BENDING OF A QUASI-ELLIPSOIDAL SHELL WITH SPECIAL REFERENCE TO RIGID AIRSHIPS[1]

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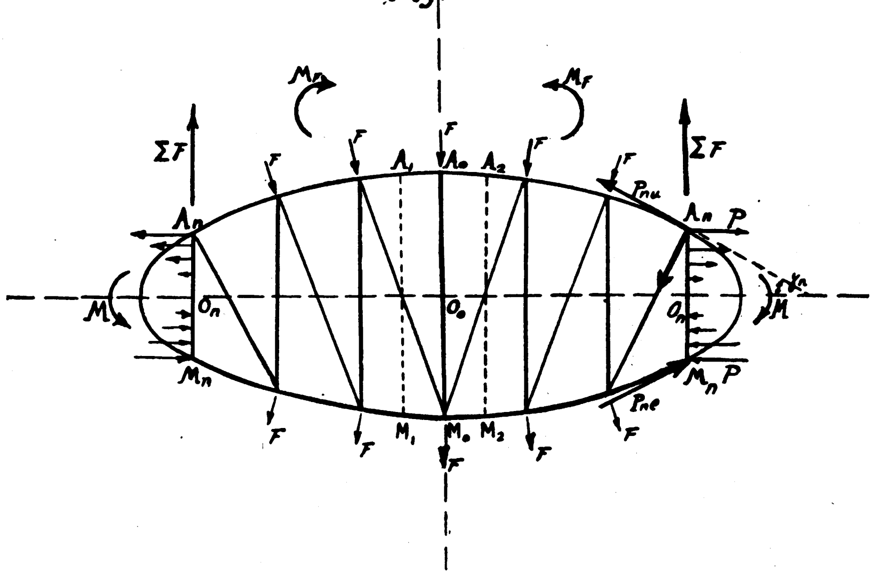
When a tubular girder with straight axis and with a thin, strongly curved shell is exposed to bending, it will behave quite differently from a plain cylindrical tube and the stress distribution will vary from that which might be expected according to the ordinary theory of bending.

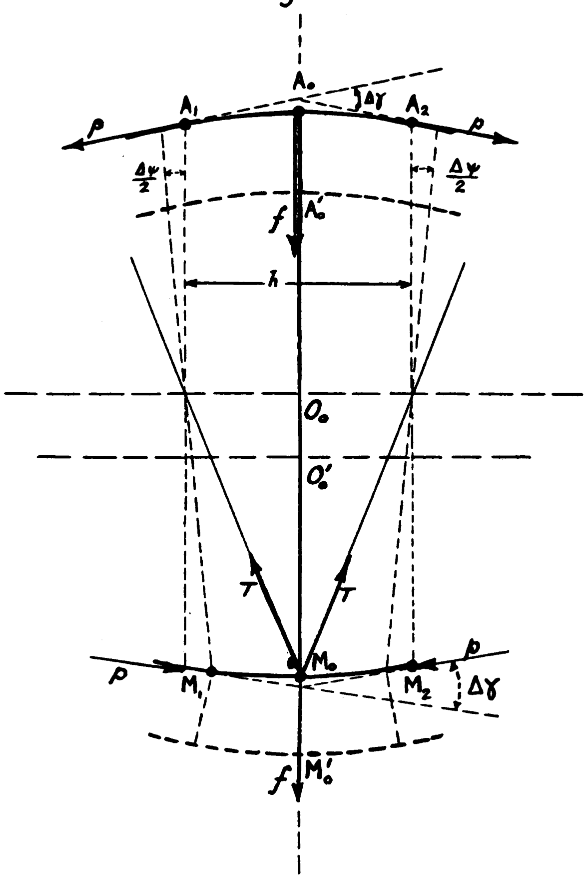
Fig. 1.



The attention of the author was drawn to this problem in a study of the behavior of curved tubes subject to bending.[2] A similar action, although different in effect, occurs in all tubular girders where the shell is curved in the plane of bending even if the axis is straight, a fact which has not, to the author's knowledge, been previously explained or discussed. In a simple ideal form the problem is that of an ellipsoidal shell, but in principle the solution applies also to plane girders with curved flanges. In the present paper the method is illustrated by application to a rigid airship.

