$(x - a)^n = {}^nC_0x^na^0 - {}^nC_1x^{n-1}a^1 + {}^nC_2x^{n-2}a^2 - \ldots + (-1)^n\, {}^nC_nx_0a^n.$ (2)

(ii) Replacing x by 1 and a by x, we get

$(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_nx^n$

(iii) Replacing x by 1 and a by $-x$, we get

$(1 - x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 - \ldots + (-1)^n\, {}^nC_nx^n.$

(iv) Adding (1) and (2), we get $(x + a)^n + (x - a)^n$

$= 2[x^n + {}^nC_2x^{n-2}a^2 + {}^nC_4x^{n-4}a^4 + \ldots]$

= 2 (sum of terms at odd places).

The last term is ${}^nC_na^{n-1}$ or ${}^nC_{n-1}xa^{n-1}$ since, n is even or odd respectively.

(v) Subtracting (2) from (1), we get,

$(x + a)^n - (x - a)^n = 2[{}^nC_1x^{n-1}a + {}^nC_3x^{n-3}a^3 + \ldots]$

= 2[sum of terms at even places]

The last term is ${}^nC_{n-1}xa^{n-1}$ or ${}^nC_na^n$ according as n is even or odd respectively.

(vi) Number of terms in the expansion of

(a) $(x + a)^n + (x - a)^n = \frac{n}{2} + 1$ when n is even.

(b) $(x + a)^n - (x - a)n = \frac{n}{2}$ when n is even

(c) $(x + a)^n \pm (x - a)^n = \left(\frac{n+1}{2}\right)$ when n is odd

(vii) Interchanging a and x in (1), we get,

$(a + x)^n = {}^nC_0a^nx^0 + {}^nC_1a^{n-1}x^1 + {}^nC_2a^{n-2}x^2 + \ldots + {}^nC_ra^{n-r}x^r + \ldots\ldots\ldots\ldots + {}^nC_nx^n.$

4. **Important Results** In the Binomial expansion of $(x + a)^n$, if the sum of odd terms be A and the sum of even terms be B, then

$(x + a)^{2n} + (x - a)^{2n} = (A + B)^2 + (A - B)^2 = 2(A^2 + B^2)$

$(x + a)^{2n} - (x - a)^{2n} = (A + B)^2 - (A - B)^2 = 4AB$

$(x^2 - a^2)^n = (A + B)(A - B) = A^2 - B^2$

SOLVED SUBJECTIVE PROBLEMS: (CBSE/STATE BOARD): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. If coefficient of x^2 and x^3 in the expansion of $[3 + ax]^9$ are equal then find the value of a.

[NCERT]

Solution

$T_{r+1} = {}^nC_rx^{n-r}ar \Rightarrow T_{r+1} = {}^9C_r(3)^9 - r(ax)^r$

$\Rightarrow T_{r+1} = {}^9C_r(3)^{9-r}a^rx^r$

For coefficient of $x2$, $r = 2$

$\therefore T_{2+1} = {}^9C_23^{9-2}a^2x^2 \Rightarrow T_3 = ({}^9C_23^7a^2)x^2$

Again, $T_{r+1} = {}^9C_r(3)^{9-1}(ax)^r$

For coefficient of x^3 $T_{3+1} = {}^9C_3(3)^{9-3}a^3x^3$

$\Rightarrow T_4 = {}^9C_3(3)^6a^3x^3$

According question, ${}^9C_23^7a^2 = {}^9C_33^6a^3$

$$\Rightarrow \frac{9 \times 8}{2} \times 3 = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} \times a$$

$$\Rightarrow a = \frac{3 \times 3}{7} \Rightarrow a = \frac{9}{7}.$$

2. By using the binomial expansion, expand $(1 + x + x^2)^3$

Solution

$(1 + x + x^2)^3 = [(1 + x) + x^2]^3$

$= {}^3C_0.(1 + x)^3 + {}^3C_1.(1 + x)^2.(x^2) + {}^3C_2.(1 + x).(x^2)^2 + {}^3C_3.(x^2)^3$

$= (1 + x)^3 + 3(1 + x)^2.x^2 + 3(1 + x).x^4 + x^6$

$= [1 + {}^3C_1.x + {}^3C_2x^2 + {}^3C_3.x^3] + 3x^2[1 + 2x + x^2] + 3x^4(1 + x) + x^6$

$= (1 + 3x + 3x^2 + x^3) + (3x^2 + 6x^3 + 3x^4) + (3x^4 + 3x^5) + x^6$

$= (x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1).$

3. Using the binomial theorem, find the value of $(102)^6$.

Solution

$(102)^6 = (100 + 2)^6$

$= {}^6C_0 \times (100)^6 + {}^6C_1 \times (100)^5 \times 2 + {}^6C_2 \times (100)^4 \times (2)^2 + {}^6C_3 \times (100)^3 \times (2)^3 + {}^6C_4 \times (100)^2 \times (2)^4 + {}^6C_5 \times 100 \times (2)^5 + {}^6C_6 \times (2)^6$

$= (100)^6 + 12 \times (100)^5 + 60 \times (100)^4 + 160 \times (100)^3 + 240 \times (100)^2 + 19200 + 64$

$= 100000000000 + 12000000000 + 6000000000 + 160000000 + 2400000 + 19200 + 64$

$= 1126162419264$

4. If the coefficients of a^{r-1}, a^r, a^{r+1} in the binomial expansion $(1 + a)^n$ are in arithmetic progression, prove that $n^2 - n(4r + 1) + 4r^2 - 2 = 0$.

[NCERT]

Solution

The general term in the expansion of $(1 + a)^n$ is given by $t_{r+1} = {}^nC_r a^r$.
Therefore coefficients of a^{r-1}, a^r and a^{r+1} in the expansion are ${}^nC_{r-1}$, ${}^nC_r$ and ${}^nC_{r+1}$, respectively.
Now, ${}^nC_{r-1}$, ${}^nC_r$ and ${}^nC_{r+1}$ are in A.P.

$\Rightarrow 2\,{}^nCr = C_{r-1} + {}^nC_{r+1} \Rightarrow \frac{{}^nC_{r-1}}{{}^nC_r} + \frac{{}^nC_{r+1}}{{}^nC_r} = 2$

$\Rightarrow \frac{n!}{(r-1)!\,.(n-r+1)!} \times \frac{(r!).(n-r)!}{n!} +$

$\frac{n!}{(r-1)!\,.(n-r+1)!} \times \frac{(r!).(n-r)!}{n!} = 2$

$\Rightarrow \frac{r}{(n-r+1)} + \frac{n-r}{n+1} = 2$

[$\because (n - r + 1)! = (n - r + 1).(n - r)!$ and $(r!) = r.\,(r - 1)!$]

$\Rightarrow r(r + 1) + (n - r)(n - r + 1) = 2(r + 1)(n - r + 1)$

$\Rightarrow n^2 - n(4r + 1) + 4r^2 - 2 = 0.$

5. Which is larger $(1.01)^{1000000}$ or 10,000?

[NCERT]

Solution

Splitting 1.01 and using binomial theorem, the first few terms are $(1.01)^{1000000} = (1 + 0.01)^{1000000} = {}^{1000000}C_0 + {}^{1000000}C_1(0.01)$ + other positive terms = $1 + 1000000 \times 0.01$ + other positive terms = 1 + 10000 + other positive terms > 10000 Hence, $(1.01)^{1000000}$ >10000

6. Find the term independentof x in the expansion of $\left(x^2 + 2 + \frac{1}{x^2}\right)^8$.

Solution

$\left(x^2 + 2 + \frac{1}{x^2}\right)^8. = \left[\left(x + \frac{1}{x}\right)^2\right]^8 = \left(x + \frac{1}{x}\right)^{16}$

Suppose, $(r + 1)$th term is independent of x

Now, $T_{r+1} = {}^nC_r\, x^{n-r}.\, a^r = {}^{16}C_r . x^{16-r}.\left(\frac{1}{x}\right)^r = {}^{16}C_r .\, x^{16-2r}$

$\because$ $(r + 1)$th term is independent of x $\therefore$

$x^{16} - 2r = x^0$

$$16 - 2r = 0 \Rightarrow r = 8$$

$\therefore$ Thus term independent of x is $T_9 = {}^{16}C_8$.

7. Find the fourth root of 624, correct to four places of decimal.

Solution

$624^{1/4} = (625 - 1)^{1/4} = (54 - 1)^{1/4}$

$= 5\left(1 - \frac{1}{5^4}\right)^{1/4}$

$= 5\left[1 - \frac{1}{4}\frac{1}{5^4} + \frac{\frac{1}{4}\left(\frac{1}{4} - 1\right)}{2!}\left(\frac{1}{5^4}\right)^2 - \ldots\right]$

$= 5 - \frac{1}{4}\frac{1}{5^3} - \frac{\frac{1}{4}\cdot\frac{3}{4}}{2}\frac{1}{5^7}\ldots$

$= 5 - \frac{2}{10^3} - \frac{12}{12^7} = 5 - .002 - .0000012$

$= 4.9979988 = 4.9979$

8. Expand $\left(\frac{x}{3} + \frac{1}{x}\right)^5$

[NCERT]

Solution

Using Binomial theorem for positive integral index, we have,

$\left(\frac{x}{3} + \frac{1}{x}\right)^5$

$= {}^5C_0\left(\frac{x}{3}\right)^5 + {}^5C_1\left(\frac{x}{3}\right)^4\left(\frac{1}{x}\right)$

$+ {}^5C_2\left(\frac{x}{3}\right)^3\left(\frac{1}{x}\right)^2 + {}^5C_3\left(\frac{x}{3}\right)^2\left(\frac{1}{x}\right)^3$

$+ {}^5C_4\left(\frac{x}{3}\right)^1\left(\frac{1}{x}\right)^4 + {}^5C_3\left(\frac{1}{x}\right)^5$

$= \left(\frac{x^5}{243}\right) + 5\left(\frac{x^4}{81}\right)\left(\frac{1}{x}\right) + 10\left(\frac{x^3}{27}\right)\left(\frac{1}{x^2}\right)$

$+ 10\left(\frac{x}{9}\right)^2\left(\frac{1}{x^3}\right) + 5\left(\frac{x}{3}\right)\left(\frac{1}{x^4}\right) + \frac{1}{x^5}$

$= \frac{x^5}{243} + \frac{5}{81}x^3 + \frac{10}{27}x + \frac{10}{9}\left(\frac{1}{x}\right)$

$+ \frac{5}{3}\left(\frac{1}{x^3}\right) + \frac{1}{x^5}$

9. Using Binomial theorem, indicate which number is larger $(1.1)^{10000}$ or 1000.

[NCERT]

Solution

Now $(1.1)^{10000} = (1 + 0.1)^{10000} = {}^{10000}C_0 + {}^{10000}C_1(0.1) + {}^{10000}C_2(0.1)2 + ... + {}^{10000}C_{10000}(0.1)^{10000}$

$= 1 + 10000 \times \frac{1}{10}$ + some positive terms

$= 1 + 1000$ + some positive terms > 1000

Hence, $(1.1)^{10000}$ is larger than 1000.

10. Prove that $\sum_{r=0}^{n} 3^r\, {}^nC_r = 4n$.

[NCERT]

Solution

Now $\sum_{r=0}^{n} 3^r\, {}^nC_r \sum_{r=0}^{n} {}^nC_r\, 3^r$

$= {}^nC_0 + \text{v}C_1\, 3 + {}^nC_2\, 3^2 + ... + {}^nC_n\, 3^n = (1 + 3)^n = 4^n$

$(\because (1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2\, x^2 + ... + {}^nC_n x^n)$

11. Write the general term in the expansion of $(x^2 - yx)^{12}, x \neq 0$.

[NCERT]

Solution

The given power of binomial is $(x^2 - yx)^{12}$ is $\{x^2 + (-yx)\}^{12}$

Here, the general term is $T_{r+1} = {}^{12}C_r\, (x^2)^{12-r}\, (-yx)\}^r = {}^{12}C_r\, x^{24-2r}\, (-1)^r\, y^r\, x^r = {}^{12}C_r\, (-1)^r x^{24-r} y^r$.

12. Find a positive value of m for which the coefficient of x^2 in the expansion $(1 + x)^m$ is 6.

[NCERT]

Solution

Now $(1 + x)^m = {}^mC_0 + {}^mC_1 + {}^mC_2\, x^2 + ... + {}^mC_m x^m$

we are given that coefficient of $x^2 = 6$

$\Rightarrow {}^mC_2 = 6$

$\Rightarrow \frac{m(m-1)}{2\,!} = 6 \Rightarrow m^2 - m = 6 \times 2! = 6 \times 2 = 12$

$\Rightarrow m^2 - m - 12 = 0 \Rightarrow (m - 4)\,(m + 3) = 0$

$\Rightarrow m = 4, -3$

But, m cannot be negative, therefore, $m = 4$.

13. Find a, b and n in the expansion of $(a + b)^n$, if the first three terms of its expansion are 729, 7290 and 30375, respectively.

[NCERT]

Solution

It is given that

$T_1 = 729 \quad \Rightarrow {}^nC_0\, a^n\, b^0 = 729 \Rightarrow a^n = 729$ (1)

$T_2 = 7290 \quad \Rightarrow {}^nC_1\, a^{n-1}\, b^1 = 7290$

$\Rightarrow na^{n-1}\, b = 7290$ (2)

and $T_3 = 30375 \Rightarrow {}^nC_2\, a^{n-2}\, b^2 = 30375$

$\Rightarrow \frac{n(n-1)}{2}\, a^{n-2}\, b^2 = 30375$ (3)

Multiplying (1) and (3), we get

$\frac{n(n-1)}{2}\, a^{2n-2}\, b^2 = 729 \times 30375$ (4)

Squaring (2), we get $n^2\, a^{2n-2}\, b^2 = 7290 \times 7290$

Dividing (4) by (5), we have

$$\frac{n(n-1)}{2n^2} = \frac{729 \times 30375}{7290 \times 7290}$$

$$\Rightarrow \frac{n-1}{2n} = \frac{30375}{72900} = \frac{5}{12}$$

$$\Rightarrow 12n - 12 = 10n \Rightarrow 2n = 12 \Rightarrow n = 6$$

Substituting, $n = 6$ in (1), we get $a^6 = 729$

$$\Rightarrow a^6 = 3^6 \Rightarrow a = 3 \qquad (6)$$

Substituting, $n = 6$ and $a = 3$ in (ii), we have

$$6(3)^{6-1}\,b = 7290 \;\Rightarrow b = \frac{7290}{6 \times 3^5} = \frac{7290}{6 \times 243}$$

$\Rightarrow b = 5.$

Hence, the required power of binomial $= (a + b)^n = (3 + 5)^6$.

UNSOLVED SUBJECTIVE PROBLEMS: (CBSE/STATE BOARD): TO GRASP THE TOPIC, SOLVE THESE PROBLEMS

Exercise I

1. Expand $(1 - 2x)^5$ by the binomial theorem. **[NCERT]**
2. Expand $\left(\frac{2}{x} - \frac{x}{2}\right)^5$ by the binomial theorem. **[NCERT]**
3. Expand $(2x - 3)^6$ by the binomial theorem. **[NCERT]**
4. Expand $\left(x + \frac{1}{x}\right)^6$ by the binomial theorem. **[NCERT]**
5. Using binomial theorem, evaluate $(96)^3$. **[NCERT]**
6. Find a if the 17th and 18th terms of the expansion of $(2 + a)^{50}$ are equal. **[NCERT]**
7. The coefficients of three consecutive terms in the expansion of $(1 + a)^n$ are in the ratio 1:7:42. Find n. **[NCERT]**
8. Expand $(x + y)^5$.
9. Expand the following $(1 - x + x^2)^4$.
10. Expand the following expressions.

    (i) $(1 - x)^6$

    (ii) $\left(x - \frac{1}{y}\right)^{11}$, $y \neq 0$

Exercise II

1. Expand $(x^2 + 2y)^5$ by the binomial theorem.
2. Expand $\left(2x - \frac{3}{y}\right)^5$ by the binomial theorem.
3. Find the value of r, if the coefficients of $(2r + 4)$th and $(r - 2)$th terms in the expansion of $(1 + x)^{18}$ are equal.
4. Using binomial theorem, evaluate $(101)^4$. **[NCERT]**
5. Using binomial theorem, evaluate $(99)^5$. **[NCERT]**
6. Find the coeffcient of $x^6 y^3$ in the expansion of $(x + 2y)^9$. **[NCERT]**
7. Find the number of terms in the expansions of the following.

   $(2x - 3y)^9$
8. Find the 7th term in the expansion of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$
9. If the coefficients of $(r - 1)$th, rth and $(r + 1)$th terms in the expansion of $(x + 1)^n$ are in the ratio 1 : 3 : 5 find n and r. **[NCERT]**
10. Expand $(x^2 + 2a)^5$ by binomial theorem.
11. The 3rd, 4th and 5th terms in the expansion of $(x+a)^n$ are, respectively 84, 280 and 560. Find the values of x, a and n.

ANSWERS

Exercise I

1. $1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$
2. $\frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5}{8}x^3 - \frac{x^5}{32}$
3. $64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729.$
4. $x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$

5. 884736

6. $a = 1$

7. $n = 55$

8. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

9. $1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8$.

10. (i) $1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$

(ii) $x^{11} - 11x^{10}y^{-1} + 55x^9y^{-2} - 165x8\,y^{-3} + 330x^7y^{-4} - 462x^6y^{-5} + 462x^5y^{-6} - 330x^4y^{-7} + 165x^3y^{-8} - 55x^2y^{-9} + 11xy^{-10}$

Exercise II

1. $x^{10} + 10x^8y + 40x^6y^2 + 80x^4y^3 + 80x^2y^4 + 32y^5$

2. $\left[32x^5 \frac{240x^4}{y} + \frac{720x^3}{y^2} \frac{1080x^2}{y^3} + \frac{810x}{y^4} \frac{243}{y^5}\right]$

3. $r = 6$

4. 104060401

5. 9509900499

6. 672

7. 10 terms

8. $\frac{10500}{x^3}$

9. $n = 7$ and $r = 3$

10. $x^{10}\ 10x^8a + 40x^6a^2 + 80x^4a^3 + 80x^2a^4 + 32a^5$

11. $x = 1$, $a = 2$ and $n = 7$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. Subjective the square root of 999 correct to three decimal places is

(a) 31.607 (b) 31.706

(c) 32.607 (d) 32.706

Solution

(a) $999^{1/2} = (900 + 99)^{1/2} = 900^{1/2}\left(1 + \frac{99}{900}\right)^{1/2}$

$= 30\,(1 + .11)^{1/2}$

$$= 30\left[1 + \frac{1}{2}(.11) + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2\,!}(.11)^2 + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)}{3\,!}(.11)^2 + \ldots\right]$$

$$= 30\left[1 + \frac{.11}{2} - \frac{1}{8}(.\,11)^2 + \frac{1}{6}(.\,11)^3 \ldots\right]$$

$= 30\,[1 + 0.055000 - 0.001512 + 0.000083 \ldots.]$

$= 30\,[1.053571] = 31.60713$

$= 31.607$ (Correct to three places of decimal)

2. The positive integer just greater than $(1 + 0.0001)^{10000}$ is

(a) 4 (b) 5

(c) 2 (d) 3

[AIEEE – 2002]

Solution

(d) We know that $e = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n$ and $2 < e < 3$

$\therefore (1 + 0.0001)^{10000} < 3$ (By putting $n = 10000$)

Also, $(1 + 0.0001)^{10000} = 1 + 10000 \times 10^{-4} + \frac{1000 \times 9999}{2\,!} \times 10^{-8} + \ldots$ upto 10001 terms

$\Rightarrow (1 + 0.0001)^{10000} > 2.$

Hence, 3 is the positive integer just greater than $(1 + 0.0001)^{10000} > 2$.

Hence, (d) is the correct option.

3. If the coefficients of second, third and fourth term in the expansion of $(1 + x)^{2n}$ are in A.P., then $2n^2 - 9n + 7$ is equal to

(a) – 1 (b) 0 (c) 1 (d) 3/2

[AMU – 2001; MP PET – 2004]

Solution

$T_2 = {}^{2n}C_1\,x$, $T_3 = {}^{2n}C_2\,x^2$, $T_4 = {}^{2n}C_3\,x^3$

Coefficient of T_2, T_3, T_4 are in A.P.

$$\Rightarrow 2.\ {}^{2n}C_2 = {}^{2n}C_1 + {}^{2n}C_3 \Rightarrow \frac{2n\,!}{2\,!(2n - 2)\,!}$$

$$= \frac{2n\,!}{(2n - 1)\,!} + \frac{2n\,!}{3\,!(2n - 3)\,!}$$

$$\Rightarrow \frac{2.2n(2n - 1)}{2} = 2n + \frac{2n(2n - 1)(2n - 1)}{6}$$

$\Rightarrow n(2n-1) = n + \dfrac{(n)(2n-1)(2n-2)}{6}$

$\Rightarrow 6(2n^2 - n) = 6n + 4n^3 - 6n^2 + 2n$

$\Rightarrow 6n(2n-1) = 2n(2n^2 - 3n + 4)$

$\Rightarrow 6n - 3 = 2n^2 - 3n + 4$

$\Rightarrow 0 = 2n^2 - 9n + 7 \Rightarrow 2n^2 - 9n + 7 = 0.$

4. Subjective the value of is equal to

(a) 1 (b) 2 (c) 2 (d) 4

$$\frac{(18^3 + 7^3 + 3.18.7.25)}{3^6 + 6.243.2 + 15.81.4 + 20.27.8 + 15.9.16 + 6.3.32 + 64}.$$

Solution

(a) The numerator is of the form $a^3 + b^3 + 3ab(a+b) = (a+b)^3$ where $a = 18$ and $b = 7$.

Therefore, $N^r = (18+7)^3 = 25^3$

For D^r, $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, $3^5 = 243$

${}^6C_1 = 6$, ${}^6C_2 = 15$, ${}^6C_3 = 20$, ${}^6C_4 = {}^6C_2 = 15$, ${}^6C_5 = {}^6C_1 = 6$, ${}^6C_6 = 1$

$\therefore D^r = 3^6 + {}^6C_1 3^5 . 2^1 + {}^6C_2 3^4 . 2^2 + {}^6C_3 3^3 . 2^3 + {}^6C_5\, 3.2^5 + {}^6C_6 2^6.$

This is clearly the expansion of $(3+2)^6 = 5^6 = (25)^3$

$$\therefore \frac{N^r}{D^r} = \frac{(25)^3}{(25)^3} = 1$$

5. The value of x for which the sixth term in the expansion of

$$\left[2^{\log_2 \sqrt{9^{x-1+7}}} + \frac{1}{21/5 \log_2(3^{x-1}+1)}\right]^7 \text{ is } 84$$

is equal to

(a) 1 (b) 2 (c) 1, 2 (d) 3

[DCE – 1993, 1995]

Solution

(c) Exp. $= \left[\sqrt{9^{x-1}+7}\ \dfrac{1}{(3^{x-1}+1)^{1/5}}\right]^7$

Now, $T_6 = 84$

$\Rightarrow {}^7C_5 \left(\sqrt{9^{x-1}+7}\right)^2 \left(\dfrac{1}{(3^{x-1}+1)^{1/5}}\right)^5 = 84$

$\Rightarrow 21\,(9^{x-1}+7).\ \dfrac{1}{(3x-1+1)} = 84$

$\Rightarrow 9^{x-1} + 7 = 4\,(3^{x-1}+1)$

$\Rightarrow \dfrac{3^{2x}}{9} + 7 = \dfrac{4}{3} 3^x + 4 \Rightarrow 3^{2x} - 12.3^x + 27 = 0$

$\Rightarrow (3^x - 3)(3^x - 9) = 0 \Rightarrow 3^x = 3$ or $3^x = 9$

$\Rightarrow x = 1, 2$

6. If the coefficient of mth, $(m+1)$th and $(m+2)$th terms in the expansion of $(1+x)^m$ are in AP, then

(a) $n^2 + n(4m+1) + 4m^2 - 2 = 0$

(b) $n^2 + n(4m+1) + 4m^2 + 2 = 0$

(c) $(n - 2m)^2 = n + 2$

(d) $(n + 2m)^2 = n + 2$

[AIEEE – 2005]

Solution

(c) $2.{}^nC_m = {}^nC_m - 1 + {}^nC_{m+1}$

$\Rightarrow 2.\ \dfrac{n!}{m!\,(n-m)!} = \dfrac{n!}{(m-1)!\,(n-m+1)!} + \dfrac{n!}{(m+1)!\,(n-m+1)!}$

$\Rightarrow 2(m+1)(n-m+1) = (m+1)m + (n-m+1)(n-m)$

$\Rightarrow 4mn - 4m^2 + n - n^2 + 2 = 0$

$\Rightarrow (n-2m)2 = n+2$

7. $1\ 1! + 2\ 2! + 3\ 3! + \ldots + n\ n!$ is equal to

(a) $(n+1)!$ (b) $(n+1)! - 1$

(c) $(n+1)! + 1$ (d) none of these

Solution

(b) $1\ 1! + 2\ 2! + 3\ 3! + \ldots + n\ n!$

$= \sum_{r=1}^{n} rr! = \sum_{r=1}^{n} \{(r+1) - 1\}\, r!$

$= \sum_{r=1}^{n} \{(r+1)r! - r!\}$

$= \sum_{r=1}^{n} ((r+1)! - r!)$

$= (2! - 1!) + (3! - 2!) + (4! - 3!) + \ldots + ((n+1)! - n!)$

$= (n+1)! - 1! = (n+1)! - 1$

8. The value of $\{1.3.5.....(2n-3)(2n-1)\}$ is

(a) $(2n)!/n!$ (b) $(2n)!/2n$

(c) $n!/(2n)!$ (d) None of these

Solution

(a) $1.3.5........(2n-1)^{2n} =$

$\dfrac{1.2.3.4.5.6.....(2n-1)(2^n)2^n}{2.4.6.....2n}$

$= \dfrac{(2n)!\,2^n}{2^n(1.2.3....n)} = (2n)!/n!$

9. If in the binomial expansion of $\left[\sqrt{2^{\log(10-3x)}} + \sqrt[5]{2^{(x-2)\log 3}}\right]^m$ 6th term is equal to 21 and coefficients of 2nd, 3rd, 4th terms are 1st, 3rd and 5th term of *A*.P., then the value of *x* (where log is defined at the base of 10)

(a) 0 (b) 1 (c) 2 (d) 3

[MPPET – 2007]

Solution

(c) As the coefficients ${}^mC_1$, ${}^mC_2$ and ${}^mC_3$ of T_2, T_3, T_4 are the first, third and fifth term of an A.P. whose common difference is $2d$, therefore $2\,.\,{}^mC_2 = {}^mC_1 + {}^mC_3 \Rightarrow (m-2)(m-7) = 0$

As the sixth term is 21 and $m = 2$ violates the rule, therefore, we will take $m = 7$ and $T_6 = 21$

$= {}^7C_5\left[\sqrt{2^{\log(10-3x)}}\right]^{7-5} \times \left[\sqrt[5]{2^{(x-2)\log 3}}\right]^5$

$= 21 = 21.2^{\log(10-3x)+\log 3x-2} = 2^{\log[(10-3x)3x-2]}$

$= 1 = 2^0$

On solving, we get, $x = 0, 2$.

10. In the expansion of $(2 - 3x^3)^{20}$, if the ratio of 10th term of 11th term is 45/22, then $x =$

(a) 2/3 (b) 3/2 (c) – 2/3 (d) – 3/2

[Orissa JEE – 2007]

Solution

(c) Given expansion $(2 - 3x3)^{20}$

$\therefore T_{r+1} = {}^{20}C_r\, 2^{20-r}(-3x^3)^r$

$\therefore$ putting $r = 9, 10 \;\therefore\; t_{10} = {}^{20}C_9\, 2^{11}(-3x^3)^9$

$t_{11} = {}^{20}C_{10}\, 2^{10}(-3x^3)^{10} \because \dfrac{t_{10}}{t_{11}} = \dfrac{45}{22}$

$\Rightarrow \dfrac{10}{11} \times \left(\dfrac{2}{1}\right)\dfrac{1}{-3x^3} = \dfrac{45}{22}\left(\because \dfrac{{}^{20}C_9}{{}^{10}C_{10}} = \dfrac{10}{11}\right)$

$\Rightarrow x^3 = \dfrac{-8}{27} \;\therefore\; x = \dfrac{-2}{3}$

11. $(\sqrt{3}+1)^4 + (\sqrt{3}-1)^4$ is equal to

(a) a rational number

(b) an irrational number

(c) a negative integer (d) none of these

Solution

(a) $(\sqrt{3}+1)^4 + (\sqrt{3}-1)^4$

$= 2\{{}^4C_0(\sqrt{3})^4 + {}^4C_2(\sqrt{3})^2 + {}^4C_4\}$,

which is positive integer and hence a rational number.

12. The coefficient of x^{53} in the following expansion

$$\sum_{m=0}^{100} {}^{100}C_m(x-3)^{100-m}.2^m \text{ is}$$

(a) ${}^{100}C_{47}$ (b) ${}^{100}C_{53}$

(c) ${}^{-100}C_{53}$ (d) ${}^{-100}C_{100}$

[IIT Sc. – 1992]

Solution

(c) The given sigma is expansion of

$[(x-3)+2]^{100} = (x-1)^{100} = (1-x)^{100}$

$\therefore x^{53}$ will occur in T_{54}. $T_{54} = {}^{100}C_{53}(-x)^{53}$

$\therefore$ Coefficient is $-\,{}^{100}C_{53}$.

13. If in the expansion of $(1+x)^m(1-x)^n$, the coefficient of x and x^2 are 3 and – 6 respectively, then m is

(a) 6 (b) 9 (c) 12 (d) 24

[IIT – 1999; MP PET – 2000]

Solution

(c) $(1+x)^m(1-x)^n$

$= \left[1 + mx + \dfrac{m(m-1)x^2}{2!} + \ldots\right]$

$\left(1 - \dfrac{nx + n(n-1)}{2!}x^2 - \ldots\right)$

$= 1 + (m-n)x$

$+ \left(\dfrac{n^2-n}{2} - mn + \dfrac{(m^2-m)}{2}\right)x^2 + \ldots$

Given $m - n = 3$ or $n = m - 3$

Hence, $\dfrac{n^2-n}{2} - mn + \dfrac{m^2-m}{2} = -6$

$\Rightarrow \dfrac{(m-3)(m-4)}{2} - m(m-3) + \dfrac{m^2-m}{2} = -6$

$\Rightarrow m^2 - 7m + 12 - 2m^2 + 6m + m^2 - m + 12 = 0$

$\Rightarrow -2m + 24 = 0$

$\Rightarrow m = 12$

14. If a_1, a_2, a_3 are coefficients of any four consecutive terms in the expansion of $(b + x)^n$, then $\frac{a_1}{a_1 + a_2} + \frac{a_3}{a_3 + a_4}$ is equal to

(a) $\frac{a_2}{a_2 + a_3}$ (b) $\frac{2a_2}{a_2 + a_3}$

(c) $\frac{-a^2}{a_2 + a_3}$ (d) $\frac{a_2}{2(a_2 + a_3)}$

[IIT – 1975]

Solution

(b) Let given coefficients are those of T_r, T_{r+1}, T_{r+2}, T_{r+3}. Then

$$\frac{a_2}{a_1} = \frac{\text{coef. of } T_{r+1}}{\text{coef. of } T_r} = \frac{C_r}{C_{r-1}} = \frac{n-r+1}{r}$$

$$\Rightarrow 1 + \frac{a_2}{a_1} = \frac{n+1}{r} \Rightarrow \frac{a_1}{a_1 + a_2} = \frac{r}{n+1} \quad (1)$$

Similarly, $\frac{a_2}{a_2 + a_3} = \frac{r+1}{n+1}$ (2)

$$\frac{a_3}{a_3 + a_4} = \frac{r+2}{n+1} \quad (3)$$

Now, (1) + (3)

$$\Rightarrow \frac{a_2}{a_1 + a_2} + \frac{a_3}{a_3 + a_4} = \frac{2(r+1)}{n+1} = \frac{2a_2}{a_2 + a_3}$$

[by (2)]

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. If the coefficient of 7th and 13th term in the expansion of $(1 + x)^n$ are equal, then $n =$

(a) 10 (b) 15

(c) 18 (d) 20

2. In the expansion of $(1 - x)^5$, coefficient of x^5 will be

(a) 1 (b) – 1

(c) 5 (d) – 5

3. If the ratio of the coefficient of third and fourth term in the expansion of $\left(x - \frac{1}{2x}\right)^n$ is 1: 2, then the value of n will be

(a) 1 8 (b) 1 6

(c) 12 (d) – 10

4. If the coefficients of rth term and $(r + 4)$th term are equal in the expansion of $(1 + x)^{20}$, then the value of r will be

(a) 7 (b) 8 (c) 9 (d) 10

[MPPET – 2002]

5. Sum of odd terms is A and sum of even terms is B in the expansion $(x + a)^n$, then

[RPET – 1987, 1992; UPSEAT – 2004; Roorkee – 1986]

(a) $AB = \frac{1}{4}[(x - a)^{2n} - (x + a)^{2n}]$

(b) $2AB = (x + a)^{2n} - (x - a)^{2n}$

(c) $4AB = (x + a)^{2n} - (x - a)^{2n}$

(d) none of these

6. 9th term in the expansion of $\left(\frac{y}{2} + 2x\right)^{12}$ is

(a) $7920\, x^7x^5$ (b) $7920\, x^6y^6$

(c) $7920\, x^8y^4$ (d) $7816\, x^8x^4$

7. If A and B are the coefficient of x^n in the expansions of $(1 + x)^{2n}$ and $(1 + x)^{2n-1}$ respectively, then

(a) $A = B$ (b) $A = 2B$

(c) $2A = B$ (d) none of these

[NCERT]

8. The total number of terms in the expansion of $(x + a)^{100} + (x - a)^{100}$ after simplification will be

(a) 202 (b) 51

(c) 50 (d) none of these

9. If p and q be positive, then the coefficient of x^p and x^q in the expansion of $(1 + x)^{p+q}$ will be

(a) equal

(b) equal in magnitude but opposite in sign

(c) reciprocal to each other

(d) none of these

[NCERT]

10. If the coefficients of 5th, 6th and 7th terms in the expansion of $(1 + x)^n$ be in A.P.m then $n =$

(a) 7 only (b) 14 only
(c) 7 or 14 (d) none of these

[Roorkee – 1984]

11. The value of $(\sqrt{5} + 1)^5 - (\sqrt{5} - 1)^5$

(a) 252 (b) 352
(c) 452 (d) 532

[MPPET – 1985]

12. If the three consecutive coefficient in the expansion of $(1 + x)^n$ are 28, 56 and 70, then the value of n is

(a) 6 (b) 4 (c) 8 (d) 10

[MPPET – 1985]

13. In the expansion of $(x^2 - 2x)^{10}$, the coefficient of x^{16} is

(a) – 1680 (b) 1680
(c) 3360 (d) 6720

14. If T_2/T_3 in the expansion of $(a + b)^n$, and T_3/T_4 in the expansion of $(a + b)^{n+3}$ are equal, then $n =$

(a) 3 (b) 4 (c) 5 (d) 6

[RPET – 1987, 1996]

15. If the coefficients of x^7 and x^8 in $\left(2 + \frac{x}{3}\right)^n$ are equal, then n is

(a) 56 (b) 55
(c) 45 (d) 15

16. If the coefficient of $(2r + 4)$th and $(r - 2)$th terms in the expansion of $(1 + x)^{18}$ are equal, then $r =$

(a) 12 (b) 10 (c) 8 (d) 6

[PCET – 2008; MPPET – 1997]

17. If the second,third and fourth term in the expansion of $(x + a)^n$ are 240, 720 and 1080, resapectively, then the value of n is:

(a) 15 (b) 20 (c) 10 (d) 5

[Kurukshetra CEE – 1991; DCE – 1995, 2001]

18. If the coefficients of T_r, $T_r + 1$, $T_r + 2$ terms of $(1 + x)^{14}$ are in A.P., then $r =$

(a) 6 (b) 7 (c) 8 (d) 9

19. The expansion $\left[x+(x^3-1)^{\frac{1}{2}}\right]^5+\left[x+(x^3-1)^{\frac{1}{2}}\right]^5$ is *a* polynomial of degree

(a) 5 (b) 6
(c) 7 (d) 8

[IIT – 1992; DCE – 1996, 2006]

20. In the expansion of $(x + a)^n$, the sum of odd terms is P and Sum of even terms is Q, then the value of $(P^2 - Q^2)$ will be:

(a) $(x^2 + a^2)^n$ (b) $(x^2 - a^2)^n$
(c) $(x - a)^{2n}$ (d) $(x + a)^{2n}$

[RPET – 1997; Pb CET – 1998]

SOLUTIONS

1. (c) ${}^nC_6 = {}^nC_{12}$

$\Rightarrow n = 6 + 12$

$\therefore n = 18$

2. (b) $(1 - x)^5$, coefficient of x^5

$\therefore T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

$\therefore$ in the expansion of $(1 - x)^5$

We have

$T_{r+1} = {}^5C_r\,(1)^{5-r}\,(-x)^r$

$= (-1)^r\, {}^5C_r\, x^r$

$\because T_{r+1}$ contains x^5

$\therefore r = 5$

Hence, the coefficient of x^5 in the expansion

$= {}^5C_5\,(-1)^5$

$= 1 \times -1$

$= -1$

3. (d) $T_3 = {}^nC_2\,(x)^{n-2}\left(-\frac{1}{2x}\right)^2$ and

$T_4 = {}^nC_3(x)^{n-3}\left(-\frac{1}{2x}\right)^2$

But according to the condition,

$$\frac{-n(n-1)\times 3\times 2\times 1\times 8}{n(n-1)(n-2)\times 2\times 1\times 4} = \frac{1}{2} \Rightarrow n = -10$$

4. (c) $T_{r+1} = {}^nC_r\, x^r$ for $(1 + x)^n$

Here the coefficient is ${}^nC_r$.

Given ${}^nC_{r-1} = {}^nC_{r+3} \Rightarrow {}^{20}C_{r-1} = {}^{20}C_{r+3}$
$\Rightarrow (r-1) + (r+3) = 20 \Rightarrow r = 9.$

5. (c) $(x+a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a + {}^nC_2\,x^{n-2}a^2 + {}^nC_3x^n - 3a^3 + \ldots.. + a^n$ (1)

$(x-a)^n = {}^nC_0x^n - {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 - {}^nC_3x^{n-3}a^3 + \ldots.. + (-1)^na^n$ (2)

By assumption,

$A = {}^nC_0x^n + {}^nC_2x^{n-2}a^2 + {}^nC_4x^{n-4}a^4 + \ldots\ldots..$

$B = {}^nC_1x^{n-1}a + {}^nC_3x^{n-3}a^3 + {}^nC_5x^{n-5}a^5 + \ldots\ldots..$

This $\Rightarrow A + B = (x+a)^n, A - B = (x-a)^n$
$\Rightarrow 4AB = (A+B)^2 - (A-B)^2 = (x+a)^{2n} - (x-a)^{2n}.$

6. (c) We know that in the expansion of $(a+b)^n$, we have $(r+1)$th term $T_{r+1} = {}^nCr\, a^{n-r}b^r$

$\therefore$ in the expansion of $\left(\frac{y}{2} + 2x\right)^{12}$

We have

9th term, $T_9 = T_{8+1}$

$= {}^{12}C_8\left(\frac{y}{2}\right)^{12-8}(2x)^8$ [Here $a = \frac{y}{2}$, $b = 2x$]

$= {}^{12}C_8\left(\frac{y}{2}\right)^4(2x)^8$

$= {}^{12}C_8\frac{y^4}{2^4} \times 2^8 \times x^8$

$= {}^{12}C_4 \times 2^4 \times x^8 \times y^4$

$= \frac{12 \times 11 \times 10 \times 9 \times 16}{4 \times 3 \times 2 \times 1}x^8y^4$

$= 72 \times 100\, x^8y^4$

$= 7920\, x^8y^4$

7. (b) A = Coefficient of x^n in $(1+x)^{2n} = {}^{2n}Cn$.
B = coefficient of x^n in
$(1+x)^{2n-1} = {}^{2n-1}C_n = {}^{2n-1}C_{(2n-1)-n} = {}^{2n-1}C_{n-1}$
Now A $= \left(\frac{2n}{n}\right)({}^{2n-1}C_{n-1}) = 2B$

8. (b) $(x+y)^{100} + (x-y)^{100} = 2[C_0x^n + C_2x^{n-2}y^2 + C_4x^{n-4}y^4 + \ldots\ldots.. + C_ny^n]$, where $n = 100$

Total terms $= \left(\frac{100}{2}\right) + 1 = 51.$

9. (a) $T_{r+1} = {}^{p+q}C_r\,x^r$
$\Rightarrow$ coefficient of $x^r = {}^{p+q}C_r$.

Hence, coefficient of $x^p = {}^{p+q}C_p$ and that of x^q is ${}^{p+q}C_q$.

Note that ${}^{p+q}C_p = {}^{p+q}C_q$ as ${}^nC_r = {}^nC_{n-r}$.

10. (c) Coefficient of T_5, T_6, T_7 are in A.P. for $(1+x)^n$
$\Rightarrow {}^nC_4, {}^nC_5, {}^nC_6$ are in A.P.
$\Rightarrow 2({}^nC_5) = {}^nC_4 + {}^nC_6$

$\Rightarrow \frac{2n!}{5!(n-5)!} = \frac{n!}{4!(n-4)!} = \frac{n!}{6!(n-6)!}$

$\Rightarrow \frac{2}{5(n-5)} = \frac{1}{(n-4)(n-5)} + \frac{1}{6.5}$

If we put $n = 7$ in (1), $\frac{2}{5(2)} = \frac{1}{3.2} + \frac{1}{30}$ or $\frac{1}{5} = \frac{1}{5}$ (True)

$n = 14$ in (1) $\Rightarrow \frac{2}{5 \times 9} = \frac{1}{10 \times 9} + \frac{1}{30}$ (True)

11. (b) $(\sqrt{5}+1)^5 - (\sqrt{5}-1)^5 = (1+\sqrt{5})^5 + (1-\sqrt{5})^5$
$= 2[1 + {}^5C_2(\sqrt{5})^2 + {}^5C_4(\sqrt{5})^4]$

$= 2\left[1 + \frac{5.4}{2!}.5 + 5(5^2)\right] = 352$

12. (c) Coefficient in $T_{r+1} = 28$, $T_{r+2} = 56$, $T_{r+3} = 70$.

This $\Rightarrow {}^nC_r = 28$ (1)
${}^nC_{r+1} = 56$ (2)
${}^nC_{r+3} = 70$ (3)

Here we apply

$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-(r-1)}{r}$,

we get

$\frac{(2)}{(1)} \Rightarrow \frac{{}^nC_{r+1}}{{}^nC_r} = \frac{56}{28} \Rightarrow \frac{n-r}{r+2} = 2$

$\Rightarrow n = 3r + 2$ (4)

$\frac{(3)}{(2)} \Rightarrow \frac{{}^nC_{r+2}}{{}^nC_{r+1}} = \frac{70}{56} \Rightarrow \frac{n-(r+1)}{r+2} = \frac{5}{4}$

$\Rightarrow \frac{(3r+2)-(r+1)}{r+2} = \frac{5}{4}$

$\Rightarrow r = 2 \Rightarrow n = 8$ by (4)

13. (c) $(x^2 - 2x)^{10} = x^{10}(x-2)^{10}$ (1)

For coefficient of x^{16} in (1), we consider coefficient of x^6 in

$(x-2)^{10}$, $T_{r+1} = {}^{10}C_r (x)^{10-r}(-2)^r$ (2)

$10 - r = 6, r = 4$

By (2), coefficient $= {}^{10}C_4(-2)^4 = \frac{10.9.8.7}{4!}.(16)$

$= 3360$

14. (c) For $(a+b)^n$: $\frac{T_2}{T_3} = \frac{{}^nC_1 a^{n-1}b}{{}^nC_2 a^{n-2}b^2} = \left(\frac{2}{n-1}\right)\left(\frac{a}{2}\right)$ (1)

For $(a+b)^{n+3}$: $\frac{T_3}{T_4} = \frac{{}^{n+3}C_2 a^{n+3-2}b^2}{{}^{n+3}C_3 a^{n+3-2}b^3}$

$= \frac{(n+3)(n+2).3!}{2!.(n+3)(n+2)(n+1)}\cdot\left(\frac{a}{b}\right)$

or, $\frac{T_3}{T_4} = \left(\frac{a}{b}\right)\times\left(\frac{3}{n+1}\right)$ (2)

By asssumption and by (1) and (2),

$\left(\frac{a}{b}\right)\left(\frac{2}{n-1}\right) = \left(\frac{a}{b}\right)\times\left(\frac{3}{n+1}\right)$ or, $\frac{2}{n-1} = \frac{3}{n+1}$

or, $2n + 2 = 3n - 3$ or, $n = 5$.

15. (b) $T_{r+1} = {}^nC_r 2^{n-r}\left(\frac{x}{3}\right)^r = {}^nC_r \frac{2^{n-r}}{3^r}x^r$.

We are given ${}^nC_7 \frac{2^{n-7}}{3^7} = {}^nC_8 \frac{2^{n-8}}{3^8}$

$\Rightarrow \frac{{}^nC_8}{{}^nC_7} = 6 \Rightarrow \frac{n!}{(n-8)!\,8!}\cdot\frac{(n-7)!\,7!}{n!} = 6$

$\Rightarrow \frac{n-7}{8} = 6 \Rightarrow n = 55.$

16. (d) Coefficient in T_{2r+4} = Coefficient in $T_{(r-2)}$ for $(1+x)^{18}$

This $\Rightarrow {}^{18}C_{2r+3} = {}^{18}C_{r-3}$

$\Rightarrow (2r+3) + (r-3) = 18 \Rightarrow r = 6$

17. (d) $T_2 = {}^nc_1 x^{n-1}a = 240$,

$T_3 = {}^nc2x^{n-2}a^2 = 720$

$T_4 = {}^nc_3 x^{n-3}a^3 = 1080$

$\therefore \frac{nx^{n-1}a}{\frac{n(n-1)}{2}x^{n-2}a^2} = \frac{240}{720} \Rightarrow \frac{2x}{(n-1)a} = \frac{1}{3}$ (1)

and $\frac{\frac{n(n-1)}{1.2}x^n - 2a^2}{\frac{n(n-1)(n-2)}{1.2.3}x^{n-3}a^3} = \frac{720}{1080}$

$\Rightarrow \frac{x}{(n-2)a} = \frac{2}{3}$ (2)

Divide (1) by (2) $\frac{2(n-2)}{3(n-1)} = \frac{1}{2}$

$\Rightarrow 4n - 8 = 3n - 3 \Rightarrow n = 5$

18. (d) $T_r = {}^{14}c_{r-1}x^{r-1}$; $T_{r+1} = {}^{14}c_r x^r$; $T_{r+2} = {}^{14}c_{r+1}x^{r+1}$

By the given condition

$2\,.\,{}^{14}c_r = {}^{14}c_{r-1} + {}^{14}c_{r+1}$

$\Rightarrow 2.\frac{14!}{r!(14-r)!} = \frac{14!}{(r-1)!(15-r)!} + \frac{14!}{(r+1)!(13-r)!}$

$\Rightarrow \frac{2}{r(r-1)!(14-r)(13-r)!} = \frac{1}{(r-1)!(15-r)(14-r)(13-r)!}$

$+ \frac{1}{(r+1)r(r-1)!(13-r)!}$

$\Rightarrow \frac{2}{r(14-r)} = \frac{1}{(15-r)(14-r)} + \frac{1}{(r+1)^r}$

$\Rightarrow \frac{(15-r)-r}{r(5-r)(14-r)} = \frac{(14-r)-(r+1)}{(r+1)r(14-r)}$

$\Rightarrow \frac{15-2r}{15-r} = \frac{13-2r}{r+1}$

$\Rightarrow 15r + 15 - 2r^2 - 2r = 195 - 30r - 13r + 2r^2$

$\Rightarrow 4r^2 - 56r + 180 = 0 \Rightarrow r^2 - 14r + 45 = 0$

$(r-5)(r-9) = 0 \Rightarrow r = 5, 9$

But 5 is not given. Hence, $r = 9$.

19. (c) $[x + (x^3-1)^{1/2}]^5 + [x - (x^3-1)^{1/2}]^5$

$= 2\,[{}^5C_0x^5 + {}^5C_2\,x^3\,(x^3-1) + {}^5C_4\,x^1\,(x^3-1)^2]$

Max. power of x is 7.

20. (b) $C_0\,x^n + C_2\,x^{n-2}\,a^2 + C_4\,x^{n-4}\,a^4 + \ldots\ldots = 0$

= Sum of odd terms $= p$

$C_1x^{n-1}a + C_3x^{n-3}a^3 + C_5x^{n-5}a^5 + \ldots.. = E =$ sum of even terms $= q$

This $\Rightarrow p + q = (x+a)^n, p - q = (x-a)^n$

$\Rightarrow p^2 - q^2 = [(x+a)(x-a)]^n$

$= (x^2 - a^2)^n$.

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. If the coefficients of $(2r + 1)$th and $(r + 5)$th terms in the expansion of $(1 + x)^{25}$ are equal, then the value of r is:

(a) 4 or 7 (b) 4 or 6 (c) 4 (d) 6

2. If in the expansion of $(1 - x)^n$ the coefficient of x^2 be 3, then the values of n are:

(a) 3, 2 (b) − 3, 2
(c) 3, − 2 (d) − 3, − 2

3. If for positive integers $r > 1$, $n > 2$, the coefficient of the $(3r)$th and $(r + 2)$th powers of x in the expansion of $(1 + x)^{2n}$ are equal, then

(a) $n = 2r$ (b) $n = 3r$
(c) $n = 2r + 1$ (d) none of these

4. The number of non-zero terms in the expansion of $(1 + 3\sqrt{2}\,x)^9 + (1 - 3\sqrt{2}\,x)^9$ is

(a) 9 (b) 0 (c) 5 (d) 10

[EAMCET – 1991]

5. If coefficient of $(2r + 3)$th and $(r - 1)$th terms in the expansion of $(1 + x)^{15}$ are equal, then value of r is

(a) 5 (b) 6 (c) 4 (d) 3

[RPET – 1995, 2003; UPSEAT – 2001]

6. If coefficients of 2nd, 3rd and 4th terms in the binomial expansion of $(1 + x)^n$ are in A.P., then $n^2 - 9n$ is equal to

(a) − 7 (b) 7 (c) 14 (d) − 14

[RPET – 1999; UPSEAT – 2002]

7. The coefficients of three successive terms in the expansion of $(1 + x)^n$ are 165, 330 and 462 respectively, then the value of n will be

(a) 11 (b) 10 (c) 12 (d) 8

8. If the coefficient of 4th term in the expansion of $(a + b)^n$ is 56, then n is

(a) 12 (b) 10 (c) 8 (d) 6

[AMU – 2000]

9. The coefficient of x^5 in the expansion of $(x + 3)^6$ is

(a) 18 (b) 6 (c) 12 (d) 10

[DCE – 2002]

10. In the expansion of $(1 + x)^n$, coefficients of 2nd, 3rd, and 4th terms are in A.P., then n is equal to

(a) 7 (b) 9
(c) 11 (d) none of these

11. What is the approximate value of $(1.02)^8$?

(a) 1.171 (b) 1.175
(c) 1.177 (d) 1.179

[NDA – 2008]

12. The coefficient of x^{12} in the expansion of $(x^2 + 2x)^{10}$ is:

(a) 11520 (b) 13410
(c) 16520 (d) 23040

[SCRA – 2007]

WORK SHEET: TO CHECK PREPARATION LEVEL

Important Instructions:

1. The answer sheet is immediately below the work sheet.
2. The test is of 8 minutes.
3. The test consists of 8 questions. The maximum marks are 24.
4. Use blue/black ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. If coefficients of $(2r + 1)$th term and $(r + 2)$th expansion of $(1 + x)^{43}$, then the value of r will be

(a) 14 (b) 15 (c) 13 (d) 16

[UPSEAT – 1999]

2. In the expansion of $(1 + x)^{11}$, the fifth the third term. Then the value of x^2 is
(a) 4 (b) 9
(c) 16 (d) 24

3. After simplifi cation, what is the number of terms in the expansion of $[(3x + y)^5]^4 - [(3x + y)^4]^5$?
(a) 4 (b) 5
(c) 10 (d) 11
[NDA – 2007]

4. In the expansion of $(1 + x)^n$ the coefficient of pth and $(p + 1)$th terms are respectively p and q. Then $p + q =$
(a) $n + 3$ (b) $n + 1$
(c) $n + 2$ (d) n

5. The first 3 terms in the expansion of $(1 + ax)^n$ $(n \neq 0)$ are 1, $6x$ and $16x^2$. Then, the value of a and n are; respectively
(a) 2 and 9 (b) 3 and 2
(c) 2/3 and 9 (d) 3/2 and 6
[Kerala Engg. – 2002]

6. If t_r is the rth term in the expansion of $(1 + x)^{101}$, then what is the ratio $\frac{t_{20}}{t_{19}}$ equal to?
(a) $\frac{20x}{19}$ (b) $83x$
(c) $19x$ (d) $\frac{83x}{19}$
[NDA – 2008]

7. What is the coefficient of x^3y^4 in $(2x + 3y^2)^5$?
(a) 240 (b) 360
(c) 720 (d) 1080
[NDA – 2008]

8. The coefficient of the $(m + 1)$th term and the $(m + 3)$th in the expansion of $(1 + x)^{20}$ are equal then value of m is
(a) 10 (b) 8
(c) 9 (d) none of these
[UP-SEE – 2007]

ANSWER SHEET

1.	ⓐ	ⓑ	ⓒ	ⓓ	4.	ⓐ	ⓑ	ⓒ	ⓓ	7.	ⓐ	ⓑ	ⓒ	ⓓ
2.	ⓐ	ⓑ	ⓒ	ⓓ	5.	ⓐ	ⓑ	ⓒ	ⓓ	8.	ⓐ	ⓑ	ⓒ	ⓓ
3.	ⓐ	ⓑ	ⓒ	ⓓ	6.	ⓐ	ⓑ	ⓒ	ⓓ					

HINTS AND EXPLANATIONS

1. $T_{2r+1} = T_{r+2} \Rightarrow {}^{43}C_{2r} = {}^{43}C_{r+1}$

$\therefore 2r = r + 1$ or $2r + r + 1 = 43$

$r = 1$ or $r = 14$

3. $((3x + y)^5)^4 - [(3x - y)^4)^5 = (3x + y)^{20} - (3x - y)^{20}$

$\therefore$ Number of terms $= \frac{20}{2} = 10$

5. $T_1 = 1, T_2 = 6x, T_3 = 16x^2$

${}^nC_0\ (ax)^0 = 1;\ {}^nC_1\ (ax)^1 = 6x;\ {}^nC_2\ (ax)^2 = 16x^2$

$\therefore an = 6$ and $\frac{n(n-1)}{2}a^2 = 16$

Solving, $a = \frac{2}{3}$ and $n = 9$

6. $\frac{b_{20}}{b_{19}} = \frac{{}^{101}C_{19}x^{19}}{{}^{101}C_{18}x^{18}} = \frac{101 - 19 + 1}{19} = \frac{83x}{19}$

LECTURE

2 Binomial Co-efficient

BASIC CONCEPTS

1. Term of the greatest coeffi cients in $(1 \pm x)^n$

(i) greatest value of ${}^nC_r$ is ${}^nC_{n/2}$ when n is even.

(ii) greatest value of ${}^nC_r$ is $\frac{{}^nC_{n-1}}{2}$ or $\frac{{}^nC_{n+1}}{2}$ when n is odd.

(iii) Terms with the greatest coefficients are as follows.

(a) $T_{\frac{n}{2}+1}$ for even n (b) $T_{\frac{n+1}{2}}$ and $T_{\frac{n+3}{2}}$ for odd n.

2. Middle Term in the Binomial Expansion of $(x + a)^n$

The middle term in the binomial expansion of $(x + a)^n$ depends upon the value of n.

(i) If n is even, then there is only one middle term, i.e., $\left(\frac{n}{2} + 1\right)$ th term and its binomial coefficient is ${}^nC_{n/2}$.

(ii) If n is odd, then there are two middle terms i.e., $\left(\frac{n+1}{2}\right)$ th and $\left(\frac{n+3}{2}\right)$ th terms.

Note: When there are two middle terms in the expansion, then their coeffi cients are equal to ${}^nC_{(n-1)/2}$ or $C(_{n+1)/2}$.

3. rth Term from the End in the Binomial Expansion of $(x + a)^n$ rth term from the end in the expansion of $(x + a)^n$ is $(n - r + 2)$th term from the beginning.

OR

$(r + 1)$th term from end = $(n - r + 1)$th term from beginning

i.e., $T_{r+1}(E) = T_{n-r+1}(B) \Rightarrow {}^nC_r = {}^nC_{n-r}$

$\therefore$ $T_r(E) = T_{n-r+2}(B)$

4. Properties of ${}^nC_r$

If $0 < r < n$, $n, r \in N$, then

(i) $r \cdot {}^nC_r = n \cdot {}^{n-1}C_{r-1}$.

(ii) $\frac{{}^nC_r}{r+1} = \frac{{}^{n+1}C_{r+1}}{n+1}$

(iii) ${}^nC_r = \frac{n}{r}\, {}^{n-1}C_{r-1}$.

(iv) $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$. (v) $\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{r+1}{n-r}$.

(vi) ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$.

(vii) ${}^nC_r = {}^nC_{n-r}$.

(viii) ${}^nC_r = {}^nC_s \Rightarrow r = s$ or $r + s = n$.

(ix) ${}^nC_{r-1}$, ${}^nC_r$ and ${}^nC_{r+1}$ are in A.P., then $n = 7$ or 14 and $r = 2$ or 5.

5. Properties of Binomial Coefficients

In the binomial expansion of $(1 + x)^n$, the coefficients ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ...${}^nC_n$ are denoted by $C_0, C_1, C_2, \ldots C_n$, respectively.

(i) Sum of all the binomial coefficients is obtained by putting all the variable x_i equal to 1 and it is equal to 2^n.

(ii) $C_0 + C_1 + C_2 + ... + C_n = 2^n$.

(iii) $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$.

Note: Sum of the binomial odd coefficients is equal to sum of the binomial even coefficients and each sum is equal to 2^{n-1}.

(iv) $C_0 - C_1 + C_2 - C_3 + C_4 - .... + (-1)^n C_n = 0$.

(v) $C_0^2 + C_1^2 + C_2^2 + ...C_n^2 = \dfrac{(2n)!}{(n!)^2} = {}^{2n}C_n$.

(vi) $C_0^2 - C_1^2 + C_2^2 - C_3^2 + ...$

$$= \begin{cases} 0, & \text{if } n \text{ odd.} \\ (-1)^{n/2}, {}^nC_{n/2}, & \text{if } n \text{ is even.} \end{cases}$$

(vii) $C_0 C_1 + C_1 C_2 + C_2 C_3 + ... + C_{n-1} C_n = {}^{2n}C_{n-1}$.

(viii) $C_0 C_r + C_1 C_{r+1} + ... C_{n-r} C_n = {}^{2n}C_{n-r}$ or ${}^{2n}C_{n+r}$.

(ix) $C_0 C_r + C_1 C_{r+1} + ... + C_{n-r} C_n = {}^{2n}C_{n-r} = \dfrac{(2n)!}{(n+r)!\,(n-r)!}$.

(x) $C_1 + 2C_2 + 3C_3 + ... + {}^nC_n = n \cdot 2^{n-1}$.

(xi) $C_1 - 2C_2 + 3C_3 - .... = 0$.

(xii) $C_0 + 2C_1 + 3C_2 + ... + (n+1) C_n = 2^{n-1}(n+2)$.

(xiii) $\dfrac{C_0}{1} + \dfrac{C_1}{2} + \dfrac{C_2}{3} + ... + \dfrac{C_n}{n+1} = \dfrac{2^{n+1} - 1}{n+1}$.

(xiv) $C_0 - \dfrac{C_1}{2} + \dfrac{C_3}{3} + \dfrac{C_3}{4} + ... + \dfrac{C_n}{n+1}\ \dfrac{(-1)^n C_n}{n+1} = \dfrac{1}{n+1}$.

(xv) $\dfrac{C_1}{C_0} + \dfrac{C_2}{C_1} + 3 \cdot \dfrac{C_3}{C_2} + ...+ n \cdot \dfrac{C_n}{C_{n-1}} = \dfrac{n \cdot (n+1)}{2}$

Explanation On putting $x = 1$ and -1 in the expansion of $(1 + x)^n$ we shall get results (2) and (4), respectively.

Also (2) + (4) and (2) − (4) will give result (3).

Further, $(1 + x)^{2n} = (1 + x)^n (x + 1)^n = (C_0 + C_1 x + ... + C_n x^n)(C_0 x^n + C_1 x^{n-1} + .. + C_n)$

Equating coefficients of x^n on both sides, we shall get result (viii). Similarly, on equating coefficients of x^{n-1}, x^{n-r} on both sides, we shall get results (vii), (viii) respectively.

Now on integrating both sides of the expansion $(1 + x)^n$ from 0 to 1, we shall get result (xii).

6. If $(1 + x + x^2)^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + .. + a_{2n} x^{2n}$, then

$a_0 + a_1 + a_2 + ... + a_{2n} = 3^n$

$a_0 - a_1 + a_2 - .... + a_{2n} = 1$

$a_0 + a_2 + a_4 + ... + a_{2n} = \dfrac{3^n + 1}{2}$

$a_1 + a_3 + a_5 + ... + a_{2n-1} = \dfrac{3^n - 1}{2}$

$a_0 + a_3 + a_6 + ... = a_1 + a_4 + a_7 + ...$

$= a_2 + a_5 + a_8 + ... = 3^{n-1}$

${}^nC_0 \cdot a_r - {}^nC_1 \cdot a_{r-1} + {}^nC_2 \cdot a_{r-2} - ......$

$$+ (-1)^r\, {}^nC_r\, a_0 = \begin{cases} 0, \text{ if } r \text{ is not multiple of 3} \\ (-1)^m \cdot {}^nC_m, \text{ if } r = 3m \end{cases}$$

(i) Middle term in the expansion of $\left(x + \frac{1}{x}\right)^{2n}$ is ${}^{2n}C_n = \dfrac{1.3.5..... (2n-1)}{n!} \times 2^n$

(ii) For all $m, n, r \in N$ and $r < m$ or n, then ${}^mC_r \cdot {}^nC_0 + {}^mC_{(r-1)} \cdot {}^nC_1 + {}^mC_{(r-2)} \cdot {}^nC_2 + ... + {}^mC_0 \cdot {}^nC_r = {}^{(m+n)}C_r$

(iii) ${}^{(2n+1)}C_0 + {}^{(2n+1)}C_1 + {}^{(2n+1)}C_2 + ... + {}^{(2n+1)}C_n = 2^{2n}$

(iv) ${}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + ... + {}^{2n+1}C_n = \dfrac{2^{2n+1} - 2}{2}$

(v) $2[{}^{2n}C_0 + {}^{2n}C_1 + {}^{2n}C_2 + ... + {}^{2n}C_{(n-1)}] + {}^{2n}C_n = 2^{2n}$

(vi) ${}^rC_r + {}^{r+1}C_r + ... {}^nC_r = {}^{n+1}C_{r+1}$.

SOLVED SUBJECTIVE PROBLEMS: (CBSE/STATE BOARD): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Prove that

$$\frac{1}{1!(n-1)!}+\frac{1}{3!(n-3)!}+\frac{1}{5!(n-5)!}+\ldots\ldots=\frac{2^{n-1}}{n!}.$$

Solution

$$\frac{1}{1!(n-1)!}+\frac{1}{3!(n-3)!}+\frac{1}{5!(n-5)!}+\ldots\ldots$$

$$=\frac{1}{n!}\left[\frac{n!}{1!(n-1)!}+\frac{n!}{3!(n-3)!}+\frac{n!}{5!(n-5)!}+\ldots\ldots\right]$$

$$=\frac{1}{n!}[{}^nC_1+{}^nC_3+{}^nC_5+\ldots\ldots]$$

$$=\frac{2^{n-1}}{n!}.$$

2. The sum of the coefficients of the first three terms in the expansion of $\left(x-\frac{3}{x^2}\right)^m$, $x \neq 0$ m being a natural number is 559. Find the term of the expansion containing x^3.

[NCERT]

Solution

The coefficients of the first three terms of $\left(x-\frac{3}{x^2}\right)^m$ are ${}^mC_0$, $(-3)\,{}^mC_1$ and $9\,{}^mC_2$. Therefore, by the given condition, we have ${}^mC_0 - 3\,{}^mC_1 + 9\,{}^mC_2 = 559$, i.e., $1-3m+\frac{9m(m-1)}{2}=559$ which gives $m=12$ (m being a natural number).

Now, $T_{r+1} = {}^{12}C_r x^{12-r}\left(-\frac{3}{x^2}\right)^r = {}^{12}C_r(-3)^r.x^{12-3r}$

Since, we need the term containing x^3, put $12-3r=3$ i.e., $r=3$.

Thus, the required term is ${}^{12}C_3(-3)^3x^3$ i.e., $-5940x^3$.

3. Prove that $C_0C_r+C_1C_{r+1}+\ldots..+C_{n-r}C_n=\frac{(2n)!}{(n-r)!(n+r)!}$

[Haryana CET – 1998; BIT Ranchi – 1986]

Solution

We know that

$$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots..+C_nx^n \quad (1)$$

Replace x by $1/x$,

$$\left(1+\frac{1}{x}\right)^n = C_0+\frac{C_1}{x}+\frac{C_2}{x^2}+\frac{C_3}{x^3}+\ldots.+\frac{C_n}{x^n} \quad (2)$$

Multiplying Equations (1) and (2),

$$(C_0+C_1x+C_2x^2+C_3x^3+\ldots\ldots+C_nx^n)\times\left(C_0+\frac{C_1}{x}+\frac{C_2}{x^2}+\frac{C_3}{x^3}+\ldots..+\frac{C_n}{x^n}\right)$$

$$=(1+x)^n\left(1+\frac{1}{x}\right)^n=\frac{(1+x)^{2n}}{x^n}$$

$$=\frac{1}{x^n}(1+{}^{2n}C_1x+{}^{2n}C_2x^2+\ldots\ldots+{}^{2n}C_nx^n)$$

Comparing coefficients of $\frac{1}{x^r}$ in both the sides,

$$C_0C_r+C_1C_{r+1}+C_2C_{r+2}+\ldots\ldots..+C_{n-r}C_n$$

$$=\text{Coefficients of }\left\{\frac{1}{x^n}x^{n-r}\right\}\text{ in R.H.S.}$$

$$={}^{2n}C_{n-r}=\frac{(2n)!}{(n-r)!(2n-n+r)!}$$

$$=\frac{(2n)!}{(n-r)!(n+r)!}$$

Proved

4. Find $(a+b)^4-(a-b)^4$. Hence evaluate $(\sqrt{3}+\sqrt{2})^4-(\sqrt{3}-\sqrt{2})^4$.

[NCERT]

Solution

Using binomial theorem for positive integral index, we have

$(a+b)^4 - (a-b)^4 =$

$= ({}^4C_0 a^4 + ={}^4C_1 a^3 b + {}^4C_2 a^2 b^2 + {}^4C_3 ab^3 + {}^4C_4 b^4)$

$- ({}^4C_0\, a^4 - {}_4C_1\, a^3 b + {}^4C_2 a^2\, b^2 - {}^4C_3\, ab^3 + {}^4C_4 b^4)$

$= 2\{{}^4C_1\, a^3 b + {}^4C_3\, ab^3\} = 2\,\{4a^3\, b + 4ab^3\}$

$= 8\, ab\, (a^2 + b^2)$

Substituting $a = \sqrt{3}$ and $b = \sqrt{2}$ in the above result,

we obtain $(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4$

$= 8\sqrt{3}\sqrt{2}\,((\sqrt{3})^2 + (\sqrt{2}))^2$

$= 8\sqrt{6}\,(3 + 2) = 40\sqrt{6}$

5. Find $(x + 1)^6 + (x - 1)^6$. Hence or otherwise evaluate $(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6$.

[NCERT]

Solution

Using binomial theorem for positive integral index, we have

$(x + 1)^6 + (x - 1)^6$

$= \{{}^6C_0\, x^6 + {}^6C_1\, x^5 + {}^6C_2\, x^4 + {}^6C_3\, x^3 + {}^6C_4\, x^2 + {}^6C_5\, x + {}^6C_6\,\} +$

$\{{}^6C_0\, x^6 - {}^6C_1\, x^5 + {}^6C_2\, x^4 - {}^6C_3\, x^3 + {}^6C_4\, x^2 - {}^6C_5\, x + {}^6C_6\}$

$= 2\{{}^6C_0 x^6 + {}^6C_2\, x^4 + {}^6C_4\, x^2 + {}^6C_6\}$

$= 2\,\{1x^6 + 15x^4 + 15x^2 + 1\}$

$= 2x^6 + 30x^4 + 30x^2 + 2$

Substituting $x = \sqrt{2}$ in the above result, we get,

$(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6 = 2(\sqrt{2})^6 + 30(\sqrt{2})^4 + 30(\sqrt{2})^2 + 2$

$= 2(8) + 30(4) + 30(2) + 2$

$= 16 + 120 + 60 + 2 = 198$

6. Find the middle terms in the expansions of $\left(3 - \frac{x^3}{6}\right)^7$

[NCERT]

Solution

In this case, exponent 7 is an odd number, therefore, there are two middle terms, namely, $\frac{7+1}{2}$ th and $\frac{7+3}{2}$ th i.e., 4th and 5th terms.

Two write down these terms, we write the general term.

$T_{r+1} = {}^7C_r\,(3)^{7-r}\left(-\frac{x^3}{6}\right)^r$

Substituting, $r = 3$ and 4, we obtain the required middle terms as

$T_4 = {}^7C_3\,(3)^{7-3}\left(-\frac{x^3}{6}\right)^3$

$= \left(\frac{7\times 6\times 5}{3\times 2\times 1}\right)3^4\left(\frac{-x^9}{6^3}\right) = -35 \times \frac{3}{2^3}x^9$

$= -\frac{105}{8}x^9$

and $T_5 = {}^7C_4\,(3)^{7-4}\left(-\frac{x^3}{6}\right)^4$

$= {}^7C_3\, 3^3\left(\frac{x^{12}}{6^4}\right) = \frac{7\times 6\times 5}{3\times 2\times 1}\cdot\frac{x^{12}}{2^4\times 3} = \frac{35x^{12}}{48}$

7. Find n, if the ratio of the fifth term from the beginning to the fifth term from the end in the expansion $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}}\right)$ is $\sqrt{6}: 1$

[NCERT]

Solution

In the expansion of $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}}\right)^n$, 5th term from the beginning is $T_5 = T_{4+1} = {}^nC_4$ $(2^{1/4})^{n-4}\left(\frac{1}{3^{1/4}}\right)^4 = {}^nC_4\frac{2^{(n-4)/4}}{3}$ (1)

Also, the 5th term from the end in the expansion of $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}}\right)^n$ is same as the 5th term from the beginning in the expansion of $\left(\frac{1}{\sqrt[4]{3}} + \sqrt[4]{2}\right)^n$ and is equal to

$T_5' = {}^nC_4\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4}(\sqrt[4]{2}\,)^4$

$= {}^nC_4\,\frac{2}{3^{(n-4)/4}}$ (2)

We are given that $T_5 : T_5' :: \sqrt{6}: 1$

$\Rightarrow \frac{T_5}{T_5'} = \frac{\sqrt{6}}{1}$

$\Rightarrow \frac{2^{(n-4)/4}}{3}\cdot\frac{3^{(n-4)/4}}{2} = \frac{\sqrt{6}}{1}$

$\Rightarrow 6^{(n-4)/4} = 6\sqrt{6} = 6^{1+\frac{1}{2}} \Rightarrow 6^{(n-4)/4} = 6^{3/2}$

$\Rightarrow \frac{n-4}{4} = \frac{3}{2}$ $\Rightarrow 2n - 8 = 12$

$\Rightarrow 2n = 20$ $\Rightarrow n = 10$

UNSOLVED SUBJECTIVE PROBLEMS (CBSE/STATE BOARD): TO GRASP THE TOPIC, SOLVE THESE PROBLEMS

Exercise I

1. Write the general term in the expansion of $(x^2 - y)^6$.

[NCERT]

2. Find the 4th term in the expansion of $(x - 2y)^{12}$.

[NCERT]

3. Find the general term in the expansion of

(i) $\left(x^2 - \frac{1}{x}\right)^{12}$ (ii) $(1 - x^2)^{12}$

4. Find the terms independent of x, $x \neq 0$ in the expansion of

(i) $\left(x - \frac{1}{x}\right)^{14}$ (ii) $\left(x^2 + \frac{1}{x}\right)^{12}$

(iii) $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^6$

5. Find the middle terms in the expansion of $\left(\frac{x}{9} + 9y\right)^{10}$.

6. Find the rth term from the end in $(x + a)^n$.

Exercise II

1. Show that the coefficients of the middle term in the expansion of $(1 + x)^{2n}$ is the sum of the coefficients of two middle terms in the expansion of $(1 + x)^{2n-1}$.

[NCERT]

2. If P be the sum of odd terms and Q that of even terms in the expansion of $(x + a)^n$, prove that

(i) $(P^2 - Q^2) = (x^2 - a^2)^n$

(ii) $4PQ = [(x + a)^{2n} - (x - a)^{2n}]$

(iii) $2\,(P^2 + Q^2) = [(x + a)^{2n} + (x - a)^{2n}]$

3. Find the number of terms in the expansions of the following.

(a) $(\sqrt{x} + \sqrt{y})^{10} + (\sqrt{x} - \sqrt{y})^{10}$

(c) $(3x + y)^8 - (3x - y)^8$

4. Prove that $\frac{C_1}{C_0} + \frac{2.C_2}{C_1} + \frac{3.C_3}{C_2} + - - - - + \frac{n.C_n}{C_{n-1}} = \frac{n\,(n+1)}{2}$.

5. If the sum of coefficients in the expansion of $(a^2x^2 - 2ax + 1)^{51}$ is zero, then find the value of a.

6. Prove that $(-1)^n \frac{1.3.5......(2^n - 1)}{n!} 2^n$ is the middle term in the expansion of $\left(x - \frac{1}{x}\right)^{2n}$.

7. Find the 5th term from end, in the expansion of $\left(x^2 - \frac{1}{2x}\right)^{10}$.

[MP – 1983]

8. Find the middle term of $\left[\frac{a}{x} + bx\right]^{12}$

9. Find the middle term of $\left[x^2 + \frac{1}{x}\right]^9$

10. Find the value of ${}^{13}C_2 + {}^{13}C_3 + ... + {}^{13}C_{13}$.

11. Prove that

$$\frac{C_0}{1} + \frac{C_2}{3} + \frac{C_4}{5} + \frac{C_6}{7} + \ldots\ldots = \frac{2n}{n+1}.$$

ANSWERS

Exercise I

$(-1)^r\, {}^6C_r \,.\, x^{12-2r} \,.\, y^r$

2. $-1760\, x^9 y^3$

3. (i) ${}^{12}C_r\, x^{24-3r} (-1)^r$ (ii) ${}^{12}C_r (-x^2)^r$

4. (i) -3432 (ii) 495

(iii) 5/12

5. $61236\, x^5 y^5$

6. ${}^nC_{n-r}\, x^{r-1} a^{n+1-r}$

Exercise II

3. (a) 6 terms (b) 4 terms
5. $a = 1$
7. $\frac{105}{32} x^2$
8. $924\, a^6 b^6$
9. $126x^6$, $126x^3$
10. $2^{13} - 14$.

SOLVED OBJECTIVE QUESTIONS: HELPING HAND

1. If $C_0, C_1, C_2, \ldots\ldots\ldots, C_n$ are the coefficients in the expansion of $(1 + x)^n$, then

$C_0 + 2C_1 + 3C_2 + \ldots\ldots\ldots (n + 1)C_n =$

(a) $(n + 2)2^{n-1}$ (b) $(n + 2)2^{n+1}$

(c) $(n - 2)2^{n-1}$ (d) None of these

[MPPET – 1996; RPET – 1997; DCE – 1995; AMU – 1995 EAMCET – 2001; IIT – 1971]

Solution

(a) $C_0 + 2C_1 + 3C_2 + \ldots\ldots\ldots (n+1)C_n$

$= (C_0 + C_1 + C_2 + \ldots + C_n) + (C_1 + 2C_2 + \ldots\ldots\ldots + {}^nC_n)$

$$= 2^n + \left\{ n + 2\frac{n(n-1)}{2!} + \frac{3n(n-1)(n-2)}{n!} + \ldots\ldots n \right\}$$

$= 2^n + n(1+1)^{n-1}\{\text{In}(1+x)^{n-1} \text{ put } x = 1\}$

$= 2^n + n2^{n-1}$

$= (2 + n)\, 2^{n-1}$

2. These are not Objective questions n is odd or even the value of $C_0^2 - C_1^2 + C_2^2 - \ldots + (-1)^n (C_n)^2 =$

(a) 0

(b) $(-1)^{n/2}$

(c) $(-1)^{n/2} \dfrac{n!}{(n/2)!\,(2/n)!}$

(d) None of these

Solution

$(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$

$(x - 1)^n = C_0 x^n - C_1 x^{n-1} + C_2 x^{n-2} + \ldots + (-1)^n C_n$

Multiplying both sides, $(-1)^n (1 - x^2)^n = ()\,()$

Now, $c_0^2 - c_1^2 + c_2^2 - \ldots\ldots$ is the coefficients of x^n in the product in R.H.S.

Hence, it is the coefficient of x^n in $(-1)^n (1 - x^2)^n$, or coefficient of $(x^2)^{n/2}$ in $(1 - x^2)^n$ which will appear in $T_{n/2+1}$.

Therefore, $(-1)^n\, n C_{n/2} (-1)^{n/2} (x^2)^{n/2}$

This is possible only when $n/2$ is an integer, i.e., n is even and in case n is odd, then the term x^n will not occur. Also, when n is even, then $(-1)^n = 1$.

$\therefore (-1)^{n/2} \dfrac{n!}{\frac{n}{2}!\,\frac{n}{2}!}$ is the required answer.

3. If x, y, r are positive integers, then ${}^xC_r + {}^xC_{r-1}\,{}^yC_1 + {}^xC_{r-2}\,{}^yC_2 + \ldots + {}^yC_r$ is equal to

(a) ${}^{xy}C_r$ (b) ${}^{x+y}C_r$

(c) $(x + y)!\,/r!$ (d) $x!\,y!\,/\,r!$

[CET (Karnataka) – 1993 ; PET (Raj.) – 2001]

Solution

(b) Since, $(1 + a)^x (1 + a)^y = (1 + a)^{x+y}$

$\Rightarrow (1 + {}^xC_1 a + {}^xC_2 a^2 + \ldots\ldots + {}^xC_{r-1} a^{r-1} + {}^xC_r a^r + \ldots + {}^xC_x a^x)$

$(1 + {}^yC_1 a + {}^yC_2 a^2 + \ldots + {}^yC_{r-1} a^{r-1} + {}^yC_r a^r + \ldots + {}^yC_y a^y)$

$= (1 + a)^{x+y}$

Now equating coefficients of ar on both sides, we get,

${}^xC_r + {}^xC_{r-1}\,{}^yC_1 + {}^xC_{r-2}\,{}^yC_2 + \ldots\ldots + {}^yC_r = {}^{x+y}C_r$

4. If $(1 + x + x^2)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_{2n} x^{2n}$, then $C_0 C_1 - C_1 C_2 + C_2 C_3 - \ldots$ is equal to

[MNR – 1998]

(a) 0 (b) 3^n

(c) $(-1)^n$ (d) 2^n

Solution

(a) Given $(1+x+x^2)^n = C_0 + C_1x + C_2x^2 + \ldots + C_{2n}x^{2n}$ (1)

Replacing x by $-1/x$ on both sides, we get,

$$\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n = C_0 - C_1\frac{1}{x} + C_2\frac{1}{x^2} - \ldots + C_{2n}\frac{1}{x^{2n}} \quad (2)$$

Multiplying (1) and (2), we get,

$$\left(1+x^2+\frac{1}{x^2}\right)^n = (C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots C_{2n}x^{2n})$$

$$\left(C_0 - C_1\frac{1}{x} + C_2\frac{1}{x^2} - C_3\frac{1}{x^3} + \ldots + C_{2n}\frac{1}{x^{2n}}\right)$$

Now, expansion of $\left(1+\frac{1}{x}+\frac{1}{x^2}\right)^n$ has no term containing x, so equating coefficients of x on both sides, we get,

$C_0C_1 - C_1C_2 + C_2C_3 - \ldots = 0.$

5. In the expansion of $(1+x)^n\left(1+\frac{1}{x}\right)^n$, the term independent of x is

[EAMCET – 1989]

(a) $C_0^2 + C_1^2 + \ldots + (n+1)C_n^2$
(b) $(C_0 + C_1 + \ldots + C_n)^2$
(c) $C_0^2 + C_1^2 + \ldots + C_n^2$
(d) none of these

Solution

(c) Exp. $= (C_0 + C_1x + C_2x^2 + \ldots + C_nx^n)$

$$\left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \ldots + \frac{C_n}{x^n}\right)$$

Therefore, the term independent of $x = C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2$

Alternative Method: Exp. $= x^n\left(1+\frac{1}{x}\right)^{2n}$.

Thus, the required term

= coefficient of $\frac{1}{x^n}$ in the expansion of $\left(1+\frac{1}{x}\right)^{2n}$

$= {}^{2n}C_n$

$= C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2$.

6. If $(1+x-2x^2)^6 = 1 + a_1x + a_2x^2 + \ldots + a_{12}x^{12}$, then the value of the expression $a_2 + a_4 + a_6 + \ldots + a_{12}$ is

(a) 32 (b) 31
(c) 63 (d) 64

[PET (Raj.) – 1986,1999; UPSEAT – 2003]

Solution

(b) Putting $x = 1$ and -1 in he given relation, we get

$0 = 1 + a_1 + a_2 + \ldots + a_{12}$ (1)

$64 = 1 - a_1 + a_2 - \ldots + a_{12}$ (2)

Adding (1) and (2), we get,

$64 = 2(1 + a_2 + a_4 + \ldots + a_{12})$

$\Rightarrow 1 + a_2 + a_4 + \ldots + a_{12} = 32$

$\Rightarrow a_2 + a_4 + \ldots + a_{12} = 31$

7. $\sum_{k=0}^{10} {}^{20}C_k$ is equal to

[JEE (Orissa) – 2004]

(a) $2^{19} + \frac{1}{20}\,{}^{20}C_{10}$ (b) 2^{10}
(c) ${}^{20}C_{10}$ (d) $2^{19} - \frac{1}{2}\,{}^{20}C_{10}$

Solution

(a) Since, $({}^{20}C_0 + {}^{20}C_1 + \ldots + {}^{20}C_9) + {}^{20}C_{10} + ({}^{20}C_{11} + {}^{20}C_{12} + \ldots + {}^{20}C_{20}) = 2^{10}$

$\Rightarrow 2({}^{20}C_0 + {}^{20}C_1 + \ldots + {}^{20}C_9) + {}^{20}C_{10} = 2^{10}$

$\Rightarrow 2\sum_{k=0}^{10} {}^{20}C_k = 2^{10} + {}^{20}C_{10}$

$\Rightarrow \sum_{k=0}^{10} {}^{20}C_k = 2^{19} + \frac{1}{2}\,{}^{20}C_{10}$

8. $C_1^2 + 2C_2^2 + 3C_3^2 + \ldots + n\,.C_n^2$ is equal to

(a) $\frac{(2n-1)}{[(n-1)!]^2}$ (b) $\frac{(2n+1)!}{[(n-1)^2!]}$
(c) $\frac{(2n+1)!}{[(n+1)!]^2}$ (d) $\frac{(2n-1)!}{[(n+1)!]^2}$

Solution

(a) $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ (1)

and $C_1x + 2C_2x^2 + 3C_3x^3 + ... + nC_nx^n$

$= nx + 2.\dfrac{n(n-1)}{2!}x^2 + 3.\dfrac{n(n-1)(n-2)}{2!}$

$x^3 + ... + n.1x^n$

$= nx[1 + (n-1)]x$

$+ \dfrac{(n-1)(n-2)}{2!}x^2 + ... + x^{n-1}\Big]$

$= nx(1+x)^{n-1}$

Replace x by $1/x$,

$\dfrac{C_1}{x}\dfrac{2C_2}{x^2} + \dfrac{3C_3}{x^3} + ... + \dfrac{nC_n}{x^n}$

$= \dfrac{n}{x^n}\left(1 + \dfrac{1}{x}\right)^{n-1}$

$\Rightarrow \dfrac{C_1}{x} + \dfrac{2C_2}{x^2} + \dfrac{3C_3}{x^3} + ... + \dfrac{nC_n}{x^n}$

$= \dfrac{n}{x^n}(1+x)^{n-1}$ (2)

Multiplying corresponding sides of Equations (1) and (2),

$(C_0 + C_1x + C_2x^2 + C_3x^3 + ... + C_nx^n)$

$\times \left(\dfrac{C_1}{x} + \dfrac{2C_2}{x^2} + \dfrac{3C_3}{x^3} + ... + \dfrac{nC_n}{x^n}\right)$

$= (1 + x)^n \times \dfrac{n}{x^n}(1 + x)^{n-1}$

$= \dfrac{n}{x^n}(1 + x)^{2n-1}$

Comparing the terms independent of x on both the sides. $C_1^2 + 2C_2^2x + 2C_3^2 + ... + {}^nC_n^2$ = The term independent of x in the expansion of $\dfrac{n}{x^n}(1 + x)^{2n-1}$

= Coefficient of x^n in the expansion of $n(1+x)^{2n-1}$

$= n \times {}^{2n-1}C_n = \dfrac{n \times (2n-1)!}{n!(2n-1-n)!}$

$= \dfrac{(2n-1)!}{(n-1)!(n-1)!}$

$= \dfrac{(2n-1)!}{[(n-1)!]^2}$

Proved

9. ${}^2C_0 + \dfrac{2^2}{2}C_1 + \dfrac{2^3}{3}C_2 + ... + \dfrac{2^{11}}{11}C_{10}$ is equal to

(a) $\dfrac{3^{11}+1}{11}$ (b) $\dfrac{3^{11}-1}{11}$

(c) $\dfrac{3^{11}-2}{11}$ (d) $\dfrac{c^{11}+2}{11}$

Solution

(a) $2C_0 + \dfrac{2^2}{2}C_1 + \dfrac{2^3}{3}C_2 + ... + \dfrac{2^{11}}{11}C_{10}$

$= 2.1 + \dfrac{2^2}{2}.10 + \dfrac{2^3}{3}.\dfrac{10.9}{1.2} + ... + \dfrac{2^{11}}{11}.1$

$= \dfrac{1}{11}\Big[2.11 + \dfrac{11 \times 10}{1.2}.2^2$

$+ \dfrac{11.10.9}{1.2.3}.2^3 + ... + 2^{11}\Big]$

$= \dfrac{1}{11}\Big[\Big\{1 + 11.2 + \dfrac{11 \times 10}{3!}.2^2$

$+ \dfrac{11.10.9}{2!}2^3 + ... + 2^{11}\Big\} - 1\Big]$

$= \dfrac{1}{11}[(1+2)^{11} - 1]$

$= \dfrac{3^{11}-1}{11}$

Proved

10. If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ then $C_0C_1 + C_1C_2 + ... + C_{n-1}C_n$ is

[Orissa JEE – 2008]

(a) $\dfrac{(2n)!}{(n-1)!(n+1)!}$

(b) $\dfrac{n!}{(n-1)!(n+1)!}$

(c) $\dfrac{(n-1)!}{n!(n+1)!}$

(d) none of these

Solution

(a) $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ (1)

and $\left(1+\frac{1}{x}\right)^n = C_0 + C_1\frac{1}{x} + C_2\frac{1}{x^2}$ (2)

$+ \ldots.. C_{n-1}\frac{1}{x^{n-1}} + C_n\frac{1}{x^n}$

Multiplying (1) and (2) and equating the coefficient of x in $\frac{1}{x^n}(1+x)^{2n}$

Therefore, $C_0C_1 + C_1C_2 + C_2C_3 + \ldots. + C_{n-1}C_n = {}^{2n}C_{n+1}$

$$= \frac{(2n)!}{(2n-n+1)!\,(n-1)!} = \frac{(2n)!}{(n-1)!\,(n+1)!}.$$

11. If the sum of the coefficients in the expansion of $(x + y)^n$ is 1024, then the value of the greatest coefficient in the expansion is

(a) 356 (b) 252
(c) 210 (d) 120

Solution

(b) Given sum of coefficients = 1024

$\Rightarrow 2^n = 1024 \Rightarrow 2^n = 2^{10} \Rightarrow n = 10$

Hence, the greatest coefficients = ${}^{10}C_5 = 252$

($\because$ ${}^nC_r$ is greatest for $r = \frac{n}{2}$ when n is even)

12. ${}^nc_0 - \frac{1}{2}{}^nc_1 + \frac{1}{3}{}^nc_2 - \ldots\ldots + (-1)^n\frac{{}^nc_n}{n+1} =$

(a) n (b) $1/n$
(c) $1/(n+1)$ (d) $1/n - 1$

Solution

(c) Put $n = 1, 2$

At $n = 1$, ${}^1C_0 - \frac{1}{2}{}^1C_1 = 1 - \frac{1}{2} = \frac{1}{2}$

At $n = 2$,

${}^2C_0 - \frac{1}{2}{}^2C_1 + \frac{1}{3}{}^2C_2 = 1 - 1 + \frac{1}{3} = \frac{1}{3}$

Therefore, ${}^nC_0 - \frac{1}{2}{}^nC_1 + \frac{1}{3}{}^nC_2 - \ldots\ldots$

$$+ (-1)^n\frac{{}^nC_n}{n+1} = \frac{1}{1+n}$$

13. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots. + \ldots\ldots C_nX^n$, then $\frac{c_1}{c_0} + \frac{2c_2}{c_1} + \frac{3c_3}{c_2} + \ldots. \frac{nc_n}{c_{n-1}} =$

(a) $\frac{n(n-1)}{2}$ (b) $\frac{n(n+2)}{2}$

(c) $\frac{n(n+1)}{2}$ (d) $\frac{(n-1)(n-2)}{2}$

[BIT, RANCHI – 1986; RPET – 1996, 1997]

Solution

(c) $\frac{{}^nC_r}{c_{r-1}} = \frac{r\,n!}{(n-r)!\,r!}\,\frac{(n-r+1)\,(r-1)!}{n!}$

$= n - r + 1$

$\therefore \frac{C_1}{C_0} + 2\frac{C_2}{C_1} + 3\frac{C_3}{C_2} + \ldots.. + n\frac{C_n}{C_{n-1}}$

$= \sum_{r=1}^{n} r\frac{C_r}{C_{r-1}} = \sum_{r=1}^{n}(n-r+1)$

$= n + (n-1) + (n-2) + \ldots\ldots.. + 1$

$= \frac{n(n+1)}{2}.$

14. If $(1 + x - 3x^2)^{10} = 1 + a_1x + a_2x^2 + .. + a_{20}x^{20}$, then $a_2 + a_4 + a_6 + \ldots\ldots + a_{20}$ equal to

[Kerala PET – 2007]

(a) $\frac{3^{10}+1}{2}$ (b) $\frac{3^9+1}{2}$

(c) $\frac{3^{10}-1}{2}$ (d) $\frac{3^9-1}{2}$

Solution

(c) Put $x = 1$ is given expansion

$(1 + 1 - 3)^{10} = 1 + a_1 + a_2 + a_3 + \ldots\ldots.. + a_{20}$

$(-1)^{10} = 1 + a_1 + a_2 + a_3 + \ldots\ldots.. + a_{20}$

$1 = 1 + a_1 + a_2 + a_3 + \ldots\ldots.. + a_{20}$ (1)

Put $x = -1$ in given expansion

$(1 - 1 - 3)^{10} = 1 - a_1 + a_2 - a_3 + \ldots\ldots\ldots.. + a_{20}$

$(-3)^{10} = 1 - a_1 + a_2 - a_3 + \ldots\ldots\ldots.. + a_{20}$

$(3)^{10} = 1 - a_1 + a_2 - a_3 + \ldots\ldots\ldots.. + a_{20}$ (2)

Adding equation (1) and (2), we get,

$1 + (3)^{10} = 2(1 + a_2 + a_4 + \ldots\ldots.. + a_{20})$

$a_2 + a_4 + a_6 + \ldots\ldots\ldots\ldots. + a_{20} = \frac{(3)^{10}+1}{2} - 1$

$= \frac{(3)^{10}-1}{2}.$

15. The value of

$$\left(\frac{{}^{50}C_0}{1} + \frac{{}^{50}C_2}{3} + \frac{{}^{50}C_4}{5} + \ldots.. \frac{{}^{50}C_{50}}{51}\right)$$

[Kerala PET – 2007]

(a) $\frac{2^{50}}{51}$ (b) $\frac{2^{50}-1}{51}$

(c) $\frac{2^{50}-1}{50}$ (d) $\frac{2^{51}-1}{51}$

Solution

(a) We know that

$$^nC_0 a + \frac{^nC_2 a^3}{3} + \frac{^nC_4 a^5}{5} + \ldots\ldots$$

$$= \frac{(1+a)^{n+1} - (1-a)^{n+1}}{2(n+1)}$$

Substituting $n = 50$, $a = 1$, we get,

$$\frac{^{50}C_0}{1} + \frac{^{50}C_2}{3} + \frac{^{50}C_4}{5} + \ldots + \frac{^{50}C_{50}}{51}$$

$$= \frac{(1+1)^{50+1} - (1-1)^{50+1}}{2(50+1)}$$

$$= \frac{2^{51}-0}{2 \times 51} = \frac{2^{50}}{51}$$

16. If third term in the expansion of $(x + x \log_{10} x)^5$ is 10^6, then x is equal to

[Roorkee – 1992 ; UPSEAT – 1999]

(a) $10^{5/2}$ (b) 10

(c) $10^{1/2}$ (d) none of these

Solution

(b) Let $\log_{10} x = z$, then exp. $= (x + x^z)^5$.

Now, $T_3 = 10^6 \Rightarrow {}^5C_2\, x^3 (x^z)^2 = 10^6$

$\Rightarrow x^{3+2z} = 10^5$

$\Rightarrow (3 + 2z) \log_{10} x = 5$ (on taking log)

$\Rightarrow (3 + 2z)z = 5 (\because z = \log_{10} x)$

$\Rightarrow 2z^2 + 3z - 5 = 0$

$\Rightarrow (z - 1)(2z + 5) = 0$

$\Rightarrow z = 1, -5/2$

$\therefore \log_{10} x = 1, -5/2$

$\Rightarrow x = 10, 10^{-5/2}$

17. If $(1 + x)^n = \sum_{r=0}^{n} C_r x^r$, then

$\frac{C_0}{2} - \frac{C_1}{3} + \frac{C_2}{4} - \frac{C_3}{5} + \ldots$ is equal to

[UPSEAT – 1999]

(a) $\frac{1}{n+1}$ (b) $\frac{1}{n(n+1)}$

(c) $\frac{1}{(n+1)(n+2)}$ (d) none of these

Solution

(c) Since, $(1 - x)^n = C_0 - C_1 x + C_2 x^2 - C_3 x^3 + \ldots\ldots + (-1)^n C_n x^n$

$\Rightarrow x(1 - x)^n = x C_0 - C_1 x^2 + C_2 x^3 - \ldots\ldots + (-1)^n C_n x^{n+1}$

Now, integrating both sides with respect to x in [0, 1], we have,

$$\left[-\frac{x(1-x)^{n+1}}{n+1} - \frac{(1-x)^{n+2}}{(n+1)(n+2)}\right]_0^1$$

$$= \left[\frac{x^2}{2} C_0 - \frac{x^3}{3} C_1 + \frac{x^4}{4} C_2 - \ldots\right]_0^1$$

$$\Rightarrow \frac{1}{(n+1)(n+2)} = \frac{C_0}{2} - \frac{C_1}{3} + \frac{C_2}{4} - \ldots$$

18. In the expansion of $(1 + x)^n \left(1 + \frac{1}{x}\right)^n$, the term independent of x is

[EAMCET – 1989]

(a) $C_0^2 + 2C_1^2 + \ldots + (n + 1) C_n^2$

(b) $(C_0 + C_1 + \ldots + C_n)^2$

(c) $C_0^2 + C_1^2 + \ldots + C_n^2$

(d) none of these

Solution

(c) Exp. $= (C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n)$

$$\left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \ldots + \frac{C_n}{x^n}\right)$$

Therefore, the term independent of $x = C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2$

Alternative Method: Exp. $= x^n \left(1 + \frac{1}{x}\right)^{2n}$.

Thus, the required term

= coefficient of $\frac{1}{x^n}$ in the expansion of $\left(1 + \frac{1}{x}\right)^{2n}$

$= {}^{2n}C_n = C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2$

19. If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ...+ C_nx^n$, then $\sum \sum C_i . C_j$, where $0 \le I \le j \le n$, is equal to

[IIT – 1983 ; MNR – 1992]

(a) $2^{n-1} \frac{(2n)!}{2(n!)^2}$ (b) $\frac{(2n)!}{(n!)^2}$

(c) $2^{2n-1} - \frac{(2n)!}{2(n!)^2}$ (d) none of these

Solution

(c) Exp. $= \sum_{0 \le i <} \sum_{j \le n} C_iC_j$

$= \frac{1}{2}\left[\left(\sum C_i\right)^2 - \sum C_i^2\right]$

$= \frac{1}{2}[(C_0 + C_1 + \ldots + C_n]^2 - (C_0^2 + C_1^2 + \ldots + C_n^2)$

$= \frac{1}{2}\left[2^{2n} - \frac{(2n)!}{(n!)^2}\right] = 2^{2n-1} - \frac{(2n)!}{2(n!)^2}$

20. If $A = 99^{50} + 100^{50}$, and $B = 101^{50}$, then

[IIT – 1982]

(a) $A = B$ (b) $A < B$

(c) $A > B$ (d) none of these

Solution

(b) $(101)^{50} = (100 + 1)^{50}$

$= 100^{50} + 50 . 100^{49} + \frac{50.49}{2.1} 100^{48} +.....$ (1)

$(99)^{50} = (100 - 1)^{50}$

$= 100^{50} - 50 . 100^{49} + \frac{50.49}{2.1} 100^{48} -....$ (2)

(1) – (2) $\Rightarrow (101)^{50} - (99)^{50}$

$= 2[50.10049 + \frac{50.49.48}{3.2.1} 10047 + ...]$

$= 100^{50} + \frac{50.49.48}{3} 100^{47} + ... > (100)^{50}$

$\Rightarrow (101)^{50} > (100)^{50} + (99)^{50}$

Therefore, $B > A$ or $A < B$

21. If a_1, a_2, a_3 are in AP and $(1 + x^2)^2 (1 + x)^n = \sum_{k=0}^{n+4} a_kx^k$, then n is equal to

[Roorkee – 1996]

(a) 2 (b) 3

(c) 4 (d) all the above

Solution

(d) For the given relation

L.H.S. $= (1+ 2x^2 + x^4) (1 + C_1x + C_2x^2 + ...)$

R.H.S. $= a_0 + a_1x + a_2x^2 + a_3x^3 + ...$

Equating coefficients of x, x^2, x^3 on both sides, we get, $a_1 = c_1$, $a_2 = c_2 + 2$, $a_3 = c_3 + 2c_1$

Also, a_1, a_2, a_3 are in AP , so $2a_2 = a_1 + a_3$

$\Rightarrow 2(c_2 + 2) = c_1 + c_3 + 2c_1$

$\Rightarrow n(n - 1) + 4 = 3n + \frac{n(n-1)(n-2)}{6}$

$\Rightarrow n^3 - 9n^2 + 26n - 24 = 0$

which is satisfied by 2, 3 and 4.

22. $\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} - \ldots + \binom{30}{20}\binom{30}{30}$ is equal to

[IIT (Screening) – 2005]

(a) $\binom{30}{15}$ (b) $\binom{30}{10}$ (c) $\binom{60}{30}$ (d) $\binom{31}{10}$

Solution

(b) Consider $(1 + x)^{30} (1 - x)^{30} = (1 - x^2)^{30}$

$\Rightarrow (1 + x)^{30} (x - 1)^{30} = (1 - x^2)^{30}$

$\Rightarrow ({}^{30}C_0 + {}^{30}C_1 x + {}^{30}C_2 x^2 + ... + {}^{30}C_{20} x^{20} + ... + {}^{30}C_{30} x^{30})$

$({}^{30}C_0 x^{30} - .... + {}^{30}C_{10} x^{20} - {}^{30}C_{11} x^{19} + ... + {}^{30}C_{30})$

$= ({}^{30}C_0 - {}^{30}C_1 x^2 + {}^{30}C_2 (x^2)^2 - .... + {}^{30}C_{10} (x^2)^{10} - .. + {}^{30}C_{30} (x^2)^{30})$

Now, equating, the coefficients of x^{20} on both the sides, we get,

${}^{30}C_0 . {}^{30}C_{10} - {}^{30}C_1 . {}^{30}C_{11} + {}^{30}C_2 . {}^{30}C_{12} - .... + {}^{30}C_{20}.$

${}^{30}C_{30} = {}^{30}C_{10}.$

$\Rightarrow \binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} +$

$\binom{30}{2}\binom{30}{12} - ..... = \binom{30}{10}$

23. The coefficient of x^{17} in the expansion of $(x-1)(x-2)(x-3)\ldots(x-18)$ is

[IIT – 1990]

(a) 342 (b) 171/2

(c) −171 (d) 684

Solution

(c) Coefficients of $x^{17} = -1-2-3-\ldots-18$

$$= \frac{-(18)(19)}{2} = -171.$$

24. If $n \in N$, then $a - {}^nC_1(a-1) + {}^nC_2(a-2) - \ldots + (-1)^n(a-n)$ equals

[IIT – 1972]

(a) 0
(b) na
(c) $n(n-1)a$
(d) none of these

Solution

(a) Exp. $= a[C_0 - C_1 + C_2 - \ldots + (-1)^n C_n] + [C_1 - 2C_2 + 3C_3 - \ldots + (-1)^{n-1} nC_n] = 0 + 0 = 0.$

25. If C_k denotes ${}^nC_k$, then $\sum_{k=1}^{n} k^3\left(\frac{C_k}{C_{k-1}}\right)^2$ is equal to

(a) $\frac{n(n+1)(n+2)}{12}$

(b) $\frac{n(n+1)(n+2)^2}{12}$

(c) $\frac{n(n+1)^2(n+2)}{12}$

(d) none of these

[Roorkee – 1991]

Solution

(c) Since, $\frac{{}^nC_k}{{}^nC_{k-1}} = \frac{n-k+1}{k}$, Thus,

$$\text{Exp.} = \sum_{k=1}^{n} k^3\left(\frac{n-k+1}{k}\right)^2$$

$$= \sum_{k=1}^{n} k\left[(n+1)^2 - 2k(n+1) + k^2\right]$$

$$= (n+1)^2\sum_{1}^{n} - 2(n+1)\sum_{1}^{n} k^2 + \sum_{1}^{n} k^3$$

26. If n is odd integer, then $\sum_{r=0}^{n} \frac{(-1)^r}{{}^nC_r}$ is equal to

[IIT – 1998]

(a) 0 (b) $1/n$

(c) $n/2^n$ (d) none of these

Solution

(a) Exp $= \frac{1}{C_0} - \frac{1}{C_1} + \frac{1}{C_2} - \ldots + \frac{1}{C_{n-1}} - \frac{1}{C_n}$

[$\because$ n is odd]

$= \left(\frac{1}{C_0} - \frac{1}{C_n}\right) + \left(\frac{1}{C_1} - \frac{1}{C_{n-1}}\right) + \ldots = 0 + 0 + \ldots = 0$

[$\because {}^nC_r = {}^nC_{n-r}$]

27. The sum of the coefficients of all the integral powers of x in the expansion of $(1+2\sqrt{x})^{40}$ is

(a) $3^{40}+1$ (b) $30^{40}-1$

(c) $\frac{1}{2}(30^{40}-1)$ (d) $\frac{1}{2}(3^{40}+1)$

Solution

(d) The coefficients of the integral powers of x are

${}^{40}C_0, {}^{40}C_2 \,.\, 2^2, {}^{40}C_4 \,.\, 2^4, \ldots, {}^{40}C_{40}\, 2^{40}$.

$(1+2)^{40} = {}^{40}C_0 + {}^{40}C_1 \,.\, 2 + {}^{40}C_2 \,.\, 2^2 + \ldots + {}^{40}C_{40} \,.\, 2^{40}$

$(1-2)^{40} = {}^{40}C_0 - {}^{40}C_1 \,.\, 2 + {}^{40}C_2 \,.\, 2^2 - \ldots + {}^{40}C_{40} \,.\, 2^{40}$

Adding, we get, $3^{40} + 1 = 2$ (required term)

Therefore, required term $= \frac{1}{2}(3^{40}+1)$.

Hence, (*d*) is the correct answer.

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. Two middle terms in the expansion of $(x - 1/x)^{11}$ are

(a) $231\,x$ and $231/x$
(b) $462\,x$ and $462/x$
(c) $-462\,x$ and $462/x$
(d) none of these

2. The middle term in the expansion of $(1 + x)^{2n}$ is

[Pb CET – 1998]

(a) $\dfrac{1.3.5........(5n-1)}{n!}x^n$

(b) $\dfrac{2.4.6........2n}{n!}n^{2n+1}$

(c) $\dfrac{1.3.5........(2n-1)}{n!}x^n$

(d) $\dfrac{1.3.5........(2n-1)}{n!}2^n x^n$

3. ${}^{14}C_1 + {}^{14}C_2 + {}^{14}C_3 + ... + {}^{14}C_{14} =$

(a) 2^{14} (b) $2^{14} - 1$
(c) $2^{14} + 2$ (d) $2^{14} - 2$

4. $C_0 - C_1 + C_2 - C_3 + ... + (-1)^n C_n$ is equal to

[MNR – 1991; RPET – 1995; UPSEAT – 2000]

(a) 2^n (b) $2^n - 1$
(c) 0 (d) 2^{n-1}

5. If $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$, then $a_0 + a_2 + a_4 + ... + a_{2n} =$

[MNR – 1992]

(a) $\dfrac{3^n+1}{2}$ (b) $\dfrac{3^n-1}{2}$

(c) $\dfrac{1-3^n}{2}$ (d) $3^n + \dfrac{1}{2}$

6. The sum of the coefficients in $(x + 2y + z)^{10}$ is

[RPET – 2003]

(a) 2^{10} (b) 3^{10}
(c) 1 (d) none of these

7. If the sum of the coefficients in the expansion of $(\alpha^2 x^2 - 2\alpha x + 1)^{51}$ vanishes, then the value of α is

[IIT – 1991; Pb CET – 1988]

(a) 2 (b) -1
(c) 1 (d) -2

8. The sum of the coefficients in the expansion of $(x + y)^n$ in 4096. The greatest coefficients in the expansion is

(a) 1024 (b) 924
(c) 824 (d) 724

[Kurukshetra CEE – 1998; AIEEE – 2002]

9. he greatest coefficients in the expansion of $(1 + x)^{2n+1}$ is

[RPET – 1997]

(a) $\dfrac{(2n+1)!}{n!(n+1)!}$ (b) $\dfrac{(2n+2)!}{n!(n+1)!}$

(c) $\dfrac{(2n+1)!}{[(n+1)!]^2}$ (d) $\dfrac{(2n)!}{(n!)}$

10. If the expansion of $(1 + x)^{50}$, the sum of the coefficient of odd powers of x is

[MNR – 1998; Roorkee – 1993]

(a) 0 (b) 2^{49}
(c) 2^{50} (d) 2^{51}

11. If the sum of the coefficients in the expansion of $(1 - 3^x + 10x^2)^n$ is a and if the sum of the coefficients in the expansion of $(1 + x^2)^n$ is b, then

[UPSEAT – 2001]

(a) $a = 3b$ (b) $a = b^3$
(c) $b = a^3$ (d) none of these

12. If $(1 + x)^n = C_0 + C_1 x + C_2x^2 + ... + C_nx^n$, then $C_0^2 + C_1^2 + C_2^2 + C_3^2 .... + C_n^2 =$

(a) $\dfrac{n!}{n!\,n!}$ (b) $\dfrac{2n!}{n!\,n!}$

(c) $\dfrac{2n!}{n!}$ (d) none of these

[MPPET – 1985; Karnataka CET – 1995; MNR – 1999]

13. $\frac{C_1}{C_0}+\frac{2C_2}{C_1}+\frac{3C_3}{C_1}+\ldots\ldots+\frac{15C_{15}}{C_{14}}=$

[IIT – 1962]

(a) 100 (b) 120
(c) – 120 (d) none of these

14. $\frac{C_0}{1}+\frac{C_1}{2}+\frac{C_2}{3}+\ldots\ \frac{C_n}{n+1}=$

[RPET – 1996]

(a) $\frac{2^n}{n+1}$ (b) $\frac{2^n-1}{n+1}$

(c) $\frac{2^{n+1}-1}{n+1}$ (d) none of these

15. The sum to $(n+1)$ terms of the following series $\frac{C_0}{2}-\frac{C_1}{3}+\frac{C_2}{4}-\frac{C_3}{5}+\ldots\ldots$ is

[Andhara – 1993; KCET– 1998]

(a) $\frac{1}{n+1}$ (b) $\frac{1}{n+2}$

(c) $\frac{2^{n+1}-1}{n+1}$ (d) none of these

16. The sum of $C_0^2-C_1^2+C_2^2-\ldots\ldots+(-1)^n C_n^2$ where n is an even integer is
(a) ${}^{2n}C_n$ (b) $(-1)^n\,{}^{2n}C_n$
(c) ${}^{2n}C_{n-1}$ (d) none of these

17. If the coefficients of the middle term in the expansion of $(1+x)^{2n+2}$ is p and the coefficients of middle terms in the expansion of $(1+x)^{2n+1}$ are q and r, then

(a) $p+q=r$ (b) $p+r=q$
(c) $p=q+r$ (d) $p+q+r=0$

18. If $C_0, C_1, C_2, \ldots, C_n$ are the binomial coefficients, then $2.C_1+2^3C_3+2^5.\,C_5+\ldots$ equals

[AMU – 1999]

(a) $\frac{3^n+(-1)^n}{2}$ (b) $\frac{3^n-(-1)^n}{2}$

(c) $\frac{3^n+1}{2}$ (d) $\frac{3^n-1}{2}$

19. The 14th term from the end in the expansion of $(\sqrt{x}-\sqrt{y})^{17}$ is

(a) ${}^{17}C_5-x^6(-\sqrt{y})^5$ (b) ${}^{17}C_6(\sqrt{x})^{11}y^3$

(c) ${}^{17}C_4\,.\,x^{13/2}\,y^2$ (d) none of these

20. The sum of the series ${}^{20}C_0-{}^{20}C_1+{}^{20}C_2-{}^{20}C_3\ldots.-\ldots.+\ldots.{}^{20}C_{10}$ is

[AIEEE – 2007]

(a) $\frac{1}{2}\,{}^{20}C_{10}$ (b) 0

(c) $\frac{1}{2}\,{}^{20}C_0$ (d) ${}^{20}C_{10}$

21. The value of $({}^7C_0+{}^7C_1)+({}^7C_1+{}^7C_2)+\ldots\ldots+({}^7C_6+{}^7C_7)$ is

(a) 2^8-1 (b) 2^8+1
(c) 2^8 (d) 2^8-2

[Kerala PET – 2008]

22. In the binomial expansion of $(a-b)^n$, $n\geq 5$, the sum of the 5th and 6th terms is zero. Then $\frac{a}{b}$ is equal to

[IIT – 2001; AIEEE – 2007; Orissa JEE – 2007]

(a) $\frac{1}{6}(n-5)$

(b) $\frac{1}{5}(n-4)$

(c) $\frac{5}{(n-4)}$

(d) $\frac{6}{(n-5)}$

23. The term independent of x in $\left[\sqrt{x}-\frac{2}{x}\right]^{18}$ is

[MPPET – 2009]

(a) ${}^{18}C_{12}\,2^8$ (b) ${}^{18}C_6\,2^{12}$
(c) ${}^{18}C_6\,2^4$ (d) ${}^{18}C_{12}\,2^6$

24. Let $\sum_{j=1}^{10} j(j-1)\ {}^{10}C_j S_2=\sum_{j=1}^{10} j\,{}^{10}C_j$ and $S_3=\sum_{j=1}^{10} j^2\ {}^{10}C_j$

[MPPET – 2009]

SOLUTIONS

1. (c) Here $n = 11$

Total number. of terms = 12

Therefore, Middle terms be $\frac{12}{2}$ th and $\left(\frac{12}{2}+1\right)^{th}$ i.e. 6th and 7th terms

Now $T_6 = {}^{11}C_5\,(x)^6\left(-\frac{1}{x}\right)^5$

$= -\,{}^{11}C_5\,x$

$= \frac{-11\times 10\times 9\times 8\times 7}{5\times 4\times 3\times 2\times 1}x$

$= -\,11\times 42\,x$

$= -\,462\,x$

and $T_7 = {}^{11}C_6\,(x)^5\left(-\frac{1}{x}\right)^6$

$= {}^{11}C_6 \times \frac{1}{x}$

$= \frac{{}^{11}C_5}{x} = \frac{462}{x}$

Hence, the two middle terms are $-\,462x$ and $\frac{462}{x}$

2. (d) Middle term in $(1+x)^{2n} = T_{n+1}$

$= {}^{2n}C_n\,x^n = \frac{2n\,!}{n\,!\,n\,!}$

$= \frac{[\,1.3.5......(2n-1)\,][2.4.6....2n]}{n\,!\,n\,!}$

$= \frac{[\,1.3.5.....(2n-1)\,]2^n}{n\,!}x^n$

3. (b) Step 1 ${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots\ldots + {}^nC_n = 2n - 1$ Step 2 ${}^{14}C_1 + {}^{14}C_2 + {}^{14}C_3 + \ldots\ldots + {}^{14}C_{14} = 2^{14} - 1$

4. (c) We know that sum of odd terms of coefficients = sum of even terms of coeffi cient. $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2\,x^2 + \ldots\ldots\ldots + {}^nC_n\,x^n$ Putting, $x = -\,1$, we get,

$(1-1)^n = {}^nC_0 - {}^nC_1 + {}^nC_2 - \ldots\ldots\ (-\,1)^n\,{}^nC_n$

Therefore $C_0 - C_1 + C_2 - C_3 + \ldots..\ (-\,1)\,{}^nC_n = 0$

5. (a) $(1 - x + x^2)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots + a_{2n}\,x^{2n}$ (1)

Putting $x = 1$, $x = -\,1$ in (1), we get, respectively, $a_0 + a_1 + a_2 + a_3 + a_4 + \ldots\ldots\ldots + a_{2n}$

$= (1 - 1 + 1)^n = 1$

and $a_0 - a_1 + a_2 - a_3 + a_4 - \ldots\ldots\ldots\ldots + a_{2n}$

$= (1 + 1 + 1)^n = 3^n$

Adding the two equations,

$2\,(a_0 + a_2 + a_4 + \ldots\ldots\ldots + a_{2n}) = 3^n + 1$

6. (d) Sum of the coefficients, in $(x + 2y + z)^{10}$

$= (1 + 2 + 1)^{10} = 4^{10}$

7. (c) Sum of the coefficients can be obtained by substituting

$x = 1$, therefore,

$(a^2\,x^2 - 2ax + 1)^{51} = 0$ for $x = 1$

$\Rightarrow (a^2 - 2a + 1)^{51} = 0$

$\Rightarrow (a - 1)^{102} = 0$

$\Rightarrow a = 1.$

8. (b) Sum of coefficients = 4096

$\therefore$ when a, b are each 1 $(a + b)^n = 4096$

$\Rightarrow (1 + 1)^n = 4096 = 2^{12}$

$\Rightarrow n = 12$

Here, n is even.

$\Rightarrow$ Greatest coefficients $= {}^nC_{n/2} = {}^{12}C_6$

$= \frac{12\,!}{6\,!\,6\,!} = 924.$

9. (a) $\frac{T_{r+1}}{T_r} = \frac{N - r + 1}{r}\cdot x$

Here, $N = 2n + 1$

$\Rightarrow \frac{T_{r+1}}{T_r} = \frac{2n + 2 - r}{r}\cdot x$

$\therefore\ T_{r+1} \geq T_r$

$\Rightarrow 2n = 2 - r \geq r$

$\Rightarrow 2n + 2 \geq 2r$

$\Rightarrow r \leq n + 1$

Therefore, $r = n$

$T_{r+1} = T_{n+1} = 2^{n+1}C_{n+1} = \frac{(2n+1)\,!}{(n+1)\,!\,n\,!}$

10. (b) We know that

$C_0 + C_2 + C_4 + C_6 + \ldots\ldots. = C_1 + C_3 + C_5 \ldots\ldots..$

$= 2^{n-1}$ in expansion of $(1 + x)^n$.

Therefore, $C_1 + C_3 + C_5 + \ldots\ldots\ldots = 2^{50-1} = 2^{49}$.

11. (b) Sum of the coefficients $(1 - 3x + 10x^2)^n$ is $8^n = a$ (1)

and sum of the coefficients in $(1 + x^2)^n$ is $2^n = b$ (2)

From Equation (1) and (2)

$(2^3)^n = a$

$(2^n)^3 = a$

$b^3 = a$

12. (c) $(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + ...+ C_nx^n$ (1)

$$\left(1 + \frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \frac{C_3}{x^3} + ... + \frac{C^n}{x^n}$$ (2)

Multiplying both sides and equating terms of independent of x

$C_0^2 + C_1^2 + C_2^2 + ... = C_n^2$.

= Term independent of x in

$$(1 + x)^n . \frac{(1 + x)^n}{x^n} = \frac{(1 + x)^{2n}}{x^n}$$

= Coefficients of x^n in $(1 + x)^{2n} = {}^{2n}C_n$ as $T_{r+1} = {}^{2n}C_r\, x^{2n-r}$.

13. (b) We know that

$$\frac{C_1}{C_0} + \frac{2(C_2)}{C_1} + \frac{3(C_3)}{C_2} + ... + \frac{n(C_n)}{C_{n-1}} = \frac{n}{2}(n + 1)$$

In the present case, we put $n = 15$, sum $= 15.\frac{16}{2} = 120$.

14. (c) $(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + ...$

Integrating with respect to x from $x = 0$ to $x = 1$,

$$\left[C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + C_3\frac{x^4}{4} + C_4\frac{x^5}{5} + ...\right]_{x=0}^{1}$$

$$= \left\{\frac{(1 + x)^{n+1}}{n + 1}\right\}_0^1$$

$$\text{or, } C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + \frac{C_4}{5} + ... = \frac{2^{n+1} - 1}{n + 1}$$

15. (d) Since, $(1 - x)^n = C_0 - C_1x + C_2x^2 - C_3x^3 + ...$

Therefore, $x(1 - x)^n = C_0x - C_1x^2 + C_2x^3 - C_3x^4 + ...$

$$\Rightarrow \int_0^1 x(1 - x)^n dx$$

$$= \int (C_0x - C_1x^2 + C_2x^2 - C_3x^4 + ...)\, dx$$

$$= \frac{C_0}{2} - \frac{C_1}{3} + \frac{C_2}{4} - .... \text{ upto } (n + 1) \text{ terms}$$

For L.H.S. put $1 - x = t$. Therefore, $dx = - dt$

$$\text{L.H.S.} = \frac{1}{(n + 1)(n + 2)}$$

16. (d) We have $(1 + x)^n$

$= C_0 + C_1x + C_2x^2 + ... + C_nx^n$ (1)

$$\left(1 - \frac{1}{x}\right)^n = C_0 - \frac{C_1}{x} + \frac{C_2}{x_2} + ... + (-1)^n\frac{C_n}{x^n}$$ (2)

Therefore, $C_0^2 - C_1^2 + C_2^2$ .... $+ (-1)^n C_n^2$.

= coefficients of the term independent of x in product of R.H.S. (1) and (2)

= coefficients of term independent of x in $(1+x)^n\left(1 - \frac{1}{x}\right)^n$

= coefficients of x^n in $(-1)^n(1 - x^2)^n = {}^nC_{n/2}$ [$\because$ n is even]

$= (-1)^n\, {}^nC_{n/2}(-1)^{n/2} = (-1)^{n/2}\, {}^nC_{n/2}$, if n is even $= 0$, if n is odd.

17. (c) Since, $(n + 2)$th term is the middle term in the expansion of $(1 + x)^{2n+2}$

Therefore, $p = {}^{2n+2}C_{n+1}$.

Since, $(n + 1)$th and $(n + 2)$th terms are middle terms in the expansion of $(1 + x)^{2n+1}$

Therefore, $q = {}^{2n+1}c_n$ and $r = {}^{2n+1}c_{n+1}$

But, ${}^{2n+1}c_n + {}^{2n+1}c_{n+1} = {}^{2n+2}c_{n+1}$ $\therefore$ $q + r = p$

18. (a) $(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + ... + C_nx^n$ $(1 - x)^n = C_0 - C_1x + C_2x^2 - C_3x^3 + ... + (-1)^n C_nx^n$

$[(1 + x)^n - (1 - x)^n] = 2[C_1x + C_3x^3 + C_5x^5 + ...]$

$$\frac{[(1 + x)^n - (1 - x)^n]}{2} = C_1x + C_3x^3 + C_5x^5 + ...$$

Put $x = 2$,

$2 . C_1 + 2^3 . C_3 + 2^5 . C_5 + ... = \frac{3n - (-1)^n}{2}$

19. (c) rth term from the end in the binomial expansion is $(n - r + 2)$th term from the beginning 5th term from the beginning in the expansion of $(\sqrt{x} - \sqrt{y})^{17}$ which is equal to ${}^{17}C_4 (-\sqrt{y})^4 (\sqrt{x})^{17-4}$ i.e. ${}^{17}C_4 y^2 x^{13/2}$

20. (a) ${}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + ... + {}^{20}C_{20} = 0$

$\Rightarrow 2({}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - ...... - {}^{20}C_9) + {}^{20}C_{10} = 0$

$\Rightarrow 2({}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - ...... - {}^{20}C_9 + {}^{20}C_{10}) = {}^{20}C_{10}$

$\Rightarrow {}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - ...... + {}^{20}C_{10} = \frac{1}{2}\,{}^{20}C_{10}.$

Note 1 ${}^{20}C_{11} = {}^{20}C_9$, ${}^{20}C_{12} = {}^{20}C_8$, ${}^{20}C_{13} = {}^{20}C_7$ etc and so on and adding ${}^{20}C_{10}$ on the either side.

21. (d) Given expression $= {}^8C_1 + {}^8C_2 + {}^8C_3 + ... + {}^8C_7$ $(\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r)$

$= ({}^8C_0 + {}^8C_1 + ... + {}^8C_7 + {}^8C_8) - ({}^8C_0 + {}^8C_8)$

$= 2^8 - (1 + 1) = 2^8 - 2.$

22. (c) Since, $T_5 + T_6 = 0$

$\Rightarrow {}^nC_4 (a)^{n-4} (-b)^4 + {}^nC_5 (a)^{n-5} (-b)^5 = 0$

$\Rightarrow {}^nC_4\, a = {}^nC_5\, b$

$\Rightarrow \frac{a}{b} = \frac{{}^nC_4}{{}^nC_5}$

$\Rightarrow \frac{a}{b} = \frac{n!}{(n-4)!\,4!} \times \frac{5!\,(n-5)!}{n!}$

$\Rightarrow \frac{a}{b} = \frac{n-4}{5}.$

Note We may also use the formula $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-(r-1)}{r}$

23. (d) The general term of $\left[\sqrt{x} - \frac{2}{x}\right]^{18}$ is

$T_{r+1} = {}^{18}C_r (\sqrt{x})^{18-r} \left(-\frac{2}{x}\right)^r$

$= {}^{18}C_r x^{\frac{18-r}{2} - r} (-1)^r (2)^r$

$= (-1)^r.\ {}^{18}C_r x^{\frac{18-3r}{2}} 2^r$

For independent term of x, put $\frac{18-3r}{2} = 0$

$\Rightarrow r = 6 \quad \therefore T_7\ {}^{18}C_6 2^6 = {}^{18}C_{12} 2^6$

24. (b)

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. The middle term in the expansion of $(x + 1/x)^{10}$ is

(a) ${}^{10}C_4\, 1/x$ (b) ${}^{10}C_5$

(c) ${}^{10}C_5$ (d) ${}^{10}C_7 x^4$

[BIT, RANCHI – 1991; RPET – 2002; Pb CET – 1991]

2. The greatest coefficients in the expansion of $(1 + x)^{2n+2}$ is

(a) $\frac{(2n)!}{(n!)^2}$ (b) $\frac{(2n+2)!}{\{(n+1)!\}^2}$

(c) $\frac{(2n+2)!}{n!\,(n+1)!}$ (d) $\frac{2n!}{n!\,(n+1)!}$

[BIT, RANCHI – 1992]

3. ${}^{15}C_0 + {}^{15}C_1 + {}^{15}C_2 + {}^{15}C_3 + ...\ {}^{15}C_{15} =$

(a) 2^{15} (b) $2^{15} - 1$

(c) $2^{15} - 2$ (d) none of these

4. (4) ${}^{10}C_1 + {}^{10}C_3 + {}^{10}C_5 + {}^{10}C_7 + {}^{10}C_9 =$

[MPPET – 1982]

(a) 2^9 (b) 2^{10}

(c) $2^{10} - 1$ (d) none of these

5. Middle term in the expansion of $(3x - 7)^{14}$ is

(a) 6th (b) 7th

(c) 8th (d) none of these

6. The sum of all the coefficients in the binomial expansion of $(x^2 + x - 3)^{319}$ is

(a) 1 (b) 2

(c) −1 (d) 0

7. The sum of coefficients in the expansion of $(x + 2y + 3z)^8$ is

(a) 3^8 (b) 5^8

(c) 6^8 (d) none of these

8. The coefficients of middle term in the expansion of $(1 + x)^{10}$ is

(a) $10! / 5!6!$ (b) $10! / (5!)^2$
(c) $10! / 5!7!$ (d) none of these

[UPSEAT – 2001]

9. $\frac{C_0}{1} + \frac{C_2}{3} + \frac{C_4}{5} + \frac{C_6}{7} + \ldots =$

[RPET – 1999]

(a) $\frac{2^{n+1}}{n+1}$ (b) $\frac{2^{n+1}-1}{n+1}$
(c) $\frac{2^n}{n+1}$ (d) none of these

10. The value of $\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \ldots$ is equal to

[Karnataka CET – 2000]

(a) $\frac{2^{n-1}}{n+1}$ (b) $n.2^n$
(c) $\frac{2^n}{n}$ (d) $\frac{2^n+1}{n+1}$

11. If $n = 5$, then $({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \ldots + ({}^nC_5)^2$ equal to

(a) 250 (b) 254
(c) 245 (d) 252

[Kerala PET – 2007]

12. In the expansion of $(1 + x)^5$, the sum of the coefficients of the term is

(a) 80 (b) 16
(c) 32 (d) 64

[RPET – 1992, 1997; Kurukshetra CEE – 2000]

13. If n is odd, then $C_0^2 - C_1^2 + C_2^2 - C_3^2 + \ldots + (-1)^n C_n^2 =$

(a) 0 (b) 1
(c) ∞ (d) $\frac{n!}{(n/2)^2!}$

14. If $m = {}^nC_2$, then ${}^mC_2$ equal to

[Kerala PET – 2007]

(a) $3\,{}^nC_4$ (b) ${}^{n+1}C_4$
(c) $3\,{}^{n+1}C_4$ (d) $3\,{}^{n+1}C_3$

15. The middle term in the expansion of $\left(x + \frac{1}{2x}\right)^{2n}$ is

(a) $\frac{1.3.5....(2n-3)}{n!}$

(b) $\frac{1.3.5....(2n-1)}{n!}$

(c) $\frac{1.3.5....(2n+1)}{n!}$

(d) none of these

[MPPET – 1995]

WORK SHEET: TO CHECK PREPARATION LEVEL

Important Instructions:

1. The answer sheet is immediately below the work sheet.
2. The test is of 15 minutes.
3. The test consists of 15 questions. The maximum marks are 45.
4. Use blue/black ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. Find the middle term in the expansion of $\left(x^2 + \frac{1}{x^2} + 2\right)^n$

(a) $\frac{(2n)!}{(n!)(n!)}$ (b) $\frac{(n!)(n!)}{(2n!)}$
(c) $-\frac{(n!)(n!)}{(2n!)}$ (d) $-\frac{(2n)!}{(n!)(n!)}$

2. If the sum of the coefficients in the expansion of $(ax^2 - 2x + 1)^{35}$ is equal to the sum of the coefficients in the expansion of $(x - \alpha y)^{35}$, then find the value of α

(a) 1 (b) 2
(c) 3 (d) 4

3. Evaluate the sum of the ${}^8C_1 + {}^8C_3 + {}^8C_5 + {}^8C_7$

(a) 128 (b) 127
(c) 130 (d) 126

4. In the expansion of $(3x+2)^4$, the coefficients of middle term is

(a) 81 (b) 54
(c) 216 (d) 36

5. The largest coefficient in the expansion of $(1+x)^{24}$ is

(a) $^{24}c_{24}$ (b) $^{24}c_{13}$ (c) $^{24}c_{12}$ (d) $^{24}c_{11}$

6. What is the middle term in the expansion of $\left(\frac{x\sqrt{y}}{3}-\frac{3}{y\sqrt{x}}\right)^{12}$?

[NDA – 2007]

(a) $C(12,7)X^3Y^{-3}$
(b) $C(12,6)X^{-3}Y^3$
(c) $C(12,7)X^{-3}Y^3$
(d) $C(12,6)X^3Y^{-3}$

7. $C_1+2C_2+3C_3+4C_4+....+nC_n$

(a) 2^n (b) $n.2^n$
(c) $n.2^{n-1}$ (d) $n.2^{n+1}$

8. The sum of coefficients in $(1+x-3x^2)^{2134}$ is

(a) -1 (b) 1
(c) 0 (d) 2^{2134}

[Kurukshetra CEE – 2001]

9. The sum of coefficients in the expansion of $(1+x+x^2)^n$ is

(a) 2 (b) 3^n
(c) 4^n (d) 2^n

[EAMCET – 2002]

10. The middle term in the expansion of $(1+x)^{2n}$ is

(a) $\frac{(2n)!}{n!}x^2$ (b) $\frac{(2n)!}{n!(n-1)!}x^{n+1}$

(c) $\frac{(2n)!}{(n!)^2}x^n$

(d) $\frac{(2n)!}{(n+1)!(n-1)!}x^n$

11. In the expansion of $(1+x)^n$ the sum of coefficients of odd powers of x is

(a) 2^n+1 (b) 2^n-1
(c) 2^n (d) 2^{n-1}

[MPPET – 1986, 1993, 2003]

12. If the sum of the coefficients in the expansion of $(x+y)^n$ is 256, then the value of n is

(a) 6 (b) 7
(c) 8 (d) 9

13. What is the sum of the coefficients in the expansion of $(5x-4y)^{100}$?

(a) 1 (b) -1
(c) 5^{100} (d) -2^{100}

14. In the expansion of $(x^2+3a/x)^{15}$, the coefficient of x^{18} will be

(a) $^{15}C_4(3a)^{11}$ (b) $^{15}C_4a^4$
(c) $^{15}C_4(3a)^4$ (d) none of these

15. In the expansion of $(x-1)^{10}$ the middle term is

(a) $-252x^5$ (b) $252x^5$
(c) x^5 (d) $x^5/252$

[MPPET – 2007]

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	6. ⓐ ⓑ ⓒ ⓓ	11. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	7. ⓐ ⓑ ⓒ ⓓ	12. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	8. ⓐ ⓑ ⓒ ⓓ	13. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	9. ⓐ ⓑ ⓒ ⓓ	14. ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ	10. ⓐ ⓑ ⓒ ⓓ	15. ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

1. $\left(x^2 + \frac{1}{x^2} + 2\right)^n = \left(\left(x + \frac{1}{x}\right)^2\right)^n = \left(x + \frac{1}{x}\right)^{2n}$

   Therefore, the middle term = ${}^{2n}C_n = \frac{(2n)!}{n!\,n!}$.

2. For the sum of coefficients, put $x = 1$, $y = 1$ in both expansion,

   $(\alpha - 2 + 1)^{35} = (1 - \alpha)^{35} \Rightarrow (\alpha - 1)^{35} = -(\alpha - 1)^{35}$

   $\therefore \alpha - 1 = 0$ or $\alpha = 1$

7. $C_1 + 2C_2 + 3C_3 + 4C_4 + \ldots\ldots\ldots + nC_n = ?$

   Consider $(1 + x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + \ldots\ldots\ {}^nC_n\, x^n$ Differentiating $n\,(1 + x)^{n-1} = 1\,.\,{}^nC_1 + 2.\,{}^nC_2\, x + \ldots\ldots\ n\,{}^nC_n\, x^{n-1}$

   put $x = 1$, $n\,.2^{n-1} = 1\,.\,{}^nC_1 + 2\,.\,{}^nC_2 + \ldots\ {}^nC_n$.

11. Sum of coefficients of odd power of x

    $= C_1 + C_3 + C_5 + \ldots\ldots\ldots\ldots\ldots = 2^{n-1}$

LECTURE

3 Particular Term and Divisibility Theorem

BASIC CONCEPTS

1. To Determine a Particular term in the Expansion

In the expansion of $\left(x^{\alpha} \pm \frac{1}{x^{\beta}}\right)^n$, if x^m occurs in T_{r+1},

$$\Rightarrow \quad r = \frac{n\alpha - m}{\alpha + \beta} \qquad (1)$$

1.1 Thus, in the above expansion if constant term, i.e., the term which is independent of x, occurs in T_{r+1} then

$$\Rightarrow r = \frac{n\alpha}{\alpha + \beta} \qquad (2)$$

2. Greatest Term (Numerically) in the Expansion of $(a + x)^n$ Method $T_{r+1} = {}^nC_r\, a^{n-r} x^r$

(i) Let T_r (the rth term) be the greatest term.

(ii) Find T_{r-1}, T_r, T_{r+1} from the given expansion.

(iii) Put $\frac{T_{r+1}}{T_r} \geq 1$

$\Rightarrow \frac{n-r+1}{r}\left|\frac{x}{a}\right| \geq 1$, $r \leq k + f$, $0 \leq f < 1$ and k is positive integer.

Note 1: If f is not zero, then $r = k$, i.e. $(k + 1)$th or $(r + 1)$th term is the greatest.

Note 2: If $f = 0$ i.e. $r \leq k$, then we find two greatest terms for $r = k - 1$ and k i.e. T_r and T_{r+1} are the greatest terms.

3. Divisibility Using Binomial Theorem

(i) Expression $(1 + x)^n - 1$ is divisible by x because $(1 + x)^n - 1 = x[{}^nC_1 + {}^nC_2\, x + \ldots\ldots + {}^nC_n\, x^{n-1}]$

(ii) $(1 + x)^n - nx - 1$ is divisible by x^2 because $(1 + x)^n - nx - 1 = x^2[{}^nC_2 + {}^nC_3\, x + \ldots\ldots + {}^nC_n\, x^{n-2}]$

4. Number of rational terms In $(a^{1/p} + b^{1/q})^n$ where a, b are rational numbers and are co-prime in nature, p, q are integers

$\therefore$ Number of rational terms =

$$\left[\frac{n}{LCM\,(p, q)}\right] + 1 = m \text{ (say)}$$

where [ ] represents integral part.

Also, number of irrational terms = $(n + 1) - m$

SOLVED SUBJECTIVE PROBLEMS: (CBSE/STATE BOARD): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Find the coefficient of x^5 in the expansion of the product $(1 + 2x)^6(1 - x)^7$.

[NCERT]

Solution

$(1 + 2x)^6(1 - x)^7$

$= [1 + {}^6C_1.(2x) + {}^6C_2.(2x)^2 + {}^6C_3.(2x)_3 + {}^6C_4.(2x)^4. + {}^6C_5.(2x)^5 + \ldots] \times [1 - {}^7C_1.x + {}^7C_2.x^2 - {}^7C_3.x^3 + {}^7C_4.x^4 - {}^7C_5.x^5 + \ldots]$

$= [1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + \ldots] \times [1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + \ldots]$.

Therefore, coefficient of x^5 in the product

$= [1 \times (-21) + (12 \times 35) + 60 \times (-35) + 160 \times 21 + 240 \times (-7) + 192 \times 1]$

$= (-21 + 420 - 2100 + 3360 - 1680 + 192)$

$= 171$.

2. Using binomial theorem, prove that 6^n-5n always leaves remainder 1, when divided by 25.

[NCERT]

Solution

For two numbers a and b if we can find numbers q and r such that $a = bq + r$, then we say that b divides a with q as quotient and r as remainder. Thus, in order to show that 6^n-5n leaves remainder 1 when divided by 25, we prove that $6^n-5n = 25k + 1$, where k is some natural number.

We have

$(1 + a)^n = {}^nC_0 + {}^nC_1a + {}^nC_2a^2 + \ldots + {}^nC_na^n$

For, $a = 5$, we get $(1 + 5)^n = {}^nC_0 + {}^nC_15 + {}^nC_25^2 + \ldots + {}^nC_n5^n$

i.e. $(6)^n = 1 + 5n + 5^2.\,{}^nC_2 + 5^3.{}^nC_3 + \ldots + 5^n$

i. e. $6^n - 5n = 1 + 5^2({}^nC_2 + .{}^nC_35 + \ldots + 5^{n-2})$

or $6^n - 5n = 1 + 25({}^nC_2 + 5.{}^nC_3 + \ldots + 5^{n-2})$

or $6^n - 5n = 25k + 1$

Where $k = {}^nC_2 + 5.{}^nC_3 + \ldots + 5^{n-2}$

This shows that when divided by 25, 6^n-5n leaves remainder 1.

3. Write down the binomial expansion of $(1 + x)^{n+1}$, when $x = 8$. Deduce that $9^{n+1}-8n-9$ is divisible by 64, where n is a positive integer.

[NCERT]

Solution

We have, $(1 + x)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1x + {}^{n+1}C_2x^2 + {}^{n+1}C_3x^3 + \ldots + n^{+1}C_{n+1}x^{n+1}$

Putting $x = 8$ we get

$(1+8)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1(8) + {}^{n+1}C_2(8)^2 + {}^{n+1}C_3(8)^3 + \ldots + {}^{n+1}C_{n+1}(8)^{n+1}$ (1)

$\Rightarrow 9^{n+1} = 1 + (n+1) \times 8 + {}^{n+1}C_2(8)^2 + {}^{n+1}C_3(8)^3 + \ldots + {}^{n+1}C_{n+1}(8)^{n+1}$

$\Rightarrow 9^{n+1} - 8n - 9 = (8)^2[{}^{n+1}C_2 + {}^{n+1}C_3(8) + {}^{n+1}C_4(8)^2 + \ldots + {}^{n+1}C_{n+1}(8)^{n-1}]$

$\Rightarrow 9^{n+1} - 8n - 9 = 64 \times$ an integer

$\Rightarrow 9^{n+1}-8n - 9$ is divisible by 64.

4. Find the coefficient of x^5 in $(x + 3)^8$.

[NCERT]

Solution

Here, $T_{r+1} = {}^8C_rx^{8-r}3^r$

This will contain x^5; if $8 - r = 5$, i.e., if $r = 8 - 5 = 3$

Substituting $r = 3$ in (1), we get, $T_4 = {}^8C_3\,x^5\,3^3 = ({}^8C_3\,3^3)\,x^5$

Hence, coefficient of $x^5 = {}^8C_3\,3^3$

$= \left(\frac{8 \times 7 \times 6}{3 \times 2 \times 1}\right) 3^3 = 56 \times 27 = 1512.$

5. Find the coefficient of $a^5\,b^7$ in $(a - 2b)^{12}$.

[NCERT]

Solution

Here, the general term $T_{r+1} = {}^{12}C_r(a)^{12-r}(-2b)^r = {}^{12}C_r\,a^{12-r}\,b^r\,(-2)^r$

This term will contain $a^5\,b^7$ if $12 - r = 5$ and $r = 7$

i.e., if $r = 7$

Substituting, $r = 7$ in (1),

we get $T_8 = {}^{12}C_7\, a^5\, b^7\, (-2)^7$

Hence, coefficient of $a^5 b^7 = (-2)^7\, {}^{12}C_7 = -2^7 \left\{\frac{12!}{5!\quad 7!}\right\}$.

$= -128 \left\{\frac{12 \times 11 \times 10 \times 9 \times 8 \times 7!}{5 \times 4 \times 3 \times 2 \times 1 \times 7!}\right\}$

$= -128 \times 792$

$= -101376.$

6. Find the 13th term in the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$ $x \neq 0$

[NCERT]

Solution

The general term in the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$ is

$T_{r+1} = {}^{18}C_r\, (9x)^{18-r} \left(-\frac{1}{3\sqrt{x}}\right)^r$

For the 13th term, we put $r + 1 = 13$, i.e., $r = 12$ in the above term.

$\therefore\ T_{13} = {}^{18}C_{12} (9x)^6 \left(-\frac{1}{3\sqrt{x}}\right)^{12}$

$= {}^{18}C_6\, 9^6 x^6 \left\{\frac{1}{3^{12}\,(\sqrt{x})^{12}}\right\}$

$(\because\ {}^nC_r = {}^nC_{n-r})$

$= \frac{18 \times 17 \times 16 \times 15 \times 14 \times 13}{4 \times 6 \times 5 \times 3 \times 2 \times 1} \cdot \frac{(3^2)^6 x^6}{3^{12}(x^{1/2})^{12}} = 18564.$

7. If a and b are distinct integers, prove that $a - b$ is a factor of $a^n - b^n$, whenever n is a positive integer.

[NCERT]

Solution

Writing a as $(a - b) + b$ in $a^n - b^n$ and applying binomial theorem for positive integral index,

we obtain,

$a^n - b^n = \{(a - b) + b\}^n - b^n$

$= {}^nC_0 (a - b)^n + {}^nC_1 (a - b)^{n-1} b^1 + {}^nC_2 (a - b)^{n-2} b^2 + \ldots\ldots\ldots + {}^nC_{n-1} (a - b)^1 b^{n-1} + {}^nC_n b^n - b^n$

$= (a - b)\, \{{}^nC_0 (a - b)^{n-1} + {}^nC_1 (a - b)^{n-2} b + {}^nC_2 (a - b)^{n-3} b^2 + \ldots\ldots\ldots + {}^nC_{n-1} b^{n-1}\} + b^n - b^n$

$= (a - b)$ {some integer}

($\because\ {}^nC_0, {}^nC_1, {}^nC_2, \ldots\ldots, {}^nC_{n-1}$ are integers and also all nonnegative powers of $a - b$ and b are integers)

Hence, $a - b$ is a factor of $a^n - b^n$.

8. Find the value of

$(a^2 + \sqrt{a^2 - 1})^4 + (a^2 - \sqrt{a^2 - 1})^4.$

[NCERT]

Solution

Using Binomial theorem for positive integral index, we have

$(a^2 + \sqrt{a^2 - 1})^4 + (a^2 - \sqrt{a^2 - 1})^4$

$= (x + y)^4 + (x - y)^4$

where $x = a^2$, $y = \sqrt{a^2 - 1}$

$= \{{}^4C_0 x^4 + {}^4C_1 x^3 y + {}^4C_2 x^2 y^2 + {}^4C_3 xy^3 + {}^4C_4 y^4\} + \{{}^4C_0 x^4 - {}^4C_1 x^3 y + {}^4C_2 x^2 y^2 - {}^4C_3 xy^3 + {}^4C_4 y^4\}$

$= 2\, \{{}^4C_0 x^4 + {}^4C_2 x^2 y^2 + {}^4C_4 y^4\}$

$= 2\, \{1\, (a^2)^4 + 6\, (a^2)^2 (\sqrt{a^2 - 1})^2 + 1 (\sqrt{a^2 - 1})^4\}$

$= 2\, \{a^8 + 6a^4\, (a^2 - 1) + (a^2 - 1)^2\}$

$= 2\, \{a^8 + 6a^6 - 5a^4 - 2a^2 + 1\}.$

9. Expand using Binomial Theorem $\left(1 + \frac{x}{2} - \frac{2}{x}\right)^4$, $x \neq 0$.

[NCERT]

Solution

Writing $1 + \frac{x}{2} - \frac{2}{x}$ as $1 + \left(\frac{x}{2} - \frac{2}{x}\right)$ and binomial theorem for positive integral index, we have

$\left(1 + \frac{x}{2} - \frac{2}{x}\right)^4 = \left\{1 + \left(\frac{x}{2} - \frac{2}{x}\right)\right\}^4 (1 + y)^4,$

where

$y = \frac{x}{2} - \frac{2}{x}$

$= {}^4C_0 + {}^4C_1 y + {}^4C_2 y^2 + {}^4C_3 y^3 + {}^4C_4 y^4$

$= 1 + 4\left(\frac{x}{2} - \frac{2}{x}\right) + 6\left(\frac{x}{2} - \frac{2}{x}\right)^2$

$+ 4\left(\frac{x}{2} - \frac{2}{x}\right)^3 + 1\left(\frac{x}{2} - \frac{2}{x}\right)^4$

$= 1 + 2x - \frac{8}{x} + 6\left(\frac{x^2}{4} - 2 + \frac{4}{x^2}\right)$

$+4\left\{\begin{matrix}\left(\frac{x}{2}\right)^3 + 3\left(\frac{x}{2}\right)^2\left(-\frac{2}{x}\right) \\ +3\left(\frac{x}{2}\right)\left(-\frac{2}{x}\right)^2 + \left(-\frac{2}{x}\right)^3\end{matrix}\right\}$

$+\left\{\begin{matrix}1\left(\frac{x}{2}\right)^4 + 4\left(\frac{x}{2}\right)^3\left(-\frac{2}{x}\right) + 6\left(\frac{x}{2}\right)^2\left(-\frac{2}{x}\right)^2 \\ +4\left(\frac{x}{2}\right)\left(-\frac{2}{x}\right)^3 + 1\left(-\frac{2}{x}\right)^4\end{matrix}\right\}$

$= 1 + 2x - \frac{8}{x} + \frac{3}{2}x^2 - 12 + \frac{24}{x^2} + \frac{x^3}{2} - 6x$

$+ \frac{24}{x} - \frac{32}{x^3} + \frac{x^4}{16} - x^2 + 6 - \frac{16}{x^2} + \frac{16}{x^4}$

$= \frac{16}{x^4} - \frac{32}{x^3} + \frac{8}{x^2} + \frac{16}{x} - 5$

$- 4x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{x^4}{16}$

10. Find the expansion of $(3x^2 - 2ax + 3a^2)^3$ using binomial theorem.

[NCERT]

Solution

Writing $3x^2 - 2ax + 3a^2$ as $(3x^2 - 2ax) + 3a^2$ and using Binomial Theorem for positive integral index.

We have

$(3x^2 - 2ax + 3a^2)^3 = \{(3x^2 - 2ax) + 3a^2\}^3$

$= 1\,(3x^2 - 2ax)^3 + 3(3x^2 - 2ax)^2\,(3a^2)^1$

$+ 3(3x^2 - 2ax)^1\,(3a^2)^2 + 1(3a^2)^3$

$= \{1\,(3x^2)^3 + 3\,(3x^2)^2\,(-2ax)$

$+ 3(3x^2)\,(-2ax)^2 + 1\,(-2ax)^3\}$

$+ 9a^2\,(9x^4 - 12ax^3 + 4a^2x^2)$

$+ 27a^4\,(3x^2 - 2ax) + 27a^6$

$= 27x^6 - 54ax^5 + 36a^2x^4 - 8a^3x^3$

$+ 81a^2x^4 - 108\,a^3x^3 + 36\,a^4x^2$

$+ 81a^4x^2 - 54\,a^5x + 27a^6$

$= 27x^6 - 54ax^5 + 117\,a^2x^4 - 116\,a^3x^3$

$+ 117\,a^4x^2 - 54\,a^5x + 27a^6.$

UNSOLVED SUBJECTIVE PROBLEMS: (CBSE /STATE BOARD): TO GRASP THE TOPIC, SOLVE THESE PROBLEMS

Exercise I

1. In the expansion of $(1 + a)^{m+n}$, prove that coefficients of a^m and a^n are equal.

[NCERT]

2. Prove that the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$.

[NCERT]

3. Evaluate $(\sqrt{3} + \sqrt{2})^6 - (\sqrt{3} - \sqrt{2})^6$.

[NCERT]

4. Find the coefficients of

(i) x^5 in $(x + 3)^9$ (ii) x^6y^3 in $(x + y)^9$

5. Find n, if the coefficients of 4th and 13th terms in the expansion of $(a + b)^n$ are equal.

6. Find a, if the coefficients of x^2 and x^3 in the expansion of $(3 + ax)^9$ are equal.

7. The binomial coefficient of the third term from the end in the expansion $(y^{2/3} + x^{4/5})^{15}$ is 91.

8. If three consecutive coefficients in the expansion of $(1 + x)^n$ are in the ratio 6: 33: 110, find x.

9. If the coefficients of $(p + 1)$th and $(p + 3)$rd terms in the binomial expansion of $(1 + x)^{2n}$ are equal, then prove that $p = n - 1$.

10. In the binomial expansion of $(1 + x)^{43}$, the coefficient of $(2r + 1)$th and $(r + 2)$th terms are equal. Find r.

11. Show that $2^{3n} - 7n - 1$ is divisible by 49, where $n \in N$.

Exercise II

1. Find the 10th term of $\left(2x^2+\frac{1}{x}\right)^{12}$.

2. Find the 6th term in the expansion of $\left(\frac{4x}{5}-\frac{5}{2x}\right)^9$

3. Find the coefficient of x^{32} and x^{-17} in the expansion of $\left(x^4-\frac{1}{x^3}\right)^{15}$.

4. Find the term independent of x in the expansion of

   (a) $\left(x^2+\frac{1}{x}\right)^9$ (b) $\left(2x-\frac{1}{x}\right)^{10}$

5. Prove that there is no term involving x^6 in the expansion of $\left(2x^2-\frac{3}{x}\right)^{11}$ where $x \neq 0$

6. Prove that the ratio of the coefficient of x^{10} and constant term in the expansions of $(1-x^2)$ and $\left(x-\frac{2}{x}\right)^{10}$ respectively is 1 : 32.

7. If the coefficients of x^7 and x^{-7} in the expansion $\left(ax^2+\frac{1}{bx}\right)^{11}$ and $\left(ax^2-\frac{1}{bx^2}\right)^{11}$, respectively are equal, then prove that $ab = 1$.

8. If x^4 appears in the rth term of the expansion $\left(x^4+\frac{1}{x^3}\right)^{15}$, then find the value of r.

9. If the coefficient of x in the expansion of $\left[x^2+\frac{\lambda}{x}\right]^5$ is 270, then find the value of λ.

10. Find the coefficient of x^{32} in the expansion of $\left[x^4+\frac{1}{x^3}\right]$

    [MP – 1995]

11. Find the value of λ in the expansion of $\left[\sqrt{x}-\frac{\lambda}{x^2}\right]^{10}$ when independent term of x is 405.

12. Find the coefficient of a^4 in the product $(1+2a)^4\,(2-a)^5$ using binomial theorem.

    [NCERT]

13. Find the greatest term in the expansion $(x+y)^{12}$ for $x = 2, y = 1$.

14. Find the greatest term in the expansion of $(3-5x)^9$ when $x = 1/5$. Ans. .

ANSWERS

Exercise I

3 $396\sqrt{6}$

4 (i) 10206 (ii) 84

6 $\frac{9}{7}$

7 $n = 15$

8 $x = 12$

10 $r = 14$

Exercise II

1 $\frac{1760}{x^3}$

2 $-5040/x$

3 1365 and −1365

4 (a) 84 (b) −8064

8 $r = 9$

10 1365

9 $\lambda = 3$

11 $\lambda = \pm 3$

12 438

13 Greatest term is 5th, i.e., $T_5 \therefore T_5 = 155380$

14 T_3 is greatest $T_3 = 78732$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. Find the number of integral terms in the expansion of $(5^{1/2} + 7^{1/8})^{1024}$.

(a) 129 (b) 128

(b) 130 (d) none

Solution

(a) The general term T_{r+1} in the expansion of $(5^{1/2} + 7^{1/8})^{1024}$ is given by

$T_{r+1} = {}^{1024}C_r\,(5^{1/2})^{1024-r}\,(7^{1/8})^r$

$\Rightarrow T_{r+1} = {}^{1024}C_r\,5^{512-r/2}\,7^{r/8}$

$\Rightarrow T_{r+1} = \{{}^{1024}C_r 5^{512-r}\} \times 5^{r/2} \times 7^{r/8}$

$\Rightarrow T_{r+1} = \{{}^{1024}C_r 5^{512-r}\} \times (5^4 \times 7)^{r/8}$

Clearly, T_{r+1} will be an integer, if $\frac{r}{8}$ is an integer such that $0 \le r \le 1024$

$\Rightarrow$ r is a multiple of 8 satisfying $0 \le r \le 1024$

$\Rightarrow r = 0,8,16,24,\ldots,1024$

$\Rightarrow r$ can assume 129 values.

Hence, there are 129 integral terms in the expansion of $(5^{1/2} + 7^{1/8})^{1024}$.

2. Find the coefficient of the term independent of x in the expansion of $\left[\frac{x+1}{x^2/3 - x^1/3 + 1} - \frac{x-1}{x - x^{1/2}}\right]^{10}$.

(a) 21 (b) –210

(c) 210 (d) none

Solution

(c) We have $\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}}$

$= \frac{(x^{1/3})^3 + 1^3}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x^{1/2}(x^{1/2} - 1)}$

$= \frac{(x^{1/3} + 1)(x^{2/3} - x^{1/3} + 1)}{x^{2/3} - x^{1/3} + 1} - \frac{x^{1/2} + 1}{x^{1/2}}$

$= x^{1/3} + 1 - 1 - x^{-1/2}$

$= x^{1/3} - x^{-1/2}$

$\therefore \left(\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}}\right)^{10}$

$= (x^{1/3} - x^{-1/2})^{10}$

Let T_{r+1} be the general term in $(x^{1/3} - x^{-1/2})^{10}$.

Then, $T_{r+1} = {}^{10}C_r (x^{1/3})^{10-r}\,(-1)^r\,(x^{-1/2})^r$

For this term to be independent of x, we must have $\frac{10-r}{3} - \frac{r}{2} = 0$

$\Rightarrow 20 - 2r - 3r = 0 \Rightarrow r = 4$

So, required coefficient $= {}^{10}C_4(-1)^4 = 210$.

3. Find the coefficient of x^n in the expansion of $(1 + x)(1 - x)^n$

(a) $(-1)^n(1 - n)$ (b) $(-1)^n(n - 1)$

(c) $(-1)^{n-1}(1 - n)$ (d) $1 - n$

Solution

(a) We have, coefficient of x^n in $(1 + x)(1 - x)^n$

= Coefficient of x^n in $(1-x)^n$

+ Coefficient of x^{n-1} in $(1 - x)^n$

$= (-1)^n\,{}^nC_n + (-1)^{n-1}\,{}^nC_{n-1} = (-1)^n(1 - n)$

4. Find the greatest value of the term independent of x in the expansion of $\left(x \sin\alpha + \frac{\cos\alpha}{x}\right)^{10}$, where $\alpha \in R$.

(a) $\frac{10!}{(5!)^2}$ (b) $\frac{10!}{(2!)^2}$

(c) $\frac{10!}{5!}$ (d) $\frac{10!}{2^5(5!)^2}$

Solution

(d) Let $(r+1)$th term be independent of x.

We have, $T_{r+1} = {}^{10}C_r(x\sin\alpha)^{10-r}\left(\frac{\cos\alpha}{x}\right)^r$

$= {}^{10}C_r x^{10-2r} (\sin\alpha)^{10-r} (\cos\alpha)^r$

If it is independent of x, then $r = 5$.

Therefore, Term independent of x

$= T_6 = {}^{10}C_5 (\sin\alpha\cos\alpha)^5$

$= {}^{10}C_5 \times 2^{-5}(\sin 2\alpha)^5$

Clearly, it is greatest when $2\alpha = \pi/2$ and its greatest value is ${}^{10}C_5 \times 2^{-5} = \frac{10\,!}{2^5(5\,!)^2}$.

5. The last digit in 7^{300} is

[Karnataka CET – 2004]

(a) 7 (b) 9 (c) 1 (d) 3

Solution

(c) We have $7^2 = 49 = 50-1$

Now, $7^{300} = (7^2)^{150} = (50-1)^{150}$

$= {}^{150}C_0(50)^{150}(-1)^0 + {}^{150}C_1(50)^{149}(-1)^1 + \ldots + {}^{150}C_{150}(50)^0(-1)^{150}$

Thus, the last digits of 7^{300} are ${}^{150}C_{150}.1.1$ i.e., 1.

6. If x^m occurs in the expansion of $\left(x + \frac{1}{x^2}\right)^{2n}$, then the coefficient of x^m is

[UPSEAT – 1999]

(a) $\frac{(2n)!}{(m)\,!\,(2n-m)\,!}$

(b) $\frac{(2n)\,!\,3\,!\,3\,!}{(2n-m)\,!}$

(c) $\frac{(2n)\,!}{\left(\frac{2n-m}{3}\right)!\left(\frac{4n+m}{3}\right)!}$

(d) none of these

Solution

(c) $T_{r+1} = {}^{2n}C_r x^{2n-r}\left(\frac{1}{x^2}\right)^r = {}^{2n}C_r x^{2n-3r}$

This contains x^m, if $2n - 3r = m$ i.e. if $r = \frac{2n-m}{3}$,

$\therefore$ Coefficient of $x^m = {}^{2n}C_r$, $r = \frac{2n-m}{3}$

$$= \frac{2n!}{(2n-r)\,!\,r\,!} = \frac{2n\,!}{\left(2n - \frac{2n-m}{3}\right)!\left(\frac{2n-m}{3}\right)!}$$

$$= \frac{2n\,!}{\left(\frac{4n+m}{3}\right)!\left(\frac{2n-m}{3}\right)!}$$

7. The remainder when 5^{99} is divided by 13 is

(a) 6 (b) 8

(c) 9 (d) 10

Solution

(b) $5^{99} = (5)(5^2)^{49} = 5(25)^{49} = 5(26-1)^{49} = 5 \times (26) \times$ (Positive terms)-5, so when it is divided by 13 it gives the remainder -5 or $(13-5)$, i.e., 8.