Consider a rigid airship as shown diagrammatically in figure 1. The structure consists of a number of longitudinal girders which perform the function of flanges, while transverse circular or polygonal frames, stiffened by transverse radial and chord wires, and a system of diagonal or shear wires perform the function of the web. The structure is not unlike a lattice girder, since the transverse frames play the part of vertical posts and the shear wires take the place of the diagonal tie-rods. In order to simplify, it is assumed that the structure is perfectly symmetrical about the axis and the longitudinals are placed at the apices of a regular polygon with the top girder A and the bottom girder M in the center line. For the sake

Fig. 2.



of clearness the longitudinals are omitted in figure 1 except the top and bottom girders, and the shear wires are represented symbolically by diagonal lines. Vertical lines represent the transverse frames. Terminal couples M are assumed to act at certain stations A_nM_n near the ends, produced by a system of horizontal forces, which act on each girder and are proportional to their distance y from the horizontal plane through the axis.

In a straight cylindrical solid bar or tube the transverse stresses are of the second order so long as the strains are within the elastic limit and are usually neglected, but in the case here under consideration they should be taken into account, first because the curvature of the longitudinals, which constitute the flanges, causes the transverse stresses to attain appreciable magnitude and, second, because the resistance of the shear wires, as also of the transverse frames which act as stiffening bulkheads, is relatively small in airships and permit certain peculiar deformations to take place. In a curved tube of cylindrical section, the transverse stresses above and below the axis counteract and balance each other, but in a straight tube with curved shell, we have the remarkable condition that the resultant of the transverse stresses both above and below the axis act in the same direction.

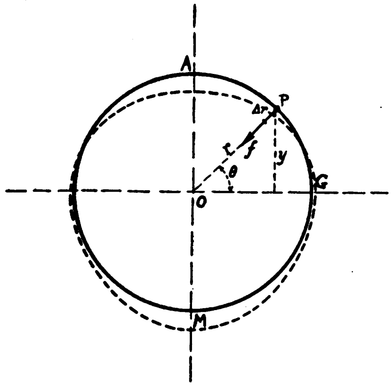
In figure 2 is shown diagrammatically a section amidships of an airship, one frame space in length with a transverse frame in the middle. We assume it to be exposed to a hogging moment, so that the A girder is under a tensile stress p, and due to the curvature the stress forces have a downward directed resultant which we denote by f. Similarly, the lower or M-girder, being under compression, will be subject to an f-force, which likewise acts downward. In fact, all the longitudinals will be subject to f-forces which have a downward component and thus the whole section becomes loaded with a vertical force, F, which causes it to move down until F is counterbalanced by the vertical component of the tension T in the diagonals.

Similar F forces act on every frame and the load so produced is through the diagonals transmitted to the terminal frames where they are counterbalanced by the vertical components P_{nu} and P_{nl} of the longitudinals acting at A_n and M_n, respectively. If it were not for the diagonal at A_n, this point as well as M_n would tend to move upward. The restraining force is equal to ΣF, the sum of the F-forces on one-half of the length of the ship.

We may, therefore, conceive the internal stress system to be equivalent to a distributed load on the longitudinals, everywhere acting in meridional planes, inward directed above the neutral axis and outward below that axis. This load produces vertical shear forces, which, being supported at the terminal stations by the upward forces ΣF, produce internal stress couples M_P acting in a direction contrary to the external couples M. Yet, at any station the resulting bending moment is necessarily M and the bending deflections of the axis may be calculated on this basis without any regard to M_P, but in determining the shearing deflections the f-forces must be taken into account. It is a curious fact that shearing is here produced by pure couples and this shearing, although entirely due to concealed internal reactions, has a very real and tangible effect in that it gives to the axis a flexure contrary to and superposed on that produced by M. It may be, in fact, so great as to overpower the deflections due to bending.