8. When 2^{301} is divided by 5, the least positive remainder is

[Karnataka CET – 2005]

(a) 4 (b) 8

(c) 2 (d) 6

Solution

(c) $2^4 \equiv 1 \pmod 5$;

$\Rightarrow (2^4)^{75} = (1)^{75} \pmod 5$ i.e., $2^{300} \equiv 1 \pmod 5 \Rightarrow 2^{300} \times 2 \equiv (1 \times 2) \pmod 5$

$\Rightarrow 2^{301} \equiv 2 \pmod 5$,

Therefore, Least positive remainder is 2.

9. $10^n + 3(4^{n+2}) + 5$ is divisible by $(n \in N)$

[Kerala (Engg.) – 2005]

(a) 7 (b) 5

(c) 9 (d) 17

Solution

(c) $10^n + 3(4^{n+2}) + 5$;

Taking, $n = 2$; $10^2 + 3 \times 4^4 + 5 = 100 + 768 + 5 = 873$

Therefore, this is divisible by 9.

10. The remainder on dividing 7^{30} by 5 is

[UPSEAT–1999]

(a) 1 (b) 4

(c) 3 (d) 2

Solution

(b) $7^{30} = (7^2)^{15} = (49)^{15} = (50-1)^{15}$

$= 50^{15} - {}^{15}C_1\, 50^{14} + {}^{15}C_2\, 50^{13} - \ldots + {}^{15}C_{14}\, 50^1 - {}^{15}C_{15}\,(50)^0$

= (a multiple of 50) − 1

= (a multilpe of 50 − 5) + 4

= (a multiple of 5) + 4

Therefore, required remainder = 4.

11. In how many terms in the expansion of $(x^{1/5} + y^{1/10})^{55}$ do not have fractional power of the variable

[CET (Pb.) – 1992]

(a) 6 (b) 7
(c) 8 (d) 10

Solution

(a) $T_{r+1} = {}^{55}C_r (x^{1/5})^{55-r} (y^{1/10})^r$
$= {}^{55}C_r\, x^{(55-r)/5}\, y^{r/10}$
Now, powers of x and y are not fractional when $r = 0, 10, 20, 30, 40, 50$.
Hence, six terms do not have fractional power.

12. The coefficient of x^5 in the expansion of $(1 + x^2)^5 (1 + x)^4$ is

[EAMCET – 1996]

(a) 50 (b) 60
(c) 55 (d) None

Solution

(b) $(1 + x^2)^5 (1 + x)^4$
$= (1 + 5x^2 + 10x^4 + ..)(1 + 4x + 6x^2 + 4x^3 + x^4)$
Therefore, coefficient of $x^5 = 5(4) + 10(4) = 60$.

13. The remainder on dividing 2^{60} by 7 is

[CET (Karnataka) – 2001]

(a) 2 (b) 0
(c) 1 (d) 3

Solution

(c) $2^{60} = (2^3)^{20} = (1 + 7)^{20}$
$= {}^{20}C_0 + {}^{20}C_1\, 7 + {}^{20}C_2\, 7^2 + \ldots\ldots + {}^{20}C_{20} \cdot 7^{20}$
$= 1 + 7\,({}^{20}C_1 + {}^{20}C_2\, 7 + \ldots\ldots + {}^{20}C_{20}\, 7^{19})$
$= 1 +$ (a multiple of 7)
Therefore, required remainder = 1

14. The coefficient of x^5 in the expansion of $(x^2 - x - 2)^5$ is

[EAMCET – 2003]

(a) 0 (b) − 83
(c) − 82 (d) − 81

Solution

(d) Exp. $= (x - 2)^5 (1 + x)^5$
Now, write expansion and find the coefficient of x^5.

15. The digit in the unit place in the number 7^{289} is

[CET (Karnataka) – 2005]

(a) 1 (b) 3
(c) 9 (d) 7

Solution

(d) $7^{289} = 7\,(7^{288})$
Now $7^{288} = (7^2)^{144} = (50 - 1)^{144}$
$= 50^{144} - {}^{144}C_1\, 50^{143} + \ldots. + {}^{144}C_{144} (-1)^{144}$
$= (50^{144} - {}^{144}C_1\, 50^{143} + \ldots - {}^{144}C_{143} \cdot 50) + 1$
= (a multiple of 50) + 1 = (a multiple of 10) + 1
$\Rightarrow 7.7^{288} =$ (a multiple of 10) + 7
Therefore, digit in the unit place = 7

16. For natural numbers m, n if $(1 - y)^m (1 + y)^n = 1 + a_1 y + a_2 y^2 + \ldots.$, and $a_1 = a_2 = 10$; then (m, n) is equal to

[AIEEE – 2006]

(a) (35, 20) (b) (45, 35)
(c) (35, 45) (d) (20, 45)

Solution

(c) $(1 - y)^m (1 + y)^n = 1 + a_1 y + a_2 y^2 + \ldots.$
$= 1 + 10y + 10y^2 + \ldots. \quad (1)$

Also, $(1 - y)^m (1 + y)^n$

$$= \left(1 - my + \frac{m(m-1)}{2} y^2 - \ldots\right)$$

$$= \left(1 - ny + \frac{n(n-1)}{2} y^2 + \ldots\right)$$

$$= 1 + (n - m)y$$

$$+ \left[\frac{n(n-1)}{2} - mn + \frac{m(m-1)}{2}\right] y^2 \quad (2)$$

Comparing (1) and (2), we have

$$n - m = 10 \quad (3)$$

$$\frac{n(n-1)}{2} - nm + \frac{m(m-1)}{2} = 10 \quad (4)$$

Now, (4) $\Rightarrow (n - m)^2 - (n + m) = 20$
$\Rightarrow 100 - (n + m) = 20$ [Using (3)]

$\Rightarrow n + m = 80$ (5)

(3), (4) $\Rightarrow m = 35, n = 45.$

17. The coefficient of x^4 in the expansion of $\left(\frac{x}{2} - \frac{3}{x^2}\right)^{10}$ is

[DCE – 2000; MPPET – 2006]

(a) $\frac{504}{259}$ (b) $\frac{450}{263}$

(c) $\frac{405}{256}$ (d) None

Solution

(c) In the expansion of $\left(\frac{x}{2} - \frac{3}{x^2}\right)^{10}$

$(r + 1)$th term is

$$T_{r+1} = {}^{10}C_r \left(\frac{x}{2}\right)^{10-r} \left(-\frac{3}{x^2}\right)^r$$

$$= {}^{10}C_r \frac{x^{10-r}}{2^{10-r}} \cdot \frac{(-1)^r .3^r}{x^{2r}}$$

$$= {}^{10}C_r \cdot \frac{x^{10-3r} . (-1)^r .3^r}{2^{10-r}}$$

$\therefore$ $10 - 3r = 4$ (coefficient of x^4)

$\Rightarrow r = 2$ Hence, coefficient of x^4 is

$$C_2 \cdot \frac{3^2}{2^8} = \frac{405}{256}.$$

18. If coefficient of x^7 in $\left(ax^2 \frac{1}{bx}\right)^{11}$ and the coefficient of x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$ are equal, then

[MP – 1999; AIEEE – 2005]

(a) $ab = 1$ (b) $a + b = 1$

(c) $a/b = 1$ (d) $a - b = 1$

Solution

(a) Suppose x^7 occurs in T_{r+1} in the first expansion, then

$$r = \frac{11(2) - 7}{2 + 1} = 5$$

Also, suppose x^{-7} occurs in $T_{r'+1}$ in the second expansion, then $r' = \frac{11(1) + 7}{1 + 1} = 6$

As given coefficients of there two terms are equal, so

$${}^{11}C_5 (a)^6 \left(\frac{1}{b}\right) = {}^{11}C_6 (a)^5 \left(-\frac{1}{b}\right)^6$$

$$\Rightarrow \frac{462a^6}{b^5} = \frac{462a^5}{b^6}$$

$\Rightarrow ab = 1.$

Note: Use formula $r = \frac{n\alpha - m}{\alpha + \beta}$

19. The coefficient of x^4 in the expansion of $(1 + x + x^2 + x^3)^n$ is

[MNR – 1993; RPET – 2001; DCE – 1998]

(a) ${}^nC_4$

(b) ${}^nC_4 + {}^nC_2$

(c) ${}^nC_4 + {}^nC_2 + {}^nC_4 . {}^nC_2$

(d) ${}^nC_4 + {}^nC_2 + {}^nC_1 . {}^nC_2$

Solution

(d) $(1 + x + x^2 + x^3)^n = \{(1 + x)^n (1 + x^2)^n\}$

$= (1 + {}^nC_1 x + {}^nC_2 x^2 + \ldots\ldots + {}^nC_n x^n)$

$\times (1 + {}^nC_1 x^2 + {}^nC_2 x^4 + \ldots\ldots + {}^nC_n x^{2n})$

Therefore, the coefficient of x^4

$= {}^nC_2 + {}^nC_2 . {}^nC_1 + {}^nC_4 = {}^nC_4 + {}^nC_2 + {}^nC_1 . {}^nC_2$

20. The coefficient of $\frac{1}{x}$ in the expansion of $(1 + x)^n \left(1 + \frac{1}{x}\right)^n$ is

(a) $\frac{n!}{(n-1)!\,(n+1)!}$

(b) $\frac{(2n)!}{(n-1)!\,(n+1)!}$

(c) $\frac{(2n)!}{(2n-1)!\,(2n+1)!}$

(d) None of these

Solution

(b) $(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots\ldots + {}^nC_n x^n) \left(1 + \frac{1}{x}\right)^n = {}^nC_0 + {}^nC_1 \frac{1}{x} + {}^nC_2 \frac{1}{x^2} + \ldots + {}^nC_n \left(\frac{1}{x}\right)^n$

Obviously, required coefficient of $\frac{1}{x}$ can be given by

$$^nC_0{}^nC_1 + {}^nC_1.{}^nC_2 + \ldots\ldots + {}^nC_{n-1}.{}^nC_n$$
$$= \frac{(2n)!}{(n-1)!\,(n+1)!}$$

21. In the expansion of $(1+3x+2x^2)^6$ the coefficient of x^{11} is

[Kerala (Engg.) – 2005]

(a) 144 (b) 288
(c) 216 (d) 576

Solution

(d) $(1 + 3x + 2x^2)^6 = [1+ x(3 + 2x)]^6$
$= 1 + {}^6C_1x(3 + 2x) + {}^6C_2x^2(3 + 2x)^2$
$+ {}^6C_3x^3(3 + 2x)^3 + {}^6C_4x^4(3 + 2x)^4$
$+ {}^6C_5x^5(3 + 2x)^5 + {}^6C_6x^6(3 + 2x)^6$

only x^{11} gets from ${}^6C_6x^6(3 + 2x)^6$
Since, ${}^6C_6x^6(3 + 2x)^6 = x^6(3 + 2x)^6$
Therefore, coefficient of x^{11} = ${}^6C_5 3.2^5$ = 576.

22. The coefficient of the term independent of x in the expansion of $(1 + x + 2x^3)\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$

[DCE – 1994]

(a) 1/3 (b) 19/54
(c) 17/54 (d) 1/4

Solution

(c) The general term in the expansion of $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ is

$$T_{r+1} = {}^9C_r\left(\frac{3}{2}x^2\right)^{9-r}\left(-\frac{1}{3x}\right)^r$$

$$= {}^9C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)x^{18-3r} \quad (1)$$

Now, the coefficient of the term independent of x in the expansion of $(1 + x + 2x^3)\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ (2)

= Sum of the coefficient of the terms x^0, x^{-1} and x^{-3} in $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$.

For x^0 in (1), $18-3r = 0 \Rightarrow r = 6$. For x^{-1} in (1), there exists no value of r and hence no such term exists.
For x^{-3} in (1), $18 - 3r = -3 \Rightarrow r = 7$
Therefore, for term independent of x, in (2) the coefficient

$$= 1 \times C_6(-1)^6\left(\frac{3}{2}\right)^{9-6}\left(\frac{1}{3}\right)^6$$

$$+ 2 \times {}^9C_7(-1)^7\left(\frac{3}{2}\right)^{9-7}\left(\frac{1}{3}\right)^7$$

$$= \frac{9.8.7}{1.2.3}\cdot\frac{3^3}{2^3}\cdot\frac{1}{3^6} + 2\,\frac{9.8}{1.2}(-1)\,\frac{3^2}{2^2}\cdot\frac{1}{3^7}$$

$$= \frac{7}{18} - \frac{2}{27} = \frac{17}{54}.$$

23. If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots C_nx^n$, then $C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2$ is equal to

(a) ${}^{2n}C_n$ (b) ${}^{2n}C_{n+1}$
(c) ${}^{2n}C_{n+2}$ (d) none of these

[IIT – 1973; MP – 1985; PET(Raj.) – 1997; UPSEAT – 1999]

Solution

(a) $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots.. + C_nx^n$, (1)

Also,
$(1 + x)^n = C_n + C_{n-1}x + C_{n-2}x^2 + \ldots.. + C_0x^n$, (2)

$[\because C_0 = C_n, C_1 = C_{n-1}, C_2 = C_{n-2}$ etc.]
Multiplying (1) and (2), we get
$(1 + x)^{2n} = (C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n)$
$\times (C_n + C_{n-1}x + C_{n-2}x^2 + \ldots + C_0x^n)$
Comparing the coefficient of x^n on both sides,
we get,
${}^{2n}C_n = C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2$

24. The sum of the rational terms in the expansion of $(\sqrt{2} + 3^{1/5})^{10}$ is

(a) 41 (b) 46
(c) 39 (d) None of these

(I.I.T. Re-ex. – 1997)

Solution

(a) $T_{r+1} = {}^{10}C_r(\sqrt{2})^{10-r}(3^{1/15})^r$ where r varies from 0 to 10 as there will be only 11 terms.

In T_{r+1} the powers of 2 and 3 are $\frac{10-r}{2}$ and $\frac{r}{5}$ where $0 \le r \le 10$.

$\frac{r}{5}$ will be an integer for r = 0, 5, 10 but $\frac{10-r}{2}$ will not be an integer for r = 5. Thus, both powers are integers for r = 0 and 10.

Hence, T_1 and T_{11} will have rational coefficients whose sum is

${}^{10}C_0(\sqrt{2})^{10}.\ 1 + {}^{10}C_{10}.\ 1\ .\ 3^2 = 32 + 9 = 41.$

26. Let n be an odd natural number greater than 1. Then the number of zeros at the end of the sum $99^n + 1$ is

(a) 3 (b) 4
(c) 2 (d) None

Solution

(c) $1 + 99^n = 1 + (100 - 1)^n$
$= 1 + [{}^nC_0\, 100^n - {}^nC_1\, 100^{n-1}$
$+ {}^nC_2\, 100^{n-2} \ldots - {}^nC_n]$ [$\because n$ is odd]
$= 100\,[{}^nC_0\, 100^{n-1} - {}^nC_1\, 100^{n-2} \ldots + {}^nC_{n-1}]$
= 100 × Integer whose units place is different from zero
[n having odd digit in the unit place]
Hence, number of zeros at the end of the sum $99^n + 1$ is 2.
Hence, (c) is the correct answer.

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. In the expansion of $\left(\frac{3x}{4} - \frac{4}{3x}\right)^5$, 4th term will be

(i) ${}^5C_3\left(\frac{3x}{4}\right)^2\left(\frac{-4}{3x}\right)^3$ (ii) ${}^5C_3\left(\frac{3x}{4}\right)^2\left(\frac{-4}{x}\right)^3$

(iii) $-{}^5C_2\,\frac{4}{x}$ (iv) none of these

2. The middle term in the expansion of $\left(\frac{x}{a} + \frac{9}{x}\right)^{10}$ is

(a) ${}^{20}C_{11}\,\frac{x}{a}$ (b) ${}^{20}C_{11}\,\frac{a}{x}$
(c) ${}^{20}C_{10}$ (d) none of these

3. If 9th term in the expansion of $\left(3x^2 - \frac{2}{x^3}\right)^n$ is independent of x, then the value of n is:

(a) 18 (b) 20
(c) 24 (d) 32

4. In the expansion of $\left(y^2 + \frac{c}{y}\right)^5$, the coefficient of y will be

[MNR – 1983]

(a) $20\,c$ (b) $10\,c$
(c) $10\,c^3$ (d) $20\,c^2$

5. The term independent of y in the expansion of $(y^{-1/6} - y^{1/3})^9$ is

[BITS, RANCHI – 1980]

(a) 84 (b) 8.4
(c) 0.84 (d) – 84

6. In $\left[\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right]^n$ if the ratio of 7th term from the beginning to the 7th term form the end is 1/6, then n =

(a) 7 (b) 8
(c) 9 (d) none of these

7. The term independent of x in the expansion of $\left(2x + \frac{1}{3x}\right)^6$ is

[MNR – 1995]

(a) 160/9 (b) 80/9
(c) 160/27 (d) 80/3

8. If x^4 occurs in the rth term in the expansion of $\left[x^4 + \frac{1}{x^3}\right]^{15}$, then r =

[MPPET – 1995; Pb CET – 2002; NDA – 2007]

(a) 7 (b) 8
(c) 9 (d) 10

9. The largest term in the expansion of $(3 + 2x)^{50}$ where $x = 1/5$ is

[IIT Sc. – 1993]

(a) 5th (b) 51th
(c) 7th (d) 6th and 7th

10. The term independent of x in $\left[\sqrt{\frac{x}{3}} + \frac{\sqrt{3}}{x^2}\right]^{10}$ is

[EAMCET – 1994; RPET – 2000; DCE – 2004]

(a) 2/3 (b) 5/3
(c) 4/3 (d) none of these

11. The greatest integer which divides the number $101^{100} - 1$ is

[MPPET – 1998]

(a) 100 (b) 1000
(c) 10000 (d) 100000

12. The coefficient of x^{-9} in the expansion of $\left(\frac{x^2}{2} - \frac{2}{x}\right)^9$ is

[Kerala Engg. – 2001]

(a) 512 (b) – 512
(c) 521 (d) 251

13. $49^n + 16n - 1$ is divisible by

[Kurukshetra CEE – 2001]

(a) 3 (b) 19
(c) 64 (d) 29

14. What are the last two digits of the number 9^{200}?

(a) 19 (b) 21
(c) 41 (d) 01

15. For any positive integer n, if $4^n - 3n$ is divided by 9, then what is the remainder?

(a) 8 (b) 6
(c) 4 (d) 1

16. The number of integral terms in the expansion of $(\sqrt{3} + \sqrt[8]{5})^{256}$ is

[AIEEE – 2003]

(a) 32 (b) 33
(c) 34 (d) 35

17. In pascal's Δ, each row is bounded by

(a) 1 (b) 0
(c) 2 (d) – 1

18. If the expansion of $\left(\frac{3\sqrt{x}}{7} - \frac{5}{2x\sqrt{x}}\right)^{13n}$ contains a term independent of x, then n should be a multiple of

[Kerala PET – 2008]

(a) 10 (b) 5
(c) 6 (d) 4

19. What is the last digit of $3^{3^{4n}} + 1$, where n is a natural number?

[NDA – 2008]

(a) 2
(b) 7
(c) 8
(d) none of the above

20. If this sum of the coefficient in the expansion of $(1 + 2x)^n$ is 6561, the greatest term in the expansion for $x = 1$ is

(a) 4th (b) 5th
(c) 6th (d) none of these

SOLUTIONS

1. (a) 4th term, $T_4 = T_{3+1}$

$= {}^5C_3 \left(\frac{3x}{4}\right)^{5-3} \left(\frac{-4}{3x}\right)^3$

$= {}^5C_3 \left(\frac{3x}{4}\right)^2 \left(\frac{-4}{3x}\right)^3$

2. (c) Here $n = 20$

Total number of terms = 21

$\therefore$ Middle term $= \frac{21 + 1}{2} = 11$th

$\therefore$ T_{11} in the expansion of $\left(\frac{x}{a} + \frac{a}{x}\right)^{20}$

$= {}^{20}C_{10} \left(\frac{x}{a}\right)^{10} \left(\frac{a}{x}\right)^{10}$

$= {}^{20}C_{10} \times 1^{10}$

$= {}^{20}C_{10}.$

3. (b) $T_9 = T_{8+1} = {}^nC_8\,(3x^2)^{n-8}\left(-\frac{2}{x^3}\right)^8$

$T_9 = {}^nC_8 \,.\, 3^{n-8} \,.\, 2^8 \,.\, x^{2n-40}$

For independent of x: $2n - 40 = 0$ or, $n = 20$.

4. (c) $T_{r+1} = {}^5C_r (y^2)^{5-r}\left(\frac{c}{y}\right)^r = {}^5C_r y^{10-3r} C^r$.

which contains y^1 if $10 - 3r = 1$

i.e., if $r = 3$.

5. (d) $T_{r+1} = {}^9C_r\left(y^{\frac{-1}{6}}\right)^{9-r} = \left(-y^{\frac{+1}{3}}\right)^r$

$= {}^9c_r y^{-\frac{9-r}{6}} . (-1)^r y^{\frac{+r}{3}}$

$\therefore \left(\frac{9-6}{6}\right) + \frac{r}{3} = 0 \Rightarrow 9 - r - 2r = 0$

$\Rightarrow 3r = 9 \Rightarrow r = 3$

Therefore, required term $= {}^9c_3\,(-1)^3 = -{}^9c_3$

$= -\frac{9 \times 8 \times 7}{1 \times 2 \times 3} = -3 \times 4 \times 7$

$= -84$

6. (c) $\frac{1}{6} = \frac{{}^nC_6 (2^{1/3})^{n-6}(3^{-3/3})^6}{{}^nC_{n-6}(2^{1/3})^6(3^{-1/3})^{n-6}}$

or $6^{-1} = 6^{-4} \,.\, 6^{n/3} = 6^{(n/3-4)}$

$\therefore \frac{n}{3} - 4 = -1 \Rightarrow n = 9$

7. (c) $\left(2x + \frac{1}{3x}\right)^6$ $(T_{r+1} = {}^nC_r x^{n-r}.\, a^r)$ gives

$T_{r+1} = {}^6C_r (2x)^{6-r}\left(\frac{1}{3x}\right)^r = {}^6C_r . \frac{2^{6-r}}{3^r} . x^{6-2r}$ (1)

Term is independent of x if power of $x = 0$,

This $\Rightarrow 6 - 2r = 0 \Rightarrow r = 3$.

$\therefore T_{r+1} = {}^6C_3 . \left(\frac{2}{3}\right)^3 = \frac{6.5.4}{3\,!} . \frac{8}{27} = \frac{160}{27}$

8. (c) $T_r = {}^{15}C_{r-1}(x^4)^{15-(r-1)}\left(\frac{1}{x^3}\right)^{(r-1)}$

For coefficient of x^4: $4\,(16 - r) - 3\,(r - 1) = 4$, $r = 9$.

9. (d) $(3 + 2x)^{50}$. Let T_{r+1} be largest term

Apply $\frac{T_{r+1}}{T_r} = \left(\frac{n-r+1}{r}\right)\frac{a}{x}$ for $(x + a)^n$.

In the present case,

$\frac{T_{r+1}}{T_r} = \left(\frac{50-r+1}{r}\right)\left(\frac{2x}{3}\right)$

$= \left(\frac{51-r}{r}\right)\left(\frac{2}{15}\right)$ as $x = \frac{1}{5}$

$\frac{T_{r+1}}{T_r} \geq 1 \Rightarrow \frac{2(51-r)}{15r} \geq 1$

$\Rightarrow 102 - 2r \geq 15r \Rightarrow r \leq 6$

T_6 and T_7 are greatest.

10. (b) $\left(\left(\frac{x}{3}\right)^{\frac{1}{2}} + \frac{\sqrt{3}}{x^2}\right)^{10}$

Coefficient of the independent term is

$T_{r+1} = {}^{10}C_r\left(\left(\frac{x}{3}\right)^{\frac{1}{2}}\right)^{10-r}\left(\frac{\sqrt{3}}{x^2}\right)^r$

$x^{5-\frac{r}{2}} x^{-2r} = x^0$

$5 - \frac{5r}{2} = 0$

$r = 2$

$T_{2+1} = {}^{10}C_2\left(\frac{1}{\sqrt{3}}\right)^8 (\sqrt{3})^2$

$= 45 \times \left(\frac{1}{\sqrt{3}}\right)^8 = \frac{45}{3 \times 3 \times 3} = \frac{5}{3}$

11. (c) $(101)^{100} = (1 + 100)^{100}$

$= 1 + {}^{100}C_1\,(100) + {}^{100}C_2\,(100)^2 + {}^{100}C_3\,(100)^3 + \ldots\ldots$

or,

$(101)^{100} - 1 = (10)^4\,[1 + {}^{100}C_2 + {}^{100}C_3\,(10)^2 + \ldots\ldots]$

This $\Rightarrow (101)^{100} - 1$ is divisible by greatest integer $(10)^4$.

12. (b) Let x^{-9} occur in T_{r+1}

Now, $T_{r+1} = {}^9C_r\left(\frac{x^2}{2}\right)^{9-r}\left(-\frac{2}{x}\right)^r$

$= {}^9C_r \frac{x^{18-3r}}{2^{9-2r}} (-1)^r$

Since, x^{-9} occur in T_{r+1}

$\therefore 18 - 3r = -9 \Rightarrow 3r = 27 \Rightarrow r = 9$

$\therefore$ required coefficient

$= \frac{{}^9C_9(-1)^9}{2^{-9}} = -2^9 = -521$

13. (c) $49^n + 16n - 1$

$= (1 + 48)^n + 16n - 1$

$= 1 + {}^nC_1\, 48 + {}^nC_2\, (48)^2 + \ldots\ldots + {}^nC_n\, (48)^n + 16n - 1$

$= (48n + 16n) + {}^nC_2\, (48)^2 + {}^nC_3\, (48)^3 + \ldots + {}^nC_n\, (48)^n$

$= 64n + 8^2\{{}^nC_2\, 6^2 + {}^nC_3 \,.\, 6^3 \,.\, 8 + {}^nC_4 .\, 6^4 \,.\, 8^2 + \ldots + {}^nC_n\, 6^n\, 8^{n-2}\}$

Hence, $49^n + 16n - 1$ is divisible by 64.

14. (d) $(10 - 1)^{200} = {}^{200}C_0\, 10^{200} - {}^{200}C_1\, 10^{199} + \ldots - {}^{200}C_{199}\, 10 + {}^{200}C_{200}\, 10°$

$= {}^{200}C_0\, 10^{200} - {}^{200}C_1\, 10^{199} + \ldots\ldots - (200)\, 10 + 1 = 100\,\lambda + 1$

Therefore, last 2 digit = 01

15. (d) For $n = 1$, $4^n - 3n = 4 - 3 = 1 = 0 + 1 = 9 \times 0 + 1$

For $n = 2$, $4^n - 3n = 16 - 6 = 10 = 9 \times 1 + 1 = 9 \times 1 + 1$

For $n = 3$, $4^3 - 3 \times 3 = 64 - 9 = 55 = 54 + 1 = 9 \times 6 + 1$

$\therefore$ By induction, when $4^n - 3n$ is divided by 9, we get remainder is 1.

16. (b) $T_{r+1} = {}^{256}C_r\, (\sqrt{3})^{225-r}\, (\sqrt[8]{5})^r$

$= {}^{256}C_r\, 3^{\frac{256}{2}}\, 5^{\frac{r}{5^8}}$

$\therefore T_{r+1}$ is an integer if $\frac{256 - r}{2}$ and $\frac{r}{8}$ are both nonnegative integers.

$\therefore$ Possible values of r are 0, 8, 16, 24, … 256

Therefore, number of integral terms = 1 + 32 = 33 ($\therefore$ 256 = 8 × 32)

17. (a) In Pascal's triangle, each row begins with 1 and ends with 1.

18. (d) General term in the expansion of $\left(\frac{3\sqrt{x}}{7} - \frac{5}{2x\sqrt{x}}\right)^{13n}$ is

$${}^{13n}C_r \left(\frac{3\sqrt{x}}{7}\right)^{13n-r} \left(\frac{-5}{2x\sqrt{x}}\right)^r$$

$= {}^{13n}C_r \left(\frac{3}{7}\right)^{13n-r} \left(\frac{-5}{2}\right)^r x^{\frac{13n-4r}{2}}$; this term will be without x, if

$$\frac{13n - 4r}{2} = 0$$

i.e., if $13n = 4r$ for some $r \in \{0, 1, 2, \ldots, 13\}$

i.e., if n is equal to $\frac{4r}{13}$ for some $r \in \{0, 1, 2, \ldots, 13\}$

Therefore, n must be a multiple of 4.

19. (d) For $n = 0$

Last digit of $3^{3^{4n}} + 3^{3^0} + 1 = 3 + 1 = 4$

For $n = 1$

Last digit of $3^{3^{4n}} + 1 = 3^{3^4} + 1 = 3^{81} + 1$

= Last digit of $3^{4 \times 16} \,.\, 3 + 1$

$= 3 + 1 = 4$

Thus, it is clear that the last digit of $3^{3^{4n}} + 1$ is 4.

20. (c) Sum of the coefficients in the expansion of $(1 + 2x)^n = 6561$

$\Rightarrow (1 + 2x)^n = 6561$, when $x = 1$

$\Rightarrow 3^n = 6561 \Rightarrow 3^n = 3^8 \Rightarrow n = 8$

Now, $\frac{T_{r+1}}{T_r} = \frac{{}^8C_r\, (2x)^r}{{}^8C_{r-1} (2x)^{r-1}} = \frac{9 - 1}{r} \times 2r$

$$\therefore \frac{T_{r+1}}{T_r} = \frac{9 - r}{r} \times 2x$$

$$\Rightarrow \frac{T_{r+1}}{T_r} = \frac{(9 - r) \times 2}{r} \qquad [\because x = 1]$$

Now, $\frac{T_{r+1}}{T_r} \geq 1 \Rightarrow 18 - 2r \geq r \Rightarrow r \leq 6$

Thus, 6th and 7th terms are greatest and are equal in magnitude.

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. In the expansion of $\left(x^3 + 1/x^2\right)^8$, the term containing x^4 is

(a) $70x^4$ (b) $60x^4$

(c) $56x^4$ (d) none of these

2. The term indpendent of x in the expansion of $(1/2\, x^{1/3} + x^{-1/5})^8$ will be

(a) 5 (b) 6

(c) 7 (d) 8

3. In the expansion of $(3x^2/2 - 1/3x)^9$, the term independent of x is

(a) ${}^9C_3\ 1/6^3$ (b) ${}^9C_3\ (3/2)^3$

(c) ${}^9C_3$ (d) none

4. The coefficient of x^7 in the expansion of $(x^2/2 - 2/x)^8$ is

[MNR – 1975]

(a) – 56 (b) 56

(c) −14 (d) 14

5. $(1 + x)^n - nx - 1$ is divisible (where $n \in N$)

(a) by $2x$ (b) by x^2

(c) by $2x^3$ (d) all of these

6. The ratio of the coefficient of x^{15} to the term independent of x in $[x^2 + (2 / x)]^{15}$ is

(a) 1: 32 (b) 32: 1

(c) 1: 16 (d) 16: 1

7. The coefficient of x^{39} in the expansion of $\left(x^4 - \frac{1}{x^8}\right)^8$ is

[MPPET – 2001]

(a) – 455 (b) – 105

(c) 105 (d) 455

8. The coefficient of x^6 in the expansion of $\left(3x^2 - \frac{1}{3x}\right)^9$ is

(a) 126 (b) 378

(c) 504 (d) 830

9. Find the $(n + 1)$th term from the end in the expansion of $(x - 1/x\,)^{3n}$ is

(a) $\frac{(3n)\,!}{(2n)\,!\,n\,!}x^n$ (b) $\frac{(3n)\,!}{(2n)\,!\,n\,!}x^{2n}$

(c) $\frac{(2n)\,!}{(3n)\,!\,n\,!}x^{5n}$ (d) none

10. The numerically greatest term in the expansion of $(4-3x)^7$ when $x = 1$

(a) 21×4^3 (b) 21×4^6

(c) 37×4^8 (d) none of these

11. If $x = 1/3$, then the greatest term in the expansion of $(1 + 4x)^8$ is

(a) 56 (b) $56\,(4/3)^3$

(c) $56.(4/3)^5$ (d) none of these

12. The digit at unit place in the number 7^{126} is

[CET (Karnataka) – 2000]

(a) 9 (b) 5

(c) 3 (d) 1

13. The remainder obtained when 5^{124} is divided by 124 is

[Karnataka CET – 2007]

(a) 5 (b) 0

(c) 2 (d) 1

14. In the expansion of $(x + 2/x^2)^{15}$, the term independent of x is

[MPPET – 1993; Pb CET – 2002]

(a) ${}^{15}C_6\,2^6$ (b) ${}^{15}C_5\,2^5$

(c) ${}^{15}C_4\,2^4$ (d) ${}^{15}C_8\,2^8$

15. The coefficient of x^{32} in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$ is

[Karnataka – 2003; Pb CET – 2000]

(a) ${}^{15}C_5$ (b) ${}^{15}C_6$

(c) ${}^{15}C_4$ (d) ${}^{15}C_7$

WORK SHEET: TO CHECK PREPARATION LEVEL

Important Instructions:

1. The answer sheet is immediately below the work sheet.

2. The test is of 15 minutes.

3. The test consists of 15 questions. The maximum marks are 45.

4. Use blue/black ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. Find numerically the greatest term in the expansion of $(2 + 3x)^9$, when $x = 3/2$

(a) $\frac{7 \times 3^{13}}{2}$ (b) $-\frac{7 \times 3^{13}}{2}$

(c) $\frac{2}{7 \times 3^{13}}$ (d) $-\frac{2}{7 \times 3^{13}}$

2. Find the 9th term in the expansion of $\left(\frac{x}{a} - \frac{3a}{x^2}\right)^{12}$

(a) $^{12}C_4 . x^{-12} . a^4 . 3^8$
(b) $-{}^{12}C_4 . x^{-12} . a^4 . 3^8$
(c) $x^{-12} . a^4 . 3^7 . {}^{12}C_4$
(d) $-x^{-12} . a^3 . 3^8 . {}^{12}C_4$

3. If the 4th term in the expansion of $(px + x^{-1})^m$ is 2. 5 for all $x \in R$, then

(a) $p = \frac{5}{2}, m = 3$ (b) $p = \frac{1}{2}, m = 6$
(c) $p = \frac{1}{2}, m = 6$ (d) noe of these

4. If the absolute term in the expansion of $\left(\sqrt{x} - \frac{k}{x^2}\right)^{10}$ is 405, then k is equal to

(a) ± 1 (b) ± 2
(c) ± 3 (d) none of these

5. Which term in the expansion of the binomial $\left[\sqrt[3]{\frac{a}{\sqrt{b}}} + \frac{\sqrt{b}}{\sqrt[6]{a}}\right]^{21}$ contains a and b to one and the same power:

(a) 9th (b) 10th
(c) 11th (d) 12th

6. If the expansion of $\left(x^2 + \frac{2}{x}\right)^n x$ for positive integer n has a term independent of x, then n is

[Roorkee – 1991]

(a) 23 (b) 18
(c) 16 (d) 0

7. The largest term in the expansion of $(3 + 2x)^{50}$ where $x = 1/5$ is

[IIT – 1993]

(a) 5th (b) 51st
(c) 7th (d) 8th

8. The co–efficient of x^{-9} in the expansion of $\left(\frac{x^2}{2} - \frac{2}{x}\right)^9$

(a) 512 (b) – 512
(c) 521 (d) – 251

[Kerala PET – 2001]

9. The number of irrational terms in the expansion of $(5^{1/6} + 2^{1/8})^{100}$ is

[Him. CET – 2006]

(a) 96 (b) 97
(c) 98 (d) 99

10. The co–efficient of x^7 in the expansion of $(1 - x^4)(1 + x)^9$ is

(a) 27 (b) – 24
(c) 48 (d) – 48

11. The term independent of x in $(2x - 1/2x^2)^{12}$ is

[RPET – 1985]

(a) – 7930 (b) – 445
(c) 445 (d) 7920

12. The coefficient of x^3 in the expansion of $\left(x - \frac{1}{x}\right)^7$ is

[MPPET – 1997]

(a) 14 (b) 21
(c) 28 (d) 35

13. The 5th term from the end in the expansion of $\left(\frac{x^2}{2} - \frac{2}{x^2}\right)^{12}$ is

(a) $1853 / x^3$ (b) $7920 / x^4$
(c) $1258 / x^5$ (d) none of these

14. If the coefficient of x in the expansion of $(x^2 + k/x)^5$ is 270, then k =

(a) 1 (b) 2
(c) 3 (d) 4

15. Coefficient of x^2 in the expansion of $(x - 1/2x)^8$ is

[UPSEAT – 2002]

(a) 1 / 7 (b) – 1 / 7
(c) – 7 (d) 7

ANSWER SHEET

1.	ⓐ	ⓑ	ⓒ	ⓓ	6.	ⓐ	ⓑ	ⓒ	ⓓ	11.	ⓐ	ⓑ	ⓒ	ⓓ
2.	ⓐ	ⓑ	ⓒ	ⓓ	7.	ⓐ	ⓑ	ⓒ	ⓓ	12.	ⓐ	ⓑ	ⓒ	ⓓ
3.	ⓐ	ⓑ	ⓒ	ⓓ	8.	ⓐ	ⓑ	ⓒ	ⓓ	13.	ⓐ	ⓑ	ⓒ	ⓓ
4.	ⓐ	ⓑ	ⓒ	ⓓ	9.	ⓐ	ⓑ	ⓒ	ⓓ	14.	ⓐ	ⓑ	ⓒ	ⓓ
5.	ⓐ	ⓑ	ⓒ	ⓓ	10.	ⓐ	ⓑ	ⓒ	ⓓ	15.	ⓐ	ⓑ	ⓒ	ⓓ

HINTS AND EXPLANATIONS

3. (c) $T_4 = {}^mC_3\,(px)^{m-3}\,(x^{-1})^3 = {}^mC_3\,p^{m-3}\,x^{m-3}\,x^{-3}$

$= {}^mC_3\,p^{m-3}\,x^{m-6} = 2.5$

$\therefore\ m = 6,\ {}^6C_3\,p^3 = 2.5 \Rightarrow p = \frac{1}{2}$

5. (b) $\left[\sqrt[3]{\frac{a}{\sqrt{b}}} + \frac{\sqrt{b}}{\sqrt[6]{a}}\right]^{21}, T_{r+1}\ {}^{21}C_r\left(\frac{a^{1/3}}{b^{1/6}}\right)^{21-r}\left(\frac{b^{1/2}}{a^{1/6}}\right)^r$

power of a = power of b

$\frac{21-r}{3} - \frac{r}{6} = \frac{r}{2} - \frac{21-r}{6}$

$42 - 2r - r = 3r - 21 + r \Rightarrow 42 - 3r = 4r - 21\ r = 9$

9. (b) $(5^{1/6} + 2^{1/8})^{100}$, General term $= {}^{100}C_r$ $(5^{1/6})^{100-r}\,(2^{1/8})^r, = {}^{100}C_r\,5^{\frac{100-r}{6}}\,2^{r/8}$

for rational term, $r = 16, 40, 64, 88$

No. of irrational terms

= total term – (no. of ratioal terms)

$= 101 - 4 = 97$

10. (d) $(1 - x^4)\,(1 + x)^9$

coefficient of $x^7 = {}^9C_7 - {}^9C_3 = -48$

13. (b) 5th term then end = 9 term beginning

$= {}^{12}C_8\left(\frac{x^3}{2}\right)^4\left(\frac{-2}{x^2}\right)^8 = \frac{7920}{x^4}$

This page is intentionally left blank.

LECTURE

4 Multinomial Expansion and Pascal's Triangle

BASIC CONCEPTS

1. The number of terms in the expansion of $(x + y + z)^n$, where n is a positive integer, is

   $\frac{1}{2}(n+1)(n+2)$.

2. The total number of terms in the multinomial expansion $(x_1+x_2+x_3+\ldots+x_k)^n$ is = Number of non negative integral solutions of the equation $r_1 + r_2 + \ldots. + r_k$ is n is ${}^{n+k-1}C_n$ or ${}^{n+k-1}C_{k-1}$.

3. ***Pascal Triangle*** In the expansion of $(x + a)^n$, the binomial coefficients ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, …. etc, when n = 0, 1, 2, ….. can easily be found in the rows of the following triangular figure

   Binomial Binomial Coefficients

$n = 0$	1
$n = 1$	1 1
$n = 2$	1 2 1
$n = 3$	1 3 3 1
$n = 4$	1 4 6 4 1
$n = 5$	1 5 10 10 5 1
$n = 6$	1 6 15 20 15 6 1
$n = 7$	1 7 21 35 35 21 7 1
…………	………………………..

Note 1 **Pascal's Triangle** is bordered by one (1).

Note 2 **Pascal's triangle** is symmetrical about a line through vertex 1.

Note 3 Binomial coefficients of $(x + a)^n$ are present in $(n + 1)$th row of the Pascal's triangle.

Explaination In the above figure, every row has 1 as its first and last number and numbers in each row at equidistant from two ends are equal. Every number in each row can be obtained by adding two numbers of its previous row which lie at left and right to this number. For example, when n = 6, then the second number of this row can be obtained by adding 1 and 5 which lie in the previous row at left and right to 6. So, this second number will be 1 + 5 = 6. Similarly, third number of this row = 5 + 10 = 15 and so on. Thus, we can obtain any number of any row. From different rows of the Pascal triangle, we can directly obtain values of binomial coefficients for different values of n. For example, when n = 4, then numbers 1, 4, 6, 4, 1 lying in 5th row are values of binomial coefficients ${}^4C_0$, ${}^4C_1$, ${}^4C_2$, ${}^4C_3$, ${}^4C_4$.

4. Binomial Theorem for Any index If n is rational number and x is real number such that $|x| < 1$, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots \text{ to } \infty \quad (1)$$

4.1 Observations

(i) In the above expansion, the first term must be a unity. In the expansion of $(a + x)^n$, where n is either a negative integer or a fraction, we proceed as follows

$$(a+x)^n = \left[a\left(1+\frac{x}{a}\right)\right]^n = a^n\left(a+\frac{x}{a}\right)^n$$

$$= a^n\left[1 + n\cdot\frac{x}{a} + \frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2 + \ldots\right] \text{ and}$$

the expansion is valid when $\left|\frac{x}{a}\right| < 1$ i.e. $|x| < |a|$.

(ii) There are infinite number of terms in the expansion of $(1 + x)^n$, when n is a negative integer or a fraction.

(iii) If x is so small that its square and higher powers may be neglected, then approximate value of $(1 + x)^n = 1 + nx$.

4.2 General Term in the Expansion of $(1 + x)^n$

The $(r + 1)$th term in the expansion of $(1 + x)^n$ is given by

$$T_{r+1} = \frac{n(n-1)(n-2)..(n-r+1)}{r}!\, x^r$$

5. Some Important Deductions from $(1 + x)^n$

5.1. Replacing n by $-n$ in (1), we get,

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \ldots + (-1)^r\frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}x^r + \ldots$$

$$T_{r+1} = (-1)^r\frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}x^r$$

(General term).

5.2. Replacing x by $-x$ in (1), we get,

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + (-1)^r\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots$$

$$T_{r+1} = (-1)^r\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r$$

(General term)

5.3. Replacing x by $-x$ and n by $-n$ in (1), we get,

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots + \frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}x^r + \ldots$$

$$T_{r+1} = \frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}$$

(General term)

5.4 Sume Useful Expansions

(i) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots + (-1)^r x^r$

(ii) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots + x^r + \ldots$

(iii) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots + \ldots + (-1)^r(r+1)x^r + \ldots$

(iv) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots + (r+1)x^r +$

(v) $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \ldots + \frac{(r+2)(r+1)}{2}x^y + \ldots$

(vi) $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots + \frac{(r+2)(r+1)}{2}x^y + \ldots$

6. Hence the following table should be remembered for the general term

Binomial Expansion General Term T_{r+1}

$(1-x)^{-4}$ $\quad \frac{(r+1)(r+2)(r+3)}{3!}x^r$

$(1+x)^{-4}$ $\quad \frac{(r+1)(r+2)(r+3)}{3!}(-x)^r$

$(1-x)^{-5}$ $\quad \frac{(r+1)(r+2)(r+3)(r+4)}{4!}x^r$

$(1+x)^{-5}$ $\quad \frac{(r+1)(r+2)(r+3)(r+4)}{4!}(-x)^r$

7. If ${}^nC_{r-1}$, ${}^nC_{r0}$ and ${}^nC_{r+1}$ are in A.P., then
   (i) $n = 7$, $r = 2$ or 5 (ii) $n = 14$, $r = 5$ or 9
8. **Sum of Binomial Series**
   Terms of the given series are compared with the term of the following standard binomial series
   $(1+x)^n = 1 + nx + \frac{n(n-1)}{1\times 2}x^2 +$ ………………
   By this procedure we find values of x and n and hence sum of the series by substituting values of x and n in $(1+x)^n$ is obtained.
   Number of terms in expansion of $\left(x^\alpha + \frac{1}{x^\alpha} + 1\right)^n$ is $2n + 1$.

Note:

(i) $(x+1)(x+2) = x^2 + (1+2)x + 1.2$

(ii) $(x+1)(x+2)(x+3)$
$= x^3 + (1+2+3)x^2$
$+ (1.2 + 2.3 + 1.3)x + 1.2.3$

(iii) $(x+1)(x+2)(x+3)$ -------------
$(x+n) = x^n + (1+2+3+\ldots+n)x^{n-1}$
$+ \{(1.2 + 1.3 + \ldots + 1(n-1)\}x^{n-2}$
$+ \ldots\ldots\ldots + 1.2.3.$ -----------$(n-1)n$

(iv) $1.2 = \frac{1}{2}[(1+2)^2 - (1^2 + 2^2)]$

$1.2 + 2.3 + 1.3 = \frac{1}{2}[(1+2+3)^2 - (1^2 + 2^2 + 3^2)]$

9. **An Important Theorem**
   If $(\sqrt{A} + B)^n = I + f$ where I and n are positive integers, n being odd and $0 \le f < 1$, then show that $(I+f)f = k^n$ where $A - B^2 = k > 0$ and $\sqrt{A} - B < 1$

Proof: Given $\sqrt{A} - B < 1$

$\therefore\ 0 < (\sqrt{A} - B)^n < 1$

Now, let $(\sqrt{A} - B)^n = f'$ where $0 < f' < 1$

$\therefore\ I + f' - f = (\sqrt{A} - B)^n - (\sqrt{A} - B)^n$

Since, R.H.S. contains even powers of $\sqrt{A}$, hence R.H.S. is an integer.

Therefore, L.H.S. is also integer ($\because$ n is odd) but I is an integer, then $f - f'$ is also integer

$\Rightarrow f - f' = 0$

($\because -1 < f - f' < 1$)

or $f - f'$

$\therefore\ (I+f)f = (I+f)f'$

$= (\sqrt{A} + B)^n(\sqrt{A} - B)^n$

$= (A - B^2)^n = k^n$

Note: If n is even integer then

$(\sqrt{A} + B)^n + (\sqrt{A} - B)^n = I + f + f'$

Hence, L.H.S. and I are integers.

$\therefore\ f + f'$ is also integers

$\Rightarrow f + f' = 1 \qquad (\because 0 < f + f' < 2)$

$\therefore\ f' = (1 - f)$

Hence, $(I+f)(I-f) = (I+f)f'$

$= (\sqrt{A} + B)^n(\sqrt{A} - B)^n$

$= (A - B^2)^n$

$= k^n$

SOLVED SUBJECTIVE PROBLEMS: (CBSE/STATE BOARD): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Expand the following
   (i) $(1-2x)^{-1}$ up to 5 terms
   (ii) $(1-x^2)^{7/3}$ up to 4 terms

Solution

(i) $(1-2x)^{-1} = 1 + (2x) + (2x)^2 + (2x)^3 + (2x)^4 + \ldots\ldots = 1 + 2x + 4x^2 + 8x^3 + 16x^4$, (5terms).

(ii) $(1-x^2)^{7/3}$

$= 1 + \frac{7}{3}(-x^2) + \frac{\frac{7}{3}\left(\frac{7}{3}-1\right)}{2!}(-x^2)^2$

$+ \frac{\frac{7}{3}\left(\frac{7}{3}-1\right)\left(\frac{7}{3}-2\right)}{3!}(-x^2)^3 + \ldots$

$= 1 - \frac{7}{3}x^2 + \frac{14}{9}x^4 - \frac{14}{81}x^6$, (4term)

2. Prove that

$(1+x)^n = 2^n$

$$\left[1 - n\left(\frac{1-x}{1+x}\right) + \frac{n(n+1)}{1.2}\left(\frac{1-x}{1+x}\right)^2 + \ldots..\infty\right]$$

Solution

$$(1+x)^n = \frac{1}{(1+x)^{-n}} = \left(\frac{1}{2}\cdot\frac{2}{1+x}\right)^{-n}$$

$$= \left[\frac{1}{2}\cdot\left\{1 + \frac{1-x}{1+x}\right\}\right]^{-n} = 2^n\left\{1 + \frac{1-x}{1+x}\right\}^{-n}$$

$$= 2^n\left[1 - n\left(\frac{1-x}{1+x}\right) + \frac{n(n+1)}{1.2}\left(\frac{1-x}{1+x}\right)^2 + \ldots\infty\right]$$

Proved

3. If $y = \frac{2}{5} + \frac{1.3}{2!}\left(\frac{2}{5}\right)^2 + \frac{1.3.5}{3!}\left(\frac{2}{5}\right)^3 + \ldots\infty$, then prove that $y^2 + 2y - 4 = 0$.

Solution

$$y = \frac{2}{5} + \frac{1.3}{2!}\left(\frac{2}{5}\right)^2 + \frac{1.3.5}{3!}\left(\frac{2}{5}\right)^3 + \ldots\ldots$$

Let R.H.S. is expansion of $(1+x)^n$.

$$\therefore\ 1 + \frac{2}{5} + \frac{1.3}{2!}\left(\frac{2}{5}\right)^2 + \ldots.$$

$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots.$$

Comparing 2nd and 3rd terms in both the sides, $\frac{2}{5} = nx$ (1)

and $\frac{1.3}{2!}\left(\frac{2}{5}\right)^2 = \frac{n(n-1)}{2!}x^2$ (2)

Dividing Equation (2) by square of the Equation (1).

$$\frac{n(n-1)}{2!\,n^2} = \frac{1.3}{2!}\left(\frac{2}{5}\right)^2 \times \left(\frac{5}{2}\right)^2$$

$$\Rightarrow \frac{n-1}{n} = 3 \Rightarrow n - 1 = 3n$$

$$\Rightarrow -2n = 1 \Rightarrow n = \frac{-1}{2}$$

From Equation (1),

$$\frac{2}{5} = -\frac{1}{2} \times x \Rightarrow x = -\frac{4}{5}\ \therefore\ 1+y$$

$$= \left(1 - \frac{4}{5}\right)^{-\frac{1}{2}}$$

$$\Rightarrow 1 + y = \left(\frac{1}{5}\right)^{-\frac{1}{2}}$$

$$\Rightarrow (1+y)^2 = \left(\frac{1}{5}\right)^{-1} = 5$$

$1 + y^2 + 2y = 5 \quad \Rightarrow y^2 + 2y - 4 = 0.$

Proved

4. Find the value of $(0.98)^{-3}$ up to 2 decimal places.

Solution

$$(0.98)^{-3} = (1 - 0.02)^{-3}$$

$$= \left[1 + (-3).(-0.02) + \frac{(-3)(-4)}{2!}\cdot(-0.02)^2 + \ldots\right]$$

$$= (1 + 0.06 + 0.0024 + \ldots) = 1.0624.$$

UNSOLVED SUBJECTIVE PROBLEMS: (CBSE /STATE BOARD): TO GRASP THE TOPIC, SOLVE THESE PROBLEMS

Exercise-I

1. Find an approximation of $(0.99)^5$ using the first three terms of its expansion. **[NCERT]**

2. How many terms are there in the expansion of $[(3x + y^2)^9]^4$?

3. The first three terms in the expansion of a binomial are 1, 10, 40. Find the expansion.

4. The first three terms in the expansion of $(1 - ax)^n$, where x is a positive integer are $1 - 4x + 7x^2$. Find the values of a and n.

5. For what value of m, the coefficients of the $(2m + 1)$th and $(4m + 5)$th terms in the expansion of $(1 + x)^{10}$ are equal.

6. Find the greatest term in the expansion of $(1 + 4x)^8$, when $x = \frac{1}{3}$

Exercise-II

1. Expand $(2 + 3x)^{-5}$ upto four terms in (i) ascending powers of x (ii) descending powers of x. Mention the condition for validity of expansion in each case.
2. Mention the condition for validity of the expansion and expand $(2 - 3x)^{-3}$ as far as the term contains x^4.
3. Expand each of the following upto 4 terms
   (i) $(2 + 3x)^{5/3}$ (ii) $(1 - x^2)^{7/3}$.
   Mention the condition for validity of the expansion in each case.
4. Find the coefficient of x^6 in the expansion of $(1 - 2x)^{-5/2}$.
5. Prove that $(1 + x + x^2 + x^3 + \ldots)(1 - x + x^2 - x^3 \ldots) = (1 + x^2 + x^4 + x^6 \ldots)$.
6. Find the number of terms in the expansion of the following $(1 + 2x + x^2)^{20}$.
7. Find the number of terms in the expansion of the following.
   (i) $(1 + 5\sqrt{2}x)^9 + (1 - 5\sqrt{2}x)^9$
   (ii) $(2x + 3y - 4z)^n$
8. Find the coefficient of x in the expansion of $[\sqrt{1 + x^2} - x]^{-1}$. When $|x| < 1$.

ANSWERS

Exercise-I

1. 0.9510
2. 37
3. $(1 + 2)^5$
4. $a = 1/2, n = 8$
5. 1
6. $T_6 = \dfrac{57344}{243}$

Exercise-II

1. (i) $\dfrac{1}{32}\left[1 - \dfrac{15}{2}x + \dfrac{135}{4}x^2 - \dfrac{945}{8}x^3 + \ldots\right]$

   (ii) $\dfrac{1}{243}\left[\dfrac{1}{x^5} - \dfrac{10}{3}\cdot\dfrac{1}{x^6} + \dfrac{20}{3}\cdot\dfrac{1}{x^7} - \dfrac{280}{27}\cdot\dfrac{1}{x^8} + \ldots\right]$

2. $\left[\dfrac{1}{8} + \dfrac{9x}{16} + \dfrac{27x^2}{16} + \dfrac{135x^2}{32} + \dfrac{1215}{128}x^4 + \ldots\right]$
3. (i) $2^{5/3}\left[1 + \dfrac{5}{2}x + \dfrac{5}{4}x^2 - \dfrac{5}{24}x^3 + \ldots\right]$

   (ii) $\left[1 - \dfrac{7}{3}x^2 + \dfrac{14}{9}x^4 - \dfrac{14}{81}x^6 + \ldots\right]$
4. 15015/16
6. 41 terms
7. (i) 5 terms

   (ii) $\dfrac{(n+1)(n+2)}{2}$
8. Coefficient of x is 1

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. If x is so small that its square and higher powers may be neglected, prove that $\left(\dfrac{1+x}{1-x}\right) = 1 + 2x$

(a) $1 + 2x$ (b) $1 - 2x$
(c) $2 - 2x$ (d) $x - 2$

Solution

(a) $\dfrac{1+x}{1-x} = (1 + x)(1 - x)^{-1}$

But $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots$

$= 1 + x$ (neglecting higher powers of x)

$\therefore \dfrac{1+x}{1-x} = (1 + x)(1 + x) = 1 + 2x + x^2 = 1 + 2x$ [neglecting x^2]

2. If x is very near to 1, then the approximate $\dfrac{ax^b - bx^a}{x^b - x^a}$ value of is equal to

[AIEEE – 2002]

(a) $\frac{a+b}{1-x}$ (b) $\frac{1}{1-x}$

(c) $\frac{1}{1+x}$ (d) $\frac{a+b}{1+x}$

Solution

(b) Exp. $= \frac{a(1+h)^b - b(1+h)^a}{(1+h)^b - (1+h)^a}$,

where h is very small

$= \frac{a(1+bh) - b(1+ah)}{(1+bh) - (1+ah)}$,

neglecting higher powers of h

$= \frac{a-b}{-h(a-b)} = -\frac{1}{h} = \frac{1}{1-x}$ [$\because$ x = 1 + h]

3. If in the expansion in powers of x, the function $\frac{1}{(1-ax)(1-bx)}$ is $a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$, then a_n is equal to

[AIEEE – 2006]

(a) $\frac{a^n - b^n}{b-a}$ (b) $\frac{a^{n+1} - b^{n+1}}{b-a}$

(c) $\frac{b^{n+1} - a^{n+1}}{b-a}$ (d) $\frac{b^n - a^n}{b-a}$

Solution

(c) Exp. = $(1 - ax)^{-1}(1 - bx)^{-1}$
$= (1 + ax + a^2x^2 + \ldots + a^nx^n + \ldots)(1 + bx + b^2x^2 + \ldots + b^nx^n + \ldots)$
Now, a_n = Coefficient of $x^n = 1\,.\,b^n + ab^{n-1} + a^2b^{n-2} + \ldots + a^{n-1}b + a^n\,.\,1$ = Sum of a GP of $(n + 1)$ terms with $T_1 = b^n$, $r = a/b$

$= \frac{b^n[1-(a/b)^{n+1}]}{1-a/b} = \frac{b^{n+1} - a^{n+1}}{b-a}$

4. $\left(1 + \frac{1}{3}x + \frac{1.4}{3.6}x^2 + \frac{1.4.7}{3.6.9}x^3 + \ldots\right)^3$ equals

[PET (Raj.) – 1986]

(a) $(1 + x)^{-1}$ (b) $(1 + x)^{-2}$

(c) $(1 - x)^{-1}$ (d) $(1 - x)^{-2}$

Solution

(c) Exp. = $[(1 - x)^{-1/3}]^3 = (1 - x)^{-1}$

5. If x is so small that its second and higher powers can be neglected and if $(1 - 2x)^{-1/2}(1 - 4x)^{-5/2} = 1 + kx$, then k is equal to

[PET (Raj.) – 1993]

(a) 1 (b) – 2

(c) 10 (d) 11

Solution

(d) On expansion, we have $[1 - 1/2(-2x) + \ldots][1 - 5/2(-4x) + \ldots] = 1 + kx$

$\Rightarrow (1 + x)(1 + 10x) = 1 + kx$

$\Rightarrow 1 + 11x = 1 + kx \Rightarrow k = 11$

6. The coefficient of x^{n-2} in the expression $(x - a)(x - b)(x - c) \ldots\ldots (x - n)$($n$ factors) is equal to

[DCE – 1997]

(a) $\sum ab$ (b) $\sum a$

(c) $\sum a^2$ (d) $\sum a^2b^2$

Solution

(a) $(x - a)(x - b) \ldots\ldots (x - n) = x^n - (a + b + c + ..)\,x^{n-1} + (\sum ab)x^{n-2} + \ldots$

Therefore, coefficient of $x^{n-2} = \sum ab$

7. If $(a + bx)^{-3} = \frac{1}{8} + \frac{9}{16}x + \ldots\ldots\ldots$, then (a, b) is equal to

(a) (2, 3) (b) (2, – 3)

(c) (3, 2) (d) (– 3, 2)

[UPSEAT – 2002]

Solution

(b) $(a + bx)^{-3} = \frac{1}{a^3}\left(1 + \frac{b}{a}x\right)^{-3}$

$= \frac{1}{a^3}\left(1 - \frac{3b}{a}x + \ldots\right)$

Comparing it with given expansion, we have,

$\frac{1}{a^3} = \frac{1}{8}$ and $\frac{3b}{a^4} = \frac{9}{16} \Rightarrow a = 2, b = -3$

8. The coefficient of x^{100} in the expansion of $\sum_{j=0}^{200}(1+x)^j$ is

(a) $\binom{200}{100}$ (b) $\binom{201}{102}$

(c) $\binom{200}{101}$ (d) $\binom{201}{100}$

[DCE – 2005; MPPET – 2006]

Solution

(a) We know that $(1+x)^j = 1 + {}^jC_1x + {}^jC_2x^2 + {}^jC_3x^3 + \ldots\ldots + {}^jC_{100}x^{100} + {}^jC_{101}x^{101} + \ldots\ldots\ldots + {}^jC_{200}x^{200}$

Therefore, coefficient of x^{100} in the expansion of

$\sum_{j=0}^{200}(1+x)^j$ will be $\sum_{j=0}^{200} {}^jC_{100}$

$= [{}^{100}C_{100} + {}^{101}C_{100} + {}^{102}C_{100} + \ldots\ldots + {}^{200}C_{100}]$

$= \binom{200}{100}$

9. Coefficient of x^{19} in the polynomial $(x-1)(x-2)\ldots\ldots(x-20)$ is equal to

(a) 210 (b) – 210

(c) 20! (d) none

[MPPET – 2006]

Solution

(b) Given polynomial is

$(x-1)(x-2)(x-3)\ldots\ldots(x-19)(x-20)$
$= x^{20} - (1+2+3+\ldots\ldots+19+20)x^{19} + (1\times2+2\times3+\ldots\ldots+19\times20)x^{18} - \ldots\ldots + (1\times2\times3\times4\times\ldots\ldots\times19\times20)$

Therefore, coefficient of x^{19}

$= -(1+2+3+\ldots\ldots+19+20)$

$= -\left[\frac{20}{2}(1+20)\right] = -10\times21 = -210$

10. $\sum_{r=0}^{m} {}^{n+r}C_n$ is equal to

[UPSEAT – 2005; Pb. CET – 2003]

(a) ${}^{n+m+1}C_{n+1}$ (b) ${}^{n+m+2}C_n$

(c) ${}^{n+m+3}C_{n-1}$ (d) none of these

Solution

(a) $\sum_{r=0}^{m} {}^{n+r}C_n = {}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + \ldots\ldots + {}^{n+m}C_n$

$= ({}^{n+1}C_{n+1} + {}^{n+2}C_n) + ({}^{n+2}C_n + {}^{n+3}C_n + \ldots\ldots + {}^{n+m}C_n)$

$= ({}^{n+2}C_{n+1} + {}^{n+2}C_n) + ({}^{n+3}C_n + \ldots + {}^{n+m}C_n)$

$= {}^{n+3}C_{n+1} + {}^{n+3}C_n + \ldots\ldots + {}^{n+m}C_n$

…… … …… …….

$= {}^{n+m+1}C_{n+1}$

11. If $S_n = \sum_{r=0}^{m}\frac{1}{{}^nC_r}$ and $t_n = \sum_{r=0}^{n}\frac{1}{{}^nC_r}$ then $\frac{t_n}{s_n} =$

(a) $n-1$ (b) $\frac{1}{2}n-1$

(c) $\frac{1}{2}n$ (d) $\frac{2n-1}{2}$

Solution

$$s_n = \frac{1}{{}^nC_0} + \frac{1}{{}^nC_1} + \frac{1}{{}^nC_2} + \ldots + \frac{1}{{}^nC_n} \quad (1)$$

$$t_n = \frac{0}{{}^nC_0} + \frac{1}{{}^nC_1} + \frac{2}{{}^nC_2} + \frac{n}{{}^nC_n} \quad (2)$$

After reversing (2) we find

$$t_n = \frac{n}{{}^nC_n} + \frac{n-1}{{}^nC_{n-1}} + \frac{n-2}{{}^nC_{n-2}} + \ldots + \frac{0}{{}^nC_0} \quad (3)$$

Adding (2) and (3)

$$2t_n = n\left[\frac{1}{{}^nC_0} + \frac{1}{{}^nC_1} + \ldots + \frac{1}{{}^nC_n}\right] = n\cdot s_n$$

12. The remainder left out when $8^{2n} - (62)^{2n+1}$ is divided by 9 is

(a) 0 (b) 2

(c) 7 (d) 8

[AIEEE – 2009]

Solution

(b) $8^{2n} - (62)^{2n+1} = (1 + 63)^n - (63 - 1)^{2n+1}$

$(1 + 63)^n + (1 - 63)^{2n+1}$

$= (1 + {}^nC_1 63 + {}^nC_2(63)^2 + ... + (63)^n)$
$+ (1 - {}^{(2n+1)}C_1 63 + {}^{(2n+1)}C_2(63)^2$
$+ .... + (-1)(63)^{(2n+1)}$

$= 2 + 63({}^nC_1 + {}^nC_2(63).......$
$+(63)^{n-1} - {}^{(2n+1)}C_1$
$+{}^{(2n+1)}C_2(63) + ... -(63)^{(2n)}$

Therefore, remainder is 2.

13. The greatest integral value of $(\sqrt{2} + 1)^6$ is

(a) 195 (b) 197

(c) 199 (d) none

[UPSEAT – 1999; PET(Raj.) – 2000]

Solution

(b) $(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6 = 198$

$\Rightarrow (\sqrt{2} + 1)^6 = 198 - (\sqrt{2} - 1)^6$

But $\sqrt{2} - 1 = 0.41 \Rightarrow 0 < (\sqrt{2} - 1)^6 < 1$

Hence, the greater integral value of $(\sqrt{2} + 1)^6 = 197$.

14. The coefficient of x^{10} in the expansion of $(1 + x^2 - x^3)^8$ is

(a) 476 (b) 496

(c) 506 (d) 528

[UPSEAT – 1999]

Solution

(a) $(1 + x^2 - x^3)^8 = [1 + x^2 (1 - x)]^8$

$= 1 + {}^8C_1x^2 (1 - x) + {}^8C_2x^4 (1 - x)^2$
$+ {}^8C_3x^6(1 - x)^3 + {}^8C_4x^8(1 - x)^4$
$+ {}^8C_5x^{10} (1 - x)^5 + ...$

Obviously x^{10} occurs only in

${}^8C_4x^8 (1 - x)^4$ and ${}^8C_5x^{10} (1 - x)^5$.

Hence the coefficient of x^{10} in the given expansion

$= {}^8C_4$[coefficient of x^2 in $(1 - x)^4$] $+ {}^8C_5$ [1]

$= {}^8C_4({}^4C_2) + {}^8C_5 = (70)(6) + 56 = 476.$

15. If n is a positive integer, then integral part of $(3 + \sqrt{7})^n)$ is

(a) an even number (b) on odd number

(c) a prime number (d) none of these

[Bihar (CEE) – 2000]

Solution

Let $(3 + \sqrt{7})^n = I + F$ where I is integral part and F is fractional part.

Further, let $(3 - \sqrt{7})^n = f$

Obviously, $0 < f < 1$ ($\because 0 < (3 - \sqrt{7}) < 1$)

Now, $I + F = 3^n + {}^nC_1\, 3^{n-1}\sqrt{7}$
$+ {}^nC_2\, 3^{n-2}(\sqrt{7})^2 +$ (1)

$f = 3^n - {}^nC_1 3^{n-1}\sqrt{7} + {}^nC_2\, 3^{n-2}(\sqrt{7})^2 -$ (2)

$(1) + (2) \Rightarrow I + F + f$

$= 2\,[3^n + {}^nC_2\, 3^{n-2}(\sqrt{7})^2 + \ldots]$

$\Rightarrow I + F + f =$ an even number (3)

Further, $0 < F < 1$ and $0 < f < 1$

$\therefore\ 0 < F + f < 2$ (4) (4)

Now, (3) and (4) show that

$F + f = 1$ (5)

Also, (3), (5) $\Rightarrow I + 1 =$ an even integer

$\Rightarrow I =$ an odd integer.

16. The coefficient of x^5 in the expansion of $(1 + x^2)^5 (1 + x)^4$ is

(a) 50 (b) 60

(c) 55 (d) none of these

[EAMCET – 1996]

Solution

(b) $(1 + x^2)^5 (1 + x)^4 = (1 + 5x^2 + 10x^4 + ...)$
$(1 + 4x + 6x^2 + 4x^3 + x^4)$

$\therefore$ coefficient of $x^5 = 5(4) + 10(4) = 60.$

17. For a positive integer n,

[IIT – 1999]

Let $a(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{(2^n) - 1}$.

Then

(a) $a(100) \le 100$ (b) $a(100) > 100$

(c) $a(200) \le 100$ (d) $a(200) > 100$

Solution

(a, d) it can be proved with the help of mathematical induction that $n/2 > a(n) \le n$.

Therefore, $\frac{200}{2} < a(200)$

$\Rightarrow a(200) > 100$ and $a(100) \leq 100$

18. Let $P(n)$ denote the statement that $n^2 + n$ is odd. It is seen that $P(n) \Rightarrow P(n + 1)$, P_n is true for all.

[IITJEE – 1996]

(a) $n > 1$ (b) n

(c) $n > 2$ (d) none of these

Solution

(d) $P(n) = n^2 + n$. It is always odd (statement) but square of any odd number is always odd and also, sum of two odd number is always even. So, for no any 'n' for which this statement is true.

19. If p and q are approximately equal, $n > 1$ and $\frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q} = \left(\frac{p}{q}\right)^k$, then k is equal to

[Roorkee–1999]

(a) n (b) $1/n$

(c) $n + 1$ (d) $1/(n + 1)$

Solution

(b) Let $p = q + \lambda$ where λ is very small. Then, from given

relation $\frac{(n+1)(q+\lambda)+(n-1)q}{(n-1)(q+\lambda)+(n+1)q}$

$$= \left(\frac{q+\lambda}{q}\right)^k$$

$$\Rightarrow \frac{2nq+(n+1)\lambda}{2nq+(n-1)\lambda} = \left(1+\frac{\lambda}{q}\right)^k$$

$$\Rightarrow \frac{1+\left(\frac{n+1}{2n}\right)\frac{\lambda}{q}}{1+\left(\frac{n-1}{2n}\right)\frac{\lambda}{q}} = 1 + \frac{k\lambda}{q}$$ (neglecting higher powers of λ)

$$\Rightarrow \left[1+\left(\frac{n+1}{2n}\right)\frac{\lambda}{q}\right]\left[1+\left(\frac{n-1}{2n}\right)\frac{\lambda}{q}\right]^{-1} = 1+\frac{k\lambda}{q}$$

$$\Rightarrow \left[1+\left(\frac{n+1}{2n}\right)\frac{\lambda}{q}\right]\left[1-\left(\frac{n-1}{2n}\right)\frac{\lambda}{q}\right] = 1+\frac{k\lambda}{q}$$

Now, comparing coefficients of $\frac{\lambda}{q}$ on both sides, we get

$$\frac{n+1}{2n} - \frac{n-1}{2n} = k \quad \Rightarrow k = 1/n.$$

20. Let $R = (5\sqrt{5} + 11)^{2n+1}$ and $f = R - [R]$ where [ ] denotes the greatest integer function, then Rf is equal to

[IIT – 1988]

(a) 2^{2n+1} (b) 4^{2n+1}

(c) 2^{4n+1} (d) none

Solution

(b) Let I be the integral part of R, then

$R = I + f = (5\sqrt{5} + 11)^{2n+1}$, $0 < f < 1$

Also let $f' = (5\sqrt{5} - 11)^{2n+1}$. Then $0 < f' < 1$

$\therefore (I + f) - f' = (5\sqrt{5} + 11)^{2n+1} - (5\sqrt{5} - 11)^{2n+1}$

$= 2[T_2 + T_4 + T_6 + \ldots + T_{2n+2}]$

$= 2\,[{}^{2n+1}C_1(5\sqrt{5})^{2n}.11 + {}^{2n+1}C_3 (5\sqrt{5})^{2n-2}.11^3 + \ldots + (11)^{2n+1}]$

= An even integer

$\Rightarrow f - f'$ must also be an integer

$\Rightarrow f - f' = 0$ [$\because 0 < f < 1, 0 < f' < 1$]

$\Rightarrow f = f'$

$\therefore Rf = Rf' = (5\sqrt{5} + 11)^{2n+1}.(5\sqrt{5} - 11)^{2n+1}$

$= (125 - 121)^{2n+1} = 4^{2n+1}$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. If $(1 + ax)^n = 1 + 8x + 24x^2 + \ldots$, then value of a and n is

(a) 2, 4 (b) 2, 3
(c) 3, 6 (d) 1, 2

2. The sum of $1 + n\left(1 - \frac{1}{x}\right) + \frac{n(n+1)}{2!}\left(1 - \frac{1}{x}\right)^2 + \ldots \infty$ will be

[Roorkee – 1975]

(a) x^n (b) x^{-n}
(c) $\left(1 - \frac{1}{x}\right)^n$ (d) none of these

3. If number of terms in the expansion of $(x - 2y + 3z)^n$ are 45, then $n =$

(a) 7 (b) 8
(c) 9 (d) none of these

4. The coeffi cient of x in the expansion of $[\sqrt{1 + x^2} - x]^{-1}$ in ascending powers if x, when $|x| < 1$, is

[MPPET – 1996; AMU – 2002]

(a) 0 (b) 1/2
(c) −1/2 (d) 1

5. Middle term in the expansion of $(1 + 3x + 3x^2 + x^3)^6$ is

[MPPET – 1997]

(a) 4th (b) 3rd
(c) 10th (d) none of these

6. If $|x| < 1/2$, what is the value of

$$1 + n\left[\frac{x}{1-x}\right] + \left[\frac{n(n+1)}{2!}\right]\left[\frac{x}{1-x}\right]^2 + \ldots \infty ?$$

(a) $\left[\frac{1-x}{1-2x}\right]^n$ (b) $(1 - x)^n$
(c) $\left[\frac{1-2x}{1-x}\right]^n$ (d) $\left(\frac{1}{1-x}\right)^n$

7. If x is positive, the first negative term in the expansion of $(1 + x)^{27/5}$ is

[AIEEE – 2003]

(a) 7th (b) 5th (c) 8th (d) 6th

8. The sum of the series

$1 + \frac{1}{5} + \frac{1.3}{5.10} + \frac{1.3.5}{2.10.15} + \ldots$ is equal to

[Roorkee – 1998]

(a) $1/\sqrt{5}$ (b) $1/\sqrt{2}$
(c) $\sqrt{5/3}$ (d) $\sqrt{5}$

9. The fourth term in the expansion of $(1 - 2x)^{3/2}$ will be

(a) $-3/4\, x^4$ (b) $x^3/2$
(c) $-x^3/2$ (d) $3/4\, x^4$

10. In the expansion of $\left(\frac{1+x}{1-x}\right)^2$, the coefficient of x^n will be

(a) $4n$ (b) $4n - 3$
(c) $4n + 1$ (d) none of these

11. $1 + \frac{1}{3}x + \frac{1.4}{3.6}x^2 + \frac{1.4.7}{3.6.9}x^3 + \ldots =$

(a) n (b) $(1 + x)^{1/3}$
(c) $(1 - x)^{1/3}$ (d) $(1 - x)^{-1/3}$

12. To expand $(1 + 2x)^{-1/2}$ as an infinite series, the range of x should be

[AMU – 2002]

(a) $[-1/2, 1/2]$ (b) $(-1/2, 1/2)$
(c) $[-2, 2]$ (d) $(-2, 2)$

13. What is the coefficient of x^{10} in the expansion of
$(1 + x + x^2 + x^3 + x^4)^{-2}$?

(a) 1 (b) 2
(c) 3 (d) 10

14. If x is so small that x^3 and higher powers of x may be neglected, then,

$\frac{(1+x)^{3/2} - (1 + 1/2\,x)^3}{(1-x)^{1/2}}$ may be approximated as:

[AIEEE – 2005]

(a) $3x + 3/8x^2$
(b) $1 - 3/8x^2$
(c) $x/2 - 3/8x^2$
(d) $-3/8x^2$

15. $\left(\frac{a}{a+x}\right)^{\frac{1}{2}} + \left(\frac{a}{a-x}\right)^{\frac{1}{2}} =$

[AIEEE – 2002; DCE – 1994; HCET – 2002]

(a) $2+\frac{3x^2}{4a^2}+\ldots$

(b) $1+\frac{3x^2}{8a^2}+\ldots$

(c) $2+\frac{x}{a}+\frac{3x^2}{4a^2}+\ldots$

(d) $2-\frac{x}{a}+\frac{3x^2}{4a^2}+\ldots$

16. The coefficient of x in the expansion of **[Kerala PET – 2008]**
$(1+x)(1+2x)(1+3x)$ …….. $(1+100x)$ is

(a) 5050 (b) 10100
(c) 5151 (d) 4950

17. Coefficient of x^5 in $(1+2x+3x^2+\ldots)^{3/2}$ is **[Orissa JEE – 2008]**

(a) 19 (b) 20
(c) 21 (d) 22

SOLUTIONS

1. (a) $na=8 \Rightarrow n^2a^2=64, \frac{n(n-1)}{2}a^2=24$

$\Rightarrow \frac{n(n-1)}{2}a^2=24$

$\therefore \frac{2n}{n-1}=\frac{8}{3} \Rightarrow 6n=8n-8$

$\Rightarrow 2n=8 \Rightarrow n=4 \therefore a=2.$

2. (b) $1+n\left(1-\frac{1}{x}\right)+\frac{n(n+1)}{2\,!}\cdot\left(1-\frac{1}{x}\right)+\ldots$

$=\left[1-\left(1-\frac{1}{x}\right)\right]^{-n}=\left(\frac{1}{x}\right)^{-n}=x^n$

For $(1-x)^{-n}=1+nx+\frac{n(n+1)}{2\,!}x^2+\ldots\ldots$

3. (c) ${}^{n+3-1}C_n={}^{n+2}C_2=45$

$\Rightarrow (n+2)(n+1)=90$

$\Rightarrow n^2+3n-88=0$

$\Rightarrow n=-11, 8$

Therefore, maximum ${}^nC_r$ = maximum ${}^8C_r$ = ${}^8C_4=70$.

4. (d) $[\sqrt{1+x^2}-x]^{-1}=\frac{1}{\sqrt{1+x^2}-x}$ (After rationalization) $=(1+x^2)^{1/2}+x$

$=\left[1+\frac{1}{2}x^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2\,!}x^4+\ldots\right]+x$

Cofficient of x in this expansion is one.

5. (c) $(1+3x+3x^2+x^3)^6=[(1+x)^3]^6=(1+x)^{18}$

Total number of terms = 18 + 1 = 19.

The middle term is $\frac{19-1}{2}+1=10$th.

6. (a) We know that

$1+nx+\frac{n(n+1)}{2}x^2+\ldots\ldots=(1-x)^{-n}$

$\therefore 1+n\left(\frac{x}{1-x}\right)+\frac{n(n+1)}{2\,!}\left(\frac{x}{1-x}\right)^2+\ldots$

$=\left(1-\frac{x}{1-x}\right)^{-n}=\left(\frac{1-x}{1-2x}\right)^n.$

7. (c) T_{r+1}

$=\frac{\frac{27}{5}\cdot\frac{22}{5}\cdot\frac{17}{5}\cdots\left(\frac{27}{5}-r+1\right)}{r\,!}x^r$

$\therefore$ Least possible value of $r=7$

$\therefore$ First (negative) term is the 8th term.

8. (c) $S=1+\frac{1}{5}+\frac{1.3}{5.10}+\frac{1.3.5}{5.10.15}+\ldots\ldots$

$(1+x)^n=1+\frac{nx}{1\,!}+\frac{n(n-1)}{2\,!}x^2+\frac{n(n-1)(n-2)}{3\,!}x^3+\ldots$

$\Rightarrow nx=\frac{1}{5}$ and $\frac{n(n-1)x^2}{2\,!}=\frac{1.3}{5.10}$

$\Rightarrow n=-\frac{1}{2}$ and $x=\frac{-2}{5}$

$\therefore S=\left(1-\frac{2}{5}\right)^{-1/2}=\left(\frac{3}{5}\right)^{-1/2}=\sqrt{\frac{5}{3}}.$

9. (a) $T_{r+1} = {}^nC_r x^r$ for $(1+x)^n$

$$= \frac{n(n-1)(n-2)..........[n-(r-1)]\,x^r}{r\,!}$$

$$T_4 = T_{3+1} = \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}{3\,!}\cdot(-2x)^3$$

$$= \frac{x^3}{2} \text{ for } (1-2x)^{3/2}.$$

10. (1) $\left(\frac{1+x}{1-x}\right)^2 = (1+x)^2(1-x)^{-2}$

$= (1+2x+x^2)[1+2x+3x^2+4x^3+........+(n-1)\,x^{n-2}+nx^{n-1}+(n+1)x^n+..........]$

Coefficient of x^n is $(n+1)+2n+(n-1)=4n$.

11. (d) If the given series is identical with the expansion of $(1+y)^n$

i.e., with $1+ny+\frac{n(n-1)}{2\,!}y^2+........$, then

$$ny = \frac{1}{3}x \text{ and } \frac{n(n-1)}{2}y^2 = \frac{2}{9}x^2$$

$$\Rightarrow n = -\frac{1}{3}, y = -x$$

12. (b) $(1+2x)^{-1/2}$ can be expanded, if

$$|2x| < 1 \text{ i.e., } |x| < \frac{1}{2}$$

i.e., if $-\frac{1}{2} < x < \frac{1}{2}$ i.e., if $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$.

13. (c) $(1+x+x^2+x^3+x^4)^{-2} =$

$$= \left(\frac{1-x^5}{1-x}\right) = (1-x)^2\,(1-x^5)^{-2}$$

$$= (1+x^2-2x)(1+2x^5+3x^{10}+......) = 3.$$

14. (d) $\dfrac{(1+x)^{3/2}-\left(1+\frac{1}{2}x\right)^3}{(1-x)^{1/2}}$

$$1+\frac{3}{2}x+\frac{3/2.1/2}{2}x^2$$

$$= \frac{-\left(1+\frac{3}{2}x+\frac{3.2}{2}\cdot\frac{x^2}{4}\right)}{(1-x)^{1/2}}$$

$$= \frac{-\frac{3}{8}x^2}{(1-x)^{1/2}} = -\frac{3}{8}x^2\,(1-x)^{-1/2}$$

$$= \frac{-3}{8}x^2\left(1+\frac{x}{a}+.....\right) = \frac{-3}{8}x^2.$$

15. (a) $\left(\frac{a}{a+x}\right)^{1/2} + \left(\frac{a}{a-x}\right)^{1/2}$

$$= \left(1+\frac{x}{a}\right)^{-1/2} + \left(1-\frac{x}{a}\right)^{-1/2}$$

$$= \left(1-\frac{1}{2}\cdot\frac{x}{a}+\frac{(-1/2).(-3/2)}{2}\left(-\frac{x}{a}\right)^2......\right)$$

$$+\left(1+\frac{1}{2}\cdot\frac{x}{a}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(-\frac{x}{a}\right)^2\right)$$

[$\because$ x is small so that $\frac{x^3}{a^3}$ etc., can be neglected]

$$= 1+\frac{3}{8}\frac{x^2}{a^2}+1+\frac{3}{8}\frac{x^2}{a^2} = 2\left[1+\frac{3x^2}{8a^2}\right]$$

$$= 2+\frac{3x^2}{4a^2}$$

16. (a) Required coefficient $= 1+2+3+.....+100$ ($\because$ The given product $= 1+x\,(1+2+3+...+100)+.....$)

$$= \frac{100(100+1)}{2} = 5050.$$

17. (c) $(1+2x+3x^2+...)^{3/2}$
$= \{(1-x)^{-2}\}^{3/2} = (1-x)^{-3}$
Coefficient of x^5 in $(1-x)^{-3}$ is

$${}^{3+5-1}C_5 = {}^7C_5 = {}^7C_2 = \frac{7\times 6}{2\,!} = 21$$

($\because$ Coefficient of x^r in $(1-x)^{-k}$ is ${}^{k+r-1}C_r$, where k is any positive integer).

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. If $|x| < 1$, then in the expansion of $(1 + 2x + 3x^2 + 4x^3 + \ldots\ldots)^{1/2}$, the coefficient of x^n is

(a) n (b) $n + 1$
(c) 1 (d) -1

2. $1 + \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.12} + \ldots \infty =$

[RPET – 1996; EAMCET – 2001]

(a) $\sqrt{2}$ (b) $1/\sqrt{2}$
(c) $\sqrt{3}$ (d) $1/\sqrt{3}$

3. The expansion of $\frac{1}{(4-3x^2)^{1/2}}$ by binomial theorem will be valid if

(a) $x < 1$ (b) $|x| < 1$
(c) $-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$ (d) none of these

4. If $(r + 1)$th term is the first negative term in the expansion of $(1 + x)^{7/2}$, then the value of r is

(a) 5 (b) 6
(c) 4 (d) 7

5. The coefficient of x^n in the expansion of $(1 - 2x + 3x^2 - 4x^3 + \ldots.)^{-n}$ is

(a) $\frac{(2n)!}{n!}$

(b) $\frac{(2n)!}{(n!)^2}$

(c) $\frac{1}{2}\frac{(2n)!}{(n!)^2}$

(d) none of these

6. The coefficient of x^n in the expansion of $(1 + x + x^2 + \ldots.)^{-n}$ is

(a) 1 (b) $(-1)^n$
(c) n (d) $n + 1$

7. If the third term in the binomial expansion of $(1 + x)^m$ is $-\frac{1}{8}x^2$, then the rational value of m is

(a) 2 (b) 1/2
(c) 3 (d) 4

8. How many terms are there in the expansion of $(x + y + z)^{10}$?

(a) 11 (b) 33
(c) 66 (d) 310

[MPPET – 2005; NDA – 2004]

9. If $|x| > 1$, then $(1 - x)^{-2} =$

(a) $1 - 2x + 3x^2 - \ldots$
(b) $1 + 2x + 3x^2 + \ldots$
(c) $1 - 2/x + 3/x^2 - \ldots$
(d) $1/x^2 + 2/x^3 + 3/x^4 - \ldots$

[MNR – 1981; AMU – 1983; JMI EEE – 2001]

10. The coefficient of x^{14} in the expansion of $(1 + x + x^2 + x^3)^6$ is

[Kerala PET – 2007]

(a) 130 (b) 120
(c) 128 (d) 125

11. If $(a + bx)^{-2} = \frac{1}{4} - 3x + \ldots.$, then $(a, b) =$

[UPSEAT – 2002]

(a) (2, 12) (b) (– 2, 12)
(c) (2, – 12) (d) none of these

12. If $|x| < 1$, then the value of

$$1 + n\left(\frac{2x}{1+x}\right) + \frac{n(n+1)}{2!}\left(\frac{2x}{1+x}\right)^2 + \ldots \infty$$

will be

(a) $\left(\frac{1+x}{1-x}\right)^n$ (b) $\left(\frac{2x}{1-x}\right)^n$

(c) $\left(\frac{1+x}{2x}\right)^n$ (d) $\left(\frac{1-x}{1+x}\right)^n$

[AMU – 1983]

WORK SHEET: TO CHECK PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the work sheet
2. The test is of 08 minutes.
3. The test consists of 08 questions. The maximum marks are 24.
4. Use blue/black ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. The expansion of $(9 - 4x^2)^{1/2}$ is valid only when

(a) $-1 < x < 1$

(b) $-2 < x < 2$

(c) $-\frac{3}{2} < x < \frac{3}{2}$

(d) none of these

2. The expansion of $(8 - 3x)^{3/2}$ in terms of powers of x is valid only if

(a) $x < \frac{3}{8}$ (b) $x < \frac{8}{3}$

(c) $x > \frac{8}{3}$ (d) $|x| < \frac{8}{3}$

3. The number of dissimilar terms in the expansion of $(x + y)^n$ is $n + 1$. Therefore, number of dissimilar terms in the expansion of $(x + y + z)^{12}$ is

(a) 13 (b) 39 (c) 78 (d) 91

4. The co-efficient of x^4 in the expansion of $\frac{(1-3x)^2}{(1-2x)}$ is

(a) 1 (b) 2

(c) 3 (d) 4

5. The fi rst three terms in the expansion of $(4 + x)^{3/2}$ are

(a) $4 + \frac{3x}{2} + \frac{3x^2}{8}$ (b) $1 + \frac{3x}{8} + \frac{3x^2}{8^2}$

(c) $8 + 3x + \frac{3x^2}{16}$ (d) none of these

6. What is the coefficient of x^5 in the expansion $(1 - 2x + 3x^2 - 4x^3 + \ldots\ldots\ldots\ldots\infty)^{-5}$?

(a) (10!)/(5!)2 (b) 5 – 5

(c) 55 (d) (10!)/{(6!)(4!)}

[NDA – 2007]

7. The expansion of $(1 - 4x)^{-1/2}$ is valid

(a) $|x| < 1$ (b) $|x - 1| > 1$

(c) $|x| < 1/4$ (d) $|x - 1| > 1/4$

8. If $|x| < 1$, then the coefficient of x^n in the expansion of $(1 + x + x^2 + \ldots)^2$ will be

(a) 1 (b) n

(c) $n + 1$ (d) none of these

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ 4. ⓐ ⓑ ⓒ ⓓ 7. ⓐ ⓑ ⓒ ⓓ

2. ⓐ ⓑ ⓒ ⓓ 5. ⓐ ⓑ ⓒ ⓓ 8. ⓐ ⓑ ⓒ ⓓ

3. ⓐ ⓑ ⓒ ⓓ 6. ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

1. $(9 - 4x^2)^{1/2} = 3\left(1 - \frac{4x^2}{9}\right)^{1/2}$

is valid when $-1 < \frac{4x^2}{9} < 1$

$\Rightarrow \frac{-3}{2} < x < \frac{3}{2}$

3. Number of dissimilar term

$= {}^{12+3-1}C_{3-1} = {}^{14}C_2 = 91$

4. $(1 - 3x)^2 (1 - 2x)^{-1}$

$(1 + 9x^2 - 6x)(1 + 2x + 4x^2 + ...)$

Coefficient of $x^4 = 2^4 + 9.2^2 - 6.2^3 = 4$

8. $(1 + x + x^2 + ...)^2 = [(1 - x)^{-1}]^2$

$= (1 - x)^{-2} = (1 + 2x + 3x^2 + ...)$

$\therefore$ Coefficient oif $x^n = n + 1$

This page is intentionally left blank.

LECTURE

5 Test Your Skills

MENTAL PREPARATION TEST

1. Expand $(1 + x + x^2)^4$ to powers of x.

2. Using binomial theorem evaluate $(103)^5$.0

3. Prove that $(101)^{50} > (99)^{50} + (100)^{50}$.

4. Find the remainder when 5^{99} is divided by 13.

5. Prove that integral part of $(8 + 3\sqrt{7})\ n$ is an odd integer.

6. If n is a positive integer, prove that $33n - 26n - 1$ is divisible by 676.

7. First three terms in the expansion of $(x + a)n$ are respectively 64, 576 and 2160. Find x, a and n.

8. If the 6th, 7th and 8th terms in the expansion of $(x + a)^n$ are 112, 7 and $\frac{1}{4}$, respectively; find x, a and n.

9. If 3rd, 4th, 5th and 6th terms in the expansion of $(x + y)^n$ are a, b, c and d, respectively, prove that $\frac{b^2 - ac}{c^2 - bd} = \frac{4a}{3c}$

10. Find coefficient of x^3 in the expansion of $(1 + 2x)^6 (1 - x)^7$.

11. Write the 5th term in the expansion of $\left(2x^2 - \frac{1}{3x^3}\right)^{10}, x \neq 0$.

12. Find the coefficient of $x^6 y^3$ in the expansion of $(x + 2y)^9$.

13. Find the middle term in $(1 - 2x + x^2)^n$.

14. If three successive coefficients in the expansion of $(1 + x)^n$ are 220, 495 and 792, then find n.

15. Determine the value of x in the expansion $\left(x + x^{\log_{10} x}\right)$, if the third term in the expansion is 1, 000, 000.

16. Write down the general term in the expansion of $(x^3 - y^3)^6$.

17. Find $(n+ 1)$th term from the end in the expansion of $\left(x - \frac{1}{x}\right)^{3n}$.

18. Find the coefficient of x^3 in the expansion of $\left(\frac{x}{8} + y^2\right)^5$.

19. Find the coefficient of x10 in the expansion of $(x^2 - 2)^{11}$.

20. Find n and x, if in the expansion of $(1 + x)^n$, fifth term is 4 times the fourth term and the fourth term is 6 times the third term.

21. Find the coefficient of x^4 in $(1 + x)^n (1 - x)^n$ and, show that $C_0 C_4 - C_1 C_3 + C_2 C_2 - C_3 C_1 + C_4 C_0 = C_2$.

22. Compute $^{18}C_2 + {}^{18}C_4 + {}^{18}C_6 + \ldots\ldots\ldots + {}^{18}C_{18}$.

23. Evaluate $\sum_{r=0}^{n} {}^nC_r 3^{n-r}$.

24. If the coefficients of pth and qth terms in the expansion of $(1 + x)^n$ are equal, prove that $p + q = n + 2, p \neq q$.

If $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \ldots\ldots\ldots + C_n x^n$, then prove that

25. $C_0 C_n + C_1 C_{n-1} + C_2 C_{n-2} + \ldots\ldots + C_n C_0 = C(2n, n)$.

26. $C_0 C_1 + C_1 C_2 + C_2 C_3 + \ldots\ldots\ldots\ldots + C_{n-1} C_n = \dfrac{2n\,!}{(n-1)\,!\,(n-1)\,!}$

27. Expand $\left(ax - \frac{b}{x}\right)^6$.

28. Compute $(11)^5$ by using binomial theorem.

29. Simplify the 7th term of $(a + 2x)^{11}$.

30. Simplify the 5th term from the last $\left(2x^3 - \dfrac{1}{2x^3}\right)^8$.

31. Find out the middle term/terms in the expansion of $\left(1 - \frac{x^2}{2}\right)^{14}$

32. Find the coefficient of x^{18} in the expansion of $\left(x^2 - \frac{3a}{x}\right)^{15}$

33. Find the coefficient of $\frac{1}{x^3}$ in $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$

34. Find the greatest term in the expansion of the $(x - 4a)^8$, if $x = \frac{1}{2}, a = \frac{1}{3}$.

35. Find the greatest term in the expansion of the (5x + 4)7, if $x = 1$.

36. Find the maximum term in the expansion of $(2 + 5x)^6$ for $x = \frac{2}{3}$.

37. Find the value of n if the coefficients of three consecutive terms in the expansion of $(1 + x)^n + x$ are 6, 15, 20.

38. Find the value of$^{13}C_2 + {}^{13}C_3 + \ldots\ldots\ldots + {}^{13}C_{13}$.

39. If $C_0, C_1, C_2, \ldots\ldots\ldots, C_n$ denote the coefficients in the expansion of $(1 + x)^n$, prove that $C_2 + 2C_3 + 3C_4 + \ldots\ldots + (n - 1)C_n = (n - 2)$ $2n - 1 + 1$

40. Expand $(2 - 3x^2)^{-4/5}$

41. Find the coefficient of xr in the expansion of $(1 - 4x)^{-1/2}$.

42. Show that the coefficient of xn in the expansion of $(1 + x)^2 (1 - x)^2$ is $4n$.

43. Evalute $\sqrt[3]{999}$ to 3 decimal places.

44. Simplify the fraction

$$\frac{(1 + x)^{1/2} + (1 - x)^{5/3} + (1 - 2x)^{1/4}}{(1 + 3x)^{1/6} + (1 + 5x)^{1/10}}, x < 1.$$

45. Compute (1.003)1/10 correct upto 5 places of decimals.

46. If square and higher powers of x may be neglected, show that

$$\frac{(1 - 3x)^{1/2} + (1 - x)^{5/3}}{(4 - x)^{1/2}} = -\frac{35x}{24}$$

47. If the coefficients of 3rd and 4th terms in the expansion of $\left(x - \frac{1}{2x}\right)$ are in the ratio 1 : 2, find the value of n.

48. If the sum of odd terms in the expansion of $(1 + x)^n$ is A and sum of even terms is B, prove that $A^2 - B^2 = (1 - x^2)^n$.

49. Find the coefficient of xn in the expansion of $\dfrac{(1 + x)^2}{(1 - x)^3}$. Also, find the coefficient of x^5 and x^7.

ASSERTION/REASONING

Assertion and Reasoning Type Questions

Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.

(a) **Assertion** is True, **Reason** is True and **Reason** is a correct explanation for **Assertion.**

(b) **Assertion** is True, **Reason** is True and **Reason** is NOT a correct explanation for **Assertion.**

(c) **Assertion** is True and **Reason** is False.

(d) **Assertion** is False and **Reason** is True.

1. **Assertion (A):** Greatest coefficient in the expansion of $(1 + 5x)^8$ is ${}^8C_4 5^4$.

   Reason (R): Greatest coefficient in the expansion of $(1 + x)^{2n}$ is the middle term.

2. **Assertion (A):** Number of the dissimilar terms in the sum of expansion $(x + a)^{102} + (x - a)^{102}$ is 206.

   Reason (R): Number of terms in the expansion of $(x + b)^n$ is $n + 1$.

3. **Assertion (A):** The term independent of x in the expansion of $\left(x + \frac{1}{x} + 2\right)^{21}$ is ${}^{42}C_{21}$.

   Reason (R): In a binomial expansion, middle term is independent of x.

4. **Assertion (A):** The sum of the last ten coefficients in the expansion of $(1 + x)^{19}$, when expanded in ascending powers of x is 2^{18}.

   Reason (R): ${}^nC_r = {}^nC_{n-r}\left(r > \frac{n}{2}\right)n \ \forall \ N$ and $r \in$ whole number.

5. **Assertion (A):** The third term in the expansion of $\left(2x + \frac{1}{x^2}\right)^m$ does not contain x. The value of x for which that term equals to the second term in the expansion of $(1 + x^3)^{30}$ is 4.

   Reason (R): $(a + x)n = \sum_{r=0}^{n} {}^nC_r a^{n-r} x^r$

6. **Assertion (A):** In the expansion of $(x + x - 2)^n$, the coefficient of eighth term and nineteenth term are equal, then $n = 25$.

   Reason (R): Middle term in the expansion of $(x + a)^n$ has greatest coefficient.

7. **Assertion (A):** The number of terms in the expansion of $\left(x + \frac{1}{x} + 1\right)^n$ is $2n + 1$.

   Reason (R): The number of terms in the expansion of $(a_1 + a_2 + a_3 + \ldots\ldots + a_m)^n$ is $n + {}^{m-1}C_{m-1}$

8. **Assertion (A):** If $\sum_{r=1}^{n} r^3 \left(\frac{{}^nC_r}{{}^nC_{r-1}}\right)^2$ 196, then the sum of the coefficients of power x in the expansion of the polynomial $(x - 3x^2 + x^3)^n$ is -1.

   Reason (R): $\frac{{}^nC_r}{{}^nC_r} = \left(\frac{n - r + 1}{r}\right) \forall n \in \mathrm{N}$ amd $r \in$ whole number.

9. **Assertion:** $\sum_{r=0}^{n} (r + 1)^n C_r = (n + 2)2^{n-1}$

   Reason: $\sum_{r=0}^{n} (r + 1)\, {}^nC_r x^r = (1 + x)^n + nx\,(1 + x)^{n-1}$

   [AIEEE – 2008]

ASSERTION/REASONING: SOLUTIONS

1. (d) Greatest coefficient in the expansion of $(1 + 5x)^8$ is ${}^8C_4$.

2. (d) Therefore, $(x + a)^{102} + (x - a)^{102} = 2$ $\{x^{102} + {}^{102}C_2 x^{100}.\ a^2 + {}^{102}C_4 x^{98}.\ a^4 + \ldots\ldots\ldots + {}^{102}C_{102} x 0\, a^{102}\}$

   Therefore, number of terms = 52.

3. (c) Since, $\left(x+\frac{1}{x}+2\right)^{21}=\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^{42}$

$T_{r+1}={}^{42}C_r(\sqrt{x})^{42-r}.\left(\frac{1}{\sqrt{x}}\right)={}^{42}C_r\cdot(x)^{21-r}$

For independent of x, $21-r=0$

$\Rightarrow r=21$

Since, $T_{21+1}={}^{42}C_{21}$ (which is independent of x) also it is a middle term.

Also, here $\sqrt{x}\times\frac{1}{\sqrt{x}}=1$.

In a binomial expansion $(x+a)^n$ (say) middle term is independent of x which is possible only when $x.\,a=1$.

4. (a) Since, $(1+x)^{19}={}^{19}C_0+{}^{19}C_1x+{}^{19}C_2+\ldots\ldots+{}^{19}C_{10}x^{10}+{}^{19}C_{11}x^{11}+\ldots\ldots.+{}^{19}C_{18}x^{18}+{}^{19}C_{19}x^{19}$

Put $x=1$, then

$2^{19}={}^{19}C_0+{}^{19}C_1+{}^{19}C_2+\ldots\ldots.+{}^{19}C_9+{}^{19}C_{10}+{}^{19}C_{11}+\ldots+{}^{19}C_{19}$

$=({}^{19}C_{19}+{}^{19}C_{18}+{}^{19}C_{17}+\ldots\ldots\ldots+{}^{19}C_{10})+({}^{19}C_{10}+{}^{19}C_{11}+\ldots\ldots..+{}^{19}C_{19})$

$=2({}^{19}C_{10}+{}^{19}C_{11}+\ldots\ldots..+{}^{19}C_{19})$

Therefore, ${}^{19}C_{10}+{}^{19}C_{11}+\ldots\ldots..+{}^{19}C_{19}=2^{18}$

5. (d) ${}^{m}C_2(2x)^{m-2}.\left(\frac{1}{x^2}\right)^2={}^{m}C_2(2)^{m-2}x^{m-6}$.

Since, the third term in the expansion of $\left(2x+\frac{1}{x^2}\right)^m$ does not contain x.

and, $m-6=0\Rightarrow m=6$

Therefore, $T_3={}^{6}C_2\,(2)^6-2=15\times16=240$

According to the question, ${}^{30}C_1x^3=240$ (given)

$x^3=8$

Therefore, $x=2$

6. (b) According to the Assertion,

${}^{n}C_7={}^{n}C_{18}\;n=7+18=25$

Reason is always true.

7. (b) Given expression $=\left\{1+\left(x+\frac{1}{x}\right)\right\}^n$

$=1+{}^{n}C_1\left(x+\frac{1}{x}\right)^2+{}^{n}C_2+{}^{n}C_3\left(x+\frac{1}{x}\right)^3+\ldots\ldots..+\left(x+\frac{1}{x}\right)^n$

This will be of the form

$=a_0+a_1x+a_2x^2+\ldots.+a_nx^n+\frac{b_1}{x}+\frac{b_2}{x^2}+\frac{b_3}{x^3}+\frac{b_n}{x^n}$

Therefore, number of terms $=1+n+n=2n+1$

8. (d) $\sum_{r=1}^{n}r3\left(\frac{n-r+1}{r}\right)^2=\sum_{r=1}^{n}r\,(n-r+1)^2$

$=\sum_{r=1}^{n}\{(n+1)^2\,(n+1)\,r+r^2\}$

$=(n+1)^2\sum n-2\,(n+1)\sum n^2+\sum n^2$

$=(n+1)^2\,n-2(n+1)n^2+n^3$

$=\frac{(n+1)^2.n.\,(n+2)}{12}=14^2$ (given)

$=7^2\times2^3$

$=\frac{72.6.8}{12}$

$\because\quad n=6$

Then, $(x-\;3x^2+x^3)^6$

Sum of coefficients $=(1-3+1)^6$

$=(-1)^6$

$=1$

9. (a) $\sum_{r=0}^{n}(r+1)\,{}^{n}C_r=\sum_{r=0}^{n}r.{}^{n}C_r=+\sum_{r=0}^{n}{}^{n}C_r$

$=\sum_{r=0}^{n}r.\frac{n}{r}.{}^{n-1}C_{r-1}+\sum_{r=0}^{n}{}^{n}C_r$

$=n.\,2^{n-1}+2^n=2^n-1(n+2)$

Thus, **Assertion** is true.

Again $\sum_{r=0}^{n}(r+1)\,{}^{n}C_rx^r+\sum_{r=0}^{n}r\,{}^{n}C_rx^r+\sum{}^{n}C_rx^r$

$=\sum_{r=0}^{n}{}^{n-1}C_{r-1}x^r=\sum_{r=0}^{n}{}^{n}C_rx^r$

$=nx\,(1+x)^{n-1}+1+x)^n$

Substitute $x=1$ in the above identity to get

$\sum(r+1)\,{}^{n}C_r=n.2n-1+2^n$

Reason is also true and explains **Assertion** also.

TOPICWISE WARMUP TEST

1. The coefficient of the middle term in the binomial expansion in powers of x of $(1 + ax)^4$ and of $(1 - ax)^6$ is the same if α equals

(a) $\frac{3}{5}$ (b) $\frac{10}{3}$ (c) $\frac{-3}{10}$ (d) $\frac{-3}{10}$

[AIEEE – 2004]

2. The number of integral terms in the expansion of $(\sqrt{3} + \sqrt[8]{5})^{265}$ is

(a) 32 (b) 33 (c) 34 (d) 35

[AIEEE – 2003]

3. The positive integer just greater than $(1 + 0.0001)^{10000}$ is

(a) 4 (b) 5 (c) 2 (d) 3

[AIEEE – 2002]

4. If p and q be positive, then the coefficients of x^p and x^q in the expansion of $(1 + x)^{p+q}$ will be

(a) equal

(b) equal in magnitude but opposite in sign

(c) reciprocal to each other

(d) none of these

[AIEEE – 2002]

5. The sum of the coefficients in the expansion of $(x + y)n$ is 4096. The greatest coefficient in the expansion is

(a) 1024 (b) 924 (c) 824 (d) 724

[AIEEE – 2002]

6. If ${}^{n-1}C_r = (k^2 - 3)nC_{r+1}$, then $k \in$

(a) $(-, \infty -2]$ (b) $[2, \infty)$

(c) ${}^{12}C_6$ (d) ${}^{12}C_7$

[IIT – 2004]

7. The coefficient of t^{24} in the expansion of $(1 + t2)^{12} (1 + t^{12}) (1 + t^{24})$ is

(a) ${}^{12}C_6 + 2$ (b) ${}^{12}C_5$

(c) ${}^{12}C_6$ (d) ${}^{12}C_7$

[IIT – 2003]

8. ${}^{20}C_4 + C_3 + {}^{20}C_2 \, {}^{-22}C_{18}$ is equal to

(a) 0 (b) 1234

(c) 7315 (d) 6345

[MPPET – 2005]

9. If x = [729 + 6 (2) (243) + 15 (4) (81) 20 (8) × (27) + 15 (16) (9) + 6 (32) 3 64] /[1 + 4 (4) 6 (16) 4 (64) + 256]

then $\sqrt{x} - \frac{1}{\sqrt{x}}$ is equal to

10. If n is a natural number, then $4n - 3n - 1$ is divisible by which one of the following?

(a) 2 (b) 9

(c) 18 (d) 27

[NDA – 2005]

11. How many terms are there in the expansion of $(4x + 7y)^{10} + (4x - 7y)^{10}$?

(a) 5 (b) 6

(c) 11 (d) 22

[NDA – 2005]

12. What are the values of k if the term independent of x in thevexpansion of

$\left(\sqrt{x} + \frac{k}{x^2}\right)^{10}$ is 405?

(a) ±3 (b) ±6

(c) ±5 (d) ±4

[NDA – 2004]

13. If in the binomial expansion of $(1 + x)^n$ where n is a natural number, the coefficients of the 5th, 6th and 7th terms are in A.P., then n is equal to

(a) 7 or 13 (b) 7 or 14

(c) 7 or 15 (d) 7 or 17

[NDA – 2003]

14. The value of ${}^nC_0 - {}^nC_1 + {}^nC_3 + \ldots + -1 \, {}^nC(-1)$ ${}_nC_n =$

(a) 0 (b) $2^n - 1$

(c) 2^n (d) 2^{n-1}

[NDA – 2002]

15. If x is so small that its square and higher powers may be neglected, then $\left(\frac{1-x}{1+x}\right)^{1/2}$ is approximately equal to

(a) $1-x$ (b) $1+x$

(c) $2-x$ (d) $1-\frac{1}{2}x$

[NDA – 2002]

16. The last digit, that is, the digit in the units place of thev number (67)25 – 1 is

(a) 6 (b) 8

(c) 0 (d) none of these

[NDA – 2000]

17. The sum of coefficients of the expansion $\left(\frac{1}{x}+2x\right)^n$ is 6561. The coefficient of term independent of x is

(a) 16 (b) 8

(c) ${}^8C_5$ (d) none of these

[DCE – 2006]

18. If the second term in the expansion is $\left[{}^{13}\sqrt{a} = \frac{a}{\sqrt{a^{-1}}}\right]$ then the value of $\frac{{}^nC_3}{{}^nC_2}$ is

(a) 4 (b) 3

(c) 12 (d) 6 **[DCE – 2006]**

19. If C_0, C_1, C_2, ... C_n denote the oefficients of the binomial expansion $(1+x)^n$, then the value of $C_1 + 3C_3 + 5C_5 + \ldots +$ is

(a) $n2^{n-2}$ (b) $n2^{n-1}$

(c) $(n+1)2^n$ (d) $(n+2)2^{n-1}$

[DCE – 2004]

20. In the expansion of $(1+x)^{30}$, the sum of the coefficients of odd powers of x is

(a) 230 (b) 231

(c) 0 (d) 229

[DCE – 2004]

21. The term independent of x in $[(\sqrt{x/3}) + \sqrt{3}/x^2]$ is

(a) 5/3 (b) 4/5

(c) 6 (d) 1/2 **[DCE – 2004]**

22. Coefficient of x^6 in the expansion $\left(x+\frac{1}{x^2}\right)^6$ is equal to

(a) 10 (b) 15

(c) 16 (d) none of these

[DCE – 2003]

23. If $(1+x)^n = C + C_1 C_1x + C_2x^2 + C_nx^n$, then the value of $C_0 + \frac{1}{2}C_1 + \frac{1}{3}C_2 + \ldots + \frac{1}{(n+1)}$ C_n is

(a) $\frac{2^{n-1}}{(n+1)}$ (b) $\frac{2^{n+1}}{(n+1)}$

(c) $\frac{2^{n-1}1}{(n+1)}$ (d) $\frac{2^{n+1}-1}{(n+1)}$

[DCE – 2002]

24. If the coefficient of x^7 and x^8 in $\left(2+\frac{x}{3}\right)^n$ are equal, then n is

(a) 56 (b) 55 (c) 45 (d) 15

[DCE – 2000]

25. The 9th term of the expansion $\left(3x-\frac{1}{2x}\right)^n$ is

(a) $\frac{1}{512x^9}$ (b) $\frac{-1}{512x^9}$

(c) $\frac{-1}{256.x^8}$ (d) $\frac{1}{256.x^8}$

[Karnataka CET – 2007]

TOPICWISE WARMUP TEST: SOLUTION

1. (c) Middle term in expansion of $(1+a\alpha)^4 = {}^4C_2\,(a\,x)^2$

Middle term in expansion of $(1-\alpha x)^6 = {}^6C_3\,(-ax)^3$

According to the question, ${}^4C_2 a\,x^2 = {}^6\,C_3 a\,3$

$\Rightarrow a = -3\ 1/10.$

2. $T_{r+1} = {}^{256}Cr\,(\sqrt{3})^{256-r}\,(\sqrt[8]{5})^r$

$= {}^{256}C_r\,(3)\,\frac{256-r}{2}\,(5)^{r/8}$

Terms would be integral, if $\frac{256-r}{2}$ and $\frac{r}{8}$ both are positive integer.

As $0 \le r \le 256$, $\therefore$ $r = 0, 8, 16, 24, \ldots, 256$

For above values of r, $\left(\frac{256-r}{2}\right)$ is also an integer.

Therefore, total number of values of $r = 33$.

3. (d) We know that $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ and $2 < e < 3$.

Therefore, $(1 + 0.0001)^{10000} < 3$ (By putting $n = 10000$)

Also, $(1 + 0.0001)^{10000}$

$= 1 + 10000\times 10^{-4}$

$+ \frac{10000\times 9999}{2\,!}\ 5\ 10^{-8} + \ldots\ldots$

upto 10001 terms

$\Rightarrow (1 + 0.0001)^{10000} > 2$.

Hence, 3 is the positive integer just greater than $(1 + 0.0001)^{10000} > 2$. Hence, (d) is the correct option.

4. (a) Coefficients of x^p is ${}^{(p+q)}C_p$ and coefficients of x_q is ${}^{(p+q)}C_q$, $(\because {}^nC_r = {}^nC_{n-r})$

5. (b) By hypothesis, $2^n = 4096 = 2^{12}$ $n = 12$ since n is even, hence the greatest coefficient is

$= {}^nC_{n/2} = {}^{12}C_6 = \frac{12.\ 11.\ 10.9.8.7}{1.2.3.4.5.6} = 924.$

6. $\Rightarrow {}^{n-1}C_r = (k^2-3).\ \frac{n}{r+1}\ {}^{n-1}C_r$

$\Rightarrow k^2 - 3\ \frac{r+1}{n}$ (Since, $n \ge r \Rightarrow \frac{r+1}{n} < 1$ and $n, r > 0$)

$\Rightarrow 0 < k^2 - 3 \le$ or $3 < k^2 \le 4$

$\Rightarrow k \in [-2, -\sqrt{3}\,] \cup (\sqrt{3}, 2)$

7. (a) $(1+t^2)^{12}(1+t^{12})(1+t^{24})$

$= (1 + {}^{12}C_1t^2 + {}^{12}C_2t^4 + \ldots + {}^{12}C_4t^8 + \ldots + + {}^{12}C_{10}t^{20} + \ldots)(1 + t^{12} + t^{24} + t^{36})$

Therefore, coefficient of $t^{24} = {}^{12}C_6 + 2$.

8. (a) ${}^{20}C_4 + 2{}^{20}C_3 + {}^{20}C_2 - {}^{22}C_{18}$

$= {}^{21}C_4 + {}^{20}C_3 + {}^{20}C_2 - {}^{22}C_{18}$

$= {}^{21}C_4 + {}^{21}C_3 - {}^{22}C_{18} = {}^{22}C_4 - {}^{22}C_{18}$

$= {}^{22}C_{18} - {}^{22}C_{18} = 0.$

9. (b) $x =$

$$\frac{729 + 6(2)(243) + 15(4)(8) + 20(8)(27) + 15(16)(9) + 6(32)3 + 64}{1 + 4(4)\ 6(16)\ 4(64)\ 256} \times$$

$$= \frac{{}^6C_0(3)6 + {}^6C_13^5.2 + {}^6C_23^4.2^2 + {}^5C_33^3.2^2 + {}^6C_4.3^22^4\ {}^6C_53.2^5 + {}^6C_62^6}{{}^6C_01^4 + {}^4C_14 + {}^4C_24^2 + {}^4C_34^3 + {}^4C4^4}$$

$x = \frac{(3+2)^6}{(1+4)^4} = 5^2$ $\qquad \therefore \sqrt{x} = 5$

Therefore, $\sqrt{x} - \frac{1}{\sqrt{x}} = 5 - \frac{1}{5} = \frac{24}{5} = 4.8.$

10. (b) Putting, $n = 2$, we get $4^n - 3n - 1 = 16 - 6 - 1 = 9$

Putting, $n = 3$, we get $4^n - 3n - 1 = 64 - 9 - 1 = 54$

Putting, $n = 4$, we get $4^n - 3n - 1 = 256 - 12 - 1 = 243$

Hence, for any value of n, $4^n - 3n - 1$ is always divisible by 9.

11. (a) $(4x + 7y)^{10} + (4x - 7y)^{10}$

$$= \begin{bmatrix}(4x)^{10} + {}^{10}C_1(4x)^9(7y)\\ + {}^{10}C_2(4x)^8(7y)^2 + .. + (7y)^{10}\end{bmatrix}$$

$$- \begin{bmatrix}(4x)^{10} - {}^{10}C_1(4x)^9(7y)\\ + {}^{10}C_2(4x)^8(7y)^2 + \ldots + (7y)^{10}\end{bmatrix}$$

$$= \begin{bmatrix}{}^{10}C_1{}^{10}(4x)^9(7y) + {}^{10}C_3(4x)^7\\ + {}^{10}C_2(4x)^8(7y)^2 + .. + (7y)^{10}\end{bmatrix}$$

Therefore, required number of terms = 5.

12. (a) $= \left(\sqrt{x + \frac{k}{x^2}}\right)^{10}$,

$T_{r+1} = {}^{10}C_r(\sqrt{x})^{10-r}\left(\frac{k}{x^2}\right)^r$

$= {}^{10}C_r.k^r(x)\ \frac{10-r}{2}\ 2r = {}^{10}C_rk^r.x\ \frac{10-5r}{2}$

For independent of x, $\frac{10-5r}{2} = 0$

Therefore, $r = 2$

Putting, $r = 2$ in the expansion. Then

$\Rightarrow {}^{10}C_2 k^2 = 405$. (given)

$\Rightarrow k^2 = \dfrac{405 \times 2}{10 \times 9} = 9 \therefore k = \pm 3$.

13. (b) $T_5 = {}^nC_4 x$, $T_6 = {}^nC_5 x^5$ and $T_7 = {}^nC_6 x^6$

Since, the coefficients of these terms are in A.P.

Therefore, ${}^nC_4 + {}^nC_6 = 2 \times {}^nC_5$

$$\Rightarrow \frac{n!}{(n-4)!\,4!} + \frac{n!}{(n-6)!\,6!} = \frac{2 \times n}{(n-5)!\,5!}$$

$$\Rightarrow \frac{n(n-1)(n-2)(n-3)}{4!}$$

$$+ \frac{n(n-1)(n-2)(n-3)(n-4)}{6!}$$

$$\frac{2n(n-1)(n-2)(n-3)(n-4)}{5!}$$

$$\Rightarrow \frac{1}{1} + \frac{(n-4)(n-5)}{5 \times 6} = \frac{2(n-4)}{5}$$

$\Rightarrow n^2 - 9n + 50 - 12n + 48 = 0$

$\Rightarrow n^2 - 21n + 98 = 0$

$\Rightarrow (n-7)(n-14) = 0 \;\; n = 7$ or 14.

14. (a) $(1 + x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n$

Put $x = -1$ on both sides, we get

$0 = 1 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots + (-1)\,{}^nC_n$.

15. (a) $\dfrac{1-x}{1+x} = (1-x)^{1/2}(1+x)^{-1/2}$

$$\left(1 - \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 + \ldots\right)$$

$$\times \left(1 - \frac{1}{2}x + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{2}!\,x^2 + \ldots\right)$$

$$= \left(1 - \frac{1}{2}x\right)\left(1 - \frac{1}{2}x\right)$$

$= -1 - \dfrac{1}{2}x - \dfrac{1}{2}x$ (other will be left) $= 1 - x$.

16. (a) On dividing 25 by 4 the remainder $= 1$ Therefore, unit digit in $(67)^{25} = 7^1 = 7$ Therefore, unit digit in $(67)^{25} - 1 = 7 - 1 = 6$.

17. (a) Since, the sum of coefficient of the expansion $\left(\frac{1}{x} + 2x\right)^n = 6561$

Therefore, $(1 + 2)^n = 3^8$

$\Rightarrow 3^n = 3^8 \;\; n = 8$

Let $(r + 1)$th term is independent of x.

Therefore, $T_{r+1} = {}^8C_r \left(\frac{1}{x}\right)(2x)^{8-r}$

$$= {}^8C_r 2^{8-r} x^{8-2r}$$

Since this term is independent of x, then

$8 - 2r = 0 \Rightarrow r = 4$

Therefore, coefficient of $T_5 = {}^8C_4.2^4 = 16.{}^8C_4$.

18. (a) We have, $T_2 = 14a^{5/2}$

$\Rightarrow {}^nC_1(a^{1/13})^{n-1} = (a^{3/2})^1 = 14a^{5/2}$

$\Rightarrow na\,\dfrac{n-1}{13} + \dfrac{3}{2} = 14a^{5/2}$

$\Rightarrow n = 14$

$$\Rightarrow \frac{{}^nC_3}{{}^nC_2} = \frac{{}^{14}C_3}{{}^{14}C_2} = \frac{12}{3} = 4.$$

19. (a) Let $C_1 + 2C_2 + 3C_3 + \ldots\ldots + {}^nC_n$

$$= \sum_{r=1}^{n} rC_r = \sum_{r=1}^{n} r\,{}^nC_r = \sum_{r=1}^{n} r\left(\frac{n}{r}\,{}^{n-1}C_{r-1}\right)$$

$$= n\sum_{r=1}^{n} {}^{n-1}C_{r-1} = n\left({}^{n-1}C_0 + {}^{n-1}C_1 + \ldots + {}^{n-1}C_{n-1}\right) = n.2^{n-1}$$

$\therefore C_1 + 2C_2 + 3C_3 + \ldots. + {}^nC_n = n.2^{n-1}$ (1)

Also, $C_1 - 2C_2 + 3C_3 \ldots\ldots$

$= C_1 + (-1)^1 2C_2 + (-1)^2 3C_3 + \ldots. (-1)^{n-1} nC_n$

$$= \sum_{r=1}^{n} (-1)^{r-1} rC_r = \sum_{r=1}^{n} (-1)^{r-1} r\,{}^nC_r$$

$$= \sum_{r=1}^{n} (-1)^{r-1} r \left(\frac{n}{r}\, {}^{n-1}C_{r-1}\right)$$

$$= \sum_{r=1}^{n} (-1)^{n-1}\, {}^{n-1}C_{r-1}$$

$$= \begin{pmatrix} n\, {}^{n-1}C_0 - {}^{n-1}C_1 + {}^{n-1}C_2 \ldots . \\ +(-1)^{n-1}\, n-1C_{n-1} \end{pmatrix}$$

$$= n(1-1)^{n-1} = 0$$

$$\therefore C_1 - 2C_2 + 3C_3 - \ldots .. + (-1)^{n-1}\, {}^nC_n = 0 \quad (2)$$

On adding Equations (1) and (2), we get,

$2(C_1 + 3C_3 + 5C_5 + \ldots) = n.2^{n-1} + 0$

Therefore, $C_1 + 3C_3 + 5C_5 + \ldots .. = n.2^{n-1}$.

20. (d) We have,

$$(1+x)^{30} = 1 + {}^{30}C_1x + {}^{30}C_2x^2 + {}^{30}C_3x^3 + \ldots + {}^{30}C_3 + x^{29} + {}^{30}C_{30}x^{30}$$

Therefore, sum of coefficient of odd powers of x is

$${}^{30}C_1 + {}^{30}C_3 + \ldots + {}^{30}C_{29} = 2^{30-1} = 2^{29}$$

21. (a) The general term is,

$$T_{r+1} = {}^{10}C_r \left(\frac{x}{3}\right)^{\frac{10-r}{2}} \cdot \frac{3^{r/2}}{x^{2r}}$$

$$= {}^{10}C_r \left(\frac{1}{3}\right)^{\frac{10-r}{2}} \times 3^{r/2} \times x^{\left(\frac{10-r}{2} - 2r\right)}$$

Therefore, for independent term of x, we have, $\frac{10-r}{2} - 2r = 0 \Rightarrow r = 0$

Therefore, term independent of x

$$= {}^{10}C\left(\frac{1}{3}\right)^4 \times 3 = 45 \times \frac{1\times 3}{81} = \frac{45}{27} = \frac{5}{3}$$

22. (b) We have, $\left(x + \frac{1}{x^2}\right)^6$

Therefore, the coefficient of x^6 in the expansion of

Therefore, is given by, $\left(x + \frac{1}{x^2}\right)^6$

$$T_{r+1} = {}^nC_r.x^{n-r}\, a^r = {}^6C_r x^{6-r}\left(\frac{1}{x^2}\right)^r$$

$$= {}^6C_r x^{6-r} x^{-2r} = {}^6C_r x^{6-3r}$$

Now, equating the coefficient of x on both sides, we get,

$6 - 3r = 0 \Rightarrow r = 2$

$\therefore T_3 = {}^6C_2 x^{6-2}\, x^{-4} = 6C_2$

$$= \frac{6\,!}{2\,!\,4\,!} = \frac{6\times 5}{2}$$

$\Rightarrow T_3 = 15.$

23. (d) $\therefore (1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$

$$\int_0^1 (1+x)^n\, dx = \int_0^1 (C_0 + C_1x + C_2x^2 + \ldots + C_nx^n)dx$$

$$\frac{2^{n+1}}{n+1} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots + \frac{C_n}{n+1}.$$

24. (b) $\left(2 + \frac{x}{3}\right)^n$. General term in above expansion

$$T_{r+1} = {}^nC_r 2^{n-r}\left(\frac{x}{3}\right)^r = {}^nC_r\,(2)^{n-r}\left(\frac{1}{3}\right)^r x^r$$

Coefficient of x^7 = Coefficient of x^8

$${}^nC_7 2^{n-7}\left(\frac{1}{3}\right)^7 = {}^nC_8\,(2)^{n-8}\left(\frac{1}{3}\right)^8$$

$$\Rightarrow {}^nC_7 \times 2 = {}^nC_8 \cdot \frac{1}{3} \Rightarrow \frac{{}^nC_7}{{}^nC_8} = \frac{1}{6}$$

$$\frac{n\,!\times 8\,!\; n-8}{7\,!\times n-7\,!\times !\,6} = \frac{1}{6}$$

$$\Rightarrow \frac{8}{n-7} = \frac{1}{6}$$

$\Rightarrow n - 7 = 48$

$\Rightarrow n = 55.$

25. (d) $\left(3x - \frac{1}{2x}\right)^8$

When we expand the given binomial, we get 9 terms, 9th term is the last term.

$T_{r+1} = {}^nC_r \cdot x^{n-r}\, a^r$. for $(x+a)^r$

$$T_9 = {}^8C_8\,(3x)^0\left(\frac{-1}{2x}\right)^8 = \frac{+1}{256x^8}.$$

QUESTION BANK: SOLVE THESE TO MASTER

1. The number of ways in which 5 rings can be worn on the 4 fingers of one hand is
(a) 45 (b) $5C_4$
(c) 54 (d) none of these

2. If ${}^{n-1}C_3 + {}^{n-1}C_4 > {}^nC_3$, then the value of n can be
(a) 5 (b) 6 (c) 7 (d) 8

3. If coefficient of $(2r + 3)$th and $(r - 1)$th terms in the expansion of $(1 + x)^{15}$ are equal, then value of r is
(a) 5 (b) 6 (c) 4 (d) 3

4. In the expansion of $\left(x - \frac{1}{x}\right)^6$, the constant term is
(a) – 20 (b) 20
(c) 30 (d) – 30

5. If the coefficient of the middle term in the expansion of $(1 + x)^{2n+2}$ is α and the coefficients of middle terms in the expansion of $(1 + x)^{2n+1}$ are β and γ, then
(a) $\alpha + \beta = \gamma$ (b) $\beta + \gamma = \alpha$
(c) $\alpha = \beta + \gamma$ (d) $\alpha + \beta + \gamma = 0$

6. Which of the following term is not numerically greatest term in the expansion of $(3 - 5x)^{15}$, when $x = 1/5$ is
(a) T_4 (b) T_6
(c) T_5 (d) none of these

7. $\left(x - \frac{1}{x}\right)^6$, can be expanded by binomial theorem, if
(a) $x < 1$ (b) $|x| < 1$
(c) $|x| < \frac{5}{4}$ (d) $|x| < \frac{4}{5}$

8. The number of non-zero terms in the expansion of
$(1 + 3\sqrt{2}x)^9 + (1 - 3\sqrt{2}x)^9$ is
(a) 9 (b) 0 (c) 5 (d) 10

9. In the expansion of $(1 + x)^{50}$, the sum of the coefficients of odd powers of x is
(a) 0 (b) 2^{49} (c) 2^{50} (d) 2^{51}

10. Sum of the series $\frac{{}^nC_1}{2} + \frac{{}^nC_5}{4} + \frac{{}^nC_5}{6} + \ldots$ is
(a) $\frac{2^{n-1} - 1}{n + 1}$ (d) $\frac{2^n - 1}{n}$
(c) $\frac{2^n - 1}{n + 1}$ (d) None of these

11. The greatest term (numerically) in the expansion of $(3 - 5x)^{11}$ when $x = \frac{1}{5}$ is
(a) 55×3^9 (b) 46×3^9
(c) 55×3^6 (d) none of these

12. If 7^{103} is divided by 25, then the remainder is
(a) 20 (b) 16
(c) 18 (d) 15

13. The sum of rational terms in the expansion of $(\sqrt{2} + 3^{1/5})^{10}$ is
(a) 31 (b) 41
(c) 51 (d) None of these

14. If the second term in the expansion $\left(\sqrt[13]{a} + \frac{a}{\sqrt{a^{-1}}}\right)^n$ is 14 $a^{5/2}$, then the value of $\frac{{}^nC_3}{{}^nC_2}$ is
(a) 8 (b) 12
(c) 4 (d) none of these

15. If the rth term in the expansion of $\left(\frac{x}{3} - \frac{2}{x^2}\right)^{10}$ contains x^4, then r is equal to
(a) 2 (b) 3 (c) 4 (d) 5

16. If $(7 + 4\sqrt{3}) = p + \beta$, where n and p are positive integers and β is a proper fraction, then $(1 - \beta)(p + \beta)$ equals
(a) 1 (b) 2 (c) 3 (d) 4
[Roorkee – 1989]

17. Coefficient of $\frac{1}{x^{12}}$ in the expansion of $\left(x - \frac{1}{x}\right)^{32}$ is equal to
(a) ${}^{33}C_{33}$ (b) ${}^{32}C_{22}$
(c) ${}^{32}C_{18}$ (d) ${}^{22}C_{20}$
[TS Rajendra – 1992]

18. If n is an integer 3 , then $[(n + 1) /n]^n$ is
(a) $> n$ (b) $< n$
(c) n (d) n
[AMU – 1996]

19. The approximate value of $(7.995)^{1/3}$ correct to four decimal places is
[UPSEAT – 1991]
(a) 1.9995 (b) 1.9996
(c) 1.9990 (d) 1.9991

20. If $(\sqrt{5} - \sqrt{3})^3\ x\sqrt{5} + y\sqrt{3}$, find the value of $(x - y)$?
(a) 504 (b) 405
(c) 305 (d) 503
[TS Rajendra – 1991]

21. If the polynomial $(x^3 + px^2 + qx + 4)$ leaves remainder '7' and '18' when divided by $(x - 1)$ and $(x - 2)$ respectively, then
(a) $p = 3, q = -1$ (b) $p = 7, q = 3$
(c) $p = 1, q = 1$ (d) $p = 8, q = 4$
[SCRA – 1991]

22. When $(2x^3 - 17x + 11)$ is divided by $(x - 5)$, then the quotient and remanders are
(a) $(2x^2 + x + 1)$ and 180
(b) $(2x^2 - x + 2)$ and 30
(c) $(2x^2 + 10x + 3)$ and 20
(d) $(2x^2 + 10x + 33)$ and 176
[SCRA – 1991]

23. Let n (> 1) be a positive integer, then the largest integer m such that $(n^m + 1)$ divides $(1 + n + n^2 + ... + n^{127})$ is
(a) 127 (b) 63
(c) 64 (d) 32
[IIT Screening Test 1995]

24. For positive integers n_1, n_2, the value of the expansion
$(1\ i^2)^{n1}\ (1\ i^3)^{n1}\ (1\ i^5)^{n2}\ (1\ i^7)^{n2}$ where $i = \sqrt{-1}$ is a real number, if and only if
[IIT – 1996]
(a) $n_1 = n_2 + 1$ (b) $n_1 = n_2 - 1$
(c) $n_1 = n_2$ (d) $n_1 > 0, n_2 > 0$

25. If the expansions of $\left(x + \frac{a}{x}\right)^n$ and $\left(x + \frac{b}{x^2}\right)^n$ in powers of x have one term independent of x, then n is divisible by
(a) 6 (b) 4
(c) 3 (d) 2
[REE Qualifying Exam –1999]

26. The expansion $(2 + \sqrt{2})^2$ has value lying between
(a) 134 and 135 (b) 135 and 136
(c) 136 and 137 (d) none of these
[AMU – 2001]

27. In the binomial expansion $(a + bx)^{-3} = \frac{1}{8} + \frac{9}{8}x + ...$ the value of a and b are
[UPSEAT – 2002]
(a) $a = 3, b = 3$ (b) $a = 2, b = -3$
(c) $a = 3, b = 2$ (d) $a = -3, b = 2$

28. The degree of the polynomial
$[x + (x^3 - 1)^{1/2}]^6 + [x - (x^3 - 1)^{1/2}]$ is equal to
(a) 9 (b) 8
(c) 10 (d) none of these
[TS Rajendra – 1993]

29. If the numerical coefficient of the pth term in the expansion of $(2x + 3)^6$ is 4860, then the value(s) of p is (are)
(a) 2 (b) 3
(c) 4 (d) 5
[REE Qualifying Exam – 1994]

30. If the sum of middle terms is S in the expansion of $\left(2a - \frac{a^2}{4}\right)^9$, then the value(s) of S is (are)
(a) $\left(\frac{63}{32}\right) a^{14} (8 + a)$ (b) $\left(\frac{63}{32}\right) a^{13} (8 + a)$
(c) $\left(\frac{63}{32}\right) a^{14} (8 - a)$ (b) $\left(\frac{63}{32}\right) a^{13} (8 - a)$
[REE Qualifying Exam 1994]

31. The value of ${}^{15}C_0^{\,2} - {}^{15}C_1^{\,2} + {}^{15}C_2^{\,2} - \ldots - {}^{15}C_{15}^{\,2}$ is

(a) 15 (b) – 15

(c) 0 (d) 51

[MPPET – 1996]

32. If $(1 + x)(1 + x + x^2)(1 + x + x^2 + x^3)\ldots(1 + x + x^2 + \ldots + x^n)$
$= C_0 + C_1 x + C_2 x^2 + C_N x^N$

$N = \frac{n(n+1)}{2}$, then the sum of even coefficients $C_0 + C_2 + C_4 + \ldots$ equals

[AMU – 1999]

(a) $2^n - 1$ (b) 2^N

(c) $(n + 1)!$ (d) $\frac{1}{2}(n + 1)!$

33. $P(1 + r)^n + P(1 + r)^{n-1} + \ldots\ldots + P(1 + r)$, where $r \neq 0$, is equal to

[AMU – 2000]

(a) $P\left(1 + \frac{1}{r}\right)(1 + r)^n$

(b) $P\left(1 + \frac{1}{r}\right)(1 + r)^n - 1$

(c) $P(1 + r)^n [1 - (1 + r)^n]$

(d) $P\frac{(1 + r)^{n+1}}{r}$

34. The value of is $\frac{{}^{21}C_1}{{}^{21}C_0} + \frac{21.{}^{21}C_2}{{}^{21}C_1} + \frac{3.{}^{21}C_3}{{}^{21}C_2}$

[NDA – 2003]

(a) 213 (b) 231

(c) 312 (d) 321

35. When n is a positive integer, the expansion $(x + a)^n = {}^nC_0 x^n + nC_1 x^{n-1} a^1 + \ldots\ldots + {}^nC_n a^n$ is valid only when

(a) $|a| < 1$

(b) $|x| < 1$

(c) $x| < 1$ and $|a| < 1$

(d) x and a are any two numbers.

36. The number of dissimilar terms in the expansion of $(a + b + c)^n$ is

(a) $n + 1$

(b) n

(c) $n + 2$

(d) $1 + 2 + 3 + \ldots.. + (n + 1)$

ANSWERS

Lecture–1: Unsolved Objective Problems (Identical Problems For Practice): For Improving Speed With Accuracy

1. (a)	4. (c)	7. (a)	10. (a)
2. (c)	5. (a)	8. (c)	11. (a)
3. (c)	6. (d)	9. (a)	12. (a)

Lecture–1: Work Sheet: To Check Preparation Level

1. (a)	3. (c)	5. (c)	7. (c)
2. (a)	4. (b)	6. (d)	8. (c)

Lecture–2:Unsolved Objective Problems (Identical Problems For Practice): For Improving Speed With Accuracy

1. (b)	5. (c)	9. (c)	13. (a)
2. (b)	6. (c)	10. (a)	14. (c)
3. (a)	7. (c)	11. (d)	15. (b)
4. (a)	8. (b)	12. (c)	

Lecture–2: Work Sheet: To Check Preparation Level

1. (a)	5. (c)	9. (b)	13. (a)
2. (a)	6. (d)	10. (c)	14. (c)
3. (a)	7. (c)	11. (d)	15. (a)
4. (c)	8. (b)	12. (c)	

Lecture–3:Unsolved Objective Problems (Identical Problems For Practice): For Improving Speed With Accuracy

1. (a)	5. (b)	9. (d)	13. (a)
2. (c)	6. (a)	10. (d)	14. (b)
3. (a)	7. (a)	11. (c)	15. (c)
4. (c)	8. (b)	12. (a)	

Lecture–3: Work Sheet: To Check Preparation Level

1. (a)	5. (b)	9. (a)	13. (b)
2. (a)	6. (b)	10. (d)	14. (c)
3. (b)	7. (c)	11. (d)	15. (c)
4. (c)	8. (b)	12. (b)	

Lecture–4:Unsolved Objective Problems (Identical Problems For Practice): For Improving Speed With Accuracy

1. (c)	4. (a)	7. (b)	10. (b)
2. (a)	5. (b)	8. (c)	11. (a)
3. (c)	6. (b)	9. (d)	12. (a)

Lecture–4: Work Sheet: To Check Preparation Level

1. (c)	3. (d)	5. (c)	7. (c)
2. (d)	4. (d)	6. (a)	8. (c)

Lecture–5: Mental Preparation Test

1. $1 + 4x + 10x^2 + 16x^3 + 19x^4 + 16x^5 + 10x^6 + 4x^7 + x^8$.
2. 11592740743
4. 8
7. $x = 2, a = 3, n = 6$
8. $x = 4, a = \frac{1}{2}, n = 8$
10. -43
11. $\frac{4480}{27}$
12. 672
13. $(-1)^n \frac{(2n)!}{(n!)^2} x^n$
14. $n = 12$
15. $x = 10^{-5/2}$
16. ${}^6C_r (-1)^r x^{12} - 2^r y^{3r}$
17. $\frac{3!}{n!\,2n!} \cdot \frac{1}{x^n}$
18. $\frac{5}{256} y^4$
19. 29568
20. 11, 2.
22. $2^{17} - 1$.
23. 4^n
27. $a^6x^6 - 6a^5x^4b + 15a^4x^2b^2 - 20a^3b^3 - 15\frac{a^2b^4}{x^2} - \frac{6ab^5}{x^4} + \frac{b^6}{x^6}$
28. 161051
29. $29568\, x^6 a^5$
30. 70
31. $\frac{429}{16} x^{14}$
32. $110565\, a^4$
33. $10500\, x^{-3}$
34. $\frac{28672}{729} (\cdots)$
35. 1400000 (sixth)
36. $\frac{200000}{27}$ (fifth)
37. 6
38. $2^{13} - 14$.
40. $2^{-4/5}\left[1 + \frac{6}{5}x^2 + \frac{81}{50} + \frac{567}{25}x^6\right]$
41. $\frac{[1.2........2\ 1]\ 2^r}{r!}$
43. 9.997
44. $1 - \frac{x}{2}$
45. 1.000299
47. 14
49. 61, 113

QUESTION BANK: SOLVE THESE TO MASTER

1. (a)	2. (d)	3. (a)	4. (a)
5. (b)	6. (b)	7. (c)	8. (c)
9. (b)	10. (c)	11. (d)	12. (c)
13. (b)	14. (b)	15. (a)	16. (a)

17. (b)	18. (b)	19. (b)	20. (a)
21. (c)	22. (d)	23. (c)	24. (d)
25. (c,d)	26. (b)	27. (b)	28. (a)
29. (b)	30. (d)	31. (c)	32. (a)
33. (c)	34. (b)	35. (d)	36. (d)

PART B

Complex Numbers

This page is intentionally left blank.

LECTURE

1 Algebra of Complex Numbers

BASIC CONCEPTS

1. **General form of Complex Number (*z*)**
   $z = x + iy$ where, $i = \sqrt{-1}$,
   $i^2 = -1$, $i^3 = -i$, $i^4 = 1$
   x = Real part of z = Re(z) and, y = − imaginary part of z = Im(z)
   Note: Euler was the first mathematician to introduce the symbol i for the square root of −1.
2. **Complex Number as an Ordered Pair**
   A complex number is defined as an ordered pair (x, y) of real numbers x and y. Thus, $z = (x, y)$, where,
   abscissa = x = real part = Re(z) and
   ordinate = y = imaginary part = Im(z)
3. z = purely real, if $y = 0$ i.e., imaginary part is zero = x.
4. z = purely imaginary if $x = 0$ i.e., Real part is zero = iy.
5. Integral powers of iota 'i'
   $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$,
   $i^{4n} = (i^4)^n = (1)^n = 1$,
   $i^{4n+1} = (i^{4n})(i) = i$, $i^{4n+2} = -1$,
   $\frac{1}{i} = -i$, $-\frac{1}{i} = i$,
   Similarly, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^{-1} = -i$, $i^{-2} = -1$, $i^{-3} = i$, $i^{-4} = 1$
   Similarly i^m, for calculating integral power of i, dividing m by 4 and according to the remainder, find the value of i^m as follows

Imaginary Numbers	i^{4n-3}	i^{4n-2}	i^{4n-1}	i^{4n}	i^{4n+1}	i^{4n+2}	i^{4n+3}
Remainder	−3	−2	−1	0	1	2	3
Value	i	−1	$-i$	1	i	−1	$-i$

6. **Order Relation** There is no order relation between any two complex numbers i.e., if z_1 and z_2 are any two complex numbers, then, either $z_1 = z_2$ or $z_1 \neq z_2$, but not $z_1 > z_2$ or $z_1 < z_2$. i.e., $5 + 4i > 2 + 3i$ and $5 + 4i < 2 + 3i$ have no meaning.
7. Zero is only a number which is both real as well as purely imaginary.
8. **Equality of Two Complex Numbers** Two complex numbers are said to be equal if their corresponding real parts and imaginary parts are separately equal.
   Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, if $z_1 = z_2 \Leftrightarrow x_1 = x_2$, $y_1 = y_2$
9. **Algebra of Complex Numbers** $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be any two complex numbers, then
   (i) Addition:
   $$z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)$$
   $$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

(ii) Subtraction:

$$z_1 - z_2 = x_1 - x_2 + i(y_1 - y_2)$$

$$(x_1, y_1) - (x_2, y_2) = (x_1 - x_2, y_1 - y_2)$$

(iii) Multiplication of Complex number:

$$z_1 = x_1 + iy_1;\ z_2 = x_2 + iy_2;$$

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$

$$= (x_1x_2 - y_1y_2) + i\,(x_1y_2 + x_2y_1)$$

(iv) Division of Complex number:

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2}$$

$$= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\left(\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}\right)$$

Note: Algebraic operations involving complex numbers are performed according to the same rule as in the operations involving real numbers with the convention that i^2 is replaced by − 1.

10. $\sqrt{a}\sqrt{b} = \sqrt{ab}$, if at least one of a and b is nonnegative.

11. $\sqrt{a}\,\sqrt{b} = -\sqrt{ab}$, if both a and b are negative. e.g., $\sqrt{-2}\,\sqrt{-3} = -\sqrt{6}$

12. Negative of a Complex Number If $z = x + iy$, then

$-z = -x - iy.$

13. Integral powers of a Complex Number If $z = x + iy$, then

(*i*) $z^2 = (x + iy)^2 = x^2 - y^2 + i\,(2xy)$

(ii) $z^3 = (x + iy)^3 = x^3 + i\,(3x^2y) - 3xy^2 - iy^3$

(iii) $z.z.z_3$ ..... k times $= z^k$

(iv) $z^0 = 1$

14. Reciprocal of a Complex Number $z = x + iy$, then reciprocal of z is denoted by

$\frac{1}{z}$ and $\frac{1}{z} = \frac{1}{x + iy}.$

It is also known as multiplicative inverse of complex number.

15. Conjugate of a Complex Numbers If $z = x + iy$, then the complex number $x - iy$ is called the complex conjugate of $\bar{z}$ and it is denoted by $\bar{z}$ which is as follows: $\bar{z} = \overline{x + iy} = x - iy$.

15.1 A complex number is purely real, if $z = \bar{z}$ ($y = 0$) and purely imaginary, if $\bar{z} = -z$ ($x = 0$).

Note: The sum and product of a complex number with its conjugate are both real.

16. Properties of Complex Numbers

If $z = x + iy$ then $\bar{z} = x - iy$

(i) $z + \bar{z} = 2x = 2\text{Re}\,(z) = 2\text{Re}\,(\bar{z})$

= Sum of a complex number with its conjugate.

(ii) $z - \bar{z} = 2iy = 2i\,\text{Im}(z) = -\,2i\,\text{Im}(\bar{z})$

(iii) $z\,.\,\bar{z} = x^2 + y^2 = |\,z\,|^2$

= Product of a complex number with its conjugate.

Note: The addition and multiplication of anytwo complex numbers are a real number, thenboth are a pair of conjugate complex numbers.

(iv) $\overline{(\bar{z})} = z$

(v) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(vi) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(vii) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$

(viii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

(ix) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

(*x*) $\overline{\overline{z_1} z_2} = z_1\,\overline{z_2}$

(xi) $z_1\,\overline{z_2} + \overline{z_1}\,z_2 = 2\text{Re}\,(\overline{z_1}\,z_2)$

$= 2\text{Re}(z_1\,\overline{z_2})$

17. Properties of Addition and Multiplication of Complex

Numbers

(*i*) $z_1 + z_2 = z_2 + z_1$ = one complex number

(ii) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

(iii) $z + 0 = 0 + z = z$

(iv) $z + (-z) = (-z) + z = 0$

(v) $z_1 + z_2 = z_1 + z_3 \Rightarrow z_2 = z_3$

(vi) $z_1 . z_2 = z_2 . z_1$ = one complex number

(vii) $(z_1 z_2)(z_3) = z_1(z_2 z_3)$

(viii) $z . 1 = 1 . z = z$

(ix) $z_1 z_2 = z_1 z_3 \Rightarrow z_2 = z_3$ or $z_1 = 0$

(x) $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

(xi) $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$

SOLVED SUBJECTIVE PROBLEMS: (CBSE/STATE BOARD): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Find real values of x and y for which the complex numbers $-3 + i x^2 y$ and $x^2 + y + 4i$ are conjugate of each other.

Solution

Since $-3 + ix^2y$ and $x^2 + y + 4i$ are complex conjugates.

Therefore, $-3 + i x^2 y = x^2 + y + 4i$

$\Rightarrow 3 + i x^2 y = x^2 + y - 4i$

$\Rightarrow -3 = x^2 + y$ (1)

and $x^2 y = -4$ (2)

$\Rightarrow -3 = x^2 - \frac{4}{x^2}$

[Putting $y = \frac{-4}{x^2}$ from (2) in (1)]

$\Rightarrow x^4 + 3x^2 - 4 = 0 \quad \Rightarrow (x^2 + 4)(x^2 - 1) = 0$

$\Rightarrow x^2 - 1 = 0$ [$\because x^2 + 4 \neq 0$ for any real x]

$\Rightarrow x = \pm 1$

From (2), $y = -4$, when $x = \pm 1$

Hence, $x = 1, y = -4$ or $x = -1, y = -4$

2. If $x = -5 + 2\sqrt{-4}$, find the value of $x^4 + 9x^3 + 35x^2 - x + 4$.

Solution

We have $x = -5 + 2\sqrt{-4}$

$\Rightarrow x + 5 = 4i \Rightarrow (x + 5)^2 = 16i^2$

$\Rightarrow x^2 + 10x + 25 = -16 \Rightarrow x^2 + 10x + 41 = 0$

Now $x^4 + 9x^3 + 35x^2 - x + 4$

$= x^2(x^2 + 10x + 41) - x(x^2 + 10x + 41) + 4(x^2 + 10x + 41) - 160$

$= x^2(0) - x(0) + 4(0) - 160$

$= -160$ [$\because x^2 + 10x + 41 = 0$]

3. If $a^2 + b^2 = 1$, then prove that $\frac{1 + b + ia}{1 + b + ia} = b + ia$.

Solution

$$\text{L.H.S.} = \frac{1 + b + ia}{1 + b - ia} = \left(\frac{1 + b + ia}{1 + b - ia}\right)\left(\frac{b + ia}{b + ia}\right)$$

$$= \frac{(1 + b + ia)(b + ia)}{b + ia + b^2 + iab - iab + a^2}$$

$$= \frac{(1 + b + ia)(b + ia)}{a^2 + b^2 + b + ia}$$

$$= \frac{(1 + b + ia)(b + ia)}{1 + b + ia} \quad [\text{Given } a^2 + b^2 = 1]$$

$= b + ia =$ R.H.S.

Proved

4. If $(x + iy)^3 = u + iv$, then show that

$\frac{u}{x} + \frac{v}{y} = 4(x^2 - y^2)$.

[NCERT]

Solution

Given $(x + iy)^3 = u + iv$

$\Rightarrow x^3 + i^3 y^3 + 3x(iy)(x + iy) = u + iv$

$\Rightarrow x^3 + (-i) y^3 + 3x^2 yi - 3xy^2 = u + iv$

$\Rightarrow x^3 - 3xy^2 + i(3x^2y - y^3) = u + iv$

Equating real and imaginary parts, we get

$u = x^3 - 3xy^2$

$\Rightarrow u = x(x^2 - 3y^2)$ and $v = 3x^2 y - y^3$

$\Rightarrow v = y(3x^2 - y^2)$

$\because \frac{u}{x} + \frac{v}{y} = x^2 - 3y^2 + 3x^2 - y^2$

or $\frac{u}{x} + \frac{v}{y} = 4(x^2 - y^2)$

5. Find the real numbers x and y if $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.

[NCERT]

Solution

Given, $(x - iy)(3 + 5i) = \overline{-6 - 24i}$

$\Rightarrow x - iy = \dfrac{-6 + 24i}{3 + 5i} \times \dfrac{3 - 5i}{3 - 5i}$

$= \dfrac{-18 - 120(-1) + 72i + 30i}{9 - 25(-1)} = \dfrac{102 + 102i}{34}$

$= 3 + 3i$

$\Rightarrow x - iy = 3 + 3i$

$\Rightarrow x = 3$ and $-y = 3$

$\Rightarrow x = 3, y = -3.$

(Equating real and imaginary parts)

6. If $\left(\dfrac{1+i}{1-i}\right)^m = 1$, then find the least + ve integral value of m.

[NCERT]

Solution

Given $\left(\dfrac{1+i}{1-i}\right)^m = 1$

$\Rightarrow \left\{\dfrac{(1 \times i)}{(1-i)} \times \dfrac{(1+i)}{(1+i)}\right\}^m = 1$

$\Rightarrow \left\{\dfrac{(1+i)^2}{1^2 - i^2}\right\}^m = 1$

$\Rightarrow \left\{\dfrac{1 + i^2 + 2i}{1 - (-1)}\right\}^m = 1$

$\Rightarrow \left(\dfrac{2i}{2}\right)^m = 1 \Rightarrow i^m = 1$

$\Rightarrow$ least positive integral value of m is 4.

7. Express each of the complex number given in the form $a + ib$.

$$\left[\left(\frac{1}{3} + i\frac{7}{3}\right) + \left(4 + i\frac{1}{3}\right) - \left(-\frac{4}{3} + i\right)\right.$$

Solution

Given complex number

$= \left[\left(\frac{1}{3} + i\frac{7}{3}\right) + \left(4 + i\frac{1}{3}\right) - \left(-\frac{4}{3} + i\right)\right.$

$= \left[\left(\frac{1}{3} + 4\right) + \left(\frac{7}{3} + \frac{1}{3}\right)i\right] - \left(-\frac{4}{3} + i\right)$

$= \left(\frac{13}{3} + \frac{8}{3}i\right) - \left(-\frac{4}{3} + i\right)$

$= \left\{\frac{13}{3} - \left(-\frac{4}{3}\right)\right\} + \left(\frac{8}{3} - 1\right)i = \frac{17}{3} + \frac{5}{3}i$

which is in the form $a + ib$, where $a = \frac{17}{3}$ and $b = \frac{5}{3}$.

8. Express each of the complex number givern in the form $a + ib$ $\left(\frac{1}{3} + 3i\right)^3$

Solution

Given complex number

$= \left(\frac{1}{3} + 3i\right)^3 = \left(\frac{1}{3}\right)^3 + (3i)^3 + 3\left(\frac{1}{3}\right)(3i)\left(\frac{1}{3} + 3i\right)$

$= \frac{1}{27} + 27i^3 + 3i\left(\frac{1}{3} + 3i\right)$

$= \frac{1}{27} + 27(-i) + i + 9i^2$

$(\because i^3 = i^2 \cdot i = -i)$

$= \frac{1}{27} - 27i + i + 9(-1) = \left(\frac{1}{27} - 9\right) - 26i$

$= \left(-\frac{242}{27}\right) + (-26)\,i =$

which is in the form $a + ib$, where,

$a = -\frac{242}{27}, b = -26$

9. Express the complex number given in the form $a + ib\left(-2 - \frac{1}{3}i\right)^3$

Solution

Given complex number

$= \left(-2 - \frac{1}{3}i\right)^3 = \left\{-1\left(2 + \frac{1}{3}i\right)\right\}^3$

$= -\left\{2^3 + \frac{1}{27}i^3 + 3(2)\left(\frac{1}{3}i\right)\left(2 + \frac{1}{3}i\right)\right\}$

$= -\left\{8 + \frac{1}{27}(-i) + 2i\left(2 + \frac{1}{3}i\right)\right\}$

$= -8 + \frac{1}{27}i - \left(4i + \frac{2}{3}i^2\right)$

$= -8 + \frac{1}{27}i - 4i + \frac{2}{3}\ (\because i^2\ -1)$

$= \left(-8 + \frac{2}{3}\right) + \left(\frac{1}{27} - 4\right)i$

$= \left(-\frac{22}{3}\right) + \left(\frac{107}{27}\right)i$

which is in the form $a + ib$ where $a = -\frac{22}{3}$ and $b = -\frac{107}{27}$

10. Evaluate $\left[i^{18} + \left(\frac{1}{i}\right)^{25}\right]^3$

Solution

Given number

$= \left[i^{18} + \left(\frac{1}{i}\right)^{25}\right]^3 = \{i^{16}\, i^2 + (-i)^{25}\}^3$

$\left(\because \frac{1}{i} = \frac{1}{i^2} = \frac{1}{-1}\right)$

$= \{(i^4)^4 (-1) + (-1)(i^4)^6 i^1\}^3$

$= (1(-1) - i)^3 = (-1 - i)^3 = (-1)^3 (1 + i)^3$

$= -1\{1^3 + i^3 + 3i(1 + i)\}$

$= -\{1 + (-i) + 3i + 3(-1)\}$

$(\because i^2 = -1 \text{ and } i^3 = -i)$

$= -\{-2 + 2i\} = 2 - 2i.$

11. Prove that $\left[\frac{1+i}{1-i}\right]^n = i^n$

[MP – 1997]

Solution

$\left[\frac{1+i}{1-i}\right]^n = \left[\frac{(1+i)(1+i)}{(1-i)(1+i)}\right]^n$

$= \left[\frac{1 + i + 2i}{1 - i^2}\right]^n = \left[\frac{1 - 1 + 2i}{1 + 1}\right]^n, \quad [\because i^2 = 1]$

$= \left[\frac{2i}{2}\right] = i^n.$

12. Express each of the following in the standard form $a + ib$

$\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$

Solution

$\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$

$= \frac{(6+6) + i(-4+9)}{(2+2) + i(4-1)} = \frac{12+5i}{4+3i}$

$= \frac{12+5i}{4+3i} \cdot \frac{4-3i}{4-3i}$

$= \frac{(48+15) + i(-36+20)}{16 - 9i^2}$

$= \frac{63}{25} - \frac{16}{25} i.$

13. Reduce $\left\{\frac{1}{1-4i} - \frac{2}{1+i}\right\}\left(\frac{3-4i}{5+i}\right)$ to the standard form.

Solution

Given number

[NCERT]

$= \left\{\frac{1}{1-4i} - \frac{2}{1+i}\right\}\left(\frac{3-4i}{5+i}\right)$

$= \left\{\frac{1 + i - 2(1-4i)}{(1-4i)(1+i)}\right\} \frac{3-4i}{5+i}$

$= \left(\frac{-1+9i}{1+4-3i}\right)\left(\frac{3-4i}{5+i}\right)$

$= \frac{-3 + 36 + 27i + 4i}{25 + 3 - 15i + 5i}$

$= \frac{33+31i}{28-10i} \times \frac{28+10i}{28+10i}$

$= \frac{33 \times 28 - 31 \times 10 + (31 \times 28 + 33 \times 10)i}{(28)^2 - (10i)^2}$

$= \frac{924 - 310 + (868 + 330)i}{784 - 100(-1)}$

$= \frac{614 + 1198i}{884} = \frac{614}{884} + \frac{1198}{884} i$

$= \frac{307}{442} + \frac{559}{442} i$

14. Find real θ such that $\frac{3 + 2i\sin\theta}{1 - 2i\sin\theta}$ is purely real.

Solution

We have $\frac{3 + 2i\sin\theta}{1 - 2i\sin\theta}$

$= \frac{(3 + 2i\sin\theta)(1 + 2i\sin\theta)}{(1 - 2i\sin\theta)(1 + 2i\sin\theta)}$

$= \frac{(3 + 6i\sin\theta + 2i\sin q - 4\sin^2\theta)}{1 + 4\sin^2\theta}$

$= \frac{3 - 4\sin^2\theta}{1 + 4\sin^2\theta} + i\frac{8\sin\theta}{1 + 4\sin^2\theta}$

We are given the complex number to be real.

Therefore,

$\frac{8\sin\theta}{1 + 4\sin^2\theta} = 0$, i.e., $\sin\theta = \theta$. Thus $\theta = n\pi$, $n \in Z$.

15. Find the value of $x^3 + 7x^2 - x + 16$, when $x = 1 + 2i$.

Solution

We have, $x = 1 + 2i$

$\Rightarrow x - 1 = 2i$

$\Rightarrow (x - 1)^2 = 4i^2$

$\Rightarrow x^2 - 2x + 1 = -4$

$\Rightarrow x^2 - 2x + 5 = 0$

Now, $x^3 + 7x^2 - x + 16$

$= x(x^2 - 2x + 5) + 9(x^2 - 2x + 5) + (12x - 29)$

$= x(0) + 9(0) + 12x - 29$ $[\because x^2 - 2x + 5 = 0]$

$= 12(1 + 2i) - 29$ $[\because x = 1 + 2i]$

$= -17 + 24i$

Hence, the value of the polynomial when $x = 1 + 2i$ is $-17 + 24i$.

UNSOLVED SUBJECTIVE PROBLEMS (CBSE/STATE BOARD): TO GRASP THE TOPIC, SOLVE THESE PROBLEMS

Exercise I

1. Evaluate the following

(i) i^{135} (ii) i^{-999}

(iii) $\left(i^{37} + \frac{1}{i^{67}}\right)$

2. Compute the following

(i) $\sqrt{-144}$ (ii) $\sqrt{-4}\sqrt{-\frac{9}{4}}$

3. Show that

(i) $\left[i^{19} + \left(\frac{1}{i}\right)^{25}\right]^2 = -4$

(ii) $i^{107} + i^{112} + i^{117} + i^{122} = 0$

4. Add $-1 + 3i$ and $5 - 8i$

5. Subtract $7 - 3i$ from $6 + 5i$

6. Find the real values of x and y, if

(i) $(3x - 7) + 2iy = -5y + (5 + x)i$

(ii) $\frac{x-1}{3+i} + \frac{y-1}{3-i} = i$

7. Express each of the following in the form $a + ib$

(i) $(3 + 4i)^2$

(ii) $(-2 + \sqrt{-3})(-3 + 2\sqrt{-3})$

8. Express each of the following in the standard form $a + ib$

(i) $\frac{1}{3 - 4i}$

(ii) $\left(\frac{5 + \sqrt{2}i}{1 - \sqrt{2}i}\right)$

9. Prove that the following complex numbers are purely real:

$\left(\frac{2+3i}{3+4i}\right)\left(\frac{2-3i}{3-4i}\right)$

10 Find the conjugate of

(i) i^3

(ii) $\frac{2 - 5i}{3 - 2i}$

11. Find real values of x and y for which the following equalities hold. $(1 + i)y^2 + (6 + i) = (2 + i)x$

Exercise II

1. Evaluate the following:

(i) $(-\sqrt{-1})^{4n+3}$

(ii) $(-i)(3i)\left(-\frac{1}{6}i\right)^3$

2. Compute the following

$\sqrt{-16} + 3\sqrt{-25} + \sqrt{-36} - \sqrt{-625}$

3. Show that $(1 + i)^4 \times \left(1 + \frac{1}{i}\right)^4$ for all $n \in N$.

4. Find $Z_1 + Z_2$ and $Z_1 - Z_2$ if $Z_1 = 3 + 5i$ and $Z_2 = -5 + 2i$

5. Find the real values of x and y if

(i) $(1 - i)x + (1 + i)y = 1 - 3i$

(ii) $(x + iy)(2 - 3i) = 4 + i$

6. Express each of the following in the form $a + ib$

(i) $(2 + 3i)(4 - 5i)$ (ii) $(4 - 3i)^3$

7. Express each of the following in the standard form $a + ib$

(i) $\frac{(1+i)^2}{3-i}$ (ii) $(-1 + \sqrt{3}i\ i)^{-1}$

8. Prove that the following complex numbers are purely real.

$$\left(\frac{3+2i}{2-3i}\right) + \left(\frac{3-2i}{2+3i}\right)$$

9. Find the conjugate of $(6 + 5i)^2$

ANSWERS

Exercise-I

1. (i) $-i$ (ii) i (iii) $2i$
2. (i) $12i$ (ii) -3
4. $4 - 5i$
5. $-1 + 8i$
6. (i) $x = -1, y = 2$ (ii) $x = -4, y = 6$
7. (i) $-7 + 24i$ (ii) $-7\sqrt{3}$
8. (i) $\frac{3}{25} + \frac{4}{25}i$ (ii) $(1 + 2\sqrt{2}\ i)$
10. (i) i (ii) $\frac{16}{13} + \frac{11}{13}i$
11. Ans $x = 5$ and $y = 2$ or $x = 5$ and $y = -2$

Exercise-II

1. (i) i (ii) $i/72$
2. 0
4. $Z_1 + Z_2 = -2 + 7i$, $Z_1 - Z_2 = 8 + 3i$
5. (i) $x = 2, y = -1$
(ii) $x = 5/13, y = 14/13$
6. (i) $23 + 2i$
(ii) $-44 - 117i$
7. (i) $-\frac{1}{5} + \frac{3}{5}i$ (ii) $\left(-\frac{1}{4} - \frac{\sqrt{3}}{4}i\right)$
9. $11 - 60i$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. The conjugate of the complex number $\frac{(1+i)^2}{1-i}$ is

(a) $1 - i$ (b) $1 + i$
(c) $-1 + i$ (d) $-1 - i$

[Karnataka CET – 2007]

Solution

(d) $\frac{(1+i)^2}{1-i} = \frac{1 + i^2 + 2i}{1-i} = \frac{1 - 1 + 2i}{1-i}$

$= \frac{2i}{1-i} \times \frac{1+i}{1+i}$

$= \frac{2i(1+i)}{1-(i)^2} = \frac{2i(1+i)}{1-(-1)} = \frac{2i(1+i)}{2}$

$= i + i^2 = i - 1$

Therefore, the required conjugate is $-i - 1$

2. The real part of $(1 - \cos\theta + 2\,i\sin\theta)^{-1}$ is

[IIT – 1978, 1986]

(a) $\frac{1}{3 + 5\cos\theta}$ (b) $\frac{1}{5 - 3\cos\theta}$

(c) $\frac{1}{3 - 5\cos\theta}$ (d) $\frac{1}{5 + \cos\theta}$

Solution

(d) $\{(1 - \cos\theta) + i.2\sin\theta\}^{-1} =$

$\left\{2\sin^2\frac{\theta}{2} + i\cdot 4\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right\}^{-1}$

$= \left(2\sin\frac{\theta}{2}\right)^{-1}\left\{\sin\frac{\theta}{2} + i2\cos\frac{\theta}{2}\right\}^{-1}$

$= \left(2\sin\frac{\theta}{2}\right)^{-1}$

$\frac{1}{\sin\frac{\theta}{2} + i.2\cos\frac{\theta}{2}} \times \frac{\sin\frac{\theta}{2} - i.2\cos\frac{\theta}{2}}{\sin\frac{\theta}{2} - i.2\cos\frac{\theta}{2}}$

The real part is

$$\frac{\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}\left(1+3\cos^2\frac{\theta}{2}\right)}$$

$$=\frac{1}{2\left(1+3\cos^2\frac{\theta}{2}\right)}$$

$$=\frac{1}{5+3\cos\theta}$$

Alternative Method: Put $\theta = \pi$ to verify (d)

3. If $(a + ib)(x + iy) = (a^2 + b^2)i$, then (x, y) is equal to

(a) (a, b) (b) (b, a)
(c) $(-a, b)$ (d) $(a, -b)1$

[VIT – 2004]

Solution

(b) $(ax - by) + i(ay + bx) = (a^2 + b^2)i$

equating real and imaginary parts

$ax - by = 0 \Rightarrow ax = by$ (1)

$a^2 + b^2 = ay + bx$ (2)

$a^2 + b^2 = ay + \frac{b \times by}{2}$ from equation (1)

$\frac{a(a^2 + b^2)}{a^2 + b^2} = y \Rightarrow y = a$ and $x = b$

$\therefore (x, y) = (b, a)$

4. If $x = 2 + 5i$ (where $i^2 = -1$), then $x^3 - 5x^2 + 33x - 19 =$

(a) 6 (b) 8
(c) 10 (d) 12

Solution

(c) Given $x = 2 + 5i$

$\Rightarrow x - 2 = 5i$

$\Rightarrow (x - 2)^2 = 25(-1)$

$\Rightarrow x^2 - 4x + 4 + 25 = 0$

$\Rightarrow x^2 - 4x + 29 = 0$

Hence, $x^3 - 5x^2 + 33x - 10$

$= x(x^2 - 4x + 29) - x^2 + 4x - 19$

$= x(x^2 - 4x + 29) - 1(x^2 - 4x + 29) + 29 - 19$

$= x(x^2 - 4x + 29) - (x^2 - 4x + 29) + 10$

$= 0 + 10$ when $x = 2 + 5i$

($\because x^2 - 4x + 29 = 0$ when $x = 2+5i$)

5. The conjugate of a complex number is $\frac{1}{i-1}$. Then that complex number is

[AIEEE – 2008]

(a) $\frac{1}{i-1}$ (b) $\frac{-1}{i-1}$
(c) $\frac{1}{i+1}$ (d) $\frac{-1}{i+1}$

Solution

(d) $\bar{z} = \frac{1}{i-1}$ (given)

We have $z = (\bar{z})$ giving

$$z = \frac{1}{i-1} = \frac{1}{-i-1} = \frac{-1}{i+1}$$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. $2\sqrt{-9}\,\sqrt{-16}$ is equal to

(a) 24 (b) – 24
(c) 48 (d) – 48

2. If $x, y \in R$, then $x + yi$ is a non-real complex number, if

(a) $x = 0$ (b) $y = 0$
(c) $y \neq 0$ (d) $x \neq 0$

3. If n is any integer, then $i^n + i^{n+1} + i^{n+2} + i^{n+3}$ is equal to

(a) i (b) $-i$
(c) 1 (d) 0

[WB JEE – 2009]

4. If $\left(\frac{1+i}{1-i}\right)^m = 1$, then the least positive integral value of m is

(a) 2 (b) 4
(c) 8 (d) none of these

[IIT – 1982; MNR – 1984; UPSEAT – 2001; MPPET – 2002]

5. If the conjugate of $(x + iy)(1 - 2i)$ be $1 + i$, then **[MPPET – 1996]**

(a) $x = \frac{1}{5}$

(b) $y = \frac{3}{5}$

(c) $x + iy = \frac{1 - i}{1 - 2i}$

(d) $x - iy = \frac{1 - i}{1 + 2i}$

6. If $z = x - iy$ and $z^{1/3} = p + iq$, then $\frac{(x/p + y/q)}{(p^2 + q^2)}$ is equal to

(a) 2 (b) – 1

(c) 1 (d) – 2

[AIEEE – 2004]

7. Let z_1, z_2 be two complex numbers such that $z_1 + z_2$ and $z_1 z_2$ both are real, then **[RPET – 1996]**

(a) $z_1 = -z_2$ (b) $z_1 = \bar{z}_2$

(c) $z_1 = -\bar{z}_2$ (d) $z_1 = z_2$

8. The real part of $\frac{1}{1 - \cos\theta + i \sin\theta}$ is equal to

(a) 1/4 (b) 1/2

(c) $\tan \theta/2$ (d) $1/1 - \cos\theta$

[Karnataka CET – 2001, 2005; MPPET – 2006]

9. Multiplicative inverse of non-zero complex number $a + ib$ $(a, b \in R)$ is

(a) $\frac{a}{a - b} - \frac{b}{a + b} \times i$

(b) $\frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i$

(c) $-\frac{a}{a^2 + b^2} + \frac{b}{a^2 + b^2} i$

(d) $\frac{a}{a + b} + \frac{b}{a + b} i$

10. If $\frac{i^4 + i^9 + i^{16}}{2 - i^8 + i^{10} + i^3} = a + ib$, then (a, b) is **[Kerala PET – 2008]**

(a) (1, 2) (b) (– 1, 2)

(c) (2, 1) (d) (– 2, – 1)

11. If Z is a complex number such that $Z = -\bar{z}$ then

(a) Z is any complex number.

(b) Real part of Z is the same as its imaginary part.

(c) Z is purely real.

(d) Z is purely imaginary.

[Karnataka CET – 2008]

12. If $z = 3 + 5i$, then $z^3 + \bar{z} + 198 =$

(a) $-3 - 5i$ (b) $-3 + 5i$

(c) $3 + 5i$ (d) $3 - 5i$

[EAMCET – 2002]

SOLUTIONS

1. (b) $2\sqrt{-9}\,\sqrt{-16} = 2\,(3i)(4i) = 24\, i^2 = -24.$

2. (c) $x + yi;\ x, y \in R$ is non real if $y \neq 0$.

3. (d) $i^n + i^{n+1} + i^{n+2} + i^{n+3}$
$= i^n\,[1 + i + i^2 + i^3]$
$= i^n\,[1 + i - 1 - i] = 0.$

4. (b) $1 = \left(\frac{1 + i}{1 - i}\right)^m = \left[\frac{i(1 - i)}{(1 - i)}\right]^m = i^m$

$\Rightarrow i^m = 1 \Rightarrow m = 4.$

5. (c) Given

$\overline{(x + iy)(1 - 2i)} = 1 + i$ (1)

using formula: $\overline{(\bar{z})} = z$ we find from (1)

$(x + iy)(1 - 2i) = \overline{1 + i} = 1 - i$

$(x + iy)(1 - 2i) = 1 - i$

$x + iy = \frac{1 - i}{1 - 2i}.$

6. (d) Given $z^{1/3} = p + iq \Rightarrow z = (p + iq)^3$

$\Rightarrow x - iy = p^3 + (iq)^3 + 3piq\,(p + iq)$

$\Rightarrow x - iy = p^3 - 3pq^2 + (3p^2q - q^3)\, i$

Equating real and imaginary parts,

$x = p^3 - 3pq^2 \quad \Rightarrow \frac{x}{p} = p^2 - 3q^2$ (1)

and $-y = 3p^2q - q^3$

$\Rightarrow \frac{y}{q} = q^2 - 3p^2$ (2)

Add (1) and (2) to obtain

$\frac{x}{p} + \frac{y}{q} = -2(p^2 + q^2)$

7. (b) Let $z_1 = a + ib$, $z_2 = c + id$, then

$z_1 + z_2$ is real $\Rightarrow (a + c) + i(b + d)$ is real

$\Rightarrow b + d = 0 \Rightarrow d = -b$ (3)

z_1z_2 is real $\Rightarrow (ac - bd) + i\,(ad + bc)$ is real

$\Rightarrow ad + bc = 0$

$\Rightarrow a(-b) + bc = 0 \Rightarrow a = c.$

Therefore, $\Rightarrow z_1 = a + ib = c - id = \bar{z}_2$

$(\because a = c$ and $b = -d)$

8. (b) $\frac{1}{(1-\cos\theta) + i\sin\theta}$

$$\frac{\{(1-\cos\theta) - i\sin\theta\}}{\{(1-\cos\theta)+i\sin\theta)\}\{(1-\cos\theta)-i\sin\theta)\}}$$

$$= \frac{(1-\cos\theta) - i\sin\theta}{(1-\cos\theta)^2 - i^2\sin^2\theta}$$

$$= \frac{(1-\cos\theta) - i\sin\theta}{1 + \cos^2\theta - 2\cos\theta + \sin^2\theta}$$

$$= \frac{(1-\cos\theta) - i\sin\theta}{2(1-\cos\theta)}$$

$$= \frac{1}{2} - \frac{i\sin\theta}{2(1-\cos\theta)}$$

Real part of $\left(\frac{1}{(1-\cos\theta) + i\sin\theta}\right) = \frac{1}{2}.$

9. (b) $(a + bi)^{-1}$

$$= \frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)}$$

$$= \frac{a-bi}{a^2+b^2}.$$

10. (b) $\frac{i^4 + i^9 + i^{16}}{2 - i^8 + i^{10} + i^3} = a$

$$\frac{1 + i + 1}{2 - 1 + i^2 + i \times i^2} = a + ib$$

$$\frac{2+i}{1-1-i} = a + ib$$

$$\frac{(2+i)}{(-i)} \times \frac{(i)}{i} = a + ib$$

$2i - 1 = a + ib$

Comparing real and imaginary part $a = -1$, $b = 2$

$(a, b) = (-1, 2).$

11. (d) $z = -\bar{z} \Rightarrow x + iy = -(x - iy) = -x + iy$

i.e., $2x = 0$ or $x = 0$

i.e., $z = iy$ = purely imaginary.

12. (c) $z = 3 + 5i$

$\bar{z} = 3 - 5i$

$z^3 = (3 + 5i)^3$

$= 27 + 125i^3 + 45i\,(3 + 5i)$

$= 27 - 125\,i + 135i + 225\,i^2$

$= 27 - 225 + 10i$

$= -198 + 10i$

$z^3 + \bar{z} + 198$

$= -198 + 10i + 3 - 5i + 198$

$= 3 + 5i$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. $\left(\frac{1+i}{1-i}\right)^2 + \left(\frac{1-i}{1+i}\right)^2$ is equal to

(a) $2i$

(b) $-2i$

(c) -2

(d) 2

2. The conjugate of the complex number $\frac{2+5i}{4-3i}$ is **[MPPET – 1994]**

(a) $\frac{7-26i}{25}$ (b) $\frac{-7-26i}{25}$

(c) $\frac{-7+26i}{25}$ (d) $\frac{7+26i}{25}$

3. $\left\{\frac{2i}{1+i}\right\}^2 =$

[MNR – 1984; BIT Ranchi – 1992]

(a) 1 (b) $2i$
(c) $1 - i$ (d) $1 - 2i$

4. The imaginary part of $\frac{(1+i)^2}{2-i}$ is

(a) 1/5 (b) 3/5
(c) 4/5 (d) none of these

5. If $\frac{5(-8+6i)}{(1+i)^2} = a + ib$, then (a, b) equals

[RPET – 1986]

(a) (15, 20) (b) (20, 15)
(c) (– 15, 20) (d) none of these

6. The true statement is

[Roorkee – 1989]

(a) $1 - i < 1 + i$
(b) $2i + 1 > -2i + 1$
(c) $2i > 1$
(d) none of these

7. If $z = x + iy$, $z^{1/3} = a - ib$ and $\frac{x}{a} - \frac{y}{b} = k(a^2 - b^2)$ then value of k equals

[DCE – 2005,MPPET–2009]

(a) 2 (b) 4
(c) 6 (d) 1

8. If $x = 2 + 3i$ and $y = 2 - 3i$, then value of $x^3 + y^3$ is

(a) 92 (b) – 92
(c) 46 (d) – 46

9. Real part of $\frac{i}{3+2i}$ is

(a) 3 (b) 2/13
(c) 13/2 (d) 13

10. Conjugate of $\frac{2+3i}{-i+1}$ is

(a) $\frac{2-3i}{i+1}$ (b) $\frac{2-3i}{1-i}$
(c) $\frac{2+3i}{1+i}$ (d) $\frac{2+3i}{1+i}$

WORK SHEET: TO CHECK PREPARATION LEVEL

Important Instructions:

1. The answer sheet is immediately below the work sheet
2. The test is of 13 minutes.
The test consists of 13 questions.
The maximum marks are 39.
3. Use blue / black ball point pen only for writing particulars / marking responses. Use of pencil is strictly prohibited.
4. Rough work is to be done on the space provided for this purpose on the worksheet – 1 sheet only.

1. If $(x + iy)^{1/3} = a + ib$, then $\frac{x}{a} + \frac{y}{a}$ is equal to

(a) $4(a^2 + b^2)$
(b) $4(a^2 - b^2)$
(c) $4(b^2 - a^2)$
(d) none of these

[IIT – 1982, Karnataka CET – 2000]

2. The number $\frac{(1-i)^3}{(1-i^3)}$ is equal to

[Pb. CET – 1991]

(a) i (b) $-i$
(c) – 1 (d) – 2

3. The value of $-i^{51}$ is

(a) $-i$ (b) 1
(c) – 1 (d) i

4. Which of the following is not applicable for a complex number?

(a) addition (b) subtraction
(c) division (d) inequality

5. The value of

$\frac{i^{592} + i^{590} + i^{588} + i^{586} + i^{584}}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}} =$

(a) – 1 (b) – 2
(c) – 3 (d) – 4

6. The least positive integer n for which $(1 + i)^{2n} = (1 - i)^{2n}$ is

(a) 2 (b) 4
(c) 1 (d) 8

7. $i^{57} + \frac{1}{i^{125}}$ is equal to

(a) 0 (b) $2i$
(c) $-2i$ (d) 2

8. If $2x = 3 + 5i$, then what is the value of $2x^3 + 2x^2 - 7x + 72$?

(a) 4 (b) -4
(c) 8 (d) -8

[NDA–2009]

9. If $x, y \in R$, then the complex number $x + yi$ is purely imaginary, if

(a) $x = 0, y \neq 0$
(b) $x \neq 0, y = 0$
(c) $x \neq 0, y \neq 0$
(d) $x = 0, y = 0$

10. $1 + i + i^2 + i^3$ is equal to

(a) i (b) 0
(c) $-i$ (d) 1

11. If z is a complex number such that $z \neq 0$ and Re $z = 0$, then

(a) Re $(z^2) = 0$
(b) Re $(z^2) =$ Im (z^2)
(c) Im $(z^2) = 0$
(d) none of these

12. Which of the following is correct?

(a) $5 + 3i > 6 + 4i$
(b) $5 + 3i = 6 + 4i$
(c) $5 + 3i < 6 + 4i$
(d) none of these

13. The conjugate of complex number $\frac{2-3i}{4-i}$ is **[MPPET – 2003]**

(a) $\frac{3i}{4}$
(b) $\frac{11+10i}{17}$
(c) $\frac{11-10i}{17}$
(d) $\frac{2+3i}{4i}$

ANSWER SHEET

1.	(a)	(b)	(c)	(d)	6.	(a)	(b)	(c)	(d)	11.	(a)	(b)	(c)	(d)
2.	(a)	(b)	(c)	(d)	7.	(a)	(b)	(c)	(d)	12.	(a)	(b)	(c)	(d)
3.	(a)	(b)	(c)	(d)	8.	(a)	(b)	(c)	(d)	13.	(a)	(b)	(c)	(d)
4.	(a)	(b)	(c)	(d)	9.	(a)	(b)	(c)	(d)					
5.	(a)	(b)	(c)	(d)	10.	(a)	(b)	(c)	(d)					

HINTS AND EXPLANATIONS

5. (b) $\frac{i^{584}(i^8 + i^6 + i^4 + i^2 + 1)}{i^{574}(i^8 + i^6 + i^4 + i^2 + 1)} = i^{584-574}$

$= (i)^{10} = (i^2)^5 = (-1)^5 = -1$

8. (a) $\because x = \frac{3+5i}{2}$

$2x - 3 = 5i$

squaring we get

$2x^2 - 6x + 17 = 0$

Now

$$\begin{array}{r} x + 4 \\ 2x^2 - 6x + 17 \,\overline{)\, 2x^3 + 2x^2 - 7x + 72} \\ 2x^2 - 6x^2 + 17x \\ \hline 8x^2 - 24x + 72 \\ 8x^2 - 24x + 68 \\ \hline 4 \end{array}$$

$\therefore\ 2x^3 + 2x^2 - 17x + 72$

$= (2x^2 - 6x + 12)(x + 4) + 4$

$= 4$

9. (a) $x + yi$; $x, y \in R$ is purely imaginary if $x = 0$ and $y \neq 0$.

10. (b) $1 + i + i^2 + i^3$

$= 1 + i - 1 + (-1)\,i$

$= 1 + i - 1 - i$

$= 0$

11. (c) Let $z = iy$; $y \in R, y \neq 0$

then $z^2 = (iy)^2 = -y^2 \Rightarrow \text{Im}(z^2) = 0$.

13. (b) $\frac{(2-3i)}{(4-i)} \times \frac{(4+i)}{(4+i)} = \frac{8 + 2i - 12i - 3i^2}{(4)^2 - i^2}$

$\frac{11 - 10i}{16 + 1} = \frac{11 - 10i}{17}$

Conjugate of complex number $= \frac{11 + 10i}{17}$

This page is intentionally left blank.

LECTURE

2 Argand Plane Modulus and Amplitude

BASIC CONCEPTS

1. **Geometrical Representation of a Complex Number** The plane on which complex numbers are represented is known as the complex plane or Argand's plane or Gaussian plane. In this representation all real numbers lie on x-axis and imaginary numbers lie on y-axis. The x-axis is called the real axis and y-axis is known as the imaginary axis.

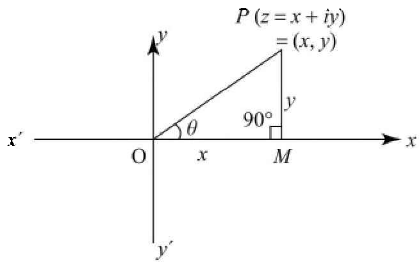
$-\bar{z} = -(\bar{z}) = -x + iy$ $= (-x, y)$	*y-axis* $z - x + iy = (x, y)$
Origin $-z = -x - iy$ $= (-x, y)$	Real axis $\bar{z} = x - iy$ $= (x, -y)$

The complex numbers $z = x + iy$ may be represented by a unique point in xy-plane the coordinates of which are (x, y). One-to-one correspondence is defi ned between the set of complex numbers and set of all point of Argand's plane or xy-plane.

(i) Distance between two points z_1 and z_2 = $|z_1 - z_2|$

(ii) Complex numbers are defined as vectors. The magnitude and direction of vectors are called magnitude and amplitude of complex numbers.

2. **Modulus—Amplitude Form or Polar Form of a Complex Number**



(i) $z = r(\cos\theta + i\sin\theta)$

(ii) The modulus of z is denoted by $|z|$ and $|z| = \sqrt{x^2 + y^2} = r \geq 0$

(iii) The argument or amplitude of z is denoted by arg (z) or amp (z). In the first quadrant $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$ and in other quadrants, arg (z) is defined by the solution of the following two equations which are as follows

$$\sin\theta = \frac{y}{\sqrt{x^2 + y^2}};\ \cos\theta = \frac{x}{\sqrt{x^2 + y^2}}$$

(iv) The principal value of argument of a complex number lies between $-\pi$ and π. i.e., the value of θ of the argument which satisfies the inequality $-\pi < \theta \leq \pi$ is called the principal value of the argument.

3. **Quicker Method for Finding Amplitude of a Complex Number Quadrantwise** In this method first of all quadrant of

complex number is known as follows, Here quadrantwise complex number with corresponding argument are given

$(-\bar{z}) = -(\bar{z}) = -x + iy$ amp $(-\bar{z})$ = amp $(-(\bar{z}))$ $= \pi - \theta$	$z = (x, y) = x + iy$ if amp $(z) = \theta$, then
amp $(-z) = -(\pi - \theta)$ $-z = -x - iy = (-x, -y)$	$\bar{z} = x - iy = (x, y)$ amp $(\bar{z}) = -\theta$

Note: $|z| = |-z| = |(-\bar{z})| = (\bar{z})| = |-(\bar{z})|$

$$= \sqrt{x^2 + y^2}$$

Note: In this method corresponding form of a given complex number in first quadrant is obtained and argument of this complex number is obtained by the formula $\theta = \tan^{-1}\frac{y}{x}$. After this, amplitude of given complex number is obtained quadrant-wise

(v) If amp (z) i.e., θ is greater than π, then principal value of argument is equal to $\theta - 2\pi$.

(vi) If amp (z) i.e., θ is less than $-\pi$, then principal value of amplitude is $\theta + 2\pi$.

(vii) If θ is the principal value of the argument of a complex number, then its general value is denoted by $\theta + 2n\pi$, where n is any integer + ve or − ve.

(viii) Argument of the complex number 0 is not defined.

(ix) Argument of the positive real number = 0°.

(x) Argument of the negative real number = π radian = 180°

(xi) Amplitude of a positive purely imaginary number, positive imaginary part = $\pi/2$.

(xii) Amplitude of the negative purely imaginary number = $-\frac{\pi}{2}$

4. Properties of Complex Number Connectedc with Magnitudes of Complex Numbers

(i) $|z| \geq 0$

(ii) $|z_1 z_2| = |z_1|\,|z_2|$

(iii) $\frac{|z_1|}{|z_2|} = \frac{|z_1|}{|z_2|}$

(iv) $|z| = |-z| = |\bar{z}| = |-\bar{z}|$

(v) $|z\bar{z}| = |z|^2$

(vi) $\frac{|z|}{|\bar{z}|} = 1$

(vii) $\left|\frac{z_1 z_2 z_3}{z_4 z_5}\right| = \frac{|z_1|\,|z_2|\,|z_3|}{|z_4|\,|z_5|}$

(viii) $|z^n| = |z|^n$

(ix) $|z| = 1 \Leftrightarrow \bar{z} = \frac{1}{z}$

5. Properties of Complex Numbers Connected with the Amplitude of Complex Numbers

(i) If amp $(z) = \theta$, then the general value of amp (z) is $2n\pi + \theta$, $n = 0, 1, 2, ...$ and principle value lies between $-\pi$ and $\pi(-\pi <$ amp $(z) \leq \pi)$

(ii) amp (z) = amp$(1/z)$ = − amp(z)

(iii) amp$(-z) = -\pi +$ amp$(z) = -(\pi -$ amp $(z))$

(iv) amp$(z^n) = n$ (amp (z))

(v) amp$(z_1 z_2)$ = amp (z_1) + amp (z_2)

(vi) amp $\left(\frac{z_1}{z_2}\right)$ = amp (z_1) − amp (z_2)

(vii) amp $(iz) = \frac{\pi}{2}$ + amp (z)

amp$(-iz) = -\frac{\pi}{2}$ + amp (z)

(ix) amp (z) + amp $(\bar{z})$ = 0 or $2n\pi$

(x) The argument of the complex number 0 is not defined.

(xi) If k is real number,

amp $(k) = 0$	$k > 0$
$= \pi$	$k > 0$
= not defined	$k = 0$

SOLVED SUBJECTIVE PROBLEMS: (CBSE/STATE BOARD): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Prove that the points represent the complex numbers $3 + 3i$, $-3 - 3i$, $-3\sqrt{3} + 3\sqrt{3}i$ form an equilateral triangle. Also, find the area of the triangle.

Solution

Let the complex numbers $3 + 3i$, $-3 - 3i$, and $-3\sqrt{3} + 3\sqrt{3}i$ be represented by points $A(3, 3)$, $B(-3, -3)$ and $C(-3\sqrt{3}, 3\sqrt{3})$, respectively on the Argand plane.

Then, $AB = \sqrt{(3+3)^2 + (3+3)^2}$

$= \sqrt{36\ 36} = 6\sqrt{2}$

$BC = \sqrt{(-3 + 3\sqrt{3})^2 + (-3 - 3\sqrt{3})^2}$

$= \sqrt{9 + 27 - 18\sqrt{3} + 27 + 9 + 18\sqrt{3}}$

$\Rightarrow \sqrt{72} = 6\sqrt{2}$

and $CA = \sqrt{(-3\sqrt{3} - 3)^2 + (3\sqrt{3} - 3)^2}$

$= \sqrt{27 + 9 + 18\sqrt{3} + 27 + 9 - 18\sqrt{3}}$

$\Rightarrow \sqrt{72} = 6\sqrt{2}$

Obviously, $AB = BC = CA$

Therefore, the given points form an equilateral triangle.

Therefore,

area of this triangle $= \frac{\sqrt{3}}{4} \times (\text{Side})^2$

$= \frac{\sqrt{3}}{4} \times (6\sqrt{2})^2 = 18\sqrt{3}$ square unit.

2. If $z_1 = 2 - i$, $z_2 = 1 + i$, find $\left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + i}\right|$

[NCERT]

Solution

Given $z_1 = 2 - i$, $z_2 = 1 + i$.

$\therefore \left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + i}\right|$

$= \frac{|(2-i) + (1+i) + 1|}{|2 - i - (1+i) + i|} \left(\because \left|\frac{z_1}{z_2}\right| = \left|\frac{z_1}{z_2}\right|\right)$

$= \frac{|4|}{|1 - i|} = \frac{4}{\sqrt{1^2 + (-1)^2}} = \frac{4}{\sqrt{2}}$

$= 2\sqrt{2}$

3. If $a + ib = \frac{(x+i)^2}{2x^2 + 1}$, prove that

$a^2 + b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$.

[NCERT]

Solution

Given $a + ib = \frac{(x+i)^2}{2x^2+1}$

$\Rightarrow |a + ib| = \left|\frac{(x+i)^2}{2x^2+1}\right|$

$\Rightarrow |a + ib| = \left|\frac{(x+i)^2}{2x^2+1}\right|$

$\Rightarrow \sqrt{a^2 + b^2} = \frac{|x+i|^2}{\sqrt{(2x^2+1)^2 + 0^2}}$

$\Rightarrow \sqrt{a^2 + b^2} = \frac{\left(\sqrt{x^2 + 1^2}\right)^2}{\sqrt{(2x^2+1)^2}}$

Squaring the two sides, we get $a^2 + b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$

4. Let $z_1 = 2 - i$, $z_2 = -2 + i$. Find

[NCERT]

(i) $\text{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right)$ (ii) $\text{Im}\left(\frac{1}{z_1 \bar{z}_2}\right)$

Solution

Given $z_1 = 2 - i$, $z_2 = -2 + i$

(i) Therefore, $z_1 z_2 = (2 - i)(-2 + i)$

$= -4 - (-1) + 2i + 2i$

$= -3 + 4i$

$\text{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right) = \text{Re}\left(\frac{-3 + 4i}{2 - i}\right)$

$= \text{Re}\left(\frac{-3 + 4i}{2 + i}\right)$

$= \text{Re}\left(\frac{-3 + 4i}{2 + i} \times \frac{2 - i}{2 - i}\right)$

$= \text{Re}\left(\frac{-6 - 4(-1) + 11i}{4 - (-1)}\right)$

$= \text{Re}\left(\frac{-2+11i}{5}\right)$

$= \text{Re}\left(-\frac{2}{5}+\frac{11}{5}i\right) = -\frac{2}{5}$

(ii) $z_1\bar{z}_2 = (2-i)\overline{(-2+i)}$

$= (2-i)(-2-i)$

$= (-i+2)(-i-2)$

$= (-i^2) - 2^2 = -1 - 4 = -5$

Therefore, $I_m\left(\frac{1}{z_1\bar{z}_2}\right) = I_m\left(\frac{1}{-5}\right)$

$= I_m\left(-\frac{1}{5}+0i\right) = 0$

5. Find the number of non–zero integral solutions of the equation $|1-i|^x = 2^x$.

[NCERT]

Solution

Given equation is $|1-i|^x = 2^x$

$\Rightarrow \left(\sqrt{1^2+(-1)^2}\right)^x = 2^x$

$\Rightarrow (2^{1/2})^x = 2^x \Rightarrow 2^{x/2} = 2^x$

$\Rightarrow \frac{x}{2} = x$

$\Rightarrow 2x = x \qquad \Rightarrow x = 0.$

Hence, the given equation has no nonzero integral solution.

6. If $(a+ib)(c+id)(e+if)(g+ih) = A + iB$, then show that $(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = A^2+B^2$.

[NCERT]

Solution

Given $(a+ib)(c+id)(e+if)(g+ih) = A+iB$

$\Rightarrow |(a+ib)(c+id)(e+if)(g+ih)|$

$= |A+iB|$

$\Rightarrow |a+ib|\,|c+id|\,|e+if|\,|g+ih|$

$= |A+iB|$

$\Rightarrow \sqrt{a^2+b^2}\sqrt{c^2+d^2}\sqrt{e^2+f^2}\sqrt{g^2+h^2}$

$= \sqrt{A^2+B^2}$

$\Rightarrow (a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2)$

$= A^2+B^2$

Alternatively,

$(a+ib)(c+id)(e+if)(g+ih) = A+iB$ (1)

$\Rightarrow \overline{(a+ib)(c+id)(e+if)(g+ih)} = \overline{A+iB}$

(Taking conjugates on the two sides)

$\Rightarrow (a-ib)(c-id)(e-if)(g-ih)$

$= A - iB$ (2)

Multiply (1) and (2), we get

$(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = A^2+B^2$

$(\because (a+ib)(a-ib) = a^2+b^2$ *etc.*)

Note: In this problem it should have been given that $a, b, c, d, e, f, g, h, A, B$ are real numbers.

7. For any two complex numbers z_1 and z_2, prove that $\text{Re}(z_1z_2) = \text{Re}\, z_1\, \text{Re}\, z_2 - \text{Im}\, z_1\, \text{Im}\, z_2$.

Solution

Let $z_1 = a+bi$ and $z_2 = c+di$, where a, b, c, d are real numbers, then $z_1z_2 = (a+bi)(c+di) = (ac-bd) + i(bc+ad) \Rightarrow \text{Re}(z_1z_2) = ac - bd = (\text{Re}\, z_1)(\text{Re}\, z_2) - (\text{Im}\, z_1)(\text{Im}\, z_2)$.

8. If $(a+ib)(c+id) = x+iy$, then prove that $(a-ib)(c-id) = x-iy$ and $(a^2+b^2)(c^2+d^2) = x^2+y^2$.

Solution

Given, $(a+ib)(c+id) = x+iy$

$\Rightarrow (ac-bd) + i(ad+bc) = x+iy$

Comparing the imaginary and real parts of both the sides,

$ac - bd = x.$ (1)

and $ad + bc = y$ (2)

Now, $(a-ib)(c-id) = (ac-bd) - i(ad+bc) = x - iy$ [From Equations (1) and (2)]

Proved

Again, $(a+ib)(c+id) = x+iy$ and $(a-ib)(c-id) = x-iy$

Multiplying them,

$[(a+ib)(c+id)][c-ib)(c-id)]$

$= (x+iy)(x-iy)$

$\Rightarrow [(a+ib)(a-ib)]\,[(c+id)(c-id)]$

$= (x+iy)(x-iy)$

$\Rightarrow (a^2 - i^2b^2)(c^2 - i^2d^2) = x^2 - i^2y^2$

$\Rightarrow (a^2 + b^2)(c^2 + d^2) = x^2 + y^2$

9. Find the locus of a complex variable z in the Argand plane, satisfying $|z - (3 - 4i)| = 7$.

Solution

Let $z = (x + iy)$ Then, $|z - (3 - 4i)| = 7$

$\Rightarrow |(x + iy) - (3 - 4i)|^2 = 7^2$

$\Rightarrow |(x - 3) + i(y + 4)|^2 = 7^2$

$\Rightarrow (x - 3)^2 + (y + 4)^2 = 7^2$

$\Rightarrow x^2 + 9 - 6x + y^2 + 16 + 8y = 49$

$\Rightarrow x^2 + y^2 - 6x + 8y - 24 = 0$

10. If $\frac{2z_1}{3z_2}$ be a purely imaginary number, then prove that

$$\left|\frac{z_1 - z_2}{z_1 + z_2}\right| = 1$$

[Similar to MPPET – 1993]

Solution

Let $\frac{2z_1}{3z_2} = ai$,

$\left[\because \frac{2z_1}{3z_1} \text{ is a purely imaginary number}\right]$

$\Rightarrow \frac{z_1}{z_2} = \frac{3ai}{2} \Rightarrow \frac{z_1 - z_2}{z_1 + z_2} = \frac{3ai - 2}{3ai + 2}$

$\Rightarrow \left|\frac{z_1 - z_2}{z_1 + z_2}\right| = \left|\frac{3ai - 2}{3ai + 2}\right|$

$\Rightarrow \left|\frac{z_1 - z_2}{z_1 + z_2}\right| = \frac{\sqrt{9a^2 + 4}}{\sqrt{9a^2 + 4}}$

$\Rightarrow \left|\frac{z_1 - z_2}{z_1 + z_2}\right| = 1$

Proved

11. Express the following expression in the form $a + ib$

$$\frac{(3 + i\sqrt{5})(3 - i\sqrt{5})}{(\sqrt{3} + \sqrt{2}i) - (\sqrt{3} - i\sqrt{2})}$$

Solution

Given complex number

$$= \frac{(3 + i\sqrt{5})(3 - i\sqrt{5})}{(3 + \sqrt{2}i) - (\sqrt{3} - i\sqrt{2})}$$

$$\frac{3^2 - (i\sqrt{5})^2}{(\sqrt{3} - \sqrt{3}) + (\sqrt{2} + \sqrt{2})i}$$

$$= \frac{9 - 5i^2}{0 + 2\sqrt{2}i} = \frac{9 - 5(-1)}{2\sqrt{2}i} \times \frac{i}{i}$$

$$= \frac{7i}{\sqrt{2}i^2} = \frac{7i}{\sqrt{2}(-1)} = 0 + \left(-\frac{7}{\sqrt{2}}\right)i$$

which is in the form $a + ib$

when $a = 0$, $b = -\frac{7}{\sqrt{2}}$.

12. Find the modulus and argument of the complex number $\frac{1 + 2i}{1 - 3i}$.

[NCERT]

Solution

Given complex number =

$$\frac{1 + 2i}{1 - 3i} = \frac{1 + 2i}{1 - 3i} \times \frac{1 + 3i}{1 + 3i}$$

$$= \frac{1 + 6(-1) + 2i + 3i}{1 - 9(-1)}$$

$$= \frac{-5 + 5i}{10} = -\frac{1}{2} + \frac{1}{2}i$$

Let $-\frac{1}{2} + \frac{1}{2}i = r(\cos\theta + i\sin\theta)$

$\Rightarrow -\frac{1}{2} = r\cos\theta$ (1)

and $\frac{1}{2} = r\sin\theta$ (2)

squaring (1) and (2) and adding

$\frac{1}{4} + \frac{1}{4} = r^2(\cos^2\theta + \sin^2\theta)$

$\Rightarrow r^2 = \frac{1}{2}$

$\Rightarrow r = \frac{1}{\sqrt{2}}$ $(\because r \nless 0)$

substituting this value of r in (1) and (2), we get

$-\frac{1}{2} = \frac{1}{\sqrt{2}}\cos\theta$ and $\frac{1}{2} = \frac{1}{\sqrt{2}}\sin\theta$

$\Rightarrow \cos\theta = -\frac{\sqrt{2}}{2} = -\frac{1}{\sqrt{2}}$ and

$\sin\theta = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

As θ lies in the second quadrant, $(\because \cos\theta < 0$ and $\sin\theta > 0)$ therefore, we can write,

$\cos\theta = -\frac{1}{\sqrt{2}} = -\cos 45°$ and

$\sin\theta = \frac{1}{\sqrt{2}} = \sin 45°$

i.e $\cos\theta = (180° - 45°) \sin\theta$

$= \sin(180° - 45°)$

$\Rightarrow \theta = 135°$ or $\frac{3\pi}{4}$

Therefore, modulus of the given complex number $= \frac{1}{\sqrt{2}}$

and its amplitude $= \theta = \frac{3\pi}{4}$

13. Convert the following expression in the polar form $-1 - i$.

Solution

Let $-1 - i = r(\cos\theta + i\sin\theta)$

$\Rightarrow (-1) + (-1)i = (r\cos\theta) + i(r\sin\theta)$

$\Rightarrow r\cos\theta = -1$ (1)

and $\sin\theta = -1$ (2)

Squaring (1) and (2) and adding

$r^2(\cos\theta + \sin^2\theta) = 1 + 1$

$\Rightarrow r^2 = 2 \quad \Rightarrow r = \sqrt{2}$ $(\because r \not< 0)$

Then from (1) and (2)

$\cos\theta = -\frac{1}{r} = -\frac{1}{\sqrt{2}}$ and

$\sin\theta = -\frac{1}{r} = -\frac{1}{\sqrt{2}}$

As θ lies in the third quadrant,
($\because \cos\theta < 0$ and also $\sin\theta < 0$)
therefore, we write

$\cos\theta = -\frac{1}{\sqrt{2}} = -\cos\frac{\pi}{4}$

$= \cos\left(\pi + \frac{\pi}{4}\right) = \cos\frac{5\pi}{4}$

and $\sin\theta = -\frac{1}{\sqrt{2}} = -\sin\frac{\pi}{4}$

$= \sin\left(\pi + \frac{\pi}{4}\right) = \sin\frac{5\pi}{4}$

$\Rightarrow \theta = \frac{5\pi}{4}$ Hence, $-1 - i$

$= \sqrt{2}\left\{\cos\left(\frac{5\pi}{4}\right) + \sin\left(\frac{5\pi}{4}\right)\right\}$

$= \sqrt{2}\left\{\cos\left(\frac{5\pi}{4} - 2\pi\right) + \sin\left(\frac{5\pi}{4} - 2\pi\right)\right\}$

$= \sqrt{2}\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)$

14. If $z = (\sqrt{2} - \sqrt{-3})$, find Re($z$), Im($z$), $\bar{z}$ and $|z|$.

Solution

$z = (\sqrt{2} - \sqrt{-3}) = \sqrt{2} - i\sqrt{3}$

$\therefore$ Re(z) $= \sqrt{2}$, Im(z) $= -\sqrt{3}$, $\bar{z} = (\sqrt{2} + i\sqrt{3})$

and $|z| = \sqrt{(\sqrt{2})^2 + (-\sqrt{3})^2} = \sqrt{2 + 3} = \sqrt{5}$

SUBJECTIVE UNSOLVED PROBLEMS: (CBSE/STATE BOARD): TO GRASP THE TOPIC, SOLVE THESE PROBLEMS

Exercise I

1. If $\frac{(a+i)^2}{2a - i} = p + iq$, show that $p^2 + q^2 = \frac{(a^2+1)^2}{4a^2 - 1}$.

2. Prove that the sum and product of two complex numbers are real if and only if they are conjugate to each other.

3. If $(a + ib) = \sqrt{\frac{1+i}{1-i}}$, then prove that $(a^2 + b^2) = 1$.

4. Find the multiplicative inverse of the following complex numbers.
(i) $3 + 2i$ (ii) $(2 + \sqrt{3}i)^2$ (iii) $3 - 2i$

5. Separate real and imaginary parts of $\frac{4+3i}{3+i}$ and find the modulus.

6. If Z_1, Z_2 are $1 - i$, $-2 + 4i$, respectively, find $I_m\left(\frac{z_1 z_2}{\bar{z}_1}\right)$

7. If $\left|\frac{z - 5i}{z + 5i}\right| = 1$, show that z is a real number.

8. For all $Z \in C$, prove that

(i) $(\bar{\bar{z}}) = z$

(ii) $z\bar{z} = |z|^2$

(iii) $(z + \bar{z})$ is real

(iv) $(z - \bar{z})$ is 0 or imaginary.

9. Show that the points respresented by complex numbers $-4 + 3i$, $2 - 3i$ and $-i$ are collinear.

10. Write the following complex numbers in the polar form

(i) $-3\sqrt{2} + 3\sqrt{2}i$ (ii) 1+ι (iii) −3

Exercise II

1. If $z = 2 + 3i$, then show that $z^2 - 4z + 13 = 0$.

2. Find the modulus of $\left(\frac{1+i}{1-i} - \frac{1-i}{1+i}\right)^n$.

3. Find the multiplicative inverse of $\frac{3+4i}{3i}$.

4. If $z_1, z_2 \in C$ prove that
$\text{Im}(z_1 z_2) = \text{Re}(z_1) \cdot \text{Im}(z_2) + \text{Im}(z_1) \cdot \text{Re}(z_2)$.

5. For all $z_1, z_2 \in C$, prove that

(i) $(\overline{z_1 + z_2}) = \overline{z_1} + \overline{z_2}$

(ii) $(\overline{z - z_2}) = (\overline{z_1} - \overline{z_2})$

(iii) $(\overline{z_1 z_2}) = \overline{z_1 z_2}$.

6. Prove that $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$, where $z_2 \neq 0$, for all $z_1, z_2 \in C$.

7. Express $6(\cos 120° + i \sin 120°)$ in the form of $x + iy$.

8. Find the magnitude of $(1 + i)(1 + 2i)(1 + 3i)$.

9. Find the magnitude of (1 + i) (1 + 2i) (1 +3i).

ANSWER SHEET

Exercise I

4. (i) $\frac{3}{13} - \frac{2}{13}i$

(b) $\frac{1}{49} - \frac{4\sqrt{3}}{49}i$

(iii) $\frac{3}{13} + \frac{2}{13}i$

5. $\frac{3}{2} + \frac{1}{2}i$ and $\sqrt{\frac{5}{2}}$

6. $I_m\left(\frac{z_1 z_2}{\overline{z_1}}\right) = 2$

10. (i) $6\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

(ii) $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ (iii) $3(\cos\pi + i\sin\pi)$

Exercise II

2. $|z| = 2$

3. $\frac{12}{25} + \frac{9}{25}i$

7. $Z = -3 + 3\sqrt{3}\,i$

8. 10

9. 10

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. Let $\arg z < 0$ then $\arg(-z) - \arg z =$

[Orissa JEE – 2007]

(a) π (b) $\pi/2$

(c) $\pi/3$ (d) none of these

Solution

(a) $\arg z < 0$ (given), Therefore, $\arg z = -\theta < 0$

$z = |z|(\cos(-\theta) + i\sin(-\theta))$

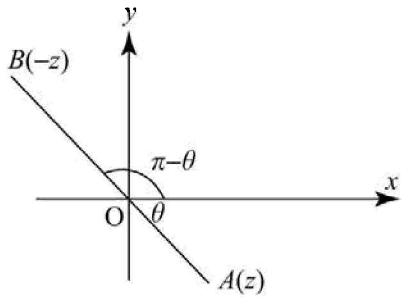
$= |z|(\cos\theta - i\sin\theta)$

$\therefore -z = |z|(-\cos\theta + i\sin\theta)$

$= |z|(\cos(\pi - \theta) + i\sin(\pi - \theta))$

$\therefore \arg(-z) = \pi - \theta = \pi + (-\theta)$

$\arg(-z) = \pi + \arg z \therefore \arg(-z) - \arg z = \pi$



OR

Clearly arg $(-z)$ − arg (z) = arg $\left(\frac{-z}{z}\right)$

= arg(−1) $\Rightarrow 1\,(\cos\pi + i\sin\pi) = -1$

$= \pi$

2. The amplitude of $(1 + i)^5$ is

[Karnataka CET − 2007

(a) $3\pi/4$ (b) $-3\pi/4$

(c) $-5\pi/4$ (d) $5\pi/4$**]**

Solution

(d) $(1 + i) = (\sqrt{2})^5\left(\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}\right)^5$

$(\sqrt{2})^5\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$= (\sqrt{2})^5\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

Therefore, Amplitude = $\frac{5\pi}{4}$

3. The locus of the point $z = x + iy$ satisfying $\left|\frac{z-2i}{z+2i}\right| = 1$ is

(a) $y = 0$ (b) $x = 0$

(c) $y = 2$ (d) $x = 2$

[EAMCET − 2007]

Solution

(a) $z = x + iy$ and $|z - 2i| = |z + 2i|$

Squaring and simplifying

$\Rightarrow x^2 + (y-2)^2 = x^2 + (y+2)^2$

Locus of $P(z)$ is $y = 0$

4. If $(\sqrt{8} + i)^{50} = 3^{49}(a + ib)$ then $a^2 + b^2$ is

[Kerala Engg. − 2005]

(a) 3 (b) 8

(c) 9 (d) $\sqrt{8}$

Solution

(c) $(\sqrt{8} + i)^{50} = 3^{49}(a + ib)$

Taking modulus and squaring on both the sides, we get

$(8+1)^{50} = 3^{98}(a^2+b^2) \Rightarrow 9^{50} = 3^{98}(a^2+b^2)$

$\Rightarrow 3^{100} = 3^{98}(a^2 + b^2)$

$\Rightarrow (a^2 + b^2) = 9.$

5. If $w = \left(\frac{z-i}{1+iz}\right)^n$, $n \in i$ then $|w| = 1$ for

(a) only even n (b) only odd n

(c) only positive n (d) all n.

Solution

(d) $|w| = 1$

$\Rightarrow \left|\left(\frac{z-i}{1+iz}\right)^n\right| = r$

$\Rightarrow \left|\frac{z-i}{i\left(z+\frac{1}{i}\right)}\right|^n = 1$

$\Rightarrow \left|\frac{z-i}{i(z-i)}\right|^n = 1 \quad \Rightarrow \left|\frac{1}{i}\right|^n = 1 \Rightarrow |-i|^n =$

which is true for all n as $|-i| = 1$.

6. If $\left|\frac{z-25}{z-1}\right| = 5$, the value of $|z|$ is

[VITEEE − 2008]

(a) 3 (b) 4 (c) 5 (d) 6

Solution

(c) Let $x + iy = z \Rightarrow (x-25)^2 + y^2$

$= 25\,[(x-1)^2 + y^2]$

$\Rightarrow x^2 + y^2 - 50x + 625$

$= 25\,[x^2 + y^2 - 2x + 1]$

$\Rightarrow 24x^2 + 24y^2 = 600$

$\Rightarrow x^2 + y^2 = \frac{600}{25}$

$|z|^2 = 25 \Rightarrow |z| = 5$

7. If z is a complex number such that $iz^3 + z^2 - z + i = 0$, then $|z|$ is equal to

[IIT − 1995]

(a) 2 (b) 1

(c) 1/2 (d) none of these

Solution

(b) $iz^3 + z^2 - z + i = 0$

$\Rightarrow z^3 - iz^2 + iz - i^2 = 0$ [on multiplying by $-i$]

$\Rightarrow z^2(z - i)(z - i) = 0$

$\Rightarrow (z^2 + i)(z - i) = 0$

$\Rightarrow z^2 = -i$ or $z = i$ If $\Rightarrow z^2 = -i$, then $|z| = 1$

If $z = i$, then $|z| = 1$ $\therefore$ $|z| = 1$.

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. Amplitude of 0 is

[RPET −2000]

(a) 0 (b) π

(c) $\pi/2$ (d) not defined

2. Distance of the point representing the complex number $1 + i$ in the Argand's plane from the origin is equal to

(a) 1 (b) 2

(c) $\sqrt{2}$ (d) none of these

3. If $|z| = 4$ and Amp $z = \frac{5\pi}{6}$, then $z =$

(a) $2\sqrt{3} + 2i$ (b) $2\sqrt{3} - 2i$

(c) $-2\sqrt{3} + 2i$ (d) $-\sqrt{3} + i$

4. For any complex number z, which of the following is not true?

(a) $z.\bar{z} = |z|^2$

(b) $|z^2| = |z|^2$

(c) $|z| = \sqrt{z^2}$

(d) $z = \text{Re}(z) + i\,\text{Im}(z)$

5. If $\left|\frac{z-2}{z-4}\right| = 1$, then Re($z$) is equal to

(a) 3 (b) 0

(c) − 3 (d) none of these

6. Amp $\left\{\sin\frac{8\pi}{5} + i\left(1 + \cos\frac{8\pi}{5}\right)\right\}$ is equal to

(a) $\frac{3\pi}{5}$ (b) $\frac{7\pi}{10}$

(c) $\frac{4\pi}{5}$ (d) $\frac{3\pi}{10}$

7. The complex number $\frac{1+2i}{1-i}$ lies in which quadrant of the complex plane.

[MPPET − 2001]

(a) First (b) Second

(c) Third (d) Fourth

8. If z is a complex number such that $\frac{z-1}{z+1}$ is purely imaginary then

[MPPET − 1998, 2002]

(a) $|z| = 0$ (b) $|z| = 1$

(c) $|z| > 1$ (d) $|z| < 1$

9. $\left|(1+i)\frac{(2+i)}{(3+i)}\right| =$

[MPPET−1995, 1999]

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) 1 (d) − 1

10. If arg (z) = θ, then arg ($\bar{z}$) =

[MPPET − 1995]

(a) θ (b) $-\theta$

(c) $\pi - \theta$ (d) $\theta - \pi$

11. The argument of the complex number $-1 + i\sqrt{3}$ is

[MPPET − 1994]

(a) − 60° (b) 60° (c) 120° (d) −120°

12. If $\frac{c+i}{c-i} = a + ib$ where a, b, c are real then $a^2 + b^2 =$

(a) 1 (b) − 1 (c) c^2 (d) $-c^2$

[MPPET − 1996]

13. Which of the following is true

[MPPET − 2006]

(a) $|3 + 4i| > |5 + i|$

(b) $|6 + 7i| < |5 + 7i|$

(c) $|7 + 8i| > |6 + 8i|$

(d) $|2 + 4i| < |1 + i|$

14. If $\frac{5z_2}{7z_1}$ is purely imaginary, then the value of $\left|\frac{2z_1 + 3z_2}{2z_1 - 3z_2}\right|$ is

(a) 37/33 (b) 2 (c) 1 (d) 3

[Kerala PET − 2008]

15. The modulus of the complex number $\frac{i}{1-i}$ is

(a) $\sqrt{2}$ (b) $\frac{1}{2\sqrt{2}}$

(c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{2}}$

16. The amplitude of the complex number $z = \sin\alpha + i(1 - \cos\alpha)$ is $\alpha \in (0, \pi)$

(a) $2\sin\frac{\alpha}{2}$ (b) $\frac{\alpha}{2}$

(c) α (d) none of these

17. If $x + iy = \sqrt{\frac{a+ib}{c+id}}$, then $(x^2 + y^2)^2 =$

[IIT–1979, RPET–1997, Karnataka CET–1999; Orissa JEE –2009]

(a) $\frac{a^2+b^2}{c^2+d^2}$ (b) $\frac{a+b}{c+d}$

(c) $\frac{c^2+d^2}{a^2+b^2}$ (d) $\left(\frac{a^2+b^2}{c^2+d^2}\right)^2$

18. The points $1 + 3i$, $5 + i$ and $3 + 2i$ in the complex plane are

(a) Vertices of a right angled triangle

(b) Collinear

(c) Vertices of an obtuse angled triangle

(d) Vertices of an equilateral triangle

[MPPET –1987]

19. The sum of amplitude of z and another complex number is π. The other complex number can be written

(a) $\bar{z}$ (b) $-\bar{z}$ (c) z (d) $-z$

[Orissa JEE –2004]

20. Let $z = \cos\theta + i\sin\theta$. Then the value of $\sum_{m=1}^{15} Im\,(z^{2m-1})$ at $\theta = 2$ is

[IIT –2009]

(a) $\frac{1}{\sin 2°}$ (b) $\frac{1}{3\sin 2°}$

(c) $\frac{1}{2\sin 2°}$ (d) $\frac{1}{4\sin 2°}$

SOLUTIONS

1. (d) Let $0 = r(\cos\theta + i\sin\theta)$

$\therefore r\cos\theta = 0;\ r\sin\theta = 0$

$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) = 0 \Rightarrow r = 0$

$\therefore \theta$ can have any value

$\therefore$ amp (0) can have any value.

2. (c) distance $= |1 + i| = \sqrt{1^2+1^2} = \sqrt{2}$

3. (c) $z = |z|$ cis (Arg z)

$= 4\cos\left(\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$= 4\left[\cos\left(\pi - \frac{\pi}{6}\right) + i\sin\left(\pi - \frac{\pi}{6}\right)\right]$

$= 4\left[-\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right] = 4\left[\frac{-\sqrt{3}}{2} + \frac{i}{2}\right]$

$= -2\sqrt{3} + 2i$

4. (c) By definition

5. (a) Let $z = x + yi;\ x, y \in R$, then

$\left|\frac{z-2}{z-4}\right| = 1 \Rightarrow |z-2| = |z-4|,\ z \neq 4$

$\Rightarrow |x + yi - 2| = |x + yi - 4|,\ z \neq 4$

$\Rightarrow \sqrt{(x-2)^2 + y^2} = \sqrt{(x-4)^2 + y^2}$

$\Rightarrow -4x + 4 \Rightarrow -8x + 16$

$\Rightarrow 4x = 12 \Rightarrow x = 3$

6. (b) Now $\sin\frac{8\pi}{5} + i\left(1 + \cos\frac{8\pi}{5}\right)$

$= 2\sin\frac{4\pi}{5}\cos\frac{4\pi}{5} + i\left(2\cos^2\frac{4\pi}{5}\right)$

$= -2\cos\frac{4\pi}{5}\left\{-\sin\frac{4\pi}{5} - i\cos\frac{4\pi}{5}\right\}$

Note that $\cos\frac{4\pi}{5} < 0$

$= \left(-2\cos\frac{4\pi}{5}\right)\left\{\begin{matrix}\cos\left(\frac{3\pi}{2} - \frac{4\pi}{5}\right) + \\ i\sin\left(\frac{3\pi}{2} - \frac{4\pi}{5}\right)\end{matrix}\right\}$

$\left(\because -\sin\theta = \cos\left(\frac{3\pi}{2} - \theta\right)\right)$

and $-\cos\theta = \sin\left(\frac{3\pi}{2} - \theta\right)$

$= \left(-2\cos\frac{4\pi}{5}\right)\left\{\cos\left(\frac{7\pi}{10}\right) + i\sin\left(\frac{7\pi}{10}\right)\right\}$

$= r\cos\theta$, where $r = -2\cos\frac{4\pi}{5} > 0$

and $\theta = \frac{7\pi}{10} \in (-\pi, \pi]$

Hence, amplitude of the given number is $\frac{7\pi}{10}$.

7. (b) $\frac{1+2i}{1-i} = \frac{(1+2i)\times(1+i)}{(1-i)(1+i)}$

$\frac{1+2i}{1-i} = \frac{1+2i+i+2i^2}{1+1}$

or

$\frac{1+2i}{1-i} = \frac{-1}{2} + \frac{3i}{2}$

Comparing with $x + iy$

$x \Rightarrow$ negative, $y \Rightarrow$ positive $(-, +)$ are in 2nd quadrant.

8. (b) $\frac{z-1}{z+1}$ = purely imaginary

Let $z = x + iy$ then $\frac{x+iy-1}{x+iy+1} = ai$

$$\left[\frac{(x-1)+iy}{(x+1)+iy}\right]\left[\frac{(x+1)-iy}{(x+1)-iy}\right] = ai$$

On comparing real parts,

$$\frac{(x-1)(x+1)+y^2}{(x+1)^2+y^2}$$

$x^2+y^2-1=0$ or $x^2+y^2=1$ or $|z|=1$

9. (c) $\left|\frac{(1+i)(2+i)}{3+i}\right| = \frac{|1+i|\cdot|2+i|}{|3+i|}$

$= \frac{\sqrt{2}\sqrt{5}}{\sqrt{10}} = 1$

10. (b) Let $z = a + ib$ then $\arg z, \theta = \tan^{-1}\frac{b}{a}$ and $\bar{z} = a - ib$

or $\arg \bar{z}, \phi = -\tan^{-1}\frac{b}{a} = -\theta$

11. (c) Argument of $z = a + ib$ is $\theta = \tan^{-1}\frac{-b}{a}$

Therefore arg of $(-1+\sqrt{3})$ is $\theta = \tan^{-1}\frac{\sqrt{3}}{-1}$ $= 120°$

(since) $x < 0, y > 0$, θ is in 4th quadrant

12. (a) $a + ib = \frac{c+i}{c-i} \Rightarrow |a+ib| = \frac{|c+i|}{|c-i|}$

$\Rightarrow a^2+b^2 = \left(\frac{c^2+1}{c^2+1}\right) \Rightarrow a^2+b^2 = 1$

13. (c) $|7+8i| = \sqrt{7^2+8^2} = \sqrt{49+64} = \sqrt{113}$

$|6+8i| = \sqrt{6^2+8^2} = 10$

$\therefore |7+8i| > |6+8i|$

14. (c) Let $\frac{5z_2}{7z_1} = i\alpha;\ \alpha \in R, \alpha \neq 0 \Rightarrow \frac{z_2}{z_1} = \frac{7\alpha i}{2}$

$$\Rightarrow \left|\frac{2z_1+3z_2}{2z_1-3z_2}\right| = \left|\frac{2+3\frac{z_2}{z_1}}{2-3\frac{z_2}{z_1}}\right| \left|\frac{2+3\left(7\frac{i\alpha}{5}\right)}{2-3\left(\frac{7i\alpha}{5}\right)}\right|$$

$$= \frac{|10+21\alpha)i}{|10-(21\alpha)i|} = \frac{\sqrt{10^2+(21\alpha)^2}}{\sqrt{10^2+(21\alpha)^2}} = 1$$

15. (d) $\left|\frac{i}{1-i}\right| = \frac{|i|}{|1-i|} = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}$

16. (b) $z = \sin\alpha + i(1-\cos\alpha)$

$$\text{amp}(z) = \tan^{-1}\left(\frac{1-\cos\alpha}{\sin a}\right)$$

$$= \tan^{-1}\left(\frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\right)$$

$$= \tan^{-1}\tan\left(\frac{\alpha}{2}\right) = \frac{\alpha}{2}$$

17. (a) $x + iy = \left(\frac{a+ib}{c+id}\right)^{1/2} \Rightarrow |x+iy|^2$

$$= \left|\frac{a+ib}{c+id}\right|$$

$\Rightarrow x^2+y^2 = \left(\frac{a^2+b^2}{c^2+d^2}\right)^{1/2}$ or $(x^2+y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$

18. (b) Let $z_1 = 1 + 3i$, $z_2 = 5 + i$ and $z_3 = 3 + 2i$ Then the area of the triangle

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 5 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 0$$

Hence, z_1, z_2 and z_3 are collinear.

19. (b) We have $z = x + iy$ and let other complex number be z_2 and given that $\arg(z) + \arg(z_2) = \pi$ $\arg z_2 = \pi - \arg(z)$; $\arg(z_2) = \pi + \left[-\tan^{-1}\frac{y}{x}\right]$ $\arg z_2 = \pi + [\arg(\bar{z})]$

which lies in second quadrant i.e. $-\bar{z}$.

20. (d) $X = \sin\theta + \sin 3\theta + \ldots\ldots + \sin 29\theta$

$2(\sin\theta) \times = 1 - \cos 2\theta + \cos 2\theta - \cos 4\theta + \ldots\ldots + \cos 28\theta - \cos 30\theta$

$$X = \frac{1-\cos 30\theta}{2\sin\theta} = \frac{1}{4\sin 2°}$$

UNSOLVED OBJECTIVE PROBLEMS: (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. If $z_1 = (4, 5)$ and $z_2 = (-3, 2)$ then $\frac{z_1}{z_2}$ equals **[RPET – 1996]**

(a) $\left(\frac{-23}{12}, \frac{-2}{13}\right)$ (b) $\left(\frac{2}{13}, \frac{-23}{13}\right)$

(c) $\left(\frac{-2}{13}, \frac{-23}{13}\right)$ (d) $\left(\frac{-2}{13}, \frac{23}{13}\right)$

2. If z is a complex number, then which of the following is not true **[MPPET – 1987]**

(a) $|z^2| = |z|^2$ (b) $|z^2| = |\bar{z}|^2$

(c) $z = \bar{z}$ (d) $\bar{z}^2 = \bar{z}^2$

3. For any complex number z, $\bar{z} = \left(\frac{1}{z}\right)$ if and only if

(a) z is a pure real number

(b) $|z| = 1$

(c) z is a pure imaginary number

(d) $z = 1$

[RPET – 1995]

4. The value of $|z - 5|$ if $z = x + iy$, is **[RPET – 1995]**

(a) $\sqrt{(x-5)^2 + y^2}$ (b) $x^2 + \sqrt{(y-5)^2}$

(c) $\sqrt{(x-y)^2 + 5^2}$ (d) $\sqrt{x^2 + (y-5)^2}$

5. $\frac{1-i}{1+i}$ is equal to **[RPET – 1984]**

(a) $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$ (b) $\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$

(c) $\sin\frac{\pi}{2} + i\cos\frac{\pi}{2}$ (d) none of these

6. Modulus of $\left(\frac{3+2i}{3-2i}\right)$ is **[RPET – 1996]**

(a) 1 (b) 1/2

(c) 2 (d) $\sqrt{2}$

7. $\arg(5 - \sqrt{3}\, i) =$

(a) $\tan^{-1}\frac{5}{\sqrt{3}}$ (b) $\tan^{-1}\left(-\frac{5}{\sqrt{3}}\right)$

(c) $\tan^{-1}\frac{\sqrt{3}}{5}$ (d) $\tan^{-1}\left(-\frac{\sqrt{3}}{5}\right)$

8. If $\bar{z}$ be the conjugate of the complex number z, then which of the following relations is false **[MPPET – 1987]**

(a) $|z| = |\bar{z}|$ (b) $z \cdot \bar{z} = |\bar{z}|^2$

(c) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (d) $\arg z = \arg \bar{z}$

9. If z is a purely real number such that $\text{Re}(z) < 0$, then $\arg(z)$ is equal to

(a) π (b) $\pi/2$

(c) 0 (d) $-\pi/2$

10. If $z_1, z_2 \in C$, then amp $\left(\frac{z_1}{z_2}\right) =$

(a) amp $(z_1 \bar{z}_2)$ (b) amp $(\bar{z}_1 z_2)$

(c) amp $\left(\frac{\bar{z}_2}{\bar{z}_1}\right)$ (d) amp $\left(\frac{\bar{z}_1}{\bar{z}_2}\right)$

11. If A, B, C are represented by $3 + 4i$, $5 - 2i$, $-1 + 16i$, then A, B, C are

(a) Collinear

(b) Vertices of equilateral triangle

(c) Vertices of isosceles triangle

(d) Vertices of right angled triangle

[RPET – 1986]

12. If $z = 1 - \cos\alpha + i\sin\alpha$, then amp $z =$

(a) $\frac{\alpha}{2}$ (b) $-\frac{\alpha}{2}$

(c) $\frac{\pi}{2} + \frac{\alpha}{2}$ (d) $\frac{\pi}{2} - \frac{\alpha}{2}$

13. If $z = \frac{1 - i\sqrt{3}}{1 + i\sqrt{3}}$, then arg (z) **[Roorkee – 1990, UPSEAT – 2004]**

(a) 60° (b) 120°

(c) 240° (d) 300°

14. The modulus of the complex number $\frac{(1+i)}{\left(1 + \frac{1}{i}\right)}$ is

(a) $\sqrt{2}$ (b) 1

(c) $3/\sqrt{2}$ (d) none of these

15. Amplitude of $\left(\frac{1-i}{1+i}\right)$ is

(a) $-\pi/2$ (b) $\pi/2$ (c) $\pi/4$ (d) $\pi/6$

[RPET – 1996]

16. $(z+1)(\bar{z}+1)$ can be expressed as

(a) $z\bar{z}+1$ (b) $|z|^2+1$

(c) $|z+1|^2$ (d) $|z|^2+2$

17. The argument of the complex number $\frac{13-5i}{4-9i}$ is

(a) $\pi/3$ (b) $\pi/4$

(c) $\pi/5$ (d) $\pi/6$

[MPPET – 1997]

WORK SHEET: TO CHECK PREPARATION LEVEL

Important Instructions:

1. The answer sheet is immediately below the work sheet
2. The test is of 13 minutes.
3. The test consists of 13 questions. The maximum marks are 39.
4. Use blue/black Ball point pen only for writing particulars / marking responses. Use of pencil is strictly prohibited.

1. The amplitude of $\sin\frac{\pi}{5}+i\left(1-\cos\frac{\pi}{5}\right)$

(a) $\pi/5$ (b) $2\pi/5$

(c) $\pi/10$ (d) $\pi/15$

2. Argument of $-1-i\sqrt{3}$ is

(a) $2\pi/3$ (b) $\pi/3$

(c) $-\pi/3$ (d) $-2\pi/3$

3. $\left|\frac{1}{(2+i)^2}-\frac{1}{(2-i)^2}\right|=$

(a) $\frac{\sqrt{8}}{5}$ (b) $\frac{25}{8}$

(c) $\frac{5}{\sqrt{8}}$ (d) $\frac{8}{25}$

4. If a complex number lies in the IIIrd quadrant then its conjugate lies in quadrant number

(a) I (b) II

(c) III (d) IV

5. Let z_1 and z_2 be two complex numbers with α and β as their principal arguments such that $\alpha+\beta>\pi$, then principal $\arg(z_1 z_2)$ is given by

(a) $\alpha+\beta+\pi$ (b) $\alpha+\beta-\pi$

(c) $\alpha+\beta-2\pi$ (d) $\alpha+\beta$

6. If z is a complex number, then

(a) $|z^2|>|z|^2$ (b) $|z^2|=|z|^2$

(c) $|z^2|<|z|^2$ (d) $|z^2|\geq|z|^2$

7. The complex number $i+\sqrt{3}$ in polar form can be written as

(a) $\frac{1}{\sqrt{2}}\left(\sin\frac{\pi}{6}+i\cos\frac{\pi}{6}\right)$

(b) $2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

(c) $\frac{1}{2}\left(\sin\frac{\pi}{6}+i\cos\frac{\pi}{6}\right)$

(d) $4\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

8. If $|z_1|=|z_2|$ and amp z_1 + amp $z_2=0$, then

[MPPET – 2006]

(a) $z_1=z_2$ (b) $\overline{z_1}=\overline{z_2}$

(c) $z_1+z_2=0$ (d) $|\overline{z_1}|\,|\overline{z_2}|$

9. Value of $|1-\cos\alpha+i\sin\alpha|$ is

[MPPET – 2007]

(a) $2\sin\frac{\alpha}{2}$ (b) $2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

(c) $2\cos\frac{\alpha}{2}$ (b) $2\sin^2\frac{\alpha}{2}$

10. Let z be a purely imaginary number such that Im $(z)>0$. Then arg (z) is equal to:

(a) π (b) $\frac{\pi}{2}$

(c) 0 (d) $-\frac{\pi}{2}$

11. If $(\sqrt{5}+\sqrt{3}i)^{33}=2^{49}z$, then modulus of the complex number z is equal to

[Kerala PET – 2008]

(a) 1 (b) $\sqrt{2}$

(c) $2\sqrt{2}$ (d) 4

12. If $\frac{z-1}{z+1}$ is purely imaginary, then

(a) $|z| = 1$ (b) $|z| = 0$

(c) $|z| < 1$ (d) $|z| > 1$

[MPPET – 1998, 2002]

13. If $(x + iy) = \sqrt{\frac{1+2i}{3+4i}}$, then $(x^2 + y^2)^2 =$

[MPPET – 2009]

(a) 5 (b) $\frac{1}{5}$

(c) 2 (d) $\frac{5}{2}$

ANSWER SHEET

1.	ⓐ	ⓑ	ⓒ	ⓓ
2.	ⓐ	ⓑ	ⓒ	ⓓ
3.	ⓐ	ⓑ	ⓒ	ⓓ
4.	ⓐ	ⓑ	ⓒ	ⓓ
5.	ⓐ	ⓑ	ⓒ	ⓓ
6.	ⓐ	ⓑ	ⓒ	ⓓ
7.	ⓐ	ⓑ	ⓒ	ⓓ
8.	ⓐ	ⓑ	ⓒ	ⓓ
9.	ⓐ	ⓑ	ⓒ	ⓓ
10.	ⓐ	ⓑ	ⓒ	ⓓ
11.	ⓐ	ⓑ	ⓒ	ⓓ
12.	ⓐ	ⓑ	ⓒ	ⓓ
13.	ⓐ	ⓑ	ⓒ	ⓓ

HINTS AND EXPLANATIONS

13. (b) Given, $x + iy = \sqrt{\frac{1+2i}{3+4i}} \times \frac{\sqrt{3-4i}}{\sqrt{3-4i}}$

$= \frac{1}{5}\sqrt{11+2i}$

Taking mod in both sides, and squaring

$\sqrt{x^2 + y^2} = \frac{1}{25}\sqrt{125}$

Again squaring,

Therefore, $(x^2 + y^2)^2 = \frac{125}{(25)^2} = \frac{1}{5}$

LECTURE

3

Euler's Formula

BASIC CONCEPTS

1. Euler's Formula $z = \cos\theta + i\sin\theta = e^{i\theta}$

$\frac{1}{z} = \cos(-\theta) + i\sin(-\theta) = e^{-i\theta}$

e.g. $i = 0 + i \times 1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2}$

$-i = 0 - i \times 1$

$= \cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right) = e^{-i\pi/2}$

$\frac{1}{2} + \frac{i\sqrt{3}}{2} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = e^{i2\pi/3}$

2. De–Moivre's Theorem

(i) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(ii) $(\sin\theta + i\cos\theta)^n = i^n\{\cos(-n\theta) + i\sin(-n\theta)\}$

Note:

$\frac{1}{z} = \frac{1}{(\cos\theta + i\sin\theta)} = (\cos\theta + i\sin\theta)^{-1}$

$= \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$

3. Exponential Form of Complex Number $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$; $z = re^{i\theta}$ is exponential form.

4. Multiplication and Division of a Complex numbers When it is in Polar Form ($z = (r(\cos\theta + i\sin\theta))$)

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1}$
and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2 e^{i\theta_2}$

(i) $z_1 z_2 = r_1 r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$
$= r_1 r_2 e^{i(\theta_1 + \theta_2)}$

$r_1 = |z_1|$, $r_2 = |z_2|$, amp $(z_1) = \theta_1$, amp $(z_2) = \theta_2$.

$|z_1 z_2| = r_1 r_2 = |z_1|\,|z_2|$,

$\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$

i.e., the magnitude of the product of two complex numbers is equal to the product of their magnitudes and the amplitude of product of two complex numbers is equal to the sum of their amplitudes.

4.1. Product of two complex numbers can be generalized for more than two complex numbers, if $z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1))$; $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$; $z_3 = r_3(\cos\theta_3 + i\sin\theta_3)$ .... and........ $z_n = r_n(\cos\theta_n + i\sin\theta_n)$.

Then $z_1, z_2, z_3 .... z_n = r_1 r_2 ...... r_n\{\cos(\theta_1 + \theta_2 + ... + \theta_n) + i\sin(\theta_1 + \theta_2 + ... + \theta_n)\}$

Hence, $|z_1 z_2 z_3 .... z_n| = |z_1|\,|z_2|\,|z_3| .... |z_n|$

$\arg(z_1 z_2 z_3 ... z_n) = \theta_1 + \theta_2 + \theta_3 + ... + \theta_n = \arg(z_1) + \arg(z_2) + ... + \arg(z_n)$

Note: If $z_1 = z_2 = z_3 = ... = z_n = z$, then

(i) $|z_n| = |z|^n$

(ii) $\arg(z^n) = n\arg(z)$

(iii) $\frac{z_1}{z_2} = \frac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$

$= \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

where $\frac{|z_1|}{|z_2|} = \frac{|r_1|}{|r_2|} = \frac{|z_1|}{|z_2|}$ arg $\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 =$ arg $(z_1) -$ arg (z_2)

The amplitude of the quotient of two complex numbers is equal to the difference of their amplitudes.

Note: If θ_1 and θ_2 are the principal values of arg z_1 and arg z_2 then $\theta_1 + \theta_2$ is not necessarily the principal value of arg $(z_1 z_2)$ nor is $\theta_1 - \theta_2$ necessarily the principal value of arg (z_1/z_2).

5. Properties of Complex Number Connected with Magnitudes of Complex Numbers

(i) $|z_1 + z_2|^2 = |z_1|^2 + |z^2|^2 + 2$ Re $(z_1 \bar{z}_2)$ OR $= |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \cos(\theta_1 - \theta_2)$

(ii) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2$Re $(z_1 \bar{z}_2)$ OR $= |z_1|^2 + |z_2|^2 - 2|z_1||z_2| \cos(\theta_1 - \theta_2)$

(iii) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$ **(MPPET−2006)**

(iv) $|az_1 + bz_2|^2 + |bz_1 - az_2|^2 = (a^2 + b^2)\{|z_1|^2 + |z_2|^2\}$

(v) $|z_1 + z_2|^2 - |z_1 - z_2|^2 = 4|z_1||z_2| \cos(\theta_1 - \theta_2)$

6. Properties of Complex Numbers Connected with the Amplitude of Complex Numbers

(i) If $|z_1 + z_2| = |z_1 - z_2| \Leftrightarrow$ arg $(z_1) -$ arg $(z_2) = \left(\frac{\pi}{2}\right)$

(ii) $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow$ arg$(z_1) =$ arg (z_2)

(iii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow$ arg $\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2}$ $\Rightarrow \frac{z_1}{z_2}$ Purely imaginary

7. Properties of Complex Number Connected with Magnitudes of Complex Numbers

(i) Triangle's Inequality

$||z_1| - |z_2|| < |z_1 + z_2| < |z_1| + |z_2|$

The sum of any two sides is greater than the third side and the difference of any two sides is less than the third side.

(ii) If z_1 and z_2 are collinear, then

$|z_1 + z_2| = |z_1| + |z_2| = ||z_1| - |z_2||$

(iii) From inequalities (i) and (ii), we get

(iv) $||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|$

Hence, maximum value of $|z_1 + z_2| = |z_1| + |z_2|$ and minimum value of $= |z_1 + z_2|$ is $||z_1| - |z_2||$

(v) $|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)}$

$= (z_1 + z_2)(z_1 + z_2)$

(vi) $-|z| \le$ Re $(z) \le |z|$

(vii) $-|z| \le$ Im $(z) \le |z|$

(viii) $|z_1 + z_2 + z_3 + \ldots + z_n| = |z_1| + |z_2| + \ldots + |z_n|$

if and only if: amp $(z_1) =$ amp $(z_2) = \ldots\ldots =$ amp (z_n)

i.e. $z_1, z_2, z_3, \ldots, z_n$ are collinear.

8. Square Root of a Complex Number

(i) $\sqrt{a + ib} = \pm\left\{\sqrt{\frac{1}{2}\{\sqrt{a^2 + b^2} + a\}}\right.$

$\pm i\sqrt{\left\{\frac{1}{2}\sqrt{a^2 + b^2} - a\right\}}$

where $b > 0$

$\sqrt{a + ib} = \pm\left\{\sqrt{\frac{1}{2}\{\sqrt{a^2 + b^2} + a\}}\right.$

$-i\left\{\sqrt{\frac{1}{2}\{\sqrt{a^2 + b^2} - a\}}\right.$

where $b < 0$ i.e., in square root, the sign of imaginary part is same as the sign of y.

(ii) $\sqrt{a + ib} + \sqrt{a - ib} = \sqrt{2a + 2\sqrt{a^2 + b^2}}$

(iii) $\sqrt{a + ib} - \sqrt{a - ib} = i\sqrt{2\sqrt{a^2 + b^2} - 2a}$

Note: When asked $\sqrt{a + ib}$, then $\sqrt{a + ib} = \pm(x \pm iy)$ can be verified by inspection method from the given four alternatives.

9. Logarithm of a Complex Number

$\log z = \log(a + ib) = (\log|z| + i \text{ amp}(z))$ or $\left(\frac{1}{2}\log(a^2 + b^2) + i\tan^{-1}\frac{b}{a}\right)$

10. Cube Roots of Unity (i) $(1)^{1/3} = 1, \omega, \omega^2$

where $\omega = \frac{-1 + \sqrt{-3}}{2}$ or $\frac{-1 + i\sqrt{3}}{2}$

or $\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$ or $e^{i\frac{2\pi}{3}}$

and $\omega^2 = \frac{-1 - \sqrt{-3}}{2}$ or $\frac{-1 - i\sqrt{3}}{2}$

or $\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$ or $e^{\frac{2\pi}{3}}$ (ii) $-\omega^2 = \frac{1 + i\sqrt{3}}{2}$

(iii) $i\omega = \frac{-i-\sqrt{3}}{2}$,

$-i\omega = \frac{i+\sqrt{3}}{2}$, $-i\omega^2 = \frac{i-\sqrt{3}}{2}$, $i\omega^2$

$= \frac{\sqrt{3}-i}{2}$

(iv) $-2\omega = 1 - i\sqrt{3}$, $-2\omega^2$

$= 1 + i\sqrt{3}$

(v) $z^3 - 1 = (z-1)(z-\omega)(z-\omega^2)$

Note: Where $\arg(\omega) = \frac{2\pi}{3}$. $\arg(\omega^2) = \frac{4\pi}{3}$

(vi) ω and ω^2 are the roots of equation $z^2 + z + 1 = 0$.

Hence, $1 + \omega + \omega^2 = 0$

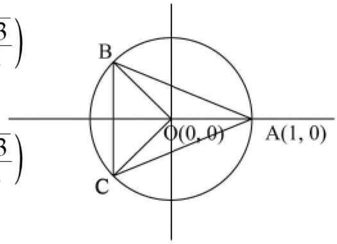
i.e. sum of cube roots = 0

(vii) $(-1)^{1/3} = -1, -\omega, -\omega^2$

(viii) Each complex cube root of unity is the square of the other and $\omega.\ \omega^2 = \omega^3 = 1$; $\omega + \omega^2 = -1$.

(ix) Each complex cube root is the reciprocal of other complex cube root.

(x)



If these points A, B and C which are cube roots of unity represented on a complex plane form the vertices of an equilateral triangle of area $\frac{3\sqrt{3}}{4}$ square unit and 1, ω, ω^2 are lying on the unit circle on a complex plane.

(xi) $1 + \omega^n + \omega^{2n}$

$= \begin{cases} 3, \text{ when } n \text{ is multiple of } 3 \\ 0, \text{ when } n \text{ is not a multiple of } 3 \end{cases}$

(xii) $a^3 + b^3 = (a+b)(a+b\omega)(a+b\omega^2)$

(xiii) $a^3 - b^3 = (a-b)(a-b\omega)(a-b\omega^2)$

(xiv) $a^2 + b^2 + c^2 - ab - bc - ca$

$= (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

(xv) $a^3 + b^3 + c^3 - 3abc$

$= (a+b+c)(a + b\omega + c\omega^2)$

$(a + b\omega^2 + c\omega)$

(xvi) $1 - \omega + \omega^2 = -2\omega$

(xvii) $1 + \omega - \omega^2 = -2\omega^2$

SOLVED SUBJECTIVE PROBLEMS: (CBSE /STATE BOARD): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. If $z = \frac{\sqrt{3}+i}{-2}$, then z^{69} is equal to **Prove.**

Solution

$$z = \frac{\sqrt{3}+i}{-2} \Rightarrow iz = -\frac{-1 \pm \sqrt{3}i}{2} = -\omega$$

$$\Rightarrow z = \frac{-\omega}{i} = i\omega$$

$$\Rightarrow 2^{69} = i^{69}.\omega^{69} = i\ (\because \omega^{3n} = i^{4n} = 1)$$

2. Using de Moivre's theorem, find the value of $(\sqrt{3} - i)^8$.

Solution

To express $\sqrt{3} - i$ in the trigonometric form,

Let $\sqrt{3} - i = r(\cos\theta + i\sin\theta)$

$r = \sqrt{3+1} = \sqrt{4} = 2$

$\cos\theta = \frac{x}{r} = \frac{\sqrt{3}}{2}$ and

$\sin\theta = \frac{y}{r} = \frac{-1}{2} \Rightarrow \theta = -30°$

Therefore $(\sqrt{3} - i)^8 = \left[2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)\right]^8$

$= 2^8 [\cos(-30°) + i \sin(-30°)]^8$

$= 2^8 [\cos(-240°) + i \sin(-240°)]$

("By de Moiver's theorem")

$= 2^8 [\cos(360° - 240°) + i \sin(360° - 240°)$

$= 2^8 [\cos 120° + i \sin 120°]$

$= 2^8 \left[\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right]$

$= 2^7 [-1 + i\, 3]$ **Ans**

3. $\frac{(\cos\alpha + i\sin\alpha)^4}{(\sin\beta + i\cos\beta)^5} = \cos(4\alpha + 5\beta) + i\sin(4\alpha + 5\beta)$ Prove.

Solution

$$\frac{(\cos\alpha + i\sin\alpha)^4}{(\sin\beta + i\cos\beta)^5} = \frac{(\cos 4\alpha + i\sin 4\alpha)}{i^5(\cos\beta - i\sin\beta)^5}$$

$= -i(\cos 4\alpha + i\sin 4\alpha)(\cos\beta - i\sin\beta)^{-5}$

$= -i[\cos 4\alpha + i\sin 4\alpha][\cos 5\beta + i\sin 5\beta]$

$= -i[\cos(4\alpha + 5\beta) + i\sin(4\alpha + 5\beta)]$

$= \sin(4\alpha + 5\beta) - i\cos(4\alpha + 5\beta)$

4. Prove that $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$
or $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2)$.

Solution

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore |z_1| = r_1$ and $|z_2| = r_2$ and $\bar{z}_2 = r_2(\cos\theta_2 - i\sin\theta_2)$

$\therefore z_1\bar{z}_2 = \rho_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 - i\sin\theta_2)$

$= r_1r_2[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

$\therefore \text{Re}(z_1\bar{z}_2) = r_1r_2\cos(\theta_1 - \theta_2)$

$\Rightarrow \text{Re}(z_1 z_2) = |z_1||z_2|\cos(\theta_1 - z_2)$ (1)

Again, $z_1 + z_2 = r_1(\cos\theta_1 + i\sin\theta_1) + r_2(\cos\theta_2 + i\sin\theta_2)$

$\Rightarrow z_1 + z_2 = (r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)$

$\therefore |z_1 + z_2| = \sqrt{(r_1\cos\theta_1 + r_2\cos\theta_2)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2}$

Squaring both the sides,

$\Rightarrow |z_1 + z_2|^2 = r_1^2\cos^2\theta_1 + r_2^2\cos^2\theta_2 + 2r_1r_2\cos\theta_1\cos\theta_2 + r_1^2\sin^2\theta_1 + r_2^2\sin^2\theta_2 + 2r_1r_2\sin\theta_1\sin\theta_2$

$\Rightarrow |z_1 + z_2|^2 = r_1^2(\cos^2\theta_1 + \sin^2\theta_1) + r_2^2(\cos^2\theta_2 + \sin^2\theta_2) + 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$

$\Rightarrow |z_1 + z_2|^2 = r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)$

$\Rightarrow |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

Or $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2)$.

From Eqation (1).

5. Prove that: $x^4 + 4 = (x + 1 + i)(x + 1 - i)(x - 1 + i)(x - 1 - i)$.

Solution

We have $(x + 1 + i)(x + 1 - i)(x - 1 + i)(x - 1 - x)$

$= \{(x + 1)^2 - i^2\}\{(x - 1)^2 - i^2\} = \{(x + 1)^2 + 1\}\{(x - 1)^2 + 1\}$

$= \{x^2 + 2x + 2\}\{x^2 + 2 - 2x\} = \{x^2 + 2 + 2x\}\{x^2 + 2 - 2x\}$

$= (x^2 + 2)^2 - (2x)^2 = x^4 + 4x^2 + 4 - 4x^2 = x^4 + 4.$

6. If 1, ω, ω^2 are the cube roots of unity, then prove that
$(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)$...... $2n$ factors $= 2^{2n}$.

[MP – 1990]

Solution

L.H.S. $= (1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)$... $2n$ factors

$= (1 + \omega^2 - \omega)(1 + \omega^4 - \omega^2)(1 + \omega^8 - \omega^4)$.... $2n$ factors

$= (-\omega - \omega)(1 + \omega - \omega^2)(1 + \omega^6.\omega^2 - \omega^4)$.. $2n$ factors

$= (-2\omega)(-2\omega^2)(-2\omega)$...... $2n$ factors,

$$\left[\because \begin{array}{c} 1 + \omega^2 = -\omega \\ 1 + \omega = -\omega^2 \\ \omega^3 = 1 \end{array}\right]$$

$= (-2)^{2n}.(\omega^3)^n = 2^{2n} =$ R.H.S. **Proved.**

7. Find the square root of $4ab - 2(a^2 - b^2)i$.

Solution

$4ab - 2(a^2 - b^2)i = 4ab - 2(a + b)(a - b)i$
$= (a + b)^2 - (a - b)^2 - 2(a + b)(a - b)i$
$= (a + b)^2 + (a - b)^2 i^2 - 2(a + b)(a - b)i$
$= [(a + a) - (a - a)i]^2$

$\therefore \sqrt{4ab - 2(a^2 - b^2)i}$

$= \sqrt{[(a + b) - (a - b)i]^2}$

$= \pm [(a + b) - (a - b)i]$

8. Find the square root of $a^2 - 1 + 2ai$.

Solution

$a^2 - 1 + 2ai = a^2 + i^2 + 2ai = (a + i)^2$

$\therefore \sqrt{a^2 - 1 + 2ai} = \sqrt{(a + i)^2} \pm (a + i)$

9. Prove that $(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8)$... up to $2n$ factors = 1.

Solution

L.H.S.= $(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8)$... up to $2n$ factors
$= \{(1+\omega)(1+\omega^2)\}.\{(1+\omega^4)(1+\omega^8)\}$........ up to n factors
$= \{(1 + \omega)(1 + \omega^2)\}.\ \{(1 + \omega^4)(1 + \omega^8)\}$...... up to n factor
$= \{(-\omega^2)(-\omega)\}.\ \{(-\omega^2)(-\omega)\}$........ up to n factors
$= (\omega^3).(\omega^3)$............... up to n factors
$= 1.\ 1$............... up to n factors = 1
= R.H.S. **Proved.**

10. Prove that $(x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z) = x^3 + y^3 + z^3 - 3xyz$.

Solution

L.H.S. $= (x + y + z)[x^2 + xy(\omega + \omega^2) + xz(\omega + \omega^2)$
$+ \omega^3 y^2 + yz(\omega^2 + \omega^4) + \omega^3 z^2]$
$= (x + y + z)[x^2 + xy(-1) + xz(-1) + y^2 + yz(\omega^2 + \omega) + z^2]$
$= (x + y + z)[x^2 - xy - xz + y^2 - yz + z^2]$
$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$
$= x^3 + y^3 + z^3 - 3xyz$
= R.H.S. **Proved.**

11. Prove that $(x + y)^2 + (x\omega + y\omega^2)^2 + (x\omega^2 + y\omega)^2 = 6xy$

[MP PET – 2008]

Solution

L.H.S. $= (x + y)^2 + (x\omega + y\omega^2)^2 + (x\omega^2 + y\omega)^2$
$= (x^2 + y^2 + 2xy) + (x^2\omega^2 + y^2\omega^4 + 2xy\omega^3)$
$+ (x^2\omega^4 + y^2\omega^2 + 2xy\omega^3)$
$= \xi^2(1 + \omega^2 + \omega^4) + y^2(1 + \omega^4 + \omega^2) + 2xy(1 + \omega^3 + \omega^3)$
$= \xi^2(1 + \omega + \omega^2) + y^2(1 + \omega + \omega^2) + 2xy(1 + 1 + 1)$, $[\because \omega^3 = 1]$
$= x^2(0) + y^2(0) + 2xy(3) = 6xy$
= R.H.S. **Proved.**

12. Prove that

$$\frac{a + b\omega + c\omega^2}{c + a\omega + b\omega^2} + \frac{a + b\omega + c\omega^2}{b + c\omega + a\omega^2} = -1$$

Solution

$$\text{L.H.S.} = \frac{a + b\omega + c\omega^2}{c + a\omega + b\omega^2} + \frac{a + b\omega + c\omega^2}{b + c\omega + a\omega^2}$$

$$= \frac{\omega^2(a + b\omega + c\omega^2)}{\omega^2(c + a\omega + b\omega^2)} + \frac{\omega(a + b\omega + c\omega^2)}{\omega(b + c\omega + a\omega^2)}$$

$$= \frac{\omega^2(a + b\omega + c\omega^2)}{c\omega^2 + a + b\omega} + \frac{\omega(a + b\omega + c\omega^2)}{b\omega + c\omega^2 + a\omega^2}$$

$$= \frac{\omega^2(a + b\omega + c\omega^2)}{(c\omega^2 + a + b\omega^4)} + \frac{\omega(a + b\omega + c\omega^2)}{(b\omega + c\omega^2 + a)}$$

$= \omega^2(1) + \omega(1) \quad = \omega^2 + \omega$

$= -1\ [\because 1 + \omega + \omega^2 = 0]$

13. If $i = \sqrt{-1}$, then prove that

$$4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{365} = i\sqrt{3}$$

Solution

L.H.S. = 4 + 5

$$\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{365}$$

$= 4 + 5(\omega)^{334} + 3(\omega)^{365}$,

$$\left[\because \omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2}\right]$$

$= 4 + 5(\omega^3)^{111}.\ \omega + 3(\omega^3)^{121}.\ \omega^2$

$= 4 + 5(1)^{111}.\ \omega + 3\ (1)^{121}.\ \omega^2$

$= 4 + 5\ \omega + 3\ \omega^2 = 1 + 3 + 3\ \omega + 2\ \omega + 3\ \omega^2$

$= 1 + 2\ \omega + 3(1 + \omega + \omega^2)$

$= 1 + 2\ \omega + 3(0)$

$[\because\ 1 + \omega + \omega^2 = 0]$

$= 1 + 2\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$

$= 1 - 1 + i\sqrt{3}$

$= i\ \sqrt{3}$ **Proved.**

14. Prove that $\sqrt{15 + 8i} + \sqrt{15 - 8i} = 8$

Solution

$\sqrt{15 + 8i} + \sqrt{15 - 8i}$

$= \sqrt{15 + 2 \times 4i} + \sqrt{15 - 2 \times 4i}$

$= \sqrt{16 - 1 + 2 \times 4i} + \sqrt{16 - 1 - 2 \times 4i}$

$= \sqrt{4^2 + i^2 + 2 \times 4i} + \sqrt{4^2 + i^2 - 2 \times 4i}$

$= \sqrt{(4 + i)^2} + \sqrt{(4 - i)^2} = 4 + i + 4 - i = 8$

Proved.

15. If $z = x + iy$ and $w = \frac{1 - iz}{z - i}$, show that $|w| = 1 \Rightarrow z$ is purely real.

Solution

We have, $|w| = \left|\frac{1 - iz}{z - i}\right| = 1$

$\Rightarrow |1 - iz| = |z - i|$

$\Rightarrow |1 - i\ (x + \text{i}y)| = |\ x + iy - i|$, where $z = x + iy$

$\Rightarrow |1 + y - ix| = |x + i(y - 1)|$

$\Rightarrow \sqrt{(1 + y)^2 + (-x)^2} = \sqrt{x^2 + (y - 1)^2}$

$\Rightarrow (1 + y)^2 + x^2 = x^2 + (y - 1)^2 \Rightarrow y = 0$

$\Rightarrow z = x + i^0 = x$, which is purely real.

16. Prove that one of the values of

$\sqrt{8 + 6i} - \sqrt{8 - 6i}$ is $2i$

Solution

$\sqrt{8 + 6i} - \sqrt{8 - 6i}$

$= \sqrt{8 + 2 \times 3i} - \sqrt{8 - 2 \times 3i}$

$= \sqrt{9 - 1 + 2 \times 3i} - \sqrt{9 - 1 - 2 \times 3i}$

$= \sqrt{3^2 + i^2 + 2 \times 3i} - \sqrt{3^2 + i^2 - 2 \times 3i}$

$= \sqrt{(3 + i)^2} - \sqrt{(3 - i)^2}$

$= 3 + i - 3 + i = 2i$

Proved.

17. Prove that

(i) $\sqrt{a + ib} = \pm \frac{1}{\sqrt{2}}\left[\sqrt{(\sqrt{a^2 + b^2} + a)} + i\ \sqrt{(\sqrt{a^2 + b^2} + a)}\right]$

(ii) $\sqrt{a - ib} = \pm \frac{1}{\sqrt{2}}\left[\sqrt{(\sqrt{a^2 + b^2} + a)} - i\ \sqrt{(\sqrt{a^2 + b^2} + a)}\right]$

(iii) $\sqrt{a + ib} + \sqrt{a - ib} = \sqrt{2(\sqrt{a^2 + b^2} + a)}$

Solution

Let $\sqrt{a + ib} = x + iy$ Then, $a + ib = (x + iy)^2$

$\Rightarrow a + ib = (x^2 - y^2) + 2ixy$

$\Rightarrow x^2 - y^2 = a$ (1)

and $2xy = b$ (2)

$(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2 = a^2 + b^2$

$x^2 + y^2 = \sqrt{a^2 + b^2}$ (3)

Solving equation (2) and (3), we have

$x = \pm \frac{1}{2}\sqrt{[\sqrt{a^2 + b^2} + a]}$ and

$x = \pm \frac{1}{2}\sqrt{[\sqrt{a^2 + b^2} - a]}$

$\therefore\ \sqrt{a + ib} = \pm \frac{1}{\sqrt{2}}\left[\sqrt{(\sqrt{a^2 + b^2} + a)} + i\sqrt{(\sqrt{a^2 + b^2} - a)}\right]$ (4)

Similarly, $\sqrt{a - ib} = \pm \frac{1}{2}\left[\sqrt{(\sqrt{a^2 + b^2} + a)} - i\sqrt{(\sqrt{a^2 + b^2} - a)}\right]$ (5)

Adding equation (4) and (5), we get

$\sqrt{a+ib}+\sqrt{a-ib}$

$= \pm \frac{1}{\sqrt{2}}\left[2\sqrt{(\sqrt{a^2+b^2}+a)}\right]$

$= \pm \sqrt{2(\sqrt{a^2+b^2}+a)}$

UNSOLVED SUBJECTIVE PROBLEMS: (CBSE /STATE BOARD): TO GRASP THE TOPIC, SOLVE THESE PROBLEMS

Exercise I

1. Write the following complex numbers in the polar form

(i) $-1-i$ (ii) $\frac{2+6\sqrt{3}i}{5+\sqrt{3}i}$

2. Find the modulus and argument of each of the following complex numbers

(i) $1+i\sqrt{3}$ (ii) -4

(iii) $\frac{1}{1+i}$ (iv) $1+i$

3. If points p represents the complex number $z = x + iy$ on the argand plane, then find the locus of the point p such that arg $(z) = 0$.

4. If z_1 and z_2 be two complex numbers such that $|z_1+z_2| = |z_1-z_2|$, then prove that arg (z_1) − arg $(z_2) = \pi/2$.

5. If arg $z < 0$, then prove that arg $(-z)$ − arg $z = \pi$

6. Prove that $\left(\frac{\sqrt{3}+i}{2}\right)^6 + \left(\frac{i-\sqrt{3}}{2}\right)^6 = -2$

7. Find the square roots of the following

(i) $7-24i$ (ii) $-8i$

8. If ω is one cube root of unity, then prove that $(1-\omega+\omega^2)^5 + (1+\omega-\omega^2)^5 = 32$

9. Find the cube roots of 27.

10. If ω_1 and ω_2 are complex cube roots of unity, then prove that $\omega_1^2 + \omega_2^4 = -\frac{1}{\omega_1\omega_2}$

Exercise ii

1. Write the following complex numbers in the polar form

(i) i (ii) $\frac{1+i}{1-i}$ (iii) $\frac{-16}{1+i\sqrt{3}}$

2. Find the modulus and argument of each of the following complex numbers

(i) $-2+2i\sqrt{3}$ (ii) $-\sqrt{3}-i$

(iii) $2\sqrt{3}-2i$

3. Express $(1-\cos\theta+i\sin\theta)$ in polar form.

4. Prove that $|z_1+z_2|^2 + |z_1-z_2|^2 = 2\,[\,|z_1|^2 + |z_2|^2]$

5. If z_1, z_2 are two non− zero complex numbers such that $|z_1+z_2| = |z_1| + |z_2|$. Then, prove that arg z_1 − arg $z_2 = 0$.

6. If $z = \frac{\sqrt{3}+i}{2}$ then prove that $Z^{69} = -i$.

7. Find the square roots of the following

(i) $5+12i$ (ii) $-15-8i$

8. If $1, \omega, \omega^2$ are the cube roots of unity, then prove that

$(1-\omega)^3 - (1+\omega^2)^3 = 0.$

9. Prove that

(i) $1+\omega^n+\omega^{2n} = 0$, if n is not a multiple of 3.

(ii) $1+\omega^n+\omega^{2n} = 3$, if n is a multiple of 3.

10. Prove that

$(1+5\omega^2+\omega^4)(1+5\omega+\omega^2)(5+\omega+\omega^2) = 64$

ANSWERS

Exercise – I

1. (i) $\sqrt{2}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$

(ii) $2\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$

2. (i) $|z| = 2$, arg $(z) = \pi/3$

(ii) $|z| = 4$, arg $(z) = \pi$
(iii) $|z| = 2$, arg $(z) = -\pi/2$
(iv) $|z| = \sqrt{2}$, arg $(z) = \frac{\pi}{4}$

3. $y = 0$ which is an equation of x – axis.

7. (i) $\pm(4 - 3i)$ (ii) $\pm 2(1 - i)$

9. $3, 3\omega, 3\omega^2$

Exercise – II

1. (i) $1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$
(ii) $1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$
(iii) $8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

2. (i) $|z| = 4$, arg $(z) = 2\pi/3$
(ii) $|z| = 2$, arg $(z) = -5\pi/6$
(iii) $|z| = 4$, arg $(z) = -\pi/6$

3. $2\sin\frac{\theta}{2}\left[\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) + i\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right]$

7. (i) $\pm(3 + 2i)$ (ii) $\pm(1 - 4i)$

SOLVED OBJECTIVE QUESTIONS: HELPING HAND

1. $\left(\frac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta}\right)^n$ is equal to

[PET (Raj.) – 1998]

(a) $\cos n\theta + i\sin n\theta$
(b) $\sin n\theta + i\cos n\theta$
(c) $\cos n(\pi/2 - \theta) + i\sin n(\pi/2 - \theta)$
(d) none of these

Solution

(c) Exp. =

$$\left[\frac{1 + \cos(\pi/2 - \theta) + i\sin(\pi/2 - \theta)}{1 + \cos(\pi/2 - \theta) - i\sin(\pi/2 - \theta)}\right]^n$$

$$= \cos n(\pi/2 - \theta) + i\sin n(\pi/2 - \theta)$$

2. If $\sqrt{x} + \frac{1}{\sqrt{x}} = 2\cos\theta$, then $x^6 + \frac{1}{x^6}$ is equal to

[CET (Karnataka) – 2003]

(a) $2\cos 6\theta$ (b) $2\cos 12\theta$
(c) $2\sin 6\theta$ (d) $2\sin 12\theta$

Solution

(b) $\sqrt{x} + \frac{1}{\sqrt{x}} = 2\cos\theta$

$\Rightarrow \sqrt{x} = \cos\theta + i\sin\theta$

$\Rightarrow (\sqrt{x})^{12} = x^6 = \cos 12\theta + i\sin 12\theta$

$\Rightarrow 1/x^6 = \cos 12\theta - i\sin 12\theta$

$\therefore x^6 + \frac{1}{x^6} = 2\cos 12\theta$

3. If $i z^4 + 1 = 0$, then z can take the value

[MPPET – 2006]

(a) $\frac{1 + i}{\sqrt{2}}$ (b) $\cos\frac{\pi}{8}\, i\sin\frac{\pi}{8}$
(c) $\frac{1}{4i}$ (d) i

Solution

(b) Given that $i z^4 + 1 = 0$

$\Rightarrow z^4 = \frac{1}{i} = \frac{i^2}{i} = i$

Let $z^4 = \cos\frac{\pi}{2}\, i\sin\frac{\pi}{2}$

$\therefore z = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/4}$

By using de Moivre's theorem, we get

$z = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}$

4. For any integer n, arg $z = \frac{(\sqrt{3} + i)^{4n+1}}{(1 - i\sqrt{3})^{4n}}$ is

(a) $\pi/6$ (b) $\pi/3$
(c) $\pi/2$ (d) $2\pi/3$

Solution

$$z = \frac{2^{4n+1}}{2^{4n}} \frac{e^{i(4n+1)\pi/6}}{e^{-i4n\pi/3}} = = 2e^{i(12n+1)\pi/6}$$

$$= 2e^{2n\pi i}\, e^{i\pi/6} = 2e^{i\pi/6}$$

$\therefore$ are $z = \pi/6$

5. If $2\cos\theta = x + \frac{1}{x}$ and $2\cos\phi = y + \frac{1}{y}$, then $\frac{x}{y} + \frac{y}{x}$ is equal to

[MNR – 1987]

(a) $2\cos(\theta - \phi)$ (b) $2\cos(\theta + \phi)$
(c) $2\sin(\theta - \phi)$ (d) $2\sin(\theta + \phi)$

Solution

(a) $x = \cos\theta + i\sin\theta$, $y = \cos\phi + i\sin\phi$

$\Rightarrow \frac{x}{y} = \cos(\theta - \phi) + i\sin(\theta - \phi)$

$\therefore \frac{x}{y} + \frac{y}{x} = 2\cos(\theta - \phi)$

6. A value of n such that $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^n = 1$ is

[EAMCET – 2007]

(a) 12 (b) 3 (c) 2 (d) 1

Solution

(a) $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = 1 \Rightarrow \left(cis\,\frac{\pi}{6}\right) = 1$

only $12 = n$, satisfies in the given answers.

7. If $\left(\frac{1 + \cos\theta + i\sin\theta}{i + \sin\theta + i\cos\theta}\right)^4 = \cos n\theta + i\sin n\theta$, then n is equal to

[EAMCET – 1986]

(a) 1 (b) 2 (c) 3 (d) 4

Solution

(d) $D^r = i(1 + \cos 0) + \sin\theta$

$= 2i\cos^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

Therefore,

$$\text{L.H.S.} = \left[\frac{\cos(\theta/2) + i\sin(\theta/2)}{i\cos(\theta/2) + \sin(\theta/2)}\right]^4$$

$$= \frac{1}{i^4}(\cos\theta + i\sin\theta)^4$$

$$= \cos 4\theta + i\sin 4\theta$$

8. For any two complex numbers z_1, z_2 we have $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$ then:

(a) $\text{Re}\left(\frac{z_1}{z_2}\right) = 0$ (b) $\text{Im}\left(\frac{z_1}{z_2}\right) = 0$

(c) $\text{Re}(z_1 z_2) = 0$ (d) $\text{Im}(z_1 z_2) = 0$

Solution

(a) We have $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$

$\Rightarrow |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$
$= |z_1|^2 + |z_2|^2$

where $\theta_1 = \arg(z_1)$, $\theta_2 = \arg(z_2)$

$\Rightarrow \cos(\theta_1 - \theta_2) = 0$

$\Rightarrow \theta_1 - \theta_2 = \frac{\pi}{2}$

$\Rightarrow \text{are}\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2} \Rightarrow \text{Re}\left(\frac{z_1}{z_2}\right)$

$= \frac{|z_1|}{|z_2|}\cos\left(\frac{\pi}{2}\right) = 0$

Note: Also $\text{Re}\left(\frac{z_1}{z_2}\right) = 0 \Rightarrow \text{Re}(z_1\bar{z}_2) = 0$

$\Rightarrow z_1\bar{z}_2$ is purely imaginary.

9. Given $z = (1 + i\sqrt{3})^{100}$, then $\frac{\text{Re}(z)}{\text{Im}(z)}$ equals

[AMU – 2002]

(a) 2^{100} (b) 2^{50} (c) $1/\sqrt{3}$ (d) $\sqrt{3}$

Solution

(c) Let $z = (1 + i\sqrt{3})$

$r = \sqrt{3 + 1} = 2$ and $r\cos\theta = 1$, $r\sin\theta = \sqrt{3}$

$\tan\theta = \sqrt{3} = \tan\frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{3}$

$\Rightarrow z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$$\Rightarrow z^{100} = \left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^{100}$$

$$= 2^{100}\left(\cos\frac{100\pi}{3} + i\sin\frac{100\pi}{3}\right)$$

$$= 2^{100}\left(-\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$$

$$= 2^{100}\left(-\frac{1}{2}\frac{i\sqrt{3}}{2}\right)$$

$$\therefore \frac{\text{Re}(z)}{\text{Im}(z)} = -\frac{1/2}{-\sqrt{3}/2} = \frac{1}{\sqrt{3}}$$

10. If $(1 + i\sqrt{3})^9 = a + ib$, then b is equal to

[RPET – 1995]

(a) 1 (b) 256
(c) 0 (d) 93

Solution

(c) $1 + i\sqrt{3} = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= 2\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right] = 2e^{i\pi/3}$

$\therefore\ (1 + i\sqrt{3})^9 = (2e^{i\pi/3})^9 = 2^9.e^{i(3\pi)}$

$= 2^9(\cos 3\pi + i\sin 3\pi) = -2^9$

$\therefore\ a + ib = (1 + i\sqrt{3})^9 = -2^9;\quad \therefore\ b = 0$

11. If $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$ $z = \cos\gamma + i\sin\gamma$ and $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$, then xyz is equal to

(a) i (b) 1 or −1
(c) −1 but not 1 (d) 0

[Kerala (CEE) – 2003]

Solution

(b) $\because \tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$

$\Rightarrow \alpha + \beta + \gamma = \pi$ or 0

$\therefore\ xyz = \cos(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma)$

$= 1$ or -1.

12. If z is any complex number satisfying $|z - 1| = 1$, then which of the following is correct:

(a) $\arg(z - 1) = 2\arg z$
(b) $2\arg(z) = 2/3\arg(z^2 - z)$
(c) $\arg(z - 1) = \arg(z + 1)$
(d) $\arg z = 2\arg(z + 1)$

Solution

(a) Therefore, $|z - 1| = 1\ \therefore\ z - 1 = e^{i\theta} = \cos\theta + i\sin\theta$ where $\arg(z - 1) = \theta$

$\therefore\ z = 1 + \cos\theta + i\sin\theta$

$= 2\cos^2(\theta/2) + 2i\sin\theta/2\cos\theta/2$

$= 2\cos\frac{\theta}{2}\left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right]$

$= 2\cos\left(\frac{\theta}{2}\right).\ e^{i\pi/2}$

$\therefore\ \arg z = \theta/2 = 1/2\arg(z - 1)$

Thus, $\arg(z - 1) = 2\arg z$.

13. If $|z| = 1$ and $|z| \neq \pm 1$, then all the values of $\frac{z}{1 - z^2}$ lie on

(a) a line not passing through the origin
(b) $|z| = 2$ (c) z − axis
(d) y − axis

[IIT – 2007]

Solution

(d) $|z| = 1, z \neq \pm 1$

Let z be $e^{i\theta}$

$\therefore\ \frac{z}{1 - z^2} = \frac{e^{i\theta}}{1 - e^{2i\theta}} = \frac{e^{i\theta}}{1 - \cos 2\theta - i\sin 2\theta}$

$= \frac{e^{i\theta}}{2\sin^2\theta - 2i\sin\theta\cos\theta}$

$= \frac{e^{i\theta}}{2\sin\theta(\sin\theta - i\cos\theta)}$

$= \frac{e^{i\theta}}{2\sin\theta e^{i(\theta - \pi/2)}} = \frac{i}{2\sin\theta}$

where $\sin\theta \neq 0$ ($\because z \neq \pm 1$)

Hence, $\frac{z}{1 - z^2}$ always lies on y − axis.

14. If $\left(\frac{1 + i\sqrt{3}}{1 - i\sqrt{3}}\right)^n \in Z$, then the least positive integral value of n is:

[UPSEAT – 2002]

(a) 1 (b) 2 (c) 3 (d) 4

Solution

(c) $\left(\frac{1 + i\sqrt{3}}{1 - i\sqrt{3}}\right)^n = \left(\frac{-1 + i\sqrt{3}}{2}\right)^{2n}$

$= \cos n\left(\frac{4\pi}{3}\right) + i\sin n\left(\frac{4\pi}{3}\right)$

It is an integer for $n = 3$ (the least positive integral value of n)

15. The square root of i is

[NDA – 2004]

(a) $\pm\frac{1}{\sqrt{2}}(1 + i)$ (b) $\pm\frac{1}{\sqrt{2}}(1 - i)$

(c) $\pm\sqrt{2}(1 + i)$ (d) $\pm\sqrt{2}(1 - i)$

Solution

(a) $(i)^{1/2} = \pm(e^{i\pi/2})^{1/2}$

$= \pm e^{i\pi/4} = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$= \pm \frac{1}{\sqrt{2}}(1 + i)$

16. If a = $\frac{1 - i\sqrt{3}}{2}$ then the correct matching of List – I from List – II is

[EAMCET – 2007]

List – I	List – II
(i) $a\bar{a}$	(A) $2\pi/3$
(ii) arg $(1/\bar{a})$	(B) $-i\sqrt{3}$
(iii) $a - \bar{a}$	(C) $2i/\sqrt{3}$
(iv) Im $(4/3a)$	(D) 1
	(E) $\pi/3$ (F) $2/\sqrt{3}$

The correct match is

	(i)	(ii)	(iii)	(iv)
(a)	D	E	C	B
(b)	D	A	B	F
(c)	F	E	B	C
(d)	D	A	B	C

Solution

(b) $a = \frac{1}{2} - \frac{\sqrt{3}}{2} i = 1cis\left(-\frac{\pi}{3}\right)$

$\therefore\ \bar{a} = 1.cis\left(+\frac{\pi}{3}\right)$

(i) $a\bar{a} = cis\left(-\frac{\pi}{3}\right) cis\left(\frac{\pi}{3}\right) = cis0 = 1$

(ii) $Arg\left(\frac{1}{\bar{a}}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

(iii) $a - \bar{a} = cis\left(-\frac{\pi}{3}\right) - cis\ \frac{\pi}{3}$

$= -2i \sin\frac{\pi}{3} = -i\sqrt{3}$

(iv) $I_m\left(\frac{4}{3a}\right) = I_m\left(\frac{4}{3}\left(\cos\frac{\pi}{3}\right)\right)$

$= \frac{4}{3} \times \frac{\sqrt{3}}{2} = \frac{2}{\sqrt{3}}$

Note: $\text{cis}\theta = \cos\theta + i\sin\theta$

17. Find the value of the expression

$2\left(1 + \frac{1}{\omega}\right)\left(1 + \frac{1}{\omega^2}\right) + 3\left(2 + \frac{1}{\omega}\right)\left(2 + \frac{1}{\omega^2}\right)$

$+ \ldots.. + (n + 1)\left(n + \frac{1}{\omega}\right)\left(n + \frac{1}{\omega^2}\right)$

[Orissa JEE – 2007]

(a) $n + \frac{n^2(n+1)^2}{4}$ (b) $n - \frac{n^2}{4(n+1)^2}$

(c) $1 - \frac{n^2}{4(n+1)^2}$ (d) $1 + \frac{n^2}{4(n+1)^2}$

Solution

(a) We have $(z + 1)(z + \omega)(z + \omega^2) = z^3 + 1$

Therefore, the given expression

$= \sum_{r=1}^{r=n} (r + 1)(r + \omega)(r + \omega^2)$

$\left(\because \left(1 + \frac{1}{\omega}\right)\left(1 + \frac{1}{\omega^2}\right) = (1 + \omega^2)(1 + \omega)\right)$

$= \sum_{r=1}^{r=n} (r^3 + 1) = \sum_{r=1}^{r=n} r^3 + \sum_{r=1}^{r=n}$

$1 = \frac{n^2(n+1)^2}{4} + n$

18. If 1, ω, ω^2 are the cube roots of unity then $(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8)$ is equal to

[Karnataka CET – 2007]

(a) 1 (b) 0

(c) ω^2 (d) ω

Solution

(a) If 1, ω, ω^2 are the cube roots of unity then $1 + \omega + \omega^2 = 0$ and ω 3 = 1.

$(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8)$

$= (1 + \omega)(-\omega)(1 + \omega)(-\omega)$

$= -\omega^2(-\omega)(-\omega^2)(-\omega)\,\omega^3.\ \omega^3 = 1.\ 1 = 1$

19. If ω ($\neq$ 1) is a cube root of unity and $(1 + \omega)^7 = a + b\,\omega$, then A and b are respectively, the numbers

[IIT – 1995]

(a) 0, 1 (b) 1, 0

(c) 1, 1 (d) – 1, 1

Solution

(c) $(1 + \omega)^7 = a + b\,\omega$

$\Rightarrow (-\omega\ 2)^7 = a + b\,\omega$

$\Rightarrow \omega^{14} = -A - b\,\omega$

$\Rightarrow \omega^2 . \omega\ 12 = -A - b\,\omega$

$\Rightarrow A + b\,\omega + \omega\ 2 = 0 \Rightarrow A = 1,\ b = 1$

$(\because 1 + \omega + \omega^2 = 0)$

20. The maximum value of $|z|$ where z satisfies the condition $\left|z + \frac{2}{z}\right| = 2$ is

(a) $\sqrt{3} - 1$ (b) $\sqrt{3} + 1$

(c) $\sqrt{3}$ (d) $\sqrt{2} + \sqrt{3}$

Solution

(b) $\left|z + \frac{2}{z}\right| = 2 \Rightarrow |z| - \frac{2}{|z|} \le 0$

$\Rightarrow |z|2 - 2|z| - 2 \le 0$

$|z| \le 2 \pm \frac{\sqrt{4+8}}{2} \le 1 \pm \sqrt{3}$

Hence maximum value of $|z|$ is $1 + \sqrt{3}$

21. If $x = a + b$, $y = a\alpha + b\beta$ and $z = \alpha\beta + b$, where α and β are the cube roots of unity then xyz is

[MP PET – 2005]

(a) $a^2 + b^2$ (b) $a^3 + b^3$

(c) a^3b^3 (d) none of these

Solution

(b) Suppose here $a = \omega$, $\alpha = \omega^2$, then

$xyz = (a + b)\ (a\,\omega + b\,\omega^2)\ (a\,\omega^2 + b\,\omega)$

$= (a + b)\ (a^2\,\omega^3 + ab\,\omega^2 + ba\,\omega^4 + b^2\,\omega^3)$

$= (a + b)\ [a^2 + ab\ (\omega^2 + \omega) + b^2]$

$= (a + b)\ (a^2 - ab + b^2) = a^3 + b^3$

22. $\frac{(-1 + i\sqrt{3})^{15}}{(1 - i)^{20}} + \frac{(-1 - i\sqrt{3})^{15}}{(1 + i)^{20}}$ is equal to

(a) – 64 (b) – 32

(c) – 16 (d) 1/16

Solution

(a) $2^{15}\left[\frac{\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{15}}{(1 - i)^{20}} + \frac{\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^{15}}{(1 + i)^{20}}\right]$

$= 2^{15}\left[\frac{\omega^{15}}{(1 - i)^{20}} + \frac{\omega^{30}}{(1 + i)^{20}}\right]$

$= 2^{15}\left[\frac{1}{(1 - i)^{20}} + \frac{1}{(1 + i)^{20}}\right]$

$= 2^{15}\,\frac{(1 + i)^{20} + (1 - i)^{20}}{(i - 12)^{20}}$

$= \frac{2^{15}}{2^{20}}\,[(1 + i)^{20} + (1 - i)^{20}]$

$= \frac{1}{32}\,[\{(1 + i)^2\}^{10} + \{(1 - i)^{10}]$

$= \frac{1}{32}\,[(2i)^{10} + (2i)^{10}]$

$= \frac{1}{32}\,[(2^{10}\,\{i^{10} + i^{10}\}] = 32\,[-1 - 1] = -64.$

23. If $z + z^{-1} = 1$, then $z^{100} + z^{-100}$ is equal to

(a) i (b) $-i$

(c) 1 (d) – 1

Solution

(d) $z + z^{-1} = 1$

$\Rightarrow z^2 - z + 1 = 0$

$\Rightarrow z = \omega$ or $-\omega^2$

For $z = -\omega$, $z^{100} + z^{-100} = (-\omega)^{100} + (-\omega)^{-100}$

$= \omega + \frac{1}{\omega} = \omega + \omega^2 = -1$

For $z = -\omega^2$, $z^{100} + z^{-100} = (-\omega^2)^{100} + (-\omega^2)^{-100}$

$= \omega^{200} + \frac{1}{\omega^{200}}$

$= \omega^2 + \frac{1}{\omega^2} = \omega^2 + \omega + = -1$

24. If z_1 and z_2 be complex numbers such that $z_1 \ne z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then $\frac{z_1 + z_2}{z_1 - z_2}$ may be

[IIT – 1986; CET (Pb.) – 1992]

(a) real and positive

(b) real and negative

(c) purely imaginary

(d) zero or purely imaginary

Solution

(d) Let $z_1 = x + iy$ and $z_2 = p + iq$, Then

$|z_1| = |z_2| \Rightarrow x^2 + y^2 = p^2 + q^2$ (1)

Now $\frac{z_1 + z_2}{z_1 - z_2} = \frac{(x+p)+i(y+q)}{(x-p)+i(y-q)}$

$= \frac{2i(xq+yp)}{(x-p)^2+(y-q)^2}$ (2)

If $xq + yp \neq 0$, then given expression is purely imaginary.

If $xq + yp = 0$, then $\frac{x}{p} = \frac{y}{-q} = \lambda$ (say) then from (1)

$p^2 + q^2 = 12(p^2 + q^2) \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = 1, -1$

For both values of λ, $z_1 \neq z_2$ but $|z_1| = |z_2|$.

$\Rightarrow z_1 = -z_2$ So in this case, the given expression is zero.

25. Solutions of the equation $z_2 + |z| = 0$ are

(a) $0, \pm 1, \pm i$ (b) $0, \pm i$

(c) $1 + i$ (d) $1 - i$

Solution

$(x+iy)^2 + \sqrt{x^2+y^2} = 0$

$x^2 - y^2 + \sqrt{x^2+y^2} + 2ixy = 0$

$x^2 - y^2 + \sqrt{x^2+y^2} = 0$ (1)

$2xy = 0$ (2)

From (2), $x = 0$ or $y = 0$. Then, from (1)

$x = 0 \Rightarrow y = 0, \pm 1$

$\therefore z = 0, \pm i$ **Ans:** *b*

26. $\tan\left[i \log \frac{a-ib}{a+ib}\right]$ is equal to

[DCE – 1996]

(a) $\frac{2ab}{a^2+b^2}$ (b) $\frac{2ab}{a^2-b^2}$

(c) $\frac{a^2-b^2}{2ab}$ (d) ab

Solution

(b) $\because \log\left(\frac{a-ib}{a+ib}\right) = \log(a-ib) - \log(a+ib)$

$= \log r + i\tan^{-1}\left(\frac{-b}{a}\right) - \left(\log r + i\tan^{-1}\frac{b}{a}\right)$

where $r = \sqrt{a^2+b^2}$

$= 2i\tan^{-1}(-b/a)$

$= 2i\tan^{-1} b/a$

$= -i \tan-1 \frac{2ab}{a^2-b^2}$

$\Rightarrow \tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \frac{2ab}{a^2-b^2}$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. For any complex number z, which of the following is not true?

(a) $\text{Re}(z) = \frac{z+\bar{z}}{2}$ (b) $\text{Im}(z)\frac{z-\bar{z}}{2i}$

(c) $|z|^2 = z\bar{z}$

(d) $-|\text{Re}(z)| \leq |z| \leq |\text{Re}(z)|$

2. If $|z-i| < |z+i|$, then

(a) $\text{Re}(z) > 0$ (b) $\text{Re}(z) < 0$

(c) $\text{Im}(z) > 0$ (d) $\text{Im}(z) < 0$

3. $(\cos\theta + i\sin\theta)^2$ is equal to

(a) $\cos 2\theta + i\sin 2\theta$ (b) $\sin 2\theta + i\cos 2\theta$

(c) $\cos 2\theta - i\sin 2\theta$ (d) none of these

4. $\left(\frac{1}{\sqrt{3}+i}\right)^{24} =$

(a) 2^{24} (b) -2^{24}

(c) $\frac{1}{2^{24}}$ (d) none of these

5. $\left(\frac{\sqrt{3}+i}{2}\right)^{69}$ is equal to

(a) 1 (b) −1 (c) $-i$ (d) i

6. If $\left(\frac{1-i}{1+i}\right)^{100} = a + ib$, then

[MPPET – 1998]

(a) $a = 2, b = -1$ (b) $a = 1, b = 0$

(c) $a = 0, b = 1$ (d) $a = -1, b = 2$

7. If z_1 and z_2 are any two complex numbers then $|z_1 + z_2|^2 + |z_1 - z_2|^2$ is equal to

[MPPET – 1993; RPET – 1997]

(a) $2|z_1|^2|z_2|^2$ (b) $2|z_1|^2 + 2|z_2|^2$
(c) $|z_1|^2 + |z_2|^2$ (d) $2|z_1||z_2|$

8. If z_1 and z_2 are two non-zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1) - \arg(z_2)$ is equal to

(a) $-\pi$ (b) $-\pi/2$
(c) $\pi/2$ (d) 0

[IIT – 1979, 1987; EAMCET – 1986; RPET – 1997; MPPET – 2001, 2007; AIEEE – 2005]

9. If z and ω are two non-zero complex numbers such that $|z| = |\omega|$ and $\arg(z) + \arg(\omega) = \pi$, then z is equal to

(a) ω (b) $-\omega$
(c) $\overline{\omega}$ (d) $-\overline{\omega}$

[IIT – 1995; AIEEE – 2002; JEE (Orissa) – 2004]

10. $-1 + \sqrt{-3} = re^{i\theta}$, then θ =

[MP PET – 1999; RPET – 1989]

(a) $\frac{2\pi}{3}$ (b) $-\frac{2\pi}{3}$
(c) $\frac{\pi}{3}$ (d) $-\frac{\pi}{3}$

11. If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then $\left(\frac{z_1}{z_2}\right)$ + are $\left(\frac{z_2}{z_3}\right)$ equals

(a) 0 (b) $-\frac{\pi}{2}$
(c) $\frac{3\pi}{2}$ (d) π

12. If $z \neq 0$ is a complex number such that $\text{Arg}(z) = \pi/4$, then

(a) $\text{Im}(z^2) = 0$ (b) $\text{Re}(z^2) = 0$
(c) $\text{Re}(z) = i\text{m}(z^2)$ (d) none of these

13. The value of $(i)^i$ is

[AMU – 1998]

(a) ω (b) ω^2
(c) $e^{-\pi/2}$ (d) $2\sqrt{2}$

14. Argument of the complex number $\left(\frac{-1-3i}{2+i}\right)$ is

(a) 45° (b) 135°
(c) 225° (d) 240°

[VITEEE – 2008]

15. What is Arg (bi) where $b > 0$?

[NDA – 2008]

(a) 0 (b) $\frac{\pi}{2}$
(c) π (d) $\frac{3\pi}{2}$

16. Let c be the set of complex numbers and z_1, z_2 are in C.

(i). Argument z_1 = argument $z_2 \Rightarrow z_1 = z_2$
(ii). $|z_1| = |z_2| \Rightarrow z_1 = z_2$

Which of the statements given above is/are correct?

[NDA – 2008]

(a) 1 only (b) 2 only
(c) both 1 and 2 (d) neither 1 nor 2

17. If $y = \cos\theta + i\sin\theta$, then the value of $y + \frac{1}{y}$ is

(a) $2\cos\theta$ (b) $2\sin\theta$
(c) $2\text{cosec}\theta$ (d) $2\tan\theta$

[RPET – 1995]

18. The imaginary part of i^i is

(a) 0 (b) 1
(c) 2 (d) -1

[Karnataka CET – 2007]

19. If z_1 and z_2 are any two complex numbers, then which of the following is not true.

(a) $|z_1 + z_2| \leq |z_1| + |z_2|$
(b) $|z_1 - z_2| \leq |z_1| + |z_2|$
(c) $|z_1 - z_2| \geq ||z_1| - |z_2||$
(d) $|z_1| - |z_2| \geq |z_1 - z_2|$

20. If $z^2 = -i$, then z =

(a) $\frac{1}{\sqrt{2}}(1+i)$ (b) $\frac{1}{\sqrt{2}}(1+i)$
(c) $\pm\frac{1}{\sqrt{2}}(1-i)$ (d) none of these

21. If ω is a non real cube root of unity, then $\frac{a + b\omega + c\omega^2}{a\omega + c + b\omega^2}$

(a) 1 (b) ω^2
(c) ω (d) none of these

22. $(3 + 3\omega + 5\omega^2)^3 - (2 + 4\omega + 2\omega^2)^3$ is equal to

(a) 0 (b) 3
(c) 2 (d) 1

23. If α and β are imaginary cube roots of unity, then the value of $\alpha^4 + \beta^{28} + \frac{1}{\alpha\beta}$, is

[MPPET – 1998]

(a) 1 (b) −1
(c) 0 (d) none of these

24. If z_1 and z_2 are two complex numbers, then $|z_1 + z_2|$ is

(a) $\le |z_1| + |z_2|$ (b) $\le |z_1| - |z_2|$
(c) $< |z_1| + |z_2|$ (d) $> |z_1| + |z_2|$

[RPET – 1985; MPPET – 1987, 2004; Kerala Engg. – 2002]

25. If ω is an imaginary cube root of unity, $(1 + \omega - \omega^2)^7$ equals

(a) $128\,\omega$ (b) $-128\,\omega$
(c) $128\,\omega^2$ (d) $-128\,\omega^2$

[IIT – 1998; MPPET – 2000]

26. If z is any complex number such that $|z + 4| \le 3$, then the greatest value of $|z + 1|$ is

[AIEEE – 2007]

(a) 6 (b) 4
(c) 5 (d) 3

27. If n is a positive integer not a multiple of 3, then $1 + \omega^n + \omega^{2n} =$

[MPPET – 2004]

(a) 3 (b) 1
(c) 0 (d) none of these

28. If 1, ω, ω^2 are the three cube roots of unity, then $(3 + \omega^2 + \omega^4)^6 =$

[MPPET – 1995]

(a) 64 (b) 729
(c) 2 (d) 0

29. If ω is a cube root of unity, then $(1 + \omega - \omega^2)(1 - \omega + \omega^2) =$

(a) 1 (b) 0
(c) 2 (d) 4

[MNR – 1990; MPPET – 1993, 2002]

30. One of the cube roots of unity is

[MPPET – 1994, 2003]

(a) $\frac{-1 + i\sqrt{3}}{2}$ (b) $\frac{1 + i\sqrt{3}}{2}$

(c) $\frac{1 - i\sqrt{3}}{2}$ (d) $\frac{\sqrt{3} - i}{2}$

31. If ω is the cube root of unity, then $(3 + 5\,\omega + 3\,\omega^2)^2 + (3 + 3\,\omega + 5\,\omega^2)^2 =$

[MPPET – 1999]

(a) 4 (b) 0
(c) − 4 (d) none of these

32. If ω is cube root of unity then the value of $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^8) =$

(a) 0 (b) 1
(c) − 1 (d) 9

[MP PET – 2006]

33. $\left(\frac{-1 + i\sqrt{3}}{2}\right)^{20} + \left(\frac{-1 - i\sqrt{3}}{2}\right)^{20} =$

(a) $20\sqrt{3}i$ (b) 1

(c) $\frac{1}{2^{19}}$ (d) − 1

34. If $\left|Z - \frac{4}{z}\right| = 2$, then the maximum value of $|z|$ is equal to

[AIEEE – 2009]

(a) $\sqrt{3} + 1$ (b) $\sqrt{5} + 1$
(c) 2 (d) $2 + \sqrt{2}$

35. What is the value of

$$\left(\frac{-1 + i\sqrt{3}}{2}\right)^{900} + \left(\frac{-1 + i\sqrt{3}}{2}\right)^{301}?$$

[NDA – 2009]

(a) $\frac{-1 + i\sqrt{3}}{2}$ (b) $\frac{1 - i\sqrt{3}}{2}$

(c) $\frac{-1 - i\sqrt{3}}{2}$ (b) $\frac{1 + i\sqrt{3}}{2}$

36. If $z^2 + z + 1 = 0$, where z is a complex number, then the value of $\left(z + \frac{1}{z}\right)^2 + \left(z^2 + \frac{1}{z^2}\right)^2 + \left(z^3 + \frac{1}{z^3}\right)^2 + \ldots + \left(z^6 + \frac{1}{z^6}\right)^2$ is

(a) 6 (b) 12
(c) 18 (d) 24

[MPPET – 2009, AIEEE – 2006]

37. The number of solutions of the equation $z^2 + \bar{z} = 0$ is

[MPPET – 2009]

(a) 1 (b) 2

(c) 3 (d) 4

38. If x is a positive integer, then $(1 + i\sqrt{3})^n + (1 - i\sqrt{3})$n is equal to

[Orissa JEE – 2009]

(a) $2^{n-1} \cos \frac{n\pi}{3}$ (a) $2^n \cos \frac{n\pi}{3}$

(c) $2^{n+1} \cos \frac{n\pi}{3}$ (d) none of these

SOLUTIONS

1. (d) If $(1 - i)^n = 2^n \Rightarrow n = 0$ clearly.

2. (c) $(1 + i)^8 = ((1 + i)^2)^4 = (1 + i^2 + 2i)^4$
$= (2i)^4 = 16$
$(1 - i)^8 = ((1 - i)^2)^4 = (1 + i^2 - 2i)^4$
$= (-2i)^4 = 16$
$(1 + i)^8 + (1 - i)^8 = 32$

3. (a) $|z_1 + z_2|$

$$= \sqrt{|z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)} \quad (1)$$

$|z_1 - z_2|$

$$= \sqrt{|z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)} \quad (2)$$

If $|z_1 + z_2| = |z_1 - z_2| \Rightarrow \cos(\theta_1 - \theta_2) = \cos 90°$ $= 0$ i.e., $\theta_1 - \theta_2 = 90°$

or $\text{amp}(z_1) - \text{amp}(z_2) = 90°$

4. (c) $\theta = \text{amp}(x + iy) = \tan^{-1}\frac{y}{x}$

$$\theta = \tan^{-1}\left(\frac{\frac{\sqrt{2}}{\sqrt{3}+1}}{\frac{1}{\sqrt{3}+1}}\right) = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

5. (c) $\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^{69} = \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{69}$

$$= \left(\cos 69 \times \frac{\pi}{6} + i \sin 69 \times \frac{\pi}{6}\right)$$

$$= \cos\left(\frac{23\pi}{2}\right) + i\sin\left(\frac{23\pi}{2}\right)$$

$$= 0 + i\left(\sin\left(10\pi + \frac{3\pi}{2}\right)\right)$$

$$= \text{i} \sin\frac{3\pi}{2} = -i$$

6. (b) $\because \frac{1-i}{1+i} = \frac{(1-i)(1-i)}{(1+i)(1-i)} = \frac{-2i}{2} = -i$

Therefore

$$\left(\frac{(1-i)}{1+i}\right)^{100} = (-i)^{100} = i^{100} = (i^4)^{25} = 1$$

Therefore, given $\left(\frac{1-i}{1+i}\right)^{100} = a + ib$ from 1, $a + ib = 1$ on comparing $a = 1, b = 0$.

7. (b) Important Formula
$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\text{Re}\,(z_1\,\bar{z}_2)$
and $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\text{Re}\,(z_1\,\bar{z}_2)$

On adding, we get

$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2\,|z_2|^2$

Alternative solution

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2) \quad (1)$

$(z_1 - z_2) = (x_1 - x_2) + i\,(y_1 - y_2) \quad (2)$

$|z_1 + z_2|^2 = (x_1 + x_2)^2 + (y_1 + y_2)^2$

$|z_1 - z_2|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

On adding, we get

$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,[x_1^2 + y_1^2\,x_2^2 + y_2^2]$

$= 2\,\{|z_1|^2 + |z_2|^2\}$

8. (d) Let $z_1 = r_1\,(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2\,(\cos\theta_2 + i\sin\theta_2)$

$\therefore |z_1 + z_2| = [(r_1\cos\theta_1 + r_2\cos\theta_2)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2]^{1/2}$
$= [r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)]^{1/2}$
$= [(r_1 + r_2)^2]^{1/2}$
$(\because |z_1 + z_2| = |z_1| + |z_2|\,)$

Therefore, $\cos(\theta_1 - \theta_2) = 1 \Rightarrow \theta_1 - \theta_2 = 0$
$\Rightarrow \theta_1 = \theta_2$
$\arg(z_1) - \arg(z_2) = 0$

Alternative Solution

$|z_1 + z_2| = |z_1| + |z_2|$
z_1, z_2 are on the same line
$\therefore \arg z_1 = \arg z_2 \Rightarrow \arg z_1 - \arg z_2 = 0.$

9. (d) Let $|z| = |\omega| = r$ and Arg $\omega = \theta$

then $\omega = r \text{ cis } \theta$ and Arg $z = \pi - \theta$

Hence $z = r \text{ cis } (\pi - \theta)$
$= r\{\cos(\pi - \theta) + i \sin(\pi - \theta)\}$
$= r(-\cos\theta + i\sin\theta) = -r(\cos\theta - i\sin\theta)$
$= -x + iy$
$= -\overline{\omega}$

OR

Quadrantwise complex numbers of equalmagnitude with corresponding argument

$-(z) = (-z) = -x + iy$ $amp(-(z)) = amp(-z)$ $= \pi - \theta$	$z = x + iy$ $amp(z) = \theta$
$-z = -x - iy$ $amp(z) = -(\theta - \theta)$	$z = x - iy$ $amp(z) = -\theta$

Clearly, amp (z) + amp$(-(\overline{z})) = \pi$
i.e., amp (z) + amp$(\omega) = \pi$
or $\omega = -(\overline{z})$

Note: $|z| = |\overline{z}| = |-z| = |-(\overline{z})|$

$$= |(-\overline{z})| = \sqrt{x^2 + y^2}$$

10. (a) Therefore, $e^{ix} = \cos x + i \sin x$
Therefore $-1 + \sqrt{-3} = re^{i\theta}$ may be written as $-1 + \sqrt{3}\, i = r(\cos\theta + i\sin\theta)$

Comparing real and imaginary part

$r\cos\theta = -1,\ r\sin\theta = \sqrt{3}$

or $\frac{r\sin\theta}{r\cos\theta} = \frac{\sqrt{3}}{-1} \Rightarrow \tan\theta = -\sqrt{3}$ or

$\theta = \frac{3\pi}{3}$

11. (a) We have $z_2 = \overline{z}_1$ and $z_4 = \overline{z}_3$, therefore, $z_1 z_2 = |z_1|^2$ and $z_3 z_4 = |z_3|^2$

Now, $\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right) = \arg\left(\frac{z_1 z_2}{z_4 z_3}\right)$

$= \arg\left(\frac{|z_1|^2}{|z_3|^2}\right) = \arg\left(\left|\frac{z_1}{z_3}\right|\right)^2 = 0$

[$\because$ argument of a positive real number is zero]

12. (b) Given that $\arg(z) = \frac{\pi}{4}$
i.e. $\tan^{-1}\frac{y}{x} = \frac{\pi}{4}$
$\Rightarrow \frac{y}{x} = 1 \Rightarrow y = x$

i.e. $z = 1 + i$ (for example)
Therefore, $z^2 = 1 - 1 + 2i = 2i$ i.e., $Re(z^2)$ = real part of $z^2 = 0$

13. (c) Let $A = i^i \Rightarrow \log A = i \log i$
$\Rightarrow \log A = i \log(0 + i) = i[\log 1 + i \tan^{-1} 1/0]$
$\Rightarrow \log A = i[0 + i\pi/2] = -\pi/2$
$\Rightarrow A = e^{-\pi/2}$

14. (c) $z = \frac{-1-3i}{2+i} \times \frac{2-i}{2-i} = \frac{-2+i-6i-3}{5}$

$z = \frac{-5-5i}{5} = -1 - i$
$\arg(z)$ is given by $\sin\theta = -\frac{1}{\sqrt{2}}$, $\cos\theta = -\frac{1}{\sqrt{2}}$
$\Rightarrow \theta = 225°$

15. (b) $\arg(bi) = \frac{\pi}{2}$ ($\because b > 0$)

OR

$\arg(bi) = \tan^{-1}\left(\frac{b}{0}\right)$

$= \tan^{-1}(\infty)\ \frac{\pi}{2}$

16. (d) None of the given statements is correct.
1. their magnitudes may be different.
2. their argument may be different.

17. (a) $y = \cos\theta + i\sin\theta = e^{i\theta}, \frac{1}{y} e^{-i\theta}$

$y + \frac{1}{y} = e^{i\theta} + e^{-i\theta} = 2\cos\theta$

18. (a) $i = e^{\frac{i\pi}{2}} \therefore i^i = \left(e^{\frac{i\pi}{2}}\right)^i = e^{\frac{i^2\pi}{2}} = e^{-\frac{\pi}{2}}$

19. (c) Third side is greater than or equal to the difference of two sides.

20. (c) $(1^2 + i^2 - 2i)/2 = - i$ (By verification ethod).

21. (b) $\dfrac{a + b\omega + c\omega^2}{a\omega + c + b\omega^2} = \dfrac{a\omega^3 + b\omega^4 + c\omega^2}{a\omega + c + b\omega^2}$

$= \dfrac{\omega^2(a\omega + b\omega^2 + c)}{a\omega + c + b\omega^2} = \omega^2.$

22. (a) $(3(1 + \omega) + 5\,\omega^2)^3 - (2(1 + \omega^2) + 4\omega)^3$

$= (-3\,\omega^2 + 5\,\omega^2)^3 - (-2\,\omega + 4\,\omega)^3$

$= (2\,\omega^2)^3 - (2\omega)^3$

$= 8 - 8 = 0.$

23. (c) Take $\alpha = \omega, \beta = \omega^2$, then $\omega^3 = 1$

$\alpha^4 + \beta^{28} + \dfrac{1}{\alpha\beta} + \omega^4 + \omega^{56} + \dfrac{1}{\omega^3}$

$= \omega^3.\ \omega + (\omega^2)^{18}\ .\ \omega^2 + 1\ \omega + \omega^2 + 1 = 0$

24. (a) By triangle identity, $|z_1 + z_2| \le |z_1| + |z_2|$ The modulus of sum of two complex numbers is always less than or equal to the sum of modulii of two complex numbers.

25. (d) $(1 + \omega - \omega^2)^7 = (1 + \omega + \omega^2 - 2\omega^2)^7$

$= (0 - 2\,\omega^2)^7$

$= -2^7.\ \omega^{14} = -128\,\omega^2.$

26. (a) Given $|z + 4| \le 3$ (1)

$\therefore |z + 1| = |(z + 4) + (-3)|$

$\le |z + 4| + |(-3)|$

$\le 3 + 3$

$\Rightarrow |z + 1| \le 6$

$\therefore$ Maximum value of $|z + 1| = 6$.

Alternatively note that $z = -7$ satisfies (1) and for this z,

$|z + 1| = |-7 + 1| = 6.$

27. (c) If n is not the multiple of 3, then

$n = 3K + 1$ or $3K + 2$

$\therefore\ 1 + \omega^n + \omega^{2n} = 1 + \omega^{3K+1} + \omega^{2(3K+1)}$

$= 1 + \omega^{3K}\ .\ \omega + \omega^{3\times 2K}\,\omega^2$

$= 1 + \omega + \omega^2 = 0$

For $n = 3K + 2$ also $1 + \omega^n + \omega^{2n} = 0$

28. (a) $(3 + \omega^2 + \omega^4)^6 = (3 + \omega^2 + \omega)^6$

$= (3 - 1)^6$

$= 26$

$= 64.$

29. (d) $(1 + \omega - \omega^2)(1 - \omega + \omega^2)$

$= (-2\omega^2)(-2\omega) = 4\omega^3 = 4.$

30. (a) Let $x = (1)^{1/3}$ or $x^3 = 1$

Therefore, $x^3 - 1 = 0$ or $(x - 1)(x^2 + x + 1) = 0$

Possible roots $x = 1$ or $x = \dfrac{-1 \pm \sqrt{1 - 4}}{2}$

$x = 1$ or $x = \dfrac{-1 + \sqrt{3}i}{2}, \dfrac{-1 - \sqrt{3}i}{2}$

31. (c) Formula $1 + \omega + \omega^2 = 0$

$\therefore\ (3 + 5\,\omega + 3\,\omega^2)^2$

$= (2\,\omega + 3 + 3\,\omega + 3\,\omega^2)^2 = (2\omega)^2$ (1)

and $(3 + 3\,\omega + 5\,\omega^2)$

$= (2\,\omega^2 + 3 + 3\,\omega + 3\,\omega^2)^2 = (2\,\omega^2)^2$ (2)

Adding 1 and 2

$(3 + 5\,\omega + 3\,\omega^2)^2 + (3 + 3\,\omega + 5\,\omega^2)^2$

$= 4\,\omega^2 + 4\,\omega^4 = 4(\omega^2 + \omega)\ (\because\ \omega^3 = 1)$

$= -4\ (\because\ \omega^2 + \omega = -1)$

32. (d) $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^8)$

$\{\omega^4 = \omega^3.\ \omega = \omega$ and $\omega^8 = \omega^6\,\omega^2 = \omega^2\}$

Given expression

$= (1 - \omega)(1 - \omega^2)(1 - \omega)(1 - \omega^2)$

$= (1 - \omega)^2(1 - \omega^2)^2$

$= (1 + \omega^2 - 2\,\omega)(1 + \omega^4 - 2\,\omega^2)$

$= (-\omega - 2\,\omega)(1 + \omega - 2\,\omega^2)$

$= (-3\,\omega)(-3\,\omega^2) = 9\,\omega^3 = 9.$

33. (d) As $\dfrac{-1 + i\sqrt{2}}{2} = \omega$ and $\dfrac{-1 - i\sqrt{3}}{2} = \omega^2$

$\therefore\ (\omega)^{20} + (\omega^2)^{20} = \omega^{18}.\ \omega^2 + \omega^{39}.\ \omega = \omega^2 + \omega$

$= -1.$

34. (b): $|Z| = \left|\left(Z - \frac{4}{Z}\right) + \frac{4}{Z}\right| \Rightarrow |Z|$

$= \left|Z - \frac{4}{Z} + \frac{4}{Z}\right|$

$\Rightarrow |z| \le \left|z - \frac{4}{z}\right| + \dfrac{4}{|z|} \Rightarrow |z| \le 2 + \dfrac{4}{|z|}$

$\Rightarrow |Z|^2 - 2\,|Z| - 4 \le 0$

$\Rightarrow \left(|z| - (\sqrt{5} + 1)\right)\left(|z| - (1 - \sqrt{5})\right) \le 0$

$\Rightarrow 1 - \sqrt{5} \le |z| \le \sqrt{5} + 1$

35. (b) $\left(\frac{-1+i\sqrt{3}}{2}\right)^{900}+\left(\frac{-1-i\sqrt{3}}{2}\right)^{301}$

$(\omega)^{900}+(\omega^2)^{301}$

$(\omega^3)^{300}+\omega^{602}$

$(1)^{300}(\omega^3)^{200}\times\omega^2$

$=1+\omega^2=-\omega=-\left(\frac{-1+i\sqrt{3}}{3}\right)$

36. (b) Given, $z^2+z+1=0$

$\Rightarrow z=\omega,\omega^2$

Take $z=\omega$,

$\therefore\left(z+\frac{1}{z}\right)^2+\left(z^2+\frac{1}{z^2}\right)^2+\left(z^3+\frac{1}{z^3}\right)^2$

$+\left(z^4+\frac{1}{z^4}\right)^2+\left(z^5+\frac{1}{z^5}\right)^5+\left(z^6+\frac{1}{z^6}\right)^2$

$=\left(\omega+\frac{1}{\omega}\right)^2+\left(\omega^2+\frac{1}{z^2}\right)^2+\left(\omega^3+\frac{1}{z^3}\right)^2$

$+\left(\omega^4+\frac{1}{z^4}\right)^2+\left(\omega^5+\frac{1}{z^5}\right)^5+\left(\omega^6+\frac{1}{z^6}\right)^2$

$=(\omega+\omega^2)^2+(2+)^2+(1+1)^2$

$+(+^2)^2+(2+)^2+(1+1)^2$

$=1+1+4+1+1+4=12$

Similarly, for $z=2$, we get the same result.

37. (d) Given, $z^2+\bar{z}=0$

Let $z=z+iy$

$\therefore(x+iy)^2+x-iy=0$

$\Rightarrow x^2-y^2+2ixy+x-iy=0$

$\Rightarrow(x^2+x-y^2)+i(2xy-y)=0$

On equating real and imaginary part, we get

$\Rightarrow x^2+x-y^2=0$ (1)

and $2xy-y=0$ $y=0$ or $x=½$

If $y=0$, then equation (1) gives $x^2+x=0$

$\Rightarrow x=0$ or -1

and if $x=\frac{1}{2}$, then equation (1) gives $\frac{1}{4}+\frac{1}{2}-y^2=0$

$y^2=\frac{3}{4};y=\pm\frac{\sqrt{3}}{4}$

Hence, there are four solutions of given equation.