Shearing flexure will take place even if the transverse frames are absolutely rigid, provided only the diagonals are elastic, but if, as in an airship, also the frames are elastic and of relatively small stiffness, the latter will suffer a peculiar deformation due to the fact that the upper and lower longitudinals are under greater stress and are subject to greater f-forces than those on the sides. Hence the radial or meridional deflection of the longitudinals at and near the top and bottom may be much greater than that of the longitudinals farther from the vertical center plane and nearer to the neutral plane. The consequence is that the top of the frame section will flatten out and the bottom will take on a sharp bulge, giving to the contour a pear-shaped form as indicated in figure 3. The upper girders straighten and the lower girders bulge out, being thus enabled to shirk their duty in opposing the external couples. In order to establish equilibrium these couples must turn through a greater angle than if this action did not exist. The top and bottom girders may not now be the most strained. Longitudinals on the sides nearer the neutral plane may be subject to greater stresses, which may be in excess even of those which the A and M girders should have according to the ordinary theory of bending.

Fig. 3.



It is seen that we have to deal with two distinctly different deformations besides that due to simple bending. One is a bodily downward movement of the frames due to shearing deflections of the whole ship. The other is a deformation of the transverse frames due to unequal loading. The magnitude of the former deformation depends on shearing stiffness in the longitudinal plane, the latter on stiffness in the transverse plane.

The effects here pointed out are most marked in short airships of full form and show the necessity for a good stiffness of the transverse frames.

In non-rigids, where the stiffening element provided by transverse frames is absent, the effects are probably much greater than in rigid airships.

Appendix.—The displacement of a longitudinal normal to the surface is denoted by Δn, the radial transverse displacement by Δr and the transverse tangential displacement by Δt. Let γ be the inclination of the tangent to the axis at any point P, then $\Delta r =$

$\Delta n \cos \gamma$. The relation between Δr and Δt is found under the assumption of inextensibility of the transverse frame.

Referring to the midship frame section, figure 2, each of the longitudinals is subject to a stress p, which causes it to stretch or contract as in an ordinary girder, producing an angular deflection $\Delta\psi$ of the boundary planes relative to each other, but at the same time, due to the curvature of the girders, the f-forces will cause the longitudinals to move down to the points A_0' and M_0' whereby the strain is reduced. The movement of the transverse frame consists of two parts; first, its form is changed from circular to pear-shaped without any displacement of the center; second, it moves bodily downward, without change of form, a distance Δy_s, as allowed by the tension in the diagonals. Due to the latter deflection the longitudinals above the axis straighten out and those below the axis bulge down, thus relieving the stresses, but the bending couples at once turn the bounding planes through an additional angle $\Delta\psi_s$ until equilibrium is established, when the stress distribution will be the same as before the shearing deflection took place.

It is first assumed that the diagonals are rigid while the frames are elastic and we consider in particular the deformation of the midship frame in its own plane, when exposed to a hogging moment.

Let

h = frame spacing

ρ = radius of curvature of the longitudinals

r = radius of the transverse frame

y = vertical ordinate at any point P of the frame referred to a horizontal plane through the axis

s = length of a longitudinal, within the frame space considered. Amidships we reckon $s = h$

$\Delta\gamma = \frac{s}{\rho}$ = angle subtended by s. For the midship frame $\Delta\gamma = \frac{h}{\rho}$ approximately.

$\Delta\psi$ = angular deflection of a transverse plane due to bending, i.e., the angle between the two bounding transverse planes of the section in the strained condition.

θ = angle between the radius OP and the horizontal radius OG. Figure 3.

The elongation due to longitudinal strains, assuming that the bounding planes remain plane, is $y\Delta\psi$ and that due to the radial displacement of P is approximately $\Delta r\Delta\gamma$. Thus the total elongation of A_1A_2 is $\epsilon = y\Delta\psi + \Delta r\Delta\gamma$ and this expression holds for any girder both above and below the axis.

Following Ritz's method the unknown deflection Δr was expressed in a series of trigonometrical or exponential terms with a corresponding number of constants.

It was found that the expression which best fulfilled the limiting conditions was:

$$\Delta r = \zeta r \sin\theta - \beta r \sin^3\theta \qquad (1)$$

where ζ is a coefficient to be determined so that the downward parallel displacement due to shear is zero, and β, which characterizes the deformation of the frame, is determined so as to make the internal elastic work a minimum, not including the work done on the diagonals.