38. (c) $(1+i\sqrt{3})^n(1-i\sqrt{3})^n$

$=2^n\left[\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^n+\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^n\right]$

$=2^n\left[\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^n+\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)^n\right]$

$=2^n\left[2\cos\frac{n\pi}{3}\right]=2^{n+1}\cos n\pi/3$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. If $(1-i)^n=2^n$, then $n=$ **[RPET – 1990]**

(a) 1 (b) 0

(c) − 1 (d) none of these

2. The value of $(1+i)^8=(1-i)^8$ is: **[RPET – 2001, KCET – 2001]**

(a) 16 (b) − 16 (c) 32 (d) − 32

3. If $|z_1+z_2|=|z_1-z_2|$, then the difference in the amplitudes of z_1 and z_2 is **[EAMCET – 1985]**

(a) $\pi/4$ (b) $\pi/3$ (c) $\pi/2$ (d) 0

4. The amplitude of $\frac{1+\sqrt{3}i}{\sqrt{3}+1}$ is: **[Karnataka CET – 1992; Pb. CET – 2001]**

(a) $\frac{\pi}{3}$ (b) $-\frac{\pi}{3}$ (c) $\frac{\pi}{3}$ (d) $-\frac{\pi}{6}$

5. The value of $\frac{(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)}{(\cos\gamma+i\sin\gamma)(\cos\delta+i\sin\delta)}$ is **[RPET – 2001]**

(a) $\cos(\alpha+\beta-\gamma-\delta)-i\sin(\alpha+\beta-\gamma-\delta)$

(b) $\cos(\alpha+\beta-\gamma-\delta)+i\sin(\alpha+\beta-\gamma-\delta)$

(c) $\sin(\alpha+\beta-\gamma-\delta)-i\cos(\alpha+\beta-\gamma-\delta)$

(d) $\sin(\alpha+\beta-\gamma-\delta)+i\cos(\alpha+\beta-\gamma-\delta)$

6. The modulus and amplitude of $\frac{1+2i}{1-(1-i)^2}$ are: **[Karnataka CET – 2005]**

(a) $\sqrt{6}$ and $\pi/6$

(b) 1 and 0

(c) 1 and $\pi/3$

(d) 1 and $\pi/4$

7. (sin + i cos)^n is equal to
(a) cos $n\theta + i$ sin $n\theta$
(b) sin $n + i$ cos $n\theta$
(c) cos $n\left(\frac{\pi}{2} - \theta\right) + i$ sin $n\theta$
(d) none of these
[RPET – 2001]

8. Which of the following are correct for any two complex numbers z_1 and z_2
[Roorkee – 1998]
(a) $|z_1 z_2| = |z_1|\,|z_2|$
(b) arg $(z_1 z_2) = (\arg z_1)(\arg z_2)$
(c) $|z_1 + z_2| = |z_1| + |z_2|$
(d) $|z_1 - z_2| \geq |z_1| - |z_2|$

9. The amplitude of $\frac{1+\sqrt{3}}{\sqrt{3}-i}$ is:
[RPET – 2001]
(a) 0 (b) $\pi/6$ (c) $\pi/3$ (d) $\pi/2$

10. If $z = \frac{-2}{1+\sqrt{3}}$ then the value of arg(z) is
[Orissa JEE – 2002]
(a) π (b) $\pi/3$ (c) 2/3 (d) $\pi/4$

11. $(1 + i)^{10}$, where $i^2 = -1$, is equal to
[AMU – 2001]
(a) 32 i
(b) 64 + i
(c) 24 i – 32
(d) none of these

12. $\sqrt{-8-6i} =$
(a) $1 \pm 3i$
(b) $\pm(1 - 3i)$
(c) $\pm(1 + 3i)$
(d) $\pm(3 - i)$
[Roorkee – 1979, RPET – 1992]

13. The value of will
$\frac{a + b\omega + c\omega^2}{b + c\omega + a\omega^2} + \frac{a + b\omega + c\omega^2}{c + a\omega + b\omega^2}$ be
(a) 1
(b) – 1
(c) 2
(d) – 2
[BIT Ranchi – 1989; Orissa JEE –2003]

14. A value of $\sqrt{i} + \sqrt{-i}$ is
[NDA – 2007]
(a) 0 (b) $\sqrt{2}$ (c) $-i$ (d) i

15. If z is a complex number, then the minimum value of $|z| + |z - 1|$ is
[Roorkee – 1992]
(a) 1
(b) 0
(c) 1/2
(d) none of these

16. If $i = \sqrt{-1}$, then $4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{335}$ is equal to
[IIT – 1999]
(a) $1 - i\sqrt{3}$
(b) $-1 + i\sqrt{3}$
(c) $i\sqrt{3}$
(d) $-i\sqrt{3}$

17. If $(-7 - 24\,i)^{1/2} = x - iy$, then $x^2 + y^2 =$
[RPET – 1989]
(a) 15
(b) 25
(c) – 25
(d) none of these

18. The square root of $3 - 4i$ is
[RPET – 1999]
(a) $\pm(2 + i)$
(b) $\pm(2 - i)$
(c) $\pm(1 - 2i)$
(d) $\pm(1 + 2i)$

19. If $\sqrt{a + ib} = x + iy$, then possible value of $\sqrt{a - ib}$ is
[Kerala (Engg.) – 2002]
(a) $x^2 + y^2$
(b) $\sqrt{x^2 + y^2}$
(c) $x + iy$
(d) $x - iy$

20. If $a = \sqrt{2}i$ then which of the following is correct
[Roorkee – 1989]
(a) $a = 1 + i$
(b) $a = 1 - i$
(c) $a = -i$
(d) None of these

21. $(27)^{1/3} =$
[NDA – 2007]
(a) 3
(b) $3, 3i, 3i^2$
(c) $3, 3\omega, 3\omega^2$
(d) None of these

22. $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{1000} =$
(a) $\frac{1}{2} + \frac{\sqrt{3}}{2}i$
(b) $\frac{1}{2} - \frac{\sqrt{3}}{2}i$
(c) $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$
(d) none of these

23. If $z = \sqrt{2i} - \sqrt{-2i}$, then $|z| =$
(a) 2
(b) $\sqrt{2}$
(c) 0
(d) $2\sqrt{2}$

24. If the roots of the equation $x^3 - 1 = 0$ are 1, ω,and ω^2, then the value of $(1 - \omega)(1 - \omega^2)$ is

(a) 0 (b) 1 (c) 2 (d) 3

25. If cube root of 1 is, then the value of (3 + + 2^2)^4 is

(a) 0 (b) 16
(c) $9\omega^2$ (d) $16\omega^2$

26. z and are two non–zero complex numbers such that $|z| = |\omega|$ and Arg z + Arg $\omega = \pi$, then z =

[AIEEE – 2002]

(a) $\overline{\omega}$ (b) $-\overline{\omega}$
(c) ω (d) $-\omega$

27. The value of $(1 - \omega + \omega^2)(1 - \omega^2 + \omega)^6$, where ω, ω^2 are cube roots of unity

[DCE – 2001]

(a) $128\,\omega$ (b) $-128\,\omega^2$
(c) $-128\,\omega$ (d) $128\,\omega^2$

28. The value of $(1 + i)^6 + (1 - i)^6$ is

[RPET – 2002]

(a) 0 (b) 2^7
(c) 2^6 (d) none of these

WORK SHEET: TO CHECK PREPARATION LEVEL

Important Instructions:

1. The answer sheet is immediately below the work sheet.
2. The test is of 17 minutes.
3. The test consists of 17 questions. The maximum marks are 51.
4. Use blue/black ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited

1. The least positive integer n such that $\left(\frac{2i}{1+i}\right)^n$ is a positive integer is

(a) 2 (b) 4 (c) 8 (d) 16

2. arg z + arg $\bar{z}$ $(z \neq 0)$ is

(a) 0 (b) π
(c) $\pi/2$ (d) none of these

3. If $|z_1| = |z_2|$ and arg $(z_1/z_2) = \pi$, then $z_1 + z_2$ is equal to

(a) 0
(b) purely imaginary
(c) purely real
(d) none of these

4. The polar form of the complex number $(i^{25})^3$ is

(a) $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$ (b) $\cos\pi + i\sin\pi$
(c) $\cos\pi - i\sin\pi$ (d) $\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$

5. Which one of the following is correct? If z and are complex numbers and denotes the conjugate of, then $|z +| = |z -|$ holds only if

(a) $z = 0$ or $\omega = 0$
(b) $z = 0$ and $\omega = 0$
(c) $z.\,\overline{\omega}$ is purely real
(d) $z.\overline{\omega}$ is purely imaginary

[NDA – 2008]

6. The value of $\left(\frac{-1+i\sqrt{3}}{2}\right)^{3n} + \left(\frac{-1-i\sqrt{3}}{2}\right)^{3n}$ is equal to

(a) 3 (b) 3/2
(c) 0 (d) 2

7. $(-1 + i\sqrt{3})^{20}$ is equal to

[RPET – 2003]

(a) $2^{20}(-1 + i\sqrt{3})^{20}$ (b) $220\,(1 - i\sqrt{3})^{20}$
(c) $2^{20}(-1 - i\sqrt{3})^{20}$ (d) none of these

8. If z_1 and z_2 are two complex number then $|z_1 + z_2|$

[MP PET – 2007]

(a) $|z_1| + |z_2|$ (b) $|z_1| - |z_2|$
(c) $< |z_1| + |z_2|$ (d) $> |z_1| + |z_2|$

9. If $2\alpha = -1 - i\sqrt{3}$ and $2\beta = -1 + i\sqrt{3}$, then $5\alpha^4 + 5\beta^4 + 7\alpha^{-1}\beta^{-1}$ is equal to

[Kerala PET – 2008]

(a) −1 (b) −2 (c) 0 (d) 2

10. What is the square root of $\frac{1}{2} - i\frac{\sqrt{3}}{2}$?

[NDA – 2008]

(a) $\pm\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$ (b) $\pm\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$

(c) $\pm\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$ (d) $\pm\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

11. If cube root of 1 is ω, then the value of $(3 + \omega + 3\omega^2)^4$ is

(a) 0 (b) 16

(c) $16\,\omega$ (d) $16\omega^2$

[Karnataka CET – 2004; Pb CET – 2000; MPPET – 2001]

12. If Z_1 and Z_2 are two complex numbers, then $|z_1 - Z_2|$ is

(a) $\geq |Z_1| - |Z_2|$ (b) $\leq |Z_1| - |Z_2|$

(c) $\geq |Z_1| + |Z_2|$ (d) $\leq |Z_2| - |Z_1|$

[MPPET – 1994]

13. If α is an imaginary cube root of unity, then for $n \in N$ the value of $\alpha^{3n+1} + \alpha^{3n+3} + \alpha^{3n+5}$ is

[MP PET – 1996]

(a) – 1 (b) 0

(c) 1 (d) 3

14. If ω is a cube root of unity, then $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5 =$

(a) 32 (b) – 32 (c) – 16 (d) 8

[IIT – 1965; RPET – 1997; MP PET – 1997]

15. $(-8)^{1/3}$ is equal to

[MP PET – 2006]

(a) 2 (b) $1 + i\sqrt{3}$

(c) $\frac{1}{2}(1 + i\sqrt{3})$ (d) $\frac{1}{2}(1 - i\sqrt{3})$

16. $|z_1 + z_2| = |z_1| + |z_2|$ is possible if:

(a) $z_2 = 1$ (b) $z_2 = \frac{1}{z_1}$

(c) $\arg(z_1) = \arg(z_2)$ (d) $|z_1| = |z_2|$

[MPPET– 1999; 2007; Pb.CET – 2002]

17. If $z_1, z_2\ C$, then

[MPPET – 1995]

(a) $|z_1 + z_2| \geq |z_1| + |z_2|$

(b) $|z_1 - z_2| \geq |z_1| + |z_2|$

(c) $|z_1 - z_2| \geq ||z_1| - |z_2||$

(d) $|z_1 + z_2| \geq ||z_1| - |z_2||$

ANSWER SHEET

1.	ⓐ	ⓑ	ⓒ	ⓓ	7.	ⓐ	ⓑ	ⓒ	ⓓ	13.	ⓐ	ⓑ	ⓒ	ⓓ
2.	ⓐ	ⓑ	ⓒ	ⓓ	8.	ⓐ	ⓑ	ⓒ	ⓓ	14.	ⓐ	ⓑ	ⓒ	ⓓ
3.	ⓐ	ⓑ	ⓒ	ⓓ	9.	ⓐ	ⓑ	ⓒ	ⓓ	15.	ⓐ	ⓑ	ⓒ	ⓓ
4.	ⓐ	ⓑ	ⓒ	ⓓ	10.	ⓐ	ⓑ	ⓒ	ⓓ	16.	ⓐ	ⓑ	ⓒ	ⓓ
5.	ⓐ	ⓑ	ⓒ	ⓓ	11.	ⓐ	ⓑ	ⓒ	ⓓ	17.	ⓐ	ⓑ	ⓒ	ⓓ
6.	ⓐ	ⓑ	ⓒ	ⓓ	12.	ⓐ	ⓑ	ⓒ	ⓓ					

HINTS AND EXPLANATIONS

11. (c) $(3 + \omega + 3\,\omega^2)^4 = (3 + 3\omega + 3\omega^2 - 2\omega)^4$

$= [3\,(1 + \omega + \omega^2) - 2\omega]^4$ [$\because$ as $1 + \omega + \omega^2 = 0$]

$= (-2\omega)^4$

$= 16\,\omega^4 = \omega$

12. (a) Third side of a triangle is greater than equal to difference of two sides.

13. (b) If is an imaginary cube root of unity then

$\alpha^3 = 1,\ 1 + \alpha + \alpha^2 = 0$

$\alpha^{3n+1} + \alpha^{3n+3} + \alpha^{3n+5} = \alpha^{3n}\,[\alpha^1 + \alpha^3 + \alpha^5]$

$= (\alpha^3)^n\,[\alpha + 1 + \alpha^3.\alpha^2]$

$= (1)^n\,[1 + \alpha + 2]$ $(\because \alpha^3 = 1)$

$= 1 + \alpha + \alpha^2 = 0$

14. (a) $1 + \omega + \omega^2 = 0$

$A = (1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5 = (-2\omega)^5 + (-2\omega^2)^5$

or, $A = -32[\omega^5 + \omega^{10}] = -32(\omega^2 + \omega) = -32(-1) = 32$

For $\omega^3 = 1$

15. (b) Let $x = (-8)^{1/3}$

or $x^3 = -8$ or $x^3 + 2^3 = 0$

$(x + 2)(x^2 - 2x + 4) = 0$

$x = -2$ or $x^2 - 2x + 4 = 0$

$$x = \frac{+2 \pm \sqrt{4 - 16}}{2}$$

$$x = \frac{+2 \pm 2\sqrt{3}i}{2} \Rightarrow x = 1 \pm \sqrt{3}i$$

This page is intentionally left blank.

LECTURE

4 Geometry of Complex Numbers

BASIC CONCEPTS

1. *n*th Roots of Unity

(i) $(1)^{\frac{1}{n}} = \left\{1, e^{i\frac{2\pi}{n}}. \left(e^{i\frac{2\pi}{n}}\right)^2, \ldots \left(e^{i\frac{2\pi}{n}}\right)^{n-1}\right\}$

$= \{1, \omega, \omega^2, \ldots\ldots, \omega^{n-1}\}$

(ii) $1 + \omega + \omega^2 + \ldots\ldots\ldots \omega^{n-1} = 0, \omega^n = 1.$

(iii) The *n*th roots of unity form a G.P. whose common ratio is $e^{i\frac{2\pi}{n}}$.

(iv) The sum of *n*th roots of unity = 0 and product of *n*th roots of unity is $(-1)^{n-1}$.

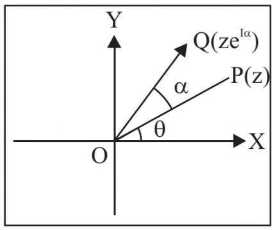
(v) *n*th roots of unity lie on a unit circle in Argand plane whose centre is origin. These root divide the circumference of circle into *n* equal parts and each part inscribed an angle $\frac{2\pi}{n}$at the centre.

(vi) *n*th roots of unity form a *n*-sided regular polygon.

2. Complex Number as a Rotating Arrow in the Argand Plane

Let $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ be a complex number represented by a point *P* in the Argand plane. Then, complex number represented by

Q is: $z = re^{i\alpha} = re^{i\theta}\, e^{i\alpha} = re^{i(\theta+\alpha)}$



(i) Multiplication by $e^{i\alpha}$ to *z* rotates the vector OP in anticlockwise direction through an angle α.

(ii) Similarly, multiplication by $e^{-i\alpha}$ to *z* rotates the vector OP in clockwise direction through an angle *a*.

Note **i** $\cos\frac{\pi}{2}\, i \sin\frac{\pi}{2} = e^{i\frac{\pi}{2}}$ Hence, angle between *z* and $iz = 90° = \pi/2$

3. Geometry of Complex Numbers

Let $z = x + iy$ be a complex number represented by a point in the Argand plane. Then, we say that the affix of *p* is *z*. The use of the word affix is similar to the position vector in the vectors.

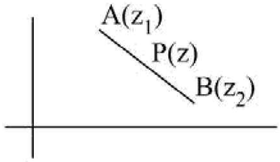
(A) Distance Between two Points Let *P* and *Q* be two points in the Argand plane having affixes z_1 and z_2 respectively.

Then $PQ = |z_2 - z_1|$ = |affix of *Q*-affix of *P*|

Modulus of a complex number *z* represented by a plane in the Argand plane is its distance from the origin

If z_1 and z_2 are two fixed points in the argand plane; then the locus of a point z in each of the following cases

(i) $|z - z_1| + |z - z_2| = |z_1 - z_2|$, $AP + BP = AB \Rightarrow P$ lies on the line segment joining $A(z_1)$ and $B(z_2)$.



(ii) $|z - z_1| = |z - z_2| \Rightarrow AP = BP \Rightarrow P$ is equidistant from A and $B \Rightarrow P$ lies on the $\perp^r$ bisector of the line segment AB.

(iii) $|z - z_1| = k|z - z_2|, k \in r^+, k \in 1. AP = KBP$ $\Rightarrow P$ lies on a circle of a point such that the ratio of its distance from two fixed points is always constant. (Recall that circle is also defi ned as the locus.)

(iv) $|z - z_1| + |z - z_2|$ = constant ($\neq |z_1 - z_2|$) $AP + BP$ = constant $\Rightarrow$ ellipse

(v) $|z - z_1| - |z - z_2|$ = constant ($\neq |z_1 - z_2|$) $\Rightarrow P$ lies on a hyperbola having its foci at A and B, respectively.

(B) 1. (i) If two points P and Q have affixes z_1 and z_2, repectively in the Argand plane, then the affix of a point R dividing

PQ internally in the ratio $m : n$ is $\dfrac{mz_2\ nz_1}{m+n}$

(ii) If R is the mid point of PQ, then affix of R is $\dfrac{z_1 + z_2}{2}$

2. If z_1, z_2, z_3 are affixes of the vertices of a Δ, then affix of its centroid is $\dfrac{z_1 + z_2 + z_3}{3}$

3. If z_1, z_2, z_3, z_4 are the affixes of the points A, B, C and D respectively.

Then $ABCD$ is a parallelogram if $z_1 + z_3 = z_2 + z_4$

4. If z_1, z_2, z_3 are the affixes of the vertices of a triangle having its circumcentre at the origin and z is the affix of its orthocentre, then $z = z_1 + z_2 + z_3$

5. Centroid G divides line join of circumcentre and orthocentre in the ratio 1 : 2, since affix of G is $\dfrac{z_1\ z_2\ z_3}{3}$ and 0 is the origin.

6. (i) Equation of a circle having centre at z_0 and radius r is $|z - z_0| = R$

(ii) $z\bar{z} + az + \bar{a}z + b = 0$ where $b \in r$ represent a circle having centre at $(-a)$ and radius $\sqrt{|a|^2 - b}$

(C) 1. General Equation of a Straight Line

The general equation of a straight line is of the form $a\bar{z} + \bar{a}z + b = 0$, where a is a complex number and b is a real number.

2. Complex slope of the line segment joining two points z_1 and z_2. $\omega = \dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$

3. If ω_1 and ω_2 are the complex slope of two lines on the Argand plane

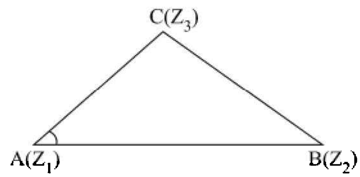
(i) If lines are $\perp^r$, if $\omega_1 + \omega_2 = 0$ (ii) parallel if $\omega_1 = \omega_2$.

4. The slopes of the two lines are and $\dfrac{-\alpha}{\bar{\alpha}}$ and $\dfrac{-\beta}{\bar{\beta}}$ respectively. The lines will ber $\perp^r$,

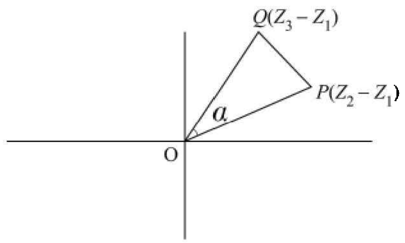
if $\dfrac{-\alpha}{\bar{\alpha}} + -\dfrac{\beta}{\bar{\beta}} = 0 \Rightarrow \alpha\bar{\beta} + \bar{\alpha}\beta = 0$.

5. If z_1, z_2, z_3 are the affixes of the points A, B and C in the Argand plane, then

(i) $\angle BAC = \arg\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right)$



(ii) $\dfrac{z_3 - z_1}{z_2 - z_1} = \dfrac{|z_3 - z_1|}{|z_2 - z_1|}(\cos\alpha + i\sin\alpha)$

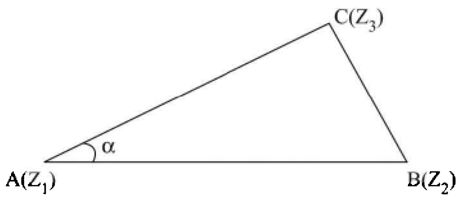


Let P and Q be two points in the Argand plane representing $z_2 - z_1$ and $z_3 - z_1$, respectively, then $\Delta OPQ = \Delta ABC$

$\therefore \angle BAC = \angle POQ = \angle XOQ - \angle XOP$

$= \arg(z_3 - z_1) - \arg(z_2 - z_1) = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$

6. Angle Between Two Lines



Angle between AC and AB:

$\angle BAC$ = amplitude of AC − amplitude of AB = amp of $(z_3 - z_1)$ − amp of $(z_2 - z_1)$

$= \arg(z_3 - z_1) - \arg(z_2 - z_1) = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$

***Note* 1**: If z_1, z_2, z_3 are collinear, then $\angle BAC = 0$

Hence, $\frac{z_3 - z_1}{z_2 - z_1}$ is purely treal because amplitude of real number = 0° i.e., $(z_3 - z_1)$ = $\arg(z_2 - z_1)$

***Note* 2**: If AC and AB are perpendicular each other then

$\alpha = \frac{\pi}{2} = \angle BAC$ Hence $\frac{z_3 - z_1}{z_2 - z_1}$ is perfectly imaginary.

***Note* 3**: If z_1, z_2, z_3 are in A.P. then they are collinear.

7. Complex numbers z_1, z_2, z_3 are vertices of an equilateral triangle iff $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$ or

$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_4} = 0.$

8. Some Important Results

(i) If $x + 1/x = 2\cos\theta$ or $x - 1/x = 2i\sin\theta$; then $x = \cos\theta + i\sin\theta$, $1/x = \cos\theta - i\sin\theta$ $x^n + 1/x^n = 2\cos n\theta$, $x^n - 1/x^n = 2i\sin n\theta$.

(ii) If $a = \cos\theta + i\sin\theta$, $b = \cos\beta + i\sin\beta$, $c = \cos\gamma + i\sin\gamma$ and $a + b + c = 0$, then

$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$

$ab + bc + ca = 0$

$a^2 + b^2 + c^2 = 0$

$a^3 + b^3 + c^3 = 3abc$ etc.

(iii) $(x \pm 1)(x \pm \omega)(x \pm \omega^2) = x^3 \pm 1$

(iv) $(x \pm y)(x \pm \omega y)(x \pm \omega^2 y) = x^3 \pm y^3$

(v) $(x \pm y)(x\omega \pm y\omega^2)(x\omega^2 \pm y\omega) = x^3 \pm y^3$

(vi) $(x + y + z)(x + y\omega + z\omega^2)(x + y\omega^2 + z\omega) = x^3 + y^3 + z^3 - 3xyz$

(vii) $(1 + i)^2 = 2i$, $(1 - i)^2 = -2i$

SOLVED SUBJECTIVE PROBLEMS: (CBSE/STATE BOARD): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. If $|z_1| = |z_2| = |z_3| = \ldots\ldots = |z_n| = 1$, then prove that

$\left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \ldots + \frac{1}{z_n}\right|$

$= |z_1 + z_2 + z_3 + \ldots + z_n|$

Solution

We have $|z_1| = |z_2| = |z_3| = \ldots\ldots = |z_n| = 1$

$\Rightarrow |z_1|^2 = |z_2|^2 = |z_3|^2 = \ldots\ldots = |z_n|^2 = 1$

$\Rightarrow z_1\bar{z}_1 = 1, z_2\bar{z}_2 = 1, z_3\bar{z}_3 = 1, \ldots, z_n\bar{z}_n = 1$

$\Rightarrow \left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}+ \ldots\ldots + \frac{1}{zn}\right|$

$= |\bar{z}_1 + \bar{z}_2 + \bar{z}_3 + \ldots\ldots + \bar{z}_n|$

$= |\overline{z_1 + z_2 + z_3 + \ldots\ldots + z_n}|$

$= |z_1 + z_2 + z_3 + \ldots\ldots + z_n|$ [$\because |\bar{z} = |z|$].

$\therefore \left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}+\ldots+\frac{1}{z_n}\right| = |z_1 + z_2 + z_3 \ldots + z_n|.$

2. If point p represents the complex number $z = x + iy$ on the Argand plane, then fi nd the locus of the point p such that

$\left|\frac{z-2}{z+2}\right| = 5.$

Solution

Given $\left|\frac{z-2}{z+2}\right| = 5 \Rightarrow \left|\frac{z-2}{z+2}\right| = 5$

$\Rightarrow |z - 2| = 5 |z + 2|$

$\Rightarrow |x + iy - 2| = 5 |x + iy + 2|$

$\Rightarrow \sqrt{(x-2)^2 + y^2} = 5\sqrt{(x+2)^2 + y^2}$

$\Rightarrow (x - 2)^2 + y^2 = 25 [(x + 2)^2 + y^2]$

$\Rightarrow 25 [x^2 + 4x + 4 + y^2] = x^2 - 4x + 4 + y^2$

$\Rightarrow 24x^2 + 24y^2 + 104x + 96 = 0$

$\Rightarrow 3x^2 + 3y^2 + 13x + 12 = 0$

which is an equation of a circle.

3. If point p represents the complex number $z = x + iy$ on the Argand plane, then find the locus of the point p such that arg $(z - 2 - 3i) = \pi/4$.

Solution

Given arg $(z - 2 - 3i) = \pi/4$

$\Rightarrow$ arg $(x + iy - 2 - 3i) = \pi/4$

$\Rightarrow \arg(\overline{x-2} + \overline{iy-3}) = \pi/4$

$\Rightarrow \tan \frac{y-3}{x-2} = \frac{\pi}{4}$

$\Rightarrow \frac{y-3}{x-2} \tan \frac{\pi}{4} = 1$

$\Rightarrow x - 2 = y - 3$

$\Rightarrow x - y = -1$

4. If α and β are different complex numbers with $|\beta| = 1$, then fi nd $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$ **[NCERT]**

Solution

Given $|\beta| = 1$, $\therefore \beta\bar{\beta} = |\beta|^2 = 1$, then

$\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right| = \left|\frac{\beta-\alpha}{\beta\bar{\beta}-\bar{\alpha}\beta}\right| = \left|\frac{\beta-\alpha}{\beta(\bar{\beta}-\bar{\alpha})}\right|$

$= \frac{|\beta-\alpha|}{|\beta|\,|\overline{\beta-\alpha}|} = \frac{1}{|\beta|}\left|\frac{\beta-\alpha}{\overline{\beta-\alpha}}\right| = \frac{1}{1} = 1$

$\because \left|\frac{z}{\bar{z}}\right| = \frac{|z|}{|\bar{z}|} = 1.$

5. If $z = 3 - 5i$, then prove that $z^3 - 10z^2 + 58z - 136 = 0$.

Solution

Given $z = 3 - 5i$

$\Rightarrow z - 3 = -5i \Rightarrow (z - 3)^2 = (-5i)^2$

$\Rightarrow z^2 - 6z + 9 = 25i^2$

$\Rightarrow z^2 - 6z + 9 = -25$

$\Rightarrow z^2 - 6z + 34 = 0$ (1)

Now, $z^3 - 10z^2 + 58z - 136$

$= z^3 - 4z^2 - 6z^2 + 24z + 34z - 136$

$= z^2 (z - 4) - 6z (z - 4) + 34 (z - 4)$

$= (z - 4) (z^2 - 6z + 34)$,

$= (z - 4) \times 0$ [from Equation (1)]

$= 0$ **Proved**

6. Prove that

$\sqrt{-1-\sqrt{-1-\sqrt{-1-\sqrt{1-1-\ldots\ldots\ldots\ldots\ldots \infty}}}}$

$= \omega$ and ω^2.

Solution

Let $x = \sqrt{-1-\sqrt{-1-\sqrt{-1-\sqrt{1-1-\ldots\ldots\ldots\ldots\ldots \infty}}}}$

$\Rightarrow x = \sqrt{-1-x}$

Squaring both the sides, we have $x^2 = -1 - x$

$\Rightarrow x^2 + x + 1 = 0$

$\Rightarrow x = \frac{-1 \pm \sqrt{1^2 - 4(1)}}{2(1)} \Rightarrow x = \frac{-1 \pm \sqrt{-3}}{2}$

$\Rightarrow x = \frac{-1 \pm i\sqrt{3}}{2} = \Rightarrow x = \omega, \omega^2$. **Proved**

7. Find the area of the triangle whose vertices are represented by the points of complex numbers z, $z + iz$, iz.

Solution

Let $z = x + iy = (x, y)$

Then, $iz = i(x + iy) = -y + ix = (-y, x)$

and $z + iz = (x + iy) + i(x + iy)$

$= x + iy + ix - y$

$= (x - y) + i(x + y) = (x - y, x + y)$

Therefore, area of the triangle

$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$

$= \frac{1}{2}[x(x - \overline{x + y}) - y(x + \overline{y - y}) + (x - y)(y - x)]$

$= \frac{1}{2}[x(-y) - y(x) - (x - y)^2]$

$= \frac{1}{2}[-xy - xy - x^2 - y^2 + 2xy]$

$= -\frac{1}{2}(x^2 + y^2) = \frac{1}{2}|z|^2.$

(Since, area is positive neglect negative sign)

8. If z_1, z_2 are two complex numbers and a, b are two real numbers, then prove that

$|az_1 - bz_2|^2 + |bz_1 + az_2|^2$

$= (a^2 + b^2)[|z_1|^2 + |z_2|^2]$

Solution

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore |z_1| = r_1$ and $|z_2| = r_2$

$\therefore az_1 - bz_2 = ar_1(\cos\theta_1 + i\sin_1\theta)$

$- br_2(\cos\theta_2 + i\sin\theta_2)$

$= (ar_1\cos\theta_1 - br_2\cos\theta_2)$

$+ i(ar_1\sin\theta_1 - br_2\sin\theta_2)$

$\therefore |az_1 - bz_2|^2 = (ar_1\cos\theta_1 - br_2\cos\theta_2)^2$

$+ (ar_1\sin\theta_1 - br_2\sin\theta_2)^2$

$= a^2r_1^2(\cos^2\theta_1 + \sin^2\theta_1)$

$+ b^2r_2^2(\cos^2\theta_2 + \sin^2\theta_2)$

$- 2ab\, r_1r_2\cos(\theta_1 - \theta_2)$

$\Rightarrow |az_1 - bz_2|^2 = a^2r_1^2 + b_2r_2^2$

$- 2ab\, r_1r_2\cos(\theta_1 - \theta_2)$ (1)

Similarly,

$|bz_1 + az_2|^2$

$= b^2r_1^2 + a^2r_2^2 + 2abr_1r_2\cos(\theta_1 - \theta_2)$ (2)

Adding equations (1) and (2), we get

$|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)r_1^2 + (b^2 + a^2)r_2^2$

$= (a^2 + b^2)(r_1^2 + r_2^2)$

$= (a^2 + b^2)[|z_1|^2 + |z_2|^2]$

Proved

UNSOLVED SUBJECTIVE PROBLEMS: (CBSE/STATE BOARD): TO GRASP THE TOPIC, SOLVE THESE PROBLEMS

Exercise I

1. If point p represents the complex number $z = x + iy$ on the Argand plane, then fi nd the locus of the point p such that $\arg(z) = 0$.

2. Prove that the points represent the complex numbers $3 + 3i$, $-3 - 3i$, $-3\sqrt{3} + 3\sqrt{3}\,i$ form an equilateral triangle. Also, find the area of triangle.

3. If $z = (\lambda + 3) + i\sqrt{5 - \lambda^2}$, then prove that the locus of the point z is a circle.

4. Show that points represented by the complex number $1 + i$, $-2 + 3i$ and $\frac{5}{3}i$ are collinear.

5. Show that the vertices represented by the complex numbers $6 - i$, $7 + 3i$, $8 + 2i$ and $7 - 2i$ form a parallelogram.

6. If the points represented by the complex numbers z, $z + iz$, iz form a triangle of the area 50 square unit, then prove that $|z| = 10$.

Exercise II

1. For example values of z, solve $|z| + z = (2 + i)$.

2. If $(1 + i)(1 + 2i)(1 + 3i) \ldots\ldots (1 + ni) = (x + iy)$, then show that $2.5.10 \ldots\ldots (1 + n^2) = x^2 + y^2$.

3. If z_1, z_2, z_3 are three complex numbers such that $|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$, then prove that $|z_1 + z_2 + z_3| = 1$.
4. Show that the points represented by complex numbers $-4 + 3i$, $2 - 3i$ and $-i$ are collinear.
5. A, B and C are three vertices of a parallelogram $ABCD$ represented by the complex.
6. Find the area of a triangle whose vertices are represented by the complex numbers $3i$, $3 + 2i$, $2 - i$.

ANSWERS

Exercise I

2. $18\sqrt{3}$ square unit.

Exercise II

1. $z = \left(\frac{3}{4} + i\right)$

5. $10 + 15i$

6. 5 units.

SOLVED OBJECTIVEQUESTIONS: HELPING HAND

1. If $\cos\alpha \cos\beta \cos\gamma = 0 = \sin\alpha + \sin\beta \sin\gamma$, then $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is equal to

[PET (Raj.) – 1998]

(a) 0 (b) 1
(c) 1/2 (d) 3/2

Solution

(d) If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$, $c = \cos\gamma + i\sin\gamma$

then

$$a + b + c = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$$

$$= 0 + i \times 0 = 0$$

and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = (\cos\alpha + i\sin\alpha)^{-1} + (\cos\beta + i\sin\beta)^{-1} + (\cos\gamma + i\sin\gamma)^{-1}$

$= (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma) = 0 - i \times 0 = 0$

$\therefore a^2 + b^2 + c^2 = (a + b + c)^2 + 2abc$

$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 0$

$a^2 = (\cos\alpha + i\sin\alpha)^2$

$\Rightarrow \cos 2\alpha + i\sin 2\alpha$ (1)

$b^2 = (\cos\beta + i\sin\beta)^2$

$\Rightarrow \cos 2\beta + i\sin 2\beta$ (2)

$c^2 = (\cos\gamma + i\sin\gamma)^2$

$\Rightarrow \cos 2\gamma + i\sin 2\gamma$ (3)

$(\cos 2\alpha + \cos 2\beta + \cos 2\gamma) + i(\sin 2\alpha + \sin 2\beta + \sin 2\gamma) = 0 + i \times 0 = 0$

Therefore, $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$ (4)

$\sin 2\alpha \sin 2\beta \sin 2\gamma = 0$

By (4) $1 - 2\sin^2\alpha + 1 - 2\sin^2\beta + 1 - 2\sin^2\gamma = 0$

$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \frac{3}{2}$

2. If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$, then $\sin 3\alpha + \sin 3\beta + \sin 3\gamma$ is equal to
(a) $\sin 3(\alpha + \beta + \gamma)$
(b) $3\sin 3(\alpha + \beta + \beta\gamma)$
(c) $3\sin 3(\alpha + \beta + \gamma)$
(d) $3\cos(\alpha + \beta + \gamma)$

[PET (Raj.) – 1989, 1991; Bihar (CEE) – 2000]

Solution

(b) If $a = \cos\alpha + i\sin\alpha, \beta = \cos\beta + i\sin\beta, c = \cos\gamma + i\sin\gamma$,

then $a + b + c = (\cos\alpha + \cos\beta + \cos\gamma) + (\sin\alpha + \sin\beta + \sin\gamma)$

$= 0 + i0 = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc$

$\Rightarrow \sum (\cos a + i\sin a)^3$

$= 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \times (\cos\gamma + i\sin\gamma)$

$\Rightarrow \sum \cos 3\alpha + i\sum \sin 3\alpha$

$= 3\cos(\alpha + \beta + \gamma) + i3\sin(\alpha + \beta + \gamma)$

$\Rightarrow \sin 3\alpha + \sin 3\beta + \sin 3\gamma$

$= 3\sin(\alpha + \beta + \gamma)$.

3. $(-\sqrt{-1})^{8n+1} + (-\sqrt{-1})^{8n+3}$ $(n \in N)$ equals

[NDA – 2005]

(a) 0 (b) 1

(c) $2\sqrt{-1}$ (d) $-2\sqrt{-1}$

Solution

(a) Exp. $= (-i)^{8n+3} + (-i)^{8n+3}$

$= -i + (-i)^3 = -i + i = 0$

4. The value of is

$$\sum_{k=1}^{10}\left(\sin\frac{2k\pi}{11} + i\cos\frac{2k\pi}{11}\right)$$

[AIEEE – 2006]

(a) 1 (b) − 1

(c) i (d) $-i$

Solution

(c) We have

$$\sum_{k=1}^{10}\left(\sin\frac{2k\pi}{11} - i\cos\frac{2k\pi}{11}\right)$$

$$\sum_{k=1}^{10} i\left(\cos\frac{2k\pi}{11} - i\sin\frac{2k\pi}{11}\right)$$

when

$$= i\sum_{k=1}^{10} e^{-i\frac{2k\pi}{11}} = i\sum_{k=1}^{10} a^k \text{ when } a = e^{-(2\pi i/11)}$$

$$= i\alpha\frac{(1-\alpha^{10})}{(1-\alpha)} = i\frac{(\alpha-\alpha^{11})}{1-\alpha} = i\left(\frac{\alpha-1}{1-\alpha}\right) = -i$$

$(\therefore \alpha^{11} = e^{-i2\pi} = \cos 2\pi - i\sin 2\pi = 1)$;

$\cos\alpha + i\sin\alpha = e^{i\alpha}$.

5. Let z_1, z_2 be two roots of the equation $z^2 + az + b = 0$, z being complex. Further assume that the origin, z_1 and z_2 form an equilateral triangle, Then,

[AIEEE – 2003]

(a) $a^2 = 4b$ (b) $a^2 = b$

(c) $a^2 = 2b$ (d) $a^2 = 3b$

Solution

(d) $z_1 + z_2 = -a, z_1z_2 = b$

$\therefore 0^2 + z_1^2 + z_2^2 = z_1z_2$

(Put $z_3 = 0$ in formula $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$)

$\Rightarrow (z_1 + z_2)^2 - 2z_1z_2 = z_1z_2 \Rightarrow a^2 = 3b$

6. $(((i)^i)^i)^{i \text{.........110 times}}$ equals

[AITSE – 1999]

(a) $e^{i\pi/2}$ (b) $e^{\pi/2}$

(c) $e^{-\pi/2}$ (d) $e^{-i\pi/2}$

Solution

(d) Exp $(i)^{i^{110}} = (i)^{-1} = 1/i = -i = e^{-i\pi/2}$

7. If P, Q, R, S are represented by the complex numbers $4 + i$, $1 + 6i$, $-4 + 3i$, $-1 - 2i$ respectively, then $PQRS$ is a

[Orissa (JEE) – 2003]

(a) rectangle (b) square

(c) rhombus (d) parallelogram

Solution

(b) $|PQ| = |QR| = |RS| = |SP|$ and $\angle PQR = 90°$.

8. If arg. $(z - a) = \frac{\pi}{4}$, where $a \in R$, then the locus of $z \in c$ is a

[MPPET – 1997]

(a) hyperbola (b) parabola

(c) ellipse (d) straight line

Solution

(d) Let $z = x + iy$ then $z - a = x + iy - a =$ $(x - a) + iy$ given arg. $(z - a) = \frac{\pi}{4}$

Therefore, $\tan^{-1} \frac{y}{x - a} = \frac{\pi}{4}$

or $\frac{y}{x - a} = \tan \frac{\pi}{4}$ or $y = x - a$ (straight line)

9. If $|z - 4i| + |z + 4i| = 10$, then z is the locus of

[MPPET – 2006]

(a) circle (b) parabola
(c) ellipse (d) none of these

Solution

(c) $|z - 4i| + |z + 4i| = 10$

Let $z = x + iy$

$|x + iy - 4i| + |x + iy + 4i| = 10$

$|x + i(y - 4)| + |x + i(y + 4)| = 10$

$\sqrt{x^2 + (y - 4)^2} + \sqrt{x^2 + (y + 4)^2} = 10$

$\{\sqrt{x^2 + (y - 4)^2}\}^2 = \{10 - \sqrt{x^2 + (y + 4)^2}\}^2$

$x^2 + (y - 4)^2 = 100 + x^2 + (y + 4)^2 - 2 \times 10\sqrt{x^2 + (y + 4)^2}$

$y^2 + 16 - 8y = 100 + y^2 + 16 + 8y - 20\sqrt{x^2 + (y + 4)^2}$

$-16y - 100 = -20\sqrt{x^2 + (y + 4)^2}$

$4y + 25 = 5\sqrt{x^2 (y + 4)^2}$

$(4y + 25)^2 = (5\sqrt{x^2 + (y + 4)^2})^2$

$16y^2 + 625 + 200y = 25(x^2 + y^2 + 16 + 8y)$

$16y^2 + 625 + 200y = 25x^2 + 25y^2 + 400 + 200y$

$25x^2 + 9y^2 = 225$

$\frac{25x^2}{225} + \frac{9y^2}{225} = \frac{225}{225} \Rightarrow \frac{x^2}{9} + \frac{y^2}{25} = 1$

which is ellipse

10. If $|z - 4| < |z - 2|$, its solution is given by

[AIEEE – 2002]

(a) Re $(z) > 0$ (b) Re $(z) < 0$
(c) Re $(z) > 3$ (d) Re $(z) > 2$

Solution

(c) $|z - 4| < |z - 2|$

or $|a - 4 + ib| < |(a - 2) + ib|$ by taking $z = a + ib$

$\Rightarrow (a - 4)^2 + b^2 < (a - 2)^2 + b^2$

$\Rightarrow -8a + 4a < -16 + 4$

$\Rightarrow 4a > 12$

$\Rightarrow a > 3$

$\Rightarrow \text{Re}(z) > 3$

11. Locus of $|z| = 1$ is

(a) $x + y = 1$ (b) $x^2 + y^2 = 1$
(c) $x^2 - y^2 = 1$ (d) $y^2 - x = 0$

[MPPET – 2007]

Solution

(b) Let $z = x + iy$ then from $|z| = 1$

$x + iy = 1$ or $x^2 + y^2 = 1$or $x^2 + y^2 = 1$

It is a circle whose radius is 1.

12. If $z = (\lambda + 3) + \sqrt{5 - \lambda^2}\, i$, then locus of z is a

[MPPET – 2006]

(a) $(x - 3)^2 + y^2 = 5$ (b) $(x - 3)^2 = 5 - y$
(c) $x - y = 8$ (d) none of these

Solution

(a) On putting $z = x + iy$

$x + iy = (\lambda + 3) + i\sqrt{5 - \lambda^2}$

On comparing real and imaginary parts

$x = \lambda + 3$ (1)

$y = \sqrt{5 - \lambda^2}$ (2)

By Equation (1) $x - 3 = \lambda$ or $\lambda^2 = (x - 3)^2$

By Equation (2) $y = \sqrt{5 - (x - 3)^2}$

or $(x - 3)^2 + y^2 = 5$

13. If z is a complex number, then $|3z - 1| = 3|z - 2|$ respresents

(a) $x = 0$ (b) $x^2 + y^2 = 3x$
(c) $y = 0$ (d) $x = 7/6$

Solution

(d) Let $z = x + iy$; where x, yr, then $|3z - 1|$

$= 3\,|z - 2|$

$\Rightarrow |3\,(x + yi) - 1| = 3\,|x + iy - 2|$

$\Rightarrow |(3x - 1) + (3y)\,i| = 3\,|x - 2 + yi|$

$\Rightarrow \sqrt{(3x - 1)^2 + (3y)^2} = 3\sqrt{(x - 2)^2 + y^2}$

Squaring, we get $9x^2 - 6x + 1 + 9y^2 = 9\,(x^2 - 4x + 4 + y^2)$

$\Rightarrow -6x + 1 = -36x + 36 \Rightarrow 30x = 35$

$\Rightarrow x = \frac{7}{6}$, , which means z is always at a constant distance $\frac{7}{6}$ from y–axis.

14. If z and ω are two non–zero complex numbers such that $|z\omega| = 1$ and $\arg(z) - \arg(\omega) = \frac{\pi}{2}$, then $\bar{z}\omega$ is equal to

(a) $-i$ (b) 1 (c) -1 (d) i

[AIEEE – 2003]

Solution

(a) $|z\omega| = 1 \Rightarrow z = 1/|\omega|$, so let

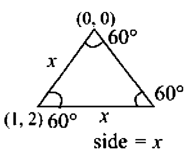
$z = (r, \theta)$, $\omega = (1/r, \theta - \pi/2)$

$\Rightarrow \bar{z} = (r_2, -\theta)$ $\therefore$ $\bar{z}\omega = (1, -\pi/2) = -i$

15. The centre of a hexagon is the origin. If its one vertex is the point $(1 + 2i)$, then its perimeter is

[PET (Raj.) – 1999]

(a) $\sqrt{5}$ (b) $4\sqrt{5}$` (c) $6\sqrt{5}$ (d) $6\sqrt{2}$



Solution

(c) Side $= \sqrt{1^2 + 2^2} = \sqrt{5}$

Therefore, perimeter $= 6\sqrt{5}$

16. If z_1, z_2 are two such nth roots of unity which subtend right angle at the origin, then n must be $(k \in x)$

[IIT (Screening) – 2001]

(a) $4k$ (b) $4k + 1$

(c) $4k + 2$ (d) $4k + 3$

Solution

(a) $1 = (\cos 2r\pi + i \sin 2r\pi)$

$\therefore$ $1^{1/n} = (\cos 2r\pi + i \sin 2r\pi)^{1/n}$

$= e^{\frac{i2r\pi}{n}}$, $r = 0, 1, 2, \ldots.(n - 1)$

$r = 0, 1, 2....(n - 1)$

we get $1, e^{i(2\pi/n)}, e^{i(4\pi/n)}, \ldots\ldots\ldots.e^{i.2(n-1)\pi/n}$

Let given two roots be

$z_1 = e^{i2m_1\pi/n}$, $z_2 = e^{im_2\pi/n}$

Since z_1, z_2 subtend right angle at the origin, so

$\left|\frac{2m_1\pi}{n} - \frac{2m_2\pi}{n}\right| = \frac{\pi}{2}$

$\Rightarrow n = 4\,|m_1 - m_2| = 4k, k \in Z$

17. A point z moves on the Argand diagram such that $|z - 3i| = 2$ then it's locus is

[MP PET – 2002]

(a) y – axis (b) a straight line

(c) a circle (d) none of these

Solution

(c) Let $z = x + iy$

$\therefore$ $z - 3i = x + iy - 3i = x + (y - 3)i$

$\therefore$ $|z - 3i| = 2$, $|z - 3i| = |x + (y - 3)i| = 2$

or $\sqrt{x^2 + (y - 3)^2} = 2$

or $x^2 + (y - 3)^2 = 4$

It is the equation of a circle

Centre of circle (0, 3) Radius = 2

18. If $z = x + iy$ is a variable complex number such that $\arg \frac{z - 1}{z + 1} = \frac{\pi}{4}$, then

[MP PET – 2004]

(a) $x^2 - y^2 - 2x = 1$ (b) $x^2 + y^2 - 2y = 1$

(c) $x^2 - 2y = 1$ (d) $y^2 + 2x = 1$

Solution

(b) $\frac{z - 1}{z + 1} = \frac{x + iy - 1}{x + iy + 1} = \frac{x^2 + y^2 - 1 + 2iy}{(x + 1)^2 + y^2}$

Multiplying and dividing by $x + 1 - iy$

$\therefore \arg \frac{z-1}{z+1} = \tan^{-1} \frac{2y}{x^2+y^2-1}$

$\therefore \tan^{-1} \frac{2y}{x^2+y^2-1} = \frac{\pi}{4}$

or $2y = x^2 + y^2 - 1$ or $x^2 + y^2 - 2y = 1$

19. If $|z| = 2$, then the complex number $-1 + 5z$ is situated on the

(a) circle (b) straight line
(c) parabola (d) ellipse

[MP PET – 2005]

Solution

(a) Let $\omega = -1 + 5z$, then $\omega + 1 = 5z$

$\Rightarrow |\omega + 1| = = |5z| = 5|z| = 5.2$

$\Rightarrow |\omega + 1| = 10.$

Therefore, is a circle whose centre is -1 and radius is $\sqrt{10}$.

20. If z be a complex number, then the locus represented by $iz - 1 + z - i = 2$ is

[Roorkee (Screening) – 1999]

(a) a line
(b) a circle
(c) a pair of straight lines
(d) a coordinate axis

Solution

(d) If $z = x + iy$, then from the given relation, we have

$|i(x+iy) - 1| + |x + iy - i| = 2$

$\Rightarrow |ix - y - 1| + |x + i(y-1)| = 2$

$\Rightarrow [(y+1)^2 + x^2]^{1/2} + [x^2 + (y-1)^2]^{1/2} = 2$

$\Rightarrow (y+1)^2 + x^2 = x^2 + (y-1)^2 + 4 - 4[x^2 + (y-1)^2]^{1/2}$

$\Rightarrow (y-1)^2 = x^2 + (y-1)^2 \Rightarrow x^2 = 0$

$\Rightarrow x = 0$ which is y-axis.

21. Vertices A, B, C of an isosceles triangle ABC are represented by complex numbers z_1, z_2, z_3 respectively. If $\angle C = 90°$, then correct statement is

[PET (Raj.) – 1999; IIT, 1986;

Delhi (EEE) – 1998]

(a) $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$
(b) $(z_1 - z_2)^2 = (z_1 - z_3)(z_3 - z_2)$
(c) $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 z_3$
(d) none of these

Solution

(a) A, B, C are represented by z_1, z_2, z_3 respectively, so $\overline{CA} = z_1 - z_3$, $\overline{CB} = z_2 - z_3$

Also $\angle C = \pi/2$ and $CA = CB$,

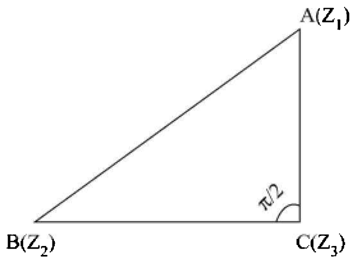
So $\overline{CB} = \overline{CA} i$ [$\because CB = CA$ and CB has been given a rotation of $-\pi/2$ with respect to CA]

$\Rightarrow z_2 - z_3 = (z_1 - z_3)i$

$\Rightarrow (z_2 - z_3)^2 = -(z_1 - z_3)^2$

$\Rightarrow z_2^2 + z_3^2 - 2z_2z_3 = -z_1^2 - z_3^2\, 2z_1z_3$

$\Rightarrow z_1^2 + z_2^2 - 2z_1z_2 = 2z_2z_3 + 2z_1z_3 - 2z_3^2 - 2z_1z_2$



$\Rightarrow (z_1 - z_2)^2 = 2[(z_1z_3 - z_3^2) - (z_1z_2 - z_2z_3)]$
$= 2(z_3 - z_2)(z_1 - z_3)$

22. The centre of a regular polygon of n sides is located at the point $z = 0$ and one of its vertex z_1 is known. If z_2 be the vertex adjacent to z_1, then z_2 is equal to

(a) $z_1\left(\cos\frac{2\pi}{n} \pm i \sin\frac{2\pi}{n}\right)$

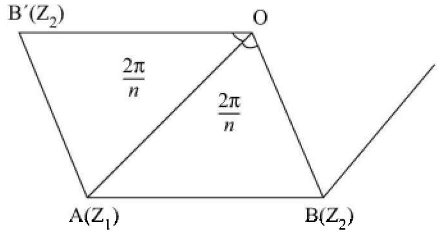
(b) $z_1\left(\cos\frac{\pi}{n} \pm i \sin\frac{\pi}{n}\right)$

(c) $z_1\left(\cos\frac{\pi}{2n} \pm i \sin\frac{\pi}{2n}\right)$

(d) none of these

Solution

(a) Let A be the vertex with affix z_1. There are two possibilities of z_2, i.e., z_2 can be obtained by rotating z_1 through $\frac{2\pi}{n}$ either in clockwise or in anticlockwise direction.



$$\therefore \frac{z_2}{z_1} = \left|\frac{z_2}{z_1}\right| e^{\pm\frac{i2\pi}{n}} \Rightarrow z_2$$

$$= z_1 e^{\pm\frac{i2\pi}{n}}, (\because |z_2| = |z_1|)$$

$$\Rightarrow z_2 = z_1\left(\cos\frac{2\pi}{n} \pm i\sin\frac{2\pi}{n}\right)$$

23. The vertices B and D of a parallelogram are $1 - 2i$ and $4 + 2i$, If the diagonals are at right angles and $AC = 2BD$, the complex number representing A is

(a) $\frac{5}{2}$ (b) $3i - \frac{3}{2}$

(c) $3i - 4$ (d) $3i + 4$

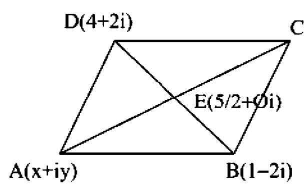
Solution

(b) We have

$|\overrightarrow{BD}| = |(4 + 2i) - (1 - 2i)| = \sqrt{9 + 16} = 5$

Let the affix of A be $z = x + iy$ The affix of the mid-point of BD is $\left(\frac{5}{2}, 0\right)$

Since the diagonals of a parallelogram bisect each other, therefore, the affix of the point of intersection of the diagonals is $\left(\frac{5}{2}, 0\right)$



We have

$|\overrightarrow{AE}| = 5\left(\because BD = \frac{1}{2}\, AC = AE\right)$

which is satisfied by option (b).

24. When $\frac{z+i}{z+2}$ is purely imaginary, the locus described by the point z in the Argand diagram is a

(a) circle of radius $\sqrt{\frac{5}{2}}$

(b) circle of radius $\frac{5}{4}$

(c) straight line

(d) parabola

Solution

(a) Given that $\text{Im}\left(\frac{z+i}{z+2}\right)$

Let $z = x + iy \Rightarrow \frac{x = iy + i}{x = iy + 2} = \frac{x + i(y+1)}{(x+2) + iy}$

$$= \frac{[x + i(y+1)][x+2) - iy]}{[(x+2) + iy][(x+2) - iy]}$$

$$\left[\frac{x^2 + 2x + y^2 + y}{(x+2)^2 + y^2}\right] + i\left[\frac{(y+1)(x+2) - xy}{(x+2)^2 + y^2}\right]$$

If it is purely imaginary, then the real part must be equal to zero.

$$= \frac{x^2 + y^2 + 2x + y}{(x+2)^2 + y^2} = 0 \Rightarrow x^2 + y^2 + 2x + y = 0$$

which is a circle and its radius is given by

$$\sqrt{g^2 + f^2 - c} = \sqrt{1 + \frac{1}{4} - 0} = \frac{\sqrt{5}}{2}$$

Therefore, Argand diagram is circle of radius $\frac{\sqrt{5}}{2}$.

25. If the point z_1, z_2, z_3 are the vertices of an equilateral triangle in the Argand plane, then

(a) $\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$

(b) $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

(c) $(z_1 - z_1)^2 + (z_2 - z_3)^2\,(z_3 - z_1)^2 = 0$

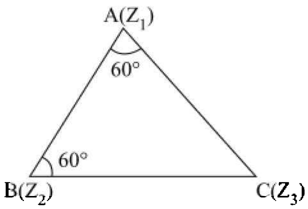
(b) $z_1^3 + z_3^3 + 3z_1z_2z_3 = 0$

Ans:*a*, *b*, *c*

Solution

Let the vertices of the ABC be represented by z_1, z_2 and z_3. By rotation in anticlockwise direction about A and B,

we get $AC = ABe^{\pi i/3}$, $BA = BCe^{\pi i/3}$



or $(z_3 - z_1) = (z_2 - z_1)e^{\pi i/3}$

and $(z_1 - z_2) = (z_3 - z_2)e^{\pi i/3}$ hence on dividing, we get

$\frac{z_3 - z_1}{z_1 - z_2} = \frac{z_2 - z_1}{z_3 - z_2}$ or $(z_3 - z_1)(z_3 - z_2)$

$= -(z_2 - z_1)^2$

or $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$ (1)

Above ⇒ (b). This equation can also be written as

$\frac{1}{2}[\Sigma (z_1 - z_2)^2] = 0$

$\Rightarrow \Sigma (z_1 - z_2)^2 = 0 \Rightarrow (c)$

Again, $\frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} = 0$ can be written as.

$\Sigma (z_3 - z_1)(z_1 - z_2) = 0$ or $\Sigma z_1(z_3 - z_1)$

$- \Sigma z_2(z_3 - z_1) = 0$

The second sigma will be zero and the first gives $\Sigma z_1^2 = \Sigma z_1z_2$ i.e.(1) ⇒ (*a*)

26. The shaded region where $P(-1,0)$, $Q(-1+\sqrt{2}, \sqrt{2})$, $R(-1+\sqrt{2}, \sqrt{-2})$, $S(1,0)$ is represented by

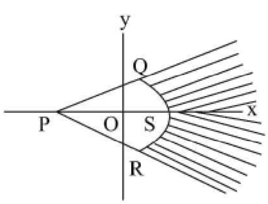
[IIT – 2005]

(a) $|z+1| > 2, |\arg(z+1)| < \frac{\pi}{4}$

(b) $|z+1| < 2, |\arg(z+1)| < \frac{\pi}{2}$

(c) $|z+1| > 2, |\arg(z+1)| < \frac{\pi}{4}$

(d) $|z-1| < 2, |\arg(z+1)| > \frac{\pi}{2}$



Solution

(a) As $|PQ| = |PS| = |PR| = 2$

Therefore, shaded part represents the external part of circle having centre $(-1,0)$ and radius 2.

As we know equation of circle having centre z_0 and radius r,is

$|z - z_0| = r \therefore |z - (-1 + 0i)| > 2$

$\Rightarrow |z+1| > 2$ (1)

Also, argument of $z + 1$ with respect to positive direction of

x − axis is $\frac{\pi}{4}$. $\therefore \arg(z+1) \le \frac{\pi}{4}$

and argument of $z + 1$ in anticlockwise direction is

$\therefore -\frac{\pi}{4} \le \arg(z+1)$ (2)

or

$|\arg(z+1)| \le \frac{\pi}{4}$

27. The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$ are the vertices of a triangle which is

[IIT – 2001;DCE – 2005]

(a) of area zero

(b) right−angled isosceles

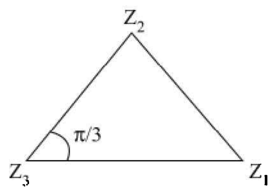
(c) equilateral

(d) obtuse−angled isosceles

Solution

(c) $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2} = \frac{(1 - i\sqrt{3})(1 + i\sqrt{3})}{2(1 + i\sqrt{3})}$

$= \frac{1 - i^2 3}{2(1 + i\sqrt{3})}$



$= \frac{4}{2(1 + i\sqrt{3})} = \frac{2}{(1 + i\sqrt{3})}$

$\Rightarrow \frac{z_2 - z_3}{z_1 - z_3} = \frac{1 + i\sqrt{3}}{2} = \cos\frac{\pi}{3} = i\sin\frac{\pi}{3}$

$\Rightarrow \left|\frac{z_2 - z_3}{z_1 - z_3}\right| = \text{and } \arg\left(\frac{z_2 - z_3}{z_1 - z_3}\right) = \frac{\pi}{3}$

Hence, the triangle is equilateral.

Passage based questions

Let A, B, C be three sets of complex numbers as defi ned here.

$A = \{z : \text{Im } z \geq 1\}$

$B = \{z : |z - 2 - i| = 3\}$

$C = \{z : \text{Re } (1 - i) z = \sqrt{2}\}$

28. Let x be any point in $A \cap B \cap C$. Then, $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between

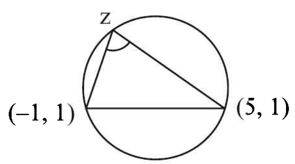
[IIT – 2008]

(a) 25 and 29 (b) 30 and 34

(c) 35 and 39 (d) 40 and 44

Solution

(c) We know that $A \cap B \cap C$ contains just one point. So z is fixed, Also, z is on the circle. The points $(-1, 1)$ and $(5, 1)$ are the ends of a diameter.



Thus, $|z + 1 - i|^2 + |z - 5 - i|^2 = (\text{diameter})^2 = 6 = 36$.

Remark: This is not a problem on finding the greatest and least values of an expression which is the first impression of students after reading "lies between"

29. Let z be any point in $A \cap B \cup C$ and let ω be any point satisfying $|\omega - 2 - i| < 3$. Then, $|z| - |\omega| + 3$ lies between

(a) − 6 and 3 (b) − 3 and 6

(c) − 6 and 6 (d) − 3 and 9

[IIT – 2008]

Solution

(a, b, c, d)

B: $|z - 2i| = 3 \Rightarrow (x - 2)^2 + (y - 1)^2 = 3$

C: $\text{Re } (1 - i) z = \sqrt{2} \Rightarrow x + y = \sqrt{2}$

$(x - 2)^2 + (\sqrt{2} - x - 1)^2 = 3$

$x^2 - x(1 + \sqrt{2}) + 2 - 2\sqrt{2} = 0$

$x = -2, \sqrt{2} - 1;$

Corresponding $y = 2 - \sqrt{2}$, 1

Since, $y \geq 1$; $(x, y) = (\sqrt{2} - 1, 1)$

$|z| = \sqrt{(\sqrt{2} - 1)^2 + 1^2} = \sqrt{4 - 2\sqrt{2}} \cong 1.1$

$|z| - |\omega| + 3 = 1.1 = |\omega| + 3 = 4.1 - |\omega|$

Also, $|\omega - 2 - i| < |3|$

$-3 < |\omega| - |2 + i| < 3$

$\Rightarrow \sqrt{5} - 3 < |\omega| < 3 + \sqrt{5}$

since, $|\omega| \geq 0 \Rightarrow 0 < |\omega| < 3 + \sqrt{5}$

or $0 < |\omega| < 5.2$

Therfore, $|z| - |\omega| + 3 = 4.1 - (\omega)$

lies between − 1.1 to 4.1

Therefore, Ans: (a), (b), (c), (d)

Note: Though the question came in single choice, Answer given by IIT JEE had more than one option correct.

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. In the Argand diagram, if O, P and Q represents, respectively the origin, the complex numbers z and $z + iz$, then the angle $\angle OPQ$ is

[MPPET – 2000]

(a) $\pi/4$ (b) $\pi/3$ (c) $\pi/2$ (d) $2\pi/3$

2. If $x = a, y = b, z = c\omega^2$ where ω is a complex cube root of unity, then $\frac{x}{a}\frac{y}{b}\frac{z}{c} =$

(a) 3 (b) 1

(c) 0 (d) none of these

[AMU – 1983]

3. If $z_i = \cos\frac{i\pi}{10} + i\sin\frac{i\pi}{10}$, then $z_1 z_2 z_3 z_4$ is equal to

(a) − 1 (b) 1 (c) − 2 (d) 2

4. Multiplication of a complex number z by i corresponds to ($z \neq 0$)

(a) clockwise rotation of the line joining z to the origin in Argand diagram through an angle $\pi/2$.

(b) anticlockwise rotation of the line joining z to the origin in Argand diagram through an angle $\pi/2$.

(c) rotation of the line joining z to origin in the Argand diagram through an angle $\pi/2$.

(d) no rotation.

5. If $a = \text{cis}\,\alpha$, $b = \text{cis}\,\beta$, $c = \text{cis}\,\gamma$ and $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 1$, then $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) =$

(a) $\frac{3}{2}$ (b) $-\frac{3}{2}$ (c) 0 (d) 1

[RPET – 2001; Orissa JEE – 2007]

6. If $|z_1| = |z_2| = ... = |z_n| = 1$, then $\frac{|z_1 + z_2 + ... + z_n|}{|z_1^{-1} + z_2^{-1} + ... + z_n^{-1}|}$ is equal to

(a) n (b) 1

(c) $1/n$ (d) none of these

7. The solution of the equation $|z| - z = 1 + 2i$ is

[MPPET – 1993]

(a) $2 - \frac{3}{2}i$ (b) $\frac{3}{2} + 2i$

(c) $\frac{3}{2} - 2i$ (d) $-2 + \frac{3}{2}i$

8. If $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$, then

(a) $\text{Re}(z) = 0$

(b) $\text{Im}(z) = 0$

(c) $\text{Re}(z) > 0$, $\text{Im}(z) > 0$

(d) $\text{Re}(z) > 0$, $\text{Im}(z) < 0$

[MPPET – 1997]

9. If $z = x + iy$, then area of the triangle whose vertices are points z, iz, $z + iz$ is

[MP PET – 1997]

(a) $3/2\,|z|^2$ (b) $|z|^2$

(c) $1/2\,|z|^2$ (d) $1/4\,|z|^2$

10. The points representing the complex numbers z, for which $|z - a|^2 + |z + a|^2 = b^2$ lie on

(a) $x + y = \frac{b - a}{2}$

(b) $x^2 + y^2 = \frac{b^2 - a^2}{2}$

(c) $y^2 + 2bx$

(d) $x^2 - y^2 = 2ab$

[MPPET – 2008]

11. Let $z = x + iy$ be a complex number where x and y are integers. Then, the area of the rectangle whose vertices are the roots of the equation $\bar{z}z^3 + z\bar{z}^3 = 350$ is

[IIT – 2009]

(a) 48 (b) 32

(c) 40 (d) 80

SOLUTIONS

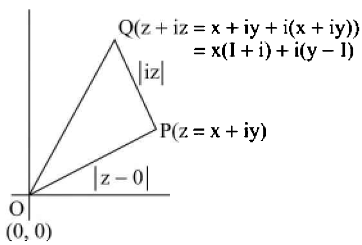
1. (c) Let $z = r(\cos\theta + i\sin\theta)$, then

$\overline{PQ}$ = Affix of Q − Affix of P

$= z + iz - z = iz$

Also, $\overline{OP} = z$

Clearly, angle between z and iz is 90°.



2. (c) Given that $x = a, y = b, z = c\omega^2$

Then, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \frac{a}{a} + \frac{b\omega}{b} + \frac{c\omega^2}{c}$

$= 1 + \omega + \omega^2 = 0$

3. (c) $z_1 z_2 z_3 z_4 \cos\left(\frac{\pi}{10}\right) + i \sin\left(\frac{\pi}{10}\right)$

$\left(\cos\left(\frac{2\pi}{10}\right) + i \sin\left(\frac{2\pi}{10}\right)\right)$

$\left(\cos\left(\frac{3\pi}{10}\right) + i \sin\left(\frac{3\pi}{10}\right)\right)$

$\left(\cos\left(\frac{4\pi}{10}\right) + i \sin\left(\frac{4\pi}{10}\right)\right)$

$$\left\{\begin{array}{l} \cos\left(\frac{\pi}{10} + \frac{2\pi}{10} + \frac{3\pi}{10} + \frac{4\pi}{10}\right) \\ + i \sin\left(\frac{\pi}{10} + \frac{2\pi}{10} + \frac{3\pi}{10} + \frac{4\pi}{10}\right)\end{array}\right\}$$

$= \cos(\pi) + i \sin \pi = -1 + i \times 0 = -1$

4. (b) $\because i = e^{i\pi/2}$ $\therefore$ Multiplying by i, z gets shaded by $\pi/2$ in anticlockwise direction

5. (d) $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 1$

$\Rightarrow \frac{cis\alpha}{cis\beta} + \frac{cis\beta}{cus\gamma} + \frac{cis\gamma}{cis\alpha} = 1$

$\Rightarrow cis(a - \beta) + cis(\beta - \gamma) + cis(\gamma - \alpha) = 1$

$\Rightarrow \cos(\alpha - \beta) + \cos(b - \gamma) + \cos(\gamma - \alpha) = 1$

(Equating real parts)

6. (b) $\because |z| = 1\ z^{-1} = \bar{z}$, so

$\text{Exp.} = \frac{|z_1 + z_2 + \ldots.. + z_n|}{|\bar{z}_1 + z_2 + \ldots.. + \bar{z}_n|}$

$= \frac{|z_1 + z_2 + \ldots.. + z_n|}{|z_1 + z_2 + \ldots.. + z_n|} = 1$

$\because |z| = |\bar{z}|$

7. (c) Let $z = a + ib$ then

$|z| = |a + ib| = \sqrt{a^2 + b^2}$

$\therefore |z| - z = 1 + 2i = \frac{3}{2} - 2i$

$\Rightarrow (\sqrt{a + b^2} - a)^2 - ib = 1 + 2i$

Comparing real and imaginary parts of the both sides

$\Rightarrow \sqrt{a^2 + b^2} - a = 1, -b = 2$

$\Rightarrow a = \frac{3}{2}, b = -2$

Therfore, $z = a + ib = \frac{3}{2} - 2i$

8. (b) de-Movier's Theorem

$(\cos\theta \pm i \sin\theta)^n = \cos n\,\theta \pm i \sin n\,\theta$

$\left(\frac{\sqrt{3} + i}{2}\right)^5 = (\cos 30 + i \sin 30)^5$

$= \cos 150 + i \sin 150$ (1)

and $\left(\frac{\sqrt{3} - i}{2}\right)^5 = \cos 150 - \text{i} \sin 150$ (2)

Adding $\left(\frac{\sqrt{3} + i}{2}\right)^5 + \left(\frac{\sqrt{3} - i}{2}\right)^5 = 2 \cos 150°$

$= 2 - \left(\frac{\sqrt{3}}{2}\right) = -\sqrt{3}$

Clearly, $(-\sqrt{3})$ is a real number. Therefore, $i_z = 0$

9. (c) Let, $z = x + iy \Rightarrow (x, y)$

$iz, i(x + iy) = -y + ix \Rightarrow (-y, x)$

$z + iz = x + iy - y + ix \Rightarrow (x - y, x + y)$

Area of $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$\Delta = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ -y & x & 1 \\ x - y & x + y & 1 \end{vmatrix}$

Applying, $R_3 \to R_3 - R_1 - R_2$

$\Delta = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ -y & x & 1 \\ 0 & 0 & -1 \end{vmatrix}$

$\Delta = \left|-\frac{1}{2}(x^2 + y^2)\right|$ or $\Delta = \frac{|z|^2}{2}$

10. (b) $|z - a|^2 + |z + a|^2 = b^2$

Let $z = x + iy$

$\therefore |x + iy - a|^2 + |x + iy + a|^2 = b^2$

$(x-a)^2+y^2+(x+a)^2+y^2=b^2$

$2x^2+2y^2+2a^2=b^2$

$\therefore x^2+y^2=\dfrac{b^2}{2}-a^2$

It is a circle.

11. (a) $z\bar{z}(\bar{z}^2+z^2)=350$

Put $z=x+iy$

$(x^2+y^2)(x^2-y^2)=175$

$(x^2+y^2)(x^2-y^2)=25.7$

$x^2+y^2=25$

$x^2-y^2=7$

$x=\pm4, y=\pm3$

$x, y \in I$

Area = 8 × 6 = 48 sq. units.

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. The two numbers such that each one is square of the other are

[MPPET – 1987]

(a) ω, ω^3 (b) $-i, i$
(c) $-1, 1$ (d) ω, ω^2

2. $\left(\dfrac{\cos\theta+i\sin\theta}{\sin\theta+i\cos\theta}\right)^4$ equals

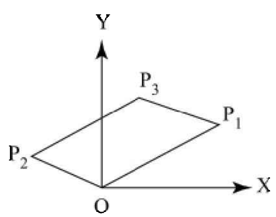
[RPET – 1996]

(a) $\sin 8\theta - i\cos 8\theta$
(b) $\cos 8\theta - i\sin 8\theta$
(c) $\sin 8\theta + i\cos 8\theta$
(d) $\cos 8\theta + i\sin 8\theta$

3. Let z be a complex number. Then, the angle between vectors z and $-iz$ is

(a) π (b) 0
(c) $-\pi/2$ (d) none of these

4. If the points P_1 and P_2 represent two complex numbers z_1 and z_2, then the point P_3 represents the number



(a) z_1+z_2
(b) z_1-z_2
(c) $z_1\times z_2$
(d) $z_1\div z_2$

5. The point represented by the complex number $2-i$ is rotated about origin through an angle of $\frac{\pi}{2}$ in clockwise direction. The new position of the point is

(a) $1+2i$
(b) $-1-2i$
(c) $2+i$
(d) $-1+2i$

6. Let O be the origin and point p represents complex number z in a complex plane. If OP be rotated anticlockwise at an angle $\pi/2$, then the new position of p is represented by the complex number

[NDA – 2007]

(a) $z-i$ (b) $z+i$
(c) iz (d) $-iz$

7. If the amplitude of $z-2-3i$ is $\frac{\pi}{4}$, then the locus of $z=x+iy$ is

[EAMCET – 2003]

(a) $x+y-1=0$
(b) $x-y-1=0$
(c) $x+y+1=0$
(d) $x-y+1=0$

8. If the areaof the triangle formed by the points z, $z+iz$ and iz on the complex plane is 18, then the value of $|z|$ is

[MPPET – 2001]

(a) 6 (b) 9
(c) $3\sqrt{2}$ (d) $2\sqrt{3}$

WORK SHEET: TO CHECK PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the work sheet
2. The test is of 5 minutes.
3. The test consists of 5 questions
   The maximum marks are 15.
4. Use blue/black ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. The value of $\left[\dfrac{1-\cos\frac{\pi}{10}\, i\sin\frac{\pi}{10}}{1-\cos\frac{\pi}{10}-i\sin\frac{\pi}{10}}\right]^{10}$

(a) 0 (b) − 1
(c) 1 (d) 2

[Karnataka CET – 2001]

2. We express

$$\frac{(\cos 2\theta - i\sin 2\theta)^4(\cos 4\theta + i\sin 4\theta)^{-5}}{(\cos 3\theta + i\sin 3\theta)^{-2}(\cos 3\theta - i\sin 3\theta)^{-9}}$$

in the form of $x + iy$, we get

[Karnataka CET – 2001]

(a) $\cos 49\theta - i\sin 49\theta$
(b) $\cos 23\theta - i\sin 23\theta$
(c) $\cos 49\theta + i\sin 49\theta$
(d) $\cos 21\theta + i\sin 21\theta$

3. If $z = \dfrac{7-i}{3-4i}$, then $z^{14} =$

[Kerala (Engg.) – 2005]

(a) 2^7 (b) $2^7 i$
(c) $2^{14} i$ (d) $-2^7 i$

4. If $x = 2 + 3i$, and $y = 2 - 3i$, then value of $\dfrac{x^2 + xy + y^2}{x^2 - xy + y^2}$ is

(a) $\dfrac{3}{23}$ (b) $-\dfrac{3}{23}$
(c) $\dfrac{5}{23}$ (d) $-\dfrac{5}{23}$

5. If 1, ω, ω^2 are cube root of unity, then the value of $(3 + 3\omega + 5\omega^2)^3 - (2 + 4\omega + 2\omega^2)^3$ is

[Pb.CET – 1998]

(a) 0 (b) 3
(c) 2 (d) 1

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ

This page is intentionally left blank.

LECTURE

5

Test Your Skills

ASSERTION/REASONING

Assertion and Reasoning Type Questions

Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.

(a) **Assertion** is True, **Reason** is True and **Reason** is a correct explanation for **Assertion**

(b) **Assertion** is True, **Reason** is True and **Reason** is NOT a correct explanation for **Assertion**

(c) **Assertion** is True and **Reason** is False

(d) **Assertion** is False and **Reason** is True

1. **Assertion (A):** If $\alpha = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$, $c = \cos\gamma + i$ sing and $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} - 1$, then $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -1$

   Reason (R): $(\cos\alpha_1 + i\sin\alpha_1)(\cos\alpha_2 + i\sin\alpha_2) = \cos(\alpha_1 + \alpha_2) + i\sin(\alpha_1 + \alpha_2)$

2. **Assertion (A):** If the area of the triangle on the Argand plane formed by the complex numbers $-z$, iz, $z - iz$ is 600 square units, then $|z| = 20$

   Reason (R): Area of the triangle on the argand plane formed by the complex numbers $-z$, iz, $z - iz$ is $\frac{3}{2}|z^2|$.

3. **Assertion (A):** The greatest value of the moduli of complex numbers z satisfying the equation is $\left|z - \frac{4}{z}\right| = 2$ is $\sqrt{5} + 1$

   Reason (R): For any two complex numbers z_1 and z_2, $|z_1 - z_2| \geq |z_1| - |z_2|$

4. **Assertion (A):** $7 + 4i > 5 + 3i$, where $i = \sqrt{-1}$ **Reason (R):** $7 > 5$ and $4 > 3$

5. **Assertion (A):**

   $\sqrt{(-1)}\sqrt{(-3)} = \sqrt{(-2)(-3)} = \sqrt{6}$

   Reason (R): If a and b both negative, then $\sqrt{a}\sqrt{b} \neq \sqrt{ab}$

6. **Assertion (A):** $\sum_{r=1}^{4n=11} i^r = i$, $i = \sqrt{-1}$

   Reason (R): Sum of the four consecutive powers of i is zero.

7. **Assertion (A):** If $\frac{5z_2}{11z_1}$ is purely imaginary, then $\left|\frac{2z_1 + 3z_2}{2z_1 - 3z_2}\right| = 1$

   Reason (R): $|z| = |\bar{z}|$.

8. **Assertion (A):** If $z = \sqrt{(5+12i)} + \sqrt{(12i-5)}$, then the principal values of arg (z) are $\pm\frac{\pi}{4}$, $\pm\frac{3\pi}{4}$, where $i = \sqrt{-1}$.

Reason (R): If $z = a + ib$, then and for

$$\sqrt{z} = \pm\left\{\sqrt{\left(\frac{|z|+a}{2}\right)} - i\left(\frac{|z|-a}{2}\right)\right\}\ b < 0$$

9. **Assertion (A):** If $|z - 3 + 2i| \le 4$, then the sum of least and greatest value of $|z|$ is 8.

   Reason (R): $||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|$.

10. **Assertion (A):** The value of

    $$\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right) \text{ is } i.$$

    Reason (R): It forms an A.P. series.

11. **Assertion (A):** For a complex number z the equation $|3z - 1| = 3|z - 2|$ represents a straight line.

    Reason (R): General equation of straight line is $ax + by + c = 0$.

12. **Assertion (A):** If $e^{i\theta} = \cos\theta + i\sin\theta$ and the value of $e^{iA}.\ e^{iB}.\ e^{iC}$ is equal to -1.

    Reason (R): $e^{i\theta} = \cos\theta + i\sin\theta$ and in any $\Delta ABC, A + B + C = 180°$.

13. **Assertion (A):** $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$, where z_1, z_2, z_3 and z_4 are the fourth roots of unity.

    Reason (R): $(1)^{1/4} = (\cos 2r\pi + i\sin 2r\pi)^{1/4}$

14. **Assertion (A):** For any four complex numbers z_1, z_2, z_3 and z_4. it is given that the four points are concyclic.

    Reason (R): $|z_1| = |z_2| = |z_3| = |z_4|$

15. **Assertion (A):**The points denoted by the complex number z lies inside the circle with radius 2 and is at the origin.

    Reason (R): $|z| > 2$ represents a straight line.

16. **Assertion (A):** The expression $\left(\frac{2i}{1+i}\right)^n$ is a positive integer for all the values of n.

    Reason (R): Here $n = 8$ is the least positive for which the above expression is a positive integer.

17. **Assertion (A):** we have an equation involving the complex number z is $\left|\frac{z-3i}{z+3i}\right| = 1$ which lies on the x-axis.

    Reason (R): The equation of the x-axis is $y = 3$.

18. **Assertion (A):** The equation $|z + 1| = \sqrt{3}\,|z - 1|$ represents a circle.

    Reason (R): The equation of straight line is $ax + by + c = 0$.

19. **Assertion (A):** The value of i^{4m+3}, when $m \in I$ is equal to $-i$.

    Reason (R): $i^4 = 1$

20. **Assertion (A):** The roots of the equation $(x - 1)^3 + 8 = 0$ are $-1, 1 - 2\omega, 1 - 2\omega^2$.

    Reason (R): $1, \omega, \omega^2$ are the cube roots of unity where $1 + \omega + \omega^2 = 0$ and $\omega^3 \ne 1$.

21. **Assertion (A):** If z is a complex number $(z \ne 1)$, then $\left|\frac{z}{|z|} - 1\right| < |\arg(z)|$.

    Reason (R): In a unit circle, chord $AP \le$ arc (AP).

22. **Assertion (A):** The least value of $|z - 3| + 5|z - 8|$, $z \in C$ is got by setting $z = \frac{8+3}{2}$

    Reason (R): The least value of $|z - 3| + 5|z - 8|$ is same as that of $PA + 5PB$, where $P = z(x, y)$ and $A = (3, 0)$, $B = (8, 0)$ and P ranges over all points in $x - y$ plane.

23. **Assertion (A):** If $x + \frac{1}{x} = 1$ and $p = x^{100} + \frac{1}{x^{100}}$ and q be the digit at unit place in $2^{(2n)} + 1$, $n \in N$, $n > 1$, then $p + q = 6$.

    Reason (R): If $x + \frac{1}{x} = -1$, then $x^2 + \frac{1}{x^2} = -1$ and $x^3 + \frac{1}{x^3} = 2$

24. **Assertion (A):** Let z_1, z_2, z_3 be three points in complex plane with nonzero imaginary parts such that $z_1 + z_2 + z_3 = 0$. Then, $z_1 + z_2 + z_3$ must be vertices on an equilateral triangle.

    Reason (R): If z_1, z_2, z_3 are vertices of an equilateral triangle, then $z_1^3 + z_2^3 + z_3^3 = 3z_1z_2z_3$

25. **Assertion (A):** $ABCD$ is a parallelogram on the Argand plane. The affixes of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are $8 + 5i, -7 - 5i, -5 + 5i$ respectively. Then, the affix of D is $10 + 15i$.

    Reason (R): The diagonals AC and BD bisect each other.

26. **Assertion (A):** If the principal argument of a complex number z is α then principal argument of z^2 is 2α.
Reason (R): $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$

27. **Assertion (A):** The modulus of the complex number $z = \frac{1-i}{3+i} + 4i$ is $\sqrt{13}$
Reason (R): Argument of z is $\tan^{-1}1\left(\frac{3}{4}\right)$

28. **Assertion (A):** If $z_1 = 3 - 4i$, $z_2 = -5 + 2i$ are two complex numbers such that $z_1 < z_2$.
Reason (R): $|z_1| < |z_2|$

29. **Assertion (A):** If $\left|\frac{zz_1 - z_2}{zz_1 + z_2}\right| = k\ (z_1, z_2 \neq 0)$, then locus of z is circle.
Reason (R): $\left|\frac{z - z_1}{z - z_2}\right| = \lambda$, represents a circle if, $\{0, 1\}$.

30. **Assertion (A):** The equation $|z - i| + |z + i| = k$, $k > 0$ can represent an ellipse, if $k > 2i$.
Reason (R): $|z - z_1| + |z - z_2| = k$, represents ellipse, if $|k| > |z_1 - z_2|$.

31. **Assertion (A):** If $1, \omega, \omega^2, ...\omega^{n-1}$ are the nth roots of unity, then $(2 - \omega)(2 - \omega^2)......(2 - \omega^{n-1})$ equals $2^n - 1$.
Reason (R): ${}^nC_1 + {}^nC_2 + {}^nC_3 + ... + {}^nC_n = 2^n$.

32. **Assertion(A):** If ω is an imaginary cube root of unity, then the value of $\sin\left\{\pi + (\omega^{10} + \omega^{23})\frac{\pi}{4}\right\}$ is $\frac{1}{\sqrt{2}}$
Reason (R): $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$

ASSERTION/REASONING: SOLUTIONS

1. (a) we have,

$\frac{1}{a} = \cos a - i\sin a$, $\frac{1}{b} = \cos\beta - i\sin\beta$

Now, $\frac{a}{b} = (\cos a + i\sin a)(\cos\beta - i\sin\beta)$

or, $\frac{a}{b} = \cos(a - \beta) + i\sin(a - \beta)$

Similarly, $\frac{b}{c} = \cos(\beta - \gamma) + i\sin(\beta - \gamma)$

and $\frac{c}{a} = \cos(\gamma - \alpha) + i\sin(\gamma - \alpha)$

Putting these values in $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = -1$,

We get $[\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha)] + i[\sin(\alpha - \beta) + \sin(\beta - \gamma) + \sin(\gamma - \beta)] = -1 = -1 + 0i$.

Comparing real and imaginary parts, we get, $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -1$

2. (a) Area of the triangle on the Argand plane formed by the complex numbers $-z$, iz, $z - iz$ is $\frac{3}{2}|z|^2$.

$\therefore \frac{3}{2}|z|^2 = 600\ |z| = 20$.

3. (a) We have, $z =$

$\left|z - \frac{4}{z}\right| \geq |z| - \left|\frac{4}{z}\right| \Rightarrow |z| - \left|\frac{4}{z}\right| \leq 2$

$\Rightarrow |z|^2 - 2|z| - 40$ or $(|z| - 1)^2 - 50 \leq 0$

$\Rightarrow (|z| - 1)^2 \leq 5$ or $|z| - 1 \leq \sqrt{5}$

$\Rightarrow |z| \leq \sqrt{5} + 1$

Hence, the greatest value of $|z|$ is $\sqrt{5} + 1$.

4. (d) Property of order i.e., $(a + ib) \lesssim (c + id)$ is not defined. The statement $7 + 4i > 5 + 3i$ makes no sense.

5. (d) If both a and b are negative then $\sqrt{a}\sqrt{b} = -\sqrt{ab}$

$\therefore \sqrt{(-2)}\sqrt{(-3)} = -\sqrt{(-2)}\sqrt{(-3)} = -\sqrt{6}$.

6. $\sum_{r=1}^{4n+11} i^r = (i + i^2 + i^3 i^4) + (i^5 + i^6 + \dots i^8)$

$+ \dots (i^{4n+5} + i^{4n+6} + i^{4n+7} + i^{4n+8})$
$+ i^{4n+9} + i^{4n+10}\, i^{4n+11}$

$= i - 1 - i + 0 = 1$

(Since, sum of four consecutive powers of i is zero)

7. (a) Let $\frac{5z_2}{11z_1} = i\lambda\ (\lambda \neq 0) \Rightarrow \frac{z_2}{z_1} = \frac{11i\lambda}{5}$

$$\left|\frac{2z_1 + 3z_2}{2z_1 - 3z_2}\right| = \left|\frac{2 + 3\frac{z_2}{z_1}}{2 - 3\frac{z_2}{z_1}}\right| = \left|\frac{2 + \frac{33i\lambda}{5}}{2 - \frac{33i\lambda}{5}}\right|$$

$$= \left|\frac{10 + 33i\lambda}{10 - 33i\lambda}\right| = 1.$$

8. (b) Let $z = z_1 + z_2$

Since, $z_1 = \sqrt{(5 + 12i)} = \pm \left\{ \sqrt{\left(\frac{13+5}{2}\right)} + i\sqrt{\left(\frac{13-5}{2}\right)} \right\}$

$$= \pm (3 + 2i)$$

and $z_2 = \sqrt{(12i - 5)} = \sqrt{(-5 + 12i)}$

$$= \pm \left\{ \sqrt{\left(\frac{13-5}{2}\right)} + i\sqrt{\left(\frac{13+5}{2}\right)} \right\}$$

$$= \pm (2 + 3i)$$

$z = \pm (3 + 2i) \pm (2 + 3i)$

$\Rightarrow z = 5 + 5i,\ 1 - i,\ -1 + i,\ -5 - 5i.$

Hence, principal values of z are

$\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, -\frac{3\pi}{4}$

9. (a) Since, $|z_1 + z_2| = |z_1 - z_2|$

Therefore, $|z - 3 + 2i| = |z - (3 - 2i)| \geq ||z| - |3 - 2i||$

$= ||z| - \sqrt{13}|$

Since, $|z - 3 + 2i| \geq ||z| - \sqrt{13}|$

Since, $|z - 3 + 2i| \leq 4$

$\Rightarrow |z|\sqrt{13}| \leq |z - 3 + 2i| \leq 4$

$\Rightarrow ||z| - \sqrt{13}| \leq 4$

$-4 \leq |z| - \sqrt{13}| \leq 4$

$\Rightarrow$ or $4 - \sqrt{13} \leq |z| \leq 4 + \sqrt{13}$

$\therefore$ Greatest value of $|z| = 4 + \sqrt{13}$ and Least value of $|z| = 4 - \sqrt{13}$

$\therefore$ Sum = 8.

10. (c) $\sum_{k=1}^{6} \left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right)$

$$= -i\sum_{k=1}^{6} \left(\cos\frac{2\pi k}{7} + i\sin\frac{2\pi k}{7}\right)$$

$$= -i\left[\left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right) + \left(\cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7}\right) + \ldots\ldots + \left(\cos\frac{12\pi}{7} + i\sin\frac{12\pi}{7}\right)\right]$$

$$= -i\left[\left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right) + \left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^2 + \ldots\ldots + \left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^6\right]$$

$= -i\left[\frac{x(x^6 - 1)}{x - 1}\right]$ if $x = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$

$= -i\left[\frac{x^7 - x}{x - 1}\right] = -i\left[\frac{1 - x}{x - 1}\right]$

$\left[\because x^7 = \left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^7\right]$

$= \cos 2\pi + i\sin 2\pi = 1$

$= i\left(\frac{x - 1}{x - 1}\right) = i$

11. (b) Let $z = x + iy$, then $|3z - 1| = 3|z - 2|$

$\Rightarrow |3(x + iy) - 1| = 3|x + iy - 2|$

$\Rightarrow |(3x - 1) + 3iy| = 3|x - 2 + iy|$

$\Rightarrow (3x - 1)^2 + 9y^2 = 9[(x - 2)^2 + y^2]$

$\Rightarrow 9x^2 + 1 - 6x + 9y^2 = 9x^2 + 36 - 36x + 9y^2$

$\Rightarrow 30x = 35 \Rightarrow x = \frac{7}{6}$

i.e. a straight line parallel to y-axis.

12. (a) $e^{iA} \cdot e^{iB} \cdot e^{iC} = e^{i(A+B+C)} = e^{\pi i}$

[Since, $A + B + C =$ for ΔABC]

$= \cos\pi + i\sin\pi$

$= -1 + i(0) = -1.$

13. (a) $(1)^{1/4} = (\cos 2\pi r + i\sin 2\pi r)^{1/4}$

$\cos\frac{\pi r}{2} + i\sin\frac{\pi r}{2}$

where $r = 0, 1, 2, 3$

$\therefore (1)^{1/4} = 1, i, -1, -i$

$\therefore z_1^2 + z_2^2 + z_3^2 + z_4^2 + 1 + i^2$

$= 2 - 1 - 1 = 0.$

14. (a) $|z_1|$ = the distance of the point representing z_1 from the origin. Therefore, the distances of the four points from the origin are equal. Therefore, points are concyclic.

15. (c) Since $|z| < 2$

$\therefore |z|^2 < 4$

$\Rightarrow x^2 + y^2 < 4$

16. (d) $\left(\frac{2i}{1+i}\right)^2 = \frac{4i}{(1+i)^2} = \frac{-4}{1+i^2+2i}$

$= \frac{-4}{2i} = \frac{-2}{i} = 2i$

$\therefore \left(\frac{2i}{1+i}\right)^4 = 4i^2 = -4$

$\therefore \left(\frac{2i}{1+i}\right)^8 = (-4)^2 = 16$

Hence, $n = 8$ is the last poitive integer

17. (c) $\left|\frac{z-3i}{z+3i}\right| = 1 = \Rightarrow \left|\frac{x+iy-3i}{x+iy+3i}\right| = 1$

$\Rightarrow \left|\frac{x+(y-3)i}{x+(y+3)i}\right| = 1 \Rightarrow \frac{x^2+(y-3)^2}{x^2+(y+3)^2} = 1$

$\Rightarrow x^2 + y^2 - 6y + 9 = x^2 + y^2 + 6y + 9$

$\Rightarrow 12y = 0$

$\Rightarrow y = 0$ which is x-axis.

Therefore, z lies on x-axis.

18. (b) $|z+1|^2 = 3\,|z-1|^2$

$\Rightarrow (x+1)^2 + y^2 = 3\,[(x-1)^2 + y^2]$

$\Rightarrow 2(x^2+y^2) - 8x + 2 = 0$

$\Rightarrow x^2 + y^2 - 4x + 1 = 0$

which is a circle.

19. (a) $i^n = i^{4m+3} = i^{4m}.\, i^3 = (i^4)^m\,(-i) = (1)^m\,(-i) = -i$

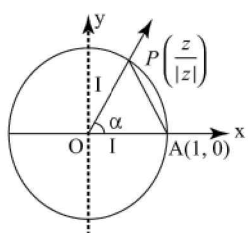
20. (a) Since $(x-1)^3 = -8 = (-2)^3$

$\therefore x - 1 = -2, -2\omega, -2\omega^2$

$\therefore x = -1, 1-2\omega, 1-2\omega^2$

21. (a) The number $\frac{z}{|z|}$ lies on a unit circle centred at origin

From the figure



Chord $AP = \left|\frac{z}{|z|} - 1\right| \leq arc\ (AP)$

$\frac{\text{arc }(AP)}{\text{radius}} = 1 = \alpha$

$\therefore \left|\frac{z}{|z|} - 1\right| \leq |\arg z|$

22. (b) $|z_2 z_3 + 8z_3 z_1 + 27z_1z_2|$

$= |z_1z_2z_3|\left|\frac{1}{z_1} + \frac{8}{z_2} + \frac{27}{z_3}\right|$

$= |z_1|\,|z_2|\,|z_3|\left|\frac{\bar{z}}{|z_1|^2} + \frac{8\bar{z}_2}{|z_2|^2} + \frac{27\bar{z}_3}{|z_3|^2}\right|$

$= 6\,|\bar{z} + 2\bar{z}_2 + 3\bar{z}_3|$

$= 6\,|z_1 + 2z_2 + 3z_3| = 6 \times 6 = 36$

$\therefore S_1$ is true, S_2 is also true but not the correct explanation for S_1.

23. (b) $x + \frac{1}{x} = 1$

$\Rightarrow x^2 - x + 1 = 0$

$\Rightarrow x = -\omega, -\omega^2$, ω is imaginary cube root of unity.

$\therefore p\ x^{100} = + \frac{1}{x^{100}} = \omega + \omega^2 = -1$

For $n > 1$, $2^n = 4m$, $m \in N$

$\Rightarrow 2^{(2^n)} = 2^{4m} = 16^m$

$\Rightarrow$ unit place of $2^{(2^n)} = 6$

$\therefore\ q =$ unit place at $2^{(2^n)} + 1 = 7$

Hence, $p + q = 7 - 1 = 6$

Also, roots of $x + \frac{1}{x} = -1$ are ω and ω^2

$\Rightarrow x^2 + \frac{1}{x^2} = -1$ and $x^3 + \frac{1}{x^3} = 2$

Hence, both S_1 and S_2 are true, but S_2 is not correct explanation for S_1.

24. (d) z_1, z_2, z_3 vertices of an equilateral triangle then

$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$ which

$z_1^3 + z_2^3 + z_3^3 - 3z_1z_2z_3 = 0$

$\therefore S_2$ is true.

But $z_1 + z_2 + z_3 = 0$ even when z_1, z_2, z_3 are collinear For example, i, $2i$ and $-3i$.

$\therefore S_1$ is false.

25. (a) Let z be the affix of D

Therefore,

$\frac{(8+5i)+(-5+5i)}{2} = \frac{(-7-5i)+z}{2}$

$z = 10 + 15i$

So, both assertion and reason are true and reason is correct explanation of assertion.

26. (d) If principal arg of z is α then argument of z^2 is $2a$. Note that it may not be principal argument

For example, Let $z = -1 + i$

$\Rightarrow \text{Arg}(z) = \frac{3\pi}{4}$

$\text{Arg}(z^2) = \frac{3\pi}{4} \times 2 = \frac{3\pi}{2}$

but principal arg $(z^2) = \frac{\pi}{4}$

So, assertion is false, reason is true.

27. (c) Converting to $a + ib$ form

$z = \left(\frac{1-i}{3+i}\right)\left(\frac{3-i}{3-i}\right) + 4i$

$z = \frac{1}{5} + \frac{18}{5}i$

$|z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{18}{5}\right)^2}$

$\arg(z) = \tan^{-1}\left|\frac{\frac{18}{5}}{\frac{1}{5}}\right| = \tan^{-1}(18)$

So, assertion is false, reason is true.

28. (d) In the set of complex number, the order relation is not defned. As such $z_1 > z_2$ or $z_1 < z_2$ has no meaning but $|z_1| > |z_2|$ or $|z_1| < |z_2|$ has got its meaning since $|z_1|$ and $|z_2|$ are real numbers.

So, assertion is false, reason is true.

29. (d) If $\left|\frac{z_1 z - z_2}{z_1 z + z_2}\right| = k$

$\Rightarrow \left|\frac{z - \frac{z_2}{z_1}}{z + \frac{z_2}{z_1}}\right| = k$

Clearly, if $k \neq 0, 1$ then z would lie on a circle.

Case I If $k = 1$, z would be on a perpendicular bisector of line segment.

Case II If $k = 0$, $\frac{z_2}{z_1}$ and $-\frac{z_2}{z_1}$ represents a point. So, assertion is false and reason is true.

30. (d) As, we know $|z - z_1| + |z - z_2| = k$ represents an ellipse,

if $|k| > |z_1 - z_2|$

Thus, $|z - i| + |z + i| = k$ represents ellipse, if $|k| > |i + i|$ or $|k| > 2$.

So, assertion is false but reason is correct.

31. (c) Let

$x = (1)^{1/n}$

$x^n - 1 = 0$

has n roots i.e. $1, \omega, \omega^2, \ldots\ldots \omega^{n-1}$

$(x^n - 1) = (x - 1)(x - \omega)(x - \omega^2) \ldots\ldots (x - \omega^{n-1})$

$\frac{2^n - 1}{(2-1)} = (2 - \omega)(2 - \omega^2) \ldots\ldots (2 - \omega^{n-1})$

(put $x = 2$)

i.e., $\because (2 - \omega).(2 - \omega^2) \ldots\ldots (2 - \omega^{n-1}) = 2^n - 1$

As, ${}^nC_0 + {}^nC_1 + \ldots\ldots + {}^nC_n = 2^n$

So, assertion is true but reason is false.

32. (a) $\left[\sin \pi + (\omega + \omega^2)\frac{\pi}{4}\right] = \sin\left[\pi - \frac{\pi}{4}\right]$

$= \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$

So, both assertion and reson are true and asserition follows reason.

MENTAL PREPARATION TEST

1. If $(x + iy)^{1/3} = a + ib, x, y, a, bR$.

Show that $\frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2)$

2. If $= \frac{a + ib}{c + id} = x + iy$,

prove that $\frac{a - ib}{c - id} = x - iy$ and $\frac{a^2 + b^2}{c^2 + d^2} = x^2 + y^2$

3. If $(a + ib)(c + id) = x + iy$, then prove that $(a - ib)(c - id) = x - iy$ and $(a^2 + b^2)(c^2 + d^2) = x^2 + y^2$.

4. If $z = 3 - 5i$, then prove that $z^3 - 10z^2 + 58z - 136 = 0$.

5. Find the modulus of $\left(\frac{1+i}{1-i}, \frac{1-i}{1+i}\right)$.

6. Find the locus of a complex variable z in the argand plane, satisfying
$|z-(3-4i)| = 7$.

7. Write the complex numbers $-1-i$ in the polar form.

8. If $\frac{2z_1}{3z_2}$ be a purely imaginary number, then prove that $\left|\frac{z_1 - z_2}{z_1 + z_2}\right| = 1$

9. If $i = \sqrt{-1}$, then prove that
$$4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) = i\sqrt{3}$$

10. Find the square roots of the following
(i) $4ab - 2(a^2 - b^2)i$
(ii) $a^2 - 1 + 2ai$.

11. If $\frac{3}{2 + \cos\theta + i\sin\theta} = a + ib$ them prove that $a^2 + b^2 = 4a - 3$.

12. Express the complex number $\frac{3-\sqrt{-16}}{1-\sqrt{-9}}$ in the form $a + ib$.

13. Prove that $\left(\frac{3+2i}{2-5i}\right) + \left(\frac{3-2i}{2+5i}\right)$ rational.

14. Find the values of x and y, for which the $(3x - 2iy)(2 + i)^2 = 10(1 + i)$ equalities hold.

15. Prove that $x^4 + 4 = (x + 1 + i)(x + 1 - i)(x - 1 + i)(x - 1 - i)$.

16. If $z = x + iy$ and $\omega = \frac{1-iz}{z-i}$, show that $|\omega| = 1 \Rightarrow z$ is purely real.

17. If $z = -5 + 2\sqrt{-4}$, show that $z^2 + 10z + 41 = 0$ and hence, find the value of $z^4 + 9z^3 + 35z^2 - z + 4$

18. If $\frac{a-ib}{a+ib} = \frac{1+i}{1-i}$, then show that $a + b = 0$.

19. If $1, \omega, \omega^2$ be the cube roots of unity, prove that
$(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11}) = 49$.

20. If α and β are imaginary cube roots of unity, show that
$\alpha^4 + \beta^4 + \alpha^{-1}.\beta^{-1} = 0$.

21. Express the numbers $\frac{1+2i}{1-3i}$ in polar form.

22. Express the $\sin 120° - i\cos 120°$ in polar form.

23. Find the radius and centre of the circle $|z + 3 + i| = 5$ where z is a complex variable.

24. Show that the points representing the complex numbers
$(3 + 2i)$, $(2 - i)$ and $-7i$ are collinear.

25. A variable complex number $z = x + iy$ is such that $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$, show that $x^2 + y^2 - 1 = 0$.

TOPICWISE WARMUP TESTS

1. $\left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^6 + \left(\frac{1-i\sqrt{3}}{1+i\sqrt{3}}\right)^6 =$

(a) 2 (b) 1
(c) 0 (d) 4

2. The area of the triangle obtained by joining complex numbers z, iz and $z + iz$ in argand diagram is

[PET (Raj.), – 1998, 2000; MP – 1997; EAMCET – 1996; IIT – 1980; DCE – 1999; UPSEAT – 2002]

(a) $2|z|^2$ (b) $|z|^2/2$
(c) $|z|^2$ (d) none of these

3. If $\frac{z-1}{z+1}$ is purely imaginary number, then

[MP – 1998, 2002]

(a) $|z| = 1$ (b) $|z| > 1$
(c) $|z| < 1$ (d) none of these

4. If z_1, z_2, z_3 be three complex numbers such that
$|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$ then
$|z_1 + z_2 + z_3|$ is equal to

[IIT (Screening) – 2000]

(a) 1 (b) less than 1
(c) greater than 3 (d) 3

5. If 1, ω, ω^2 are cube roots of unity and $a + b + c = 0$ then $(a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3$ is equal to

(a) 0 (b) $3abc$
(c) $27abc$ (d) none of these

6. If z and ω are two nonzero complex numbers such that $|z| = |\omega|$ and arg (z) + arg $(\omega) = \pi$, then z is equal to

[IIT – 1995; AIEEE – 2002; JEE (Orissa) – 2004]

(a) ω (b) $-\omega$
(c) $\overline{\omega}$ (d) $-\overline{\omega}$

7. If z and ω are complex numbers such that $\bar{z} + i\overline{\omega} = 0$ and arg $(z\omega) = \pi$, then arg (z) is equal to

[AIEEE – 2004]

(a) $3\pi/4$ (b) $\pi/2$
(c) $\pi/4$ (d) $5\pi/4$

8. The polar form of

[Roorkee – 1981]

(a) $\sqrt{2}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$
(b) $\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$
(c) $\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$
(d) none of these

9. If z be multiplied by $1 + i$, then in complex plane vector z will be rotated at an angle

[ICS – 2001]

(a) 90° clockwise
(b) 45° clockwise
(c) 90° anti-clockwise
(d) 45° anti-clockwise

10. If w is imaginary cube root of unity, then

$$\frac{a + b\omega + c\omega^2}{c + a\omega + b\omega^2} + \frac{c + a\omega + b\omega^2}{a + b\omega + c\omega^2} + \frac{b + c\omega + a\omega^2}{b + c\omega^4 + a\omega^5}$$

is equal to

[Kerala (CEE) – 2003]

(a) 1 (b) −1 (c) 0 (d) ω

11. $\frac{1 + 2\omega + 3\omega^2}{2 + 3\omega + \omega^2} + \frac{2 + 3\omega + \omega^2}{3 + \omega + 2\omega^2}$ is equal

[Orissa (JEE) – 2003]

(a) 0 (b) − 1
(c) 2ω (d) -2ω

12. If α, β are roots of the equation $x^2 + x + 1 = 0$, then $\alpha^{2001} + \beta^{2001}$ equal to

[ICS (Pre) – 2004]

(a) −2 (b) 2
(c) 0 (d) − 1

13. The modulus and amplitude of $\frac{1 + 2i}{1 - (1 - i)^2}$ are

[CET (Karnataka) – 2005]

(a) 1, 0 (b) $2, \pi$
(c) 1/2, 0 (d) $3, \pi/2$

14. If $\omega \neq 1$ be a cube root of unity and
$(1 + \omega^2)^n = (1 + \omega^4)^n$
then the least + ive value of n is

[IIT (Screening) – 2004]

(a) 2 (b) 3
(c) 4 (d) 5

15. Let z_1 and z_2 be complex numbers, then $|z_1 + z_2|^2 + |z_1 - z_2|^2$ is equal to

[MP PET – 2006]

(a) $|z_1|^2 + |z_2|^2$ (b) $2(|z_1|^2 + |z_2|^2)$
(c) $2(z_1^2 + z_2^2)$ (d) $4z_1z_2$

16. If ωis an imaginary cube root of unity, then the value of $\sin\left[(\omega^{10} + \omega^{23})\pi - \frac{\pi}{4}\right]$ is:

[IIT (Screening) – 1994]

(a) $\frac{-\sqrt{3}}{2}$ (b) $\frac{-1}{\sqrt{2}}$
(c) $\frac{1}{\sqrt{2}}$ (d) $\frac{\sqrt{3}}{2}$

17. What is the value of
$[(-1 + i\sqrt{3})/2]^{10} + [(-1 - i\sqrt{3})/2]^{10}$

[NDA – 2007]

(a) 1 (b) −1
(c) 2 (d) 0

18. Real part of $\frac{1}{1 + \cos\theta + i\sin\theta}$ is

[MP PET – 2006]

(a) 1/3 (b) 1/5
(c) 1/2 (d) 1/8

19. Value of $|1 - \cos\alpha + i\sin - \alpha|$is

[MP PET – 2007]

(a) $2\sin\frac{\alpha}{2}$ (b) $2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

(c) $2\cos\frac{\alpha}{2}$ (d) $2\sin^2\frac{\alpha}{2}$

20. $\left[\dfrac{1+\cos(\pi/8)+i\sin(\pi/8)}{1+\cos(\pi/8)-i\sin(\pi/8)}\right]^8$ is equal to

[RPET – 2001]

(a) −1 (b) 0

(c) 1 (d) 2

21. If for complex numbers z_1 and z_2, arg $(z_1/z_2) = 0$, then $|z_1 - z_2|$ is equal to

(a) $|z_1| + |z_2|$ (b) $|z_1| - |z_2|$

(c) $||z_1| - |z_2||$ (d) 0

22. If $x = \cos\theta + i\sin\theta$, then $x^4 + \dfrac{1}{x^4} =$

[MP PET – 2006]

(a) $2\cos 4\theta$ (b) $2i\sin 4\theta$

(c) $-2i\sin 4\theta$ (d) $-2\cos 4\theta$

23. If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha$ then
$+ \sin\beta + \sin\gamma$
$\cos 3\alpha + \cos 3\beta + \cos 3\gamma$ is equal to

[Bihar (CEE) – 2000; EAMCET – 1995]

(a) $3\cos(\alpha+\beta+\gamma)$

(b) $\cos(3\alpha + 3\beta + 3\gamma)$

(c) $\cdots$ $(\alpha+\beta+\gamma)$

(d) $3\sin(\alpha+\beta+\gamma)$

24. A complex number z is such that arg $\left\{\dfrac{z-2}{z+2}\right\} = \dfrac{\pi}{2}$. The points representing this complex number will lie on

[MP PET – 2001]

(a) $\dfrac{x^2}{4} + \dfrac{y^2}{4\sqrt{3}} = 1$

(b) $y^2 = 4\sqrt{3}x$

(c) $x^2 + y^2 - 4y - 4 = 0$

(d) $x + y = 4\sqrt{3}$

25. If $|z^2 - 1| = |z|^2 + 1$, then z lies on

[AIEEE – 2004]

(a) a circle

(b) the imaginary axis

(c) the real axis

(d) an ellipse

TOPICWISE WARMUP TESTS: SOLUTION

1. (a) L.H.S. $= \left(\dfrac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^6 + \left(\dfrac{1-i\sqrt{3}}{1+i\sqrt{3}}\right)^6$

$$= \left(\dfrac{\frac{-1-i\sqrt{3}}{2}}{-\frac{-1+i\sqrt{3}}{2}}\right)^6 + \left(\dfrac{\frac{-1+i\sqrt{3}}{2}}{\frac{-1-i\sqrt{3}}{2}}\right)^6$$

$$= \left(\dfrac{-\omega^2}{-\omega}\right)^6 + \left(\dfrac{-\omega}{-\omega^2}\right)^6$$

$$= \left[\because \omega = \dfrac{-1+i\sqrt{3}}{2}, \omega^2 = \dfrac{-1-i\sqrt{3}}{2}\right]$$

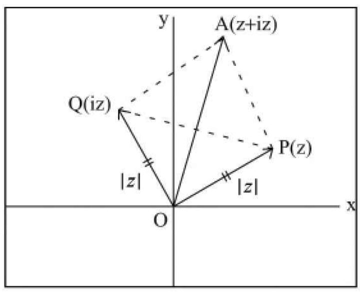
$$= \omega^6 + \dfrac{1}{\omega^6} = (\omega^3)^2 + \dfrac{1}{(\omega^3)^2} = 1 + \dfrac{1}{1}$$

$= 1 + 1 = 2$ = R.H.S. **Proved**

2. (b) Let $z = \overrightarrow{OP}$, $iz = \overrightarrow{OQ}$, $z + iz = \overrightarrow{OA}$.

Then obviously $OP \perp OQ$ and $OP = OQ$

$[\because |z| = = |iz|]$, amp (z) − amp $(iz) = -\pi/2]$
$\Rightarrow OPAQ$ is a square



Therefore, area of given $\Delta = \dfrac{1}{2}$ (area of the square)

$$= \dfrac{1}{2}|z|^2$$

3. (a) Let $\dfrac{z-1}{z+2} = \dfrac{i\lambda}{1}$ where ω is a real number.

$\Rightarrow \frac{2z}{2} = \frac{1+i\lambda}{1-i\lambda}$

$\Rightarrow |z| = \frac{1+i\lambda}{1-i\lambda}$

$= \frac{\sqrt{1+1^2}}{\sqrt{1+1^2}} =$

4. (a) $\because |z| = 1 \Rightarrow \frac{1}{2} = \bar{z}$

Hence $\frac{1}{z_1} = \bar{z}_1, \frac{1}{z_2} = \bar{z}_2, \frac{1}{z_3} = \bar{z}_3$

$\because \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$

$\Rightarrow |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 1$

$\Rightarrow |\overline{z_1 + z_2 + z_3}| = 1$

$\Rightarrow |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 1$ [$\because |z| = |\bar{z}|$].

5. (c) Let $x = a + b\omega + c\omega^2$, $y = a + b\omega^2 + c\omega$.

$x^3 + y^3 = (x+y)(x\omega^2 + y\omega)(x\omega + y\omega^2)$

$= [2a + (w + \omega^2)\, b + (\omega^2 + \omega)c]$

$[(\omega^2 + \omega)\, a + 2b + (\omega + \omega^2)\, c]$

$[(\omega^2 + \omega)\, a + (\omega^2 + \omega)\, b + 2c]$

$= (2a - b - c)(-a + 2b - c)(-a - b + 2c)$

$= (3a)\ (3b)\ (3c)$

[$\because a + b + c = 0$] $= 27abc$.

6. (d) Let $\omega = r(\cos\theta + i\sin\theta)$, then

$z = r\{\cos(\pi - \theta) + i\sin(\pi - \theta)\}$

[$\because |z| = |\omega|$ and $\arg(z) + \arg(\omega) = \pi$]

$= r(\cos\theta\ i\sin\theta) - r(\cos\theta\ i\sin\theta) = -\bar{\omega}$

7. (a) $= \bar{z} + i\bar{\omega} = 0$

$\Rightarrow z - i\bar{\omega} \Rightarrow z = i\omega$

$\Rightarrow \arg(z) - \arg(\omega) + \pi/2$ (1)

But.

$\arg(z\omega) = \pi \Rightarrow \arg(z) + \arg(\omega) = \pi$ (2)

$(1) + (2) \Rightarrow 2\arg(z) = 3\pi/2$

$\Rightarrow \arg(z) = 3\pi/4$

8. (b) $\frac{1+7i}{(2-i)^2} = \frac{1+7i}{3-4i} = \frac{(1+7i)(3+4i)}{25}$

$= -1 + i$

$= \sqrt{2}\,(\cos 3\pi/4 + i\sin 3\pi/4)$.

9. (d) amp $(1+i) = 45°$,

so z will be rotated at an angle 45° in anticlockwise.

10. (c) Exp $\frac{1}{\omega} + \frac{1}{\omega^2} + 1 = \omega^2 + \omega + 1 = 0$.

11. (c) Exp $\omega + \omega = 2\omega$.

12. $x = \omega, \omega^2$.

So exp. $\omega^{2001} + \omega^{4002} = 1 + 1^2 = 2$

13. (a) $\frac{1+2i}{1} - (1-i) = \frac{1+2i}{1+2i} = 1$

$\Rightarrow$ its modulus = 1,

amplitude = 0.

14. (b) Given $(1+\omega^2)^n = (1+\omega)^n \because \omega^4 = \omega$

or $(-\omega)^n = (-\omega^2)^n$ or $\omega^n = \omega^{2n}$

Clearly $n = 3$ is the least value of n satisfying above $\because \omega^3 = \omega^6 = 1$.

15. (b) Let z_1 and z_2 be complex numbers as follows

$z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$

$\therefore = |z_1 + z_2|^2 + |z_1 - z_2|^2$

$= (x_1 + x_2)^2 + (y_1 + y_2)^2 + (x_1 - x_2)^2 + (y_1 + y_2)^2$

$= x_1^1 + x_2^2 + 2x_1x_2 + y_1^2 + y_2^2 - 2y_1y_2$

$= 2\,(x_1^2 + y_1^2 + x_2^2 + y_2^2) = 2\,(|z_1|^2 + |z_2|^2)$

16. (c) Given $\sin\left[(\omega^{10} + \omega^{23})\,\pi - \frac{\pi}{4}\right]$

$= \sin\left[(\omega + \omega^2)\,\pi - \frac{\pi}{4}\right]$

$= \sin\left(-\pi - \frac{\pi}{4}\right) = \sin\left(\pi + \frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

17. (b) $(\omega)^{10} + (\omega^2)^{10} = +\,\omega^2 = -1$.

18. (c) $= \frac{1}{1 + \cos\theta + i\sin\theta}$

$$= \frac{1}{2\cos^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$\frac{1}{2\cos\frac{\theta}{2}\left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right]}$$

$$= \frac{1}{2\cos\frac{\theta}{2}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^{-1}$$

$$\frac{1}{2\cos\frac{\theta}{2}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

$$= \text{Real part} = \frac{\cos\frac{\theta}{2}}{2\cos\frac{\theta}{2}} = \frac{1}{2}.$$

19. (a) $|(1 - \cos\alpha) + i\sin\alpha|$

$$= \sqrt{(1-\cos\alpha)^2 + (\sin\alpha)^2}$$

$$\sqrt{\left(2\sin^2\frac{\alpha}{2}\right)^2 + \left(2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)^2}$$

$$= \sqrt{4\sin^2\frac{\alpha}{2} + \left(\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}\right)^2}$$

$$= 2\sin\frac{\alpha}{2}.$$

20. (a) $\left[\frac{1+\cos(\pi/8) + i\sin(\pi/8)}{1+\cos(\pi/8) - i\sin(\pi/8)}\right]^8$

$$= \left[\frac{2\cos^2(\pi/16) + 2i\sin(\pi/16)\cos(\pi/16)}{2\cos^2(\pi/16) - 2i\sin(\pi/16)\cos(\pi/16)}\right]^8$$

$$= \frac{[\cos(\pi/16) + i\sin(\pi/16)]^8}{[\cos(\pi/16) - i\sin(\pi/16)]^8}$$

$$= \left[\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right]^8 \left[\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right]^8$$

$= [\cos(\pi/16) + i\sin(\pi/16)]16$

$= \cos 16(\pi/16) + i\sin 16(\pi/16) = \cos\pi = -1.$

21. (c) We have $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2| \cos(\theta_1 - \theta_2)$ where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$ Since $\arg z_1 - \arg z_2 = 0$

$\therefore |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2| = (|z_1| - |z_2|)^2$

$\Rightarrow |z_1 - z_2| = ||z_1| - |z_2||.$

22. (a) $x = \cos\theta + i\sin\theta$

$x^4 = (\cos\theta + i\sin\theta)^4$

$x^4 = (\cos 4\theta + i\sin 4\theta)$ (By Demoivers theorem)

$\frac{1}{x^4} = x^{-4} = (\cos\theta + i\sin\theta)^{-4}$

$= \cos(-4\theta) + i\sin(-4\theta)$

$= \cos 4\theta - i\sin 4\theta$

$x^4 + \frac{1}{x^4} = \cos 4\theta + i\sin 4\theta + \cos 4\theta - i\sin 4\theta$

$x^4 + \frac{1}{x^4} = 2\cos 4\theta.$

23. (a) Let $x = (1, \alpha), y = (1, \beta), z = (1, \gamma)$ then

$x + y + z = \Sigma\cos\alpha + i\,\Sigma\sin\alpha = 0$

$\Rightarrow x^3 + y^3 + z^2 = 3xyz$

$\Sigma(\cos\alpha + i\sin\alpha)^3 = 3[\cos(\alpha+\beta+\gamma)] + i\sin(a+\beta+\gamma)$

$\Rightarrow \cos 3\alpha + \cos 3\beta + \cos\gamma$

$= 3\cos(\alpha+\beta+\gamma).$

24. (c) Let $z = x + iy$ $\arg\frac{(x+iy-2)}{x+iy-2} = \frac{\pi}{3}$

are $|(x-2) + iy|$ – are $|(x+2) + iy| = \frac{\pi}{3}$

Therefore, $4y = x^2 + y^2 - 4$

or $x^2 + y^2 - 4y - 4 = 0$ points representing the given complex number will lie on CIRCLE.

25. (b) $|z^2 - 1| = |z|^2 + 1$ Let $z = x + iy$

$\Rightarrow |x^2 - y^2 + 2ixy - 1| = x^2 + y^2 + 1$

$\Rightarrow (x^2 - y^2 - 1)^2 + 4x^2y^2 = z(x^2 + (y^2 + 1)^2$

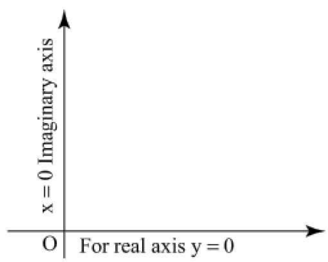
$\Rightarrow 4x^2y^2 = 4x^2(y^2 + 1)\; x = 0$

$|(x + iy)^2 - 1| = x^2 + y^2 + 1$

$\sqrt{(x^2 - y^2 - 1)^2 + (2ixy)^2} = x^2 + y^2 + 1$

Solving, we get $4x^2 = 0$

$x = 0.$



$\Rightarrow z$ lies on imaginary axis.

QUESTION BANK: SOLVE THESE TO MASTER

1. If a complex number satisfies z, $|z - 5i| \le 1$ and the argument of z is minimum then, $z =$

(a) $\frac{2}{5}\sqrt{6} + \frac{24}{5}i$ (b) $\frac{2}{5}\sqrt{6} - \frac{24}{5}i$

(c) $-\frac{2}{5}\sqrt{6} + \frac{24}{5}i$ (d) $-\frac{2}{5}\sqrt{6} - \frac{24}{5}i$

2. The conjugate of complex number $\frac{2-3i}{4-i}$ is

(a) $\frac{11+10i}{17}$ (b) $\frac{5+3i}{4-i}$

(c) $\frac{5-3i}{4-9i}$ (d) $\frac{11-10i}{17}$

3. The argument of the complex number $\frac{13-5i}{4-9i}$ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{5}$ (d) $\frac{\pi}{6}$

4. If the conjugate of $(x + iy)(1 - 2i)$ be $1 + i$ then

(a) $x = \frac{1}{5}$ (b) $y = \frac{1}{3}$

(c) $x + iy = \frac{1-i}{1+2i}$ (d) $x - iy = \frac{1-i}{1+2i}$

5. Value of $|1 - \cos\alpha + i\sin\alpha|$ is:

(a) $|2\sin\alpha/2|$ (b) $\left|2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right|$

(c) $2\cos\frac{\alpha}{2}$ (d) $\left|2\sin^2\frac{\alpha}{2}\right|$

6. z and ω are two nonzero complex numbers such that $|z| = |\omega|$ and Arg z + Arg ω = then z equals

(a) $\overline{\omega}$ (b) $-\overline{\omega}$ (c) ω (d) $-\omega$

7. If $z = x - iy$ and $z^{1/3} = p + iq$, then $\left(\frac{x}{p} + \frac{y}{p}\right)/(p^2 + q^2)$ is equal to

(a) -2 (b) -1 (c) 1 (d) 2

8. The conjugate of a complex number $\frac{1}{i-1}$ is then that complex number is

(a) $\frac{-1}{i-1}$ (b) $\frac{1}{i+1}$

(c) $\frac{-1}{i+1}$ (d) $\frac{1}{i-1}$

9. If $\left|\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}\right| = 1$, $|z_2| \ne 1$, then $|z_1| =$

(a) 4 (b) 2

(c) 1 (d) None of these

10. The value of $\left(\frac{1+i}{\sqrt{2}}\right)^8 + \left(\frac{1-i}{\sqrt{2}}\right)^8$ is equal to

(a) 4 (b) 6 (c) 8 (d) 2

11. If $\sqrt[3]{a - ib} = x - iy$, then $= \sqrt[3]{a + ib} =$

(a) $x + iy$ (b) $x - iy$

(c) $y + ix$ (d) $y - ix$

12. For any integer n, the argument of $z = \frac{(\sqrt{3}+i)^{4n+1}}{(1-i\sqrt{3})^{4n}}$ is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$

13. The maximum value of $|z|$ when z satisfies the condition $\left|z + \frac{2}{z}\right|$ is

(a) $\sqrt{3} - 1$ (b) $\sqrt{3} + 1$

(c) $\sqrt{3}$ (d) $\sqrt{2} + \sqrt{3}$

14. The real values of x and y for which the complex numbers $-3 + ix^2y$ and $x^2 + y + 4i$ are conjugate of each other are

(a) 1, 4 (b) 1, − 4

(c) − 1, − 6 (d) − 1, 4

15. Let z be any non-zero complex number. Then, arg (z) + arg $(\bar{z})$ is equal to

(a) π (b) $-\pi$ (c) 0 (d) $\pi/2$

16. Let z be a complex number. Then the angle between vectors z and iz is:

(a) π (b) 0

(c) $\pi/2$ (d) None of these

17. If $x = 3 + i$, then $x^3 - 3x^2 - 8x + 15 =$

(a) 6 (b) 10 (c) − 18 (d) − 15

18. If $|z| \le 4$, then the maximum value of $|iz + 3 - 4i|$ is equal to:

(a) 2 (b) 4

(c) 3 (d) 9

19. If $z_1 = (4, 5)$ and $z_2 = (-3, 2)$, then $\frac{z_1}{z_2}$ equals

(a) $\left(\frac{-23}{12}, \frac{-2}{13}\right)$ (b) $\left(\frac{2}{13}, \frac{-23}{12}\right)$

(c) $\left(\frac{-2}{13}, \frac{-23}{13}\right)$ (d) $\left(\frac{-2}{13}, \frac{23}{13}\right)$

20. If $|z_1| = |z_2|$ and arg $\left(\frac{z_1}{z_2}\right) = \pi$, then $z_1 + z_2$ is equal to

(a) 0
(b) purely imaginary
(c) purely real
(d) none of these

21. $(\cos 2\theta + i \sin 2\theta) - 5 (\cos 3\theta - i \sin 3\theta)^6 (\sin - i \cos\theta)^3$ in the form of $A + iB$ is
(a) $(\cos 25\theta + i \sin 25\theta)$
(b) $i (\cos 25\theta + i \sin 25\theta)$
(c) $i (\cos 25\theta - i \sin 25\theta)$
(d) $(\cos 25\theta - i \sin 25\theta)$

22. The value of $\sum_{k=1}^{6} \left(\sin \frac{2\pi k}{7} - i \cos \frac{2\pi k}{7}\right)$ is
(a) -1 (b) 0 (c) $-i$ (d) i

23. If $i = \sqrt{-1}$, then $4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$ is equal to
(a) $1 - i\sqrt{3}$ (b) $-1 + i\sqrt{3}$
(c) $i\sqrt{3}$ (d) $-i\sqrt{3}$

ANSWERS

Lecture–1: Unsolved Objective Problems (Identical Problems For Practice): For Improving Speed With Accuracy

1.	(c)	4.	(c)	7.	(b)	10.	(a)
2.	(b)	5.	(a)	8.	(b)		
3.	(b)	6.	(d)	9.	(b)		

Lecture–1: Work Sheet: To Check Preparation Level

1.	(b)	5.	(b)	9.	(a)	13.	(b)
2.	(d)	6.	(a)	10.	(b)		
3.	(d)	7.	(a)	11.	(c)		
4.	(d)	8.	(a)	12.	(d)		

Lecture–2: Unsolved Objective Problems (Identical Problems For Practice): For Improving Speed With Accuracy

1.	(c)	6.	(a)	11.	(a)	16.	(a)
2.	(c)	7.	(d)	12.	(d)	17.	(c)
3.	(b)	8.	(d)	13.	(a)	18.	(b)
4.	(a)	9.	(a)	14.	(c)		
5.	(b)	10.	(c)	15.	(b)		

Lecture–2: Work Sheet: To Check Preparation Level

1.	(d)	5.	(c)	9.	(a)	13.	(b)
2.	(d)	6.	(b)	10.	(b)		
3.	(d)	7.	(b)	11.	(b)		
4.	(b)	8.	(c)	12.	(a)		

Lecture–3: Unsolved Objective Problems (Identical Problems For Practice): For Improving Speed With Accuracy

1.	(b)	8.	(d, a)	15.	(a)	22.	(c)
2.	(c)	9.	(d)	16.	(c)	23.	(a)
3.	(c)	10.	(c)	17.	(b)	24.	(d)
4.	(a)	11.	(a)	18.	(a)	25.	(c)
5.	(b)	12.	(b)	19.	(d)	26.	(b)
6.	(b)	13.	(b)	20.	(a)	27.	(c)
7.	(c)	14.	(b)	21.	(c)		

Lecture–3: Work Sheet: To Check Preparation Level

1.	(c)	5.	(a)	9.	(d)	13.	(b)
17.	(d)	2.	(a)	6.	(d)	10.	(a)
14.	(a)	3.	(a)	7.	(d)	11.	(c)
15.	(b)	4.	(d)	8.	(a)	12.	(a)
16.	(c)						

Lecture–4: Unsolved Objective Problems (Identical Problems For Practice): For Improving Speed With Accuracy

1.	(d)	3.	(c)	5.	(b)	7.	(d)
2.	(d)	4.	(a)	6.	(c)	8.	(a)

Lecture–4: Work Sheet: To Check Preparation Level

1.	(b)	3.	(d)	5.	(a)
2.	(a)	4.	(b)		

Lecture–5: Mental Preparation Test

(5) $|z| = 2$

(6) $x^2 + y^2 - 6x + 8y - 24 = 0$

(7) $\sqrt{2}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$

(10) (i) $\pm[(a+b) - (a-b)i]$ (ii) $\pm(a+i)$

(12) $\frac{3}{2} + \frac{1}{2}i$

(14) $x = \frac{14}{15}, y = \frac{1}{5}$

(17) -160

(21) $\frac{1}{\sqrt{2}}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

(22) $\cos 30^\circ + i\sin 30^\circ$.

(23) radius = 5, centre = $(-3, -1)$.

QUESTION BANK: SOLVE THESE TO MASTER

1.	(a)	2.	(a)	3.	(b)
4.	(c)	5.	(a)	6.	(b)
7.	(a)	8.	(c)	9.	(a)
10.	(a)	11.	(b)	12.	(c)
13.	(d)	14.	(b)	15.	(c)
16.	(c)	17.	(d)	18.	(d)
19.	(c)	20.	(a)	21.	(c)
22.	(d)	23.	(c)		

Part C

Quadratic Equations

This page is intentionally left blank.

LECTURE

1 Formation of Quadratic Equations

BASIC CONCEPTS

1. General Form of Quadratic Equation

An equation of the form $Ax^2 + Bx + C = 0$ where $A \neq 0$ and x is A variable is called A quadratic equation and A, B, C are constants (real or complex) which are called coefficient of x^2, coeffi cient of x and constant term, respectively.

2. Pure Quadratic Equation

In pure quadratic equation fi rst degree term does not appear. e.g., $Ax^2 + C = 0$, $x^2 - 4 = 0$ etc

3. Roots of an Equation

The values of x that satisfy the given quadratic equation are called the roots of the given equation.

Note:

(i) Every polynomial equation of degree n has exactly n roots. (real or complex).

(ii) If A quadratic equation has more than two roots then it must be an identity, i.e., $A = B = C = 0$.

(iii) If α is any one root of the quadratic equation $Ax^2 + Bx + C = 0$, then $A\alpha^2 + B\alpha + C = 0$ and $(x - \alpha)$ is one factor of the expression $Ax^2 + Bx + C$.

(iv) If α and β are the roots of the equation $Ax^2 + Bx + C = 0$ then $Ax^2 + Bx + C = A(x - \alpha)(x - \beta)$

or $x^2 + \frac{B}{A}x + \frac{C}{A} = x^2 - (\alpha + \beta)x + \alpha\beta$

Hence $\alpha + \beta = -\frac{B}{A}$, $\alpha\beta = \frac{C}{A}$

(v) The quadratic equation whose roots are α and β is given by $(x - \alpha)(x - \beta) = 0$ or $x^2 - (\alpha + \beta)x + \alpha\beta = 0$ i.e., $x^2 -$ (sum of the roots) x + product of roots = 0.

(vi) Difference of roots i.e., $|\alpha - \beta|$

$$= \sqrt{(\alpha + \beta)^2 - 4\alpha\beta} = \frac{\sqrt{B^2 - 4AC}}{A}$$

(vii) **Every** equation of odd degree has at least one real root.

4. Some Important Formulas Connected with Roots of Quadratic Equation

$Ax^2 + Bx + C = 0$

If α and β are the roots of the equation $Ax^2 + Bx + C = 0$, then

$$\alpha + \beta = -\frac{B}{A} = -\left(\frac{\textit{Coefficient of } x}{\textit{Coefficient of } x^2}\right)$$

$$\alpha\beta = \frac{C}{A} = \frac{\textit{Constant term}}{\textit{Coefficient of } x^2}$$

(i) If $C = 0$ then $\alpha\beta = 0$, i.e., one of the two roots is zero.

(ii) If $B = 0$ then $\alpha + \beta = 0$, i.e., both roots are equal in magnitude but opposite in signs.

(iii) If $B = C = 0$ then both roots are zero.

(iv) If $A = 0$, then the quadratic equation reduces to linear equation $Bx + C = 0$; $x = -\frac{C}{B}$.

(v) $A = B = C = 0$, then the quadratic equation $Ax^2 + Bx + C = 0$ becomes an identity.

(vi) Roots are reciprocal of each other, if $\alpha\beta = 1 = \frac{C}{A}$, i.e., $A = C$.

5. Value of Symmetric Functions

Let α and β are the roots of the equation $Ax^2 + Bx + C = 0$, then

$\alpha + \beta = -\frac{B}{A}$ and $\alpha\beta = \frac{C}{A}$.

Formula for finding the values of following symmetric functions:

(i) $\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta)$

(ii) $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

(iii) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}$

(iv) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

(v) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$

(vi) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

(vii) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3 + \beta^3}{\alpha\beta}$

(viii) $(A\alpha + B)^{-1} = -\frac{\alpha}{C}$; $(A\beta + B)^{-1} = \frac{-\beta}{C}$

(ix) $\alpha^5 + \beta^5 = (\alpha^2 + \beta^2)(\alpha^3 + \beta^3) - \alpha^2\beta^2(\alpha + \beta)$

(x) $(1 + \alpha + \alpha^2)(1 + \beta + \beta^2) = 1 + (\alpha + \beta) + (\alpha^2 + \beta^2) + \alpha\beta + \alpha^2\beta^2 + \alpha\beta(\alpha + \beta)$

SOLVED SUBJECTIVE ROBLEMS: (CBSE/STATE BOARD): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. If α, β be the roots of the equation $ax^2 + bx + c = 0$. then find the value of the following (i) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$ (ii) $\frac{\alpha^3}{\beta} + \frac{\beta^3}{\alpha}$.

Solution

α, β are roots of the equation $ax^2 + bx + c = 0$.

$\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

(i) $$\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3 + \beta^3}{\alpha\beta} = \frac{(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)}{\alpha\beta}$$

$$= \frac{\left(-\frac{b}{a}\right)^3 - 3\frac{c}{a}\left(-\frac{b}{a}\right)}{c/a} = \left(\frac{-b^3 + 3abc}{a^3}\right) \times \frac{a}{c}$$

$$\therefore \frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{b(3ac - b^2)}{a^2c}$$

Ans.

(ii) $$\frac{\alpha^3}{\beta} + \frac{\beta^3}{\alpha} = \frac{\alpha^4 + \beta^4}{\alpha\beta} = \frac{(\alpha^2 + \beta^2)^2 - 2\alpha^2}{\alpha\beta}$$

$$= \frac{[(\alpha + \beta)^2 - 2\alpha\beta]^2 - 2\alpha^2\beta^2}{\alpha\beta}$$

$$= \frac{\left[\left(-\frac{b}{a}\right)^2 - 2\frac{c}{a}\right]^2 - 2\left(\frac{c}{a}\right)^2}{c/a}$$

$$= \frac{\left(\frac{b^2}{a^2} - \frac{2c}{a}\right)^2 - \frac{2c^2}{a^2}}{c/a}$$

$$= \frac{\left[\frac{(b^2 - 2ac)^2}{a^4} - \frac{2c^2}{a^2}\right]}{c/a}$$

$$= \frac{(b^4 + 4a^2c^2 - 4ab^2c - 2a^2c^2)}{a^4} \times \frac{a}{c}$$

$$\frac{\alpha^3}{\beta} + \frac{\beta^3}{\alpha} = \frac{b^4 - 4ab^2c + 2a^2c^2}{a^3c}$$

Ans.

2. If 8 and 2 are roots of the quadratic equation tion $x^2 + ax + \beta = 0$ and 3, 3 are roots of the quadratic equation $x^2 + ax + b = 0$. Then, find the roots of the quadratic equation $x^2 + ax + b = 0$.

Solution

Since, 8 and 2 are roots of the quadratic equation $x^2 + ax + \beta = 0$

Therefore, $8 + 2 = -a/1 \Rightarrow a = -10$

Again, 3, 3 are roots of the quadratic equation $x^2 + ax + b = 0$

Therefore, $3 \times 3 = b \Rightarrow b = 9$

Therefore, $x^2 + ax + b = x^2 - 10x + 9 = 0$, [by substituting values of a and b]

$\Rightarrow x^2 - 9x - x + 9 = 0 \Rightarrow x(x - 9) - 1(x - 9) = 0$

$\Rightarrow (x - 9)(x - 1) = 0 \Rightarrow x - 9 = 0, x - 1 = 0$

$\Rightarrow x = 9, x = 1$

Thus, 1 and 9 are required roots.

3. If the difference of the roots of equation $x^2 - lx + m = 0$ is 1, then prove that $l^2 = 1 + 4m$.

Solution

Let $\alpha, \alpha + 1$ be the roots of equation $x^2 - lx + m = 0$

Therefore,

sum of the roots $= \alpha + \alpha + 1 = -(-l) = l$ (1)

and Product of the roots $= \alpha(\alpha + 1) = m$ (2)

Form equation (1), $\alpha + \alpha + 1 = l$

$\Rightarrow 2a = l - 1 \therefore a = \frac{l-1}{2}$

Putting the value of α in equation (2), we get,

$\left(\frac{l-1}{2}\right)\left(\frac{l-1}{2} + 1\right) = m$

$\Rightarrow \left(\frac{l-1}{2}\right)\left(\frac{l+1}{2}\right) = m$

$\Rightarrow \frac{l^2 - 1^2}{4} = m$

Therefore, $l^2 = 4m + 1$

Proved

4. If α, β are roots of the quadratic equation $ax^2 + 2bx + c = 0$, then prove that $\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} = -\frac{2b}{\sqrt{ac}}$

[BITS RANCHI – 1990]

Solution

Since α, β are roots of the quadratic equation $ax^2 + 2bx + c = 0$.

Therefore, $\alpha + \beta = -\frac{2b}{a}$ and $\alpha\beta = \frac{c}{a}$

Thus

$$\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} = \frac{\alpha + \beta}{\sqrt{\alpha\beta}} = \frac{-2b}{a} \times \sqrt{\frac{a}{c}} = \frac{-2b}{\sqrt{ac}}$$

Proved

5. If the roots of the equation $x^2 - (1 + m^2)x + \frac{1 + m^2 + m^4}{2} = 0$ are α, β then prove that $\alpha^2 + \beta^2 = m^2$

[MPPET – 2008, NDA – 2009]

Solution

Since, roots of given equation

$x^2 - (1 + m^2)x + \frac{1 + m^2 + m^4}{2} = 0$ are α, β

$\therefore \alpha + \beta = 1 + m^2$ and $\alpha.\beta = \frac{1 + m^2 + m^4}{2}$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$(1 + m^2)^2 - 2\frac{(1 + m^2 + m^4)}{2}$

$= 1 + 2m^2 + m^4 - 1 - m^2 - m^4$

$= m^2$

Proved

6. If the roots of the equation $x^2 - 3ax + a^2 = 0$ are α, β and $\alpha^2 + \beta^2 = 1.75$, then find the value of a.

Solution

As α and β are roots of equation $x^2 - 3ax + a^2 = 0$

Therefore $\alpha + \beta = 3\alpha$ and $\alpha\beta = \alpha^2$

given, $\alpha^2 + \beta^2 = 1.75$

$\Rightarrow (\alpha + \beta)^2 - 2\alpha\beta = 1.75$

$\Rightarrow (3\alpha)^2 - 2\alpha^2 = 1.75$

$\Rightarrow 9\alpha^2 - 2\alpha^2 = 1.75$

$\Rightarrow \quad 7\alpha^2 = 1.75$

$\Rightarrow \quad \alpha^2 = 0.25$

$\Rightarrow \quad \alpha = \pm 0.5$

7. If α, β are the roots of the equation $x^2 + x + 1 = 0$, then prove that the equation whose roots are $m\alpha + n\beta$ and $m\beta + n\alpha$ is $x^2 + (m + n)x + (m^2 - mn + n^2) = 0$.

Solution

α and β are roots of the equation $x^2 + x + = 0$

$\therefore \quad \alpha + \beta = -1 \alpha$ and $\beta = 1$

Now, adding the given roots

$= (m\alpha + n\beta) + (m\beta + n\alpha)$

$= (m\alpha + m\beta) + (n\alpha + n\beta)$

$= m(\alpha + \beta) + n(\alpha + \beta)$

$= (\alpha + \beta)(m + n)$

$= -(m + n) \qquad (\because \alpha + \beta = -1)$

and their product $= (m\alpha + n\beta) \times (m\beta + n\alpha)$

$= (m^2 + n^2)\alpha\beta + m\,n(\alpha^2 + \beta^2)$

$= (m^2 + n^2)\alpha\beta + m\,n[(\alpha^2 + \beta)^2 - 2\alpha\beta]$

$= (m^2 + n^2)1 + m\,n[(-1)^2 - 21]$

$= (m^2 + n^2) + m\,n(-1)$

$= (m^2 + n^2) - m\,n$

$= m^2 - m\,n + n^2$

Therefore, required equation is

$x^2 -$ (sum of the roots)$x +$ product of roots $= 0$

$x^2 - \{-(m + n)\}x + m^2 - m\,n + n^2 = 0$

$x^2 + (m + n)x + m^2 - mn + n^2 = 0$

Proved

(8) Find the value of k for which the roots α, β of the equation $x^2 - 6x + k = 0$ satisfy the relation $3\alpha + 2\beta = 20$.

Solution

Clearly, $\alpha + \beta = -(-6) = 6$ and $\alpha\beta = k$.

Now $3\alpha + 2\beta = 20$

$\Rightarrow \alpha + 2(\alpha + \beta) = 20$

$\Rightarrow \alpha + 2 \times 6 = 20$

$\Rightarrow \alpha = 8.$

but $\alpha + \beta = 6, \alpha = 8$

$\Rightarrow \beta = -2$

$\therefore k = \alpha\beta = 8 \times (-2) = -16.$

UNSOLVED SUBJECTIVE PROBLEMS (CBSE/STATE BOARD): TO GRASP THE TOPIC, SOLVE THESE PROBLEMS

Exercise I

1. If $px^2 - qx + r = 0$ has α and β as its roots, evaluate

$\alpha^3\beta + \beta^3\alpha$

2. If α and β are the roots of the equation $ax^2 + bx + c = 0$, find the equation whose roots are $\frac{1}{a\alpha + b}, \frac{1}{a\beta + b}$.

3. If α, β be the roots of $ax^2 + 2bx + c = 0$ & $\alpha + \delta, \beta + \delta$ be roots of $Ax^2 + 2bx + c = 0$, then prove that

$\frac{b^2 - ac}{B^2 - AC} = \left(\frac{a}{A}\right)^2.$

4. If α and β be the roots of the equation $px^2 + qx + r = 0$. Hence, obtain the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

5. Sum of roots of the quadratic equation is 2 and sum of cube of roots is 98. Find the equation of roots.

6. If α, β are roots of the quadratic equation $x^2 + px + p^2 + q = 0$, then prove that $a^2 + \alpha\beta + \beta^2 + q = 0$.

7. If p and q are roots of the quadratic equation $x^2 + px + q = 0$ then find the value of p and q.

8. If roots of equation $\frac{1}{x + p} + \frac{1}{x + q} = \frac{1}{r}$ are equal in magnitude but opposite in sign, then show that $p + q = 2r$ and prove that product of roots is $-\frac{p^2 + q^2}{2}$.

Exercise II

1. If roots of equation $\frac{a}{x - a} + \frac{b}{x - b} = 1$ are equal in magnitude but opposite in sign, then prove that $a + b = 0$.

2. If α and β are the roots of $2x^2 - 5x + 7 = 0$, find out the equation whose roots are $2\alpha + 3\beta$ and $3\alpha + 2\beta$.
3. If α and β be roots of the equation $ax^2 + bx$ c = 0 then, form the equation whose roots are $\alpha + \frac{1}{\beta}$ and $\beta + \frac{\beta}{\alpha}$
4. If α and β are roots of the quadratic equation $x^2 - px + 36 = 0$ and $\alpha^2 + \beta^2 = 9$, then find the value of p.
5. If one root of the equation $5x^2 + 13x + k = 0$ is reciprocal of other, then find the value of k.
6. If roots of equation $2x^2 + 3(k - 2)x + 4 - k = 15x$ are same but of opposite sign, then fi nd the value of k.
7. If 8 and 2 are roots of the quadratic equation $x^2 + \alpha x + \beta = 0$ and 3, 3 are roots of the quadratic equation $x^2 + \alpha x + b = 0$. Then, find the roots of the quadratic equation $x^2 + ax + b = 0$.

ANSWERS

Exercese I

1. $\frac{r(q^2-2pr)}{p3}$
2. $acx^2 - bx + 1 = 0$
4. $prx^2 + (2rp - q^2)x + rp = 0$
5. $x^2 - 2x - 15 = 0$
7. $p = 1$, and $q = -2$

Exrcise II

2. $2x^2 - 25x + 82 = 0$
3. $acx^2 + b(c + a)x + (c + a)^2 = 0$
4. $p = +9$
5. $k = 5$
6. $k = 7$
7. 9, 1

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. f the sum of the roots of the quadratic equation $ax^2 + bx + c = 0$ is equal to the sum of the square of their reciprocals, then a/c, b/a, c/b are in

[AIEEE – 2003; DCE – 2000]

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

Solution

(c) As given, if α, β be the roots of the quadratic equation,

then, $a + \beta = \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha^2 b^2}$

$\Rightarrow -\frac{b}{a} = \frac{(b^2/a^2) - (2c/a)}{c^2/a^2} = \frac{b^2 - 2ac}{c^2}$

$\Rightarrow \frac{2a}{c} = \frac{b^2}{c^2} + \frac{b}{a} \Rightarrow \frac{(ab^2 + bc^2)}{ac^2} = \frac{2a}{c}$

$\Rightarrow 2a^2c = ab^2 + bc^2$

$\Rightarrow \frac{2a}{b} = \frac{b}{c} + \frac{c}{a}$

$\Rightarrow \frac{c}{a}, \frac{a}{b}, \frac{b}{c}$ are in A.P

$\Rightarrow \frac{a}{c}, \frac{b}{a}, \frac{c}{b}$ are in H.P

2. If α and β are the roots of $6x^2 - 6x + 1 = 0$, then the value of $\frac{1}{2}[a + b\alpha + c\alpha^2 + d\alpha^3] + \frac{1}{2}[a + b\beta + c\beta^2 + d\beta^3]$ is

(a) $\frac{1}{4}(a + b + c + d)$
(b) $\frac{a}{1} + \frac{b}{2} + \frac{c}{3} + \frac{d}{4}$
(c) $\frac{a}{2} + \frac{b}{2} + \frac{c}{3} + \frac{d}{4}$
(d) none of these

[RPET – 2000]

Solution

(b) α, β are the roots of the equation $6x^2 - 6x + 1 = 0 \Rightarrow \alpha + \beta = 1$, $\alpha\beta = 1/6$

Therefore, $\frac{1}{2}[a + b\alpha + c\alpha^2 + d\alpha^3]$

$+\frac{1}{2}[a + b\beta + c\beta^2 + d\beta^3]$

$= a + \frac{1}{2}b(\alpha + \beta) + \frac{1}{2}c(\alpha^2 + \beta^2)$

$+\frac{1}{2}d(\alpha^3 + \beta^3)$

$= a + \frac{1}{2}b + \frac{1}{2}c[(\alpha + \beta)^2 - 2\alpha\beta]$

$+\frac{1}{2}d[(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)]$

$= a + \frac{b}{2} + \frac{1}{2}c\left[(1)^2 - 2.\frac{1}{6}\right]$

$+\frac{1}{2}d\left[(1)^3 - 3.\frac{1}{6}\right]$

$= \frac{a}{1} + \frac{b}{2} + \frac{c}{3} + \frac{d}{4}.$

3. If α, β are the roots of $ax^2 + bx + c = 0$ and $2\alpha + \beta, \alpha^2 + \beta^2, \alpha^3 + \beta^3$ are in G.P., where $\Delta = b^2 - 4ac$, then:

[IIT (Screening) – 2005]

(a) $\Delta \neq 0$ (b) $b\Delta = 0$
(c) $cb \neq 0$ (d) $c\Delta = 0$

Solution

Step 1:

Given $\alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{b^2}{a^2} - \frac{2c}{a}$

$= \frac{b^2 - 2ac}{a^2}$

$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$= -\frac{b^3}{a^3} - 3\left(+\frac{c}{a}\right)\left(-\frac{b}{a}\right)$

$= -\frac{b^3}{a^3} + \frac{3bc}{a^2} = \frac{-b^3 + 3abc}{a^3}$

Step 2: Also given

(d) $(\alpha^2 + \beta^2)^2 = (\alpha + \beta)(\alpha^3 + \beta^3)$

$\left(\frac{b^2 - 2ac}{a^2}\right)^2 = \left(\frac{-b}{a}\right)\left(\frac{-b^3 + 3abc}{a^3}\right)$

$\Rightarrow 4a^2c^2 = acb^2$

$\Rightarrow ac(b^2 - 4ac) = 0$ as $a = 0$

$\Rightarrow c\Delta = 0$

4. In a triangle PQR, $\angle R = \frac{\pi}{2}$. If $\tan\left(\frac{P}{2}\right)$ and $\tan\left(\frac{Q}{2}\right)$ are the roots of $ax^2 + bx + c = 0$, $a \neq 0$, then:

[AIEEE – 2005]

(a) $b = a + c$ (b) $b = c$
(c) $c = a + b$ (d) $a = b + c$

Solution

(c) If α and β are the roots of the equation $ax^2 + bx + c = 0$, then $\alpha + \beta = \frac{-b}{a}$ = and $\alpha\beta = \frac{c}{a}$.

Since, $\tan\left(\frac{P}{2}\right)$ and $\tan\left(\frac{Q}{a}\right)$ are roots of equaction $ax^2 + bx + c = 0$.

$\tan\frac{P}{2} + \tan\frac{Q}{2} = -\frac{b}{a}$

Therefore,

and $\tan\frac{P}{2}\tan\frac{Q}{2} = \frac{c}{a}$

Also, $\frac{P}{2} + \frac{Q}{2} + \frac{R}{2} = \frac{\pi}{2}$

(As P, Q, R are angles of a triangle)

$\Rightarrow \frac{P + Q}{2} = \frac{\pi}{2} - \frac{R}{2} = \frac{P + Q}{2} = \frac{\pi}{4}$

Now, $\tan\left(\frac{P}{2} + \frac{Q}{2}\right) = 1$

$\Rightarrow \tan\frac{\frac{P}{2} + \tan\frac{Q}{2}}{1 - \tan\frac{P}{2}\tan\frac{Q}{2}} = 1$

$\Rightarrow \frac{-\frac{b}{a}}{1 - \frac{c}{a}} = 1 \Rightarrow -\frac{b}{a} = 1 - \frac{c}{a}$

$\Rightarrow -b = I - c$

$\Rightarrow c = a + b$

5. If the roots of the equation $x^2 - 5x + 16 = 0$ are α and β, and the roots of equation $x^2 + px + q = 0$ are $\alpha^2 + \beta^2, \alpha\beta/2$, then:

[MP PET – 2001]

(a) $p = 1, q = -56$
(b) $p = -1, q = -56$
(c) $p = 1, q = 56$
(d) $p = -1, q = 56$

Solution

(b) Since roots of the equation $x^2 - 5x + 16 = 0$ are α, β.

$\Rightarrow \alpha + \beta = 5$ and $\alpha\beta = 16$

and $\alpha^2 + \beta^2 + \frac{\alpha\beta}{2} = -p$

$\Rightarrow (\alpha + \beta)^2 - 2\alpha\beta + \frac{\alpha\beta}{2} = -p$

$\Rightarrow 25 - 32 + 8 = -p$

$\Rightarrow p = -1$ and $(\alpha^2 + \beta^2)\left(\frac{\alpha\beta}{2}\right) = q$

$\Rightarrow [(\alpha + \beta)^2 - 2\alpha\beta]\left[\frac{\alpha\beta}{2}\right] = q$

$\Rightarrow q = [25 - 32]\,\frac{16}{2}$
$= -56$

So, $p = -1$, $q = -56$

6. If the roots of the equation $\frac{x^2 - bx}{ax - c} = \frac{m-1}{m+1}$ are negatives of each other then is

(a) $\frac{a+b}{a-b}$ (b) $\frac{a-b}{a+b}$

(a) $\frac{a-b}{a+b}$ (b) $\frac{a+b}{a-b}$

[MP – 1996; PET (Raj.) 1988 – 2001]

Solution

(b) The equation can be written as,

$(m + 1)(x^2 - bx) - (m - 1)(ax - c) = 0$

$\Rightarrow (m + 1)x^2 - (bm + b + am - a)\,x + c(m - 1) = 0$

Its roots are negatives of each other.

Therefore, Coefficient of $x = 0$

$\Rightarrow bm + b + am - a = 0 \Rightarrow m = \frac{a-b}{a+b}$

7. Ramesh and Mahesh solve a quadratic equation. Ramesh reads its constant term wrongly and finds its roots as 8 and 2 whereas Mahesh reads the coeffi ient of x wrongly and finds its roots as 11 and -1. The correct roots of the equation are

[BIHAR (CEE) – 1999]

(a) 11, 1 (b) − 11, 1

(c) 11, −1 (d) none of these

Solution

(c) Let equation be $x^2 + bx + c = 0$.

Ramesh reads c wrongly but b correctly, so $8 + 2 = -b \Rightarrow b = -10$.

Mahesh reads b wrongly but c correctly, so $(11).(-1) = c \Rightarrow c = -11$

correct equation is $x^2 - 10x - 11 = 0$,

Its roots are 11, − 1.

8. If α and β are the roots of the $ax^2 + bx + c = 0$ and if $px^2 + qx + r = 0$ has roots $\frac{1-\alpha}{\alpha}$ and $\frac{1-\beta}{\beta}$ then $r =$

(a) $a + 2b$ (b) $a + b + c$

(c) $ab + bc + ca$ (d) abc

[EAMCET – 2007]

Solution

(b) $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$ (1)

The quadratic equation whose roots are $\frac{1-\alpha}{\alpha}, \frac{1-\beta}{\beta}$ is

$$x^2 - \left(\frac{1-\alpha}{\alpha}, \frac{1-\beta}{\beta}\right)x + \frac{1-\alpha}{\alpha}.\frac{1-\beta}{\beta} = 0$$

Using (1) $x^2 + \frac{b+2c}{c}x + \frac{a+b+c}{c} = 0$

or $cx^2 + (b + 2c)\,x + (a + b + c) = 0$ comparing with $px^2 + qx + r = 0$

$r = a + b + c$.

9. f A is the A.M. of the roots of the equation $x^2 - 2ax + b = 0$ and G is the GM of the roots of the equation $x^2 - 2Bx + a^2 = 0$, then

[UPSEAT – 2001]

(a) $A > G$ (b) $A \neq G$

(c) $A = G$ (d) none of these

Solution

(c) Sum of the roots of $x^2 - 2ax + b^2 = 0$ is $2a$

Therefore, A = A.M. of the roots $= a$

Product of the roots of $x^2 - 2bx + a^2 = 0$ is a^2

Therefore, G.M. of the roots is $G = a$

Thus, $A = G$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. If a and b are roots of $x^2 - px + q = 0$, then $\frac{1}{a} + \frac{1}{b} =$

(a) $1/p$ (b) $1/q$
(c) $1/2p$ (d) p/q

[Orissa JEE – 2004]

2. If α, β are the roots of the equation $x^2 + 2x + 4 = 0$, then $\frac{1}{\alpha^3} + \frac{1}{\beta^3}$ is equal to

[Kerala (Engg.) – 2002]

(a) $-1/2$ (b) $1/2$
(c) 32 (d) $1/4$

3. If the sum of the roots of the equation $ax^2 + bx + c = 0$ be equal to the sum of their squares, then

(a) $a(a + b) = 2bc$
(b) $c(a + c) = 2ab$
(c) $b(a + b) = 2ac$
(d) $b(a + b) = ac$

4. If the sum of the roots of the equation $\lambda x^2 + 2x + 3\lambda = 0$ be equal to their product, then $\lambda =$

(a) 4 (b) -4
(c) 6 (d) none of these

5. If α and β are the roots of the equation $x^2 - a(x + 1) - b = 0$, then $(\alpha + 1)(\beta + 1) =$

[BIT (Mesra) 2000; HCET– 2001]

(a) b (b) $-b$
(c) $1 - b$ (d) $b - 1$

6. If the product of roots of the equation $mx^2 + 6x + (2m - 1) = 0$ is -1, then the value of m is

[Pb. CET – 1990]

(a) 1 (b) -1
(c) $1/3$ (d) $-1/3$

7. Difference between the corresponding roots of $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ is same and $a \neq b$, then

(a) $a + b + 4 = 0$ (b) $a + b - 4 = 0$
(c) $a - b - 4 = 0$ (d) $a - b + 4 = 0$

[AIEEE – 2002]

8. The equation whose roots are $\frac{1}{3+\sqrt{2}}$ and $\frac{1}{3-\sqrt{2}}$ is

(a) $7x^2 - 6x + 1 = 0$
(b) $6x^2 - 7x + 1 = 0$
(c) $x^2 - 6x + 7 = 0$
(d) $x^2 - 7x + 6 = 0$

[MPPET – 1994]

9. If p and q are the roots of $x^2 + px + q = 0$, then

(a) $p = 1, q = -2$ (b) $p = -2, q = 1$
(c) $p = 1, q = 0$ (d) $p = -2, q = 0$

[IIT-1995; AIEEE–2002; UPSEAT–2003, RPET – 2001]

10. If the root of $x^2 - bx + c = 0$ are two consecutive integers, then $b^2 - 4c$ is

[AIEEE – 2005]

(a) 1 (b) 2 (c) 3 (d) 4

11. If the roots of the equation $\frac{1}{x+p} + \frac{1}{x+q} = \frac{1}{r}$ are equal in magnitude but opposite in sign, then the product of the roots will be

[IIT–1967; RPET – 1999]

(a) $\frac{p^2+q^2}{2}$ (b) $\frac{(p^2+q^2)}{2}$
(a) $\frac{p^2-q^2}{2}$ (b) $\frac{(p^2-q^2)}{2}$

12. If α and β are the roots of the equation $ax^2 + bx + c = 0$, then

$\frac{\alpha}{a\beta+b} + \frac{\beta}{a\alpha+b} =$

(a) $2/a$ (b) $2/b$
(c) $2/a$ (d) $-2/a$

13. If the roots of the quadratic equation $\frac{x-m}{mx+1} = \frac{x+n}{nx+1}$ are reciprocal to each other, then

[MPPET – 2001]

(a) $n = 0$ (b) $m = n$
(c) $m + n = 1$ (d) $m^2 + n^2 = 1$

14. The number of values of a for which $(a^2-3a+2)x^2+(a^2-5a+6)x+a^2-4=0$ is an identity in x, is

(a) 0 (b) 2
(c) 1 (d) 3

15. Two students while solving a quadratic equation in x, one copied the constant term incorrectly and got the roots 3 and 2.

The other copied the constant term and coefficient of x^2 correctly as − 6 and 1 respectively. The correct roots are

(a) 3, − 2 (b) − 3, 2
(c) − 6, − 1 (d) 6, − 1

[EAMCET – 1991]

16. If the product of roots of the equation $x^2 - 3kx + 2e^{2\log k} - 1 = 0$ is 7, then its roots will be a real when

[IIT – 1984]

(a) $k = 1$ (b) $k = 2$
(c) $k = 3$ (d) none of these

17. Let N be the number of quadratic equations with coefficients {0, 1, 2,..., 9} such that zero is a solution of each equation.

[Kerala PET – 2003]

Then the value of N is

(a) Infinite (b) 2^9
(c) 90 (d) 900

18. If the roots of $ax^2 + bx + c = 0$ are α, β and the roots of $Ax^2 + Bx + C = 0$ are $\alpha - k, \beta - k$, then $\frac{B^2-4AC}{b^2-4ac}$ is equal to

(a) 0 (b) 1
(c) $\left(\frac{A}{a}\right)^2$ (d) $\left(\frac{a}{A}\right)^2$

[RPET – 1999]

19. If the sum of the roots of the equation $x^2 + px + q = 0$ is three times their difference, then which one of the following is true

[Dhanbad Engg. – 1968]

(a) $9p^2 = 2q$ (b) $2q^2 = 9p$
(c) $2p^2 = 9q$ (d) $9q^2 = 2p$

20. The value of 'c' for which $|\alpha^2 - \beta^2| = 7/4$, where α and β are the roots of $2x^2 + 7x + c = 0$, is

(a) 4 (b) 0
(c) 6 (d) 2

21. If α and β are the roots of the equation $ax^2 + bx + c = 0$, $\alpha\beta = 3$ and a, b, c are in A.P., then $a + b$ is equal to

(a) − 4 (b) − 1
(c) 4 (d) − 2

[Kerala PET – 2007]

22. If α and β are roots of the equation $Ax^2 + Bx + C = 0$, then value of $\alpha^3 + \beta^3$ is

[RPET–1996; DCE – 2005]

(a) $\frac{3ABC - B^3}{A^3}$ (b) $\frac{3ABC + B^3}{A^3}$

(c) $\frac{B^3 - 3ABC}{A^3}$ (d) $\frac{B^3 - 3ABC}{B^3}$

23. If 3 is a root of $x^2 + kx - 24 = 0$, it is also a root of

(a) $x^2 + 5x + k = 0$ (b) $x^2 - 5x + k = 0$
(c) $x^2 - kx + 6 = 0$ (d) $x^2 + kx + 24 = 0$

[EAMCET – 2002]

24. If α and β are the roots of $ax^2 + 2bx + c = 0$, then $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ is equal to

[MPPET – 2009]

(a) $\frac{4b^2 - 2ac}{ac}$ (b) $\frac{4b^2 - 4ac}{ac}$

(a) $\frac{2b^2 - 2ac}{ac}$ (b) $\frac{2b^2 - 4ac}{ac}$

SOLUTIONS

1. (d) Here, $a + b = p$, $ab = q$

$\Rightarrow \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{p}{q}$

2. (d) Here, $\alpha + \beta = -2$ and $\alpha + \beta = 4$

$\therefore \frac{1}{\alpha^3} + \frac{1}{\beta^3} = \frac{\alpha^3 + \beta^3}{(\alpha\beta)^3} = \frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{(\alpha\beta)^3}$

$= \frac{(-2)^3 - 3(-2)(4)}{(4)^3} = \frac{16}{64} = \frac{1}{4}$

3. (c) Let α and β are roots of equation $ax^2 + bx + c = 0$ then

$\alpha + \beta = \frac{-b}{a}\ \alpha\beta = \frac{c}{a}$

According to the questions

$\alpha + \beta = \alpha^2 + \beta^2$

$\alpha + \beta = (\alpha + \beta)^2\ \ 2\alpha\beta$

$\frac{-b}{a} = \frac{b^2}{a^2} - \frac{2c}{a}$

$\frac{-b}{a} = \frac{b^2 - 2ac}{a^2}$

$b^2 - 2ac = -ab$

$b^2 + ab = 2ac$

$b(a + b) = 2ac$

4. (d) If α and β are roots of equation $\lambda x^2 + 2x + 3\lambda = 0$ then,

$\alpha + \beta\ (sum) = \frac{-2}{\lambda}$

and $\alpha\beta$ (product) $= \frac{3\lambda}{\lambda} = 3$

According to the questions,

Sum = Product

$\frac{-2}{\lambda} = 3 \Rightarrow 1 = \frac{-2}{3}$

5. (c) $\alpha + \beta = a, \alpha\beta = -a - b$

Now, $(\alpha + 1)(\beta + 1) = \alpha\beta + (\alpha + \beta) + 1$

$-a - b + a + 1 = 1 - b$

6. (c) Product of the roots $= \alpha\beta = \frac{c}{a}$

Product of the roots $= \frac{2m - 1}{m} = -1$ (given)

$\Rightarrow 2m - 1 = -m$

$\Rightarrow 3m = 1$

$\Rightarrow m = \frac{1}{3}$

7. (a) Let the roots of $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ be

α, β and γ, δ respectively.

$\therefore |\alpha - \beta| = |\gamma - \delta|$

$\Rightarrow (\alpha - \beta)^2 = (\gamma - \delta)^2$

$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = (\gamma + \delta)^2 - 4\gamma\delta$

$\Rightarrow (-a)^2 - 4b = (-b)^2 - 4a$

$\Rightarrow a^2 - b^2 + 4a - 4b = 0$

$\Rightarrow (a - b)(a + b + 4) = 0, a - b \neq 0$ (given)

$\Rightarrow a + b + 4 = 0$

8. (a) Given roots are $\frac{1}{3 + \sqrt{2}}, \frac{1}{3 - \sqrt{2}}$

sum of roots $= \frac{1}{3 + \sqrt{2}} + \frac{1}{3 - \sqrt{2}} = \frac{6}{7}$

Product of roots $= \frac{1}{(3 + \sqrt{2})} \times \frac{1}{3 - \sqrt{2}} = \frac{1}{7}$

Quadratic equation whose roots are α and β is given by $x^2 -$ (sum of the roots)$x +$ product of roots = 0

$\therefore$ required equation, $x^2 - \frac{6}{7}x + \frac{1}{7} = 0$

or $7x^2 - 6x + 1 = 0$

9. (a) We have $p + q = -p$

and $pq = 9$

$\therefore$ $p = 1$

and $1 + q = -1$

or $q = -2$

10. (a) Let n and $(n + 1)$ be the roots of $x^2 - bx + c = 0$

Then $n + (n + 1) = b$ and $n(n + 1) = c$

$b^2 - 4c = (2n + 1)^2 - 4n(n + 1)$

$= 4n^2 + 4n + 1 - 4n^2 - 4n$

$= 1$

OR

$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$ and $|\alpha - \beta| = 1$

11. (b) On simplification the given equation is $x^2 + x(p + q - 2r) + (pq - pr - qr) = 0$

By given condition

$\beta = -\alpha$ or $\alpha + \beta = 0$ (1)

Therefore, $p + q - 2r = 0$ or $p + q = 2r$

Product of roots: $\alpha\beta = \frac{c}{a}$

$\Rightarrow$ $pq - pr - qr = pq - r(p + q)$

$\Rightarrow$ $pq - \frac{p + q}{2}(p + q)$

$\Rightarrow$ $\frac{1}{2}[2pq - (p + q)^2]$, by (1)

$\Rightarrow$ $-\frac{1}{2}[p^2 + q^2]$.

12. (d) If α and β are roots of equations $ax^2 + bx + c = 0$ then

$\alpha + \beta = \frac{-b}{a}$ and $\alpha\beta = \frac{c}{a}$

$\therefore \dfrac{\alpha}{(a\beta + b)} + \dfrac{\beta}{a\alpha + b}$

$= \dfrac{a\alpha^2 + b\alpha + a\beta^2 + b\beta}{(a\beta + b)(a\alpha + b)}$

$= \dfrac{a(\alpha^2 + \beta^2) + b(\alpha + \beta)}{a^2\alpha\beta + ab\beta + ab\alpha + b^2}$

$= \dfrac{a\{(\alpha + \beta)^2 - 2\alpha\beta\} + b(\alpha + \beta)}{a^2\alpha\beta + ab(\alpha + \beta) + b^2}$

$= \dfrac{a\left\{\frac{b^2}{a^2} - \frac{2c}{a}\right\} + b\left(\frac{-b}{a}\right)}{a^2 \times \frac{c}{a} + ab\left(\frac{-b}{a}\right) + b^2}$

$= \dfrac{\frac{b^2 - 2ac - b^2}{a}}{\frac{a^2c - ab^2 + ab^2}{a}}$

$= \dfrac{-2ac}{a^2c} = \dfrac{-2}{a}$

13. (a) If the roots of $Ax^2 + Bx + C = 0$ are reciprocal to each other then from $C = A$.

Given equation $(x - m)(nx + 1) = (x + n)(mx + 1)$

$(m - n)x^2 + (2mn + 0)x + n + m = 0$

Clearly, from $A = m - n$, $c = m + n$, $C = A$

$m - n = m + n$ or $n = 0$

14. (c) Step 1: If $ax^2 + bx + c = 0$ is identity then $a = b = c = 0$

Step 2: If given equation is an identity then,

$a^2 - 3a + 2 = 0 \Rightarrow a = 1, 2$

$a^2 - 5a + 6 = 0 \Rightarrow a = 2, 3$

$a^2 - 4 = 0$

$\Rightarrow a = -2, 2$

The common value of $a = 2$.

Therefore, the number of values of $a = 1$.

15. (d) Let α, β be the roots of given equation, then $\alpha + \beta = 5$ (from the observation of student first) and $\alpha\beta = -6$ (from the observation of student second)

So, the equation is $x^2 - 5x - 6 = 0$

$\Rightarrow (x - 6)(x + 1) = 0$

$\Rightarrow x = 6, -1.$

16. (b) Product of the roots $= \alpha\beta = \frac{c}{a} = 7$ (1) (given)

$\Rightarrow 2e^{2 \log k} - 1 = 7$

$\therefore e^{\log k2} = 4$

$\therefore k^2 = 4$ or $k = \pm 2$

But, by definition of log,

(since negative numbers do not have log),

$k \neq -2$

$\therefore k = 2$

Again $\Delta \geq 0$

$\therefore 9k^2 - 4(7) \geq 0$

This inequality is also satisfi ed when $k = 2$.

17. (c) Step 1: Clearly $ax^2 + bx = 0$ is the required quadratic whose one root is zero.

Step 2: Clearly $a \neq 0$, therefore, total number of ways of selecting a is clearly 9 and that of b is 10 with given 10 coefficients.

Step 3: Total number of ways of selection of a and b simultaneously is $= 9 \times 10$ ways in turn giving total 90 quadratic equations.

18. (c) If α and β are roots of equation $ax^2 + bx + c = 0$ then

$$\alpha + \beta = \frac{-b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

and $\alpha - k$, $\beta - k$ are roots of equation $Ax^2 + Bx + C = 0$ then

$(\alpha - k) + (\beta - k) = \dfrac{-B}{A}$

and $(\alpha - k)(\beta - k) = \dfrac{C}{A}$

Evidently

$(\alpha - k) - (\beta - k) = \alpha - \beta$

Squaring both sides and using

$(a - b)^2 = (a + b)^2 - 4ab$

we get,

$[(\alpha - k) + (\beta - k)]^2 - 4(\alpha - k)(\beta - k)$

$= (\alpha + \beta)^2 - 4\alpha\beta$

$\Rightarrow \left(\frac{-B}{A}\right)^2 - 4\left(\frac{C}{A}\right) = \left(\frac{-b}{a}\right)^2 = -\frac{4c}{a}$

$\Rightarrow \frac{B^2 - 4Ac}{A^2} = \frac{b^2 - 4ac}{a^2}$

$\Rightarrow \frac{B^2 - 4Ac}{b^2 - 4ac} = \frac{A^2}{a^2}$

$\Rightarrow \frac{B^2 - 4Ac}{b^2 - 4ac} = \left(\frac{A}{a}\right)^2$

19. (c) Let α, β are roots of $x^2 + px + q = 0$
So $\alpha + \beta = -p$ and $\alpha\beta = q$
Given that $(\alpha + \beta) = 3(\alpha - \beta) = -p$

$\Rightarrow \alpha - \beta = \frac{-p}{3}$

Now $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$\Rightarrow \frac{p^2}{9} = p^2 - 4q$ or $2p^2 = 9q$.

20. (c) Here, $\alpha + \beta = \frac{7}{2}$ and $\alpha\beta = \frac{c}{2}$

$\therefore |\alpha^2 - \beta^2| = \frac{7}{4}$

$\alpha^2 - \beta^2 = \pm\frac{7}{4}$

$(\alpha + \beta)(\alpha - \beta) = \pm\frac{7}{4}$

$\Rightarrow = \frac{-7}{2}\sqrt{\frac{49}{4} - 2c} = \pm\frac{7}{4}$

$\Rightarrow \sqrt{49 - 8c} = \mp 1$

$49 - 8c = 1$

$\Rightarrow \quad 8c = 48$

$\Rightarrow \quad c = 6$

21. (d) Here, $\alpha + \beta =$ and $\alpha\beta = \frac{c}{a} = 3$

$\Rightarrow c = 3a$

Also, a, b, c are in A.P.

$\Rightarrow 2b = a + c = a + 3a = 4a$

$\Rightarrow b = \frac{4a}{2} = 2a$

Hence, $\alpha + \beta = -\frac{b}{a} = \frac{-2a}{a} = -2$

22. (a) If α and β are roots of the equation $Ax^2 + Bx + C = 0$, then,

$a + b = \frac{-B}{A}$ and $\alpha\beta = \frac{C}{A}$

$\therefore \alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$= \frac{-B^3}{A^3} - \frac{3C}{A}\left(\frac{-B}{A}\right)$

$= \frac{-B^3 + 3BCA}{A^3}$

23. (c) One root of the equation $x^2 + kx - 24 = 0$ is 3, therefore,

$\Rightarrow 3^2 + 3k - 24 = 0$

$\Rightarrow k = 5$

Put $x = 3$ and $k = 5$ in option (c) we find:

$(3)2 - (5)(3) + 6 = 9 - 15 + 6 = 0.$

Hence option (c) is correct.

24. (a) Given equation is $ax^2 + 2bx + c = 0$

$\therefore \alpha + \beta = \frac{2b}{a}$ and $\alpha\beta = \frac{c}{a}$

Now, $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$

$= \frac{\frac{4b^2}{a^2} - \frac{2c}{a}}{\frac{c}{a}} = \frac{4b^2 - 2ac}{ac}$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. If the product of the roots of the equation $2x^2 + 6x + \alpha^2 + 1 = 0$ is $-\alpha$, then the value of a will be

(a) -1 (b) 1

(c) 2 (d) -2

2. If x_1, x_2, x_3 are distinct roots of the equation $ax^2 + Bx + c = 0$ then

(a) $a = b = 0, c \in R$

(b) $a = c = 0, b \in R$

(c) $b^2 - 4ac \geq 0$

(d) $a = b = c = 0$

3. If a and β are the roots of the equation $4x^2 + 3x + 7 = 0$,
then $\frac{1}{a} + \frac{1}{\beta} =$

[MnR – 1981; RPET – 1990]

(a) 3 / 7 (b) 3 / 7
(c) − 3 / 5 (d) 3 /5

4. If the roots of the equation $ax^2 + bx + c = 0$ are reciprocal to each other, then

[RPET – 1985]

(a) $a - c = 0$ (b) $b - c = 0$
(c) $a + c = 0$ (d) $b + c = 0$

5. If p and q are the roots of the equation $x^2 + pq = (p + 1)x$, then $q =$
(a) − 1 (b) 1
(c) 2 (d) none of these

6. If the sum of the roots of the equation $x^2 + px + q = 0$ is equal to the sum of their squares, then

[Pb. CET – 1999]

(a) $p^2 - q^2 = 0$ (b) $p^2 + q^2 = 2q$
(c) $p^2 + p = 2q$ (d) none of these

7. If the roots of the equation $\frac{\alpha}{x-\alpha} + \frac{\beta}{x-\beta} = 1$ be equal in magnitude but opposite in sign, then $\alpha + \beta =$
(a) 0 (b) 1
(c) 2 (d) none of these

8. Two candidates attempt to solve the equation $x^2 + px + q = 0$.
One starts with the wrong values of p and finds the roots to be 2 and 6 and the other starts with a wrong value of q and fi nd the roots to be 2 and − 9. The roots of the original equation are:
(a) 2, 3 (b) 3, 4
(c) − 2, − 3 (d) − 3, − 4

9. If α, β are the roots of the equation $x^2 + ax + b = 0$ then the value of $\alpha^3 + \beta^3$ is equal to

[RPET – 1989; Pb. CET – 1991]

(a) $-(a^3 + 3ab)$ (b) $a^3 + 3ab$
(c) $-a^3 + 3ab$ (d) $a^3 - 3ab$

10. If the difference of the roots of $x^2 - px + 8 = 0$ be 2, then the value of p is

[Roorkee – 1992; Haryana – 2003]

(a) ± 2 (b) ± 4
(c) ± 6 (d) ± 8

11. If the sum of the roots of equation $(m + 1)x^2 + 2mx + 3 = 0$ is 1, then the value of m is
(a) 1/2 (b) − 1/2
(c) 1/ 3 (d) − 1/3

12. If the equation $x^2 + kx + 1 = 0$ has the roots a and b, then what is the value of $(\alpha + \beta) \times (\alpha^{-1} + \beta^{-1})$?

[NDA – 08]

(a) k^2 (b) $1/k^2$
(c) $2k^2$ (d) $1/(2k^2)$

13. If the roots of $4x^2 + 5k = (5k + 1)x$ differ by unity, then the negative value of k is

[MP PET – 2008]

(a) − 3 (b) − 5
(c) − 1/5 (d) − 3/5

WORK SHEET: TO CHECK PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the work sheet
2. The test is of 16 minutes.
3. The test consists of 16 questions.
The maximum marks are 48.
4. Use blue/black ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. If the roots of the given equation $(2k + 1)x^2 - (7k + 3)x + k + 2 = 0$are reciprocal to each other, then the value of k will be
(a) 0 (b) 1
(c) 2 (d) 3

[MPPET – 1986]

2. If the roots of the equation $ax^2 + bx + c = 0$ are α, β, then the value of $\alpha\beta^2 + \alpha^2\beta + \alpha\beta$ will be

(a) $\frac{c(c-b)}{a^2}$ (b) 0

(c) $-\frac{bc}{a^2}$ (d) none of these

[EAMCET-1980; AMU – 1984]

3. If α and β are the roots of the equation $x^2 - 4x + 1 = 0$ the value of $\alpha^3 + \beta^3$ is

[MPPET – 1994]

(a) 76 (b) 52
(c) − 52 (d) − 76

4. What is the sum of the squares of roots of $x^2 - 3x + 1 = 0$

[Karnataka CET – 1998]

(a) 5 (b) 7
(c) 9 (d) 10

5. If one root of the equation $ax^2 + bx + c = 0$ be reciprocal of other, then

(a) $b = c$ (b) $a = c$
(c) $a = 0$ (d) $b = 0$

6. If $ax^2 + bx + c = 0$ is satisfied by every value of x, then

(a) $b = 0, c = 0$ (b) $c = 0$
(c) $b = 0$ (d) $a = b = c = 0$

7. The numerical difference of the roots of $x^2 - 7x - 9 = 0$

(a) 7 (b) $2\sqrt{85}$
(c) $9\sqrt{7}$ (d) $8\sqrt{5}$

8. If the roots of the equation $x^2 + px + q = 0$ differ by 1, then

[MPPET – 1999]

(a) $p^2 = 4q$ (b) $p^2 = 4q + 1$
(c) $p^2 = 4q - 1$ (d) none of these

9. If the product of the roots of the equation $(a + 1)x^2 + (2a + 3)x + (3a + 4) = 0$ be 2, then the sum of roots is

(a) 1 (b) − 1
(c) 2 (d) − 2

10. If roots of $x^2 - 7x + 6 = 0$ are a, b, then $\frac{1}{\alpha} + \frac{1}{\beta} =$

[RPET – 1995]

(a) 6/7 (b) 7/6
(c) 7/10 (d) 8/9

11. Sum of roots is − 1 and sum of their reciprocals is 1/6, then the equation is

[Karnataka CET – 1998]

(a) $x^2 + x - 6 = 0$ (b) $x^2 - x + 6 = 0$
(c) $6x^2 + x + 1 = 0$ (d) $x^2 - 6x + 1 = 0$

12. If the roots of the equation $x^2 - bx + c = 0$ and $x^2 - cx + b = 0$ differ by the same quantity, then $b + c$ is equal to

(a) 4 (b) 1
(c) 0 (d) − 4

[BIT RANCHI –1969; MPPET – 1993]

13. Suppose that two persons A and B solve the equation $x^2 + ax + b = 0$. While solving A commits a mistake in constant term and finds the roots as 6 and 3 and b commits a mistake inthe coefficient of x and finds the roots as − 7 and − 2. Then the equation is.

[Kerala PET – 2008]

(a) $x^2 + 9x + 14 = 0$
(b) $x^2 - 9x + 14 = 0$
(c) $x^2 + 9x - 14 = 0$
(d) $x^2 - 9x - 14 = 0$

14. If $x = \sqrt{7+4\sqrt{3}}$, then $1\ x + \frac{1}{x} =$

[EAMCET – 1994]

(a) 4 (b) 6
(c) 3 (d) 2

15. The sum of the roots of a equation is 2 and sum of their cubes is 98, then the equation is

[MPPET – 1986]

(a) $x^2 + 2x + 15 = 0$
(b) $x^2 + 15x + 2 = 0$
(c) $2x^2 - 2x + 15 = 0$
(d) $x^2 - 2x - 15 = 0$

16. If the roots of the given equation $2x^2 + 3(\lambda - 2)x + \lambda + 4 = 0$ be equal in magnitude but opposite in sign, then $\lambda =$

(a) 1 (b) 2
(c) 3 (d) 2/3

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	7. ⓐ ⓑ ⓒ ⓓ	12. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	8. ⓐ ⓑ ⓒ ⓓ	13. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	9. ⓐ ⓑ ⓒ ⓓ	14. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	10. ⓐ ⓑ ⓒ ⓓ	15. ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ	11. ⓐ ⓑ ⓒ ⓓ	16. ⓐ ⓑ ⓒ ⓓ
6. ⓐ ⓑ ⓒ ⓓ		

HINTS AND EXPLANATIONS

9. (b) It is given that

$\alpha\beta = 2 \Rightarrow \frac{3a+4}{a+1} = 2$

$\Rightarrow 3a + 4 = 2a + 2 \Rightarrow a = -2$

Also, $\alpha + \beta = -\frac{2a+3}{a+1}$

Putting this value of a, we get sum of roots,

$= \frac{2a+3}{a+1} = -\frac{-4+3}{-2+1} = -1$

13. (b) Sum of roots = 6 + 3 = 9

Product of roots = (− 7) (− 2) = 14

Correct equation is

$x^2 -$ (Sum of roots)x + Product = 0

$x^2 - 9x + 14 = 0$

14. (a) We have $x = \sqrt{7 + 4\sqrt{3}}$

$\therefore \frac{1}{x} = \frac{1}{\sqrt{7+4\sqrt{3}}} = \frac{\sqrt{7-4\sqrt{3}}}{\sqrt{7+4\sqrt{3}}.\sqrt{7-4\sqrt{3}}}$

$= \sqrt{7 + 4\sqrt{3}}$

$\therefore x + \frac{1}{x} = \sqrt{7+4\sqrt{3}} + \sqrt{7-4\sqrt{3}}$

$= (\sqrt{3} + 2) + (2 - \sqrt{3}) = 4$

15. (d) Let roots are α and β

$\alpha + \beta = 2$ and $\alpha^3 + \beta^3 = 98$

$\therefore \alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)$

$\Rightarrow 98 = 2[(\alpha + \beta)^2 - 3\alpha\beta]$

$\Rightarrow 49 = (4 - 3\alpha\beta)$

$\Rightarrow \alpha\beta = -15$

This equation is $x^2 - 2x - 15 = 0$

This page is intentionally left blank.

LECTURE

2 Nature of Roots

BASIC CONCEPTS

1. **Nature of the Roots** Roots of the equation $Ax^2 + Bx + C = 0$ are given by,

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

The quantity $B^2 - 4AC$ is called the discriminant of the equation. In terms of α and β roots are as follows,

$$\alpha = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

$$\beta = \frac{-B - \sqrt{B^2 - 4AC}}{2A}$$

The roots of quadratic equation $Ax^2 + Bx + C = 0$ are

(i) real, unequal and rational if $B^2 - 4AC = 1, 4, 9, ...$ i.e., positive and perfect square.

(ii) real, unequal and irrational if $B^2 - 4AC = 2, 3, 5, 6, ...$ i.e., positive and not a perfect square and A, B, C are rationals (A, B and C are not irrational)

In this case, irrational roots of a quadratic equation occur in conjugate pair of surds i.e., if one root is $p + \sqrt{q}$, then the other root must be $p - \sqrt{q}$.

Note: If $B^2 - 4AC > 0$, then the roots of the equation are real and unequal.

(iii) The roots of the equation are real and equal, if $B^2 - 4AC = 0$.

Note: If $B^2 - 4AC > 0$, then both roots of the equation are real.

(iv) The roots of the equation are imaginary and unequal if $B^2 - 4AC < 0$.

Note: If the coefficient of a quadratic equation are real numbers, then the imaginary roots always occur is conjugate pairs of the form $x \pm iy$. (A, B and C are not complex)

2. **Some Important Formulas Connected with Roots of Quadratic Equation** $Ax^2 + Bx + C = 0$

(i) If $A + B + C = 0$, then one root of the quadratic equation is always unity ($x = 1$) and other root is $\frac{C}{A}$ i.e., if $\alpha = 1$ and $\beta = \frac{C}{A}$

Examples (algebraic sum of the coefficients is zero)

(i) $(1 - m)x^2 + (m - n)x + (n - l) = 0$

(ii) $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$

(iii) $(b + c - 2a)x^2 + (c + a - 2b)x + (a + b - 2c) = 0$

Note: Other root is $\frac{C}{A}$. Hence both roots $\left(1 \text{ and } \frac{C}{A}\right)$ are rational.

SOLVED SUBJECTIVE PROBLEMS: (CBSE/STATE BOARD): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. If the roots of the quadratic equation $p(q - r)x^2 + q(r - p)x + r(p - q) = 0$ are the same, then prove that $\frac{1}{p} + \frac{1}{r} = \frac{2}{q}$.

Solution

Given equation is $p(q - r)x^2 + q(r - p)x + r(p - q) = 0$

Here, $a = p(q - r)x^2$, $b = q(r - p)$, $c = r(p - q)$

If roots of this equation are same, then

$b^2 - 4ac = 0$

$\Rightarrow q^2(r - p)^2 - 4p(q - r) . r(p - q) = 0$

$\Rightarrow q^2(r^2 + p^2 - 2rp) - 4pr(pq - pr - q^2 + qr) = 0$

$\Rightarrow q^2r^2 + q^2p^2 - 2pq^2r - 4p^2qr + 4p^2r^2 + 4pq^2r - 4pqr^2 = 0$

$\Rightarrow q^2r^2 + q^2p^2 + 2pq^2r - 4p^2qr - 4pqr^2 + 4p^2r^2 = 0$

$\Rightarrow q^2(r^2 + p^2 + 2rp) - 4pqr(p + r) + 4p^2r^2 = 0$

$\Rightarrow q^2(p + r)^2 - 4pqr(p + r) + (2pr)^2 = 0$

$\Rightarrow [q (p + r) - 2pr]^2 = 0$

$\Rightarrow q (p + r) = 2pr$

$\Rightarrow p + r = \frac{2}{q} pr$

$\Rightarrow \frac{p + r}{pr} = \frac{2}{q}$

$\Rightarrow \frac{p}{pr} + \frac{r}{p} = \frac{2}{q}$

$\Rightarrow \frac{p}{pr} + \frac{r}{pr} = \frac{2}{q}$

$\Rightarrow \frac{1}{r} + \frac{1}{p} = \frac{2}{q}$

Proved

2. If roots of the quadratic equation $(p^2 + q^2)x^2 - 2(ap + bq) x + a^2 + b^2 = 0$ are equal, then prove that $\frac{a}{b} = \frac{p}{q}$.

Solution

Given quadratic equation,

$(p^2 + q^2)x^2 - 2(ap + bq)x + a^2 + b^2 = 0$

Here, $A = p^2 + q^2$, $B = -2(ap + bq)$ and $C = a^2 + b^2$

If roots of the given equation are equal, then

$B^2 - 4AC = 0$

$\Rightarrow 4(ap + bq)^2 - 4(p^2 + q^2) (a^2 + b^2) = 0$

$\Rightarrow (ap + bq)^2 = (p^2 + q^2) (a^2 + b^2)$

$\Rightarrow a^2p^2 + b^2q^2 + 2abpq$

$= a^2p^2 + b^2q^2 + b^2p^2 + a^2q^2$

$\Rightarrow a^2q^2 + b^2p^2 -2abpq = 0$

$\Rightarrow (aq - bp)^2 = 0$

$\Rightarrow (aq - bp) = 0$

$\Rightarrow aq = bp \Rightarrow \frac{a}{b} = \frac{p}{q}$

Proved

3. If roots of the equation $(a^2 - bc)x^2 + 2(b^2 - ca) x + c^2 - ab = 0$ are equal, then prove that $a^3 + b^3 + c^3 = 3abc$.

Solution

Given equation is $(a^2 - bc)x^2 + 2(b^2 - ca)x + c^3 - ab = 0$

Here, $A = a^2 - bc$, $B = 2(b^2 - ca)$ and $C = c^2 - ab$

Since, roots of the given equation are equal, therefore $B^2 = 4AC$

$\Rightarrow 4(b^2 - ca)^2 = 4(a^2 - bc)(c^2 - ab)$

$\Rightarrow b^4 + c^2a^2 - 2ab^2c = a^2c^2 - a^3b - bc^3 + ab^2c$

$\Rightarrow b^4 + a^3b + bc^3 = 2ab^2c + ab^2c$

$\Rightarrow b (b^3 + a^3 + c^3) = 3ab^2c$

$\Rightarrow b^3 + a^3 + c^3 = 3abc$

Proved

4. The roots of the quadratic equation $x^2 + \lambda x + \mu = 0$ are equal and 2 is one root of the quadratic equation $x^2 + \lambda x - 12 = 0$.
Find the value of λ and μ.

Solution

Since 2 is one root of the quadratic equation

$x^2 + \lambda x -12 = 0$

$2^2 + \lambda(2) - 12 = 0$

$\Rightarrow 4 + 2\lambda - 12 = 0$

$\Rightarrow 2\lambda - 8 = 0$

$\Rightarrow 1 = 4$

And roots of the quadratic equation $x^2 + \lambda x + m = 0$ are equal.

$\therefore\ b^2 - 4ac = 0$

$\Rightarrow \lambda^2 - 4(1)(\mu) = 0$

$\Rightarrow 4^2 - 4\mu = 0$

$\Rightarrow 4\mu = 16 \Rightarrow \mu = 4$

Thus $\lambda = 4 = \mu$. Ans.

5. If roots of equation $x^2 + 2(p - q)x + pq = 0$ are imaginary then prove that $4x^2 + 4(p - q)x + (4p^2 + 4q^2 - 11pq) = 0$ will have real roots.

Solution

Comparing the given equation $x^2 + 2(p - q)x + pq = 0$ with $ax^2 + bx + c = 0$ we get $a = 1$, $b = 2(p - q)$, $c = pq$

Given that discriminant of Ist equation

$b^2 - 4ac < 0$

$\therefore\ [2(p - q)]^2 - 4 \times 1 \times pq < 0$

$4(p - q)^2 - 4pq < 0$

$4p^2 + 4q^2 - 8pq - 4pq < 0$

$4p^2 + 4q^2 - 12pq < 0$ (1)

Second equation is,

$4x^2 + 4(p - q)x + (4p^2 + 4q^2 - 11pq) = 0$

Let discriminant of second equation $D = b^2 - 4ac$

$= [4(p - q)]^2 - 4 \times 4(4p^2 + 4q^2 - 11pq)$

UNSOLVED SUBJECTIVE PROBLEMS: (CBSE/STATE BOARD): TO GRASP THE TOPIC, SOLVE THESE PROBLEMS

Exercise I

1. Solve $\sqrt{5}\, x^2 + x + \sqrt{5} = 0$
2. Solve $9x^2 + 10x + 3 = 0$
3. Solve $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$
4. Solve the equation $4x^2 + 9 = 0$ by factorization method.
5. Solve the equation $9x^2 - 12x + 20 = 0$ by factorization method.
6. Solve the quadratic equation $2x^2 - 4x + 3 = 0$ by using the general expressions for the roots of a quadratic equation.
7. Solve the equation $25x^2 - 30x + 11 = 0$ by using the general expression for the roots of a quadratic equation.
8. If the roots of equation $(1 + n)x^2 - 2(1 + 3n)x + (1 + 8n) = 0$ are equal, then find the value of n.
9. If roots of the quadratic equation $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ are equal, then prove that $b(c + a) = 2ac$.

Exercise II

1. Solve $x^2 + 2 = 0$.
2. Solve $x^2 + x + 1 = 0$.
3. Solve $3x^2 + 8ix + 3 = 0$.
4. Solve the following quadratic equations by factorization method.
   (i) $x^2 - 5ix - 6 = 0$
   (ii) $x^2 + 4ix - 4 = 0$
   (iii) $x^2 - \sqrt{2}\, ix + 12 = 0$
   (iv) $3x^2 + 7\, ix + 6 = 0$
   (v) $x^2 - (3\sqrt{2} + 2i)x + 6\sqrt{2}\, i = 0$
5. Solve $x^2 - (7 - i)x + (18 - i) = 0$.
6. Find the roots of the equation $(p - q)x^2 + (q - r)x + (r - p) = 0$.
7. Discuss the nature of the roots of the equation $3x^2 - 4x + 2 = 0$.
8. Examine the nature of the roots of the quadratic equation $2x^2 - 9x + 8 = 0$.
9. If roots of the equation $4x^2 + 15x + m = 0$ are equal, then find the value of m.
10. Find the equation whose roots are $2 + 3i$, $2 - 3i$.

ANSWERS

Exercise I

1. $\frac{-1 \pm \sqrt{19}i}{2\sqrt{5}}$
2. $x = \frac{-5 \pm \sqrt{2}i}{9}$
3. $\left(\frac{\sqrt{2} \pm \sqrt{34}i}{2\sqrt{3}}\right)$
4. $x = \frac{3}{2}i, -\frac{3}{2}i$
5. $x = \frac{3}{2} \pm \frac{4}{3}i$
6. $1 + \frac{1}{\sqrt{2}}i$ and $1 - \frac{1}{\sqrt{2}}i$
7. $\frac{3}{5} \pm \frac{\sqrt{2}}{5}i$
8. $n = 0, n = 3$

Exercise II

1. $x = \pm \sqrt{2}\, i$
2. $x = \frac{-1 \pm \sqrt{3}i}{2}$
3. $\frac{i}{3}, -3i$
4. (i) $x = 3i, 2i$ (ii) $x = -2i, 2i$
   (iii) $x = 3\sqrt{2}\, i, -2\sqrt{2}i$ (iv) $x = -3i, 2/3\, i$
   (v) $x = 2i, 3\sqrt{2}$
5. $4 - 3i$ and $3 + 2i$
6. $x = \frac{r - p}{p - q}, 1$
7. Roots are imaginary
8. Roots are real, unequal and irrational.
9. $m = \frac{225}{16}$
10. $x^2 - 4x + 13 = 0$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. If $x = \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots\ldots\text{to infinity}}}}$, then $x =$

(a) $\frac{1+\sqrt{5}}{2}$ (b) $\frac{1-\sqrt{5}}{2}$

(c) $\frac{1+\sqrt{5}}{2}$ (d) none of these

Solution

(a) $x = \sqrt{1+\sqrt{1+\sqrt{1+\ldots\ldots \textit{to } \infty}}}$, we have $x = \sqrt{1+x}$

$\Rightarrow x^2 = 1 + x \Rightarrow x^2 - x - 1 = 0$

$\Rightarrow x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$.

As $x > 0$, we get, $x = \frac{1+\sqrt{5}}{2}$

2. In a triangle ABC the value of $\angle A$ is given by $5 \cos A + 3 = 0$, then the equation whose roots are $\sin A$ and tan A will be

(a) $15x^2 - 8x + 16 = 0$
(b) $15x^2 + 8x - 16 = 0$
(c) $15x^2 - 8 - \sqrt{2}x + 16 = 0$
(d) $15x^2 - 8x - 16 = 0$

[Roorkee – 1972]

Solution

(b) Given that $5\cos A + 3 = 0$ or $\cos A = -3/5$

$\therefore$ A lies in II quadrant Sin $A = 4/5$,

$\tan A = -4/3$

$\therefore$ equation having roots as $\sin A$, $\tan A$ is

$$x^2 - \left(\frac{4}{5} - \frac{4}{3}\right)x - \frac{16}{15} = 0$$

$\Rightarrow 15x^2 + 8x - 16 = 0$

3. If p, q, r are +ve and are in A.P., the roots of quadratic equation $px^2 + qx + r = 0$ are all real for

[IIT– 1994]

(a) $\left|\frac{r}{p}-7\right| \le 4\sqrt{3}$ (b) $\left|\frac{p}{r}-7\right| \ge 4\sqrt{3}$

(c) all p and r (d) no p and r

Solution

(b) For real roots $q^2 - 4pr \ge 0$

$\Rightarrow \left(\frac{p+r}{2}\right)^2 - 4pr \ge 0$ ($\because p, q, r$ are in A.P.)

$\Rightarrow p^2 + r^2 - 14pr \ge 0$

$\Rightarrow \frac{p^2}{r^2} - 14\frac{p}{r} + 1 \ge 0$

$\Rightarrow \left(\frac{p}{r}-7\right)^2 - 48 \ge 0$

$\Rightarrow \left|\frac{p}{r}-7\right| \ge 4\sqrt{3}$

4. If α, β are roots of the equation $x^2 + px + 1 = 0$ and γ, δ are roots of the equation $x^2 + qx + 1 = 0$, then $(\alpha-\gamma)(\beta-\gamma)(\alpha+\delta)(\beta+\delta)$ is equal to

[IIT – 1978, DCE – 2000]

(a) $p^2 - q^2$ (b) $p^2 + q^2$

(c) $q^2 - p^2$ (d) none of these

Solution

(c) $\because \alpha+\beta = -p, \gamma+\delta = -q\ \alpha\beta = 1 = \gamma\delta$

Exp.$= [\alpha\beta + \gamma(\alpha+\beta) + \gamma^2]$

$[\alpha\beta - \delta(\alpha+\beta) + \delta^2]$

$= (1 + p\gamma + \gamma^2)(1 - p\delta + \delta^2)$

But γ, δ are roots of the second equation, so

$\gamma^2 + q\gamma + 1 = 0 \Rightarrow \gamma^2 + 1 = -q\gamma$,

$\delta^2 + q\delta + 1 = 0 \Rightarrow \delta^2 + 1 = -q\delta$

$\therefore$ Exp. $(p\gamma - q\gamma)(-p\delta - q\delta)$

$= -\gamma\delta(p^2 - q^2) = q^2 - p^2$

5. If $a, b, c \in R$ are such that $4a + 2b + c = 0$ and $ab > 0$ then equation $ax^2 + bx + c = 0$ has

[Haryana (CET) – 1993]

(a) only one root

(b) purely imaginary roots

(c) real roots

(d) complex roots

Solution

(c) $4a + 2b + c = 0 \Rightarrow x = 2$ is a solution of the given equation, so its other must also be real.

6. If $x = 2 + 2^{1/2} + 2^{2/3}$, then $x^3 - 6x^2 + 6x$ equals

(a) 2 (b) –2

(c) 0 (d) 1

[MNR 1985; PET (Raj.) – 1995]

Solution

$\Rightarrow x = 2 + 2^{1/2} + 2^{2/3} \Rightarrow (x-3)^3$

$= (2^{1/3} + 2^{2/3})^3$

$\Rightarrow x^3 - 8 - 6x(x-2) = 2 + 4 + 3.2(x-2)$

$\Rightarrow x^3 - 6x^2 + 6x = 2$

7. If $x^2 - 2x + \sin^2\alpha = 0$, then

[Roorkee(Sc.) – 1999]

(a) $x \in [-1, 1]$ (b) $x \in [0, 2]$

(c) $x \in [-2, 2]$ (d) $x \in [-1, 2]$

Solution

(b) $x = \frac{2 \pm \sqrt{4 - 4\sin^2\alpha}}{2} = 1 \pm \cos\alpha$

Now $-1 \le \pm\cos\alpha < 1$

$\Rightarrow 1 - 1 \le 1 \pm \cos\alpha \le 1 + 1$

$\Rightarrow 0 \le 1 \pm \cos\alpha \le 2$

$\Rightarrow x \in [0, 2]$

8. If α, β are roots of the equation $x^2 + x + 1 = 0$ then the equation whose roots are $\alpha^2 + \beta^2$ and $\alpha^{-2} + \beta^{-2}$ will be

[IIIT (Allahabad) –2001]

(a) $x^2 - x + 1 = 0$ (b) $x^2 - x - 1 = 0$

(c) $x^2 - 2x + 1 = 0$ (d) $(x+1)^2 = 0$

Solution

(d) $\alpha + \beta = -1, \alpha\beta = 1 \Rightarrow \alpha^2 + \beta^2 = -1$

and $\alpha^{-1} + \beta^{-2} = -1$

$\therefore$ required equation is $(x+1)^2 = 0$

Alternative Method

Obviously, $\alpha = \omega, \beta = \omega^2$

$\therefore \alpha^2 + \beta^2 = \omega^2 + \omega^4 = -1, \alpha^{-2} + \beta^{-2}$

$= \omega^{-2} + \omega^{-4} = \omega + \omega^2 = -1$

$\therefore$ The required equation is $(x+1)^2 = 0$

9. If A, b, c are integers and $b^2 = 4(ac + 5d^2)$, $d \in N$, then roots of the equation $ax^2 + bx + c = 0$ are

[ICS – 2001]

(a) irrational
(b) rational and different
(c) complex conjugate
(d) rational and equal

Solution

(a) Here, $b^2 - 4ac = 20d^2$
$[\because b^2 = 4ac + 20d^2]$
which is not a perfect square ($\because$ $d \in Z$), so roots are irrational.

10. If $0 < C < \pi/2$ and $\sin C$, $\cos C$ are roots of the equation $2x^2 - px + 1 = 0$, then possible values of p are

[NDA – 2004]

(a) 1
(b) 2
(c) 3
(d) 4

Solution

(a) $\sin C + \cos C = \frac{p}{2}$ (1), $\sin C \cdot \cos C = \frac{1}{2}$
$\Rightarrow \sin 2C = 1$
$C = 45° \left(\because C \in \left(0, \frac{\pi}{2}\right)\right) \therefore p = 2\sqrt{2}$

(By (1))

11. If the roots of the quadratic equation $x^2 + px + q = 0$ are tan 30° and tan 15° respectively, then the value of $2 + q - p$ is

(a) 0 (b) 1
(c) 2 (d) 3

[AIEEE – 2006]

Solution

(d) $\tan 30° + \tan 15° = -p$ (1)
$\tan 30° \tan 15° = q$ (2)
Now, $\tan (30° + 15°) = \frac{\tan 30° + \tan 15°}{1 - \tan 30° \tan 15°}$
$\Rightarrow 1 = \frac{-p}{1-p}$
$\Rightarrow q - p = 1$
$\Rightarrow q - p + 2 = 3.$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. The roots of $4x^2 + 6px + 1 = 0$ are equal, then the value of p is

[MPPET – 2003]

(a) $\frac{4}{5}$ (b) $\frac{1}{3}$
(c) $\frac{2}{3}$ (d) $\frac{4}{3}$

2. The quadratic equation with real coefficient with one root $1 + \sqrt{3}$ is

(a) $x^2 + 2x + 2 = 0$ (b) $x^2 - 2x + 2 = 0$
(c) $x^2 + 2x - 2 = 0$ (d) $x^2 - 2x - 2 = 0$

3. The quadratic equation whose one root is $\frac{1}{2+\sqrt{5}}$ will be

[RPET – 1987]

(a) $x^2 + 4x - 1 = 0$ (b) $x^2 + 4x + 1 = 0$
(c) $x^2 - 4x - 1 = 0$ (d) $\sqrt{2}x^2 - 4x + 1 = 0$

4. The roots of the equation $x^2 + 2\sqrt{3}\,x + 3 = 0$ are

[RPET – 1986]

(a) real and unequal
(b) rational and equal
(c) irrational and equal
(d) irrational and unequal

5. The value of k for which $2x^2 - kx + x + 8 = 0$ has equal and real roots are

[BIT Ranchi – 1990]

(a) – 9 and – 7 (b) 9 and 7
(c) – 9 and 7 (d) 9 and – 7

6. The least integer k which makes the roots of the equation $x^2 + 5x + k = 0$ imaginary is

[Kerala (Engg.) – 2002]

(a) 4 (b) 5 (c) 6 (d) 7

7. If $2 + i\sqrt{3}$ is a root of the equation $x^2 + px + q = 0$, where p and q are real, then $(p, q) =$

[IIT – 1981; MPPET – 1997]

(a) $(-4, 7)$ (b) $(4, -7)$
(c) $(4, 7)$ (d) $(-4, -7)$

8. If a root of the equation $x^2 + px + 12 = 0$ is 4, while the roots of the equation $x^2 + px + q = 0$ are same, then the value of q will be

[RPET – 1991; AIEEE – 2004; Kerala PET – 2007]

(a) 4 (b) 4/49
(c) 49/4 (d) none of these

9. The roots of the equation $ix^2 - 4x - 4i = 0$ are

(a) $-2i$ (b) $2i$
(c) $-2i, -2i$ (d) $2i, 2i$

10. If $a + b + c = 0$, then the roots of the equation $4ax^2 + 3bx + 2c = 0$ are

(a) equal
(b) imaginary
(c) real
(d) none of these

11. $x^2 + x + 1 + 2k(x^2 - x - 1) = 0$ is a perfect square for how many values of k

[Orissa JEE – 2004]

(a) 2 (b) 0 (c) 1 (d) 3

12. Roots of $ax^2 + b = 0$ are real and distinct, if

(a) $ab > 0$ (b) $ab < 0$
(c) $a, b > 0$ (d) $a, b < 0$

13. Roots of the equation $x^2 + bx - c = 0$ $(b, c > 0)$ are

(a) both positive (b) both negative
(c) of opposite sign (d) none of these

14. If roots of the equation $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ are equal, then a, b, c are in

[Roorkee – 1993; RPET – 2001]

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

15. The condition for the roots of the equation $(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$ to be equal is

[MPPET – 1995]

(a) $a = 0$ (b) $b = 0$
(c) $c = 0$ (d) none of these

SOLUTIONS

1. (c) For equal roots $B^2 = 4AC$

$\therefore (6p)^2 = 4 \times 4 \times 1$ or $p = \pm\frac{2}{3}$

2. (d) If given one root is $1 + \sqrt{3}$, then its other root must be $1 - \sqrt{3}$

Sum of roots $= (1 + \sqrt{3}) + (1 - \sqrt{3}) = 2$

Product of roots $= (1 + \sqrt{3})(1 - \sqrt{3}) = 1 - 3 = -2$

Thus, the required equation is as follows

$x^2 -$ (sum of roots) x + Product of roots = 0

$x^2 - 2x - 2 = 0$

3. (a) Other root is $\frac{1}{2 - \sqrt{5}}$ By definition

$$x^2 - x\left[\frac{1}{2 + \sqrt{5}} + \frac{1}{2 - \sqrt{5}}\right] + \frac{1}{(2 + \sqrt{5})(2 - \sqrt{5})} = 0$$

or, $x^2 - \frac{4x}{(4 - 5)} + \frac{1}{(4 - 5)} = 0$

or, $x^2 + 4x - 1 = 0$

OR

Step 1: The quadratic equation whose one root is

$\frac{1}{2 - \sqrt{5}} \times \frac{2 - \sqrt{5}}{2 - \sqrt{5}} = \frac{2 - \sqrt{5}}{4 - 5} = \sqrt{5} - 2$ then its other root must be $= -\sqrt{5} - 2$

Step 2: The desired quadratic equation is

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$x^2 - (\sqrt{5} - 2 - \sqrt{5} - 2)x + (\sqrt{5} - 2)(-\sqrt{5} - 2) = 0$

i.e., $x^2 - (-4x) - (5 - 4) = 0$

or $x^2 + 4x - 1 = 0$

4. (c) Here coefficient of x is not rational so their may be two equal irrational roots.

Further, equation is $(x + \sqrt{3})^2 = 0$ so it has two equal irrational roots each equal to $-\sqrt{3}$.

5. (d) $2x^2 + x(1 - k) + 8 = 0$

Roots are equal $\Rightarrow B^2 - 4AC = 0$

or $(k - 1)^2 - 4 . 2 . 8 = 0$

or $(k - 1)^2 = 8^2$

or $k - 1 = \pm 8$

or $k = 1 \pm 8 = 9, -7$

6. (d) Roots are non-real if disc. < 0

i.e., if $5^2 - 4.1.k < 0$

i.e., $4k > 25$

i.e., if $k > \frac{25}{4}$ $k > 6 + \frac{1}{4}$

Hence, the required least integer k is 7.

7. (a) If $x^2 + px + q = 0$ has one root $\alpha = 2 + i\sqrt{3}$, then its other root must be $\beta = 2 - i\sqrt{3}$

$\alpha + \beta = -p, \alpha\beta = q$

$\Rightarrow -p = 4, q = 2^2 + (\sqrt{3})^2 = 7$

$\Rightarrow (p, q) = (-4, 7)$

8. (c) A, 4 are roots of $x^2 + px + 12 = 0$

$\therefore \alpha + 4 = -p, 4\alpha = 12 \Rightarrow \alpha = 3$

$\Rightarrow 3 + 4 = -p \Rightarrow p = -7.$

Equation $x^2 + px + q = 0$ becomes $x^2 - 7x + q = 0$.

It has equal roots.

$\therefore B^2 - 4AC = 0$ or, $49 - 4q = 0$ or $q = \frac{49}{4}$

9. (c) $ix^2 - 4x - 4i = 0$ (Coefficients are not real)

$$x = \frac{4 \pm \sqrt{16 - 4i(-4i)}}{2 \times i}$$

$$x = \frac{4 \pm \sqrt{16 - 16}}{2i}$$

$$x = \frac{4}{2i} = \frac{2}{i} \times \frac{i}{i} = \frac{2i}{-1} = -2i$$

$= -2i, -2i$ (i.e., both roots are equal)

10. (c) $B^2 - 4AC = 9b^2 - 32ac$

$= 9(-a - c)^2 - 32ac$ $[\because a + b + c = 0]$

$= 9a^2 + 9c^2 - 14ac$

$= c^2\left[9\left(\frac{a}{c}\right)^2 - 14\frac{a}{c} + 9\right]$

which is always positive. $[\because B^2 - 4AC < 0]$

Note: Sign of quadratic $ax^2 + bx + c$ is same as a if $b^2 - 4ac \le 0$.

11. (a) Given equation is $(1 + 2k)x^2 + (1 - 2k)x + (1 - 2k) = 0$

If equation is a perfect square then its both root are equal

i.e., $b^2 - 4ac = 0$

i.e., $(1 - 2k)^2 - 4(1 + 2k)(1 - 2k) = 0$

i.e., $k = \frac{1}{2}, \frac{-3}{10}$

Hence, total number of values = 2.

12. (b) Since, roots of $ax^2 + b = 0$ are real and distinct

if $B^2 - 4AC > 0$

i.e., $0 - 4ab > 0$ i.e. $4ab < 0$ i.e., $ab < 0$

a and b are of the opposite signs.

13. (c) Since, $b, c > 0$

Therefore, $\alpha + \beta = -b < 0$ and $\alpha\beta = -c < 0$ Since, product of the roots is negative. Therefore, roots must be of opposite sign.

14. (c) $b^2(c - a)^2 - 4ac(b - c)(a - b) = 0$

or $b^2(c^2 + a^2 - 2ac) - 4ac[ab - ac - b^2 + bc] = 0$

or $b^2(c^2 + a^2 - 2ac + 4ac) + 4a^2c^2 - 4abc(c + a) = 0$

or $[b(c + a)]^2 + (2ac)^2 - 2 . 2ac . b(c + a) = 0$

or $[b(c + a) - 2ac]^2 = 0.$

$\therefore b(c + a) = 2ac$

or $b = \frac{2ac}{a + c}$

$\therefore b$ is H.M. of a and c i.e., a, b, c are in H.P.

Alternative Method

Here $\Sigma a(b - c) = 0$ i.e. algebraic sum of the coefficients is zero.

$\therefore x = 1$ is a root and since both roots are equal, therefore they are 1, 1.

$\therefore \alpha\beta = 1 = \dfrac{c\,(a-b)}{a\,(b-c)}$ or $b = \dfrac{2ac}{a+c}$

$\therefore$ a, b, c are in H.P.

15. (a) $(c^2 - ab)\,x^2 - 2\,(a^2 - bc)\,x + (b^2 - ac) = 0$

If the roots be equal, then $B^2 - 4AC = 0$

$\therefore 4\,(a^2 - bc)^2 - 4(c^2 - ab)\,(b^2 - ac) = 0$

or $[a^4 - 2a^2\,bc + b^2c^2] - [b^2c^2 - ab^3 - ac^3 + a^2bc] = 0$

or $a\,(a^3 + b^3 + c^3 - 3abc) = 0$

$\therefore$ Either $a = 0$ or $a^3 + b^3 + c^3 - 3abc = 0$.

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. If the roots of $4x^2 + px + 9 = 0$ are equal, then absolute value of p is **[MPPET – 1995]**

(a) 144 (b) 12
(c) – 12 (d) ±12

2. If the equation $(m - n)\,x^2 + (n - l)\,x + l - m = 0$ has equal roots, then l, m and n satisfy **[DCE – 2002]**

(a) $2l = m + n$
(b) $2m = n + 1$
(c) $m = n + 1$
(d) $1 = m + n$

3. The quadratic equation with real coefficients whose one root is $7 + 5i$, will be **[RPET – 1992]**

(a) $x^2 - 14x + 74 = 0$
(b) $x^2 + 14x + 74 = 0$
(c) $x^2 - 14x - 74 = 0$
(d) $x^2 + 14x - 74 = 0$

4. The quadratic equation whose one root is $2 - \sqrt{3}$ will be **[RPET – 1985]**

(a) $x^2 - 4x - 1 = 0$
(b) $x^2 - 4x + 1 = 0$
(c) $x^2 + 4x - 1 = 0$
(d) $x^2 + 4x + 1 = 0$

5. If $3 + 4i$ is a root of the equation $x^2 + px + q = 0$ (p, q are real numbers), then **[EAMCET – 1985]**

(a) $p = 6, q = 25$
(b) $p = 6, q = 1$
(c) $p = -6, q = -7$
(d) $p = -6, q = 25$

6. Roots of $x^2 + k = 0$, $k < 0$ are

(a) complex conjugates
(b) real and distinct
(c) real and equal
(d) rational

7. If p, q, r are real and $p \neq q$, then the roots of the equation $(p - q)\,x^2 + 5\,(p + q)\,x - 2\,(p - q) = r$ are

(a) real and equal
(b) unequal and rational
(c) unequal and irrational
(d) nothing can be said

8. The roots of the quadratic equation $(a + b - 2c)x^2 - (2a - b - c\,)\,x + (a - 2b + c) = 0$ are

(a) $a + b + c$ and $a - b + c$
(b) 1/2 and $a - 2b + c$
(c) $a - 2b + c$ and $1 / a + b - x$
(d) none of these

9. If $a + b + c = 0$, $a \neq 0$, $a, b, c \in Q$, then both the roots of the equation $ax^2 + bx + c = 0$ are

(a) rational
(b) non–real
(c) irrational
(d) zero

10. The roots of the quadratic equation $2x^2 + 3x + 1 = 0$, are **[IIT–1983]**

(a) irrational
(b) rational
(c) imaginary
(d) none of these

WORK SHEET: TO CHECK PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the work sheet
2. The test is of 13 minutes.
3. The test consists of 13 questions. The maximum marks are 39.
4. Use blue/black ball point pen only for writing particulars/marking responses. Use of pencil is Strictly prohibited.

1. Let one root of $ax^2 + bx + c = 0$ where a, b, c are integers be $3 + \sqrt{5}$, then the other root is

[MNR – 1982]

(a) $3 - \sqrt{5}$ (b) 3
(c) $\sqrt{5}$ (d) none of these

2. The roots of the given equation $(p - q)x^2 + (q - r)x + (r - p) = 0$ are

[RPET – 1986; MPPET – 1999; Pb. CET – 2004]

(a) $\frac{p-q}{r-p}, 1$ (b) $\frac{q-r}{p-q}, 1$
(c) $\frac{r-p}{p-q}, 1$ (d) $1, \frac{q-r}{p-q}$

3. If a and b are the odd integers, then the roots of the equation $2ax^2 + (2a + b)x + b = 0$, $a \neq 0$ will be

[Pb. CET – 1988]

(a) rational (b) irrational
(c) non–real (d) equal

4. The value of k for which the quadratic equation, $kx^2 + 1 = kx + 3x - 11x^2$ has real and equal roots are

[BIT Ranchi – 1993]

(a) $-11, -3$ (b) $5, 7$
(c) $5, -7$ (d) none of these

5. or what values of k will the equation $x^2 - 2(1 + 3k)x + 7(3 + 2k) = 0$ have equal roots

[MPPET – 1997]

(a) $1, -\frac{10}{9}$ (b) $2, -\frac{10}{9}$
(c) $3, -\frac{10}{9}$ (d) $4, -\frac{10}{9}$

6. If one root of the quadratic equation, $ix^2 - 2(i + 1)x + (2 - i) = 0$ is $2 - i$, then the other root is

(a) $-i$ (b) i (c) $2 + i$ (d) $2 - i$

7. If $1 - i$ is a root of the equation $x^2 - ax + b = 0$, then $b =$

[EAMCET – 2002]

(a) -2 (b) -1 (c) 1 (d) 2

8. Both the roots of equation $x^2 - x - 3 = 0$ are

(a) real rational (b) real irrational
(c) real and equal (d) imaginary roots

9. The equation whose roots are $\frac{3}{7}$ and $\frac{4}{5}$ is

(a) $35x^2 + 13x - 12 = 0$
(b) $35x^2 - 13x + 12 = 0$
(c) $35x^2 + 13x + 12 = 0$
(d) $35x^2 - 13x - 12 = 0$

10. One root of the equation $x^2 = px + q$ is reciprocal of the other and $p \neq \pm 1$. What is the value of q?

[NDA – 2008]

(a) $q = -1$ (b) $q = 1$
(c) $q = 0$ (d) $q = \frac{1}{2}$

11. The solution of the equation $x + \frac{1}{x} = 2$ will be

[MNR – 1983]

(a) $2, -1$ (b) $0, -1, -\frac{1}{5}$
(c) $-1, -\frac{1}{5}$ (d) none of these

12. If the roots of the equation $x^2 + 2mx + m^2 - 2m + 6 = 0$ are same, then the value of m will be

[MPPET – 1986]

(a) 3 (b) 0 (c) 2 (d) -1

13. The equation of the smallest degree with real coefficients having $1 + i$ as one of the root is

[Kerala (Engg.) – 2002]

(a) $x^2 + x + 1 = 0$
(b) $x^2 - 2x + 2 = 0$
(c) $x^2 + 2x + 2 = 0$
(d) $x^2 + 2x - 2 = 0$

ANSWER SHEET

1.	ⓐ	ⓑ	ⓒ	ⓓ	6.	ⓐ	ⓑ	ⓒ	ⓓ	10.	ⓐ	ⓑ	ⓒ	ⓓ
2.	ⓐ	ⓑ	ⓒ	ⓓ	7.	ⓐ	ⓑ	ⓒ	ⓓ	11.	ⓐ	ⓑ	ⓒ	ⓓ
3.	ⓐ	ⓑ	ⓒ	ⓓ	8.	ⓐ	ⓑ	ⓒ	ⓓ	12.	ⓐ	ⓑ	ⓒ	ⓓ
4.	ⓐ	ⓑ	ⓒ	ⓓ	9.	ⓐ	ⓑ	ⓒ	ⓓ	13.	ⓐ	ⓑ	ⓒ	ⓓ
5.	ⓐ	ⓑ	ⓒ	ⓓ										

HINTS AND EXPLANATIONS

3. (a) Given equation $2ax^2 + (2a + b)x + b = 0$, $(a \neq 0)$

Now its discriminant $D = B^2 - 4AC$

$= (2a + b)^2 - 4.2ab = (2a - b)^2$

Hence, D is a perfect square, so given equation has rational roots.

5. (b) Step 1: The equation $x^2 - 2\,(1 + 3k)\,x + 7\,(3 + 2k) = 0$

Here $a = 1$, $b = -2\,(1 + 3k)$, $c = 7(3 + 2k)$

Step 2: For equal roots $b^2 - 4ac = 0$

So, $(2 + 6k)^2 - 4\,(21 + 14k) = 0$

$\Rightarrow 4 + 36k^2 + 24\,k - 84 - 56\,k = 0$

$\Rightarrow 36k^2 - 32k - 80 = 0$

$\Rightarrow 9k^2 - 8k - 20 = 0$

using $k = \dfrac{18 \pm \sqrt{64 + 80(9)}}{18} = \dfrac{8 \pm 28}{18}$

The required value of k are 2, $\dfrac{-10}{9}$

Note: By using formula: $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

12. (a) $x^2 + 2mx + (m^2 - 2m + 6) = 0$

Here $\beta = \alpha$

$\therefore D = 0$; $(2m)^2 - 4(m^2 - 2m + 6) = 0$

$\Rightarrow 2m - 6 = 0$ or $m = 3$.

This page is intentionally left blank.

LECTURE

3 Transformation of Root

BASIC CONCEPTS

1. **Some Important Formulas Connected with Roots of Quadratic Equation:**

   $Ax^2 + Bx + C = 0$

   (i) If one root is λ times the other root (α, λ, α,), then condition is

   $AC\ (1 + \lambda)^2 = B^2\lambda$

***Note* 1:** If both roots of the quadratic be in the ratio $m: n$ then required condition is: $AC\ (m + n)^2 = m\ n\ B^2$

   (ii) If one root is square of the other then condition is $AC\ (A + C) + B^3 = 3ABC$

   (iii) If one root is n^th^ power of the other root then condition is

   $(ac^n)^{\frac{1}{n+1}} + (a^n c)^{\frac{1}{n+1}} + b = 0$

Proof:

Let one root of equation $ax^2 + bx + c = 0$ is a, therefore, second root is a^n.

Thus, $a + a = -\dfrac{b}{a}$ (1)

and $(a)(a^n) = \dfrac{c}{a}$

$$\Rightarrow a^{n+1} = \frac{c}{a} \Rightarrow a = \left(\frac{c}{a}\right)^{\frac{1}{n+1}}$$

Putting the value of a in Equation (1) we get,

$$\Rightarrow \left(\frac{c}{a}\right)^{\frac{1}{n+1}} + \left(\frac{c}{a}\right)^{\frac{1}{n+1}} = \frac{-b}{a}$$

$$\Rightarrow a\left(\frac{c}{a}\right)^{\frac{1}{n+1}} + a\left(\frac{c}{a}\right)^{\frac{1}{n+1}} = -b$$

$$\Rightarrow \left(a^{n+1}\frac{c}{a}\right)^{\frac{1}{n+1}} + \left(a^{n+1}\frac{c^n}{a^n}\right)^{\frac{n}{n+1}} + b = 0$$

$$\Rightarrow (a^n c)^{\frac{1}{n+1}} + (ac^n)^{\frac{1}{n+1}} = b = 0$$

   (iv) If difference of the roots is 1 then $B^2 - 4AC = A^2$

   (v) If both roots are of opposite sign then $\alpha\beta = \dfrac{C}{A} < 0$ i.e., A and C are of the opposite sign.

   (vi) If both the roots are positive then

   $\alpha + \beta = -\dfrac{B}{A} > 0;\ \alpha\beta = \dfrac{C}{A} > 0$ i.e.,

   $1, A > 0, B < 0, C > 0$

   or $A < 0, B > 0, C < 0$

   (vii) If both the roots are negative then A, B and C are of the same sign.

Note: If $A > 0$, $B > 0$, $C > 0$ and $B^2 - 4AC < 0$ then

   (viii) If a number a is such that $f(a) = 0$ and $f'(a) = 0$, then a is a repeated root of $f(x) = 0$ and $f(x) = A\ (x - a)^2$.

2. **Formation of Quadratic Equation in terms of the Transformation of the Roots of $Ax^2 + Bx + C = 0$**

Let $\alpha\beta$, be the roots of the equation $Ax^2 + Bx + C = 0$, then the quadratic equation.

(i) whose roots are $-\alpha$, β is $x^2 - (-\alpha - \beta)x + (-a)(-\beta) = 0$ or
$Ax^2 - Bx + C = 0$

(ii) whose roots are $1/\alpha$, $1/\beta$ is
$$x^2 - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)x + \left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right) = 0$$
$Cx^2 + Bx + A = 0$

(iii) whose roots are $-(1/\alpha)$, $-(1/\beta)$ is
$$x^2 - \left(\frac{-1}{\alpha} - \frac{1}{\beta}\right)x + \left(-\frac{1}{\alpha}\right)\left(-\frac{1}{\beta}\right) = 0$$
or
$Cx^2 - Bx + A = 0$

(iv) form a quadratic equation whose roots are k more $(\alpha + k, \beta + k)$, then the roots of given equation change x to $x - k$.
$x^2 - (\alpha + k + \beta + k)x + (\alpha + k)(\beta + k) = 0$
or $A(x - k)^2 + B(x - k) + C = 0$

(v) Form a quadratic equation whose roots are k less $(\alpha - k, \beta - k)$ than the roots of given equation change x to $x + k$.
$x^2 - \{(\alpha - k) + (\beta - k)\}x + (\alpha - k)(\beta - k) = 0$
or $A(x + k)^2 + B(x + k) + C = 0$

***Note* 1**: To find an equation whose roots are reciprocals of the roots of the given equation change to x to $1/x$. For example: case (ii)

***Note* 2:** To find an equation whose roots are with opposite signs to those of the given equations change to x to $-x$. For example: case (i)

***Note* 3:** To find an equation whose roots are square of the roots of the given equation change to x to $\sqrt{x}$

(vi) To find an equation whose roots are λ times the roots of given equation change x to x/λ and resulting equation is $Ax^2 + \lambda Bx + C\lambda^2 = 0$.

SUBJECTIVE SOLVED PROBLEMS: (CBSE/STATE BOARD): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Find the equation whose roots are double of the roots of the equation $4x^2 - 5x - 3 = 0$

Solution

Let the roots of equation $4x^2 - 5x - 3 = 0$ be α and β.

Therefore, $\alpha + \beta = \frac{-b}{a} = \frac{5}{4}$

and $\alpha\beta = \frac{c}{a} = \frac{-3}{4}$

Roots of required equation are 2α, 2β.

Therefore, required equation is, $x^2 -$ (sum of the roots) x + (product of the roots) = 0

$\Rightarrow x^2 - (2\alpha + 2\beta)x + (2\beta\alpha \times 2\beta) = 0$

$\Rightarrow x^2 - 2(\alpha + \beta)x + 4\alpha\beta = 0$

$$\Rightarrow x^2 - 2\left(\frac{5}{2}\right)x + 4\left(-\frac{3}{4}\right) = 0$$

Therefore, $2x^2 - 5x - 6 = 0$

OR

Change x to $\frac{x}{2}$ in given equation

we find $4\left(\frac{x}{2}\right)^2 - 5\left(\frac{x}{2}\right) - 3 = 0$

or $x^2 - \frac{5x}{2} - 3 = 0$

or $2x^2 - 5x - 6 = 0$

Ans.

2. If $a^2 = 5a - 3$ and $b^2 = 5b - 3$ where $a \neq b$ form a quadratic equation whose roots are $\frac{a}{b}$ and $\frac{a}{b}$.

[AIEEE – 2002]

Solution

Let $x^2 - 5x + 3 = 0$ be a quadratic equation. If a and b are roots of this quadratic equation, then a, b will satisfy this equation.

Therefore, $a^2 - 5a + 3 = 0$

and $b^2 - 5b + 3 = 0$

Which are given conditions

Now, Sum of roots = $a + b = 5$

and Product of roots = $ab = 3$

Therefore, $\frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ab}$

$= \frac{(a+b)^2 - 2ab}{ab} = \frac{(5)^2 - 2(3)}{3}$

$= \frac{25 - 6}{3} = \frac{19}{3}$

Thus, the equation of that quadratic equation whose roots are $\frac{a}{b}$ and $\frac{b}{a}$ ia

$x^2 - \left(\frac{a}{b} + \frac{b}{a}\right)x + \left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = 0$

$\Rightarrow x^2 - \frac{19}{3}x + 1 = \Rightarrow 3x^2 - 19x + 3 = 0$

UNSOLVED SUBJECTIVE PROBLEMS: (CBSE/STATE BOARD): TO GRASP THE TOPIC, SOLVE THESE PROBLEMS

Exercise I

1. In equation $x^2 + px + q = 0$, if one of the roots is double that of the other root, then prove that $2p^2 = 9q$.
2. If $3p^2 = 5p + 2$ and $3q^2 = 5q + 2$ where $p \neq q$, then find the value of pq.
3. Find the equation whose roots are double of the roots of the equation $4x^2 - 5x - 3 = 0$.
4. If the roots of the equation $ax^2 + bx + c = 0$ are in ratio $m: n$, then prove that $mnb^2 = (m + n)2\ ac$.
5. If one root of the equation $ax^2 + bx + c = 0$ be the square of the other, then prove that $b^3 + ac^2 + a^2c = 3abc$.
6. If one root of $ax^2 + 10x + 5 = 0$ is three times the other, then find the value of a.
7. If roots of the quadratic equation $x^2 + ax + 12 = 0$ are in the ratio 1: 3, then find the value of a.

Exercise II

1. Form an equation whose roots are 2 more than the roots of the equation $x^2 - bx + c = 0$.
2. If sum of roots of the equation $\frac{x^2 - bx}{ax - c} = \frac{\lambda - 1}{\lambda + 1}$ is zero, then fi nd the value of λ.
3. If one root of the equation $3x^2 + px + 3 = 0$ $(p > 0)$ is square of other, then find the value of p.
4. If c and d are roots of equation $(x - a)(x - b) - k = 0$, then prove that a and b are roots of equation $(x - c)(x - d) + k = 0$.
5. If one root is square of other of equation $x^2 - x - k = 0$, then prove that $k = 2 \pm \sqrt{5}$.

ANSWERS

Exercise I

2. $pq = -2/3$
3. $2x^2 - 5x - 6 = 0$
6. $a = 0, a = 15/4$
7. $a = \pm 8$

Exercise II

1. $x^2 - (4 + b)x + c + 2b + 4 = 0$
2. $\lambda = \frac{a - b}{a + b}$
3. $p = 3$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. If $x^2 + px + q = 0$ is the quadratic equation whose roots are $a - 2$ and $b - 2$ where a and b are the roots of $x^2 - 3x + 1 = 0$, then

[Kerala (Engg.) – 2002]

(a) $p = 1, q = 5$ (b) $p = 1, q = -5$
(c) $p = -1, q = 1$ (d) none of these

Solution

(d) If a and b are roots of equation $x^2 - 3x + 1 = 0$ then the equation with roots $a - 2$, $b - 2$ is obtained by replacing,

$x \rightarrow x + 2$

$(x + 2)^2 - 3(x + 2) + 1 = 0$

$x^2 + x - 1 = 0$

$p = 1, q = -1$ (by comparing)

2. The value of 'a' for which one root of the quadratic equation $(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$ is twice as large as the other, is

[AIEEE 2003; MPPET – 2009]

(a) 2/3 (b) – 2/3
(c) 1/3 (d) – 1/3

Solution

(a) Let the roots are a and $2a$

$\Rightarrow a + 2a = \frac{1 - 3a}{a^2 - 5a + 3a}$ and

$a.\ 2a = \frac{2}{a^2 - 5a + 3}$, Eliminating a

$\Rightarrow 2\left[\frac{1}{9}\frac{(1 - 3a)^2}{(a^2 - 5a + 3)^2}\right] = \frac{2}{a^2 - 5a + 3}$

$\Rightarrow \frac{(1 - 3a)^2}{(a^2 - 5a + 3)} = 9$

$\Rightarrow 9a^2 - 6a + 1 = 9a^2 - 45a + 27$

$\Rightarrow 39a = 26$

$\Rightarrow a = \frac{2}{3}$

OR

Using the formula: $AC(1 + \lambda)^2 = B^2\lambda$

3. If the roots of the equation $ax^2 + cx + c = 0$ are in the ratio of $p : q$ than

$\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{c}{a}} = 0$

(a) 0 (b) 1
(c) 2 (d) 3

[RPET – 1997; Pb. CET – 1992]

Solution

(a) Let the roots be pa and qa then their sum

$a(p + q) = \frac{-c}{a}$ (1)

and product $a^2 pq = \frac{c}{a}$ (2)

Substituting the value of a from (2) in (1), we get,

$\frac{\sqrt{\frac{c}{a}}(p + q)}{\sqrt{pq}} + \frac{c}{a} = 0$

$\Leftrightarrow \frac{p + q}{\sqrt{pq}} + \sqrt{\frac{c}{a}} = 0$ (dividing by $\sqrt{\frac{c}{a}}$)

$\Leftrightarrow \frac{p}{\sqrt{pq}} + \frac{q}{\sqrt{pq}} + \sqrt{\frac{c}{a}} = 0$

$\Leftrightarrow \sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{c}{a}} = 0$ Ans.

4. If $\alpha\beta$ are roots of $x^2 - 5x - 3 = 0$, then the equation with roots $\frac{1}{2a - 3}$ and $\frac{1}{2\beta - 3}$ is

[PET (Raj.) – 1998]

(a) $33x^2 + 4x - 1 = 0$
(b) $33x^2 - 4x + 1 = 0$
(c) $33x^2 - 4x - 1 = 0$
(d) $33x^2 + 4x + 1 = 0$

Solution

(a) Sum of roots

$= \frac{1}{2\alpha - 3} + \frac{1}{2\beta - 3} = \frac{2(\alpha + \beta) - 6}{4\alpha\beta - 6(\alpha + \beta) + 9}$

$= \frac{-4}{33}$

Product of roots $= \frac{1}{2\beta - 3} + \frac{1}{2\alpha - 3}$

$= \frac{2(\alpha + \beta) - 6}{4\alpha\beta - 6(\alpha + \beta) + 9} = \frac{-1}{33}$

Hence, equation is x^2 – (sum of roots) x + product of roots = 0.

Therefore $33x^2 + 4x - 1 = 0$

5. The value of k for which one of the roots of $x^2 - x + 3k = 0$ is double of one of the roots of $x^2 - x + k = 0$ is

[UPSEAT – 2001]

(a) 1 (b) – 2
(c) 2 (d) none of these

Solution

(b) Let a be a root of $x^2 - x + k = 0$, then $2a$ is a root of $x^2 - x + 3k = 0$.

$\therefore a^2 - a + k = 0$ and $4(\alpha)^2 - 2\alpha + k = 0$

$$\Rightarrow \frac{\alpha^2}{-3k+2k} = \frac{\alpha}{4k-3k} = \frac{1}{-2+4}$$

$$\Rightarrow a^2 = \frac{-k}{2} \text{ and } a = \frac{k}{2}$$

Now $a^3 = (a)^2 \Rightarrow \frac{-k}{2} = \left(\frac{k}{2}\right)^2$

$\Rightarrow k^2 + 2k = 0$

$\Rightarrow k = 0$ or -2

6. If $\alpha\beta$ are roots of the equation $ax^2 + bx + c = 0$ then the equation whose roots are $a + 1/\beta$ and $\beta + 1/\alpha$ will be

(a) $acx^2 + b(c + a)x + (c + a)^2 = 0$

[PET (Raj.) – 1991]

(b) $acx2 - b(c + a)x - (c + a)^2 = 0$

(c) $acx^2 - b(c + a)x - (c + a)^2 = 0$

(d) none of these

Solution

(a) We have $\alpha + \beta = -b/a$, $\alpha\beta = c/a$

Now sum of the roots of the required equation

$$= \left(\alpha + \frac{1}{\beta}\right) + \left(\beta + \frac{1}{\alpha}\right)$$

$$= (\alpha + \beta) + \left(\frac{\alpha + \beta}{\alpha\beta}\right)$$

$$= -\frac{b}{a} - \frac{b}{c} = \frac{-b(c+a)}{ac}$$

Product of the roots

$$= \left(\alpha + \frac{1}{\beta}\right)\left(\beta + \frac{1}{\alpha}\right)$$

$$= \alpha\beta + \frac{1}{\alpha\beta} + 2\frac{c}{a} + \frac{a}{c} + \frac{a}{c} + 2 = \frac{(c+a)^2}{ac}$$

Therefore, requied equation is

$$x^2 + \frac{b(c+a)}{ac} + \frac{(c+a)^2}{ac} = 0$$

$\Rightarrow acx^2 + b(c + a)x + (c + a)^2 = 0$

7. A polynomial whose zeros are the squares of the roots of the equation $x^2 - 3x + 1 = 0$ is

[MPPET – 2007]

(a) $x^2 - 7x + 1$ (b) $x^2 - 7x - 1$

(c) $x^2 + 7x + 1$ (d) none of these

Solution

(a) : $\alpha + \beta = 3$, $\alpha\beta = 1$

Therefore, polynomial whose roots are α^2, β^2

$= x^2 - (a^2 + \beta^2)x + \alpha^2\beta^2$

$= x^2 - [(\alpha + \beta)^2 - 2\alpha\beta]x + \alpha^2\beta^2$

$= x^2 - [(3)^2 - 2(1)]x + (1)^2$

$= x^2 - 7x + 1$

8. If one root of the equation $ax^2 + bx + c = 0$ be the square of the other, then $a(c - b)^3 = cX$, where X is

(a) $a^3 + b^3$ (b) $(a - b)^3$

(c) $a^3 - b^3$ (d) none of these

Solution

(b) If one root is square of other of the equation $ax^2 + bx + c = 0$, then $b^3 + ac^2 + a^2c = 3abc$

Which can be written in the form $a(c - b)^3 = c(a - b)^3$

Trick: Let roots be 2 and 4, then the equation is $x^2 - 6x + 8 = 0$

Here obviously,

$$X = \frac{a(c-b)^3}{c} = \frac{1(14)^3}{8} = \frac{14}{2} \times \frac{14}{2} \times \frac{14}{2} = 7^3$$

which is given by $(a - b)^3 = 7^3$. (verifi cation method)

9. Which one of the following is one of the roots of the equation $(b - c)x^2 + (c - a)x + (a - b) = 0$?

[NDA – 2009]

(a) $(c - a)/(b - c)$ (b) $(a - b)/(b - c)$

(c) $(b - c)/(a - b)$ (d) $(c - a)/(a - b)$

Solution

(b) $(b - c)x^2 + (c - a)x + (a - b) = 0$

$\Rightarrow (b - c)x^2 - (b - c - b + a)x + (a - b) = 0$

$\Rightarrow (b - c)x(x - 1) - (a - b)(x - 1) = 0$

$\Rightarrow \{(b - c)x - (a - b)\}\{x - 1\} = 0$

$$\Rightarrow x = \frac{a-b}{b-c} \text{ and } x = 1$$

OBJECTIVE PROBLEMS WITH SOLUTIONS: IMPORTANT QUESTIONS

1. If the roots of the equation $2x^2 + 3x + 2 = 0$ be α and β, then the equation whose roots are $a + 1$ and $\beta + 1$ is

(a) $2x^2 + x + 1 = 0$ (b) $2x^2 - x - 1 = 0$
(c) $2x^2 - x + 1 = 0$ (d) none of these

2. If α and β be the roots of the equation $2x^2 + 2(a + b)\,x + a^2 + b^2 = 0$, then the equation whose roots are $(\alpha + \beta)^2$ and $(\alpha - \beta)^2$ is

(a) $x^2 - 2abx - (a^2 - b^2)^2 = 0$
(b) $x^2 - 4abx - (a^2 - b^2)^2 = 0$
(c) $x^2 - 4abx + (a^2 - b^2)^2 = 0$
(d) none of these

3. If the roots of the equation $12x^2 - mx + 5 = 0$ are in the ratio 2 : 3, then m =

[RPET – 2002]

(a) $5\sqrt{10}$ (b) $3\sqrt{10}$
(c) $2\sqrt{10}$ (d) none of these

4. The condition that one root of the equation $ax^2 + bx + c = 0$ is three times the other is

[DCE – 2002]

(a) $b^2 = 8ac$ (b) $3b^2 + 16ac = 0$
(c) $3b^2 = 16ac$ (d) $b^2 + 3ac = 0$

5. Let α and β be the roots of the equation $x^2 + x + 1 = 0$. The equation whose roots are a^{19}, β^7 is

(a) $x^2 - x - 1 = 0$ (b) $x^2 - x + 1 = 0$
(c) $x^2 + x - 1 = 0$ (d) $x^2 + x + 1 = 0$

[IIT (Sc.) 1994; DCE – 2003]

6. If the roots of the equation $x^2 + x + 1 = 0$ are α, β and the roots of the equation $x^2 + px + q = 0$ are $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ then p is equal to

(a) – 2 (b) – 1
(c) 1 (d) 2

[RPET – 1987]

7. The value of p for which one root of the equation $x^2 - 30x + p = 0$ is the square of the other, are

[Roorkee – 1998]

(a) 125 only (b) 125 and– 216
(c) 125 and 215 (d) 216 only

8. Let two numbers have arithmetic mean 9 and geometric mean 4. Then, these numbers are the roots of the quadratic equation

[AIEEE – 2004]

(a) $x^2 - 18x - 16 = 0$
(b) $x^2 - 18x + 16 = 0$
(c) $x^2 + 18x - 16 = 0$
(d) $x^2 + 18x + 16 = 0$

9. If $(k \in (-\infty, -2) \cup (2, \infty))$, then the roots of the equation $x^2 + 2kx + 4 = 0$ are

[DCE – 2002]

(a) complex
(b) real and unequal
(c) real and equal
(d) one real and one imaginary

10. If α, β are roots of $x^2 - 3x + 1 = 0$, then the equation whose roots are $\frac{1}{\alpha - 2}, \frac{1}{\beta - 2}$, is

[RPET – 1999]

(a) $x^2 + x - 1 = 0$ (b) $x^2 + x + 1 = 0$
(c) $x^2 - x - 1 = 0$ (d) none of these

11. The roots of $Ax^2 + Bx + C = 0$ are r and s. For the roots of $x^2 + px + q = 0$ to be r^2 and s^2, what must be the value of p?

(a) $(B^2 - 4AC)/A^2$ (b) $(B^2 - 2AC)/A^2$
(c) $(2AC - B^2)/A^2$ (d) $B^2 - 2C$

[NDA – 2009]

12. The equation whose roots are the reciprocal of the roots of the equation $x^2 + 2x + 3 = 0$ is

(a) $x^2 + 2x + 1 = 0$ (b) $3x^2 - 2x + 1 = 0$
(c) $3x^2 + 2x + 1 = 0$ (d) none of these

SOLUTIONS

1. (c) Given α and β are roots of equations $2x^2 + 3x + 2 = 0$ then,

$\alpha + \beta - \frac{3}{2}$ and $\alpha\beta = \frac{2}{2} = 1$

Sum of given root $= \alpha + 1 + \beta + 1$

$= \alpha + \beta + 2$

$= -\frac{3}{2} + \frac{2}{1}$

$= \frac{1}{2}$

Product of given roots $= (\alpha + 1)(\beta + 1)$

$= \alpha\beta + \alpha + \beta + 1$

$= 1 + \frac{-3}{2} + 1$

$= 2 - \frac{3}{2} = \frac{1}{2}$

Required equation: $x^2 - \frac{1}{2}x + \frac{1}{2} = 0$

$\Rightarrow 2x^2 - x + 1 = 0$

OR

Required equation is obtained on replacing x by $x - 1$ in given equation as follows: $2(x - 1)^2 + 3(x - 1) + 2 = 0$

2. (b) Given α and β are roots of $2x^2 + 2(a + b)x + a^2 + b^2 = 0$ then, $\alpha + \beta = \frac{-2(a+b)}{2}$ $= -(a + b)$ and $\alpha\beta = \frac{a^2 + b^2}{2}$.

Sum of desired roots $= (\alpha + \beta)^2 + (\alpha - \beta)^2$

$= \alpha^2 + \beta^2 + 2\alpha\beta + \alpha^2 + \beta^2 - 2\alpha\beta$

$= 2(\alpha^2 + \beta^2)$

$= 2\{(\alpha^2 + \beta^2) - 2\alpha\beta\}$

$= 2\{(a + b)^2 - (a^2 + b^2)\}$

$= 2\{a^2 + b^2 + 2ab - a^2 - b^2\}$

$= 4ab$

Product of desired roots $= (\alpha + \beta)^2 (\alpha - \beta)^2$

$= \{\alpha^2 - \beta^2\}^2$

$= \alpha^4 + \beta^4 - 2\alpha^2\beta^2$

$= (\alpha^2 + \beta^2)^2 - 4\alpha^2\beta^2$

$= \{(\alpha + \beta)^2 - 2\alpha\beta\}^2 - 4(\alpha\beta)^2$

$= \{(a + b)^2 - (a^2 + b^2)\}^2 - 4\frac{(a^2 - b^2)^2}{4}$

$= 4a^2b^2 - a^4 - b^4 - 2a^2b^2$

$= -(a^4 + b^4 - 2a^2b^2)$

$= -(a^2 - b^2)^2$

The required equation is $x^2 - 4abx - (a^2 - b^2)^2 = 0$

3. (a) Let roots of 2α and $3a$

Therefore, $2\alpha + 3a = \frac{m}{12}$ and $(2\alpha)(3a) = \frac{5}{12}$

$\Rightarrow 5a = \frac{m}{12}$ and $6a^2 = \frac{5}{12}$

$6\left(\frac{m}{60}\right)^2 = \frac{5}{15}$

$m^2 = \frac{5}{12} \times \frac{60 \times 60}{6}$

$m = \pm 5\sqrt{10}$

4. (c) In the formula ac $(1 + \lambda)^2 = b^2\lambda$ we have to put 3 for λ to get the answer $16ac = 3b^2$.

5. (d) If roots of the $x^2 + x + 1 = 0$ are quite clearly ω and ω^2 therefore, $\alpha = \omega$

$\beta = \omega^2$

Thus, $\alpha^{19} = \omega^{19} = \omega$

$\beta^7 = \omega^{14} = \omega^2$

Then, the equation will remain same $x^2 + x + 1 = 0$

6. (c) Given α and β are roots of $x^2 + x + 1 = 0$

Therefore, $\alpha + \beta = -1$, $\alpha\beta = 1$

and also given , $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ are roots of $x^2 + px + q = 0$, then,

$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} + -p$

$\frac{\alpha^2 + \beta^2}{\alpha\beta} = -p$

$\frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} = -p$

$\frac{1 - 2}{1} = -p$

$-1 = -p \Rightarrow p = 1$

7. (b) We know that one root of equation $ax^2 + bx + c = 0$ is square of the other if,

$ac(a + c) + b^3 = 3abc$ (1)

Put a = 1, $b = -30$, $c = p$

$p + p^2 + 3(1)(30)(p) - (30)^3 = 0$

$\Rightarrow p^2 + 91p - 27000 = 0$

$\Rightarrow p = 125, -216$

8. (b) Let the numbers be a, b.

Therefore, $\frac{a+b}{2} = 9 = 9\sqrt{ab} = 4$

Therefore $a + b = 18$, $ab = 16$

Therefore, the equation whose roots are a, b is $x^2 - 18x + 16 = 0$

9. (b) Here, $D = 4k^2 - 16 = 4(k^2 - 4)$

For $k \in (-, \infty- 2) \cup (2, \infty)$

$k^2 - 4 > 0 \Rightarrow D > 0$.

Hence, roots will be real and distinct.

10. (c) $\alpha + \beta = 3$, $\alpha\beta = 1 = 1$

$(\alpha - 2) + (\beta - 2) = (\alpha + \beta) - 4 = -1$

$(\alpha - 2)(\beta - 2) = \alpha\beta - 2(\alpha + \beta) + 4 = -1$

Hence, equation with roots $\alpha - 2$ and $\beta - 2$ will be, $x^2 + x - 1 = 0$

$\Rightarrow$ Equation with roots $\frac{1}{\alpha - 2}, \frac{1}{\beta - 2}$, will be

$-x^2 + x + 1 = 0$ or $x^2 - x - 1 = 0$

$\because$ Equation whose roots are reciprocal of the roots of the given quadratic equation is obtained on replacing x by $\frac{1}{x}$.

Alternative Method:

$\frac{1}{x-2} = \Rightarrow x = \frac{1}{y} + 2$

hence, so apply $x \rightarrow \frac{1}{x} + 2$ transformation.

11. (c) Since, r and s are the roots of $Ax^2 + Bx + C = 0$, then

$r + s = -\frac{B}{A}$ and $rs = \frac{C}{A}$

Now, roots of $x^2 + px + q = 0$ be r^2 and s^2

Therefore, $r^2 + s^2 = -p$ and $r^2s^2 = q$

$\Rightarrow (r + s)^2 - 2rs = -p$

$\Rightarrow \frac{B^2}{A^2} - \frac{2C}{A} = -p$

$\Rightarrow \frac{B^2 - 2AC}{A^2} = -p$

$\Rightarrow p = \frac{2AC - B^2}{A^2}$

12. (c) Let α, β be roots of the equation $x^2 + 3x + 3$

Sum of roots $= \alpha + \beta = -2$

Produce of roots $= \alpha\beta = 3$

then required roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$

(as roots are reciprocal to each other)

Sum of desired roots $= \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{-2}{3}$

Product of desired roots $= \frac{1}{\alpha}\frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{1}{3}$

So, the required equation is

$x^2 - \left(\frac{-2}{3}\right)x + \frac{1}{3} = 0$

$\Rightarrow 3x^2 + 2x + 1 = 0$

UNSOLVED OBJECTIVE PROBLEMS: (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. If α, β are the roots of $ax^2 + bx + c = 0$, then the equation whose roots are $2 + \alpha$, $2 + \beta$ is

[EAMCET – 1994]

(a) $ax^2 + x(4a - b) + 4a - 2b + c = 0$

(b) $ax^2 + x(4a - b) + 4a + 2b + c = 0$

(c) $ax^2 + x(b - 4a) + 4a + 2b + c = 0$

(d) $ax^2 + x(b - 4a) + 4a - 2b + c = 0$

2. If one root of $x^2 - x - k = 0$ is square of the other, then k = 0

(a) $2 \pm \sqrt{3}$ (b) $3 \pm \sqrt{2}$

(c) $2 \pm \sqrt{5}$ (d) $5 \pm \sqrt{2}$

[EAMCET –1986, 1987]

3. The equation whose roots are two times the roots of the equation $3x^2 - 8x - 3 = 0$ is

(a) $3x^2 + 16x + 12 = 0$

(b) $3x^2 - 16x - 12 = 0$

(c) $3x^2 + 16x - 12 = 0$

(d) none of these

4. The equation formed by decreasing each root of $ax^2 + bx + c = 0$ by 1 is $2x^2 + 8x + 2 = 0$, then

(a) $a = -b$ (b) $b = -c$

(c) $c = -a$ (d) $b = a + c$

5. If $ax^2 + bx + c = 0$, then $x =$

(a) $\frac{b \pm \sqrt{b^2 - 4ac}}{2a}$ (b) $\frac{-b \pm \sqrt{b^2 - ac}}{2a}$

(c) $\frac{2c}{-b \pm \sqrt{b^2 - 4ac}}$ (d) none of these

6. If α, β are the roots of the equation $2x^2 - 7x + 6 = 0$, then the equation whose roots are a^2 and b^2 is equal to

(a) $4x^2 + 73x + 36 = 0$

(b) $4x^2 + 25x + 36 = 0$

(c) $4x^2 - 25x + 36 = 0$

(d) $4x^2 + 25x - 36 = 0$

7. If α, β are the roots of $9x^2 + 6x + 1 = 0$, then the equation with the roots $\frac{1}{\alpha}, \frac{1}{\beta}$, is

[EAMCET–2000]

(a) $2x^2 + 3x + 18 = 0$ (b) $x^2 + 6x - 9 = 0$

(c) $x^2 + 6x + 9 = 0$ (d) $x^2 - 6x + 9 = 0$

8. Let a, a^2 be the roots of $x^2 + x + 1 = 0$, then the equaton whose roots are α^{31}, α^{62} is

[AMU – 1999]

(a) $x^2 - x + 1 = 0$ (b) $x^2 + x - 1 = 0$

(c) $x^2 + x + 1 = 0$ (d) $x^{60} + x^{30} + 1 = 0$

9. The number of solutions for the equation $x^2 - 5|x| + 6 = 0$ is

[Karnataka CET – 2004]

(a) 4 (b) 3

(c) 2 (d) 1

10. For the equation $3x^2 + px + 3 = 0, p > 0$ if one of the roots is square of the other, then p is equal to

[IIT–Screening – 2000]

(a) 1/3 (b) 1

(c) 3 (d) 2/3

WORK SHEET: TO CHECK PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the work sheet.
2. The test is of 13 minutes.
3. The test consists of 13 questions. The maximum marks are 39.
4. Use blue/black ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. If α and β are the roots of the equation $2x^2 - 3x + 4 = 0$, then the equation whose roots are α^2 and β^2 is

(a) $4x^2 + 7x + 16 = 0$

(b) $4x^2 + 7x + 6 = 0$

(c) $4x^2 + 7x + 1 = 0$

(d) $4x^2 - 7x + 16 = 0$

2. The roots of the equation $x^2 + ax + b = 0$ are p, and q, then the equation whose roots are p^2q and pq^2 will be

[MPPET – 1980]

(a) $x^2 + abx + b^3 = 0$

(b) $x^2 - abx + b^3 = 0$

(c) $bx^2 + x + a = 0$

(d) $x^2 + ax + ab = 0$

3. If α, β be the roots of the equation $x^2 - 2x + 3 = 0$, then the equation whose roots are $1/\alpha^2$ and $1/\beta^2$ is

(a) $x^2 + 2x + 1 = 0$ (b) $9x^2 + 2x + 1 = 0$

(c) $9x^2 - 2x + 1 = 0$ (d) $9x^2 + 2x - 1 = 0$

4. If one root of the equation $x^2 + px + q = 0$ is $2 + \sqrt{3}$, then values of p and q are

[UPSEAT – 2002; Haryana –2004]

(a) $-4, 1$ (b) $4, -1$

(c) $2, \sqrt{3}$ (d) $-2, \sqrt{3}$

5. The real roots of the equation $x^2 + 5|x| + 4$

(a) $-1, 4$ (b) $1, 4$

(c) $-4, 4$ (d) none of these

[UPSEAT – 1993, 1999; orissa JEE – 2004]

6. If the roots of the equation $ax^2 + bx + c = 0$ be α and β, then the roots of the equation $cx^2 + bx + a = 0$ are

(a) $-\alpha, -\beta$ (b) $\alpha, 1/\beta$

(c) $1/\alpha, 1/\beta$ (d) none of these

7. If α, β are the roots of the equation $x^2 + bx + c = 0$, then $\frac{1}{\alpha} + \frac{1}{\beta}$

(a) $1/c$ (b) c/b (c) b/c (d) $-b/c$

8. If a,b are the roots of the equation $x^2 + x + 1 = 0$, then the equation whose roots are α^{22} and β^{19}, is

(a) $x^2 - x + 1 = 0$ (b) $x^2 + x + 1 = 0$
(c) $x^2 + x - 1 = 0$ (d) $x^2 - x - 1 = 0$

9. If the roots of the equation $ax^2 + bx + c = 0$ are l and $2l$, then

[MPPET–1986; MPPET – 2002]

(a) $b^2 = 9ac$ (b) $2b^2 = 9ac$
(c) $b^2 = -4ac$ (d) $a^2 = c^2$

10. The equation whose roots are reciprocal of the roots of the equation $3x^2 - 20x + 17 = 0$ is

[DCE – 2002]

(a) $3x^2 + 20x - 17 = 0$
(b) $17x^2 - 20x + 3 = 0$
(c) $17x^2 + 20x + 3 = 0$
(d) none of these

11. If the ratio of the roots of the equation $ax^2 + bx + c = 0$ be $p : q$, then

[Pb. CET – 1994]

(a) $pqb^2 + (p + q)^2 ac = 0$
(b) $pqb^2 - (p + q)^2 ac = 0$
(c) $pqa^2 - (p + q)^2 bc = 0$
(d) none of these

12. If one of the roots of equation $x^2 + ax + 3 = 0$ is 3 and one of the roots of the equation $x^2 + ax + b = 0$ is three times the other root, then the value of b is equal to

[J&K – 2005]

(a) 3 (b) 4
(c) 2 (d) 1

13. If one root of the equation $ax^2 + bx + c = 0$ be n times the other root, then

(a) $na^2 = bc(n + 1)^2$ (b) $nb^2 = ac(n + 1)^2$
(c) $nc^2 = ab(n + 1)^2$ (d) none of these

ANSWER SHEET

1.	ⓐ	ⓑ	ⓒ	ⓓ	6.	ⓐ	ⓑ	ⓒ	ⓓ	11.	ⓐ	ⓑ	ⓒ	ⓓ
2.	ⓐ	ⓑ	ⓒ	ⓓ	7.	ⓐ	ⓑ	ⓒ	ⓓ	12.	ⓐ	ⓑ	ⓒ	ⓓ
3.	ⓐ	ⓑ	ⓒ	ⓓ	8.	ⓐ	ⓑ	ⓒ	ⓓ	13.	ⓐ	ⓑ	ⓒ	ⓓ
4.	ⓐ	ⓑ	ⓒ	ⓓ	9.	ⓐ	ⓑ	ⓒ	ⓓ					
5.	ⓐ	ⓑ	ⓒ	ⓓ	10.	ⓐ	ⓑ	ⓒ	ⓓ					

HINTS AND EXPLANATIONS

1. (a) Given α and β are roots of $2x^2 - 3x + 4 = 0$ then

$\alpha + \beta \frac{3}{2}, \alpha\beta = \frac{4}{2} = 2$

Sum of given roots $= \alpha^2 + \beta^2$

$$= (\alpha + \beta)^2 - 2\alpha\beta$$

$$= \frac{9}{4} - \frac{4}{1} = \frac{-7}{4}$$

Product of given roots $= \alpha^2 . \beta^2 = (\alpha\,\beta)^2 = 4$

The required equation is,

$x^2 - \left(-\frac{7}{4}\right)x + 4 = 0$

$\Rightarrow 4x^2 + 7x + 16 = 0$

OR

Required equation is obtained on replacing x by $\sqrt{x}$ in given equation as follows

$(2\sqrt{x})^2 - 3\sqrt{x} + 4 = 0$ or $2x + 4 = 3\sqrt{x}$, On squaring

$\Rightarrow 4x^2 + 16x + 16 = 9x$

$\Rightarrow 4x^2 + 7x + 16 = 0$

2. (a) Given p and q are roots of $x^2 + ax + b = 0$ then $p + q = -$ a and $pq = b$ (1)

Sum of desired roots $= p^2q + pq^2$

$= pq\,(p + q)$

$= b(-a) = -ab$

Product of desired roots = $p^2q \cdot pq^2 = (pq)^3$ $= b^3$ The required equation is $x^2 - (-ab)\,x + b^3 = 0$

$\Rightarrow x^2 + abx + b^3 = 0$

3. (b) Given α and β are roots of the equation $x^2 - 2x + 3 = 0$ then, $\alpha + \beta = 2$, $\alpha\beta = 3$

Sum of desired roots $= \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2 + \beta^2}{\alpha^2\beta^2}$

$= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$

$\frac{4-6}{9} = \frac{-2}{9}$

Product of desired roots $= \frac{1}{\alpha^2} \times \frac{1}{\beta^2}$

$= \frac{1}{(\alpha\beta)^2} = \frac{1}{9}$

The required equation is,

$x^2 -$ (sum of roots) $x +$ product of roots $= 0$

$x^2 - \left(-\frac{2}{9}\right)x + \frac{1}{9} = 0$

$\Rightarrow 9x^2 + 2x + 1 = 0.$

11. (b) Let $p\alpha$, $q\alpha$ be the roots of the given equation $ax^2 + bx + c = 0$.

Then, $p\alpha + q\alpha = -\frac{b}{a}$ and $p\alpha \cdot q\alpha = \frac{c}{a}$

From first relation, $\alpha = -\frac{b}{a(p+q)}$

Substituting this value of α in second relation,

we get,

$\frac{b^2}{a^2(p+q)^2} \times pq = \frac{c}{a}$

$\Rightarrow pqb^2 - (p + q)^2\, ac = 0$

12. (a) Given, $3^2 + a.3 + 3 = 0$; $a = -4$ Let roots of equation $x^2 - 4x + b = 0$ is a and $3a$.

$a + 3a = 4 \Rightarrow 4a = 4 \Rightarrow a = 1$

Hence, $1 - 4 + b = 0 \Rightarrow b = 3$

13. (b) Let one root be a . Then, another root $= n^a$. Threrfore,

$\Rightarrow (1 + n)\alpha = -\frac{b}{a}$ and $n\alpha^2 = \frac{c}{\alpha}$

$\Rightarrow (1 + n)2\alpha^2 = -\frac{b^2}{a^2}$ and $n\alpha^2 = \frac{c}{\alpha}$

$\Rightarrow (1 + n)^2\alpha^2 = -\frac{b^2}{a^2}$ and $\alpha^2 = \frac{c}{n\alpha}$

$\Rightarrow (1 + n)^2 \frac{c}{n\alpha} = \frac{b^2}{a^2}\left[\because \alpha^2 = \frac{c}{n\alpha}\right]$

$\Rightarrow ac\,(1 + n)^2 - b^2n = 0$

which is the required condition.

This page is intentionally left blank.

LECTURE

4 Graph of Quadratic Equations

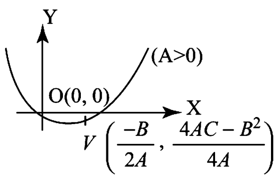
BASIC CONCEPTS

1. **Maximum and Minimum Value and Graph of a Quadratic Expression $Y = Ax^2 + Bx + C$** Let the expression $y = Ax^2 + Bx + C, A, B, C \in R, A \neq 0$. This expression can be written as follows by completing the perfect square method $\left(x + \frac{B}{2A}\right)^2 = \frac{1}{A}\left(y - \frac{4AC - B^2}{4A}\right)$

   which is parabola whose vertex is

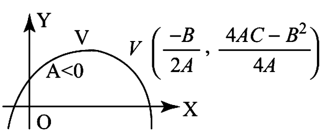
   $\left(\frac{-B}{2A}, \frac{4AC - B^2}{4A}\right)$.

   Case 1: If $A > 0$, then parabola is upward.



   and minimum value of $Ax^2 + Bx + C = \frac{4AC - B^2}{4A}$ for $x = \frac{-B}{2A}$.

   Case 2: If $A < 0$, then parabola is downward.



   and maximum value of $Ax^2 + Bx + C = 0$ $\frac{4AC - B^2}{4A}$ for $x = \frac{-B}{2A}$

Note 1: If $B^2 - 4AC > 0$, then the parabola $y = Ax^2 + Bx + C$ intersects x-axis in two real and distinct points.

Note 2: If $B^2 = 4AC$, then parabola $y = Ax^2 + Bx + C$ touches x-axis in one point.

Note 3: If $B^2 < 4AC$ then both roots are imanary and parabola does not intersect x-axis.

2. **Sign of the Expression $Ax^2 + Bx + C$**

   For all real values of x, the expression $Ax^2 + Bx + C$ has the same sign as that of a except when the roots of the equation $Ax^2 + Bx + C = 0$ are real and distinct and x has a value lying between them.

***Note*:** Let $Ax^2 + Bx + C = A(x - \alpha)(x - \beta)$ $(\alpha < \beta)$

Case 1: When x is greater than both the roots i.e., $x > \beta > \alpha$ then $Ax^2 + Bx + C = a(x - \alpha)(x - \beta) = A$ (one positive value) Hence, $Ax^2 + Bx + C$ and A are same sign $(B^2 > 4AC)$

Case 2: When x is smaller than both the roots hence $\beta > \alpha > x$ then $x - \alpha < 0$, $x - \beta < 0$ and $Ax^2 + Bx + C = A(x - \alpha)(x - \beta) = (A)(-)(-)$
$= (A)$ (one positive value) $(B^2 > 4AC)$

Case 3: When x lies between both the roots i.e., $\alpha < x < \beta$ then $Ax^2 + Bx + C = A(x - \alpha)(x - \beta) = A$ (negative value) $(B^2 > 4AC)$

Case 4: If both roots are equal i.e., $B^2 = 4AC$ then $Ax^2 + Bx + C = A(x - \alpha)(x - \alpha) = A(x - \alpha)^2 = (A)$ (positive value)

Case 5: If both roots are imaginary i.e., $B^2 - 4AC < 0$ or $4AC - B^2 > 0$

$$Ax^2+Bx+C=A\left[\left(x+\frac{B}{2A}\right)^2+\frac{4AC-B^2}{4A^2}\right]=Ax$$

A (one positive value)

Note:

1. $(x - 2)(x - 3) < 0 \Rightarrow 2 < x < 3$
2. $(x - 2)(x - 3) > 0 \Rightarrow x < 2, x > 3$, $x \in (-\infty, 2) \in$ and $x \in (3,\infty)$
3. $(x - a)(x - b) = x^2 - (a + b)x + ab < 0$; $a < b \Rightarrow x \in (a, b) \Rightarrow a < x < b$
4. $(x - a)(x - b) = x^2 - (a + b)x + ab > 0$; $a < b \Rightarrow x < \boldsymbol{a}, x > b \Rightarrow x \in (-\infty, a)$ and $x \in (b, \infty)$.

3. Limits or Maximum and Minimum or Range of the Expression $\dfrac{A_1x^2 + B_1x + C_1}{A_2x^2 + B_2x + C_2}$

are obtained by in following steps

Step 1: First of all given expression is equated to k and quadratic equation in x is obtained.

Step 2: For real x interval of k is obtained by $B^2 \geq 4AC$. Interval of k is called limits of given expression.

4. Let $y = ax^2 + bx + c$ If $D < 0$, then

(i) $y > 0$, for all real values of x if $a > 0$

(ii) $y < 0$, for all real values of x if $a < 0$

i.e., y has same sign as that of a if $D < 0$.

5. If any function $f(x)$ has $(x - a)(x - b)$ as the factor then $f(a) = f(b) = 0$.

Quadratic Equation involving modulus sign The value of modulus function is always positive but its sign depends on the sign of variable or expression or function.

Examples

(i) $|Ax^2 + Bx + C| = +(Ax^2 + Bx + C)$ if $Ax^2 + Bx + C \geq 0$

(ii) $|Ax^2 + Bx + C| = -(Ax^2 + Bx + C)$ if $Ax^2 + Bx + C < 0$

(iii) $|x| = +x$ if $x \geq 0$
$= -x$ if $x < 0$

(iv) $|x - a| = +(x - a)$ if $x - a \geq 0$
$= -(x - a)$ if $x - a < 0$

(v) $|x^2| = x^2$

(vi) $|10| = +10$, $|-9| = -(-9)$.

7. Important Formulas Related with Two Quadratic Equations

$$A_1x^2 + B_1x + C_1 = 0 \quad (1)$$

$$A_2x^2 + B_2x + C_2 = 0 \quad (2)$$

(i) If both quadratic equations have one root common, then,

$$\begin{vmatrix} C_1 & A_1 & B_1 \\ C_2 & A_2 & B_2 \end{vmatrix}$$

$$\begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}^2 = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} \text{ or}$$

$$(C_1A_2 - C_2A_1)^2 = (A_1B_2 - B_1A_2) \times (B_1C_2 - B_2C_1)$$

$$\text{and common root} = \left|\frac{\begin{matrix} C_1 & A_2 \\ C_2 & A_2 \end{matrix}}{\begin{matrix} A_1 & B_1 \\ A_2 & B_2 \end{matrix}}\right|$$

(ii) If both quadratic equations have both roots comman i.e., both quadratic equations are identical, then,

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$$

(iii) If roots of both quadratic equations are reciprocals of each other, then $\dfrac{A_1}{C_2} = \dfrac{B_1}{B_2} = \dfrac{C_1}{A_2}$

(iv) If the ratio of both roots of quadratic equations are same, then condition is

$$\left(\frac{B_1}{B_2}\right)^2 = \left(\frac{A_1}{A_2}\right)\left(\frac{C_1}{C_2}\right)$$

i.e., $\dfrac{A_1}{A_2}, \dfrac{B_1}{B_2}, \dfrac{C_1}{C_2}$ are in G.P.

(v) If the difference of roots of both quadratic equations are same, then

i.e., $\dfrac{A_1}{A_2}, \dfrac{B_1}{B_2}, \dfrac{C_1}{C_2}$ are G.P

(v) If the difference of roots of both quadratic equation are same, then

$$\frac{B_1^2 - 4A_1C_1}{B_2^2 - 4A_2C_2} = \frac{A_1^2}{A_2^2}$$

8. Interval Containing the Roots

(i) If $f(x) = Ax^2 + Bx + C$. If $f(a)$ and $f(b)$ are of opposite signs ($f(a)\,f(b) < 0$) then exactly one real root of the equation $f(x) = 0$ lies between a and b.

(ii) If $f(a)$ and $f(b)$ are of same sign ($f(a)\,f(b) > 0$), then either no root or even number of roots of the equation $f(x) = 0$ lies between a and b.

(iii) If a real number k lies between roots of the equation $Ax^2 + Bx + C = 0$, then $f(k) = Ak^2 + Bk + C < 0$ i.e., $f(k)$ is negative.

(iv) If a real number k lies outside the interval formed by the roots of the equation then $f(k) = Ak^2 + Bk + C > 0$ i.e., $f(k)$ is positive.

(v) If both roots of the equation $Ax^2 + Bx + C = 0$ are greater than the one real number k, then $B^2 \geq 4AC$; $\alpha + \beta = -\frac{B}{2A} > 2k$ and $f(k) = Ak^2 + Bk + C > 0$

(vi) If both roots of the equation $Ax^2 + Bx + C = 0$ are smaller then the one real number k, B then $B^2 \geq 4AC$; $\alpha + \beta = -\frac{B}{2A} < 2k$ and $f(k) = Ak^2 + Bk + C > 0$

(vii) If both the roots are positive, then $D > 0$, $\alpha + \beta > 0, \alpha\beta > 0\,; -\frac{b}{\alpha} > 0, \frac{c}{\alpha} > 0$

(viii) If both the roots are negative, then $D < 0$,

$\alpha + \beta < 0, \alpha\beta > 0\,; -\frac{b}{\alpha} > 0, \frac{c}{\alpha} > 0$

9. If α, β, γ are the roots of the cubic equation

$Ax^3 + Bx^2 + Cx + d = 0$, $(A \neq 0)$ then

$$\alpha + \beta + \gamma = -\frac{B}{A}$$

$$= (-1)^1\left(\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}\right)$$

$$\alpha\beta + \beta\gamma + \alpha\gamma = \frac{C}{A}$$

$$= (-1)^2\left(\frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}\right)$$

$$\alpha\beta\gamma = -\frac{D}{A} = (-1)^3\left(\frac{\text{Constant term}}{\text{Coefficient of } x^3}\right)$$

10. Descarte's Rule of Signs

The maximum number of positive real roots of a polynomial equation $f(x) = 0$ is the number of changes of signs in $f(x)$ and the maximum number of negative roots of $f(x) = 0$ is the number of changes in $f(-x)$.

Example 1: Consider $x^3 + 5x^2 + 7x - 4 = 0$

The signs of the various terms are + + + −
Since there is only one change of sign in the expression $x^3 + 5x^2 + 7x - 4$

$\therefore$ the given equation has at most one positive real root.

Example 2: Consider $f(x) = x^4 + 2x^3 + 3x^2 - 8x - 4 = 0$

$\therefore f(-x) = x^4 - 2x^3 + 3x^2 + 8x - 4$

The signs of various terms of $f(-x)$ are + − + + −

Since there are three changes of signs. Therefore, the given equation has at most three negative roots.

Example 3: Consider $f(x) = x^4 + 3x^2 + 5$

$\therefore f(-x) = x^4 + 3x^2 + 5$

Clearly, $f'(x)$ and $f(-x)$ do not have any change of signs.

$\therefore$ the equation of $f(x) = 0$ has no real roots, i.e., all roots are imaginary.

Example 4: How many real solutions does the equation $x^7 + 14x^5 + 16x^3 + 30x - 560 = 0$ have?

[AIEEE −2008]

(a) 5 (b) 7

(c) 1 (d) 3

Solution

(c) Let $f(x) = x^7 + 14x^5 + 16x^3 + 30x - 560$

$\therefore f'(x) = 7x^6 + 70x^4 + 48x^2 + 30$

$\Rightarrow f'(x) > 0 \; \forall \; x \in R$

i.e., $f(x)$ is an strictly increasing function. So, it can have at the most one solution. It can be shown that it has exactly one solution.

SOLVED SUBJECTIVE PROBLEMS: (CBSE/STATE BOARD): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Find the condition, if the roots of the equation $ax^2 + bx + c = 0$ is reciprocal of the roots of the equation $a'x^2 + b'x + c' = 0$.

Solution

Let α, β be the roots of the equation $ax^2 + bx + c = 0$ and $\frac{1}{\alpha}, \frac{1}{\beta}$ will be roots of the equation $a'x^2 + b'x + c' = 0$.

Also if we replace x by $\frac{1}{x}$ in $ax^2 + bx + c = 0$

we get $a\left(\frac{1}{x}\right)^2 + b\left(\frac{1}{x}\right) + c = 0,$

or $cx^2 + bx + a = 0$

Now, this is similar to 2nd equation. Therefore, ratio of coefficient will be same.

Therefore, $\frac{c}{a'} = \frac{b}{b'} = \frac{a}{c'}$ which is the required condition.

2. If both roots of equations $3x^2 + ax - 4 = 0$ and $bx^2 - 2x - 8 = 0$ are common, then find the value of a and b

Solution

Let α, β are common roots of given equations

$3x^2 + ax - 4 = 0$ (1)

$bx^2 - 2x - 8 = 0$ (2)

Therefore, ratio of coefficient will be same $\frac{3}{b} = \frac{a}{-2} = \frac{-4}{-8}$

Therefore, $a = -1;\ b = 6$

Ans.

3. If both roots of equations $k(6x^2 + 3) + rx + 2x^2 - 1 = 0$ and $6K(2x^2 + 1) + px + 4x^2 - 2 = 0$ are common, then prove that $2r - p = 0$

Solution

Given equations are $k(6x^2 + 3) + rx + 2x^2 - 1 = 0$

$\Rightarrow x^2(6K + 2) + rx + 3K - 1 = 0$ (1)

and $6K(2x^2 + 1) + px + 4x^2 - 2 = 0$

$\Rightarrow x^2(12K + 4) + px + 6K - 2 = 0$ (2)

Since both roots of Equations (1) and (2) are common, therefore, sum and product of roots of are respectively equal. That means Equations (1) and (2) are identical.

$$\therefore \frac{6K + 2}{12K + 4} = \frac{r}{p} = \frac{3K - 1}{6K - 2}$$

$$\Rightarrow \frac{1}{2} = \frac{r}{p} = \frac{1}{2} \Rightarrow 2r = p \Rightarrow 2r - p = 0$$

4. If one root is common of equations $x^2 + ax + bc = 0$ and $x^2 + bx + ca = 0$ then prove that their other roots will satisfy the equation $x^2 + cx + ab = 0$.

Solution

Given equations are $x^2 + ax + bc = 0$ (1)

and $x^2 + bx + ca = 0$ (2)

Let α, β and α, γ are roots of Equations (1) and (2) respectively,

then $\alpha + \beta = -a$ (3)

$\alpha\beta = bc$ (4)

$\alpha + \gamma = -b$ (5)

and $\alpha\gamma = ca$ (6)

Subtracting Equation (5) from (3), we get,

$\beta - \gamma = b - a$ (7)

Subtracting Equation (6) from (4), we get,

$\alpha(\beta - \gamma) = c(b - a)$

$\Rightarrow \alpha(b - a) = c(b - a)$ [from Equation (7)]

$\Rightarrow \alpha = c, (\because b \neq a)$

Putting the value of α in Equations (4) and (6), we get,

$c\beta = bc \Rightarrow \beta = b$ and $c\gamma = ca \Rightarrow \gamma = a$

Again, adding Equations (3) and (5), we get,

$2\alpha + \beta + \gamma = -a - b \Rightarrow \beta + \gamma = -a - b - 2\alpha$

$\Rightarrow \beta + \gamma = -a - b - 2c \qquad (8)$

Since α is one root of Equation (1) or Equation (2)

$\therefore\ a^2 + a\alpha + bc = 0 \Rightarrow c^2 + ac + bc = 0$

$\Rightarrow c + a + b = 0 \Rightarrow -a - b = c$

Putting the value of $-a-b$ in Equation (8),

$\Rightarrow \beta + \gamma = c - 2c \Rightarrow \beta + y = -c$

Thus, the Equation whose roots are β, γ is

$x^2 - (\beta + \gamma)\,x + \beta\gamma = 0$

$\Rightarrow\ x^2 - (-c)x + ab = 0 \Rightarrow x^2 + cx + ab = 0$

which is required equation.

UNSOLVED SUBJECTIVE PROBLEMS: (CBSE/STATE BOARD): TO GRASP THE TOPIC, SOLVE THESE PROBLEMS

Exercise I

1. If equations $ax^2 + bx + c = 0$ and $cx^2 + bx + a = 0$ have one root common, show that either $a + b + c = 0$ or $a - b + c = 0$.
2. Find the condition that two quadratic equation $a_1x^2 + b_1x + 4 = 0$ and $a_2x^2 + b_2x + 4 = 0$ may, have both roots common.
3. If α, β are roots of the quadratic equation $ax^2 + 2bx + c = 0$, then prove that $\sqrt{\alpha/\beta} + \sqrt{\beta/\alpha} = \frac{-2b}{\sqrt{ac}}$.
4. If ratio of the roots of $x^2 + px + q = 0$ be same as ratio of the roots of $x2 + p'x + q' = 0$, then prove that $p^2q' = p'^2q$.
5. If α, β are roots of the quadratic equation $x^2 + px + p^2 + q = 0$, then prove that $\alpha^2 + \alpha\beta + \beta^2 + q = 0$.
6. If both roots of equations $K(6x^2 + 3) + rx + 2x^2 - 1 = 0$ and $6K(2x^2 + 1) + px + 4x^2 - 2 = 0$ are common, then prove that $2r - p = 0$.

ANSWERS

Exercise I

2. $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. The expression $y = ax^2 + bx + c$ has always the same sign as c if

(a) $4ac < b^2$ (b) $4ac > b^2$

(c) $ac < b^2$ (d) $ac > b^2$

Solution

(b) Let $f(x) = ax^2 + bx + c$.

Then, $f(0) = c$. Thus, the graph of $y = f(x)$ meets y-axis at (0, c). If $c > 0$, then by hypothesis $f(x) > 0$. This means that the curve $y = f(x)$ does not meet x-axis.

If $c < 0$, then by hypothesis $f(x) < 0$, which means that the curve $y = f(x)$ is always below x-axis and so it does not intersect with x-axis. Thus, in both cases $y = f(x)$ does not intersect with x-axis i.e., $f(x) \neq 0$ for any real x. Hence, $f(x) = 0$ i.e., $ax^2 + bx + c = 0$ has imaginary roots and so $b^2 < 4ac$.

2. If $a < 0$ then the inequality $ax^2 - 2x + 4 > 0$ has the solution represented by

[AMU – 2001]

(a) $\frac{1+\sqrt{1-4a}}{a} > x > \frac{1-\sqrt{1-4a}}{a}$

(b) $x < \frac{1-\sqrt{1-4a}}{a}$

(c) $x < 2$

(d) $2 > x > \frac{1-\sqrt{1-4a}}{a}$

Solution

(a) $ax^2 - 2x + 4 > 0$

$\Rightarrow x = \frac{2 \pm \sqrt{4-16a}}{2a}$

$\Rightarrow x = \frac{1 \pm \sqrt{1-4a}}{a}$

$\therefore \frac{1-\sqrt{1-4a}}{a} < x < \frac{1+\sqrt{1-4a}}{a}$

3. The value of a ($a \geq 3$) for which the sum of the cubes of the roots of $x^2 - (a-2)x + (a-3) = 0$, assumes the least value is

[Orissa – JEE – 2002]

(a) 3 (b) 4

(c) 5 (d) none of these

Solution

(a) Let the roots be α and β

so, $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\beta(\alpha + \beta)$

$= (a-2)^3 - 3(a-3)(a-2)$

$= a^3 - 9a^2 + 27a - 26$

$= (a-3)^3 + 1$

It assumes the least value, if $(a-3)^3 = 0$ Therefore, $a = 3$.

4. The values of 'a' for which $(a^2 - 1)x^2 + 2(a-1)x + 2$ is positive for any x are

[UPSEAT – 2001]

(a) $a \geq 1$ (b) $a \leq 1$

(c) $a > -3$ (d) $a < -3$ or $a > 1$

Solution

(d) We know that the expression $ax^2 + bx + c > 0$ for all x if $a > 0$ and $b^2 < 4ac$.

$\therefore (a^2 - 1)x^2 + 2(a-1)x + 2$ is positive for all x, if $a^2 - 1 > 0$ and $4(a-1)^2 - 8(a^2 - 1) < 0$

$\Rightarrow a^2 - 1 > 0$ and $-4(a-1)(a+3) < 0$

$\Rightarrow a^2 - 1 > 0$ and $(a-1)(a+3) > 0$

$\Rightarrow a^2 > 1$ and $a < -3$ or $a > 1$ $a < -3$ or $a > 1$

5. If a, b, c are real and $x^3 - 3b^2x + 2c^3$ is divisible by $x - a$ and $x - b$, then

(a) $a = -b = -c$

(b) $a = 2b = 2c$

(c) $a = b = c$ or $a = -2b = -2c$

(d) none of these

Solution

(c) As $f(x) = x^3 - 3b^2x + 2c^3$ is divisible by $x - a$ and $x - b$

Therefore, $f(a) = 0$

$\Rightarrow a^3 - 3b^2a + 2c^3 = 0$ (1)

and $f(b) = 0 \Rightarrow b^3 - 3b^3 + 2c^3 = 0$ (2)

from (2) $b = c$; from (1), $a^3 - 3ab^2 + 2b^3 = 0$ (putting $b = c$)

$\Rightarrow (a-b)(a^2 + ab - 2b^2) = 0 \Rightarrow a = b$ or $a^2 + ab = 2b^2$

Thus, $a = b = c$ or $a^2 + ab = 2b^2$ and $b = c$ $a^2 + ab = 2b^2$ is satisfied by $a = -2b$. But $b = c$ $\therefore a^2 + ab - 2b^2$ and $b = c$ is equivalent to $a = -2b = -2c$.

6. If $x^2 + 2ax + 10 - 3a > 0$ for all $x \in R$, then

[IIT(Screening) – 2004]

(a) $-5 < a < 2$ (b) $a < -5$

(c) $a > 5$ (d) $2 < a < 5$

Solution

(a) According to given condition

$4a^2 - 4(10 - 3a) < 0$ $a^2 + 3a - 10 < 0$

$\Rightarrow (a+5)(a-2) < 0$ $-5 < a < 2$.

7. The roots of the equation $|x-2|^2 + |x-2| - 6 = 0$ are

[UPSEAT – 2003]

(a) 0, 4 (b) 2, 4

(c) −1, 3 (d) 0, 3

Solution

(b) $|x-2|^2 + |x-2| - 6 = 0$

$\Rightarrow |x-2| = 2, -3$

But $|x-2| \neq -3|$.

Therefore, $x - 2 = 2, -2$

$x = 4, 0$

8. If for every real value of b, the roots of $x^2 + (a - b)x + (1 - a - b) = 0$ are real and unequal, then

[IIT (main) – 2003]

(a) $a < 1$ (b) $a > 1$

(c) $a > 0$ (d) $a < 0$

Solution

(b) Roots are real and unequal

$\Rightarrow (a-b)^2 - 4(1 - a - b) > 0$

$\Rightarrow b^2 + (4 - 2a)b + (a^2 + 4a - 4) > 0$

This is true for all real values of b. Hence,

$(4 - 2a)^2 - 4(a^2 + 4a - 4) < 0$

$\Rightarrow -32a + 32 < 0 \Rightarrow a > 1$

9. The real roots of the equation $x^{2/3} + x^{1/3} - 2 = 0$ are

[MP PET – 2006]

(a) 1, 8 (b) $-1, -8$

(c) $-1, 8$ (d) $1, -8$

Solution

(d) The given expression is $x^{2/3} + x^{1/3} - 2 = 0$

Put, $x^{1/3} = y$, then, $y^2 + y - 2 = 0$

$\Rightarrow (y-1)(y+2) = 0$

$\Rightarrow y = 1$ or $y = -2$

$\Rightarrow x^{1/3} = 1$ or $x^{1/3} = -2$

$\therefore x = (1)^3$ or $x = (-2)^3 = -8$

Hence, the real roots of the given equation are $1, -8$.

10. The equation $x^{(3/4)(\log_2 x)^2 + (\log_2 x) - 5/4} = \sqrt{2}$ has

[IIT – 1989]

(a) at least one real solution

(b) exactly three real solutions

(c) exactly one irrational solution

(d) all the above

Solution

(d) For the given equation to be meaningful we must have $x > 0$. For $x > 0$ the given equation can be written as

$\frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4}$

$= \log_x \sqrt{2} = \frac{1}{2}\log_x 2$

$\Rightarrow \frac{3}{4}t^2 + t - \frac{5}{4} = \frac{1}{2}\left(\frac{1}{t}\right)$

By putting $t = \log_2 x$ so that $\log_x 2 = \frac{1}{t}$ because $\log_2 x \log_x 2 = 1$

$\Rightarrow 3t^3 + 4t^2 - 5t - 2 = 0$

$\Rightarrow (t-1)(t+2)(3t+1) = 0$

$\Rightarrow \log_2 x = t = 1, -2, -\frac{1}{3} \Rightarrow x = 2, 2^{-2}, 2^{-1/3}$

or $x = 2, \frac{1}{4}, \frac{1}{2^{1/3}}$

Thus, the given equation has exactly three real solutions out of which exactly one is irrational namely $\frac{1}{2^{1/3}}$.

11. The equation $\sqrt{(x+1)} - \sqrt{(x-1)} = \sqrt{4x-1}$ has

[IIT – 1997 (Cancelled)]

(a) no solution

(b) one solution

(c) two solutions

(d) more than two solutions

Solution

(a) Given $\sqrt{(x+1)} - \sqrt{(x-1)} = \sqrt{(4x-1)}$

Squaring both sides, we get $-2\sqrt{(x^2-1)} = 2x - 1$

Squaring again, we get $x = \frac{5}{4}$ which does not satisfy the given equation. Hence, the equation has no solution.

12. If $\log_2 x + \log_x 2 = \frac{10}{3} = \log_2 y + \log_y 2$ and $x \neq y$, then $x + y =$

[EAMCET – 1994]

(a) 2 (b) 65/8

(c) 37/6 (d) none of these

Solution

(d) We have $\log_2 x + \frac{1}{\log_2} = 3 + \frac{1}{3}$

$= \log_2 y + \frac{1}{\log_2 y}$

$\therefore \log_2 x = 3, \log_2 y = \frac{1}{3} \; (\because x \neq y)$

$\Rightarrow x = 2^3$ and $y = 2^{1/3} \Rightarrow x + y = 8 + 2^{1/3}$.

13. If $x^2 + px + 1$ is a factor of the expression $ax^3 + bx + c$, then

[IIT – 1980]

(a) $a^2 + c^2 = -ab$ (b) $a^2 - c^2 = -ab$
(c) $a^2 - c^2 = ab$ (d) none of these

Solution

(c) Given that $x^2 + px + 1$ is factor of $ax^3 + bx + c = 0$, then let $ax^3 + bx + c \equiv (x^2 + px + 1)(ax + \lambda)$, where λ is a constant. Then, equating the coefficient of like power of x on both sides, we get, $0 = ap + \lambda$, $b = p\lambda + a$, $c = \lambda$.

$\Rightarrow p = -\frac{\lambda}{a} = -\frac{c}{a}$. Hence, $b = (-c/a)c + a$

or $ab = a^2 - c^2$.

14. If the two equations $x^2 - cx + d = 0$ and $x^2 - ax + b = 0$ have one common root and the second has equal roots, then $2(b + d) =$

(a) 0 (b) $a + c$
(c) ac (d) $-ac$

Solution

(c) Let roots of $x^2 - cx + d = 0$ be α, β then roots of $x^2 - ax + b = 0$ be α, α

$\therefore \alpha + \beta = c, \alpha\beta = d, \alpha + \alpha = a, \alpha^2 = b$

Hence, $2(b + d) = 2(\alpha^2 + \alpha\beta) = 2\alpha(\alpha + \beta) = ac$

15. If every pair of the equations $x^2 + px + qr = 0$, $x^2 + qx + rp = 0$, $x^2 + rx + pq = 0$ have a common root, then the sum of three common roots is

(a) $\frac{-(p+q+r)}{2}$ (b) $\frac{-p+q+r}{2}$
(c) $-(p + q + r)$ (d) $-p + q + r$

Solution

(a) Let the roots be α, β; β, γ and γ, α, respectively.

$\therefore \alpha + \beta = -p, \beta + \gamma = -q, \gamma + \alpha = -r$

Adding all, we get $\sum \alpha = -(p + q + r)/2$ etc.

16. If $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have common root $a \neq 0$, then $= \frac{a^3 + b^3 + c^3}{abc}$

[IIT – 1982; Kurukshetra CEE – 1982]

(a) 1 (b) 2
(c) 3 (d) none of these

Solution

(c) It can be seen that
2 is common root,
$\therefore a + b + c = 0$
gives $a^3 + b^3 + c^3 = 3abc$

17. If α, β, γ are the roots of the equation $x^3 + x + 1 = 0$, then the value of $\alpha^3\beta^3\gamma^3$

[MP PET – 2004]

(a) 0 (b) -3
(c) 3 (d) -1

Solution

(d) We know that the roots of the equation $ax^3 + bx^2 + cx + d = 0$ follows $\alpha\beta\gamma = -d/a$. Comparing the above equation with given equation we get $d = 1$, $a = 1$

So, $\alpha\beta\gamma = -1$, or $\alpha^3\beta^3\gamma^3 = -1$

18. If a, b, c are in G.P., then the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root if $\frac{d}{a}, \frac{e}{d}, \frac{f}{c}$ are in

[IIT – 1985; Pb. CET – 2000; DCE – 2000]

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

Solution

(a) As given, $b^2 = ac \Rightarrow$ equation $ax^2 + 2bx + c = 0$ can be written as

$ax^2 + 2\sqrt{ac}\,x + c = 0$

$\Rightarrow (\sqrt{a}\,x + \sqrt{c})^2 = 0$

$\Rightarrow x = -\sqrt{\frac{c}{a}}$ (repeated root)

This must be the common root by hypothesis. So,it must satisfy the equation

$dx^2 + 2ex + f = 0$.

$\Rightarrow d\frac{c}{a} - 2e\sqrt{\frac{c}{a}} + f = 0$

$\Rightarrow \frac{d}{a} + \frac{f}{c} = \frac{2e}{c}\sqrt{\frac{c}{a}} = \frac{2e}{b}$

$\Rightarrow \frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in A.P.

19. If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + dx + c$, where, $ac \neq 0$, then $P(x)\,Q(x) = 0$ has at least two real roots.

[IIT – 1985; Jamia – 2004]

(a) True
(b) False
(c) Both true and false
(d) None of these

Solution

We have $P(x) = ax^2 + bx + c$
for which $D_1 = b^2 - 4ac$
and $Q(x) = -ax^2 + dx + c$ (1)
for which $D_2 = d^2 + 4ac$ (2)
Given, that $ac = \neq 0$ Therefore, following two cases are possible.
If $ac > 0$ then from Equation (2) D_2 is + ve
$\Rightarrow Q(x)$ has real roots.
If $ac < 0$ then from Equation (1) D_1 is + ve
$\Rightarrow P(x)$ has real roots.
Thus, $P(x)\ Q(x) = 0$ has at least two real roots.
$\therefore$ Given statement is true.

20. If only one root of the equations $2x^2 + 3x + 5\lambda = 0$ and $x^2 + 2x + 3\lambda = 0$ is common, then find the value of λ is

(a) 0, – 1 (b) 1, 1
(c) 1, –1 (d) –1, 2

Solution

(a) Given equations are $2x^2 + 3x + 5\gamma = 0$ and $x^2 + 2x + 3\gamma = 0$. If α is a common root of both quadratic equations, then $2\alpha^2 + 3\alpha + 5\gamma = 0$ and $\alpha^2 + 2\alpha + 3\gamma = 0$.

$$\therefore \frac{\alpha^2}{9\lambda - 10\lambda} = \frac{\alpha}{5\lambda - 6\lambda} = \frac{1}{4-3}$$

$$\Rightarrow \frac{\alpha^2}{-\lambda} = \frac{\alpha}{-\lambda} = \frac{1}{1}$$

$\Rightarrow \alpha^2 = -\lambda$ and $\alpha = -\lambda$
$\because (\alpha)^2 = \alpha^2$
$\Rightarrow (-\lambda)^2 = -\lambda$
$\Rightarrow \lambda^2 + \lambda = 0,$
$\Rightarrow \lambda(\lambda + 1) = 0$
$\Rightarrow \lambda = 0, -1$

21. The roots of the equation $x^2 - 2x + a = 0$ are p, q and the roots of the equation $x^2 - 18x + B = 0$ are r, s. If $p < q < r < s$ are in AP, then

[IIT – 1997]

(a) $A = 3, B = 77$
(b) $A = -3, B = 77$
(c) $A = 3, B = -77$
(d) $A = -3, B = -77$

Solution

(b) Let p, q, r, s be $a - 3d, a - d, a + d, a + 3d$, respectively.
Now $p + q = 2, r + s = 18$
$pq = A, rs = B \therefore p + q + r + s = 4a = 20$
$\Rightarrow a = 5$
Also, $p + q = 2$
$\Rightarrow 10 - 4d = 2$
$\therefore d = 2$
Thus, p, q, r, s are $-1, 3, 7, 11$, respectively.
Hence, $A = pq = -3, B = rs = 77$

22. The maximum possible number of real roots of equation $x^5 - 6x^2 - 4x + 5 = 0$ is

[EAMCET – 2002]

(a) 0 (b) 3
(c) 4 (d) 5

Solution

(b) Let $f(x) = x^5 - 6x^2 - 4x + 5 = 0$
Then the number of change of sign in $f(x)$ is 2, therefore $f(x)$ can have at most two positive real roots.
Now, $f(-x) = -x^5 - 6x^4 + 4x + 5 = 0$. Then, the number of change of sign is 1.
Hence, $f(x)$ can have at most one negative real root. So, that total possible number of real roots is 3.

23. If α, β are the roots of $x^2 - 3x + a = 0$ and γ, δ be the roots of $x^2 - 12x + b = 0$ and numbers $\alpha, \beta, \gamma, \delta$ (in order) form an increasing G.P., then

[DCE – 2000]

(a) $a = 3, b = 12$ (b) $a = 12, b = 3$
(c) $a = 2, b = 32$ (d) $a = 4, b = 16$

Solution

(c) Let $r > 1$ be the common ratio of the G.P. $\alpha, \beta, \gamma, \delta$ then

$\beta = r\alpha, \gamma = r^2\alpha$ and $\delta = r^3\alpha$

$\therefore \alpha + \beta = \alpha(1 + r) = 3$ (1)

$\alpha\beta = \alpha(\alpha r) = a$ (2)

$\gamma + \delta = a r^2 (1 + r) = 12$ (3)

and $\gamma\delta = (\alpha r^2)(\alpha r^3) = b$

$\Rightarrow \alpha^2 r^5 = b$ (4)

Dividing (3) by (1), $r^2 = 4 \Rightarrow r = 2$

Then, from (1), $\alpha = 1 \Rightarrow a = 2, b = 2^5 = 32$.

24. If the roots of $x^2 + x + a = 0$ exceed a, then

[EAMCET – 1994]

(a) $2 < a < 3$ (b) $a > 3$
(c) $-3 < a < 3$ (d) $a < -2$

Solution

(d) If the roots of the quadratic equation $ax^2 + bx + c = 0$ exceed a number k, then $ak^2 + bk + c > 0$ if $a > 0$, $b^2 - 4ac \geq 0$ and sum of the roots $> 2k$. Therefore, if the roots of $x^2 + x + a = 0$ exceed a number a, then $a^2 + a + a > 0$, $1 - 4a \geq 0$ and $-1 > 2a$

$\Rightarrow a(a + 2) > 0,\ a \leq \frac{1}{4}$ and $a < -\frac{1}{2}$

$\Rightarrow a > 0$ or $a < -2,\ a < \frac{1}{4}$ and $a < -\frac{1}{2}$

Hence, $a < -2$.

25. If both the roots of the quadratic equation $x^2 - 2kx + k^2 + k - 5 = 0$ are less than 5, then k lies in the interval

[AIEEE – 2005]

(a) $(-\infty, 4)$ (b) $[4, 5]$
(c) $(5, 6]$ (d) $(6, \infty)$

Solution

(a) $x^2 - 2kx + k^2 + k - 5 = 0$

Roots are less than 5, $D \geq 0$

$4k^2 - 4(k^2 + k - 5) \geq 0$ (1)

$\Rightarrow k \leq 5 \Rightarrow f(5) > 0$ (2)

$\Rightarrow k \in (-\infty, 4) \cup (5, \infty);\ -\left(\frac{-2k}{2}\right) < 5$

$\Rightarrow k < 5$ (3)

form (1), (2) and (3), $k\,(-\infty, 4)$

26. The values of 'a' for which $2x^2 - 2(2a + 1)x + a(a + 1) = 0$ may have one root less than a and other root greater than a are given by

[UPSEAT – 2001]

(a) $1 > a > 0$ (b) $-1 < a < 0$
(c) $a \geq 0$ (d) $a > 0$ or $a < -1$

Solution

(d) The given condition suggest that a lies between the roots. Let $f(x) = 2x^2 - 2(2a + 1)x + a(a + 1)$ For 'a' to lie between the roots we must have discriminant ≥ 0 and $f(a) < 0$.

Now, Discriminant $\geq 0 \Rightarrow 4(2a + 1)^2 - 8a(a + 1) \geq 0$

$\Rightarrow 8(a^2 + a + 1/2) \geq 0$ which is always true.

Also, $f(a) < 0$

$\Rightarrow 2a^2 - 2a(2a + 1) + a(a + 1) < 0$

$\Rightarrow -a^2 - a < 0 \Rightarrow a^2 + a > 0$

$\Rightarrow a(1 + a) > 0 \Rightarrow a > 0$ or $a < -1$

Note: For two real and distinct roots of a quadratic equation Disc. > 0.

27. Let a, b, c be real numbers $a \neq 0$. If is a root $a^2x^2 + bx + c = 0$, β is a root of $a^2x^2 - bx - c = 0$ and $0 < \alpha < \beta$, then the equation $a^2x^2 + 2bx + 2c = 0$ has a root γ that always satisfies

[IIT – 1989]

(a) $\gamma = \frac{\alpha + \beta}{2}$ (b) $\gamma = \alpha + \frac{\beta}{2}$
(c) $\gamma = \alpha$ (d) $\alpha < \gamma < \beta$

Solution

(d) Since, α and β are the roots of given equations.

So, we have $a^2\alpha^2 + b\alpha + c = 0$ and $\alpha^2\beta^2 - b\beta - c = 0$.

Let, $f(x) = a^2x^2 + 2bx + 2c = 0$

Then, $f(\alpha) = a^2\alpha^2 + 2b\alpha + 2c = 0$

$= a^2\alpha^2 + 2(b\alpha + c) = a^2\alpha^2 - 2a^2\alpha^2$

$= -a^2\alpha^2 =$ negative

and $f(\beta) = a^2\beta^2 + 2(b\beta + c)$

$= a^2\beta^2 + 2a^2\beta^2 = 3a^2\beta^2 =$ positive

Since, $f(\alpha)$ and $f(\beta)$ are of opposite signs, therefore, by theory of equations there lies a root γ of the equation $f(x) = 0$ between α and β i.e., $\alpha < \gamma < \beta$.

28. All the values of m for which both roots of the equation $x^2 - 2mx + m^2 - 1 = 0$ are greater than -2 but less than 4, lie in the interval **[AIEEE – 2006]**

(a) $m > 3$ (b) $-1 < m < 3$
(c) $1 < m < 4$ (d) $-2 < m < 0$

Solution

STEP 1: If a quadratic equation has either no root or even number of roots between a and b then $f(a)$ and $f(b)$ have same sign.

STEP 2: (b) Both roots will lie in $(-2, 4)$, if

$-2 < -b/2a < 4$ (1)

$f(-2) > 0$ (2)

$f(4) > 0$ (3)

Now (1) $\Rightarrow -2 < m < 4$ (4)

(2) $\Rightarrow m^2 + 4m + 3 > 0$

$\Rightarrow (m + 1)(m + 3) > 0$

$\Rightarrow m < -3$ or $m > -1$ (5)

(3) $\Rightarrow m^2 - 8m + 15 > 0$

$\Rightarrow (m - 3)(m - 5) > 0$

$\Rightarrow m < 3$ or $m > 5$ (6)

(4), (5) and (6) hold together when $-1 < m < 3$.

29. If roots of the equation $2x^2 - (a^2 + 8a + 1)x + a^2 - 4a = 0$ are in opposite sign, then **[Aligarh – 1998]**

(a) $0 < a < 4$ (b) $a > 0$
(c) $a < 8$ (d) $-4 < a < 0$

Solution

(a) Roots are in opposite sign

$\Rightarrow \frac{c}{a} < 0 \Rightarrow \frac{a^2 - 4a}{2} < 0$

30. If x be real then the value of $\frac{x^2 + 34x\ \ 71}{x^2 + 2x - 7}$ will not lie between

[Roorkee – 1983; Ranchi – 1999]

(a) -5 and 9 (b) 5 and 9
(c) -9 and -5 (d) 0 and 9

Solution

(b) Let $y = \frac{x^2 + 34x - 71}{x^2 + 2x - 7}$, then $(y - 1)x^2 + 2x(y - 17) + (71 - 7y) = 0$

Since x is real, so $b^2 - 4ac \geq 0 \Rightarrow 4(y - 17)^2 - 4(y - 1)(71 - 7y) \geq 0$

$\Rightarrow (y - 5)(y - 9) \geq 0 \therefore y \geq 9$ or $y \leq 5$

31. If 1, 2, 3 and 4 are the roots of the equation $x^4 + ax^3 + bx^2 + cx + d = 0$, then $a + 2b + c = 0$ **[EAMCET – 2007]**

(a) -25 (b) 0
(c) 10 (d) 24

Solution

(c) $x^4 + ax^3 + bx^2 + cx + d = (x - 1)(x - 2)(x - 3)(x - 4)$

$= x^4 - 10x^3 + 35x^2 - 50x + 24$

$a = -10, b = 35, c = -50$

$\therefore a + 2b + c = 10$

Alternative Method

$-a = S_1 = 1 + 2 + 3 + 4 = 10$

$b = S_2 = 1.2 + 2.3 + 3.4 + 1.3 + 1.4 + 2.4 = 35$

$-c = S_3 = 1.2.3 + 1.2.4 + 1.3.4 + 2.3.4 = 50$

$d = S_4 = S_4 = 1.2.3.4 = 24$ and $a + 2b + c = 10$.

32. If α, β, γ are the roots of $x^3 - 2x^2 + 3x - 4 = 0$, then the value of $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$ is **[EAMCET – 2007]**

(a) -7 (b) -5
(c) -3 (d) 0

Solution

(a) $\alpha + \beta + \gamma = 2, \alpha\beta + \beta\gamma + \gamma\alpha = 3$, $\alpha\beta\gamma = 4$

$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$

$= (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma) = -7$

33. The quadratic equations $x^2 - 6x + a = 0$ and $x^2 - cx + 6 = 0$ have one root in common. The other roots of the first and second equations are integers in the ratio 4 : 3. Then the common root is

[AIEEE – 2008]

(a) 2 (b) 1
(c) 4 (d) 3

Solution

(a) Let α and 4β be the root of $x^2 - 6x + a = 0$ and α and 3β be those of the equation $x^2 - cx + 6 = 0$

From the relation between roots and coefficients, $\alpha + 4\beta = 6$ and $4\alpha\beta = a$

$\alpha + 3\beta = c$ and $3\alpha\beta = 6$

We obtain, $\alpha\beta = 2$ giving $a = 8$

The first equation is $x^2 - 6x + 8 = 0 \Rightarrow x = 2, 4$

For $\alpha = 2$, $4\beta = 4 \Rightarrow 3\beta = 3$

For $\alpha = 4$, $4\beta = 2 \Rightarrow 3\beta = 3/2$ (not an integer)So the common root is $\alpha = 2$.

34. If the roots of the equation $x^3 - 12x^2 + 39x - 28 = 0$ are in A.P., then their common difference will be

[UPSEAT – 1994, 1999, 2001; RPET – 2001]

(a) ± 1 (b) ± 2
(c) ± 3 (d) ± 4

Solution

(c) Let $a - d, a, a + d$ be the roots of the equation $x^3 - 12x^2 + 39x - 28 = 0$

Then $(a - d) + a + (a + d) = 12$ and $(a - d)a(a + d) = 28$

$\Rightarrow 3a = 12$ and $a(a^2 - d^2) = 28$

$\Rightarrow a = 4$ and $a(a^2 - d^2) = 28$

$\Rightarrow 16 - d^2 = 7 \Rightarrow d = \pm 3$

35. The sum of squares of the roots of the equation $x^3 + x^2 + x + 1 = 0$ is

[MPPET – 2007]

(a) 1 (b) − 1
(c) 0 (d) 2

Solution

(b) $\alpha^2 + \beta^2 + \gamma = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$
$= (1)^2 - 2(1) = -1.$

36. If p, q, r, s are real numbers and $pr = 2(q + s)$, then equations $x^2 + px + q = 0$ and $x^2 + rx + s = 0$.

[IIT – 1975]

(a) both have real roots
(b) both have imaginary roots
(c) atleast one has real roots
(d) only one has real roots

Solution

(c) For first equation $B^2 - 4AC = p^2 - 4q = \lambda$ (say)

For second equation $B^2 - 4AC = r^2 - 4s = \mu$ (say)

Now $\lambda + \mu = p^2 + r^2 - 4(q + s)$
$= p^2 + r^2 - 2pr$
$= (p - r)^2 \geq 0$ [$\because 2(q + s) = pr$]

$\Rightarrow$ both λ, μ may not be negative

$\Rightarrow$ atleast one of λ, μ is positive

$\Rightarrow$ atleast one equation has real roots.

37. If roots of the equations $ax^2 + 2bx + c = 0$ and $bx^2 - 2\sqrt{acx} + b = 0$ are real, then

[Aligarh – 1998]

(a) $ac = b^2$ (b) $4b^2 - ac = 0$
(c) $a = b, c = 0$ (d) $a = b = 0$

Solution

(a) Roots are real, so

$4b^2 - 4ac \geq 0 \Rightarrow b^2 - ac \geq 0$ and

$4ac - 4b^2 \geq 0 \Rightarrow ac - b^2 \geq 0$

$\Rightarrow b^2 - ac \leq 0$

(i), (ii) $\Rightarrow b^2 - ac = 0 \Rightarrow b^2 = ac$

38. The two roots of an equation $x^3 - 9x^2 + 14x + 24 = 0$ are in the ratio 3: 2. The roots will be

[UPSEAT – 1999]

(a) 6, 4, − 1 (b) 6, 4, 1
(c) − 6, 4, 1 (d) − 6, − 4, 1

Solution

(a) Let required roots are $3\alpha, 2\alpha, \beta$

($\because$ ratio of two roots are 3: 2)

$\therefore \Sigma\alpha = 3\alpha + 2\alpha + \beta = \frac{-(-9)}{1} = 9$

$\Rightarrow 5\alpha + \beta = 9$ (1)

$\Sigma\alpha\beta = 3\alpha.2\alpha + 2\alpha.\beta + \beta.3\alpha = 14$

$\Rightarrow 5\alpha\beta + 6\alpha^2 = 14$ (2)

and $\Sigma\alpha\beta\gamma = 3\alpha.2\alpha.\beta = -24$

$\Rightarrow 6\alpha^2\beta = -24$

or $\alpha^2\beta = -4$ (3)

from (1), $\beta = 9 - 5\alpha$, put the value of β in (2)

$\Rightarrow 5\alpha(9 - 5\alpha) + 6\alpha^2 = 14$

$\Rightarrow 19\alpha^2 - 45\alpha + 14 = 0$

$\Rightarrow (\alpha - 2)(19\alpha - 7) = 0$

$\therefore \alpha = 2$ or $\frac{7}{19}$

$\therefore$ From (1), if $\alpha = 2$, then $\beta = 9 - 5 \times 2 = -1$

$\because \alpha = 2, \beta = -1$ satisfy the Equation (3) so required roots are 6, 4, – 1.

39. If the sum of two of the roots of $x^3 + px^2 + qx + r = 0$ is zero, then $pq =$

[EAMCET – 2003]

(a) $-r$ (b) r

(c) $2r$ (d) $-2r$

Solution

(b) Given that, $\alpha + \beta = 0$

$\alpha + \beta + \gamma = -p \Rightarrow \gamma = -p$

Substituting $\gamma = -p$ in the given equation

$\Rightarrow -p^3 + p^3 - pq + r = 0 \Rightarrow pq = r$

40. If the roots of the equation $8x^3 - 14x^2 + 7x - 1 = 0$ are in G.P., then the roots are

[MPPET – 1986]

(a) $1, \frac{1}{2}, \frac{1}{4}$

(b) 2, 4, 8

(c) 3, 6, 12

(d) none of these

Solution

(a) Let the roots be $\frac{\alpha}{\beta}, \alpha, \alpha, \beta, \beta, \neq 0$. Then, the product of roots is $\alpha^3 = -\frac{-1}{8} = \frac{1}{8} \Rightarrow \alpha = \frac{1}{2}$ and hence, $\beta = \frac{1}{2}$ so roots are $1, \frac{1}{2}, \frac{1}{4}$

Trick: By inspection, we get the numbers $1, \frac{1}{2}, \frac{1}{4}$ satisfying the given equation.

Note: $\alpha + \beta + \gamma = -\left(\frac{14}{8}\right)$

$$\frac{1}{2\beta} + \frac{1}{2} + \frac{1}{2}\beta = \frac{14}{8} = \frac{7}{4}$$

41. If α, β, γ are roots of equation $x^3 + ax^2 + bx + c = 0$, then $\alpha^{-1} + \beta^{-1} + \gamma^{-1} =$

[EAMCET – 2002]

(a) a/c (b) $-b/c$

(c) b/a (d) c/a

Solution

(b) α, β, gare the roots of the equation $x^3 + ax^2 + bx + c = 0$

so $\alpha + \beta + \gamma = -a$, $\alpha\beta + \beta\gamma + \gamma a = b$ and $\alpha\beta\gamma = -c$

Now, $\alpha^{-1} + \beta^{-1} + \gamma^{-1} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$$\frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}$$

$= -b/c$.

42. If x is real, the maximum value of $\frac{3x^2 + 9x + 17}{3x^2 + 9x + 7}$ is

[AIEEE – 2006]

(a) 1 (b) 41

(c) 1/4 (d) 17/7

Solution

(b) Let $= \frac{3x^2 + 9x + 17}{3x^2 + 9x + 17} = y$

Then, $(3y - 3)x^2 + (9y - 9)x + (7y - 17) = 0$

But, $x \in R$, so its roots must be real.

Hence, $(9y - 9)^2 - 4(3y - 3)(7y - 17) \geq 0$

$\Rightarrow y^2 - 42y + 41 \leq 0$

$\Rightarrow (y - 1)(y - 41) \leq 0$

$\Rightarrow 1 \leq y \leq 41$.

Hence, maximum value = 41

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. The value of a for which the sum of the squares of the roots of the equation $x^2 - (a-2)x - (a+1) = 0$ has the least value, is

[AIEEE – 2005]

(a) 0 (b) 1
(c) 2 (d) 3

2. The maximum value of $5 + 20x - 4x^2$, $x \in R$ is

(a) 25 (b) 30
(c) 5 (d) 1

3. If x be real, then least value of $3x^2 + 7x + 10$ is

(a) 10 (b) 10/3
(c) 7/3 (d) 71/12

4. If x is real, then the maximum and minimum values of the expression $\frac{x^2 - 3x + 4}{x^2 + 3x + 4}$ will be

[IIT – 1984]

(a) 2, 1 (b) 5, 1/5
(c) 7, 1/7 (d) none of these

5. The quadratic in t, such that A.M. of its roots in A and G.M. is G, is

[IIT – 1968, 1974]

(a) $t^2 - 2At + G^2 = 0$ (b) $t^2 - 2At - G^2 = 0$
(c) $t^2 + 2At + G^2 = 0$ (d) none of these

6. The expression $x^2 + 2bx + c$ has the positive value if

[Roorkee – 1975]

(a) $b^2 - 4c > 0$ (b) $b^2 - 4c < 0$
(c) $c^2 < b$ (d) $b^2 < c$

7. Let $f(x) = x^2 + 4x + 1$, then

(a) $f(x) > 0$ for all x
(b) $f(x) > 1$ when $x \geq 0$
(c) $f(x) \geq 1$ when $x \leq -4$
(d) $f(x) = f(-x)$ for all x

8. If the roots of the equation $x^2 - 8x + (a^2 - 6a) = 0$ are real, then

[RPET –1987, 1997; MPPET – 1999]

(a) $-2 < a < 8$ (b) $2 < a < 8$
(c) $-2 \leq a \leq 8$ (d) $2 \leq a \leq 8$

9. The number of roots of the equation $|x|^2 - 7|x| + 12 = 0$ is

[MNR – 1995]

(a) 1 (b) 2
(c) 3 (d) 4

10. Product of real roots of the equation $t^2 x^2 + |x| + 9 = 0$,

[AIEEE – 2002]

(a) is always positive
(b) is always negative
(c) does not exist
(d) none of these

11. The number of roots of the equation $|x| = x^2 + x - 4$ is

[Kerala PET – 2007]

(a) 4 (b) 3 (c) 1 (d) 2

12. $x^2 - 3x + 2$ be a factor of $x^4 - px^2 + q$, then $(p, q) =$

[IIT– 1974; MPPET–1995; Pb. CET – 2001]

(a) (3, 4) (b) (4, 5)
(c) (4, 3) (d) (5, 4)

13. If $(x + a)$ is a factor of both the quadratic polynomials $x^2 + px + q$ and $x^2 + lx + m$, where p, q, l and m are constants, then which one of the following is correct?

[NDA – 2009]

(a) $a = (m - q) / (l - p)$ $(l \neq p)$
(b) $a = (m + q) / (l + p)$ $(l \neq -p)$
(c) $l = (m - q) / (a - p)$ $(a \neq p)$
(d) $p = (m - q) / (a - l)$ $(a \neq l)$

14. What is the value of x satisfying the equation $16\left(\frac{a-x}{a+x}\right)^3 = \frac{a+x}{a-x}$?

[NDA–2009]

(a) $a/2$ (b) $a/3$ (c) $a/4$ (d) 0

15. If α, β be the roots of $x^2 + px + q = 0$ and $\alpha + h, \beta + h$ are the roots of $x^2 + rx + s = 0$, then

[AMU – 2001]

(a) $\frac{p}{r}=\frac{q}{s}$ (b) $2h=\left[\frac{p}{q}+\frac{r}{s}\right]$

(c) $p^2-4q=r^2-4s$ (d) $pr^2=qs^2$

16. $x^2-11x+a$ and $x^2-14x+2a$ will have a common factor, if $a=$

[Roorkee– 1981]

(a) 24 (b) 0, 24

(c) 3, 24 (d) 0, 3

17. The real root of the equation $x^3-6x+9=0$ is

[Karnataka CET – 2008]

(a) 6 (b) −3

(c) −6 (d) −9

18. If α, β are the roots of the quadratic equation $x^2+bx-c=0$, then the equation whose roots are b and c is

[Pb. CET – 1989]

(a) $x^2+\alpha x-\beta=0$

(b) $x^2-[(\alpha+\beta)+\alpha\beta]x-\alpha\beta(\alpha+\beta)=0$

(c) $x^2-[(\alpha+\beta)+\alpha\beta]x+\alpha\beta(\alpha+\beta)=0$

(d) $x^2+[\alpha\beta+(\alpha+\beta)]x-\alpha\beta(\alpha+\beta)=0$

19. If α_1, α_2 and β_1, β_2 are the roots of the equations $ax^2+bx+c=0$ and $px^2+qx+r=0$, respectively and system of equations $\alpha_1 y+\alpha_2 z=0$ and $\beta_1 y+\beta_2 z=0$ has a nonzero solution.

[IIT – 1987]

(a) $a^2 qc=p^2 br$ (b) $b^2 pr=q^2 ac$

(c) $c^2 ar=r^2 pb$ (d) none of these

20. Let α, β be the roots of the equation $ax^2+2bx+c=0$ and γ, δ be the roots of the equation $px^2+2qx+r=0$. If $\alpha, \beta, \gamma, \delta$ are in G.P., then

(a) $q^2 ac=b^2 pr$ (b) $qac=bpr$

(c) $c^2pq=r^2ab$ (d) $p^2 ab=a^2qr$

21. The value of k for which the equation $(k-2)x^2+8x+k+4=0$ has both roots real, distinct and negative is

[Orissa JEE – 2002]

(a) 0 (b) 2 (c) 3 (d) −4

22. The set of values of λ for which the equation $3x^2+2x+\lambda(\lambda-1)=0$ are of opposite signs is

(a) (0, 1) (b) [0, 1] (c) [0, 1) (d) (0, 1]

23. The values of a for which one root of the equation $x^2-(a+1)x+a^2+a-8=0$ exceeds 2 and the other is lesser than 2, are given by

(a) $a>3$ (b) $9<a<10$

(c) $-2<a<3$ (d) none of these

24. The value of p for which both the roots of the equation $4x^2-20px+(25p^2+15p-66)=0$ are less than 2, lies in the interval

(a) $(-1, -4/5)$ (b) $(-\infty, -1)$

(c) $(2, \infty)$ (d) none of these

25. If both the roots of $ax^2+bx+c=0$ are positive, then

(a) $-\frac{b}{a}>0$ (b) $\frac{c}{a}>0$

(c) $b^2\geq 4ac$ (d) $ac>0$

26. The value of a for which the quadratic equation $3x^2+2(a^2+1)x+(a^2-3a+2)=0$ passesses roots with opposite sign, lies in

(a) $(-\infty, 1)$ (b) $(-\infty, 0)$

(c) (1, 2) (d) (3/2, 2)

27. If the roots of the equation $bx^2+cx+a=0$ be imaginary, then for all real values of x, the expression $3b^2x^2+6bcx+2c^2$ is

[AIEEE – 2009]

(a) greater than $4ab$ (b) less than $4ab$

(c) greater than $-4ab$ (d) less than $-4ab$

SOLUTIONS

1. (b) Let α, β be the roots, then

$\alpha+\beta=a-2$ and $\alpha\beta=-(a+1)$

Now $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$=(a-2)^2+2(a+1)$

$=a^2-2a+6$

$=(a-1)^2+5$,

which is least when $a-1=0$, i.e., when $a=1$.

2. (b) **Step 1:** Maximum value of $ax^2 + bx + c$ is

$\frac{4ac - b^2}{4a}$ if $a < 0$

Step 2: $a = -4, b = 20, c = 5$

$$\text{Maximum value} = \frac{4 \times (-4) \times 5 - (20)^2}{4 \times (-4)}$$

$$= \frac{-480}{-16} = 30$$

3. (d) **Step 1:** Minimum value of $ax^2 + bx + c$ is

$\frac{4ac - b^2}{4a}$ if $a > 0$

Step 2: $a = 3, b = 7, c = 10.$

$$\text{Minimum value} = \frac{4 \times 3 \times 10 - 7 \times 7}{4 \times 3}$$

$$= \frac{120 - 49}{12} = \frac{71}{12}$$

4. (c) Let $y = \frac{x^2 - 3x + 4}{x^2 + 3x + 4}$, then

$x^2(y - 1) + 3x(y + 1) + 4(y - 1) = 0$

Now x is real $\Rightarrow b^2 - 4ac \geq 0$

$\Rightarrow 9(y + 1)^2 - 16(y - 1)^2 \geq 0$

$\Rightarrow (7y - 1)(7 - y) \geq 0$

$\Rightarrow (7y - 1)(y - 7) \leq 0$

$\therefore \frac{1}{7} \leq y \leq 7$

Ans: $7, \frac{1}{7}$

5. (a) $\frac{\alpha + \beta}{2} = A, \sqrt{\alpha\beta} = G$

$\therefore \alpha + \beta = 2A, \alpha\beta = G^2$

Equation is $t^2 - (\alpha + \beta)t + \alpha\beta = 0$ or $t^2 - 2At + G^2 = 0.$

6. (d) $x^2 + 2bx + c > 0$ if "$a > 0, D < 0$" or $1 > 0$, $4b^2 - 4c < 0$ or $b^2 < c$.

Note: Sign of quadratic $Ax^2 + Bx + C = 0$ is same as a if $B^2 - 4AC \leq 0$.

7. (c) $f(x) = (x + 2)^2 - 3 \geq 1 \Rightarrow x^2 + 4x \geq 0$
$x(x + 4) \geq 0 \Rightarrow x \leq -4$ and $x \leq 0$

8. (c) If roots of $Ax^2 + Bx + C = 0$ are real, then $(B^2 \geq 4AC)$ or $B^2 - 4AC \geq 0$. (1)

given $A = 1, B = -8, C = a^2 - 6a$

$\therefore 64 - 4(a^2 - 6a) \geq 0$

or $a^2 - 6a - 16 \leq 0$ or $(a + 2)(a - 8) \leq 0$

$a \in [-2, 8]$

9. (d) $|x|^2 - 7|x| + 12 = 0$ (1)

Case 1: When $x > 0$, then (1) gives $x^2 - 7x + 12 = 0$ or $(x - 4)(x - 3) = 0$ or $x = 3, 4 > 0$

$\therefore$ given Equation (1) has two roots.