The assumed inextensibility of the frame leads to:

$$\Delta t = r(\zeta - \beta)\cos\theta + \frac{1}{3} r\beta \cos^3\theta \qquad (2)$$

which for $\theta = 0$ gives the downward displacement of the center:

$$\Delta t_o = r\zeta - \frac{2}{3} r\beta$$

Since this is to be zero we must have: $\zeta = \frac{2}{3}\beta$.

Substituting and dividing the total elongation by $s = h = \rho\Delta\gamma$ we find the unit elongation

$$e = r\left(\left(\frac{\Delta\psi}{h} + \frac{2}{3}\frac{\beta}{\rho}\right)\sin\theta - \frac{\beta}{\rho}\sin^3\theta\right) \tag{3}$$

and the longitudinal stress

$$p = Er\left(\left(\frac{\Delta\psi}{h} + \frac{2}{3}\frac{\beta}{\rho}\right)\sin\theta - \frac{\beta}{\rho}\sin^3\theta\right). \tag{4}$$

Let ω be the area of the longitudinals per unit length of the contour of the transverse section, imagining their area to be distributed uniformly along the circumference. We can then determine the f-forces which result from the curvature of the longitudinals. For the midship frame space we have:

$$f = p\omega r d\theta d\gamma = \frac{\omega s r^2 E}{\rho}\left(\left(\frac{\Delta\psi}{h} + \frac{2}{3}\frac{\beta}{\rho}\right)\sin\theta - \frac{\beta}{\rho}\sin^3\theta\right)d\theta \tag{5}$$

so that the total vertical force acting on this section is:

$$F = \int_0^{2\pi} f\sin\theta = \frac{hEI}{\rho r}\left(\frac{\Delta\psi}{h} - \frac{1}{12}\frac{\beta}{\rho}\right) \tag{6}$$

where $I = \pi\omega r^3$ is the moment of inertia of the ship about the neutral axis.

This downward force will be resisted by the shear wires so that by equating F to the sum of the vertical components of the tensions T in the shear wires all round the section we can find these tensions.

We may now proceed to determine the elastic work done by the internal stresses in the various parts of the structure.

This work includes first the straining of the longitudinals under the stress p, which work we denote by W_1, second, W_2, the work done in deformation of the frame ring against its inherent stiffness and third, W_3, that done against the tension of the transverse wires. The work done against the tension of the shear wires, W_s is considered separately as it is independent of β. We have thus the internal work: $W_i = W_1 + W_2 + W_3$.

This equation takes the form:

$$W_i = \frac{\beta^2}{144\rho^2}(5\delta + 144(\mu + \lambda)) - \frac{1}{12}\frac{\delta}{h\rho}\beta\Delta\psi + \frac{\delta}{2h^2}\overline{\Delta\psi}^2. \tag{7}$$

where μ is quantity which depends on the transverse stiffness of the frame itself and λ depends on the transverse wires in the plane of the frame ring. The quantity δ is equal to hEI which expresses the stiffness of the whole section multiplied by the frame space.

In order to determine β it is argued that it must be such as to make the internal elastic work a minimum. Hence the angular deflection $\Delta\psi$ is regarded as constant and β, which determines the displacement of the longitudinals, is supposed to vary and is adjusted so as to make:

$$\frac{\partial W_i}{\partial\beta} = 0. \tag{8}$$

This is an equation of the first degree in β and gives

$$\beta = \frac{6\rho\delta}{h(5\delta + 144(\mu + \lambda))}\Delta\psi. \tag{9}$$

We next substitute the value of β so determined in the expression for the bending moment M:

$$M = \int_0^{2\pi} p y \omega r d\theta = EI\left(\frac{\Delta\psi}{h} - \frac{1}{12}\frac{\beta}{\rho}\right) \tag{10}$$

and find $\Delta\psi$ in terms of M

$$\Delta\psi = \frac{h^2 M}{\delta}\frac{(10\delta + 288(\mu + \lambda))}{(9\delta + 288(\mu + \lambda))} = A\,\frac{hM}{EI} \tag{11}$$

where A is a factor which is always greater than one, showing that the angular deflection of a girder of this type will be greater than that found by the ordinary formula. It appears that A is not much greater than unity in rigid airships if the frames are well stiffened, but it will probably attain higher values in non-rigids.