Case 2: When $x < 0$, then (1) gives $x^2 + 7x + 12 = 0$ or $(x + 4)(x + 3) = 0$ or $x = -3, -4 < 0$. $\therefore$ Given Equation (1) has two roots. Total roots = 2 + 2 = 4

10. (c) Note that for $t \in R$, $t^2x^2 + |x| + 9 \geq 9$ and hence the given equation cannot have real roots.

11. (d) Given equation is $x^2 + x - |x| - 4 = 0$ (1)

Two cases arise

Case 1: If $x \geq 0$, then $|x| = x$ and then the equation becomes,

$x^2 + x - x - 4 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

$\Rightarrow x = 2$ is a solution in this case

Case 2: If $x < 0$, then $|x| = -x$ and the equation becomes,

$x^2 + x - (-x) - 4 = 0$

$\Rightarrow x^2 + 2x - 4 = 0$

$\Rightarrow x = \frac{-2 \pm \sqrt{4 + 16}}{2}$

$\Rightarrow x = -1 \pm \sqrt{5}$, but, $x < 0$

therefore, $x = -1 - \sqrt{5}$.

12. (d) If $x^2 - 3x + 2$ is one of the factors of the expression $x^4 - px^2 + q$, then on dividing the expression by factor, remainder = 0 On dividing $x^4 - px^2 + q$ by $x^2 - 3x + 2$ Remainder = $(12 - 3p)x + (2p + q - 14) = 0$ On comparision $15 - 3p = 0$ or $p = 5$ and $2p + q - 14 = 0$ gives $q = 4$.

Trick: If $x^2 - 3x + 2$ is one of the factor of given expression then roots of $x^2 - 3x + 2 = 0$ will also satisfy $x^4 - px^2 + q$. The roots of $x^2 - 3x + 2 = 0$ are $x = 1, x = 2$.

Put $x = 1$ in expression $x^4 - px^2 + q$

$1 - p + q = 0$ (1)

and put $x = 2$ in expression $x^4 - px^2 + q = 0$, we get $16 - 4p + q = 0$ (2)

Solving (1) and (2), we find, $p = 5$, $q = 4$.

13. (a) Since, $(x + a)$ is a factor of $x^2 + px + q$ and $x^2 + lx + m$

$\therefore a^2 - ap + q = 0$ (1)

and $a^2 - la + m = 0$ (2)

From Equations (1) and (2), we get,

$-ap + q + la - m = 0$

$\Rightarrow (l - p)\, a = m - q$

$\Rightarrow a = \frac{m - q}{l - p}\ (l \neq p)$

14. (b) $16\left(\frac{a - x}{a + x}\right)^3 = \frac{a + x}{a - x}$

$\Rightarrow \left(\frac{a - x}{a + x}\right)^4 = \left(\frac{1}{2}\right)^4$

$\Rightarrow \frac{a - x}{a + x} = \frac{1}{2}$

$\Rightarrow 2a - 2x = a + x$

$\Rightarrow a = 3x$

$\Rightarrow x = \frac{a}{3}$

15. (c) Given α, β are the roots of $x^2 + px + q = 0$, then $\alpha + \beta = -p$ and $\alpha\beta = q$ and $\alpha + h, \beta + h$ are the roots of $x^2 + rx + s = 0$ then,

$\alpha + \beta + 2h = -r$

$-p + 2h = -r$

$2h = -r + p$ (1)

and $(\alpha + h)(\beta + h) = s$

$\Rightarrow \alpha\beta + \alpha h + \beta h + h^2 = s$

$\Rightarrow q + h(-p) + h^2 = s$

$\Rightarrow q + \frac{(-p)(p - r)}{2} + \frac{(p - r)^2}{4} = s$

$\Rightarrow \frac{4q + 2pr - 2p^2 + p^2 + r^2 - 2pr}{4} = s$

$\Rightarrow 4q + r^2 - p^2 = 4s$

$\Rightarrow r^2 - 4s = p^2 = 4q$

OR

Quicker Method:

Difference of the roots of both quadratic being same, therefore applying the formula of the difference of roots of the quadratic $Ax^2 + Bx + C = 0$.

$|\alpha - \beta| = \frac{\sqrt{B^2 - 4AC}}{4A}$

We find $\frac{\sqrt{p^2 - 4q}}{4} = \frac{\sqrt{r^2 - 4s}}{4}$

or $p^2 - 4q = r^2 - 4s$

16. (b) Let common factor be $x - \alpha$. Then, $\alpha^2 - 11\alpha + a = 0$ and $\alpha^2 - 14\alpha + 2a = 0$

$\Rightarrow \frac{\alpha^2}{-8a} = \frac{\alpha}{-a} = \frac{1}{-3}$

$\Rightarrow \frac{8a}{a} = \frac{a}{3} \Rightarrow a = 24$ or 0.

17. (b) $x^3 - 6x + 9 = 0$

$x^3 + 3x^2 - 3x^2 - 9x + 3x + 9 = 0$

$x^2(x + 3) - 3x(x + 3) + 3(x + 3) = 0$

$(x + 3)(x^2 - 3x + 3) = 0$

Either $x + 3 = 0 \Rightarrow x = -3$

and $x^2 - 3x + 3 = 0$

$x = \frac{9 + \sqrt{9 - 4 \times 1 \times 3}}{2 \times 1}$

$x = \frac{9 \pm \sqrt{3i}}{2}$

18. (c) $\alpha + \beta = -b$ and $\alpha\beta = -c$

$\Rightarrow b + c = -(\alpha + \beta) - \alpha\beta$

and $bc = (\alpha + \beta)\alpha\beta$

Therefore, the equation is,

$x^2 + [(\alpha + \beta) + \alpha\beta]x + (\alpha + \beta)\alpha\beta = 0$

19. (b) $\therefore \alpha_1, \alpha_2 : ax_2 + bx + c = 0$

$\therefore \alpha_1 + \alpha_2 = \frac{b}{a}, \alpha_1\alpha_2 = \frac{c}{a}$

$\beta_1, \beta_2 : px^2 + qx + r = 0$

$\therefore \beta_1 + \beta_2 = -\frac{q}{p}, \beta_1\beta_2 = \frac{r}{p}$

$\left.\begin{matrix} \alpha_1 y + \alpha_2 z = 0 \\ \beta_1 y + \beta_2 z = 0 \end{matrix}\right\}$ have non trival solution, if

$\begin{Bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{Bmatrix} = 0$ or $\alpha_1\beta_2 - \alpha_2\beta_1 = 0$

or, $\frac{\alpha_1}{\alpha_2}=\frac{\beta_1}{\beta_2}$ or $\frac{\alpha_1+\alpha_2}{\alpha_1-\alpha_2}=\frac{\beta_1+\beta_2}{\beta_1-\beta_2}$

or $\frac{(\alpha_1+\alpha_2)^2}{(\alpha_1+\alpha_2)^2-4\alpha_1\alpha_2}=\frac{(\beta_1+\beta_2)^2}{(\beta_1+\beta_2)^2-4\beta_1\beta_2}$

or $\frac{b^2}{(b^2-4ca)}=\frac{q^2}{(q^2-4rp)}$

or, $b^2(q^2-4rp)=q^2(b^2-4ca)$

or, $rpb^2=acq^2$

OR

From given equation we clearly have

$\frac{\alpha_1}{\alpha_2}=\frac{\beta_1}{\beta_2}=\frac{-z}{y}$

i.e., ratio of the both roots of the quadratic equations are same therefore, $\frac{a}{p}, \frac{b}{q}$ and $\frac{c}{r}$ are in G.P.

i.e., $\left(\frac{b}{q}\right)^2=\left(\frac{a}{p}\right)\left(\frac{c}{r}\right)$

$prb^2=acq^2$

20. (a) $\alpha+\beta=-\frac{2b}{a}, \alpha\beta=\frac{c}{a},$

$\gamma+\delta=-\frac{2q}{p}, \gamma\delta=\frac{r}{p}$

As given $\alpha, \beta, \gamma, \delta$ are in G.P., therefore,

$\frac{\alpha}{\gamma}=\frac{\beta}{\delta}$ (1)

But, $\frac{\alpha\beta}{\gamma\delta}=\frac{pc}{ar}\Rightarrow\left(\frac{\beta}{\delta}\right)^2=\frac{pc}{ar}$ [By (1)] (2)

Also,

$\frac{\alpha}{\beta}=\frac{\gamma}{\delta}\Rightarrow\frac{\alpha+\beta}{\beta}=\frac{\gamma+\delta}{\delta}$

$\Rightarrow\frac{\alpha+\beta}{\gamma+\delta}=\frac{\beta}{\delta}$

$\Rightarrow\frac{bp}{aq}=\sqrt{\frac{pc}{ar}}\Rightarrow\frac{b^2p^2}{a^2q^2}=\frac{pc}{ar}$

$\Rightarrow q^2ac=b^2pr$

OR

Since, $\alpha, \beta, \gamma, \delta$ are in G.P.; therefore $\frac{\alpha}{\beta}=\frac{\gamma}{\delta}$ i.e., ratio of the both roots of the quadratic equations are in same therfore ratio of the corresponding coefficients of both quadratic equations must be in G.P.

i.e., $\frac{a}{p}, \frac{b}{q}$ and $\frac{c}{r}$ are in G.P.

i.e., $\left(\frac{b}{q}\right)^2=\left(\frac{a}{p}\right)\left(\frac{c}{r}\right)$

or $b^2pr=q^2ac$

21. (c) For real and unequal roots, $D>0$

$\Rightarrow(k+6)(k-4)<0$

$\Rightarrow k<4$ and $k>-6$

Also, for roots to be negative $\frac{k+4}{k-2}>0$

$\Rightarrow k>2$ and $k>-4$

22. (a) Roots are in opposite sign

$\Rightarrow\frac{\lambda(\lambda-1)}{3}<0$

$\Rightarrow\lambda(\lambda-1)<0$

$\Rightarrow 0<\lambda<1$

23. (c) If α, β be the roots of the given equation $f(x)=0$ then by the given condition,

$\alpha<2<\beta$ (1)

Also, $f(x)=1.(x-\alpha)(x-\beta), \alpha<\beta$ (2)

We conclude from (1) that the roots of the given equation must be real as order relation does not exist in complex numbers. Secondly, from (2) we conclude that $f(x)$ = negative for all values of x which lie between α and β.

$\therefore f(2)$ = negative

$\therefore \Delta>0$ (distinct roots) and $f(2)$ = negative

$\therefore (a+1)^2-4(a^2+a-8)$ = positive

or $3a^2+2a-33$ = negative

$\therefore -\frac{11}{3}<a<3$ (3)

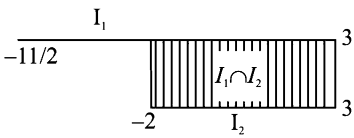
Again, $f(2)$ = negative

$\Rightarrow 4-2(a+1)+a^2+a-8$ is negative

$a^2-a-6=-$ ive or $(a+2)(a-3)$ = negative

$\therefore -2<a<3$ (4)

Hence a should be so chosen as to satisfy both (3) and (4). In other words it will satisfy the common region or intersection of the intervals given by (3) and (4). Mark them on real line as shown below and take their intersection.



$\therefore$ Common region is $-2 < a < 3$.

24. (b) Let $f(x) = 4x^2 - 20px + (25p^2 + 15p - 66) = 0$ (1)

The roots of (1) are real if,

$b^2 - 4ac = 400p^2 - 16(25p^2 + 15p - 66)$

$= 16(66 - 15p) \geq 0$

$\Rightarrow p \leq \frac{22}{5}$ (2)

Both roots of (1) are less than 2. Therefore, $f(2) > 0$ and sum of roots < 4.

$\Rightarrow 4.2^2 - 20p.2 + (25p^2 + 15p - 66) > 0$

and $\frac{20p}{4}$

$\Rightarrow p^2 - p - 2 > 0$ and $p < \frac{4}{5}$

$\Rightarrow (p + 1)(p - 2) > 0$ and $p < \frac{4}{5}$

$\Rightarrow p < -1$ or $p > 2$ and $p < \frac{4}{5}$

$\Rightarrow p < -1$ (3)

From (2) and (3), we get, $p < -1$ i.e. $p \in (-\infty, -1)$

25. (*a*, *b*, *c*, *d*) When both the roots of $ax^2 + bx + c = 0$ are positive, then $b^2 - 4ac \geq 0$ as the roots are real, $-\frac{b}{a} > 0$ as the sum of the roots is positive and $\frac{c}{a} > 0$ as the product of the roots is positive.

As c and a are of same sign, ac is also positive.

26. (c) In order that the quadratic equation may have two roots with opposite signs, it must have real roots with their product negative, i.e., if the discriminant,

$4(a^2 + 1)^2 - 12(a^2 - 3a + 2) > 0$ and

$\frac{1}{3}(a^2 - 3a + 2) < 0$

Both of these conditions get satisfied if $a^2 - 3a + 2 < 0$.

i.e., if $(a - 1)(a - 2) < 0$ or if $1 < a < 2$.

27. (c) For equation, $bx^2 + cx + a = 0$, the roots are imaginary,

so, $c^2 - 4ab < 0$

or $c^2 < 4ab$

or $-c^2 > -4ab$

Given expression $3b^2x^2 + 6bcx + 2c^2$ has minimum value.

$$\frac{4(3b^2)(2c^2) - 36b^2c^2}{4(3b^2)} = \frac{12b^2c^2}{12b^2} = -c^2$$

$= -c^2 > -4ab$ $(3b^2 > 0)$

UNSOLVED OBJECTIVE PROBLEMS: IDENTICAL PROBLEMS FOR PRACTICE: FOR IMPROVING SPEED WITH ACCURACY

1. If x be real, then the maximum value of $5 + 4x - 4x^2$ will be equal to

[MNR – 1979]

(a) 5 (b) 6

(c) 1 (d) 2

2. If x be real, the least value of $x^2 - 6x + 10$ is

[Kurukshetra CEE – 1998]

(a) 1 (b) 2

(c) 3 (d) 10

3. If x is real and $\frac{x^2 - x + 1}{x^2 + x + 1}$, then

[MNR – 1992; RPET –1997]

(a) $\frac{1}{3} \leq k \leq 3$ (b) $k \geq 5$

(c) $k \leq 0$ (d) none of these

4. If x be real, then the minimum value of $x^2 - 8x + 17$ is

(a) – 1 (b) 0 (c) 1 (d) 2

5. The number of real solutions of the equation $|x|^2 - 3|x| + 2 = 0$ are

[IIT – 1982, 1989; MPPET – 1997, 2009; DCE – 2002; AMU – 2000; UPSEAT – 1999; AIEEE – 2003]

(a) 1 (b) 2

(c) 3 (d) 4

6. Let α, β be the roots of $x^2 + (3 - \lambda) x - \lambda = 0$. The value of λ for which $\alpha^2 + \beta^2$ is minimum, is

[AMU – 2002]

(a) 0 (b) 1
(c) 2 (d) 3

7. If sec θ and tan θ are the roots of $ax^2 + bx + c = 0$ $(a, b \neq 0)$, then the value of $\sec\theta - \tan\theta$ is

[Kerala PET – 2008]

(a) $-\frac{a}{b}$ (b) $-\frac{b}{a}$
(c) $\frac{a^2}{b^2}$ (d) $1 + \frac{a^2}{b^2}$

8. If a root of the equations $x^2 + px + q = 0$ and $x^2 + \alpha x + \beta = 0$ is common, then its value will be (where $p \neq \alpha$ and $q \neq \beta$)

[IIT – 1974, 1976; RPET – 1997]

(a) $\frac{q-\beta}{\alpha - p}$ (b) $\frac{p\beta - \alpha q}{q - \beta}$
(c) $\frac{q-\beta}{\alpha - p}$ or $\frac{p\beta - \alpha q}{q - \beta}$ (d) none of these

9. If $x^2 - hx - 21 = 0$, $x^2 - 3hx + 35 = 0$ $(h > 0)$ has a common root, then the value of h is equal to

(a) 1 (b) 2
(c) 3 (d) 4

10. If the roots of the equation $qx^2 + px + q = 0$, where p, q are real, be complex, then the roots of the equation $x^2 - 4qx + p^2 = 0$ are

(a) real and unequal (b) real and equal
(c) imaginary (d) none of these

11. A real root of the equation $\log_4 \{\log_2(\sqrt{x+8} - \sqrt{x})\} = 0$ is

[AMU – 1999]

(a) 1 (b) 2
(c) 3 (d) 4

12. If both roots of $x^2 - mx + 121 = 0$ are greater than 10, then minimum value of m is

[NDA – 2004]

(a) 22 (b) 23
(c) 21 (d) 221/10

13. The least value of a so that both roots of the equation $x^2 - 2(a - 1)x + (2a + 1) = 0$ are positive will be

(a) 1 (b) 3 (c) 4 (d) 5

14. The set of values of p for which the roots of the equation $3x^2 + 2x + p(p - 1) = 0$ are of opposite signs is

[IIT – 1992]

(a) $(-\infty, 0)$ (b) $(0, 1)$
(c) $(1, \infty)$ (d) $(0, \infty)$

15. If both the roots of $ax^2 + bx + c = 0$ are negative, then

(a) $b^2 \geq 4ac$ (b) $\frac{c}{a} > 0$
(c) $\frac{b}{a} > 0$ (d) $\frac{c}{a} > 0$

16. If the ratio of the roots of $x^2 + bx + c = 0$ and $x^2 + qx + r = 0$ be the same, then

(a) $r^2c = b^2q$ (b) $r^2b = c^2q$
(c) $rb^2 = cq^2$ (d) $rc^2 = bq^2$

17. The minimum value of $x^2 + 8x + 17$ is

(a) – 1 (b) 0 (c) 1 (d) 17

18. The equation $\log_e x + \log_e (1 + x) = 0$ can be written as

(a) $x^2 + x - e = 0$ (b) $x^2 + x - 1 = 0$
(c) $x^2 + x + 1 = 0$ (d) $x^2 + xe - e = 0$

19. If α, β be the roots of the quadratic equation $ax^2 + bx + c = 0$ and k be a real number, then the condition so that $\alpha < k < \beta$ is given by

(a) $ac > 0$
(b) $ak^2 + bk + c = 0$
(c) $ac < 0$
(d) $a^2k^2 + abk + ac < 0$

20. The product of all real roots of the equation $x^2 - |x| - 6 = 0$ is

[Roorkee – 2000]

(a) – 9 (b) 6
(c) 9 (d) 36

21. For what value of λ the sum of the squares of the roots of $x^2 + (2 + \lambda)x - \frac{1}{2}(1 + \lambda) = 0$ is minimum

[AMU – 1999]

(a) 3/2 (b) 1
(c) – 5/2 (d) 11/4

22. If one of the roots of the equation $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ is coincident, then the numerical value of $(a + b)$ is

[IIT – 1986; RPET – 1992; EAMCET – 2002]

(a) 0 (b) -1
(c) 2 (d) 5

23. The value of 'a' for which the equations $x^2 - 3x + a = 0$ and $x^2 + ax - 3 = 0$ have a common root is

[Pb. CET – 1999]

(a) 3 (b) 1
(c) -2 (d) 2

24. If $(x + 1)$ is a factor of $x^4 - (p - 3)x^3 - (3p - 5)x^2 + (2p - 7)x + 6$, then $p =$

(a) 4 (b) 2
(c) 1 (d) none of these

25. The conditions that the equation $ax^2 + bx + c = 0$ has both the roots positive are that:

(a) a, b, c are of same sign; $b^2 - 4ac \geq 0$
(b) a and b are of same sign; $b^2 - 4ac \geq 0$
(c) b and c have the same sign opposite to that of a; $b^2 - 4ac \geq 0$.
(c) a and c have the same sign opposite to that of b; $b^2 - 4ac \geq 0$.

WORK SHEET: TO CHECK PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the work sheet
2. The test is of 15 minutes.
3. The test consists of 15 questions. The maximum marks are 45.
4. Use blue/black ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. If x is real, then the value of $x^2 - 6x + 13$ will not be less than

[RPET – 1986]

(a) 4 (b) 6
(c) 7 (d) 8

2. If $x^2 + 2x + n > 10$ for all real number x, then which of the following conditions is true?

[Kerala PET – 2008]

(a) $n < 11$ (b) $n = 10$
(c) $n = 11$ (d) $n > 11$

3. The smallest value of $x^2 - 3x + 3$ in the interval $(-3, 3/2)$ is

[EAMCET – 1991, 1993]

(a) 3/4 (b) 5
(c) -15 (d) -20

4. For the equation $|x^2| + |x| - 6 = 0$, the roots are

[EAMCET – 1988, 1993]

(a) one and only one real number
(b) real with sum one
(c) real with sum zero
(d) real with product zero

5. The solution set of the equation $x^{\log_x (1-x)^2} = 9$ is

[Pb. CET – 2003]

(a) $\{-2, 4\}$ (b) $\{4\}$
(c) $\{0, -2, 4\}$ (d) none of these

6. If $x^{2/3} - 7x^{1/3} + 10 = 0$, then $x =$

[BIT RANCHI – 1992]

(a) $\{125\}$ (b) $\{8\}$
(c) ϕ (d) $\{125, 8\}$

7. If $a > 0, b > 0, c > 0$, then both the roots of the equation $ax^2 + bx + c = 0$

(a) are real and negative
(b) have negative real parts
(c) are rational numbers
(d) none of these

8. The roots of the equation $2^{2x} - 10.2^x + 16 = 0$ are

(a) 2, 8 (b) 1, 3
(c) 1, 8 (d) 2, 3

9. The solution of the equation $2x^2 + 3x - 9 \leq 0$ is given by

(a) $3/2 \leq x \leq 3$ (b) $-3 \leq x \leq 3/2$
(c) $-3 \leq x \leq 3$ (d) $3/2 \leq x \leq 2$

10. If the equations $x^2 - 5x + 6 = 0$ and $x^2 + mx + 3 = 0$ have a common root, then

(a) $m = -4$
(b) $m = -\frac{7}{2}$
(c) $m = -4$ and $m = -\frac{7}{2}$
(d) none of these

11. If the equation $x^2 + px + q = 0$ and $x^2 + qx + p = 0$, have a common root, then $p + q + 1 =$

[Orissa JEE – 2002]

(a) 0 (b) 1
(c) 2 (d) – 1

12. If $\sin\alpha$, $\cos\alpha$ are the roots of the equation $ax^2 + bx + c = 0$, then

(a) $a^2 - b^2 + 2ac = 0$ (b) $(a - c)^2 = b^2 + c^2$
(c) $a^2 + b^2 - 2ac = 0$ (d) $a^2 + b^2 + 2ac = 0$

13. If the roots of the equation $ax^2 + x + b = 0$ be real, then the roots of the equation $x^2 - 4\sqrt{ab}\,x + 1 = 0$ will be

(a) rational (b) irrational
(c) real (d) imaginary

14. How many roots does the equation $\frac{x-2}{x-1} = 1 - \frac{2}{x-1}$ have?

[IIT – 1984; UPSEAT – 1999; Pb. CET – 2003; Jamia – 2004]

(a) one (b) two
(c) infinite (d) none of these

15. If a roots of the equation $ax^2 + bx + c = 0$ be reciprocal of a root of the equation $a'x^2 + b'x + c' = 0$, then,

[IIT – 1968]

(a) $(cc' - aa')^2 = (ba' - cb')(ab' - bc')$
(b) $(bb' - aa')^2 = (ca' - bc')(ab' - bc')$
(c) $(cc' - aa')^1 = (ba' + cb')(ab' + bc')$
(d) none of these

ANSWER SHEET

1.	ⓐ	ⓑ	ⓒ	ⓓ	6.	ⓐ	ⓑ	ⓒ	ⓓ	11.	ⓐ	ⓑ	ⓒ	ⓓ
2.	ⓐ	ⓑ	ⓒ	ⓓ	7.	ⓐ	ⓑ	ⓒ	ⓓ	12.	ⓐ	ⓑ	ⓒ	ⓓ
3.	ⓐ	ⓑ	ⓒ	ⓓ	8.	ⓐ	ⓑ	ⓒ	ⓓ	13.	ⓐ	ⓑ	ⓒ	ⓓ
4.	ⓐ	ⓑ	ⓒ	ⓓ	9.	ⓐ	ⓑ	ⓒ	ⓓ	14.	ⓐ	ⓑ	ⓒ	ⓓ
5.	ⓐ	ⓑ	ⓒ	ⓓ	10.	ⓐ	ⓑ	ⓒ	ⓓ	15.	ⓐ	ⓑ	ⓒ	ⓓ

HINTS AND EXPLANATIONS

3. (a) $x^2 - 3x + 3 = \left(x - \frac{3}{2}\right)^2 + 3 - \frac{9}{4}$

$= \left(x - \frac{3}{2}\right)^2 + \frac{3}{4}$

minimum value $= \frac{3}{4}$.

OR

Use the formula is $\frac{4ac - b^2}{4a}$

on factorisation given

4. (c) Equation is $(|x| + 3)(|x| - 2) = 0$

$\Rightarrow |x| = -3$ which is not possible

or $|x| = 2$

$\therefore x = \pm 2$

5. (a) $\log_x (1 - x)^2 = \log_x 9$ ($\because a^{\log_a N} = N$)

$(1 - x)^2 = 9$

$1 - x = \pm 3x = -2, 4$

6. (d) Take $x^{1/3} = y$.

Then $y^2 - 7y + 10 = 0$

or $(y - 5)(y - 2) = 0$ or, $x^{1/3} = y = 2, 5$

$\Rightarrow x = 2^3, 5^3 = 8, 125$.

7. (b) The roots of the equations are given by,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(i) Let $b^2 - 4ac > 0$,

Now if $a > 0, b > 0, c > 0$ and $b^2 - 4ac < b^2$

$\Rightarrow$ the roots are negative

(ii) Let $b^2 - 4ac < 0$, then the roots are given by,

$$x = \frac{-b \pm i\sqrt{(4ac - b^2)}}{2a}, (i = \sqrt{-1})$$

Which are imaginary and have negative real part ($\because b > 0$)

$\therefore$ In each case, the roots have negative real

8. (b) Step 1: Given equation is quadratic in 2^x therefore,

$$2^x = \frac{10 \pm \sqrt{100 - 4 \times 1 \times 16}}{2 \times 1} = \frac{10 \pm 6}{2}$$

$2^x = 8, 2$ hence $x = 3, 1$

9. (b) $2x^2 + 3x - 9 \leq 0$

$2x^2 + 6x - 3x - 9 \leq 0$

$2x(x + 3) - 3(x + 3) \leq 0$

$(2x - 3)(x + 3) \leq 0$

$-3 \leq x \leq 3/2$

10. (c) Let the common root be

$x^2 - 5x + 6 = 0$ (1)

and $x^2 + m + 3 = 0$ and $(m + 5)(6m + 15) = -9$

$6m^2 + 15m + 30m + 75 + 9 = 0$

$6m^2 + 45m + 84 = 0$

$2m^2 + 15m + 28 = 0$

$2m^2 + 8m + 7m + 28 = 0$

$2m(m + 4) + 7(m + 4) = 0$

$(m + 4)(2m + 7) = 0$

$m = -4, m = -\frac{7}{2}$

11. (a) Let α be common root.

$\alpha^2 + p\alpha + q = 0$ (1)

$\alpha^2 + qx + p = 0$ (2)

(1) – (2) gives

$(p - q)\alpha + q - p = 0$

$(p - q)\alpha = p - q = \alpha = 1$

Putting in (1), we get,

$(1)^2 + p + q = 0\ p + q = -1$

$p + q + 1 = 0$

12. (a) $\sin\alpha + \cos\alpha = \frac{-b}{a}$ (1)

$\sin\alpha \cos\alpha =$ (2)

Now eliminate by squaring (1).

$\sin^2\alpha + \cos^2\alpha + 2\sin\alpha\cos\alpha = \frac{b^2}{a^2}$

$a^2 = b^2 - 2ac$

$a^2 - b^2 + 2ac = 0$

13. (d) $ax^2 + x + b = 0$ has real roots

$(1)^2 - 4ab0 - 4ab - 1$ or $4ab\ 1$ (1)

Now second equation is $x^2 - 4x + 1 = 0$ Therefore, $D = 16ab - 4$, from (1) $D\ 0$ Hence, roots are imaginary.

14. (d) $x^2 - x - 2 = x - 3$, provided $x\ 1$

$x^2 - 2x + 1 = 0$

$(x - 1)^2 = 0$

$x = 1, 1$

Since, $x = 1$, does not satisfy the equation therefore, given quadratic has no root.

15. (a) If is a root of $ax^2 + bx + c = 0$ (1)

then is a root of (2)

$a^2 + b + c = 0$

and (2)

By (1) and (2),

or

Now gives

$(cc - aa)^2 = (ab - bc)(ab - bc)$.

This page is intentionally left blank.

LECTURE

5

Test Your Skills

ASSERTION/REASONING

Assertion–Reasoning type questions

Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.

(a) **Assertion** isTrue, **Reason** is True and **Reason** is a correct explanation for **Assertion.**

(b) **Assertion** is True, **Reason** is True and **Reason** is NOT a correct explanation for **Assertion.**

(c) **Assertion** is True and **Reason** is False.

(d) **Assertion** is False and **Reason** is True.

1. Assertion (A): The quadratic equation whose one root is $\frac{1}{2+\sqrt{5}}$ will be $x^2 + 4x - 1 = 0$.

Reason (R): The irrational and complex roots of a quadratic equation with rational coefficient always occurs in pairs. Therefore, if one root is $3 + 4i$ then other root is $3 - 4i$.

2. Assertion (A): If the product of the roots of the equation $(a + 1)x^2 + (2a + 3)x + (3a + 4) = 0$ be 2, then the sum of the roots is -1.

Reason (R): In any quadratic equation, sum of the roots is always greater than product of the roots of the quadratic equation.

3. If x be real, then the minimum value of $(x^2 - 8x + 17)$ is 0.

Reason (R): The graph of $(ax^2 + bx + c)$ extends upwardly accordingly to $a > 0$. Now, when graph extends upwardly, then the vertex V determines the minimum value is $\left(-\frac{D}{4a}\right)$ or $\frac{4ac - b^2}{4a}$

4. Assertion (A): The roots of $4x^2 + 6px + 1 = 0$ are equal, then the value of p is $\frac{1}{3}$

Reason (R): The equation $(a, b, c \in R)$ $ax^2 + bx + c = 0$ has non – real roots if $b^2 - 4ac < 0$.

5. Assertion (A): If one root of the equation $8x^2 - 6x - a - 3 = 0$ is the square of the other, then a are -4 and 24.

Reason (R): If $ax^2 + bx + c > 0$ for all x if $a > 0$ and $b^2 - 4ac < 0$.

6. Assertion (A): The roots of the equation $ax^2 + bx + c = 0$ will be imaginary if $a > 0$, $b = 0$, $c < 0$.

Reason (R): $ax^2 + bx + c = 0$ is a quadratic equation. Suppose $a, b, c \in Q$ and $a \neq 0$. If $d > 0$ and d is not a perfect square, then roots are irrational and unequal.

7. Assertion (A): The equation $x^3 + 6x^2 + 11x - 6 = 0$ has at most one positive real root.

Reason (R): The maximum number of positive real roots of a polynomial equation $f(x) = 0$ is the number of changes of sign from positive to negative and negative to positive in $f(x)$. It is called Descartes rule of signs.

8. Assertion (A): The roots of the equation $\log_2(x^2 - 4x + 5) = (x - 2)$ are 2 and 3.

Reason (R): Every equation of an even degree whose last term is negative and coefficient of first term positive has at least two real roots, one positive and one negative.

9. Assertion (A): If the roots of the equation $x^2 - bx + c = 0$ are two consecutive integer, then $b^2 - 4c$ is equal to 1.

Reason (R): If the coefficient of x^2 in a quadratic equation is unity, then its roots must be integers.

10. Assertion (A): The equation $2x^2 + kx - 5 = 0$ and $x^2 - 3x - 4 = 0$ have one common root if, $k = -3$ or $k = -\frac{27}{4}$

Reason (R): The required condition for one root to be common of two quadratic equation $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ is $(a_1b_2 - b_1a_2) \cdot (b_1c_2 - b_2c_1) = (c_1a_2 - c_2a_1)^2$.

11. Assertion (A): $9^x + 6^x = 2.4^x$ has no solution.

Reason (R): $\log_2(9 - 2^x) = 10^{\log(3-x)}$ has only one solution.

12. Assertion (A): The remainder obtained on dividing the polynomial $P(x)$ by $(x - 3)$ is equal to $P(3)$.

Reason (R): $f(x)$: $(x - 8)^3(x + 4) \Rightarrow f'(x)$ may not be divisible by $(x^2 - 16x + 64)$.

13. Assertion (A): $f(x) = ax^2 + b + c$, then $f(x) = 0$ has integral roots only when $a = 1$, $b, c \in I$ and $b^2 - 4ac$ is a perfect square of integer.

Reason (R): $x^3 + 1 = 0$ has only one integral root.

14. Assertion (A): $(|x| + 1)^2 = 4|x| + 9$ has only two real solutions.

Reason (R): $\frac{x-8}{n-10} = \frac{n}{x}$ has no solutions for some (more than one) values of $n \in N$.

15. Assertion (A): If $a, b, c \in R$ and $a + b + c = 0$, then the quadratic equation $3ax^2 + 2bx + c = 0$ has at least one real root in $[0, 1]$.

Reason (R): If a, b and c all are positive real numbers, then both the roots of the equation $ax^2 + bx + c = 0$ have positive real parts.

16. Assertion (A): If $a_1 < a_2 < a_3 < a_4$, then $(x - a_1)(x - a_3) + \lambda(x - a_2)(x - a_4) = 0$ has real roots, $(\lambda \in R)$.

Reason (R): If $f(a).f(b) < 0$ for polynomial $f(x)$ then $f(x) = 0$ must have at least one real root between a and b.

17. Assertion (A): $x^2 + bx + c = 0$ has distinct roots and both greater than 2 if $b^2 - 4c > 0$, $b < -4$ and $2b + c + 4 > 0$.

Reason (R): $x^2 + 2x + c = 0$ has distinct roots and both less than 1 if $c \in (-3, 1)$.

18. Assertion (A): We can get the equation whose roots are 2 more than the roots of equation $ax^2 + bx + c = 0$ by replacing x by $(x + 2)$.

Reason (R): $x^2 + |x| + 5 = 0$ has no real roots.

19. Assertion (A): The number of positive roots of $x^3 + 3x^2 + 7x - 11 = 0$ is at most 1.

Reason (R): The number of positive real roots of polynomial equation $f(x) = 0$ is the number of changes of the signs of coefficients from positive to negative and negative to positive.

20. Assertion (A): If $a > 0$, then minimum value of $ax^2 + bx + c$ is $\frac{4ac - b2}{4a}$.

Reason (R): If $a < 0$, then minimum value of $ax^2 + bx + c$ is $\frac{4ac - b^2}{4a}$

21. Assertion (A): If $f(x)$ is a polynomial of degree one or more and a is any number real or complex, then $x - \alpha$ divides $f(x)$ if $f(\alpha) = 0$.

Reason (R): Let a, b, c, d, e be real numbers such that $a + b + c + d + e = 0$, then $x + 1$ is a factor of $ax^4 - bx^3 + cx^2 - dx + e$.

22. **Assertion (A):** If $b^2 - 4ac \geq 0$, then the roots of the equation $ax^2 + bx + c = 0$ are real and if $b^2 - 4ac < 0$ then roots of $ax^2 + bx + c = 0$ are nonreal.

Reason (R): The equation $ix^2 - 3ix + 2i = 0$ has nonreal roots as "$b^2 - 4ac$" is $9i^2 - 4i(2i) = -9 + 8 = -1$ is negative.

ASSERTION/REASONING: SOLUTIONS

1. (a) One root is $\frac{1}{2+\sqrt{5}}$ means

$$\frac{2-\sqrt{5}}{(2+\sqrt{5})(2-\sqrt{5})} = \sqrt{5} - 2$$

Second root is $-2 - \sqrt{5}$

S = Sum of the roots $= -4$

P = Product of the roots $= -1$

$x^2 - (S)x + p = 0$

$x^2 + 4x - 1 = 0$

Because imaginary and irrational roots always. Thus, assertion and reason both are true and assertion follows from reason.

2. (c) We have $(a + 1)x^2 + (2a + 3)x + (3a + 4) = 0$

$\alpha\beta = P$

$$\Rightarrow \frac{3a+4}{a+1} = 2$$

$3a + 4 = 2a + 2$

$a = -2$

$\alpha + \beta = S$

$$\Rightarrow \frac{-(2a+3)}{(a+1)} = \frac{-[2(-2)+3]}{(-2+1)} = \frac{1}{-1} = -1$$

Assertion is true. But reason is not ture.

3. (d)

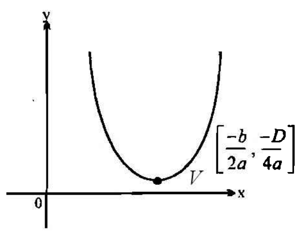


Figure 1

Minimum value of $x^2 - 8x + 17$

$a > 0, D > 0$

$$-\frac{D}{4a} = \frac{4ac - b^2}{4a}$$

$$= \frac{4(1)(17) - (8)^2}{4(1)} = \frac{68 - 64}{4} = 1$$

So, minimum of $x^2 - 8x + 17$ is one. Thus, assertion is false but reason is correct.

4. (d) We have $4x^2 + 6px + 1 = 0$ roots are equal

$D = 0$

$36p^2 - 4(4)(1) = 0$

$36p^2 - 16 = 0$

$$p^2 = \frac{16}{36}$$

$$P = \frac{4}{6} = \frac{2}{3}$$

So, assertion is not true. In the quadratic equation $d < 0$ then roots are imaginary $(b^2 - 4ac < 0)$

Beacause $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Thus, reason is true.

5. (b) Let α and α^2 are the roots of the equation

$8x^2 - 6x - a - 3 = 0$

$$\alpha + \alpha^2 = \frac{6}{8}, (\alpha)(\alpha^2) = -\frac{(a+3)}{8}$$

$$\alpha + \alpha^2 = \frac{3}{8}, \alpha^3 = -\left(\frac{a+3}{8}\right)$$

$$\because \alpha = -\frac{3}{2} \text{ and } \frac{1}{2}$$

$$\alpha^2 = -\frac{(a+3)}{8}$$

Put $\alpha = -\frac{3}{2}$ amd $\frac{1}{2}$

We get $a = -4$ and 24

So, assertion is true.

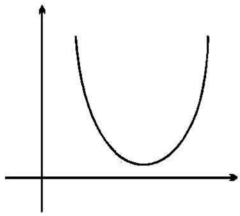


Figure 2

If $(ax^2 + bx + c) > 0$ for all x, then $a > 0$ and $(b^2 - 4ac) < 0$

Because graph does not cuts the real axis and gives imaginary roots, thus the expression has always positive sign.

Thus, reason is also true.

6. (d) The given equation is $ax^2 + bx + c = 0$ If roots of the equation are imaginary then $d < 0$

$b^2 - 4ac < 0$

If $b = 0, a > 0, c < 0$

Then 0 – 4 (positive) (negative) < 0

$4 < 0$

Thus, assertion is not true. But reason is true.

7. (a) We have the equation $x^3 + 6x^2 + 11x - 6 = 0$

The signs of the various terms are

+ + + –

Clearly, there is only one change of sign in the expression

$x^3 + 6x^2 + 11x - 6 = 0$. So, the given equation has at most one positive real root. Thus assertion and reason both are correct and assertion follows from reason.

8. (b) $\log_2(x^2 - 4x + 5) = (x - 2)$ $x^2 - 4x + 5 = 2^{x-2}$

Clearly, $x = 2$ and $x = 3$ satisfy the equation.

Then, assertion is true.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ if } a > 0 \text{ and } c < 0$$

Then, $\sqrt{b^2 - 4ac}$ is a positive term. It means $d > 0$.

Thus, Quadratic equation gives two real roots. One is positive and the other is nagative.

But, assertion does not follow from reason.

9. (c) $|\alpha - \beta| = 1$

$(\alpha - \beta)^2 = 1$

$(\alpha + \beta)^2 - 4\alpha\beta = 1$

$b^2 - 4c = 1$

Thus, assertion is true but reason is not true.

10. (a) If $2x^2 + kx - 5 = 0$ and $x^2 - 3x - 4 = 0$ are the two quadratic equations. If α is a common root between then

So, $2\alpha^2 + k\alpha - 5 = 0$ (1)

$\alpha^2 - 3\alpha - 4 = 0$ (2)

$$\frac{\alpha^2}{-4k + 15} = \frac{\alpha}{-5 + 8} = \frac{1}{-6 - k}$$

$$\frac{\alpha^2}{-4k + 15} = \frac{\alpha}{3} = \frac{1}{-(6 + k)}$$

$$\alpha = -\frac{3}{(6 + k)} \quad (3)$$

$$\alpha^2 = \left(\frac{4k - 15}{6 + k}\right) \quad (4)$$

$(1)^2 = (2)$

$$\frac{9}{(6 + k)^2} = \left(\frac{4k - 15}{6 + k}\right)$$

After solving, $k = -3$ or $\frac{27}{4}$

Thus, assertion and reason both true and assertion follows from reason.

11. (d) $x = 0$ is a solution of (A), (A) is false.

12. (c) In R, $f'(x)$ is divisible by $(x - 8)^2$. A is true.

13. (d) When $a \neq 1$ then integral roots are also possible.

e.g., $4x^2 - 8x + 4 = 0$.

14. (b) $|x|^2 + 2|x| + 1 = 4|x| + 9$

$\Rightarrow |x|^2 - 2|x| - 8 = 0$

$\Rightarrow (|x| - 4)(|x| + 2) = 0$

$\Rightarrow |x| = 4, -2$ But $|x| \neq -2$

$\because |x| = 4$

$\Rightarrow x = \pm 4,$

i.e., has only two real solutions.

$\therefore$ (A) is correct.

In R, for $n = 10$, it has no solution $x^2 - 8x = n(n - 10)$

$$\therefore x = \frac{8 \pm \sqrt{64 + 4n^2 - 40n}}{2}$$

$$= 4 \pm \sqrt{16 + n^2 - 10n}$$

$n^2 - 10n + 16 = (n - 8)(n - 2)$.

$n^2 - 10n + 16 < 0$ for $2 < n < 8$.

i.e., for $n = 3, 4, 5, 6, 7$ it has not solutions

$\therefore r$ is correct.

But r is not the correct explanation of A.

15. (c) Let $G(x) = ax^3 + bx^2 + cx + d$. $G(x)$ is continuous in [0, 1] and differentiable in (0, 1). $G(0) = d, G(1) = a + b + c + d = 0 + d = d$ i.e., $G(0) = G(1)$

$\therefore$ According to Rolle's theorem, $G'(x) = 0$ has at least one real root in (0, 1)

$\Rightarrow 3ax^2 + 2bx + c = 0$ has at least one real root

$\therefore$ (A) is true

In R, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\because b^2 - 4ac < b^2$ $(a > 0, c > 0)$

$\therefore$ real parts negative.

If $d < 0$, then $x = \dfrac{-b \pm i\sqrt{-D}}{2a}$

which have real parts negative.

$\therefore r$ is false.

16. (a) Let $f(x) = (x - a_1)(x - a_3) + \lambda(x - a_2)(x - a_4)$

$f(a_1) = \lambda(a_1 - a_2)(a_1 - a_4) = (+\text{ real no})\lambda$

$f(a_3) = \lambda(a_3 - a_2)(a_3 - a_4) = (-\text{ real no})\lambda$

$\because f(x) = 0$ has one root between a_1 and a_3.

$f(x) = 0$ has all coefficients real and its roots also real.

$\therefore A$ is correct

R is correct and it explains A.

17. (b) Step 1: Clearly $\alpha + \beta = -b$ in A and Discriminant $= b^2 - 4c = 4 - 4c$ in R

Step 2: In A, $\dfrac{-b}{1} > 4 \Rightarrow b < -4$, $x^2 + bx + c$ at $x = 2$ must be positive i.e., $4 + 2b + c > 0$

$\therefore A$ is true.

In R, $4 - 4c > 0 \Rightarrow c < 1$

$x^2 + 2x + c$ at $x = 1$ must be positive

$\because 1 + 2 + c > 0 \Rightarrow c > -3$

$\therefore c \in (-3, 1)$ $\therefore$ true

But R does not explain A

18. (d) In A, it should be $(x - 2)$ in place of $(x + 2)$.

$\therefore A$ is true

R is true.

19. (a) In A, the expresion $x^3 + 3x^2 + 7x - 11$ has in no. of changes of sign is 1.

$\therefore A$ is true.

R is true and it explains A

20. (c) We note that

$$ax^2 + bx + c = a\left(\frac{x + b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

$$\geq \frac{4ac - b^2}{4a} \text{ if } a > 0$$

Again, $ax^2 + bx + c$

$$= a\left(\frac{x + b}{2a}\right)^2 = \frac{4ac - b^2}{4a} \text{ if } a < 0.$$

$\Rightarrow$ When $a < 0$, max. value of $ax^2 + bx + c$ is $\leq \dfrac{4ac - b^2}{4a}$

21. (a) Assertion is a standard result, known as 'factor theorem'

If $\phi(x) = ax^4 - bx^3 + cx^2 - dx + e$

then $\phi(-1) = a + b + c + d + e = 0$

$\Rightarrow x - (-1)$ is a factor of $\phi(x)$.

22. (d) Assertion is not a correct. The correct statement is "If a, b, c are real and $b^2 - 4ac \geq 0$, then roots are real and if $b^2 - 4ac < 0$, then roots are non-real".

$ix^2 - 3ix + 2i = 0$

or $i(x^2 - 3x + 2) = 0$

$\Rightarrow x^2 - 3x + 2 = 0$

$\Rightarrow x = 1, 2$, which are real.

MENTAL PREPARATION TEST

1. Solve the equation $x^2 - 4x + 13 = 0$ by the factorization method. roots.

2. If roots of equation $x^2 + 2(p - q)x + pq = 0$ are imaginary then prove that $4x^2 + 4(p - q)x + (4p^2 + 4q^2 - 11q) = 0$ will have real roots.

3. If the roots of equation $(1 + n)x^2 - 2(1 + 3n)x + (1 + 8n) = 0$ are equal, then find the value of n.

4. Prove that roots of equation $(a + c - b)x^2 + 2cx + (b + c - a) = 0$ are real, rational and unequal.

5. Find the equation whose roots are $a + \sqrt{-b}$ and $a - \sqrt{-b}$

6. If $px^2 - qx + r = 0$ has α and β as its roots, evaluate $\alpha^3\beta + \beta^3\alpha$.

7. If α, β are the roots of the equation $x^2 + x + 1 = 0$, then prove that the equation whose roots are $m\alpha + n\beta$ and $mb + n\alpha$ is $x^2 + (m + n)x + (m^2 - mn + n^2) = 0$.

8. If α and β be the roots of the equation $px^2 + qx + r = 0$. Hence obtain the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

9. If the difference of roots of equation $x^2 - px + q = 0$ is 1, then prove that $p^2 + 4q^2 = (1 + 2q)^2$.

[MP–1992, 1997, 1998]

10. If ratio of roots of equation $x^2 + px + q = 0$ be same as ratio of roots of equation $x^2 + lx + m = 0$, then prove that $p^2m = l^2q$.

11. If ratio of roots of equation $ax^2 + bx + c = 0$ is m: n, then show that $\sqrt{\frac{m}{n}} + \sqrt{\frac{n}{m}} = \sqrt{\frac{b^2}{ac}}$.

12. If only one root is common to the equations $x^2 - px + q = 0$, and $x^2 - qx + p = 0$, then prove that $p + q + 1 = 0$

13. If only one root is common to the equations $x^2 - kx - 21 = 0$ and $x^2 - 3kx + 35 = 0$, then find the value of k. Ans: ± 4

14. If only one root of equations $ax^2 + bx + c = 0$ and $a^2x^2 + b^2x + c^2 = 0$ is common, then prove that $b^2(a - b)(b - c) = ac(a - c)^2$.

15. If only one root of equation $x^2 + ax + 10 = 0$ and $x^2 + bx - 10 = 0$ is common, then prove that $a^2 - b^2 = 40$.

16. The coefficient of x in the equation $x^2 + px + q = 0$ was taken as 17 in place of 13 its roots were found to be -2 and -15. Prove that the roots of the original equation are $-10, -3$.

17. If $\alpha + \beta = 3$ and $\alpha^3 + \beta^3 = 7$, then show that α, β are roots of the equation $9x^2 - 27x + 20 = 0$.

18. If $ax^2+bx+c = 0$ has roots $\frac{K+1}{K}$ and $\frac{K+2}{K+1}$, then prove that: $(a + b + c)^2 = b^2 - 4ac$.

19. If $\sin\theta$, $\cos\theta$ are roots of equation $ax^2 + bx + c = 0$, then prove that $(a + c)^2 = b^2 + c^2$.

20. If α, β are roots of equation $x^2 - 4x + 1 = 0$, then prove that $\alpha^3 + \beta^3 = 52$.

21. If $\tan\alpha$, $\tan\beta$ are roots of equation $x^2 - px + q = 0$, then prove that

$$\sin^2(\alpha + \beta) = \frac{p^2}{p^2 + (1 - q)^2}$$

TOPICWISE WARMUP TEST

1. If the roots of the equation $Ax^2 + Bx + c = 0$ are α, β and the roots of the equation $x^2 + px + q = 0$ are α^2, β^2, then valueof p will be **[R PET – 1986]**

(a) $\frac{B^2 - 2AC}{A^2}$ (b) $\frac{2AC - B^2}{A^2}$

(c) $\frac{B^2 - 4AC}{A^2}$ (d) none of these

2. If α, β be the roots of the equation $2x^2 - 35x + 2 = 0$ then the value of $(2\alpha - 35)^3 \cdot (2\beta - 35)^3$ is equal to

(a) 1 (b) 64
(c) 8 (d) none of these

[Bihar CEE – 1994]

3. If A.M. of the roots of a quadratic equation is 8/5 and A.M. of their reciprocals is 8/7, then the equation is

[AMU – 2001]

(a) $5x^2 - 16x + 7 = 0$
(b) $7x^2 - 16x + 5 = 0$
(c) $7x^2 - 16x + 8 = 0$
(d) $3x^2 - 12x + 7 = 0$

4. $2x^2 - (p + 1)x + (p - 1) = 0$. If $\alpha - \beta = \alpha\beta$, then what is the value of p

[Orissa JEE – 2005]

(a) 1 (b) 2
(c) 3 (d) – 2

5. If $x^2 + ax + 10 = 0$ and $x^2 + bx - 10 = 0$ have a common root, then $a^2 - b^2$ is equal to

[Kerala (Engg.) – 2002

(a) 10 (b) 20
(c) 30 (d) 40

6. If x is real, the expression $\dfrac{x+2}{2x^2+3x+6}$ takes all value inthe interval

[IIT – 1969]

(a) $\left(\frac{1}{13}, \frac{1}{3}\right)$ (b) $\left[-\frac{1}{13}, \frac{1}{3}\right]$
(c) $\left(-\frac{1}{3}, \frac{1}{13}\right)$ (d) none of these

7. If the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are real and less than 3, then

[IIT – 1999; MPPET– 2000]

(a) $a < 2$ (b) $2 \le a \le 3$
(c) $3 < a \le 4$ (d) $a > 4$

8. The coefficient of x in the equation $x^2 + px + q = 0$ was taken as 17 in place of 13, its roots were found to be – 2 and – 15. The roots of the original equation are

[IIT – 1977, 1979]

(a) 3, 10 (b) – 3, – 10
(c) – 5, – 18 (d) none of these

9. If $3x^2 + 2(\lambda^2 + 1)x + (\lambda^2 - 3\lambda + 2) = 0$ has roots of opposite signs then λ lies in the interval

[CET (Karnataka) – 1993]

(a) $(-\infty, 0)$ (b) $(-\infty, 1)$
(c) $(1, 2)$ (d) $(3/2, 2)$

10. If number 3 lies between the roots of the equation $x^2 + (1 - 2k)x + (k^2 - k - 2) = 0$, then

(a) $k < 2$ (b) $k > 5$
(c) $2 < k < 3$ (d) $2 < k < 5$

[CET (Karnataka) – 95]

11. If α, β are roots of the equation $8x^2 - 3x + 27 = 0$, then the value of $(\alpha^2/\beta^2)^{1/3} + (\beta^2/\alpha)^{1/3}$ is equal to

[Haryana (CET) – 2000]

(a) 4 (b) 1/3
(c) 1/4 (d) 7/2

12. If α, β are roots of the equation $x^2 - 6x + 4 = 0$, then $(\alpha^2 - \beta^2)^2$ is equal to

[Ranchi – 2001]

(a) 720 (b) 1008
(c) 360 (d) 504

13. Real roots of the equation $7^{\log_7(x^2 - 4x + 5)} = x - 1$ are

(a) 1, 2 (b) 3, 4
(c) 2, 3 (d) 4, 5

[DCE – 2001]

14. If α, β are roots of the equation $x^2 + 2x + 5 = 0$, then the equation whose roots are $1/\alpha + 1/\beta$ and $\alpha + \beta$ will be

(a) $5x^2 + 12x - 4 = 0$
(b) $5x^2 + 12x + 4 = 0$
(c) $5x^2 - 12x + 4 = 0$
(d) none of these **[AMU – 2001]**

15. The equation whose roots are –5 times those of the equation $x^2 - x + 2 = 0$ *is*

[PET (Raj.) – 2002]

(a) $x^2 - 5x - 10 = 0$
(b) $x^2 + 5x + 50 = 0$
(c) $x^2 + 25x + 50 = 0$
(d) $x^2 - 25x - 50 = 0$

16. If $x = \sqrt{2+\sqrt{2+\sqrt{2+\ldots\ldots\ldots\ldots}}}$ then

[UPSEAT – 2002]

(a) $x = -1$ (b) $-1 < x < 2$
(c) $x = -2$ (d) $x = 3$

17. If α, β are roots of the equation $ax^2 + bx + c = 0$ and $\beta = \alpha^{1/3}$, then $(a^3b)^{1/4} + (ac^3)^{1/4}$ is equal to

[Kerala (CEE) – 2003]

(a) b (b) $-b$
(c) c (d) $-c$

18. If the equation $x^2 + k^2 = 2(k+1)x$ has equal roots, then what is the value of k

[NDA – 2007]

(a) $-1/3$ (b) $-1/2$
(c) 0 (d) 1

19. $(x + 2)$ is a common factor of expressions $(x^2 + ax + b)$ and $(x^2 + bx + a)$. The ratio a/b is equal to

[NDA – 2002]

(a) 1 (b) 2
(c) 3 (d) 4

20. If $(x + 3)$ is a factor of $3x^2 + ax + 6$, then the value of 'a' is

(a) 6 (b) 11
(c) –6 (d) 9

[NDA – 2000]

21. The roots of the equation $(x + 3)(x - 3) = 25$ are

[NDA – 2000]

(a) 5 and –5 (b) 3 and –3
(c) $\sqrt{34}$ and $-\sqrt{34}$ (d) 8 and 2

22. If α, β are the roots of the equation $ax^2 + bx + c = 0$, then what is the value of $(a\alpha + b)^{-1} + (a\beta + b)^{-1}$

[NDA – 2007]

(a) $a/(bc)$ (b) $b/(ac)$
(c) $-b/(ac)$ (d) $-a/(bc)$

23. If α, β are the roots of the equation $x^2 - 2x - 1 = 0$, then what is the value of $\alpha^2\beta^{-2} + \alpha^{-2}\beta^2$

[NDA – 2007]

(a) –2 (b) 0
(c) 30 (d) 34

24. If the sum of the roots of the quadratic equation $ax^2 + bx + c = 0$ is equal to the sum of the squares of their reciprocals, then $\frac{b^2}{ac} + \frac{bc}{a^2} =$

[BIT Ranchi – 1996]

(a) 2 (b) –2
(c) 1 (d) –1

25. If $3p^2 = 5p + 2$ and $3q^2 = 5q + 2$ where $p \neq q$, then the equation whose roots are $3p - 2q$ and $3q - 2p$ is

(a) $3x^2 - 5x - 100 = 0$
(b) $5x^2 + 3x + 100 = 0$
(c) $3x^2 - 5x + 100 = 0$
(d) $5x^2 - 3x - 100 = 0$

[Kerala (Engg.) – 2005]

26. If the sum of the roots of the equation $ax^2 + bx + c = 0$ is equal to the sum of the squares of their reciprocals then bc^2, ca^2, ab^2 are in

[IIT – 1996; DCE – 1996; PET (Raj.) – 2000;AIEEE – 2002, 2003]

(a) AP (b) GP
(c) HP (d) none of these

27. If the sum of the two roots of the equation $4x^3 + 16x^2 - 9x - 36 = 0$ is zero, then the roots are **[MPPET – 1986]**

(a) 1, 2, –2 (b) $-2, \frac{2}{3}, -\frac{2}{3}$
(c) $-3, \frac{3}{2}, -\frac{3}{2}$ (d) $-4, \frac{3}{2}, -\frac{3}{2}$

28. If x is real, then the maximum and minimum values of expression $\frac{x^2 + 14x + 9}{x^2 + 2x + 3}$ will be

[Dhanbad Engg. – 1968]

(a) 4, –5 (b) 5, –4
(c) –4, 5 (d) –4, –5

29. If α, β and γ are the roots of $x^3 + px + q = 0$, then the value of $\alpha^3 + \beta^3 + \gamma^3$ is equal to

[Pb. CET – 2002]

(a) $-3q$ (b) $-p$
(c) $-pq$ (d) $3pq$

30. If α, β are the roots of $x^2 + px + 1 = 0$ and γ, δ are the roots of $x^2 + qx + 1 = 0$, then $q^2 - p^2 =$

[IIT – 1978; DCE – 2000]

(a) $(\alpha-\gamma)(\beta-\gamma)(\alpha+\delta)(\beta+\delta)$
(b) $(\alpha+\gamma)(\beta+\gamma)(\alpha-\delta)(\beta+\delta)$
(c) $(\alpha+\gamma)(\beta+\gamma)(\alpha+\delta)(\beta+\delta)$
(d) none of these

31. If α and β are the roots of the equation $ax^2+bx+c=0$ ($a\neq 0$; a, b, c being different), then $(1+\alpha+\alpha^2)(1+\beta+\beta^2)=$

(a) zero (b) positive
(c) negative (d) none of these

[DCE – 2000]

32. If the roots of $10x^3-cx^2-54x-27=0$ are in harmonic progression, then find c and all the roots.

[Roorkee – 1995]

TOPICWISE WARMUP TEST: SOLUTIONS

1. (b) α, β, are the roots of $Ax^2+Bx+c=0$.

So, $\alpha+\beta=-\frac{B}{A}$ and $\alpha\beta=\frac{C}{A}$

Again α^2, β^2 are the roots of $x^2+px+q=0$ then

$\alpha^2+\beta^2=-p$ and $(\alpha\beta)^2=q$

Now, $\alpha^2+\beta^2=(\alpha\beta)^2-2\alpha\beta$

$\Rightarrow \alpha^2+\beta^2=\left(-\frac{B}{A}\right)^2-2\frac{C}{A}$

$\Rightarrow -p=\frac{B^2-2AC}{A^2} \Rightarrow p=\frac{2AC-B^2}{A^2}$

2. (b) Since, α, β are the roots of the equation $2x^2-35x+2=0$. Also $\alpha\beta=1$

$\therefore\ 2\alpha^2-35\alpha=-2$ or $2\alpha-35=\frac{-2}{\alpha}$

$2\beta^2-35\beta=-2$ or $2\beta-35=\frac{-2}{\beta}$

Now, $(2\alpha-35)^3(2\beta-35)^3=\left(\frac{-2}{\alpha}\right)^3\left(\frac{-2}{\beta}\right)^3$

$=\frac{8.8}{\alpha^3\beta^3}=\frac{64}{1}=64$

3. (a) Let the roots are α and β

$\Rightarrow \frac{\frac{\alpha+\beta}{2}}{2}=\frac{8}{5} \Rightarrow \alpha+\beta=\frac{16}{5}$ (1)

$\frac{\frac{1}{\alpha}+\frac{1}{\beta}}{2}=\frac{8}{7} \Rightarrow \frac{\alpha+\beta}{2\alpha\beta}=\frac{8}{7}$

and

$\Rightarrow \frac{(16/5)}{2(8/7)}=\alpha\beta$

$\Rightarrow \alpha\beta=\frac{7}{5}$

$\therefore$ Equation is $x^2-\left(\frac{16}{5}\right)x+\frac{7}{5}=0$

$\Rightarrow 5x^2-16x+7=0$

4. (b) $2x^2-(p+1)x+(p-1)=0$

Given $\alpha-\beta=\alpha\beta \Rightarrow (\alpha+\beta)^2-4\alpha\beta=\alpha^2\beta^2$

$\Rightarrow \frac{(p-1)^2}{4}\ \frac{(p+1)^2}{4}-\frac{4(p-1)}{2}$

$\Rightarrow 2(p-1)=p$

$\Rightarrow p=2$

5. (d) Let α be a common root, then $a^2+a\alpha+10=0$ (1)

and $\alpha^2+b\alpha-10=0$ (2)

from (1) – (2),

$(a-b)\alpha+20=0 \Rightarrow \alpha=-\frac{20}{a-b}$

Substituting the value of α in (1), we get

$\left(-\frac{20}{a-b}\right)^2+a\left(-\frac{20}{a-b}\right)+10=0$

$\Rightarrow 400-20a(a-b)+10(a-b)^2=0$

$\Rightarrow 40-2a^2+2ab+a^2+b^2-2ab=0$

$\Rightarrow a^2-b^2=40.$

6. (b) If the given expression be y, then,

$y=2x^2y+(3y-1)x+(6y-2)=0$

If $y\neq 0$ then $\Delta\geq 0$ for real x i.e., $B^2-4AC\geq 0$ or $-39y^2+10y+1\geq 0$ or $(13y+1)(3y-1)\leq 0$

$\Rightarrow -1/13\leq y\leq 1/3$

If $y=0$ then $x=-2$ which is real and this value of y is included in the above range.

7. (a) Given equation is $x^2-2ax+a^2+a-3=0$

If roots are real, then $d\geq 0$

$\Rightarrow 4a^2-4(a^2+a-3)\geq 0 \Rightarrow -a+3\geq 0$

$\Rightarrow a-3\leq 0$

$\Rightarrow a \le 3$

As roots are less than 3, hence $f(3) > 0$

$9 - 6a + a^2 + a - 3 > 0 \Rightarrow a^2 - 5a + 6 > 0$

$\Rightarrow (a-2)(a-3) > 0$

$\Rightarrow$ either $a < 2$ or $a > 3$

Hence $a < 2$ satisfy all.

8. (b) Let the equation (in correctly written form) be $x^2 + 17x + q = 0$. Roots are -2, -15. So $30 = q$, so correct equation is $x^2 + 13x + 30 = 0$. Hence roots are -3, -10.

9. (c) Roots are of opposite sign

$\Rightarrow \frac{\lambda^2 - 3\lambda + 2}{3} < 0$

$\Rightarrow (\lambda - 1)(\lambda - 32) < 0$

$\Rightarrow 1 < \lambda < 2$

10. (d) Number 3 will lie between the roots of the given equation

$f(x) = 0$ if $f(3) < 0$

$\Rightarrow 9 + (1 - 2k)3 + (k^2 - k - 2) < 0$

$\Rightarrow k^2 - 7k + 10 < 0$; $(k-2)(k-5) < 0$

$\Rightarrow 2 < k < 5$

11. (c) Exp $= \frac{\alpha + \beta}{(\alpha\beta)^{1/3}} = \frac{3/8}{(27/8)^{1/3}}$

$= \left(\frac{3}{8}\right)\left(\frac{2}{3}\right) = \frac{1}{4}$

12. (a) $(\alpha^2 - \beta^2) = (\alpha + \beta)^2 (\alpha - \beta)$

$= (\alpha + \beta)^2 \left[(\alpha + \beta)^2 - 4\alpha\beta\right]$

$= 36(36 - 16) = 720$

13. (c) Given equation is

$x^2 - 4x + 5 = x - 1$

$\Rightarrow x^2 - 5x + 6 = 0 \Rightarrow x = 2, 3$

14. (b) $\alpha + \beta = -2, \alpha\beta = 5 \Rightarrow \frac{1}{\alpha} + \frac{1}{\beta} = 2$

$\therefore$ required equation is $(x + 2)(x + 2/5) = 0$

$\Rightarrow 5x^2 + 12x + 4 = 0$

15. (b) required equation is

$x^2 - 5(-1)x + (-5)^2 2 = 0$

$\Rightarrow x^2 + 5x + 50 = 0$

16. (a) $x = \sqrt{2 + x} \Rightarrow x^2 - x - 2$

$= 0 \Rightarrow x = \frac{1 \pm 3}{2} = 2, -1$

17. (b) $\alpha = \beta^3 \Rightarrow \beta^3 + \beta = -b$ and $\beta^4 = c/a$

i.e. $\beta = (c/a)^{1/4}$

$\Rightarrow (c/a)^{3/4} + (c/a)^{1/4} = -b/a$

$\Rightarrow (ac^3)^{1/4} + (a^3c)^{1/4} = -b$

18. (b) $x^2 - 2(k+1)x + k^2 = 0$

Condition for equal roots $b^2 - 4ac = 0$

$\Rightarrow 4(k+1)^2 - 4k^2 = 0$

$\Rightarrow 4k^2 + 4 + 8k - 4k^2 = 0$

$\Rightarrow 8k = -4$

$\Rightarrow k = -1/2$.

19. (a) $4 - 2a + b = 0$ (1)

$4 - 2b + a = 0$ (2)

or, $2a - b = 4$ (1)

$2b - a = 4$, $4b - 2a = 8$ (2)

on addition, $3b = 12$, $b = 4$; $2a = 8$, $a = 4$

$\therefore \frac{a}{b} = \frac{4}{4} = 1$

20. (b) Putting $x = -3$ in the given expression, we get,

$= 3(-3)^2 + a(-3) + 6 = 0$

$\Rightarrow 27 - 3a + 6 = 0 \therefore a = 11$

21. (c) $(x + 3)(x - 3) = 25 \Rightarrow x^2 = 34$

$\therefore x = \pm\sqrt{34}$

22. (b) $\alpha + \beta = \frac{-b}{a}$ and $\alpha \cdot \beta = \frac{c}{a}$

$\therefore \frac{1}{a\alpha + b} + \frac{1}{a\beta + b}$

$= \frac{a\beta + b + a\alpha + b}{(a\alpha + b)(a\beta + b)}$

$= \frac{a(\alpha + \beta) + 2b}{a^2\alpha\beta + ab(\alpha + \beta) + b^2}$

$\frac{a\left(-\frac{b}{a}\right) + 2b}{a^2\left(\frac{c}{a}\right) + ab\left(-\frac{b}{a}\right) + b^2}$

$= \frac{b}{ac - b^2 + b^2} = \frac{b}{ac}$

23. (d) Given $\alpha + \beta = 2, \alpha \cdot \beta = -1$

$\therefore \frac{\alpha^2}{\beta^2} + \frac{\beta^2}{\alpha^2} = \frac{\alpha^4 + \beta^4}{\alpha^2\beta^2} = -\frac{2\alpha^2\beta^2}{\alpha^2\beta^2}$

$\frac{\{(\alpha + \beta)^2 - 2\alpha\beta\}^2 - 2\alpha^2\beta^2}{\alpha^2\beta^2}$

$\frac{\{(4 + 2)^2 - 2 \times 1\}}{1}$

$= 36 - 2 = 34$

24. (a) Let α, β be the roots of the equation $ax^2 + bx + c = 0$

then $\alpha + \beta = -\frac{\alpha}{\beta}, \alpha\beta = \frac{c}{a}.$

Given $\alpha + \beta = \frac{1}{\alpha^2} + \frac{1}{\beta^2} \Rightarrow -\frac{b}{a} = \underline{a^2\beta}$

$\Rightarrow -\frac{b}{a}\ \frac{c^2}{a^2} = (\alpha+\beta)^2 - 2\alpha\beta$

$\Rightarrow -\frac{bc^2}{a^3} = \frac{b^2}{a^2} - \frac{2c}{a}$

$\Rightarrow \frac{2}{a} = \frac{b^2}{a^2c} + \frac{bc}{a^3} \Rightarrow 2 = \frac{b^2}{ac} + \frac{bc}{a^2}$

25. (a) Given roots are $3p - 2q$ and $3q - 2p\ pq$ are roots of equation $3x^2 = 5x + 2$ Sum of roots $= (3p - 2q) + (3q - 2p) = (p + q) = \frac{5}{3}$

Product of roots $= (3p - 2q)(3q - 2p)$

$= 9pq - 6q^2 - 6p^2 + 4pq = 13pq - 2(3p^2 + 3q^2)$

$= 13\left(-\frac{2}{3}\right) - 2\ (5p + 2 + 5q + 2$

$= 13\left(\frac{-2}{3}\right) - 2\left[5\left(\frac{5}{3}\right) + 4\right]$

$= -26/3 - 2\left[\frac{25}{3} + 4\right] = -\frac{100}{3}$

Hence, equation is $3x^2 - 5x - 100 = 0$.

26. (a) Let the roots be $\alpha\ \beta$. Then

$\alpha + \beta = -b/a, \alpha\beta = c/a$

Now as given $\alpha + \beta = \frac{1}{\alpha^2} + \frac{1}{\beta^2}$

$\Rightarrow \alpha + \beta = \frac{\alpha^2 + \beta^2}{\alpha^2\beta^2} + \Rightarrow (\alpha\beta)^2(\alpha + \beta)$

$= (\alpha + \beta)^2 - 2\alpha\beta$

$\frac{c^2}{a^2}\left(-\frac{b}{a}\right) = \frac{b^2}{a^2} - \frac{2c}{\alpha} \Rightarrow -bc^2 - b^2a - 2ca^2$

$\Rightarrow bc^2 + ab^2 = 2ca^2 \therefore bc^2, ca^2, ab^2$ are in A.P.

27. (d) Given equation $4x^3 + 16x^2 - 9x - 36 = 0$. Putting, $x = -4 \Rightarrow -4 \times 64 + 256 + 36 - 36 = 0$ Hence, $x = -4$ is a root of the equation Now, reduced equation is $4x^2(x + 4) - 9(x + 4) = 0$

$\Rightarrow (x + 4)(4x^2 - 9) = 0 \Rightarrow x = -4, x = \pm\frac{3}{2}$

Thus, roots are $-4, -\frac{3}{2}, \frac{3}{2}$

28. (a) Let $y = \frac{x^2 + 14x + 9}{x^2 + 2x + 3}$

$\Rightarrow y(x^2 + 2x + 3) - x^2 - 14x - 9 = 0$

$\Rightarrow (y - 1)x^2 + (2y - 14)x + 3y - 9 = 0$

For real x, its discriminant ≥ 0

i.e. $4(y - 7)^2 - 4(y - 1)\ 3\ (y - 3) \geq 0$

$\Rightarrow y^2 + y - 20 \leq 0$ or $(y - 4)(y + 5) \leq 0$

Now, the product of two factors is negative, if these are of opposite signs. So following two cases arise

Case I: $y - 4 \geq 0$ or $y \geq 4$ and $y + 5 \geq 0$ or $y \geq -5$

This is not possible.

Case II: $y - 4 \geq 0$ or $y \leq 4$ and $y + 5 \geq 0$ or $y \geq -5$

Both of these are satisfied if $-5 \leq y \leq 4$

Hence maximum value of y is 4 and minimum value is -5.

29. (a) We have, $x^3 + px + q = 0$ (1)

$\therefore$ The roots of equation (i) is α, β and γ

$\therefore$ The sum of roots $= \alpha + \beta + \gamma$

$\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3} = \frac{-0}{1} = 0$

and the product of any two roots

$= \alpha\beta + \beta\gamma + \gamma\alpha = \frac{\text{Coefficient of } x}{\text{Coefficient of } x^3} = p$

$\therefore$ Product of all three roots $= \alpha, \beta\ \gamma = -q$

$\therefore \alpha + \beta + \gamma = 0$

$\therefore \alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma = -3q.$

30. (a) As given $\alpha + \beta = -p, \alpha\beta$
$= 1, \gamma + \delta = -q$

Now $\gamma\ \delta = 1 = (\alpha - \gamma)(\beta - \gamma)$

$(\alpha + \delta)(\beta + \delta)$

(Since γ is a root of $x^2 + qx + 1 = 0$)

$\gamma^2 + q\gamma + 1 = 0 \Rightarrow \gamma^2 + 1 = -q\gamma$ and similarly

$\delta + 1 = -q\delta = -\gamma\delta(p - q)(p + q)$

$= q^2 - p^2.$

31. (b) Given equation is $ax^2 + bx + c = 0$

α, β are the roots of this equation

$\alpha + \beta = -\frac{b}{a}$ (1)

$\alpha\beta = \frac{c}{a}$ (2)

then, $(1 + \alpha + \alpha^2)(1 + \beta + \beta^2)$

$= 1 + (\alpha + \beta) + \alpha^2 + \beta^2 + \alpha\beta(1 + \alpha + \beta) + \alpha^2\beta^2$

Putting the value of $\alpha + \beta$ and $\alpha\beta$

$$= 1 - \frac{b}{a} + \left(-\frac{b}{a}\right)^2 - \frac{2c}{a} + \frac{c}{a}\left(1 - \frac{b}{a}\right) + \frac{c^2}{a^2}$$

$$= 1 \frac{b}{a} + \frac{b^2}{a^2} - \frac{c}{a} - \frac{bc}{a^2} + \frac{c^2}{a^2}$$

$$= \frac{a^2 - ab + b^2 - ac - bc + c^2}{a^2}$$

$$= \frac{(a^2 + b^2 - 2ab) + (b^2 + c^2 - 2abc) + (c^2 + a^2 - 2ac)}{2a^2}$$

$(a - b)^2 + (b - c)^2 + (c - a)^2$

Here all terms are in square, therefore, it is always positive.

QUESTION BANK: SOLVE THESE TO MASTER

1. Let two numbers have arithmetic mean 9 and geometric mean 4. Then, these numbers are the roots of the quadratic equation

(a) $x^2 - 18x - 16 = 0$ (b) $x^2 - 18x + 16 = 0$
(c) $x^2 + 18x - 16 = 0$ (d) $x^2 + 18x + 16 = 0$

2. If one root of the equation $x^2 + px + 12 = 0$ is 4. While the equation $x^2 + px + q = 0$ has equal roots, then the value of 'q' is

(a) 4 (b) 12
(c) 3 (d) $\frac{49}{4}$

3. The roots of the equation $4x^2 - 2\sqrt{5}x + 1 = 0$ are

(a) cos 18°, cos 36° (b) sin 36°, cos 18°
(c) sin 18°, cos 36° (d) sin 36°, sin 18°

4. If r be the ratio of the roots of the equation $ax^2 + bx + c = 0$, then $\frac{(r+1)^2}{r} =$

(a) $\frac{a^2}{bc}$ (b) $\frac{b^2}{ca}$
(c) $\frac{c^2}{ab}$ (d) none of these

5. The greatest negative integer satisfying $x^2 - 4x - 77 < 0$ and $x^2 > 4$ is

(a) – 4 (b) – 6
(c) – 7 (d) none of these

6. The value of k for which the number 3 lies between the roots of the equation $x^2 + (1 - 2k)x + (k^2 - k - 2) = 0$ is given by

(a) $2 < k < 5$ (b) $k < 2$
(c) $2 < k < 3$ (d) $k > 5$

7. If α, β are the roots of the equation $8x^2 - 3x + 27 = 0$, then the value of $\left(\frac{\alpha^2}{\beta}\right)^{1/3} + \left(\frac{\beta^2}{\alpha}\right)^{1/3}$ is

(a) $\frac{1}{3}$ (b) $\frac{7}{2}$
(c) 4 (d) $\frac{1}{4}$

8. If $x^2 + 3x + 5 = 0$ and $ax^2 + bx + c = 0$ have a common root and $a, b, c \in N$, then minimum value of $a + b + c$ is equal to

(a) 9 (b) 3
(c) 6 (d) 12

9. If $(a^2 - 1)x^2 + (a - 1)x + a^2 - 4a + 3 = 0$ be an identity in x, then the value of a is

(a) – 1 (b) 1 (c) 2 (d) 0

10. For all $x \in R$, if $\lambda x^2 - 9\lambda x + 5\lambda + 1 > 0$, then λ lies in the interval

(a) [0, 4/61) (b) [0, 4/61]
(c) (– 4/ 61, 0) (d) none of these

11. If the equation $(a - 5)x^2 + 2(a - 10)x + a + 10 = 0$ has roots of opposite sign, then the values of a are

(a) $-10 < a < 5$ (b) $-5 < a < 10$
(c) $0 < a < 5$ (d) none of these

12. If the product of the roots of the equation $(a + 1)x^2 + (2a + 3)x + (3a + 4) = 0$ be 2, then the sum of roots is

(a) 1 (b) 2
(c) – 1 (d) – 2

13. If the roots of the equation $12x^2 - mx + 5 = 0$ are in the ratio 2: 3, then m =

(a) $2\sqrt{10}$ (b) $5\sqrt{10}$

(c) $3\sqrt{10}$ (d) none of these

14. If the roots of the equation $ax^2 + bx + c = 0$ are reciprocal to each other, then

(a) $a + c = 0$

(b) $b + c = 0$

(c) $a - c = 0$

(d) $b - c = 0$

15. If α and β are the roots of the equation $x^2 + 6x + \lambda = 0$ and $3\alpha + 2\beta = -20$, then λ =

(a) 16 (b) – 8 (c) – 16 (d) 8

16. If A.M. of the roots of a quadratic equation is 8/5 and A.M. of their reciprocals is 8/7, then the equation is

(a) $7x^2 - 16x + 8 = 0$

(b) $3x^2 - 12x + 7 = 0$

(c) $5x^2 - 16x + 7 = 0$

(d) $7x^2 - 16x + 5 = 0$

17. The quadratic equation whose roots are reciprocal of the roots of the equation $ax^2 + bx + c = 0$ is

(a) $cx^2 + ax + b = 0$ (b) $bx^2 + ax + c = 0$

(c) $cx^2 + bx + a = 0$ (d) $bx^2 + cx + a = 0$

18. If the ratio of the roots of $a_1x^2 + b_1x + c_1 = 0$ be equal to the ratio of the roots of $a_2x^2 + b_2x + c_2 = 0$, then $\frac{a_1}{a_2}, \frac{b_1}{b_2}, \frac{c_1}{c_2}$ are in

(a) H.P. (b) A.P.

(c) G.P. (d) none of these

19. If $ax^2 + bx + c = a(x - \alpha)(x - \beta)$, then $a(\alpha x + 1)(\beta x + 1)$ is equal to

(a) $ax^2 + bx + c$ (b) $cx^2 - bx + a$

(c) $cx^2 - bx - a$ (d) $cx^2 + bx + a$

20. If x is real, then the expression $\frac{x^2 + 34x - 71}{x^2 + 2x - 7}$ can have no value between

(a) 3 and 7 (b) 4 and 8

(c) 5 and 9 (d) 6 and 10

ANSWERS

Lecture–1: Unsolved Objective Problems (Identical Problems For Practice): For Improving Speed With Accuracy

1.	(a)	5.	(b)	9.	(c)	13.	(c)
2.	(d)	6.	(c)	10.	(c)		
3.	(a)	7.	(a)	11.	(d)		
4.	(a)	8.	(d)	12.	(a)		

Lecture–1: Work Sheet: To Check Preparation Level

1.	(b)	5.	(b)	9.	(d)	13.	(b)
2.	(a)	6.	(d)	10.	(b)	14.	(a)
3.	(b)	7.	(d)	11.	(a)	15.	(d)
4.	(b)	8.	(b)	12.	(d)	16.	(b)

Lecture–2: Unsolved Objective Problems (Identical Problems For Practice): For Improving Speed With Accuracy

1.	(b)	4.	(b)	7.	(d)	10.	(b)
2.	(b)	5.	(d)	8.	(d)		
3.	(a)	6.	(b)	9.	(a)		

Lecture–2: Work Sheet: To Check Preparation Level

1.	(a)	5.	(b)	9.	(a)	13.	(b)
2.	(c)	6.	(a)	10.	(a)		
3.	(a)	7.	(d)	11.	(d)		
4.	(c)	8.	(b)	12.	(a)		

Lecture–3: Unsolved Objective Problems (Identical Problems For Practice): For Improving Speed With Accuracy

1.	(d)	4.	(b)	7.	(c)	10.	(c)
2.	(c)	5.	(c)	8.	(c)		
3.	(b)	6.	(c)	9.			

Lecture–3: Work Sheet: To Check Preparation Level

1.	(a)	5.	(d)	9.	(b)	13.	(b)
2.	(a)	6.	(c)	10.	(b)		
3.	(b)	7.	(d)	11.	(b)		
4.	(a)	8.	(b)	12.	(a)		

Lecture–4: Unsolved Objective Problems (Identical Problems for Practice: For Improving Speed with Accuracy

1.	(b)	8.	(c)	15.	(a)(b)(c)(d)	22.	(b)
2.	(a)	9.	(d)	16.	(c)	23.	(d)
3.	(a)	10.	(a)	17.	(c)	24.	(a)
4.	(c)	11.	(a)	18.	(b)	25.	(d)
5.	(d)	12.	(d)	19.	(d)		
6.	(c)	13.	(c)	20.	(a)		
7.	(a)	14.	(b)	21.	(c)		

Lecture–4: Work Sheet: To Check Preparation Level

1.	(a)	5.	(a)	9.	(b)	13.	(d)
2.	(d)	6.	(d)	10.	(c)	14.	(d)
3.	(a)	7.	(b)	11.	(a)	15.	(a)
4.	(c)	8.	(b)	12.	(a)		

Lecture–5: Mental Preparation Test

1. $x = 2 \pm 3i$
3. $n = 0, n = 3$
7. $x^2 - 2ax + a^2 - b = 0$
8. $\frac{r(q^2 - 2pr)}{p^3}$
10. $acx^2 - bx + 1 = 0$ (3)
11. $pq = -2/3$
12. $prx^2 + (2rp - q^2)x + rp = 0$
17. ± 4

QUESTION BANK: SOLVE THESE TO MASTER

1.	(b)	6.	(d)	11.	(a)	16.	(c)
2.	(d)	7.	(c)	12.	(c)	17.	(c)
3.	(a)	8.	(a)	13.	(b)	18.	(b)
4.	(b)	9.	(b)	14.	(c)	19.	(b)
5.	(b)	10.	(a)	15.	(c)	20.	(c)

Part D

Progression

This page is intentionally left blank.

LECTURE

1 Arithmetic Progression

BASIC CONCEPTS

1. **Arithmetic Progression** A sequence whose terms increase or decrease by the same constant is called an Arithmetic Progression. This constant is called the common difference of the Airthmetic progression and is usually denoted by '*d*'.

   In short form the arithmetic progression is denoted by A.P.

   For example Following sequence is in A.P.

   1. 1, 3, 5, 7, ... (common difference = 2)
   2. 54, 51, 48, ... (common difference = − 3)

Note: If T_1, T_2, T_3 ....Tn_1 .... are in A.P., then $d = T_2 - T_1 = T_3 - T_2 =$ .... and $-d = T_1 - T_2 = T_2 - T_3 =$ ....

where

T_n = *n*th term of an A.P.

2. **The *n* terms (General term) of Arithmetic Progression**

   $a, a + d, a + 2d, ..., a + (n - 1) d, ...$

Note: (i) First term of A.P. = a

(ii) 1st, 2nd, 3rd and *n*th terms of an A.P. are denoted by T_1, T_2, T_3 and T_n respectively.

(iii) *n*th terms of an A.P. $T_n = a + (n - 1) d = l$ (*say*)

(iv) *n*th term from the end of an A.P. = $(m - n + 1)$th term

m = Total number of terms of an A.P.

(v) Three numbers a, b, c are in A.P. if and only if

$2b = a + c$ i.e., $b - a = c - b$.

3. **Middle Term** The middle term depends upon the number of terms.

   1, 3, 5, 7, 9, 11 ⇒ number of terms = $n = 6$

   1, 3, 5, 7, 9, 11, 13 ⇒ number of terms = $n = 7$

   (i) If the total number of terms of an A.P. is even, then there are two middle term i.e., $\frac{n}{2}$th and $\left(\frac{n}{2} + 1\right)$th

   where n = number of terms

   (ii) If the total number of terms of an A.P. is odd, then there is only one middle term i.e., $\left(\frac{n+1}{2}\right)$th that is $T_{\frac{n+1}{2}}$

   where n = number of terms.

Note 1: If the number of terms of an A.P. is $2n$, then *n*th and $(n + 1)$ th are two middle terms.

Note 2: If the number of terms of an A.P. is $(2n + 1)$, then $(n + 1)$th term is only middle term.

4. Sum of n terms of an AP

(i) $S_n = a + (a + d) + ... + (a + (n - 1)d) = \frac{n}{2}[2a + (n - 1)d]$ (1)

or

$\frac{n}{2}[a + l]$ where $l = a + (n - 1)d = t_n = n$th term

Notations: (i) S_n = sum of n terms

(ii) S_{2n} = sum of $2n$ terms

(iii) S_{2n+1} = sum of $(2n + 1)$ terms

(iv) S_{3n} = sum of $3n$ terms

(ii) nth term (T_n) in terms of S_n (Sum)

$\because S_n = T_1 + T_2 + ... + T_{n-1} + T_n$

$S_n = S_{n-1} + T_n \Rightarrow T_n = S_n - S_{n-1}$

(iii) Common difference (d) in terms of S_n

S_1 = Sum of one term = T_1

S_2 = Sum of two terms = $T_1 + T_2$

Thus, $T_2 = S_2 - S_1$ again common difference $d = T_2 - T_1$

$d = S_2 - S_1 - S_1 = S_2 - 2S_1$ $d = S_2 - 2S_1$

(iv) If the sum of n, $2n$ and $3n$ terms of A.P. are S_n, S_{2n}, S_{3n} respectively, then $S_{3n} = 3(S_{2n} - S_n)$.

5. Selection of terms in A.P.

(i) Three consecutive terms of an A.P.

$a - d, a, a + d$ or $a, a + d, a + 2d$

(ii) Four consecutive term: $a - 3d, a - d, a + d, a + 3d$ or $a, a + d, a + 2d, a + 3d$

(iii) Last four consecutive term if l = last term

$l - 3d, l - 2d, l - d, l$

6. Arithmetic Mean

(i) If p, q, r, s is in A.P., then $q - p = r - q = s - r$ = common difference = d, then $q = \frac{p + r}{2}$ => q is called Arithmetic mean of numbers p and r, similarly r is called Arithmetic mean of numbers q and s. That is if three terms p, q, r are in A.P., then the middle one is called the Arithmetic mean between the other two.

Note: Similarly the A.M. of n number $x_1, x_2, ...x_n$ is given by A.M. $= \frac{x_1 + x_2 + x_3 ...... + x_n}{n}$

(ii) n, A.M.'S between two numbers a and b If $a, A_1, A_2, A_3, ...A_n, b$ are in A.P., then the numbers $A_1, A_2, A_3 ....A_n$ are called the n-Arithmetic means between a and b. Also last term b is (n + 2)th term of A.P..

Note: nth Arithmetic mean = $(n + 1)$th term of A.P. Common difference for inserting arithmetic means between two numbers a and b: Common difference $= d = \frac{b - a}{n + 1}$

Note: Between any two numbers, infinite sets of arithmetic means can be inserted.

(iii) In an A.P. the sum of two terms equidistant from the beginning and the end is constant and is equal to the sum of the first and the last terms.

That is if

(i) a, A_1, A_2, b is an A.P., then $a + b = A_1 + A_2$.

(ii) $a_1, a_5, a_{10}, a_{15}, a_{20}, a_{24}$ in an A.P., then $a_1 + a_{24} = a_5 + a_{20} = a_{10} + a_{15}$ where a = nth term of A.P.

(iv) Sum of n, A.M.'S between A and B: If $a, A_1, A_2, A_3, ...A_n, b$ is an A.P., then

$A_1 + A_2 + A_3 + ...+ A_n = n\left(\frac{a + b}{2}\right)$

That is the sum of n Arithmetic means between two given numbers is equal n times the single A.M. between them.

7. Properties of A.P.

(i) If nth term of any series is a linear expression i.e., $T_n = An + B$ (A and B are constants) Then the series is an A.P. In this case the common difference of an A.P. is $d = T_2 - T_1 = A$

(ii) If sum of n terms of any series is a quadratic expression in n, i.e., $S_n = An^2 + Bn$, then the series is an A.P. and in this case the common difference $d = S_2 - 2S_1 = 2A$.

(iii) If each term of an A.P. is increased or decreased, multiplied or divided by the same non zero number then the resulting series is also an A.P. i.e., if a_1, a_2, a_3, ...a_n are in A.P., then

(i) $a_1 \pm k$; $a_2 \pm k$, ...$a_n \pm k$, ... are also in A.P. its common difference $= d$

(ii) $ka_1, ka_2, ..., ka_n, ...$ is an A.P. and its common difference $= kd$

(iii) $\frac{a_1}{k}, \frac{a_2}{k}, ... \frac{a_n}{k} ....$ A.P. is an A.P. and its common difference $= d/k$

(iv) Term by term addition or subtraction of two arithmetic progressions is also in A.P.

i.e., If $a_1, a_2, a_3, ...a_n, ...$ and $b_1, b_2, b_3,....b_n$, are in A.P, then $a_1 \pm b_1$; $a_2 \pm b_2$; $a_3 \pm b_3$, ...are also in A.P.

SOLVED SUBJECTIVE PROBLEMS: (CBSE/STATE BOARD): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. If a, b and $2a$ are respectively first, second and last terms of an A.P., then show that sum of the series is $\frac{3ab}{2(b-a)}$

Solution

Since, first term is a and b is second term.

$\therefore$ Common difference, $d = b - a$

Let there are n terms in this series Then,

According to question, $l = 2a$

$\Rightarrow a + (n - 1)\, d = 2a$

$\Rightarrow a + (n - 1)\,(b - a) = 2a$

$\Rightarrow (n - 1)\,(b - a) = 2a - a \Rightarrow n - 1 = \frac{a}{b-a}$

$\Rightarrow n = \frac{a}{b-a} + 1 \Rightarrow n = \frac{a+b-a}{b-a}$

$\Rightarrow n = \frac{b}{b-a}$

$\therefore$ Sum of the series $= \frac{n}{2}[a + l]$

$= \frac{b}{2(b-a)}[a + 2a] = \frac{3ab}{2(b-a)}$

Proved

2. First and last terms of an A.P. are a and l respectively. If S denotes the sum of all terms, then show that common difference is

$\frac{l^2 - a^2}{2S - (l + a)}$.

[DCE – 1998]

Solution

Let d is common difference. Then according to question

$S = \frac{n}{2}(a + l)$ (1)

and $l = a + (n - 1)d$ (2)

$\Rightarrow (n - 1)d = l - a$

$\Rightarrow \left[\frac{2S}{a+1} - 1\right]d = l - a$, [from Equation (1)]

$\Rightarrow \frac{2S - (a+1)}{a+1} d = l - a$

$\Rightarrow d = \frac{(l-a)(l+a)}{2S - (a+l)}$

$\Rightarrow d = \frac{l^2 - a^2}{2S - (a+l)}$ **Proved**

3. If in an A.P., $S_n = n^2p$ and $S_m = m^2 p$ $(m \neq n)$, then prove that $S_p = p^3$.

Solution

Let first term is a and common difference is d.

Then, $S_n = n^2 p$ and $S_m = m^2 p \Rightarrow S_r = r^2 p$, where $S_r \in N$

$\therefore S_1 = a = 1^2 \times p = p$ and $S_2 = 2^2 \times p = 4p$

$\therefore$ 2nd term of this A.P. $= S_2 - S_1$

$= 4p - p = 3p$

$\therefore d = 3p - p = 2p$

$\therefore S_p = \frac{p}{2}[2 \times p + (p - 1) \times 2p]$

$= p\,[p + p^2 - p] = p^3$ **Proved**

4. An A.P. has even number of terms. Sum of odd terms is 24 and sum of even terms is 30.

Last term is $10\frac{1}{2}$ more than first term. Find the number of terms and A.P.

Solution

Let first term is a, common difference is d and number of terms is $2n$.

$\because$ Sum of odd terms = 24

$\therefore a + (a + 2d) + (a + 4d) + \ldots\ldots$ to n terms = 24

$\Rightarrow \frac{n}{2}[2a + (n-1) \times 2d] = 24$

$\Rightarrow n[a + nd - d] = 24$ (1)

Again, sum of even terms = 30

$\Rightarrow (a + d) + (a + 3d) + \ldots\ldots$ to n terms = 30

$\Rightarrow \frac{n}{2}[2(a + d) + (n-1) \times 2d] = 30$

$\Rightarrow n[a + d + nd - d] = 30$

$\Rightarrow n[a + nd] = 30$ (2)

$\because$ Last term = First term + $10\frac{1}{2}$

$\therefore a + (2n-1)d = a + 10\frac{1}{2}$

$\Rightarrow (2n-1)d = \frac{21}{2}$ (3)

Now, subtracting Equation (1) from Equation (2), we get, $nd = 6 \Rightarrow d = \frac{6}{n}$

Putting the value d in Equation (3), we get,

$(2n-1)\left(\frac{6}{n}\right) = \frac{21}{2}$

$\Rightarrow 12(2n-1) = 21n \Rightarrow 4(2n-1) = 7n$

$\Rightarrow 8n - 4 = 7n \Rightarrow n = 4$

$\therefore d = \frac{6}{n} = \frac{6}{4} = \frac{3}{2}$ and from Equation (2).

$4\left(a + 4 \times \frac{3}{2}\right) = 30$

$\Rightarrow a + 6 = \frac{30}{4} \Rightarrow a = \frac{15}{2} - 6 = \frac{3}{2}$

Therefore, there are $2n = 8$ terms in A.P. and

A.P. is $\frac{3}{2}, 3, 4\frac{1}{2}, \ldots\ldots$

5. If a, b, c are in A.P., then prove that

(i) $(a-c)^2 = 4(b^2 - ac)$

(ii) $a^3 + 4b^3 + c^3 = 3b(a^2 + c^2)$

Solution

(i) a, b, c are in A.P. $\Rightarrow 2b = a + c$

$\Rightarrow b = \frac{a+c}{2}$

Putting $b = \frac{a+c}{2}$ on RHS, we get,

RHS $= 4(b^2 - ac) = 4\left\{\left(\frac{a+c}{2}\right)^2 - ac\right\}$

$= 4\left\{\frac{(a+c)^2 - 4ac}{4}\right\}$

$= (a+c)^2 - 4ac = (a-c)^2 =$ LHS.

(ii) Putting $b = \frac{a+c}{2}$ on RHS, we get,

RHS $= \frac{3}{2}(a+c)(a^2 + c^2)$

$= \frac{3}{2}(a^3 + c^3 + ac^2 + a^2c)$

LHS $= a^3 + 4\left(\frac{a+c}{2}\right)^2 + c^3$

$= \frac{3}{2}(a^3 + c^3 + ac^2 + a^2c)$

$\therefore$ LHS = RHS

6. If a, b, c are in A.P. and x, y are arithmetic means of a, b and b, c respectively, then prove that a, x, b, y, c are in A.P.

Solution

Since a, b, c are in A.P., therefore,

$b = \frac{1}{2}(a + c)$ (1)

Again, x is arithmetic mean of a, b

$\therefore x = \frac{1}{2}(a + b)$ (2)

and y is arithmetic mean of b, c

$\therefore y = \frac{1}{2}(b + c)$ (3)

Adding Equations (2) and (3), we get,

$x + y = \frac{1}{2}(a + b) + \frac{1}{2}(b + c)$

$= \frac{1}{2}[a + b + b + c] = \frac{1}{2}[a + c + 2b]$

$= \frac{1}{2}(a + c) + \frac{1}{2}(2b) = b + b,$

[from Equation (1)]

$= 2b$ i.e., $b = \frac{1}{2}(x + y)$

$\therefore b$ is arithmetic mean of x and y.

Therefore, a, x, b, y, c are in A.P.

Proved

7. If $\frac{(b+c-a)}{a}, \frac{(c+a-b)}{b}, \frac{(a+b-c)}{c}$, are in A.P., prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in AP.

Solution

$\frac{(b+c-a)}{a}, \frac{(c+a-b)}{b}, \frac{(a+b-c)}{c}$, are in A.P.

$\Rightarrow \left\{\frac{(b+c-a)}{a}+2\right\}, \left\{\frac{(c+a-b)}{b}+2\right\}, \left\{\frac{(a+b-c)}{c}+2\right\}$ are in AP

[adding 2 to each term]

$\Rightarrow \frac{(a+b+c)}{a}, \frac{(a+b+c)}{b}, \frac{(a+b+c)}{c}$ are in AP

$\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP

[dividing each term by $(a+b+c)$]

8. The income of a person is Rs. 3,00,000 in the first year and he receives an increase of Rs. 10,000 to his income per year for the next 19 years. Find the total amount, he received in 20 years.

Solution

Here, we have an A.P. with $a = 3, 00, 000$, $d = 10, 000$, and $n = 20$. Using the sum formula, we get,

$$S_{20} = \frac{20}{2}[600000 + 19 \times 10000]$$

$=10\ (790000) = 79, 00, 000.$

Hence, the person received Rs. 79, 00, 000 as the total amount at the end of 20 years.

9. If $a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P., show that either a, b, c are in A.P. or $ab + bc + ca = 0$.

Solution

$a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P.

$\Rightarrow b^2(c+a) - a^2(b+c) = c^2(a+b) - b^2(c+a)$

$\Rightarrow (b^2a - a^2b) + (b^2c - a^2c)$

$= (c^2b - b^2c) + (c^2a - b^2a)$

$\Rightarrow (b-a)(ab+bc+ac) = (c-b)(ab+bc+ca)$

$\Rightarrow (ab+bc+ca)(2b-a-c) = 0$

$\Rightarrow ab+bc+ac = 0$ or $2b = a+c$

$\Rightarrow ab+bc+ac = 0$ or a, b, c are in A.P.

10. Two cars start together in the same direction from the same place. The first goes with uniform speed of 10 km/h. The second goes at a speed of 8 km/h in the first hour and increases the speed by 1/2 km each succeeding hour. After how many hours will the second car overtake the first car if both cars go non-stop?

Solution

Suppose the second car overtakes the first car after t hours. Then the two cars travel the same distance in t hours.

Distance travelled by the first car in t hours $= 10\,t$ km.

Distance travelled by the second car in t hours

= Sum of t terms of an A.P. with first term 8 and common difference 1/2.

$= \frac{t}{2}\left[2 \times 8 + (t-1) \times \frac{1}{2}\right] = \frac{t(t+31)}{4}$

When the second car overtakes the first car, we have

$= 10t = \frac{t(t+31)}{4} \Rightarrow t(t-9) = 0 \Rightarrow t = 9$

$[\because t \neq 0]$

Thus, the second car will overtake the fi rst car in 9 hours.

11. A man replays a loan of Rs 3250 by paying Rs 20 in the first month and then increases the payment by Rs 15 every month. How long will it take him to clear the loan ?

Solution

Suppose the loan is cleared in n months. Clearly, the amounts form an A.P. with first term 20 and the common difference 15.

$\therefore$ Sum of the amounts = 3250

$\Rightarrow \frac{n}{2}[2 \times 20 + (n-1) \times 15] = 3250$

$\Rightarrow 3n^2 + 5n - 1300 = 0$

$\Rightarrow (n-20)(3n+65) = 0$

$\Rightarrow n = 20$ or $n = \frac{-65}{3}$

$\Rightarrow n = 20$

$\because n \neq \frac{-65}{3}$

12. Sum of two numbers is $2\frac{1}{6}$. Some A.M.s are inserted between them whose numbers are even and sum of their is 1 more than the numbers. Find the number of inserted A.M.s

Solution

Let the two numbers are a and b and $2n$ A.M.s are inserted between them.

According to question, $a + b = 13/6$ (1)

and sum of A.M.s $= 2n + 1$ (2)

Obviously, there will be $(2n + 2)$ terms in this A.P. whose first term is a and last term is b.

$\therefore$ Sum of A.P. $= \frac{2n+2}{2}(a + b)$

$= (n + 1)(a + b)$

$= \frac{13}{6}(n + 1)$ [from (1)]

$\therefore$ Sum of A.M.s = Sum of A.P. $- (a + b)$

$\therefore 2n + 1 = \frac{13}{6}(n + 1) - \frac{13}{6}$

[from (1) and (2)]

$\Rightarrow 12n + 6 = 13n + 13 - 13$

$\Rightarrow 12n + 6 = 13n \Rightarrow n = 6$

13. Find the number of terms common to the two A.P.'s 3, 7, 11, ... 407 and 2, 9, 16,... 709.

Solution

It is easy to observe that both the series consist of 102 terms.

Let $T_p = 3 + 4(p - 1) = 4p - 1$ and $T_q = 2 + 7(q - 1) = 7q - 5$ be the general terms of the two series where both p and q lie between 1 and 102. We have to find the the values of p and q for which $T_p = T_q$.

i.e., $4p - 1 = 7q - 5$ or $4(p + 1) = 7q$ (1)

Now p and q are + ive integers and hence from (1) we conclude that q is multiple of 4 and so let $q = 4s$ and as q lies between 1 and 102,

therefore, s lies between 1 and 25.

$\therefore \frac{p+1}{7} = \frac{q}{4}\lambda = p + 1 = 7\lambda$ and $q = 4\lambda$

both p and q vary from 1 to 102

$\therefore \lambda$ varies from 1 to 14 or from 1 to 25.

Hence, we choose λ to vary from 1 to 14.

Thus, there are only 14 common terms.

$T_p = 4p - 1 = a(7\lambda - 1) = 28\lambda - 5$

Put $\lambda = 1, 2, 3, ..., 14$ and common terms are 23, 51, 79, ...

14. Prove that there are 17 identical terms in the two A.P.'s 2, 5, 8, 11, ... 60 terms and 3, 5, 7, 9 .... 50 terms.

Solution

$T_p = T_q \Rightarrow 3p - 1 = 2q + 1.$

Subtract 5 from both sides $3(p - 2) = 2(q - 2)$

or $\frac{p-2}{2} = \frac{q-2}{3} = k$ say.

$\therefore p = 2k + 2$ and $q = 3k + 2$

Now p varies from 1 to 60 and q varies from 1 to 50. Hence, we have the following

$1 \leq 2k + 2 \leq 60$ and $1 \leq 3k + 2 \leq 50$

$\therefore -\frac{1}{2} \leq k \leq 29$ and $-\frac{1}{3} \leq k \leq 16.$

Clearly $k = 0, 1, 2, 3, ..., 16$ for common values and hence there will be 17 common terms.

UNSOLVED SUBJECTIVE PROBLEMS: (CBSE/STATE BOARD): TO GRASP THE TOPIC, SOLVE THESE PROBLEMS

Exercise I

1. Find the 10th term of the A.P. 1, 5, 9, 13, ...

2. Find 21st term of sequence 16, 11, 6, ...

3. Which term of 4, 7, 10 ..... is 148.

4. Find the sum of all natural numbers lying between 100 and 1000, which are multiples of 5.
5. How many terms of the series, 24 + 20 + 16 + ... should be added so that the sum may be 72?
6. Prove that the sum of the terms of an A.P. equidistant from its first and last terms, is constant.
7. The 5*th* and 9*th* term of an A.P. are 11 and 17 respectively. Find the sum of 20 terms.
8. Find the sum of odd numbers between 100 and 200.

   If the sum of the first n terms of a progression is a quadratic expression in n, show that it is an AP.
9. If $< a_n >$ is an A.P. and $a_1 + a_4 + a_7 + ... + a_{16} = 147$, then find the value of $a_1 + a_6 + a_{11} + a_{16}$.
10. If S_1, S_2, S_3 are the sum of three arithmetic progressions. If first term of each series is 1 and common differences are 1, 2, 3 respectively, then prove that $S_1 + S_3 = 2S_2$.
11. If S_n denotes sum of n terms of an A.P. and if $S_1 = 6$, $S_7 = 105$, then prove that $S_n : S_{n-3} = (n + 3) : (n - 3)$.
12. If l is last term and d is common difference of an A.P., then prove that sum of its n terms is $\frac{n}{2}[2l - (n - 1)d]$
13. Find the sum of all three digit natural numbers, which are divisible by 7.
14. Find the AM between

    (i) 14 and 18 (ii) $(a - b)$ and $(a + b)$
15. If the AM between pth and qth terms of an AP be equal to the AM between rth and sth terms of the AP, then show that

    $(p + q) = (r + s)$.
16. The angles of a quadrilateral are in A.P. Their common difference is 15. Find the smallest angle.
17. If the progressions 3, 10, 17, ... and 63, 65, 67, ... are such that their nth terms are equal, then find the value of n.

Exercise II

1. Is 301 a term of the AP 5, 11, 17, 23, ...?
2. If the nth term of a progression is a linear expression in n then Show that it is an AP.
3. Is 302 any term of series 3, 8, 13, 18, ... Since here n is not an integer, Hence 302 is not a term of the given series.
4. Show that the sequence 9, 12, 15, 18, ... is an A.P. find its 16th term and the general term.
5. $3x$, $x + 2$ and 8 are three continuous terms of an A.P. Find its fourth term.
6. Find the sum of 23 terms of the AP 5, 9, 13, 17, ...
7. How many terms of the AP $-6, \frac{-11}{2}, -5, ...$ are needed to give the sum -25? Explain the double answer.
8. Find the 19th term from the end of the AP2, 6, 10, 14, ..., etc.
9. The sum of 10 terms of an A.P. is four times the sum of 5 terms. Find the ratio of first term and common difference.
10. Insert 6 numbers between 3 and 24 such that the resulting sequence is an AP.
11. Insert 5 A.M. between 11 and − 7.
12. Prove that in an A.P. whose number of terms are even, A.M. of two middle terms is equal to A.M. of 1st and last terms.
13. Between 7 and 49 there are n A.M.s and $\frac{\text{4th A.M}}{(n-2)\text{th A.M}} = \frac{5}{4}$, then find out the value of n.
14. The A.M. of two numbers is 7 and their product is 45. Find the numbers.
15. The length of sides of a right angled triangle are in A.P. Prove that they are proportional to 3, 4, 5.
16. Divide 32 into four parts such that they are in A.P. and the ratio of product of first term and fourth terms to the product of second and third terms is equal to 7 : 15
17. Find the sum of first 24 terms of A.P. $a_1, a_2, a_3, a_4, ...$ if $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$.

18. If a, b, c are in AP show that

(i) $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in AP

(ii) $\frac{a(b+c)}{bc}, \frac{b(c+a)}{ca}, \frac{c(a+b)}{ab}$, are in AP

ANSWERS

Exercise I

1. 37
2. − 84
3. 49th term is 148.
4. 99550.
5. $n = 4$ or $n = 9$
7. 385
8. 7500
10. 98
14. 70336
15. (i) 16 (ii) a
17. (a) = 67.50°
18. $n = 13$.

Exercise II

1. 301 is not a term of the given AP.
3. $n = 304/5$
4. $a_{16} = 54$ and $a_n = 3n + 6$
5. $t_4 = 18$
6. Hence, the sum of 23 terms of the given AP is 1127.
7. Thus, the sum of first 5 terms as well as the sum of first 20 terms is − 25.
8. 19th term from the end = 14
9. $a : d = 1 : 2$
10. Hence, the required numbers are 6, 9, 12, 15 and 18 and 21.
11. The required numbers are 8, 5, 2, − 1 and − 4.
13. $n = 5$
14. 5 and 9.
16. 2, 6, 10, 14 and 14, 10, 6, 2

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. If $\log_3 2$, $\log_3 (2^x - 5)$ and $\log_3 \left(2^x - \frac{7}{2}\right)$ are in A.P., then x is equal to

(a) 1, 1/2 (b) 1, 1/3
(c) 1, 3/2 (d) none of these

[IIT – 1990]

Solution

(d) $\log_3 2$, $\log_3 (2^x - 5)$ and $\log_3 \left(2^x - \frac{7}{2}\right)$ are in A.P.

$\Rightarrow 2\log_3(2^x - 5) = \log_3\left[(2)\left(2^x - \frac{7}{2}\right)\right]$

$\Rightarrow (2^x - 5)^2 = 2^{x+1} - 7$

$\Rightarrow 2^{2x} - 12.\ 2^x + 32 = 0$

$\Rightarrow x = 2, 3$

But $x = 2$ does not hold, hence $x = 3$.

2. If a, b, c, d, e are in A.P. then the value of $a + b + 4c - 4d + e$ in terms of a, if possible is

[RPET – 2002]

(a) $4a$ (b) $2a$
(c) 3 (d) none of these

Solution

(d) It is not possible to express $a + b + 4c - 4d + e$ in terms of a.

3. The sum of the integers from 1 to 100 which are not divisible by 3 or 5 is

[MP PET – 2000]

(a) 2489 (b) 4735
(c) 2317 (d) 2632

Solution

(d) Let $S = 1 + 2 + 3 + \ldots\ldots\ldots\ldots\ldots + 100$

$= \frac{100}{2}(1 + 100) = 50\,(101) = 5050$

Let $S_1 = 3 + 6 + 9 + 12 + \ldots\ldots\ldots\ldots + 99$

$= 3(1 + 2 + 3 + 4 + \ldots\ldots\ldots\ldots\ldots + 33)$

$= 3.\frac{33}{2}(1 + 33) = 99 \times 17 = 1683$

Let $S_2 = 5 + 10 + 15 + \ldots\ldots\ldots\ldots + 100$

$= 5(1 + 2 + 3 + \ldots\ldots\ldots\ldots\ldots + 20)$

$= 5.\frac{20}{2}(1 + 20) = 50 \times 21 = 1050$

Let $S_3 = 15 + 30 + 45 + \ldots\ldots\ldots\ldots + 90$

$= 15\,(1 + 2 + 3 + \ldots\ldots\ldots\ldots\ldots\ldots + 6)$

$= 15\,.\frac{6}{2}(1 + 6) = 45 \times 7 = 315$

$\therefore$ Required sum $= S - S_1 - S_2 + S_3$

$= 5050 - 1683 - 1050 + 315$

$= 2632.$

4. If sum of n terms of an A.P. is $3n^2 + 5n$ and $T_m = 164$, then $m =$

[RPET – 1991, 1995; DCE – 1999]

(a) 26 (b) 27

(c) 28 (d) none of these

Solution

(b) Obviously

$164 = (3m^2 + 5m) - \{3\,(m - 1)^2 + 5\,(m - 1)\}$

$= (3m^2 + 5m) - 3m^2 + 6m - 3 - 5m + 5$

$\Rightarrow 164 = 6m + 2 \Rightarrow m = 27$

5. If the sum of the 10 terms of an A.P. is 4 times to the sum of its 5 terms, then the ratio of first term and common difference is

(a) 1: 2 (b) 2: 1

(c) 2: 3 (d) 3: 2

[RPET – 1986]

Solution

(a) Under conditions, we get,

$\frac{10}{2}\{2a + (10 - 1)d\}$

$= 4\left[\frac{5}{2}[2a + (5 - 1)\,d]\right]$

$\Rightarrow 2a + 9d = 4a + 8d$ or $\frac{a}{d} = \frac{1}{2}$

Hence, $a : d = 1 : 2$.

6. Let a_1, a_2, a_3 .... be terms of an AP.

If $\frac{a_1 + a_2 + \ldots\ldots + a_p}{a_1 + a_2 + \ldots\ldots + a_q} = \frac{p^2}{q^2}, p \neq q$, then $\frac{a_6}{a_{21}}$ equals

(a) $\frac{2}{7}$ (b) $\frac{7}{2}$ (c) $\frac{11}{41}$ (d) $\frac{41}{11}$

[AIEEE – 2006]

Solution

(c) Let d be the common difference of given AP. Then,

$$\frac{\frac{p}{2}[2a_1 + (p - 1)d]}{\frac{q}{2}[2a_1 + (q - 1)d]} = \frac{p^2}{q^2}$$

$$\Rightarrow \frac{a_1 + \left(\frac{p - 1}{2}\right)d}{a_1 + \left(\frac{q - 1}{2}\right)d} = \frac{p}{q} \qquad (1)$$

Now when $\frac{p - 1}{2} = 5$ i.e., $p = 11$ and $\frac{q - 1}{2} = 20$ i.e., $q = 4$

We shall have $\frac{a_1 + 5d}{a_1 + 20d} = \frac{11}{41} \Rightarrow \frac{a_6}{a_{21}} = \frac{11}{41}$

7. The ratio of sum of m and n terms of an A.P. is $m^2 : n^2$, then the ratio of mth and nth term will be

[Roorkee – 1963; MPPET – 1995; Pb. CET – 2001]

(a) $\frac{m - 1}{n - 1}$ (b) $\frac{n - 1}{m - 1}$

(c) $\frac{2m - 1}{2n - 1}$ (d) $\frac{2n - 1}{2m - 1}$

Solution

(c) Given that $\frac{\frac{m}{2}[2a + (m - 1)d]}{\frac{n}{2}[2a + (n - 1)d]} = \frac{m^2}{n^2}$

$$\Rightarrow \frac{2a + (m - 1)d}{2a + (n - 1)d} = \frac{m}{n}$$

$$\Rightarrow \frac{a + \frac{1}{2}(m - 1)d}{a + \frac{1}{2}(m - 1)d} = \frac{m}{n}$$

$\Rightarrow an + \frac{1}{2}(m-1)nd = am + \frac{1}{2}(n-1)md$

$\Rightarrow a(n-m) + \frac{d}{2}[mn - n - mn + m] = 0$

$\Rightarrow a(n-m) + \frac{d}{2}(m-n) = 0 \Rightarrow a = \frac{d}{2}$

or $d = 2a$

So, required ratio,

$\frac{T_m}{T_n} = \frac{a+(m-1)+d}{a+(n-1)d} = \frac{a+(m-1)2a}{a+(n-1)2a}$

$= \frac{1+2m-2}{1+2n-2} = \frac{2m-1}{2n-1}$

Trick: Replace m by $2m - 1$ and n by $2n - 1$.

Obviously if S_m is of degree 2, then T_m is of 1 i.e., linear.

8. If tan $n\theta$ = tan $m\theta$ then the different values of θ will be in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these

[Karnataka CET – 1998]

Solution

(a) We have tan $n\theta$ = tan $m\theta$

$\Rightarrow n\theta = N\pi + (m\theta)$

$\Rightarrow \theta = \frac{N\pi}{n-m}$, putting $N = 1, 2, 3$ ....,

we get,

$\frac{\pi}{n-m}, \frac{2\pi}{n-m}, \frac{3\pi}{n-m}$ .... which are obviously in A.P.

Since, common difference $d = \frac{\pi}{n-m}$

9. If $\frac{1}{p+q}, \frac{1}{r+p}, \frac{1}{q+r}$ are in A.P., then,

(a) q, r are in A.P.

(b) p^2, q^2, r^2 are in A.P.

(c) $\frac{1}{p}, \frac{1}{q}, \frac{1}{r}$ are in A.P.

(d) none of these

[RPET – 1995]

Solution

(b) Since, $\frac{1}{p+q}, \frac{1}{r+q}$, and $\frac{1}{q+r}$ are in A.P

$\therefore \frac{1}{r+q} - \frac{1}{p+q} = \frac{1}{q+r} - \frac{1}{r+p}$

$\Rightarrow \frac{p+q-r-p}{(r+p)(p+q)} = \frac{r+p-q-r}{(q+r)(r+p)}$

$\Rightarrow \frac{q-r}{p+q} = \frac{p-q}{q+r}$

$\therefore$ or $q^2 - r^2 = p^2 - q^2$

$\therefore 2q^2 = r^2 + p^2$

$\therefore p^2, q^2, r^2$ are in A.P.

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. If $2x, x+8, 3x+1$ are in A.P., then the value of x will be

(a) 3 (b) 7

(c) 5 (d) −2

[MPPET – 1984]

2. If the sum of n terms of an A.P. is $nA + n^2B$, where A, B are constants, then its common difference will be

[MNR – 1977]

(a) $A - B$ (b) $A + B$ (c) $2A$ (d) $2B$

3. If the 9th term of an A.P. is 35 and 19th is 75, then its 20th terms will be

(a) 78 (b) 79 (c) 80 (d) 81

[RPET – 1989]

4. If mth terms of the series 63 + 65 + 67 + 69 ...... and 3 + 10 + 17 + 24 + ... be equal, then m =

(a) 11 (b) 12

(c) 13 (d) 15

[Kerla (Engg.) – 2002]

5. The number of terms in the series 101 + 99 + 97 + ...+ 47 is

(a) 25 (b) 28 (c) 30 (d) 20

6. If a, b, c are in A.P., then $\frac{(a-c)^2}{(b^2-ac)} =$

[Roorkee – 1975]

(a) 1 (b) 2

(c) 3 (d) 4

7. If p times the pth term of an A.P. is equal to q times the qth term of an A.P., then $(p + q)$ th term is

(a) 0 (b) 1 (c) 2 (d) 3

[MPPET – 1997; Karnataka CET – 2002]

8. If twice the 11th term of an A.P. is equal to 7 times of its 21st term, then its 25th term is equal to

[J & K – 2002]

(a) 24 (b) 120
(c) 0 (d) none of these

9. The sum of first n natural numbers is

[MPPET – 1984; RPET –1995]

(a) $n(n-1)$ (b) $\frac{n(n-1)}{2}$
(c) $n(n+1)$ (d) $\frac{n(n+1)}{2}$

10. The first term of an A.P. is 2 and common difference is 4. The sum of its 40 terms will be

[MNR – 1978; MPPET – 2002]

(a) 3200 (b) 1600
(c) 200 (d) 2800

11. If nth terms of two A.P.'s are $3n + 8$ and $7n + 15$, then the ratio of their 12th terms will be

[MPPET – 1986]

(a) 4/9 (b) 7/16 (c) 3/7 (d) 8/15

12. If the sum of the series 2 + 5 + 8 + 11 ...... is 60100, then the number of terms are

(a) 100 (b) 200
(c) 150 (d) 250

[MNR – 1991, DCE – 2001; MPPET – 2009]

13. The nth term of an A.P. is $3n - 1$. Choose from the following the sum of its first five terms

[MPPET – 1983]

(a) 14 (b) 35 (c) 80 (d) 40

14. The sum of integers from 1 to 100 that are divisible by 2 or 5 is

[IIT – 1984]

(a) 3000 (b) 3050 (c) 4050 (d) None

15. There are 15 terms in an arithmetic rogression. Its first term is 5 and their sum is 390. The middle term is

[MPPET – 1994]

(a) 23 (b) 26
(c) 29 (d) 32

16. If the sum of n terms of an A.P. is $2n^2 + 5n$, then the nth term will be

[RPET – 1992]

(a) $4n+3$ (b) $4n+5$
(c) $4n+6$ (d) $4n+7$

17. The sums of n terms of two arithmetic series are in the ratio $2n + 3: 6n + 5$, then the ratio of their 13th terms is

[MPPET – 2004]

(a) 53 : 155 (b) 27 : 77
(c) 29 : 83 (d) 31 : 89

18. Let T_r be the rth term of an A.P. for $r = 1, 2, 3, \ldots$ If for some positive integers m, n we have $T_m = \frac{1}{n}$ and $T_n = \frac{1}{m}$, then T_{mn} equals

(a) $\frac{1}{mn}$ (b) $\frac{1}{m} + \frac{1}{n}$
(c) 1 (d) 0

[IIT – 1998]

19. If $a_1, a_2, a_3, \ldots\ldots\ldots\ldots a_{24}$ are in arithmetic progression and $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$, then $a_1 + a_2 + a_3 + \ldots + a_{23} + a_{24} =$

[MPPET – 1999; AMU – 1997]

(a) 909 (b) 75 (c) 750 (d) 900

20. Three number are in A.P. such that their sum is 18 and sum of their squares is 158. The greatest number among them is

(a) 10 (b) 11
(c) 12 (d) none of these

[UPSEAT – 2004]

21. Three numbers are in A.P. whose sum is 33 and product is 792, then the smallest number from these numbers is

(a) 4 (b) 8
(c) 11 (d) 14

[RPET – 1988]

22. The sum of the fi rst four terms of an A.P. is 56. The sum of the last four terms is 112. If its first term is 11, the number of terms is

(a) 10 (b) 11
(c) 12 (d) none of these

23. Four numbers are in arithmetic progression. The sum of first and last term is 8 and the product of both middle terms is 15. The least number of the series is

[MPPET – 2001]

(a) 4 (b) 3
(c) 2 (d) 1

24. If a, b, c, d, e, f are in A.P., then the value of $e - c$ will be

(a) $2(c - a)$ (b) $2(f - d)$
(c) $2(d - c)$ (d) $d - c$

[Pb. CET – 1989, 1991]

25. The arithmetic mean of first n natural number

[RPET – 1986]

(a) $(n - 1)/2$ (b) $(n + 1)/2$
(c) $n/2$ (d) n

26. After inserting n A.M's between 2 and 38, the sum of the resulting progression is 200. The value of n is

[MPPET – 2001]

(a) 10 (b) 8
(c) 9 (d) none of these

27. The four arithmetic means between 3 and 23 are

[MPPET – 1985]

(a) 5, 9, 11, 13 (b) 7, 11, 15, 19
(c) 5, 11, 15, 22 (d) 7, 15, 19, 21

28. If $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ be the A.M. of a and b, then $n =$

[MPPET – 1995]

(a) 1 (b) – 1
(c) 0 (d) none of these

29. If the angles of a quadrilateral are in A.P. whose common difference is 10°, then the angles of the quadrilateral are

(a) 65°,85°,95°,105°
(b) 75°,85°,95°,105°
(c) 65°,75°,85°,95°
(d) 65°,95°,105°,115°

30. If A_1, A_2 be two arithmetic means between 1/3 and 1/24, then their values are

(a) 7/72, 5/36 (b) 17/72, 5/36
(c) 7/36, 5/72 (d) 5/72,17/72

31. If a, b, c are in A.P. then $1/bc$, $1/ca$, $1/ab$ will be in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

[MPPET – 1985; Roorkee – 1975; DCE – 2002]

32. If the pth, qth and rth term of an arithmetic sequence are a, b and c respesctively, then the value of

$[a(q - r) + b(r - p) + c(p - q)] =$

[MPPET – 1985]

(a) 1 (b) – 1 (c) 0 (d) 1/2

33. The interior angles of a polygon are in A.P. If the smallest angle be 120° and the common difference be 5°, then the number of sides is

[IIT – 1980; MPPET – 2007]

(a) 8 (b) 10 (c) 9 (d) 6

34. If S_n denotes the sum of n terms of an arithmetic progression, then the value of $(S_{2n} - S_n)$ is equal to

(a) $2S_n$ (b) S_{3n} (c) $\frac{1}{2} S_{3n}$ (d) $\frac{1}{2} S_n$

35. The sum of all two digit numbers which, when divided by 4, yield unity as a remainder is

(a) 1190 (b) 1197
(c) 1210 (d) None

36. Let S_n denotes the sum of n terms of an A.P. If $S_{2n} = 3S_n$, then ratio $\frac{S_{3n}}{S_n} =$

[MNR–93; UPSEAT – 2001]

(a) 4 (b) 6
(c) 8 (d) 10

37. Third term of an A.P. is 7 and 7th term is – 9 then find sum of n terms

[MP PET 07]

(a) $2n^2 - 17n$ (b) $2n^2 - 17$
(c) $17 - 2n^2$ (d) $17n - 2n^2$

38. If $\frac{5+9+13 \text{ .... } n \text{ terms}}{7+9+11 \text{ ..... } 12 \text{ terms}} = \frac{5}{12}$, then $n =$

[MPPET – 2009]

(a) 5 (b) 6 (c) 9 (d) 12

39. If the first, second and last terms of an arithmetic series are a, b and c respectively, then the number of terms is

(a) $\frac{b+c-2a}{b-a}$ (b) $\frac{b+c+2a}{b-a}$

(c) $\frac{b+c-2a}{b+a}$ (d) $\frac{b+c+2a}{b+a}$

[MPPET – 2009]

40. If the difference between the roots of the equation $x^2 + ax + 1 = 0$ is less than $\sqrt{5}$, then the set of possible values of a is

[AIEEE – 2007]

(a) $(-3, 3)$ (b) $(-3, \infty)$

(c) $(3, \infty)$ (d) $(-\infty, -3)$

SOLUTIONS

1. (c) **Step 1:** If a, b and c are in A.P. then $2b = a + c$

Step 2: Given $2x$, $x + 8$, $3x + 1$ are in AP

$2(x + 8) = 2x + 3x + 1$

$\Rightarrow 2x + 16 = 5x + 1$

$\Rightarrow 3x = 15 \Rightarrow x = 5$

2. (d) **Step 1:** If sum of n terms be S_n then $d = S_2 - 2S_1$

Step 2: $S_n = nA + n^2 B$

$S_1 = A + B$

$S_2 = 2A + 4B$

the common difference $(d) = S_2 - 2S_1$

$= 2A + 4B - 2A - 2B$

$= 2B$

3. (b) **Step 1:** nth term of an A.P. $= a + (n-1)d$

Step 2: 9th term = 35

$a + 8d = 35$ (1)

and 19th term = 75

$a + 18d = 75$ (2)

Solving equation (1) and (2) we get,

$d = 4$ and $a = 3$

$\therefore$ 20th term $= a + 19d$

$= 3 + 19 \times 4 = 79$

4. (c) **Step 1:** mth term of an A.P. $= a + (m-1)d$

Given series

$63 + 65 + 67 + 69 + \text{.......}$ mth term (1)

$3 + 10 + 17 + 24 + \text{.........}$ mth term (2)

Step 2: mth term of 1st series

$= 63 + (m-1)2$

$= 63 + 2m - 2 = 61 + 2m$

and mth term of 2nd series

$= 3 + (m-1)7$

$= 3 + 7m - 7 = 7m - 4$

Therefore,

$61 + 2m = 7m - 4$

$-5m = -65$

$m = 13$

5. (b) **Step 1:** The last term of an AP

$l = a + (n-1)d$

Step 2: $47 = 101 + (n-1)(-2)$

$47 = 101 - 2n + 2$

$47 - 103 = -2n$

$\Rightarrow -2n = -56$

$\Rightarrow n = 28$

6. (d) a, b, c in A.P. $\Rightarrow 2b = a + c$

$$\Rightarrow \frac{(a-c)^2}{b^2 - ac} = \frac{(a-c)^2}{\left(\frac{a+c}{2}\right)^2 - ac}$$

$$= \frac{4(a-c)^2}{(a-c)^2} = 4$$

7. (a) **Step 1:** Given that $pT_p = qT_q$ of an A.P. and we have to find $T_{p+q} = a + (p + q - 1)d$

Step 2: $p[a + (p-1)d] = q[a + (q-1)d]$

$\Rightarrow a(p - q) = d[q^2 - q - p^2 + p]$

$a(p - q) = d[(q - p)(q + p) - (q - p)]$

$a(p - q) = d(q - p)(q + p - 1)$

$a = d(1 - p - q)$

$a + (p + q - 1)d = 0$

$\therefore$ $(p + q)$th term = 0.

8. (c) 2 × 11th term = 7 × 21st term

$2(a + 10d) = 7(a + 20d)$

$2a + 20d = 7a + 140d$

$5a + 120d = 0$

$a + 24d = 0$

25th term = 0

9. (d) The Sum of 1st natural number = 1 + 2 + 3 + ... + n

can be verified for $n = 1$.

as $S_1 = T_1$. i.e., sum of one term is same as 1st term.

Now we put $n = 1$ in each of the four options and verify with that option which will give 1 for $n = 1$.

(a) $(1 - 1) = 0$ (b) 0

(c) 2 (d) 1

Hence, option (d) is correct.

10. (a) $a = 2, d = 4$ in A.P.

$S_{40} = \frac{n}{2}[2a + (n - 1)d]$

$= \left(\frac{40}{2}\right)[4 + 39 \times 4]$

$= 3200$

11. (a) $\frac{T_n}{T'_n} = \frac{3n + 8}{7n + 15} = \frac{3 \times 12 + 8}{7 \times 12 + 15} = \frac{4}{9}$

for $n = 12$

12. (b) $S = 2 + 5 + 8 + 11 + ... + t_n$

and $S = 2 + 5 + 8 + ... + t_{n-1} + t_n$

Subtracting, $0 = (2 + 3 + 3 + 3 + ...\ n \text{ terms}) - t_n$

or, $t_n = 2 + 3(n - 1) = 3n - 1$

$S_n = \sum t_n = \sum 3n - \sum 1 = 3\frac{n}{2}(n + 1) - n$

$= \frac{n}{2}(3n + 1)$

By verification Method

$60100 = \frac{n}{2}(3n + 1)$

$\Rightarrow 3n^2 + n - 2 \times 60100 = 0$

$\Rightarrow 3n^2 + n - 120200 = 0$

$\Rightarrow 3n^2 + 601n - 600n - 120200 = 0$

$\Rightarrow n(3n + 601) - 200(3n + 601) = 0$

$\Rightarrow (n - 200)(3n + 601) = 0$

$\Rightarrow n = 200$ or $n = -601$

$\Rightarrow n$ can not be negative

$\therefore n = 200$.

13. (d) $S_n = \sum t_n = \sum(3n - 1) = 3\sum n$

$= \sum 1 = \frac{3n}{2}(n + 1) - n$

$\therefore S_5 = 3 \times 5 \times \frac{6}{2} - 5 = 40$

14. (b) Sum of numbers = $S_1 + S_2 - S_3$, where

S_1 = Sum of numbers divisible by 2

$= 2 + 4 + 6 + 8 + 10 + ... + 100$

$T_n = 100 = 2 + (n - 1)2$ or $n = 50$,

$S_n = \frac{n}{2}(a + l)$ gives

$S_1 = \frac{50}{2}(2 + 100) = 2550$,

S_2 = Sum of numbers divisible by 5

$= 5 + 10 + 15 + ... + 100$;

$T_n = 100 = 5 + (n - 1)5$ or $n = 20$

$S_2 = \frac{20}{2}[5 + 100] = 1050$

S_3 = Sum of numbers divisible by 2 and 5 both i.e., by 10

$= 10 + 20 + 30 + ... + 100$;

$T_n = 100 = 10 + (n - 1)10$

or $n = 10$. $S_3 = \frac{10}{2}[10 + 100] = 550$.

$S = S_1 + S_2 - S_3$

$S = 2550 + 1050 - 550 = 3050$

15. (b) Given $n = 15$

$a = 5$

Sum = 390

$S = \frac{n}{2}[2a + (n - 1)d]$

$390 \times 2 = 15[2 \times 5 + 14d]$

$52 = 10 + 14d \Rightarrow d = 3$

$\therefore$ Middle term = 8th term

$= a + 7d = 5 + 21 = 26$.

OR

Note: $S_n = n$ (one middle term) if n is odd

$390 = 15 \times$ middle term

26 = middle term

16. (a) **Step 1:** T_n = Sum of n terms − Sum of $(n + 1)$ terms.

Step 2: $S_n = 2n^2 + 5n$

$\Rightarrow T_n = S_n - S_{n-1}$

$= (2n^2 + 5n) - \{2(n-1)^2 + 5(n-1)\}$

$= 4n + 3.$

17. (a) **Step 1:** We have $\frac{S_n^1}{S_n^2} = \frac{2n+3}{6n+5}$

$$\Rightarrow \frac{\frac{n}{2}[2a_1 + (n-1) + d_1]}{\frac{n}{2}[2a_2 + (n-1) + d_2]} = \frac{2n+3}{6n+5}$$

$$\Rightarrow \frac{2\left[a_1 + \left(\frac{n-1}{2}\right)d_1\right]}{2\left[a_2 + \left(\frac{n-1}{2}\right)d_2\right]} = \frac{2n+3}{6n+5}$$

$$\Rightarrow \frac{a_1 + \left(\frac{n-1}{2}\right)d_1}{a_2 + \left(\frac{n-1}{2}\right)d_2} = \frac{2n+3}{6n+5} \quad (1)$$

Now replacing $\frac{n-1}{2} = N - 1$ in Equation (1) we find $n = 2N - 1$ and consequently Equation (1) becomes

$$\frac{a_1 + (N-1)d_1}{a_2 + (N-1)d_2} = \frac{2(2N-1)+3}{6(2N-1)+5}$$

$$= \frac{4N+1}{12N-1} \quad (2)$$

Step 2: On replacing $N = 13$ we get,

$$\frac{a_1 + 12d_1}{a_2 + 12d_2} = \frac{4 \times 13 + 1}{12 \times 13 - 1}$$

$$\Rightarrow \frac{T_{13}^1}{T_{13}^2} = \frac{53}{155}$$

18. (c) Let the A.P. be $a + (a + d) + (a + 2d) + \ldots..$

Now $T_m = \frac{1}{n} \Rightarrow a + (m-1)\,d = \frac{1}{n}$ (1)

and $T_n = \frac{1}{m} \Rightarrow a + (n-1)\,d = \frac{1}{m}$ (2)

Solving (1) and (2), for a, d; we get,

$a = \frac{1}{mn}$, $d = \frac{1}{mn}$

$\therefore T_{mn} = a + (mn - 1)\,d$

$= \frac{1}{mn} + (mn-1)\,d = \frac{1}{mn}$

19. (d) Given $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$

$\Rightarrow (a_1 + a_{24}) + (a_5 + a_{20}) + (a_{10} + a_{15}) = 225$

$\Rightarrow 3\,(a_1 + a_{24}) = 225 \Rightarrow a_1 + a_{24} = 75$ (1)

($\because$ Sum of terms of an A.P. equidistant from beginning and end is constant and equal to the sum of first and last terms)

$\therefore$ Sum of the A.P.

$= \frac{24}{2}(a_1 + a_{24}) = 12\,(75) = 900$ by Equation (1)

20. (b) The three numbers in A.P. are $\alpha - \beta$, α, $\alpha + \beta$

Sum $= (\alpha - \beta) + \alpha + (\alpha + \beta) = 18$

$\Rightarrow 3\alpha = 18 \Rightarrow \alpha = 6$

Sum of squares = 158

$\Rightarrow (\alpha - \beta)^2 + \alpha^2 + (\alpha + \beta)^2 = 158$

$\Rightarrow (6 - \beta)^2 + 6^2 + (6 + \beta)^2 = 158$

$\Rightarrow \beta^2 = 25 \Rightarrow \beta = \pm 5$

Greatest number $= \alpha + \beta = 6 + 5 = 11$ if $\alpha\,\beta > 0$.

21. (a) Suppose that three numbers are $a + d$, a, $a - d$, therefore, $a + d + a + a - d = 33$

$\Rightarrow a = 11$ $a(a + d)\,(a - d) = 792$

$\Rightarrow 11(121 - d^2) = 792 \Rightarrow d = 7$

Then required numbers are 4, 11, 18

Hence, smallest number is 4.

22. (b) **Step 1:** Let last term of an A.P. is l.

Also first term $a = 11$ (given)

$11 + (11 + d) + (11 + 2d) + (11 + 3d) = 56$ (1)

$l + (l - d) + (l - 2d) + (l - 3d) = 112$ (2)

Step 2: $44 + 6d = 56 \Rightarrow 6d = 12 \Rightarrow d = 2$

$4l - 6d = 112 \Rightarrow 4l - 12 = 112$

$4l = 124 \Rightarrow l = 31$

Step 3: $l = a + (n - 1)d$

$31 = 11 + (n - 2)^2$

$20 = 2n - 2$

$2n = 22 \Rightarrow n = 11$

23. (d) Four numbers in A.P. are $a - 3d$, $a - d$, $a + d$, $a + 3d$.

According to the assumption, $T_1 + T_4 = 8$; $T_2\,T_3 = 15$

This $\Rightarrow (a - 3d) + (a + 3d) = 8$,

$(a-d)(a+d) = 15$

$\Rightarrow 2a = 8, a^2 - d^2 = 15 \Rightarrow a = 4, 4^2 - d^2 = 15$ or $d = 1$.

$a - 3d, a - d, a + d, a + 3d = 4 - 3, 4 - 1, 4 + 1, 4 + 3$

$= 1, 3, 5, 7$ Least number = 1.

24. (c) **Step 1:** Given $b - a = c - b = d - c = e - d = f - e$ = common difference.

Step 2: $e - c = (e - d) + (d - c)$

= Common difference + common difference

= 2 × common difference

25. (b) $\text{A.M} = \dfrac{1+2+3+\ldots.+n}{n}$

$$= \frac{\frac{1}{2}n(n+1)}{n} = \frac{n+1}{2}$$

26. (b) If n arithmetic means are inserted between 2 and 38 then total number of terms $= n + 2$

$\therefore\ a = 2, l = 38, N = n + 2$ and $s = 200$

$s = \dfrac{N}{2}(a + l)$

$200 = \dfrac{n+2}{2}(2 + 38) \Rightarrow n = 8$

27. (b) $3, A_1, A_2, A_3, A_4, 23$ are in AP

$23 = T_6 = 3 + (6-1)d \Rightarrow d = \dfrac{20}{5} = 4$

$A_1 = 3 + 4 = 7, A_2 = 7 + 4 = 11,$

$A_3 = 11 + 4 = 15, A_4 = 15 + 4 = 19.$

28. (c) $(a + b)(a^n + b^n) = 2(a^{n+1} + b^{n+1})$

$a^{n+1} + ab^n + ba^n + b^{n+1} = 2a^{n+1} + 2b^{n+1}$

$ab^n + ba^n = a^{n+1} + b^{n+1}$

$ab^n - b^{n+1} = a^{n+1} - ba^n$

$b^n(a - b) = a^n(a - b)$

$(a - b)\{a^n - b^n\} = 0$

Since, $a - b \neq 0$

$\therefore\ a^n - b^n = 0$

$\Rightarrow a^n = b^n$

$\Rightarrow n = 0$

OR

Verification Method

$$\frac{a+b}{2} = \frac{a^{n+1}+b^{n+1}}{a^n+b^n}$$

$\Rightarrow (a - b)(a^n - b^n) = 0$

Hence, $n = 0$.

29. (b) Suppose that $\angle A = x^0$, then $\angle B = x + 10°$,

$\angle C = x + 20°$ and $\angle D = x + 30°$

So, we know that $\angle A + \angle B + \angle C + \angle D = 2\pi$

Putting these values, we get,

$(x°) + (x° + 10°) + (x° + 20°) + (x° + 30°) = 360°$

$\Rightarrow x = 75°$

Hence, the angles of the quadrilateral are $75°, 85°, 95°, 105°$.

Trick: In these type of questions, students should satisfy the conditions through options. Here (b) satisfies both the conditions i.e., angles are in A.P. with common difference 10° and sum of angles is 360°.

30. (b) **Step 1:** Arithmetic mean between two numbers a and b is

$\text{AM} = \dfrac{a+b}{2}$

Step 2: $\dfrac{1}{3}, A_1, A_2, \dfrac{1}{24}$

Given $a = \dfrac{1}{3}, l = \dfrac{1}{24}, n = 4$

$\therefore\ l = a + (n-1)d$

$\dfrac{1}{24} = \dfrac{1}{3} = 3d$

$d = \dfrac{1-8}{3 \times 24} = \dfrac{-7}{72}$

First arithmetic

$A_1 = a + d = \dfrac{1}{3} - \dfrac{7}{12} = \dfrac{24-7}{72} = \dfrac{17}{72}$

Second arithmetic

$A_2 = a + 2d = \dfrac{1}{3} - \dfrac{14}{72} = \dfrac{24-14}{72} = \dfrac{10}{72} = \dfrac{5}{36}$

31. (a) Given a, b, c in AP

$\Rightarrow \dfrac{a}{abc}, \dfrac{b}{abc}, \dfrac{c}{abc}$ in A.P

$\Rightarrow \dfrac{1}{bc}, \dfrac{1}{ca}, \dfrac{1}{ab}$ in A.P

32. (c) **Step 1:** $a = T_p = x + (p-1)y$ (1)

$b = T_q = x + (q-1)y$ (2)

$c = T_r = x + (r-1)y$ (3)

Step 2: On solving (1) − (2), (2) − (3) and (3) − (1) we get,

$a - b = (p - q)y$ (4)

$b - c = (q - r)y$ (5)

$c - a = (r - p)y$ (6)

Step 3: Multiplying equations (4), (5) and (6) by c, a and b and adding we get,

$a(q - r) + b(r - p) + c(p - q) = 0$

33. (c) **Step 1:** If number of sides in a polygon is n, then sum of total number of internal angles

$= (2n - 4)\,90° = (n - 2)\pi = S_n$ (say)

Step 2:

$S_n = \frac{n}{2}[2a + (n - 1)d]$ gives $90(2n - 4)$

$= \frac{n}{2}[2 \times 120 + (n - 1)5]$

or $n^2 - 25n + 144 = 0$ or $(n - 16)(n - 9) = 0$

or $n = 16, 9$.

Also $T_{16} = 120 + (16 - 1)(5) = 195$.

But any internal angle is always less than 180°.

$\therefore$ $n = 16$ is not possible.

34. (c) $S_{2n} - S_n = \frac{2n}{2}\{2a + (2n - 1)d\}$

$-\frac{n}{2}\{2a + (n - 1)d\}$

$= \frac{n}{2}\{4a + 4nd - 2d - 2a - nd + d\}$

$= \frac{n}{2}\{2a + (3n - 1)d\}$

$= \frac{1}{3}\cdot\frac{3n}{2}\{2a + (3n - 1)d\} = \frac{1}{3}S_{3n}$

35. (c) Number of type $= 4n + 1$ for $n = 1, 2, 3$

Two digit numbers 13, 17, 21,.... 97.

$T_n = 97 = 13 = 13 + (n - 1)4$ or $n = 22$

$S_{22} = \frac{22}{2}[13 + 97] = 11\,(110) = 1210$

36. (b) Let the A.P. be $a + (a + d) + (a + 2d) + ...$

Given $\frac{S_{2n}}{S_n} = 3$

$\Rightarrow \frac{\frac{2n}{2}\{2a + (2n - 1)d\}}{\frac{n}{2}\{2a + (n - 1)d\}} = 3$

$\Rightarrow 4a + 4nd - 2d = 6a + 3nd - 3d$

$\Rightarrow 2a = nd + d = (n + 1)d$ (1)

$\therefore \frac{S_{2n}}{S_n} = \frac{\frac{3n}{2}\{2a + (3n - 1)d\}}{\frac{n}{2}\{2a + (n - 1)d\}}$

$= \frac{3\{(n + 1)\,d + (3n - 1)d\}}{(n + 1)\,d + (n - 1)d}$ (using (1))

$= \frac{3(4nd)}{2nd} = 6$

37. (d) $T_n = a + (n - 1)d$

Given $T_3 = a + 2d = 7$ (1)

and $T_7 = a + 6d = -9$ (2)

From Equation (1) and (2)

$4d = -16$ or $d = -4$ and $a = 15$

$S_n = \frac{n}{2}[2 \times 15 + (n - 1)(-4)]$

$= n(15 - 2n + 2)$

$= n(17 - 2n) = 17n - 2n^2$

38. (b) Let $S_1 = 5 + 9 + 13 + ... + n$ terms

$\Rightarrow S_1 = \frac{n}{2}[2 \times 5 + (n - 1)4] = n(3 + 2n)$

and $S_2 = 7 + 9 + 11 + ...+ 12$ terms

$= \frac{12}{2}[2 \times 7 + (12 - 1)2]$

$= 6\,[36]$

$= 216$

Since, $\frac{S_1}{S_2} = \frac{5}{12}$ (given)

$\Rightarrow \frac{n(3 + 2n)}{216} = \frac{5}{12}$

$\Rightarrow 2n^2 + 3n - 90 = 0$

$\Rightarrow (2n + 15)(n - 6) = 0$

$\Rightarrow n = 6$ ($\because$ n cannot be negative)

39. (a) Since, $l = A + (n - 1)d$

$\therefore c = a + (n - 1)(b - a)$

$\Rightarrow (n - 1) = \frac{c - a}{b - a}$

$\Rightarrow n = \frac{b + c - 2a}{b - a}$

40. (a) $(\alpha + \beta)^2 - 4\alpha\beta < 5 \Rightarrow a^2 - 4 < 5$

$\Rightarrow a \in (-3, 3)$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. If the pth term of an A.P. be q and qth term be p, then its rth term will be **[RPET – 1999]**

(a) $p + q + r$ (b) $p + q - r$
(c) $p + r - q$ (d) $p - q - r$

2. The sum of all natural numbers between 1 and 100 which are multiples of 3 is **[MPPET – 1984]**

(a) 1680 (b) 1683 (c) 1681 (d) 1682

3. The sum of 1 + 3 + 5 + 7 + ... upto n terms is **[MPPET–84]**

(a) $(n + 1)^2$ (b) $(2n)^2$
(c) n^2 (d) $(n - 1)^2$

4. If $S_n = nP + \frac{1}{2} n (n - 1)Q$, where S_n denotes the sum of the first n terms of an A.P., then the common difference is **[WBJEE – 1994]**

(a) $P + Q$ (b) $2P + 3Q$
(c) $2Q$ (d) Q

5. The number of terms of the A.P. 3, 7, 11, 15........... to be taken so that the sum is 406 is **[Kerala Engg. – 2002]**

(a) 5 (b) 10 (c) 12 (d) 14

6. If the sum of the series 54 + 51 + 48 + ... is 513, then the number of terms are

(a) 18 (b) 20
(c) 17 (d) none of these

[Roorkee – 1970]

7. The sum of the numbers between 100 and 1000 which is divisible by 9 will be **[MPPET – 1982]**

(a) 55350 (b) 57228
(c) 97015 (d) 62140

8. The sum of numbers from 250 to 1000 which are divisible by 3 is **[RPET – 1997]**

(a) 135657 (b) 136557
(c) 161575 (d) 156375

9. If the sum of the first $2n$ terms of 2, 5, 8...... is equal to the sum of the first n terms of 57, 59, 61 ....., then n is equal to **[IIT Screening – 2001]**

(a) 10 (b) 12 (c) 11 (d) 13

10. 7th term of an A.P. is 40, then the sum of first 13 terms is **[Karnataka CET – 2003]**

(a) 53 (b) 520
(c) 1040 (d) 2080

11. The sum of n terms of an A.P. is $3n^2 - n$, 10th term is

(a) 62 (b) 56
(c) 74 (d) 290

12. If the pth term of an A.P. be $1/q$ and qth term be $1/p$, then the sum of its pq terms will be

(a) $\frac{pq-1}{2}$ (b) $\frac{1-pq}{2}$
(c) $\frac{pq+1}{2}$ (d) $-\frac{pq+}{2}$

13. Sum of the first 50 positive integers will be

(a) 1200 (b) 1300
(c) 1375 (d) 1275

14. In an A.P. the sum of the terms equidistant from the beginning and end is equal to

(a) first term
(b) second term
(c) sum of first and last term
(d) last term

15. The sum of the first and third term of an arithmetic progression is 12 and the product of first and second term is 24, then first term is

(a) 1 (b) 8
(c) 4 (d) 6

[MPPET – 2003]

16. The sum of n arithmetic means between a and b, is

[RPET–1986]

(a) $n(a + b)/2$ (b) $n(a + b)$

(c) $(n + 1)(a + b)/2$ (d) $(n + 1)(a + b)$

17. The mean of the series $a, a + nd, a + 2nd$ is

[DCE – 2002]

(a) $a + (n - 1)d$ (b) $a + nd$

(c) $a + (n + 1)d$ (d) none of these

18. If the sum of three numbers of a arithmetic sequence is 15 and the sum of their squares is 83, then the numbers are

[MPPET – 1985

(a) 4, 5, 6 (b) 3, 5, 7

(c) 1, 5, 9 (d) 2, 5, 8

19. If the sum of three consecutive terms of an A.P. is 51 and the product of last and first term is 273, then the numbers are

[MPPET – 1986]

(a) 21, 17, 13 (b) 20, 16, 12

(c) 22, 18, 14 (d) 24, 20, 16

20. The sums of n terms of three A.P's whose fi rst term is 1 and common differences are 1, 2, 3 are S_1, S_2, S_3 respectively. The true relation is

(a) $S_1 + S_3 = S_2$ (b) $S_1 + S_3 = 2S_2$

(c) $S_1 + S_2 = 2S_3$ (d) $S_1 + S_2 = S_3$

21. If a, b and c are in A.P., then which one of the following is not true?

(a) $\frac{k}{a}, \frac{k}{b}$ and $\frac{k}{c}$ are in H.P

(b) $a + k$, $b + k$ and $c + k$ are in A.P.

(c) ka, kb and kc are in A.P

(d) a^2, b^2 and c^2 are in A.P.

22. If the first, second and last terms of an arithmetic series are a, b and c repectively, then the number of terms will be

(a) $\frac{1}{2}(a + b + c)$ (b) $\frac{1}{2}(b + c - 2a)$

(c) $\frac{b + c - 2a}{b - a}$ (d) None

23. If the sides of a right angled triangle are in A. P., then the sides are proportional to

[Roorkee – 1974]

(a) 1: 2: 3 (b) 2: 3: 4

(c) 3: 4: 5 (d) 4: 5: 6

24. If A is one AM between two numbers a and b, and the sum of n AM's between them is S, then S / A depends on

[CET (Pb.) – 1992]

(a) n, a, b (b) n, b

(c) n, a (d) n

25. If the fourth term of an A.P. 13, then the sum of first seven terms is

(a) 13^7 (b) 7×13

(c) 7^{13} (d) none of these

26. The sum of first n (odd) terms of an A.P. whose middle term is m is

(a) mn (b) m^n

(c) n^m (d) None of these

27. $(p + q)$th term of an A.P. is m and $(p - q)$th term is n, then the pth term is

(a) $\frac{1}{2}(m - n)$ (b) mn

(c) $\sqrt{mn}$ (d) $\frac{1}{2}(m + n)$

28. If a, b, c are in A.P. then $\frac{a - b}{b - c}$ is equal to

(a) a/a (b) a/b

(c) a/c (d) none of these

29. If a^{-1}, b^{-1}, c^{-1} are in A.P. $(abc \neq 0)$, then $\frac{a - b}{b - c}$ is equal to

(a) a/a (b) a/b

(c) a/c (d) none of these

WORK SHEET: TO CHECK PREPARATION LEVEL

Important Instructions:

1. The answer sheet is immediately below the work sheet
2. The test is of 15 minutes.
3. The test consists of 15 questions.The maximum marks are 45.

4. Use blue/black ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. If the sum of two extreme numbers of an A.P. with four terms is 8 and product of remaining two middle term is 15, then greatest number of the series will be

[Roorkee – 1965]

(a) 5 (b) 7
(c) 9 (d) 11

2. There are n A.Ms. between 1 and 31. If the ratio of 7th A.M. to $(n-1)$th A.M. is 5 : 9, then the value of n is.

(a) 13 (b) 14
(c) 15 (d) none of these

3. If the sum of first n natural numbers is 1/5 times the sum of their squares, then n equals

[IIT – 1992]

(a) 5 (b) 6
(c) 7 (d) 8

4. If $a\left(\frac{1}{b}+\frac{1}{c}\right)$, $b\left(\frac{1}{c}+\frac{1}{a}\right)$ and $c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in AP, then

(a) a, b, c are in AP (b) a, b, c are in HP
(c) $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in GP (d) none of these

[DCE – 1997; Delhi (EEE) – 1998]

5. If the sum of 40 A.M.s between two numbers is 120, then the sum of 50 A.M.s between them is equal to

(a) 130 (b) 160
(c) 150 (d) none of these

6. If $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in A.P. then

[Kerala PET – 2007]

(a) a, b, c are in A.P.
(b) c, a, b are in A.P.
(c) a^2, b^2, c^2 are in A.P
(d) a, b, c are in G.P.

7. If the sum of the first n terms of a series be $5n^2 + 2n$, then its second term is

[MPPET – 1996]

(a) 7 (b) 17 (c) 24 (d) 42

8. The sum of the series $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}+\ldots\ldots+$ to 9 terms is

[MNR – 1985]

(a) $-\frac{5}{6}$ (b) $-\frac{1}{2}$
(c) 1 (d) $-\frac{3}{2}$

9. If $\frac{3+5+7+\ldots\ldots \text{to } n \text{ terms}}{5+8+11+\ldots\ldots \text{to } 10 \text{ terms}} = 7$, then the value of n is

[MNR – 1983; Pb CET – 2000]

(a) 35 (b) 36 (c) 37 (d) 40

10. If a, b, c are in A.P., then $(a + 2b - c)(2b + c - a)(c + a - b)$ equals

[Pb. CET – 1999]

(a) $abc/2$ (b) abc
(c) $2\,abc$ (d) $4\,abc$

11. If the ratio of the sum of n terms of two A.P.'s be $(7n + 1) : (4n + 27)$, then the ratio of their 11th terms will be

[AMU – 1996]

(a) 2 : 3 (b) 3 : 4
(c) 4 : 3 (d) 5 : 6

12. The solution of the equation
$(x+1)+(x+4)+(x+7)+\ldots+(x+28)=155$
is given by x is equal to

(a) 1 (b) 2
(c) 3 (d) 4

13. The first term of an A.P. consecutive integers is $p^2 + 1$. The sum of $(2p + 1)$ terms of this series can be expressed as

(a) $(p + 1)^2$ (b) $(p + 1)^3$
(c) $(2p + 1)(p + 1)^2$ (d) $p^3 + (p + 1)^3$

14. If a^2, b^2, c^2 are in A.P., then $(b + c)^{-1}$, $(c + a)^{-1}$ and $(a + b)^{-1}$ will be in

[Roorkee – 1968, RPET – 1996]

(a) H.P. (b) G.P.
(c) A.P . (d) none of these

15. If the sum of 12th and 22nd terms of an A.P. is 100, then the sum of the first 33 terms of the A.P. is

[Kerala PET – 2008]

(a) 1700 (b) 1650 (c) 3300 (d) 3400
(e) 3500

ANSWER SHEET

1. (a) (b) (c) (d)	6. (a) (b) (c) (d)	11. (a) (b) (c) (d)
2. (a) (b) (c) (d)	7. (a) (b) (c) (d)	12. (a) (b) (c) (d)
3. (a) (b) (c) (d)	8. (a) (b) (c) (d)	13. (a) (b) (c) (d)
4. (a) (b) (c) (d)	9. (a) (b) (c) (d)	14. (a) (b) (c) (d)
5. (a) (b) (c) (d)	10. (a) (b) (c) (d)	

HINTS AND EXPLANATIONS

1. (b) Four numbers in A.P.: $a - 3d, a - d, a + d, a + 3d$ .... (1)

Under condition I: $(a - 3d) + (a + 3d) = 8$ or $a = 4$

Under condition II: $(a - d)(a + d) = 15$

$\Rightarrow 4^2 - d^2 = 15$

$\Rightarrow d = 1$. Put $a = 4$, $d = 1$ in (1), we get,

$4 - 3, 4 - 1, 4 + 1, 4 + 3 = 1, 3, 5, 7$

Hence, the greatest number = 7.

2. (b) Suppose between 1 and 31 there are n A.M.'s $A_1, A_2, A_3, \ldots, A_n$ from let $1, A_1, A_2, A_3, \ldots, A_n, 31$ are in AP.

31 being the $(n + 2)$th term of the AP

Let common difference be d then

$a + (n + 2 - 1)d = 31$

$\therefore 1 + (n + 1)d = 31$

$(n + 1)d = 30$

$d = \dfrac{30}{n+1}$

$\therefore$ 7th AM = 8th term = $a + 7d$

$= 1 + 7\left(\dfrac{30}{n+1}\right) = \dfrac{n+211}{n+1}$

$\therefore$ $(n - 1)$th AM = nth term

$= a + (n - 1)d$

$= 1 + \dfrac{(n-1)\times 30}{n+1} = \dfrac{31n - 29}{n+1}$

As per question,

$\Rightarrow \dfrac{7th\ Question}{(n-1)^{th} AM} = \dfrac{5}{9}$

$\Rightarrow \dfrac{211 + n}{31n - 29} = \dfrac{5}{9}$

$\Rightarrow 155n - 145 = 1899 + 9n$

$\Rightarrow 155n - 9n = 1899 + 145$

$\Rightarrow 146\,n = 2044 \Rightarrow n = 14.$

3. (c) $\sum\limits_{n=1}^{n} n = \dfrac{1}{5}\sum\limits_{n=1}^{n} n^2$

$\Rightarrow \dfrac{n}{2}(n + 1) = \dfrac{1}{5}\cdot\dfrac{n}{6}(n + 1)(2n + 1)$

$\Rightarrow 15 = 2n + 1 \Rightarrow n = 7.$

4. (a) Since,

$a\left(\dfrac{1}{b}+\dfrac{1}{c}\right), b\left(\dfrac{1}{c}+\dfrac{1}{a}\right), c\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$ are in A.P

$\therefore \dfrac{a(b + c)}{bc}, \dfrac{b(c + a)}{ca}, \dfrac{c(a + b)}{ab}$ are in A.P.

$\Rightarrow a^2(b + c), b^2(c + a), c^2(a + b)$ are in AP

$\Rightarrow b^2(c + a) - a^2(b + c) = c^2(a + b) - b^2(c + a)$

$\Rightarrow b^2c - a^2c + b^2a - a^2b$

$= c^2a - b^2a + c^2b - b^2c$

$\Rightarrow c(b^2 - a^2) + ab(b - a)$

$= a(c^2 - b^2) + cb(c - b)$

$\Rightarrow (b - a)(c(b + a) + ab)$

$= (c - b)[a(c + b) + cb]$

$\Rightarrow (b - a)(bc + ca + ab)$

$= (c - b)(ac + ab + bc)$

$\Rightarrow b - a = c - b \Rightarrow a, b, c$ are in A.P.

5. (c) The sum of n A.M.'s between two numbers is n times the single A.M. between them.