If we now substitute the value of $\Delta\psi$ in (9) we obtain

$$\beta = \frac{12\rho h M}{9\delta + 288(\mu + \lambda)} \tag{12}$$

which gives β in terms of the bending moment.

The solution here obtained satisfies the condition that the internal and external work shall be equal, $W_i = \frac{1}{2} M\Delta\psi$.

It remains to show that W_s, the work done against the shear wires, has its equivalent in work done by M turning through the additional angle $\Delta\psi_s$.

Assume that the frames are rigid but the longitudinals and diagonals elastic, then the shearing deflection calculated by the method ordinarily used in rigid airships is found:

$$\Delta y_s = 2Kr\rho\delta\,\frac{\Delta\psi_1}{h} \tag{13}$$

where K is a factor depending on the resistance to shear and $\Delta\psi_1$ is the angular deflection caused by the strains in the longitudinals as in ordinary bending.

$$\Delta\psi_1 = \frac{Mh}{EI} = \frac{Mh^2}{\delta} = \frac{\Delta\psi}{A}. \tag{14}$$

The work done is: $W_s = Kh^2M^2$ which is equal to $\frac{1}{2} F\Delta y_s$ and is seen to be independent of the deformation of the frame.

As explained above, the shearing deflection causes the bounding plane to rotate through an angle $\Delta\psi_s$ taking up, so to speak, the slack in the longitudinals. It can be shown geometrically that

$$\Delta\psi_s = 2K\delta\,\Delta\psi_1 = 2K\delta\,\frac{\Delta\psi}{A}\,. \tag{15}$$

Hence the external work is $\frac{1}{2} M\Delta\psi_s = K\delta M\Delta\psi_1 = Kh^2M^2 = W_s$ and we have complete equality between the external and the total internal work:

$$W_e = W_i + W_s = \frac{1}{2} M\Delta\psi + \frac{1}{2} M\Delta\psi_s = \frac{1}{2} M\Delta\psi_1(A + 2K\delta) \tag{16}$$

The total angle through which M turns is seen to be greater than that in ordinary bending in the ratio $(A + 2K\delta)$.

The formulas for other frame sections where the hull is tapering gives analogous formulas differing from the above only in the occurrence of cos γ and sec γ in the expressions for β and $\Delta\psi$.

The total downward shear deflection of the amidship frame and the total rotation of the end frames in the ship are the sum of the respective deflections of all the frame sections on one side of amidships.

[1] A more detailed account will probably appear in *Trans. Institution of Naval Architects*, London, 1927.

[2] *Proc. Nat. Acad. Sci.*, **12**, No. 6, June, 1926.

INFLUENCE OF IRON CONTENT ON MORTAR STRENGTH

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Those most intimately concerned with the use of concrete for bridges, highways or other work freely admit that the most important problem is the variation in mortar strength developed by mixtures of different kinds of sands. This variation in actual test conditions lies between 100 and 600 pounds per square inch in the 28-day tension test. In the attempt to find the cause for some of this variation and, therefore, its control the authors have been investigating several different elements of sand and mortar structure, chemical content, etc., which might lead to or influence mortar strength. One of the elements which has come under consideration is that of the iron found in the native sand. This element is usually present as an oxide, due to the weathering or disintegration of some of the component materials found in the sand aggregate. The iron may be present as a disintegration of several of the common component minerals or due to the leaching of ferruginous material from other sources into the sand. It should be noticed that the makeup of the sand aggregate with regard to iron is quite different under these two methods of allocating the iron in the aggregate. It is, however, difficult to differentiate between iron derived from one source as contrasted with that derived from the other. The material will, therefore, be treated as though the iron content came from the same source.

After some experiment a delicate test for determining the presence and the approximate amount of iron was developed in this laboratory.[1] Through the use of this test it is possible to arrange sands on the basis of the amount of iron present in them and to contrast this with the strength developed by these sands in the ordinary mortar test. While the amount of the iron detected in this manner is extremely small in total amount the data show that its effect on strength is significant for the 284 native Maine sands involved. The percentages of the sand's iron content are arranged on a geometric basis.