$\Rightarrow$ Sum of 50 A.M. between two numbers = $50 \times 3 = 150$

6. (c) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in A.P

$\Rightarrow \frac{a}{a+c}+1, \frac{b}{c+a}+1, \frac{c}{a+b}+1$ are in A.P

$\Rightarrow \frac{a+b+c}{b+c}, \frac{b+c+a}{c+a}, \frac{c+a+b}{a+b}$, are in A.P

$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

(Assuming that $a+b+c \neq 0$)

$\Rightarrow \frac{1}{c+a}-\frac{1}{b+c}=\frac{1}{a+b}-\frac{1}{c+a}$

$\Rightarrow \frac{b-a}{(c+a)(b+c)}=\frac{c-b}{(a+b)(c+a)}$

$\Rightarrow b^2-a^2=c^2-b^2$.

7. (b) $\because$ Sum of n terms $= 5n^2+2n$

put $n = 1$, Sum of 1st term, (i.e.)

$T_1 = 5+2 = 7$

put $n = 2$ Sum of 2 terms i.e., $T_1 + T_2 = 24$

$\therefore T_2$ = Sum of 2 terms − sum of 1 term

$T_2 = 24 - 7 = 17$

8. (d) The given series $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}+\ldots\ldots$ is an A.P. series

$\left[Here\ a=\frac{1}{2}, d=\frac{1}{3}-\frac{1}{2}=-\frac{1}{6}\right]$

$S_9 = \frac{9}{2}[2a+(9-1)d]$

$= \frac{9}{2}\left[2\times\frac{1}{2}+8\left(-\frac{1}{6}\right)\right]$

$= \frac{9}{2}\left[1-\frac{3}{4}\right]=\frac{9}{2}\left(-\frac{1}{3}\right)=-\frac{3}{2}$

9. (a) Since, $\frac{3+5+7+\ldots\ldots n \text{ terms}}{5+8+11+\ldots\ldots 10 \text{ terms}}=7$

$\therefore \frac{\frac{n}{2}[2\times 3+(n-1)2]}{\frac{10}{2}[2\times 5+(10-1)3]}$

or $\frac{n[6+2n-2]}{[10(10+27)]}=7$

or $n[2n+4] = 7 \times 370 = 2590$

or $n(n+2) = 1295$

or $n^2+2n-1295 = 0$

or $(n-35)(n+37) = 0$

or $n = 35$

[$\because n = -37$ is not possible]

10. (d) Given a, b, c are in AP

$\therefore 2b = a+c \Rightarrow b = \frac{a+c}{2}$

$(a+2b-c)(2b+c-a)(c+a-b)$

$= (a+a+c-c)(a+c+c-a)(2b-b)$

$2a \times 2c \times b = 4abc$

11. (c) $\frac{S_n}{S'_n}=\frac{a+[(n-1)/2]d}{a+[(n-1)/2]d'}=\frac{7n+1}{4n+27}$ (1)

$\therefore \frac{T_p}{T'_p}=\frac{a+(p-1)d}{a'+(p-1)d'}$

Put $\frac{n-1}{2}=p-1$

or $n = 2p-1$ in (1)

$\therefore \frac{T_p}{T'_p}=\frac{7(2p-1)+1}{4(2p-1)+27}=\frac{14p-6}{8p+23}$

Now replacing p by n in the above

$\frac{T_n}{T'_p}=\frac{14n-6}{8n+23}$

Put $n = 11$ in above equation

$=\frac{14\times 11-6}{8\times 11+23}=\frac{154-6}{88+23}=\frac{148}{111}=\frac{4}{3}$

12. (a) We have $(x+1)+(x+4)+\ldots\ldots+(x+28) = 155$

Let n be the number of terms in the A.P. on L.H.S.

Then $x+28 = (x+1)+(n-1)3 \Rightarrow n = 10$

$\therefore (x+1)+(x+4)+\ldots\ldots+(x+28) = 155$

$\Rightarrow \frac{10}{2}[(x+1)+(x+28)] = 155 \Rightarrow x = 1.$

13. (d) $S_{2p+1}=\frac{2p+1}{2}$

$[2(p^2+1)+(2p+1-1)1]$

$=\frac{2p+1}{2}[2(p^2+1)+2p]$ (Here $d = 1$)

$= (2p+1)(p^2+p+1)$

$= 2p^3+2p^2+2p+p^2+p+1$

$= 2p^3+3p^2+3p+1$

$= p^3+(p^3+3p^2+3p+1) = p^3+(p+1)^3$

14. (c) $a^2 + (ab + bc + ca)$, $b^2 + (ab + bc + ca)$,
$c^2 + (ab + bc + ca)$ in A.P.
$\Rightarrow (a + b)(a + c), (a + b)(b + c), (c + b), (c + a)$ in A.P.
Dividing, by $(a + b)(b + c)(c + a)$,
we get, $(b + c)^{-1}, (c + a)^{-1}, (a + b)^{-1}$ in A.P.

15. (b) We know that
$a_1 + a_{33} = a_{12} + a_{22}$
$\therefore a_1 + a_{33} = 100$

$$S_{33} = \frac{33}{2}(a_1 + a_{33}) = \frac{33}{2} \times 100$$

$= 1650$

This page is intentionally left blank.

LECTURE

2 Geometric Progression

BASIC CONCEPTS

1. **Geometric Progression**

A series in which each term is same multipe of the preceeding term is called a geometric progression i.e. a series in which the ratio of successive terms is constant is called a G.P. This constant ratio is called common ratio and is denoted by r.

For Example

(1) 2, $\sqrt{2}$, 1, ... (Common ratio = $1/\sqrt{2}$)

(2) 1, 3, 9, 27, ... (Common ratio = 3)

(3) $1, -\frac{1}{3}, \frac{1}{9}, \frac{1}{27}$ .... (Common ratio $= -\frac{1}{3}$)

Note: If T_1, T_2, T_3, ..., T_n, ...are in G.P., then the common ratio $= r = \frac{T_2}{T_1} = \frac{T_3}{T_2} = \ldots$ and $\frac{1}{r} = \frac{T_1}{T_2} = \frac{T_2}{T_3} = \cdots$

2. ***n* term of General G.P.** a, ar, ar^2, ar^3, ..., ar^{n-1}, ...

Note:

(i) First term of G.P. = a

(ii) 1st, 2nd, 3rd and nth terms of an G.P. are denoted by T_1, T_2, T_3 and T_n respectively.

(iii) nth term of an G.P. = $T_n = ar^{n-1} = 1$

(iv) nth term from the end of an G.P. = $(m - n + 1)$th term from the beginning. Where m = Total number of terms of an G.P.

(v) Three numbers a, b, c are in G.P. if and only if $b^2 = ac$. i.e. $\frac{b}{a} = \frac{c}{b}$.

3. **Middle term: Same as A.P.**

4. **Sum of n terms of a G.P.** (i) $S_n = a + ar + ar^2 + \ldots\ldots\ldots. + ar^{n-1} = a \cdot \frac{r^n - 1}{r - 1}$, $r > 1$ or $r < 1$

(ii) If $T_n = ar^{n-1} = l$, then $S_n = \frac{lr - a}{r - 1}$; $r > 1$ or $r < 1$

5. **Sum of an infinite number of terms of an G.P.**

$S_\infty = a + ar + ar^2 + \ldots.. \infty$, $S_\infty = \frac{a}{1 - r}$; $|r| < 1$

Note: where $r > 1$ or $r = 1$, then the sum of an infinite terms of an G.P. is also infinite.

6. **Selection of Terms in G.P.**

(i) Three consecutive term: $\frac{a}{r}$, a, ar or a, ar, ar^2

(ii) Four consecutive term: $\frac{a}{r^3}$, $\frac{a}{r}$, ar, ar^3 or a, ar, ar^2, ar^3

(iii) If the terms of a given G.P. are chosen at regular intervals (i.e. in A.P.) then they are also in G.P.

7. **GEOMETRIC MEAN** If the three numbes a, b, c are in G.P., then b is called the Geometric mean between numbers a and b.

Thus, $\frac{b}{a} = \frac{c}{b} \Rightarrow b^2 = ac \Rightarrow \sqrt{ac} = b$

Similarly the Geometric mean of n numbers $x_1, x_2, ...x_n$ is G.M. $= (x_1 x_2 x_3 .... x_n)^{1/n}$

Note: (i) a, b and $\sqrt{ab}$ i.e. a and b and their Geometric mean are of the same sign.

(ii) If a and b are opposite sign then their geometric mean is not defined.

(iii) If $G_1, G_2, ... G_n$ be the n geometric means between two numbers a and b.

Then $a, G_1, G_2, G_3, ... G_n, b$ are in G.P. and $G_n = (n + 1)$th term $= n$th G.M.

$b = (n + 2)$th term and the common ratio is

$r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$ i.e., the common ratio for inserting n Geometric means between two numbers a and $b = r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

(iv) Product of n geometric means between a and $b = (\sqrt{ac})^n =$ (only one G.M. between a and $b)^n$

8. Relation Between A.M. and G.M.

If A and G are respectively A.M. and G.M. between two numbers a and b i.e.

$\sim A = \frac{a+b}{2}$ and $G = \sqrt{ab}$, then

(i) $A > G$ if $a \neq b$

(ii) $A = G$ if $a = b$

(iii) $A \geq G$ for all a and b

(iv) $a = A + \sqrt{A^2 - G^2}$; $b = A - \sqrt{A^2 - G^2}$ and $a : b = (A + \sqrt{A^2 - G^2})\,(A - \sqrt{A^2 - G^2})$

(v) If a and b are positive, then $a + b \geq 2\sqrt{ab}$ i.e., the minimum value of $a + b = 2\sqrt{ab}$

(vi) In an finite G.P. the product of two terms equidistant from the beginning and the end is constant and it is equal to the product of the first and the last terms i.e., if

(i) a, G_1, G_2, b is an G.P., then $ab = G_1 G_2$

(ii) $a_r \cdot a_{n-(r-1)} = a_1 \cdot a_n$

(iii) The each term of GP (except the first term) is equal to the square root of the product of eqidostant terms.

i.e. $a_n = \sqrt{a_{n-r}\, a_{n+r}}$

9. Properties of G.P.

(i) If each term of a G.P. be multiplied or divided by the same non–zero number, then the resulting series is also a G.P. i.e. if $g_1, g_2, g_3, ..., g_n, ...$are in G.P. and k is a non–zero number, then

(a) $kg_1, kg_2, kg_3, ... kg_n ...$ are in G.P.

(b) $\frac{g_1}{k}, \frac{g_2}{k}, \frac{g_3}{k}, ... \frac{g_n}{k}$ are also in G.P.

(ii) The reciprocals of the term of a G.P. also form a G.P. i.e. if a, b, c are in a G.P., then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in G.P.

(iii) If each term of a G.P. be raised to the same power, the resulting numbers also forms a G.P. i.e. if a, b, c are in a G.P., then a^n, b^n, c^n are also in G.P.

(iv) Three numbers a, b, c are in G.P. if and only if $b^2 = ac$

(v) If the set of positive numbers $a_1, a_2, a_3, ..., a_n, ...$are in G.P., then $\log a_1, \log a_2, \log a_3, ..., \log a_n, ...$are in A.P. and vice– versa.

(vi) Term by term multiplication or division of two G.P.'s are also in G.P. i.e. $a_1, a_2, a_3, \ldots, a_n$, and $b_1, b_2, b_3, \ldots, b_n$, are in G.P., then (i) $a_1b_1, a_2b_2, a_3b_3, ...$, and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{a_3}, ...$ are also in G.P.

(vii) **Solution by inspection**: Problems based on n i.e. number of terms of sequence should be solved by the method of verification using four alternative or options.

SOLVED SUBJECTIVE PROBLEMS: (CBSE/STATE BOARD): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. pth terms of the series 1,2,4,8, ... and 256, 128, ... are equal. Find the value of p.

Solution

Given G.P.s. are: 1, 2, 4, 8,........and 256, 128, ... whose pth terms are respectively

$T_p = (1)\left(\frac{2}{1}\right)^{p-1} = 2^{p-1}$ and

$t_p = (256)\left(\frac{128}{256}\right)^{p-1} = (256)\left(\frac{1}{2}\right)^{p-1} = \frac{256}{2^{p-1}}$

According to question, $T_p = t_p$

$\Rightarrow 2^{p-1} = \frac{256}{2^{p-1}} \Rightarrow 2^{p-1} \times 2^{p-1} = 256$

$\Rightarrow 2^{2p-2} = 2^8$

$\Rightarrow 2p - 2 = 8 \Rightarrow 2p = 10 \Rightarrow p = 5$ Ans.

2. In an increasing G.P., the sum of the first and the last term is 66, the product of the second and the last but one term is 128, and the sum of all the term is 126. How many terms are there in the progression?

[M.N.R. −1993]

Solution

(a) $n = 6$. G.P. is increasing $r > 1$,

$a + ar^{n-1} = 66$ (1)

$ar - ar^{n-2} = 128$ or $a^2 r^{n-1} = 128$

Put $ar^{n-1} = \frac{128}{a}$ in (1)

Putting in (1), we get $a + \frac{128}{a} = 66$

$\therefore a^2 - 66a + 128 = 0$ or $(a - 2)(a - 64) = 0$

$\therefore a = 2, 64, r^{n-1} = 32, 1/32$

We reject the second value as $r > 1$

$\therefore r^{n-1} = 32$

Sum $= \frac{a(r^n - 1)}{r - 1} = 126$ or $\frac{2(32r - 1)}{r - 1} = 126$

$r^{n-1} = 32 \therefore 32r - 1 = 63r - 63 \therefore r = 2$ and

$r^{n-1} = 32$ gives $2^{n-1} = 2^5 \therefore n - 1 = 5$ or $n = 6$

3. If a, b, c are in G.P., then

[RPET −1995]

(a) a^2, b^2, c^2 are in G.P.

(b) $a^2(b + c), c^2(a + b), b^2(a + c)$ are in G.P.

(c) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in G.P.

(d) none of the above

Solution

(a) a, b, c are in G.P.

$\therefore \frac{b}{a}, \frac{c}{b} = r \Rightarrow \frac{b^2}{a^2} = \frac{c^2}{b^2} = r^2$

$\Rightarrow a^2, b^2, c^2$ are in G.P.

4. A ball is dropped from the height of 48 metres and it bounced $\frac{2}{3}$ of this height. It falls and bounced in same way continuously. Find the total distance covered by the ball before coming to rest.

Solution

When ball is dropped from the height of 48 metres the covered distance = 48 metres

The covered distance in first bounce $\frac{2}{3}(48)$ metres

The covered distance in second bounce =

$\frac{2}{3} \times \frac{2}{3}(48)$ metres

The covered distance in third bounce =

$\frac{2}{3} \times \left(\frac{2}{3}\right)^2 (48)$ metres

This process is being infinitely.

Therefore, total distance

= 48 + 2

$\left[\left(\frac{2}{3}\right)(48) + \left(\frac{2}{3}\right)^2 (48) + \left(\frac{2}{3}\right)^2 48 + ...\infty\right]$

$= 48 + 2 \times 48 \times \frac{2}{3}\left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + ...\infty\right]$

$= 48 + 64 \times \frac{1}{1 - \frac{2}{3}}\left[\because s_\infty = \frac{a}{1} - r\right]$

$= 48 + 64 \times \frac{1}{1/3} = 48 + 64 \times 3 = 48 + 192$

= 240 metres.

5. If $A = 1 + r^a + r^{2a} + ... \infty$ and $B = 1 + r^b + r^{2b} + ... \infty$, then prove that

$$r = \left(\frac{A-1}{A}\right)^{1/a} = \left(\frac{B-1}{B}\right)^{1/b} \quad (1)$$

[DCE −1999]

Solution

$A = 1 + r^a + r^{2a} + ... \infty$

$\therefore A = 1 + r^a + (r^a)^2 + ... \infty$

$\Rightarrow A = \frac{1}{1 - r^a} \quad \therefore \left[s_\infty = -\frac{a}{1-r}\right]$

$\Rightarrow 1 - r^a = \frac{1}{A} \Rightarrow r^a = 1 - \frac{1}{A} \Rightarrow r^a = \frac{A-1}{A}$

$\Rightarrow r = \left(\frac{A-1}{A}\right)^{1/a} \quad (1)$

Again, $B = 1 + r^b + r^{2b} + ... \infty \Rightarrow B = 1 + r^b + (r^b)^2 + ... \infty$

$\Rightarrow B = \frac{1}{1 - r^b} \Rightarrow 1 - r^b = \frac{1}{B}$

$\Rightarrow r^b = 1 - \frac{1}{B}$

$\Rightarrow r^b = \frac{B-1}{B}$

$\Rightarrow r = \left(\frac{B-1}{B}\right)^{1/b} \quad (2)$

Therefore, from equations (1) and (2), we have

$r = \left(\frac{A-1}{A}\right)^{1/a} = \left(\frac{B-1}{B}\right)^{1/b}$

Proved

UNSOLVED SUBJECTIVE PROBLEMS XII (CBSE/STATE BOARD): TO GRASP THE TOPIC, SOLVE THESE PROBLEMS

Exercise I

1. Find the 6th term of sequence −6, − 3, − 3/2 ....

2. Find 8th term of progression 3, 3^2, 3^3, 3^4, ...

3. Which term of series $\frac{-1}{27}, \frac{1}{9}, -\frac{1}{2}$ is 729.

4. Find the sum of the geometric series
$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$to 12 terms.

5. How many terms of the geometric series 1 + 4 + 16 + 64 + ... will make the sum 5461?

6. Find the sum of the series 2 + 6 + 18 + 54 + ...+ 4374.

7. If the first and the nth terms of a GP are a and b respectively and p is the product of its first n terms then prove that $p^2 = (ab)^n$.

8. Find the sum of the infinite geometric series
$(\sqrt{2} + 1) + 1 + (\sqrt{2} - 1) + ... \infty$

9. The sum of an infinite GP is $\frac{80}{9}$ and its common ratio is $\frac{-4}{5}$. Find its first term.

10. Find the infinite geometric series in which the sum of first two terms is 5 and first term is equal to 3 times the sum of the terms following the first term.

11. Insert 5 geometric means between $3\frac{5}{9}$ and $40\frac{1}{2}$.

12. If S_1, S_2 and S_3 be respectively the sum of n, $2n$ and $3n$ terms of a GP, prove that $S_1(S_3 - S_2) = (S_2 - S_1)^2$.

13. If a, b, c, d are in GP prove that $a + b, b + c, c + d$ are also in GP.

14. If $a^2 + b^2$, $ab + bc$ and $b^2 + c^2$ are in G.P., prove that a, b, c are also in G.P.

15. Calculate the third term from the end of the series $\frac{2}{27}, \frac{2}{9}, \frac{2}{3}, ...162$.

16. The pth term of the series 1, 2, 4, 8,........ and 256, 128, 64, ... are equal. Find the value of p.

17. If S be the sum, P the product and R the sum of the reciprocals of n terms in a GP, prove that $P^2 = \left(\frac{S}{R}\right)^n$.

Exercise II

1. Find out the seventh term of the series 0.4, 0.8, 1.6, …
2. Which term of the series $\frac{1}{4}, \frac{-1}{2}, 1, ...,$ is -128?
3. If the 4th and 9th terms of a G.P. are 54 and 13122 respectively, find the G.P. Also find its general term.
4. Find the 8th term from the end of the GP 3, 6, 12, 24, ..., 12288.
5. Find the sum of 8 terms of the GP 3, 6, 12, 24, ...
6. Find the sum of the infinite series
   (i) $1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + ... \infty$
   (ii) $-\frac{5}{4} + \frac{5}{16} - \frac{5}{64} + ... \infty$
7. The product of three consecutive terms of a G.P. is 8. The sum of product of these terms taken in pairs is 14. Find the numbers.
8. The sum of an infinite GP is 57 and the sum of their cubes is 9747, find the G.P.
9. If each term of an infinite GP is twice the sum of the terms following it, then find the common ratio of the GP.
10. Find three numbers in GP whose sum is 13 and the sum of whose squares is 91.
11. Find three numbers in GP whose sum is 52 and the sum of whose products in pairs is 624.
12. Insert three numbers between 1 and 256 so that the resulting sequence is a GP.
13. Find the GM of 6 and 9.
14. Sum the series 5 + 55 + 555 + ... to n terms.
15. The sum of two numbers is 6 times their geometric means. Show that the numbers are in the ratio $(3 + 2\sqrt{2})$: $3 - 2\sqrt{2})$.

ANSWERS

Exercise I

1. $-3/16$
2. 6561
3. $n = 10$
4. $\frac{4095}{2048}$
5. $n = 7$
6. 6560
8. $\frac{(4 + 3 + \sqrt{2})}{2}$
9. first term = 16
10. $4 + 1 + \frac{1}{4} + \frac{1}{16} \cdots\cdots$
11. $\pm 16/3, 8, \pm 8, 18, \pm 27$
15. 18
16. $p = 5$

Exercise II

1. 25.6
2. 10th term
3. Required G.P. is 2, 6, 18, 54, ... General term of the G.P. is given by $a_n = 2 \times (3)^{n-1}$.
4. 96
5. 765
6. (i) 3/4 (ii) -1
7. (1, 2, 4) or (4, 2, 1)
8. 19, 38/3, 76/9,.........
9. Common ratio (r) = 1/3
10. (1, 3, 9) or (9, 3, 1)
11. (36, 12, 4) or (4, 12, 36)
12. (4, 16, 64) or (-4, 16, -64)
13. $\pm 3\sqrt{6}$
14. $\frac{5}{18} \times 10^{n+1} - 9n - 10$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. If nth term of a positive term GP, is a_n and $\sum_{n=1}^{100} a_{2n} = \alpha \sum_{n=1}^{100} a_{2n-1} = \beta$, $(\alpha \neq \beta)$, then its common ration is

(a) α/β

(b) β/α

(c) $\sqrt{\alpha/\beta}$

(d) $\sqrt{\beta/\alpha}$

[IIT– 1992; MNR – 1998]

Solution

(a) Let a be the first term and r be the common ratio of given GP.

Then

$$\sum_{n=1}^{100} a_{2n} = a_2 + a_4 + \ldots\ldots. \text{ 100 terms}$$

$$= ar + ar^3 + \ldots\ldots.. \text{ 100 terms}$$

$$= ar(1 + r^2 + r^4 + \ldots\ldots.. \text{ 100 terms})(1)$$

$$\beta = \sum_{n=1}^{100} a_{2n-1} = a_1 + a_3 + a_5 + \text{100 terms}$$

$$= a + ar^2 + ar^4 + \ldots. \text{ 100 terms}$$

$$= a(1 + r^2 + r^4 + \ldots. \text{ 100 terms}) \qquad (2)$$

$$(1) \div (2) \Rightarrow \frac{\alpha}{\beta} = r$$

2. The sum of 10 terms of the series $\left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)^2 + \left(x^3 + \frac{1}{x^3}\right)^2 + \ldots\ldots..$ is

(a) $\left(\frac{x^{20}-1}{x^2-1}\right)\left(\frac{x^{22}+1}{x^{20}}\right) + 20$

(b) $\left(\frac{x^{18}-1}{x^2-1}\right)\left(\frac{x^{11}+1}{x^9}\right) + 20$

(c) $\left(\frac{x^{18}-1}{x^2-1}\right)\left(\frac{x^{11}+1}{x^9}\right) + 20$

(d) none of these

[IIT –1968]

Solution

(a) Sum $= (x^2 + x^4 + x^6 + ..10 \text{ terms}) + \left(\frac{1}{x^2} + \frac{1}{x^4} + \ldots..10 \text{ terms}\right) + 20$

$$\frac{x^2(1 - x^{20})}{1 - x^2} + \frac{1 - x^{20}}{(1 - x^2)x^{20}} + 20$$

$$\left(\frac{x^{20}-1}{x^2-1}\right)\left(\frac{x^{22}+1}{x^{20}}\right) + 20$$

3. The number of terms of a GP is even. If the sum of all terms is 5 times the sum of its odd terms, then its common ratio is

(a) 2 (b) 3

(c) 4 (d) 5

[Roorkee –1990]

Solution

(c) Step 1: Let total number of terms be $2n$ out of which n items are odd with common ratio n^2.

Step 2: Given $a + ar + ar^2 + \ldots\ldots.. = 5(a + ar^2 + ar^4 + \ldots..)$

$$\frac{a(1 - r2^n)}{1 - r} = \frac{a(1 - r^{2n})}{1 - r^2} \Rightarrow r + 1 = 5$$

$$\Rightarrow r = 4$$

4. a, b, c are three distinct real numbers and they are in GP. If $a + b + c = xb$, then

[JEE (WB) –1992]

(a) $x < -3$ or $x > 2$

(b) $x < -4$ or $x > 3$

(c) $x < -1$ or $x > 3$

(d) none of these

Solution

(c) Let r be the corresponding common ratio. Then $b = ar$, $c = ar^2$ $\therefore$ $a + b + c = xb$

$\Rightarrow a + ar + ar^2 = x(ar)$

$\Rightarrow r^2 + (1 - x)r + 1 = 0$

Since, r is real so $(1 - x)^2 - 4 \leq 0 \Rightarrow x^2 - 2x - 3 \geq 0$

$\Rightarrow (x + 1)(x - 3) \geq 0 \Rightarrow x \leq -1$ or $x \geq 3$.

5. In a geometric progression consisting of positive terms, each term equals the sum of the next two terms. Then the common ratio of this progression equals

[AIEEE –2007]

(a) $\frac{1}{2}(1-\sqrt{5})$ (b) $\frac{1}{2}\sqrt{5}$

(c) $\sqrt{5}$ (d) $\frac{1}{2}(\sqrt{5}-1)$

Solution

(d) Given $ar^{n-1} = ar^n + ar^{n+1} \Rightarrow 1 = r + r^2$

$\therefore\ r = \dfrac{\sqrt{5}-1}{2}$

6. $2.\dot{3}5\dot{7} =$

[IIT –1983; RPET –1995; DCE –2000]

(a) $\dfrac{2355}{1001}$ (b) $\dfrac{2370}{997}$

(c) $\dfrac{2355}{999}$ (d) none of these

Solution

(c) Given that $2.\dot{3}5\dot{7} = 2.357357357357......= 2 + 0.357 + 0.000357 + 0.000000357 + ...$

$= 2 + \dfrac{357}{10^3} + \dfrac{357}{10^6} + \dfrac{357}{10^9} + ...$

$\therefore\ s_\infty = 2 + \dfrac{\frac{357}{10^3}}{1 - \frac{1}{10^3}}$

$= 2 + \dfrac{357}{10^3} \times \dfrac{10^3}{999} = \dfrac{2355}{999}$

Aliter: Let $x = 2.\dot{3}5\dot{7} = 2.357357$ (1)

$\Rightarrow 1000\,x = 2357.357357$ (2)

$(2) - (1) \Rightarrow 999x = 2355\ \therefore\ x = \dfrac{2355}{999}$

7. If the pth, qth and rth term of a G.P. are a, b, c respectively, then $a^{q-r}.\ b^{r-p}.\ c^{p-q}$ is equal to

(a) 0 (b) 1

(c) abc (d) pqr

[Roorkee –1955, 1963, 1973; Pb. CET – 1991, 1995]

Solution

(b) Let $AR^{p-1} = a$ (1)

$AR^{q-1} = b$ (2)

and $AR^{r-1} = c$ (3)

So $a^{q-r}\, b^{r-p}\, c^{p-q} = \{AR^{p-1}\}^{q-r} \{AR^{q-1}\}^{r-p} \{AR^{r-1}\}^{p-q}$

$= A^{(q-r+r-p+p-q)} R^{(pq-pr-q+r+qr-pq-r+p+pr-rq-p+q)}$

$= A^0R^0 = 1.$

Note: Such type of questions i.e. containing terms of powers in cyclic order associated with negative sign, reduce to 1 mostly.

8. If the sum of first 6 term is 9 times to the sum of first 3 terms of the same G.P., then the common ratio of the series will be

(a) – 2 (b) 2 (c) 1 (d) 1/2

[RPET –1985]

Solution

(b) Under given conditions, we get

$\Rightarrow \dfrac{a(r^6-1)}{(r-1)} = 9.\ \dfrac{a(r^3-1)}{(r-1)}\ (\because\ r > 1)$

$\Rightarrow r^6 - 1 = 9r^3 - 9 \Rightarrow (r^3)^2 - 9(r^3) + 8 = 0$

$(r^3 - 1)\,(r^3 - 8) = 0$

$\Rightarrow r = 1, \omega, \omega^2$ and $r = 2$. But $r = 1, \omega, \omega^2$ can not satisfy the given condition. Hence, $r = 2$.

9. If the geometric mean between a and b is $\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$, then the value of n is

(a) 1 (b) −1/2 (c) 1/2 (d) 2

Solution

(b) As given $\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$

$\Rightarrow a^{n+1} - a^{n+1/2}\, b^{1/2} + b^{n+1} - a^{1/2}\, b^{n+1/2} = 0$

$\Rightarrow (a^{n+1/2} - b^{n+1/2})\,(a^{1/2} - b^{1/2}) = 0$

$\Rightarrow a^{n+1/2} - b^{n+1/2} = 0\ (\because\ a \neq b \Rightarrow a\frac{1}{2} \neq b^{1/2})$

$\Rightarrow \left(\frac{a}{b}\right)^{n+1/2} = 1 = \left(\frac{a}{b}\right)^n \Rightarrow n = \frac{1}{2} = 0$

$\Rightarrow n = -\dfrac{1}{2}$

10. If the product of three consecutive terms of G.P. is 216 and the sum of product of pair–wise is 156, then the numbers will be

(a) 1, 3, 9 (b) 2, 6, 18

(c) 3, 9, 27 (d) 2, 4, 8

[MNR –1978]

Solution

(b) Let numbers are $\frac{a}{r} \cdot a.\ ar$ Under conditions, we get $\frac{a}{r}.\ a\ .\ ar = 216 \Rightarrow a = 6.$

And sum of product pair wise =156

$\Rightarrow \frac{a}{r} . a + \frac{a}{r} . ar + a . ar = 156$

$\Rightarrow r = 3$ Hence, numbers are 2, 6, 18.

Trick: Since $2 \times 6 \times 18 = 216$ (as given) and no other option gives the value.

11. The sum of infinite terms of a G.P. is x and on squaring the each term of it, the sum will be y, then the common ratio of this series is

[RPET –1988]

(a) $\frac{x^2 - y^2}{x^2 + y^2}$ (b) $\frac{x^2 + y^2}{x^2 - y^2}$

(c) $\frac{x^2 - y}{x^2 + y}$ (d) $\frac{x^2 + y}{x^2 - y}$

Solution

(c) We have $\frac{a}{1-r} = x$

and $\frac{a^2}{1-r^2} = \frac{a}{1-r} \frac{a}{1+r} = y$

$\Rightarrow y = x. \frac{a}{1+r} = x. \frac{x(1-r)}{1-r} \Rightarrow \frac{y}{x^2} = \frac{1-r}{1+r}$

$\Rightarrow \frac{x^2}{y} = \frac{1+r}{1-r} \Rightarrow \frac{x^2}{y}(1-r) = 1 + r$

$\Rightarrow r\left[1 + \frac{x^2}{y}\right] = -\frac{x^2}{y} \Rightarrow r = \frac{x^2 - y}{x^2 + y}$

12. If S is the sum to infinity of a G.P., whose first term is a, then the sum of the first n terms is

[UPSEAT – 2002]

(a) $\left(1 - \frac{a}{s}\right)^n$ (b) $s\left[1 - \left(1 - \frac{a}{s}\right)^n\right]$

(c) $a\left[1 - \left(1 - \frac{a}{s}\right)^n\right]$ (d) none of these

Solution

(b) Let r be the common ratio of the G.P. Then

$S = \frac{a}{1-r} \Rightarrow r = 1 - \frac{a}{s}$ Now S = sum of n terms

$= a\frac{1-r^2}{1-r} = \frac{a}{1-r}(1 - r^n) = s\left[1 - \left(1 - \frac{a}{s}\right)^n\right]$

13. 0.14189189189 ... can be expressed as a rational number

(a) $\frac{7}{3700}$ (b) $\frac{7}{50}$ (c) $\frac{525}{1111}$ (d) $\frac{21}{48}$

[AMU – 2000]

Solution

(d) 0.14189189189 ... = 0.14 + 0.00189 + 0.00000189 + ...

$= \frac{14}{100} + 189\left[\frac{1}{10^5} + \frac{1}{10^8} + ...\infty\right]$

$= \frac{7}{50} + 189\left[\frac{1/10^5}{1 - (1/10^8)}\right]$

$= \frac{7}{50} + 189\left[\frac{1}{10^5} \times \frac{10^3}{999}\right]$

$= \frac{7}{50} + \frac{189}{999 \times 100} \frac{7}{50} + \frac{7}{3700}$

$= \frac{7}{50} + \frac{7}{25 \times 148} \frac{21}{148}$

14. The value of $4^{1/3} . 4^{1/9} . 4^{1/27}$ .... ∞ is

[RPET – 2003]

(a) 2 (b) 3 (c) 4 (d) 9

Solution

(a) $4^{1/3}. 4^{1/9}. 4^{1/27}........\infty$

$\therefore S = 4^{1/3 + 1/9 + 1/27} ... \infty$

$\Rightarrow S = 4^{\left(\frac{1/3}{1-1/3}\right)} = 4^{\frac{1}{3/2/3}}$

$\Rightarrow S = 4^{1/2} \Rightarrow S = 2$

15. The sum of infinite terms of the geometric progression

$\frac{\sqrt{2}+1}{\sqrt{2}-1}, \frac{1}{2-\sqrt{2}}, \frac{1}{2}$ ...... is

[Kerala (Engg.) – 2002]

(a) $\sqrt{2}(\sqrt{2}+1)^2$ (b) $(\sqrt{2}+1)^2$

(c) $5\sqrt{2}$ (d) $3\sqrt{2} + \sqrt{5}$

Solution

(a) $\frac{\sqrt{2}+1}{\sqrt{2}-1}, \frac{1}{\sqrt{2}(\sqrt{2}-1)}, \frac{1}{2}$.......

Common ratio of the series = $\frac{1}{\sqrt{2}(\sqrt{2}+1)}$

Therefore, sum =

$\frac{a}{1-r} = \left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\left(\frac{1-1}{\sqrt{2}(\sqrt{2}+1)}\right)$

$= \frac{(\sqrt{2}+1}{(\sqrt{2}-1)} \cdot \frac{\sqrt{2}(\sqrt{2}+1)}{(1-\sqrt{2})} = \sqrt{2}(\sqrt{2}+1)^2$

16. If $\frac{a+bx}{a-bx}=\frac{b+cx}{b-cx}=\frac{c+dx}{c-dx}$ a, b, c, d are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

[RPET – 1986]

Solution

$\frac{a+bx}{a-bx}=\frac{b+cx}{b-cx}=\frac{c+dx}{c-dx}$

Applying componendo and dividendo, we get

$\frac{2a}{2bx}=\frac{2b}{2cx} \Rightarrow b^2 = ac$ and $c^2\ bd$

$\Rightarrow a, b, c$ and b, c, d are in G.P.

Therefore a, b, c, d are in G.P.

17. If $1 + \cos\alpha + \cos^2\alpha + ... \infty = 2 - \sqrt{2}$, then α, $(0 < \alpha < \pi)$ is

[Roorkee – 2000; AMU – 2005]

(a) $\pi/8$ (b) $\pi/6$
(c) $\pi/4$ (d) $3\pi/4$

Solution

(d) $1 - \cos\alpha = \frac{1}{2-\sqrt{2}} = 1 + \frac{1}{\sqrt{2}}$

$\Rightarrow \cos\alpha = -\frac{1}{\sqrt{2}} = \cos\frac{3\pi}{4} \Rightarrow \alpha = \frac{3\pi}{4}$

18. The first term of an infinite geometric progression is x and its sum is 5. Then

[IIT Screening –2004]

(a) $0 \le x \le 10$ (b) $0 < x < 10$
(c) $-10 < x < 0$ (d) $x > 10$

Solution

(b) $5 = \frac{x}{1-r} \Rightarrow 5 - 5r = x \Rightarrow r = 1 - \frac{x}{5}$

As $|r| < 1$ i.e., $\left|1-\frac{x}{5}\right| < 1$ $-1 < 1 - \frac{x}{5} < 1$

$-5 < 5 - x < 5 = -10 < -x < 0 = 10 > x > 0$ i.e. $0 < x < 10$.

19. Suppose a, b, c are in A.P. and a^2, b^2, c^2 are in G.P.. If $a < b < c$ and $a + b + c = \frac{3}{2}$, then the value of a is

[IIT Screening –2002]

(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{2\sqrt{3}}$
(c) $\frac{1}{2}-\frac{1}{\sqrt{3}}$ (d) $\frac{1}{2}-\frac{1}{\sqrt{2}}$

Solution

(d) Let 3 numbers a, b, c, be $p - q, p, p + q$ then $(p - q) + p + (p + q) = \frac{3}{2}p = \frac{1}{2}$; then numbers are $\frac{1}{2} - q, \frac{1}{2}, \frac{1}{2} + q$

a^2, b^2, c^2 are in G.P.

$\therefore \left(\left(\frac{1}{2}\right)^2\right)^2 = \left(\frac{1}{2}-q\right)^2\left(\frac{1}{2}+q\right)^2$

$\Rightarrow \frac{1}{16} = \left(\frac{1}{4}-q^2\right)^2 \Rightarrow \left(\frac{1}{4}-q^2\right) = \pm\frac{1}{4}$

$\neq 0, q^2 = 1/2 \Rightarrow q = \pm 1/\sqrt{2}$

Since, $a < b < c$, $a = -\frac{1}{2} - \frac{1}{\sqrt{2}}$

20. What is the ratio of corresponding terms of two Geometric series where G_1 and G_2 are geomertic means of the two series

[NDA –2007]

(a) $\log G_1 - \log G_2$
(b) $\log G_1 + \log G_2$
(c) G_1/G_2
(d) G_1G_2

Solution

(c) Let the two series be

$a, ar, ar^2, ar^3,$.............

$b, br_1, br_1^2, br_1^3,$............

$\frac{a}{b}, \frac{a}{b}\left(\frac{r}{r_1}\right), \frac{a}{b}\left(\frac{r}{r_1}\right)^2$, ...common ratio $\frac{r}{r_1}$.

21. $0.4\dot{2}\dot{3}$

[Roorkee –1961; IIT –1973]

(a) $\frac{419}{990}$ (b) $\frac{419}{999}$ (c) $\frac{417}{990}$ (d) $\frac{417}{999}$

Solution

(a) We have $0.4\dot{2}\dot{3} = 0.4232323$ ....

$= 0.4 + 0.023 + 0.00023 + 0.0000023 + ... \infty$

$= \frac{4}{10} + \frac{23}{10^3} + \frac{23}{10^5} + \frac{23}{10^7}$ ....∞

$= \frac{4}{10} + \frac{23}{10^3}\left[1 + \frac{1}{10^2} + \frac{1}{10^4} \text{.........} \infty\right]$

$= \frac{4}{10} + \frac{23}{1000}\left(\frac{1}{1-\frac{1}{10^2}}\right) = \frac{4}{10} + \frac{23}{990} = \frac{419}{990}$

22. The number 111 .... 1 (91 times) is a:

(a) even number
(b) prime number
(c) not prime
(d) none of these

Solution

(c) Step 1: $11 = 1 + 10^1$

Similarly, $111 = 1 + 10^1 + 10^2$

Step 2: $S = 1 + 10 + 10^2 + ... + 10^{90}$ (91terms)

$$\frac{1.(10^{91} - 1)}{10 - 1} = \frac{(10^{13})^7 - 1}{10^{13} - 1} \times \frac{10^{13} - 1}{10 - 1}$$

$$= [(10^{13})^6 + (10^{13})^5 + (10^{13})^4 + ... 1]\ (10^{12} + 10^{11} + ... +1)$$

It is the product of two integers and hence not prime.

23. If $a_n = \frac{3}{4} - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + ... + (-1)^{n-1}\, k\left(\frac{3}{4}\right)^n$ and $b_n = 1 - a_n$, then the minimum natural number n_0 such that $b_n > a_n\ \forall\ n > n_0$ is **[IIT –JEE –2006]**

(a) 4
(b) 5
(c) 6
(d) 12

Solution

(c) Step 1: $a = \frac{3}{4}$, c.r. $= r = -\frac{3}{4}$

$$\frac{3}{4}\frac{\left[1 - \left(-\frac{3}{4}\right)^n\right]}{1 + 3/4} \quad \text{[by GP sum formula]}$$

$$\frac{3}{7}[1 - (-3/4)^n]$$

Now $b_n > a_n$ and $b_n = 1 - a_n$

$\Rightarrow = 1 - a_n > a_n \Rightarrow 2a_n < 1$

$\Rightarrow \frac{6}{7}[1 - (-3/4)^n] < 1$ [by Equation (1)]

$\Rightarrow 1 - (-3/4)^n < 7/6$

$\Rightarrow -(-3/4)^n < \frac{1}{6}$, (2)

or $-\left(\frac{3}{4}\right)^n > \frac{-1}{6}$ (3)

Now we observe that for $n = 1, 3, 5$ this inequality does not hold. But it is true for $n = 6, 7, 8,.....$ Hence required minimum natural value n_0 of n is 6.

24. If a, b, c, d are such unequal real numbers that $(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)p + (b^2 + c^2 + d^2) \le 0$, then a, b, c, d are in **[IIT –1987]**

(a) AP
(b) GP
(c) HP
(d) none of these

Solution

(b) $(a^2p^2 - 2abp + b^2) + (b^2p^2 - 2bcp + c^2) + (c^2p^2 - 2cdp + d^2) \le 0$

$\Rightarrow (ap - b)^2 + (bp - c)^2 + (cp - d)^2 \le 0$

$\Rightarrow ap - b = 0, bp - c = 0, cp - d = 0$

$\Rightarrow p = \frac{b}{a} = \frac{c}{b} = \frac{d}{c}$

25. The first two terms of a geometric progression add up to 12. The sum of the third and the fourth terms is 48. If the terms of the geometric progression are alternately positive and negative, then the first term is **[AIEEE –2008]**

(a) 4
(b) – 4
(c) – 12
(d) 12

Solution

(c) Let the GP be $a, ar, ar^2, ar^3, ...$

we have $a + ar = 12$ (1)

$ar^2 + ar^3 = 48$ (2)

on division we have

$$\frac{ar^2(1 + r)}{a(1 + r)} = \frac{48}{12}$$

$\Rightarrow r^2 = 4$

$\therefore r = \pm 2$

But the terms are alternately positive and negative,

$\therefore r = -2$

Now $a = \frac{12}{1+r} = \frac{12}{1-2} = \frac{12}{-1} = -12$ From Equation (1)

27. The sum to infinity of the series

$1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4}$

(a) 2

(b) 3

(c) 4

(d) 6

[AIEEE – 2009]

Solution

(b) Let $S = 1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4}$ (1)

Multiplying the series (1) by $\frac{1}{3}$ on either side and shifting the series on the R.H. side by one column to the right.

or $\frac{1}{3}S = \frac{1}{3} + \frac{2}{3^2} + \frac{6}{3^3} + \frac{10}{3^4} + \ldots\ldots$ (2)

Subtracting Equations (2) from (1)

$S\left(1 - \frac{1}{3}\right) = 1 + \frac{1}{3} + \frac{4}{3^2} + \frac{4}{3^3} + \frac{4}{3^4} + \ldots\ldots$

or $\frac{2}{3}S = \frac{4}{3} + \frac{4}{3^2}\left(1 + \frac{1}{3} + \frac{1}{3^3} + \ldots\ldots\right)$

$\frac{2}{3}S = \frac{4}{3} + \frac{4}{3} + \frac{4}{3^2}\left(\frac{1}{1 - \frac{1}{3}}\right)$

$= \frac{4}{3} + \frac{4}{3^3} \cdot \frac{3}{2} = \frac{4}{3} + \frac{2}{3} = \frac{6}{3}$

$\Rightarrow \frac{2}{3}S = \frac{6}{3} \Rightarrow S = 3$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. If x, $2x + 2$, $3x + 3$, are in G.P., then the fourth term is

(a) 27 (b) −27

(c) 13.5 (d) −13.5

[MNR – 1981]

2. The third term of a G.P. is the square of fi rstterm. If the second term is 8, then the 6th term is

[MPPET – 1997]

(a) 120 (b) 124 (c) 128 (d) 132

3. 7th term of the sequence $\sqrt{2}$, $\sqrt{10}$, $5\sqrt{2}$, ... is

(a) $125\sqrt{10}$

(b) $25\sqrt{2}$

(c) 125

(d) $125\sqrt{2}$

4. If the 4th, 7th and 10th terms of a G.P. be a, b, c respectively, then the relation between a, b, c is

[MNR –1995; Karnataka CET –1999]

(a) $b = \frac{a+c}{2}$ (b) $a^2 = bc$

(b) $b^2 = ac$ (c) $c^2 = ab$

5. If the 5th term of a G.P. is $\frac{1}{3}$ and 9th term is $\frac{16}{243}$, then the 4th term will be

(a) 3/4 (b) 1/2

(c) 1/3 (d) 2/5

[MPPET – 1982]

6. Fifth term of *a* G.P. is 2, then the product of its 9 terms is

(a) 256 (b) 512

(c) 1024 (d) none of these

[Pb. CET – 1990, 1994; AIEEE – 2002]

7. If $(p + q)$th term of a G.P. be m and $(p - q)$th term be n, then the pth term will be

[RPET – 1997; MPPET – 1985, 1999, 2008]

(a) m/n (b) $\sqrt{mn}$

(c) mn (d) 0

8. If the first term of a G.P. be 5 and common ratio be − 5, then which term is 3125

(a) 6th (b) 5th
(c) 7th (d) 8th

9. The sum of first two terms of a G.P. is 1 and every term of this series is twice of its following terms, then the first term will be

[RPET − 1988]

(a) 1/4 (b) 1/3
(c) 2/3 (d) 3/4

10. The first term of a G.P. is 7, the last term is 448 and sum of all terms is 889, then the common ratio is

[MPPET − 2003]

(a) 5 (b) 4
(c) 3 (d) 2

11. If the sum of n terms of a G.P. is 255 and nth terms is 128 and common ratio is 2, then first term will be

[RPET − 1990]

(a) 1 (b) 3
(c) 7 (d) none of these

12. The sum of the series 6 + 66 + 666 + ... upto n terms is

(a) $(10^{n-1} - 9n + 10)/81$
(b) $2(10^{n+1} - 9n - 10)/27$
(c) $2(10^{n} - 9n - 10)/27$
(d) none of these

[IIT − 1974]

13. The number which should be added to the numbers 2, 14, 62 so that the resulting numbers may be in G.P., is

(a) 1 (b) 2
(c) 3 (d) 4

14. The first term of a G.P. is 7, the last term is 48 and sum of all terms is 89, then the common ratio is

(a) 5 (b) 4
(c) 3 (d) 2

15. If $\frac{b+a}{b-a} = \frac{b+c}{b-c}$, then a, b, c are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

16. The sum of infi nity of a geometric progression is 4/3 and the first term is 3/4. The common ratio is

[MPPET − 1994]

(a) 7/16 (b) 9/16
(c) 1/9 (d) 7/9

17. If $3 + 3\alpha + 3\alpha^2 + \ldots\ldots\ldots\ldots \infty = 45/8$, then the value of α will be

(a) 15/23 (b) 7/15
(c) 7/8 (d) 15/7

[Pb. CET − 1989]

18. If sum of infinite terms of a G.P. is 3 and sum of squares of its terms is 3, then its first and common ratio are

(a) 3/2, 1/2 (b) 1, 1/2
(c) 3/2, 2 (d) none of these

[0Roorkee − 1972; RPET − 1999]

19. If in an infinite G.P. first term is equal to the twice of the sum of the remaining terms, then its common ratio is

[RPET − 2002]

(a) 1 (b) 2
(c) 1/3 (d) − 1/3

20. Consider an infi nite G.P. with first term a and common ratio r, its sum is 4 and the second term is 3/4, then

(a) $\alpha = \frac{7}{4}, r = \frac{3}{7}$ (b) $\alpha = \frac{3}{2}, r = \frac{1}{2}$
(c) $\alpha = 2, r = \frac{3}{8}$ (d) $\alpha = 3, r = \frac{1}{4}$

[IIT Screening − 2000; DCE − 2001]

21. Three numbers are in G.P. such that their sum is 38 and their product is 1728. The greatest number among them is

(a) 18 (b) 16
(c) 14 (d) none of these

[UPSEAT − 2004; MPPET − 1994]

22. The G.M. of roots of the equation $x^2 - 18x + 9 = 0$ is

(a) 3 (b) 4
(c) 2 (d) 1

[RPET − 1997]

23. The G.M. of the numbers 3, 3^2, 3^3, ...3^n is

[DCE – 2002]

(a) $\frac{2}{3^n}$ (b) $3^{\frac{n+1}{2}}$

(c) $3^{\frac{n}{2}}$ (d) $3^{\frac{n-1}{2}}$

24. The two geometric means between the number 1 and 64 are

(a) 1 and 64 (b) 4 and 16

(c) 2 and 16 (d) 8 and 16

[Kerala (Engg.) – 2002]

25. If n geometric mean be inserted between a and b, then the nth geometric mean will be

(a) $(b/a)^{n/n-1}$ (b) $a(b/a)^{n-1/n}$

(c) $a(b/a)^{n/n+1}$ (d) $a(b/a)^{1/n}$

26. If G be the geometric mean of x and y, then

$$\frac{1}{G^2-x^2}+\frac{1}{G^2-y^2}=$$

(a) G^2 (b) $1/G^2$

(c) $2/G^2$ (d) $3G^2$

27. If three geometric means be inserted between 2 and 32, then the third geometric mean will be

(a) 8 (b) 4

(c) 16 (d) 12

28. If five G.M.'s are inserted between 486 and 2/3 then fourth G.M. will be:

[RPET –1999]

(a) 4 (b) 6

(c) 12 (d) –6

29. $x = 1 + a + a^2 + \ldots\ldots \infty (a < 1)$, $y = 1 + b + b^2 + \ldots\ldots \infty (b < 1)$ Then the value of $1 + ab + a^2 b^2 + \ldots\ldots\ldots\ldots \infty$ is

(a) $\frac{xy}{x+y-1}$ (b) $\frac{xy}{x+y+1}$

(c) $\frac{xy}{x-y-1}$ (d) $\frac{xy}{x-y+1}$

[MNR–1980; MPPET –1985]

30. The terms of a G.P. are positive. If each term is equal to the sum of two terms that follow it, then the common ratio is

(a) $\frac{\sqrt{5}-1}{2}$ (b) $\frac{1-\sqrt{5}}{2}$

(c) 1 (d) $1\sqrt{2}$

31. The sum to infinity of the following series

$$2+\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^3}\ \ldots\ldots\ldots,$$

will be

[AMU –1984]

(a) 3 (b) 4 (c) 7/2 (d) 9/2

32. The numbers $(\sqrt{2}+1)$, 1, $(\sqrt{2}-1)$ will be in

[AMU –1983]

(a) A.P. (b) G.P.

(c) H.P. (d) None of these

33. If n geometric means between a and b be $G_1, G_2, \ldots G_n$ and a geometric mean be G, then the true relation is

(a) $G_1.G_2\ldots\ldots G_n = G$

(b) $G_1.G_2\ldots\ldots G_n = G^{1/n}$

(c) $G_1.G_2\ldots\ldots G_n = G^n$

(d) $G_1.G_2\ldots\ldots G_n = G^{2/n}$

34. The Geometric mean of 1,2, 2^2, ...,2^n is

[MPPET –2009]

(a) $2^{n/2}$ (b) $2^{(n+1)/2}$

(c) $2^{n(n+1)/2}$ (d) $2^{(n-1)/2}$

35. If the sum of the first two terms and the sum of the first four terms of a geometric progression with positive common ratio are 8 and 80 respectively, then what is the 6th term?

[N.D.A –2009]

(a) 88 (b) 243 (c) 486 (d) 1458

36. If $x > 1$ and $\log_2 x$, $\log_3 x$, $\log_x 16$ are in GP, then what is x equal to?

[N.D.A –2009]

(a) 9 (b) 8 (c) 4 (d) 2

37. In a geometric progression with first term a and common ratio r, what is the arithmetic mean of first five terms?

[N.D.A –2009]

(a) $a + 2r$ (b) ar^2

(c) $a(r^5 - 1)/(r - 1)$

(d) $a(r^5 - 1)/[5(r - 1)]$

SOLUTIONS

1. (d) x, $2(x+1)$, $3(x+1)$ in G.P. (1)
$[2(x+1)]^2 = x \cdot 3(x+1)$ or $x^2 + 5x + 4 = 0$
or, $(x+4)(x+1) = 0$ or $x = -1, -4$.
This $\Rightarrow x + 1 = 0$ which gives no result, according to (1).
If $x = -4$ in (1), then terms are $-4, -6, -9$,
here $r = \frac{3}{2}$, $T_4 = -9r = -9\left(\frac{3}{2}\right)$
Note: No term of G.P. can be zero.

2. (c) In G.P. $T_n = ar^{n-1}$
given $T_3 = T_1^2$ or $ar^2 = a^2$ (1)
and $T_2 = 8$ or $ar = 8$ (2)
From (1) and (2) $r = 2$, $a = 4$
$\therefore T_6 = ar^5$ or $T_6 = 4(2)^5$
$\therefore T_6 = 128$.

3. (d) First term (a) = $\sqrt{2}$
Common ratio $(r) = \frac{\sqrt{10}}{\sqrt{2}} = \sqrt{5}$
7th term $= ar^6$
$= \sqrt{2}\,(\sqrt{5})^6$
$= \sqrt{2} \times 125 = 125\sqrt{2}$

4. (c) Given 4th term of GP = a
$Ar^3 = a$ (Here $A = T_1$ = first term of G.P.) (1)
7th term of GP = b $Ar^6 = b$ (2)
and 10th term of GP = c
$Ar^9 = c$ (3)
multiplying Equations (1) and (3), we get
$a.c = Ar^3 \times Ar^9$
$= A^2r^{12}$
$= (Ar^6)^2 = b^2$
$\therefore ac = b^2$

5. (b) $\frac{1}{3} = T_5 = ar^4$ (1)
$\frac{16}{243} = T_9 = $ ar8(2)
$\frac{(2)}{(1)} \Rightarrow r^4 = \frac{16}{243} \times 3 \Rightarrow r = \frac{2}{3}$
$\Rightarrow a\left(\frac{2}{3}\right)^4 = \frac{1}{3} \Rightarrow a = \left(\frac{3}{2}\right)^4 \frac{1}{3}$
$T_4 = ar^3 = \left(\frac{3}{2}\right)^4 \cdot \frac{1}{3} \cdot \left(\frac{2}{3}\right)^4 = \frac{1}{2}$

6. (b) Given 5th term = 2
$\therefore ar^4 = 2$ ($T_1 = a_1$ common ratio = r) (1)
Product of its 9 terms = $a \times ar \times ar^2 \times ar^3 \times ar^4 \times ar^5 \times ar^6 \times ar^7 \times ar^8$
$= 2 \times r^{1+2+3+......+8}$
$= a^9\, r^{8(8+1)/2}$
$= a^9 \times r^{36}$
$= (ar^4)^9$
$= (2)^9$
$= 512$ by Equation (1)

7. (b) In Geometrical progression
nth term $T_n = ar^{n-1}$
given $T_{p+q} \Rightarrow ar^{p+q-1} = m$ (1)
& $T_{p-q} \Rightarrow ar^{p-q-1} = n$ (2)
multiplying Equations (1) and (2) sidewise we get
$a^2\, r^{2p-2} = mn \Rightarrow (ar^{p-1})^2 = mn$
or $ar^{p-1} = \sqrt{mn}$ $\therefore$ pth term = $\sqrt{mn}$

8. (b) Given first term $a = 5$
common ratio $r = -5$
$l = 3125 = T_n = n$th term (Let)
$\therefore 1 = ar^{n-1}$
$3125 = 5(-5)^{n-1}$
$625 = (-5)^{n-1}$
$(-5)^4 = (-5)^{n-1}$
$\Rightarrow n - 1 = 4 \Rightarrow n = 5$

9. (d) As given $a + ar = 1$ (1)
and $a = 2(ar + ar^2 + ar^3 + \ldots\ldots\ldots.\ \infty)$
$a = 2\left(\frac{ar}{1-r}\right)$ (2)
From Equation (2) $1 - r = 2r$ $\therefore$ $r = 1/3$
So from Equation (1) $a = 3/4$

10. (d) $a = 7$, and $ar^{n-1} = 448$
Now, sum of 'n' terms $= \frac{a(r^n - 1)}{r - 1} = 889$
$\Rightarrow \frac{ar^n - a}{r - 1} = 889 \Rightarrow \frac{448r - 7}{r - 1} = 889$
Now, $r = 2$

11. (a) Let a be the first term. Then as given $r = 2$, $T_n = 128$ and $S_n = 255$. Also $ar^n = (ar^{n-1})\, r = rT_n$.

But $S_n = \dfrac{rT_n - a}{r - 1}$

$\Rightarrow 255 = \dfrac{2(128) - a}{2 - 1}$

$\Rightarrow a = 1$

12. (b) Sum $= \frac{6}{9}[9 + 99 + 999 + \ldots.\ n$ terms]

$= \frac{2}{3}[(10 - 1) + (10^2 - 1) + (10^3 - 1) + \ldots.\ n$ terms]

$= \dfrac{2}{3}\left[\dfrac{1(10^n - 1)}{10 - 1} - n\right]$

$= \dfrac{2}{27}[10^{n+1} - 9n - 10]$

13. (b) Let x should be added to the numbers so that numbers $2 + x$, $14 + x$, $62 + x$ are in GP

$(14 + x)^2 = (2 + x)(62 + x)$

$196 + x^2 + 28x = 124 + 2x + 62x + x^2$

$196 - 124 = 64x - 28x$

$\Rightarrow 36x = 72 \Rightarrow x = 2$

14. (d) First term (a) = 7

Last term (l) $= 48 = ar^{n-1} = 7r^{n-1}$

$l = ar^{n-1}$

$\therefore\ 48 = \dfrac{7(r)^n}{r} = \dfrac{48r}{7} = r^n$ (1)

and $S_n = 89$

$\dfrac{a(r^n - 1)}{r - 1} = 89$

$\dfrac{7\left(\frac{48r}{7} - 1\right)}{r - 1} = 89$

$\left\{\text{from Equation (1) } r^n = \dfrac{48r}{7}\right\}$

$48r - 7 = 89r - 89$

$82 = 41r$

$r = 2.$

15. (d) Given $\dfrac{b + a}{b - a} = \dfrac{b + c}{b - c}$

$(b + a)(b - c) = (b - a)(b + c)$

$b^2 - bc + ab - ac = b^2 + bc - ab - ac$

$2ab = 2bc$

$c = a.$

16. (a) Series $= a + ar + ar^2 + \ldots\ldots. = S_\infty$

$s_\infty = \dfrac{a}{1 - r} = \dfrac{4}{3}$ or $a = \dfrac{4}{3}(1 - r)$

Also $a = \frac{3}{4}$ (given).

$\Rightarrow \dfrac{3}{4} = \dfrac{4}{3}(1 - r) \Rightarrow r = \dfrac{7}{16}.$

17. (b) **Step 1:** Given $a = 3$, $r = \alpha$ and

$S_\infty = \dfrac{a}{1 - r}$

Given $3 + 3 + 3\alpha^2 + \ldots\ldots\ldots. + \infty = \dfrac{45}{8}$

Step 2: $\dfrac{3}{1 - \alpha} = \dfrac{45}{8}$

$24 = 45 - 45\alpha$

$45\alpha = 21$

$\alpha = \dfrac{21}{45} = \dfrac{7}{15}$

18. (a) First series is $a + ar + ar^2 + \ldots\ldots\ldots. = S_\infty$

$S_\infty = \dfrac{a}{1 - r} = 3 = \Rightarrow 3(1 - r) = a$ (1)

Second series is $a^2 + (ar)^2 + (ar^2)^3 + \ldots\ldots$

$S = \dfrac{a^2}{1 - r^2} = 3 \Rightarrow a^2 = 3(1 - r^2)$ (2)

Using (1), $9(1 - r)^2 = 3(1 - r^2)$ or $3(1 - r) = 1 + r$

or $2 = 4r$ or $r = \frac{1}{2}.$

19. (c) Given $a = 2(ar + ar^2 + ar^3 + \ldots\ldots\ldots.\ \infty)$

$a = 2\left(\dfrac{ar}{1 - r}\right)$

$\Rightarrow 1 - r = 2r \Rightarrow r = \dfrac{1}{3}$

20. (d) **Step 1:** Clearly sum of the series $= \dfrac{a}{1 - r} = 4$ (1)

Second term $= \dfrac{3}{4} \Rightarrow ar = \dfrac{3}{4}$ 2)

$\Rightarrow 4\,ar = 3$ (3)

Also, $a = 4 - 4r$

$\Rightarrow a^2 = 4a - 4ar = 4a - 3$ from equation (3)

$\Rightarrow a^2 - 4a + 3 = 0$

$\Rightarrow (a - 3)(a - 1) = 0$

$\Rightarrow a = 3$ or $a = 1$

Step 2: If $a = 3$, then $3r = \dfrac{3}{4} \Rightarrow r = \dfrac{1}{4}$

If $a = 1$, then $r = \dfrac{3}{4}$

Then, $a = 3$, $r = \dfrac{1}{4}$

21. (a) Given three numbers $\frac{a}{r}$, a, ar are in G.P.

Sum = $\frac{a}{r} + a + ar = 38$ (1)

Product = $\frac{a}{r} \cdot a \cdot ar = 1728$ (2)

$\Rightarrow a^3 = 1728 = (12)^3 \Rightarrow a = 12$

Now (1) $\Rightarrow 12 \Rightarrow \left(\frac{1}{r} + 1 + r\right) = 38$

$\Rightarrow 6\,(1 + r + r^2) = 19r$

$6r^2 - 13r + 6 = 0$

$\Rightarrow r = \frac{3}{2}, \frac{2}{3}$

when $r = \frac{3}{2}$, then $\frac{a}{r}, a, ar = 12 \times \frac{3}{2}, 12\ 12 \times \frac{3}{2}$

$= 8, 12, 18$

Greatest number = 18.

22. (a) Let α and β be the roots of equation $x^2 - 18x + 9 = 0$

Therefore, G.M. of α and $\beta = \sqrt{\alpha\beta} = \sqrt{9} = 3$.

23. (b) Step 1: Geometric mean of n, numbers $x_1, x_2, x_3, \ldots, x_n$ is $(x_1 x_2 x_3 \ldots\ldots x_n)^{1/n}$.

Step 2: $a = 3$, $r = 3$

G.M. $= (3 \cdot 3^2 \cdot 3^3 \ldots\ldots 3^n)^{1/n}$

$= (3^{1+2+3 \ldots\ldots +n})^{1/n}$

$= \left(3^{\frac{n(n+1)}{2}}\right)^{1/n} = 3^{\frac{(n+1)}{2}}$

24. (b) If g_1, g_2 be the two geometric means between 1 and 64, then 1, g_1, g_2, 64 are in G.P.

$\Rightarrow 64 = 1\, r^{4-1}$, r being the common ratio

$\Rightarrow r^3 = 64 \Rightarrow r = 4$

$\therefore g_1 = 1\, r = 4$ and $g_2 = 1\, r^2 = 4^2 = 16$.

Alternatively, note that 1, 4, 16, 64 are in G.P.

25. (c) If n geometric means $g_1, g_2 \ldots\ldots g_n$ are to be inserted between two positive real numbers a and b, then

$a, g_1, g_2 \ldots\ldots g_n, b$ are in G.P. Then

$g_1 = ar, g_2 = ar^2 \ldots\ldots g_n = ar^n$

So $b = ar^{n+1} \Rightarrow r = \left(\frac{b}{a}\right)^{1/(n+1)}$

$(n+1)$th term = nth geometric mean.

T_{n+1} = Now nth geometric mean $(g_n) = ar^n = a\left(\frac{b}{a}\right)^{n/(n+1)}$

26. (b) $G^2 = xy$. so L.H.S. $= \frac{1}{xy - x^2} + \frac{1}{xy - y^2}$

$= \frac{1}{x-y}\left(\frac{1}{y} - \frac{1}{x}\right) = \frac{1}{xy} = \frac{1}{G^2}$

27. (c) Let the three geometric mean between 2 and 32 be as follows 2, G_1, G_2, G_3, 32

$a = 2, l = 32, n = 5$

$l = ar^{n-1}$

$32 = 2\,r^4 \Rightarrow r^4 = 16 = 2^4$

$\Rightarrow r = 2$

T_4 = Third geometric mean $= G_3 = ar^3 = 2 \times (2)^3 = 16$

28. (b) Let the five G.M. between 486 and 2/3 are as follows

486, G_1, G_2, G_3, G_4, G_5, 2/3

$a = 486, n = 7, l = 2/3$

$l = ar^{n-1} \Rightarrow \frac{2}{3} = 486(r)^{7-1}$

$\frac{2}{3 \times 486} = r^6$

$\left(\frac{1}{3}\right)^6 (r)^6 \Rightarrow r = \frac{1}{3}$

T_5 = Fourth G.M. $= G_4 = ar^4 = 486 \times \left(\frac{1}{3}\right)^4 = 486 \times \frac{1}{86} = 6$

29. (a) $x = 1 + a + a^2 + \ldots \Rightarrow x = \frac{1}{1-a}$

$\Rightarrow a = \frac{x-1}{x}$

$y = 1 + b + b^2 + \ldots\ldots \Rightarrow y = \frac{1}{1-b}$

$\Rightarrow b = \frac{y-1}{y}$

$1 + ab + (ab)^2 + (ab)^3 + \ldots\ldots = \frac{1}{-ab}$

$= \frac{1}{1 - \left(\frac{x-1}{x}\right)\left(\frac{y-1}{y}\right)}$

$= \frac{xy}{x+y-1}$

30. (a) $a = ar + ar^2$

$\Rightarrow r^2 + r + 1 = 0$

$\Rightarrow r = \dfrac{-1 \pm \sqrt{1+4}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

but $r > 0$, $\therefore$ $r = \sqrt{5} - 1/2$

31. (c) $2 + \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ...\infty\right) + \left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + ...\infty\right)$

$= 2 + \dfrac{\frac{1}{2}}{1-\frac{1}{2}} + \dfrac{\frac{1}{3}}{1-\frac{1}{3}}$

$= 2 + \frac{1}{2} \times \frac{2}{1} + \frac{1}{3} \times \frac{3}{2}$

$= 2 + 1 + \frac{1}{2} = 3 + \frac{1}{2} = \frac{7}{2}$

32. (b) The numbers $(\sqrt{2} + 1)$, 1, $(\sqrt{2} - 1)$ will be in G.P.

$\therefore$ $(1)^2 = (\sqrt{2} + 1)(\sqrt{2} - 1) = (\sqrt{2})^2 - (1)^2 = 2 - 1 = 1$.

33. (c) Here $G = (ab)^{1/2}$ and $G_1 = ar$, $G_2 = ar^2$,........... $G_n = ar^n$

Therefore $G_1.G_2.\ G_3.......G\ n = a^n\ r^{1+2+.......+n} = a^n\ r^{n(n+1)/2}$

But $ar^{n+1} = b \Rightarrow r = \left(\frac{b}{a}\right)^{1/(n+1)}$

Therefore, the required product is

$a^n \left(\frac{b}{a}\right)^{1/(n+1).\ n(n+1)/2} = (ab)^{n/2}$

$= \{(ab)^{1/2}\}^n = G^n$

34. (b) G.M. $=(2\ .2^2\ .\ ....\ .\ 2^n)^{1/n}$

$= 2^{\frac{1+2+.......+n}{n}} = 2^{\frac{n(n+1)}{2n}} = 2^{\frac{n+1}{2}}$

35. (c) Since $a + ar = 8$

$\Rightarrow a(1 + r) = 8$ (1)

and $a + ar + ar^2 + ar^3 = 80$

$\Rightarrow a(1 + r) + ar^2(1 + r) = 80$

$\Rightarrow a(1 + r)(1 + r^2) = 80$

$\Rightarrow 1 + r^2 = \frac{80}{8} = 10$ (2)

$\Rightarrow r = 3\ (\because r > 0)$

From Equation (1), $a(1 + 3) = 8$

$\Rightarrow a = 2$

Now, $T_6 = ar^5 = 2(3)^5 = 2 \times 243 = 486$

36. (a) $\because$ $\log_2 x$, $\log_3 x$, $\log_x 16$ are in GP

$\therefore$ $(\log_3 x)^2 = \log_2 x\ .\ \log_x 16$

$\Rightarrow (\log_3 x)^2 = \log_2 16$

$\Rightarrow (\log 3\ x)^2 = 4 \log_2 2 = 4$

$\Rightarrow \log_3 x = 2$

$\Rightarrow x = 32 = 9$

37. (d) First five terms of a geometric progression are as follows: a, ar, ar^2, ar^3, ar^4.

Therefore, Mean

$= \dfrac{a + ar + ar^2 + ar^3 + ar^4}{5} = \dfrac{a(r^5 - 1)}{5(r - 1)}$.

UNSOLVED OBJECTIVE PROBLEMS: (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. If the 10th term of a geometric progression is 9 and 4th term is 4, then its 7th term is **[MPPET −1996]**

(a) 6 (b) 36 (c) 4/9 (d) 9/4

2. The sum of the series 3 + 33 + 333 +......+n terms is

(a) $\frac{1}{27}(10^{n+1} + 9n - 28)$

(b) $\frac{1}{27}(10^{n+1} + 9n - 10)$

(a) $\frac{1}{27}(10^{n+1} + 9n - 9)$

(d) none of these

[RPET −2000]

3. The sum of a G.P. with common ratio 3 is 364, and last term is 243, then the number of terms is

[MPPET −2003]

(a) 6 (b) 5

(c) 4 (d) 10 4.

4. If the third term of a G.P. is 4 then the product of its first 5 terms is

[IIT –1982; RPET –1991]

(a) 4^3 (c) 4^4
(c) 4^5 (d) none of these

5. The sum of few terms of any ratio series is 728, if common ratio is 3 and last term is 486, then first term of series will be

[UPSEAT –1999]

(a) 2 (b) 1 (c) 3 (d) 4

6. If x is added to each of numbers 3, 9, 21 so that the resulting numbers may be in G.P., then the value of x will be

[MPPET –1986]

(a) 3 (b) 1/2
(c) 2 (d) 1/3

7. If $2k+2, 5k-11, 7k-13$ be the consecutive terms of a G.P., then $k =$

(a) 11/21 (b) 1/7
(c) 7 (d) 14

8. If the sum of three terms of G.P. is 19 and product is 216, then the common ratio of the series is

[Roorkee –1972]

(a) – 3/2 (b) 3/2
(c) 2 (d) 3

9. The first term of a G.P. whose second term is 2 and sum to infinity is 8, will be

[MNR – 1979; RPET – 1992, 1995]

(a) 6 (b) 3 (c) 4 (d) 1

10. If $y = x - x^2 + x^3 - x^4 + \ldots\ldots\ldots \infty$, then value of x will be

[MNR – 1975; RPET – 1988; MPPET – 2002]

(a) $y + \frac{1}{y}$ (b) $\frac{y}{1+y}$

(c) $y - \frac{y}{y}$ (d) $\frac{xy}{1-y}$

11. If $y = x + x^2 + x^3 + \ldots\ldots\ldots\ldots \infty$, then $x =$

[DCE – 1999]

(a) $\frac{y}{1+y}$ (b) $\frac{1-y}{y}$

(c) $\frac{y}{1-y}$ (d) none of these

12. The 6th term of *a* G.P. is 32 and its 8th term is 128, then the common ratio of the G.P. is

[Pb. CET – 1999]

(a) –1 (b) 2
(c) 4 (d) – 4

13. If $x, G_1 . G_2, y$ be the consecutive terms of a G.P., then the value of $G_1 \cdot G_2$ will be

(a) y/x (b) x/y
(c) xy (d) xy

14. The sum can be found of a infinite G.P. whose common ratio is r

(a) for all values of r
(b) for only positive value of r
(c) only for $0 < r < 1$
(d) only for $-1 < r < 1$ $(r \neq 0)$

[AMU – 1982]

15. If s is the sum of an infinite G.P., the first term a then the common ratio r given by

[J & K – 2005]

(a) $\frac{a-s}{s}$ (b) $\frac{s-a}{s}$

(c) $\frac{a}{1-s}$ (d) $\frac{s-a}{a}$

16. The first term of an infi nite G.P. is 1 and each term is twice the sum of the succeeding terms, then the sum of the series is

[Kerala PET – 2007]

(a) 2 (b) 5/2
(c) 7/2 (d) 3/2

17. In an infinite geometric series the first term is a and common ratio is r. If the sum of the series is 4 and the second term is 3/4 then (a, r) is

[Kerala PET – 2007]

(a) (4/7, 3/7) (b) (2, 3/8)
(c) (3/2, 1/2) (d) (3, 1/4)

18. Sum of infinite number of terms in G.P. is 20 and sum of their square is 100. The common ratio of G.P. is

[AIEEE – 2002]

(a) 5 (b) 3/5
(c) 8/5 (d) 1/5

19. The sum of three decreasing numbers in A.P. is 27. If − 1, − 1, 3 are added to them respectively, the resulting series is in G.P. The numbers are

[AMU − 1999]

(a) 5, 9, 13
(b) 15, 9, 3
(c) 13, 9, 5
(d) 17, 9, 1

20. If p, q, r are in one geometric progression and a, b, c in another geometric progression, then cp, bq, ar are in

[Roorkee − 1998]

(a) A.P. (b) H.P.
(c) G.P. (d) none of these

21. If the third term of a G.P. be 6, then the product of first five terms is

(a) 6^3 (b) 6^4 (c) 6^5 (d) 6^6

WORK SHEET: TO CHECK PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the work sheet
2. The test is of 15 Minutes.
3. The test consists of 15 questions. The maximum marks are 45.
4. Use blue/black Ball point pen only for writing particulars / marking responses. Use of pencil is Strictly prohibited.

1. If the ratio of the sum of first three terms and the sum of first six terms of a G.P. be 125: 152, then the common ratio r is

[IIT − 1974]

(a) 3/5 (b) 5/3
(c) 2/3 (d) 3/2

2. If the nth term of geometric progression $5, \frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \ldots$ is $\frac{5}{1024}$, then the value of n is

[Kerala (Engg.) − 2002]

(a) 11 (b) 10
(c) 9 (d) 4

3. 0.5737373 .... =

[Karnataka CET −2004; AMU − 2006]

(a) 284/497 (b) 284/495
(c) 568/990 (d) 567/990

4. If the arithmetic mean of two numbers be A and geometric mean be G, then the numbers will be

[CET Karnataka − 1994]

(a) $A \pm (A^2 - G^2)$
(b) $\sqrt{A} \pm \sqrt{A^2 - G^2}$
(c) $A \pm \sqrt{(A+G)(A-G)}$
(d) $\frac{A + \sqrt{(A+G)2(A-G)}}{2}$

5. If the product of three terms of G.P. is 512. If 8 added to first and 6 added to second term, so that number may be in A.P., then the numbers are

[Roorkee − 1964]

(a) 2, 4, 8 (b) 4, 8, 16
(c) 3, 6, 12 (d) none of these

6. If r is one AM and p, q are two GM's between two given numbers, then $p^3 + q^3$ is equal to

[IIT − 1997]

(a) $2pqr$ (b) $2p^2q^2r$
(c) $2pq/r$ (d) none of these

7. The value of $9^{1/3} \times 9^{1/9} \times 9^{1/27} \times \ldots\ldots\infty$ is

(a) 9 (b) 1
(c) 3 (d) none of these

[MP PET − 2006]

8. If $x = \sqrt[4]{4}.\sqrt[8]{4}.\sqrt[16]{4}\ldots\ldots\infty$, then

[Kerala CEE − 2003]

(a) $x^2 - 4x + 6 = 0$
(b) $x^2 - 3x + 2 = 0$
(c) $x^2 - 5x + 4 = 0$
(d) $x^2 + 5x + 4 = 0$

9. The product (32) $(32)^{1/6}$ $(32)^{1/36}$ .... to ∞ is

(a) 16 (b) 32 (c) 64 (d) 0

[Roorkee – 1991; KCET–1993; Kerala (Engg.) – 2005]

10. The product of first nine terms of a G.P. is, in general, equal to which one of the following?

(a) The 9th power of the 4th term
(b) The 4th power of the 9th term
(c) The 5th power of the 9th term
(d) The 9th power of the 5th term

[NDA – 2008]

11. If in a geometric progression $\{a_n\}$, $a_1 = 3$, $a_n = 96$ and $S_n = 189$ then the value of n is

(a) 5 (b) 6 (c) 7 (d) 8

12. If the sum of the series $4 + \frac{2}{x} + \frac{4}{x^2} + \frac{8}{x^3}$ ....∞ is a finite num ber, then

[UPSEAT – 2002]

(a) $x > 2$ (b) $x > -2$
(c) $x > 1/2$ (d) none of these

13. The sum to infinity of the progression

$9 - 3 + 1 - \frac{1}{3} + ...$ is

[Karnataka CET –2005]

(a) 9 (b) 9/2
(c) 27/4 (d) 15/2

14. The value of 0.037 where .037 stands for the number 0.037037037 .... is

[MP PET –2004]

(a) 37/1000
(b) 1/27
(c) 1/37
(d) 37 / 999

15. If $a^{1/x} = b^{1/y} = c^{1/z}$ and a, b, c are in geometrical progression, then x, y, z are

[MPPET –2008]

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

ANSWER SHEET

1.	ⓐ	ⓑ	ⓒ	ⓓ	6.	ⓐ	ⓑ	ⓒ	ⓓ	11.	ⓐ	ⓑ	ⓒ	ⓓ
2.	ⓐ	ⓑ	ⓒ	ⓓ	7.	ⓐ	ⓑ	ⓒ	ⓓ	12.	ⓐ	ⓑ	ⓒ	ⓓ
3.	ⓐ	ⓑ	ⓒ	ⓓ	8.	ⓐ	ⓑ	ⓒ	ⓓ	13.	ⓐ	ⓑ	ⓒ	ⓓ
4.	ⓐ	ⓑ	ⓒ	ⓓ	9.	ⓐ	ⓑ	ⓒ	ⓓ	14.	ⓐ	ⓑ	ⓒ	ⓓ
5.	ⓐ	ⓑ	ⓒ	ⓓ	10.	ⓐ	ⓑ	ⓒ	ⓓ					

HINTS AND EXPLANATIONS

1. (a) Series is $a + ar + ar^2 + ...$

Given $= \frac{S_3}{S_6} = \frac{125}{152} \Rightarrow 152\,\frac{a(r^3-1)}{r-1}$

$= 125\,\frac{a(r^6-1)}{r-1}$

$\Rightarrow 152 = 125(r^3+1) \Rightarrow r^3 + 1 = \frac{152}{125}$

$\Rightarrow r^3 = \frac{152}{125} - 1 = \frac{27}{125}$

$\Rightarrow r = \frac{3}{5}$

2. (a) $5\left(-\frac{1}{2}\right)^{n-1} = \frac{5}{1024}$

$\Rightarrow \left(-\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^{10}$

$\Rightarrow \left(-\frac{1}{2}\right)^{n-1} = \left(-\frac{1}{2}\right)^{10}$

$\Rightarrow n - 1 = 10$

$\Rightarrow n = 11$

3. (c) $x = 0.5737373........$

$1000\,x = 573.7373........$

$10x = 5.7373$ ....

By (1) − (2) $990x = 568$

$x = \frac{568}{990}$

4. (c) Let the numbers be a and b then A = A.M.

$= \frac{a+b}{2}$ or $a + b = 2A$ (1)

G = G.M. $= \sqrt{ab} \therefore ab = G^2$ (2)

From (1) and (2) we find that a and b are the roots of $t^2 - 2At + G^2 = 0$

$\therefore t = \frac{2A \pm \sqrt{4A^2 - AG^2}}{2}$

$= A \pm \sqrt{(A-G)(A+G)}$

$= A \pm \sqrt{A^2 - G^2}$

5. (b) Three number in G.P. $\frac{a}{r}, a, ar$ .... (1)

Under condition I: $\frac{a}{r} . a . ar = 512 \Rightarrow a^3 = 8^3$

$\Rightarrow a = 8$

Under condition II: $\frac{a}{r} + 8, a + 6, ar$ in A.P.

This $\Rightarrow 2(a+6) = \left(\frac{a}{r} + 8\right) + ar$

$\Rightarrow 14 \times 2 = \frac{8}{r} + 8 + 8r$

$\Rightarrow 2r^2 - 5r + 2 = 0 \Rightarrow r = 2, \frac{1}{2}$.

But $r > 1$. Put $r = 2$ in (1).

$\frac{8}{2}$, 8, 8, 2 = 4, 8, 16

6. (a) Since y, z are two geometric means between a and b, therefore, a, y, z, b are in G.P.

$\Rightarrow \frac{y}{a} = \frac{z}{y} = \frac{b}{z}$

$\Rightarrow \frac{y^2}{z} = a$ and $\frac{z^2}{y} = b$

$\Rightarrow \frac{y^2}{z} + \frac{z^2}{y} = a + b = 2x$

$\left(\because x = \frac{a+b}{2}\right)$

7. (c) $9^{\frac{1}{3}} . 9^{\frac{1}{9}} . 9^{\frac{1}{27}} ......\infty = 9^{\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+......\infty}$

where $S = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + ...\infty$

$\frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$

Therefore, required value $= 9^{\frac{1}{2}} = \sqrt{9} = 3$

8. (c) $4^{\frac{1}{2}} . 4^{\frac{1}{4}} . 4^{\frac{1}{8}} .....4^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+......\infty} = 4^{\frac{\frac{1}{2}}{1-\frac{1}{2}}} = 4^1 = 4$

This is a root of $x^2 - 5x + 4 = 0$.

9. (c) The given product $= (32)^{1+\frac{1}{6}+\frac{1}{36}+......\infty}$

$= (32)^t$

where $t = 1 + \frac{1}{6} + \frac{6}{36}$ ....$\infty = \frac{1}{1-\frac{1}{6}} = \frac{5}{6}$

Therefore, given product $= (2^5)^{6/5} = 2^6 = 64$.

10. (d) Product of first 9 terms of GP

$= a \times ar \times ar^2 \times ar^3 \times ar^4 \times ar^5 \times ar^6 \times ar^7 \times ar^8$

$= a^9 r^{36}$

$= (ar^4)^9$

$= (5\text{th term})^9$

11. (b) $a_1 = 3, a_n = 96 \Rightarrow a_1 r^{n-1} = 96$

$\Rightarrow r^{n-1} = 32$ (i)

$S_n = \frac{a_1(r^n - 1)}{r - 1} = 186$

$\Rightarrow \frac{3(32r - 1)}{r - 1} = 186$

Hence, $r = 2$ and $n = 6$.

12. (a) $S_\infty = \frac{a}{1-r}$

where $|r| < 1$

$\frac{2}{x} < 1$

$\because x > 2$

13. (c) $9 - 3 + 1 - \frac{1}{3} +$.......... ∞

$a = 9$

$r = \frac{-3}{9} = \frac{-1}{3}$

$S_\infty = \frac{a}{1-r} = \frac{9}{1+\frac{1}{3}}$

$= \frac{\frac{9}{4}}{3} = \frac{9 \times 3}{4} = \frac{27}{4}$

14. (d) 0.37037037 =.037 +.00037 +.0000037

$= \frac{37}{10^3} + \frac{37}{10^5} + \frac{37}{10^7}$ ....

$= \frac{37}{10^3}\left[1 + \frac{1}{100} ....\right]$

$= \frac{37}{10^3}\left[\frac{1}{1-\frac{1}{100}}\right] = \frac{37}{990}$

15. (a) Let $a^{1/x} = b^{1/y} = c^{1/z} = k$

then $a = k^x$, $b = k^y$, $c = k^z$

as a, b, c are in GP

$\therefore b^2 = ac$ or $k^{2y} = k^x$

$k^z k^{2y} = k^{x+z}$

$\frac{x+z}{2} = y$

LECTURE

3 Harmonic Progression

BASIC CONCEPTS

1. Harmonic Progression

1. A sequence of numbers is said to be in a harmonical progression if the reciprocals of its terms are in arithmetic progression. In short form it is denoted by H.P.

For example

H.P.	Corresponding A.P.
1, 1/2, 1/3, ...	1, 2, 3, ...
$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, ...$	$a, a+d, a+2d, ...$

2. The *n* Term of General H.P.

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, ...\frac{1}{a+(n-1)d}, ...$$

$$T_n = n\text{th term of H.P.} = \frac{1}{a+(n-1)d}, ... ,$$

where first term of corresponding A.P. = a; and common difference = d.

Note: Problems based on H.P. first of all are solved for corresponding A.P.

3. Harmonic Mean

(1) The harmonic mean between two number a and b H.M. $= \frac{2ab}{a+b} = \frac{2}{\frac{1}{a}+\frac{1}{b}}$

(2) Harmonic mean of the n numbers $x_1, x_2, x_3, ...x_n$ H.M.

$$= \frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+......+\frac{1}{x_n}}$$

(3) If $a, H_1, H_2, H_3, ...H_n, b$ are in H.P., then $H_1, H_2, H_3,...........H_n$, be n H.M.'s between a and b., and

$$\frac{1}{H_n} = \frac{1}{a} + \frac{n(a-b)}{(n+1)ab} \text{ or } H_n = \frac{ab(n+1)}{b+na}$$

Example Two H.M. between a and b

$$H_1 = \frac{3ab}{a+2b}, H_2 = \frac{3ab}{b+2a}$$

4. Relation among A.M., G.M. and H.M.

(i) The three quantities a, b, c are in A.P., G.P. and H.P. if $= \frac{a-b}{b-c} = \frac{a}{a}, \frac{a}{b}$ and $\frac{a}{c}$ repectively.

(ii) If a, b, c are in H.P. then $a - \frac{b}{2}, \frac{b}{2}, c - \frac{b}{2}$, are in G.P.

(iii) 6, 3, 2 are in H.P.

(iv) 1, 25, 49 are in A.P.

(v) $1 \pm \sqrt{3}, -2, 1 \pm \sqrt{3}$ are in H.P.

(vi) $4 - 2\sqrt{3}$, 4, $4 + 2\sqrt{3}$ are in A.P.

5. Relation among A.M., G.M. and H.M. If A.M., G.M., and H.M. be arithmetic, geometric and Harmonic means between two numbers a and b. then

(i) $A > G > H$ if $a \neq b$

(ii) $A = G = H$ if $a = b$

(iii) $G^2 = AH$, i.e., A, G, H are in G.P.

(iv) If the ratio of A.M. and G.M. between two numbers a and b is λ, then

$$a : b = (\lambda + \sqrt{\lambda^2 - 1}) : (\lambda - \sqrt{\lambda^2 - 1})$$

(v) If the Arithmetic mean of two numbers a and b is the m times of their harmonic mean, then $a : b = \{\sqrt{m} + \sqrt{m-1}\} : \{\sqrt{m} - \sqrt{m-1}\}$

(vi) If the geometric mean of a and b is the n times of their harmonic mean, then $a : b = (n + \sqrt{n^2 - 1}) : (n - \sqrt{n^2 - 1})$

(vii) Let two arithematic mean A_1, A_2 two geometric mean G_1, G_2 and two Harmonic mean H_1, H_2 be between two numbers a and b, then $\dfrac{A_1 + A_2}{H_1 + H_2} = \dfrac{G_1G_2}{H_1H_2}$

(ix) Between the two numbers a and b the quantity $\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$ being A.M., G.M. and H.M respectively according as n is 0, $-\frac{1}{2}$ and -1 respectively.

(x) No term of G.P. and H.P. can be zero (0).

(xi) Common ratio of G.P. can not be zero.

SOLVED SUBJECTIVE PROBLEMS: (CBSE/STATE BOARD): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. $a_1, a_2, a_3, ..., a_{10}$ are in A.P. and $h_1, h_2, h_3, ..., h_{10}$ are in H.P. If $a_1 = h_1 = 2$ and $a_{10} = h_{10} = 3$, then prove that $a_4h_7 = 6$.

[IIT – 1999]

Solution

Given A.P. is $a_1, a_2, a_3, ..., a_{10}$

$\therefore a_{10} = a_1 + (10 - 1)\, d$, [$\because = a + (n - 1)d$]

$\Rightarrow 3 = 2 + 9d \Rightarrow d = \frac{1}{9}$

Given H.P. is $h_1, h_2, h_3, ..., h_{10}$

$\therefore \frac{1}{h_{10}} = \frac{1}{h_1} + (10 - 1)\, d_1 \Rightarrow -\frac{1}{3} = \frac{1}{2} + 9d_1$

$\Rightarrow 9d_1 = \frac{1}{3} - \frac{1}{2} \Rightarrow 9d_1 = -\frac{1}{6}$

$\Rightarrow d_1 = -\frac{1}{54} \therefore \frac{1}{h_7} = \frac{1}{h_1} + (7 - 1)d_1$

$\Rightarrow \frac{1}{h_7} = \frac{1}{2} + 6\left(-\frac{1}{54}\right)$

$\Rightarrow \frac{1}{h_7} = \frac{1}{2} - \frac{1}{9} = \frac{7}{18} \Rightarrow h_7 = \frac{18}{7}$

and $a_4 = a_1 + (4 - 1)d$

$\Rightarrow a_4 = 2 + 3 \times \frac{1}{9} \Rightarrow a_4 = 2 + \frac{1}{3} \Rightarrow a_4 = \frac{7}{3}$

$\therefore a_4h_7 = \frac{7}{3} \times \frac{18}{7} = 6$

Proved

2. If the ratio of A.M. and G.M. between a and b is $m : n$, then prove that: $\dfrac{a}{b} = \dfrac{m + \sqrt{m^2 - n^2}}{m - \sqrt{m^2 - n^2}}$.

[Kerala (Engg.) – 2005]

Solution

A.M. of a and $b = \frac{a + b}{2}$ and G.M. of a and $b = \sqrt{ab}$

According to question, $\frac{a + b}{2} : \sqrt{ab} = m : n$

$\Rightarrow \dfrac{a + b}{2\sqrt{ab}} = \dfrac{m}{n}$ (1)

$\Rightarrow \dfrac{(a + b)^2}{4ab} = \dfrac{m^2}{n^2}$

$\Rightarrow \dfrac{(a + b)^2}{4ab} - 1 = \dfrac{m^2}{n^2} - 1$

$$\Rightarrow \frac{(a+b)^2 - 4ab}{4ab} = \frac{m^2 - n^2}{n^2}$$

$$\Rightarrow \frac{(a-b)^2}{4ab} = \frac{m^2 - n^2}{n^2}$$

$$\Rightarrow \frac{a-b}{2\sqrt{ab}} = \frac{\sqrt{m^2 - n^2}}{n} \qquad (2)$$

Dividing Equation (1) by Equation (2), we have

$$\frac{a+b}{a-b} = \frac{m}{\sqrt{m^2 - n^2}}$$

By componendo and dividendo law

$$\frac{a+b+a-b}{a+b-a+b} = \frac{m + \sqrt{m^2 - n^2}}{m - \sqrt{m^2 - n^2}}$$

$$\Rightarrow \frac{a}{b} = \frac{m + \sqrt{m^2 - n^2}}{m - \sqrt{m^2 - n^2}}$$

Proved

3. If the roots of $10x^3 - cx^2 - 54x - 27 = 0$ are in harmonic progression, then find c and all the roots.

[Roorkee – 1995]

Solution

(b) If α, β, γ be the roots of the given equation in H.P. then p, q, r will be the roots of the equation $\frac{10}{x^3} - \frac{c}{x^2} - \frac{54}{x} - 27 = 0$

or $27x^3 + 54x^2 + cx - 10 = 0$

Clearly p, q and r are in A.P. (1)

$\therefore\ 2q = p + r$ or $3q = p + q + r = -54/27 = -2$

$\therefore\ q = -2/3$. Hence, $(-2/3)$ is a root of (1)

Putting in (1) we get, $c = 9$

$\therefore\ 27x^3 + 54x^2 + 9x - 10 = 0$

has a root $-2/3$ or $3x + 2$ is its factor.

$\therefore\ (3x + 2)(9x^2 + 12x - 5) = 0$

or $(3x + 2)(3x + 5)(3x - 1) = 0$

$\therefore\ x = 1/3, -2/3, -5/3$ which are in A.P. Hence, the roots of the given equation in H.P. are $3, -3/2, -3/5$.

UNSOLVED SUBJECTIVE PROBLEMS (CBSE/STATE BOARD): TO GRASP THE TOPIC, SOLVE THESE PROBLEMS

Exercise I

1. Find the 10th term of the series $8 + 2\frac{2}{3} + 1\frac{3}{5} + \ldots\ldots.$
2. Fifth and eighth term of an H.P. are $\frac{1}{11}$ and $\frac{1}{17}$ respectively. Find the progression.
3. Which term of the sequence $\frac{1}{10}, \frac{1}{8}, \frac{1}{6}, \frac{1}{4}$ .... will be $\left(-\frac{1}{26}\right)$?
4. Find the harmonic mean between $\frac{a}{b}$ and $\frac{b}{a}$.
5. If H is the harmonic mean between a and b, prove that $\frac{H+a}{H-a} + \frac{H+b}{H-b} = 2$
6. A.M. between two numbers is 6 and H.M. between them is $\frac{16}{3}$. Find the numbers.
7. If a, b, c are in H.P. prove that $\frac{a}{c} = \frac{a-b}{b-c}$
8. If mth term of an H.P. is n and nth term is m, then find its (i) rth and (ii) $(m + n)$th term
9. If a, b, c are in A.P. and b, c, d are in H.P., then prove that $ad = bc$.

Exercise II

1. Find nth term of $1, \frac{2}{3}, \frac{1}{2}, \frac{2}{5}, \ldots\ldots.$
2. If a, b, c are in H.P., prove that $\frac{1}{b-a} + \frac{1}{b-c} + \frac{1}{a} + \frac{1}{c}$
3. Insert four harmonic means between $\frac{3}{7}$ and $\frac{1}{4}$.
4. Find the 4th term of the sequence $2, 2\frac{1}{2}, 3\frac{1}{3}, \ldots.$
5. Find the nth term of $4, 4\frac{2}{7}, 4\frac{8}{5}, 5, \ldots\ldots.$
6. Second term of an H.P. is $\frac{1}{6}$ and its 6th term is $\left(-\frac{1}{6}\right)$ find its 20th term.

7. If $b + c, c + a, a + b$ are in H.P. prove that a^2, b^2, c^2 are in A.P.

8. Insert three Harmonic means between $\frac{9}{2}$ and $\frac{3}{2}$.

9. Three numbers are in H.P. Their sum is 11 and sum of their reciprocals is 1. Find the numbers.

10. If a, b, c are in H.P., prove that

$\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are also in H.P.

ANSWERS

Exercise I

1. $\frac{8}{19}$

2. $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots\ldots$

3. 19th

4. $\frac{2ab}{a^2 + b^2}$

6. 8, 4

8. (i) $\frac{mn}{r}$ (ii) $\frac{mn}{m+n}$

Exercise II

1. $\frac{1}{2}(n + 1)$

3. $\frac{3}{8}, \frac{1}{3}, \frac{3}{10}, \frac{3}{11}$

4. 5

5. $\frac{60}{16 - n}$

6. $-\frac{1}{48}$

8. $3, \frac{9}{4}, \frac{9}{5}$

9. 2, 3, 6, or 6, 3, 2

SOLVED OBJECTIVE QUESTIONS: HELPING HAND

1. If $4a^2 + 9b^2 + 16c^2 = 2\,(3ab + 6bc + 4ac)$, where a, b, c are non−zero numbers. Then a, b, c are in

[AMU – 2005]

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

Solution

(c) $(2a - 3b)^2 + (3b - 4c)^2 + (4c - 2a)^2 = 0$

$\therefore\ 2a = 3b = 4c = k$ (Assume)

a, b, c are $k/2, k/3, k/4$ which are in H.P.

2. If x, y, z are in H.P., then the value of expression $\log(x + z) + \log(x - 2y + z)$ will be

[RPET – 1985, 2000]

(a) $\log(x - z)$ (b) $2\log(x - z)$
(c) $3\log(x - z)$ (d) $4\log(x - z)$

Solution

(b) If x, y, z are in H.P., then $y = \frac{2xz}{x+z}$

Now, $\log_e(x + z) + \log_e(x - 2y + z)$

$= \log_e\{(x + z)(x - 2y + z)\}$

$= \log_e\left[(x + z)\left(x + z - \frac{4xz}{x + b}\right)\right]$

$= \log_e[(x + z)^2 - 4xz] = \log_e(x - z)^2$

$= 2\log_e(x - z)$.

3. If the harmonic mean between a and b be H, then the value of

$\frac{1}{H-a} + \frac{1}{H-b}$ is

(a) $a + b$ (b) ab

(c) $\frac{1}{a} + \frac{1}{b}$ (d) $\frac{1}{a} - \frac{1}{b}$

Solution

(c) Putting $H = \frac{2ab}{a+b}$, we have,

$\frac{1}{H-a} + \frac{1}{H-b}$

$= \frac{1}{\left(\frac{2ab}{a+b} - a\right)} + \frac{1}{\left(\frac{2ab}{a+b} - b\right)}$

$$= \frac{a+b}{ab-a^2} + \frac{a+b}{ab-b^2}$$

$$\left(\frac{a+b}{b-a}\right)\left(\frac{1}{a}-\frac{1}{b}\right) = \left(\frac{a+b}{b-a}\right)\left(\frac{b-a}{ab}\right)$$

$$= \frac{a+b}{ab} = \frac{1}{a} + \frac{1}{b}$$

4. If a, b, c are in H.P., then $\frac{c}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in:

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

[Roorkee – 1980]

Solution

(c) If a, b, c are in H.P. $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in A.P.

$\Rightarrow \frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c}$ are in A.P

$\Rightarrow \frac{b+c}{a}, \frac{a+c}{b}, \frac{a+b}{c}$ are in H.P

$\Rightarrow \frac{a}{b+c}, \frac{b}{a+c}, \frac{c}{a+b}$ are in H.P

5. If $\frac{x+y}{2}, y, \frac{y+z}{2}$ are in H.P., then x, y, z are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

[RPET – 1989; MPPET – 2003]

Solution

(b) If $\frac{x+y}{2}, y, \frac{y+z}{2}$ are in H.P., then

$$y = \frac{2\left(\frac{x+y}{2} \cdot \frac{y+z}{2}\right)}{\frac{x+y}{2} + \frac{y+z}{2}} = \frac{\frac{2}{4}(x+y)(y+z)}{\frac{1}{2}(x+2y+z)}$$

$$y = \frac{xy + xz + y^2 + yz}{x + 2y + z}$$

$\Rightarrow xy + 2y^2 + yz = xy + xz + y^2 + yz \Rightarrow y^2 = xz$ Thus, x, y, z will be in G.P.

6. If three unequal numbers p, q, r are in H.P. and their squares are in A.P., then the ratio $p : q : r$ is

(a) $1 - \sqrt{3} : 2 : 1 + \sqrt{3}$
(b) $1 : \sqrt{2} : -\sqrt{3}$
(c) $1 : -\sqrt{2} : \sqrt{3}$
(d) $1 \pm \sqrt{3} : -2 : 1 \mp \sqrt{3}$

Solution

(d) By hypothesis,

$$q = \frac{2pr}{p+r} \Rightarrow \frac{q}{2} = \frac{pr}{p+r} = K \text{ (say)}$$

$\Rightarrow q = 2K, pr = (p + r) K$. Also p^2, q^2, r^2 are in A.P.

$\therefore 2q^2 = p^2 + r^2 = (p + r)^2 - 2pr$
$\Rightarrow 8K^2 = (p + r)^2 - 2(p + r) K$
$\Rightarrow (p + r)^2 - 2 (p + r) K - 8K^2 = 0$
$\Rightarrow p + r = 4K, -2K$
when $p + r = 4K$, then $pr = 4K^2$
$\therefore (p - r)^2 = (p + r)^2 - 4pr = 16K^2 - 16K^2 = 0$
$\Rightarrow p = r$
But this is not possible ($\because p \neq r$)
$\therefore p + r = -2K \Rightarrow pr = -2K \cdot K = -2K^2$
Now $(p - r)^2 = (p + r)^2 - 4pr$
$= 4K^2 - 4(-2K^2) = 12K^2$
$\Rightarrow p - r = \pm 2\sqrt{3} K$, also $p + r = -2K$
$\therefore 2p = (-2 \pm 2\sqrt{3}) K \Rightarrow p = (-1 \pm \sqrt{3})K$
and $2r = -2K \mp 2\sqrt{3} K \Rightarrow r = (-1 \mp \sqrt{3})K$
$\therefore p : q : r = (-1 \pm \sqrt{3}) K : 2K : (-1 \mp \sqrt{3})K$
$= -1 \pm \sqrt{3} : 2 : 1 \mp 3 = 1 \mp \sqrt{3} : -2 : 1 \pm \sqrt{3}$.

7. $\log_3 2, \log_6 2, \log_{12} 2$ are in

[RPET – 1993, 2001]

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

Solution

(c) If the numbers are $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, then $x = \log_2 3$, $y = \log_2 2 \cdot 3 = 1 + \log_2 3$ and $z = 2 + \log_2 3$. Therefore x, y, z are in A.P. with common difference 1.

Hence, $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ i.e., the given numbers are in H.P.

8. If a, b, c are in A.P., then

$\frac{1}{\sqrt{a}+\sqrt{b}}, \frac{1}{\sqrt{a}+\sqrt{c}}, \frac{1}{\sqrt{b}+\sqrt{c}}$ are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

[Roorkee – 1999; Kerala (Engg.) – 2005]

Solution

(a) a, b, c are in A.P. i.e., $2b = a + c$

Let

$$\frac{1}{\sqrt{a}+\sqrt{c}} - \frac{1}{\sqrt{a}+\sqrt{b}} = \frac{1}{\sqrt{b}+\sqrt{c}} - \frac{1}{\sqrt{a}+\sqrt{c}}$$

$$\Rightarrow \frac{\sqrt{b}-\sqrt{c}}{\sqrt{a}+\sqrt{b}}, \frac{\sqrt{a}-\sqrt{b}}{\sqrt{b}+\sqrt{c}} \Rightarrow b - c = a - b$$

$\Rightarrow 2b = a + c$

$\therefore \frac{1}{\sqrt{a}+\sqrt{b}}, \frac{1}{\sqrt{a}+\sqrt{c}}, \frac{1}{\sqrt{b}+\sqrt{c}}$ are in A.P

9. Given $a + d > b + c$ where a, b, c, d are real numbers, then

(a) a, b, c, d are in A.P.

(b) $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$ are in A.P

(c) $(a + b), (b + c), (c + d), (a + d)$ are in A.P.

(d) $\frac{1}{a+b}, \frac{1}{b+c}, \frac{1}{c+d}, \frac{1}{a+d}$ are in A.P

[Kurukshetra CEE – 1998]

Solution

(b) $a + d > b + c$

$\Rightarrow a + b + c + d > 2b + 2c \Rightarrow \frac{a+c}{2} + \frac{b+d}{2} > b + c$

$\because \frac{a+c}{2} > b$ and $\frac{b+d}{2} > c$ [$\because A > H$]

b is the H.M. of a and c and their A.M. is $\frac{a+c}{2}$

c is H.M. of b and d and their A.M. is $\frac{b+d}{2}$

Hence, a, b, c, d are in H.P. $\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$ are in A.P.

10. If $\frac{b}{a}, \frac{c}{b}, \frac{a}{c}$ are in H.P., then

[UPSEAT – 2002]

(a) a^2b, c^2a, b^2c are in A.P.

(b) a^2b, b^2c, c^2a are in H.P.

(c) a^2b, b^2c, c^2a are in G.P.

(d) none of these

Solution

(a) $\frac{b}{a}, \frac{c}{b}, \frac{a}{c}$ are in A.P.

$\Rightarrow \frac{2c}{b} = \frac{b}{a} + \frac{a}{c} \Rightarrow \frac{2c}{b} = \frac{bc + a^2}{ac} \Rightarrow 2ac^2 = b^2c + ba^2$

$\therefore a^2b, c^2a$ and b^2c are in A.P.

11. A boy goes to school from his home at a speed of x km/hour and comes back at a speed of y km/hour, then the average speed is given by

(a) A.M. (b) G.M.
(c) H.M. (d) none of these

[DCE – 2002]

Solution

(c) Let, the distance of school from home $= d$ and time taken are t_1 and t_2.

$\therefore t_1 = \frac{d}{x}$ and $t_2 = \frac{d}{y}$;

Avg. velocity $= \frac{\textit{Total}\text{ distance}}{\textit{Total}\text{ time}}$

$$= \frac{2d}{\left(\frac{d}{x}+\frac{d}{y}\right)} = \frac{2xy}{x+y},$$

which is the H.M. of x and y.

12. If $\cos(x - y)$, $\cos x$ and $\cos(x + y)$ are in HP, then $\cos x \sec(y/2)$ equals

[IIT – 1997]

(a) 1 (b) 2
(c) $\sqrt{2}$ (d) none of these

Solution

(c) $\frac{2}{\cos x} = \frac{1}{\cos(x-y)} + \frac{1}{\cos(x+y)}$

$$= \frac{2\cos x \cos y}{\cos^2 x - \sin^2 y}$$

$\Rightarrow \cos^2 x - \sin^2 y = \cos^2 x \cos y$

$\Rightarrow \cos^2 x (1 - \cos y) = \sin^2 y = 1 - \cos^2 y$

$\Rightarrow \cos^2 x (1 - \cos y) = (1 - \cos y)(1 + \cos y)$

$\Rightarrow \cos^2 x = 1 + \cos y = 2\cos^2 y/2 \therefore \cos x \sec(y/2) = \pm\sqrt{2}$.

13. a, b, c are in G.P. with $1 < a < b < n$, and $n > 1$ is an integer. $\log_a n, \log_b n, \log_c n$ form a sequence . This sequence is which one of the following

(a) harmonic progression
(b) arithmetic progression
(c) geometric progression
(d) none of these

[NDA – 2007]

Solution

(a) Given a, b, c are in G.P. $\therefore$ $b^2 = ac$ taking log on both sides $2\log b = \log a + \log c$
$\log a, \log b, \log c$ are in A.P.

$\therefore \frac{1}{\log a}, \frac{1}{\log b}, \frac{1}{\log c}$ are in H.P.

$\therefore \log_a n, \log_b n, \log_c n$ are in H.P.

14. If a, b, c are in A.P. and a^2, b^2, c^2 are in H.P., then

(a) $a = b = c$ (b) $2b = 3a + c$
(c) $b^2 = \sqrt{(ac/8)}$ (d) none of these

[MNR – 1986, 1988; IIT – 1977, 2003]

Solution

(a) Given that a, b, c are in A.P.

$\Rightarrow 2b = a + c$ (1)

and a^2, b^2, c^2 are im H.P. $\Rightarrow b^2 = \frac{2a^2c^2}{a^2 + c^2}$

$\Rightarrow b^2 (a^2 + c^2) = 2a^2c^2$

$\Rightarrow b^2\{(a + c)^2 - 2ac\} = 2a^2c^2$

$\Rightarrow b^2 \{4b^2 - 2ac\} = 2a^2c^2$, from (1)

$\Rightarrow 4b^4 - 2acb^2 = 2a^2c^2$

$\Rightarrow (b^2 - ac)(2b^2 + ac) = 0$

$\Rightarrow$ Either $b^2 - ac = 0$ or $2b^2 + ac = 0$

If $b^2 - ac = 0$, then $b^2 = ac \Rightarrow \left\{\frac{1}{2}(a + c)\right\}^2 = ac$ from (1)

$\Rightarrow (a + c)^2 = 4ac \Rightarrow (a - c)^2 = 0$.

Therefore $a = c$ and if $a = c$ then from $b^2 = ac$, we get $b^2 = a^2$ or $b = a$. Thus, $a = b = c$.

15. If a, b, c are in G.P. and x, y are the arithmetic means between a, b and b, c respectively, then $\frac{a}{x} + \frac{c}{y}$ is equal to

(a) 0 (b) 1
(c) 2 (d) 1/2

[Roorkee – 1969]

Solution

(c) Given that a, b, c are in G.P.

So, $b^2 = ac$ (1)

$x = \frac{a + b}{2}$ (2)

$y = \frac{b + c}{2}$ (3)

Now,

$$\frac{a}{x} + \frac{c}{y} = \frac{2a}{a+b} + \frac{2c}{b+c} = \frac{2(ab + bc + 2ca)}{ab + ac + b^2 + bc}$$

$$= \frac{2(ab + bc + 2ca)}{(ab + ac + ac + bc)} = 2, \{\because b^2 = ac\}$$

Trick: Let $a = 1, b = 2, c = 4$, then obviously $x = \frac{3}{2}$ and $y = 3$, then $\frac{1}{3/2} + \frac{4}{3} = 2$

16. If the $(m + 1)$th, $(n + 1)$th and $(r + 1)$th terms of an A.P. are in G.P. and m, n, r are in H.P., then the value of the ratio of the common difference to the first term of the A.P. is

(a) $-\frac{2}{n}$ (b) $\frac{2}{n}$
(c) $-\frac{n}{2}$ (d) $\frac{n}{2}$

[MNR – 1989; Roorkee – 1994]

Solution

(a) Let a be the first term and d be the common difference of the given A.P. Then as given the $(m + 1)$th, $(n + 1)$th, and $(r + 1)$th terms are in G.P.

$\Rightarrow a + md, a + nd, a + rd$ are in G.P.

$\Rightarrow (a + nd)^2 = (a + md)(a + rd)$

$\Rightarrow a(2n - m - r) = d(mr - n^2)$

or $\frac{d}{a} = \frac{2n - (m + r)}{mr - n^2}$ (1)

Next, m, n, r in H.P.

$\Rightarrow n = \frac{2mr}{m + r}$ (2)

Form (1) and (2)

$$\frac{d}{a} = \frac{2n(m + r)}{mr - n^2} = \frac{2}{n}\left(\frac{2n - (m + r)}{(m + r) - 2n}\right) = -\frac{2}{n}$$

17. $x + y + z = 15$ if $9, x, y, z, a$ are in A.P. while $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{5}{3}$ if $9, x, y, z, a$ are in H.P., then the value of a will be

(a) 1 (b) 2
(c) 3 (d) 9

[IIT – 1978]

Solution

(a) $x + y + z = 15$, if 9, x, y, z, a are in A.P.
Sum $= 9 + 15 + a = \frac{5}{2}(9 + a) \Rightarrow 24 + a = \frac{5}{2}(9 + a)$
$\Rightarrow 48 + 2a = 45 + 5a \Rightarrow 3a = 3$
$\Rightarrow a = 1$ (1)

and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{5}{3}$ if 9, x, y, z, a are in H.P.

Sum $= \frac{1}{9} + \frac{5}{3} + \frac{1}{a} = \frac{5}{2}\left[\frac{1}{9} + \frac{1}{a}\right] \Rightarrow a = 1$

18. If the ratio of A.M. between two positive real numbers a and b to their H.M. is m: n, then

(a) $\frac{\sqrt{m-n}+\sqrt{n}}{\sqrt{m-n}-\sqrt{n}}$ (b) $\frac{\sqrt{n}+\sqrt{m-n}}{\sqrt{n}-\sqrt{m-n}}$

(c) $\frac{\sqrt{m}+\sqrt{m-n}}{\sqrt{m}-\sqrt{m-n}}$ (d) none of thess

Solution

(c) $\dfrac{\frac{a+b}{2}}{\frac{2ab}{a+b}} = \frac{m}{n} \Rightarrow \frac{(a+b)^2}{4ab} = \frac{m}{n}$

$\Rightarrow \frac{(a+b)^2}{2ab} = \frac{2m}{n}$

Applying dividendo, we get

$\frac{a^2 + b^2}{2ab} = \frac{2m - n}{n}$ Applying componendo and dvidendo

We get,

$\frac{(a+b)^2}{(a-b)^2} = \frac{m}{m-n} \Rightarrow \frac{a+b}{a-b} = \frac{\sqrt{m}}{\sqrt{m-n}}$

Again, applying componendo and dividend

We get, $\frac{a}{b} = \frac{\sqrt{m}+\sqrt{m-n}}{\sqrt{m}-\sqrt{m-n}}$

19. If a, b, c are in H.P., then for all $n \Rightarrow N$ the true statement is

(a) $a^n + c^n < 2b^n$ (b) $a^n + c^n > 2b^n$
(c) $a^n + c^n = 2b^n$ (d) none of these

[RPET – 1995]

Solution

(b) For two numbers a and c

$\frac{a^n + c^n}{2} > \left(\frac{a+c}{2}\right)^n$ (Where $n \Rightarrow N$, $n > 1$)

$\because$ A.M. > G.M. > H.M. $\therefore \frac{a+c}{2} > b$

($\because$ a, b, c are in H.P.)

$\Rightarrow \left(\frac{a+c}{2}\right)^n > b^n$

$\Rightarrow \frac{a^n + c^n}{2} > \left(\frac{a+c}{2}\right)^n > b^n$.

20. If first three terms of sequence 1/16, a, b, 1/6 are in geometric series and last three terms are in harmonic series, then the values of a and b will be

[UPSEAT – 1999]

(a) $a = -1/4$, $b = 1$
(b) $a = 1/12$, $b = 1/9$
(c) (a) and (b) both are
(d) none of these true

Solution

(c) If 1/16, a, b are in G.P., then $a^2 = \frac{b}{16}$
or $16a^2 = b$ (1)

and if a, b, 1/6 are in H.P., then $b = \dfrac{2.a.\frac{1}{6}}{a + \frac{1}{6}}$

$= \frac{2a}{6a+1}$ (2)

From (1) and (2), $16a^2 = \frac{2a}{6a+1}$

or $2a\left(8a - \frac{1}{6a+1}\right) = 0$

or $8a(6a + 1) - 1 = 0$
or $48a^2 + 8a - 1 = 0$, ($\because$ $a \neq 0$)

or $(4a + 1)(12a - 1) = 0$ $\because$ $a = -\frac{1}{4}, \frac{1}{12}$

When $a = -\frac{1}{4}$ then from (1),

$b = 16\left(-\frac{1}{4}\right)^2 = 1$

When $a = \frac{1}{12}$ then from (1),

$b = 16\left(\frac{1}{12}\right)^2 = \frac{1}{9}$

Therefore, $a = -\frac{1}{4}$, $b = 1$ or $a = \frac{1}{12}$, $b = \frac{1}{9}$

21. If $(y-x)$, $2(y-a)$ and $(y-z)$ are in H.P., then $x-a, y-a, z-a$ are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

[RPET – 2001]

Solution

(b) $(y-x)$, $2(y-a)$; $(y-z)$ are in H.P.

$$\Rightarrow \frac{1}{y-x}, \frac{1}{2(y-a)}, \frac{1}{y-z} \text{ are in A.P.}$$

$$\Rightarrow \frac{1}{2(y-a)} - \frac{1}{y-x} = \frac{1}{y-z} - \frac{1}{2(y-a)}$$

$$\Rightarrow \frac{y-x-2y+2a}{(y-x)} = \frac{2y-2a-y+z}{(y-a)-(z-a)}$$

$$\Rightarrow \frac{-x-y+2a}{(y-x)} = \frac{y+z+2a}{(y-z)}$$

$$\Rightarrow \frac{(x-a)+(y-a)}{(x-a)-(y-a)} = \frac{(y-a)+(z-a)}{(y-a)-(z-a)}$$

$$\Rightarrow \frac{(x-a)}{(y-a)} = \frac{(y-a)}{(z-a)}$$

$\Rightarrow (y-a)^2 = (x-a)(z-a)$

i.e., $(x-a)$, $(y-a)$, $(z-a)$ are in G.P.

22. If the sum of the n terms of G.P. is S product is P and sum of their inverse is R, then p^2 is equal to

[IIT – 1996; Roorkee – 1981]

(a) $\frac{R}{S}$ (b) $\frac{S}{R}$

(c) $\left(\frac{R}{S}\right)^n$ (d) $\left(\frac{S}{R}\right)^n$

Solution

(d) Given that sum

$$S = \frac{a(r^n-1)}{r-1} = \frac{a(r-1^n)}{1-r} \quad (1)$$

$P = a(ar)(ar^2) \ldots\ldots (ar^{n-1}) = a^n r^{1+2+\ldots+(n-1)}$

$= a^n r^{(n-1)n/2}$ i.e., $P^2 = a^{2n} r^{n(n-1)}$ (2)

and $R = \frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \ldots\ldots$ up to n terms

$= \frac{1}{a}\left(1 + \frac{1}{r} + \frac{1}{r^2} + \ldots\ldots \text{upto } n \text{ terms}\right)$

$$= \frac{\frac{1}{a}\left[\left(\frac{1}{r}\right)^n - 1\right]}{\left(\frac{1}{r}-1\right)} \left(\because \frac{1}{r} > 1\right)$$

if $r < 1 = \frac{(1-r^n)}{ar^{n-1}(1-r)}$ (3)

Therefore,

$$\left(\frac{S}{R}\right) = \frac{a(1-r^n)}{1-r} \times \frac{ar^{n-1}(1-r)}{(1-r^n)} = a^2 r^{n-1}$$

or $\left(\frac{S}{R}\right)^n = (a^2 r^{n-1})^n = a^{2n} r^{n(n-1)} = P^2$

23. If a, b, c, d are positive real numbers such that $a+b+c+d = 2$, then $M = (a+b)(c+d)$ satisfies the relation

(a) $0 < M \leq 1$ (b) $1 \leq M \leq 2$
(c) $2 \leq M \leq 3$ (d) $3 \leq M \leq 4$

[IIT Screening – 2000]

Solution

(a) **Step 1:** A.M. $\geq$ G.M. (Arithmetic mean of positive numbers is always greater than equal to G.M.)

$$\frac{(a+b)+(c+d)}{2} \geq \sqrt{(a+b)(c+d)} \text{ or}$$

$\frac{2}{2} > \sqrt{M}$ or $1 \geq M$

Also $M > 0$. So, $0 < M \leq 1$.

24. If $a_1, a_2, a_3, \ldots\ldots\ldots, a_n$ are in H.P., then $a_1a_2 + a_2a_3 + .. + a_{n-1}a_n$ will be equal to

(a) $a_1 a_n$ (b) $na_1 a_n$
(c) $(n-1)\, a_1a_n$ (d) none of these

[AMU – 2003; AIEEE – 2006; IIT – 1975]

Solution

(c) Since, $a_1, a_2, a_3, \ldots\ldots, a_n$ are in H.P.

Therefore, $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3} \ldots\ldots \frac{1}{a_n}$ will be in A.P.

Which gives

$$\frac{1}{a_2}, \frac{1}{a_1}, \frac{1}{a_3}, \frac{1}{a_2} = \ldots\ldots \frac{1}{a_n} - \frac{1}{a_{n-1}} = d$$

$$\Rightarrow \frac{a_1-a_2}{a_1a_2} = \frac{a_2-a_3}{a_2a_3} = \ldots\ldots = \frac{a_{n-1} - a_n}{a_{n-1}a_n} = d$$

$\Rightarrow a_1 - a_2 = da_1 a_2$

$a_2 - a_3 = da_2 a_3$

----

and $a_{n-1} - a_n = da_n a_{n-1}$

Adding these, we get, $d(a_1a_2 + a_2a_3 + ...+ a_n a_{n-1})$

$= (a_1 + a_2 + ... + a_{n-1}) - (a_2 + a_3 + ... + a_n)$

$= a_1 - a_n$ (1)

Also nth term of this A.P. is given by

$$\frac{1}{a_n} = \frac{1}{a_1} + (n-1)d$$

$$\Rightarrow d = \frac{a_1 - a_n}{a_1 a_n (n-1)}$$

Substituting this value of d in (1)

$$(a_1 - a_n) = \frac{a_1 - a_n}{a_1 a_n (n-1)}(a_1a_2 + a_2a_3 + ... + a_n a_{n-1})$$

$$(a_1 a_2 + a_2a_3 + ... + a_n a_{n-1}) = a_1a_n (n-1)$$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. If 5th term of a H.P. is 1/45 and 11th term is 1/69, then its 16th term will be

(a) 1/89 (b) 1/85 (c) 1/80 (d) 1/79

[RPET – 1987, 1997]

2. If the 7th term of a H.P. is 1/10 and the 12th term is 1/25, then the 20th term is

(a) 1/37 (b) 1/41

(c) 1/45 (d) 1/49

[MPPET – 1997]

3. The 9th term of the series $27 + 9 + 5\frac{2}{5} + 3\frac{6}{7}$ +....will be

(a) $1\frac{10}{17}$ (b) 10/ 17

(c) 16 /27 (d) 17/ 27

[MPPET – 1983]

4. If a, b, c are in H.P., then which one of the following is true

(a) $\frac{1}{b-a} + \frac{1}{b-c} = \frac{1}{b}$ (b) $\frac{ac}{a+c} = b$

(c) $\frac{b+a}{b-a} + \frac{b+c}{b-c} = 1$ (d) none of these

[MNR – 1985]

5. If a, b, c are in H.P., then the value of $\left(\frac{1}{b} + \frac{1}{c} - \frac{1}{a}\right)\left(\frac{1}{c} + \frac{1}{a} - \frac{1}{b}\right)$, is

(a) $\frac{2}{bc} + \frac{1}{b_2}$ (b) $\frac{3}{c^2} + \frac{2}{ca}$

(c) $\frac{3}{b^2} + \frac{2}{ab}$ (d) none of these

[MPPET – 1998; Pb. CET – 2000]

6. If the harmonic mean between a and b be $\frac{H+a}{H-a} + \frac{H+b}{H-b} =$

[AMU – 1998]

(a) 4 (b) 2

(c) 1 (d) $a + b$

7. If there are n harmonic means between 1 and 1/31 and the ratio of 7th and $(n-1)$th harmonic means is 9: 5, then the value of n will be:

(a) 12 (b) 13 (c) 14 (d) 15

[RPET – 1986]

8. Let the positive numbers a, b, c, d be in A.P., then abc, abd, acd, bcd are

(a) not in A.P./G.P./H.P

(b) in A.P.

(c) in G.P.

(d) in H.P.

[IIT Screening – 2001]

9. If $\log_a x$, $\log_b x$, $\log_c x$ be in H.P., then a, b, c are in

(a) A.P. (b) H.P.

(c) G.P. (d) none of these

10. If a, b, c are in A.P., then 3^a, 3^b, 3^c shall be in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these

[Pb. CET – 1990]

11. If a, b, c are in A.P., then 10^{ax+10}, 10^{bx+10}, 10^{cx+10} will be in

(a) A.P$\int$.
(b) G.P. only when $x > 0$
(c) G.P. for all values of x
(d) G.P. for $x < 0$

[Pb. CET – 1989]

12. If a, b, c are in A.P., as well as in G.P then

[MNR – 1981]

(a) $a = b \neq c$ (b) $a \neq b = c$
(c) $a \neq b \neq c$ (d) $a = b = c$

13. If x, 1, z are in A.P. and x, 2, z are in G.P., then x, 4, z will be in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

[IIT – 1965]

14. If a, b, c are in A.P., b, c, d are in G.P. and c, d, e are in HP, then a, c, e are in

[AMU – 1988, 2001; MPPET – 1993]

(a) no particular order (b) A.P.
(c) G.P. (d) H.P.

15. The sixth H.M. between 3 and 6/13 is

[RPET – 1996]

(a) 63/120 (b) 63/12
(c) 126/105 (d) 120/63

16. The fifth term of the H.P.., 2, $2\frac{1}{2}$, $3\frac{1}{3}$, ... will be

(a) $5\frac{1}{5}$ (b) 3
(c) 1/10 (d) 10

[MPPET – 1984]

17. The first term of a harmonic progression is 1/7 and the second term is 1/9. The 12th term is

[MPPET – 1994]

(a) 1/19 (b) 1/29
(c) 1/17 (d) 1/27

18. If the 7th term of a harmonic progression is 8 and the 8th, term is 7, then its 15th term is

[MPPET – 1996]

(a) 16 (b) 14
(c) 27/14 (d) 56/15

19. In a H.P.., pth term is q and the qth term is p. Then pqth term is

(a) 0 (b) 1
(c) pq (d) $pq(p + q)$

[Karnataka CET – 2002]

20. If H is the harmonic mean between p and q, then the value of $\frac{H}{p} + \frac{H}{q}$ is

[MNR – 1990; UPSEAT – 2000, 2001; VIT – 2007]

(a) 2 (b) $\frac{pq}{p+q}$
(c) $\frac{p+q}{pq}$ (d) none of these

21. If the A.M. and H.M. of two numbers is 27 and 12 respectively, then G.M. of the two numbers will be

(a) 9 (b) 18 (c) 24 (d) 36

[RPET – 1987]

22. If A_1, A_2 are the two A.M's between two numbers a and b and G_1, G_2 be two G.M.'s between same two numbers, then $= \frac{A_1 + A_2}{G_1 . G_2}$

[Roorkee – 1983; DCE – 1998]

(a) $\frac{a+b}{ab}$ (b) $\frac{a+b}{2ab}$
(c) $\frac{2ab}{a+b}$ (d) $\frac{ab}{a+b}$

23. If the A.M is twice the G.M. of the numbers a and b, then $a : b$ will be

(a) $\frac{2-\sqrt{3}}{2+\sqrt{3}}$ (b) $\frac{2+\sqrt{3}}{2-\sqrt{3}}$
(c) $\frac{\sqrt{3}-2}{\sqrt{3}+2}$ (d) $\frac{\sqrt{3}-2}{\sqrt{3}+2}$

[Roorkee – 1953]

24. If the ratio of H.M. and G.M. between two numbers a and b is 4 : 5, then the ratio of the two numbers will be:

(a) 1: 2 (b) 2 : 1
(c) 4 : 1 (d) 1 : 4

[IIT – 1992; MPPET – 2000]

25. The harmonic mean between two numbers is$14\frac{2}{5}$ and the geometric mean 24. The greater number between them is

[UPSEAT – 2004]

(a) 72 (b) 54
(c) 36 (d) none of these

26. The harmonic mean of two numbers is 4 and the arithmetic and geometric means satisfy the relation $2A + G^2 = 27$, the numbers are

[MNR – 1987; UPSEAT – 1999, 2000]

(a) 6, 3 (b) 5, 4
(c) 5, – 2.5 (d) – 3, 1

27. A boy goes to school from his home at a speed of x km/hour and comes back at a speed of y km/hour,then the average speed is 406 is

(a) AM. (b) G.M.
(c) H.M. (d) None

28. If the A.M. and G.M. of roots of a quadratic equations are 8 and 5 respectively, then the quadratic equation will be

(a) $x^2 - 16x - 25 = 0$
(b) $x^2 - 8x + 5 = 0$
(c) $x^2 - 16x + 25 = 0$
(d) $x^2 + 16x - 25 = 0$

[PbCET – 1990]

29. If a be the arithmetic mean of b and c and G_1, G_2 be the two geometric means between them, then $G_1^3 + G_2^3 =$

(a) G_1G_2a (b) $2G_1G_2a$
(c) $3G_1G_2a$ (d) None of these

30. In a G.P. the sum of three numbers is 14, if 1 is added to first two numbers and subtracted from third number, the series becomes A.P., then the greatest number is

(a) 8 (b) 4
(c) 24 (d) 16

[Roorkee – 1973]

31. If a, b, c are the positive integers, then $(x + y)(y + z)(z + x)$ is

(a) $< 8\,xyz$ (b) $> 8\,xyz$
(c) $= 8\,xyz$ (d) none of these

[MPPET – 2005, DCE – 2000]

32. If $a_1, a_2, \ldots\ldots a_n$ are positive real numbers whose product is a fixed number c, then the minimum value of $a_1 + a_2 + \ldots + a_{n-1} + 2a_n$ is

[IIT (Sc.) – 2002]

(a) $n(2c)^{1/n}$ (b) $(n + 1)c^{1/n}$
(c) $2nc^{1/n}$ (d) $(n + 1)(2c)^{1/n}$

33. $2^{\sin\theta} + 2^{\cos\theta}$ is greater than

[AMU – 2000]

(a) 1/2 (b) $\sqrt{2}$
(c) $2^{\frac{1}{\sqrt{2}}}$ (d) $2^{\left(1-\frac{1}{\sqrt{2}}\right)}$

34. The product of n positive numbers is unity. Their sum is

(a) a positive integer (b) equal to $n + \frac{1}{n}$
(c) divisible by n (d) never less than n

[MPPET – 2000]

35. If a, b, c are positive real numbers, then minimum value of $(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$ is

(a) 9 (b) 8
(c) 10 (d) 11

36. If a, b, c are in A.P. and $|a|, |b|, |c| < 1$ and

$x = 1 + a + a^2 + \ldots \infty$, $y = 1 + b + b^2 + .. \infty$,
$z = 1 + c + c^2 + \ldots \infty$

Then x, y, z shall be in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

[Kurukshetra CET – 1995; AIEEE – 2005; Kerala PET – 2008]

37. The common difference of an A.P. whose first term is unity and whose second, tenth and thirty fourth terms are in G.P. is

(a) 1/5 (b) 1/3 (c) 1/6 (d) 1/9

[AMU – 2000]

38. If a, b, c are three unequal numbers such that a, b, c are in A.P. and $b - a$, $c - b$, a are in G.P., then $a : b : c$ is

[UPSEAT – 2001]

(a) 1 : 2 : 3 (b) 2 : 3 : 1
(c) 1 : 3 : 2 (d) 3 : 2 : 1

39. If a, b, c are positive real numbers, then minimum value of $\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}$ is

(a) 3 (b) 4 (c) 5 (d) 6

40. The difference between two numbers is 48 and the difference between their arithmetic mean and their geometric mean is 18. Then the greater of two numbers is

(a) 96 (b) 60
(c) 54 (d) 49

[MPPET – 2008]

41. If H_1, H_2 are two harmonic means between two positive numbers a and b $(a \neq b)$, A and G are the arithmetic and geometric means between a and b $\frac{H_2 + H_2}{H_2 H_1}$ is

(a) A/G
(b) $2A/G$
(c) $A/2G^2$
(d) $2A/G^2$

[Kerala PET – 2007]

42. If in a ΔABC, the altitude from the vertices A, B, C on opposite sides are in H.P., then $\sin A$, $\sin B$, $\sin C$ are in

[MPPET – 2009]

(a) G.P.
(b) arithmetic geometric progression
(c) A.P.
(d) H.P.

43. The harmonic mean of two numbers is 21.6. If one the numbers is 27, then what is the other number?

(a) 16.2 (b) 17.3 (c) 18 (d) 20

[N.D.A – 2009]

SOLUTIONS

1. (a) 5th and 11th terms of the corresponding AP are 45 and 69.

$\therefore a + 4d = 45$ and $a + 10d = 69$

$\Rightarrow a = 29, d = 4$

Hence, $T_{16} = \frac{1}{a + 15d} = \frac{1}{29 + 60} = \frac{1}{89}$

2. (d) Terms of H.P. are reciprocal of A.P.

$\therefore (T_7)_{HP} = \frac{1}{10}$

$\Rightarrow (T_7)_{AP} = 10$

= seventh term of corresponding A.P.

$(T_{12})_{HP} = \frac{1}{25}$

$\therefore (T_{12})_{HP} = 25$ = Twelth term of orresponding A.P.

In A.P., $T_n = a + (n - 1)d$

$\therefore T_7 \Rightarrow a + 6d = 10$ (1)

$T_{12} \Rightarrow a + 11\,d = 25$........ (2)

solving (1) & (2) $a = -8, d = 3$

$\therefore (T_{20})_{AP} \Rightarrow a + 19d = -8 + 57 = 49$

$\because (T_{20})_{AP} = 49 \therefore (T_{20})_{HP} = \frac{1}{49}$.

3. (a) $27 + 9 + \frac{27}{5} + \frac{27}{7} + \ldots$

$= \frac{27}{1} + \frac{27}{3} + \frac{27}{5} + \frac{27}{7} + \ldots$

$T_n = \frac{27}{1 + 2(n-1)} = \frac{27}{2n - 1}$

$\therefore T_9 = \frac{27}{2 \times 9 - 1} = \frac{27}{17}$

4. (d) a, b, c in H.P. $\Rightarrow b = \frac{2ac}{a + c}$

(a) $b - a = \frac{2ac}{a+c} - a = \frac{a(c-a)}{a+c}$, $b - c$

$= \frac{2ac}{a+c} - c = \frac{c(a-c)}{a+c}$

$\frac{1}{b-a} + \frac{1}{b-c} = \frac{a+c}{a(c+a)} - \frac{a+c}{c(c-a)}$

$= \frac{a+c}{ac} = \frac{2}{b}$ (False)

(b) (False)

(c) $\frac{b+a}{b-a} + \frac{b+a}{b-a} = \frac{a(3c+a)}{a(c-a)}$

$+ \frac{c(3a+c)}{c(a-c)} = 1$ (False)

5. (c) a, b, c in H.P. $\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ in A.P.

$\Rightarrow \frac{2}{b} = \frac{1}{a} + \frac{1}{c} +$ .... (1) and $\frac{2}{b} - \frac{1}{a} = \frac{1}{c}$ (1)

Now $\left(\frac{1}{b} + \frac{1}{c} - \frac{1}{a}\right)\left(\frac{1}{c} + \frac{1}{a} - \frac{1}{b}\right)$

$= \left(\frac{1}{c} - \left(\frac{1}{a} - \frac{1}{b}\right)\right)\left(\frac{1}{c} + \left(\frac{1}{a} - \frac{1}{b}\right)\right)$

$= \frac{1}{c^2}\left(\frac{1}{a} + \frac{1}{b}\right)^2$

$= \frac{1}{c^2}\left\{\frac{1}{a^2}+\frac{1}{b^2}-\frac{2}{ab}\right\}$

$= \left(\frac{2}{b}-\frac{1}{b}\right)^2 - \frac{1}{a^2} - \frac{1}{b^2} + \frac{2}{ab}$

$= \frac{4}{b^2}+\frac{1}{a^2}-\frac{4}{ab}-\frac{1}{a^2}-\frac{1}{b^2}+\frac{2}{ab}$

$= \frac{3}{b^2}-\frac{2}{ab}$

6. (b) $\frac{2a}{a+b}, H + a$

$= \frac{2ab}{a+b} + a = \frac{a^2+3ab}{a+b}$

$H - a = \frac{a(b-a)}{a+b}$. Dividing,

$\frac{H+a}{H-a} = \frac{a+3b}{b-a}$. $H + b = \frac{2ab}{a+b} + b$

$\frac{3ab+b^2}{a+b} = \frac{b(a+3b)}{a+b}$ $H - b = \frac{b(a-b)}{a+b}$

This $\Rightarrow \frac{H+a}{H-b} = \frac{3a+b}{a-b}$. Now

$\frac{H+a}{H-a} + \frac{H+b}{H-b} = \frac{a+3b}{b-a} - \frac{(3a+b)}{b-a}$

$= \frac{-2a+2b}{b-a} = 2$

7. (c) According to the condition

$\frac{\frac{1}{a+7d}}{\frac{1}{a+(n-1)d}} = \frac{9}{5}$ (1)

and $d = \frac{31-1}{n+1} = \frac{30}{n+1}, a = 1.$ (2)

On solving (1) and (2) we get $n = 14$ and $d = 2$

Also $\frac{1}{a+(n+1)d} = \frac{1}{31}$ (3)

Put $a = 1, d = 2$ in equation (3) $n = 14$.

8. (d) Take $k = abcd$.

given a, b, c, d are in A.P.

$\Rightarrow \frac{a}{k}, \frac{b}{k}, \frac{c}{k}, \frac{d}{k}$ are also in A.P.

$\Rightarrow \frac{k}{a}, \frac{k}{b}, \frac{k}{c}, \frac{k}{d}$ in H.P.

$\Rightarrow$ *bcd*, *acd*, *abd*, *abc* in HP. Writing in reverse order we get,

$\Rightarrow$ *abc*, *abd*, *acd*, *bcd* in H.P.

9. (c) If $\log_a^x, \log_b^x, \log_c^x$ are in H.P.

$\frac{\log x}{\log a}, \frac{\log x}{\log b}, \frac{\log x}{\log c},$ are in H.P.

$\frac{\log a}{\log x}, \frac{\log b}{\log x}, \frac{\log c}{\log x}$ are in A.P.

$\frac{2\log b}{\log x} = \frac{\log a + \log c}{\log x}$

$\log b^2 = \log (ac)$

$b^2 = ac$

$\therefore a, b, c$ are in G.P.

10. (b) Since, a, b, c are in A.P.

$\therefore 2b = a + c \quad \therefore 3^{2b} = 3^{a+c} = 3^a \cdot 3^c$

$\Rightarrow (3^b)^2 = 3^a \times 3^c$

$\therefore 3^a, 3^b, 3^c$ are in G.P.

11. (c) Here, $\frac{10^{bx+10}}{10^{ax+10}} = 10^{(b-a)x}$ and

$\frac{10^{cx+10}}{10^{bx+10}} = 10^{(c-b)x}$

Since, a, b, c are in A.P., therefore

$b - a = c - b$

$\Rightarrow (b-a)x = (c-b)x\ \forall x$

$\Rightarrow$ given numbers are in G.P.

OR

Given $2b = a + c$

Multiplying both sides by x and adding 20 on either side we get

$20 + 2bx = (a + c)\,x + 20$

$\Rightarrow 2\,(bx + 10) = (ax + 10) + (c + 10)$

$\therefore 10^{(bx+\cdots)} = 10^{(ax+)+cx+}$

$\Rightarrow 10^{ax+10}, 10^{bx+10}, 10^{cx+10}$

$\Rightarrow 10^{ax+10}, 10^{bx+10}, 10^{cx+10}$ are in G.P

12. (d) By assumption $b = \frac{a+c}{2}, b^2$ (1)

$\Rightarrow \sqrt{ac} = \frac{a+c}{2} \Rightarrow a + c = -2\sqrt{ac} = 0$

This $\Rightarrow (\sqrt{a} - \sqrt{b})^2 = 0$

$\Rightarrow a = c$

Puting $a = c$ in (1) we get, $a = b = c$.

13. (c) $x, 1, z$ are in A.P.

$\Rightarrow 2 = x + z$ (1), $x, 2, z$ are in G.P.

$\Rightarrow 4 = xz$ .... (2) Now $x, 4, z$ are in H.P.

if

$4 = \frac{2xz}{x+z} = \frac{2(4)}{2} = 4$ (True)

14. (c) Given a, b, c are in A.P., b, c, d are in G.P., c, d, e are in H.P.

$\Rightarrow 2b = a + c$ (1)

$c^2 = bd$ (2)

$d = b = \frac{2ce}{c+e}$ (3)

Putting (1) & (3) in (2),

$c^2 = \left(\frac{a+c}{2}\right)\left(\frac{2ce}{c+e}\right) = \frac{(a+c)(ce)}{c+e}$

or, $c(c+e) = (a+c)e$

or $c^2 + ce = ae + ce$ or $c^2 = ae$

$\therefore$ a, c, e in G.P.

15. (a) $x_n = \frac{(n+1)ab}{na+b}$

Sixth H.M. $x_6 = \frac{7.3.6/13}{\left(6.3 + \frac{6}{13}\right)} = \frac{126}{240} = \frac{63}{120}$

16. (d) $2, 2\frac{1}{2}, 3\frac{1}{3}, \ldots\ldots$ in H.P.

$\Rightarrow 2, \frac{5}{2}, \frac{10}{3}, \ldots..$

$\Rightarrow \frac{1}{2}, \frac{2}{5}, \frac{3}{10} \ldots\ldots$

$\Rightarrow a = \frac{1}{2}, d = \frac{2}{5} - \frac{1}{2} = -\frac{1}{10}$

$T = a + 4d = \frac{1}{2} + 4\left(-\frac{1}{10}\right) = \frac{1}{10}$

$\Rightarrow T_5 = \frac{10}{1}$ in H.P.

17. (b) $T_1 = \frac{1}{7}, T_2 = \frac{1}{9}$ in H.P.

$\Rightarrow T_1 = 7, T_2 = 9$ in corresponding A.P.

$\Rightarrow a + d = 9 \Rightarrow 7 + d = 9 \Rightarrow d = 2$

$T_{12} = a + 11d = 7 + 11(2) = 29$

$\Rightarrow T_{12}$ for H.P. is $\frac{1}{29}$

18. (d) $T_7 = 8, T_8 = 7$ in H.P.

$\Rightarrow T_7 = \frac{1}{8}, T_8 = \frac{1}{7}$ in A.P.

$\Rightarrow a + 6d = \frac{1}{8}, a + 7d = \frac{1}{7}$, Subtracting

$d = \frac{1}{7} - \frac{1}{8} = \frac{1}{56}$

Now $a + 6d = \frac{1}{8}, d = \frac{1}{56} \Rightarrow a = \frac{1}{56}$

$T_{15} = a + 14d = a + 14a = 15a = \frac{15}{56}$

$\Rightarrow T_{15}$ in H.P. is $\frac{56}{15}$

19. (b) In the corresponding A.P., pth term $= \frac{1}{q}$ and qth term $= \frac{1}{p}$

Let a be the first term and d be the C.D. of this A.P., then $a + (p-1)d = \frac{1}{q}$ and $a + (q-1)d = \frac{1}{p}$

$\Rightarrow d = \frac{1}{pq}$ and $a = \frac{1}{qp}$

Hence (pq)th term in the corresponding A.P. is $a + (pq-1)d$

$= \frac{1}{pq} + (pq-1)\frac{1}{qp} = 1$

$\Rightarrow$ In the given H.P., (pq)th term is 1.

20. (a) $H = \frac{2PQ}{P+Q} \Rightarrow \frac{2}{H} = \frac{P+Q}{PQ}$

$\Rightarrow \frac{2}{H} = \frac{1}{P} + \frac{1}{Q}$

$\Rightarrow 2 = \frac{H}{P} + \frac{H}{Q}$

21. (b) Given A.M. = 27, H.M. = 12

$(GM)^2 = (A.M.)(H.M.)$

G.M. $= \sqrt{27 \times 12}$

$= \sqrt{9 \times 3 \times 3 \times 4}$

$= 3 \times 3 \times 2$

$= 18$

22. (a) **I.** a, A_1, A_2, b in A.P. $b = T_4 = a + (4-1)d$

$\Rightarrow \frac{b-a}{3} = d; A_1 = T_2 = a + d = a + \left(\frac{b-a}{3}\right) = \frac{b+2a}{3}$

$A_2 = T_3 = a + 2d = a + \frac{2}{3}(b-a) = \frac{a+2b}{3}$

$A_1 + A_2 = \left(\frac{b+2a}{3}\right) + \left(\frac{a+2b}{3}\right) = a + b$

II. a, G_1, G_2, b in G.P. $b = T_4 = ar^3$ or

$\left(\frac{b}{a}\right)^{1/3} = r$

$G_1 = ar = a\left(\frac{b}{a}\right)^3 = (a^2 b)^{1/3}, G_2 = ar^2 = (ab^2)^{1/3},$

$G_1 G_2 = [(a^2 b)(ab^2)]^{1/3} = ab$ or

$\frac{A_1 + A_2}{G_1 G_2} + \frac{a+b}{ab}$

OR

Step 1: If a, A_1, A_2, b are in A.P. then sum of the equidistant terms from beginning and end is equal to sum of the first and last term therefore $a + b = A_1 + A_2$ (1)

Step 2: If a, G_1, G_2, b are in A.P. then we know that product of the equidistant terms from beginning and end is equal to product of the first and last terms therefore $ab = G_1 G_2$ (2)

Step 3: On dividing (1) by (2) we get the answer.

23. (b) A.M. = 2 G.M. $\Rightarrow$

$\frac{a+b}{ab} = 2\sqrt{ab} \Rightarrow \frac{a+b}{2\sqrt{ab}} = \frac{2}{1}$

$\Rightarrow \frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{2+1}{2-1} = \left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\right)^2$

$= \frac{3}{1} \Rightarrow \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{3}}{1}$

$\Rightarrow \frac{(\sqrt{a}+\sqrt{b})+(\sqrt{a}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})-(\sqrt{a}-\sqrt{b})} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

$\Rightarrow \frac{2\sqrt{a}}{2\sqrt{b}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

$\Rightarrow \frac{a}{b} = \left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) \Rightarrow \frac{4+2\sqrt{3}}{4-2\sqrt{3}} = \frac{2+\sqrt{3}}{2-\sqrt{3}}$

24. (c, d) By assumption, $\frac{H}{G} = \frac{4}{5}$

$\Rightarrow 5H = 4G$

$\Rightarrow \frac{5(2ab)}{a+b} = 4\sqrt{ab}$

$\Rightarrow 5\sqrt{ab} = 2(a+b) \Rightarrow 5 = 2\left(K + \frac{1}{K}\right),$

where $K = \left(\frac{a}{b}\right)^{1/2}$

$\Rightarrow 2K^2 - 5K + 2 = 0 \Rightarrow K = 2, \frac{1}{2}$. If $K = 2$,

then $\left(\frac{a}{b}\right)^{1/2} = 2$

or $\frac{a}{b} = \frac{4}{1}$ · This $\Rightarrow a : b = 4 : 1$ or $1 : 4$.

25. (a) $\frac{2ab}{a+b} = \frac{72}{5}$ (1), $\sqrt{ab} = 24$ or $ab = 576$

$\Rightarrow 10\,(576) = 72\,(a+b) \Rightarrow a + b = 80$

$(a-b)^2 = (a+b)^2 - 4ab = (80)^2 - 4 \times (24)^2$

$= (16)^2\,(16)$

$\Rightarrow a - b = 64$. Also, $a + b = 80$

$\Rightarrow a = 72, b = 8$

26. (a) Given $\frac{2ab}{a+b} = 4$ (1)

$\frac{a+b}{2} = A.$ (2)

$\sqrt{ab} = G$ (3)

$2A + G^2 = 27$ (4)

Putting values of A and G in (4), we get,

$(a+b) + ab = 27$, using (1), $\frac{ab}{2} + ab = 27$

or $ab = 18$ (5)

Put this in (1), $a + b = 9$

$(a-b)^2 = (a+b)^2 - 4ab = 81 - 4(18) = 9$

or $a - b = 3$. Also a + b = 9. This

$\Rightarrow a = 6, b = 3$.

27. (c) Let, the distance of school from home = d and time taken are t_1 and t_2.

$\therefore t_1 = \frac{d}{x}$ and $t_2 = \frac{d}{y}$

Avg. velocity = $\frac{\text{Total distance}}{\text{Total time}}$

$= \frac{2d}{\left(\frac{d}{x}+\frac{d}{y}\right)} = \frac{2xy}{x+y}$, which is the H.M. of x and y.

28. (c) Given that A.M. = 8 and G.M. = 5, if α, β are roots of quadratic equation, then quadratic equation is

$x^2 - x((\alpha+\beta) + (\alpha\beta)) = 0$ (1)

A.M. $= \frac{\alpha+\beta}{2} = 8 \Rightarrow \alpha + \beta = 16$ (2)

and G.M. $= \sqrt{\alpha\beta} = 5 \Rightarrow \alpha\beta = 25$ (3)

So the required quadratic equation will be $x^2 - 16x + 25 = 0$.

29. (b) Quicker Method: Put $b = 1$ and $c = 8$ so that $a = 4.5$ and $G_1 = 2, G_2 = 4$.

Now $G_1^3 + G_2^3 = 72$. Also option (b) gives this value i.e., $2 \times 2 \times 4 \times \frac{9}{2} = 72$.

30. (a) Three numbers in G.P. are $\frac{a}{r}, a, ar$ (1)
Under given condition I
$\frac{a}{r} + a + ar = 14$ or $a(r^2 + r + 1) = 14r$

$\Rightarrow \frac{a}{r}(1 + r^2) = 14 - a$ (2)

Under condition II: $\frac{a}{r} + 1, a + 1, ar - 1$ in A.P.
This $\Rightarrow 2(a + 1) = \left(\frac{a}{r} + 1\right) + (ar - 1)$

$\Rightarrow 2(a + 1) = \frac{a}{r}(1 + r^2)$ (3)

Using (2) in (3),

$2(a + 1) = 14 - a$ or $a = 4$.

Put in (2), $4(r^2 + r + 1) = 14r$

or $2r^2 - 5r + 2 = 0$ or $r = 2, \frac{1}{2}$.

Put $r = 2, a = 4$ in (1),

$\frac{4}{2}, 4, 4 \times 2 = 2, 4, 8$.

$\therefore$ Greatest number = 8.

31. (a) A.M. between x and $y = \frac{x + y}{2}$
G.M. between x and $y = \sqrt{xy}$

Since, A.M. > G.M. $\Rightarrow \frac{x + y}{2} > \sqrt{xy}$

$\Rightarrow x + y > 2\sqrt{xy}$ (1)

Similarly, $y + z > 2\sqrt{yz}$ (2)

and $z + x > 2\sqrt{zx}$ (3)

on multiplication above three equations sidewise, we get, $(x + y)(y + z)(z + x) > 8xyz$

32. (a) Now, $a_1, a_2, a_3, \ldots, a_{n-1}, 2a_n$ are n positive numbers, therefore,

$$\frac{a_1 + a_2 + \ldots + a_{n-1} + 2a_n}{n} \geq \{(a_1 a_2 \ldots a_{n-1})(2a_n)\}^{1/n}$$

($\because$ A.M. $\geq$ G.M.)

$\Rightarrow a_1 + a_2 + \ldots + a_{n-1} + 2a_n$

$\Rightarrow n(2a_1 a_2 \ldots a_n)^{1/n}$

$\Rightarrow a_1 + a_2 + \ldots + a_{n-1} + 2a_n \Rightarrow n(2c)^{1/n}$.

33. (d) $\frac{1}{2}[2^{\sin\theta} + 2^{\cos\theta}] \geq \sqrt{2^{\sin\theta}\, 2^{\cos\theta}}$
($\therefore$A.M. $\geq$ G.M.)

$\Rightarrow 2^{\sin\theta} + 2^{\cos\theta} > 2.2^{(\sin\theta + \cos\theta)/2}$ (1)

Since, $(\sin\theta + \cos\theta) = \sqrt{2}\sin(\theta + ^{\pi} \geq/4)$

$\Rightarrow -\sqrt{2}$ for all real θ therefore

$2^{\sin\theta} + 2^{\cos\theta} \geq 2.2^{(\sin\theta + \cos\theta)/2} > 2.2^{-\sqrt{2}/2}$

$\Rightarrow 2^{\sin\theta} + 2^{\cos\theta} \geq 2^{1-(1/\sqrt{2})}$

34. (d) Given $a_1, a_2, a_3 \ldots\ldots a_n = 1$ (1), where $a_n > 0 \ \forall_n$
A.M. $\geq$ G.M. $\Rightarrow \frac{1}{n}(a_1 + a_2 + a_3 + \ldots + a_n)$
$(a_1 a_2 \ldots a_n)^{1/n} = 1^{1/n} = 1$

$\Rightarrow a_1 + a_2 + \ldots + a_n n \Rightarrow a_1 + a_2 + \ldots + a_n \not< n$

35. (a) Since A.M. between two positive numbers cannot be less than the G.M. between them, therefore, ab

$$\frac{\frac{a}{b} + \frac{b}{a}}{2} \geq \sqrt{\left(\frac{a}{b}\right)\left(\frac{b}{a}\right)}$$

$\Rightarrow \frac{a}{b} + \frac{b}{a} \geq 2$ (1)

Similarly, $\frac{b}{c} + \frac{c}{b} + \geq 2$ and $\frac{c}{a} + \frac{a}{c} \geq 2$

Hence, $(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$

$= 3 + \left(\frac{a}{b} + \frac{b}{a}\right) + \left(\frac{b}{c} + \frac{c}{b}\right) + \left(\frac{c}{a} + \frac{a}{c}\right)$

$\geq 3 + 2 + 2 + 2 = 9$

OR

A.M. of $a, b, c \geq$ H.M. of a, b, c

$\geq \frac{a + b + c}{3} \geq 3/\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

$\geq (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9$

36. (c) p, q, r are in A.P. $\Rightarrow 2q = p + r$

$x = 1 + p + p^2 + p^3 + \ldots\ldots \Rightarrow x = \frac{1}{1 - p}$

$\Rightarrow 1 - p - \frac{1}{x}$

$\geq p = 1 - \frac{1}{x}$

Similarly $q = 1 - \frac{1}{y}$ and $r = 1 - \frac{1}{z}$

$\because p, q, r$ are in A.P. therefore

$1-\frac{1}{x}, 1-\frac{1}{y}, 1-\frac{1}{z}$ are also in A.P.

or $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P.

$\therefore$ x, y, z are in H.P.

37. (b) First term of an A.P. = 1, let common difference = d

$\therefore T_2 = a + d, T_{10} = a + 9d, T_{34} = a + 33d$

$\therefore (a + 9d)^2 = (a + d)(a + 33d)$

$\Rightarrow a^2 + 81d^2 + 18ad = a^2 + ad + 33ad + 33d^2$

Put $a = 1 \Rightarrow 1 + 81d^2 + 18d$

$= 1 + d + 33d + 33d^2$

$\Rightarrow 48d^2 - 16d = 0 \Rightarrow 16d(3d - 1) = 0$

$\Rightarrow d = 0, d = 1/3.$

38. (a) We have a, b, c are in AP then

$b - a = c - b$ (1)

and $(b - a), (c - b)$ and a are in GP

$\frac{c-b}{b-a} = \frac{a}{c-b}$ (2)

from equation (2), we get,

$\Rightarrow (c - b)^2 = a(b - a)$

$\Rightarrow (b - a)^2 = a(b - a)$ [Using (1)]

$\Rightarrow (b - a)(b - a - a) = 0 \Rightarrow b = a$ or $2a$

$\therefore b = 2a$ ($\because$ a, b, c are unequal)

(1) $\Rightarrow 2a - a = c - 2a \Rightarrow c = 3a$

$\therefore a : b : c = a : 2a : 3a = 1 : 2 : 3$

39. ()

40. (d) Let two numbers be x and y

Given $x - y = 48$ (1)

and $\frac{x+y}{2} - \sqrt{xy} = 18$

from equation (1) $y = x - 48$

$\therefore \frac{2x-48}{2} - \sqrt{x(x-48)} = 18$

$x - 24 - \sqrt{x(x-48)} = 18$

$-\sqrt{x(x-48)} = 42 - x$

or $x(x - 48) = (x - 42)$ (2)

$x^2 - 48x = x^2 + 42^2 - 84x$

$36x = 42 \times 42 \Rightarrow x = 49$

and $y = 1$ by equation (1)

41. (d) Here, $A = A = \frac{a+b}{2}$ (1)

and $G = \sqrt{ab}$ (2)

Also, a, H_1, H_2, b are in H.P.

($\because H_1, H_2$ are two H.M.s between a and b)

$\Rightarrow \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{b}$ are in A.P.

$\Rightarrow \frac{1}{H_1} + \frac{1}{H_2} = 2\left\{\frac{\frac{1}{a}+\frac{1}{b}}{2}\right\}$

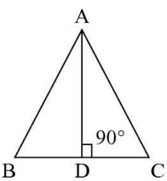
($\because$ Sum of n A.M.s between two numbers is n times the single A.M. between them)

$\Rightarrow \frac{H_2 + H_1}{H_1H_2} = \frac{b+a}{ab}$

$\Rightarrow \frac{H_1 + H_2}{H_1H_2} = \frac{a+b}{ab} = \frac{2A}{G^2}$

(Using (1) and (2))

42. (c) **Step 1:**



Area of $\Delta ABC = \Delta = \frac{1}{2} \times BC \times P_1$

$P_1 = \frac{2\Delta}{BC} = \frac{2\Delta}{a}$

Step 2: The altitudes from the vertices A, B and C are

$\frac{2\Delta}{a}, \frac{2\Delta}{b}$ and $\frac{2\Delta}{c}$ respectively.

Aslo, these are in HP

$\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in HP

$\Rightarrow a, b, c$ are in AP

$\Rightarrow \sin A, \sin B, \sin C$ are in AP

43. (c) $\because$ $H = 21.6$ and $a = 27$

We know that

$H = \frac{2ab}{a+b} \Rightarrow 216 = \frac{2 \times 27 \times b}{27 \times b}$

$\Rightarrow 583.2 = 54b - 21.6b$

$\Rightarrow b = \frac{583.2}{32.4} = 18$

UNSOLVED OBJECTIVE PROBLEMS: (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. H.M. between the roots of the equation $x^2 - 10x + 11 = 0$ is

(a) 1/5 (b) 5/21 (c) 21/20 (d) 11/5

[MPPET – 1995]

2. The harmonic mean of $\frac{a}{1-ab}$ and $\frac{a}{1+ab}$ is

(a) $\frac{a}{\sqrt{1-a^2b^2}}$ (b) $\frac{a}{1-a^2b^2}$

(c) a (d) $\frac{a}{1-a^2b^2}$

[MPPET – 1996; Pb. CET – 2001]

3. If a, b, c are in A.P., then $\frac{a}{bc}, \frac{1}{c}, \frac{2}{b}$ are in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these

[MNR – 1982; MPPET – 2002]

4. If three numbers be in G.P., then their logarithms will be in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these

[BIT – 1992]

5. If $\frac{1}{b-c} + \frac{1}{b-c} = \frac{1}{a} + \frac{1}{c}$, then a, b, c are in

(a) A.P.

(b) G.P.

(c) H.P.

(d) in G.P. and H.P. both

[MNR – 1984;MPPET – 1997; UPSEAT – 2000]

6. If sixth term of a H.P. is 1/61 and its tenth term is 1/105, then first term of that H.P. is

[Karnataka CET – 2001]

(a) 1/28 (b) 1/39 (c) 1/6 (d) 1/17

7. If $a^x = b^y = c^z$ and a, b, c are in G.P. then x, y, z are in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these

[Pb.CET – 1993;DCE – 1999; AMU – 1999]

8. If a, b, c are in A.P., then $2^{ax+1}, 2^{bx+1}, 2^{cx+1}$, $x \neq 0$ are in:

(a) A.P.

(b) G.P. only when $x > 0$

(c) G.P. if $x < 0$

(d) G.P. for all $x \neq 0$

[DCE – 2000; Pb. CET – 2000]

9. If a, b, c are in G.P., then $\log_a x, \log_b x, \log_c x$ are in

(a) A.P. (b) G.P.

(c) H.P. (d) None

[RPET – 2002]

10. If $x^a = x^{b/2} z^{b/2} = z^c$, then a, b, c are in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these

11. Which of the following statements is correct:

(a) If to each term of an A.P. a number is added or subtracted, then the series so obtained is also an A.P.

(b) The nth term of geometric series whose first term is a common ratio r, is ar^{n-1}

(c) If each term of a G.P. be raised to the same power, the resulting terms are in G.P.

(d) All the above

12. Three numbers form a G.P.. If the 3rd term is decreased by 64, then the three numbers thus obtained will consitute an A.P.. If the second term of this A.P. is decreased by 8, a G.P. will be formed again, then the numbers will be:

(a) 4, 20, 36 (b) 4, 12, 36

(c) 4, 20, 100 (d) none of these

[MPPET – 2007]

13. Harmonic mean of 5 and 25 is

(a) 15 (b) $\frac{25}{3}$

(c) $\frac{3}{25}$ (d) $5\sqrt{5}$

14. If G.M. = 18 and A.M. = 27, then H.M. is **[RPET – 1996]**

(a) 1/18 (b) 1/12
(c) 12 (d) $9\sqrt{6}$

15. The geometric mean of two numbers is 6 and their arithmetic mean is 6.5. The numbers are **[MPPET – 1994]**

(a) (3,12) (b) (4, 9)
(c) (2, 18) (d) (7, 6)

16. If the A.M., G.M. and H.M. between two positive numbers a and b are equal, then **[RPET – 2003]**

(a) $a = b$ (b) $ab = 1$
(c) $a > b$ (d) $a < b$

17. If arithmetic mean of two positive numbers is A, their geometric mean is G and harmonic mean is H, then H is equal to **[MPPET – 2004]**

(a) $\frac{G^2}{A}$ (b) $\frac{G}{A^2}$
(c) $\frac{A^2}{G}$ (d) $\frac{A}{G^2}$

18. a, g, h are arithmetic mean, geometric mean and harmonic mean between two positive numbers x and y respectively. Then identify the correct statement among the following

(a) h is the harmonic mean between a and g
(b) no such relation exists between a, g and h
(c) g is the geometric mean between a and h
(d) A is the arithmetic mean between g and h

[Karnataka CET – 2001]

19. If the arithmetic, geometric and harmonic means between two distinct positive real numbers be A, G and H respectively, then the relation between them is **[MPPET – 1984; Roorkee – 1995]**

(a) $A > G > H$ (b) $A > G < H$
(c) $H > G > A$ (d) $G > A > H$

20. If the arithmetic, geometric and harmonic means between two positive real numbers be A, G and H, then

(a) $A^2 = GH$ (b) $H^2 = AG$
(c) $G = AH$ (d) $G^2 = AH$

[AMU – 1979,1982; MPPET – 1993]

21. If a and b are two different positive real numbers, then which of the following relations is true

(a) $2\sqrt{ab} > (a + b)$ (b) $2\sqrt{ab} < (a + b)$
(c) $2\sqrt{ab} = (a + b)$ (d) none of these

[MPPET – 1982; MPPET – 2002]

22. If A.M. of two terms is 9 and H.M. is 36, then G.M. will be

(a) 18 (b) 12
(c) 16
(d) none of the above

[RPET – 1995]

23. If the A.M. of two numbers is greater than G.M. of the numbers by 2 and the ratio of the numbers is 4 : 1,then the numbers are

(a) 4, 1 (b) 12, 3
(c) 16, 4 (d) none of these

[RPET – 1988]

24. The A.M., H.M. and G.M. between two numbers are 144/15, 15 and 12, but not necessarily in this order. Then H.M., G.M. and A.M. respectively are

(a) 15, 12, 144/15
(b) 144/15, 12, 15
(c) 12, 15, 144/15
(d) 144/15, 15,12

25. If the ratio of two numbers be 9 : 1, then the ratio of geometric and harmonic means between them will be

(a) 1 : 9 (b) 5 : 3
(c) 3 : 5 (d) 2 : 5

26. If the ratio of H.M. and G.M. of two quantities is 12: 13, then the ratio of the numbers is **[RPET – 1990]**

(a) 1: 2 (b) 2: 3
(c) 3: 4 (d) none of these

27. Which number should be added to the numbers 13, 15, 19 so that the resulting numbers be the consecutive terms of a H.P. is

(a) 7 (b) 6 (c) − 6 (d) − 7

28. If the 7th terms of a harmonic progression is 8 and the 8th term is 7, then its 15th terms is

(a) 16 (b) 14

(c) 27/14 (d) 56/15

29. In a H.P., pth term is q and the qth term is p. Then pqth term is

(a) 0 (b) 1

(c) pq (d) $pq(p+q)$

30. H.M. between the roots of the equation $x^2 - 10x + 11 = 0$ is

(a) 1/5 (b) 5/21

(c) 21/20 (d) 11/5

31. $4^3 + 5^3 + 6^3 + \ldots\ldots\ldots\ldots + 10^3$

(a) 2980 (b) 2985

(c) 2989 (d) none of these

32. $2 + 3 + 5 + 6 + 8 + 9 + \ldots\ldots\ldots$ to $2n$ terms

(a) $3n^2 + 2n$ (b) $4n^2 + 2n$

(c) $4n^2$ (d) none of these

33. 30th term of the series 3 + 5 + 9 + 15 + 23 +..... is

(a) 873 (b) 872

(c) 810 (d) none of these

34. If $y = 3^{x-1} + 3^{-x-1}$ (x real), then the least value of y is

(a) 2 (b) 6

(c) 2/3 (d) none of these

[MP PET – 2006]

35. If the 7th term of a H.P. is 1/10 and the 12th term is 1/25, then the 20th term is

[MPPET – 1997]

(a) 1/37 (b) 1/41

(c) 1/45 (d) 1/49

36. Two arithmetic mean's A_1 and A_2 two geometric mean's G_1 and G_2 and two harmonic mean's H_1 and H_2 are inserted between any two numbers. Then $\frac{1}{H_1} + \frac{1}{H_2}$ is equal to

(a) $\frac{1}{A_1} + \frac{1}{A_2}$ (b) $\frac{1}{G_1} + \frac{1}{G_2}$

(c) $\frac{G_1G_2}{A_1 + A_2}$ (d) $\frac{A_1 + A_2}{G_1G_2}$

37. If a is positive and if A and G are the arithmetic mean and the geometric mean of the roots of $x^2 - 2ax + a^2 = 0$ respectively, then

[Kerala PET – 2008]

(a) $A = G$ (b) $A = 2G$

(c) $2A = G$ (d) $A^2 = G$

38. The H.M. of two numbers is 4. Their AM. is A and G.M. is G. If $2A + G^2 = 27$ then A is equal to

(a) 9 (b) 9/2

(c) 18 (d) 27

[Kerala PET – 2008]

WORK SHEET: TO CHECK PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the work sheet
2. The test is of 20 minutes.
3. The test consists of 20 questions. The maximum marks are 60.
4. Use blue/black ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. When $\frac{1}{a} + \frac{1}{c} + \frac{1}{a-b} + \frac{1}{c-b}$ and $b \neq a \neq c$, then a, b, c are

[MPPET – 2004; MNR – 2000]

(a) in H.P. (b) in G.P.

(c) in A.P. (d) none of these

2. If a, b, c, d are in HP then

[IIT – 1970; PET (Raj.) – 1991]

(a) $ab > cd$ (b) $ac > bd$

(c) $ad > bc$ (d) none of these

3. a, b, c are first three terms of a GP. If HM of a and b is 12 and that of b and c is 36, then a equals

[Roorkee – 1998]

(a) 24 (b) 8
(c) 72 (d) 1/3

4. If a, b, c are in HP, then

[PET (Raj.) – 1994]

(a) $a^2 + c^2 > b^2$
(b) $a^2 + c^2 > 2b^2$
(c) $a^2 + c^2 < 2b^2$
(d) $a^2 + c^2 = 2b^2$

5. Five numbers a, b, c, d, e are such that a, b, c are in A.P; b, c, d are in GP and c, d, e are in HP. If $a = 2$, $e = 18$; then values of b, c, d are

(a) 2, 6, 18
(b) 4, 6, 9
(c) 4, 6, 8
(d) − 2, − 6, 18

[IIT – 1976]

6. If a, b, c, d are in H.P., then $ab + bc + cd$ is equal to

(a) $3ad$
(b) $(a + b)(c + d)$
(c) $3ac$
(d) none of these

7. The difference between the nth term and $(n - 1)$th term of a sequence is independent of n. Then the sequence follows which one of the following?

[NDA – 2008]

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

8. If $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ be the harmonic mean between a and b, then the value of n is

(a) 1 (b) − 1
(c) 0 (d) 2

[Assam PET – 1986]

9. If pth, qth, rth and sth terms of an A.P. be in G.P., then $(p - q), (q - r), (r - s)$ will be in

(a) G.P. (b) A.P.
(c) H.P. (d) none of these

[MPPET – 1993]

10. If $a^{1/x} = b^{1/y} = c^{1/z}$ and a, b, c are in G.P., then x, y, z will be in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

[IIT – 1969; UPSEAT – 2001; MPPET – 2008]

11. If $x > 1$, $y > 1$, $z > 1$ are in G.P., then $\frac{1}{1 + \ln x}, \frac{1}{1 + \ln y}, \frac{1}{1 + \ln}$ are in

[IIT – 1998; UPSET – 2001]

(a) A.P. (b) H.P.
(c) G.P. (d) none of these

12. In the four numbers first three are in G.P. and last three are in A.P., whose common difference is 6. If the first and last numbers are same, then first will be

(a) 2 (b) 4
(c) 6 (d) 8

[IIT – 1974]

13. If A_1, A_2, G_1, G_2 and H_1, H_2 be two A.M.s, G.M.s and H.M.s between two numbers respectively, then

$$\frac{G_1G_2}{H_1H_2} \times \frac{H_1 + H_2}{A_1 + A_2} =$$

(a) 1 (b) 0
(c) 2 (d) 3

[RPET – 1997]

14. If a, b, c are in A.P. and a, b, d in G.P., then $a, a - b, d - c$ will be in:

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

[Ranchi BIT – 1968]

15. Given $a^x = b^y = c^z = d^u$ and a, b, c, d are in G.P., then x, y, z, u are in

[ISM Dhanbad – 1972; Roorkee – 1984; RPET – 2001]

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

ANSWER SHEET

1.	ⓐ	ⓑ	ⓒ	ⓓ	6.	ⓐ	ⓑ	ⓒ	ⓓ	11.	ⓐ	ⓑ	ⓒ	ⓓ
2.	ⓐ	ⓑ	ⓒ	ⓓ	7.	ⓐ	ⓑ	ⓒ	ⓓ	12.	ⓐ	ⓑ	ⓒ	ⓓ
3.	ⓐ	ⓑ	ⓒ	ⓓ	8.	ⓐ	ⓑ	ⓒ	ⓓ	13.	ⓐ	ⓑ	ⓒ	ⓓ
4.	ⓐ	ⓑ	ⓒ	ⓓ	9.	ⓐ	ⓑ	ⓒ	ⓓ	14.	ⓐ	ⓑ	ⓒ	ⓓ
5.	ⓐ	ⓑ	ⓒ	ⓓ	10.	ⓐ	ⓑ	ⓒ	ⓓ	15.	ⓐ	ⓑ	ⓒ	ⓓ

HINTS AND EXPLANATIONS

1. (c) $\frac{1}{b-c}+\frac{1}{b-c}=\frac{1}{a}+\frac{1}{c}$

Trick I: If a, b, c in A.P., then $b=\frac{a+c}{2}$ and so

$b-a=\frac{a+c}{2}-a=\frac{c-a}{2}$, $b-c=\frac{a+c}{2}-c$

$=\frac{a-c}{2}=-\frac{1}{2}(c-a)$

Put in (1), $\frac{2}{c-a}-\frac{2}{c-a}=\frac{1}{a}+\frac{1}{c}$ or $\frac{1}{a}+\frac{1}{c}=0$ which is not true.

II. If a, b, c are in G.P., then $b^2=ac$ and so

$b-a=\sqrt{ac}-a$

$=\sqrt{a}(\sqrt{c}-\sqrt{a})$

or $b-c=\sqrt{ac}-c=\sqrt{c}(\sqrt{a}-\sqrt{c})$

$=\sqrt{c}(\sqrt{c}-\sqrt{a})$

Now (1) $\Rightarrow\left(\frac{1}{\sqrt{c}-\sqrt{a}}\right)\left(\frac{1}{\sqrt{a}}-\frac{1}{\sqrt{c}}\right)$

$=\frac{1}{a}+\frac{1}{c}\Rightarrow\frac{1}{\sqrt{ac}}$

$=\frac{1}{a}+\frac{1}{c}$

which is not ture.

III. If a, b, c are in H.P., then

$b=\frac{2ac}{a+c}$, $b-a=\frac{2ac}{a+c}$

$=\frac{2ac-a^2-ac}{a+c}$

or $b-a=\frac{2ac-a^2}{c+a}=\frac{a(c-a)}{c+a}$

$=b-c=\frac{2ac}{a+c}-c=\frac{c(a-c)}{a+c}$. Put in (1)

$\left(\frac{c+a}{c-a}\right)\left(\frac{1}{a}-\frac{1}{c}\right)=\frac{1}{a}+\frac{1}{c}$

or $\left(\frac{c+a}{c-a}\right)\left(\frac{c-a}{ac}\right)=\frac{a+c}{ac}$

or $\frac{1}{ac}=\frac{1}{ac}$, which is true.

2. (c) HM between a and $c=b$ and $GM=\sqrt{ac}$

Also HM between b and $d=c$ and $GM=\sqrt{bd}$

But GM > HM

$\therefore\ \sqrt{ac}>b$ and $\sqrt{bd}>c$

$\Rightarrow\sqrt{ac}\sqrt{bd}>bc$

$\Rightarrow ad>bc$

3. (b)

4. (b) AM > HM $\Rightarrow\frac{a+c}{2}>b$

$\Rightarrow\left(\frac{a+c}{2}\right)^2>b^2$ (1)

Also $\frac{a^2+c^2}{2}>\left(\frac{a+c}{2}\right)^2$ (2)

(1) and (2) $\Rightarrow a^2+c^2>2b^2$

5. (b) $b=\frac{2+c}{2}$ (1)

$c^2=bd$ (2)

$d=\frac{36c}{c+18}$ (3)

Eliminate d from (2) and (3) we get $c=\pm 6$

Now from (1) $b=4,-2$ from (3) $d=9,-18$

$\therefore\ b=4, c=6, d=9$

6. (a) Since a, b, c, d are in H.P., therefore b is the H.M. of a and c

i.e., $b=\frac{2ac}{a+c}$ and c is the H.M. of b and d

i.e., $c=\frac{2bd}{b+d}$,

$\therefore (a+c)(b+d)\frac{2ac}{b}\cdot\frac{2bd}{c}$
$\Rightarrow ab+ad+bc+cd=4ad$
$\therefore ab+bc+cd=3ad$

Trick: Check for $a=1, b=\frac{1}{2}, c=\frac{1}{3}, d=\frac{1}{4}$.

7. (a) Let nth term is $an+b$.
$\therefore (n-1)$th term is a $(n-1)+b$.
Difference between these terms
$= an+b-an+a-b=a$
Hence, the sequence is in AP for which difference between the nth term and $(n-1)$th term is independent of n.

8. (b) Let $K=\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$, H.M. $=H=\frac{2ab}{a+b}$

If $n=1, K=\frac{a^2+b^2}{a+b}\neq H$.

If $n=-1, K=\frac{a^0+b^0}{a^{-1}+b^{-1}}=\frac{2ab}{a+b}=H$

9. (a) T_p, T_q, T_r of A.P. in GP
$\Rightarrow a+(p-1)d=T_p, a+(q-1)d$
$=T_q, a+(r-1)d=T_r$ in G.P.

$R=\frac{T_q}{T_p}=\frac{T_r}{T_q}$

$\Rightarrow R=\frac{T_q-T_r}{T_p-T_q}=\frac{(q-r)d}{(p-q)d}=\frac{q-r}{p-q}$

Similarly $R=\frac{r-s}{q-r}$

$\therefore \frac{r-s}{q-r}=\frac{q-r}{p-q}$

or $(q-r)^2=(r-s)(p-q)$

10. (a) $a^{1/x}=b^{1/y}=c^{1/z}$. Taking log,

$\frac{1}{x}\log a=\frac{1}{y}\log b=\frac{1}{z}\log c=K$, Say

$\therefore \frac{1}{x}\log a=K$ (1)

$\frac{1}{y}\log b=K$ (2)

$\frac{1}{z}\log c=K$ (3)

But a, b, c in G.P. $\Rightarrow b^2=ac$.

Now (2) $\Rightarrow \frac{1}{y}\log\sqrt{(ac)}=K$

$\Rightarrow \frac{1}{2y}\log(ac)=K$ (4)

(1) + (3) $\Rightarrow \log a+\log c=K(x+z)$
$\Rightarrow \log(ac)=K(x+z)$
$\Rightarrow 2yK=K(x+z)$, by (4)
$\Rightarrow 2y=x+z\Rightarrow x, y, z$ in A.P.

11. (b) Given $y^2=xz\Rightarrow 2\log y=\log x+\log z$
$\Rightarrow \log x, \log y, \log z$ are in A.P.
$\Rightarrow 1+\log x, 1+\log y, 1+\log z$ are in A.P.
$\Rightarrow \frac{1}{1+\log x}, \frac{1}{1+\log y}, \frac{1}{1+\log z}$ are in H.P.

12. (d) According to assumption, four numbers are

$\frac{a}{r}, a, ar, \frac{a}{r}$ (1).

Last three numbers are in A.P. with common difference = 6.
$\therefore ar=a+6, \frac{a}{r}=a+12$.
Onmultiplying, $a^2=(a+6)(a+12)$
or, $0=18a+72$
or, $a=-4$
Now $ar=a+6, a=-4$
$\Rightarrow -4r=2\Rightarrow r=-\frac{1}{2}$
First number $=\frac{a}{4}(-4)(-2)=8$

13. (a) a, A_1, A_2, b in A.P.

$T_4=b=a+3d\Rightarrow \frac{b-a}{3}=d$

$A_1=a+d=a+\left(\frac{b-a}{3}\right)=\frac{b+2a}{3}$,

$A_2=a+2d=\frac{a+2b}{3}$,

$A_1+A_2=a+b$

a, G_1, G_2, b in G.P.

$T_4=b=ar^3\Rightarrow r=\left(\frac{b}{a}\right)^{1/3}$

$G_1=ar=a\left(\frac{b}{a}\right)^{1/3}, G_2=ar^2=a\left(\frac{b}{a}\right)^{1/3}$

Also $G_1G_2=ab$

and a, H_1, H_2, b in H.P.

$\Rightarrow \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{b}$ in A.P.

$T_4=\frac{1}{b}=\frac{1}{a}+3d\Rightarrow d=\frac{1}{3}\left(\frac{1}{b}-\frac{1}{a}\right)=\frac{ab}{3ab}$

$$\frac{1}{H_1} = \frac{1}{a} + d = \frac{1}{a} + \left(\frac{a-b}{3ab}\right) = \frac{a+2b}{3ab}$$

$$\frac{1}{H_2} = \frac{1}{a} + 2d = \frac{2a+b}{3ab}, \frac{1}{H_1} + \frac{1}{H_2}$$

$$= \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$$

Now $\frac{H_1+H_2}{H_1H_2} = \frac{a+b}{ab} = \frac{A_1+A_2}{G_1G_2}$

$$\Rightarrow \frac{G_1G_2}{H_1H_2} = \frac{A_1+A_2}{H_1+H_2}$$

14. (b) Given that a, b, c are in A.P.

$\Rightarrow b = \frac{a+c}{2}$ (1)

and $b^2 = ad$

Hence, a, $a - b$, $d - c$ are in G.P. because $(a - b)^2 = a^2 - 2ab + b^2 = a(a - 2b) + ad$

$= a(a - a - c) + ad = ad - ac.$

Trick: Take $a = 1$, $b = 2$, $c = 3$ and $d = 4$ and check.

15. (c) Given $a^x = b^y = c^z = d^u$ (1)

a, b, c, d in G.P.

This $\Rightarrow b = ar$, $c = ar^2$, $d = ar^3$.

Taking log in (1), $x \log a = y \log b = z \log c = u \log d = K$, or $x \log a = y \log (ar) = z \log (ar^2) = u \log (ar^3) = K$

$\therefore x = \frac{K}{\log a}$ = etc. Now $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{u}$

$= \frac{1}{K}$ [$\log a$, $\log a + \log r$, $\log a + 2 \log r$, $\log a + 3 \log r$]

which is clearly in A.P.

$\therefore x, y, z, u$ in H.P.

This page is intentionally left blank.

LECTURE

4

Arithmetico–Geometric Series

BASIC CONCEPTS

1. Arithmetico−Geometric Series

Series whose terms are formed by the product of the corresponding terms of an A.P. and a G.P. is called an Arithmetico−geometric series and is written in short as A.G.S. or A.G. Series.

For example

$a, a+d, a+2d, \ldots \{a+(n-1)d\} \ldots$ is an A.P. and $1, r, r^2, \ldots\ldots\ldots\ldots r^{n-1}$ is a G.P., then the general form of an Arithmetico−geometric series in terms of n is $a+(a+d)r+(a+2d)r^2+\ldots\ldots\ldots\ldots+\{a+(n-1)d\}r^{n-1}$ and the sum of n terms of an AGP is

$$S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r}$$

$n \in N$ (not necessary to memorise)

2. Sum of an infinite Arithmetico Geometric Series

$$S = a + (a+d)r + \ldots\ldots\ldots + \ldots\ldots\ldots + \infty.$$

$$\text{Sum} = S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

where $|r| < 1$ i.e., $-1 < r < 1$

Note 1: The nth term of A.G.S. = nth term of A.P. × nth term of G.P.

Note 2: To find the sum of such a progression its sum is taken equal to S and multiply both sides by the common ratio of the corresponding G.P. and then subtract after shifting the terms of r s by one column on the right hand side.

3. Method of Difference

If terms of the sequence are neither in A.P. nor in G.P. but difference between successive terms are either in A.P. or in G.P., then nth term of the sequence is determined by the method of difference which is as follows

Suppose $a_1, a_2, a_3, \ldots$ is a sequence such that the sequence $a_2 - a_1, a_3 - a_2, \ldots$ is either an A.P. or a G.P. The nth term 'a'_n of this sequence is obtained as follows

$$S = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n$$

$$S = a_1 + a_2 + \ldots\ldots + a_{n-2} + a_{n-1} + a_n$$

$$\Rightarrow a_n = a_1 + [(a_2 - a_1) + (a_3 - a_2) + \ldots + (a_n - a_{n-1})]$$

Since the terms within the brackets are either in an A.P. or in a G.P., we can find the value of a_n, the nth term, We can now find the sum of the n terms of the sequence as

$$S = \sum_{k=1}^{n} a_k$$

4. Some Important Formula

(i) The sum of first n natural numbers

$$\sum n = 1 + 2 + 3 + \ldots\ldots + n = \frac{n(n+1)}{2}$$

$\sum$ means it is adding the series and it is called the sigma.

(ii) The sum of first n odd natural numbers

$$\sum_{r=1}^{n} (2r-1) = 1 + 3 + 5 + \ldots\ldots + (2n-1) = n^2$$

(iii) The sum of first n even numbers

$$\sum_{r=1}^{n} 2r = 2 + 4 + 6 + \ldots\ldots + 2n = n(n+1)$$

$$= n^2 + n$$

(iv) The sum of squares of first n natural numbers

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + \ldots\ldots + n^2$$

$$= \frac{n(n+1)(2n+1)}{6}$$

Proof

Let S_n (or Σn) denote the sum of the squares of the first n natural numbers, then

$S_n = 1^2 + 2^2 + 2^3 + \ldots\ldots\ldots\ldots\ldots + n^2$.

Consider the identity

$$x^3 - (x-1)^3 = 3x^2 - 3x + 1 \quad (1)$$

Putting $x = 1, 2, 3, \ldots\ldots, n$ in (1), we obtain

$$1^3 - 0^3 = 3 \cdot 1^2 - 3 \cdot 1 + 1$$
$$2^3 - 1^3 = 3 \cdot 2^2 - 3 \cdot 2 + 1$$
$$3^3 - 2^3 = 3 \cdot 3^2 - 3 \cdot 3 + 1$$

....

....

$$n^3 - (n-1)^3 = 3n^2 - 3n + 1$$

Adding these, we get,

$$n^3 - 0^3 = 3 \,.\, (1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + n^2) - 3 \cdot (1 + 2 + 3 + \ldots + n) + n$$

$$\Rightarrow n^3 = 3 \cdot S_n - 3 \cdot \frac{n(n+1)}{2} + n$$

$$\left(\because 1 + 2 + 3 + \ldots\ldots + n = \frac{n(n+1)}{2}\right)$$

$$\Rightarrow 3S_n = n^3 + \frac{3}{2} n(n+1) - n$$

$$= \frac{n}{2}[2n^2 + 3(n+1) - 2]$$

$$= \frac{n}{2}(2n^2 + 3n + 1) = \frac{n(n+1)(2n+1)}{2}$$

$$\Rightarrow S_n = \frac{n(n+1)(2n+1)}{6}$$

$$\Rightarrow \sum n^2 = \frac{n(n+1)(2n+1)}{6}$$

(v) The sum of cubes of the first n natural numbers

$$\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + \ldots\ldots + n^3$$

$$= \left[\frac{n(n+1)}{2}\right]^2$$

Proof

Let $\Sigma n^3 = 1^3 + 2^3 + 3^3 + \ldots\ldots + (n-1)^3 + n^3$

We know that $x^4 - (x-1)^4 = 4x^3 - 6x^2 + 4x - 1$

Putting $x = 1, 2, 3, \ldots\ldots, (n-1), n$ successively,

$$1^4 - 0^4 = 4.1^3 - 6.1^2 + 4.1 - 1$$
$$2^4 - 1^4 = 4.2^3 - 6.2^2 + 4.2 - 1$$

.... =................................

.... =................................

$$(n-1)^4 - (n-2)^4 = 4(n-1)^3 - 6(n-1)^2 + 4(n-1) - 1$$

$$n^4 - (n-1)^4 = 4n^3 - 6n^2 + 4n - 1$$

Adding columnwise,

$$n^4 = 4[1^3 + 2^3 + 3^3 + \ldots\ldots + (n-1)^3 + n^3] - 6(1^2 + 2^2 + 3^2 + \ldots\ldots + (n-1)^2 + n^2] + 4[1 + 2 + 3 + \ldots\ldots + (n-1) + n] - n$$

$$\Rightarrow n^4 = 4\sum n^3 - 6\sum n^2 + 4\sum n - n$$

$$\Rightarrow 4\sum n^3 = n^4 + 6\sum n^2 - 4\sum n + n$$

$$= n^4 + \frac{6n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n$$

$$= n^4 + n(n+1)(2n+1) - 2n(n+1) + n$$
$$= n[n^3 + (n+1)(2n+1) - 2(n+1) + 1]$$
$$= n[n^3 + 2n^2 + 3n + 1 - 2n - 2 + 1]$$
$$= n[n^3 + 2n^2 + n]$$
$$= n^2[n^2 + 2n + 1]$$

$$\Rightarrow 4\sum n^3 = [n(n+1)]^2 = [n(n+1)]^2$$

$$\Rightarrow \sum n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

SUBJECTIVE PROBLEMS: (CBSE/STATE BOARD): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Find the sum to infinity of the series

$1-\frac{3}{2}+\frac{5}{4}-\frac{7}{8}+\frac{9}{10}-\ldots\ldots$

Solution

$S=1-\frac{3}{2}+\frac{5}{4}-\frac{7}{8}+\frac{9}{10}-\ldots\ldots$

$\therefore S=1+3\left(-\frac{1}{2}\right)+5\left(-\frac{1}{2}\right)^2+7\left(-\frac{1}{2}\right)^3+9\left(-\frac{1}{2}\right)^4+$ (1)

Multiplying both sides of Equation (1) by (−1/2),

$S\left(-\frac{1}{2}\right)=\left(-\frac{1}{2}\right)+3\left(-\frac{1}{2}\right)^2+5\left(-\frac{1}{2}\right)^3+7\left(-\frac{1}{2}\right)^4+$ (2)

Subtracting equation (2) from Equation (1),

$S\left[1+\frac{1}{2}\right]=1+2\left[\left(-\frac{1}{2}\right)+\left(-\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)^3+\ldots\ldots\right]$

or $\frac{3}{2}S=1+2\left[\frac{\left(-\frac{1}{2}\right)}{1-\left(-\frac{1}{2}\right)}\right],\left[S_\infty=\frac{a}{1-r}\right]$

$\Rightarrow \frac{3}{2}S=1-\frac{2}{3}=\frac{1}{3}$

$\Rightarrow S=\frac{2}{9}$

2. Find the value of $1^3-2^3+3^3-4^3+\ldots\ldots\ldots-9^3$.

Solution

$1^3-2^3+3^3-4^3+\ldots\ldots\ldots\ldots-9^3$

$=(1^3+2^3+3^3+\ldots\ldots\ldots\ldots 9^3)-2(2^3+4^3+6^3+8^3)$

$=\left[\frac{9(9+1)}{2}\right]^2-2\times 2^3(1^3+2^3+3^3+4^3)$

$=\left[\frac{9\times 10}{2}\right]^2-2\times 8\left[\frac{4(4+1)}{2}\right]^2$

$=(45)^2-16(10)^2$

$=2025-1600$

$=425$ Ans

3. Prove that the next term of the sequence 1, 5, 14, 30, 55, ... is 91.

Solution

(b) $T_1=1=l^2+0,$

$T_2=5=2^2+1=2^2+T_1,$

$T_3=14=3^2+5=3^2+T_2,$

$T_4=30=4^2+14=4^2+T_3,$

$T_5=55=5^2+30=5^2+T_4$

$\therefore T_6=6^2+T_5=36+55=91$

4. Balls are arranged in rows to form an equilateral triangle. The first row consists of one ball, the second row of two balls and so on. If 669 more balls are added then all the balls can be arranged in the shapes of a square and each of the sides then contains 8 balls less than each side of the triangle did. Determine the initial numbers of balls.

[Bihar C.E.T. – 1999]

Solution

$S=1+2+3+4+\ldots\ldots+n=\sum n=\frac{n(n+1)}{2};$

$S+669=(n-8)^2$

or $\frac{n(n+1)}{2}+669=n^2-16n+64$

or $n^2-33n-1210=0$

or $n^2-55n+22n-1210=0$

$\therefore n=55.$

$\therefore$ Number of balls is 55 . 56/2 =1540.

Check: $1540+669=2209=(55-8)^2=47^2$.

5. Along a road lie an odd number of stones placed at intervals of 10 metres. These stones have to be assembled around the middle stone.

A person can carry only one stone at a time. A man carried the job with one of the end stones by carrying them in succession. In carrying all the stones he covered a distance of 3 km. Find the number of stones.

[MP PET −2005]

Solution

(b) Let the number of stones be $2n + 1$ so that there is one mid-stone and n stones each on either side of it. If P be mid stone and A, B be last stones on the left and right of P respectively.

There will be $(n + 1)$ stones on the left and $(n + 1)$ stones on right side of P (P being common to both sides) or n intervals each of 10 metres both on the right and left side of mid-stone. Now he starts from one of the end stones, picks it up, goes to mid-stone, drops it and goes to last stone on the other side, picks it and comes back to mid-stone. In all he travels n intervals of 10 metres each 3 times. Now from centre he will go to 2nd stone on L.H.S. then come back and then go to 2nd last on R.H.S. and again come back.

Thus, he will travel $(n - 1)$ intervals of 10 metres each 4 times. Similarly $(n - 2)$ intervals of 10 metres each 4 times for 3rd and so on for the last.

Hence the total distance covered as given = 3 k.m. = 3000 m.

or $3.10\, n + 4\,[10\,(n - 1) + 10(n - 2) + ... + 10]$

$\Rightarrow 30n + 40[\,1 + 2 + 3 + ...\,(n - 1)] = 3000$

$\Rightarrow 30n + 40[(n - 1)/2]\,[1 + n - 1] = 3000$

or $2n^2 + n - 300 = 0$

or $(n - 12)\,(2n + 25) = 0$.

$\therefore n = 12$.

Hence the number of stones = $2n + 1 = 25$.

6. 150 workers were engaged to finish a piece of work in a certain number of days. Four workers dropped the second day, four more workers dropped the third day and so on. It takes 8 more days to finish the work now. Find the number of days in which the work was completed.

Solution

Suppose the work is completed in n days when the workers started dropping. Since 4 workers are dropped on every day except the first day. Therefore, the total number of workers who worked all the n days is the sum of n terms of an A.P. with first term 150 and common difference -4 i.e.,

$\frac{n}{2}\,[2 \times 150 + (n - 1) \times -4] = n\,(152 - 2n)$

Had the workers not dropped then the work would have finished in $(n - 8)$ days with 150 workers working on each day. Therefore, the total number of workers who would have worked all the n days is $150\,(n - 8)$.

$\therefore n(152 - 2n) = 150\,(n - 8)$

$\Rightarrow n^2 - n - 600 = 0$

$\Rightarrow (n - 25)\,(n + 24) = 0$

$\Rightarrow n = 25$.

Thus, the work is completed in 25 days.

UNSOLVED SUBJECTIVE PROBLEMS: (CBSE/STATE BOARD): TO GRASP THE TOPIC, SOLVE THESE PROBLEMS

Exercise I

Find the sum to n terms of each of the following series

1. $1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + ...$

2. $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + ...$

3. $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + ...$

4. $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + ...$

5. Show that

$$\frac{1 \times 2^2 + 2 \times 3^2 + ... + n \times (n + 1)^2}{1^2 \times 2 + 2^2 \times 3 + ... + n^2 \times (n + 1)} = \frac{3n + 5}{3n + 1}$$

6. Find the nth term and sum to n terms of the following series

$1 + (1 + 2) + (1 + 2 + 3)$
$+ (1 + 2 + 3 + 4) + ...$

7. Find the nth term of the series $1^2 + 4^2 + 7^2 + 10^2 + ...$ and also sum of n terms of it.

8. Find the sum to n term of the series
$1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+......+\frac{2n-1}{2^n-1}$

9. Find the sum up to infinity $1 + 2x + 3x^2 + 4x^3 + ... \infty$.

Exercise II

Find the sum to n terms of each of the following series

1. $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 +$..........

2. $3 \times 8 + 6 \times 11 + 9 \times 14 +$.......

3. $(1 \times 2 \times 4) + (2 \times 3 \times 7) + (3 \times 4 \times 10) +$ .... to n terms.

4. Find the nth term as well as sum to n terms of the following series 2.5 + 3.8 + 4.11 + 5.14 + ...

5. Find the sum of the following series $1.2 + 2.2^2 + 3.2^3 + 4.2^4 +$ .... to n terms.

6. Find the sum to n terms of the series

$1+\left(1+\frac{1}{2}\right)+\left(1+\frac{1}{2}+\frac{1}{2^2}\right)+.........$

7. If $x < 1$, then find the sum of the series $1 - 5x + 9x^2 - 13x^2 +$ .... up to ∞

[MP – 1993, 1997]

ANSWERS

Exercise I

1. $\frac{n}{3}(n+1)(n+2)$

2. $\frac{n}{6}(n+1)(3n^2+5n+1)$

3. $\frac{n(n+1)^2(n+2)}{12}$

4. $\frac{n}{n+1}$

6. $\frac{n(n+1)(n+2)}{6}$

7. $S_n = \frac{n}{2}[6n^2 - 3n - 1]$

8. $S_n = 6 - \frac{1}{2^{n-3}} - \frac{2n-1}{2^{n-1}}$

9. $S = \frac{1}{(1-x)^2}$

Exercise II

1. $\frac{n(n+1)(n+2)(n+3)}{4}$

2. $3n(n+1)(n+3)$

3. $\frac{n(n+1)(3n^2+19n+17)}{12}$

4. $S_n = \frac{n}{2}[2n^2 + 8n + 10]$

5. $S_n = 2 + (n-1)2^{n+1}$

6. $2n - 2 + \frac{1}{2^{n-1}}$

7. $\frac{1-3x}{(1+x)^2}$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. The sum of the series $1 + 2x + 3x^2 + 4x^3 +$ ....up to n terms is

(a) $\frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}$

(b) $\frac{1-x^n}{1-x}$ (c) x^{n+1}

(d) none of these **[EAMCET –1998]**

Solution

(a) Let S_n be the sum of the given series to n terms, then

$S = 1 + 2x + 3x^2 + 4x^3 + ... + nx^{n-1}$ (1)

$xS_n = x + 2x^2 + 3x^2 + ... + nx^n$ (2)

Subtracting (2) from (1) after shifting the terms of xS_n by one column on the right hand side.

$(1-x)S_n = 1 + x + x^2 + x^3 + \ldots\ldots\ldots \text{to } n \text{ terms} - nx^n$

$$= \left(\frac{(1-x^n)}{(1-x)}\right) - nx^n$$

$$\Rightarrow S_n = \frac{(1-x^n) - nx^n(1-x)}{(1-x)^2}$$

$$= \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}$$

2. The sum of $(n + 1)$ terms of $\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots\ldots$ is

(a) $\frac{n}{n+1}$ (b) $\frac{2n}{n+1}$

(c) $\frac{2}{n(n+1)}$ (d) $\frac{2(n+1)}{n+2}$

[RPET – 1999]

Solution

(d) $T_n = \frac{1}{\left[\frac{n(n+1)}{2}\right]} = 2\left[\frac{1}{n} - \frac{1}{n+1}\right]$

Put $n = 1, 2, 3, \ldots\ldots, (n + 1)$

$T_1 = 2\left[\frac{1}{1} - \frac{1}{2}\right], T_2 = 2\left[\frac{1}{2} - \frac{1}{3}\right], \ldots.,$

$T_{n+1} = 2\left[\frac{1}{n+1} - \frac{1}{n+2}\right]$

Hence sum of $(n + 1)$ terms $= \sum_{k=1}^{n+1} T_k$

$\Rightarrow S_{n+1} = 2\left[1 - \frac{1}{n+2}\right]$

$\Rightarrow S_{n+1} = \frac{2(n+1)}{(n+2)}.$

3. The sum of first n terms of the given series $1^2 + 2\,.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + \ldots\ldots\ldots\ldots$ is $\frac{n(n+1)^2}{2}$, when n is even. When n is odd, the sum will be

(a) $\frac{n(n+1)^2}{2}$ (b) $\frac{1}{2}n^2(n+1)$

(c) $n(n + 1)^2$ (d) none of these

[IIT – 1988]

Solution

(b) When n is odd, the last term i.e., the nth term will be n^2 in this case $n - 1$ is even and so the sum of the first $n - 1$ terms of the series is obtained by replacing n by $n - 1$ in the given formula and so is $\frac{1}{2}(n-1)n^2$. Hence the sum of the n terms

= (the sum of $n - 1$ terms) + the nth term

$= \frac{1}{2}(n-1)n^2 + n^2 = \frac{1}{2}(n+1)n^2.$

Trick: Check for $n = 1, 3$. Here $S_1 = 1$, $S_3 = 18$ which gives (b).

4. The sum of the series $3.6 + 4.7 + 5.8 + \ldots..$ upto $(n - 2)$ terms

(a) $n^3 + n^2 + n + 2$

(b) $\frac{1}{6}(2n^3 + 12n^2 + 10n - 84)$

(c) $n^3 + n^2 + n$

(d) none of these

[EAMCET – 1980]

Solution

(b) $S = 3\,.\,6 + 4.7 + \ldots\ldots\ldots$ upto $n - 2$ terms

$= (1.4 + 2.5 + 3.6 + 4.7 + \ldots\ldots.. \text{ upto } n \text{ terms}) - 14$

$= \sum n(n+3) - 14$

$= \frac{1}{2}(2n^3 + 12n^2 + 10n) - 14$

$= \left(\frac{2n^3 + 12n^2 + 10n - 84}{6}\right)$

where $n = 3, 4, 5 \ldots\ldots\ldots$

Trick: $S_1 = 18$, $S_2 = 46$. Now put in options $(n - 2) = 1, 2$ i.e $n = 3, 4$

Obviously (b) gives the values.

5. $\frac{\frac{1}{2}.\frac{2}{2}}{1^3} + \frac{\frac{2}{2}.\frac{3}{2}}{1^3+2^3} + \frac{\frac{3}{2}.\frac{4}{2}}{1^3+2^3+3^3} + \ldots\ldots$ n terms =

(a) $\left(\frac{n}{n+1}\right)^2$ (b) $\left(\frac{n}{n+1}\right)^3$

(c) $\left(\frac{n}{n+1}\right)$ (d) $\left(\frac{1}{n+1}\right)$

[EAMCET – 2000]

Solution

(c) $T_n = \dfrac{\dfrac{n(n+1)}{2.2}}{1^3 + 2^3 + 3^3 + \ldots\ldots + n^3}$

$= \dfrac{\dfrac{n(n+1)}{4}}{\left(\dfrac{n(n+1)}{2}\right)^2}$

$= \dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$

$\therefore S_n = \sum \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right)$

$+ \ldots\ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$= 1 - \dfrac{1}{n+1} = \dfrac{n}{n+1}$

6. If $t_n = \frac{1}{4}(n+2)(n+3)$ for $n = 1, 2, 3, \ldots\ldots$

then $\dfrac{1}{t_1} + \dfrac{1}{t_2} + \dfrac{1}{t_3} + \ldots\ldots + \dfrac{1}{t_{2003}} =$

(a) $\dfrac{4006}{3006}$ (b) $\dfrac{4003}{3007}$

(c) $\dfrac{4006}{3008}$ (d) $\dfrac{4006}{3009}$

[EAMCET – 2003]

Solution

(d) $t_n = \frac{1}{4}(n+2)(n+3)$, them

$\dfrac{1}{t_1} + \dfrac{1}{t_2} + \dfrac{1}{t_3} + \ldots\ldots + \dfrac{1}{t_{2003}} =$

$= 4\left[\dfrac{1}{3.4} + \dfrac{1}{4.5} + \dfrac{1}{5.6} + \ldots\ldots + \dfrac{1}{(2005).(2006)}\right]$

$= 4\left[\dfrac{1}{3} - \dfrac{1}{2006}\right] = 4.\ \dfrac{2003}{3(2006)} = \dfrac{4006}{3009}$

7. If $\dfrac{1}{1^4} + \dfrac{1}{2^4} + \dfrac{1}{3^4} + \ldots\ldots + \infty = \dfrac{\pi^4}{90}$, then the value of $\dfrac{1}{1^4} + \dfrac{1}{3^4} + \dfrac{1}{5^4} + \ldots\ldots + \infty$ is

(a) $\dfrac{\pi^4}{96}$ (b) $\dfrac{\pi^4}{45}$

(c) $\dfrac{89}{90}\pi^4$ (d) none of these

[AMU – 2005]

Solution

(a) $\dfrac{1}{1^4} + \dfrac{1}{2^4} + \dfrac{1}{3^4} + \ldots\ldots + \infty = \dfrac{\pi^4}{90}$

$\dfrac{1}{1^4} + \dfrac{1}{3^4} + \dfrac{1}{5^4} + \ldots\ldots + \infty$

$+ \dfrac{1}{2^4}\left(\dfrac{1}{1^4} + \dfrac{1}{2^4} + \dfrac{1}{3^4} + \dfrac{1}{4^4} + \ldots\ldots\infty\right) = \dfrac{\pi^4}{90}$

$\dfrac{1}{1^4} + \dfrac{1}{3^4} + \dfrac{1}{5^4} + \dfrac{1}{7^4} + \ldots\ldots + \infty + \dfrac{1}{16} \times \dfrac{\pi^4}{90} = \dfrac{\pi^4}{90}$

$\therefore \dfrac{1}{1^4} + \dfrac{1}{3^4} + \dfrac{1}{5^4} + \dfrac{1}{7^4} + \ldots\ldots + \infty$

$= \dfrac{\pi^4}{90} - \dfrac{1}{16}\left(\dfrac{\pi^4}{90}\right)$

$= \dfrac{15}{16}\left(\dfrac{\pi^4}{90}\right) = \dfrac{\pi^4}{69}$

8. The value of $\dfrac{1}{(1+a)(2+a)}$

$+ \dfrac{1}{(2+a)(3+a)} + \dfrac{1}{(3+a)(4+a)} + \ldots\ldots$

$+ \infty$ is, (where a is a constant)

(a) $\dfrac{1}{1+a}$ (b) $\dfrac{2}{1+a}$

(c) ∞ (d) none of these

[AMU – 2005]

Solution

(a) $\dfrac{1}{(1+a)(2+a)} + \dfrac{1}{(2+a)(3+a)}$

$+ \dfrac{1}{(3+a)(4+a)} + \ldots\ldots + \infty$ nth term of series

$T_n = \dfrac{1}{(n+a)(n+1+a)} = \dfrac{1}{n+a} - \dfrac{1}{n+1+a}$

$T_1 = \dfrac{1}{1+a} - \dfrac{1}{2+a}$; $T_2 = \dfrac{1}{2+a} - \dfrac{1}{3+a}$,

$T_3 = \dfrac{1}{3+a} - \dfrac{1}{4+a}$

$T_{n-1} = \dfrac{1}{n-1+a} - \dfrac{1}{n+a}$,

$T_n = \dfrac{1}{n+a} - \dfrac{1}{n+1+a}$

$\therefore S_n = T_1 + T_2 + T_3 + \ldots\ldots + T_n$

$= \dfrac{1}{1+a} - \dfrac{1}{n+1+a}$

$$= \frac{n}{(1+a)(n+1+a)},$$

$$S_n = \frac{1}{(1+a)\left(1+\frac{1}{n}+\frac{a}{n}\right)}$$

$S_\infty = S_n$, where $n \to \infty$

$$\therefore S_\infty = \frac{1}{(1+a)}$$

9. Jairam purchased a house in Rs 15000 and paid Rs 5000 at once. Rest money he promised to pay in annual installment of Rs 1000 with 10% per annum interest. How much money is to be paid by Jairam.

(a) Rs 21555 (b) Rs 20475
(c) Rs 20500 (d) Rs 20700

[UPSEAT – 1999]

Solution

(c) It will take 10 years for Jairam to pay off Rs 10000 in 10 yearly installments.

$\because$ He pays 10% annual interest on remaining amount

$\therefore$ Money given in first year

$$= 1000 + \frac{10000 \times 10}{100} = \text{Rs } 2000$$

Money given in second year

= 1000 + interest of (10000 − 1000) with interest rate 10% per annum

$$= 1000 + \frac{9000 \times 10}{100} = \text{Rs } 1900$$

Money paid in third year = Rs 1800 etc. So money given by Jairam in 10 years will be Rs 2000, Rs 1900, Rs 1800, Rs 1700, ..., which is in arithmetic progression, whose first term a = 2000 and d = − 100 Total money given in 10 years = sum of 10 terms of arithmetic progression

$$= \frac{10}{2}[2(2000) + (10-1)(-100)]$$

= Rs 15500

Therefore, total money by jairam = 5000 + 15500 = Rs 20500

10. If a_1, a_2, a_3 .... a_n are in A.P., where $a_i > 0$ for all i, then the value of

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \ldots\ldots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}$$

(a) $\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$ (b) $\frac{n+1}{\sqrt{a_1}+\sqrt{a_n}}$

(c) $\frac{n-1}{\sqrt{a_1}-\sqrt{a_n}}$ (d) $\frac{n+1}{\sqrt{a_1}+\sqrt{a_n}}$

[IIT −1982]

Solution

(a) As given $a_2 - a_1 = a_3 - a_2 = \ldots\ldots$

$= a_n - a_{n-1} = d$

Where d is the common difference of the given A.P. Also $a_n = a_1 + (n-1)d$. Then by rationalising each term,

$$\frac{1}{\sqrt{a_2}+\sqrt{a_1}} + \frac{1}{\sqrt{a_3}+\sqrt{a_2}} + \ldots\ldots + \frac{1}{\sqrt{a_n}+\sqrt{a_{n-1}}}$$

$$= \frac{\sqrt{a_2}-\sqrt{a_1}}{a_2-a_1} + \frac{\sqrt{a_3}-\sqrt{a_2}}{a_3-a_2} + \ldots\ldots + \frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{a_n-a_{n-1}}$$

$$= \frac{1}{d}\{\sqrt{a_2}-\sqrt{a_1}+\sqrt{a_3}-\sqrt{a_2}+\ldots\ldots+\sqrt{a_n}-\sqrt{a_{n-1}}\}$$

$$= \frac{1}{d}\{\sqrt{a_n}-\sqrt{a_1}\} = \frac{1}{d}\left\{\frac{an-a_1}{\sqrt{a_n}+\sqrt{a_1}}\right\}$$

$$= \frac{1}{d}\left\{\frac{(n-1)d}{\sqrt{a_n}+\sqrt{a_1}}\right\} = \frac{n-1}{\sqrt{a_n}+\sqrt{a_1}}$$

11. If a_1, a_2, ..., a_n are in A.P. with common difference, d, then the sum of the following series is sin d(cosec a_1. cosec a_2 + cosec a_2. cosec a_3 +.............+ cosec a_{n-1} cosec a_n)

(a) sec a_1 − sec a_n
(b) cot a_1 − cot a_n
(c) tan a_1 − tan a_n
(d) cosec a_1 − cosec a_n

[RPET – 2000]

Solution

(b) As given $d = a_2 - a_1 = a_3 - a_2$

$= \ldots\ a_n - a_{n-1}$

$\therefore$ sin d {cosec a_1 cosec a_2 +..........+ cosec a_{n-1} cosec a_n}

$$= \frac{\sin(a_2-a_1)}{\sin a_1 . \sin a_2} + \ldots\ldots + \frac{\sin(a_n-a_{n-1})}{\sin a_{n-1} \sin a_n}$$

$= (\cot a_1 - \cot a_2) + (\cot a_2 - \cot a_3)$
$+ \ldots + (\cot a_{n-1} - \cot a_n) = \cot a_1 - \cot a_n$.

12. If $a_1, a_2, \ldots, a_{n+1}$ are in A.P., then $\frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \ldots + \frac{1}{a_na_{n+1}}$ is

[Ranchi – 1985; AMU – 2002]

(a) $\frac{n-1}{a_na_{n+1}}$ (b) $\frac{1}{a_1a_{n+1}}$

(c) $\frac{n+1}{a_1a_{n+1}}$ (d) $\frac{n}{a_1a_{n+1}}$

Solution

(d) $a_1, a_2, a_3, \ldots, a_{n+1}$ are in A.P. and common difference = d

Let $S = \frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \ldots + \frac{1}{a_na_{n+1}}$

$\Rightarrow S = \frac{1}{d}\left\{\frac{d}{a_1a_2} + \frac{d}{a_2a_3} + \ldots + d/a_na_{n+1}\right\}$

$\Rightarrow S = \frac{1}{d}\left\{\frac{a_2-a_1}{a_1a_2} + \frac{a_3-a_2}{a_2a_3} + \ldots + \frac{a_{n+1}-a_n}{a_na_{n+1}}\right\}$

$\Rightarrow S = \left\{\frac{1}{d}\frac{1}{a_1} - \frac{1}{a_2} + \frac{1}{a_2} - \frac{1}{a_3} + \ldots + \frac{1}{a_n} - \frac{1}{a_{n+1}}\right\}$

$\Rightarrow S = \frac{1}{d}\left\{\frac{1}{a_1} - \frac{1}{a_{n+1}}\right\} = \frac{a}{d}\left\{\frac{a_{n+1}-a_1}{a_1a_{n+1}}\right\}$

$\Rightarrow S = \frac{1}{d}\left(\frac{nd}{a_1a_{n+1}}\right) = \frac{n}{a_1a_{n+1}}$

Trick: Check for $n = 2$

13. Let the sequence $a_1, a_2, a_3, \ldots a_{2n}$ form an A.P. Then $a_1^2 - a_2^2 + a_3^2 - \ldots + a_{2n-1}^2 - a_{2n}^2 =$

(a) $\frac{n}{2n-1}(a_1^2 - a_{2n}^2)$ (b) $\frac{2n}{n-1}(a_{2n}^2 - a_1^2)$

(c) $\frac{n}{n+1}(a_1^2 + a_{2n}^2)$ (d) none of these

Solution

(a) Since, $a_1, a_2, a_3, \ldots, a_n$ form an A.P. therefore $a_2 - a_1 = a_4 - a_3 = \ldots = a_{2n} - a_{2n-1} = d$

Here $a_1^2 - a_2^2 + a_3^2 - a_4^2 + \ldots + a_{2n-1}^2 - a_{2n}^2$
$= (a_1 - a_2)(a_1 + a_2) + (a_3 - a_4)(a_3 + a_4)$
$+ \ldots + (a2_{n-1} - a_{2n})(a_{2n-1} + a_{2n})$

$= -d(a_1 + a_2 + \ldots + a_{2n})$

$= -d\left\{\frac{2n}{2}(a_1 + a_{2n})\right\}$

Also we know $a_{2n} = a_1 + (2n-1)d$

$\Rightarrow d = \frac{a_{2n} - a_1}{2n-1}$

$\Rightarrow -d = \frac{a_1 - a_{2n}}{2n-1}$

Therefore, the sum is

$= \frac{n(a_1 + a_{2n})(a_1 + a_{2n})}{2n-1} = \frac{n}{2n-1}(a_1^2 - a_{2n}^2)$

14. The sum of n terms of the series

$= \frac{1}{1+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{7}} + \ldots$ is

(a) $\sqrt{2n+1}$ (b) $\frac{1}{2}\sqrt{2n+1}$

(c) $\sqrt{2n+1} - 1$ (d) $\frac{1}{2}(\sqrt{2n+1} - 1)$

[UPSEAT – 2002]

Solution

(d) Putting $n = 1$;

$s = \frac{1}{1+\sqrt{3}} = \frac{\sqrt{3}-1}{2}$

On checking, (d) is the correct option.

15. The sum of all the products of the first n natural numbers taken two at a time is

(a) $\frac{1}{24}n(n-1)(n+1)(3n+2)$

(b) $\frac{n^2}{48}(n-1)(n-2)$

(c) $\frac{1}{6}n(n+1)(n+2)(n+5)$

(d) none of these

Solution

(a) We know that $\left\{\frac{n}{2}(n+1)\right\}^2$

$= (1 + 2 + \ldots + n)^2 = \sum_1^n r^2 + 2\sum_{s<t} st$

$\Rightarrow \sum_{s<t} st = \frac{1}{2}\left\{\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}\right\}$

$= \frac{n}{24}(n-1)(n+1)(3n+2)$

Trick: $S_n = 1.2 + 2.3 + 3.4 + ... + (n-1).n$
Check by putting $(n-1) = 1, 2$ i.e., $n = 2, 3$ in the options.

16. The sum of the series $1.3^2 + 2.5^2 + 3.7^2 +$ ...... upto 20 terms is
(a) 188090 (b) 189080
(c) 199080 (d) none of these

[IIT – 1973]

Solution

(a) Here T_n of the A.P. 1, 2, 3, ... $= n$ and T_n of the A.P. 3, 5, 7, ... $= 2n + 1$

$\therefore$ T_n of given series $= n(2n+1)^2$

$= 4n^3 + 4n^2 + n$

Hence, $S = \sum_1^{20} T_n = 4\sum_1^{20} n^3 + 4\sum_1^{20} n^2 + \sum_1^{20} n$

$= 4\cdot\frac{1}{4}20^2\cdot 21^2 + 4\cdot\frac{1}{6}20\cdot 21\cdot 41 + \frac{1}{2}20\cdot 21$

$= 188090.$

17. Let $n(> 1)$ be a positive integer, then the largest integer m such that $(n^m + 1)$ divides $(1 + n + n^2 + ... + n^{127})$, is
(a) 32 (b) 63
(c) 64 (d) 127

[IIT – 1995]

Solution

(c) Since, $n^m + 1$ divides $1 + n + n^2 + ... + n^{127}$

Therefore $\frac{1 + n + n^2 + ... + n^{127}}{n^m + 1}$ is an integer

$\Rightarrow \frac{1 - n^{128}}{1 - n} \times \frac{1}{n^m + 1}$ is an integer

$\Rightarrow \frac{(1 - n^{64})(1 + n^{64})}{(1 - n)(n^m + 1)}$

is an integer when largest $m = 64$.

Comprehension Type

Let V_r denote the sum of the first r terms of an arithmetic progression (A.P.) whose first term is r and the common difference is $(2r - 1)$.

Let $T_r = V_{r+1} - V_r - 2$ and $Q_r = T_{r+1} - T$ for $r = 1, 2$ .....

18. The sum $V_1 + V_2 + ... + V_n$ is

(a) $\frac{1}{12}n(n+1)(3n^2 - n + 1)$

(b) $\frac{1}{12}n(n+1)(3n^2 + n + 2)$

(c) $\frac{1}{2}n(2n^2 - n + 1)$

(d) $\frac{1}{3}(2n^3 - 2n + 3)$

[IIT – 2007]

Solution

(b) $V_r = \frac{r}{2}[2r + (r-1)(2r-1)]$

$= \frac{1}{2}(2r - r + r)$

$\sum_{r=1}^{n} V_i = \sum_{r=1}^{n} V_r = \sum_{r=1}^{n} r^3 - \frac{1}{2}\sum_{r=1}^{n} r^2 + \frac{1}{2}\sum_{r=1}^{n}$

$= \frac{n^2(n+1)}{4 - n(n+1)(2n+1)/12 + n(n+1)/4}$

$= \sum_{r=1}^{n} V_i = \frac{n(n+1)(3n^2 + n + 2)}{12}$

19. T_r is always
(a) an odd number
(b) an even number
(c) a prime number
(d) a composite number

[IIT –2007]

Solution

(d) $T_r = V_{r+1} - V_r - 2$

$= (r+1)^3 - r^3 - \frac{1}{2}[(r+1)^2 - r^2] + \frac{1}{2} - 2$

$= 3r^2 + 3r + 1 - (r) - 2 = 3r^2 + 2r - 1$

$= (3r - 1)(r + 1)$

$\therefore$ T_r is a composite number.

20. Which one of the following is a correct statement
(a) Q_1, Q_2, Q_3, ... are in A.P. with common difference 5.

(b) $Q_1, Q_2, Q_3,$ .... are in A.P. with common difference 6.

(c) $Q_1, Q_2, Q_3,$ .... are in A.P. with common difference 11.

(d) $Q_1 = Q_2 = Q_3 =$.............

[IIT −2007]

Solution

(b) $Q_r = T_{r+1} - T_r = 3((r+1)^2 - r^2) + 2$
$= 3(2r+1) + 2 = 6r + 5$ $Q_{r+1} - Q_r$
$= 6 =$ constt.

$\therefore Q_1, Q_2$ .... are in A.P. with to common difference 6.

Comprehension Type

Let A_1, G_1, H_1 denote the arithmetic, geometric and harmonic means, respectively, of two distinct positive numbers. For $n \geq 2$, let A_{n-1} and H_{n-1} have arithmetic, geometric and harmonic means A_n, H_n, H_m respectively.

21. Which one of the following statements is correct

(a) $G_1 > G_2 > G_3 >$ ....

(b) $G_1 < G_2 < G_3 <$ ....

(c) $G_1 = G_2 = G_3 =$...........

(d) $G_1 < G_3 < G_5 <$ ..... and $G_2 > G_4 > G_6 >$ ....

[IIT −2007]

Solution

(c) Let the two numbers be a and b

$$A_1 = \frac{a+b}{2}, G_1 = \sqrt{ab},$$

$$H_1 = \frac{2ab}{a+b}$$

$$A_n = \frac{A_{n-1} + H_{n-1}}{2}, G_1 = \sqrt{A_{n-1} \cdot H_{n-1}},$$

$$H_n = \frac{2A_{n-1} \cdot + H_{n-1}}{A_{n-1} + H_{n-1}}$$

$G_2 = \sqrt{A_1 \cdot H_1} = \sqrt{ab}$. Similarly, $G_3 = \sqrt{ab}$

$\therefore G_1 = G_2 =$.........$G_n = \sqrt{ab}$

22. Which one of the following statements is correct

[IIT −2007]

(a) $A_1 > A_2 > A_3 >$ ....

(b) $A_1 < A_2 < A_3 <$ ....

(c) $A_1 > A_3 > A_5 >$ .... and $A_2 < A_4 < A_6 <$ ....

(d) $A_1 < A_3 < A_5 <$ .... and $A_2 > A_4 >$ $A_6 >$ ....

Solution

(a) A_2 is the arithmetic mean of A_1 and H_1 and $A_1\, H_1\, A_1 > A_2 > H_2$

Similarly, A_3 is arithmetic mean of A_2 and $H_2 A_2 > A_3 > H_2$ Proceeding same way, we get $A_1 > A_2 > A_3 >$ ....

23. Which one of the following statements is correct

[IIT −2007]

(a) $H_1 > H_2 > H_3 >$ ....

(b) $H_1 < H_2 < H_3 <$ ....

(c) $H_1 > H_3 > H_5 >$ .... and $H_2 < H_4 > H_6 <$ ....

(d) $H_1 < H_3 < H_5 <$ .... and $H_2 > H_4 > H_6 >$ ....

Solution

(b) H_2 is the H.M. of A_1 and H_1 and $A_1 > H_1$

$$\left(\frac{1}{A_1} < \frac{1}{H_1}\right)$$

$\therefore A_1 > H_2 > H_1$. Similarly, $A_2 > H_3 > H$

$\therefore H_1 < H_2 < H_3$ ....

24. The sum of the series $1 + \frac{1.3}{6} + \frac{1.3.5}{6.8} + \ldots\ldots \infty$ is

(a) 1 (b) 0

(c) ∞ (d) 4

Solution

(d) Let, $S = 1 + \frac{1.3}{6} + \frac{1.3.5}{6.8} + \ldots\ldots \infty$ is

[UPSEAT – 2001]

$$\Rightarrow \frac{S}{4} = \frac{1}{4} + \frac{1.3}{4.6} + \frac{1.3.5}{4.6.8} + \ldots\ldots \infty$$

$$\Rightarrow \frac{1}{2} - \frac{S}{4}$$

$$= \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{4} - \frac{1}{2} \cdot \frac{1.3}{4.6} - \frac{1}{2} \frac{1.3.5}{4.6.8} + \ldots\ldots \infty$$

$$\Rightarrow \frac{1}{2}-\frac{S}{8}=1-\frac{1}{2}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1.2}-\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1.2.3}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{1.2.3.4}\ldots\ldots\infty$$

$\Rightarrow 1/2 - S/8 = (1-1)1/2 = 0$

$\Rightarrow S/8 = 1/2$

$\Rightarrow S = 4.$

25. If the sum of first n terms of an A. P. is cn^2, then the sum of squares of these n terms is **[IIT – 2009]**

(a) $\frac{n(4n^2-1)c^2}{6}$

(b) $\frac{n(4n^2+1)c^2}{3}$

(c) $\frac{n(4n^2-1)c^2}{3}$

(b) $\frac{n(4n^2+1)c^2}{6}$

Solution

(c) $t_n = c\{n^2-(n-1)^2\} = c(2n-1)$

$\Rightarrow t_n^2 = c^2(4n^2-4n+1)$

$$\Rightarrow \sum_{n=1}^{n} t_n^2 = c^2\left\{\frac{4n(n+1)(2n+1)}{6}-\frac{4n(n+1)}{2}+n\right\}$$

$= \frac{c^2n}{6}\{4(n+1)(2n+1)-12(n+1)+6\}$

$= \frac{c^2n}{3}\{4n^2+6n+2-6n-6+3]$

$= \frac{c^2}{3}n(4n^2-1)$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. The sum of $1+\frac{2}{5}+\frac{3}{5^2}+\frac{4}{5^3}+\ldots\ldots$ upto n terms is

(a) $\frac{25}{16}-\frac{4n+5}{16\times 5^{n-1}}$ (b) $\frac{3}{4}-\frac{2n+5}{16\times 5^{n+1}}$

(c) $\frac{3}{7}-\frac{3n+5}{16\times 5^{n+1}}$ (d) $\frac{1}{2}-\frac{5n+1}{3\times 5n+2}$

[MPPET –1982]

2. If $|x|<1$, then the sum of the series $1+2x+3x^2+4x^3+\ldots\ldots\infty$ will be

(a) $\frac{1}{1-x}$ (b) $\frac{1}{1+x}$

(c) $\frac{1}{(1+x)^2}$ (d) $\frac{1}{(1-x)^2}$

3. $2^{1/4}.4^{1/8}.8^{1/16}.16^{1/32}$............ is equal to

(a) 1 (b) 2

(c) 3/2 (d) 5/2

[MNR–1984; MPPET–1998; AIEEE –2002]

4. The sum of the series 1 + (1 + 2) + (1 + 2 + 3) +. . upto n terms, will be

[MPPET –1986]

(a) n^2-2n+6

(b) $\frac{n(n+1)(2n-1)}{6}$

(c) n^2+2n+6

(d) $\frac{n(n+1)(n+1)}{6}$

5. 2 + 4 + 7 + 11 + 16 +.. to n terms =

[Roorkee –1977]

(a) $\frac{1}{6}(n^2+3n+8)$

(b) $\frac{n}{6}(n^2+3n+8)$

(c) $\frac{1}{6}(n^2-3n+8)$

(d) $\frac{n}{6}(n^2-3n+8)$

6. The sum to n terms of the series $2^2+4^2+6^2+\ldots\ldots$is

(a) $\frac{n(n+1)(2n+1)}{3}$

(b) $\frac{2n(n+1)(2n+1)}{3}$

(c) $\frac{n(n+1)(2n+1)}{6}$

(d) $\frac{n(n+1)(2n+1)}{9}$

[MPPET – 1994]

7. $11^2 + 12^2 + 13^2 + \ldots\ldots\ldots\ldots 20^2 =$

[MPPET – 1995]

(a) 2481 (b) 2483
(c) 2485 (d) 2487

8. The sum of n terms of the following series 1.2 + 2.3 + 3.4 + 4.5 + ... shall be

[MNR – 1980]

(a) n^3
(b) $1/3\, n(n+1)(n+2)$
(c) $1/6\, n(n+1)(n+2)$
(d) $1/2\, n(n+1)(2n+1)$

9. The sum to n terms of the infinite series $1.3^2 + 2.5^2 + 3.7^2 + \ldots\ \infty$ is

[AMU – 1982]

(a) $\frac{n}{6}(n+1)(6n^2+14n+7)$
(b) $\frac{n}{6}(n+1)(2n+1)(3n+1)$
(c) $4n^3 + 4n^2 + n$
(d) none of these

10. The nth term of series $\frac{1}{1} + \frac{1+2}{2} + \frac{1+2+3}{3} + \ldots\ldots$ will be

(a) $\frac{n+1}{2}$ (b) $\frac{n-1}{2}$
(c) $\frac{n^2+1}{2}$ (d) $\frac{n^2-1}{2}$

[AMU – 1982]

11. The sum to n terms of $(2n-1) + 2(2n-3) + 3(2n-5) + \ldots$ is

(a) $(n+1)(n+2)(n+3)/6$
(b) $n(n+1)(n+2)/6$
(c) $n(n+1)(2n+3)$
(d) $n(n+1)(2n+1)/6$

[AMU – 2001]

12. The sum to infinity of the following series $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots$ shall be

[MNR – 1982]

(a) ∞ (b) 1
(c) 0 (d) none of these

13. The sum of the series $\frac{1}{3\times 7} + \frac{1}{7\times 11} + \frac{1}{11\times 15} + \ldots$ is

(a) 1/3 (b) 1/6
(c) 1/9 (d) 1/12

[MNR – 84; UPSEAT – 2000]

14. $\frac{1^3+2^3+3^3+4^3+\ldots 12^3}{1^2+2^2+3^2+4^2+\ldots 12^2} =$

[MPPET – 1998]

(a) $\frac{234}{25}$ (b) $\frac{243}{35}$
(c) $\frac{263}{27}$ (d) none of these

15. If 1, $\log_9(3^{1-x}+2)$, $\log_3(4.3^x - 1)$ are in A.P., then x equals

(a) $\log_3 4$ (b) $1 - \log_3 4$
(c) $1 - \log_4 3$ (d) $\log_4 3$

[AIEEE – 2002]

16. If p, q, r are in A.P. and are positive, the roots of the quadratic equation $px^2 + qx + r = 0$ are all real for

[IIT – 1995]

(a) $\left|\frac{r}{p} - 7\right| \geq 4\sqrt{3}$
(b) $\left|\frac{p}{r} - 7\right| \geq 4\sqrt{3}$
(c) all p and r
(d) no p and r

17. Let $f(n)$ be the sum of n terms of an A.P., then $f(n+3) + 3f(n+1) = f(n) + ?$

(a) $f(n+2)$ (b) $2f(n+2)$
(c) $3f(n+2)$ (d) none of these

18. The sum of the series $1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^2} \ldots$ upto ∞, is

(a) $\frac{35}{16}$ (b) $\frac{37}{16}$
(c) $\frac{39}{16}$ (d) 3

[DCE–1996, 2000; IIT Sc.–1992; MPPET – 2009]

SOLUTIONS

1. (a) Given series, let $S_n = 1 + \frac{2}{5} + \frac{3}{5^2} + \frac{4}{5^2} + \ldots.. + \frac{n}{5^{n-1}}$

$\frac{1}{5}S_n = \frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \ldots\ldots + \frac{n}{5^n}$

Subtracting, after shifting the terms of $\frac{1}{5}S_n$ by one column on the right hand side.

$\left(1 + \frac{1}{5}\right) S_n = \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \ldots..$ + upto n terms $- \frac{n}{5^n}$

$\Rightarrow 4/5\ Sn = \dfrac{1 - \frac{1}{5^n}}{\frac{4}{5}} - \frac{n}{5^n}$

$\Rightarrow S_n = \frac{25}{16} - \frac{4n+5}{16 \times 5^{n-1}}$

2. (d) Let $S = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots.$

$xS = x + 2x^2 + 3x^3 \ldots\ldots\ldots$

$\therefore S(1-x) = 1 + x + x^2 + x^3 + \ldots\ldots\ldots.$

$= \frac{1}{1-x} \qquad \therefore S = \frac{1}{(1-x)^2}$

3. (b) The given product

$2^{\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{4}{32}+\ldots\ldots} = 2^s$ (say)

Now $S = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \ldots..$ (1)

$\Rightarrow \frac{1}{2}S = \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \ldots..$ (2)

Subtracting (2) from (1)

$\Rightarrow \frac{1}{2}S = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots..$

$= \dfrac{1/4}{1 - \frac{1}{2}} = \frac{1}{2} \qquad \therefore S = 1$

$\Rightarrow$ Product $= 2^1 = 2$

4. (d) $T_n = \sum n = \frac{n}{2}(n+1) = \frac{1}{2}(n^2+n)$

$S_n = \sum T_n \Rightarrow 2S_n = \sum n^2 + \sum n$

$= \frac{n}{6}(n+12)(n+1) + \frac{n}{2}(n+1)$

5. (b) $2 + 4 + 7 + 11 + 16 + \ldots.$ to n terms.

Quicker Method: $S_3 = 2 + 4 + 7 = 13$. Put $n = 3$ in (a), (b), (c), (d), we get,

(a) $\rightarrow \frac{1}{6}(26)$ (b) $\rightarrow \frac{1}{2}() =$

(c) $\rightarrow \frac{1}{6}(8)$ (d) $\rightarrow \frac{1}{2}()8 = 4$

But 13 is required: (verification method)

6. (b) $2^2 + 4^2 + 6^2 \ldots..$ upto n term

$= 2^2 + 4^2 + 6^2 \ldots\ldots\ldots (2n)^2$

$\sum (2n)^2 = 4\sum n^2$

required sum $= 4\,\frac{n(n+1)(2n+1)}{6}$

$\left(\therefore \sum n^2 = \frac{n(n+12)(2n+1)}{6}\right)$

Required sum $= n(n+1)(2n+1)$

7. (c) $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$

Let sum $S = 11^2 + 12^2 + 13^2 \ldots\ldots 20^2$

$S = (1^2 + 2^2 + 3^2 + \ldots.. + 20^2) - (1^2 + 2^2 + \ldots\ldots + 10^2)$

or $S = \sum_{n=1}^{n=20} n^2 \sum_{n=1}^{n=20} n^2$

By Equation (1)

$S = \frac{20(21)(41)}{6} - \frac{10(11)(21)}{6} = 2485$

8. (b) $S_n = 1.2 + 2.3 + 3.4 + 4.5 + \ldots\ldots$ to n terms Series formed by first factors = 1, 2, 3, 4, ………, $t_n = n$ Series formed by second factors = 2, 3, 4,….., $t_n = n + 1$

$S_n = \sum t_n = \sum n(n+1) = \sum n^2$

$= \frac{n}{6}(n+1)(2n+1) + \frac{n}{2}(n+1)$

$= \frac{n}{3}(n+1)(n+2)$

9. (a) This is an series whose nth term is equal to $T_n = n(2n+1)^2 = 4n^3 + 4n^2 + n$

$$\therefore Sn = \sum_1^n T_n = \sum_1^n (4n^3 + 4n^2 + n)$$

$$= 4\sum_1^n n^3 + 4\sum_1^n n^2 + 4\sum_1^n n$$

$$= 4\left\{\frac{n}{2}(n+1)\right\}^2 + \frac{4}{6}n(n+1)$$

$$\times(2n+1) + \frac{n}{2}(n+1)$$

$$= \frac{n}{6}(n+1)(6n^2 + 14n + 7)$$

10. (a) $T_n = \dfrac{1+2+3+\ldots n}{n}$

$$= n\frac{(n+1)}{2n} = \frac{n+1}{2}$$

11. (d) $S = (2n-1) + 2(2n-3) + 3(2n-5) + \ldots..$

$S = [2n + 2.2n + 3.2n + \ldots\ldots\ldots + n.2n]$

$[1 + 2.3 + 3.5 + \ldots\ldots\ldots. + n.(2n-1)]$

Let $S_1 = 2n(1 + 2 + 3 + \ldots. + n)$

$$= \frac{2n.n(n+1)}{2} = n^2(n+1)$$

and $S_2 = 1 + 2.3 + 3.5 + \ldots\ldots\ldots. + n.(2n-1)$

$T_n = n(2n-1) = 2n^2 - n$

$\therefore\ S_2 = \Sigma(2n^2 - n) = 2\Sigma(n^2) - \Sigma(n)$

$$= \frac{2n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$\therefore\ S = S_1 - S_2$

$$= n^2(n+1)\left[n - \frac{2n+1}{3} + \frac{1}{2}\right]$$

$$= n(n+1)\left[\frac{6n - 4n - 2 + 3}{6}\right]$$

$$= \frac{n(n+1)(2n+}{6}$$

12. (a) $Sn = \dfrac{1}{1.2} + \dfrac{1}{2.3} + \dfrac{1}{3.4} + \ldots\ldots + \dfrac{1}{n(n+1)}$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) +$$

$$\ldots. + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

13. (d) $S_\infty = \dfrac{1}{3.7} + \dfrac{1}{7.11} + \dfrac{1}{11.15} + \ldots.$
then =

$$\frac{1}{4}\left[\left(\frac{1}{3} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{11}\right) + \left(\frac{1}{11} - \frac{1}{15}\right) + \ldots.\right]$$

$$= \frac{1}{4}\frac{1}{3} = \frac{1}{12}$$

14. (a) $\left\{\sum_{n=1}^{12} n^3 + \sum_{n=1}^{12} n2\right\}$

$$= \left\{\frac{n}{2}(n+1)\right\}^2_{n=12} + \left\{\frac{n}{6}(n+1)(2n+1)\right\}_{n=12}$$

$$= (6 \times 13)^2 = 2 \times 13 \times 25 = \frac{18 \times 13}{25} = \frac{234}{25}$$

15. (b) 1, $\log_9(3^{1-x} + 2)$, $\log_3(4.3^x - 1)$ are in A.P.

$\Rightarrow 2\log_9(3^{1-x} + 2) = 1 + \log_3(4.3^x - 1)$

$\Rightarrow \log_3(3^{1-x} + 2) = \log_3 3 + \log_3(4.3^x - 1)$

$\Rightarrow \log_3(3^{1-x} + 2) = \log_3[3(4.3^x - 1)]$

$\Rightarrow 3^{1-x} + 2 = 3(4.3^x - 1)$

$\Rightarrow 3.\ 3^{-x} + 2 = 12.3^x - 3$

Put $3^x = t$

$\Rightarrow 3 + 2 = 12t - 3$ or $12t^2 - 5t - 3 = 0$

Hence, $t = -\dfrac{1}{3}, \dfrac{3}{4}$

$\Rightarrow 3^x = \dfrac{3}{4}$ (as $3^x \neq$ negative)

$\Rightarrow x = \log_3\dfrac{3}{4}$ or $x = \log_3 3 - \log_3 4$

$\Rightarrow x = 1 - \log_3 4$

16. (b) Since p, q, r are in A.P., then $q = \dfrac{p+r}{2}$
The roots of the equation $px^2 + qx + r = 0$ are real

$\Leftrightarrow q^2 - 4pr \geq 0$

$\Leftrightarrow \dfrac{p+r}{2} - 4pr \geq 0$

$\Leftrightarrow p^2 + r^2 - 14pr \geq 0$

$\Leftrightarrow \dfrac{p^2}{r^2} - 14\dfrac{p}{r} + 1 \geq 0$

$\Leftrightarrow \left(\dfrac{p}{7} - 7\right)^2 - 48 \geq 0$

$\Leftrightarrow \left|\dfrac{p}{r} - 7\right| 4\sqrt{3}$

OR

$\Rightarrow \frac{r}{p2} - 14\frac{r}{p} + 1 \geq 0$

$\Rightarrow \left|\frac{r}{p} - 7\right|^2 48 \left|\frac{r}{p} - 7\right| \geq 4\sqrt{3}$

17. ()

18. (c) $S = 1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} +$ (1)

$\frac{1}{5}S = \frac{1}{5} + \frac{4}{5^2} + \frac{7}{5^3} +$ (2)

On Subtracting second from (1) by method of difference we find

$\frac{4}{5}S = 1 + \frac{3}{5} + \frac{3}{5^2} + \frac{3}{5^3} + ...$

$= 1 + \frac{3}{5}\left(1 + \frac{1}{5} + \frac{1}{5^2} + ...\right)$

$= 1 + \frac{3}{5} \cdot \frac{1}{1 - \left(\frac{1}{5}\right)}$

$1 + \frac{3}{5} \cdot \frac{5}{4} = \frac{7}{4}$ or $S = \frac{7}{4} \cdot \frac{5}{4} = \frac{35}{16}$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. The sum infinite terms of the following series $1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + ...$ will be

(a) 3/16 (b) 35/8
(c) 35/4 (d) 35/16

[MPPET - 1981; RPET -1997; Roorkee – 1992;DCE – 1996,2000]

2. The sum of the series $1 + 3x + 6x^2 + 10x^3 + ... \infty$ will be

(a) $\frac{1}{(1-x)^2}$ (b) $\frac{1}{1-x}$
(c) $\frac{1}{1+x)^2}$ (d) $\frac{1}{1+x)^3}$

3. Sum of n terms of series 12 + 16 + 24 + 40 + ... will be

[UPSEAT – 1999]

(a) $2(2^n - 1) + 8n$ (b) $2(2^n - 1) + 6n$
(c) $3(2^n - 1) + 8n$ (d) $4(2^n - 1) + 8n$

4. $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ... + ... \frac{1}{n(n+1)}$ equals

(a) $\frac{1}{n(n+1)}$ (b) $\frac{n}{(n+1)}$
(d) $\frac{2n}{(n+1)}$ (d) $\frac{2}{n(n+1)}$

[AMU – 1995; RPET – 1996; UPSEAT – 1999,2001]

5. The sum of $1^3 + 2^3 + 3^3 + 4^3 + ... + 15^3$, is

[MPPER – 2003]

(a) 22000 (b) 10,000
(c) 14,400 (d) 15,000

6. $\sum_{m=1}^{n} m^2$ is equal to

(a) $\frac{m(m+1)}{2}$

(b) $\frac{m(m+1)(2m+1)}{6}$

(c) $\frac{n(n+1)(2n+1)}{6}$

(d) $\frac{n(n+1)}{2}$ **[RPET – 1995]**

7. If the nth term of a series be $3 + n(n - 12)$, then the sum of n terms of the series is

(a) $\frac{n^2+n}{3}$ (b) $\frac{n^3+8n}{3}$
(c) $\frac{n^2+8n}{5}$ (d) $\frac{n^2-8n}{3}$

8. $1 + \frac{3}{2} + \frac{5}{2^2} + \frac{7}{2^3} + ... \infty$ is equal to

(a) 3 (b) 6
(c) 9 (d) 12

[DCE – 1999]

9. 1 + 3 + 7 +15 + 31 + ...to n terms =

[IIT – 1963]

(a) $2^{n+1} - n$ (b) $2^{n+1} - n - 2$
(c) $2^n - n - 2$ (d) none of these

10. For all positive integral values of n, the value of $3.1.2 + 3.2.3 + 3.3.4 + ... + 3.n.(n + 1)$ is **[RPET – 1999]**

(a) $n(n+1)(n+2)$
(b) $n(n+1)(2n+1)$
(c) $(n-1)n(n+1)$
(d) $(n-1)n(n+1)/2$

WORK SHEET: TO CHECK PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the work sheet
2. The test is of 12 minutes.
3. The test consists of 12 questions. The maximum marks are 36.
4. Use blue/black ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. $1.3 + 2.4 + 3.5 + ...$to n terms

(a) $\frac{n(n+1)(2n+3)}{6}$
(b) $\frac{n^2(n+1)^2}{4}$
(c) $\frac{n(n+1)(2n+1)}{6}$
(d) none of these

2. The sum of $(n - 1)$ terms of $1 + (1 + 3) + (1 + 3 + 5) + ...$ is

(a) $\frac{n(n+1)(2n+3)}{6}$
(b) $\frac{n^2(n+1)^2}{4}$
(c) $\frac{n(n-1)(2n-1)}{6}$
(d) n^2 **[RPET – 1999]**

3. The sum of the series $1.2.3 + 2.3.4 + 3.4.5 + ...$ to n terms is

(a) $n(n+1)(n+2)$
(b) $(n+1)(n+2)(n+3)$
(c) $\frac{1}{4}n(n+1)(n+2)(n+3)$
(d) $\frac{1}{4}(n+1)(n+2)(n+3)$

[Kurukshetra CEE – 1998]

4. $11^3 + 12^3 + ... + 20^3$

[Pb. CET – 1997; RPET – 2002]

(a) is divisible by 5
(b) is an odd integer divisible by 5
(c) is an even integer which is not divisible by 5
(d) is an odd integer which is not divisible by 5

5. The sum to n terms of the series $3 + 15 + 35 + 63 + ...$ is

(a) $\frac{n}{3}(4n^2 + 6n - 1)$ (b) $\frac{n}{3}(2n^2 + 6n - 1)$
(c) $\frac{n}{3}(4n^2 + 4n - 1)$ (d) none of these

6. Sum of the n terms of the series $\frac{3}{1^2} + \frac{5}{1^2 + 2^2} + \frac{7}{1^2 + 2^2 + 3^2} + ...$ is

(a) $\frac{2n}{n+1}$ (b) $\frac{4n}{n+1}$
(c) $\frac{6n}{n+1}$ (d) $\frac{9n}{n+1}$

[Pb. CET – 1999; RPET – 2001]

7. Sum of the series $\frac{2}{3} + \frac{8}{9} + \frac{26}{27} + \frac{80}{81} + ...$ to n terms is

(a) $n - \frac{1}{2}(3^n - 1)$ (b) $n + \frac{1}{2}(3^n - 1)$
(c) $n + \frac{1}{2}(1 - 3^n)$ (d) $n + \frac{1}{2}(3^{-n} - 1)$

[Karnataka CET – 2001]

8. The sum of all numbers between 100 and 10,000 which are of the form $n^3 (n \in N)$ is equal to

[IIT – 1989]

(a) 55216 (b) 53261
(c) 51261 (d) none of these

9. $\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + ...$ upto n terms–

(a) $\frac{n}{6n+4}$ (b) $\frac{n}{3n+7}$

(c) $\frac{n}{4n+6}$ (d) $\frac{1}{6n+4}$

[Karnataka CET – 2007, 2008]

10. Sum of the squares of first n natural numbers exceeds their sum by 330, then n =

[Karnataka CET – 1998]

(a) 8 (b) 10

(c) 15 (d) 20

11. The sum of the first n terms of the series $\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+ \ldots$ is

(a) $2^n - n - 1$

(b) $1 - 2^{-n}$

(c) $n + 2^{-n} - 1$

(d) $2^n - 1$

[IIT – 1988; MPPET – 1996; RPET – 1996, 2000; Pb.CET–1994; DCE–1995, 1996, 2006]

12. If $3 + \frac{1}{4}(3+d) + \frac{1}{4^2}(3+2d) + \ldots \infty = 8$, then the value of d is

(a) 6 (b) 7

(c) 8 (d) 9

ANSWER SHEET

1.	ⓐ	ⓑ	ⓒ	ⓓ	5.	ⓐ	ⓑ	ⓒ	ⓓ	9.	ⓐ	ⓑ	ⓒ	ⓓ
2.	ⓐ	ⓑ	ⓒ	ⓓ	6.	ⓐ	ⓑ	ⓒ	ⓓ	10.	ⓐ	ⓑ	ⓒ	ⓓ
3.	ⓐ	ⓑ	ⓒ	ⓓ	7.	ⓐ	ⓑ	ⓒ	ⓓ	11.	ⓐ	ⓑ	ⓒ	ⓓ
4.	ⓐ	ⓑ	ⓒ	ⓓ	8.	ⓐ	ⓑ	ⓒ	ⓓ	12.	ⓐ	ⓑ	ⓒ	ⓓ

HINTS AND EXPLANATIONS

4. (b) $11^3 + 12^3 + 13^3 + \ldots + 20^3$

$= (1^3 + 2^3 + \ldots + 20^3) - (1^3 + 2^3 + \ldots + 10^3)$

$$= \left[\frac{20(20+1)}{2}\right]^2 - \left(\frac{10(10+1)}{2}\right)^2$$

$$\left[\because \sum n^3 = \left(\sum n\right)^2 = \left(\frac{n(n+1)}{2}\right)^2\right]$$

$= (10 \times 21)^2 - (5 \times 11)^2 = 44100 - 3025 = 41075$

= an odd integer divisible by 5.

6. (c) Let T_n denote the nth term, then

$$T_n = \frac{nth\ term\ of\ A.P.\ 3, 5, 7\ldots}{1^2 + 2^2 + 3^2 + \ldots + n^2}$$

$$= \frac{3 + (n-2) \times 2}{\frac{n(n+1)(2n+1)}{6}} = \frac{6}{n(n+1)}$$

$$= 6\left\{\frac{n+1-n}{(n+1)\,n}\right\} = 6\left\{\frac{1}{n} - \frac{1}{n+1}\right\}$$

$\therefore\ S_n = T_1 + T_2 + T_3 + \ldots + T_n$

$$= 6\left(\frac{1}{1}-\frac{1}{2}\right) + 6\left(\frac{1}{2}-\frac{1}{3}\right) + 6\left(\frac{1}{3}-\frac{1}{4}\right) + \ldots + 6\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$= 6\left\{\left(\frac{1}{1}-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \ldots + \left(\frac{1}{n}-\frac{1}{n+1}\right)\right\}$$

$$= 6\left(1 - \frac{1}{n+1}\right) \qquad (*)$$

$$= \frac{6n}{n+1}.$$

7. (d) $S = \frac{2}{3} + \frac{8}{9} + \frac{26}{27} + \frac{80}{81} + \ldots n\ terms$

$$= 1 - \frac{1}{3} + 1 - \frac{1}{3^2} + 1 - \frac{1}{3^3} + 1 - \frac{1}{3^4} + \ldots + n\ terms$$

$= (1 + 1 + 1 + 1 + \ldots\ldots n \text{ terms})$

$-\left[\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots + n\ terms\right]$

$$= n - \frac{\frac{1}{3}\,1 - \left(\frac{1}{3}\right)^n}{1 - \frac{1}{3}}$$

$$= n - \frac{1}{3} \times \frac{3}{2}\left(1 - \frac{1}{3^n}\right)$$

$$= n - \frac{1}{2}(1 - 3^{-n}) = n + \frac{1}{2}(3^{-n} - 1)$$

8. (b) The smallest and the largest numbers between 100 and 10, 000 which can be written in the form x^3 are

$5^3 = 125$ and $21^3 = 9261$

$\therefore$ the required sum $= 5^3 + 6^3 + 7^3 + \ldots\ldots + 21^3$

$$\sum_{n=1}^{21} n^3 + \sum_{n=1}^{4} n^3 = \left(\frac{n^2(n+1)}{4}\right)_{n=21} - \left(\frac{n2\ (n+1)}{4}\right)_{n=4}$$

$$= \frac{441 \times 484}{4} - \frac{16 \times 25}{4}$$

$= 441 \times 121 - 100 = 53361 - 100 = 53261$

9. (a) Let T_n denote the nth term, then $= T_n$

$$\frac{1}{(n\text{th term of A.P.. } 2, 5, 8, \ldots)(n\text{th term of A.P. } 5, 8, 11, \ldots)}$$

$$\text{or } T_n = \frac{1}{\{2 + (n-1) \times 3\}\{5 + (n-1) \times 3\}}$$

$$= \frac{1}{(3n-1)(3n+2)}$$

$$\therefore S_n = \frac{1}{2 \times 5} + \frac{1}{5 \times 8} + \frac{1}{8 \times 11} + \ldots + \frac{1}{(3n-1)(3n+2)}$$

$$= \frac{1}{3}\left\{\frac{5-2}{2 \times 5} + \frac{8-5}{5 \times 8} + \frac{11-8}{8 \times 11} + \ldots\ldots + \frac{(3n+2)-(3n-1)}{(3n-1)(3n+2)}\right.$$

$$= \frac{1}{3}\left\{\frac{1}{2} - \frac{1}{5} + \frac{1}{5} - \frac{1}{8} + \ldots + \frac{1}{3n-1} - \frac{1}{3n+2}\right\}$$

$$= \frac{1}{3}\left\{\frac{1}{2} - \frac{1}{3n+2}\right\}$$

$$= \frac{n}{2(3n+2)} = \frac{n}{6n+4}$$

10. (b) $\sum_{n=1}^{n} n^2 \sum_{n=1}^{n} n = 330$

$\Rightarrow \frac{n}{6}(n+^{\cdot\cdot})^{\cdot}(n+) - \frac{n}{2}(n+)$

$\Rightarrow n(n + 1)\,[(2n + 1) - 3] = 330 \times 6$

$\Rightarrow n^2 - n - 990 = 0$ (1)

If we put $n = 8$ in (1)

$512 - 8 - 990 = 0.$

If we pur $n = 10$ in (1),

$1000 - 10 - 990 = 0$, which is true.

$\therefore$ (b) is true.

11. (c) Required sum

$$S = \frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \frac{15}{16} \ldots\ldots n \text{ terms}$$

$$S = \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{4}\right) + \left(1 - \frac{1}{8}\right) + \left(1 - \frac{1}{16}\right)\ldots.$$

$$S = n - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ldots \text{terms}\right)$$

$$\left(\therefore \sum_{1}^{n} 1 = n\right)$$

$$S = n - \frac{1}{2}\frac{\left[1 - \left(\frac{1}{2}\right)^n\right]}{1 - \frac{1}{2}}$$

$$\left(\text{By } \frac{(1 - r^n)}{1 - r} = S \textit{ for G.P}\right)$$

$$S = n - \left(1 - \frac{1}{2^n}\right) = n - + 2^{-n}$$

12. (d) Let $S = 3 + \frac{1}{4}(3 + d) + \frac{1}{4^2}(3 + 2d) + \ldots\ldots \infty$

$$\therefore \frac{1}{4}\ S = \frac{3}{4} + \frac{3+d}{4^2} + \ldots\ldots \infty$$

$$S\left(1 - \frac{1}{4}\right) = 3 + \frac{1}{4}d + \frac{1}{4^2}d + \ldots \infty$$

$$= 3 + \frac{\frac{1}{4}d}{1 - \frac{1}{4}} = 3 + \frac{\frac{1}{4}d}{\frac{3}{4}}$$

$$\frac{3}{4}S = 3 = \frac{d}{3} \therefore S = 4 + \frac{4}{3}\cdot\frac{d}{3} = 4 + \frac{4d}{9}$$

But $S = 8 \quad \therefore 4 + \frac{4d}{9} = 8$

$$\frac{4d}{9} = 4 \Rightarrow d = 9$$

LECTURE

5

Test Your Skills

ASSERTION/REASONING

Assertion−Reasoning type questions

Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.

(a) **Assertion** is True, **Reason** is True and **Reason** is a correct explanation for **Assertion**

(b) **Assertion** is True, **Reason** is True and **Reason** is NOT a correct explanation for **Assertion**

(c) **Assertion** is True and **Reason** is False

(d) **Assertion** is False and **Reason** is True

1. **Assertion (A):** The sums of n terms of two arithmetic progressions are in the ratio $(7n + 1) : (4n + 17)$, then the ratio of their nth terms is 7 : 4.

   Reason (R): If $S_n = ax^2 + bx + c$, then $T_n = S_n - S_n - 1$

2. **Assertion (A):** $a + b + c = 12 (a, b, c > 0)$, then maximum value of abc is 64.

   Reason (R): Maximum value occurs when $a = b = c$.

3. **Assertion (A):** 3, 6, 12 are in GP, then 9, 12, 18 are in HP

   Reason (R): If middle term is added in three consecutive terms of a GP, resultant will be in HP.

4. **Assertion (A):** If sum of n terms of a series is $6n^2 + 3n + 1$, then the series is in AP.

   Reason (R): Sum of n terms of an AP is always of the form $an^2 + bn$

5. **Assertion (A):** If a, b, c are three positive numbers in GP, then

   $$\left(\frac{a+b+c}{3}\right)\cdot\left(\frac{3abc}{ab+bc+ca}\right) = (\sqrt[3]{abc})^2$$

   Reason (R): (AM) (HM) = $(\text{GM})^2$ is true for positive numbers.

6. **Assertion (A):** If positive numbers a^{-1}, b^{-1}, c^{-1} are in AP, then product of roots of equation $x^2 - \lambda x + 2b^{101} - a^{101} - c^{101} = 0$ $(\lambda \varepsilon R)$ has −ve sign.

   Reason (R): If a, b, c are in HP, then

   $\frac{a^n + b^n}{2} > b^n$

   $(\because \text{AM} > \text{HM}) \Rightarrow 2b^n - a^n - c^n < 0$

   $\therefore 2b^{101} - a^{101} - c^{101} < 0$

7. Suppose four distinct positive numbers a_1, a_2, a_3, a_4 are in G.P. Let $b_1 = a$, $b_2 = b_1 + a_2$, $b_3 = b_2 + a_3$ and $b_4 = b_3 + a_4$.

   Assertion (A): The numbers b_1, b_2, b_3, b_4 are neither in A.P. nor in G.P.

   Reason (R): The numbers b_1, b_2, b_3, b_4 are in H.P.

   [IIT −2008]

8. **Assertion (A):** Let
$f(n) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$
Then $\sum_{r=1}^{n} f(r) = (n + 1) f(n) - n$
Reason (R):
$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \geq \frac{2n}{n+1} \cdot \forall\, n \in N$

9. **Assertion (A):** If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ $(a, b, c, \varepsilon R^+)$, are in A.P. $(a \neq c)$ then product of roots of the equation $x^2 + \mu x + (2b^{51} - a^{51} - c^{51}) = 0$, μ is negative.
Reason (R): $\frac{a^n + b^n}{2} > b^n$

10. **Assertion (A):** If a, b, c, d are positive and distinct numbers in H.P., then $a + d > b + c$.
Reason (R): If a, b, c, d are in H.P., then $\frac{a+d}{ad} + \frac{b+c}{bc}$

11. **Assertion (A):** If $x, y > 0$ and $x^2y^3 = 6$, then least value of $3x + 4y$ is 10.
Reason (R): Least value occurs when $9x = 8y$

12. **Assertion (A):** For $n \in N$ $n > 1$, $2^n > 1 + n2^{(n-1)/2}$.
Reason (R): A.M. of distinct positive numbers is greater than their G.M.

13. **Assertion (A):** If a, b, c are in A.P., then $2b = a + c$.
Reason (R): If a, b, c are in A.P., then $10^a, 10^b, 10^c$ are in G.P.

14. **Assertion (A):** Three non-zero real numbers a, b, c are in G.P. iff $b^2 = ac$.
Reason (R): If the quadratic equation $(a^2 + b^2)x^2 - 2(ab + bc)x + (b^2 + c^2) = 0$ has real roots, then a, b, c are in G.P.; a, b, c being non-zero real numbers.

15. **Assertion (A):** Three numbers a, b, c are in A.P. iff $s - a, s - b, s - c$ are in A.P.; s being any number.
Reason (R): In any triangle ABC, if a, b, c are in A.P., then
$\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are also in A.P.

16. **Assertion (A):** If a, b, c are in H.P., then $b = \frac{2ac}{a+c}$.
Reason (R): If in a triangle ABC, $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in A.P., then $\cot\frac{C}{2} = 3\tan\frac{A}{2}$

17. **Assertion (A):** If $\log_3 2$, $\log_3(2^x - 5)$ and $\log_3\left(2^x - \frac{7}{2}\right)$ are in A.P., then $x = 2$.
Reason (R): If $\log_{10} 2$, $\log_{10}(2^x - 1)$ and $\log_{10}(2^x + 3)$ be three consecutive terms of an A.P., then $x = \log_2 5$.

18. **Assertion (A):** $1 + 3 + 7 + 13 + \dots\dots$up to n terms $= \frac{n(n^2 + 2)}{3}$
Reason (R): $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ is Harmonic mean of a and b if $n = -\frac{1}{2}$

19. **Assertion (A):** The first term of an infinite G.P. is 1 and any term is equal to the sum of all the succeeding terms. Such type of series is not possible.
Reason (R): Product of n geometric means between a and b is $(\sqrt{ab})^n$

20. **Assertion (A):** 1111..........1 (up to 91 terms) is a prime number.
Reason (R): If
$\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P., then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in A.P.

21. **Assertion (A):** If A and G be the A.M. and G.M. between two positive real numbers a and b then a, b are given by
$A \pm \sqrt{(A+G)(A-G)}$
Reason (R): Using $x^2 - (a + b)x + ab = 0$; where $a + b = 2A$, $ab = G^2$, we calculate x.

22. **Assertion (A):** The sum of all numbers of the form n^3 which lie between 100 and 10,000 is 53261.
Reason (R): If $\frac{a-b}{b-c} = \frac{a}{c}$ then a, b, c are in G.P.

23. Assertion (A): The number of terms of the A.P. 3, 7, 11, 15,... to be taken so that the sum is 465 is 15.

Reason (R): The sum of the integers from 1 to 100 which are not divisible by 3 or 5 is 2632.

24. Assertion (A): The sum of all two digit numbers which when divided by 4, yield unity as remainder is 1200.

Reason (R): The fourth term of a G.P. is 3. The product of its first seven terms is 3^7.

25. Assertion (A): If $a_1, a_2, a_3,.......$ is an A.P. such that $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$ and $a_1 + a_2 +..... + a_{23} + a_{24}$ is equal to 625.

Reason (R): The sum of terms equidistant from the beginning and end in an A.P. is equal to sum of first and the last term.

26. Assertion (A): If three positive real numbers in GP represents sides of a triangle then the common ratio of GP must be between 2 sin 18° and 2cos 36°.

Reason (R): Three positive real numbers can form a triangle if sum of any two sides is greater than the third.

27. Assertion (A): The A.M., G.M. and H.M. between two given real positive numbers are 5, 4 and H respectively.

Then H is $\frac{18}{5}$.

Reason (R): The relation A, G and H is $AH = G^2$

28. Assertion (A): Inserted three geometric means between 4 and $\frac{1}{4}$ so, middle G.M. is 1.

Reason (R): In a fi nite G.P., the number of terms be odd then its middle term is the G.M.of the fi rst and last term.

29. Assertion (A): If the terms of a given G.P. are chosen at regular intervals, then new sequence is also a G.P.

Reason (R): If $a_1, a_2, a_3,........ a_n$ are in G.P. then $\log a_1, \log a_2,........ \log a_n$ are in A.P.

30. Assertion (A): If $x > 1, y > 1, z > 1$ are in G.P. then $\frac{1}{1+\ln x}, \frac{1}{1+\ln y}, \frac{1}{1+\ln z}$ are in A.P.

Reason (R): Arithmetic mean, Geometric mean and Harmonic mean are in G.P.

31. Assertion (A): The number of terms common to two A.P.'s 3,7, 11,........87 and 2, 9, 16,.......86 is 4.

Reason (R): If d_1 and d_2 are the common difference of two given A.P.'s so the L.C.M. of d_1 and d_2 is the common difference of common terms.

ASSERTION/REASONING: SOLUTIONS

1. (d) $\because \frac{S_n}{S'_n} = \frac{(7n+1)}{(4n+17)} = \frac{n(7n+1)}{n(4n+17)}$

$\therefore S'_n = (7n^2 + n)\lambda, S'_n = (4n^2 + 17n)\lambda$

Then,

$$\frac{T_n}{T'_n} = \frac{S_n - S_n - 1}{S'_n - S_{n-1}} = \frac{7(2n-1)+1}{4(2n-1)+17}$$

$$= \frac{14n - 6}{8n + 13}$$

$\Rightarrow T_n : T'_n = (14n - 6) : (8n + 13)$

2. (a) $\because$ AM $\geq$ GM

$\Rightarrow \frac{a+b+c}{3} \geq (abc)^{1/3} \Rightarrow \frac{12}{3} \geq (abc)^{1/3}$

$\because abc \leq 64$

Maximum value of abc is 64 only when $a = b = c$.

3. (a) If a, b, c are in GP, then $a + b$, $b + b$, $c + b$ are in HP,

$\Rightarrow (2b) = \frac{2(a+b)(b+c)}{(a+b)+(c+b)}$

$\Rightarrow b(a + 2b + c) = (a + b)(b + c)$

$\Rightarrow b(a + c) + 2b^2 = ab + ac + b^2 + bc$

$\Rightarrow b^2 = ac$ ($\because$ a, b, c are in GP)

4. (d) $\because$ Sum of n terms of an AP is $S_n = \frac{n}{2}\{2A + (n-1)D\}$

where A and D are first term and common difference.

Hence, sum always of the form $an^2 + bn$

5. (c) For two positive numbers a and b, (AM)(HM) = $(GM)^2$.

This result will be true for n positive numbers if they are in GP.

6. (c) $\Rightarrow a^{-1}, b^{-1}, c^{-1}$ are in AP

$\because$ a, b, c are in HP $\quad \because$ AM > GM

$\Rightarrow \frac{a^n + c^n}{2} > \sqrt{a^n c^n}$

But GM > HM (i)

$\because \sqrt{ac} > b$ or $(\sqrt{ac})^n > b^n$

$\Rightarrow \sqrt{a^n c^n} > b^n$ (ii)

From equations (i) and (ii),

$\frac{a^n + c^n}{2} > \sqrt{a^n c^n} > b^n$

$\Rightarrow a^n + c^n > 2b^n$

$\Rightarrow a^{101} + c^{101} - 2b^{101} > 0$

$\Rightarrow 2b^{101} - a^{101} - c^{101} < 0$

7. (c) Take four distinct positive numbers as 1, 2, 4, 8.

Now $b_1 = 1, b_2 = 3, b_3 = 7, b_4 = 15$.

It can be easily seen that the numbers are neither in A.P. nor in G.P. The numbers are not in H.P. even.

Thus Assertion is true and Reason is false.

Remark: We have used a 'counter example' to support our validity − Instead of taking numbers a_1, a_2, a_3, a_4 and a, ar, ar^2, ar^3, above approach of working with concrete numbers saves time.

8. (b) $\sum_{r=1}^{n} f(r) = \sum_{r=1}^{n} \left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{r}\right)$

$= \text{I}.n + \frac{1}{2}(n-1) + \frac{1}{3}(n-2) + ... + \frac{1}{n}.\text{I}$

$= n\left\{1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}\right\}$

$= nf(n) - (n - f(n))$

$= (n+1)f(n) - n$

$\therefore$ A is true. Also R is true obviously but R is not correct reason for A.

9. (a) a, b, c are H.P., $a \neq c$, $a, b, c \in R^+$

$\Rightarrow \frac{a^n + c^n}{2} > (\sqrt{ac})^n > b^n$

[A.M. > G.M.> H.M.]

$\Rightarrow 2b^n - a^n - c^n < 0$

$\forall n \in \text{N}$

$\therefore$ Product of roots < 0.

$\therefore$ A is true, R is true and R is correct reason for A.

10. (b) $\frac{a+c}{3} > b, \frac{b+d}{2} > c \Rightarrow a + d > b + c$

[$\because$A.M. > H.M.]

$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$ in A.P. $\Rightarrow \frac{1}{b} - \frac{1}{a} = \frac{1}{d} - \frac{1}{c}$

$\Rightarrow \frac{1}{a} + \frac{1}{d} = \frac{1}{b} + \frac{1}{c}$

$\Rightarrow \frac{a+d}{ad} = \frac{b+c}{bc}$

$\therefore$ A and R both are true but R is not correct reason for A.

11. (a)

$$\frac{\frac{3x}{2} + \frac{3x}{2} + \frac{4y}{3} + \frac{4y}{3} + \frac{4y}{3}}{5} \geq \left[\frac{3x^2}{2}\left(\frac{4y}{3}\right)^3\right]^{1/5}$$

$\Rightarrow 3x + 4y \geq 5\left(\frac{16}{3}x^2 y^3\right)^{1/5}$

$\Rightarrow 3x + 4y \geq 10.$ [A.M. ≥ G.M.]

$\therefore$ Least value of $3x + 4y = 10$ which occurs when all numbers are equal i.e., $\frac{3x}{2} = \frac{4y}{3}$

i.e., $9x = 8y$.

12. (a) R is obviously true.

$\frac{1 + 2 + 2^2.... + 2^{n-1}}{n} > (1.2.2^2....2^{n-1})^{1/n}$

[A.M. > G.M.]

$\Rightarrow 2^n - 1 > n\cdot\left(2^{\frac{n(n-1)}{2}}\right)^{1/5} = 2^{(n-1)2}$

$\Rightarrow 2^n > 1 + n\cdot 2^{(n-1)/2}$

For $n = 1$. LHS = 2, RHS = 1 + 1 = 2

$\therefore$ $2^n > 1 + n.2^{(n-1)/2}$ is true for $n > 1$.

13. (a) When a, b, c are in A.P., then $b - a = c - b$

$\Rightarrow 2b = a + c$

So, Assertion is true.

Again, when a, b, c are in A.P., then

$\therefore 10^a, 10^b, 10^c$ are in G.P.

if $\frac{10^b}{10^a} = \frac{10^c}{10^b}$ i.e. if $10^{b-a} = 10^{c-b}$

i.e. if $b - a = c - b$ i.e. if $2b = a + c$

14. (a) a, b, c are in G.P. iff $\frac{b}{a} = \frac{c}{b}$
i.e. iff $b^2 = ac$. So assertion is true
The given quadratic in Reason has real roots
$\Rightarrow$ discriminant of this equation ≥ 0
$\Rightarrow \{-2(ab + bc)\}^2 - 4(a^2 + b^2)(b^2 + c^2) \geq 0$
$\Rightarrow -(b^4 + a^2c^2 + 2ab^2c) \geq 0$
$\Rightarrow (b^2 - ac)^2 \geq 0$
$\Rightarrow b^2 - ac = 0$
($\because$ square of a real number cannot be negative)
$\Rightarrow a, b, c$ are in G.P.

15. (a) a, b, c are in A.P.
$\Rightarrow -a, -b, -c$ are in A.P.
$\Rightarrow s - a, s - b, s - c$ are in A.P.
So, Assertion is correct.
Again $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in A.P.
$\Leftrightarrow 2\cot\frac{B}{2} = \cot\frac{A}{2} + \cot\frac{C}{2}$

$$\Leftrightarrow 2\sqrt{\frac{a(s-b)}{(s-a)(s-c)}}$$

$$= \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$\Rightarrow 2(s - b) = (s - a) + (s - c)$
(Multiplying both sides by

$$\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$\Rightarrow s - a, s - b, s - c$ are in A.P.
$\Rightarrow a, b, c$ are in A.P.
So Reason is true and the Assertion is correct explanation for the Reason.

16. (b) a, b, c are in H.P.
$\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.
$\Rightarrow \frac{2}{a} = \frac{1}{a} + \frac{1}{c} \Rightarrow b = \frac{2ac}{a+c}$
$\therefore$ Assertion is true.
Again $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in A.P.
$\Rightarrow a, b, c$ are in A.P.
$\Rightarrow 2b = a + c$

$\therefore s = \frac{a+b+c}{2} = \frac{2b+b}{2}$ and hence

$s - b = \frac{3b}{2} - b = \frac{b}{2}$

Now $\cot\frac{C}{2} = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$

$$= \sqrt{\frac{\frac{3b}{2}(s-c)}{(s-a)\left(\frac{3b}{2} - b\right)}}$$

$$= \sqrt{\frac{3b(s-c)}{(s-a)b}} = 3\sqrt{\frac{b(s-c)}{(s-a)(3b)}}$$

$$= 3\sqrt{\frac{\frac{b}{2}(s-c)}{(s-a)\left(\frac{3b}{2}\right)}}$$

$$= 3\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = 3\tan\frac{A}{2}$$

So, Reason is true
But Assertion is not a correct explanation for Reason.
Alternatively, for the truth of reason, you may show that $\cot\frac{A}{2}\cot\frac{C}{2} = 3$

17. (d) When $x = 2$, $2^x - 5 = 2^2 - 5 = -1$ and logarithm of a negative number is not defined. So, the Assertion cannot be valid. However, if $\log_{10}2, \log_{10}(2^x - 1), \log_{10}(2^x + 3)$ are in A.P. then $2\log_{10}(2^x - 1) = \log_{10}2 + \log_{10}(2^x + 3)$
$\Rightarrow \log_{10}(2^x - 1)2^2 = \log_{10}\{2(2^x + 3)\}$
$\Rightarrow (2^x - 1)^2 = 2(2^x + 3)$
$\Rightarrow 2^{2x} - 2 \times 2^x + 1 = 2 \times 2^x + 6$
$\Rightarrow 2^{2x} - 4.2^x - 5 = 0$

$\Rightarrow 2^x = \frac{4 \pm \sqrt{16 + 20}}{2} \; \frac{4 \pm 6}{2}$

$\Rightarrow 2^x = 5, -1$
But $2^x = -1$ is not possible, therefore, $2^x = 5$
$\Rightarrow x = \log_2 5$. So, reason is true.

18. (c) $S_n = 1 + 3 + 7 + 13 + \ldots\ldots\ldots\ldots + t_n$
$S_n = 1 + 3 + 7 + \ldots\ldots + t_{n-1} + t_n$
....
....
$0 = 1 + (2 + 4 + 6 + \ldots\ldots$ to $(n - 1)$ term$) - t_n$

$\therefore\ t_n = 1 + \frac{n-1}{2}\{4 + (n-1-1)2\}$

$= 1 + (n-1)n$

$t_n = n^2 - (n+1)n$

$\therefore\ S_n = \sum n^2 - \sum n + \sum 1$

$= \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + n$

$= \frac{n}{6}[2n^2 + 3n + 1 - 3n - 3 + 6]$

$= \frac{n}{6}(2n^2 + 4) = \frac{n}{3}(n^2 + 2)$ $\therefore$ A is correct

In R, for $n = -\frac{1}{2}$, $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ is the geometric mean of a and b.

$\therefore$ R is false

19. (d) $t_p = (t_{p+1} + t_{p+2} + \ldots\ldots\ldots \text{ to } \infty)$

$\Rightarrow 1.\ r^{p-1} = (r^p + r^p + 1 + \ldots\ldots\ldots\ldots \text{to } \infty)$

$\Rightarrow r^{p-1} = \frac{r^p}{1-r}$

$\Rightarrow 1 - r = r \Rightarrow r = \frac{1}{2}$

$\therefore$ series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ to ∞

$\therefore$ series is possible hence assertion is false.

R is correct.

20. (d) 111...........1 (upto 91 terms) =

$10^{90} + 10^{89} + 10^{88} + \ldots. + 10 + 10°$

$= 1.\frac{10^{91} - 1}{(10-1)}$ = divisible by 9

$\therefore$ the given number is not prime. $\therefore$ A is false

$\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P.

$\Rightarrow \frac{b+c-a}{a}, 2\frac{c+a-b}{b}, 2\frac{a+b-c}{c} \rightarrow$ A.P.

$\Rightarrow \frac{a+b+c}{a}, \frac{a+b+c}{a}, \frac{a+b+c}{a}, \rightarrow$ A.P.

$\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \rightarrow$ A.P. $\therefore$ R is true

21. (a) $x = \frac{(a+b) \pm \sqrt{(a+b)^2 - 4ab}}{2}$

$= \left(\frac{a+b}{2}\right) \pm \sqrt{\frac{(a+b)^2}{4} - ab}$

$= A \pm \sqrt{A^2 - G^2}$

$= A \pm \sqrt{(A+G)(A-G)}$

$\therefore$ A and R are both correct and R is the correct explanation of A.

22. (c) Required sum $= 5^3 + 6^3 + 7^3 + \ldots\ldots.. + 21^3$

$= (1^3+2^3+3^3+\ldots\ldots\ldots+21^3) - (1^3+2^3+3^3+4^3)$

$\left(\frac{21 \times 22}{2}\right)^2 - \left(\frac{4 \times 5}{2}\right)^2$

$= (231 - 10)(231 + 10) = 53261$

$\therefore$ A is true

If $\frac{a-b}{b-c} = \frac{a}{c}$, then a, b, c are in H.P.

$\therefore$ R is false.

23. (b) $465 = \frac{n}{2}\{6 + (n-1)4\}$

$\Rightarrow 465 = 3n + 2n^2 - 2n$

$\Rightarrow 2n^2 + n - 465 = 0$

$\Rightarrow 2n^2 + 31n - 30n - 465 = 0$

$\Rightarrow n(2n + 31) - 15(2n + 31) = 0$

$\Rightarrow (n - 15)(2n + 31) = 0$ $\therefore$ $n = 15$

$\Rightarrow A$ is correct

Sum of integers from 1 to 100 which are not divisible by 3 or 5.

$= (1 + 2 + \ldots\ldots\ldots + 100) - (3 + 6 + 9 + 12 + \ldots\ldots, 99)$

$-(5+10+15+\ldots..+100)+(15+30+\ldots.,90)$

$= \frac{100 \times 101}{2} - \frac{33}{2}(6 + 32 \times 3) - \frac{20}{2}$

$(10 + 19 \times 5) + \frac{6}{2}(30 + 5 \times 15)$

$= 5050 - 1683 - 1050 + 315$

$= 5365 - 2733 = 2632$

$\therefore$ R is true

But R is not the correct explanation of A.

24. (d) Required sum $= 13 + 17 + 21 + \ldots\ldots.. + 97$

$= \frac{22}{2}\{26 + 21 \times 4\} = 22 \times 55 = 1210$

$\therefore$ A is false

For R, $t_4 = ar^3 = 3$

$t_1.\ t_2.\ t_3.\ t_4.\ t_5.\ t_6.\ t_7$

$= a.\ ar.\ ar^2.\ ar^3\ldots\ldots.\ ar^6 = a^7.\ r(1 + 2 + ..+ 6)$

$= a^7.\ r^{\frac{7 \times 6}{2}} = a^7.\ r^{21} = (ar^3)^7 = 3^7$

$\therefore$ R is true

25. (d) $a_1 + a_{24} = a_5 + a_{20} = a_{10} + a_{15}$

So, $3(a_1 + a_{24}) = 225$

$a_1 + a_{24} = 75$

$\text{sum} = \frac{24}{2}[a + l] = 12 \times 75 = 900$

Then Assertion is not true but Reason is true by defi nition of A.P.

26. (a) Let a, ar, ar^2 are the sides of a triangle proved by taking the intersection of the inequalities $a > 0$, $ar > 0$, $ar^2 > 0$, $a + ar > ar^2$, $ar + ar^2 > a$, $ar^2 + a > ar$

By $(r^2 - r + 1) > 0$

$\frac{\sqrt{5} - 1}{2} < r < \frac{\sqrt{5} + 1}{2}$

Assertion is true and follow from Reason.

27. (d) $A = 5, G = 4, H = H$

$AH = G^2$

$5H = 16$

$H = \frac{16}{5}$

Assertion is not true but Reason is true.

28. (a) If $4, G_1, G_2, G_3, \frac{1}{4}$ are in G.P.

$r = \left(\frac{\frac{1}{4}}{4}\right)^{\frac{1}{3+1}} = \left(\frac{1}{16}\right)^{\frac{1}{4}} = \frac{1}{2}$

$G_1 = 2, G_2 = 1, G_3 = 1$

G.M. of first and last term is also equal to 1. Middle G.M. is 1.

So Assertion and Reason both are correct and Assertion follow from Reason.

29. (b) Clearly shows Assertion and Reason both are correct but Assertion does not follow from Reason.

30. (d) x, y, z are in G.P.

The $\ln x$, $\ln y$, $\ln z$ are in A.P. (Adding one in each term)

Now, $(1 + \ln x), (1 + \ln y), (1 + \ln z)$ are also in A.P.

So $\frac{1}{1 + \ln x}, \frac{1}{1 + \ln y}, \frac{1}{1 + \ln z}$ are in H.P.

Assertion is not true but Reason is correct.

31. (d) First common term between them is 23.

d_1(common difference of first A.P.) = 4

d_2(common difference of second A.P.) = 7

Let n be number of common terms between two A.P.'s

L.C.M. of d_1 and $d_2 = 28$

Thus $a + (n - 1)d \leq 86$

$\Rightarrow 23 + (n - 1)28 \leq 86$

MENTAL PREPARATION TEST I

1. If $a = 2$, $d = 2$ and $n = 50$. Find the last term.

Ans. 100

2. Find the $(n - 3)$th term of 5, 11, 17,.................

Ans. $6n - 19$

3. Which term of the progression 27, 24, 21, 18,......... is zero ?

Ans. $n = 10$ or 10th term.

4. If the mth term of an AP be $(1/n)$ and its nth term be $(1/m)$ then show that its (mn)th term is 1.

5. The 4th term of an AP is 14 and its 10th term is 32. Find its 7th term.

Ans. 7th term = 23.

6. Seven times of the seventh term of an arithmetic series is equal to eleven times of its eleventh term. Find the eighteenth term of the series.

Ans. 0

7. The nth term of the A.P. is $19 - 5n$. Find its 35th term.

Ans. $T_{35} = -156$

Ans: Thus, the sum of first 5 terms as well as the sum of first 20 terms is − 25.

8. Find the sum of all odd integers from 1 to 1001.

Ans. 251001.

9. Find the value of x, when $1 + 6 + 11 + ... + x = 148$

Ans. $x = 36$

10. The sum of four terms of an A.P. is 24, their product is 945.
Find the terms.

Ans. 3,5,7,9 or 9,7,5,3

11. The sum of n–terms of an A.P. is $2n + 3n^2$.
Find its rth term

Ans: $Tr = 6r - 1$.

12. The sum of 8th terms and 19th terms of an A.P. are 64 and 361 respectively, then fi nd the sum of n terms.

Ans: $S_n = n^2$

13. In an A.P., first term is 12, common difference is 4 and sum of n terms is 132, find the value of n.

Ans: $n = 6$ [$\because n \neq -11$]

14. If the sum of first n, $2n$, $3n$ terms of an AP be S_1, S_2, S_3 respectively, then prove that $S_3 = 3(S_2 - S_1)$.

15. The sum of three numbers in A.P. is − 3 and their product is 8. Find the numbers.

Ans: − 4, − 1, 2 or 2, − 1, − 4

16. Show that the sum of n arithmetic means between two given numbers is n times of AM of these numbers.

17. If interior angles of a polygon are in A.P., whose shortest angle is 88° and common difference is 10°. Find the number of sides.

Ans: $n = 5$

18. If a, b, c are in AP, show that: $(b + c)$, $(c + a)$ and $(a + b)$ are in AP.

MENTAL PREPARATION TEST II

1. Calculate the third term from the end of the series $\frac{2}{27}, \frac{2}{9}, \frac{2}{3}$, ...162

Ans: 18

2. Second term of a GP in 2 and its 5th term is 16, find the series and its 8th term.

Ans: 128

3. Find the 10th and nth terms of the G.P. 5, 25, 125,..

Ans: 10th term = 5^{10} and $a_n = 5^n$

4. Which term of the GP 2, 8, 32, .. up to n terms is 131072

Ans: $n = 9$

5. In a GP, the 3rd term is 24 and 6th term is 192. Find the 10th term.

Ans: 3072

6. Find the 9th and nth terms of the GP 3, 6, 12, 24,........

Ans: 768 and nth term = $3 \times 2\ (n - 1)$

7. Which term of the G.P. 5, 10, 20, 40,... is 5120 ?

Ans: 11th term

8. The first term of a GP is 1. The sum of its third and fifth terms is 90. Find the common ratio of the G.P. Ans: ± 3

9. The 4th, 7th and 10th terms of a GP are a, b, c respectively. Prove that $b^2 = ac$.

10. Find the sum of the infinite geometric series $\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots \infty\right)$

Ans: 3 /2

11. The sum of first two terms of an infinite geometric series is 15 and each term of the series is equal to the sum of all the terms following it. Find the series.

Ans: $10 + 5 + \frac{5}{2} + \frac{5}{4} + \ldots \infty$

12. The sum of an infinite geometric series is 8. If its second term is 2, find its common ratio.

Ans: 1 / 2

13. The sum of an infinite geometric series is 15 and the sum of the squares of these terms is 45. Find the series.

Ans: $5 + \frac{10}{3} + \frac{20}{9} + \frac{40}{27} + \ldots \infty$

14. The 2nd term of a G.P. is 2 and its sum up to infinity is 8. Find the common ratio and first term.

Ans: first term = 4, C.R. = 1/2

15. The product of three consecutive terms of a GP is 216, their sum is 19. Find the terms.

Ans: 4, 6, 9 or 9, 6, 4

16. If the number of terms in GP are even and if the sum of all the terms is 5 times the sum of the numbers at odd place, then fi nd the common ratio.

Ans: Required common ratio = 4

17. Find a GP for which the sum of first two terms is – 4 and the fifth term is 4 times the third term.

Ans: 4, – 8, 16, – 32

18. The sum of first three terms of a GP is $\frac{13}{12}$ and their product is – 1. Find these numbers.

Ans: $\left\{\frac{3}{4}, -1, \frac{4}{3}\right\}$ or $\left\{\frac{4}{3}, -1, \frac{3}{4}\right\}$

19. Insert two numbers between 3 and 81 so that the resulting sequence is GP.

Ans: $G_1 = 9$, $G_2 = 27$

20. The sum of two numbers is 6 times their geometric means. Show that the numbers are in the ratio $(3 + 2\sqrt{2}) : (3 - 2\sqrt{2})$

21. Insert 3 geometric means between 2 and 32. Ans: 4, 8, 16 or – 4, 8, – 16

22. The $(m + n)$th and the $(m - n)$th terms of a GP are p and q respectively. Show that the mth and nth terms of the GP are $\sqrt{pq}$ and $p.(q / p)^{(m/2n)}$ respectively.

23. If the pth, qth and rth terms of a GP be a, b, c respectively, prove that $a^{(q-r)}. b^{(r-p)}. c^{(p-q)} = 1$.

24. If S be the sum, P the product and R the sum of the reciprocals of n terms in a GP, prove that $P^2 = \left(\frac{S}{R}\right)^n$

25. If a, b, c, d are in GP, prove that $(b - c)^2 + (c - a)^2 + (d - b)^2 = (a - d)^2$

26. Sum the series. 4 + .44 + .444 +........... to n terms.

Ans. $\frac{4}{81}\left\{9n - 1 + \frac{1}{10^n}\right\}$

TOPICWISE WARMUP TEST

1. If the sum of two extreme numbers of an A.P. with four terms is 8 and product of remaining two middle term is 15, then greatest number of the series will be:

[Roorkee – 1965]

(a) 5 (b) 7
(c) 9 (d) 11

2. Let the positive numbers a, b, c, d be in A.P., then abc, abd, acd, bcd are:

(a) Not in A.P./G.P./H.P
(b) In A.P.
(c) In G.P.
(d) In H.P.

[IIT Screening – 2001, NDA – 2007]

3. There are n A.Ms. between 1 and 31. If the ratio of 7th A.M. to (n – 1)th A.M. is 5: 9, then the value of n is:

(a) 14 (b) 15
(c) 16 (d) None of these

4. If A is one AM between two numbers a and b, and the sum of n AM's between them is S, then S / A depends on:

[CET (Pb.) – 1992]

(a) n, a, b (b) n, b
(c) n, a (d) n

5. If the sum of fi rst n natural numbers is 1/5 times the sum of their squares, then n equals:

[IIT – 1992]

(a) 5 (b) 6 (c) 7 (d) 8

6. If $a\left(\frac{1}{b} + \frac{1}{c}\right)$, $b\left(\frac{1}{c} + \frac{1}{a}\right)$ and $c\left(\frac{1}{a} + \frac{1}{b}\right)$ are in AP, then:

(a) a, b, c are in AP
(b) a, b, c are in HP
(c) $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in GP
(d) None of these

[DCE – 1997; Delhi (EEE) – 1998]

7. Three numbers are in AP such that their sum is 18 and sum of their squares is 158. The greatest number among them is:

[MP PET – 2006]

(a) 10 (b) 11
(c) 12 (d) None of these

8. If the ratio of the sum of first three terms and the sum of first six terms of a G.P. be 125 : 152, then the common ratio *r* is:

(a) 3 /5 (b) 5 / 3
(c) 2 /3 (d) 3 /2

9. If the nth term of geometric progression $5, -\frac{5}{2}, \frac{5}{4}, \frac{5}{8}$, ... is $\frac{5}{1024}$, then the value of *n* is

[Kerala (Engg.) 2002]

(a) 11 (b) 10
(c) 9 (d) 4

10. The value of 0.234 is

[MNR 1986; UPSEAT – 2000]

(a) $\frac{232}{990}$ (b) $\frac{232}{9990}$
(c) $\frac{232}{999}$ (d) $\frac{232}{9909}$

11. Sum of infinite number of terms in G.P. is 20 and sum of their square is 100. The common ratio of G.P. is:

[AIEEE – 2002]

(a) 5 (b) 3/5
(c) 8/5 (d) 1/5

12. 0.5737373....... =

[Karnataka CET – 2004]

(a) 284/497 (b) 284/495
(c) 568/990 (d) 567/990

13. The value of 0.037 where .037 stands for the number 0.037037037......... is:

[MP PET – 2004]

(a) 37/1000 (b) 1/27
(c) 1/37 (d) 37/999

14. If the arithmetic mean of two numbers be *A* and geometric mean be *G*, then the numbers will be:

[CET Karnataka – 1994]

(a) $A \pm (A^2 - G^2)$
(b) $\sqrt{A} \pm \sqrt{A^2 - G^2}$
(c) $A \pm \sqrt{(A+G)(A-G)}$
(d) $A \pm \frac{\sqrt{(A+G)(A-G)}}{2}$

15. If the product of three terms of G.P. is 512. If 8 added to first and 6 added to second term, so that number may be in A.P., then the numbers are:

[Roorkee 1964]

(a) 2, 4, 8 (b) 4, 8, 16
(c) 3, 6, 12 (d) None of these

16. If G_1 and G_2 are two geometric means and *A* the arithmetic mean inserted between two numbers, then the value of $\frac{G_1^2}{G_2} + \frac{G_2^2}{G_1}$ is

(a) $\frac{A}{2}$ (b) A
(c) 2A (d) None

17. The sum of three decreasing numbers in A.P. is 27.
If − 1, − 1, 3 are added to them respectively, the resulting series is in G.P. The numbers are:

[AMU – 1999]

(a) 5, 9, 13 (b) 15, 9, 3
(c) 13, 9, 5 (d) 17, 9, 1

18. If *p*, *q*, *r* are in one geometric progression and *a*, *b*, *c* in another geometric progression, then *cp*, *bq*, *ar* are in:

(a) A.P. (b) H.P.
(c) G.P. (d) None

[Roorkee 1998]

19. If *r* is one AM and *p*, *q* are two GM's between two given numbers, then $p^3 + q^3$ is equal to:

[IIT – 1997]

(a) $2pqr$ (b) $2p^2q^2r$
(c) $2pq/r$ (d) None of these

20. The value of $9^{1/3} \times 9^{1/9} \times 9^{1/27} \times$..........$\infty$ is:

(a) 9 (b) 1
(c) 3 (d) none

[MP PET 2006, NDA – 2007]

21. When $\frac{1}{a}+\frac{1}{c}+\frac{1}{c-b}+\frac{1}{c-b}=0$ and $b \neq a \neq c$, then a, b, c are:

[MPPET 2004]

(a) In H.P. (b) In G.P.
(c) In A.P. (d) None of these

22. If a, b, c, d are in HP then:

[IIT –70; PET (Raj.), 91]

(a) $ab > cd$ (b) $ac > bd$
(c) $ad > bc$ (d) None of these

23. a, b, c are first three terms of a GP. If HM of a and b is 12 and that of b and c is 36, then a equals:

[Roorkee – 1998]

(a) 24 (b) 8
(c) 72 (d) 1/3

24. If a, b, c are in HP, then

[PET (Raj.) – 1994]

(a) $a^2 + c^2 > b^2$ (b) $a^2 + c^2 > 2b^2$
(c) $a^2 + c^2 < 2b^2$ (d) $a^2 + c^2 = 2b^2$

25. Five numbers a, b, c, d, e are such that a, b, c are in A.P; b, c, d are in GP and c, d, e are in HP.

If $a = 2$, $e = 18$; then values of b, c, d are:

[IIT – 1976]

(a) 2, 6, 18 (b) 4, 6, 9
(c) 4, 6, 8 (d) – 2, – 6, 18

26. The sumof the series

$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+......+\frac{1}{\sqrt{n^2-1}+\sqrt{n^2}}$$ equals:

[AMU – 2002]

(a) $\frac{(2n+1)}{\sqrt{n}}$ (b) $\frac{\sqrt{n}+1}{\sqrt{n}+\sqrt{n-1}}$

(c) $\frac{\left(n+\sqrt{n^2-1}\right)}{2\sqrt{n}}$ (d) $n - 1$

27. The sum of 10 terms of the series. 7 +.77 +.777 +..... is:

(a) $\frac{7}{9}\left(89+\frac{1}{10^{10}}\right)$ (b) $\frac{7}{81}\left(89+\frac{1}{10^{10}}\right)$

(c) $\frac{7}{8}\left(189+\frac{1}{10^{9}}\right)$ (d) None

[Aligarh, 1983]

28. Certain numbers appear in both arithmetic progressions 17, 21, 25,... and 16, 21, 26,.... Find the sum of first hundred numbers appearing in both progressions.

29. Find the sum to n term of the series:

$$1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+......+\frac{2n-1}{2^{n-1}}$$

TOPICWISE WARMUP TEST: SOLUTION

1. (b) Let four numbers are $a - 3d$, $a - d$, $a + d$, $a + 3d$.

Now $(a - 3d) + (a + 3d) = 8 \Rightarrow a = 4$ and $(a - d)(a + d) = 15 \Rightarrow a^2 - d^2 = 15 \Rightarrow d = 1$

Thus required numbers are 1, 3, 5, 7.

Hence greatest number is 7.

2. (d) a, b, c, d are in A.P.

$\Rightarrow \frac{a}{abcd}, \frac{c}{abcd}, \frac{c}{abcd}, \frac{d}{abcd}$ are in A.P.

$\therefore \frac{1}{bcd}, \frac{1}{acd}, \frac{1}{abd}, \frac{1}{abc}$ are in A.P.

$\therefore$ bcd, acd, abd, abc are in H.P.

$\therefore$ In reverse order abc, abd, acd, bcd are in H.P.

3. (a) Suppose between 1and 31 there are n A.Ms.

$A_1, A_2, A_3, .., A_n$ .Then, $1, A_1, A_2, A_3, ..., A_n$, 31 are in A.P.,

whose first term is 1 and $(n + 2)$th term is 31. Let common difference is d. Then, $1 + (n + 2 - 1)\,d = 31$

$\Rightarrow (n + 1)d = 31-1 \Rightarrow d = \frac{30}{n+1}$

$\therefore \frac{7^{th}\text{ A.M}}{(n-1)^{th}\text{ A.M}} = \frac{5}{9}$

$\Rightarrow \frac{8^{th}\ term}{n^{th}\ term} = \frac{5}{9}$

$\Rightarrow \frac{1+7d}{1+(n-1)d} = \frac{5}{9}$

$\Rightarrow 5 + 5(n-1)d = 9 + 63d$
$\Rightarrow (5n - 5 - 63)\,d = 9 - 5 \Rightarrow (5n - 68)\,d = 4$

$\Rightarrow (5n - 68) \times \frac{30}{n+1} = 4$ [from Equation (1)]

$\Rightarrow 15\,(5n - 68) = 2(n + 1$

$\Rightarrow 75n - 2n = 2 + 1020$

$\Rightarrow 73n = 1022 \Rightarrow n = \frac{1022}{73}$

$\Rightarrow n = 14$ Ans

4. (d) $\frac{S}{A} = \frac{n(A)}{A} = n \Rightarrow$ it depends on n.

5. (c) As given $\sum n = \frac{1}{5}\sum n^2$

$\Rightarrow 5\,\frac{n(n+1)}{2} = \frac{n(n+1)(2n+1)}{6}$

$\Rightarrow n = 7$

6. (a) $\frac{ac+ab}{bc}, \frac{ab+bc}{ca}, \frac{bc+ca}{ab}$ are in A.P.

$\Rightarrow \frac{ab+bc+ca}{bc}, \frac{ab+bc+ca}{ca},$

$\frac{ab+bc+ca}{ab}$ are in A.P.

$\Rightarrow \frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in A.P

$\Rightarrow a, b, c$ are in A.P.

7. (a) Let the three numbers $(a - d)$, a and $(a + d)$ are in AP.
According to problem $(a - d) + a + (a + d) = 18$
$3a = 18$, $a = 6$...................(i) and $(a - d)^2 + a^2 + (a + d)^2 = 158$
$\Rightarrow 3a^2 + 2d^2 = 158 \Rightarrow 2d^2 = 158 - 3 x 36$
$\Rightarrow d^2 = 25 \Rightarrow d = \pm 5$
Hence the required numbers are 1,6,11
$\therefore$ Greatest number is 11.

8. (a) Here

$\frac{S_3}{S_6} = \frac{125}{152} \Rightarrow \frac{a(r^3-1)/(r-1)}{a(r^6-1)/(r-1)} = \frac{125}{152}$

$\Rightarrow (r^3 - 1)152 = 125\,(r^6 - 1) \Rightarrow r^3 = \frac{27}{125}$

$\Rightarrow r = \frac{3}{5}$

9. (a) $T_n = ar^{n-1} \Rightarrow \frac{5}{1024} = 5.\left(\frac{-1}{2}\right)^{n-1}$

$\Rightarrow \left(\frac{-1}{2}\right)^{10} = \left(\frac{-1}{2}\right)^{n-1}$

$\Rightarrow 10 = n - 1 \Rightarrow n = 11.$

10. (a) 0.234 = 0.2343434............
= 0.2 + 0.034 + 0.00034 + 0.0000034 +............

$0.2 + \frac{34}{1000} + \frac{34}{1000} + \frac{34}{1000} + ...\infty$

$= \frac{2}{10} + 34\left[\frac{1}{10^3} + \frac{1}{10^5} + \frac{1}{10^7} + ...\infty\right]$

$= \frac{2}{10} + 34\left[\frac{1/10^3}{1 - 1/1000}\right]$

$= \frac{2}{10} + 34 \times \frac{1}{1000} \times \frac{100}{99}$

$= \frac{2}{10} + \frac{34}{990} = \frac{232}{990}$ Ans

11. (b) $\frac{a}{1-r} = 20$...... (i) $\frac{a^2}{1-r^2} = 100$.........(ii)
From (i) and (ii), $\frac{a}{1-r} = 5$, [$\because a = 20\,(1 - r)$ by (i)]

$\Rightarrow \frac{20(1-r)}{1+r} = 5$

$\Rightarrow 5r = 3 \Rightarrow r = 3/5$

12. (c) Given series 0.5737373.......

= 0.5 + 0.073 + 0.00073 +..........

$= 0.5 + 73\left[\frac{1}{1000} + \frac{1}{100000} + ...\right]$

= 0.5 + 73

$\left[\frac{1/10000}{1 - \frac{1}{100}}\right] = 0.5 + \frac{73}{1000}\cdot\frac{100}{99} = \frac{5}{10} + \frac{73}{990}$

$= \frac{495 + 73}{990} = \frac{568}{990}$

13. (d) Given series 0.037037037..........
= 0.037 + 0.000037 + 0.0000000037 +.........

$= \frac{37}{10^3} + \frac{37}{10^6} + \frac{37}{10^9} + ...$

$= 37\left[\frac{1}{10^3} + \frac{1}{10^6} + \frac{1}{10^9} + \ldots\ldots\right]$

$= 37\left[\frac{1/10^3}{1 - 1/10^3}\right] = 37\left[\frac{1}{10^3} \cdot \frac{10^3}{999}\right] = \frac{37}{999}$

14. (c) A.M. $= \frac{a+b}{2}$ A and G.M. $= \sqrt{ab} = G$ n solving a and b are given by the values

$A \pm \sqrt{(A+G)(A-G)}$

Trick: Let the numbers be 1, 9. Then $A = 5$ and $G = 3$, Now put these values in options.

Here (c) $\Rightarrow 5 \pm \sqrt{8 \times 2}$ i.e. 9 and 1.

15. (b) Let three terms of a G.P. are $\frac{a}{r}, a, ar$

So $\frac{a}{r} \cdot a \cdot ar = 512 \Rightarrow a^3 = 83 \Rightarrow a = 8$

From second condition, we get $\frac{a}{r} + 8, a + 6, ar$ will be in A.P.

$\Rightarrow 2(a+6) = \frac{a}{r} + 8 + ar$

$\Rightarrow 28 = 8\left\{\frac{1}{r} + 1 + r\right\}$

$\Rightarrow \frac{1}{r} + r + 1 = \frac{7}{2} \Rightarrow 1/r + r - \frac{5}{2} = 0$

$\Rightarrow r^2 - \frac{5}{2}r + 1 \Rightarrow 2r^2 - 5r + 2 = 0$

$\Rightarrow (2r-1)(r-2) = 0$

$\Rightarrow r = \frac{1}{2}, r = 2\ (\because r > 1)$

$\Rightarrow r = 2$. Hence required numbers are 4, 8, 16.

Trick: Check for (a) 2 + 8, 4 + 6, 8 are not in A.P.

(b) 4 + 8, 8 + 6, 16 i.e. 12, 14, 16 are in A.P.

16. (c) Let the two numbers be p and q

$\therefore G_1 = p^{2/3} q^{1/3}, G_2 = p^{1/3} q^{2/3}$

$\therefore \frac{G_1^2}{G_2} + \frac{G_2^2}{G_1} = \frac{p^{2/3} q^{4/3}}{p^{2/3} q^{1/3}}$

$= p + q = 2 \times \left(\frac{p+q}{2}\right) = 2A.$

17. (d) Let the three terms of the series is $a + d, a, a - d$

$\therefore a + d + a + a - d = 27$

$\Rightarrow 3a = 27 \Rightarrow a = 9$

Now, $(a + d - 1), (a - 1), (a - d + 3)$ are in G.P.

$\Rightarrow (a-1)^2 = (a+d-1)(a-d+3)$

$\Rightarrow 64 = (8+d)(12-d)$

$\Rightarrow 64 = -d^2 + 4d + 96$

$\Rightarrow d^2 - 4d - 32 = 0$

$\Rightarrow d^2 - 8d + 4d - 32 = 0$

$\Rightarrow (d-8)(d+4) = 0, \therefore d = -4, 8$

Series is 5, 9, 13 (for $d = -4$) and 17, 9, 1 (for $d = 8$)

$\therefore$ Decreasing A.P. is 17, 9, 1

18. (c) As p, q, r are in G.P. $\therefore q^2 = pr$ (1)

and a, b, c are also in G.P. $\therefore b^2 = ac$ (2)

From (1) and (2), $q^2 b^2 = (pr)(ac)$

$\Rightarrow (bq)^2 = (cp).(ar).$

Hence cp, bq, ar are in G.P.

19. (a) Let given numbers be x and y. Then

$r = \frac{1}{2}(x+y)$ .... (1)

$\left.\begin{array}{l} p = (x^2y)^{1/2} \\ q = (xy^2)^{1/3} \end{array}\right] \Rightarrow pq = xy$

$\therefore p^3 + q^3 = x^2y + xy^2 = xy(x+y)$

$= pq(2r) = 2pqr$

20. (c) $9^{1/3} \times 9^{1/9} \times 9^{1/27} \times \ldots\ldots.\infty$

$9^{\frac{1}{3} + \frac{1}{9} + \frac{1}{27} \ldots\ldots\infty} = 9^{\frac{1/3}{1 - 1/3}} = 9^{\frac{1}{2}} = 3$

21. (a) We have $\frac{1}{a} + \frac{1}{c} + \frac{1}{a-b} + \frac{1}{c-b} = 0$

$\frac{1}{a} + \frac{1}{c-b} = \frac{1}{b-a} - \frac{1}{c}$

$\Rightarrow \frac{c-b+a}{a(c-b)} = \frac{c-b+a}{(b-a)c}$

$\Rightarrow ac - ab = bc - ac$

$\Rightarrow 2ac = ab + bc$

$\Rightarrow \frac{2ac}{a+c} = b$ i.e. a, b, c are in H.P.

22. (c) HM between a and $c = b$ and GM $= \sqrt{ac}$

Also HM between b and $d = c$ and GM $= \sqrt{bd}$

But GM > HM

$\therefore \sqrt{ac} > b$ and $\sqrt{bd} > c$

$\Rightarrow \sqrt{ac}\ \sqrt{bd} > bc \Rightarrow ad > bc$

23. (b) Let given three terms be br, b, b/r

$\therefore\ 12 = \dfrac{2(br)b}{br + b} = \dfrac{2br}{r+1}$ .... (1)

and $36 = \dfrac{2b(b/r)}{b + (b/r)} = \dfrac{2b}{r+1}$ ....(2)

(1) ÷ (2) $\Rightarrow r = 1/3$. Then from (2) b = 24 $\therefore$ $a = br = 8$

24. (b) AM > HM $\Rightarrow \dfrac{a+c}{2} > b$

$\Rightarrow \left(\dfrac{a+c}{2}\right)^2 > b^2$ Also

$\dfrac{a^2+b^2}{2} > \left(\dfrac{a+c}{2}\right)^2$

(1) and (2) $\Rightarrow a^2 + c^2 > 2b^2$

25. (b) $b = (2 + c) / 2$ (1)

$c^2 = bd$ (2)

$d = 36\,c/(c + 18)$ (3)

Eliminate d from (2) and (3) we get $c = \pm 6$

Now from (1) $b = 4, -2$ from (3) $d = 9, -18$

$\therefore\ b = 4, c = 6, d = 9$

26. (d) $\dfrac{1}{\sqrt{1}+\sqrt{2}} + \dfrac{1}{\sqrt{2}+\sqrt{3}} + \dfrac{1}{\sqrt{3}+\sqrt{4}}$

$+ \ldots + \dfrac{1}{\sqrt{n^2-1}+\sqrt{n^2}}$

Rationalization of D^r

$\therefore S = (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + \ldots + \left(\sqrt{n^2} - \sqrt{n^2} - 2\right)$

$S = n - 1.$

27. (b) Sum $= \dfrac{7}{9}$ [.9 + .99 + .999 +... 10 terms]

$= \dfrac{7}{9}\left[\left(1 - \dfrac{1}{10}\right) + \left(1 - \dfrac{1}{10^2}\right) + \left(1 - \dfrac{1}{10^3}\right) + \ldots\ldots 10\ terms\right]$

$= \dfrac{7}{9}\left[10 - \dfrac{1}{10}\left(\dfrac{1 - \dfrac{1}{10^{10}}}{1 - \dfrac{1}{10}}\right)\right]$

$= \dfrac{7}{18}\left[89 + \dfrac{1}{10^{10}}\right]$

28. Denoting the nth and mth terms of the two progressions by T_n and T'_m, we have $T_n = 17 + (n - 1).\,4 = 4n + 13$ and $T'_m = 16 + (m - 1).\,5 = 5m + 11$

For common terms, we must have

$T_n T'_m \Rightarrow 4n + 13 = 5m + 11$

$\Rightarrow 5m = 2(2n + 1)$

This shows that $2n + 1 = 5k$, $k = 1, 3, 5, \ldots$

Hence the common terms are given by

$T'_{2k} = 5.\,2k + 11 = 10k + 11$, $k = 1, 3, 5, \ldots$

$\therefore$ sum of the first 100 common terms = 21 + 41 + 61 +.... to 100 terms

$= \dfrac{100}{2}\,[2 \times 21 + (100 - 1)20] = 101100$

29. Let S_n denotes the sum of n terms of the series, then

$S_n = 1 + \dfrac{3}{2} + \dfrac{5}{4} + \dfrac{7}{8} + \ldots\ldots + \dfrac{2n-1}{2n-1}$

and

$\dfrac{1}{2}S_n = \dfrac{1}{2} + \dfrac{3}{4} + \dfrac{5}{8} + \ldots\ldots + 2n - 3/2n - 1 + 2n$

$\dfrac{1}{2}S_n = \dfrac{1}{2} + \dfrac{2}{3} + \dfrac{5}{8} + \ldots\ldots + \dfrac{2n-3}{2^{n-1}} + \dfrac{2n-1}{2n}$

$\therefore\ S_n = \dfrac{1}{2}S_n = 1 + \left(\dfrac{3}{2} - \dfrac{1}{2}\right) + \left(\dfrac{5}{4} - \dfrac{3}{4}\right) + \left(\dfrac{7}{8} - \dfrac{5}{8}\right) + \ldots\ldots + \left(\dfrac{2n-1}{2^{n-1}} - \dfrac{2n-3}{2^{n-1}}\right) - \dfrac{2n-1}{2^n}$

$\Rightarrow \dfrac{1}{2}S_n = 1 + \dfrac{2}{2} + \dfrac{2}{4} + \dfrac{2}{8} + \ldots + \dfrac{1}{2^{n-1}} - \dfrac{2n-1}{2^n}$

$\Rightarrow \dfrac{1}{2}S_n = 1 + 2\,\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \ldots + \dfrac{1}{2^{n-1}}\Big] - \dfrac{2n-1}{2^n}$

$\Rightarrow \dfrac{1}{2}S_n = 1 + 2\left[\dfrac{\dfrac{1}{2}\left\{1 - \left(\dfrac{1}{2}\right)^{n-1}\right\}}{1 - \dfrac{1}{2}}\right] - \dfrac{2n-1}{2^n}$

$\Rightarrow S_n = 2 + 4\left[1 - \dfrac{1}{2^{n-1}}\right] - \dfrac{2n-1}{2n}$

$\therefore\ S_n = 6 - \dfrac{1}{2^{n-3}} - \dfrac{2n-1}{2^{n-1}}.$

QUESTION BANK: SOLVE THESE TO MASTER

1. Let a_1, a_2, ... a_{10} be in A.P. and h_1, h_2,......... h_{10} be in H.P. If $a_1 = h_1 = 2$ and $a_{10} = h_{10} = 3$, then a_4h_7 is:

(a) 2 (b) 3
(c) 5 (d) 6

2. If the sum of the first $2n$ terms of the A.P. 2, 5, 8......., is equal to the sum of the n terms of the A.P. 57, 59, 61......, then n equals

(a) 10 (b) 12
(c) 11 (d) 13

3. Let two numbers have arithmetic mean 9 and geometric mean 4. Then these numbers are the roots of the quadratic equation:

(a) $x^2 - 18x - 16 = 0$
(b) $x^2 - 18x + 16 = 0$
(c) $x^2 + 18x - 16 = 0$
(d) $x^2 + 18x + 16 = 0$

4. Which term of the series 17, 21, 25............417 and 16, 21, 26..........466 is common

(a) 20 (b) 19
(c) 21 (d) 18

5. The sum of 11 terms of an A.P. whose middle term is 30, is:

(a) 320 (b) 330
(c) 340 (d) 350

6. The minimum number of terms from the beginning of the series $20 + 22\frac{2}{3} + 25\frac{1}{3}$ +........., so that the sum may exceed 1568, is:

(a) 25 (b) 27
(c) 28 (d) 29

7. The maximum sum of the A.P. 40, 38, 36, 34,...... is:

(a) 390 (b) 420
(c) 460 (d) 210

8. Between two numbers whose sum is $2\frac{1}{6}$, an even number of arithmetic means are inserted. If the sum of these means exceeds their number by unity, then the number of means are:

(a) 12 (b) 10
(c) 8 (d) none of these

9. Sum of the three arithmetic means between 3 and 19 is:

(a) 26 (b) 33
(c) 28 (d) 34

10. If sum of the infi nite G.P. $p, 1, 1/p, 1/p^2$,...... is 9/2, then value of p is:

(a) 2 (b) 9/2
(c) 3 (d) None of these

11. Three numbers form an increasing G.P. If the middle number is doubled, then the new numbers are in A.P. The common ratio of the G.P. is:

(a) $2 - \sqrt{3}$ (b) $2 + \sqrt{3}$
(c) $\sqrt{3} - 2$ (d) $3 + \sqrt{2}$

12. If the first two terms of an H.P. are 2/5 and 12/13, respectively. Then the third term is:

(a) $\frac{15}{17}$ (b) -3
(c) $\frac{49}{13}$ (d) None of these

13. If the first two terms of harmonic progression be $\frac{1}{2}, \frac{1}{4}$, then the harmonic mean of first four numbers is:

(a) 5 (b) 1/5
(c) 10 (d) 1/10

14. If the sum of the first n terms of a series be $5n^2 + 2n$, then its second term is:

(a) 16 (b) 17
(c) $\frac{27}{14}$ (d) $\frac{56}{15}$

15. In a geometric progression consisting of positive terms, each term equals the sum of the next two terms. Then the common ratio of this progression equals:

(a) $\frac{1}{2}$ (b) $\sqrt{2}$

(c) $\frac{1}{2}(\sqrt{5}-1)$ (d) $\frac{1}{2}-\sqrt{5}$

16. In an A.P., the pth term is q and the $(p+q)$th term is 0. Then the qth term is:

(a) $-p$ (b) p

(c) $p+q$ (d) $p-q$

17. Let S_n denote the sum to n terms of an A.P. Let $Sn = n^2 p$, $Sm = m^2p$ where m, n, p are + veintegers and $m \neq n$. Then $Sp =$

(a) $\frac{(m+n)p^2}{mn}$ (b) $\frac{mnp^2}{m+n}$

(c) p^3 (d) $\frac{(p+m+n)^2}{p\,mn}$

18. Given two numbers a and b. Let A denote the single A.M.between these and S denote the sum of n A.M.'s between them. Then S/A depends upon:

(a) n, a, b (b) n, a

(c) n, b (d) n

19. Let a, b, c be A.P., then $\frac{1}{c}, \frac{b}{ac}, \frac{1}{a}$ are in:

(a) A.P. (b) G.P.

(c) H.P. (d) None

20. If the roots of the equation $(b-c)x^2 + (c-a)x + (a-b) = 0$ are equal, then a, b, c are in:

(a) A.P. (b) G.P.

(c) H.P. (d) None

21. If x, $2x+2$, $3x+3$ are in G.P., then the fourth term is:

(a) 27 (b) -27 (c) $\frac{27}{2}$ (d) $-\frac{27}{2}$

22. $3 + 3\alpha + 3\alpha^2 + ...\infty$ is equal to $\frac{45}{8}$, $\alpha > 0$

(a) 15/23 (b) 7/15

(c) 7/8 (d) 15/7

23. The number of terms common to two A.P.'s 3, 7, 11,........, 407 and 2, 9, 16,..........., 709 is:

(a) 14 (b) 21

(c) 28 (d) None

24. The sum of first n terms of two A.P. are $3n + 8$, $7n + 15$. Then the ratio of their twelfth term is:

(a) $\frac{7}{16}$ (b) $\frac{8}{15}$

(c) $\frac{4}{9}$ (d) $\frac{3}{7}$

25. If a, b, c are in A.P., then which one of the following is not true:

(a) $\frac{K}{a}, \frac{K}{b}, \frac{K}{c}$ are in H.P.

(b) $a+K, b+K, c+K$ are in A.P.

(c) Ka, Kb, Kc are in A.P.

(d) a^2, b^2, c^2 are in A.P.

26. If $1^2 + 2^2 + ... + n^2 = 1015$, then value of n is:

(a) 12 (b) 14

(c) 15 (d) None

27. Let α be the A.M. and β, γ be two G.M.'s between two positive numbers the value of $\frac{\beta^3+\gamma^3}{\alpha\beta\gamma}$ is:

(a) 1 (b) 2

(c) 0 (d) 3

ANSSWERS

Lecture–1: Unsolved Objective Problems (Identical Problems For Practice): For Improving Speed With Accuracy

1. (b)	9. (c)	17. (b)	25. (b)	4. (d)	12. (c)	20. (b)	28. (a)
2. (b)	10. (b)	18. (b)	26. (a)	5. (d)	13. (d)	21. (d)	29. (c)
3. (c)	11. (b)	19. (a)	27. (d)	6. (a)	14. (c)	22. (c)	
				7. (a)	15. (c)	23. (c)	
				8. (d)	16. (a)	24. (d)	

Lecture–1: Work Sheet: To Check Preparation Level

1.	(b)	5.	(c)	9.	(a)	13.	(d)
2.	(b)	6.	(c)	10.	(d)	14.	(c)
3.	(c)	7.	(b)	11.	(c)	15.	(b)
4.	(a)	8.	(d)	12.	(a)		

Lecture–2: Unsolved Objective Problems (Identical Problems For Practice): For Improving Speed With Accuracy

1.	(a)	7.	(c)	13.	(c)	19.	(d)
2.	(b)	8.	(b)	14.	(d)	20.	(c)
3.	(a)	9.	(c)	15.	(b)	21.	(c)
4.	(c)	10.	(d)	16.	(d)		
5.	(a)	11.	(a)	17.	(d)		
6.	(a)	12.	(b)	18.	(b)		

Lecture–2: Work Sheet: To Check Preparation Level

1.	(a)	5.	(b)	9.	(c)	13.	(c)
2.	(a)	6.	(a)	10.	(d)	14.	(d)
3.	(c)	7.	(c)	11.	(b)	15.	(a)
4.	(c)	8.	(c)	12.	(a)		

Lecture–3: Unsolved Objective Problems (Identical Problems For Practice): For Improving Speed With Accuracy

1.	(d)	11.	(d)	21.	(b)	31.	(c)
2.	(c)	12.	(c)	22.	(a)	32.	(a)
3.	(d)	13.	(b)	23.	(c)	33.	(a)
4.	(a)	14.	(c)	24.	(b)	34.	(c)
5.	(c)	15.	(b)	25.	(b)	35.	(d)
6.	(c)	16.	(a)	26.	(d)	36.	(d)
7.	(c)	17.	(a)	27.	(d)	37.	(a)
8.	(d)	18.	(c)	28.	(d)	38.	(b)
9.	(c)	19.	(a)	29.	(b)		
10.	(c)	20.	(d)	30.	(d)		

Lecture–3: Work Sheet: To Check Preparation Level

1.	(a)	5.	(b)	9.	(a)	13.	(a)
2.	(c)	6.	(a)	10.	(a)	14.	(b)
3.	(b)	7.	(a)	11.	(b)	15.	(c)
4.	(b)	8.	(b)	12.	(d)		

Lecture–4: Unsolved Objective Problems (Identical Problems for Practice): For Improving Speed with Accuracy

1.	(d)	4.	(b)	7.	(b)	10.	(a)
2.	(d)	5.	(c)	8.	(d)		
3.	(d)	6.	(c)	9.	(b)		

Lecture–4: Work Sheet: To Check Preparation Level

1.	(d)	4.	(b)	7.	(d)	10.	(b)
2.	(c)	5.	(a)	8.	(b)	11.	(c)
3.	(c)	6.	(c)	9.	(a)	12.	(d)

Lecture–5: Mental Preparation Test I

1. 100 2. $6n - 19$
3. $n = 10$ or 10th term.
5. 7th term = 23.
6. 0 7. $T_{35} = -156$
8. 251001. 9. $x = 36$
10. 3,5,7,9 or 9,7,5,3
11. $T_r = 6r - 1$.
12. $S_n = n^2$
13. $n = 6$ [$\therefore$ $n \neq -11$]
15. $-4, -1, 2$ or $2, -1, -4$ 17. $n = 5$

Lecture–5: Mental Preparation Test II

1. 18 2. 128
3. 10th term = 5^{10} and $a_n = 5^n 4$. $n = 9$
5. 3072 6. 768 and nth term = $3 \times 2^{(n-1)}$
7. 11th term 8. ± 3
10. 3 /2
11. $10 + 5 + \frac{5}{2} + \frac{5}{4}$ ...... ∞
12. 1 / 2
13. $5 + \frac{10}{3} + \frac{20}{9} + \frac{40}{27} + ... \infty$ 14. First term = 4, C.R. = 1/2
15. 4, 6, 9 or 9, 6, 4 16. Required common ratio = 4
17. $4, -8, 16, -32$
18. $\left\{\frac{3}{4}, -1, \frac{4}{3}\right\}$ or $\left\{\frac{4}{3}, -1\ \frac{3}{4}\right\}$,
19. $G_1 = 9, G_2 = 27$ 21. 4, 8, 16 or -4, 8,

QUESTION BANK: SOLVE THESE TO MASTER

1.	(d)	8.	(c)	15.	(c)	22.	(b)
2.	(c)	9.	(b)	16.	(b)	23.	(a)
3.	(b)	10.	(c)	17.	(c)	24.	(a)
4.	(c)	11.	(b)	18.	(d)	25.	(d)
5.	(c)	12.	(b)	19.	(a)	26.	(b)
6.	(b)	13.	(b)	20.	(a)	27.	(c)
7.	(d)	14.	(b)	21.	(d)		